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Contents

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Data dependence of solutions for Fredholm-Volterra integral equations in $L^2[a, b]$

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Abstract. In this paper we study the continuous dependence and the differentiability with respect to the parameter $\lambda \in [\lambda_1, \lambda_2]$ of the solution operator $S : [\lambda_1, \lambda_2] \to L^2[a, b]$ for a mixed Fredholm-Volterra type integral equation. The main tool is the fiber Picard operators theorem (see [9], [8], [11], [3] and [2]).

1 Introduction

We study the solution operator of the equations

$$y(x) = f(x) + \int_a^x K_1(x, s, y(s); \lambda) ds + \int_a^b K_2(x, s, y(s); \lambda) ds, \tag{1}$$

and

$$y(x) = f(x) + \int_a^x K_1(x, s, y(g_1(s)); \lambda) ds + \int_a^b K_2(x, s, y(g_2(s)); \lambda) ds, \tag{2}$$

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where $\lambda \in [\lambda_1, \lambda_2]$ is a real parameter. The existence and uniqueness of the solutions of such equations in $C[a,b]$ was studied by many authors [5], [6], [1], we recall the results from [1]. If the functions K_i and f satisfy the conditions under which the existence and uniqueness (in $C[a,b]$) is guaranteed then the differentiability of the functions K_i with respect to the parameter guarantees the differentiability of the solution. This property was proved in [1] using the following fiber Picard operator theorem:

Theorem 1 (Fiber Picard operator's) [9] *Let* (V, d) *be a generalized metric space with* $d(v_1, v_2) \in \mathbb{R}^p_+$*, and* (W, ρ) *a complete generalized metric space with* $\rho(w_1, w_2) \in \mathbb{R}^m_+$*. Let* $A : V \times W \to V \times W$ *be a continuous operator. If we suppose that:*

a) $A(v, w) = (B(v), C(v, w))$ *for all* $v \in V$ *and* $w \in W$*;*

b) *the operator* $B : V \to V$ *is a weakly Picard operator;*

c) *there exists a matrix* $Q \in M_m(\mathbb{R}_+)$ *convergent to zero, such that the operator* $C(v, \cdot) : W \to W$ *is a* Q *contraction for all* $v \in V$*,*

then the operator A *is a weakly Picard operator. Moreover, if* B *is a Picard operator, then the operator* A *is a Picard operator.*

In this paper we use the same technique to give some modified Carathéodory type conditions which guarantee the continuity and differentiability with respect to the parameter of the solution operator. We study these equations both in bounded and unbounded intervals.

2 Fredholm-Volterra equations on a compact interval

We need the following lemma.

Lemma 1 *If* $I = [a, b]$, $k \in L^2(I^2)$ *and the function* $u \in L^2(I)$ *has nonnegative values then the inequality*

$$u(t) \leq \alpha + \int_a^b k(t,s)u(s)ds, \ \text{a.e.}\, t \in I, \tag{3}$$

where $\alpha > 0$ *and* $\|k\|_{L^2(I^2)} < 1$*, implies*

$$\|u\|_{L^2(I)} \leq \frac{\alpha\sqrt{2(b-a)}}{1 - \|k\|_{L^2(I^2)}}.$$

Proof. Consider the sets

$$A = \{t \in I \,|\, u(t) \leq \alpha\} \text{ and } B = \{t \in I \,|\, u(t) > \alpha\}.$$

These sets are measurable because u is measurable. If $t \in B$, from the Cauchy-Buniakovski inequality we have

$$(u(t) - \alpha)^2 \leq \left(\int_a^b k(t,s)u(s)ds \right)^2 \leq \int_a^b k^2(t,s)ds \cdot \int_a^b u^2(s)ds.$$

By integrating on B we deduce

$$\begin{aligned}
\int_B u^2(s)ds \quad &\leq 2\alpha \textstyle\int_B u(t)dt - \alpha^2 \cdot \mu(B) + \int_B \int_a^b k^2(t,s)dsdt \cdot \|u\|^2_{L^2(I)} \\
&\leq 2\alpha \textstyle\int_B u(t)dt - \alpha^2 \cdot \mu(B) + \int_a^b \int_a^b k^2(t,s)dsdt \cdot \|u\|^2_{L^2(I)} \\
&\leq 2\alpha \sqrt{\mu(B) \textstyle\int_a^b u^2(t)dt} - \alpha^2 \cdot \mu(B) + \|k\|^2_{L^2(I^2)} \cdot \|u\|^2_{L^2(I)}.
\end{aligned}$$

By the other hand $u^2(t) \leq \alpha^2$, for $t \in A$, so

$$\int_A u^2(t)dt \leq \alpha^2 \cdot \mu(A).$$

From these inequalities we have

$$\left(\|u\|_{L^2(I)} - \alpha\sqrt{\mu(B)} \right)^2 \leq \alpha^2 \mu(A) + \|k\|^2_{L^2(I^2)} \cdot \|u\|^2_{L^2(I)},$$

so

$$\|u\|_{L^2(I)} - \alpha\sqrt{\mu(B)} \leq \sqrt{\alpha^2 \mu(A) + \|k\|^2_{L^2(I^2)} \cdot \|u\|^2_{L^2(I)}}.$$

From

$$\sqrt{\alpha^2 \mu(A) + \|k\|^2_{L^2(I^2)} \cdot \|u\|^2_{L^2(I)}} \leq \alpha\sqrt{\mu(A)} + \|k\|_{L^2(I^2)} \cdot \|u\|_{L^2(I)}$$

and

$$\sqrt{\mu(A)} + \sqrt{\mu(B)} \leq \sqrt{2(b-a)}$$

we deduce the desired inequality. □

Remark 1 *By using both the Minkovski and the Cauchy-Buniakovski inequality we can prove a sharpened version:*

$$\|u\|_{L^2(I)} \leq \frac{\alpha\sqrt{(b-a)}}{1 - \|k\|_{L^2(I^2)}}.$$

Indeed (3) *implies*

$$\|u\|_{L^2(I)} \le \left\| \alpha + \sqrt{\int_a^b k^2(t,s)ds \cdot \int_a^b u^2(s)ds} \right\|_{L^2(I)} \le \alpha\sqrt{b-a} + \|k\|_{L^2(I^2)} \cdot \|u\|_{L^2(I)}.$$

By an analogous reasoning we have the following property: If $k \in L^2(I^2)$, $g \in L^2(I)$ *and the function* $u \in L^2(I)$ *has nonnegative values then the inequality*

$$u(t) \le g(t) + \int_a^b k(t,s)u(s)ds, \ \ a.e.\, t \in I,$$

where $\|k\|_{L^2(I^2)} < 1$, *implies*

$$\|u\|_{L^2(I)} \le \frac{\|g\|_{L^2(I)}}{1 - \|k\|_{L^2(I^2)}}.$$

These inequalities are in fact Gronwall type inequalities and they can be proved also by using the abstract Gronwall lemma from [10].

We use the usual definition of differentiability for functions with values in a Banach space and a generalized Weierstrass type theorem. To avoid any misunderstanding we recall this definition and we prove the above mentioned theorem.

Definition 1 *If* $S : [\lambda_1, \lambda_2] \to L^2(I)$ *is a continuous function then we call it differentiable at the point* λ, *if exists* $z_\lambda \in L^2(I)$ *such that*

$$\lim_{\overline{\lambda} \to \lambda} \frac{\|S(\overline{\lambda}) - S(\lambda) - (\overline{\lambda} - \lambda)z_\lambda\|_{L^2(I)}}{\overline{\lambda} - \lambda} = 0.$$

For the simplicity we identify the function $t \to tz_\lambda$ *(the differential) with the element* z_λ.

Theorem 2 *If the sequence* $y_n(\cdot, \lambda) \in L^2(I)$, $n \ge 0$ *converges in* $L^2(I)$ *to* $y^*(\cdot, \lambda)$ *for all* $\lambda \in [\lambda_1, \lambda_2]$, *the operators* $S_n : [\lambda_1, \lambda_2] \to L^2(I)$ *defined by* $S_n(\lambda)(t) = y_n(t, \lambda)$, $\forall\, t \in I$, $\forall\, \lambda \in [\lambda_1, \lambda_2]$ *are differentiable, their differentials converge in* $L^2(I)$ *to* $z^*(\cdot, \lambda)$, *and these convergencies are uniform with respect to* λ, *then the operator* $S : [\lambda_1, \lambda_2] \to L^2(I)$ *defined by* $S(\lambda)(t) = y^*(t, \lambda)$, $\forall\, t \in I$, $\forall\, \lambda \in [\lambda_1, \lambda_2]$ *is differentiable and* $z^*(\cdot, \lambda)$ *is its differential in* λ.

Proof. Due to the mean theorem for functions with values in a Banach space (see [4] 2-5) we have the following inequality:

$$\frac{\|[y_m(\cdot,\overline{\lambda})-y_n(\cdot,\overline{\lambda})]-[y_m(\cdot,\lambda)-y_n(\cdot,\lambda)]\|_{L^2(I)}}{\overline{\lambda}-\lambda}$$
$$\leq \sup_{\lambda\in[\lambda_1,\lambda_2]}\|z_m(\cdot,\lambda)-z_n(\cdot,\lambda)\|_{L^2(I)},$$

where $z_m(\cdot,\lambda)$ is the differential of $S_n(\lambda)(\cdot)$.

The condition $\|z_n(\cdot,\lambda)-z^*(\cdot,\lambda)\|_{L^2(I)}\to 0$ uniform with respect to λ, implies that for every $\varepsilon>0$ exists $n_1(\varepsilon)\in\mathbb{N}$ such that

$$\frac{\|[y^*(\cdot,\overline{\lambda})-y^*(\cdot,\lambda)]-[y_n(\cdot,\overline{\lambda})-y_n(\cdot,\lambda)]\|_{L^2(I)}}{\overline{\lambda}-\lambda}\leq\frac{\varepsilon}{3},\ \forall n\geq n_1(\varepsilon). \tag{4}$$

By the other hand for all $\varepsilon>0$ exists $n_2(\varepsilon)\in\mathbb{N}$ such that

$$\|z_n(\cdot,\lambda)-z^*(\cdot,\lambda)\|_{L^2(I)}\leq\frac{\varepsilon}{3},\ \forall n\geq n_2(\varepsilon) \tag{5}$$

and there exists $\delta>0$ such that

$$\frac{\|y_n(\cdot,\overline{\lambda})-y_n(\cdot,\lambda)-(\overline{\lambda}-\lambda)z_n(\cdot,\lambda)\|_{L^2(I)}}{\overline{\lambda}-\lambda}\leq\frac{\varepsilon}{3}, \tag{6}$$

if $|\overline{\lambda}-\lambda|<\delta$. From these relations we deduce

$$\lim_{\overline{\lambda}\to\lambda}\frac{\|y^*(\cdot,\overline{\lambda})-y^*(\cdot,\lambda)-(\overline{\lambda}-\lambda)z^*(\cdot,\lambda)\|_{L^2(I)}}{\overline{\lambda}-\lambda}=0,$$

so S is differentiable in λ and its differential is $z^*(\cdot,\lambda)$. □

For equation (1) we have the following theorem (some parts of this theorem are classical):

Theorem 3 *If*

I. (*Carathéodory type conditions*) *the functions* $K_i : I^2\times[\lambda_1,\lambda_2]\times\mathbb{R}\to\mathbb{R}$, $i\in\{1,2\}$ *with* $I=[a,b]$ *satisfy the conditions*

a) $K_i(\cdot,\cdot,\lambda,u)$ *is measurable on* $I^2=[a,b]\times[a,b]$ *for all* $u\in\mathbb{R}$ *and* $\lambda\in[\lambda_1,\lambda_2]$;

b) $K_i(x,s,\lambda,\cdot)$ *is continuous on* $\mathbb{R}$ *for almost every pairs* $(x,s)\in I^2$ *and every* $\lambda\in[\lambda_1,\lambda_2]$.

II. (*space invariance*) $f \in L^2(I)$, $K_i(\cdot,\cdot,\lambda,0) \in L^2(I^2)$ *for all* $\lambda \in [\lambda_1,\lambda_2]$, $i \in \{1,2\}$ *and exists* $M_1 > 0$ *such that* $\|K_i(\cdot,\cdot,\lambda,0)\|_{L^2(I^2)} < M_1$ *for all* $\lambda \in [\lambda_1,\lambda_2]$;

III. (*Lipschitz type conditions*) *exists* $k_i \in L^2(I^2)$, $i \in \{1,2\}$, *such that*

$$|K_i(t,s,\lambda,u) - K_i(t,s,\lambda,v)| \leq k_i(t,s)|u-v|, \ \forall\, t,s \in I, \lambda \in [\lambda_1,\lambda_2], u,v \in \mathbb{R};$$

IV. (*contraction condition*)

$$L^2 := \int_a^b\int_a^t (k_1(t,s)+k_2(t,s))^2 dsdt + \int_a^b\int_t^b k_2^2(t,s)dsdt < 1 \tag{7}$$

then

1. *for all* $\lambda \in [\lambda_1,\lambda_2]$ *exists a unique solution* $y^*(\cdot,\lambda) \in L^2(I)$ *of the equation (1);*

2. *the sequence of successive approximation*

$$y_{n+1}(x) = f(x) + \int_a^x K_1(x,s,\lambda,y_n(s))ds + \int_a^b K_2(x,s,\lambda,y_n(s))ds$$

   converges in $L^2(I)$ *to* $y^*(\cdot,\lambda)$, *for all* $y_0(\cdot) \in L^2(I)$ *and every* $\lambda \in [\lambda_1,\lambda_2]$*;*

3. *for every* $n \in \mathbb{N}$ *we have*

$$\|y_n(\cdot) - y^*(\cdot,\lambda)\|_{L^2(I)} \leq \frac{L^n}{1-L}\|y_1(\cdot) - y_0(\cdot)\|_{L^2(I)}.$$

Moreover if

I.c) *the functions* $(K_i(x,s,\cdot,u))_{x,s\in I, u\in\mathbb{R}}$ *are equally continuous,*

then the operator $S : [\lambda_1,\lambda_2] \to L^2(I)$ *defined by* $S(\lambda)(x) = y^*(x,\lambda)$, $\forall\, x \in I$, $\forall\, \lambda \in [\lambda_1,\lambda_2]$ *is continuous.*

If instead of I.b), I.c) *and* III. *we have the conditions*

I.b') $K_i(x,s,\lambda,\cdot)$ *is in* $C^1(\mathbb{R})$ *for all* $\lambda \in [\lambda_1,\lambda_2]$, *a.e.* $(x,s) \in I^2$, *and there exist* $k_i \in L^2(I^2)$, $i \in \{1,2\}$, *such that*

$$\left|\frac{\partial K_i(t,s,\lambda,u)}{\partial u}\right| \leq k_i(t,s), \ \forall\, t,s \in I, \forall\, \lambda \in [\lambda_1,\lambda_2], \forall\, u \in \mathbb{R};$$

I.c') $K_i(x,s,\cdot,u)$ *is in* $C^1[\lambda_1,\lambda_2]$ *for all* $u \in \mathbb{R}$, *a.e.* $(x,s) \in I^2$, *the partial derivatives satisfy condition* I., $\frac{\partial K_i}{\partial \lambda}(\cdot,\cdot,\lambda,u) \in L^2(I^2), i \in \{1,2\}$ *and there exists* $M_2 > 0$ *such that*

$$\left\| \frac{\partial K_i}{\partial \lambda}(\cdot,\cdot,\lambda,u) \right\|_{L^2(I^2)} < M_2,\ \forall \lambda \in [\lambda_1,\lambda_2],\ \forall u \in \mathbb{R},$$

then the operator S *is differentiable.*

Proof. First we prove that for a fixed λ the operator $T : L^2(I) \to L^2(I)$ defined by

$$T[y](x) = f(x) + \int_a^x K_1(x,s,\lambda,y(s))ds + \int_a^b K_2(x,s,\lambda,y(s))ds$$

is a contraction. From the Lipschitz condition we have

$$\int_a^b K_2(t,s,\lambda,y(s))ds \le \int_a^b K_2(t,s,\lambda,0) + k_2(t,s)|y(s)|ds.$$

Due to Minkovski and Cauchy-Buniakovski inequality we deduce

$$\int_a^b \left(\int_a^b K_2(t,s,\lambda,y(s))ds \right)^2 dt$$

$$\le \left(\sqrt{b-a} \|K_2(\cdot,\cdot,\lambda,0)\|_{L^2(I^2)} + \sqrt{b-a}\|k_2\|_{L^2(I^2)} \cdot \|y\|_{L^2(I)} \right)^2 < \infty.$$

Analogously

$$\int_a^b \left(\int_a^t K_1(t,s,\lambda,y(s))ds \right)^2 dt < \infty,$$

so because of $f \in L^2(I)$ we have $T[y] \in L^2(I)$. On the other hand

$$\begin{aligned}
|T[y_1](t) - T[y_2](t)| &\le \int_a^t |K_1(t,s,\lambda,y_1(s)) - K_1(t,s,\lambda,y_2(s))|ds \\
&\quad + \int_a^b |K_2(t,s,\lambda,y_1(s)) - K_2(t,s,\lambda,y_2(s))|ds \\
&\le \int_a^t k_1(t,s)|y_1(s) - y_2(s)|ds + \int_a^b k_2(t,s)|y_1(s) - y_2(s)|ds \\
&= \int_a^b (\overline{k}_1(t,s) + k_2(t,s))|y_1(s) - y_2(s)|ds,
\end{aligned}$$

where $\overline{k}_1(t,s) = \begin{cases} k_1(t,s), & t \geq s \\ 0, & t < s \end{cases}$. From the Cauchy-Buniakovski inequality we obtain

$$\|T[y_1](\cdot) - T[y_2](\cdot)\|^2_{L^2(I)} \leq L^2 \cdot \|y_1(\cdot) - y_2(\cdot)\|^2_{L^2(I)},$$

where L^2 is defined by (7). Hence T is a contraction and from the contractions principle we have the conclusions.

If we have condition I.c), then for every $\varepsilon > 0$ there exists $\varepsilon_1 = \frac{(1-L)\varepsilon}{2(b-a)\sqrt{2(b-a)}}$ and $\delta > 0$ such that for $|\lambda - \overline{\lambda}| < \delta$ we have

$$|K_i(t,s,\lambda,u) - K_i(t,s,\overline{\lambda},u)| \leq \varepsilon_1,$$

for all $u \in \mathbb{R}$ and a.e. $(t,s) \in I^2$. If y^*_λ and $y^*_{\overline{\lambda}}$ are the corresponding unique solutions to λ, and $\overline{\lambda}$, then

$$\begin{aligned} |y^*_\lambda(t) - y^*_{\overline{\lambda}}(t)| &\leq \int_a^t |K_1(t,s,\lambda,y^*_\lambda(s)) - K_1(t,s,\overline{\lambda},y^*_{\overline{\lambda}}(s))|ds \\ &\quad + \int_a^b |K_2(t,s,\lambda,y^*_\lambda(s)) - K_2(t,s,\overline{\lambda},y^*_{\overline{\lambda}}(s))|ds \\ &\leq 2(b-a)\varepsilon_1 + \int_a^t |K_1(t,s,\lambda,y^*_\lambda(s)) - K_1(t,s,\lambda,y^*_{\overline{\lambda}}(s))|ds \\ &\quad + \int_a^b |K_2(t,s,\lambda,y^*_\lambda(s)) - K_2(t,s,\lambda,y^*_{\overline{\lambda}}(s))|ds \\ &\leq 2(b-a)\varepsilon_1 + \int_a^b (\overline{k}_1(t,s) + k_2(t,s))|y^*_\lambda(s) - y^*_{\overline{\lambda}}(s)|ds. \end{aligned}$$

From this inequality and Lemma 1 we obtain

$$\|y^*_\lambda(\cdot) - y^*_{\overline{\lambda}}(\cdot)\|_{L^2(I)} \leq \frac{2(b-a)\varepsilon_1\sqrt{2(b-a)}}{1-L},$$

where L is defined in (7). So for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\lambda - \overline{\lambda}| < \delta \Rightarrow \|y^*_\lambda(\cdot) - y^*_{\overline{\lambda}}(\cdot)\|_{L^2(I)} < \varepsilon,$$

this is the continuity of the operator S.

If we have I.b') and I.c'), we use the fiber Picard theorem to study the differentiability of the operator S. Consider the spaces $V = W = L^2(I)$ and the operators $B : V \to V$, $C : V \times W \to W$ defined by the following relations

$$B[v](t) = f(t) + \int_a^t K_1(t,s,\lambda,y(s))ds + \int_a^b K_2(t,s,\lambda,y(s))ds$$

and

$$C[(v,w)](t) = \int_a^t \frac{\partial K_1(t,s,v(s);\lambda)}{\partial \lambda} ds + \int_a^b \frac{\partial K_2(t,s,v(s);\lambda)}{\partial \lambda} ds$$

$$+ \int_a^t \frac{\partial K_1(t,s,v(s);\lambda)}{\partial v} w(s)ds + \int_a^b \frac{\partial K_2(t,s,v(s);\lambda)}{\partial v} w(s)ds.$$

Due to the given conditions the operator B is a Picard operator (condition I.b') implies condition III.) and the operator C satisfies

$$\|C[(v,w_1)] - C[(v,w_2)]\|_{L^2(I)} \leq L_1 \|w_1 - w_2\|_{L^2(I)},$$

where $L_1 = \sqrt{\int_a^b \int_a^t (k_1(t,s) + k_2(t,s))^2 dsdt + \int_a^b \int_t^b k_2^2(t,s)dsdt}$. Theorem 1 implies that the triangular operator $A[v,w] = (B[v], C[v,w])$ is a Picard operator and so the sequence of successive approximations constructed by the relations $(y_{n+1}, z_{n+1}) = A[y_n, z_n]$ converges in $(L^2(I))^2$ to the unique fixed point. If we choose for $y_0(\cdot,\lambda)$ a C^1 function in its last variable and $z_0 = \frac{\partial y_0}{\partial \lambda}$, then from the definition of the operator C we deduce $z_n = \frac{\partial y_n}{\partial \lambda}$. By the other hand the operators $S_n : [\lambda_1,\lambda_2] \to L^2(I)$ defined by $S_n(\lambda)(t) = y_n(t)$, $\forall t \in I$, $\forall \lambda \in [\lambda_1,\lambda_2]$ are differentiables and the differential of S_n in λ is z_n, hence we can apply Theorem 2 and we obtain the differentiability of the operator S. $\square$

Remark 2 *We can prove the same results working in the space*

$$Y = \left\{ y : I \times \Lambda \to \mathbb{R} \;\middle|\; y(\cdot,\lambda) \in L^2[I],\ \forall \lambda \in \Lambda,\ y(t,\cdot) \in C(\Lambda) \text{ a.e. } t \in I \right\},$$

where $\Lambda = [\lambda_1, \lambda_2]$ *and the norm is defined by* $\|y\|_Y = \max\limits_{\lambda \in \Lambda} \|y(\cdot,\lambda)\|_{L^2(I)}$.

Using the same arguments we can prove the following theorem for equation (2).

Theorem 4 *If*

a) *the functions* $K_i : I \times I \times [\lambda_1, \lambda_2] \times \mathbb{R} \to \mathbb{R}, i = \overline{1,2}$ *satisfy conditions* I.-IV. *from Theorem 3;*

b) *the functions* $g_1, g_2 : [a, b] \to \mathbb{R}$ *are injective and measurable and they satisfy the conditions* $\mathrm{Im}(g_1) = [a_1, a_2]$, $\mathrm{Im}(g_2) = [b_2, b_1]$ *with* $a_1 \le a \le a_2 \le b$, *and* $a \le b_2 \le b \le b_1$;

c) $\varphi_1 \in L^2([a_1, a])$ *and* $\varphi_2 \in L^2([b, b_1])$;

then

1) *equation* (2) *has a unique solution* $y^*(\cdot, \lambda)$ *in* $L^2(I_1)$ *for all* $\lambda \in [\lambda_1, \lambda_2]$, *where* $I_1 = [a_1, b_1]$;

2) *the sequence of successive approximations converges* $L^2(I_1)$ *to* $y^*(\cdot, \lambda)$ *for every admissible initial function* $y_0(\cdot, \lambda)$, *where the set of admissible functions is*

$$Y_a = \{ y(\cdot, \lambda) \in L^2(I_1) \,|\, y_0(t, \lambda) = \varphi_1(t), \forall\, t \in [a_1, a], y_0(t, \lambda) \\ = \varphi_2(t), \forall\, t \in [b, b_1] \};$$

3) *we have the following estimation:*

$$\|y_n(\cdot) - y^*(\cdot, \lambda)\|_{L^2(I_1)} \le \frac{L^n}{1 - L} \|y_1(\cdot) - y_0(\cdot)\|_{L^2(I_1)},$$

where L *is defined by relation* (7).

Moreover if condition I.c) *holds, then the operator* $S : [\lambda_1, \lambda_2] \to L^2(I_1)$ *defined by* $S(\lambda)(x) = y^*(x, \lambda)$, $\forall\, x \in [a_1, b_1]$, $\forall\, \lambda \in [\lambda_1, \lambda_2]$ *is continuous.*

If instead of conditions I.b), I.c) *and* III. *the conditions* I.b') *and* I.c') *are satisfied, then* S *is differentiable.*

Remark 3 *The differentiability of* S *implies the existence of the partial derivative* $\frac{\partial y^*(\cdot,\lambda)}{\partial \lambda}$ *and so from the construction of the operator* C *we deduce that this partial derivative satisfies the equation*

$$\frac{\partial y^*(t, \lambda)}{\partial \lambda} = \int_a^t \frac{\partial K_1(t, s, \lambda, y^*(s, \lambda))}{\partial \lambda} ds + \int_a^b \frac{\partial K_2(t, s, \lambda, y^*(s, \lambda))}{\partial \lambda} ds \\ + \int_a^t \frac{\partial K_1(t, s, \lambda, y^*(s, \lambda))}{\partial y^*} \frac{\partial y^*(s, \lambda)}{\partial \lambda} ds + \int_a^b \frac{\partial K_2(t, s, \lambda, y^*(s, \lambda))}{\partial y^*} \frac{\partial y^*(s, \lambda)}{\partial \lambda} ds;$$

in the case of Theorem 3 and the equation

$$\frac{\partial y^*(t,\lambda)}{\partial\lambda} = \int\limits_a^t \frac{\partial K_1(t,s,\lambda,y^*(g_1(s),\lambda))}{\partial\lambda}ds + \int\limits_a^b \frac{\partial K_2(t,s,\lambda,y^*(g_2(s),\lambda))}{\partial\lambda}ds$$
$$+ \int\limits_a^t \frac{\partial K_1(t,s,\lambda,y^*(g_1(s),\lambda))}{\partial y^*}\cdot\frac{\partial y^*(g_1(s),\lambda)}{\partial\lambda}ds$$
$$+ \int\limits_a^b \frac{\partial K_2(t,s,\lambda,y^*(g_2(s),\lambda))}{\partial y^*}\cdot\frac{\partial y^*(g_2(s),\lambda)}{\partial\lambda}ds$$

in the case of Theorem 4.

3 Fredholm-Volterra equations on an unbounded interval

If $I = [a,\infty)$, we can't use the same inequalities because in Lemma 1 and in some estimations we used it was essential the finite length of the interval. Due to this problem we need other conditions to guarantee the same properties of the solution operator.

Theorem 5 *If conditions* I.-III. *from Theorem 3 are satisfied with* $I = [a,\infty)$ *and*

$$L^2 := \int_a^\infty\int_a^t (k_1(t,s)+k_2(t,s))^2 ds dt + \int_a^\infty\int_t^\infty k_2^2(t,s) ds dt < 1, \tag{8}$$

then

1. *for every* $\lambda\in[\lambda_1,\lambda_2]$ *there exists an unique solution* $y^*(\cdot,\lambda)\in L^2(I)$;

2. *the sequence of successive approximations*

$$y_{n+1}(x) = f(x) + \int\limits_a^x K_1(x,s,\lambda,y_n(s))ds + \int\limits_a^\infty K_2(x,s,\lambda,y_n(s))ds$$

converges in $L^2(I)$ *to* $y^*(\cdot,\lambda)$, *for all* $y_0(\cdot)\in L^2(I)$;

3. *for every* $n \in \mathbb{N}$ *we have*

$$\|y_n(\cdot) - y^*(\cdot,\lambda)\|_{L^2(I)} \le \frac{L^n}{1-L}\|y_1(\cdot) - y_0(\cdot)\|_{L^2(I)}.$$

Moreover if

I.c) *there exist* $\Lambda_i : [\lambda_1,\lambda_2] \times [\lambda_1,\lambda_2] \to \mathbb{R}$, *and* $g_i : I^2 \to \mathbb{R}$, $i \in \{1,2\}$ *such that*

i)
$$|K_i(x,s,\lambda,u) - K_i(x,s,\overline{\lambda},u)| \le \Lambda_i(\lambda,\overline{\lambda}) \cdot g_i(t,s), \tag{9}$$
$\forall\, u \in \mathbb{R}, \lambda, \overline{\lambda} \in [\lambda_1,\lambda_2]$, a.e.$(t,s) \in I^2, i \in \{1,2\}$;

ii) $\lim\limits_{\overline{\lambda}\to\lambda} \Lambda(\lambda,\overline{\lambda}) = 0$;

iii) $\int\limits_a^\infty \left[\left(\int\limits_a^t g_1(s,t)ds\right)^2 + \left(\int\limits_a^\infty g_2(s,t)\right)^2\right] dt < +\infty$

then the operator $S : [\lambda_1,\lambda_2] \to L^2(I)$ *defined by* $S(\lambda)(x) = y^*(x,\lambda)$, $\forall\, x \in I$, $\forall\, \lambda \in [\lambda_1,\lambda_2]$ *is continuous.*

If instead of the conditions I.b) *and* III. *condition* I.b') *from Theorem 3 is fulfilled and*

I.c') $K_i(x,s,\cdot,u)$ *is a* $C^1[\lambda_1,\lambda_2]$ *function for all* $u \in \mathbb{R}$, *a.e.* $(x,s) \in I^2$, *the partial derivatives satisfy condition* I., *and there exists* $M_3 > 0$ *such that*

$$\int_a^\infty \left(\int_a^t \frac{\partial K_1}{\partial\lambda}(t,s,\lambda,u)ds\right)^2 dt + \int_a^\infty \left(\int_a^t \frac{\partial K_2}{\partial\lambda}(t,s,\lambda,u)ds\right)^2 dt < M_3^2,$$

for all $\lambda \in [\lambda_1,\lambda_2]$ *and for all* $u \in \mathbb{R}$,

then S *is differentiable.*

Proof. As in Theorem 3 for a fixed λ the operator $T : L^2(I) \to L^2(I)$ defined by

$$T[y](x) = f(x) + \int\limits_a^x K_1(x,s,\lambda,y(s))ds + \int\limits_a^\infty K_2(x,s,\lambda,y(s))ds$$

is a contraction with Lipschitz constant L. If y^*_λ and $y^*_{\overline{\lambda}}$ are the unique solutions corresponding to λ and $\overline{\lambda}$, from I.c) we deduce:

$$\int\limits_a^\infty\left(\int\limits_a^t |K_1(t,s,\lambda,y^*_{\overline{\lambda}}(s)) - K_1(t,s,\overline{\lambda},y^*_{\overline{\lambda}}(s))|ds\right)^2 dt \le \Lambda_1^2(\lambda,\overline{\lambda}) \cdot \int\limits_a^\infty\left(\int\limits_a^t g_1(t,s)ds\right)^2 dt$$

and

$$\int_a^\infty \left(\int_a^\infty |K_2(t,s,\lambda,y_{\overline{\lambda}}^*(s)) - K_2(t,s,\overline{\lambda},y_{\overline{\lambda}}^*(s))|ds \right)^2 dt \leq \Lambda_2^2(\lambda,\overline{\lambda}) \cdot \int_a^\infty \left(\int_a^\infty g_2(t,s)ds \right)^2 dt.$$

From

$$\begin{aligned}
|y_\lambda^*(t) - y_{\overline{\lambda}}^*(t)| &\leq \int_a^t |K_1(t,s,\lambda,y_\lambda^*(s)) - K_1(t,s,\overline{\lambda},y_{\overline{\lambda}}^*(s))|ds \\
&\quad + \int_a^b |K_2(t,s,\lambda,y_\lambda^*(s)) - K_2(t,s,\overline{\lambda},y_{\overline{\lambda}}^*(s))|ds \\
&\leq \int_a^t |K_1(t,s,\lambda,y_{\overline{\lambda}}^*(s)) - K_1(t,s,\overline{\lambda},y_{\overline{\lambda}}^*(s))|ds \\
&\quad + \int_a^b |K_2(t,s,\lambda,y_{\overline{\lambda}}^*(s)) - K_2(t,s,\overline{\lambda},y_{\overline{\lambda}}^*(s))|ds \\
&\quad + \int_a^t |K_1(t,s,\lambda,y_\lambda^*(s)) - K_1(t,s,\lambda,y_{\overline{\lambda}}^*(s))|ds \\
&\quad + \int_a^b |K_2(t,s,\lambda,y_\lambda^*(s)) - K_2(t,s,\lambda,y_{\overline{\lambda}}^*(s))|ds \\
&\leq \int_a^t |K_1(t,s,\lambda,y_{\overline{\lambda}}^*(s)) - K_1(t,s,\overline{\lambda},y_{\overline{\lambda}}^*(s))|ds \\
&\quad + \int_a^b |K_2(t,s,\lambda,y_{\overline{\lambda}}^*(s)) - K_2(t,s,\overline{\lambda},y_{\overline{\lambda}}^*(s))|ds \\
&\quad + \int_a^b (\overline{k}_1(t,s) + k_2(t,s))|y_\lambda^*(s) - y_{\overline{\lambda}}^*(s)|ds
\end{aligned}$$

we deduce (using Minkovski inequality)

$$\|y_\lambda^*(\cdot) - y_{\overline{\lambda}}^*(\cdot)\|_{L^2(I)} \leq \frac{\Lambda}{1-L},$$

where L is defined in (8) and

$$\Lambda = \Lambda_1(\lambda,\overline{\lambda}) \sqrt{\int_a^\infty \left(\int_a^t k_1(s,t)ds \right)^2 dt} + \Lambda_2(\lambda,\overline{\lambda}) \sqrt{\int_a^\infty \left(\int_a^\infty k_2(s,t) \right)^2 dt}.$$

This inequality implies the continuity of the operator S.

If conditions I.b') and I.c') are satisfied we can use the fiber Picard theorem again. Consider the spaces $V = W = L^2(I)$ and the operators $B : V \to V$, $C : V \times W \to W$ defined by the following relations

$$B[v](t) = f(t) + \int_a^t K_1(t, s, \lambda, y(s))ds + \int_a^\infty K_2(t, s, \lambda, y(s))ds$$

and

$$\begin{aligned} C[(v, w)](t) = &\int_a^t \frac{\partial K_1(t, s, v(s); \lambda)}{\partial \lambda} ds + \int_a^\infty \frac{\partial K_2(t, s, v(s); \lambda)}{\partial \lambda} ds \\ &+ \int_a^t \frac{\partial K_1(t, s, v(s); \lambda)}{\partial v} w(s) ds + \int_a^\infty \frac{\partial K_2(t, s, v(s); \lambda)}{\partial v} w(s) ds. \end{aligned}$$

Due to the given conditions B is a Picard operator (condition I.b') implies condition III.) and C satisfies the uniform contraction condition:

$$\|C[(v, w_1)] - C[(v, w_2)]\|_{L^2(I)} \leq L_1 \|w_1 - w_2\|_{L^2(I)},$$

where $L_1 = \sqrt{\int_a^\infty \int_a^t (k_1(t, s) + k_2(t, s))^2 ds dt + \int_a^\infty \int_t^\infty k_2^2(t, s) ds dt}$. Theorem 1 implies that the triangular operator $A[v, w] = (B[v], C[v, w])$ is a Picard operator. Hence the sequence of successive approximation $(y_{n+1}, z_{n+1}) = A[y_n, z_n]$ converges in $(L^2(I))^2$ to the unique fixed point. If we choose $y_0(\cdot, \lambda)$ continuously differentiable (with respect to λ) and $z_0 = \frac{\partial y_0}{\partial \lambda}$, then from the construction of the operator C we obtain $z_n = \frac{\partial y_n}{\partial \lambda}$. On the other hand the operators $S_n : [\lambda_1, \lambda_2] \to L^2(I)$ defined by $S_n(\lambda)(t) = y_n(t, \lambda)$, $\forall\, t \in I$, $\forall\, \lambda \in [\lambda_1, \lambda_2]$ are differentiables and the differential of S_n in λ is z_n, so from Theorem 2 we obtain the differentiability of S. $\square$

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Some inequalities in bicentric quadrilateral

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Dedicated to the memory of Professor Antal Bege

Abstract. In this paper we prove some results concerning bicentric quadrilaterals. We offer a new proof of the Blundon-Eddy inequality, which we use to obtain other inequalities in bicentric quadrilaterals.

1 Introduction

Let $ABCD$ be a bicentric quadrilateral with $a = AB, b = BC, c = CD, d = AD$, $d_1 = AC$, $d_2 = BD$, $s = \frac{a+b+c+d}{2}$, R the radius of the circumscribed circle of the quadrilateral $ABCD$ and r the radius of the inscribed circle, F the area.

In [1] W. J. Blundon and R. H. Eddy proved that:

$$8r\left(\sqrt{4R^2+r^2}-r\right) \le s^2 \le \left(r+\sqrt{4R+r}\right)^2.$$

In the following we give a simple proof to this double inequality using the product

$$(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2,$$

then we deduce many other important new inequalities. We mention that the result concerning the above product is new.

We denote:

$$\sigma_1 = \sum a, \sigma_2 = \sum ab, \sigma_3 = \sum abc, x_1 = bc+ad, x_2 = ab+cd, x_3 = ac+bd.$$

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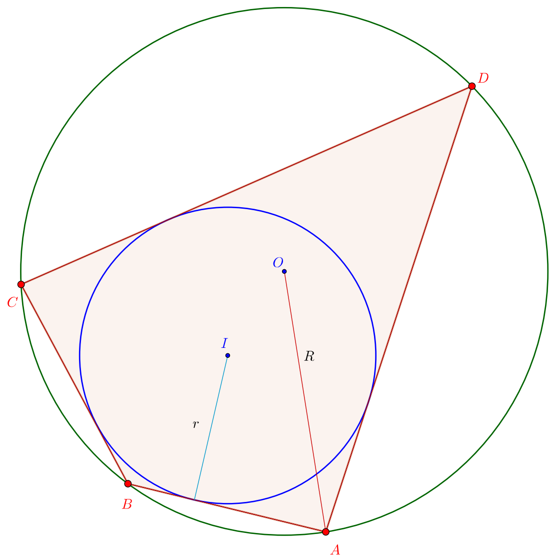

2 Main results

Lemma 1 *In every bicentric quadrilateral* ABCD *the following equalities are true:*

1) $F^2 = (s-a)(s-b)(s-c)(s-d) = abcd$;
2) $x_1x_2x_3 = 16R^2r^2s^2$;
3) $x_1 + x_2 = s^2$;
4) $x_1 + x_2 + x_3 = s^2 + 2r^2 + 2r\sqrt{r^2 + 4R^2}$;
5) $x_3 = 2r\left(r + \sqrt{4R^2 + r^2}\right)$;
6) $(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2 = (x_1-x_2)^2(x_2-x_3)^2(x_3-x_1)^2$.

Proof.

1) We have $a + c = b + d$. It results that $s - b = d$ and three similar equalities which imply

$$(s-a)(s-b)(s-c)(s-d) = abcd.$$

2) From Ptolemy's theorem it results that $x_3 = d_1d_2$. We have the equalities:

$$ad\sin A + bc\sin C = 2F, \;\; ab\sin B + dc\sin D = 2F.$$

We obtain $(ad + bc)\, d_1 = 4RF$, $(ab + dc)\, d_2 = 4RF$ which implies

$$(ad + bc)(ab + dc)\, d_1 d_2 = 16R^2F^2 \text{ or } x_1x_2x_3 = 16R^2r^2s^2. \qquad (1)$$

3) We have $x_1 + x_2 = ad + bc + ab + cd = (a + c)(d + b) = (a + c)^2 = \left(\frac{a+b+c+d}{2}\right) = s^2$.

4) From (1) it results that

$$\begin{aligned} &(ab + bc)(ad + dc)(ac + bd) = 16R^2F^2 \quad \text{or} \\ &abcd \sum a^2 + \sigma_3^2 - 2abcd\sigma_2 = 16R^2F^2 \quad \text{or} \\ &\sigma_3^2 - 4s^2r^2\sigma_2 + 4s^4r^2 = 16R^2r^2s^2v. \end{aligned} \qquad (2)$$

But $(s - a)(s - b)(s - c)(s - d) = s^2r^2$ or $-s^3 + \sigma_2 s - \sigma_3 = 0$ which implies

$$\sigma_3 = s\left(\sigma_2 - s^2\right). \qquad (3)$$

From (2) and (3) we have:

$$\begin{aligned} &s^2\left(\sigma_2 - s^2\right)^2 - 4s^2r^2\sigma_2 + 4s^4r^2 = 16R^2r^2s^2 \quad \text{or} \\ &\sigma_2^2 - \left(2s^2 + 4r^2\right)\sigma_2 + s^4 + 2s^2r^2 - 16r^2R^2 = 0. \end{aligned}$$

It results that: $\sigma_2 = s^2 + 2r^2 + 2r\sqrt{r^2 + 4R^2}$. But $\sigma_2 = x_1 + x_2 + x_3$, so it follows that

$$x_1 + x_2 + x_3 = s^2 + 2r^2 + 2r\sqrt{r^2 + 4R^2}. \qquad (4)$$

5) From 4) since $x_1 + x_2 = s^2$ it follows that $x_3 = 2r^2 + 2r\sqrt{4R^2 + r^2}$.

6) We have $(a - b)^2 (a - c)^2 (a - d)^2 (b - c)^2 (b - d)^2 (c - d)^2 =$
$[(a - b)(c - d)]^2\, [(a - c)(b - d)]^2\, [(a - d)(b - c)]^2 =$
$(x_1 - x_2)^2 (x_2 - x_3)^2 (x_2 - x_1)^2$.

□

Theorem 1 *In every bicentric quadrilateral* ABCD *the following equality is true:*

$$\begin{aligned} &(a - b)^2 (a - c)^2 (a - d)^2 (b - c)^2 (b - d)^2 (c - d)^2 \\ &= 16r^4s^2\left[s^2 - 8r\left(\sqrt{4R^2 + r^2} - r\right)\right]\left[s^2 - \left(r + \sqrt{4R^2 + r^2}\right)^2\right]^2. \end{aligned}$$

Proof. We denote $\triangle = (a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2$.
From Lemma 1 6) we have:

$$\begin{aligned}\triangle &= (x_1-x_2)^2(x_3-x_1)^2(x_3-x_2)^2\\ &= \left[(x_1+x_2)^2-4x_1x_2\right]\left[x_3^2-x_3(x_1+x_2)+x_1x_2\right]^2.\end{aligned} \tag{5}$$

From Lemma 1 2) and 5) it results that:

$$x_1x_2 = \frac{8R^2r^2s^2}{r\left(r+\sqrt{4R^2+r^2}\right)} = 2r\left(\sqrt{4R^2+r^2}-r\right)s^2. \tag{6}$$

From Lemma 1 3), 5) and equalities (5), (6) we obtain:

$$\begin{aligned}\triangle &= \left[s^4-8r\left(\sqrt{4R^2+r^2}-r\right)s^2\right]\left[4r^2\left(r+\sqrt{4R^2+r^2}\right)^2\right.\\ &\quad\left.-2s^2r\left(r+\sqrt{4R^2+r^2}\right)+2r\left(\sqrt{4R^2+r^2}-r\right)s^2\right]^2\\ &= s^2\left[s^2-8r\left(\sqrt{4R^2+r^2}-r\right)\right]\left[4r^2\left(r+\sqrt{4R^2+r^2}\right)^2-4r^2s^2\right]^2\\ &= 16r^4s^2\left[s^2-8r\left(\sqrt{4R^2+r^2}-r\right)\right]\left[s^2-\left(r+\sqrt{4R^2+r^2}\right)^2\right].\end{aligned}$$

□

Theorem 2 *In every bicentric quadrilateral* $ABCD$ *the following double inequality is true:* $8r\left(\sqrt{4R^2+r^2}-r\right) \le s^2 \le \left(r+\sqrt{4R^2+r^2}\right)^2$. *The equality holds in the case of two bicentric quadrilaterals* $A_1B_1C_1D_1$ *and* $A_2B_2C_2D_2$ *with the sides*

$$\begin{aligned}a_1 &= c_1 = \sqrt{2r\sqrt{4R^2+r^2}-2r^2}\\ b_1 &= \sqrt{2r\sqrt{4R^2+r^2}-2r^2}-\sqrt{2r\sqrt{4R^2+r^2}-6r^2}\\ d_1 &= \sqrt{2r\sqrt{4R^2+r^2}-2r^2}+\sqrt{2r\sqrt{4R^2+r^2}-6r^2}\\ a_2 &= d_2 = \frac{r+\sqrt{r^2+4R^2}-\sqrt{4R^2-2r^2-2r\sqrt{4R^2+r^2}}}{2}\\ b_2 &= c_2 = \frac{r+\sqrt{r^2+4R^2}+\sqrt{4R^2-2r^2-2r\sqrt{4R^2+r^2}}}{2}.\end{aligned}$$

Proof. We have $(x_3 - x_1)(x_3 - x_2) = (a-b)(b-c)(c-d)(d-a)$ and because $a+c = b+d$ it results that $(a-b)(b-c)(c-d)(d-a) = (a-b)^2 (b-c)^2 \geq 0$, which implies $(x_3 - x_1)(x_3 - x_2) \geq 0$ or

$$s^2 \leq \left(r + \sqrt{4R^2 + r^2}\right)^2.$$

But, from Theorem 1 since $\triangle \geq 0$, it results that

$$8r\left(\sqrt{4R^2 + r^2} - r\right) \leq s^2.$$

It remain to study the equality cases for $s_1 \leq s \leq s_2$ where

$$s_1 = \sqrt{8r\left(\sqrt{4R^2 + r^2} - r\right)},\ s_2 = r + \sqrt{4R^2 + r^2}.$$

From Theorem 1 it results that we may have the cases:

Case 1.

$$a = c.$$

We denote $a = x$. Then

$$a = x, b = y, c = x, d = 2x - y.$$

From Lemma 1 we have:

$$x_3 = 2r\left(r + \sqrt{4R^2 + r^2}\right) \text{ or } x^2 + y(2x - y) = 2r\left(r + \sqrt{4R^2 + r^2}\right).$$

But $F^2 = abcd$ or $(2x - y)y = 4r^2$. It results that $x^2 = 2r\sqrt{4R^2 + r^2} - 2r^2$. Since $s_1^2 = 4x^2 = 8r\left(\sqrt{4R^2 + r^2} - r\right)$ represents the left side of the inequality from the statement, so:

$$x = \sqrt{2r\sqrt{4R^2 + r^2} - 2r^2}$$

$$(y - x)^2 = 2r\sqrt{4R^2 + r^2} - 6r^2 \text{ or } |y - x| = \sqrt{2r\sqrt{4R^2 + r^2} - 6r^2}.$$

We denote $u_1 = 2r\sqrt{4R^2 + r^2} - 2r^2, u_2 = 2r\sqrt{4R^2 + r^2} - 6r^2$.
If $x \leq y$ we have

$$a = x = \sqrt{u_1},\ b = y = \sqrt{u_1} + \sqrt{u_2},\ c = \sqrt{u_1},\ d = 2x - y = \sqrt{u_1} - \sqrt{u_2}.$$

If $x > y$ we have

$$a = x = \sqrt{u_1},\ b = y = x - \sqrt{u_2} = \sqrt{u_1} - \sqrt{u_2},\ c = \sqrt{u_1},\ d = 2x - y = \sqrt{u_1} + \sqrt{u_2}.$$

It results that the equality from the left side of the inequality of the statement holds in the case of bicentric quadrilateral $A_1B_1C_1D_1$ with the sides

$$\sqrt{u_1}, \sqrt{u_1} - \sqrt{u_2}, \sqrt{u_1}, \sqrt{u_1} + \sqrt{u_2}.$$

Case 2.

$$a = d = x, b = c = y.$$

In this case $m(\measuredangle D) = m(\measuredangle B) = 90^\circ$, $AC = 2R$. It results that $F = sr = 2\frac{xy}{2}$ or $xy = (x + y)r$.

We denote $\alpha = x + y, \beta = xy$.

We have $\beta = \alpha r$. But $x^2 + y^2 = 4R^2$ which implies $\alpha^2 - 2\beta = 4R^2$ so we have $\alpha^2 - 2\alpha r - 4R^2 = 0$.

It results that $\alpha = r + \sqrt{r^2 + 4R^2}$.

But $s_1 = x + y = \alpha = r + \sqrt{r^2 + 4R^2}$ which represents the right side of the inequality from the statement. We have $\begin{cases} x + y = \alpha \\ xy = r\alpha \end{cases}$, so x, y are the solutions of the equation $u^2 - \alpha u + r\alpha = 0$ which implies:

$$x = \frac{\alpha - \sqrt{\alpha^2 - 4r\alpha}}{2} = \frac{r + \sqrt{r^2 + 4R^2} - \sqrt{4R^2 - 2r^2 - 2r\sqrt{4R^2 + r^2}}}{2},$$

$$y = \frac{r + \sqrt{r^2 + 4R^2} + \sqrt{4R^2 - 2r^2 - 2r\sqrt{4R^2 + r^2}}}{2}.$$

So, the equality for the right side of the inequality from the statement is true in the case of bicentric quadrilateral $A_2B_2C_2D_2$ with the sides

$$a_2 = x, b_2 = x, c_2 = y, d_2 = y.$$

$\square$

Theorem 3 *In every bicentric quadrilateral* $ABCD$ *the following inequalities are true:*

$$2r\left(r + \sqrt{4R^2 + r^2}\right) \leq \min\{ab + cd, bc + ad\} \leq 4r\left(\sqrt{4R^2 + r^2} - r\right)$$
$$\leq \max\{ab + cd + bc + ad\} \leq 4R^2.$$

Proof. We suppose that $x_1 \le x_2$, $x_1 + x_2 = s^2$, $x_1 x_2 = \alpha s^2$ where

$$\alpha = \frac{8R^2 r}{\sqrt{4R^2+r^2}+r} = 2r\left(\sqrt{4R^2+r^2}-r\right).$$

It results that: $x_1 = \frac{s^2-\sqrt{s^4-4\alpha s^2}}{2}$, $x_2 = \frac{s^2+\sqrt{s^4-4\alpha s^2}}{2}$. We consider the functions $f, g : (0, +\infty) \to R$.

$$f(s) = \frac{s^2-\sqrt{s^4-4\alpha s^2}}{2}, g(s) = \frac{s^2+\sqrt{s^4-4\alpha s^2}}{2}.$$

After differentiation we obtain:

$$f'(s) = \frac{s\left(\sqrt{s^4-4\alpha s^2}-s^2+2\alpha\right)}{\sqrt{s^4-4\alpha s^2}} \le 0,\ \ g'(s) = \frac{s\left(\sqrt{s^4-4\alpha s^2}+s^2-4\alpha\right)}{\sqrt{s^4-4\alpha s^2}} \ge 0.$$

From Theorem 2 it results that: $s^2 \ge 8r\left(\sqrt{4R^2+r^2}-r\right) = 4\alpha$.

It results that f is a decreasing and g is an increasing function. Because $s \le r + \sqrt{4R^2+r^2}$ we have $f\left(r+\sqrt{4R^2+r^2}\right) \le f(s) = x_1$. If follows that

$$\begin{aligned}
x_1 \ge \frac{1}{2}\Bigg[&\left(r+\sqrt{4R^2+r^2}\right)^2 \\
&-\left(r+\sqrt{4R^2+r^2}\right)\sqrt{\left(r+\sqrt{4R^2+r^2}\right)^2-8r\left(\sqrt{4R^2+r^2}-r\right)}\Bigg] \\
= \frac{\left(r+\sqrt{4R^2+r^2}\right)}{2}\Bigg[&r+\sqrt{4R^2+r^2} \\
&-\sqrt{r^2+4R^2+r^2+2r\sqrt{4R^2+r^2}-8r\sqrt{4R^2+r^2}+8r^2}\Bigg] \\
= \frac{\left(r+\sqrt{4R^2+r^2}\right)}{2}&\left[r+\sqrt{4R^2+r^2}-\sqrt{\left(\sqrt{4R^2+r^2}\right)^2+9r^2-6r\sqrt{4R^2+r^2}}\right] \\
= 2r\left(r+\sqrt{4R^2+r^2}\right)&.
\end{aligned}$$

It follows that

$$x_1 \ge 2r\left(r+\sqrt{4R^2+r^2}\right). \tag{7}$$

From $s \le r + \sqrt{4R^2 + r^2}$ it results also that

$$\begin{aligned}
x_2 &= g(s) \le g\left(r + \sqrt{4R^2 + r^2}\right) \\
&= \frac{1}{2}\Bigg[\left(r + \sqrt{4R^2 + r^2}\right)^2 \\
&\quad + \left(r + \sqrt{4R^2 + r^2}\right)\sqrt{\left(r + \sqrt{4R^2 + r^2}\right)^2 - 8r\left(\sqrt{4R^2 + r^2} - r\right)}\Bigg] \\
&= \left(\sqrt{4R^2 + r^2} + r\right)\left(\sqrt{4R^2 + r^2} - r\right) = 4R^2.
\end{aligned}$$

Thus we get the following inequality

$$x_2 \le 4R^2. \tag{8}$$

Since $8r\left(\sqrt{4R^2 + r^2} - r\right) \le s^2$ we have $x_1 = f(s) \le f\left(\sqrt{8r\left(\sqrt{4R^2 + r^2} - r\right)}\right)$ or in an equivalent form

$$\begin{aligned}
x_1 &\le \frac{1}{2}\Bigg[8r\left(\sqrt{4R^2 + r^2} - r\right) \\
&\quad - \sqrt{8r\left(\sqrt{4R^2 + r^2} - r\right)}\sqrt{8r\left(\sqrt{4R^2 + r^2} - r\right) - 8r\left(\sqrt{4R^2 + r^2} - r\right)}\Bigg] \\
&= 4r\left(\sqrt{4R^2 + r^2} - r\right).
\end{aligned}$$

It follows that

$$x_1 \le 4r\left(\sqrt{4R^2 + r^2} - r\right). \tag{9}$$

Because $8r\left(\sqrt{4R^2 + r^2} - r\right) \le s^2$ and g is an increasing function it follows that:

$$g\left(\sqrt{8r\left(\sqrt{4R^2 + r^2} - r\right)}\right) \le g(s) = x_2 \text{ or } x_2 \ge 4r\left(\sqrt{4R^2 + r^2} - r\right). \tag{10}$$

From (7) (8) (9) and (10) it results that:

$$x_3 = 2r\left(r + \sqrt{4R^2 + r^2}\right) \le x_1 \le 4r\left(\sqrt{4R^2 + r^2} - r\right) \le x_2 \le 4R^2.$$

□

Remark 1 *From Theorem 3 it results that* $2r\left(r+\sqrt{4R^2+r^2}\right) \le 4r\left(\sqrt{4R^2+r^2}-r\right)$ *which, after performing some calculation, represent the well-known Fejes inequality* $R \ge \sqrt{2}r$.

Theorem 4 *In every bicentric quadrilateral* ABCD *the following inequalities are true:*

$$\frac{r\left(\sqrt{4R^2+r^2}+r\right)}{R} \le \min\{d_1, d_2\} \le \frac{\sqrt{4R^2+r^2}+r}{R}\sqrt{\frac{\left(\sqrt{4R^2+r^2}-r\right)r}{2}}$$
$$\le \max\{d_1, d_2\} \le 2R.$$

Proof. We suppose that $x_1 \le x_2$.

From Ptolemy's theorem it results that $\frac{x_1}{x_2} = \frac{d_1}{d_2}$ which implies $d_1 \le d_2$.

Because $d_1 d_2 = x_3$ we have

$$d_1^2 = \frac{x_1}{x_2}x_3 = \frac{s^2-\sqrt{s^4-4\alpha s^2}}{s^2+\sqrt{s^4-4\alpha s^2}}x_3 = x_3\frac{\left(s^2-\sqrt{s^4-4\alpha s^2}\right)^2}{4\alpha s^2}$$
$$= x_3\frac{2s^4-4\alpha s^2-2s^2\sqrt{s^4-4\alpha s^2}}{4\alpha s^2} = \frac{x_3\left(s^2-2\alpha-\sqrt{s^4-4\alpha s^2}\right)}{2\alpha}$$
$$= \frac{2r\left(r+\sqrt{4R^2+r^2}\right)}{4r\left(\sqrt{4R^2+r^2}-r\right)}\left[s^2-\sqrt{s^4-4\alpha s^2}-2\alpha\right]$$
$$= \frac{\left(\sqrt{4R^2+r^2}+r\right)^2}{8R^2}\left[s^2-\sqrt{s^4-4\alpha s^2}-2\alpha\right] = B\left(2x_1-2\alpha\right),$$

where we denote $B = \frac{\left(\sqrt{4R^2+r^2}+r\right)^2}{8R^2}$.

But from Theorem 3 we have

$$4r\left(r+\sqrt{4R^2+r^2}\right) \le 2x_1 \le 8r\left(\sqrt{4R^2+r^2}-r\right).$$

We obtain

$$4r\left(r+\sqrt{4R^2+r^2}\right)-2\alpha \le 2x_1-2\alpha \le 8r\left(\sqrt{4R^2+r^2}-r\right)-2\alpha \text{ or}$$
$$8r^2 \le 2x_1-2\alpha \le 4r\left(\sqrt{4R^2+r^2}-r\right) \text{ or}$$
$$8r^2B \le B\left(2x_1-2\alpha\right) \le 4r\left(\sqrt{4R^2+r^2}-r\right)B \text{ or}$$
$$\frac{8r^2\left(\sqrt{4R^2+r^2}+r\right)^2}{8R^2} \le d_1^2 \le \frac{4r\left(\sqrt{4R^2+r^2}-r\right)\left(\sqrt{4R^2+r^2}+r\right)^2}{8R^2}.$$

It results that:

$$\frac{r\left(\sqrt{4R^2+r^2}+r\right)}{r} < d_1 \leq \frac{\sqrt{4R^2+r^2}+r}{R}\sqrt{\frac{\left(\sqrt{4R^2+r^2}-r\right)r}{2}}. \tag{11}$$

Also:

$$\begin{aligned} d_2^2 &= \frac{x_2}{x_1}x_3 = \frac{s^2+\sqrt{s^4-4\alpha s^2}}{s^2-\sqrt{s^4-4\alpha s^2}}x_3 = \frac{\left(s^2+\sqrt{s^4-4\alpha s^2}\right)^2}{4\alpha s^2}x_3 \\ &= \frac{x_3}{4\alpha s^2}\left(2s^4-4\alpha s^2+2s^2\sqrt{s^4-4\alpha s^2}\right) = \frac{x_3}{2\alpha}\left(s^2+\sqrt{s^4-4\alpha s^2}-2\alpha\right) \\ &= \frac{x_3}{2\alpha}(2x_2-2\alpha) = \frac{\left(\sqrt{4R^2+r^2}+r\right)^2(2x_2-2\alpha)}{8R^2}. \end{aligned}$$

But we have proved that $4r\left(\sqrt{4R^2+r^2}-r\right) \leq x_2 \leq 4R$. It results that:

$$\begin{aligned} &4r\left(\sqrt{4R^2+r^2}-r\right) \leq 2x_2-2\alpha \leq 2\left(4R^2+2r^2-2r\sqrt{4R^2+r^2}\right) \text{ or} \\ &4r\left(\sqrt{4R^2+r^2}-r\right)\left(\frac{\sqrt{4R^2+r^2}+r}{2\sqrt{2}R}\right)^2 \leq d_2^2 \\ &\qquad \leq 2\left(\sqrt{4R^2+r^2}-r\right)^2\frac{\left(\sqrt{4R^2+r^2}+r\right)^2}{8R^2} \text{ or} \\ &\frac{\sqrt{4R^2+r^2}+r}{R}\sqrt{\frac{\left(\sqrt{4R^2+r^2}-r\right)r}{2}} \leq d_2 \leq 2R. \end{aligned} \tag{12}$$

From (11) and (12) it results the inequalities from the statement. □

Theorem 5 *Let be* $\alpha, \beta \in R$ *so that* $s \leq \alpha R + \beta r$ *is true in every bicentric quadrilateral* $ABCD$. *Then* $2R + \left(4-2\sqrt{2}\right)r \leq \alpha R + \beta r$ *is true in every bicentric quadrilateral* $ABCD$.

Proof. We consider the case of the square with the sides $a = b = c = d = 1$. We have $2 \leq \alpha\frac{1}{\sqrt{2}} + \beta\frac{1}{2}$. It results that

$$4 \leq \sqrt{2}\alpha + \beta. \tag{13}$$

If $a = b = 1, c = d = 0$ it results that $R = \frac{1}{2}, r = 0$.

It follows that
$$1 \le \frac{\alpha}{2} \text{ or } \alpha \ge 2. \tag{14}$$

We know that
$$R \ge \sqrt{2}r. \tag{15}$$

From (13), (14) and (15) it results that
$$\begin{aligned}(\alpha - 2)R + \left(\beta - 4 + 2\sqrt{2}\right)r &\ge (\alpha - 2)\sqrt{2}r + \left(\beta - 4 + 2\sqrt{2}\right)r \\ &= \left(\alpha\sqrt{2} + \beta - 4\right)r \ge 0,\end{aligned}$$

therefore
$$\alpha R + \beta r \ge 2R + \left(4 - 2\sqrt{2}\right)r.$$

□

Theorem 6 *In every bicentric quadrilateral the following inequality is true:*
$$s \le 2R + \left(4 - 2\sqrt{2}\right)r.$$

Proof. From the Theorem 1 we have $s \le r + \sqrt{4R^2 + r^2}$. We denote $x = \frac{R}{r}$.

We prove that
$$r + \sqrt{4R^2 + r^2} \le 2R + \left(4 - 2\sqrt{2}\right)r,$$

or in an equivalent form
$$1 + \sqrt{4x^2 + 1} \le 2x + 4 - 2\sqrt{2} \text{ or } \sqrt{4x^2 + 1} \le 2x + 3 - 2\sqrt{2} \text{ or}$$
$$1 \le 4\left(3 - 2\sqrt{2}\right)x + \left(3 - 2\sqrt{2}\right)^2 \text{ or } x \ge \frac{\left(-2 + 2\sqrt{2}\right)\left(4 - 2\sqrt{2}\right)}{4\left(3 - 2\sqrt{2}\right)}.$$

After performing some calculation it results that $x \ge \sqrt{2}$ which represents just the Fejes's inequality [2]. □

Theorem 7 *In every bicentric quadrilateral* $ABCD$ *the following inequalities are true:*

1) $4r\left(3\sqrt{4R^2 + r^2} - 5r\right) \le a^2 + b^2 + c^2 + d^2 \le 8R^2$;

2) $2r\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}\left(7\sqrt{4R^2+r^2}-9r\right) \leq \sum a^2b \leq 8R^2+2r^2$;

3) $2r\left(5\sqrt{4R^2+r^2}-3r\right) \leq \sum ab \leq 4\left(R^2+r^2+r\sqrt{4R^2+r^2}\right)$;

4) $32r^2\sqrt{4R^2+r^2}\left(\sqrt{4R^2+r^2}-r\right) \leq \sum a^2bc$
$\leq 4r\sqrt{4R^2+r^2}\left(r+\sqrt{4R^2+r^2}\right)^2$;

5) $\left(2r^2+2r\sqrt{4R^2+r^2}\right)\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)} \leq \sum abc$
$\leq 2r\left(r+\sqrt{4R^2+r^2}\right)^2$.

Proof. We have $\sigma_2 = s^2+\alpha$, $\sigma_3 = s\alpha$ where $\alpha = 2r^2+2r\sqrt{r^2+4R^2}$.

1) $\sum a^2 = (2s)^2 - 2\sigma_2 = 4s^2 - 2\sigma_2 = 4s^2 - 2s^2 - 4r^2 - 4r\sqrt{4R^2+r^2}$.
It results that: $\sum a^2 = 2s^2 - 4r^2 - 4r\sqrt{4R^2+r^2}$.
From Theorem 2 we obtain

$$4r\left(3\sqrt{4R^2+r^2}-5r\right) \leq a^2+b^2+c^2+d^2 \leq 8R^2.$$

2) $a^2b = ab\,(2s-b-c-d) = 2sab - ab^2 - abc - abd$ or $a^2b + ab^2 = 2sab - abc - abd$.
It results that $\sum a^2b = 2s\sigma_2 - 3\sigma_3 = 2s^3 - s\alpha = s\left(2s^2-\alpha\right)$ which implies $\sum a^2b = s\left(2s^2-\alpha\right)$. We consider the increasing function

$$f:(0,+\infty)\to R,\, f(s) = 2s^3 - s\alpha, \text{ with } f'(s) = 6s^2-\alpha \geq 0 \text{ as}$$
$$s^2 \geq 8r\left(\sqrt{4R^2+r^2}-r\right) \geq \frac{\alpha}{6} = \frac{2r^2+2r\sqrt{r^2+4R^2}}{6}.$$

The last inequality may be written as:

$$24\sqrt{4R^2+r^2}-24r \geq r+\sqrt{4R^2+r^2} \text{ or } 23\sqrt{4R^2+r^2} \geq 25r.$$

But from inequality of Fejes it results that

$$23\sqrt{4R^2+r^2} \geq 25\sqrt{9r^2} = 75r > 25r.$$

From Theorem 2 it results that:

$$\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)\left(16\left(\sqrt{4R^2+r^2}-r\right)-2r^2-2r\sqrt{4R^2+r^2}\right)}$$
$$\leq \sum a^2b \leq \left(r+\sqrt{4R^2+r^2}\right)$$
$$\left(2r^2+8R^2+2r^2+2r\sqrt{4R^2+r^2}-2r^2-2r\sqrt{4R^2+r^2}\right)$$

which is equivalent with the inequality from the statement.

3) $\sigma_2 = \sum ab = s^2+\alpha$ or $8r\left(\sqrt{4R^2+r^2}-r\right)+2r^2+2r\sqrt{4R^2+r^2} \leq \sum ab \leq r^2+4R^2+r^2+2r\sqrt{4R^2+r^2}+2r^2+2r\sqrt{4R^2+r^2}$

which is equivalent with the inequality from the statement.

4) $a^2bc = a\ abc = (2s-b-c-d)\,abc = 2sabc - ab^2c - abc^2 - abcd$ or $a^2bc + ab^2c + abc^2 = 2sabc - abcd$ or $\sum a^2bc = 2s\sigma_3 - 4abcd = 2s\ s\alpha - 4s^2r^2$ or $\sum a^2bc = s^2\left(2\alpha - 4r^2\right) = s^2\left(4r^2+4r\sqrt{4R^2+r^2}-4r^2\right) = 4r\sqrt{4R^2+r^2}s^2$.

From Theorem 2 it results the inequality from the statement.

5) $\sum abc = s\alpha$.

According to Theorem 2 it results the inequality from the statement.

□

Theorem 8 *Let be* $\alpha, \beta, \gamma \in R$, $\beta \geq 4$ *so that* $s^2 \leq \alpha R^2 + \beta Rr + \gamma r^2$ *is true in all bicentric quadrilateral. Then*

$$4R^2 + 4Rr + \left(8-4\sqrt{2}\right)r^2 \leq \alpha R^2 + \beta Rr + \gamma r^2$$

is true in all bicentric quadrilateral.

Proof. We consider the case of the bicentric quadrilateral with $a = b = c = d = 1$. It results that $4 \leq \frac{\alpha}{2} + \frac{\beta}{2\sqrt{2}} + \frac{\gamma}{4}$ or $16 \leq 2\alpha + \sqrt{2}\beta + \gamma$.

In the case of $a = b = 1$, $c = d = 0$ it results that $R = \frac{1}{2}$, $r = 0$ and $\alpha \geq 4$. But from inequality $R \geq \sqrt{2}r$ we have:

$$\begin{aligned}
&(\alpha-4)\,R^2+(\beta-4)\,Rr+\left(\gamma-8+4\sqrt{2}\right)r^2\\
&\geq(\alpha-4)\,2r^2+\sqrt{2}\,(\beta-4)\,r^2+\left(\gamma-8+4\sqrt{2}\right)r^2\\
&\geq(\alpha-4)\,2r^2+\sqrt{2}\,(\beta-4)\,r^2+\left(\gamma-8+4\sqrt{2}\right)r^2\\
&=\left(2\alpha+\sqrt{2}\beta+\gamma-16\right)r^2\geq 0.
\end{aligned}$$

□

Theorem 9 *In every bicentric quadrilateral* ABCD *the following inequality is true:*

$$s^2\leq 4R^2+4Rr+\left(8-4\sqrt{2}\right)r^2.$$

Proof. Since $s^2\leq\left(r+\sqrt{4R^2+r^2}\right)^2$ it is sufficient to prove that:

$\left(\sqrt{4x^2+1}+1\right)^2\leq 4x^2+4x+8-4\sqrt{2}$ or

$4x^2+1+1+2\sqrt{4x^2+1}\leq 4x^2+4x+8-4\sqrt{2}$ or

$2\sqrt{4x^2+1}\leq 4x+6-4\sqrt{2}$ or

$\sqrt{4x^2+1}\leq 2x+3-2\sqrt{2}$ or $4x^2+1\leq 4x^2+\left(12-8\sqrt{2}\right)x+\left(3-2\sqrt{2}\right)^2$ or

$$x\geq\frac{\left(1-3+2\sqrt{2}\right)\left(1+3-2\sqrt{2}\right)}{4\left(3-2\sqrt{2}\right)}=\frac{\left(\sqrt{2}-1\right)\left(2-\sqrt{2}\right)}{3-2\sqrt{2}}=\sqrt{2}.$$

□

Theorem 10 *In every bicentric quadrilateral* ABCD *the following inequalities are true:*

1) $\sum abc\leq 8R^2r+8Rr^2+\left(16-8\sqrt{2}\right)r^3;$

2) $\sum ab\leq 4\left[R^2+2Rr+\left(4-2\sqrt{2}\right)r^2\right];$

3) $\sum a^2bc\leq 32R^3r+16Rr^3+\left(80-32\sqrt{2}\right)R^2r^2+\left(32-16\sqrt{2}\right)r^4.$

Proof.

1) We proved that $\sum abc \le 2r\left(r+\sqrt{4R^2+r^2}\right)^2$, and

$$\left(r+\sqrt{4R^2+r^2}\right)^2 \le 4R^2+4Rr+\left(8-4\sqrt{2}\right)r^2.$$

It results that

$$\sum abc \le 2r\left(4R^2+4Rr+\left(8-4\sqrt{2}\right)r^2\right).$$

2) Since $\sqrt{4R^2+r^2} \le 2R+\left(3-2\sqrt{2}\right)r$, from Theorem 7 3) it results that:

$$\begin{aligned}\sum ab &\le 4\left(R^2+r^2+r\sqrt{4R^2+r^2}\right)\\ &\le 4\left[R^2+r^2+r\left(2R+\left(3-2\sqrt{2}\right)r\right)\right]\\ &= 4\left[R^2+r^2+2Rr+\left(3-2\sqrt{2}\right)r^2\right] \quad \text{or}\\ \sum ab &\le 4\left[R^2+2Rr+\left(4-2\sqrt{2}\right)r^2\right].\end{aligned}$$

3) From Theorem 7 4) it results that:

$$\begin{aligned}\sum a^2bc &\le 4r\sqrt{4R^2+r^2}\left(r+\sqrt{4R^2+r^2}\right)^2\\ &= 4r\sqrt{4R^2+r^2}\left(r^2+4R^2+r^2+2r\sqrt{4R^2+r^2}\right)\\ &= 8r\sqrt{4R^2+r^2}\left(2R^2+r^2+r\sqrt{4R^2+r^2}\right)\\ &= \left(16R^2r+8r^3\right)\sqrt{4R^2+r^2}+8r^2\left(4R^2+r^2\right)\\ &\le \left(16R^2r+8r^3\right)\left[2R+\left(3-2\sqrt{2}\right)r\right]+32R^2r^2+8r^4\\ &= 32R^3r+\left(48-32\sqrt{2}\right)R^2r^2+16Rr^3+\left(24-16\sqrt{2}\right)r^4\\ &\quad +32R^2r^2+8r^4,\end{aligned}$$

which is equivalent with the inequality from the statement.

□

Theorem 11 *In every bicentric quadrilateral* ABCD *the following inequalities are true:*

1) $2r\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}\left(5\sqrt{4R^2+r^2}-11r\right)\leq\sum a^3$
$\leq 2\left(r+\sqrt{4R^2+r^2}\right)\left(4R^2-r^2-r\sqrt{4R^2+r^2}\right);$
2) $352R^2r^2+208r^4-240r^3\sqrt{4R^2+r^2}$
$\leq\sum a^3b\leq\left(r+\sqrt{4R^2+r^2}\right)^2\left(8R^2-4r^2\right).$

Proof.

1) $a^3=a^2(2s-b-c-d)=2a^2s-a^2b-a^2c-a^2d$ or $\sum a^3=2s\sum a^2-\sum a^2b=2s\left(2s^2-2\alpha\right)-2s^3+s\alpha.$

It results that $\sum a^3=2s^3-3\alpha s.$

We consider the function $f:(0,+\infty)\to\mathbb{R}$, $f(s)=2s^3-3\alpha s$, with the derivate $f'(s)=6s^2-3\alpha$. We prove that $f'(s)\geq 0$ or $s^2\geq\frac{\alpha}{2}$.

But $s^2\geq 8r\left(\sqrt{4R^2+r^2}-r\right)$. It will be sufficient to prove that:

$$8r\left(\sqrt{4R^2+r^2}-r\right)\geq r^2+r\sqrt{4R^2+r^2}\text{ or}$$
$$8\sqrt{4x^2+1}-8\geq 1+\sqrt{4x^2+1}\text{ or }\sqrt{4x^2+1}\geq\frac{9}{7},$$

which is true because $\sqrt{4x^2+1}\geq 2$ according to Fejes inequality.

Since f is an increasing function it results from Theorem 2 that:

$$\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}\left[16\left(\sqrt{4R^2+r^2}-r\right)-6r^2-6r\sqrt{4R^2+r^2}\right]$$
$$\leq\sum a^3\leq\left(r+\sqrt{4R^2+r^2}\right)\left[2r^2+8R^2+2r^2\right.$$
$$\left.+4r\sqrt{4R^2+r^2}-6r^2-6r\sqrt{4R^2+r^2}\right],$$

which is equivalent with the inequality from the statement.

2) $a^3b=ab\left(\sum a^2-b^2-c^2-d^2\right)=ab\sum a^2-ab^3-abc^2-abd^2$ or $a^3b+ab^3=ab\sum a^2-abc^2-abd^2$ or $\sum a^3b=\sum ab\sum a^2-\sum a^2bc=\left(s^2+\alpha\right)\left(2s^2-2\alpha\right)-\left(2\alpha-4r^2\right)s^2$ or $\sum a^3b=2s^4-\left(2\alpha-4r^2\right)s^2-2\alpha^2.$

We denote $s^2=t$ and consider the function: $f:(0,+\infty)\to\mathbb{R}$,

$$f(t)=2t^2-\left(2a-4r^2\right)t-2a^2$$

and

$$t_v = \frac{2a - 4r^2}{4} = \frac{a - 2r^2}{2} = r\sqrt{4R^2 + r^2}.$$

We prove that $t \geq t_v$.

$s^2 \geq r\sqrt{4R^2 + r^2}$. But $s^2 \geq 8r\left(\sqrt{4R^2 + r^2} - r\right)$. It will be sufficient to prove that

$$8r\left(\sqrt{4R^2 + r^2} - r^2\right) \geq r\sqrt{4R^2 + r^2} \text{ or } \sqrt{4R^2 + r^2} \geq \frac{8}{7}$$

which is true because $\sqrt{4R^2 + r^2} \geq 3$.

It results that f is an increasing function which implies:

$$128r^2\left(4R^2 + 2r^2 - 2r\sqrt{4R^2 + r^2}\right) - 4r\sqrt{4R^2 + r^2}8r\left(\sqrt{4R^2 + r^2} - r\right)$$
$$-2\left(2r^2 + 2r\sqrt{4R^2 + r^2}\right)^2 \leq \sum a^3b \leq 2\left(r + \sqrt{4R^2 + r^2}\right)^4$$
$$-4r\sqrt{4R^2 + r^2}\left(r + \sqrt{4R^2 + r^2}\right)^2 - 2\left(2r^2 + 2r\sqrt{4R^2 + r^2}\right)^2$$

or

$$512R^2r^2 + 256r^4 - 256r^3\sqrt{4R^2 + r^2} - 32r^2\left(4R^2 + r^2\right) + 32r^3\sqrt{4R^2 + r^2}$$
$$-8r^4 - 8r^2\left(4R^2 + r^2\right) - 16r^3\sqrt{4R^2 + r^2} \leq \sum a^3b \leq 2\left(r + \sqrt{4R^2 + r^2}\right)^2$$
$$\left(r^2 + 4R^2 + r^2 + 2r\sqrt{4R^2 + r^2} - 2r\sqrt{4R^2 + r^2}\right) - 8r^2\left(r + \sqrt{4R^2 + r^2}\right)^2$$

or

$$352R^2r^2 + 208r^4 - 240r^3\sqrt{4R^2 + r^2} \leq \sum a^3b \leq \left(r + \sqrt{4R^2 + r^2}\right)^2$$
$$\left(4r^2 + 8R^2 - 8r^2\right).$$

□

Theorem 12 *In every bicentric quadrilateral* ABCD *the following inequalities are true:*

1) $\sum a^3 \leq 16R^3 + \left(24 - 16\sqrt{2}\right)R^2r - 8Rr^2 - \left(16 - 8\sqrt{2}\right)r^3$;

2) $\sum a^3b \leq 32R^4 - 16R^2r^2 + 32R^3r + 16Rr^3 + \left(64 - 32\sqrt{2}\right)R^2r^2 - \left(32 - 16\sqrt{2}\right)r^4$;

3) $\sum a^3b \geq 352R^2r^2 + \left(480\sqrt{2} - 512\right) r^4 - 480Rr^3.$

Proof.

1) From Theorem 11 it results that:

$$\begin{aligned}\sum a^3 &\leq \left(r + \sqrt{4R^2 + r^2}\right)\left(8R^2 - 2r^2 - 2r\sqrt{4R^2 + r^2}\right)\\ &= 8R^2r - 2r^3 - 2r^2\sqrt{4R^2 + r^2} + 8R^2\sqrt{4R^2 + r^2}\\ &\quad - 2r^2\sqrt{4R^2 + r^2} - 8R^2r - 2r^3\\ &= \left(8R^2 - 4r^2\right)\sqrt{4R^2 + r^2} - 4r^3\\ &\leq \left(8R^2 - 4r^2\right)\left[2R + \left(3 - 2\sqrt{2}\right) r\right] - 4r^3\\ &= 16r^3 + \left(24 - 16\sqrt{2}\right) R^2r - 8Rr^2 - \left(12 - 8\sqrt{2}\right) r^3 - 4r^3,\end{aligned}$$

which is equivalent with inequality from the statement.

2) From Theorem 11 it results that

$$\sum a^3b \leq \left(r + \sqrt{4R^2 + r^2}\right)^2\left(8R^2 - 4r^2\right)$$

and

$$\left(r + \sqrt{4R^2 + r^2}\right)^2 \leq 4R^2 + 4Rr + \left(8 - 4\sqrt{2}\right) r^2.$$

It results that:

$$\begin{aligned}\sum a^3b &\leq \left[4R^2 + 4Rr + \left(8 - 4\sqrt{2}\right) r^2\right]\left(8R^2 - 4r^2\right)\\ &= 32R^4 - 16R^2r^2 + 32R^3r - 16Rr^3 + \left(64 - 32\sqrt{2}\right) R^2r^2\\ &\quad - (32 - 16\sqrt{2})r^4,\end{aligned}$$

which is equivalent with the inequality from the statement.

3) We prove that:

$$\begin{aligned}\sum a^3b &\geq 352R^2r^2 + 208r^4 - 240r^3\sqrt{4R^2 + r^2}\\ &\geq 352R^2r^2 + 208r^4 - 240r^3\left[2R + \left(3 - 2\sqrt{2}\right) r\right]\\ &= 352R^2r^2 + 208r^4 - 480Rr^3 - \left(720 - 480\sqrt{2}\right) r^4,\end{aligned}$$

which is equivalent with the inequality from the statement.

$\square$

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Approximation by limit q-Bernstein operator

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Dedicated to the memory of Professor Antal Bege

Abstract. We establish quantitative estimates for the limit q-Bernstein operator introduced in [3], via the second order Ditzian-Totik modulus of smoothness.

1 Introduction

The q-Bernstein operators were introduced by Phillips in [8] and they generalize the well-known Bernstein operators. A survey of the obtained results and references concerning q-Bernstein operators can be found in [6]. It is worth mentioning that the first generalization of the Bernstein operators based on q-integers was obtained by Lupaş [4].

Let $q > 0$. For each nonnegative integer k, the q-integers $[k] \equiv [k]_q$ and the q-factorials $[k]!$ are defined by

$$[k] = \begin{cases} 1 + q + \cdots + q^{k-1}, & \text{if} \quad k \geq 1 \\ 0, & \text{if} \quad k = 0 \end{cases}$$

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and

$$[k]! = \begin{cases} [1][2]\dots[k], & \text{if} \quad k \geq 1 \\ \\ 1, & \text{if} \quad k = 0. \end{cases}$$

For integers $0 \leq k \leq n$, the q-binomial coefficients are defined by

$$\left[\begin{array}{c} n \\ k \end{array} \right] = \frac{[n]!}{[k]![n-k]!}.$$

The q-Bernstein operators $B_{n,q} : C[0,1] \to C[0,1]$ are given by

$$(B_{n,q}f)(x) \equiv B_{n,q}(f,x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) p_{n,k}(q,x), \tag{1}$$

where $n = 1, 2, \dots$, $0 < q \leq 1$, $x \in [0,1]$ and

$$p_{n,k}(q,x) = \left[\begin{array}{c} n \\ k \end{array} \right] x^k (1-x)(1-xq)\dots(1-xq^{n-k-1})$$

for $k = 0, 1, \dots, n$ (an empty product is taken to be equal 1). For $q = 1$ we recover the Bernstein operators. In [8], it is proved the uniform convergence of $B_{n,q_n}f$ to f on $[0,1]$, as $n \to \infty$, when $q = q_n \in (0,1)$ and $q_n \to 1$ as $n \to \infty$.

Let $q \in (0,1)$ and $f \in C[0,1]$ be given. Il'inskii and Ostrovska proved in [3] that the sequence $\{B_{n,q}(f,x)\}$ converges to $B_{\infty,q}(f,x)$ as $n \to \infty$, uniformly for $x \in [0,1]$, where *the limit* q*-Bernstein operator* $B_{\infty,q} : C[0,1] \to C[0,1]$ is defined by

$$\begin{aligned} (B_{\infty,q}f)(x) &\equiv B_{\infty,q}(f,x) \\ &= \begin{cases} \displaystyle\sum_{k=0}^{\infty} f(1-q^k) \frac{x^k}{(1-q)^k[k]!} \prod_{s=0}^{\infty} (1-xq^s), & \text{if} \quad 0 \leq x < 1 \\ \\ f(1), & \text{if} \quad x = 1. \end{cases} \end{aligned} \tag{2}$$

The approximation of continuous functions f by $B_{\infty,q}f$ as $q \nearrow 1$, has been investigated by Videnskii in [9]. We cite the following result of Videnskii. If $0 < q < 1$, $x \in [0,1]$ and $f \in C[0,1]$, then

$$|B_{\infty,q}(f,x) - f(x)| \leq 2\,\omega(f, \frac{1}{2}\sqrt{1-q}), \tag{3}$$

where $\omega(f,\delta) = \sup\{|f(x)-f(y)| : x,y \in [0,1], |x-y| \le \delta\}$ is the usual modulus of continuity of f. For the second modulus of smoothness of f, defined by

$$\omega^2(f,\delta) = \sup_{0<h\le\delta} \sup_{x\in[0,1-2h]} |f(x+2h) - 2f(x+h) + f(x)|,$$

Wang obtained the following estimate (see [10] and [11]):

$$|B_{n,q}(f,x) - B_{\infty,q}(f,x)| \le C\,\omega^2\left(f, \sqrt{q^n}\right), \tag{4}$$

where $n = 1,2,\ldots$, $x \in [0,1]$, $0 < q < 1$ and $f \in C[0,1]$. Here we mention that $C > 0$ is a constant independent of n, x and q, which can be different at each occurrence.

The goal of the paper is to establish quantitative results for the rate of convergence of (2), obtaining similar estimates to (3) and (4). In our estimates we shall use the second order Ditzian-Totik modulus of smoothness of f, defined by

$$\omega_\varphi^2(f,\delta) = \sup_{0<h\le\delta} \sup_{x\pm h\varphi(x)\in[0,1]} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|,$$

where $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ (for details see [1]). Further, we consider the following K-functional:

$$K_{2,\varphi}(f,\delta) = \inf_{g\in W^2(\varphi)} \left\{\|f-g\| + \delta\|\varphi^2 g''\|\right\},$$

where $\|\cdot\|$ denotes the uniform norm on $C[0,1]$ and $W^2(\varphi) = \{g \in C[0,1] : g' \in AC_{loc}[0,1], \varphi^2 g'' \in C[0,1]\}$; $g' \in AC_{loc}[0,1]$ means that g is differentiable such that g' is absolutely continuous on every interval $[a,b] \subset [0,1]$. It is known (see [1, (2.1.4)]) that $\omega_\varphi^2(f,\sqrt{\delta})$ and $K_{2,\varphi}(f,\delta)$ are equivalent, i.e. there exists $C > 0$ such that

$$C^{-1}\omega_\varphi^2(f,\sqrt{\delta}) \le K_{2,\varphi}(f,\delta) \le C\omega_\varphi^2(f,\sqrt{\delta}). \tag{5}$$

2 Main results

Theorem 1 *There exists* $C > 0$ *such that*

$$\|B_{\infty,q}f - f\| \le C\,\omega_\varphi^2(f, \sqrt{1-q})$$

for all $f \in C[0,1]$ *and* $q \in (0,1)$. *Consequently,* $B_{\infty,q}f$ *converges uniformly to* f *on* $[0,1]$ *as* $q \nearrow 1$.

Proof. By [9, (7.7)-(7.8)], we have $B_{\infty,q}(1,x) = 1$ and $B_{\infty,q}(t,x) = x$. For $g \in W^2(\varphi)$, by Taylor's formula:

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\,du, \quad t,x \in [0,1],$$

we get

$$B_{\infty,q}(g,x) - g(x) = B_{\infty,q}\left(\int_x^t (t-u)g''(u)\,du, x\right).$$

Using the inequality

$$\left|\int_x^t (t-u)g''(u)\,du\right| \leq (t-x)^2\varphi^{-2}(x)\|\varphi^2 g''\| \tag{6}$$

(see [1, Lemma 9.6.1]) and $B_{\infty,q}((t-x)^2,x) = (1-q)\varphi^2(x)$ (see [9, (7.12)]), we find

$$\begin{aligned} |B_{\infty,q}(g,x) - g(x)| &\leq B_{\infty,q}\left(\left|\int_x^t |t-u||g''(u)|\,du\right|, x\right) \\ &\leq B_{\infty,q}((t-x)^2,x)\varphi^{-2}(x)\|\varphi^2 g''\| \\ &= (1-q)\|\varphi^2 g''\|. \end{aligned} \tag{7}$$

On the other hand, by (2) and $B_{\infty,q}(1,x) = 1$, we obtain $|B_{\infty,q}(f,x)| \leq \|f\|B_{\infty,q}(1,x) = \|f\|$, i.e.

$$\|B_{\infty,q}f\| \leq \|f\| \tag{8}$$

for all $f \in C[0,1]$. Now, in view of (7) and (8), we get

$$\begin{aligned} \|B_{\infty,q}f - f\| &\leq \|B_{\infty,q}f - B_{\infty,q}g\| + \|B_{\infty,q}g - g\| + \|g - f\| \\ &\leq 2\|f-g\| + (1-q)\|\varphi^2 g''\| \\ &\leq 2\left\{\|f-g\| + (1-q)\|\varphi^2 g''\|\right\}. \end{aligned}$$

Taking the infimum on the right-hand side over all $g \in W^2(\varphi)$ and using (5), we get the assertion of our theorem. □

Remark 1 The main result of [2] provides an estimate for positive linear operators that preserve linear functions. The result was improved in [7, (2.138)], which implies for the limit q-Bernstein operator that

$$\|B_{\infty,q}f - f\| \leq \frac{5}{2}\,\omega_\varphi^2(f, \sqrt{1-q}), \quad \text{where} \quad \frac{3}{4} \leq q < 1.$$

Theorem 2 *Let* $q \in (0,1)$ *be given. Then there exists* $C > 0$ *such that*

$$\|B_{n,q}f - B_{\infty,q}f\| \le \frac{C}{q(1-q)}\,\omega_\varphi^2(f,\sqrt{q^n})$$

for all $f \in C[0,1]$ *and* $n = 1,2,\ldots$.

Proof. Let $g \in W^2(\varphi)$ and $x \in [0,1]$. Then, by [5, (3.2)], we have

$$B_{n,q}(g,x) - B_{n+1,q}(g,x) = \sum_{k=1}^{n} a_{n,k}(g)\,p_{n+1,k}(q,x), \tag{9}$$

where

$$\begin{aligned} a_{n,k}(g) \;&=\; \frac{[n+1-k]}{[n+1]}\,g\left(\frac{[k]}{[n]}\right) + q^{n+1-k}\frac{[k]}{[n+1]}\,g\left(\frac{[k-1]}{[n]}\right) \\ &\quad - g\left(\frac{[k]}{[n+1]}\right). \end{aligned} \tag{10}$$

By Taylor's formula, we find

$$\begin{aligned} g\left(\frac{[k]}{[n]}\right) \;&=\; g\left(\frac{[k]}{[n+1]}\right) + \left(\frac{[k]}{[n]} - \frac{[k]}{[n+1]}\right) g'\left(\frac{[k]}{[n+1]}\right) \\ &\quad + \int_{[k]/[n+1]}^{[k]/[n]} \left(\frac{[k]}{[n]} - u\right) g''(u)\,du \end{aligned}$$

and

$$\begin{aligned} g\left(\frac{[k-1]}{[n]}\right) \;&=\; g\left(\frac{[k]}{[n+1]}\right) + \left(\frac{[k-1]}{[n]} - \frac{[k]}{[n+1]}\right) g'\left(\frac{[k]}{[n+1]}\right) \\ &\quad + \int_{[k]/[n+1]}^{[k-1]/[n]} \left(\frac{[k-1]}{[n]} - u\right) g''(u)\,du, \end{aligned}$$

respectively. Hence, by (10),

$$\begin{aligned} a_{n,k}(g) \;&=\; \frac{[n+1-k]}{[n+1]}\,g\left(\frac{[k]}{[n]}\right) + q^{n+1-k}\frac{[k]}{[n+1]}\,g\left(\frac{[k-1]}{[n]}\right) \\ &\quad - \frac{[n+1-k] + q^{n+1-k}[k]}{[n+1]}\,g\left(\frac{[k]}{[n+1]}\right) \\ &=\; \frac{[n+1-k]}{[n+1]}\left(\frac{[k]}{[n]} - \frac{[k]}{[n+1]}\right) g'\left(\frac{[k]}{[n+1]}\right) \end{aligned}$$

$$
\begin{aligned}
& + \frac{[n+1-k]}{[n+1]} \int_{[k]/[n+1]}^{[k]/[n]} \left(\frac{[k]}{[n]} - u \right) g''(u)\, du \\
& + \frac{q^{n+1-k}[k]}{[n+1]} \left(\frac{[k-1]}{[n]} - \frac{[k]}{[n+1]} \right) g' \left(\frac{[k]}{[n+1]} \right) \\
& + \frac{q^{n+1-k}[k]}{[n+1]} \int_{[k]/[n+1]}^{[k-1]/[n]} \left(\frac{[k-1]}{[n]} - u \right) g''(u)\, du \\
= \;& \frac{[n+1-k]}{[n+1]} \int_{[k]/[n+1]}^{[k]/[n]} \left(\frac{[k]}{[n]} - u \right) g''(u)\, du \\
& + \frac{q^{n+1-k}[k]}{[n+1]} \int_{[k]/[n+1]}^{[k-1]/[n]} \left(\frac{[k-1]}{[n]} - u \right) g''(u)\, du, \qquad (11)
\end{aligned}
$$

because

$$
\begin{aligned}
& \frac{[n+1-k]}{[n+1]} \left(\frac{[k]}{[n]} - \frac{[k]}{[n+1]} \right) + \frac{q^{n+1-k}[k]}{[n+1]} \left(\frac{[k-1]}{[n]} - \frac{[k]}{[n+1]} \right) \\
= \;& \frac{[k]}{[n][n+1]^2} \{ [n+1-k]([n+1]-[n]) \\
& + \; q^{n+1-k}([k-1][n+1] - [k][n]) \Big\} \\
= \;& \frac{[k]}{[n][n+1]^2} \Big\{ [n+1-k]q^n + q^{n+1-k}(-q^{k-1}[n+1-k]) \Big\} \\
= \;& 0.
\end{aligned}
$$

In view of (6) and (11), we have

$$
\begin{aligned}
|a_{n,k}(g)| \;\; \leq \;\; & \frac{[n+1-k]}{[n+1]} \left(\frac{[k]}{[n]} - \frac{[k]}{[n+1]} \right)^2 \varphi^{-2} \left(\frac{[k]}{[n+1]} \right) \|\varphi^2 g''\| \\
& + \frac{q^{n+1-k}[k]}{[n+1]} \left(\frac{[k-1]}{[n]} - \frac{[k]}{[n+1]} \right)^2 \varphi^{-2} \left(\frac{[k]}{[n+1]} \right) \|\varphi^2 g''\| \\
= \;\; & \left\{ \frac{[n+1-k][k]([n+1]-[n])^2}{[n]^2[n+1]([n+1]-[k])} \right. \\
& \left. + \; \frac{q^{n+1-k}([k-1][n+1]-[k][n])^2}{[n]^2[n+1]([n+1]-[k])} \right\} \|\varphi^2 g''\| \\
= \;\; & \left\{ \frac{[n+1-k][k]q^{2n}}{[n]^2[n+1]q^k[n+1-k]} \right.
\end{aligned}
$$

$$
\begin{aligned}
& + \frac{q^{n+1-k}(-q^{k-1}[n+1-k])^2}{[n]^2[n+1]q^k[n+1-k]}\Big\} \|\varphi^2 g''\| \\
= \; & \frac{q^{n-1}}{[n]^2[n+1]} \left\{ q^{n+1-k}[k] + [n+1-k] \right\} \|\varphi^2 g''\| \\
= \; & \frac{q^{n-1}}{[n]^2} \|\varphi^2 g''\| \le q^{n-1} \|\varphi^2 g''\|.
\end{aligned}
$$

Hence, by (9) and $B_{n+1,q}(1,x) = 1$ (see [9, (2.5)]), we find

$$
|B_{n,q}(g,x) - B_{n+1,q}(g,x)| \le q^{n-1} \|\varphi^2 g''\|
$$

for all $x \in [0,1]$. This implies that

$$
\begin{aligned}
& \|B_{n,q}g - B_{n+p,q}g\| \\
& \quad \le \; \|B_{n,q}g - B_{n+1,q}g\| + \|B_{n+1,q}g - B_{n+2,q}g\| \\
& \qquad + \cdots + \|B_{n+p-1,q}g - B_{n+p,q}g\| \\
& \quad \le \; (q^{n-1} + q^n + \cdots + q^{n+p-2}) \|\varphi^2 g''\| \\
& \quad \le \; \frac{q^{n-1}}{1-q} \|\varphi^2 g''\| \qquad (12)
\end{aligned}
$$

for $n, p = 1, 2, \ldots$ In conclusion $\{B_{n,q}g\}$ is a Cauchy-sequence in $C[0,1]$, so $\{B_{n,q}g\}$ converges to $B_{\infty,q}g$ as $n \to \infty$ (see also [3]). Now let $p \to \infty$ in (12). Then we obtain

$$
\|B_{n,q}g - B_{\infty,q}g\| \le \frac{q^n}{q(1-q)} \|\varphi^2 g''\|. \qquad (13)
$$

Further, by (1) and $B_{n,q}(1,x) = 1$ (see [9, (2.5)]), we obtain $|B_{n,q}(f,x)| \le \|f\| B_{n,q}(1,x) = \|f\|$, i.e.

$$
\|B_{n,q}f\| \le \|f\| \qquad (14)
$$

for all $f \in C[0,1]$. Then (14), (8) and (13) imply that

$$
\begin{aligned}
\|B_{n,q}f - B_{\infty,q}f\| \; \le \; & \|B_{n,q}f - B_{n,q}g\| + \|B_{n,q}g - B_{\infty,q}g\| \\
& + \|B_{\infty,q}g - B_{\infty,q}f\| \\
\le \; & 2\|f-g\| + \frac{q^n}{q(1-q)} \|\varphi^2 g''\| \\
\le \; & \frac{2}{q(1-q)} \left\{ \|f-g\| + q^n \|\varphi^2 g''\| \right\}.
\end{aligned}
$$

Taking the infimum on the right-hand side over all $g \in W^2(\varphi)$ and using (5), we get the assertion of our theorem. $\square$

Remark 2 Because $\omega_\varphi^2(f,\delta) \le C\omega^2(f,\delta) \le 2C\omega(f,\delta)$ (for details see [1]), we obtain, in view of Theorem 1 and Theorem 2, the following weaker estimates:

$$|B_{\infty,q}(f,x) - f(x)| \le C\,\omega(f,\sqrt{1-q})$$

and

$$|B_{\infty,q}(f,x) - B_{n,q}(f,x)| \le \frac{C}{q(1-q)}\,\omega^2(f,\sqrt{q^n}).$$

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Galois covering and smash product of skew categories

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Dedicated to the memory of Professor Antal Bege

Abstract. In this paper we give a new proof of the famous result of E. L. Green [3], that gradings of a finite, path connected quiver are in one-to-one correspondence with Galois coverings. Namely we prove that the inverse construction to the skew group construction has as many solutions as the number of different gradings on the starting quiver.

1 Introduction

In mathematics, a $\mathtt{k}$-category or an Abelian category is a category in which morphisms and objects can be added. The motivation for $\mathtt{k}$-categories originated from examining the category of Abelian groups, $\mathcal{A}\mathsf{b}$. The theory rises from a tentative attempt to unify several cohomology theories by A. Grothendieck.

In this paper we examine skew categories and their connection to skew group algebras. The paper has two parts. In the first part we recall the basic notions and results of this topic, for this we use as a basis literature [1] and [2]. We present the categorical machinery developed by the above mentioned authors among some reformulations of several coherence results, to fit better in our context. In the second part we give a new proof of the famous result of E. L. Green [3], that gradings of a finite, path connected quiver are in one-to-one correspondence with Galois coverings. More precisely we prove that the inverse construction to the skew group construction has as many solutions as the number of different gradings on the starting quiver.

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1.1 Basic notions

From now on we deal with small categories over a commutative field k, this means that the objects C_0 form a set and not a class, and the morphisms consist of modules over k. We set G to be an arbitrary group. We define the notions of group action and grading on k-categories by [2].

Definition 1 *A* G*-**category** is a category* C *with the propriety that* G *acts on the set of objects, in other words the elements of* G *are* k*-module morphisms, such that the following hold: for all* $s \in G$*, and for all* $x, y \in C_0$ *for* $s\colon\ {}_yC_x \to_{sy} C_{sx}$*, we have that*

- $s(gf) = (sg)(sf)$*, if* f *and* g *can be composed in* C*;*
- *If* $t, s \in G$ *and* f *is a morphism in* C*, then* $(ts)f = t(sf)$*;*
- $1f = f$*,* $1 \in G$ *the identity element.*

In other words G is a group of autofunctors of C.

Since we defined a G-action on our categories it makes sense to talk about graded categories. These gradings will play a crucial role in the inverse construction we deal with in the second section.

Definition 2 *A* G*-**graded category** is a category* C *for which the following hold:*

- *For all* $x, y \in C_0$ *we have* ${}_yC_x = \bigoplus_{s\in G}({}_yC_x^s)$ *and*
$$ {}_zC_y^t\ {}_yC_x^s \subset_y C_x^{ts} $$
- ${}_x1_x \in_x Cx^1$.

For the definition of Galois covering of categories we must define first what a quotient category is. For this we cite [2], Definition 2.1.

Definition 3 *If* C *is a free* G*-category over* k*, then the objects of the **quotient category*** C/G *are the* G*-orbits of* C_0 *and if* α, β *are two* G*-orbits, then the morphisms between them are:*

$$ {}_\beta(C/G)_\alpha = \left(\bigoplus_{x\in\alpha,\ y\in\beta} ({}_yC_x) \right) /G. $$

Now let $p : C \to C/G$ *be the projection functor, then we call* p *the **Galois covering** of* C *with Galois group* G*.*

Similarly to the above construction we are interested in the categorical definition of skew group algebras, namely skew categories ([2], Definition 2.3).

Definition 4 *Let* C *be a* G*-category. Then the objects of the* ***skew category*** $C[G]$ *are the objects of the category* C*, so we have* $(C[G])_0 = C_0$ *and the morphisms between them are*

$$_y(C[G])_x = \bigoplus_{s\in G} ({}_yC_{sx}).$$

It is natural to ask if the above construction gives back the classical notion of skew group algebras. For this we cite a coherence result with the usual skew algebra construction ([2], Proposition 2.4).

Proposition 1 *Let* G *be a finite group and let* C *be* G*-category over* k*, with finite number of objects. Let* $a(C)$ *be the* k*-algebra associated to* C*, namely*

$$a(C) = \bigoplus_{x,y\in C_0} {}_yC_x,$$

provided with the matrix product induced by the composition of morphisms. Then we have that

$$a(C[G]) \cong a(C)[G].$$

Now let us see how Galois coverings and skew group algebras are related. The following theorem is very important for the theory ([2], Theorem 2.8) and from now on we will use it implicitly without referring to it.

Proposition 2 *Let* C *be a free* G*-category over* k*. The quotient category* C/G *and the skew category* $C[G]$ *are equivalent.*

The main goal of this part is to present the necessary tools for developing the inverse construction to taking the quotient of a category, this will be the smash product category ([2], Definition 3.1).

Definition 5 *Let* G *be a group and let* C *be a* G*-graded category over* k*. Then the* ***smash product category*** $C\#G$ *has object set* $C_0 \times G$*. Let* $(x,s),(y,t) \in C_0 \times G$ *be two objects. The* k*-module morphisms are defined as follows:*

$$_{(y,t)}(C\#G)_{(x,s)} = {}_y C_x^{t^{-1}s}.$$

It is natural to ask again if this construction gives back the classical notion of the smash product of algebras. For this we present a coherence result in a form to serve better our further goals.

Proposition 3 *Let* G *be a finite group and let* C *be a* G*-category, then the* k*-algebras* $\mathfrak{a}(\mathsf{C})\#\mathsf{G}$ *and* $\mathfrak{a}(\mathsf{C}\#\mathsf{G})$ *are Morita equivalent.*

Since this is not the classical statement regarding smash product categories we present a proof for it.

Proof. If C is a G-category, then every morphism space ${}_y\mathsf{C}_x$ is a G-module, but G being finite one can also regard these spaces as $(\mathsf{kG})^*$ modules. In this setting C can be thought as a $(\mathsf{kG})^*$-module category.

Now by Theorem 2.9 of [1] we have that the k-categories $\mathsf{C}\#\mathsf{G}$ and $\mathsf{C}\#(\mathsf{kG})^*$ are Morita equivalent. Moreover we can derive from this that $\mathfrak{a}(\mathsf{C}\#\mathsf{G})$ and $\mathfrak{a}(\mathsf{C}\#(\mathsf{kG})^*)$ are Morita equivalent as k-algebras.

Combining this with Proposition 2.3 from [1], which claims that the k-algebras $\mathfrak{a}(\mathsf{C})\#(\mathsf{kG})^*$ and $\mathfrak{a}(\mathsf{C}\#(\mathsf{kG})^*)$ are isomorphic, we get that $\mathfrak{a}(\mathsf{C})\#\mathsf{G}$ and $\mathfrak{a}(\mathsf{C}\#\mathsf{G})$ are Morita equivalent. □

A last definition before we reach to the main duality theorems of this section, is of a matrix category ([1], Definition 4.1).

Definition 6 *Let* C *be a* k*-category and let* n *be a sequence of positive integers* $(n_x)_{x\in \mathsf{C}_0}$*. The object set of the* ***matrix category*** $\mathrm{M}_n(\mathsf{C})$ *remains the same objects of* C*. The set of morphisms from* x *to* y *is the vector space of* n_x *columns and* n_y *rows rectangular matrices with entries in* ${}_y\mathsf{C}_x$*. Composition of morphisms is given by the matrix product combined with the composition in* C*.*

A classical way of relating the matrix categories to the corresponding matrix algebras is to consider single object categories provided by an algebra A and then proving that the matrix category has one object with endomorphism algebra precisely the usual algebra of matrices $\mathrm{M}_n(A)$. Unfortunately this approach is not sufficient for our further goal, so we need to develop a different correspondence between these categorical and ring theoretical objects. For this we have the following lemma.

Lemma 1 *Let* C *be a* k*-category and let* n *be a positive integer, then we have the following* k*-algebra isomorphism*

$$\mathfrak{a}\left(\mathrm{M}_n(\mathsf{C})\right) \cong \mathrm{M}_n(\mathsf{kC}),$$

where in the right hand side kC *is regarded as the path algebra of the underlying quiver of C.*

Proof. Let us examine carefully the construction of the morphism spaces of the matrix category, we have that

$$\mathfrak{a}\,(M_n(C)) = \bigoplus_{x,y \in M_n(C)_0} {}_y(M_n(C))_x = \bigoplus_{x,y \in C_0} M_n({}_yC_x) =$$

$$= M_n\left(\bigoplus_{x,y \in C_0} {}_yC_x\right) = M_n(kC).$$

Here we consider the vertices as the identity morphisms on the corresponding object, hence the set of vertices is a subset of the set of all morphisms. In this respect we can consider $\mathfrak{a}(C)$ isomorphic to the path algebra kC. □

Going back for a moment to the matrix categories we want to recall the following equivalence ([1], Corollary 4.5).

Proposition 4 *Let C be a k-category and n a positive integer, then C and $M_n(C)$ are Morita equivalent.*

Now the last statement of this section is the categorical version of the Cohen-Mongomery duality ([2], Proposition 3.2).

Theorem 1 *Let C be a G-graded category over k. Then the category $(C\#G)[G]$ is equivalent to C.*

2 The inverse construction

Now that we presented the categorical machinery developed for skew categories and smash products, we pass to the main theorem of this paper, namely the inverse construction to the skew group construction. From now on we consider finite, path connected quivers as categories over k: the objects are the vertices of the quiver and morphisms between two vertices are free k-modules having a basis given by the paths between these vertices.

Theorem 2 *Let C be a finite, path connected quiver, and let G be a group acting on it. Given a G-grading on C, we have that the skew group algebra $(kC_G)[G]$ and the path algebra kC are Morita equivalent, where C_G is the quiver corresponding to $C\#G$.*

Proof. We are considering C as a k-category, then by the Cohen-Mongomery duality (Theorem 1) we have the following equivalence of categories

$$(C\#G)[G] \cong C.$$

Translating this to the language of k-algebras, via the functor $\mathfrak{a}$, we get that

$$\mathfrak{a}((\mathsf{C}\#\mathsf{G})[\mathsf{G}]) \cong \mathfrak{a}(\mathsf{C}),$$

as k-algebras. Now by the remark in the proof of Lemma 1 we can consider $\mathfrak{a}(\mathsf{C})$ to be the path algebra kC.

Applying the coherence property of the skew group construction (Proposition 1), we get the following isomorphism of algebras

$$\mathfrak{a}(\mathsf{C}\#\mathsf{G})[\mathsf{G}] \cong \mathsf{kC}.$$

From this point, by applying the coherence result of the smash product (Proposition 3), we pass to Morita equivalences. So we get that $(\mathfrak{a}(\mathsf{C})\#\mathsf{G})[\mathsf{G}]$ is Morita equivalent to kC, where $\mathfrak{a}(\mathsf{C})\#\mathsf{G}$ is a smash product of algebras.

Finally applying again the remark from Lemma 1, we get that $(\mathsf{k}(\mathsf{C}\#\mathsf{G}))[\mathsf{G}]$ is Morita equivalent to kC, here $\mathsf{k}(\mathsf{C}\#\mathsf{G})$ is viewed as the path algebra of the quiver corresponding to $\mathsf{C}\#\mathsf{G}$.

Now putting everything in our notation we get the expected result, that the skew group algebra $(\mathsf{kC_G})[\mathsf{G}]$ and the path algebra kC are Morita equivalent, where $\mathsf{C_G}$ is the quiver corresponding to $\mathsf{C}\#\mathsf{G}$. $\square$

One can see from the above result that each different grading of C will lead to a different solution $\mathsf{C}\#\mathsf{G}$ to the inverse construction problem.

Finally we give an example to illustrate our result.

Example 1 *Let* $\mathsf{G} = \langle \mathsf{g} \rangle$ *be a cyclic group of order two and let* C *be the following quiver*

$$\begin{tikzcd} \mathsf{e}\bullet \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & \bullet\mathsf{f}\ . \end{tikzcd}$$

We can consider C *to be a* G*-graded quiver by setting degree* 1 *for the elements* $\{\mathsf{e}, \mathsf{f}\}$ *and degree* g *for the elements* $\{\alpha, \beta\}$*. In this case the quiver* $\mathsf{C_G}$*, corresponding to the smash product of* C *and* G *is the following*

$$\begin{tikzcd} (\mathsf{e},1) \arrow[dr, bend left] \arrow[dr, bend right] & (\mathsf{f},1) \\ (\mathsf{e},\mathsf{g}) \arrow[ur, bend left] \arrow[ur, bend right] & (\mathsf{f},\mathsf{g}). \end{tikzcd}$$

So we get that the skew group algebra $(\mathsf{kC_G})[\mathsf{G}]$ *is Morita equivalent to the path algebra of* C*.*

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Leader election in synchronous networks

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Dedicated to the memory of my friend Professor Antal Bege

Abstract. Worst, best and average number of messages and running time of leader election algorithms of different distributed systems are analyzed. Among others the known characterizations of the expected number of messages for LCR algorithm and of the worst number of messages of Hirschberg-Sinclair algorithm are improved.

1 Introduction

We consider the problem of leader election in synchronous networks [11, 16, 30, 43, 59, 92]. The networks are modeled by directed graphs, the processors are called processes and are modeled as an automaton (see e.g. [11, 59]). In the case of the deterministic algorithms it is supposed that the processes have a *unique identifier* (UID).

The main topic of this paper is the presentation of leader election algorithms of different synchronous networks and their performance features.

It is known that if the processes are indistinguishable then there is no deterministic algorithm to solve the problem. For such *anonymous* or *symmetric* networks random algorithms are proposed by Itai and Rodeh [38, 39], by Ghaffni et al. [31], and by Kalpathi et al. [42].

Lower and upper bounds for the number of necessary messages or necessary bits are presented by Afek and Gafni, Attiya et al., Bodlaender, Frederickson and Lynch, Korach et al., and Loui et al. [1, 2, 8, 9, 26, 47, 58].

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The structure of the paper is as follows. After the introductory Section 1 in Section 2 the enumeration of some distributed systems is presented, then in Section 3 simple (as complete, chain, mesh and star networks), ring (unidirectional and bidirectional), special (such as De Bruijn, hypercube, Cayley, tree and recursively scalable networks) and general networks are analyzed.

2 Enumeration of labeled directed networks

Leader election requires that any process can inform any other process on its own data (e.g. on its own uid). In order to guarantee the participation of all processes we suppose that the investigated networks are strongly connected. It is worth to remark that there are also algorithms not requiring the strong connectedness, but these algorithms have also such output that the leader election is not solvable.

In this section we deal at first with the influence of the requirement of strong connectedness on the number of the tested networks, then with some simple networks such as complete network, star and chain.

2.1 Enumeration of connected and strongly connected networks

Let $D(n)$, $C(n)$, and $S(n)$ denote the number of labeled simple, labeled simple weakly connected and labeled simple strongly connected digraphs, respectively.

The known simple formula

$$D(n) = 2^{n(n-1)} \tag{1}$$

gives $D(n)$. The values of $D(n)$, further $C(n)/D(n)$ and $S(n)/D(n)$ are shown in Table 1 for $n = 1, \dots, 15$. Values of $D(n)$ for $n = 1, \dots, 35$ can be found in [65].

In 2012 Critzer [17] proposed the following method to determine the number $C(n)$ of the simple labeled weakly connected digraphs:

$$C(n) = D(n) - \frac{1}{n}\sum_{i=1}^{n-1} k\binom{n}{k}C(k)D(n-k). \tag{2}$$

Using (1) one can compute the $D(n)$ values necessary to get the values of $C(n)$ from (2). E.g. (1) results $D(1) = 1$ and then (2) gives $C(1) = 1$. In a similar way $D(2) = 4$ and $C(2) = 3$, further $D(3) = 64$ and $C(3) = 54$.

n	$D(n)$	$C(n)/D(n)$	$S(n)/D(n)$
1	1	1.000000	1.00000
2	4	0.750000	0.25000
3	64	0.843750	0.28125
4	4096	0.936035	0.39209
5	1 048 576	0.979500	0.53890
6	1 073 741 824	0.994008	0.68431
7	4 398 046 511 104	0.998280	0.80106
8	72 057 594 037 927 936	0.999511	0.88506
9	$\sim 4.722\,366\,483 \cdot 10^{21}$	0.999863	0.93161
10	$\sim 1.237\,940\,039 \cdot 10^{27}$	0.999962	0.96132
11	$\sim 1.298\,074\,215 \cdot 10^{33}$	0.999990	0.97843
12	$\sim 5.444\,517\,871 \cdot 10^{39}$	0.999997	0.98835
13	$\sim 9.134\,385\,523 \cdot 10^{46}$	0.999999	0.99367
14	$\sim 6.129\,982\,164 \cdot 10^{54}$	0.9999998	0.99659
15	$\sim 1.645\,504\,557 \cdot 10^{60}$	0.99999994	0.99817

Table 1: Number $D(n)$ of simple labeled directed graphs and the ratios $C(n)/(D(n)$ and $S(n)/D(n)$.

Table 2 contains $C(n)$ for $n = 1, \ldots, 15$. In [66] the values for $n = 16, \ldots, 35$ can be found.

V. A. Liskovets in 1969 [52, 100] proposed the following recursive formulas to compute $S(n)$:

$$a(n) = n(n-1) - \sum_{i=1}^{n-1} \binom{n-1}{t-1} a(t), \tag{3}$$

$$\lambda_t(m) = 2^{m(m+t-1)} - \sum_{k=0}^{m-1} \binom{m}{k} \lambda_t(k) \tag{4}$$

and

$$S(n) = a(n) + \sum_{i=1} \binom{n-1}{t-1} 2^{(m-1)(m-k)} \lambda_t(n-t) S(t). \tag{5}$$

Using (3) and (4) one can compute the $a(n)$ and $\lambda(n)$ values necessary to get the values of $S(n)$ from (5).

n	$C(n)$
1	1
2	3
3	54
4	3 834
5	1 027 080
6	1 067 308 488
7	4 390 480 193 904
8	72 022 346 388 181 584
9	4 721 717 643 249 254 751 360
10	1 237 892 809 110 149 882 059 440 768
11	1 298 060 596 773 261 804 821 355 107 253 504
12	5 444 502 293 680 983 802 677 246 555 274 553 481 984
13	91 343 781 554 246 596 956 424 128 384 394 531 707 099 632 640
14	6 129 980 884 648 631 844 901 425 521 287 946 137 183 899 295 465 755 648
15	1 645 504 465 371 454 407 878 687 557 239 154 898 196 072 267 336 301 175 996 872 704

Table 2: Number $C(n)$ of simple labeled connected digraphs.

Simplifying Liskovets's method in 1971 Wright [100] proposed the following formulas. Let $n \geq 1$,

$$\eta(n) = D(n) - \sum_{i=1}^{n-1} 2^{(n-1)(n-i)}\eta(i) \tag{6}$$

and

$$S(n) = \eta_n + \sum_{i=1}^{n} \binom{n-1}{i-1} S(i)\eta_{n-i}. \tag{7}$$

According to (6) $\eta_1 = 1$, $\eta_2 = 0$, $\eta_3 = 16$, and $\eta_4 = 1536$. Using these η values from (7) we get $S(1) = 1$, $S(2) = 1$, $S(3) = 18$, and $S(4) = 1606$.

The values of $S(n)$ are in Table 3 for $n = 1, 2, \ldots, 15$. In [75] also the values for $n = 16, 17, 18$ can be found.

In 1969 Liskovets [52] proved the following theorem.

Theorem 1 (Liskovets, 1969 [52]) *If* $n \geq 1$, *then*

$$D(n) - 2(n+4)n^{(n+1)(n+1)} \leq S(n) \leq D(n) \tag{8}$$

and

$$S(n) = D(n)\left(1 - n2^{2-n} + n(2n-1)2^{2-2n}\right) + O(n^3 n^{n(n-4)}). \tag{9}$$

Proof. See (Liskovets, 1969 [52]). □

n	$S(n)$
1	1
2	1
3	18
4	1 606
5	565 080
6	734 774 776
7	3 523 091 615 568
8	63 519 209 389 664 176
9	4 400 410 978 376 102 609 280
10	1 190 433 705 317 814 685 295 399 296
11	1 270 463 864 957 828 799 318 424 676 767 488
12	5 381 067 966 826 255 132 459 611 681 511 359 329 536
13	90 765 788 839 403 090 457 244 128 951 307 413 371 883 494 400
14	6 109 064 462 821 545 704 046 426 032 465 737 763 224 760 635 732 888 576
15	1 642 494 209 200 959 152 585 925 675 993 911 516 594 334 047 201 121 102 632 675 328

Table 3: Number $S(n)$ of simple labeled strongly connected digraph.

2.2 Generation of all strongly connected graphs

Let m and n be positive integers, $V = \{V_1 \ldots, V_n\}$ be a finite set and $A = \{a_1, \ldots, a_m\}$ be a finite family of ordered pairs $(V_i, V_j) \in V \times V$ of the elements of V. Let $D = (V, A)$ be an arbitrary directed graph [78, Volume A, page 28] and $D^T = (V, A^T)$ be the *transpose* [15, page 530, Exercise 22.1-3] of D defined by

$$A^T = \{(V_i, V_j) \in V \times V \mid (V_j, V_i) \in V \times V\}. \tag{10}$$

A *directed spanning tree* T of a directed graph $D = (V, A)$ is a rooted tree that consists entirely of arcs in A, all arcs directed from parents to children in the tree, and that contains every vertex of D. A directed spanning tree of D with root vertex $V_i \in V$ is a *breadth-first spanning tree* provided that each vertex of D at distance d from V_i appears at depth d in the tree (that is, at distance d from V_i in the tree) [59].

We enumerated the strongly connected networks. The base of the enumeration is the fact that the strong components of a directed graph D and its transpose D^T contain the same strongly connected components [15]. Therefore we choose arbitrary vertex as a root and build a BST (breath-first spanning tree) of the given D and of its transpose D^T. D is strongly connected if and only if both deep search trees contain all vertices of D.

Lemma 1 (Cormen et al., 1969 [15]) *A directed graph* D *is strongly connected* *if and only if its arbitrary vertex (e.g.* V_1*) is the root of a breadth-first spanning tree of* D *and also the root of the breadth-first search tree* D^T.

Proof. Let V_a and V_b be arbitrary vertices of a strongly connected graph D. Then D contains a directed path $V_a = V_{i_1}, \ldots, V_{i_p} = V_b$ and also a directed path $V_b = V_{j_1}, \ldots, V_{j_q} = V_a$. Therefore D^T contains the directed paths $V_a = V_{j_q}, \ldots, V_{j_1} = V_b$ and $V_b = V_{i_p}, \ldots, V_{i_1} = V_a$, therefore the given condition is necessary.

Again let V_a and V_b arbitrary vertices of D. If D contains a directed path $(V_1 = V_{i_1}, \ldots, V_{i_r} = V_a)$ and also a directed path $(V_1 = V_{j_1}, \ldots, V_{j_q} = V_b)$, further D^T contains directed paths $(V_1 = V_{k_1}, \ldots, V_{k_r} = V_a)$ and $(V_1 = V_{l_1}, \ldots, V_{l_s} = V_b)$, then D contains directed paths $(V_a = V_{k_r}, \ldots, V_{k_1} = V_1 = V_{i_1}, \ldots, V_{i_r} = V_b)$ and $(V_b = V_{l_s}, \ldots, V_{l_1} = V_1 = V_{i_1}, \ldots, V_{i_r} = V_a)$, therefore the given condition is sufficient. □

Algorithm Strong is based on Lemma 1. It decides if a given directed graph D is strongly connected.

Input parameters are: $n > 1$: the number of processes; $B = (b_1, \ldots, b_{n^2})$: the adjacency matrix of the current graph as a vector.

Output parameter is L: if D is strongly connected then $L = 1$, otherwise $L = 0$.

Working parameters are i (current number of the vertices); j, k: cycle variables; m: the current number of vertices in the tree; $Q = (Q_1, \ldots, Q_n)$: a queue for the waiting vertices; $h(Q) = h$: the *head* index of the queue; $t(Q) = t$: the *tail* index of the queue; $p = (p_1, \ldots, p_n)$: the presence vector of the vertices ($p_i = 1$, if V_i is in the tree, and $p_i = 0$ otherwise).

```
Strong(n, B)
01  p_1 = 1                                   // line 01–08: Initialization.
02  m = 1
03  h = 1
04  Q_1 = 1
05  t = 2
06  L = 1
07  for j = 2 to n
08      p_j = 0                                  // V_j is not in the tree.
09  while t > h                                  // line 09–30: Test of D.
10        u = Q_h
11        for j = 1 to n − 1
12            for k = 1 to j − 1   // line 12–20: Before the main diagonal.
```

```
13                 if b_{(u−1)n+k} == 1 and p_{(u−1)n+k} == 0
                                                  // line 13: V_j not in tree
14                    p_k = 1 // line 14–15: A new vertex of the tree is found.
15                    m = m + 1
16                 if m == n
17                    return L
18                 Q_t = j
19                 t = t + 1
20                 h = h + 1
21              for k = j + 1 to n − 1 // line 21–30: After the main diagonal.
22                 if b_{(u−1)n+k} == 1 and p_{(u−1)n+k} == 0
                                                  // line 22: V_j not in tree.
23                    p_k = 1
24                    m = m + 1
25                    return L
26                 Q_t = j
27                 t = t + 1
28                 h = h + 1
29 L = 0                  // line 30–31: The graph is not strongly connected.
30 return L
```

We remark that Strong tests *only* the existence of a breadth-first spanning tree of D. The test of the existence of a breadth-first spanning tree of D^T requires similar instructions (the only difference that in lines 13 and 22 $b_{(u-1)n+k} == 1$ must be replaced by $b_{(u-1)n+k} == 0$.

The next assertion characterizes the resource requirements of Strong.

Theorem 2 *If* $b \geq 2$, *then* Strong *requires* $\Theta(n^2)$ *memory locations in all cases and* $O(2^{b(b-1)}n^2)$ *time units in worst case.*

Proof. The memory requirement is determined by the size of the input neighborhood matrix B, therefore the maximal memory requirement is $\Theta(n^2)$ memory locations. The time requirement of Strong is determined by the facts that the algorithm investigates at most $2^{n(n-1)}$ graphs and constructs an $n \times n$ sized matrix for all investigated graphs. □

Algorithm All-Strong enumerates the strongly connected networks for $a,\ a+1,\ \ldots,\ b$ vertices. It is also based on Lemma 1.

The *input parameters* of All-Strong are $a \geq 2$ and $b \geq a$: lower and upper bound for the current size of the investigated network.

Output parameter is $S = (S(a), \ldots, S(b))$, where $S(a)$ is the number of the strongly connected networks consisting of a processes, ..., $S(b)$ is the number of the strongly connected networks consisting of b processes.

Working parameters are i (current number of the vertices) and j (both are cycle variables); $B = (b_1, \ldots, b_n)$: the adjacency matrix of the current network as a vector; b_0: help variable to stop the increasing of the adjacency vector; $Q = (Q_1, \ldots, Q_n)$: a queue for the waiting vertices; $h(Q)$: the *head* index of the queue; $t(Q) = t$: the *tail* index of the queue; L: logical variable (if the current graph is strong, then $L = 1$, otherwise $L = 0$.

```
All-Strong(a, b)
01 for i = a to b                        // line 01–04: Generation of the first graph.
02     S(i) = 0                          // line 02: Initialization of the enumeration.
03     for j = 0 to i(i − 1)
04         b_j = 0
05     Strong(i, B)                                          // line 05–07: Test of D.
06     if L == 0                                      // line 06–07: D is not strong.
07       go to 14
08     for j = 1 to i(i − 1)                                 line 08–12: Test of D^T.
09         t_j = 1 − b_j
10     Strong(i, T)
11     if L == 0                                    // line 11–12: D^T is not strong.
12       go to 14
13     S(i) = S(i) + 1                                         // line 14: D is strong
14     for j = i(i − 1) downto 1                          // line 14–18: Generation
           of the next graph.
15         if b_j == 0
16           b_j = 1
17           for k = j + 1 to i(i −−1)
18               b_k = 0
19         go to 05                       // line 19: Continue with the next graph.
20     print i, S(i)                   // line 20: Print result for the current size.
```

The next assertion characterizes the resource requirements of All-Strong.

Theorem 3 *If* $b \geq 2$, *then* All-Strong *requires* $\Theta(b(b-1))$ *memory locations in all cases and* $O(2^{b(b-1)}n^2)$ *time units in worst case.*

Proof. The memory requirement is determined by the size of the neighborhood matrices $\mathcal{B}$ and $\mathcal{T}$ defined in lines 03–04 and 08–09. The maximal size of these matrices appears in the case when the graphs contain b vertices, therefore the maximal memory requirement is $\Theta(b(b-1))$ memory locations. The time requirement of All-Strong is determined by the facts that the algorithm investigates $2^{b(b-1)}$ graphs and constructs an $n \times n$ sized matrix what according to Theorem 1 requires $O(n^2)$ time for one matrix. Multiplying these expression we get the bound $O(2^{b(b-1)}n^2)$. □

Another possible approach to generate all labeled strongly connected digraphs is to use the minimal digraphs investigated by García-López and Marijun [27].

3 Leader election

In the following sections the problem of leader election is considered. The mathematical models described in [59] are used: networks are modeled by directed (or sometimes undirected) graphs, processes by vertices. We suppose that the processors communicate and compute in synchronous rounds. The *leader election problem* is *to elect a unique leader*. Usually it is supposed that the processes are identical except for *unique identifiers* (UIDs). The size of the network is usually unknown.

In Subsection 3.1 some simple networks, then in Subsection 3.2 ring networks, in Subsection 3.3 further unidirectional networks, and finally in Subsection 3.4 further special and general networks are considered.

3.1 Leader election in simple networks

In this subsection the problem of leader election in simple networks as complete, chain, mesh and star networks is considered.

Peterson [71] in 1985, Afek and Gafni [1, 2] in 1981 and in 1985, Singh [81] in 1992 derived time and complexity bounds for mesh and complete networks.

In 1984 Korach et al. [47] proved optimal lower bounds for the number of messages in complete networks.

In 1985 Loui et al. [58] investigated the influence of the direction of the connections on the leader election algorithms.

There are known algorithms for chain [19] and star [80] networks too.

3.2 Leader election in ring networks

In this subsection comparison-based algorithms of different ring networks (in details unidirectional and bidirectional ones) are described and analyzed.

3.2.1 LCR algorithm in unidirectional ring

Figure 1 shows an unidirectional ring consisting of the processes $P_1, \ldots, P_n$.

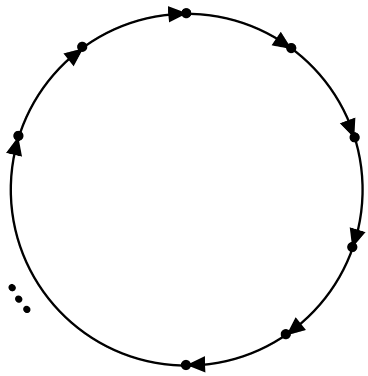

Figure 1: A ring of processes $P_1, \ldots, P_n$.

The first known leader election algorithm was proposed by Le Lann [51] in 1977 for unidirectional rings. It is a very simple algorithm. In the first step each process sends its UID to its clockwise neighbor. In the further steps each process compares the received UID with its own UID, and if they are equal, then the process declares itself the leader, otherwise sends the larger UID to the clockwise neighbor. The algorithm terminates when the process having the largest UID gets back its own UID.

This algorithm requires n steps and n^2 messages.

Chang and Roberts in 1979 [13] proposed an improved version of the previous algorithm: after the comparison of the received and own UID the processes send a message only if the received UID is the larger one. We give a formal description [59] of this algorithm called usually LCR (after Le **L**ann, **C**hang and **R**oberts) algorithm. It is supposed that the UID's are the natural numbers $1, 2, \ldots, n$.

Input parameter is n: the number of processes and $p = p_1, \ldots, p_n$: a permutation of the UID's.

Output parameter is $M_n = M$: the number of messages.

The message alphabet is $\{1, 2, \ldots, n\}$. For each i $(1 \leq i \leq n)$ the state $state_i$ consists of three components:

- `u`, a UID, initially the UID of P_i;

- $\texttt{send}_i$, a UID or `null`, initially the UID of P_i;

- `status`$_i$, having possible values {`unknown`,`leader`}.

The state of P_i consists of the single state defined by the given initial values. The message generation function `msgs`$_i$ is defined by

- send the current value of `send` to P_i.

We remark that indices are interpreted everywhere mod n.

The transition function `trans`$_i$ is defined by the following pseudocode used in [59]:

`send` := **null**
if the incoming message is `v`, then
  case
    `v` > `u`: `send` := `v`
    `v` = `u`: `status`$_i$:= `leader`
    `v` < `u`: do nothing
  endcase

Since LCR is a basic algorithm of leader election and since we execute the simulation of LCR on a sequential processor, the algorithm is described also using the pseudocode of [15, 40].

Input parameters are $n > 1$: the number of processes; $p = p_1, \ldots, p_n$: a permutation of the UID's.

Output parameters are L: the index of the elected leader; M: the number of messages.

Working parameters are $m = (m_1, \ldots, m_n)$, where m_i is the current message of P_i; i cycle variable.

LCR(n, p)

01 P_i **in parallel for** $i = 1$ **to** n // line 01–05: Initialization.
02   **read** p_i
03   $m_i = i$
04   $s_i = 0$
05 $M = n$
06 **while** all states $s_i == 0$ // line 06–13: Election.
07   P_i **in parallel for** $i = 1$ **to** n
08   **if** $m_{i-1} > p_i$
09         $m_i = m_{i-1}$
10         $M = M + 1$
11   **if** $m_{i-1} == p_i$

```
12          s_i = m_{i-1}
13          L = i
14 return L, M                    // line 14: Return of the result.
```

Let X_n be a random variable characterizing the number of messages of LCR and let M_n be the expected value of X_n at the uniform distribution of the permutations of the UID's.

Chang and Roberts in [13] not only improved the algorithm of Le Lann, but also determined M_n.

Theorem 4 (Chang, Roberts, 1979 [13]) *If the permutations of the UID's have uniform distribution, then*

$$M_n = n + \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} kP(n; i, k) = n + \sum_{k=1}^{n-1} \frac{n}{k+1} = O(n \log n), \tag{11}$$

and

$$M_n = nH_n = O(n \log n), \tag{12}$$

where H_n *is the* n*th harmonic number and* $P(n; i, k)$ *is the probability that the message* i *is passed* k *times.*

Proof. See [13]. □

$P(i, k, n)$ is the probability that the $k-1$ clockwise neighbors of i are less than i and the kth clockwise neighbor of i is larger than i. There are $i-1$ processes less than i and $n-i$ processes larger than i.

Since the place of the UID i can be fixed, the remaining identifiers can be permuted in $(n-1)!$ manner. The small UID's can be choosen in $(i-1)\cdots(i-k+1)$ manner, the kth large UID in $n-i$ manner, and the remaining UID's $(n-k)\cdots 1$ manner. So we get

$$P(n; i, k) = \frac{[(i-1)\cdots(i-k+1)](n-i)[(n-k)\cdots 1]}{(n-1)(n-2)\cdots 1}. \tag{13}$$

Using the well-known bounds

$$\frac{1}{2}\lfloor \log n \rfloor < H_n < \lceil \log n \rceil \tag{14}$$

it is easy to get the stronger assertion

$$M_n = \Theta(n \log n). \tag{15}$$

Using Leonhard Euler's following lemma we prove Lemma 3 in which (18) and (19) are stronger than (12) in Theorem 4.

Lemma 2 (Euler [22]) *If* $n \geq 1$ *then*

$$H_n = \sum_{i=1}^{n} \frac{1}{i} = \ln n + \gamma + \beta_n, \tag{16}$$

where γ *is the Euler-Mascheroni constant* ($\gamma \sim 0.577\ 215\ 665$) [22, 63, 96] *and*

$$\lim_{n\to\infty} \beta_n = 0. \tag{17}$$

Proof. See Fichtengolz [24, Volume II, page 270]. □

Lemma 3 *If* $n \geq 1$, *then*

$$M_n = n \ln n + n\gamma + n\beta_n \tag{18}$$

and

$$M_n = \Theta(n \log n). \tag{19}$$

Proof. Substitution of (16) into (12) results (18) which implies (19). □

Table 4 illustrates the accuracy of the approximation of (18).

Chen [14] in 2006 published a detailed probabilistic cost analysis of LCR algorithm. Using generating functions he proved

$$M_n = \frac{2}{n-1} \sum_{i=1}^{n-1} M_i + \frac{n}{2} \quad \text{for} \quad n \geq 2 \tag{20}$$

and remarked that $M_1 = 1$.

Using (20) Chen reproved (11) and gave a more exact characterization

$$M_n = n \log n + \gamma n + O(1) \tag{21}$$

of the mean of X_n, further determined the variance of X_n as

$$V(X_n) = \left(2 - \frac{\pi^2}{6} n^2\right) + O(n \log n). \tag{22}$$

Using Euler-Maclaurin summation [21, 60, 64, 95] D. E. Knuth [46] derived the following improved version of Lemma 2.

n	$E(M_{LCR}(n))$	$n \ln n$	$n\gamma$	$n\beta_n$
1	1.00000000000	0.000000000000	0.5772156649015	0.4227843350985
2	3.00000000000	1.386294361120	1.154431329803	0.4592743090770
3	5.50000000000	3.295836866004	1.731646994705	0.4725161392911
4	8.33333333333	5.545177444480	2.308862659606	0.4792932292476
5	11.41666666667	8.047189562171	2.886078324508	0.4833987799885
6	14.70000000000	10.75055681537	3.463293989409	0.4861491952225
7	18.15000000000	13.62137104339	4.040509654311	0.4881193023021
8	21.74285714286	16.63553233344	4.617725319212	0.4895994902062
9	25.46071428571	19.77502119603	5.194940984114	0.4907521055745
10	29.28968253968	23.02585092994	5.772156649015	0.4916749607267
11	33.21865079365	26.37684800078	6.349372313917	0.4924304789518
12	37.23852813853	29.81887979746	6.926587978818	0.4930603622537
13	41.34173881674	33.34434164700	7.503803643720	0.4935935260189
14	45.52187257187	36.94680261461	8.081019308621	0.4940506486375
15	49.77343489843	40.62075301653	8.658234973523	0.4944469083788
16	54.09166389166	44.36141955584	9.235450638425	0.4947936974029
17	58.47239288489	48.16462684896	9.812666303326	0.4950997326112
18	62.91194540753	52.02669164213	10.38988196823	0.4953717971751
19	67.40705348573	55.94434060416	10.96709763313	0.4956152484385
20	71.95479314287	59.91464547108	11.54431329803	0.4958343737632

Table 4: Concrete values of the expressions in (18).

Lemma 4 (Knuth [46]) *If* $n \geq 1$ *then*

$$H_n = \sum_{i=1}^{n} \frac{1}{i} = \ln n + \gamma + \frac{1}{2n} + \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{\Theta_{2,n}}{252n^6}, \tag{23}$$

where $0 < \Theta_{2,n} < 1$.

Proof. See [46, Page 474]. □

It is remarkable that in the *Online Encyclopedia of Integer Sequences* [83, 88] one can find further members of the series in (23). Using the ideas of the proof of Lemma 4 we get the following characterization of M_n.

Theorem 5 *If* $n \geq 1$ *then*

$$M_n = n \ln n + \gamma n + \frac{1}{2} + \frac{1}{12n^2} + \frac{1}{120n^4} + \Theta\left(\frac{1}{n^5}\right). \tag{24}$$

Proof. Using different methods in 1979 Chang and Roberts, in 2006 Chen proved (12). Substitution of the right side of (23) into (11) results

$$M_n = E(n) = n \ln n + \gamma n + \frac{1}{2} - \frac{1}{12n} + \frac{1}{120n^3} - \frac{\Theta_{2,n}}{252n^5}, \tag{25}$$

implying (24). □

Table 5 illustrates the accuracy of the approximation of (25).

n	$E(n)$	$n \ln n$	$n\gamma + \frac{1}{2}$	$-\frac{1}{12n}$	$\frac{1}{120n^3}$	$-\frac{\Theta_{2,n}}{252n^5}$
1	1.00000	0.00000	1.07722	−0.0833333	0.0083333	−0.0022157
2	3.00000	1.38629	1.65443	−0.0416667	0.0010417	−0.0001007
3	5.50000	3.29584	2.23165	−0.0277778	0.0003086	−0.0000147
4	8.33333	5.54518	2.80886	−0.0208333	0.0001302	−0.0000036
5	11.41667	8.04719	3.38608	−0.0166667	0.0000667	−0.0000012
6	14.70000	10.75056	3.96329	−0.0138889	0.0000386	−0.0000005
7	18.15000	13.62137	4.54051	−0.0119048	0.0000243	−0.0000002
8	21.74286	16.63553	5.11773	−0.0104167	0.0000163	−0.0000001
9	25.46071	19.77502	5.69494	−0.0092593	0.0000114	−0.0000001
10	29.28968	23.02585	6.27216	−0.0083333	0.0000083	−0.0000000
11	33.21865	26.37685	6.84937	−0.0075758	0.0000063	−0.0000000
12	37.23853	29.81888	7.42659	−0.0069444	0.0000048	−0.0000000
13	41.34174	33.34434	8.00380	−0.0064103	0.0000038	−0.0000000
14	45.52187	36.94680	8.58102	−0.0059524	0.0000030	−0.0000000
15	49.77343	40.62075	9.15824	−0.0055556	0.0000025	−0.0000000
16	54.09166	44.36142	9.73545	−0.0052083	0.0000020	−0.0000000
17	58.47239	48.16463	10.31267	−0.0049020	0.0000017	−0.0000000
18	62.91195	52.02669	10.88988	−0.0046296	0.0000014	−0.0000000
19	67.40705	55.94434	11.46710	−0.0043860	0.0000012	−0.0000000
20	71.95479	59.91465	12.04431	−0.0041667	0.0000010	−0.0000000

Table 5: Concrete values of the expressions in (25).

A third possibility for the proof of (12) is the application of Pascal's next formula [68] allowing the recursive computation of the sum of the kth powers of the first n positive integers.

Theorem 6 (Kovcs [49], Pascal [68], Pólya [72], Wolfram [98]) *If* $n \geq 1$ *and*

$p \geq 1$, *then*

$$S(n,p) = \sum_{i=1}^{n} i^p = \frac{1}{p+1}\left((n+1)^{p+1} - 1 - \sum_{k=1}^{p-1}\binom{p+1}{k}S(n,k)\right). \qquad (26)$$

The following Faulhaber formula [23] also allows the computation of $S(n,p)$.

Theorem 7 (Faulhaber [23], Weisstein [97]) *If* $n \geq 1$ *and* $p \geq 1$, *then*

$$S(n,p) = \frac{1}{p+1}\sum_{i=1}^{p+1}(-1)^{\delta_{i,p}}\binom{p+1}{i}B_{p+1-i}n^i, \qquad (27)$$

where $\delta_{i,p}$ *is the Kronecker-delta* [87] *and* B_i *is the Bernoulli number* [84, 85].

The following double sum gives $S(n,p)$ without recursion.

Theorem 8 (Weisstein [94]) *If* $n \geq 1$ *and* $p \geq 1$, *then*

$$S(n,p) = \sum_{i=1}^{p}\sum_{j=0}^{i-1}(-1)^j(i-j)\binom{n+p-i+1}{n-i}\binom{p+1}{j}. \qquad (28)$$

3.2.2 Hirschberg-Sinclair algorithm in bidirectional ring

Hirschberg and Sinclair [35] in 1980 proposed an algorithm (HS) for bidirectional rings which elects as leader also the process having the largest UID. HS requires in worst case only $\Theta(n \log n)$ messages instead of the $\Theta(n^2)$ requirement of LCR. Figure 2 shows a bidirectional ring.

Input parameters are $n > 1$: the number of processes; $p = p_1, \ldots, p_n$: a permutation of the UID's $1, \ldots, n$.

Output parameters: i the index of the elected leader process; $N = (N_1, \ldots, N_n)$, where N_i is the number of messages, sent by process P_i; Q: the total number of sent messages.

Working parameters are $\mathcal{M}$: the message alphabet $ml = (ml_1, \ldots, ml_n)$, where ml_i is the current message of P_i to P_{i-1}; $mr = (mr_1, \ldots, mr_n)$, where mr_i is the current message of P_i to P_{i+1}; $s = (s_1, \ldots, s_n)$: status of P_i; i is a cycle variable; *null* the empty message.

The messages are triples, consisting a UID, a *flag value(in* or *out,* and a positive integer counter (*hop-count*) h. The possible values of the status of the processes are *unknown* or *leader.*

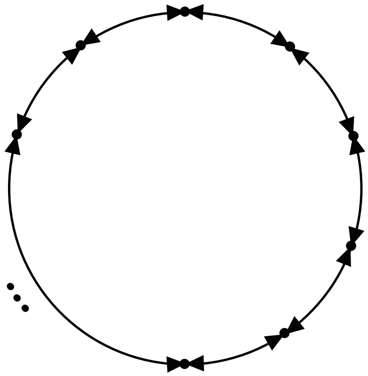

Figure 2: A bidirectional ring of n processes.

```
HS(n, p)
01 P_i in parallel for i = 1 to n                     // line 01–05: Initialization.
02     read p_i
03     ml_i = (i, out, 1)                          // line 03: First message of P_i to P_{i-1}.
04     mr_i = (i, out, 1)                          // line 04: First message of P_i to P_{i+1}.
05     s_i = unknown               // line 05: Initialization of the first state of P_i.
06 N = 2n                                               // line 06:Iinitialization of M.
07 while all states are unknown                // line 07–12: Computation of M.
07         P_i in parallel for i = 1 to n
08             mr_i = null
09             ml_i = null
10             if mr_{i-1} == (j, out, h)
11                 if j > i and h > 1
12                     mr_i = (j, out, h - 1)
13                     N_i = N_i + 1
14                 if j > i and h == 1
15                     ml_i = (j, in, 1)
16                     N_i = N_i + 1
17                 if j = i
18                     s_i = leader
19                     Q = 0          // line 17–19: Summing numbers of messages.
20                     for i = 1 to n
21                         Q = Q + N_i
22                     return i, N, Q               // line 22: Return of the results.
```

```
        if ml_{i+1} == (j, out, h)
            if j > i and h > 1
            ml_i = (j, out, h − 1)
            N_i = N_i + 1
            if j > i and h == 1
            mr_i = (j, in, 1)
            N_i = N_i + 1
            if j = i
            s_i = leader
            Q = 0 // line 17–19: Summing the numbers of the messages.
            for i = 1 to n
                Q = Q + N_i
            return i, N, Q                    // line 17: Return of the results.
        if ml_{i+1} == (j, in, 1) and i ≠ j
            mr_i = (j, in, 1)
            N_i = N_i + 1
        if ml_{i+1} == (j, in, 1) and i ≠ j
            ml_i = (j, in, 1)
            N_i = N_i + 1
        if mr_{i−1} == (i, in, 1) and ml_{i+1} == (i, in1)
            phase = phase + 1
            mr_i = (i, out, 2^phase
            ml_i = (i, out, 2^phase
```

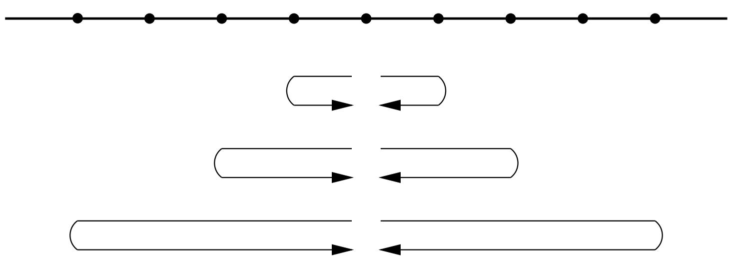

Figure 3: Paths of messages of process P_i in algorithm HS.

Hirschberg and Sinclair [35] proved the following property of their algorithm. Let W_n denote the maximal number of messages required by HS in a bidirectional synchronous ring.

Theorem 9 (Hirschberg, Sinclair [35]) *If* $n \geq 1$, *then*

$$W_n \leq 8n(\lceil \log n \rceil + 1) = \Theta(n \log n) \tag{29}$$

and

$$W_n = O(n \log n). \tag{30}$$

We proved the following, stronger assertion.

Theorem 10 *If* $n \geq 2$, *then*

$$2n\lfloor \log n \rfloor \leq W_n \leq 8n\lceil \log n \rceil \tag{31}$$

and

$$W(n) = \Theta(n \log n). \tag{32}$$

Proof. The proof follows the ideas of application of *bit reversing rings* (see [59, Example 3.6.3] and [59, Figure 3.3]. Let $n = 2^k$, for example with $k = 3$. If we choose $p_{2^0} = p_1 = n = 8$ and $p_{2^0+2^{k-1}} = p_5 = 7$, then $p_{2^0+2^{k-2}} = p_3 = 5$, $p_{2^0+2^{k-1}+2^{k-2}} = p_7 = 6$, and finally the remaining processes get the UIDS 1, 2, 3, 4, and use similar construction for larger k's then we need at least $8 \cdot 2 + 4 \cdot 2 + 2 \cdot 2 = 28$ (in general: $3{,}5n$) messages. If $2^{k-1} \leq n < 2^k$ then we suppose $n = 2^k$ processes and need at least n messages instead of $2n$. If $n = 2$ then we need only $2 \cdot 2$ (in general: $2n$) messages, therefore appears in the theorem only $2n$ as lower bound. □

Burns [12] published in 1980 a bidirectional algorithm which has a bit better worst case bound for the number of necessary messages.

3.3 Leader election is further unidirectional networks

Dolev et al. [20] and Peterson in 1982 [70] independently published an unidirectional algorithm whose worst message number is $O(n \log)n$, but their algorithm allows that the processes have arbitrary long response time that is they algorithm works only in asynchronous networks.

Rotem et al. [76] in 1987, Santoro et al. [77] in 1988 proposed an unidirectional asynchronous algorithm having $O(n \log n)$ messages in the worst case. Their algorithm elected not only the process having the largest UID, but also the processes having the k largest UID's.

Higham and Przytycka [33, 34] used a trick of Smith [89] and proposed an asynchronous algorithm what sends no more then $1.271 n \log n + O(n)$ messages in worst case.

The mentioned algorithms suppose that the processes start in the same round (otherwise they can not terminate). Recently Arrieta et al. [4] elaborated an algorithm allowing different starting rounds of the processes. The price of this property is that the guarantee for the worst message number is only $O(n^2)$.

In 1996 Alimonti et al. [3] considered the problem of choosing the minimum and maximum of the UID's when equal UID's are allowed. If the size of the ring is unknown then the problem is unsolvable. The authors describe an algorith for the unidirectional ring network containing n processes, where the processes know n. The worst bit complexity (that is the number of sent bits) of their algorithm is $O((c + \log n)n)$ with arbitrary $c > 0$ and the time bound is $O(c \cdot n \cdot x^{1/c})$, where $x = \max(|u_{\min}|, |u_{\max}|)$.

Attiya et al. [5] in 1989, Kalamboukis et al. [41] in 1991, and Pan [67] in 1994 studied the leader election in chordal rings.

Vitányi [93] in 1984 analyzed the leader election algorithms of Archimedean rings, Kranakis and Krizane [50] in 1997 of *anonymous* (in which the processes are undistinguishable) hypercube, and Mans [61] also in 1997 of unlabeled tori

Attiya et al. [6] proved lower bounds for the necessary number of messages for anonymous ring networks.

Ingram et al. [37] proposed a leader election algorithm for dynamic asynchronous network. Ingram et al. [36] described algorithms for dynamic networks with clausal clocks. Augustin et al. [7] published a robust leader election algorithm for the fast-changing world.

3.4 Leader election in further special and general networks

Peterson [71] in 1985 described efficient algorithms for mesh networks.

In 1995 Masapati and Ural [62] proposed a linear time leader election algorithms for recursively scalable networks.

Yamashita and Kameda [101], further Kranakis and Krizanc [50] investigated algorithms in anonymous hypercube networks.

Tel in 1995 [91], Flocchini and Mans [25] in 1996 analyzed the leader election algorithms of hypercube networks.

King et al. [45] in 1989, Kim and Belford [44] in 1996 proposed algorithms for unreliable networks.

In 1997 Mans [61] described an optimal distributed algorithm for unlabeled tori.

In 2001 Gavoille [29] analyzed the leader election problem of De Bruijn networks.

In 2005 Shi and Srimani [80] described an algorithm for hierarchical star networks.

In 2007 Srimani and Lafiti [90] proposed an algorithm for Cayley networks.

In 2008 Sepehri and Godarzi [79] described an algorithm for tree networks and using heap structure they proved that their algorithm in worst case requires only $O(n)$ messages.

Peterson [70] in 1952 described efficient algorithms for general networks.

In 1985 Afek and Gafni [2] proved that leader election in general networks requires $\Omega(n \log n)$ messages and $\Omega(\log n)$ time.

Peleg in 1990 [69] proposed a time optimal leader election algorithm for gereral networks which can be applied also for some special networks.

The basic algorithms of general networks are FloodMax and OptFloodMax (see e.g. [59]).

Das et al. [18] proposed effective algorithms which either elect a leader or signalize that the election is impossible.

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About a condition for starlikeness

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Dedicated to the memory of Professor Antal Bege

Abstract. A result concerning the starlikeness of the image of the Alexander operator is improved in this paper. The techniques of differential subordinations are used.

1 Introduction

Let $U(z_0, r) = \{z \in \mathbb{C} \big| \, |z - z_0| < r\}$ be the disk centred in z_0 and let $U = U(0, 1)$ be the open unit disk in $\mathbb{C}$. Let $\mathcal{A}$ be the class of analytic functions f, which are defined on the unit disc U and have the form: $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$.

The subclass of $\mathcal{A}$ consisting of functions for which the domain $f(U)$ is starlike with respect to 0, is denoted by S^*. An analytic characterization of S^* is given by

$$S^* = \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > 0, \; z \in U \right\}.$$

Another subclass of $\mathcal{A}$ we deal with is the class of close-to-convex functions denoted by C. A function $f \in \mathcal{A}$ belongs to the class C if and only if there is a starlike function $g \in S^*$, so that $\Re \frac{zf'(z)}{g(z)} > 0, \quad z \in U$. We note that C and

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S^* contain univalent functions. The Alexander integral operator is defined by the equality

$$A(f)(z) = \int_0^z \frac{f(t)}{t}\, dt.$$

The authors of [1] (p. 310–311) proved the following result:

Theorem 1 *Let A be Alexander operator and let $g \in \mathcal{A}$ satisfy*

$$\Re \frac{zg'(z)}{g(z)} \geq \left| \Im \frac{z(zg'(z))'}{g(z)} \right|, \ z \in U. \tag{1}$$

If $f \in \mathcal{A}$ and

$$\Re \frac{zf'(z)}{g(z)} > 0, \ z \in U, \tag{2}$$

or

$$\Re \frac{f'(z)}{g'(z)} > 0, \ z \in U, \tag{3}$$

then $F = A(f) \in S^*$.

In [1], [3], [5] improvements of the first part ($(1),\ (2) \ \Rightarrow \ A(f) \in S^*$) of this result is proved, simplifying condition (1). The aim of this paper is to give an improvement for the second part of Theorem 1. In order to do this, we need the definitions and lemmas exposed in the next section.

2 Preliminaries

Let f and g be analytic functions in U. The function f is said to be subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1, \ z \in U$ and $f(z) = g(w(z)), \ z \in U$. Recall that if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Lemma 1 [2] p. 24 (Miller-Mocanu)

Let $p(z) = a + \sum\limits_{k=n}^{\infty} a_k z^k$ be analytic in U with $p(z) \not\equiv a$, $n \geq 1$ and let $q : U \to \mathbb{C}$ be an analytic and univalent function with $q(0) = a$. If p is not subordinate to q, then there are two points $z_0 \in U$, $|z_0| = r_0$ and $\zeta_0 \in \partial U$ and a real number $m \in [n, \infty)$, so that q is defined in ζ_0, $\ p(U(0, r_0)) \subset q(U)$, and:

(i) $p(z_0) = q(\zeta_0)$,

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$,

and

(iii) $\mathrm{Re}\left(1 + \dfrac{z_0 p''(z_0)}{p'(z_0)}\right) \geq m\mathrm{Re}\left(1 + \dfrac{\zeta_0 q''(\zeta_0)_0}{q'(\zeta_0)}\right)$.

We note that $z_0 p'(z_0)$ *is the outward normal to the curve* $p(\partial U(0, r_0))$ *at the point* $p(z_0)$*.* $(\partial U(0, r_0)$ *denotes the border of the disc* $U(0, r_0)$*.)*

Lemma 2 [2] p. 26 (Miller-Mocanu) *Let* $p(z) = a + \sum\limits_{k=n}^{\infty} a_k z^k$, $p(z) \not\equiv a$ *and* $n \geq 1$.

If $z_0 \in U$ *and*

$$\mathrm{Re}\, p(z_0) = \min\{\mathrm{Re}\, p(z) : |z| \leq |z_0|\},$$

then

(i) $z_0 p'(z_0) \leq -\dfrac{n}{2}\dfrac{|p(z_0) - a|^2}{\mathrm{Re}\,(a - p(z_0))}$

and

(ii) $\mathrm{Re}\,[z_0^2 p''(z_0)] + z_0 p'(z_0) \leq 0$.

Lemma 3 *If* $d = \frac{2}{\pi}\arctan\left(\frac{1}{2.273}\right)$, *and* $k_d(z) = \int_0^z \frac{\left(\frac{1+t}{1-t}\right)^d - 1}{t} dt$, *then*

$$\left|\Im\big(k_d(z)\big)\right| \leq \frac{\pi}{6}, \quad z \in U.$$

Proof. The maximum principle for harmonic functions implies that

$$\sup_{z \in U}\left|\Im k_d(z)\right| = \sup_{\theta \in [-\pi,\pi]}\left|\Im k_d(e^{i\theta})\right|.$$

On the other hand we have:

$$v_d(\theta) = \Im k_d(e^{i\theta}) = \int_0^1 \frac{1}{x}\left(\frac{1 + x^2 + 2x\cos\theta}{1 + x^2 - 2x\cos\theta}\right)^d \sin\left(d\arctan\left(\frac{2x\sin\theta}{1 - x^2}\right)\right)dx.$$

This implies that v_n is an even function, consequently

$$\sup_{\theta \in [-\pi,\pi]}\left|\Im k_d(e^{i\theta})\right| = \sup_{\theta \in [0,\pi]}\left|\Im k_d(e^{i\theta})\right|.$$

We will prove the following equality:

$$k_d(e^{i\theta}) = \int_0^1 \frac{\left(\frac{1+xe^{i\theta}}{1-xe^{i\theta}}\right)^d - 1}{x}dx = \int_0^\infty \left[\left(\frac{e^t - 1}{e^t + 1}\right)^d - 1\right]dt + i(\pi - \theta) +$$
$$\left(\sin\left(\frac{\pi}{2}d\right) - i\cos\left(\frac{\pi}{2}d\right)\right)\int_0^{\pi-\theta} \tan^d\frac{x}{2}dx, \ \theta \in [0, \pi]. \qquad (4)$$

We begin with the observation that the change of variable $x = e^{-t}$ leads to

$$k_d(e^{i\theta}) = \int_0^{\infty} \left[\left(\frac{e^t + e^{i\theta}}{e^t - e^{i\theta}}\right)^d - 1\right] dt.$$

Let $\theta \in [0, \pi]$ and consider the function

$$f(z) = \left(\frac{e^z + e^{i\theta}}{e^z - e^{i\theta}}\right)^d - 1.$$

We integrate it on $\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where $\gamma_1(t) = t,\ t \in [0, R]$, $\gamma_2(t) = R - it,\ t \in [0, \pi - \theta]$, $\gamma_3(t) = R - t + i(\theta - \pi),\ t \in [0, R]$ and $\gamma_4(t) = i(\theta - \pi + t),\ t \in [0, \pi - \theta]$. The obtained equality $\int_\Gamma f(z)dz = 0$ leads to

$$\begin{aligned} k_d(e^{i\theta}) = \lim_{R\to\infty} \int_{\gamma_1} f(z)dz = -\lim_{R\to\infty} \Bigg[\int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz + \\ \int_{\gamma_4} f(z)dz \Bigg] = \int_0^{\infty} \left[\left(\frac{e^x - 1}{e^x + 1}\right)^d - 1\right] dx + \\ i(\pi - \theta) + \left(\sin\left(\frac{\pi}{2}d\right) - i\cos\left(\frac{\pi}{2}d\right)\right) \int_0^{\pi-\theta} \tan^d \frac{x}{2} dx. \end{aligned}$$

Thus, it follows

$$v_d(\theta) = \Im k_d(e^{i\theta}) = \pi - \theta - \cos\left(\frac{\pi}{2}d\right) \int_0^{\pi-\theta} \tan^d \frac{x}{2} dx.$$

The function $v_d : [0, \pi] \to \mathbb{R}$ has a maximum at the point $\theta_d = 2\arctan \left(\cos^{\frac{1}{d}}\left(\frac{\pi}{2}d\right)\right)$. A suitable numerical approach shows that

$$\left|\Im\big(k_d(z)\big)\right| \le v_d(\theta_d) = 0.49\dots < \frac{\pi}{6}.$$

$\square$

Lemma 4 *If* $q_d(z) = \exp\left(\int_0^z \frac{\left(\frac{1+t}{1-t}\right)^d - 1}{t} dt\right) = \exp(k_d(z))$, $p \in \mathcal{A}$, *and*

$$\frac{zp'(z)}{p(z)} \prec h(z) = \frac{zq_d'(z)}{q_d(z)},\ z \in U,$$

then $p \prec q_d$.

Proof. We have: $\frac{1}{2d}h(z) = \frac{1}{2d}\left(\left(\frac{1+t}{1-t}\right)^d - 1\right) \in S^*$. Lemma 3 implies:

$$\Re \exp\left(\int_0^z \frac{\left(\frac{1+t}{1-t}\right)^d - 1}{t} dt\right) > 0, \ z \in U.$$

On the other hand:

$$\frac{zq_d'(z)}{h(z)} = \exp\left(\int_0^z \frac{\left(\frac{1+t}{1-t}\right)^d - 1}{t} dt\right).$$

These imply $q_d \in C$, which means that q_d is univalent. If the subordination $p \prec q_d$ does not hold, then according to the Miller-Mocanu lemma it follows that there are two points, $z_0 \in U$ and $\zeta_0 \in \partial U$, and a real number $m \in [1, \infty)$ such that

$$p(z_0) = q_d(\zeta_0),$$
$$z_0 p'(z_0) = m\zeta_0 q_d'(\zeta_0).$$

Since $h(U)$ is a starlike domain with respect to 0, it follows that:

$$\frac{z_0 p'(z_0)}{p(z_0)} = m\frac{\zeta_0 q_d'(\zeta_0)}{q_d(\zeta_0)} = mh(\zeta_0) \notin h(U).$$

This contradicts the subordination $\frac{zp'(z)}{p(z)} \prec h(z), \ z \in U$. The obtained contradiction implies: $p \prec q_d$. □

Lemma 5 *If* $f \in \mathcal{A}$ *and*

$$\left|\arg\frac{zg'(z)}{g(z)}\right| < \arctan\left(\frac{1}{2.273}\right), \ z \in U,$$

then

$$\left|\arg\frac{g(z)}{z}\right| < \frac{\pi}{6}, \ z \in U.$$

Proof. The condition of the lemma is equivalent to

$$\frac{zg'(z)}{g(z)} \prec \left(\frac{1+z}{1-z}\right)^d, \ z \in U.$$

Replacing in the previous lemma $p(z) = \frac{g(z)}{z}$, we get

$$\frac{g(z)}{z} \prec q_d(z) = \exp\left(\int_0^z \frac{\left(\frac{1+t}{1-t}\right)^d - 1}{t} dt\right), \ z \in U.$$

Thus

$$\left|\arg\frac{g(z)}{z}\right| \leq \max_{\theta\in[-\pi,\pi]}\left|\Im\int_0^1 \frac{\left(\frac{1+e^{i\theta}t}{1-e^{i\theta}t}\right)^d - 1}{t}dt\right| = \nu_d(\theta_d) < \frac{\pi}{6},\ z\in U.$$

□

In [1] the following theorem is proved.

Theorem 2 *If* $f\in\mathcal{A}$, *end*

$$\Re\frac{zg'(z)}{g(z)} \geq \left|\Im\frac{z(zg'(z))'}{g(z)}\right|,\ z\in U, \tag{5}$$

then the following inequality holds:

$$\Re\frac{zg'(z)}{g(z)} > 2.273\left|\Im\frac{zg'(z)}{g(z)}\right|, \quad z\in U. \tag{6}$$

3 The main result

Theorem 3 *If* $g\in\mathcal{A}$ *satisfies* (5), *then*

$$\left|\arg(g'(z))\right| < \frac{5\pi}{17},\ z\in U. \tag{7}$$

Proof. Inequality (6) is equivalent to

$$\left|\arg\frac{zg'(z)}{g(z)}\right| < \arctan\frac{1}{2.273}, \quad z\in U. \tag{8}$$

Thus according to Lemma 5 the inequality

$$\left|\arg\frac{g(z)}{z}\right| < \frac{\pi}{6},\ z\in U$$

follows. Summarizing we get

$$|\arg g'(z)| \leq \left|\arg\frac{zg'(z)}{g(z)}\right| + \left|\arg\frac{g(z)}{z}\right| < \arctan\frac{1}{2.273} + \frac{\pi}{6} < 0.92 < \frac{5\pi}{17}.$$

□

If we could improve the previously proved result proving that $\left|\arg(g'(z))\right| < \frac{\pi}{5},\ z\in U$, then it would follow that the next theorem is an improvement of Theorem 1.

Theorem 4 *If* $f, g \in \mathcal{A}$ *and*

$$\left|\arg\left(g'(z)\right)\right| < \frac{\pi}{5}, \ z \in U, \tag{9}$$

then the condition

$$\Re\frac{f'(z)}{g'(z)} > 0, \ z \in U$$

implies that $F = A(f) \in S^*$.

Proof. The conditions of the theorem imply

$$\left|\arg f'(z)\right| \leq \left|\arg \frac{f'(z)}{g'(z)}\right| + \left|\arg g'(z)\right| \leq \frac{7\pi}{10}, \ z \in U. \tag{10}$$

Using this result, we will prove that

$$\left|\arg \frac{f(z)}{z}\right| \leq \alpha_0 = \frac{50\pi}{108}, \ z \in U. \tag{11}$$

To do this we rewrite inequality (11) in the following equivalent form:

$$\frac{f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\frac{2}{\pi}\alpha_0}, \ z \in U.$$

If this subordination does not hold, then using Lemma 1 it follows that there are two points $z_0 \in U$, $\zeta_0 = e^{i\theta_0} \in \partial U$ and a real number $m_0 \in [1, \infty)$, such that:

$$\frac{f(z_0)}{z_0} = \left(\frac{1+\zeta_0}{1-\zeta_0}\right)^{\frac{2}{\pi}\alpha_0} = \left(i \cot \frac{\theta_0}{2}\right)^{\frac{2}{\pi}\alpha_0}$$

$$\begin{aligned} z\left(\frac{f(z)}{z}\right)'\bigg|_{z=z_0} &= f'(z_0) - \frac{f(z_0)}{z_0} = \frac{2}{\pi} m_0 \alpha_0 \zeta_0 \left(\frac{1+\zeta_0}{1-\zeta_0}\right)^{\frac{2}{\pi}\alpha_0 - 1} \frac{2}{(1-\zeta_0)^2} \\ &= \frac{2}{\pi} m_0 \alpha_0 \left(i \cot \frac{\theta_0}{2}\right)^{\frac{2}{\pi}\alpha_0 - 1} \frac{-1}{2\sin^2 \frac{\theta_0}{2}}. \end{aligned}$$

Using these equalities, we deduce

$$f'(z_0) = \left(i \cot \frac{\theta_0}{2}\right)^{\frac{2}{\pi}\alpha_0} \left(1 + i\frac{2}{\pi}\alpha_0 \frac{m_0}{\sin \theta_0}\right).$$

Thus, if $\theta_0 \in [0, \pi]$, then

$$\big|\arg f'(z_0)\big| = \alpha_0 + \arctan\left(\frac{2}{\pi}\alpha_0 \frac{m_0}{\sin\theta_0}\right) \geq \alpha_0 + \arctan\left(\frac{2}{\pi}\alpha_0\right) > \frac{7\pi}{10}, \quad (12)$$

and the case $\theta \in [-\pi, 0]$ is analogous to the previous one. If $\alpha_0 = \frac{50\pi}{108}$, then (12) holds, and this contradicts (10). The contradiction shows that inequality (11) holds.

We prove in the followings that

$$\left|\arg\frac{F(z)}{z}\right| < \alpha_1 = \frac{3\pi}{10}\ z \in U. \quad (13)$$

This inequality is equivalent to the subordination

$$p(z) = \frac{F(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\frac{2}{\pi}\alpha_1} = q(z),\ z \in U.$$

If this subordination does not hold, then we use again Lemma 1 and we get that there are two points $z_1 \in U$, $\zeta_1 = e^{i\theta_1} \in \partial U$ and a real number $m_1 \in [1, \infty)$, such that

$$p(z_1) = q(\zeta_1),$$
$$z_1 p'(z_1) = m_1 \zeta_1 q'(\zeta_1).$$

These equalities imply

$$\begin{aligned}
\frac{f(z_1)}{z_1} &= z_1 p'(z_1) + p(z_1) \\
&= \left(i\cot\frac{\theta_1}{2}\right)^{\frac{2}{\pi}\alpha_1} - \frac{2}{\pi}\alpha_1 m_1 \times \left(i\cot\frac{\theta_1}{2}\right)^{\frac{2}{\pi}\alpha_1 - 1} \frac{1}{2\sin^2\frac{\theta_1}{2}} \qquad (14) \\
&= \left(i\cot\frac{\theta_1}{2}\right)^{\frac{2}{\pi}\alpha_1}\left(1 + i\frac{2}{\pi}\alpha_1\frac{m_1}{\sin\theta_1}\right).
\end{aligned}$$

If $\theta_1 \in [0, \pi]$, then

$$\arg\left(1 + i\frac{2}{\pi}\alpha_1\frac{m_1}{\sin\theta_1}\right) = \arctan\left[\frac{2}{\pi}\frac{m_1}{\sin\theta_1}\right] \geq \arctan\left(\frac{2}{\pi}\alpha_1\right),$$

and (14) implies

$$\left|\arg\frac{f(z_1)}{z_1}\right| \geq \alpha_1 + \arctan\left(\frac{2}{\pi}\alpha_1\right) > \frac{50\pi}{108}.$$

If $\theta \in [-\pi, 0]$, then the same inequality can be deduced. This contradicts (11) and the contradiction implies (13). Now we are able to prove that $F = A(f) \in S^*$. Differentiating the equality $F = A(f)$ twice, we obtain

$$F'(z) + zF''(z) = f'(z).$$

This can be rewritten using the notations $p(z) = \frac{zF'(z)}{F(z)}$, $P(z) = \frac{F(z)}{zg'(z)}$ in the form

$$P(z)(zp'(z) + p^2(z)) = \frac{f'(z)}{g'(z)} z \in U.$$

The conditions of the theorem imply that

$$\Re\big[P(z)(zp'(z) + p^2(z))\big] > 0,\ z \in U. \tag{15}$$

We observe that (9) and (13) imply $\big|\arg(P(z))\big| < \frac{\pi}{2},\ z \in U$ and this is equivalent to $\Re P(z) > 0,\ z \in U$. If $\Re p(z) > 0,\ z \in U$ is not true, then according to Lemma 2 it follows that there are two real numbers $x_2, y_2 \in \mathbb{R}$ and a point $z_2 \in U$, such that $p(z_2) = ix_2$ and $z_2 p'(z_2) = y_2 \leq -\frac{1}{2}(x_2^2 + 1)$. Thus the equality

$$P(z_2)(z_2 p'(z_2) + p^2(z_2)) = P(z_2)(y_2 - x_2^2)$$

and $\Re P(z_2) > 0$ imply that

$$\Re\left[P(z_2)(z_2 p'(z_2) + p^2(z_2))\right] \leq 0.$$

This inequality contradicts (15), hence we deduce $\Re\, p(z) = \Re \frac{zF'(z)}{F(z)} > 0, z \in U$. □

We end the paper stating a hypothesis.

Conjecture 1 *We think that Theorem 3 and Theorem 4 can be improved in such a way that the obtained result would become an improvement of the second part of Theorem 1.*

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A survey of the alternating sum-of-divisors function

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Dedicated to the memory of my friend and colleague, Professor Antal Bege

Abstract. We survey arithmetic and asymptotic properties of the alternating sum-of-divisors function β defined by $\beta(p^a) = p^a - p^{a-1} + p^{a-2} - \cdots + (-1)^a$ for every prime power p^a $(a \geq 1)$, and extended by multiplicativity. Certain open problems are also stated.

1 Introduction

Let β denote the multiplicative arithmetic function defined by $\beta(1) = 1$ and

$$\beta(p^a) = p^a - p^{a-1} + p^{a-2} - \cdots + (-1)^a \tag{1}$$

for every prime power p^a $(a \geq 1)$. That is,

$$\beta(n) = \sum_{d \mid n} d\,\lambda(n/d) \tag{2}$$

for every integer $n \geq 1$, where $\lambda(n) = (-1)^{\Omega(n)}$ is the Liouville function, $\Omega(n)$ denoting the number of prime power divisors of n.

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The function β, as a variation of the sum-of-divisors function σ, was considered by Martin [19], Iannucci [16], Zhou and Zhu [32] regarding the following problem. In analogy with the perfect numbers, n is said to be imperfect if $2\beta(n) = n$. More generally, n is said to be k-imperfect if $k\beta(n) = n$ for some integer $k \geq 2$. The only known imperfect numbers are $2, 12, 40, 252, 880, 10\,880, 75\,852$, $715\,816\,960$ and $3\,074\,457\,344\,902\,430\,720$ (sequence A127725 in [33]). Examples of 3-imperfect numbers are $6, 120, 126, 2520$. No k-imperfect numbers are known for $k > 3$. See also the book of Guy [9, p. 72].

This function occurs in the literature also in another context. Let

$$b(n) = \#\{k : 1 \leq k \leq n \text{ and } \gcd(k, n) \text{ is a square}\}.$$

Then $b(n) = \beta(n)$ $(n \geq 1)$, see Cohen [5, Cor. 4.2], Sivaramakrishnan [23], [24, p. 201], McCarthy [20, Sect. 6], [22, p. 25], Bege [3, p. 39], Iannucci [16, p. 12]. A modality to show the identity $b(n) = \beta(n)$ is to apply a familiar property of the Liouville function, namely,

$$\sum_{d|n} \lambda(d) = \chi(n) \quad (n \geq 1), \tag{3}$$

where χ is the characteristic function of the set of squares. Using (3),

$$\begin{aligned} b(n) = \sum_{k=1}^{n} \chi(\gcd(k, n)) &= \sum_{k=1}^{n} \sum_{d|\gcd(k,n)} \lambda(d) = \sum_{d|n} \lambda(d) \sum_{1 \leq k \leq n,\ d|k} 1 \\ &= \sum_{d|n} \lambda(d) \frac{n}{d} = \beta(n). \end{aligned}$$

In this paper we survey certain known properties of the function β, and give also other ones (without references), which may be known, but we could not locate them in the literature.

We point out that the function β has a double character. On the one hand, certain properties of this function are similar to those of the sum-of-divisors function σ, due to the fact that both β and σ are the Dirichlet convolution of two completely multiplicative functions. Such functions are called in the literature specially multiplicative functions or quadratic functions. Their study in connection to the Busche-Ramanujan identities goes back to the work of Vaidyanathaswamy [30]. See also [15, 21, 22, 24].

On the other hand, further properties of this function are analogous to those of the Euler's totient function φ, as a consequence of the representation of β given above.

We call the function β the *alternating sum-of-divisors function* or *alternating sigma function*. Sivaramakrishnan [24, p. 201] remarked that it may be termed the square totient function.

It is possible, of course, to define other alternating sums of the positive divisors of n. For example, let $\theta(n) = \sum_{d|n} d\,\lambda(d)$ $(n \geq 1)$. Then $\theta(n) = \lambda(n)\beta(n)$ $(n \geq 1)$. This is sequence A061020 in [33]. Another example: let $n = d_1 > d_2 > \cdots > d_{\tau(n)} = 1$ be the divisors of n, in decreasing order, and let $A(n) = \sum_{j=1}^{\tau(n)} (-1)^{j-1} d_j$, cf. [2]. Note that the function A is not multiplicative.

We do not give detailed proofs, excepting the proofs of formulae (10), (16), (17) and of the Proposition in Section 7, which are included in Section 8. We leave to the interested reader to compare the corresponding properties of the functions β, σ and φ. See, for example, the books [1, 10, 22, 24, 27].

In Section 7 we pose certain open problems. One of them is concerning super-imperfect numbers n, defined by $2\beta(\beta(n)) = n$. This notion seems not to appear in the literature. The super-imperfect numbers up to 10^7 are $n = 2, 4, 8, 128, 32\,768$. The number $2\,147\,483\,648$ is also super-imperfect.

The corresponding concept for the sigma function is the following: A number n is called superperfect if $\sigma(\sigma(n)) = 2n$. The even superperfect numbers are 2^{p-1}, where $2^p - 1$ is a Mersenne prime, cf. [26] (sequence A019279 in [33]). No odd superperfect numbers are known.

2 Basic properties

It is clear from (1) that for every prime power p^a $(a \geq 1)$,

$$\beta(p^a) = \frac{p^{a+1} + (-1)^a}{p+1} = \begin{cases} \frac{p^{a+1}-1}{p+1}, & \text{if } a \geq 1 \text{ is odd,} \\ \frac{p^{a+1}+1}{p+1}, & \text{if } a \geq 2 \text{ is even.} \end{cases} \tag{4}$$

We obtain from (2),

$$\sum_{n=1}^{\infty} \frac{\beta(n)}{n^s} = \frac{\zeta(s-1)\zeta(2s)}{\zeta(s)} \quad (\Re(s) > 2),$$

where ζ is the Riemann zeta function, leading to another convolution representation of β, namely

$$\beta(n) = \sum_{d^2 k = n} \varphi(k) \quad (n \geq 1), \tag{5}$$

cf. McCarthy [20, Sect. 6], [22, p. 25], Bege [3, p. 39].

We have $\varphi(n) \le \beta(n) \le n$ $(n \ge 1)$. More exactly, it follows from (5) that for every $n \ge 1$,

$$\beta(n) = \varphi(n) + \sum_{d^2 k = n,\, d>1} \varphi(k) \ge \varphi(n),$$

with equality for the squarefree values of n. Also,

$$\beta(n) \le \sum_{dk=n} \varphi(k) = n,$$

with equality only for $n = 1$.

Moreover, $\beta(n) \le \varphi^*(n)$ for every $n \ge 1$, with equality if and only if n is squarefree or twice a squarefree number. This follows easily from (4). Here φ^* is the unitary Euler function, which is multiplicative and given by $\varphi^*(p^a) = p^a - 1$ for every prime power p^a $(a \ge 1)$, cf. [22, 24]. Also, $\beta(n) \ge \sqrt{n}$ $(n \ge 1,\ n \ne 2,\ n \ne 6)$.

Similar to the corresponding property of the function σ, $\beta(n)$ is odd if and only if n is a square or twice a square.

The function β appears in certain elementary identities regarding the set of squares, for example in

$$\sum_{\substack{k=1 \\ \gcd(k,n) \text{ a square}}}^{n} k = \frac{n(\beta(n) + \chi(n))}{2} \quad (n \ge 1),$$

$$\prod_{\substack{k=1 \\ \gcd(k,n) \text{ a square}}}^{n} k = n^{\beta(n)} \prod_{d \mid n} (d!/d^d)^{\lambda(n/d)} \quad (n \ge 1),$$

which can be deduced from (3).

3 Generalizations

An obvious generalization of β is the function β_a $(a \in \mathbb{C})$ defined by

$$\beta_a(n) = \sum_{d \mid n} d^a \lambda(n/d) \quad (n \ge 1). \tag{6}$$

If $a = m$ is a positive integer, then the following representation can be given: $\beta_m(n) = \#\{k : 1 \le k \le n^m, (k, n^m)_m \text{ is a } 2m\text{-th power}\}$, where $(a, b)_m$

stands for the largest common m-th power divisor of a and b. See McCarthy [20, Sect. 6], [22, p. 51].

Note that if $a = 0$, then $\beta_0 = \chi$, the characteristic function of the set of squares, used above.

For an arbitrary nonempty set S of positive integers let $\varphi_S(n) = \#\{k : 1 \leq k \leq n, \gcd(k, n) \in S\}$. For $S = \{1\}$ and S the set of squares this reduces to Euler's function φ and to the function β, respectively. The function φ_S was investigated by Cohen [6]. For every set S one has

$$\varphi_S(n) = \sum_{d|n} d\mu_S(n/d) \quad (n \geq 1),$$

where the function μ_S is defined by $\sum_{d|n} \mu_S(d) = \chi_S(n)$ $(n \geq 1)$, i.e., $\mu_S = \mu * \chi_S$ in terms of the Dirichlet convolution $*$, χ_S and μ denoting the characteristic function of S and the Möbius function, respectively.

Also, let

$$B(r, n) = \sum_{\substack{k=1 \\ \gcd(k,n) \text{ a square}}}^{n} \exp(2\pi i k r/n),$$

which is an analog of the Ramanujan sum to be considered in Section 5. Then

$$B(r, n) = \sum_{d|\gcd(r,n)} d\,\lambda(n/d) \quad (r, n \geq 1),$$

see Sivaramakrishnan [23], [24, p. 202], Haukkanen [13]. For $r = n$ one has $B(n, n) = \beta(n)$.

These generalizations can also be combined. See also Haukkanen [11, 12]. Many of the results given in the present paper can be extended for these generalizations.

We consider in what follows only the functions β_a defined by (6) and do not deal with other generalizations.

4 Further properties

For every $n, m \geq 1$,

$$\beta(n)\beta(m) = \sum_{d|\gcd(n,m)} \beta(nm/d^2)\,d\,\lambda(d), \tag{7}$$

and equivalently,

$$\beta(nm) = \sum_{d|\gcd(n,m)} \beta(n/d)\beta(m/d)d\mu^2(d), \tag{8}$$

cf. [23], [22, p. 26]. Here (7) and (8) are special cases of the Busche-Ramanujan identities, valid for specially multiplicative functions. See [15, 21, 22, 24, 30] for their discussions and proofs.

Direct proofs of (7) and (8) can be given by showing that both sides of these identities are multiplicative, viewed as functions of two variables and then computing their values for prime powers. Recall that an arithmetic function f of two variables is called multiplicative if it is nonzero and $f(n_1m_1, n_2m_2) = f(n_1, n_2)f(m_1, m_2)$ holds for any $n_1, n_2, m_1, m_2 \geq 1$ such that $\gcd(n_1n_2, m_1m_2) = 1$. See [30], [29], [24, Ch. VII].

The proof of the equivalence of identities of type (7) and (8) is outlined in [14], referring to the work of Vaidyanathaswamy [30].

It follows at once from (8) that $\beta(nm) \geq \beta(n)\beta(m)$ for every $n, m \geq 1$, i.e., β is super-multiplicative. Formula (8) leads also to the double Dirichlet series

$$\sum_{n,m=1}^{\infty} \frac{\beta(nm)}{n^s m^t} = \frac{\zeta(s-1)\zeta(2s)\zeta(t-1)\zeta(2t)\zeta(s+t-1)}{\zeta(s)\zeta(t)\zeta(2(s+t-1))},$$

valid for $s, t \in \mathbb{C}$ with $\Re(s) > 2, \Re(t) > 2$.

The generating power series of β is

$$\sum_{n=1}^{\infty} \beta(n)x^n = \sum_{n=1}^{\infty} \frac{\lambda(n)x^n}{(1-x^n)^2} \quad (|x| < 1),$$

which is a direct consequence of (2).

Consider the functions β_a defined by (6). One has

$$\sum_{n=1}^{\infty} \frac{\beta_a(n)\beta_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a-b)\zeta(2s-2a)\zeta(2s-2b)}{\zeta(s-a)\zeta(s-b)\zeta(2s-a-b)}, \tag{9}$$

valid for every $s, a, b \in \mathbb{C}$ with $\Re(s) > 1 + \max(0, \Re(a), \Re(b), \Re(a+b))$.

This formula is similar to Ramanujan's well-known result for the product $\sigma_a(n)\sigma_b(n)$, where $\sigma_a(n) = \sum_{d|n} d^a$. Formula (9) is due to Chowla [4], in an equivalent form for the product $\theta_a(n)\theta_b(n)$, where $\theta_a(n) = \sum_{d|n} d^a\lambda(d)$.

Formula (9) and that of Ramanujan follow from the next more general result concerning the product of two arbitrary specially multiplicative functions.

If f, g, h, k are completely multiplicative functions, then

$$(f * g)(h * k) = fh * fk * gh * gk * w, \tag{10}$$

where $w(n) = \mu(m)f(m)g(m)h(m)k(m)$ if $n = m^2$ is a square and $w(n) = 0$ otherwise.

This result is given by Vaidyanathaswamy [30, p. 621], Lambek [17], Subbarao [25]. See also [24, p. 50]. The proof of (10) can be carried out using Euler products. This is well-known in the case of Ramanujan's result regarding $\sigma_a\sigma_b$, and is presented in several texts, cf., e.g., [10, Th. 305], [22, Prop. 5.4]. An alternative proof is given by Lambek [17].

In Section 8 we offer another less known short proof of (10).

In the case of the functions $f(n) = n^a$, $h(n) = n^b$, $g(n) = k(n) = \lambda(n)$ we obtain (9) by using the known formulae for the Dirichlet series corresponding to the right hand side of (10).

If $f(n) = n^a$, $h(n) = n^b$, $g(n) = \lambda(n)$, $k(n) = 1$, then we deduce

$$\sum_{n=1}^{\infty} \frac{\beta_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s-a)\zeta(s-a-b)\zeta(2s)\zeta(2s-2b)\zeta(2s-a-b)}{\zeta(s)\zeta(s-b)\zeta(4s-2a-2b)},$$

valid for the same region as (9).

Remark that we obtain, as direct corollaries, the next formulae:

$$\sum_{n=1}^{\infty} \frac{\beta^2(n)}{n^s} = \frac{\zeta(s)\zeta(s-2)\zeta(2s-2)}{\zeta^2(s-1)},$$

$$\sum_{n=1}^{\infty} \frac{\beta(n^2)}{n^s} = \frac{\zeta(s)\zeta(s-2)}{\zeta(s-1)}, \tag{11}$$

$$\sum_{n=1}^{\infty} \frac{\beta(n)\sigma(n)}{n^s} = \frac{\zeta(s-2)\zeta(2s)\zeta^2(2s-2)}{\zeta(s)\zeta(4s-4)},$$

all valid for $\Re(s) > 3$. Here (11) is obtained from (9) by choosing $a = 1$ and $b = 0$.

From these Dirichlet series representations we can deduce the following convolutional identities:

$$\beta^2(n) = \sum_{dk=n} d\, 2^{\omega(d)}\lambda(d)\sigma_2(k) \quad (n \geq 1), \tag{12}$$

$$\beta(n^2) = \sum_{dk=n} d\mu(d)\sigma_2(k) \quad (n \geq 1), \tag{13}$$

$$\beta(n)\sigma(n) = \sum_{d^2 k=n} d^2 2^{\omega(d)} \beta_2(k) \quad (n \geq 1), \tag{14}$$

where $\omega(n)$ denotes the number of distinct prime factors of n.

5 Asymptotic behavior

The average order of $\beta(n)$ is $(\pi^2/15)n$, more exactly,

$$\sum_{n\leq x} \beta(n) = \frac{\pi^2}{30}x^2 + \mathcal{O}\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right). \tag{15}$$

Formula (15) follows from the convolution representation (5) and from the known estimate of Walfisz regarding $\sum_{n\leq x}\varphi(n)$ with the same error term as above.

There are also other asymptotic properties of the φ function, which can be transposed to β by using that $\beta(n) \geq \varphi(n)$, with equality for n squarefree. For example,

$$\liminf_{n\to\infty} \frac{\beta(n)\log\log n}{n} = e^{-\gamma},$$

where γ is Euler's constant (cf. [10, Th. 328] concerning φ). Another example: the set $\{\beta(n)/n : n \geq 1\}$ is dense in the interval $[0, 1]$.

Let $c_r(n)$ denote the Ramanujan sum, defined as the sum of n-th powers of the primitive r-th roots of unity. Then

$$\frac{\beta(n)}{n} = \frac{\pi^2}{15}\sum_{r=1}^{\infty} \frac{\lambda(r)}{r^2} c_r(n) \tag{16}$$

$$= \frac{\pi^2}{15}\left(1 - \frac{(-1)^n}{2^2} - \frac{2\cos(2\pi n/3)}{3^2} + \frac{2\cos(\pi n/2)}{4^2} + \ldots\right),$$

showing how the values of $\beta(n)/n$ fluctuate harmonically about their mean value $\pi^2/15$, cf. [7], [22, p. 245].

A quick direct proof of formula (16) is given in Section 8. We refer to [18] for a recent survey of expansions of functions with respect to Ramanujan sums.

From the identities (12), (13) and (14) we deduce the following asymptotic formulae:

$$\sum_{n\leq x} \beta^2(n) = \frac{2\zeta(3)}{15}x^3 + \mathcal{O}\left(x^2(\log x)^2\right),$$

$$\sum_{n\le x} \beta(n^2) = \frac{2\zeta(3)}{\pi^2}x^3 + \mathcal{O}\left(x^2\log x\right),$$

$$\sum_{n\le x} \beta(n)\sigma(n) = \frac{\pi^6}{2430\zeta(3)}x^3 + \mathcal{O}\left(x^2\right).$$

We also have

$$\sum_{n\le x} \frac{1}{\beta(n)} = K_1 \log x + K_2 + \mathcal{O}\left(x^{-1+\varepsilon}\right), \tag{17}$$

for every $\varepsilon > 0$, where K_1 and K_2 are constants,

$$K_1 = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{a=1}^{\infty} \frac{1}{\beta(p^a)}\right).$$

For the proof of (17) see Section 8.

6 Unitary analog

Consider the function β^* defined by

$$\beta^*(n) = \sum_{d\|n} d\,\lambda(n/d) \quad (n \ge 1),$$

where the sum is over the unitary divisors d of n. Recall that d is a unitary divisor of n if $d \mid n$ and $\gcd(d, n/d) = 1$. Here $\beta^*(p^a) = p^a + (-1)^a$ for every prime power p^a ($a \ge 1$) and

$$\sum_{n=1}^{\infty} \frac{\beta^*(n)}{n^s} = \frac{\zeta(s-1)\zeta(2s)\zeta(2s-1)}{\zeta(s)\zeta(4s-2)} \quad (\Re(s) > 2). \tag{18}$$

The formula (18) can be derived using Euler products or by establishing the convolutional identity

$$\beta^*(n) = \sum_{dk^2=n} \beta(d)k\,q(k),$$

q standing for the characteristic function of the squarefree numbers. This leads also to the asymptotic formula

$$\sum_{n\le x} \beta^*(n) = \frac{63\zeta(3)}{2\pi^4}x^2 + \mathcal{O}\left(x(\log x)^{5/3}(\log\log x)^{4/3}\right).$$

Note the following interpretation: $\beta^*(n) = \#\{k : 1 \le k \le n$ and $\gcd(k,n)_*$ is a square$\}$, where $(a,b)_*$ is the largest divisor of a which is a unitary divisor of b.

7 Super-imperfect numbers, open problems

A number n is super-imperfect if $2\beta(\beta(n)) = n$, cf. Introduction. Observe that, excepting 4, all the other examples of super-imperfect numbers are of the form $n = 2^{2^k-1}$ with $k \in \{1,2,3,4,5\}$. The proof of the next statement is given in Section 8.

Proposition. For $k \ge 1$ the number $n_k = 2^{2^k-1}$ is super-imperfect if and only if $k \in \{1,2,3,4,5\}$.

Problem 1. Is there any other super-imperfect number?

More generally, we define n to be (m,k)-imperfect if $k\beta^{(m)}(n) = n$, where $\beta^{(m)}$ is the m-fold iterate of β. For example, 3, 15, 18, 36, 72, 255 are $(2,3)$-imperfect, 6, 12, 24, 30, 60, 120 are $(2,6)$-imperfect, 6, 36, 144 are $(3,6)$-imperfect numbers.

We refer to [8] regarding (m,k)-perfect numbers, defined by $\sigma^{(m)}(n) = kn$.

Problem 2. Investigate the (m,k)-imperfect numbers.

The numbers $n = 1, 20, 116, 135, 171, 194, 740, \ldots$ are solutions of the equation $\beta(n) = \beta(n+1)$.

Problem 3. Are there infinitely many numbers n such that $\beta(n) = \beta(n+1)$?

Remark that it is not known if there are infinitely many numbers n such that $\sigma(n) = \sigma(n+1)$ (sequence A002961 in [33]). See also Weingartner [31].

The next problem is the analog of Lehmer's open problem concerning the φ function.

Problem 4. Is there any composite number $n \neq 4$ such that $\beta(n)$ divides $n-1$?

Up to 10^6 there are no such composite numbers.

The computations were performed using Maple. The function $\beta(n)$ was generated by the following procedure:

```
beta:= proc(n) local x, i: x:= 1:
for i from 1 to nops(ifactors(n)[2])
do p_i:= ifactors(n)[2][i][1]: a_i:= ifactors(n)[2][i][2];
x:= x*((p_i^(a_i+1)+(-1)^(a_i))/(p_i+1)): od: RETURN(x) end;
# alternating sum-of-divisors function
```

8 Proofs

Proof of formula (10): Write

$$(f*g)(n)(h*k)(n) = \sum_{\substack{d\mid n\\ e\mid n}} f(d)g(n/d)h(e)k(n/e),$$

where $d \mid n, e \mid n \iff \operatorname{lcm}[d,e] \mid n$. Write $d = mu$, $e = mv$ with $\gcd(u,v) = 1$. Then $\operatorname{lcm}[d,e] = muv$ and obtain that this sum is

$$\sum_{\substack{muv\mid n\\ \gcd(u,v)=1}} f(mu)g(n/(mu))h(mv)k(n/(mv))$$

$$= \sum_{muv\mid n} f(mu)g(n/(mu))h(mv)k(n/(mv)) \sum_{\delta\mid(\gcd(u,v)} \mu(\delta).$$

Putting now $u = \delta x$, $v = \delta y$ and using that the considered functions are all completely multiplicative the latter sum is

$$\sum_{\delta^2 xymt=n} (\mu fghk)(\delta)(fk)(x)(gh)(y)(fh)(m)(gk)(t),$$

finishing the proof (cf. [30, p. 621] and [27, p. 161]).

Proof of formula (16): Let $\eta_r(n) = r$ if $r \mid n$ and $\eta_r(n) = 0$ otherwise. Applying that $\sum_{d\mid r} c_d(n) = \eta_r(n)$ we deduce

$$\frac{\beta(n)}{n} = \sum_{d\mid n} \frac{\lambda(d)}{d} = \sum_{d=1}^{\infty} \frac{\lambda(d)}{d^2}\eta_d(n) = \sum_{d=1}^{\infty} \frac{\lambda(d)}{d^2} \sum_{r\mid d} c_r(n)$$

$$= \sum_{r=1}^{\infty} \frac{\lambda(r)}{r^2} c_r(n) \sum_{k=1}^{\infty} \frac{\lambda(k)}{k^2} = \frac{\zeta(4)}{\zeta(2)} \sum_{r=1}^{\infty} \frac{\lambda(r)}{r^2} c_r(n),$$

using that λ is completely multiplicative and its Dirichlet series is $\sum_{n=1}^{\infty} \lambda(n)/n^s = \zeta(2s)/\zeta(s)$. The rearranging of the terms is justified by the absolute convergence.

Proof of formula (17): Write

$$\frac{1}{\beta(n)} = \sum_{\substack{dk=n\\ \gcd(d,k)=1}} \frac{h(d)}{\varphi(k)}$$

as the unitary convolution of the functions h and $1/\varphi$, where h is multiplicative and for every prime power p^a ($a \geq 1$),

$$\frac{1}{\beta(p^a)} = h(p^a) + \frac{1}{\varphi(p^a)}, \quad h(p^a) = -\frac{p^{a-1} + (-1)^a}{p^{a-1}(p-1)(p^{a+1} + (-1)^a)}.$$

Here

$$|h(p^a)| \leq \frac{1}{p^a(p-1)^2}, \quad |h(n)| \leq \frac{f(n)}{\varphi(n)} \ (n \geq 1),$$

with $f(n) = \prod_{p|n}(p(p-1))^{-1}$. We deduce

$$\sum_{n \leq x} \frac{1}{\beta(n)} = \sum_{d \leq x} h(d) \sum_{\substack{k \leq x/d \\ \gcd(d,k)=1}} \frac{1}{\varphi(k)},$$

and use the known estimates for the inner sum. The same arguments were applied in the proof of [28, Th. 2].

Proof of the Proposition of Section 7: The fact that the numbers n_k with $1 \leq k \leq 5$ are super-imperfect follow also by direct computations, but the following arguments reveal a connection to the Fermat numbers $F_m = 2^{2^m} + 1$.

For $n_k = 2^{2^k - 1}$ with $k \geq 1$ we have

$$\beta(n_k) = \frac{2^{2^k} - 1}{3} = F_1 F_2 \cdots F_{k-1}$$

(for $k = 1$ this is 1, the empty product). Since the numbers F_m are pairwise relatively prime,

$$\beta(\beta(n_k)) = \beta(F_1)\beta(F_2) \cdots \beta(F_{k-1}).$$

Now for $2 \leq k \leq 5$, using that F_1, F_2, F_3, F_4 are primes,

$$\beta(\beta(n_k)) = 2^{2^1} \cdot 2^{2^2} \cdot \ldots \cdot 2^{2^{k-1}} = 2^{2^k - 2} = \frac{n_k}{2},$$

showing that n_k is super-imperfect.

Now let $k \geq 6$. We use that F_5 is composite and that $\beta(n) \lneqq n - 1$ for every $n \neq 4$ composite. Hence $\beta(F_5) \lneqq 2^{2^5}$ and

$$\beta(\beta(n_k)) \lneqq \beta(F_1)\beta(F_2) \cdots \beta(F_{k-1}) = 2^{2^k - 2} = \frac{n_k}{2},$$

ending the proof.

Note that for $k \geq 2$ the number $m_k = 2^{2^k - 1} F_1 F_2 \cdots F_{k-1}$ is imperfect if and only if $k \in \{2, 3, 4, 5\}$. This follows by similar arguments. The imperfect numbers of this form are 40, $10\,880$, $715\,816\,960$ and $3\,074\,457\,344\,902\,430\,720$, given in the Introduction.

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