Acta Universitatis Sapientiae

Mathematica

Volume 4, Number 2, 2012

Sapientia Hungarian University of Transylvania
Scientia Publishing House

Contents

Acta Univ. Sapientiae, Mathematica, **4**, 2 (2012) 109–116

Metric on the hyper-octahedral group: the minimal deviation

János Gonda
Eötvös Loránd University
Department of Computer Algebra
Budapest, Hungary
email: `andog@inf.elte.hu`

Abstract. The n-dimensional hyper-octahedral group is the group of the distance-preserving transformations of the n-dimensional cube. This group, denoted by T_n, is the semi-direct product of S_2^n and S_n, where for any positive integer k, S_k is the symmetric group of degree k. On this group a metric can be defined in the following way. Let us consider the set of the distances between the images under the two transformations of every vertex of the hypercube. Then the distance between the two transformations is the maximum of this set. If we consider the vertices of the cube as the points of the n-dimensional Boolean space, that is, if we represent the vertices of the n-dimensional cube by the elements of the set of $\{0,1\}^n$, then a particular element of T_n can be given in the form of $(\pi, \underline{\alpha})$, where $\underline{\alpha} \in \{0,1\}^n$, and π is a permutation of the set of $\{k \in \mathbf{N} \,|\, k < n\}$ ($\mathbf{N}$ denotes the set of the non-negative integers, and the elements of $\{0,1\}^n$ are indexed from 0). By this representation the metric defined on T_n can be determined by an inner manner, that is, the distance of two transformations is determined by $\underline{\alpha}$ and the decomposition of π into disjoint cycles (see for instance in [2]).

This metric involves a norm on the group, the norm of a transformation being its distance from the identity of the group. This norm is a maximum, being the maximum of the set of distances between a vertex and its transformed image, for every vertex of the hypercube. However, sometimes the minimum of these distances can be interesting. In this paper we deal with this value.

2010 Mathematics Subject Classification: 11A25, 06E30, 94C10, 15A18
Key words and phrases: hyperoctahedral group, metric, norm, deviance, Boolean space

1 Introduction

Let B_n denote the set of the n-dimensional Boolean vectors. B_n is a metric space with the Hamming-distance, that is, with $d\left(\underline{x}, \underline{y}\right) = \sum_{i=0}^{n-1}\left(x_i \oplus y_i\right)$ [1] where $\underline{x} \in B_n$, $\underline{y} \in B_n$, x_i and y_i are the i-th coordinates of $\underline{x}$ and $\underline{y}$, respectively, and $\oplus$ denotes the modulo 2 sum. B_n is a representation of the abstract notion of the n-dimensional cube. The cardinality of B_n is equal to 2^n, this being the number of the vertices of an n-dimensional cube. Two vertices of the n-dimensional cube are neighbouring if and only if they are connected by an edge of the cube. We can define a similar relation, the relation of neighbourhood, between the elements of the n-dimensional Boolean vectors as follows. Let two Boolean vectors be neighbouring if and only if they differ from each other in exactly one component, that is, if and only if the Hamming-distance of the two Boolean vectors is 1. A vertex of an n-dimensional cube has n neighbouring vertices, and this is the number of the Boolean vectors having a Hamming-distance of 1 from a fixed Boolean vector. If we define the distance of two vertices of an n-dimensional cube as the minimum number of edges we have to pass from one to the other, then it is easy to see that this rule defines a distance function. There are $2^n n!$ distance-preserving bijections between the the vertices of the n-dimensional cube and the vectors of the n-dimensional Boolean space. Indeed, let us fix an arbitrary vertex of the n-dimensional cube, denoted by v_0. We have 2^n different choices for a corresponding Boolean vector. Every Boolean vector has n neighbouring vectors in the same way as every vertex of the cube has n neighbouring vertices. There are altogether $n!$ one to one mappings between these two sets of n elements, so we have a total of $2^n n!$ different bijections between $n+1$ elements of the corresponding sets. Till now we have given the image of an arbitrarily chosen vertex, together with its neighbours. Let us denote this mapping by φ and let A be the set of these $n+1$ vertices. Then it can be proved that there is exactly one extension ψ of φ such that $d\left(\psi\left(v'\right), \varphi\left(v\right)\right) = \widetilde{d}\left(v', v\right)$ for all pairs of vertices v' of the cube and $v \in A$ (see for instance [1]).

From the previously mentioned facts follows that we can study the effects of the n-dimensional hyper-octahedral group on B_n. Let T_n denote the group of the congruences of the n-dimensional cube acting on B_n. In this case $T_n = \left\{(\pi, \underline{\alpha}) \mid \pi \in S_n \text{ and } \underline{\alpha} \in \{0,1\}^n\right\}$, where S_n is the symmetric group of degree n acting on the set of the non-negative integers less than n. If $\underline{x} = \left(x_0, \dots x_{n-1}\right) \in B_n$, $u = (\pi, \underline{\alpha}) \in T_n$ and $\underline{\alpha} = \left(\alpha_0, \dots, \alpha_{n-1}\right)$, then $\underline{x}^u = \left(x_{\pi(0)}^{\alpha_0}, \dots, x_{\pi(n-1)}^{\alpha_{n-1}}\right)$ so that $x^\alpha = \alpha \oplus x$. Among all transformations on B_n, only the elements of T_n

preserve the distances between the elements of B_n, so this group is the isometric group of B_n. T_n is the wreath product of S_2 and S_n, that is, $T_n = S_2 \wr S_n$, where S_n is the symmetric group of degree n [5], [6], [7], [8].

In [2] we have dealt with an inner characterization of the metric and the norm of the hyper-octahedral group. In the following we shortly summarize the results of that article, and then, in the next section, we deal with the minimal value of the effect of a transformation of the hyper-octahedral group.

Definition 1 *Let* $n \in \mathbf{N}$*,* $u \in T_n$*,* $v \in T_n$*. Then* $\overline{d}\,(u, v) = \max\limits_{\underline{x} \in B_n} \{d\,(\underline{x}^u, \underline{x}^v)\}$.

$\overline{d}$ defines a metric on T_n (see for instance in [9]).

$\overline{d}$ is left and right invariant on T_n, that is, for any $u \in T_n$, $v \in T_n$ and $w \in T_n$,

$$\overline{d}\,(uw, vw) = \overline{d}\,(u, v) \tag{1}$$

and

$$\overline{d}\,(wu, wv) = \overline{d}\,(u, v)\,. \tag{2}$$

$\overline{d}$ can be determined in an inner manner. Let $w = (\pi, \underline{\alpha}) \in T_n$ be an arbitrary element, let

$$\pi = \prod_{t=0}^{s-1} c_t \tag{3}$$

be the disjoint cycle decomposition of the permutation π. Further, let $c_k = \left(c_{k_0}, \dots, c_{k_{m_k-1}}\right)$ be the k-th member of the product in (3), where $0 \le k < s$, m_k is the length of the k-th cycle of the previous product for $0 \le k < s$, and $\underline{\alpha} = (\alpha_0, \dots, \alpha_{n-1}) \in \{0, 1\}^n$, furthermore let $t_k = \left(m_k + \sum_{i=0}^{m_k-1} \alpha_{c_{k_i}}\right) \bmod 2$ and $\tau\,(w) = \sum_{k=0}^{s-1} t_k$.

Theorem 1 *Let* u *and* v *be two arbitrary elements from* T_n*. Then* $\overline{d}\,(u, v) = n - \tau\left(uv^{-1}\right)$*.*

Using the metric studied above, one can define the norm of the elements of T_n [2].

Definition 2 *Let* T_n *be the isometric group of the* n*-dimensional Boolean space. Then* $\|u\| = \overline{d}\,(e, u)$ *is the norm of* $u \in T_n$*.*

From this definition immediately follows that

1. $\|u\| = 0$ if and only if $u = e$;

2. $\|u\| = \left\|u^{-1}\right\|$ for every $u \in T_n$;

3. $\overline{d}(u, v) = \left\|uv^{-1}\right\|$ for every $(u, v) \in T_n^2$.

Theorem 2 *Let* $\varphi : u \mapsto \|u\|$*. Then* $\mathrm{Im}(\varphi) = \mathbf{N}_n = \{k \in \mathbf{N} \,|k < n\}$.

In Theorem 2 $\mathbf{N}$ denotes the set of the non-negative integers.

2 New results

In the previous section we characterized an element of the hyper-octahedral group by its maximal effect regarded as the distance between a vector of the Boolean space and its transformed image. But sometimes the expectation is the opposite, that is, we wish that the effect of the transformation be as little as possible. This expectation leads to the following notion.

Definition 3 *Let* T_n *be the isometric group of the* n*-dimensional Boolean space and let* $u \in T_n$*. Then* $\langle\langle u \rangle\rangle = \min\limits_{\underline{x} \in B_n} \{d(\underline{x}, \underline{x}^u)\}$.

$\langle\langle u \rangle\rangle$ shows the minimal effect of $u \in T_n$. By the definition it seems, that $\langle\langle u \rangle\rangle$ depends not only on u, but on the elements of the Boolean space. However, the next statement proves that $\langle\langle u \rangle\rangle$ can be given in a form depending only on u.

Theorem 3 *Let* $u = (\pi, \underline{\alpha}) \in T_n$*, where* $\pi \in S_n$ *and* $\underline{\alpha} \in \{0, 1\}^n$*. If* $\pi = \prod_{t=0}^{s-1} c_t$ *is the disjoint cycle decomposition of the permutation* π*, for* $0 \leq k < s$ $c_k = \left(c_{k_0}, \ldots, c_{k_{m_k-1}}\right)$ *is the* k*-th member of the previous product, then*

$$\langle\langle u \rangle\rangle = \sum_{k=0}^{s-1} t_k^{'}, \tag{4}$$

where $t_k^{'}$ *denotes the remainder of* $\sum_{i=0}^{m_k-1} \alpha_{c_{k_i}}$ *by* $\mathtt{modulo}\ 2$.

Before the precise verification of the theorem we would like to highlight the idea of the proof.

For the sake of the simplicity let us suppose that π in $u = (\pi, \alpha) \in T_n$ is a cycle, for instance the cycle of the first k elements of the indices, that is, $\pi = (0, 1, \ldots, k-1)$, where $n > k \in N$, and for $n > i \geq k, i \in \mathbf{N}$, $\alpha_i = 0$. In this case for an arbitrary element $\underline{x}$ of B_n,

$$\begin{pmatrix} \underline{x} \\ \underline{x}^u \end{pmatrix} = \begin{pmatrix} x_0 & x_1 & \ldots & x_{k-2} & x_{k-1} & x_k & \ldots & x_{n-1} \\ x_1^{\alpha_0} & x_2^{\alpha_1} & & x_{k-1}^{\alpha_{k-2}} & x_0^{\alpha_{k-1}} & x_k & \ldots & x_{n-1} \end{pmatrix}.$$

Now the number of the positions where the original and the transformed vectors differ from each other can be calculated as follows. If $n > i \geq k, i \in \mathbf{N}$, then $x_i = x_{\pi(i)}^{\alpha_i} = (\underline{x}^u)_i$, so in that part of the vector there is no position where the two vectors differ, the number of the different positions of that domain is equal to 0. Now let us consider the first part of the vectors, that is, the first k positions. We try to get as few different positions as possible. The best result is, if $x_i = x_{\pi(i)}^{\alpha_i} = x_{(i+1) \bmod k}^{\alpha_i}$ for every $k > i \in \mathbf{N}$. Then

$$\begin{array}{ccccccc}
x_0 & & & & & = & x_1^{\alpha_0} \\
x_0 & = & x_1^{\alpha_0} & = & \left(x_2^{\alpha_1}\right)^{\alpha_0} & = & x_2^{\alpha_1 \oplus \alpha_0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_0 & = & x_{k-2}^{\alpha_{k-3} \oplus \cdots \oplus \alpha_0} & = & \left(x_{k-1}^{\alpha_{k-2}}\right)^{\alpha_{k-3} \oplus \cdots \oplus \alpha_0} & = & x_{k-1}^{\alpha_{k-2} \oplus \alpha_{k-3} \oplus \cdots \oplus \alpha_0}
\end{array}$$

and finally

$$x_0 = x_{k-1}^{\alpha_{k-2} \oplus \alpha_{k-3} \oplus \cdots \oplus \alpha_0} = \left(x_0^{\alpha_{k-1}}\right)^{\alpha_{k-2} \oplus \cdots \oplus \alpha_0} = x_0^{\alpha_{k-1} \oplus \alpha_{k-2} \oplus \cdots \oplus \alpha_0}$$

($\oplus$ denotes the $\mathrm{modulo}\ 2$ sum).

All but the last conditions can be easily satisfied. As $a^b = a \oplus b$, so

$$\begin{aligned}
x_0 &= x_0^{\alpha_{k-1} \oplus \alpha_{k-2} \oplus \cdots \oplus \alpha_0} \\
&= x_0 \oplus \alpha_{k-1} \oplus \alpha_{k-2} \oplus \cdots \oplus \alpha_0.
\end{aligned}$$

This last equality is true if and only if $\alpha_{k-1} \oplus \alpha_{k-2} \oplus \cdots \oplus \alpha_0 = 0$, that is, if and only if $\alpha_{k-1} + \alpha_{k-2} + \cdots + \alpha_0$ is an even number. In this case the two vectors are identical, there is no differences, the distance of the two vectors is equal to 0. In the other case, that is, if the sum of the exponents is an odd number, then there is exactly one position where the two vectors differ, so, the distance of the two vectors, and then $\langle\langle u \rangle\rangle$ is equal to 1. That means that the minimal number of the differences, in another words, the minimal deviation caused by this transform is either 0 or 1, depending on the parity of the sum of the exponents.

Now we prove exactly the statement.

Proof. Let $x_{c_{k_1}} = x_{c_{k_0}}^{\alpha_{c_{k_0}}} = x_{c_{k_0}} \oplus \alpha_{c_{k_0}}$. Then

$$\begin{aligned}
(x^u)_{c_{k_0}} &= x_{\pi(c_{k_0})}^{\alpha_{c_{k_0}}} = x_{\pi(c_{k_0})} \oplus \alpha_{c_{k_0}} = x_{c_{k_1}} \oplus \alpha_{c_{k_0}} \\
&= \left(x_{c_{k_0}} \oplus \alpha_{c_{k_0}}\right) \oplus \alpha_{c_{k_0}} = x_{c_{k_0}}.
\end{aligned} \tag{5}$$

For every $1 \le i < m_k$ we have that

$$x_{c_{k_i}} = x_{c_{k_{i-1}}}^{\alpha_{c_{k_{i-1}}}} = x_{c_{k_{i-1}}} \oplus \alpha_{c_{k_{i-1}}} \\ = x_{c_{k_0}} \oplus \left(\bigoplus_{j=0}^{i-1} \alpha_{c_{k_j}} \right). \tag{6}$$

Then we get that

$$x_{c_{k_{m_k-1}}} = x_{c_{k_0}} \oplus \left(\bigoplus_{j=0}^{m_k-2} \alpha_{c_{k_j}} \right). \tag{7}$$

The equality $(x^u)_{c_{k_{m_k-1}}} = x_{c_{k_{m_k-1}}}$ holds if and only if $x_{\pi\left(c_{k_{m_k-1}}\right)}^{\alpha_{c_{k_{m_k-1}}}} = x_{c_{k_{m_k-1}}}$, or, in another way, if and only if

$$x_{c_{k_0}} \oplus \alpha_{c_{k_{m_k-1}}} = x_{c_{k_{m_k-1}}} = x_{c_{k_0}} \oplus \left(\bigoplus_{j=0}^{m_k-2} \alpha_{c_{k_j}} \right). \tag{8}$$

From the equation above we get that

$$\bigoplus_{j=0}^{m_k-1} \alpha_{c_{k_j}} = 0. \tag{9}$$

If this condition is fulfilled then all of the components of $\underline{x}$ and $\underline{x}^u$ belonging to the k-th cycle of the decomposition of π are the same. In the opposite case they differ exactly in one position, and then

$$\bigoplus_{j=0}^{m_k-1} \alpha_{c_{k_j}} = 1. \tag{10}$$

These results mean that if we construct a vector $\underline{x}_0$ taking into consideration the above-mentioned conditions, then the number of the different coordinates of the vectors $\underline{x}_0$ and $\underline{x}_0^u$ is exactly $\sum_{k=0}^{s-1} \left(\oplus_{j=0}^{m_k-1} \alpha_{c_{k_j}} \right)$, and this is the minimal value of the Hamming-distances between the elements of the Boolean space and their transformed images under u, according to the statement of the theorem. $\square$

The range of the values of the function $u \mapsto \langle\langle u \rangle\rangle$, where $u \in T_n$, is as follows.

Theorem 4 *The set of the values of the function* $u \mapsto \langle\langle u \rangle\rangle$*, defined on* T_n*, is equal to* $A = \{k \in \mathbf{N} \,|\, k \le n\}$.

Proof. It is obvious that the set of the values of the function is a subset of the set of $A = \{k \in \mathbf{N} \,|\, k \leq n\}$. We have to show that for every element of that set there is at least one element in T_n so, that $\langle\langle u \rangle\rangle$ is equal to the given integer. Let us consider the following transformation:

$$u = \left(\varepsilon, \left(\underbrace{1, \ldots, 1}_{k}, \underbrace{0, \ldots, 0}_{n-k}\right)\right), \tag{11}$$

where ε is the identity of S_n. Then for any $\underline{x} = (x_0, \ldots, x_{n-1}) \in B_n$ we have that

$$\begin{aligned} \underline{x}^u &= (x_0, \ldots, x_{n-1})^u \\ &= (\overline{x}_0, \ldots, \overline{x}_{k-1}, x_k, \ldots, x_{n-1}). \end{aligned} \tag{12}$$

As $d\left((x_0, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}), (\overline{x}_0, \ldots, \overline{x}_{k-1}, x_k, \ldots, x_{n-1})\right) = k$, that is, $d(\underline{x}, \underline{x}^u) = k$ for every $\underline{x} \in B_n$, so $\langle\langle u \rangle\rangle = \min\limits_{\underline{x} \in B_n} \{d(\underline{x}, \underline{x}^u)\} = k$. □

3 Conclusion

Considering two Boolean functions of the same variables, they are not essentially different if they differ only in the ordering of the variables and in assigning the 0 and 1 to the variables that is in the case when $f_2(x_0, \ldots, x_{n-1}) = f_1\left(x_{\pi(0)}^{\alpha_0}, \ldots, x_{\pi(n-1)}^{\alpha_{n-1}}\right)$, where π is a permutation of the indices of the variables, $\alpha_i \in \{0, 1\}$ and $x^\alpha = \alpha \oplus x = \begin{cases} x & \text{, if } \alpha = 0 \\ \overline{x} & \text{, if } \alpha = 1 \end{cases}$. For instance, let us suppose that we want to describe the statement

"Now it is either raining or the sky is blue, and yesterday MU won again" by the help of mathematical formalism. Then we can denote the first part of the sentence by A (A = "it is raining"), the second part of the sentence by B (B = "the sky is blue") and the third part of it by C (C = "Yesterday MU won again"). By these notations our statement is $F = (A \vee B) \wedge C$, if $\vee$ denotes the disjunction and $\wedge$ denotes the conjunction. But the meaning of $B \wedge (\neg A \vee C)$ is the same as the meaning of the previous form, if now B denotes the sentence "yesterday MU won again", C denotes "the sky is blue" and A stands for "Now it is not raining". As this simple example shows, the two forms of $(A \vee B) \wedge B$ and $B \wedge (\neg A \vee C)$ do not differ essentially, they differ only in the assignment of the variables to the original statements.

This fact explains, why the hyper-octahedral group is so important when we investigate the Boolean functions. And if it is so, then it is understandable that it is important to know, what is the maximal and the minimal impact of an element of the group on the Boolean functions. In another article [2] we examined the maximal effect, and now the minimal effect of the transformations, and stated, that this effect depends only on the transformation given, and that every possible value can be achieved by a transformation chosen in an appropriate way.

References

[1] P. J. Cameron, J. H. van Lint, Designs, Graphs, Codes and their Links, *London Mathematical Society Student Texts 22*, Cambridge University Press, Cambridge, 1991.

[2] J. Gonda, Metrics on the hyper-octahedral group, In: *Informatika a felsőoktatásban 2008*, (http://www.agr.unideb.hu/if2008/kiadvany/papers/F11.pdf, in Hungarian).

[3] M. Hall jr., The theory of groups, MacMillan Co., New York, 1959.

[4] B. Huppert, Endliche Gruppen 1. Springer, Berlin, 1967.

[5] L. A. Kaluzhnin, P. M. Beleckij, V. Z. Feinberg, Kranzprodukte, *Teubner-Texte Vol. 101*, B. G. Tebner, Leipzig, 1987.

[6] L. A. Kaluzhnin, M. Kh. Klin, V. I. Sushchanskii, Exponentiation of permutation groups I. *Izv. Vyssch. Uchebn. Zaved. Mat.* **8** (1979) 26–33 (in Russian).

[7] M. Krasner M. L. Kaloujnine, Produit complet des groupes de permutations et problème d'extension de groupes I. *Acta Sci. Math. Szeged* **13** (1950) 208–230.

[8] M. Krasner M. L. Kaloujnine, Produit complet des groupes de permutations et problème d'extension de groupes II. *Acta Sci. Math. Szeged* **14** (1951) 39–66, 69–82.

[9] L. S. Pontrjagin, Nepreryvnye gruppy, Nauka, Moscow, 1984.

Received: July 14, 2011

Acta Univ. Sapientiae, Mathematica, 4, 2 (2012) 117–131

Some multiplier lacunary sequence spaces defined by a sequence of modulus functions

Kuldip Raj
Shri Mata Vaishno Devi University
School of Mathematics
Katra - 182320, J&K, India
email: kuldipraj68@gmail.com

Sunil K. Sharma
Shri Mata Vaishno Devi University
School of Mathematics
Katra - 182320, J&K, India
email: sunilksharma42@yahoo.co.in

Amit Gupta
Shri Mata Vaishno Devi University
School of Mathematics
Katra - 182320, J&K, India
email: guptaamit796@gmail.com

Abstract. In the present paper we introduce some multiplier sequence spaces defined by a sequence of modulus functions $F = (f_k)$. We also make an effort to study some topological properties and inclusion relations between these spaces.

1 Introduction and preliminaries

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$,
3. f is increasing,
4. f is continuous from right at 0.

2010 Mathematics Subject Classification: 40A05, 40C05, 46A45
Key words and phrases: modulus function, lacunary sequence, paranorm space

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus $f(x)$ is unbounded. Subsequently, modulus function has been discussed in [1], [2], [3], [4], [5], [20], [22], [23], [24], [26] and references therein.
Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$,

2. $p(-x) = p(x)$, for all $x \in X$,

3. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,

4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [29], Theorem 10.4.2, P.183).

Let w denote the set of all real sequences $x = (x_n)$. By ℓ_∞ and c, we denote respectively the Banach space of bounded and the Banach space of convergent sequences $x = (x_n)$, both normed by $\|x\| = \sup_n |x_n|$. A linear functional $\mathcal{L}$ on ℓ_∞ is said to be a Banach limit (see [6]) if it has the properties :

1. $\mathcal{L}(x) \geq 0$ if $x \geq 0$ (i.e. $x_n \geq 0$ for all n),

2. $\mathcal{L}(e) = 1$, where $e = (1, 1, \cdots)$,

3. $\mathcal{L}(Dx) = \mathcal{L}(x)$,

where the shift operator D is defined by $(Dx_n) = (x_{n+1})$.
Let $\mathfrak{B}$ be the set of all Banach limits on ℓ_∞. A sequence x is said to be almost convergent to a number L if $\mathcal{L}(x) = L$ for all $\mathcal{L} \in \mathfrak{B}$. Lorentz [17] has shown that x is almost convergent to L if and only if

$$t_{km} = t_{km}(x) = \frac{x_m + x_{m+1} + \cdots + x_{m+k}}{k+1} \to L \text{ as } k \to \infty, \text{ uniformly in } m.$$

Let $\hat{c}$ denote the set of all almost convergent sequences. Maddox [18] and (independently) Freedman et al. [13] have defined x to be strongly almost convergent to a number L if

$$\frac{1}{k+1}\sum_{i=0}^{k}|x_{i+m}-L|\to 0 \text{ as } k\to\infty, \text{ uniformly in } m.$$

Let $[\hat{c}]$ denote the set of all strongly almost convergent sequences. It is easy to see that $[\hat{c}]\subset\hat{c}\subset\ell_\infty$. Das and Sahoo [11] defined the sequence space

$$[w(p)]=\left\{x\in w:\frac{1}{n+1}\sum_{k=0}^{n}|t_{km}(x-L)|^{p_k}\to 0 \text{ as } n\to\infty, \text{ uniformly in } m.\right\}$$

and investigated some of its properties.
The space of lacunary strong convergence have been introduced by Freedman et al. [13]. A sequence of positive integers $\theta=(k_r)$ is called "lacunary" if $k_0=0, 0<k_r<k_{r+1}$ and $h_r=k_r-k_{r-1}\to\infty$, as $r\to\infty$. The intervals determined by θ are denoted by $I_r=(k_{r-1},k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ is defined by Freedman et al. [13] as follows:

$$N_\theta=\left\{x=(x_i):\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}|x_i-s|=0, \text{ for some } s\right\}.$$

Lacunary sequence spaces were studied by many authors (see [7], [8], [9]) and references therein.
Let $F=(f_k)$ be a sequence of modulus functions, $p=(p_k)$ be a bounded sequence of positive real numbers, $u=(u_k)$ be any sequence of strictly positive real numbers and X be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q. By $w(X)$ we denote the space of all sequences $x=(x_k)$ for all k. In the present paper we define the following classes of sequences:

$$(w,\theta,F,u,p,q)=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[f_k(q(t_{km}(x-L)))\Big]^{p_k}=0,$$
$$\text{uniformly in } m, \quad \text{for some } L\Big\},$$

$$(w,\theta,F,u,p,q)_0=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[f_k(q(t_{km}(x)))\Big]^{p_k}=0,$$
$$\text{uniformly in } m\Big\}$$

and

$$(w,\theta,F,u,p,q)_\infty=\Big\{x=(x_k)\in w(X):\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[f_k(q(t_{km}(x)))\Big]^{p_k}<\infty\Big\}.$$

If we take $f(x)=x$, we have

$$(w,\theta,u,p,q)=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[(q(t_{km}(x-L)))\Big]^{p_k}=0,$$

$$\text{uniformly in } m, \quad \text{for some } L\Big\},$$

$$(w,\theta,u,p,q)_0=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[(q(t_{km}(x)))\Big]^{p_k}=0,$$

$$\text{uniformly in } m\Big\}$$

and

$$(w,\theta,u,p,q)_\infty=\Big\{x=(x_k)\in w(X):\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[(q(t_{km}(x)))\Big]^{p_k}<\infty\Big\}.$$

If we take $p=(p_k)=1$ for all $k\in\mathbb{N}$, we have

$$(w,\theta,F,u,q)=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[f_k(q(t_{km}(x-L)))\Big]=0,$$

$$\text{uniformly in } m, \quad \text{for some } L\Big\},$$

$$(w,\theta,F,u,q)_0=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[f_k(q(t_{km}(x)))\Big]=0,$$

$$\text{uniformly in } m\Big\}$$

and

$$(w,\theta,F,u,q)_\infty=\Big\{x=(x_k)\in w(X):\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[f_k(q(t_{km}(x)))\Big]<\infty\Big\}.$$

If we take $f(x)=x$ and $u=(u_k)=1$ for all $k\in\mathbb{N}$, we have

$$(w,\theta,p,q)=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[(q(t_{km}(x-L)))\Big]^{p_k}=0,$$

$$\text{uniformly in } m, \quad \text{for some } L\Big\},$$

$$(w,\theta,p,q)_0 = \Big\{x=(x_k)\in w(X): \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[(q(t_{km}(x)))\Big]^{p_k}=0,$$

$$\text{uniformly in } m\Big\}$$

and

$$(w,\theta,p,q)_\infty = \Big\{x=(x_k)\in w(X): \sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r}\Big[(q(t_{km}(x)))\Big]^{p_k}<\infty\Big\}.$$

If we take $\theta = (2^r)$, then the spaces (w,θ,F,u,p,q), $(w,\theta,F,u,p,q)_0$ and $(w,\theta,F,u,p,q)_\infty$ reduces to (w,F,u,p,q), $(w,F,u,p,q)_0$ and $(w,F,u,p,q)_\infty$. Throughout the paper Z will denote the $0,1$ or ∞. The following inequality will be used throughout the paper. If $0 < h = \inf p_k \le p_k \le \sup p_k = H$, $D = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \le D\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \le \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

In this paper we study some topological properties and prove some inclusion relations between above defined classes of sequences.

2 Main results

Theorem 1 *Let $F = (f_k)$ be a sequence of modulus functions, $u = (u_k)$ be any sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Then $(w,\theta,F,u,p,q)_Z$ are linear spaces over the field of complex numbers $\mathbb{C}$.*

Proof. We shall prove the result for $Z = 0$. Let $x = (x_k), y = (y_k) \in (w,\theta,F,u,p,q)_0$ and $\alpha,\beta \in \mathbb{C}$. Then there exist integers M_α and N_β such that $|\alpha| \le M_\alpha$ and $|\beta| \le N_\beta$. By using inequality (1.1) and the properties of modulus function, we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}u_k[f_k(q(t_{km}(\alpha x_k+\beta y_k)))]^{p_k} &\le \frac{1}{h_r}\sum_{k\in I_r}u_k[f_k(q(\alpha t_{km}x_k+\beta t_{km}y_k))]^{p_k}\\ &\le D\frac{1}{h_r}\sum_{k\in I_r}u_k[M_\alpha f_k(q(t_{km}(x_k)))]^{p_k}\end{aligned}$$

$$
\begin{aligned}
& + \quad D\frac{1}{h_r}\sum_{k\in I_r} u_k[N_\beta f_k(q(t_{km}(y_k)))]^{p_k} \\
& \leq \quad DM_\alpha^H \frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x_k)))]^{p_k} \\
& + \quad DN_\beta^H \frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(y_k)))]^{p_k} \\
& \to \quad 0, \text{ uniformly in } m.
\end{aligned}
$$

This proves that $(w, \theta, F, u, p, q)_0$ is a linear space. Similarly, we can prove that (w, θ, F, u, p, q) and $(w, \theta, F, u, p, q)_\infty$ are linear spaces. □

Theorem 2 *Let* $F = (f_k)$ *be a sequence of modulus functions. Then we have*

$$(w, \theta, F, u, p, q)_0 \subset (w, \theta, F, u, p, q) \subset (w, \theta, F, u, p, q)_\infty.$$

Proof. The inclusion $(w, \theta, F, u, p, q)_0 \subset (w, \theta, F, u, p, q)$ is obvious. Now, let $x = (x_k) \in (w, \theta, F, u, p, q)$ then

$$\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x_k)))]^{p_k} \to 0, \text{ uniformly in } m.$$

Now by using (1.1) and the properties of modulus function, we have

$$
\begin{aligned}
\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x_k)))]^{p_k} &= \sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x_k - L + L)))]^{p_k} \\
&\leq D\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x_k - L)))]^{p_k} \\
&+ D\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(L))]^{p_k} \\
&\leq D\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r} [f_k(q(t_{km}(x_k - L)))]^{p_k} \\
&+ D\max\{f_k(q(L))^h, f_k(q(L))^H\} \\
&< \infty.
\end{aligned}
$$

Hence $x = (x_k) \in (w, \theta, F, u, p, q)_\infty$. This proves that

$$(w, \theta, F, u, p, q) \subset (w, \theta, F, u, p, q)_\infty.$$

□

Theorem 3 *Let $p = (p_k)$ be a bounded sequence of positive real numbers. Then the space $(w, \theta, F, u, p, q)_0$ is a paranormed space with the paranorm defined by*

$$g(x) = \sup_{r,m} \Big(\frac{1}{h_r} \sum_{k \in I_r} u_k [f_k(q(t_{km}(x_k)))]^{p_k} \Big)^{\frac{1}{M}},$$

where $M = \max(1, \sup_k p_k)$.

Proof. Clearly $g(-x) = g(x)$. It is trivial that $t_{km}x_k = 0$ for $x = 0$. Hence we get $g(0) = 0$. Since $\frac{p_k}{M} \leq 1$ and $M \geq 1$, using the Minkowski's inequality and definition of modulus function, for each x, we have

$$\begin{aligned}
&\Big(\tfrac{1}{h_r} \sum_{k \in I_r} u_k [f_k(q(t_{km}(x_k + y_k)))]^{p_k} \Big)^{\frac{1}{M}} \\
&\leq \Big(\frac{1}{h_r} \sum_{k \in I_r} u_k [f_k(q(t_{km}(x_k))) + f_k(q(t_{km}(y_k)))]^{p_k} \Big)^{\frac{1}{M}} \\
&\leq \Big(\frac{1}{h_r} \sum_{k \in I_r} u_k [f_k(q(t_{km}(x_k)))]^{p_k} \Big)^{\frac{1}{M}} + \Big(\frac{1}{h_r} \sum_{k \in I_r} u_k [f_k(q(t_{km}(y_k)))]^{p_k} \Big)^{\frac{1}{M}}.
\end{aligned}$$

Now it follows that g is subadditive. Finally, to check the continuity of scalar multiplication, let us take any complex number λ. By definition of modulus function F, we have

$$\begin{aligned}
g(\lambda x) &= \sup_{r,m} \Big(\frac{1}{h_r} \sum_{k \in I_r} u_k [f_k(q(t_{km}(\lambda x_k)))]^{p_k} \Big)^{\frac{1}{M}} \\
&\leq K^{\frac{H}{M}} g(x),
\end{aligned}$$

where $K = 1 + [|\lambda|]$ ($[|\lambda|]$ denotes the integer part of λ). Since F is a sequence of modulus functions, we have $x \to 0$ implies $g(\lambda x) \to 0$. Similarly $x \to 0$ and $\lambda \to 0$ implies $g(\lambda x) \to 0$. Finally, we have for fixed x and $\lambda \to 0$ implies $g(\lambda x) \to 0$. $\square$

Theorem 4 *Let F' and F'' be any two sequences of modulus functions. For any bounded sequences $p = (p_k)$ and $t = (t_k)$ of strictly positive real numbers and for any two sequences of seminorms q_1 and q_2, we have*
(i) $(w, \theta, F', u, p, q)_Z \subset (w, \theta, F' o F'', u, p, q)_Z$;
(ii) $(w, \theta, F', u, p, q)_Z \cap (w, \theta, F'', u, p, q)_Z \subset (w, \theta, F' + F'', u, p, q)_Z$;

(iii) $(w, \theta, F, u, p, q_1)_Z \cap (w, \theta, F, u, p, q_2)_Z \subset (w, \theta, F, u, p, q_1 + q_2)_Z$;
(iv) If q_1 *is stronger than* q_2 *then* $(w, \theta, F, u, p, q_1)_Z \subset (w, \theta, F, u, p, q_2)_Z$;
(v) If q_1 *equivalent to* q_2 *then* $(w, \theta, F, u, p, q_1)_Z = (w, \theta, F, u, p, q_2)_Z$;
(vi) $(w, \theta, F, u, p, q)_Z \cap (w, \theta, F, u, t, q)_Z \neq \phi$.

Proof. It is easy to prove so we omit the details. □

Corollary 2.5. *Let* $F = (f_k)$ *be a sequence of modulus functions. Then* $(w, \theta, u, q)_Z \subset (w, \theta, F, u, q)_Z$.
Proof. Let $x = (x_k) \in (w, \theta, u, q)_Z$ and $\epsilon > 0$. We can choose $0 < \delta < 1$ such that $f_k(t) < \epsilon$ for every $t \in [0, \infty)$ with $0 \leq t \leq \delta$. Then, we can write

$$\frac{1}{h_r} \sum_{k \in I_r} u_k[f_k(q(t_{km}(x_k - L)))]$$

$$\begin{aligned}
&= \frac{1}{h_r} \sum_{k \in I_r, t_{km}(x_k - L) \leq \delta} u_k[f_k(q(t_{km}(x_k - L)))] \\
&+ \frac{1}{h_r} \sum_{k \in I_r, t_{km}(x_k - L) > \delta} u_k[f_k(q(t_{km}(x_k - L)))] \\
&\leq \max\left\{f_k(\epsilon), f_k(\epsilon)\right\} \\
&+ \max\left\{1, (2f_k(1)\delta^{-1})\right\} \frac{1}{h_r} \sum_{k \in I_r, t_{km}(x_k - L) > \delta} u_k[f_k(q(t_{km}(x_k - L)))].
\end{aligned}$$

Therefore $x = (x_k) \in (w, \theta, F, u, q)_Z$. This completes the proof of the theorem. Similarly, we can prove the other cases. □

Theorem 5 *Let* $F = (f_k)$ *be a sequence of modulus functions, if* $\lim_{t \to \infty} \frac{f(t)}{t} = \beta > 0$, *then* $(w, \theta, u, q)_Z = (w, \theta, F, u, q)_Z$.

Proof. By Corollary 2.5, we need only to show that $(w, \theta, F, u, q)_Z \subset (w, \theta, u, q)_Z$. Let $\beta > 0$ and $x \in (w, \theta, F, u, q)_Z$. Since $\beta > 0$, we have $f(t) \geq \beta t$ for all $t \geq 0$. Hence

$$\frac{1}{h_r} \sum_{k \in I_r} u_k[f_k(q(t_{km}(x_k - L)))] \geq \frac{\beta}{h_r} \sum_{k \in I_r} u_k(q(t_{km}(x_k - L))).$$

Therefore, $x \in (w, \theta, u, q)_Z$. □

Theorem 6 *Let* $0 < p_k \leq t_k$ *and* $\left(\frac{t_k}{p_k}\right)$ *be bounded. Then*

$$(w, \theta, F, u, t, q)_Z \subset (w, \theta, F, u, p, q)_Z.$$

Proof. Let $x = (x_k) \in (w, \theta, F, u, t, q)_Z$. Let $r_k = u_k[f_k(q(t_{km}x_k - L))]^{t_k}$ and $\lambda_k = (\frac{p_k}{t_k})$ for all $k \in \mathbb{N}$ so that $0 < \lambda \leq \lambda_k \leq 1$. Define the sequences (u_k) and (v_k) as follows:
For $r_k \geq 1$, let $u_k = r_k$ and $v_k = 0$ and for $r_k < 1$, let $u_k = 0$ and $v_k = r_k$. Then clearly for all $k \in \mathbb{N}$, we have $r_k = u_k + v_k$, $r_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, $u_k^{\lambda_k} \leq u_k \leq r_k$ and $v_k^{\lambda_k} = v_k^{\lambda}$. Therefore

$$\frac{1}{h_r}\sum_{k\in I_r} r_k^{\lambda_k} \leq \frac{1}{h_r}\sum_{k\in I_r} r_k + [\frac{1}{h_r}\sum_{k\in I_r} v_k]^{\lambda}.$$

Hence $x = (x_k) \in (w, \theta, F, u, p, q)_Z$. Thus $(w, \theta, F, u, t, q)_Z \subset (w, \theta, F, u, p, q)_Z$.
□

Theorem 7 *Let* $\theta = (k_r)$ *be a lacunary sequence. If* $1 < \liminf_r q_r \leq \lim \sup_r q_r < \infty$ *then for any modulus function* F*, we have* $(w, F, u, p, q)_0 = (w, \theta, F, u, p, q)_0$.

Proof. Suppose $\liminf_r q_r > 1$ then there exist $\delta > 0$ such that $q_r = (\frac{k_r}{k_{r-1}}) \geq 1 + \delta$ for all $r \geq 1$. Then for $x = (x_k) \in (w, F, u, p, q)_0$, we write

$$\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x)))]^{p_k}$$

$$\begin{aligned}
&= \frac{1}{h_r}\sum_{k=1}^{k_r} u_k[f_k(q(t_{km}(x)))]^{p_k} - \frac{1}{h_r}\sum_{k=1}^{k_{r-1}} u_k[f_k(q(t_{km}(x)))]^{p_k} \\
&= \frac{k_r}{h_r}\left(k_r^{-1}\sum_{k=1}^{k_r} u_k[f_k(q(t_{km}(x)))]^{p_k}\right) - \frac{k_{r-1}}{h_r}\left(k_{r-1}^{-1}\sum_{k=1}^{k_{r-1}} u_k[f_k(q(t_{km}(x)))]^{p_k}\right).
\end{aligned}$$

Since $h_r = k_r - k_{r-1}$, we have $\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}$ and $\frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}$. The terms

$$k_r^{-1}\sum_{k=1}^{k_r} u_k[f_k(q(t_{km}(x)))]^{p_k}$$

and $\frac{k_{r-1}}{h_r}\left(k_{r-1}^{-1}\sum_{k=1}^{k_{r-1}} u_k[f_k(q(t_{km}(x)))]^{p_k}\right)$ both converge to zero, uniformly in m and it follows that

$$\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x)))]^{p_k} \to 0,$$

as $r \to \infty$ uniformly in m, that is, $x \in (w, \theta, F, u, p, q)_0$.
If $\limsup_r q_r < \infty$, there exists $B > 0$ such that $\eta_r < B$ for all $r \geq 1$. Let $x \in (w, \theta, F, u, p, q)_0$ and $\epsilon > 0$ be given. Then there exists $R > 0$ such that for every $j \geq R$ and all m.

$$A_j = \frac{1}{h_j}\sum_{k\in I_j} u_k[f_k(q(t_{km}(x)))]^{p_k} < \epsilon.$$

We can also find $K > 0$ such that $A_j < K$ for all $j = 1, 2, \cdots$. Now let t be any integer with $k_{r-1} < t \leq k_r$, where $r > R$. Then

$t^{-1}\sum_{k=1}^{t} u_k[f_k(q(t_{km}(x)))]^{p_k}$

$$\begin{aligned}
&\leq k_{r-1}^{-1}\sum_{k=1}^{k_r} u_k[f_k(q(t_{km}(x)))]^{p_k}\\
&= k_{r-1}^{-1}\sum_{k\in I_1} u_k[f_k(q(t_{km}(x)))]^{p_k} + k_{r-1}^{-1}\sum_{k\in I_2} u_k[f_k(q(t_{km}(x)))]^{p_k}\\
&+ \cdots + k_{r-1}^{-1}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x)))]^{p_k}\\
&= \frac{k_1}{k_{r-1}}k_1^{-1}\sum_{k\in I_1} u_k[f_k(q(t_{km}(x)))]^{p_k} + \frac{k_2-k_1}{k_{r-1}}(k_2-k_1)^{-1}\sum_{k\in I_2} u_k[f_k(q(t_{km}(x)))]^{p_k}\\
&+ \cdots + \frac{k_R-k_{R-1}}{k_{r-1}}(k_R-k_{R-1})^{-1}\sum_{k\in I_R} u_k[f_k(q(t_{km}(x)))]^{p_k}\\
&+ \cdots + \frac{k_r-k_{r-1}}{k_{r-1}}(k_r-k_{r-1})^{-1}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x)))]^{p_k}\\
&= \frac{k_1}{k_{r-1}}A_1 + \frac{k_2-k_1}{k_{r-1}}A_2 + \cdots + \frac{k_R-k_{R-1}}{k_{r-1}}A_R\\
&+ \frac{k_{R+1}-k_R}{k_{r-1}}A_{R+1} + \cdots + \frac{k_r-k_{r-1}}{k_{r-1}}A_r
\end{aligned}$$

$$\leq (\sup_{j\geq 1} A_j)\frac{k_R}{k_{r-1}} + (\sup_{j\geq R} A_j)\frac{k_r - k_R}{k_{r-1}}$$
$$\leq K\frac{k_R}{k_{r-1}} + \epsilon B.$$

Since $K_{r-1} \to \infty$ as $t \to \infty$, it follows that $t^{-1}\sum_{k=1}^{t} u_k[f_k(q(t_{km}(x)))]^{p_k} \to 0$ uniformly in m and consequently $x \in (w, F, u, p, q)_0$. □

3 Statistical convergence

The notion of statistical convergence was introduced by Fast [12] and Schoenberg [28] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [14], Connor [10], Salat [25], Murasaleen [21], Isik [15], Savas [27], Malkosky and Savas [20], Kolk [16], Maddox [18, 19] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Cech compactification of natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. The notion depends on the density of subsets of the set $\mathbb{N}$ of natural numbers.

A subset E of $\mathbb{N}$ is said to have the natural density $\delta(E)$ if the following limit exists: $\delta(E) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_E(k)$, where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$.

Let θ be a lacunary sequence, then the sequence $x = (x_k)$ is said to be q-lacunary almost statistically convergent to the number L provided that for every $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : q(t_{km}(x - L)) \geq \epsilon\}| = 0, \text{ uniformly in } m.$$

In this case we write $[S_\theta]_q - \lim x = L$ or $x_k \to L([S_\theta]_q)$ and we define

$$[S_\theta]_q = \Big\{x \in w(X) : [S_\theta]_q - \lim x = L, \text{ for some } L\Big\}.$$

In the case $\theta = (2^r)$, we shall write $[S]_q$ instead of $[S_\theta]_q$.

Theorem 8 *Let* $F = (f_k)$ *be a sequence of modulus functions and* $0 < h = \inf_k p_k \le p_k \le \sup_k p_k = H < \infty$. *Then* $(w, \theta, F, u, p, q) \subset [S_\theta]_q$.

Proof. Let $x \in (w, \theta, F, u, p, q)$ and $\epsilon > 0$ be given. Then

$$\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x-L)))]^{p_k}$$

$$\begin{aligned}
&\ge \frac{1}{h_r}\sum_{k\in I_r, q(t_{km}(x-L))\ge\epsilon} u_k[f_k(q(t_{km}(x-L)))]^{p_k}\\
&\ge \frac{1}{h_r}\sum_{k\in I_r, q(t_{km}(x-L))\ge\epsilon} u_k[f_k(\epsilon)]^{p_k}\\
&\ge \frac{1}{h_r}\sum_{k\in I_r, q(t_{km}(x-L))\ge\epsilon} \min(u_k[f_k(\epsilon)]^h, u_k[f_k(\epsilon)]^H)\\
&\ge \frac{1}{h_r}|\{k \in I_r : q(t_{km}(x-L)) \ge \epsilon\}| \min(u_k[f_k(\epsilon)]^h, u_k[f_k(\epsilon)]^H).
\end{aligned}$$

Hence $x \in [S_\theta]_q$. □

Theorem 9 *Let* $F = (f_k)$ *be a bounded sequence of modulus functions and* $0 < h = \inf_k p_k \le p_k \le \sup_k p_k = H < \infty$. *Then* $[S_\theta]_q \subset (w, \theta, F, u, p, q)$.

Proof. Suppose that $F = f_k$ is bounded. Then there exists an integer K such that $f_k(t) < K$, for all $t \ge 0$. Then

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(q(t_{km}(x-L)))]^{p_k} &= \frac{1}{h_r}\sum_{k\in I_r, q(t_{km}(x-L))\ge\epsilon} u_k[f_k(q(t_{km}(x-L)))]^{p_k}\\
&+ \frac{1}{h_r}\sum_{k\in I_r, q(t_{km}(x-L))<\epsilon} u_k[f_k(q(t_{km}(x-L)))]^{p_k}\\
&\le \frac{1}{h_r}\sum_{k\in I_r, q(t_{km}(x-L))\ge\epsilon} \max(K^h, K^H)\\
&+ \frac{1}{h_r}\sum_{k\in I_r, q(t_{km}(x-L))<\epsilon} u_k[f_k(\epsilon)]^{p_k}\\
&\le \max(K^h, K^H)\frac{1}{h_r}|\{k \in I_r : q(t_{km}(x-L)) \ge \epsilon\}|\\
&+ \max(u_k[f_k(\epsilon)]^h, u_k[f_k(\epsilon)]^H).
\end{aligned}$$

Hence $x \in (w, \theta, F, u, p, q)$. □

Theorem 10 $[S_\theta]_q = (w, \theta, F, u, p, q)$ *if and only if* $F = (f_k)$ *is bounded.*

Proof. Let $F = (f_k)$ be bounded. By the Theorem 3.1 and Theorem 3.2, we have

$$[S_\theta]_q = (w, \theta, F, u, p, q).$$

Conversely, suppose that F is unbounded. Then there exists a positive sequence (t_n) with $f(t_n) = n^2, n = 1, 2, \cdots$.
If we choose

$$x_k = \begin{cases} t_n, & k = n^2, n = 1, 2, \cdots, \\ 0, & \text{otherwise} \end{cases}.$$

Then we have

$$\frac{1}{n}|\{k \le n : |x_k| \ge \epsilon\}| \le \frac{\sqrt{n}}{n} \to 0, n \to \infty.$$

Hence $x_k \to 0([S_\theta]q)$ for $t_{0m}(x) = x_m, \theta = (2^r)$ and $q(x) = |x|$, but $x \notin (w, \theta, F, u, p, q)$. This contradicts to $[S_\theta]_q = (w, \theta, F, u, p, q)$. □

References

[1] Y. Altin, M. Et, Generalized difference sequence spaces defined by a modulus function in a locally convex space, *Soochow J. Math.*, **31** (2005), 233–243.

[2] H. Altinok, Y. Altin, M. Isik, The sequence space $Bv_\sigma(M, p, q, s)$ on seminormed spaces, *Indian J. Pure Appl. Math.*, **39**, 1 (2008), 49–58.

[3] Y. Altin, Properties of some sets of sequences defined by a modulus function, *Acta Math. Sci. Ser. B Engl.* Ed., **29**, 2 (2009), 427–434.

[4] Y. Altin, H. Altinok, R. Çolak, On some seminormed sequence spaces defined by a modulus function, *Kragujevac J. Math.*, **29** (2006), 121–132.

[5] Y. Altin, A. Gökhan, H. Altinok, Properties of some new seminormed sequence spaces defined by a modulus function, *Studia Univ. Babeş-Bolyai Math.* **L.**, 3 (2005), 13–19.

[6] S. Banach, *Theorie operations linearies*, Chelsea Publishing Co., New York (1955).

[7] T. Bilgin, Lacunary strong A-convergence with respect to a modulus, *Studia Univ. Babeş-Bolyai Math.*, **XLV**I (2001), 39–46.

[8] T. Bilgin, Lacunary strong A-convergence with respect to a sequence of modulus functions, *Appl. Math. Comput.*, **151** (2004), 595–600.

[9] R. Colak, B. C. Tripathy, M. Et, Lacunary strongly summable sequences and q-lacunary almost statistical convergence, *Vietnam J. Math.*, **34** (2006), 129–138.

[10] J. S. Connor, A topological and functional analytic approach to statistical convergence, *Appl. Numer. Harmonic Anal.*, (1999), 403–413.

[11] G. Das, S. K. Sahoo, On some sequence spaces, *J. Math. Anal. Appl.*, **164** (1992), 381–398.

[12] H. Fast, Sur la convergence statistique, *Colloq. Math.*, **2** (1951), 241–244.

[13] A. R. Freedman, J. J. Sember, M. Raphael, Some Cesaro-type summability spaces, *Proc. London Math. Soc.*, **37** (1978), 508–520.

[14] J. A. Fridy, On the statistical convergence, *Analysis*, **5** (1985), 301–303.

[15] M. Isik, On statistical convergence of generalized difference sequence spaces, *Soochow J. Math.*, **30** (2004), 197–205.

[16] E. Kolk, The statistical convergence in Banach spaces, *Acta. Comment. Univ. Tartu*, **928** (1991), 41–52.

[17] G. G. Lorentz, A contribution to the theory of divergent series, *Acta Math.*, **80** (1948), 167–190.

[18] I. J. Maddox, A new type of convergence, *Math. Proc. Cambridge Phil. Soc.*, **83** (1978), 61–64.

[19] I. J. Maddox, Statistical convergence in a locally convex space, *Math. Proc. Cambridge Phil. Soc.*, **104** (1988), 141–145.

[20] E. Malkowsky, E. Savas, Some λ-sequence spaces defined by a modulus, *Arch. Math.*, **36** (2000), 219–228.

[21] M. Mursaleen, λ-statistical convergence, *Math. Slovaca*, **50** (2000), 111–115.

[22] K. Raj, S. K. Sharma, A. K. Sharma, Some new sequence spaces defined by a sequence of modulus functions in $\mathfrak{n}$-normed spaces, *Int. J. Math. Sci. Eng. Appl.*, **5**, 2 (2011), 395–403.

[23] K. Raj, S. K. Sharma, Difference sequence spaces defined by sequence of modulus functions, *Proyecciones*, **30** (2011), 189–199.

[24] K. Raj, S. K. Sharma, Some difference sequence spaces defined by sequence of modulus functions, *Int. Journal of Mathematical Archive*, **2** (2011), 236–240.

[25] T. Salat, On statistical convergent sequences of real numbers, *Math. Slovaca*, **30** (1980), 139–150.

[26] E. Savas, On some generalized sequence spaces defined by a modulus, *Indian J. Pure Appl. Math.*, **30** (1999), 459–464.

[27] E. Savas, Strong almost convergence and almost λ-statistical convergence, *Hokkaido Math. J.*, **29** (2000), 531–566.

[28] I. J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly*, **66** (1959), 361–375.

[29] A. Wilansky, Summability through functional analysis, *North-Holland Math. Stud.*, Amsterdam (1984).

Received: October 6, 2011

Acta Univ. Sapientiae, Mathematica, 4, 2 (2012) 132–144

Common fixed point theorems for single and set-valued maps in non-Archimedean fuzzy metric spaces

T. K. Samanta
Uluberia College
Department of Mathematics
Uluberia, Howrah, West Bengal, India
email: mumpu_tapas5@yahoo.co.in

Sumit Mohinta
Uluberia College
Department of Mathematics
Uluberia, Howrah, West Bengal, India
email: sumit.mohinta@yahoo.com

Abstract. The intent of this paper is to establish a common fixed point theorem for two pairs of occasionally weakly compatible single and set-valued maps satisfying a strict contractive condition in a non-Archimedean fuzzy metric space.

1 Introduction

The concept of fuzzy sets was first coined by Zadeh [9] in 1965 to describe the situation in which data are imprecise or vague or uncertain. Consequently, the last three decades remained productive for various authors [1, 11, 13] etc. and they have extensively developed the theory of fuzzy sets due to a wide range of application in the field of population dynamics, chaos control, computer programming, medicine, etc. Kramosil and Michalek [10] introduced the concept of fuzzy metric spaces (briefly, FM-spaces) in 1975, which opened a new avenue for further development of analysis in such spaces. Later on the concept of FM-space is modified and a few concepts of mathematical analysis have been developed in fuzzy metric space by George and Veeramani [1, 2]. In fact, the concept of fixed point theorem have been developed in fuzzy metric space in the paper [12].

2010 Mathematics Subject Classification: 03E72, 47H10, 54H25
Key words and phrases: occasionally weakly compatible maps, implicit relation, common fixed point theorems, strict contractive condition, fuzzy metric space

In recent years several fixed point theorems for single and set valued maps were proved and have numerous applications and by now there exists a considerable rich literature in this domain [4, 7].

Various authors [3, 7, 8] have discussed and studied extensively various results on coincidence, existence and uniqueness of fixed and common fixed points by using the concept of weak commutativity, compatibility, non-compatibility and weak compatibility for single and set valued maps satisfying certain contractive conditions in different spaces and they have been applied to diverse problems.

The intent of this paper is to establish a common fixed point theorem for two pairs of occasionally weakly compatible single and set-valued maps satisfying a strict contractive condition in a non-Archimedean fuzzy metric space.

2 Preliminaries

We quote some definitions and a few theorems which will be needed in the sequel.

Definition 1 [5] *A binary operation* $*: [0,1] \times [0,1] \longrightarrow [0,1]$ *is continuous* t*-norm if it satisfies the following conditions:*

(i) $*$ *is commutative and associative,*

(ii) $*$ *is continuous,*

(iii) $a * 1 = a \qquad \forall a \in [0,1]$,

(iv) $a * b \leq c * d$ *whenever* $a \leq c$, $b \leq d$ *and* $a, b, c, d \in [0,1]$.

Result 1 [6] (a) *For any* $r_1, r_2 \in (0,1)$ *with* $r_1 > r_2$, *there exist* $r_3 \in (0,1)$ *such that* $r_1 * r_3 > r_2$,

(b) *For any* $r_5 \in (0,1)$, *there exist* $r_6 \in (0,1)$ *such that* $r_6 * r_6 \geq r_5$.

Definition 2 [1] *The* 3*-tuple* $(X, \mu, *)$ *is called a fuzzy metric space if* X *is an arbitrary non-empty set,* $*$ *is a continuous* t*-norm and* μ *is a fuzzy set in* $X^2 \times (0, \infty)$ *satisfying the following conditions:*

(i) $\mu(x, y, t) > 0$;

(ii) $\mu(x, y, t) = 1$ *if and only if* $x = y$

(iii) $\mu(x, y, t) = \mu(y, x, t)$;

(iv) $\mu(x, y, s) * \mu(y, z, t) \leq \mu(x, z, s + t)$;

(v) $\mu(x, y, \cdot) : (0, \infty) \to (0, 1]$ *is continuous;*
for all x,y,z $\in X$ *and* $t, s > 0$.

Note that $\mu(x, y, t)$ *can be thought as the degree of nearness between* x *and* y *with respect to* t.

Example 1 *Let* $X = [0, \infty)$, $a * b = ab$ *for every* $a, b \in [0, 1]$ *and* d *be the usual metric defined on* X. *Define* $\mu(x, y, t) = e^{-\frac{d(x,y)}{t}}$ *for all* $x, y, t \in X$. *Then clearly* $(X, \mu, *)$ *is a fuzzy metric space.*

Example 2 *Let* (X, d) *be a metric space, and let* $a*b = ab$ *or* $a*b = \min\{a, b\}$ *for all* $a, b \in [0, 1]$. *Let* $\mu(x, y, t) = \frac{t}{t+d(x,y)}$ *for all* $x, y \in X$ *and* $t > 0$. *Then* $(X, \mu, *)$ *is a fuzzy metric space and this fuzzy metric* μ *induced by* d *is called the standard fuzzy metric [1].*

Note 1 *George and Veeramani [1] proved that every fuzzy metric space is a metrizable topological space. In this paper, they also have proved, if* (X, d) *is a metric space, then the topology generated by* d *coincides with the topology generated by the fuzzy metric* μ *of Example 2. As a result, we can say that an ordinary metric space is a special case of a fuzzy metric space.*

Note 2 *Consider the following condition:*

(iv') $\mu(x, y, s) * \mu(y, z, t) \leq \mu(x, z, \max\{s, t\})$.

If the condition (iv) in Definition 2 is replaced by the condition (iv'), *the fuzzy metric space* $(X, \mu, *)$ *is said to be a non-Archimedean fuzzy metric space.*

Remark 1 *In fuzzy metric space* X, *for all* $x, y \in X$, $\mu(x, y, \cdot)$ *is non-decreasing with respect to the variable* t. *In fact, in a non-Archimedean fuzzy metric space,*

$\mu(x, y, t) \geq \mu(x, z, t) * \mu(z, y, t)$ *for* $x, y, z \in X$, $t > 0$.

Every non-Archimedean fuzzy metric space is also a fuzzy metric space.

Throughout the paper X will represent the non-Archimedean fuzzy metric space $(X, \mu, *)$ and $CB(X)$, the set of all non-empty closed and bounded subset of X. We recall these usual notations: for $x \in X$, $A \subseteq X$ and for every $t > 0$,

$$\mu(x, A, t) = \max\{\mu(x, y, t) : y \in A\}$$

and let H be the associated Hausdorff fuzzy metric on $CB(X)$: for every A, B in $CB(X)$,

$$H(A, B, t) = \min\left\{\min_{x \in A} \mu(x, B, t), \min_{y \in B} \mu(A, y, t)\right\}.$$

Definition 3 *[4] A sequence* $\{A_n\}$ *of subsets of* X *is said to be convergent to a subset* A *of* X *if*

(i) *given* $a \in A$*, there is a sequence* $\{a_n\}$ *in* X *such that* $a_n \in A_n$ *for* $n = 1, 2, \cdots$ *, and* $\{a_n\}$ *converges to* a*.*

(ii) *given* $\epsilon > 0$*, there exists a positive integer* N *such that* $A_n \subseteq A_\epsilon$ *for* $n > N$*, where* A_ϵ *is the union of all open spheres with centers in* A *and radius* ϵ*.*

Definition 4 *A point* $x \in X$ *is called a coincidence point (resp. fixed point) of* $A : X \longrightarrow X$*,* $B : X \longrightarrow CB(X)$ *if* $Ax \in Bx$ *(resp.* $x = Ax \in Bx$*).*

Definition 5 *Maps* $A : X \longrightarrow X$ *and* $B : X \longrightarrow CB(X)$ *are said to be compatible if* $ABx \in CB(X)$ *for all* $x \in X$ *and*

$$\lim_{n\to\infty} H(ABx_n, BAx_n, t) = 1$$

whenever $\{x_n\}$ *is a sequence in* X *such that* $Bx_n \longrightarrow M \in CB(X)$ *and* $Ax_n \longrightarrow x \in M$*.*

Definition 6 *Maps* $A : X \longrightarrow X$*and* $B : X \longrightarrow CB(X)$ *are said to be weakly compatible if they commute at coincidence points ie., if* $ABx = BAx$ *whenever* $Ax \in Bx$*.*

Definition 7 *Maps* $A : X \longrightarrow X$ *and* $B : X \longrightarrow CB(X)$ *are said to be occasionally weakly compatible (**owc**) if there exists some point* $x \in X$ *such that* $Ax \in Bx$ *and* $ABx \subseteq BAx$*.*

Example 3 *Let* $X = [1, \infty[$ *with the usual metric. Define* $f : X \longrightarrow X$ *and* $F : X \longrightarrow CB(X)$*, for all* $x \in X$ *by*

$$fx = x + 1, Fx = [1, x + 1],$$

$$fx = x + 1 \in Fx \text{ and } fFx = [2, x + 2] \subset Ffx = [1, x + 2].$$

Hence, f *and* F *are occasionally weakly compatible but not weakly compatible.*

Definition 8 *Let* $F : X \longrightarrow 2^X$ *be a set-valued map on* X*.* $x \in X$ *is a fixed point of* F *if* $x \in Fx$ *and is a strict fixed point of* F *if* $Fx = \{x\}$*.*

Property 1 *Let* A *and* $B \in CB(X)$*, then for any* $a \in A$*, we have*

$$\mu(a, B, t) \geq qH(A, B, t).$$

Proof. Obvious. □

3 A strict fixed point theorem

Theorem 1 *Let* $f, g : X \longrightarrow X$ *be mappings and* $F, G : X \longrightarrow CB(X)$ *be set-valued mappings such that the pairs* $\{f, F\}$ *and* $\{g, G\}$ *are* ***owc****. Let* $\varphi : R^6 \longrightarrow R$ *be a real valued map satisfying the following conditions*

$$(\varphi_1) : \varphi \text{ is increasing invariables } t_2, t_5 \text{ and } t_6;$$
$$(\varphi_2) : \varphi(u(t), u(t), 1, 1, u(t), u(t)) > 1 \quad \forall u(t) \in [0, 1).$$

If for all x *and* $y \in X$ *for which*

$$(\star) \quad \varphi(H(Fx, Gy, t), \mu(fx, gy, t), \mu(fx, Fx, t), \mu(gy, Gy, t), \mu(fx, Gy, t), \mu(gy, Fx, t)) < 1$$

then f, g, F *and* G *have a unique fixed point which is a strict fixed point for* F *and* G*.*

Proof. (i) We begin to show the existence of a common fixed point. Since the pairs $\{f, F\}$ and $\{g, G\}$ are **owc**, there exist u, v in X such that $fu \in Fu$, $gv \in Gv$, $fFu \subseteq Ffu$ and $gGv \subseteq Ggv$. Also, using the triangle inequality and Property 1, we obtain

$$\mu(fu, gv, t) \geq H(Fu, Gv, t) \tag{1}$$

and

$$\mu(f^2u, gv, t) \geq H(Ffu, Gv, t). \tag{2}$$

First we show that $gv = fu$. The condition $(\star)$ implies that

$$\varphi(H(Fu, Gv, t), \mu(fu, gv, t), \mu(fu, Fu, t), \mu(gv, Gv, t), \mu(fu, Gv, t), \mu(gv, Fu, t)) < 1$$

$$\Longrightarrow \varphi(H(Fu, Gv, t), \mu(fu, gv, t), 1, 1, \mu(fu, Gv, t), \mu(gv, Fu, t)) < 1.$$

By (φ_1) we have

$$\varphi(H(Fu, Gv, t), H(Fu, Gv, t), 1, 1, H(Fu, Gv, t), H(Fu, Gv, t)) < 1$$

which from (φ_2) gives $H(Fu, Gv, t) = 1$.
So $Fu = Gv$ and by (1), $fu = gv$. Again by (2), we have

$$\mu(f^2u, fu, t) \geq H(Ffu, Gv, t).$$

Next, we claim that $f^2u = fu$. The condition ($\star$) implies that

$$\varphi(H(Ffu, Gv, t), \mu(f^2u, gv, t), \mu(f^2u, Ffu, t), \mu(gv, Gv, t),$$
$$\mu(f^2u, Gv, t), \mu(gv, Ffu, t)) < 1$$
$$\Longrightarrow \varphi(H(Ffu, Gv, t), \mu(f^2u, fu, t), 1, 1, \mu(f^2u, Gv, t), \mu(fu, Ffu, t)) < 1.$$

By (φ_1) we have

$$\Longrightarrow \varphi(H(Ffu, Gv, t), H(Ffu, Gv, t), 1, 1, H(Ffu, Gv, t), H(Ffu, Gv, t)) < 1$$

which, from (φ_2), gives $H(Ffu, Gv, t) = 1$.

By (2), we obtain $f^2u = fu$. Since (f, F) and (g, G) have the same role, we have $gv = g^2v$. Therefore,

$$ffu = fu = gv = ggv = gfu$$

and

$$fu = f^2u \in fFu \subset Ffu$$

So $fu \in Ffu$ and $fu = gfu \in Gfu$. Then fu is common fixed point of f, g, F and G.

(ii) Now, we show uniqueness of the common fixed point.

Put $fu = w$ and let $w^{'}$ be another common fixed point of the four maps, then we have

$$\mu(w, w', t) = \mu(fw, gw', t) \geq H(Fw, Gw^{'}, t) \tag{3}$$

by ($\star$) , we get

$$\varphi(H(Fw, Gw', t), \mu(fw, gw', t), \mu(fw, Fw, t), \mu(gw', Gw', t),$$
$$\mu(fw, Gw', t), \mu(gw', Fw, t)) < 1$$
$$\Longrightarrow \varphi(H(Fw, Gw', t), \mu(fw, gw', t), 1, 1, \mu(fw, Gw', t),$$
$$\mu(gw', Fw, t)) < 1$$

By (φ_1) we get

$$\varphi(H(Fw, Gw', t), H(Fw, Gw', t), 1, 1, H(Fw, Gw', t), H(Fw, Gw', t)) < 1$$

So, by (φ_2), $H(Fw, Gw', t) = 1$ and from (3), we have

$$\mu(fw, gw^{'}, t) = \mu(w, w', t) = 1 \Longrightarrow w = w'.$$

(iii) Let $w \in Ffu$. Using the triangle inequality and Property (1), we have

$$\mu(fu, w, t) \geq \mu(fu, Ffu, t) * H(Ffu, Gv, t) * \mu(w, Gv, t).$$

Since $fu \in Ffu$ and $H(Ffu, Gv, t) = 1$,

$$\mu(w, fu, t) \geq \mu(w, Gv, t) \geq H(Ffu, Gv, t) = 1.$$

So $w = fu$ and $Ffu = \{fu\} = \{gv\} = Ggv$.
This completes the proof. □

4 A Gregus type fixed point theorem

Theorem 2 *Let* $f, g : X \longrightarrow X$ *be mappings and* $F, G : X \longrightarrow CB(X)$ *be set-valued mappings such that that the pairs* $\{f, F\}$ *and* $\{g, G\}$ *are* ***owc****. Let* $\psi : R \longrightarrow R$ *be a non-decreasing map such that, for every* $0 \leq l < 1$, $\psi(l) > l$ *and satisfies the following condition:*

$$(\star) \qquad H^p(Fx, Gy, t) \geq \psi\left[a\mu^p(fx, gy, t) + (1-a)\mu^{\frac{p}{2}}(gy, Fx, t)\mu^{\frac{p}{2}}(fx, Gy, t)\right]$$

for all x *and* $y \in X$, *where* $0 < a \leq 1$ *and* $p \geq 1$.

Then f, g, F *and* G *have a unique fixed point which is a strict fixed point for* F *and* G.

Proof. Since $\{f, F\}$ and $\{g, G\}$ are **owc**, as in proof of Theorem 1, there exist $u, v \in X$ such that $fu \in Fu$, $gv \in Gv$, $fFu \subseteq Ffu$, $gGv \subseteq Ggv$ and (1), (2) holds.
(i) As in proof of Theorem 1, we begin to show the existence of a common fixed point. We have,

$$H^p(Fu, Gv, t) \geq \psi\left[a\mu^p(fu, gv, t) + (1-a)\mu^{\frac{p}{2}}(gv, Fu, t)\mu^{\frac{p}{2}}(fu, Gv, t)\right]$$

and by (1) and Property 1,

$$H^p(Fu, Gv, t) \geq \psi\left[aH^p(Fu, Gv, t) + (1-a)H^p(Gv, Fu, t)\right]$$

$$= \psi\left(H^p(Fu, Gv, t)\right)$$

So, if $0 \leq H(Fu, Gv, t) < 1$, $\psi(l) > l$ for $0 \leq l < 1$, we obtain

$$H^p(Fu, Gv, t) \geq \psi[H^p(Fu, Gv, t)] > H^p(Fu, Gv, t)$$

which is a contradiction, thus we have $H(Fu, Gv, t) = 1$ and hence $fu = gv$. Again, if $0 \leq H(Ffu, Gv, t) < 1$ then by (2) and ($\star$), we have

$$\begin{aligned} H^p(Ffu, Gv, t) &\geq \psi[a\mu^p(f^2u, gv, t) + (1-a)\mu^{\frac{p}{2}}(gv, Ffu, t)\mu^{\frac{p}{2}}(f^2u, Gv, t)] \\ &\geq \psi[aH^p(Ffu, Gv, t) + (1-a)H^p(Ffu, Gv, t)] \\ &= \psi(H^p(Ffu, Gv, t)) \end{aligned}$$

If $0 \leq H(Ffu, Gv, t) < 1$, we obtain

$$H^p(Ffu, Gv, t) \geq \psi[H^p(Ffu, Gv, t)] > H^p(Ffu, Gv, t)$$

which is a contradiction, thus we have $H(Ffu, Gv, t) = 1$,

$$\Longrightarrow Ffu = Gv \Longrightarrow f^2u = fu$$

Similarly, we can prove that $g^2v = gv$.
Let $fu = w$ then $fw = w = gw$, $w \in Fw$ and $w \in Gw$, this completes the proof of the existence.
(ii) For the uniqueness, let w' be a second common fixed point of f, g, F and G. Then

$$\mu(w, w', t) = \mu(fw, gw', t) \geq H(Fw, Gw', t)$$

and by assumption ($\star$), we obtain

$$\begin{aligned} H^p(Fw, Gw', t) &\geq \psi\left[a\mu^p(fw, gw', t) + (1-a)\mu^{\frac{p}{2}}(fw, Gw', t)\mu^{\frac{p}{2}}(gw', Fw, t)\right] \\ &\geq \psi(H^p(Fw, Gw', t)) > H^p(Fw, Gw', t) if 0 \\ &\leq H(Fw, Gw', t) < 1 \end{aligned}$$

which is a contradiction. So, $Fw = Gw'$. Since w and w' are common fixed point of f, g, F and G, we have

$$\mu(fw, gw', t) \geq \mu(fw, Fw, t) * H(Fw, Gw', t) * \mu(gw', Gw', t) \geq H(Fw, Gw', t)$$

So, $w = fw = gw' = w'$ and there exists a unique common fixed point of f, g, F, and G.
(iii) The proof that the fixed point of F and G is a strict fixed point is identical of that of theorem (1). □

Theorem 3 *Let* $f, g : X \longrightarrow X$ *and* $F, G : X \longrightarrow CB(X)$ *be single and set-valued maps respectively such that the pairs* $\{f, F\}$ *and* $\{g, G\}$ *are* ***owc*** *and satisfy inequality*

$$
\begin{aligned}
(\star) \quad H^p(Fx, Gy, t) \;\geq\; & a(\mu(fx, gy, t))[\min\{\mu(fx, gy, t)\mu^{p-1}(fx, Fx, t),\\
& \mu(fx, gy, t)\mu^{p-1}(gy, Gy, t), \mu(fx, Fx, t)\mu^{p-1}\\
& (gy, Gy, t), \mu^{p-1}(fx, Gy, t)\mu(gy, Fx, t)\}]
\end{aligned}
$$

for all $x, y \in X$, *where* $p \geq 2$ *and* $a : [0, 1] \longrightarrow [0, \infty)$ *is decreasing and satisfies the condition*

$$a(t) > 1 \quad \forall\ 0 \leq t < 1 \qquad \text{and} \qquad a(t) = 1 \quad \text{if } f \cdot t = 1$$

Then f, g, F *and* G *have a unique fixed point which is a strict fixed point for* F *and* G.

Proof. Since the pairs $\{f, F\}$ and $\{g, G\}$ are **owc**, then there exist two elements u and v in X such that $fu \in Fu$, $fFu \subseteq Ffu$ and $gv \in Gv$, $gGv \subseteq Ggv$.
First we prove that $fu = gv$. By property (1) and the triangle inequality we have $\mu(fu, gv, t) \geq H(Fu, Gv, t)$, $\mu(fu, Gv, t) \geq H(Fu, Gv, t)$ and $\mu(Fu, gv, t) \geq H(Fu, Gv, t)$.
Suppose that $H(Fu, Gv, t) < 1$. Then by inequality $(\star)$ we get

$$
\begin{aligned}
(\star) \quad H^p(Fu, Gv, t) \;\geq\; & a(\mu(fu, gv, t))[\min\{\mu(fu, gv, t)\mu^{p-1}(fu, Fu, t),\\
& \mu(fu, gv, t)\mu^{p-1}(gv, Gv, t), \mu(fu, Fu, t)\mu^{p-1}(gv, Gv, t),\\
& \mu^{p-1}(fu, Gv, t)\mu(gv, Fu, t)\}]\\
=\; & a(\mu(fu, gv, t))[\min\{\mu(fu, gv, t), \mu(fu, gv, t), 1,\\
& \mu^{p-1}(fu, Gv, t)\mu(gv, Fu, t)\}]\\
\geq\; & a(H(Fu, Gv, t))[\min\{H(Fu, Gv, t), 1, H^p(Fu, Gv, t)\}]\\
>\; & H^p(Fu, Gv, t)
\end{aligned}
$$

which is a contradiction. Hence $H(Fu, Gv, t) = 1$ which implies that $fu = gv$.
Again by property (1) and the triangle inequality we have

$$\mu(f^2u, fu, t) = \mu(f^2u, gv, t) \geq H(Ffu, Gv, t)$$

We prove that $f^2u = fu$. Suppose $H(Ffu, Gv, t) < 1$ and by $(\star)$, property (1) we obtain

$$\begin{aligned} H^p(Ffu, Gv, t) &\geq qa(\mu(f^2u, gv, t))[\min\{\mu(f^2u, gv, t)\mu^{p-1}(f^2u, Ffu, t), \\ &\quad \mu(f^2u, gv, t)\mu^{p-1}(gv, Gv, t), \mu(f^2u, Ffu, t)\mu^{p-1}(gv, Gv, t), \\ &\quad \mu^{p-1}(f^2u, Gv, t)\mu(gv, Ffu, t)\}] \\ &= a(\mu(f^2u, gv, t))[\min\{\mu(f^2u, gv, t), \mu(f^2u, gv, t), 1, \\ &\quad \mu^{p-1}(f^2u, Gv, t)\mu(gv, Ffu, t)\}] \\ &\geq qa(H(Ffu, Gv, t))[\min\{H(Ffu, Gv, t), H^p(Ffu, Gv, t)\}] \\ &> H^p(Ffu, Gv, t) \end{aligned}$$

which is a contradiction. Hence $H(Ffu, Gv, t) = 1$ which implies that $f^2u = gv = fu$.
Similarly, we can prove that $g^2v = gv$. Putting $fu = gv = z$, then $fz = gz = z$, $z \in Fz$ and $z \in Gz$. Therefore z is a common fixed point of maps f, g, F and G. Now, suppose that f, g, F and G have another common fixed point $z' \neq qz$. Then, by property (1) and the triangle inequality we have

$$\mu(z, z', t) = \mu(fz, gz', t) \geq H(Fz, Gz', t)$$

Assume that $H(Fz, Gz', t) < 1$. Then the use of inequality $(\star)$ gives

$$\begin{aligned} H^p(Fz, Gz', t) &\geq qa(\mu(fz, gz', t))[\min\{\mu(fz, gz', t)\mu^{p-1}(fz, Fz, t), \mu(fz, gz', t) \\ &\quad \mu^{p-1}(gz', Gz', t), \mu(fz, Fz, t)\mu^{p-1}(gz', Gz', t), \\ &\quad \mu^{p-1}(fz, Gz', t)\mu(gz', Fz, t)\}] \\ &= a(\mu(fz, gz', t))[\min\{\mu(fz, gz', t), \mu(f^2z, gz', t), 1, \\ &\quad \mu^{p-1}(f^2z, Gz', t)\mu(gz', Ffz, t)\}] \\ &\geq qa(H(Fz, Gz', t))[\min\{H(Fz, Gz', t), H^p(Fz, Gz', t)\}] \\ &> H^p(Fz, Gz', t) \end{aligned}$$

which is a contradiction. Hence $H(Fz, Gz', t) = 1$ which implies that $z' = z$.

(iii) The proof that the fixed point of F and G is a strict fixed point is identical of that of theorem (1) □

5 Another type fixed point theorem

Theorem 4 *Let* $f, g : X \longrightarrow X$ *be mappings and* $F, G : X \longrightarrow CB(X)$ *be set-valued maps and* ϕ *be non-decreasing function of* $[0, 1]$ *into itself such that*

$\phi(t) = 1$ *iff* $t = 1$ *and for all* $t \in [0,1), \phi$ *satisfies the following inequality*

$$(\star) \qquad \phi(H(Fx, Gy, t)) \geq a(\mu(fx, gy, t))\phi(\mu(fx, gy, t)) + b(\mu(fx, gy, t)) \min\{\phi(\mu(fx, Gy, t)), \phi(\mu(gy, Fx, t))\}$$

for all x *and* y *in* X*, where* $a, b : [0,1] \longrightarrow [0,1]$ *are satisfying the conditions*

$$a(t) + b(t) > 1 \qquad \forall t > 0$$

and

$$a(t) + b(t) = 1 \qquad \text{iff.} t = 1$$

If the pairs $\{f, F\}$ *and* $\{g, G\}$ *are* ***owc****, then* f, g, F *and* G *have a unique common fixed point in* X *which is a strict fixed point for* F *and* G*.*

Proof. Since $\{f, F\}$ and $\{g, G\}$ are **owc**, as in proof of theorem(1), there exist u, v in X such that $fu \in Fu$, $gv \in Gv$, $fFu \subseteq Ffu$, $gGv \subseteq Ggv$,

$$\mu(fu, gv, t) \geq H(Fu, Gv, t) \qquad (1)$$

and

$$\mu(f^2u, gv, t) \geq H(Ffu, Gv, t) \qquad (2)$$

(i) First we prove that $fu = gv$. Suppose $H(Fu, Gv, t) < 1$. By $(\star)$, Property (1), we have

$$\begin{aligned}
\phi(H(Fu, Gv, t)) &\geq a(\mu(fu, gv, t))\phi(\mu(fu, gv, t)) \\
&\quad + b(\mu(fu, gv, t)) \min\{\phi(\mu(fu, Gv, t)), \phi(\mu(gv, Fu, t))\} \\
&\geq [a(\mu(fu, gv, t)) + b(\mu(fu, gv, t)]\phi(H(Fu, Gv, t)) \\
&> \phi(H(Fu, Gv, t))
\end{aligned}$$

which is a contradiction. Hence $H(Fu, Gv, t) = 1$ and thus $fu = gv$. Now we prove that $f^2u = fu$. Suppose $H(Ffu, Gv, t) < 1$. By $(\star)$ and Property 1, we have

$$\begin{aligned}
\phi(H(Ffu, Gv, t)) &\geq qa(\mu(f^2u, gv, t))\phi(\mu(f^2u, gv, t)) \\
&\quad +b(\mu(f^2u, gv, t)) \min\{\phi(\mu(f^2u, Gv, t)), \phi(\mu(gv, Ffu, t))\} \\
&\geq [a(\mu(f^2u, fu, t)) + b(\mu(f^2u, fu, t)]\phi(H(Ffu, Gv, t)) \\
&> \phi(H(Ffu, Gv, t))
\end{aligned}$$

which is a contradiction. Hence $H(Ffu, Gv, t) = 1$ and this implies that $f^2u = fu$. Similarly, we can prove that $g^2v = gv$. So, if $w = fu = gv$ then $fw = w = gw$, $w \in Fw$ and $w \in Gw$. Existence of a common fixed point is proved.

(ii) Assume that there exists a second common fixed point w' of f, g, F and G. We see that

$$\mu(w, w', t) = \mu(fw, gw', t) \geq H(Fw, Gw', t)$$

If $H(Fw, Gw', t) < 1$, by inequality $(\star)$ we obtain

$$\begin{aligned}\phi(H(Fw, Gw', t)) &\geq a(\mu(fw, gw', t))\phi(\mu(fw, gw', t)) \\ &\quad + b(\mu(fw, gw', t)) \min\{\phi(\mu(fw, Gw', t)), \phi(\mu(gw', Fw, t))\} \\ &\geq [a(\mu(w, w', t)) + b(\mu(w, w', t)]\phi(H(Fw, Gw', t)) \\ &> \phi(H(Fw, Gw', t))\end{aligned}$$

this contradiction implies that $H(Fw, Gw', t) = 1$, hence $w' = w$

(iii) This part of the proof is analogous of that of Theorem 1. □

References

[1] A. George, P. Veeramani. On some result in fuzzy metric spaces, *Fuzzy Sets Syst.*, **64** (1994), 395–399.

[2] A. George, P. Veeramani. On some results of analysis for fuzzy metric spaces, *Fuzzy Sets Syst.*, **90** (1997), 365–368.

[3] A. Aliouche, A common fixed point theorem for weakly compatible mappings in compact metric spaces satisfying an implicit relation, *Sarajevo J. Math.*, **3**, (2007) no. 1, 123–130.

[4] B. Fisher, Common fixed points theorems of mappings and set-valued mappings, *Rostock. Math. Kolloq.*, **18** (1981), 69–77.

[5] B. Schweizer, A. Sklar, Statistical metric space, *Pacific J. Math.*, **10** (1960), 314–334.

[6] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Kluwer, Dordrecht (2000).

[7] H. Bouhadjera, A. Djoudi, B. Fisher, A unique common fixed point theorem for occasionally weakly compatible maps, *Surv. Math. Appl.*, **3** (2008), 177–182.

[8] H. Bouhadjera, C. Godet Thobie, *Common fixed point theorems for pairs of subcompatible maps*, hal- 00356516, Version 1–16 (2009).

[9] L. A. Zadeh, Fuzzy sets, *Information and Control*, **8** (1965), 338–353.

[10] O. Kramosil, J. Michalek, Fuzzy metric and statisticalmetric spaces, *Kybernetica*, **11** (1975), 326–334.

[11] O. Kaleva, S. Seikkala, On fuzzy metric spaces, *Fuzzy Sets Syst.*, **12** (1984), 215–229.

[12] R. Vasuki, P. Veeramani, Fixed point theorems and Cauchy sequences in fuzzy metric spaces, *Fuzzy Sets Syst.*, **135** (2003), 415–417.

[13] Z. Deng, Fuzzy pseudo-metric space, *J. Math. Anal. Appl.*, **86** (1982), 74–95.

Received: February 22, 2012

Acta Univ. Sapientiae, Mathematica, 4, 2 (2012) 145–153

On Cusa-Huygens type trigonometric and hyperbolic inequalities

József Sándor
Babeş-Bolyai University
Department of Mathematics
Cluj-Napoca, Romania
email: jsandor@math.ubbcluj.ro

Róbert Oláh-Gál
Sapientia – Hungarian University of Transylvania
Department of Economics
Miercurea Ciuc, Romania
email: olahgalrobert@sapientia.siculorum.ro

Abstract. Recently a trigonometric inequality by N. Cusa and C. Huygens (see e.g. [1], [6]) has been discussed extensively in mathematical literature (see e.g. [4], [6, 7]). By using a unified method based on monotonicity or convexity of certain functions, we shall obtain new Cusa-Huygens type inequalities. Hyperbolic versions will be pointed out, too.

1 Introduction

In recent years the trigonometric inequality

$$\frac{\sin x}{x} < \frac{\cos x + 2}{3}, \quad 0 < x < \frac{\pi}{2} \tag{1}$$

among with other inequalities, has attracted attention of several researchers. This inequality is due to N. Cusa and C. Huygens (see [6] for more details regarding this result).

Recently, E. Neuman and J. Sándor [4] have shown that inequality (1) implies a result due to S. Wu and H. Srivastava [10], namely

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2, \quad 0 < x < \frac{\pi}{2} \tag{2}$$

2010 Mathematics Subject Classification: 26D05, 26D07
Key words and phrases: inequalities, trigonometric functions, hyperbolic functions

called as "the second Wilker inequality". Relation (2) implies in turn the classical and famous Wilker inequality (see [9]):

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{3}$$

For many papers, and refinements of (2) and (3), see [4] and the references therein.

A hyperbolic counterpart of (1) has been obtained in [4]:

$$\frac{\sinh x}{x} < \frac{\cosh x + 2}{3}, \quad x > 0. \tag{4}$$

We will call (4) as the hyperbolic Cusa-Huygens inequality, and remark that if (4) is true, then holds clearly also for $x < 0$.

In what follows, we will obtain new proofs of (1) and (4), as well as new inequalities or counterparts of these relations.

2 Main results

Theorem 1 *Let* $f(x) = \dfrac{x(2+\cos x)}{\sin x}$, $0 < x < \dfrac{\pi}{2}$. *Then* f *is a strictly increasing function. Particularly, one has*

$$\frac{2+\cos x}{\pi} < \frac{\sin x}{x} < \frac{2+\cos x}{3}, \quad 0 < x < \frac{\pi}{2}. \tag{5}$$

Theorem 2 *Let* $g(x) = \dfrac{x\left(\dfrac{4}{\pi} + \cos x\right)}{\sin x}$, $0 < x < \dfrac{\pi}{2}$. *Then* g *is a strictly decreasing function. Particularly, one has*

$$\frac{1+\cos x}{2} < \frac{\dfrac{4}{\pi} + \cos x}{\dfrac{4}{\pi} + 1} < \frac{\sin x}{x} < \frac{\dfrac{4}{\pi} + \cos x}{2}. \tag{6}$$

Proof. We shall give a common proofs of Theorems 1 and 2. Let us define the application

$$f_a(x) = \frac{x(a+\cos x)}{\sin x}, \quad 0 < x < \frac{\pi}{2}.$$

Then, easy computations yield that

$$\sin^2 x \cdot f_a'(x) = a \sin x + \sin x \cos x - a x \cos x - x = h(x). \tag{7}$$

The function h is defined on $\left[0, \frac{\pi}{2}\right]$. We get

$$h'(x) = (\sin x)(ax - 2\sin x).$$

Therefore, one obtains that

(i) If

$$\frac{\sin x}{x} < \frac{a}{2},$$

then $h'(x) > 0$. Thus by (7) one has $h(x) > h(0) = 0$, implying $f_a'(x) > 0$, i.e. f_a is strictly increasing.

(ii) If

$$\frac{\sin x}{x} > \frac{a}{2},$$

then $h'(x) < 0$, implying as above that f_a is strictly decreasing.

Select now $a = 2$ in (i). Then $f_a(x) = f(x)$, and the function f in Theorem 1 will be strictly increasing. Selecting $a = \frac{4}{\pi}$ in (ii), by the famous Jordan inequality (see e.g. [3], [7], [8], [2])

$$\frac{\sin x}{x} > \frac{2}{\pi}, \tag{8}$$

so $f_a(x) = g(x)$ of Theorem 2 will be strictly decreasing.

Now remarking that $f(0) < f(x) < f\left(\frac{\pi}{2}\right)$ and $g(0) > g(x) > g\left(\frac{\pi}{2}\right)$, after some elementary transformations, we obtain relations (5) and (6). □

Remarks. 1. The right side of (5) is the Cusa-Huygens inequality (1), while the left side seems to be new.

2. The first inequality of (6) follows by an easy computation, based on $0 < \cos x < 1$. The inequality

$$\frac{1 + \cos x}{2} < \frac{\sin x}{x} \tag{9}$$

appeared in paper [5], and rediscovered by other authors (see e.g. [2]).

3. It is easy to see that inequalities (5) and (6) are not comparable, i.e. none of these inequalities implies the other one for all $0 < x < \pi/2$.

Before turning to the hyperbolic case, the following auxiliary result will be proved:

Lemma 1 *For all* $x \geq 0$ *one has the inequalities*

$$\cos x \cosh x \leq 1 \tag{10}$$

and

$$\sin x \sinh x \leq x^2. \tag{11}$$

Proof. Let $m(x) = \cos x \cosh x - 1$, $x \geq 0$. Then

$$m'(x) = -\sin x \cosh x + \cosh x \sinh x,$$

$$m''(x) = -2 \sin x \sinh x < 0.$$

Thus $m'(x) < m'(0) = 0$ and $m(x) < m(0) = 0$ for $x > 0$, implying (10), with equality only for $x = 0$.

For the proof of (11), let

$$n(x) = x^2 - \sin x \sinh x.$$

Then

$$n'(x) = 2x - \cos x \sinh x - \sin x \cosh x,$$

$$n''(x) = 2(1 - \cos x \cosh x) < 0$$

by (10), for $x > 0$. This easily implies (11). □

Theorem 3 *Let*

$$F(x) = \frac{x(2 + \cosh x)}{\sinh x}, \quad x > 0.$$

Then F *is a strictly increasing function. Particularly, one has inequality (4). On the other hand,*

$$\frac{2 + \cosh x}{k^*} < \frac{\sinh x}{x} < \frac{2 + \cosh x}{3}, \quad 0 < x < \frac{\pi}{2} \tag{12}$$

where $k^* = \frac{\pi}{2}(2 + \cosh \pi/2)/\sinh(\pi/2)$.

Theorem 4 *Let*

$$G(x) = \frac{x(\pi + \cosh x)}{\sinh x}, \quad x > 0.$$

Then G *is a strictly decreasing function for* $0 < x < \pi/2$. *Particularly, one has*

$$\frac{\pi + \cosh x}{\pi + 1} < \frac{\sinh x}{x} < \frac{\pi + \cosh x}{k}, \quad 0 < x < \frac{\pi}{2} \tag{13}$$

where $k = \frac{\pi}{2}(\pi + \cosh \pi/2)/\sinh(\pi/2)$.

Proof. We shall deduce common proofs to Theorem 3 and 4. Put

$$F_a(x) = \frac{x(a + \cosh x)}{\sinh x}, \quad x > 0.$$

An easy computation gives

$$(\sinh x)^2 F_a'(x) = g_a(x) = a \sinh x + \cosh x \sinh x - ax \cosh x - x.$$

The function g_a is defined for $x \geq 0$. As

$$g_a'(x) = (\sinh x)(2 \sinh x - ax),$$

we get that:

(i) If

$$\frac{\sinh x}{x} > \frac{a}{2},$$

then $g_a'(x) > 0$. This in turn will imply $F_a'(x) > 0$ for $x > 0$.

(ii) If

$$\frac{\sinh x}{x} < \frac{a}{2},$$

then $F_a'(x) < 0$ for $x > 0$.

By letting $a = 2$, by the known inequality $\sinh x > x$, we obtain the monotonicity if $F_2(x) = F(x)$ of Theorem 3. Since $F(0) = \lim_{x \to 0+} F(x) = 3$, inequality (4), and the right side of (12) follows. Now, the left side of (12) follows by $F(x) < F(\pi/2)$ for $x < \pi/2$.

By letting $a = \pi$ in (ii) we can deduce the results of Theorem 4. Indeed, by relation (11) of the Lemma 1 one can write $\dfrac{\sinh x}{x} < \dfrac{x}{\sin x}$ and by Jordan's inequality (8), we get $\dfrac{\sinh x}{x} < \dfrac{\pi}{2}$ thus $a = \pi$ may be selected. Remarking that $g(0) > g(x) > g\left(\dfrac{\pi}{2}\right)$, inequalities (13) will follow. □

Remark. By combining (12) and (13), we can deduce that:

$$3 < k^* < k < \pi + 1. \tag{14}$$

Now, the following convexity result will be used:

Lemma 2 *Let* $k(x) = \dfrac{1}{\tanh x} - \dfrac{1}{x}$, $x > 0$. *Then* k *is a strictly increasing, concave function.*

Proof. Simple computations give

$$k'(x) = \frac{1}{x^2} - \frac{1}{(\sinh x)^2} > 0$$

and

$$k''(x) = \frac{2[x^3 \cosh x - (\sinh x)^3]}{x^3 (\sinh x)^3} < 0,$$

since by a result of I. Lazarević (see e.g. [3], [4]) one has

$$\frac{\sinh x}{x} > (\cosh x)^{1/3}. \tag{15}$$

This proves Lemma 2. □

Theorem 5 *Let the function* $k(x)$ *be defined as in Lemma 2. Then one has*

$$\frac{1 + x^2 \cdot \dfrac{k(r)}{r}}{\cosh x} \leq \frac{x}{\sinh x} \text{ for any } 0 < x \leq r \tag{16}$$

and

$$\frac{x}{\sinh x} \leq \frac{1 + k(r)x + k'(r)x(x-r)}{\cosh x} \text{ for any } 0 < x, r. \tag{17}$$

In both inequalities (16) and (17) there is equality only for $x = r$.

Proof. Remark that $k(0+) = \lim\limits_{x\to 0+} k(x) = 0$, and that by the concavity of k, the graph of function k is above the line segment joining the points $A(0,0)$ and $B(r, k(r))$. Thus $k(x) \geq \dfrac{k(r)}{r} \cdot x$ for any $x \in (0, r]$. By multiplying with x this inequality, after some transformations, we obtain (16).

For the proof of (17), write the tangent line to the graph of function k at the point $B(r, k(r))$. Since the equation of this line is $y = k(r) + k'(r)(x-r)$ and writing that $y \leq k(x)$ for any $x > 0$, $r > 0$, after elementary transformations, we get relation (17).

□

For example, when $r = 1$ we get:

$$\left[x^2 \left(\frac{2}{e^2 - 1}\right) + 1\right] / \cosh x \leq \frac{x}{\sinh x} \text{ for all } 0 < x \leq 1 \tag{18}$$

and

$$\frac{x}{\sinh x} \leq \left[1 + \left(\frac{2}{e^2-1}\right)x + \left(\frac{e^4-6e^2+1}{e^4-2e^2+1}\right)x(x-1)\right] / \cosh x \tag{19}$$

for any $x > 0$.

In both inequalities (18) and (19) there is equality only for $x = 1$.

In what follows a convexity result will be proved:

Lemma 3 *Let* $j(x) = 3x - 2\sinh x - \sinh x \cos x$, $0 < x < \frac{\pi}{2}$. *Then* j *is a strictly convex function.*

Proof. Since $j''(x) = 2(\cosh x \sin x - \sinh x) > 0$ is equivalent to

$$\sin x > \tanh x, \quad 0 < x < \frac{\pi}{2} \tag{20}$$

we will show that inequality (20) holds true for any $x \in \left(0, \frac{\pi}{2}\right)$. We note that in [2] it is shown that (20) holds for $x \in (0, 1)$, but here we shall prove with another method the stronger result (20).

Inequality (20) may be written also as

$$p(x) = (e^x + e^{-x})\sin x - (e^x - e^{-x}) > 0.$$

Since $p''(x) = (e^x - e^{-x})(2\cos x - 1)$ and $e^x - e^{-x} > 0$, the sign of $p''(x)$ depends on the sign of $2\cos x - 1$. Let $x_0 \in \left(0, \frac{\pi}{2}\right)$ be the unique number such that $2\cos x_0 - 1 = 0$. Here $x_0 = \arccos\left(\frac{1}{2}\right) \approx 1.0471$. Thus, $\cos x$ being a decreasing function, for all $x < x_0$ one has $\cos x > \frac{1}{2}$, i.e. $p''(x) > 0$ in $(0, x_0)$. This implies $p'(x) > p'(0) = 0$, where

$$p'(x) = (e^x - e^{-x})\sin x + (e^x + e^{-x})\cos x - (e^x + e^{-x}).$$

This in turn gives $p(x) > p(0) = 0$.

Let now $x_0 < x < \pi/2$. Then, as $p'(x_0) > 0$ and $p'\left(\frac{\pi}{2}\right) < 0$ and p' being continuous and decreasing, there exists a single $x_0 < x_1 < \pi/2$ such that $p'(x_1) = 0$. Then p' will be positive on (x_0, x_1) and negative on $\left(x_1, \frac{\pi}{2}\right)$. Thus p will be strictly decreasing on $\left(x_1, \frac{\pi}{2}\right)$, i.e. $p(x) > p\left(\frac{\pi}{2}\right) > 0$. This means that, for any $x \in \left(0, \frac{\pi}{2}\right)$ one has $p(x) > 0$, completing the proof of (20).

□

Now, via inequality (1), the following improvement of relation (11) will be proved:

Theorem 6 *For any* $x \in \left(0, \frac{\pi}{2}\right)$ *one has*

$$\frac{\sin x}{x} < \frac{\cos x + 2}{3} < \frac{x}{\sinh x}. \tag{21}$$

Proof. The first inequality of (21) is the Cusa-Huygens inequality (1). The second inequality of (21) may be written as $j(x) > 0$, where j is the function defined in Lemma 3. As $j'(0) = 0$ and $j'(x)$ is strictly increasing, $j'(x) > 0$, implying $j(x) > j(0) = 0$. This finishes the proof of (21).

□

Finally, we will prove a counterpart of inequality (20):

Theorem 7 *For any* $x \in \left(0, \frac{\pi}{2}\right)$ *one has*

$$\sin x \cos x < \frac{(\sin x)(1 + \cos x)}{2} < \frac{(x + \sin x \cos x)}{2} < \tanh x < \sin x. \tag{22}$$

Proof. The first two inequalities are consequences of $0 < \cos x < 1$ and $0 < \sin x < x$, respectively. The last relation is inequality (20), so we have to prove the third inequality. For this purpose, consider the application

$$u(x) = \tanh x - \frac{(x + \sin x \cos x)}{2},$$

where $x \in \left[0, \frac{\pi}{2}\right]$. An easy computation implies $(\cosh x)^2 . (u'(x)) = 1 - (\cos x \cosh x)^2 \leq 0$ by relation (10) of Lemma 1. Therefore, since $u(0) = 0$, and $u(x) \leq u(0)$, the inequality follows.

□

Remark. As a corollary, we get the following nontrivial relations: For all $x \in \left(0, \frac{\pi}{2}\right)$, we have:

$$x + \sin x \cos x < 2 \sin x \tag{23}$$

and

$$\sin x \cos x < \tanh x < \sin x. \tag{24}$$

References

[1] C. Huygens, *Oeuvres completes 1888–1940*, Société Hollondaise des Sciences, Haga.

[2] R. Klén, M. Visuri, M. Vuorinen, On Jordan type inequalities for hyperbolic functions, *J. Ineq. Appl.* vol. 2010, art. ID 362548, 14 pages.

[3] D. S. Mitrinović, *Analytic inequalities*, Springer Verlag, Berlin, 1970.

[4] E. Neuman, J. Sándor, Inequalities involving trigonometric and hyperbolic functions, *Math. Ineq. Appl.*, **13**, 4 (2010), 715–723.

[5] J. Sándor, Certain trigonometric inequalities, *Octogon Math. Mag.*, **9**, 1 (2001), 331–336.

[6] J. Sándor, M. Bencze, On Huygens' trigonometric inequality, *RGMIA Res. Rep. Collection*, **8**, 3 (2005), art. 14.

[7] J. Sándor, A note on certain Jordan type inequalities, *RGMIA Res. Rep. Collection*, **10**, 1 (2007), art. 1.

[8] J. Sándor, *Selected chapters of geometry, Analysis and number theory: Classical topics in new perspectives*, Lambert Acad. Publ., 2008.

[9] J. B. Wilker, Problem E3306, *Amer. Math. Monthly*, **96** (1989), 55.

[10] S. -H. Wu, H. M. Srivastava, A weighted and exponential generalization of Wilker's inequality and its applications, *Integral Trans. Spec. Funct.*, **18**, 8 (2007), 525–535.

Received: February 19, 2012

Acta Univ. Sapientiae, Mathematica, **4**, 2 (2012) 154–167

Spaces of entire functions represented by vector valued Dirichlet series of slow growth

G. S. Srivastava
Indian Institute of Technology Roorkee
Department of Mathematics
Roorkee, India
email: girssfma@iitr.ernet.in

Archna Sharma
Indian Institute of Technology Roorkee
Department of Mathematics
Roorkee, India
email: gs91490@gmail.com

Abstract. Spaces of all entire functions f represented by vector valued Dirichlet series and having slow growth have been considered. These are endowed with a certain topology under which they become a Frechet space. On this space the form of linear continuous transformations is characterized. Proper bases have also been characterized in terms of growth parameters.

1 Introduction

Let

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \quad s = \sigma + it \ (\sigma, t \text{ are real variables}), \tag{1}$$

where $\{a_n\}$ is a sequence of complex numbers and the sequence $\{\lambda_n\}$ satisfies the conditions $0 < \lambda_1 < \lambda_2 < \lambda_3 \ldots < \lambda_n \ldots$, $\lambda_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \sup \frac{n}{\lambda_n} = D < \infty, \tag{2}$$

$$\lim_{n\to\infty} \sup(\lambda_{n+1} - \lambda_n) = h > 0, \tag{3}$$

2010 Mathematics Subject Classification: 30D15, 46A35
Key words and phrases: vector valued Dirichlet series, entire functions, spaces, bases

and

$$\lim_{n\to\infty} \sup \frac{\log|a_n|}{\lambda_n} = -\infty. \tag{4}$$

By giving different topologies on the set of entire functions represented by the Dirichlet series, Kamthan and Hussain [2] have studied various properties of this space.

Now let $a_n \in E, n = 1, 2, \ldots$, where $(E, \|\cdot\|)$ is a complex Banach space and (4) is replaced by the condition

$$\lim_{n\to\infty} \sup \frac{\log\|a_n\|}{\lambda_n} = -\infty. \tag{5}$$

Then the series in (1) is called a vector valued Dirichlet series and represents an entire function $f(s)$. In what follows, the series in (1) will represent a Vector valued entire Dirichlet series.

Let for entire functions defined as above by (1) and satisfying (2), (3) and (5),

$$M(\sigma, f) = M(\sigma) = \sup_{-\infty<t<\infty} \|f(\sigma + it)\|.$$

Then $M(\sigma)$ is called the maximum modulus of $f(s)$. The order ρ of $f(s)$ is defined as [1]

$$\rho = \lim_{\sigma\to\infty} \sup \frac{\log\log M(\sigma)}{\sigma}, \qquad 0 \leq \rho \leq \infty \tag{6}$$

Also, for $0 < \rho < \infty$ the type T of $f(s)$ is defined by [1]

$$T = \lim_{\sigma\to\infty} \sup \frac{\log M(\sigma)}{e^{\sigma\rho}}, \qquad 0 \leq T \leq \infty.$$

It was proved by Srivastava [1] that if $f(s)$ is of order ρ $(0 < \rho < \infty)$ and (2) holds then $f(s)$ is of type T if and only if

$$T = \lim_{n\to\infty} \sup \frac{\lambda_n}{\rho e}\|a_n\|^{\rho/\lambda_n}.$$

This implies

$$\lim_{n\to\infty} \sup \lambda_n^{1/\rho}\|a_n\|^{1/\lambda_n} = (T\rho e)^{1/\rho}. \tag{7}$$

We now denote by X the set of all vector valued entire functions $f(s)$ given by (1) and satisfying (2), (3) and (5) for which

$$\lim_{\sigma\to\infty} \sup \frac{\log M(\sigma)}{e^{\sigma\rho}} \leq T < \infty, \qquad 0 < \rho < \infty.$$

Then from (7), we have

$$\lim_{n\to\infty} \sup \lambda_n^{1/\rho} \|a_n\|^{1/\lambda_n} \leq (T\rho e)^{1/\rho} \ . \tag{8}$$

From (8), for arbitrary $\varepsilon > 0$ and all $n > n_0(\varepsilon)$,

$$\|a_n\|. \left[\frac{\lambda_n}{(T+\varepsilon)\, e\rho}\right]^{\lambda_n/\rho} < 1.$$

Hence, if we put

$$\|f\|_q = \sum_{n\geq 1} \|a_n\| \left[\frac{\lambda_n}{(T+q^{-1})\, e\rho}\right]^{\lambda_n/\rho} \qquad q \geq 1,$$

then $\|f\|_q$ is well defined and for $q_1 \leq q_2$, $\|f\|_{q_1} \leq \|f\|_{q_2}$. This norm induces a metric topology on X. We define

$$\lambda(f,g) = \sum_{q\geq 1} \frac{1}{2^q} \cdot \frac{\|f-g\|_q}{1+\|f-g\|_q}$$

We denote the space X with the above metric λ by X_λ. Various properties of bases of the space X_λ using the growth properties of the entire vector valued Dirichlet series have been obtained in [3]. These results obviously do not hold if the order ρ of the entire function $f(s)$ is zero. In this paper we have introduced a metric on the space of entire function of zero order represented by vector valued Dirichlet series thereby obtaining various properties of this space.

2 Main results

The vector valued entire function $f(s)$ represented by (1), for which order ρ defined by (6) is equal to zero, we define the logarithmic order ρ^* by

$$\rho^* = \lim_{\sigma\to\infty} \sup \frac{\log\log M(\sigma)}{\log \sigma}, \qquad 1 \leq \rho^* \leq \infty.$$

For $1 < \rho^* < \infty$ the logarithmic type T^* is defined by

$$T^* = \lim_{\sigma\to\infty} \sup \frac{\log M(\sigma)}{\sigma^{\rho^*}}, \qquad 0 \leq T^* \leq \infty.$$

In [4] the authors have established that $f(s)$ is of logarithmic order ρ^*, $1 < \rho^* < \infty$, and logarithmic type T^*, $0 < T^* < \infty$, if and only if

$$\lim_{n\to\infty} \sup \frac{\lambda_n \phi(\lambda_n)}{\log \|a_n\|^{-1}} = \frac{\rho^*}{(\rho^* - 1)} (\rho^* T^*)^{1/(\rho^*-1)}, \tag{9}$$

where $\phi(t)$is the unique solution of the equation $t = \sigma^{\rho^*} - 1$. The above formula can be proved on the same lines as for ordinary Dirichlet series in [5]. Let Y denote the set of all entire functions $f(s)$ given by (1) and satisfying (2), (3) and (5), for which

$$\lim_{\sigma\to\infty} \sup \frac{\log M(\sigma)}{\sigma^\rho} \le T^* < \infty, \qquad 0 < \rho^* < \infty.$$

Then from (9) we have

$$\lim_{n\to\infty} \sup \frac{\lambda_n \phi(\lambda_n)}{\log \|a_n\|^{-1}} \le \frac{\rho^*}{(\rho^* - 1)} (\rho^* T^*)^{1/(\rho^*-1)}, \tag{10}$$

where $\phi(\lambda_n) = \lambda_n^{1/\rho^*-1}$. From (10), for arbitrary $\varepsilon > 0$ and all $n > n_0(\varepsilon)$,

$$\|a_n\| \le \exp\left[-\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \varepsilon)\}^{1/(\rho^*-1)}}\right], \tag{11}$$

where $K = \{\rho^*/(\rho^* - 1)\}^{(\rho^*-1)}$ be a constant. For each $f \in Y$, we define the norm

$$\|f\|_\alpha = \sum_{n\ge 1} \|a_n\| \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \alpha^{-1})\}^{1/(\rho^*-1)}}\right], \qquad \alpha \ge 1$$

then $\|f\|_\alpha$ is well defined and for $\alpha_1 \le \alpha_2$, $\|f\|_{\alpha_1} \le \|f\|_{\alpha_2}$. This norm induces a metric topology on Y defined by

$$d(f, g) = \sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha} \cdot \frac{\|f - g\|_\alpha}{1 + \|f - g\|_\alpha}.$$

We denote the space Y with the above metric d by Y_d. Now we prove

Theorem 1 *The space Y_d is a Frechet space.*

Proof. Here, Y_d is a normed linear metric space. For showing that Y_d is a Frechet space, we need to show that Y_d is complete. Hence, let $\{f_p\}$ be a Cauchy sequence in Y_d. Therefore, for any given $\varepsilon > 0$ there exists an integer $n_0 = n_0(\varepsilon)$ such that

$$d(f_p, f_q) \ < \ \varepsilon \ \forall \ p, q > n_0.$$

Hence $\|f_p - f_q\|_\alpha < \varepsilon \ \ \forall \ p, q > n_0, \ \alpha \geq 1$.
Denoting by $f_p(s) = \sum_{n=1}^{\infty} a_n^{(p)} e^{s \cdot \lambda_n}$, $f_q(s) = \sum_{n=1}^{\infty} a_n^{(q)} e^{s \cdot \lambda_n}$, we have therefore

$$\sum_{n=1}^{\infty} \|a_n^{(p)} - a_n^{(q)}\| \cdot \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \alpha^{-1})\}^{1/(\rho^*-1)}}\right] < \varepsilon \tag{12}$$

for all $p, q > n_0$, $\alpha \geq 1$. Since $\lambda_n \to \infty$ as $n \to \infty$, therefore we have $\|a_n^{(p)} - a_n^{(q)}\| < \varepsilon \ \forall p, q \geq n_0$, and $n = 1, 2, \ldots$, i.e. for each fixed $n = 1, 2, \ldots$, $\left\{a_n^{(p)}\right\}$ is a Cauchy sequence in the Banach space E.

Hence there exists a sequence $\{a_n\} \subseteq E$ such that

$$\lim_{p \to \infty} a_n^{(p)} = a_n, \quad n \geq 1.$$

Now letting $q \to \infty$ in (12), we have for $p \geq n_0$,

$$\sum_{n=1}^{\infty} \|a_n^{(p)} - a_n\| \cdot \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \alpha^{-1})\}^{1/(\rho^*-1)}}\right] < \varepsilon \tag{13}$$

Taking $p = n_0$, we get for a fixed α in (12)

$$\|a_n\| \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \alpha^{-1})\}^{1/(\rho^*-1)}}\right] <$$
$$\|a_n^{(n_0)}\| \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \alpha^{-1})\}^{1/(\rho^*-1)}}\right] + \varepsilon$$

Now $f^{(n_0)} = \sum_{n=1}^{\infty} a_n^{(n_0)} e^{s.\lambda_n} \in Y_d$, hence the condition (11) is satisfied. For arbitrary $\alpha < \beta$, we have, $\|a_n^{(n_0)}\| < \exp\left[\frac{-\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \beta^{-1})\}^{1/(\rho^*-1)}}\right]$ for arbitrarily large n. Hence we have,

$$\|a_n\| \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \alpha^{-1})\}^{1/(\rho^*-1)}}\right] <$$
$$\exp\left[\frac{\lambda_n \phi(\lambda_n)}{(K \cdot \rho^*)^{1/(\rho^*-1)}}\left\{\frac{1}{(T^* + \alpha^{-1})^{1/(\rho^*-1)}} - \frac{1}{(T^* + \beta^{-1})^{1/(\rho^*-1)}}\right\}\right] + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary and the first term on the right hand side $\to 0$ as $n \to \infty$, we find that the sequence $\{a_n\}$ satisfies (11) and therefore $f(s) = \sum_{n=1}^{\infty} a_n e^{s\cdot\lambda_n}$ belongs to Y_d. Using (13) again, we have for α=1, 2. . . ,

$$\|f_p - f\|_\alpha < \varepsilon.$$

Hence

$$d\,(f_p, f) = \sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha} \frac{\|f_p - f\|_\alpha}{1 + \|f_p - f\|_\alpha} \leq \frac{\varepsilon}{1+\varepsilon} \sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha} < \varepsilon.$$

Since the above inequality holds for all $p > n_0$, we finally get $f_p \to f$ as $p \to \infty$ with respect to the metric d, where $f \in Y_d$. Hence Y_d is complete. This proves Theorem 1. □

Next we prove

Theorem 2 *A continuous linear transformation* $\psi : Y_d \to E$ *is of the form*

$$\psi\,(f) = \sum_{n=1}^{\infty} a_n C_n$$

if and only if

$$|C_n| \leq A \cdot \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \alpha^{-1})\}^{1/(\rho^*-1)}}\right] \qquad \textit{for all}\ n \geq 1, \alpha \geq 1, \qquad (14)$$

where A *is a finite, positive number,* $f = f\,(s) = \sum_{n=1}^{\infty} a_n e^{s\cdot\lambda_n}$ *and* λ_1 *is sufficiently large.*

Proof. Let $\psi : Y_d \to E$ be a continuous linear transformation then for any sequence $\{f_m\} \subseteq Y_d$ such that $f_m \to f$, we have $\psi\,(f_m) \to \psi\,(f)$ as $m \to \infty$. Now, let $f\,(s) = \sum_{n=1}^{\infty} a_n e^{s.\lambda_n}$ where $a_n's \in E$ satisfy (11). Then $f \in Y_d$. Also, let $f_k\,(s) = \sum_{n=1}^{k} a_n\, e^{s\lambda_n}$. Then $f_k \in Y_d$ for $k = 1, 2\ldots.$ Let α be any fixed positive integer and let $0 < \varepsilon < \alpha^{-1}$. From (11) we can find an integer m such that

$$\|a_n\| < \exp\left[\frac{-\lambda_n \phi(\lambda_n)}{\{K \cdot \rho^*(T^* + \varepsilon)\}^{1/(\rho^*-1)}}\right], \ \ \forall n > m.$$

Then

$$\left\| f - \sum_{n=1}^{m} a_n e^{s\cdot\lambda_n} \right\|_\alpha = \left\| \sum_{n=m+1}^{\infty} a_n e^{s\cdot\lambda_n} \right\|_\alpha$$
$$= \sum_{n=m+1}^{\infty} \|a_n\| \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K\cdot\rho^*(T^*+\alpha^{-1})\}^{1/(\rho^*-1)}}\right]$$
$$< \sum_{n=m+1}^{\infty} \exp\left[\frac{\lambda_n \phi(\lambda_n)}{(K\cdot\rho^*)^{1/(\rho^*-1)}}\left\{(T^*+\alpha^{-1})^{-1/(\rho^*-1)} - (T^*+\varepsilon)^{-1/(\rho^*-1)}\right\}\right] < \varepsilon,$$

for sufficiently large values of m.

Hence

$$d\,(f, f_m) = \sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha} \frac{\|f - f_m\|_\alpha}{1 + \|f - f_m\|_\alpha} \leq \frac{\varepsilon}{1+\varepsilon} < \varepsilon,$$

i.e. $f_m \to f$ as $m \to \infty$ in Y_d. Since ψ is continuous, we have

$$\lim_{m\to\infty} \psi\,(f_m) = \psi\,(f)\,.$$

Let us denote by $C_n = \psi\left(e^{s\cdot\lambda_n}\right)$. Then

$$\psi\,(f_m) = \sum_{n=1}^{m} a_n \psi\left(e^{s.\lambda_n}\right) = \sum_{n=1}^{m} a_n C_n.$$

Also $|C_n| = |\psi\left(e^{s\lambda_n}\right)|$. Since ψ is continuous on Y_d it is continuous on $Y_{\|\cdot\|_\alpha}$ for each $\alpha = 1, 2, 3 \ldots$. Hence there exists a positive constant A independent of α such that

$$|\psi\left(e^{s.\lambda_n}\right)| = |C_n| \leq A\|p\|_\alpha, \qquad \alpha \geq 1$$

where $p\,(s) = e^{s\cdot\lambda_n}$. Now using the definition of the norm for $p\,(s)$, we get

$$|C_n| \leq A \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K\cdot\rho^*(T^*+\alpha^{-1})\}^{1/(\rho^*-1)}}\right], \qquad n \geq 1, \quad \alpha \geq 1.$$

Hence we get $\psi\,(f) = \sum_{n=1}^{\infty} a_n C_n$, where the sequence $\{C_n\}$ satisfies (14).

Conversely, suppose that $\psi\,(f) = \sum_{n=1}^{\infty} a_n C_n$ and $C_n's$ satisfy (14). Then for $\alpha \geq 1$,

$$\|\psi\,(f)\| \leq A \sum_{n=1}^{\infty} \|a_n\| \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K\cdot\rho^*(T^*+\alpha^{-1})\}^{1/(\rho^*-1)}}\right]$$

i.e. $|\psi(f)| \leq A\|f\|_\alpha \ \alpha \geq 1$.

Now, since $d(f,g) = \sum_{\alpha\geq 1} \frac{1}{2^\alpha} \cdot \frac{\|f-g\|_\alpha}{1+\|f-g\|_\alpha}$, therefore ψ is continuous. This completes the proof of Theorem 2. □

3 Linear continuous transformations and proper bases

Following Kamthan and Hussain [2] we give some more definitions. A subspace X_0 of X is said to be spanned by a sequence $\{\alpha_n\} \subseteq X$ if X_0 consists of all linear combinations $\sum_{n=1}^{\infty} c_n\alpha_n$ such that $\sum_{n=1}^{\infty} c_n\alpha_n$ converges in X. A sequence $\{\alpha_n\} \subseteq X$ which is linearly independent and spans a subspace X_0 of X is said to be a base in X_0. In particular, if $e_n \in X$, $e_n(s) = e^{s\lambda_n}$, $n \geq 1$, then $\{e_n\}$ is a base in X. A sequence $\{\alpha_n\} \subseteq X$ will be called a 'proper base' if it is a base and it satisfies the condition:
"for all sequences $\{a_n\} \subseteq E$, convergence of $\sum_{n=1}^{\infty} \|a_n\|\ \alpha_n$ in X implies the convergence of $\sum_{n=1}^{\infty} a_n e_n$ in X". As defined above, for $f \in Y$, we put $\|f, T^* + \delta\| = \sum_{n\geq 1} \|a_n\| \exp\left[\frac{\lambda_n\varphi(\lambda_n)}{\{K\rho^*(T^*+\delta)\}^{1/(\rho^*-1)}}\right]$. We now prove

Theorem 3 *A necessary and sufficient condition that there exists a continuous linear transformation* $F: Y \to Y$ *with* $F(e_n) = \alpha_n$, $n = 1, 2, \ldots$, *where* $\alpha_n \in Y$, *is that for each* $\delta > 0$

$$\lim_{n\to\infty} \sup \frac{\log \|\alpha_n : T^* + \delta\|^{1/\lambda_n}}{\varphi(\lambda_n)} \leq \left(\frac{\rho^*-1}{\rho^*}\right)(\rho^* T^*)^{-1/\rho^*-1}. \tag{15}$$

Proof. Let F be a continuous linear transformation from Y into Y with $F(e_n) = \alpha_n$, $n = 1, 2, \ldots$ Then for any given $\delta > 0$, there exists a $\delta_1 > 0$ and a constant $K' = K'(\delta)$ depending on δ only, such that

$$\begin{aligned}
\|F(e_n); T^* + \delta\| \leq K'\|e_n; T^* + \delta_1\| &\Rightarrow \|\alpha_n; T^* + \delta\| \\
&\leq K' \exp\left\{\frac{(\rho^*-1)\lambda_n\varphi(\lambda_n)}{(T^*+\delta_1)^{1/\rho^*-1}\left((\rho^*)^{\rho^*/\rho^*-1}\right)}\right\} \\
&\Rightarrow \log\|\alpha_n; T^* + \delta\|^{1/\lambda_n} \\
&\leq o(1) + \frac{\varphi(\lambda_n)(\rho^*-1)}{(T^*+\delta_1)^{1/\rho^*-1}\left((\rho^*)^{\rho^*/\rho^*-1}\right)}, \\
&\Rightarrow \lim_{n\to\infty} \sup \frac{\log\|\alpha_n; T^* + \delta\|^{1/\lambda_n}}{\varphi(\lambda_n)} \leq \frac{(\rho^*-1)}{\rho^*(\rho^* T^*)^{1/\rho^*-1}}.
\end{aligned}$$

Conversely, let the sequence $\{\alpha_n\}$ satisfy (15) and let $\alpha = \sum_{n=1}^{\infty} a_n e_n$. Then we have

$$\lim_{n\to\infty} \sup \frac{\lambda_n\ \phi(\lambda_n)}{\log \|a_n\|^{-1}} \leq \frac{\rho^*(\rho^* T)^{1/\rho^*-1}}{(\rho^*-1)}.$$

Hence, given $\eta > 0$, there exists $N_0 = N_0\,(\eta)$, such that

$$\frac{\varphi(\lambda_n)}{\log \|a_n\|^{-1/\lambda_n}} \leq \frac{\rho^*}{(\rho^*-1)}\ \{\rho^*(T^*+\eta)\}^{\ 1/\rho^*-1}\quad \forall\, n \geq N_0.$$

Further, for a given $\eta_1 > \eta$, from (15), we can find $N_1 = N_1\,(\eta_1)$ such that for $n \geq N_1$

$$\frac{\log \|\alpha_n; T^* + \delta\|^{1/\lambda_n}}{\varphi(\lambda_n)} \leq \left(\frac{\rho^*-1}{\rho^*}\right) \{\rho^*(T^*+\eta_1)\}^{\ -1/(\rho^*-1)}.$$

Choose $n \geq \max(N_0, N_1)$. Then

$$\frac{\log \|\alpha_n; T^* + \delta\|^{1/\lambda_n}}{\log \|a_n\|^{-1/\lambda_n}} \leq \left(\frac{T^*+\eta}{T^*+\eta_1}\right)^{1/(\rho^*-1)}$$

$\Rightarrow\ \|a_n\|\ \|\alpha_n; T^*+\delta\|\ \leq\ \|a_n\|^{1-(T^*+\eta/T^*+\eta_1)^{1/(\rho^*-1)}} = \|a_n\|^{\beta}$ (say) where $\beta = 1-(T^*+\eta/T^*+\eta_1)^{1/(\rho^*-1)} > 0$. Now from (5) we can easily show that for any arbitrary large number $K > 0, \|a_n\| < e^{-k\lambda_n}$.

Hence we have for all large values of $n, \|a_n\|\,\|\alpha_n; T^*+\delta\| \leq e^{-K\beta\lambda_n}$.

Consequently the series $\sum_{n=1}^{\infty} \|a_n\|\ \|\alpha_n; T^*+\delta\|$ converges for each $\delta > 0$. Therefore $\sum_{n=1}^{\infty} \|a_n\|\,\alpha_n$ converges to an element of Y. For each $\alpha \in Y$, We define $F(\alpha) = \sum_{n=1}^{\infty} a_n\,\alpha_n$. Then $F(e_n) = \alpha_n$. Now, given $\delta > 0$, $\exists\, \delta_1 > 0$ such that

$$\frac{\log \|\alpha_n; T^* + \delta\|^{1/\lambda_n}}{\varphi(\lambda_n)} \leq \left(\frac{\rho^*-1}{\rho^*}\right) \{\rho^*(T^*+\eta_1)\}^{\ -1/(\rho^*-1)}$$

for all $n \geq N = N(\delta, \delta_1)$. Hence

$$\Rightarrow\ \|\alpha_n; T^*+\delta\| \leq\ K' \exp\left\{\frac{(\rho^*-1)\,\lambda_n\,\varphi(\lambda_n)}{\rho^*\{\rho^*\,(T^*+\delta_1)\}^{1/\rho^*-1}}\right\}$$

where $K' = K'(\delta)$ and the inequality is true for all $n > 0$. Now

$$\begin{aligned}\|F(\alpha); T^*+\delta\| &\leq \sum_{n=1}^{\infty} \|a_n\|\,\|\alpha_n; T^*+\delta\| \\ &\leq K' \sum_{n=1}^{\infty} \|a_n\| \exp\left\{\frac{(\rho^*-1)\lambda_n\varphi(\lambda_n)}{\rho^*\{\rho^*\,(T^*+\delta_1)\}^{1/\rho^*-1}}\right\} = K'\,\|a_n; T^*+\delta\|.\end{aligned}$$

Hence F is continuous. This proves Theorem 3. $\square$

We now give some results characterizing the proper bases.

Lemma 1 *In the space* Y_d*, the following three conditions are equivalent:*

(i) *For each* $\delta > 0$, $\lim\limits_{n\to\infty} \sup \frac{\log \|\alpha_n;T^*+\delta\|^{1/\lambda_n}}{\varphi(\lambda_n)} \le \left(\frac{\rho^*-1}{\rho^*}\right) (\rho^* T^*)^{-1/(\rho^*-1)}$.

(ii) *For any sequence* $\{a_n\}$ *in* E*, the convergence of* $\sum_{n=1}^{\infty} a_n e_n$ *in* Y *implies that* $\lim\limits_{n\to\infty} \|a_n\|\alpha_n = 0$ *in* Y.

(iii) *For any sequence*$\{a_n\}$ *in* E*, the convergence of* $\sum_{n=1}^{\infty} a_n e_n$ *in* Y *implies the convergence of* $\sum_{n=1}^{\infty} \|a_n\|\alpha_n$ *in* Y.

Proof. First suppose that (ii) holds. Then for any sequence $\{a_n\}$ $\sum_{n=1}^{\infty} a_n e_n$ converges in Y implies that $\sum_{n=1}^{\infty} \|a_n\|\,\alpha_n$ converges in Y which in turn implies that $\|a_n\|\alpha_n \to 0$ as $n \to \infty$. Hence (ii) $\Rightarrow$ (iii).

Now we assume that (iii) is true but (i) is false. Hence for some $\delta > 0$, there exists a sequence $\{n_k\}$ of positive integers such that $\forall\, n_k$, $k = 1, 2, \dots,$

$$\frac{\log \|\alpha_{n_k};T^*+\delta\|^{1/\lambda_{n_k}}}{\varphi(\lambda_{n_k})} > \left(\frac{\rho^*-1}{\rho^*}\right) \left\{\rho^*(T^*+\frac{1}{k})\right\}^{-1/(\rho^*-1)}.$$

Define a sequence $\{a_n\}$ as follows:

$$\|a_n\| = \begin{cases} \|\alpha_n;T^*+\delta\|^{-1}, & n = n_k \\ 0; & n \ne n_k \end{cases} \tag{16}$$

Then, we have for all large values of k,

$$\frac{\varphi(\lambda_{n_k})}{\log \|a_{n_k}\|^{-1/\lambda_{n_k}}} = \frac{\varphi(\lambda_{n_k})}{\log \|\alpha_{n_k};T^*+\delta\|^{1/\lambda_{n_k}}} < \left(\frac{\rho^*}{\rho^*-1}\right) \left\{\rho^*(T^*+\frac{1}{k})\right\}^{1/(\rho^*-1)}.$$

Hence,

$$\lim_{k\to\infty} \sup \frac{\varphi(\lambda_{n_k})}{\log \|a_{n_k}\|^{-1/\lambda_{n_k}}} \le \left(\frac{\rho^*}{\rho^*-1}\right) (\rho^* T^*)^{1/(\rho^*-1)}.$$

Thus $\{a_n\}$ defined by (16) satisfies the condition

$$\lim_{n\to\infty} \sup \frac{\varphi(\lambda_n)}{\log \|a_n\|^{-1/\lambda_n}} \le \left(\frac{\rho^*}{\rho^*-1}\right) (\rho^* T^*)^{1/(\rho^*-1)}$$

which in view of Theorem 1 above is equivalent to the condition that $\sum a_n e_n$ converges in Y. Hence by (iii), $\lim\limits_{n\to\infty} \|a_n\|\alpha_n = 0$. However

$$\|\, \|a_{n_k}\|\alpha_{n_k};T^*+\delta\| = \|a_{n_k}\| \cdot \|\alpha_{n_k};T^*+\delta\| = 1.$$

Hence $\lim_{n\to\infty} \|a_n\|\alpha_n \neq 0$ in $Y(\rho^*, T^*, \delta)$. This is a contradiction. Hence (iii)⇒(i). In the course of proof of Theorem 3 above, we have already proved that (i)⇒(ii). Thus the proof of Lemma 1 is complete. □

Next we prove

Lemma 2 *The following three properties are equivalent:*

(a) For all sequences $\{a_n\}$ *in* E, $\lim_{n\to\infty} a_n\alpha_n = 0$ *in* Y *implies that* $\sum_{n=1}^{\infty} a_n e_n$ *converges in* Y.

(b) For all sequences $\{a_n\}$ *in* E, *the convergence of* $\sum_{n=1}^{\infty} \|a_n\|\alpha_n$ *in* Y *implies the convergence of* $\sum_{n=1}^{\infty} a_n e_n$.

(c) $\lim_{\delta\to 0}\left\{\lim_{n\to\infty}\inf \frac{\log\|\alpha_n; T^*+\delta\|^{1/\lambda_n}}{\varphi(\lambda_n)}\right\} \geq \left(\frac{\rho^*-1}{\rho^*}\right)(\rho^* T^*)^{-1/(\rho^*-1)}$.

Proof. Obviously (a)⇒(b). We now prove that (b)⇒(c). To prove this, we suppose that (b) holds but (c) does not hold. Hence

$$\lim_{\delta\to 0}\left\{\lim_{n\to\infty}\inf \frac{\log\|\alpha_n; T^*+\delta\|^{1/\lambda_n}}{\varphi(\lambda_n)}\right\} < \left(\frac{\rho^*-1}{\rho^*}\right)(\rho^* T^*)^{-1/(\rho^*-1)}.$$

Since $\log\|\alpha_n; T+\delta\|$ increases as δ decreases, this implies that for each $\delta > 0$,

$$\left\{\lim_{n\to\infty}\inf \frac{\log\|\alpha_n; T^*+\delta\|^{1/\lambda_n}}{\varphi(\lambda_n)}\right\} < \left(\frac{\rho^*-1}{\rho^*}\right)(\rho^* T^*)^{-1/(\rho^*-1)}.$$

Hence, if $\eta > 0$ be a fixed small positive number, then for each $r > 0$, we can find a positive number n_r such that $\forall r$, we have $n_{r+1} > n_r$ and

$$\lim_{n\to\infty}\inf \frac{\log\|\alpha_{n_r}; T^*+r^{-1}\|^{1/\lambda_{n_r}}}{\varphi(\lambda_n)} < \left(\frac{\rho^*-1}{\rho^*}\right)\{\rho^*(T^*+\eta)\}^{-1/(\rho^*-1)} \tag{17}$$

Now we choose a positive number $\eta_1 < \eta$, and define a sequence $\{a_n\}$ as

$$\|a_n\| = \begin{cases} \left(\frac{T^*+\eta_1}{T^*+\eta}\right)^{\lambda_n} \exp\left\{-\left(\frac{\rho^*-1}{\rho^*}\right)\frac{\lambda_n\varphi(\lambda_n)}{\{\rho^*(T^*+\eta)\}^{1/(\rho^*-1)}}\right\}, & n = n_r \\ 0, & n \neq n_r \end{cases}.$$

Then, for any $\delta > 0$

$$\sum_{n=1}^{\infty}\|a_n\|\cdot\|\alpha_n; T^*+\delta\| = \sum_{r=1}^{\infty}\|a_{n_r}\|\cdot\|\alpha_{n_r}; T^*+\delta\|. \tag{18}$$

For any given $\delta > 0$, we omit from the above series those finite number of terms, which correspond to those number n_r for which $1/r$ is greater than δ. The remainder of the series in (18) is dominated by $\sum_{r=1}^{\infty} \|a_{n_r}\| \cdot \|\alpha_{n_r}; T^* + r^{-1}\|$. Now by (17) and (18), we find that

$$\begin{aligned}
&\sum_{r=1}^{\infty} \|a_{n_r}\| \cdot \|\alpha_{n_r}; T^* + r^{-1}\| \\
&\leq \sum_{r=1}^{\infty} \left\{ \exp\left\{ -\left(\frac{\rho^* - 1}{\rho^*}\right) \frac{\lambda_{n_r} \varphi(\lambda_{n_r})}{\{\rho^*(T^* + \eta)\}^{1/(\rho^*-1)}} \right\} \left(\frac{T^* + \eta_1}{T^* + \eta}\right)^{\lambda_{n_r}} \right\} \\
&\times \exp\left\{ \left(\frac{\rho^* - 1}{\rho^*}\right) \frac{\lambda_{n_r} \varphi(\lambda_{n_r})}{\{\rho^*(T^* + \eta)\}^{1/(\rho^*-1)}} \right\} \leq \sum_{r=1}^{\infty} \left(\frac{T^* + \eta_1}{T^* + \eta}\right)^{\lambda_{n_r}}.
\end{aligned}$$

Since $\eta_1 < \eta$, therefore the above series on the right hand side is convergent. For this sequence $\{a_n\}$, $\sum_{n=1}^{\infty} \|a_n\| \, \alpha_n$ converges in $Y(\rho^*, T^*, \delta)$ for each $\delta > 0$ and hence converges in Y.

But we have,

$$\lim_{n\to\infty} \sup \frac{\varphi(\lambda_n)}{\log \|a_n\|^{-1/\lambda_n}} = \left(\frac{\rho^*}{\rho^* - 1}\right) \{\rho^*(T^* + \eta)\}^{1/(\rho^*-1)}$$

which contradicts (10). This proves (b) $\Rightarrow$(c).

Now we prove that (c)$\Rightarrow$(a). We assume (c) is true but (a) is not true. Then there exists a sequences $\{a_n\}$ of complex numbers for which $\|a_n\|\alpha_n \to 0$ in Y, but $\sum_{n=1}^{\infty} a_n e_n$ does not converge in Y. This implies that

$$\lim_{n\to\infty} \sup \frac{\varphi(\lambda_n)}{\log \|a_n\|^{-1/\lambda_n}} > \left(\frac{\rho^*}{\rho^* - 1}\right) (\rho^* T^*)^{1/(\rho^*-1)}$$

Hence there exists a positive number ε and a sequence $\{n_k\}$ of positive integers such that

$$\frac{\varphi(\lambda_n)}{\log \|a_n\|^{-1/\lambda_n}} = \left(\frac{\rho^*}{\rho^* - 1}\right) \{\rho^*(T^* + \varepsilon)\}^{1/(\rho^*-1)}, \qquad \forall n = n_k$$

We choose another positive number $\eta < \varepsilon/2$. By assumption we can find a positive number δ i.e. $\delta = \delta(\eta)$ such that

$$\lim_{n\to\infty} \inf \frac{\log \|\alpha_n, T^* + \delta\|^{1/\lambda_n}}{\varphi(\lambda_n)} > \left(\frac{\rho^* - 1}{\rho^*}\right) \{\rho^*(T^* + \eta)\}^{-1/(\rho^*-1)}.$$

Hence there exists $N = N(\eta)$, such that

$$\frac{\log \|\alpha_n, T^* + \delta\|^{1/\lambda_n}}{\varphi(\lambda_n)} \geq \left(\frac{\rho^* - 1}{\rho^*}\right) \{\rho^*(T^* + 2\eta)\}^{-1/(\rho^*-1)}, \quad \forall n \geq N.$$

Therefore

$$\begin{aligned} \max \| \|a_n\| \alpha_n; T^* + \delta\| &= \max \{\|a_n\| \cdot \|\alpha_n; T^* + \delta\| \} \\ &\geq \max \{\|a_{n_k}\| \cdot \|\alpha_{n_k}; T^* + \delta\|\} \\ &\geq \exp \left\{ \frac{-\lambda_{n_k} \varphi(\lambda_{n_k})(\rho^* - 1)}{\rho^* \{\rho^*(T^* + \varepsilon)\}^{1/(\rho^*-1)}} \right\} \\ &\times \exp \left\{ \frac{\lambda_{n_k} \varphi(\lambda_{n_k})(\rho^* - 1)}{\rho^* \{\rho^*(T^* + 2\eta)\}^{1/(\rho^*-1)}} \right\} > 1 \end{aligned}$$

for $n_k > N$ as $\varepsilon > 2\eta$.

Thus $\{\|a_n\| \ \alpha_n\}$ does not tend to zero in $Y(\rho^*, T^*, \delta)$ for the δ chosen above. Hence $\{\|a_n\| \ \alpha_n\}$ does not tend to 0 in Y and this is a contradiction. Thus (c)$\Rightarrow$(a) is proved. This proves Lemma 2. □

Lastly we prove:

Theorem 4 *A base $\{\alpha_n\}$ in a closed subspace Y_0 of Y is proper if and only if the conditions (i) and (c) stated above are satisfied.*

Proof. Let $\{\alpha_n\}$ be a proper base in a closed subspace Y_0 of Y. Hence for any sequence of complex number $\{a_n\}$ the convergence of $\sum_{n=1}^{\infty} \|a_n\| \alpha_n$ in Y_0 implies the convergence of $\sum_{n=1}^{\infty} a_n e_n$ in Y_0. Therefore (b) and hence (c) is satisfied. Further the convergence of $\sum_{n=1}^{\infty} a_n e_n$ in Y_0 is equivalent to the condition

$$\lim_{n \to \infty} \sup \frac{\varphi(\lambda_n)}{\log \|a_n\|^{-1/\lambda_n}} = \left(\frac{\rho^*}{\rho^* - 1}\right) (\rho^* T^*)^{1/(\rho^*-1)}.$$

Now let $\alpha = \sum_{n=1}^{\infty} a_n e_n$. Then proceeding as in second part of the proof of Theorem 1, we can prove that $\sum_{n=1}^{\infty} \|a_n\| \alpha_n$ converges to an element of Y_0 and thus (ii) is satisfied. But (ii) is equivalent to (i). Hence necessary part of the theorem is proved.

Conversely, suppose that conditions (i) and (c) are satisfied, with $\{\alpha_n\}$ being a base in a closed subspace Y_0 of Y. Then by Lemma 2, we find that for any sequence $\{a_n\}$ in E, convergence of $\sum_{n=1}^{\infty} \|a_n\| \alpha_n$ in Y_0 implies the convergence of $\sum_{n=1}^{\infty} a_n e_n$ in Y_0. Therefore $\{\alpha_n\}$ is a proper base of Y_0. This concludes the proof. □

Acknowledgements

The authors are grateful to the referee for his valuable comments and suggestions.

References

[1] B. L. Srivastava, *A study of spaces of certain classes of vector valued Dirichlet series*, Thesis, I. I. T Kanpur, 1983.

[2] P. K. Kamthan and T. Hussain, Spaces of entire functions represented by Dirichlet series, *Collec. Math.*, **19**, 3 (1968), 203–216.

[3] G. S. Srivastava, A. Sharma, Spaces of entire functions represented by vector valued Dirichlet series, *J. Math. Appl.*, **34**, (2011), 97–107.

[4] G. S. Srivastava, A. Sharma, On generalized order and generalized type of vector valued Dirichlet series of slow growth, *Int. J. Math. Archive*, **2**, 12 (2011), 2652–2659.

[5] G. S. Srivastava, A note on the proximate order of entire function represented by Dirichlet series, *Bull. De L'Academie Polonaise des Sciences*, **19**, 3 (1971), 199–202.

Received: February 4, 2012

Acta Univ. Sapientiae, Mathematica, 4, 2 (2012) 168–181

Unification of extensions of zip rings

Amit Bhooshan Singh
Jamia Millia Islamia
(Central University)
Department of Mathematics
Delhi 1110025, India
email: amit.bhooshan84@gmail.com

V. N. Dixit
Jamia Millia Islamia
(Central University)
Department of Mathematics
Delhi 1110025, India
email: vn_dixit@yahoo.com

Abstract. In this note, we investigate that a ring R is a right zip ring if and only if skew monoid ring $R*G$ (induced by a monoid homomorphism $\lambda : G \to \mathrm{Aut}(R)$) is a right zip ring when R is a right (G, λ)-McCoy ring, where G be a u.p.-monoid. Moreover, we study the relationship between right zip property of a ring R and skew generalized power series ring $R[[G, \omega]]$ (induced by a monoid homomorphism $\omega : G \to \mathrm{End}(R)$) over R when R is (G, ω)-Armendariz and G-compatible, where G is a strictly ordered monoid, which provides a unified solution to the questions raised by Faith [9].

1 Introduction

Throughout this article, R and G denote an associative ring with identity and monoid, respectively. For any subset X of a ring R, $r_R(X)$ denotes the right annihilator of X in R. Faith [8] called a ring R right zip provided that if the right annihilator $r_R(X)$ of a subset X of R is zero, then there exists a finite subset $Y \subseteq X$ such that $r_R(Y) = 0$; equivalently, for left ideal L of R with $r_R(L) = 0$, there exists a finitely generated left ideal $L_1 \subseteq L$ such that $r_R(L_1) = 0$. R is zip if it is both right and left zip. The concept of zip rings was initiated by Zelmanowitz [31] and appeared in various papers [3, 4, 7, 8, 9] and

2010 Mathematics Subject Classification: 16P60, 16S86, 16S99
Key words and phrases: zip ring, (G, λ)-Armendariz ring, right (G, λ)-McCoy ring, S-compatible ring, (G, ω)-Armendariz ring, skew monoid ring, skew generalized power series ring

references therein. Zelmanowitz stated that any ring satisfying the descending chain conditions on right annihilators is a right zip ring, but the converse does not hold. Beachy and Blair [3] studied rings that satisfy the condition that every faithful right ideal I of a ring R (a right ideal I of a ring R is faithful if $r_R(I) = 0$) is cofaithful (a right ideal I of a ring R is cofaithful if there exists a finite subset $I_1 \subseteq I$ such that $r_R(I_1) = 0$). Right zip rings have this property and conversely for commutative ring R.

Extensions of zip rings were studied by several authors. Beachy and Blair [3] showed that if R is a commutative zip ring, then polynomial ring $R[x]$ over R is a zip ring. Afterwards, Cedo [4] proved that if R is a commutative zip ring, then the $n \times n$ full matrix ring $Mat_n(R)$ over R is zip; moreover, he settled negatively the following questions which were posed by Faith [8]: Does R being a right zip ring imply $R[x]$ being right zip?; Does R being a right zip imply $\mathrm{Mat}_n(R)$ being right zip?; Does R being a right zip ring imply $R[G]$ being right zip when G is a finite group? Based on the preceding results, Faith [9] again raised the following questions: When does R being a right zip ring imply $R[x]$ being right zip?; Characterize a ring R such that $Mat_n(R)$ is right zip; When does R being a right zip ring imply $R[G]$ being right zip when G is a finite group? Also he proved that if R is a commutative ring and G is a finite Abelian group, then the group ring $R[G]$ of G over R is zip.

In [14], Hong et al. studied above questions and proved that R is a right zip ring if and only if $R[x]$ is a right zip ring when R is an Armendariz ring. They also showed that if R is a commutative ring and G a u.p.-monoid that contains an infinite cyclic submonoid, then R is a zip ring if and only if $R[G]$ is a zip ring. Further, Cortes [7] studied the relationship between right (left) zip property of R and skew polynomial extensions over R by using skew versions of Armendariz rings and generalized the results of Hong et al. [14]. Later, Hashemi [10] showed that R is a right zip ring if and only if $R[G]$ is a right zip ring when R be a reversible ring and G a strictly totally ordered monoid. In this paper, we prove the above mentioned results to skew monoid ring $R * G$ (induced by a monoid homomorphism $\lambda : G \to \mathrm{Aut}(R)$) and skew generalized power series ring $R[[G, \omega]]$ (induced by a monoid homomorphism $\omega : G \to \mathrm{End}(R)$).

This paper is organized as follows. In Section 2, we introduce the concept of right (G, λ)-McCoy ring and extend the above mentioned results proved by Hong et al. [14], Cortes [7] and Hashemi [10] to skew monoid ring $R * G$ (induced by a monoid homomorphism $\lambda : G \to \mathrm{Aut}(R)$). In Section 3, we discuss a unification of the above extensions and prove that if R is (G, ω)-Armendariz ring and G-compatible, then skew generalized power series ring $R[[G, \omega]]$ (induced by a monoid homomorphism $\omega : G \to \mathrm{End}(R)$) is right zip if and only if R is right zip. This provides a unified generalization of the results due to Hong et al. [14] and Cortes [7].

2 Right zip skew monoid rings

In this section, we study the fundamental concept of a skew monoid ring and give the definition of right (G, λ)-McCoy ring which is a generalization of right G-McCoy ring. Moreover, we investigate a relationship between right zip property of a ring R and skew monoid ring $R * G$ over R, and we also extend some results of [7, 10, 14].

Definition 2.1 *A monoid G is called a unique product monoid (or a u.p.-monoid) if for any two nonempty finite subsets $A, B \subseteq G$ there exist $a \in A$ and $b \in B$ such that $ab \neq a'b'$ for every $(a', b') \in A \times B \backslash \{(a, b)\}$; the element ab is called a u.p.-element of $AB = \{cd : c \in A, d \in B\}$.*

The class of u.p.-monoids includes the right and left totally ordered monoids, submonoids of a free group, and torsion-free nilpotent groups (for details see [23, 24]).

From [15, 22], let R be a ring and G a u.p.-monoid. Assume that there exists a monoid homomorphism $\lambda : G \to \mathrm{Aut}(R)$. We denote by $\lambda^g(r)$ the image of $r \in R$ under $g \in G$. Then we can form a skew monoid ring $R * G$ (induced by the monoid homomorphism $\lambda : G \to \mathrm{Aut}(R)$) by taking its elements to be finite formal combinations $\sum\limits_{i=1}^{n} a_i g_i$, where $a_i \in R$, $g_i, \in G$ for all i, with multiplication rule defined by $gr = \lambda^g(r)g$.

It is well known that if R is a commutative ring and $f(x)$ is a zero divisor in $R[x]$, there is a nonzero element $r \in R$ with $f(x)r = 0$, as proved by McCoy [26, Theorem 2]. Based on this result, Nielsen [28] called a ring R right McCoy if for each pair of nonzero polynomials $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$ there exists a nonzero element $r \in R$ with $f(x)r = 0$. Left McCoy ring can be defined similarly. A ring R is McCoy if it is both right and left McCoy. Thus every commutative ring is McCoy. Further, Hashemi [10] generalized the concept of McCoy ring to monoid ring and called a ring R right G-McCoy ring (right McCoy ring relative to monoid) if whenever $0 \neq \alpha = a_1 g_1 + a_2 g_2 + \cdots + a_n g_n, 0 \neq \beta = b_1 h_1 + b_2 h_2 + \cdots + b_m h_m \in R[G]$, with $a_i, b_j \in R$, $g_i, h_j \in G$ satisfy $\alpha\beta = 0$ implies $\alpha r = 0$ for some nonzero $r \in R$. The left G-McCoy ring can be defined similarly. If R is both right and left G-McCoy, then R is

G-McCoy. Here, we extend the concept of McCoy ring to skew monoid ring $R * G$ (induced by a monoid homomorphism $\lambda : G \to \mathrm{Aut}(R)$) with the help of a construction of skew monoid rings $R * G$.

Definition 2.2 *Let R be a ring, G a u.p.-monoid and $\lambda : G \to \mathrm{Aut}(R)$ a monoid homomorphism. A ring R is called right (G, λ)-McCoy if whenever $\alpha = \sum_{i=1}^{n} a_i g_i$, $\beta = \sum_{j=1}^{m} b_j h_j \in R * G$, with $a_i, b_j \in R$, $g_i, h_j \in G$ satisfy $\alpha\beta = 0$, then $\alpha r = 0$ or $\lambda^{-g_i}(a_i) r = 0$ for some nonzero $r \in R$. The left (G, λ)-McCoy ring is defined similarly. If R is both right and left (G, λ)-McCoy, then R is (G, λ)-McCoy.*

Example 2.3 *We give some special cases of right (G, λ)-McCoy rings.*

(1) *Suppose G be trivial order monoid and $\lambda = 1 : G \to \mathrm{Aut}(R)$ a monoid homomorphism. Then R is right (G, λ)-McCoy if and only if R is right G-McCoy. Thus right G-McCoy [10] is special case of right (G, λ)-McCoy.*

(2) *Suppose $G = (\mathbb{N} \cup \{0\}, +)$ and $\lambda = 1 : G \to \mathrm{Aut}(R)$. Then R is right (G, λ)-McCoy if and only if R is right McCoy. Thus right McCoy [28] is special case of right (G, λ)-McCoy.*

In the following theorem, we extend the results of Hong et al. [14, Theorem 11, Corollary 13, Proposition 14, Theorem 16 and Corollary 17], Cortes [7, Theorem 2.8(i)] and Hashemi [10, Theorem 1.25 and Corollary 1.26] to skew monoid ring $R * G$ (induced by a monoid homomorphism $\lambda : G \to \mathrm{Aut}(R)$) using right (G, λ)-McCoy ring, and also provides a generalized solution to the questions posed by Faith [9] for noncommutative zip rings.

Theorem 2.4 *Let G be a u.p.-monoid and $\lambda : G \to \mathrm{Aut}(R)$ a monoid homomorphism. If R is a right (G, λ)-McCoy ring, then R is right zip if and only if $R * G$ is right zip.*

Proof. Suppose R is right zip and Y a nonempty subset of $R * G$ such that $r_{R*G}(Y) = 0$. Let V be the set of all coefficients of elements of Y and defined by $V = C_Y = \bigcup_{\alpha \in Y} C_\alpha$ such that $C_\alpha = \{\lambda^{-g_i}(a_i) : 1 \le i \le n\}$, where $\alpha = \sum_{i=1}^{n} a_i g_i \in R * G$. Take any $a \in r_R(V)$ then $a \in r_R(\bigcup_{\alpha \in Y} C_\alpha)$ which implies $a \in r_R(C_\alpha)$ for all $\alpha \in Y$. Thus $a \in r_{R*G}(\alpha) = 0$ for all $\alpha \in Y$. Therefore $r_R(V) = 0$. Since R is right zip, there exists a nonempty subset

$V_1 = \{\lambda^{-g_{i_1}}(a_{i_1}), \lambda^{-g_{i_2}}(a_{i_2}), \dots, \lambda^{-g_{i_n}}(a_{i_n})\}$ such that $r_R(V_1) = 0$. For each $\lambda^{-g_{i_j}}(a_{i_j}) \in V_1$, there exists $\alpha_{i_j} \in Y$ such that some of the coefficients of α_{i_j} are a_{i_j} for each $1 \le j \le n$. So we have $Y_0 = \{\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_n}\}$ be a nonempty subset of Y whose some of the coefficients are a_{i_j} for each $1 \le j \le n$. Suppose V_0 be a set of all the coefficients of Y_0. Then $V_1 \subseteq V_0$ which implies $r_R(V_0) \subseteq r_R(V_1) = 0$. Now we will show that $r_{R*G}(Y_0) = 0$. Consider $r_{R*G}(Y_0) \neq 0$, so $0 \neq \beta \in r_{R*G}(Y_0)$ which gives $\alpha_{i_j}\beta = 0$ for all $\alpha_{i_j} \in Y_0$. Since R is right (G, λ)-McCoy, there exists a nonzero element $r_1 \in R$ with $\alpha_{i_j} r_1 = 0$ for all $\alpha_{i_j} \in Y_0$. Thus $\lambda^{-g_{i_j}}(a_{i_j})r_1 = 0$ for each $1 \le j \le n$, it follows that $r_1 \in r_R(V_1) = 0$. Therefore $r_1 = 0$, which is a contradiction and so $r_{R*G}(Y_0) = 0$. Hence $R * G$ is a right zip ring.

Conversely, suppose $R*G$ is right zip and V a nonempty subset of R such that $r_R(V) = 0$. Then $r_{R*G}(V) = 0$. Since $R * G$ is right zip, there exists a nonempty subset V_1 of V such that $r_{R*G}(V_1) = 0$. Thus $r_R(V_1) = r_{R*G}(V_1) \cap R = 0$. Hence R is right zip. □

In 1974, Armendariz [2] proved that $a_i b_j = 0$ for every i and j whenever polynomials $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$ over a reduced ring R (a ring R with no nonzero nilpotent elements) satisfy $f(x)g(x) = 0$, where x is an indeterminate over R, following which, Rege and Chhawchharia [29] called such a ring (not necessarily reduced) an Armendariz ring.

Recall for a ring R and a ring automorphism $\sigma : R \to R$, the skew polynomial ring $R[x; \sigma]$ (skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$) consists of polynomials in the form $f(x) = \sum_{i=0}^{n} a_i x^i$ ($f(x) = \sum_{j=q}^{m} b_j x^j$), where the addition is defined as usual and multiplication defined by the rule $xa = \sigma(a)x$ ($x^{-1}a = \sigma^{-1}(a)x$) for any $a \in R$. In [13], Hong et al. extended Armendariz property to skew polynomial ring $R[x; \sigma]$ and defined that a ring R with an endomorphism σ is σ-skew Armendariz if whenever polynomials $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x; \sigma]$ satisfy $f(x)g(x) = 0$ which implies $a_i \sigma^i(b_j) = 0$ for every i and j. Further, Liu [21] introduced the concept of Armendariz ring relative to monoid as is a generalization of Armendariz ring and called a ring R G-Armendariz ring (Armendariz ring relative to monoid), if whenever elements $\alpha = a_1 g_1 + a_2 g_2 + \dots + a_n g_n, \beta = b_1 h_1 + b_2 h_2 + \dots + b_m h_m \in R[G]$ satisfy $\alpha\beta = 0$ implies $a_i b_j = 0$ for each i and j, where $a_i, b_j \in R$, $g_i, h_j \in G$ and G is a monoid. Now we define Armendariz ring to skew monoid ring $R * G$.

Definition 2.5 *Let* R *be a ring,* G *a u.p.-monoid and* $\lambda : G \to \mathrm{Aut}(R)$ *a* *monoid homomorphism. A ring* R *is called* (G,λ)*-Armendariz if whenever* $\alpha = \sum_{i=1}^{n} a_i g_i$*,* $\beta = \sum_{j=1}^{m} b_j h_j \in R * G$*, with* $a_i, b_j \in R$*,* $g_i, h_j \in G$ *satisfy* $\alpha\beta = 0$*, then* $a_i \lambda^{g_i}(b_j) = 0$ *for all* i, j*.*

Notice that all the above mentioned classes of Armendariz rings are special cases of (G,λ)-Armendariz, whereas (G,λ)-Armendariz ring is a special case of (G,ω)-Armendariz ring which was investigated by Marks et al. [24] (for details see section 3). It is also clear from the definition 2.1 and definition 2.4 that every (G,λ)-Armendariz ring is a right (G,λ)-McCoy ring. In the following example, we show that converse need not be true.

Given a ring R and a bimodule ${}_R\mathcal{M}_R$, the trivial extension of R by $\mathcal{M}$ is the ring $T(R,\mathcal{M}) = R \bigoplus \mathcal{M}$ with usual addition and the multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$, where $r_1 r_2 \in R$ and $m_1 m_2 \in M$.

Example 2.6 *There exists a right* (G,λ)*-McCoy ring which is not* (G,λ)*-Armendariz.*

Proof. Let $\mathbb{Z}_8$ be a ring of integers of modulo 8 then its trivial extension is $T(\mathbb{Z}_8, \mathbb{Z}_8)$. Suppose $\lambda : G \to \mathrm{Aut}(T)$ defined by $\lambda^g(a,b) = (b,a)$, for any $g \in G$ and $(a,b) \in T(\mathbb{Z}_8,\mathbb{Z}_8)$, where G be a u.p.-monoid. It is easy to check that λ is a monoid homomorphism. Assume $e, g \in G$ with $e \neq g$ and $\alpha = (4,0)e + (4,1)g$, $\beta = (0,4)e + (1,4)g \in R * G$. Then $\alpha\beta = 0$, while $(4,0)\lambda^g(1,4) \neq 0$. Thus T is not (G,λ)-Armendariz. Consider a nonzero element $t = (4,0) \in T$, so $\alpha t = ((4,0)e + (4,1)g)(4,0) = 0$. Therefore R is a right (G,λ)-McCoy ring. □

Now, we get the following corollary.

Corollary 2.7 *Let* G *be a u.p.-monoid and* $\lambda : G \to \mathrm{Aut}(R)$ *a monoid homomorphism. If* R *is* (G,λ)*-Armendariz then* R *is right zip if and only if* $R * G$ *is right zip.*

Proof. Since R is a (G,λ)-Armendariz ring so R is a right (G,λ)-McCoy ring. Thus by Theorem 2.4, R is right zip if and only if $R * G$ is right zip. □

We also deduce some important results as corollaries of above theorem.

Recall that a ring R is called reversible if $ab = 0$ implies $ba = 0$ for all $a, b \in R$. Let $(G,\leq)$ be an ordered monoid. If for any $g, g', h \in G, g < g'$ implies that $gh < g'h$ and $hg < hg'$, then $(G,\leq)$ is called strictly ordered monoid.

Corollary 2.8 (Hashemi [10, Theorem 1.25]) *Let* R *be a reversible ring and* G *a strictly totally ordered monoid. Then* R *is right zip if and only if* $R[G]$ *is right zip.*

Proof. Since R is reversible ring and G a strictly totally ordered monoid, by [10, Corollary 1.5], R is a G-McCoy ring. Suppose $\lambda = 1 : G \to \mathrm{Aut}(R)$ a monoid homomorphism then R is right (G, λ)-McCoy ring if and only if R is a right G-McCoy ring. Thus by Theorem 2.4, R is right zip if and only if $R[G]$ is right zip. □

Definition 2.9 *A ring* R *is called right duo if all right ideals are two sided ideals. Left duo rings are defined similarly, and a ring is called duo if it is both right and left duo.*

Corollary 2.10 *Let* R *be a right duo ring and* G *a strictly totally ordered monoid. Then* R *is right zip if and only if* $R[G]$ *is right zip.*

Proof. Since R be a right duo ring and G a strictly totally ordered monoid, by [10, Theorem 1.8], R is a right G-McCoy ring. Suppose $\lambda = 1 : G \to \mathrm{Aut}(R)$ a monoid homomorphism then R is a right (G, λ)-McCoy ring if and only if R is a right G-McCoy ring. Thus by Theorem 2.4, R is right zip if and only if $R[G]$ is right zip. □

The following definition is taken from [7].

Definition 2.11 (1) *A ring* R *satisfies* $SA1'$ *if for* $f(x) = \sum_{i=0}^{n} a_i x^i$ *and* $g(x) = \sum_{j=0}^{m} b_j x^j$ *in* $R[x;\sigma]$, $f(x)g(x) = 0$ *implies that* $a_i\sigma^i(b_j) = 0$ *for all* i *and* j, *where* σ *be an endomorphism of* R.

(2) *A ring* R *satisfies* $SA3'$ *if for* $f(x) = \sum_{i=p}^{n} a_i x^i$ *and* $g(x) = \sum_{j=q}^{m} b_j x^j$ *in* $R[x, x^{-1};\sigma]$, $f(x)g(x) = 0$ *implies that* $a_i\sigma^i(b_j) = 0$ *for all* i *and* j, *where* σ *be an automorphism of* R.

Corollary 2.12 (Cortes [7, Theorem 2.8(i)]) *Let* σ *be an automorphism of* R *and satisfies* $SA1'$. *The following conditions are equivalent:*

(1) R *is right zip;*

(2) $R[x;\sigma]$ *is right zip;*

(3) $R[x, x^{-1}; \sigma]$ *is right zip.*

Proof. (1) $\Leftrightarrow$ (2) Given σ be an automorphism of R and satisfies $SA1'$(R is a σ-skew Armendariz ring). Suppose $G = (\mathbb{N} \cup \{0\}, +)$ and $\lambda(1) = \sigma$. Then R is a σ-skew Armendariz ring if and only if R is a (G, λ)-Armendariz ring. Thus by Corollary 2.7, R is right zip if and only if $R[x, \sigma]$ is right zip.
(1) $\Leftrightarrow$ (3) Given σ be an automorphism of R and satisfies $SA3'$ then by [7, Lemma 2.3] R satisfies $SA1'$(R is a σ-skew Armendariz ring). Suppose $G = (\mathbb{Z} \cup \{0\}, +)$ and $\lambda(1) = \sigma$. Then R is a σ-skew Armendariz ring if and only if R is a (G, λ)-Armendariz ring. Thus by Corollary 2.7, R is right zip if and only if $R[x, x^{-1}; \sigma]$ is right zip. $\square$

Corollary 2.13 (Hong et al. [14, Theorem 11]) *Let* R *be an Armendariz ring. Then* R *is a right zip ring if and only if* $R[x]$ *is a right zip ring.*

Proof. Since Armendariz ring is a special case of (G, λ)-Armendariz when $G = (\mathbb{N} \cup \{0\}, +)$ and $\lambda = 1 : G \to \mathrm{Aut}(R)$. Thus by Corollary 2.7, R is a right zip ring if and only if $R[x]$ is a right zip ring. $\square$

Corollary 2.14 (Hong et al. [14, Propsition 2]) *Let* R *be a reduced ring and* G *a u.p.-monoid. Then* R *is right zip if and only if* $R[G]$ *is right zip.*

Proof. Since R be a reduced ring and G a u.p.-monoid, by [21, Proposition 1.1] R is G-Armendariz. Suppose $\lambda = 1 : G \to \mathrm{Aut}(R)$ a monoid homomorphism then R is a (G, λ)-Armendariz ring if and only if R is a G-Armendariz ring. Thus by Corollary 2.7, R is right zip if and only if $R[G]$ is right zip. $\square$

Corollary 2.15 (Hong et al. [14, Theorem 16]) *Suppose that* R *is a commutative ring and* G *a u.p.-monoid that contains an infinite cyclic submonoid. Then* R *is a zip ring if and only if* $R[G]$ *is a zip ring.*

3 Skew generalized power series rings

In this section, we study the concept of skew generalized power series rings, (G, ω)-Armendariz rings and G-compatible rings which were introduced by Mazurek et al. [25]. Moreover, we investigate a relationship between right zip property of a ring R and skew generalized power series ring $R[[G, \omega]]$ over R. This relationship generalizes some of the results of [7, 14].

Recall from [24, 25] that for a construction of skew generalized power series ring, we need some definitions. Let $(G, \leq)$ be a partially ordered set. Then $(G, \leq)$ is called artinian if every strictly decreasing sequence of elements of G is finite, and $(G, \leq)$ is called narrow if every subset of pairwise order-incomparable elements of G is finite. Thus, $(G, \leq)$ is artinian and narrow if and only if every nonempty subset of G has at least one but only a finite number of minimal elements.

An ordered monoid is a pair $(G, \leq)$ consisting of a monoid G and an order $\leq$ on G such that for all $a, b, c \in G$, $a \leq b$ implies $ca \leq cb$ and $ac \leq bc$. An ordered monoid $(G, \leq)$ is said to be strictly ordered if for all $a, b, c \in G$, $a < b$ implies $ca < cb$ and $ac < bc$.

Let R be a ring, $(G, \leq)$ a strictly ordered monoid and $\omega : G \to \mathrm{End}(R)$ a monoid homomorphism. For $s \in G$, let ω_s denote the image of s under ω, that is, $\omega_s = \omega(s)$. Let $\mathcal{A}$ be the set of all functions $\alpha : G \to R$ such that the $\mathrm{supp}(\alpha) = \{s \in G : \alpha(s) \neq 0\}$ is artinian and narrow. Then for any $s \in G$ and $\alpha, \beta \in \mathcal{A}$ the set

$$X_s(\alpha, \beta) = \{(x, y) \in \mathrm{supp}(\alpha) \times \mathrm{supp}(\beta) : s = xy\}$$

is finite. Thus one can define the product $\alpha\beta : G \to R$ of $\alpha, \beta \in \mathcal{A}$ as follows:

$$(\alpha\beta)(s) = \sum_{(x,y)\in X_s(\alpha,\beta)} \alpha(x) \cdot \omega_x(\beta(y)) .$$

With pointwise addition and multiplication as defined above, $\mathcal{A}$ becomes a ring, called the ring of skew generalized power series with coefficients in R and exponents in G (see [25]), denoted by $R[[G, \omega, \leq]]$ (or by $[[G, \omega]]$). The skew generalized power series ring $R[[G, \omega]]$ is a compact generalization of (skew) polynomial rings, (skew) Laurent polynomial rings, (skew) power series rings, (skew) group rings, (skew) monoid rings, Mal'cev Neumann Laurent series rings and generalized power series rings.

The symbol 1 denote the identity elements of the multiplicative monoid G, the ring R, and the ring $R[[S, \omega]]$, as well as the trivial monoid homomorphism $\omega = 1 : G \to \mathrm{End}(R)$ that sends every element of G to the identity endomorphism.

To each $r \in R$ and $s \in G$, we associate elements $c_r, e_s \in R[[G, \omega]]$ defined by

$$c_r(x) = \begin{cases} r & \text{if } x = 1 \\ 0 & \text{if } x \in G \setminus \{1\} \end{cases}, \quad e_s(x) = \begin{cases} 1 & \text{if } x = s \\ 0 & \text{if } x \in G \setminus \{s\}. \end{cases}$$

It is clear that $r \mapsto c_r$ is a ring embedding of R into $R[[G, \omega]]$ and $s \mapsto e_s$ is a monoid embedding of G into a multiplicative monoid of ring $R[[S, \omega]]$, and $e_s c_r = c_{\omega_s(r)} e_s$. Moreover, for each nonempty subset X of R we put $X[[G, \omega]] = \{\alpha \in R[[G, \omega]] : \alpha(s) \in X \cup \{0\}$ for every $s \in G\}$ denotes a subset of $R[[G, \omega]]$, and for each nonempty subset Y of $R[[G, \omega]]$ we put $C_Y = \{\beta(t) : \beta \in Y, t \in G\}$ denotes a subset of R.

In [18], Kim et al. studied a stronger condition than Armendariz and defined a ring R is called powerserieswise Armendariz if whenever power series $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$ in $R[[x]]$ satisfy $f(x)g(x) = 0$ then $a_i b_j = 0$ for all i and j. Further, Liu [20] generalized the definition of the powerserieswise Armendariz to (untwisted) generalized power series ring $[[R^{G \leq}]]$ (particular case of Skew generalized power series ring $R[[G, \omega]]$) and defined as follows: if R is a ring and $(G, \leq)$ is a commutative strictly ordered monoid, then R is called G-Armendariz if whenever generalized power series $\alpha, \beta \in R[[G, 1]]$ satisfy $\alpha\beta = 0$ then $\alpha(s)(\beta(t)) = 0$ for all $s, t \in S$. With the help of a construction of skew generalized power series ring $R[[G, \omega]]$, Marks et al. [24] introduced the concept of Armendariz property to skew generalized power series ring $R[[G, \omega]]$ and gave a unified approach to all classes of Armendariz property.

Definition 3.1 *Let* R *be a ring,* $(G, \leq)$ *a strictly ordered monoid and* $\omega : G \to \mathrm{End}(R)$ *a monoid homomorphism. A ring* R *is called* (G, ω)*-Armendariz if whenever* $\alpha\beta = 0$ *for* $\alpha, \beta \in R[[G, \omega]]$*, then* $\alpha(s).\omega_s(\beta(t)) = 0$ *for all* $s, t \in G$*. If* $G = \{1\}$ *then every ring is* (G, ω)*-Armendariz*

We recall the definition of compatible endomorphism from [24, Definition 2.3].

Definition 3.2 *An endomorphism* σ *of a ring* R *is compatible if for all* $a, b \in R$*,* $ab = 0 \Leftrightarrow a\sigma(b) = 0$*.*

Definition 3.3 *Let* R *be a ring,* $(G, \leq)$ *a strictly ordered monoid and* $\omega : G \to \mathrm{End}(R)$ *a monoid homomorphism. Then* R *is* G*-compatible if* ω_s *is compatible for every* $s \in G$*.*

To prove the main result of this section, we need Lemma 3.4 and Lemma 3.5 which were proved by Marks et al. [24]. Here we quote only the statements.

Lemma 3.4 *Let* R *be a ring,* $(G, \leq)$ *a strictly ordered monoid and* $\omega : G \to \mathrm{End}(R)$ *a monoid homomorphism. The following conditions are equivalent:*

(1) R *is* G*-compatible;*

(2) *for any* $a \in R$ *and any nonempty subset* $Y \subseteq R[[G, \omega]]$,

$$a \in ann_r^R(C_Y) \Leftrightarrow c_a \in ann_r^{R[[G,\omega]]}(Y).$$

Proof. See [24, Lemma 3.1]. □

Lemma 3.5 *Let* R *be a ring,* $(G, \leq)$ *a strictly ordered monoid and* $\omega : G \to \mathrm{End}(R)$ *a monoid homomorphism. If* R *is* G*-compatible, then for any nonempty subset* $X \subseteq R$, $ann_r^R(X)[[G, \omega]] = ann_r^{R[[G,\omega]]}(X[[G, \omega]])$.

Proof. See [24, Lemma 3.2]. □

Now, we are able to prove the main theorem of this section. The following theorem generalizes Corollary 2.7 (Section 2), Hong et al. [14, Theorem 11, Corollary 13 and Proposition 14], Cortes [7, Theorem 2.8] and propose a unified solution to the questions raised by Faith [9].

Theorem 3.6 *Let* R *be a ring,* $(G, \leq)$ *a strictly ordered monoid and* $\omega : G \to \mathrm{End}(R)$ *a monoid homomorphism. If* R *is* (G, ω)*-Armendariz and* G*-compatible then* $R[[G, \omega]]$ *is right zip if and only if* R *is right zip.*

Proof. Suppose that $R[[G, \omega]]$ is a right zip ring. We show that R is a right zip ring. For this consider $Y \subseteq R$ with $r_R(Y) = 0$. Since $Y \subseteq R$, so we put $Y[[G, \omega]] = \{\alpha : \alpha(s) \in R \text{ and } s \in G\} \subseteq R[[G, \omega]]$. Let any $\beta \in r_{R[[G,\omega]]}$ $(Y[[G, \omega]])$. Then $\alpha\beta = 0$ which implies $\alpha(s)\beta(t) = 0$ for all $s, t \in G$ since R is G-compatible and (G, ω)-Armendariz. Thus $\beta(t) \in r_R(\alpha(s)) = 0$ for all $\alpha(s) \in R$ which implies $\beta = 0$ for all $\beta \in R[[G, \omega]]$. It follows that $r_{R[[G,\omega]]}(Y[[G, \omega]])$ $= 0$. Since $R[[G, \omega]]$ is a right zip ring, there exists a subset $V \subseteq Y[[G, \omega]]$ such that $r_{R[[G,\omega]]}(V) = 0$. Then we put $C_V = \{\gamma(u) : u \in S \text{ and } \gamma \in V\}$ is a subset of Y. By Lemma 3.4, for any $a \in r_R(C_V) \Leftrightarrow c_a \in r_{R[[G,\omega]]}(V)$ since R is G-compatible. Thus we have $r_R(C_V) = 0$. Hence R is a right zip ring.

Conversely, suppose R is a right zip ring and a subset $U \subseteq R[[G, \omega]]$ with $r_{R[[G,\omega]]}(U) = 0$. We put $C_U = \{\beta(t) : \beta \in U \text{ and } t \in G\}$ which is nonempty subset of R. By Lemma 3.4, for any $p \in r_R(C_U) \Leftrightarrow c_p \in r_{R[[G,\omega]]}(U)$ since R is G-compatible. Thus $r_R(C_U) = 0$. Since R is a right zip ring, there exists a nonempty subset $X \subseteq C_U$ such that $r_R(X) = 0$. So we put $X[[G, \omega]] = \{\alpha \in R[[G, \omega]] : \alpha(s) \in X \cup \{0\} \text{ and } s \in G\}$. Thus by Lemma 3.5, $r_{R[[G,\omega]]}(X[[G, \omega]]) = r_R(X)[[G, \omega]] = 0$ since R is G-compatible. Therefore $R[[G, \omega]]$ is right zip. □

Now, we get following result as corollary which was proved by Cortes [7]. To get Corollary 3.8, we need the following definition.

Definition 3.7 ([7, Definition 2.2(ii)]) *A ring* R *satisfies* $SA2'$ *if for* $f(x) = \sum_{i=0}^{\infty} a_i x^i$ *and* $g(x) = \sum_{j=0}^{\infty} b_j x^j$ *in* $R[[x, \sigma]]$*,* $f(x)g(x) = 0$ *implies that* $a_i \sigma^i(b_j) = 0$ *for all* i *and* j*, where* σ *be an endomorphism of* R*.*

Corollary 3.8 (Cortes [7, Theorem 2.8(ii)]) *Let* σ *be an automorphism of* R *and* R *satisfies* $SA2'$*. The following conditions are equivalent:*

(1) R *is right zip;*

(2) $R[[x, \sigma]]$ *is right zip.*

Proof. Suppose $G = (\mathbb{N} \cup \{0\}, +)$ and $\omega(1) = \sigma$. Then R satisfies $SA2'$ if and only if R is (G, ω)-Armendariz. Thus by Theorem 3.6, R is right zip if and only if $R[x, \sigma]$ is right zip. □

Acknowledgment

The authors are deeply indebted to the referee for various valuable comments and suggestions which led to improvements of this article.

References

[1] D. D. Anderson, Camillo, Armendariz rings and Gaussian rings, *Comm. Algebra*, **27**, 7 (1998), 2265–2272.

[2] E. P. Armendariz, A note on extensions of Baer and P.P.-rings, *J. Aust. Math. Soc.*, **18** (1974), 470–473.

[3] J. A. Beachy, W. D. Blair, Rings whose faithful left ideals are cofaithful, *Pacific J. Math.*, **58** (1975), 1–13.

[4] F. Cedo, Zip rings and Mal'cev domains, *Comm. Algebra*, **19** (1991), 1983–1991.

[5] A. W. Chatters, W. Xue, On right duo P.P.-rings, *Glasgow Math. J.*, **32**, 2 (1990), 221–225.

[6] P. M. Cohn, Reversible rings, *Bull. London Math. Soc.*, **31** (1999), 641–648.

[7] W. Cortes, Skew polynomial extensions over zip rings, *Int. J. Math. Math. Sci.*, **10** (2008), 1–8.

[8] C. Faith, Rings with zero intersection property on annihilators: zip rings, *Publ. Math.*, **33** (1989), 329–332.

[9] C. Faith, Annihilator ideals, associated primes and Kasch-McCoy commutative rings, *Comm. Algebra.*, **19**, 7 (1991), 1867–1892.

[10] E. Hashemi, McCoy rings relative to a monoid, *Comm. Algebra*, **38** (2010), 1075–1083.

[11] E. Hashemi, A. Moussavi, Polynomial extensions of quasi-Baer rings, *Acta. Math. Hung.*, **107**, 3 (2005), 207–224.

[12] Y. Hirano, On annihilator ideals of a polynomial ring over a noncommutative ring, *J. Pure Appl. Algebra*, **168**, 1, (2000), 45–52.

[13] C. Y. Hong, N. K. Kim, T. K. Kwak, On skew Armendariz rings, *Comm. Algebra*, **31**, 1 (2003), 103–122.

[14] C. Y. Hong, N. K. Kim, T. K. Kwak, Y. Lee, Extension of zip rings, *J. Pure Appl. Algebra*, **195**, 3 (2005), 231–242.

[15] C. Y. Hong, N. K. Kim, Y. Lee, Extensions of McCoy theorem, *Glasgow Math. J.*, **52** (2010), 155–159.

[16] C. Huh, Y. Kim, A. Smoktunowich, Armendariz and semicommutative rings, *Comm. Algebra*, **30**, 2 (2002), 751–761.

[17] N. K. Kim, Y. Lee, Armendariz and reduced rings, *J. Algebra*, **223**, 2 (2000), 477–488.

[18] N. K. Kim, K. H. Lee, Y. Lee, Power series rings satisfying a zero divisor property, *Comm. Algebra*, **34**, 6 (2006), 2205–2218.

[19] T. K. Lee, Y. Zhou, A unified approach to the Armendariz property of polynomial rings and power series rings, *Colloq. Math.*, **113**, 1 (2008), 151–168.

[20] Z. Liu, Special properties of rings of generalized power series, *Comm. Algebra*, **32**, 8 (2004), 3215–3226.

[21] Z. Liu, Armendariz rings relative to monoid, *Comm. Algebra*, **33**, 3 (2005), 649–661.

[22] Z. Liu, Y. Xiaoyan, On annihilator ideals of skew monoid rings, *Glasgow Math. J.*, **52** (2010), 161–168.

[23] G. Marks, R. Mazurek, M. Ziembowski, A new class of unique product monoids with applications to rings theory, *Semigroup Forum*, **78**, 2 (2009), 210–225.

[24] G. Marks, R. Mazurek, M. Ziembowski, A unified approach to various generalization of Armendariz rings, *Bull. Aust. Math. Soc.*, **81** (2010), 361–397.

[25] R. Mazurek, M. Ziembowski, On Von-Neumann regular rings of skew generalized power series, *Comm. Algebra*, **36**, 5 (2008), 1855–1868.

[26] N. H. McCoy, Annihilators in polynomial rings, *Amer. Math. Monthly*, **14** (1957), 28–29.

[27] J. Okninski, *Semigroup algebras*, Monographs and textbooks in pure and applied mathematics 136, Marcel Dekker, New York, 1991.

[28] P. P. Nielsen, Semi-commutativity and the McCoy condition, *J. Algebra*, **298** (2006), 134–141.

[29] M. B. Rege, S. Chhawchharia, Armendariz rings, *Proc. Japan Acad. Sci. A Math. Sci.*, **73** (1997), 14–17.

[30] P. Ribenboim, Special properties of generalized power series, *Arc. Math.*, **54** (1990), 365–371.

[31] J. M. Zelmanowitz, The finite intersection property on annhilator right ideals, *Proc. Amer. Math. Soc.*, **57**, 2, (1976) 213–216.

Received: May 3, 2011

Acta Univ. Sapientiae, Mathematica, **4**, 2 (2012) 182–198

On contact CR-submanifolds of Kenmotsu manifolds

Mehmet Atçeken
Gaziosmanpaşa University
Faculty of Arts and Sciences
Department of Math.
60250 Tokat, Turkey
email: mehmet.atceken@gop.edu.tr

Süleyman Dírík
Gaziosmanpaşa University
Faculty of Arts and Sciences
Department of Math.
60250 Tokat, Turkey
email: suleyman.dirik@amasya.edu.tr

Abstract. In this paper, we study the differential geometry of contact CR-submanifolds of a Kenmotsu manifold. Necessary and sufficient conditions are given for a submanifold to be a contact CR-submanifold in Kenmotsu manifolds. Finally, the induced structures on submanifolds are investigated, these structures are categorized and we discuss these results.

1 Introduction

In [4], K. Kenmotsu defined and studied a new class of almost contact manifolds called Kenmotsu manifolds. The study of the differential geometry of a contact CR-submanifolds, as a generalization of invariant and anti-invariant submanifolds, of an almost contact metric manifold was initiated by A. Bejancu [3] and was followed by several geometers. Several authors studied contact CR-submanifolds of different classes of almost contact metric manifolds given in the references of this paper.

The contact CR-submanifolds are rich and interesting subject. Therefore we continue to work in this subject matter.

2010 Mathematics Subject Classification: 53C42, 53C15
Key words and phrases: Kenmotsu manifold, CR-submanifold and contact CR-submanifold

The purpose of this paper is to study the differential geometric theory of submanifolds immersed in Kenmotsu manifold. We obtain the new integrability conditions of the distributions of contact CR-submanifolds and prove some characterizations for the induced structure to be parallel.

2 Preliminaries

In this section, we give some notations used throughout this paper. We recall some necessary facts and formulas from the Kenmotsu manifolds. A $(2m+1)$-dimensional Riemannian manifold $(\bar{M}, g)$ is said to be a Kenmotsu manifold if there exist on $\bar{M}$ a $(1,1)$ tensor field φ, a vector field ξ(called the structure vector field) and 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)$$

and

$$(\bar{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad \bar{\nabla}_X \xi = X - \eta(X)\xi, \tag{2}$$

for any $X, Y \in \Gamma(T\bar{M})$, where $\bar{\nabla}$ is a Levi-Civita connection on $\bar{M}$ and $\Gamma(T\bar{M})$ denotes the set of all differentiable vectors on $\bar{M}$ [5].

A plane section π in $T_x\bar{M}$ is called a φ-section if it is spanned by X and φX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a φ-section is called a φ-holomorphic sectional curvature. A Kenmotsu manifold with constant φ-holomorphic sectional curvature c is said to be a Kenmotsu space form and it is denoted by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of a $\bar{M}(c)$ is also given by

$$\begin{aligned} \bar{R}(X,Y)Z &= \left(\frac{c-3}{4}\right)\{g(Y,Z)X - g(X,Z)Y\} + \left(\frac{c+1}{4}\right)\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi + g(X,\varphi Z)\varphi Y \\ &- g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\}, \end{aligned} \tag{3}$$

for any $X, Y, Z \in \Gamma(T\bar{M})$ [1].

Now, let M be an isometrically immersed submanifold in $\bar{M}$. In the rest of this paper, we assume the submanifold M of $\bar{M}$ is tangent to the structure vector field ξ. Then the formulas of Gauss and Weingarten for M in $\bar{M}$ are given, respectively, by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{4}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{5}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where $\bar{\nabla}$ and ∇ denote the Riemannian connections on $\bar{M}$ and M, respectively, h is the second fundamental form, $\nabla^{\perp}$ is the normal connection on the normal bundle $T^{\perp}M$ and A_V is the shape operator of M in $\bar{M}$. It is well known that the second fundamental form and the shape operator are related by formulae

$$g(A_V X, Y) = g((h(X, Y), V), \tag{6}$$

where, g denotes the Riemannian metric on $\bar{M}$ as well as M. For any submanifold M of a Riemannian manifold $\bar{M}$, the equation of Gauss is given by

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\bar{\nabla}_X h)(Y, Z) \\ &- (\bar{\nabla}_Y h)(X, Z), \end{aligned} \tag{7}$$

for any $X, Y, Z \in \Gamma(TM)$, where $\bar{R}$ and R denote the Riemannian curvature tensors of $\bar{M}$ and M, respectively. The covariant derivative $\bar{\nabla}h$ of h is defined by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y), \tag{8}$$

and the covariant derivative $\bar{\nabla}A$ is defined by

$$(\bar{\nabla}_X A)_V Y = \nabla_X (A_V Y) - A_{\nabla^{\perp}_X V} Y - A_V \nabla_X Y, \tag{9}$$

for any $X, Y, Z \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

The normal component of (7) is said to be the Codazzi equation and it is given by

$$(\bar{R}(X, Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z), \tag{10}$$

where $(\bar{R}(X, Y)Z)^{\perp}$ denotes the normal part of $\bar{R}(X, Y)Z$. If $(\bar{R}(X, Y)Z)^{\perp} = 0$, then M is said to be curvature-invariant submanifold of $\bar{M}$.

The Ricci equation is given by

$$g(\bar{R}(X, Y)V, U) = g(R^{\perp}(X, Y)V, U) + g([A_U, A_V]X, Y), \tag{11}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$, where $R^{\perp}$ denotes the Riemannian curvature tensor of the normal vector bundle $T^{\perp}M$ and if $R^{\perp}$=0, then the normal connection of M is called flat [6].

Taking into account (3) and (11), we have

$$\begin{aligned} g(R^{\perp}(X,Y)V,U) &= \left(\frac{c+1}{4}\right)\{g(X,\varphi V)g(U,\varphi Y) - g(Y,\varphi V)g(\varphi X,U) \\ &+ 2g(X,\varphi Y)g(\varphi V,U) + g([A_V,A_U]X,Y)\}, \end{aligned} \tag{12}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$.

By using (3) and (7), the Riemannian curvature tensor R of an immersed submanifold M of a Kenmotsu space form $\bar{M}(c)$ is given by

$$\begin{aligned} R(X,Y)Z &= \left(\frac{c-3}{4}\right)\{g(Y,Z)X - g(X,Z)Y\} + \left(\frac{c+1}{4}\right)\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi + g(X,\varphi Z)PY \\ &- g(Y,\varphi Z)PX + 2g(X,\varphi Y)PZ\} + A_{h(Y,Z)}X - A_{h(X,Z)}Y. \end{aligned} \tag{13}$$

From (3) and (10), for a submanifold, the Codazzi equation is given by

$$\begin{aligned} (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z) &= \left(\frac{c+1}{4}\right)\{g(X,\varphi Z)FY - g(Y,\varphi Z)FX \\ &+ 2g(X,\varphi Y)FZ\}. \end{aligned} \tag{14}$$

3 Contact CR-submanifolds of a Kenmotsu manifold

Now, let M be an isometrically immersed submanifold of a Kenmotsu manifold $\bar{M}$. For any vector X tangent to M, we set

$$\varphi X = PX + FX, \tag{15}$$

where PX and FX denote the tangent and normal parts of φX, respectively. Then P is an endomorphism of the TM and F is a normal-bundle valued 1-form of TM.

The covariant derivatives of P and F are, respectively, defined by

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y \tag{16}$$

and

$$(\nabla_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y. \tag{17}$$

In the same way, for any vector field V normal to M, φV can be written in the following way;

$$\varphi V = BV + CV, \tag{18}$$

where BV and CV denote the tangent and normal parts of φV, respectively. Also, B is an endomorphism of the normal bundle $T^{\perp}M$ of TM and C is an endomorphism of the subbundle of the normal bundle $T^{\perp}M$.

The covariant derivatives of B and C are also, respectively, defined by

$$(\nabla_X B)V = \nabla_X BV - B\nabla_X^{\perp}V \tag{19}$$

and

$$(\nabla_X C)V = \nabla_X^{\perp}CV - C\nabla_X^{\perp}V. \tag{20}$$

Furthermore, for any $X, Y \in \Gamma(TM)$, we have $g(PX, Y) = -g(X, PY)$ and $U, V \in \Gamma(T^{\perp}M)$, we get $g(U, CV) = -g(CU, V)$. These show that P and C are also skew-symmetric tensor fields. Moreover, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$ we have

$$g(FX, V) = -g(X, BV), \tag{21}$$

which gives the relation between F and B.

Definition 1 *Let M be an isometrically immersed submanifold of a Kenmotsu manifold $\bar{M}$. Then M is called a contact CR-submanifold of $\bar{M}$ if there is a differentiable distribution $D : p \longrightarrow D_p \subseteq T_p(M)$ on M satisfying the following conditions:*

i) $\xi \in D$,

ii) D is invariant with respect to φ, i.e., $\varphi D_x \subset T_p(M)$ for each $p \in M$, and

iii) the orthogonal complementary distribution $D^{\perp} : p \longrightarrow D_p^{\perp} \subseteq T_p(M)$ satisfies $\varphi D_p^{\perp} \subseteq T_p^{\perp}M$ for each $p \in M$.

For a contact CR-submanifold M of a Kenmotsu manifold, for the structure vector field $\xi \in \Gamma(D) \subseteq \Gamma(TM)$, from (1), we have

$$\varphi\xi = P\xi + F\xi = 0,$$

which is equivalent to

$$P\xi = F\xi = 0. \tag{22}$$

Furthermore, applying φ to (15), by using (1), (18), we conclude that

$$P^2 + BF = -I + \eta \otimes \xi \text{ and } FP + CF = 0. \tag{23}$$

Similarly, applying φ to (18), making use of (1), (15), we have

$$C^2 + FB = -I \text{ and } PB + BC = 0. \tag{24}$$

Proposition 1 *Let* M *be a contact CR-submanifold of a Kenmotsu manifold* $\bar{M}$*. Then the invariant distribution* D *has an almost contact metric structure* (P, ξ, η, g) *and so* $\dim(D_p) = \text{odd}$ *for each* $p \in M$*.*

Now, we denote the orthogonal distribution of $\varphi(D^{\perp})$ in $T^{\perp}M$ by ν. Then we have the direct decomposition

$$T^{\perp}M = \varphi(D^{\perp}) \oplus \nu \text{ and } \varphi(D^{\perp}) \perp \nu. \tag{25}$$

Here we note that ν is an invariant subbundle with respect to φ and so $\dim(\nu) = \text{even}$.

Theorem 1 *Let* M *be an isometrically immersed submanifold of a Kenmotsu manifold* $\bar{M}$*. Then* M *is a contact CR-submanifold if and only if* $FP = 0$*.*

Proof. We assume that M is a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. We denote the orthogonal projections on D and $D^{\perp}$ by R and S, respectively. Then we have

$$R + S = I, \quad R^2 = R, \quad S^2 = S \text{ and } RS = SR = 0. \tag{26}$$

For any $X \in \Gamma(TM)$, we can write

$$X = RX + SX \text{ and } \varphi X = \varphi RX + \varphi SX = PRX + FRX + PSX + FSX. \tag{27}$$

Since D is invariant distribution, it is clear that

$$FR = 0 \text{ and } SPR = 0. \tag{28}$$

On the other hand, we can easily verify that

$$RP = P = PR.$$

From the second side of (23), we reach

$$FPR + CFR = 0. \tag{29}$$

Since $FR = 0$, (29) reduces to

$$FP = 0. \tag{30}$$

By virtue of (23) and (30), we arrive at

$$CF = 0. \tag{31}$$

Conversely, let M be a submanifold of a Kenmotsu manifold $\bar{M}$ such that (30) is satisfied. For any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, by direct calculations, we have

$$\begin{aligned} g(X, \varphi^2 V) &= g(\varphi^2 X, V) \\ g(X, \varphi BV) &= g(\varphi FX, V), \\ g(X, PBV) &= g(CFX, V) = 0. \end{aligned}$$

Thus we get

$$PB = 0. \tag{32}$$

Making use of the equations (23), (24) and (32), we have $P^3 + P = 0$ and $C^3 + C = 0$ which show that P and C are f-structures on TM and $T^{\perp}M$, respectively. Here if we put $R = -P^2 + \eta \otimes \xi$ and $S = I + P^2 - \eta \otimes \xi$, then we can easily see that

$$R + S = I, \quad R^2 = R, \quad S^2 = S \text{ and } RS = SR = 0, \tag{33}$$

that is, R and S are orthogonal projections and they define orthogonal complementary distributions such as D and $D^{\perp}$. Since $R = -P^2 + \eta \otimes \xi$ and $P^3 + P = 0$, we get $PR = P$ and $PS = 0$. Taking account of P being skew-symmetric and S being symmetric, we have

$$\begin{aligned} g(SPX, Y) &= g(PX, SY) \\ &= -g(X, PSY) = 0, \end{aligned}$$

for any $X, Y \in \Gamma(TM)$. Thus we have

$$SP = 0.$$

It implies that

$$SPR = 0.$$

Since $R = -P^2 + \eta \otimes \xi$, $P\xi = F\xi = 0$ and from (30), it is clear that

$$FR = 0. \tag{34}$$

(33) and (34) tell us that D and $D^{\perp}$ are invariant and anti-invariant distributions on M, respectively. Furthermore, from the definitions of R and S, we have

$$R\xi = \xi \text{ and } S\xi = 0,$$

that is, the distribution D contains ξ. On the other hand, setting

$$R = -P^2 \quad \text{and} \quad S = I + P^2,$$

we can easily see that projections R and S define orthogonal distributions such as D and $D^\perp$, respectively. Thus we have

$$PR = P, \quad SP = 0, \quad FR = 0 \quad \text{and} \quad PS = 0,$$

that is, D is an invariant distribution, $D^\perp$ is an anti-invariant distribution and

$$R\xi = 0 \quad \text{and} \quad S\xi = \xi.$$

This tell us that ξ belongs to $D^\perp$. Hence the proof is complete. □

Now, let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. Then for any $X, Y \in \Gamma(TM)$, by using (2), (4), (5), (15) and (18), we have

$$\begin{aligned} (\bar{\nabla}_X \varphi)Y &= \bar{\nabla}_X \varphi Y - \varphi \bar{\nabla}_X Y \\ g(\varphi X, Y)\xi - \eta(Y)\varphi X &= \bar{\nabla}_X PY + \bar{\nabla}_X FY - \varphi \nabla_X Y - \varphi h(X, Y). \end{aligned}$$

From the tangent and normal components of this last equations, respectively, we have

$$(\nabla_X P)Y = A_{FY}X + Bh(X, Y) + g(\varphi X, Y)\xi - \eta(Y)PX \tag{35}$$

and

$$(\nabla_X F)Y = Ch(X, Y) - h(X, PY) - \eta(Y)FX. \tag{36}$$

In the same way, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, we have

$$\begin{aligned} (\bar{\nabla}_X \varphi)V &= \bar{\nabla}_X \varphi V - \varphi \bar{\nabla}_X V \\ g(\varphi X, V)\xi &= (\nabla_X B)V + (\nabla_X C)V + h(X, BV) - A_{CV}X + PA_V X \\ &+ FA_V X. \end{aligned} \tag{37}$$

From the normal and tangent components of (37), respectively, we have

$$(\nabla_X C)V = -h(X, BV) - FA_V X, \tag{38}$$

and

$$(\nabla_X B)V = g(FX, V)\xi + A_{CV}X - PA_V X. \tag{39}$$

On the other hand, since M is tangent to ξ, making use of (2) and (6) we obtain

$$A_V \xi = h(X, \xi) = 0 \tag{40}$$

for all $V \in \Gamma(T^{\perp}M)$ and $X \in \Gamma(TM)$. It is well-known that $Bh = 0$ plays an important role in the geometry of submanifolds. This means that the induced structure P is a Kenmotsu structure on M. Then (35) reduces to

$$(\nabla_X P)Y = g(PX, Y)\xi - \eta(Y)PX, \tag{41}$$

for any $X, Y \in \Gamma(D)$. This means that the induced structure P is a Kenmotsu structure on M. Moreover, for any $Z, W \in \Gamma(D^{\perp})$ and $U \in \Gamma(TM)$, also by using (2) and (6), we have

$$\begin{aligned} g(A_{FZ}W - A_{FW}Z, U) &= g(h(W, U), FZ) - g(h(Z, U), FW) \\ &= g(\bar{\nabla}_U W, \varphi Z) - g(\bar{\nabla}_U Z, \varphi W) \\ &= g(\varphi\bar{\nabla}_U Z, W) - g(\bar{\nabla}_U \varphi Z, W) = -g((\bar{\nabla}_U \varphi)Z, W) \\ &= g(\varphi Z, U)\eta(W) - g(\varphi W, U)\eta(Z) = 0. \end{aligned}$$

It follows that

$$A_{FZ}W = A_{FW}Z, \tag{42}$$

for any $Z, W \in \Gamma(D^{\perp})$.

Hence we have the following theorem.

Theorem 2 *Let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. Then the anti-invariant distribution $D^{\perp}$ is completely integrable and its maximal integral submanifold is an anti-invariant submanifold of $\bar{M}$.*

Proof. For any $Z, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(D)$, by using (2) and (42) we have

$$\begin{aligned} g([Z, W], X) &= g(\bar{\nabla}_Z W, X) - g(\bar{\nabla}_W Z, X) \\ &= g(\bar{\nabla}_W X, Z) - g(\bar{\nabla}_Z X, W) = g(\varphi\bar{\nabla}_W X, \varphi Z) - g(\varphi\bar{\nabla}_Z X, \varphi W) \\ &= g(\bar{\nabla}_W \varphi X - (\bar{\nabla}_W \varphi)X, \varphi Z) - g(\bar{\nabla}_Z \varphi X - (\bar{\nabla}_Z \varphi)X, \varphi W) \\ &= g(h(\varphi X, W), \varphi Z) - g(h(\varphi X, Z), \varphi W) - g(g(\varphi W, X)\xi \\ &- \eta(X)\varphi W, \varphi Z) + g(g(\varphi Z, X)\xi - \eta(X)\varphi Z, \varphi W) \\ &= g(A_{\varphi Z}W - A_{\varphi W}Z, \varphi X) = 0. \end{aligned}$$

Thus $[Z, W] \in \Gamma(D^{\perp})$ for any $Z, W \in \Gamma(D^{\perp})$, that is, $D^{\perp}$ is integrable. Thus the proof is complete. □

Theorem 3 *Let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. Then the invariant distribution D is completely integrable and its maximal* *integral submanifold is an invariant submanifold of* $\bar{M}$ *if and only if the shape operator* A_V *of* M *satisfies*

$$A_V P + P A_V = 0, \tag{43}$$

for any $V \in \Gamma(T^{\perp}M)$.

Proof. In [1], it was proved that D is integrable if and only if the second fundamental form h of M satisfies the condition $h(X, PY) = h(PX, Y)$, for any $X, Y \in \Gamma(D)$. We can easily verify that this condition is equivalent to (43). So we omit the proof. □

Theorem 4 *Let* M *be a contact CR-submanifold of a Kenmotsu manifold* $\bar{M}$. *If the invariant distribution* D *is integrable, then* M *is* D*-minimal submanifold in* $\bar{M}$.

Proof. Let $\{e_1, e_2, \ldots, e_p, \varphi e_1, \varphi e_2, \ldots, \varphi e_p, \xi\}$ be an orthonormal frame of $\Gamma(D)$ and we denote the second fundamental form of M in $\bar{M}$ by h. Then the mean curvature tensor H of M can be written as

$$H = \frac{1}{2p+1}\left\{\sum_{i=1}^{p}\{h(e_i, e_i) + h(\varphi e_i, \varphi e_i)\} + h(\xi, \xi)\right\}. \tag{44}$$

By using (2) we mean that $h(\xi, \xi) = 0$. Since D is integrable, we have

$$\begin{aligned} H &= \frac{1}{2p+1}\left\{\sum_{i=1}^{p}\{h(e_i, e_i) + h(P^2 e_i, e_i)\right\} \\ &= \frac{1}{2p+1}\left\{\sum_{i=1}^{p}\{h(e_i, e_i) + h(-e_i + \eta(e_i)\xi, e_i)\right\} \\ &= \frac{1}{2p+1}\left\{\sum_{i=1}^{p}\{h(e_i, e_i) - h(e_i, e_i)\right\} = 0. \end{aligned}$$

This proves our assertion. □

Theorem 5 *Let* M *be a contact CR-submanifold of a Kenmotsu manifold* $\bar{M}$. *If the second fundamental form of the contact CR-submanifold* M *is parallel, then* M *is a totally geodesic submanifold.*

Proof. If the second fundamental form h of M is parallel, then by using (8), we have

$$\nabla_X^{\perp} h(Y, Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y) = 0,$$

for any $X, Y, Z \in \Gamma(TM)$. Here, choosing $Y = \xi$ and taking into account (2) and (40), we conclude that $h(X, Z) = 0$. This proves our assertion. □

Theorem 6 *Let M be a submanifold of a Kenmotsu manifold $\bar{M}$. Then M is a contact CR-submanifold if and only if the endomorphism C defines an f-structure on ν, that is, $C^3 + C = 0$.*

Proof. If M is a contact CR-submanifold, then from Theorem 1, we know that C is an f-structure on ν.

Conversely, if C is an f-structure on ν, from (24) we can derive $CFB = 0$. So for any $V \in \Gamma(T^{\perp}M)$, by using (21), we have

$$\begin{aligned} g(BCV, BCV) &= g(\varphi CV, BCV) = -g(CV, FBCV) \\ &= g(V, CFBCV) = 0. \end{aligned}$$

This implies that $BC = 0$ which is equivalent to $PB = 0$. Also, from Theorem 3.1 we conclude that M is a contact CR-submanifold. □

Theorem 7 *Let M be a submanifold of a Kenmotsu manifold $\bar{M}$. If the endomorphism P on M is parallel, then M is anti-invariant submanifold in $\bar{M}$.*

Proof. If P is parallel, from (35) and (40), we have

$$\begin{aligned} 0 &= g(\varphi X, Y) + g(A_{FY}X, \xi) + g(Bh(X, Y), \xi) \\ &= g(\varphi X, Y) + g(h(X, \xi), FY) \\ &= g(\varphi X, Y), \end{aligned}$$

for any $X, Y \in \Gamma(TM)$. This implies that M is anti-invariant submanifold. □

Theorem 8 *Let M be a submanifold of a Kenmotsu manifold $\bar{M}$. If the endomorphism F is parallel, then M is invariant submanifold in $\bar{M}$.*

Proof. If F is parallel, then from (36), we have

$$Ch(X, Y) - h(X, PY) - \eta(Y)FX = 0,$$

for any $X, Y \in \Gamma(TM)$. Here, choosing $Y = \xi$ and taking into account that $h(X, \xi) = 0$, we conclude that $FX = 0$. This proves our assertion. □

Theorem 9 *Let* M *be a submanifold of a Kenmotsu manifold* $\bar{M}$. *Then the structure* F *is parallel if and only if the structure* B *is parallel.*

Proof. Making use of (36) and (39), we have

$$\begin{aligned} g((\nabla_X F)Y, V) &= g(Ch(X,Y), V) - g(h(X, PY), V) - \eta(Y)g(FX, V) \\ &= -g(h(X,Y), CV) - g(A_V X, PY) - g(FX, V)\eta(Y) \\ &= -g(A_{CV}Y, X) + g(PA_V X, Y) - g(FX, V)\eta(Y) \\ &= -g((\nabla_X B)V, Y), \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. This proves our assertion. □

From Theorem 8 and Theorem 9, we have the following corollary.

Corollary 1 *Let* M *be a submanifold of a Kenmotsu manifold* $\bar{M}$. *If the structure* B *is parallel, then* M *is invariant submanifold.*

For a contact CR-submanifold M, if the invariant distribution D and anti-invariant distribution $D^{\perp}$ are totally geodesic in M, then M is called contact CR-product. The following theorems characterize contact CR-products in Kenmotsu manifolds.

Theorem 10 *Let* M *be a contact CR-submanifold of a Kenmotsu manifold* $\bar{M}$. *Then* M *is a contact CR-product if and only if the shape operator* A *of* M *satisfies the condition*

$$A_{\varphi W}\varphi X + \eta(X)W = 0, \tag{45}$$

for all $X \in \Gamma(D)$ *and* $W \in (D^{\perp})$.

Proof. Let us assume that M is a contact CR-submanifold of $\bar{M}$. Then by using (2) and (4), we obtain

$$\begin{aligned} g(A_{\varphi W}\varphi X + \eta(X)W, Y) &= g(h(\varphi X, Y), \varphi W) = g(\bar{\nabla}_Y \varphi X, \varphi W) \\ &= g((\bar{\nabla}_Y \varphi)X + \varphi\bar{\nabla}_Y X, \varphi W) \\ &= g(g(\varphi Y, X)\xi - \eta(X)\varphi Y, \varphi W) + g(\nabla_Y X, W) \\ &= g(\nabla_Y X, W) \end{aligned}$$

and

$$\begin{aligned} g(A_{\varphi W}\varphi X + \eta(X)W, Z) &= g(h(\varphi X, Z), \varphi W) + \eta(X)g(Z, W) \\ &= g(\bar{\nabla}_Z \varphi X, \varphi W) + \eta(X)g(Z, W) \\ &= g((\bar{\nabla}_Z \varphi)X + \varphi\bar{\nabla}_Z X, \varphi W) \\ &= g(g(\varphi Z, X)\xi - \eta(X)\varphi Z, \varphi W) + g(\bar{\nabla}_Z X, W) \\ &+ \eta(X)g(Z, W) = -g(\nabla_Z W, X), \end{aligned}$$

for all $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^{\perp})$. So $\nabla_X Y \in \Gamma(D)$ and $\nabla_Z W \in \Gamma(D^{\perp})$ if and only if (45) is satisfied. This proves our assertion. □

Theorem 11 *Let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. Then M is contact CR-product if and only if*

$$Bh(X, U) = 0, \tag{46}$$

for all $U \in \Gamma(TM)$ and $X \in \Gamma(D)$.

Proof. For a contact CR-product M in [1], it was proved that $A_{\varphi W} X = 0$, for all $X \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$. This condition implies (46).

Conversely, we suppose that (46) is satisfied. Then we have

$$\begin{aligned} g(\nabla_X Y, W) &= g(\varphi \bar{\nabla}_X Y, \varphi W) = g(\bar{\nabla}_X \varphi Y, \varphi W) - g((\bar{\nabla}_X \varphi)Y, \varphi W) \\ &= g(h(X, PY), \varphi W) - g(g(\varphi X, Y)\xi - \eta(Y)\varphi X, \varphi W) \\ &= -g(Bh(X, PY), W) \end{aligned}$$

and

$$\begin{aligned} g(\nabla_Z W, \varphi X) &= -g(\bar{\nabla}_Z \varphi X, W) = -g((\bar{\nabla}_Z \varphi)X + \varphi \bar{\nabla}_Z X, W) \\ &= -g(g(\varphi Z, X)\xi - \eta(X)\varphi Z, W) + g(\bar{\nabla}_Z X, \varphi W) \\ &= -g(Bh(X, Z), W), \end{aligned}$$

for all $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^{\perp})$. This proves our assertion □

Theorem 12 *Let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. The structure C is parallel if and only if the shape operator A_V of M satisfies the condition*

$$A_V BU = A_U BV, \tag{47}$$

for all $U, V \in \Gamma(T^{\perp}M)$.

Proof. From (21) and (38), we have

$$\begin{aligned} g((\nabla_X C)V, U) &= -g(h(X, BV), U) - g(FA_V X, U) = -g(A_U BV) + g(A_V X, BU) \\ &= g(A_V BU - A_U BV, X), \end{aligned}$$

for all $X \in \Gamma(TM)$. The proof is complete. □

Theorem 13 *Let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. If C is parallel, then M is totally geodesic submanifold of $\bar{M}$.*

Proof. If C is parallel, from (38), we have

$$\varphi A_V X + h(X, BV) = 0, \tag{48}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. Applying φ to (48) and taking into account (2) and (40), we obtain

$$-A_V X + Bh(X, BV) = 0. \tag{49}$$

On the other hand, also by using (24) and (47), we conclude that

$$g(Bh(X, BV), Z) = -g(h(X, BV), FZ) = -g(A_{FZ}BV, X) = -g(A_V BFZ, X) = 0,$$

for all $Z \in \Gamma(D^{\perp})$. So arrive at $A_V = 0$, that is, M is totally geodesic in $\bar{M}$. □

4 Contact CR-submanifolds in Kenmotsu space forms

Theorem 14 *Let M be a contact CR-submanifold of a Kenmotsu space form $\bar{M}(c)$ such that $c \neq -1$. If M is a curvature-invariant contact CR-submanifold, then M is invariant or anti-invariant submanifold.*

Proof. We suppose that M is a curvature-invariant contact CR-submanifold of a Kenmotsu space form $\bar{M}(c)$ such that $c \neq -1$. Then from (14) we have

$$g(X, PZ)FY - g(Y, PZ)FX + 2g(X, PY)FZ = 0, \tag{50}$$

for any $X, Y, Z \in \Gamma(TM)$. Taking $Z = X$ in equation (50), we have

$$3g(PY, X)FX = 0.$$

This implies that $F = 0$ or $P = 0$, that is, M is invariant or anti-invariant submanifold. Thus the proof is complete. □

Thus we have the following corollary.

Corollary 2 *There isn't any curvature-invariant proper contact CR- submanifold of a Kenmotsu space form $\bar{M}(c)$ such that $c \neq -1$.*

Theorem 15 *Let M be a contact CR-submanifold of a Kenmotsu space form $\bar{M}(c)$ with flat normal connection such that $c \neq -1$. If $PA_V = A_V P$ for any vector V normal to M, then M is an anti-invariant or generic submanifold of $\bar{M}(c)$.*

Proof. If the normal connection of M is flat, then from (12) we have

$$\begin{aligned} g([A_U, A_V]X, Y) &= \left(\frac{c+1}{4}\right)\{g(X, \varphi V)g(\varphi Y, U) - g(Y, \varphi V)g(\varphi X, U) \\ &+ 2g(X, \varphi Y)g(\varphi V, U)\}, \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$. Here, choosing $X = PY$ and $V = CU$, by direct calculations, we conclude that

$$g(A_U A_{CU} PY - A_{CU} A_U PY, Y) = (\frac{c+1}{2})\{g(P^2 Y, Y)g(CU, CU)\}.$$

If $PA_U = A_U P$, then we can easily see that $(c+1)Tr(P^2)g(CU, CU) = 0$. This tells us that $P = 0$ (that is, M is anti-invariant submanifold) or $CU = 0$(that is, M is generic submanifold). □

Theorem 16 *Let M be a proper contact CR-submanifold of a Kenmotsu space form $\bar{M}(c)$. If the invariant distribution D is integrable, then $c < -1$.*

Proof. If the invariant distribution D is integrable, the from (43), we have

$$PA_V Y + A_V PY = 0. \tag{51}$$

It follows that

$$g(A_V PY, BU) = 0, \tag{52}$$

for any $Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$. By differentiating the covariant derivative in the direction of $X \in \Gamma(T\bar{M})$ of (52), and by using (9), (19), we get

$$\begin{aligned} 0 &= g(\bar{\nabla}_X A_V PY, BU) + g(A_V PY, \bar{\nabla}_X BU) \\ &= g((\nabla_X A)_V PY + A_{\nabla^{\perp}_X V} PY + A_V(\nabla_X PY), BU) \\ &+ g((\nabla_X B)U + B\nabla^{\perp}_X U, A_V PY). \end{aligned}$$

Again, by using (35), (39) and taking into account (51), we obtain

$$\begin{aligned}
-((\nabla_X A)_V PY, BU) &= -g((\nabla_X h)(PY, BU), V)\\
&= g(A_V\{A_{FY}X + Bh(X,Y) + g(\varphi X, Y)\xi - \eta(Y)PX\}, BU)\\
&+ g(g(FX,U)\xi + A_{CU}X - PA_U X, A_V PY)\\
&= g(A_V A_{FY}X + A_V Bh(X,Y), BU) + g(A_{CU}X, A_V PY)\\
&+ g(A_U PX, A_V PY)\\
-g((\nabla_X h)(PY, BU), V) &= g(A_{FY}X, A_V BU) + g(A_V BU, Bh(X,Y))\\
+g(A_{CU}X, A_V PY) &\\
&+ g(A_U PX, A_V PY).
\end{aligned}$$

Here, if PX is taken instead of X in this last equation, we have

$$\begin{aligned}
-g((\bar{\nabla}_{PX}h)(PY, BU), V) &= g(A_{FY}PX, A_V BU) + g(A_V BU, Bh(PX,Y))\\
&+ g(A_{CU}PX, A_V PY) + g(A_U P^2X, A_V PY).
\end{aligned}$$

Also, by using (51) and taking into account that M is a contact CR-submanifold in $\bar{M}(c)$, by direct calculations we have

$$\begin{aligned}
g((\bar{\nabla}_{PY}h)(PX, BU) &- (\bar{\nabla}_{PX}h)(PY, BU), V) = g(A_{CU}A_V PY, PX)\\
&- g(A_{CU}A_V PX, PY) - g(A_U P^3X, A_V Y)\\
&- g(A_U Y, A_V P^3X)\\
&= g(A_U PX, A_V Y) + g(A_U Y, A_V PX)\\
&+ g(A_{CU}PX, A_V PY) - g(A_{CU}PY, A_V PX). \qquad (53)
\end{aligned}$$

Also, from (14), we get

$$\begin{aligned}
\left(\frac{c+1}{2}\right) g(PY, X) g(BU, BV) &= g((\bar{\nabla}_{PY}h)(PX, BU)\\
&- (\bar{\nabla}_{PX}h)(PY, BU), V). \qquad (54)
\end{aligned}$$

Substituting (53) into (54), we obtain

$$\begin{aligned}
\left(\frac{c+1}{4}\right) g(PY, X) g(BU, BV) &= g(A_U PX, A_V Y) + g(A_U Y, A_V PX)\\
&+ g(A_{CU}PX, A_U PY) - g(A_{CU}PY, A_V PX),
\end{aligned}$$

which implies that

$$\left(\frac{c+1}{4}\right) g(PY, PY) g(U, U) = -g(A_U PY, A_U PY).$$

This proves our assertion. □

Acknowledgments

The authors are grateful to the referee(s) for the useful comments, which improved the presentation of the manuscript.

References

[1] M. Atçeken, Contact CR-submanifolds of Kenmotsu manifolds, *Serdica Math. J.*, **37** (2011), 67–78.

[2] M. Atçeken, Contact CR-warped product submanifolds in cosymplectic space forms, *Collect. Math.* (2011) 62:1726 DOI 10.1007/s13348-010-0002-z

[3] A. Bejancu, *Geometry of CR-submanifolds*, D. Reidel Publ. Co., Dordrecht, Holland, 1986.

[4] K. Kenmotsu, A class of almost contact Riemannian manifold, *Tohoku Math. J.*, **24** (1972), 93–103.

[5] V. A. Khan , K. A. Khan, Sirajuddin, Contact CR-warped product submanifolds of Kenmotsu manifolds, *Thai J. Math.*, **6**, 1 (2008), 139–154.

[6] Y. Kentaro, K. Masahiro, Contact CR-Submanifolds, *Kodai Math. J.*, **5** (1982), 238–252.

[7] S. Sibel, C. Özgür, On Some Submanifolds of Kenmotsu Manifolds, *Chaos, Solitons and Fractal* , **42** (2009), 1990–1995.

Received: April 24, 2012

Acta Univ. Sapientiae, Mathematica, 4, 2 (2012) 199–205

Improvement of some results concerning starlikeness and convexity

Róbert Szász
Sapientia - Hungarian University of Transylvania
Department of Mathematics and Informatics
Târgu Mureş, Romania
email: rszasz@ms.sapientia.ro

Abstract. We prove sharp versions of several inequalities dealing with univalent functions. We use differential subordination theory and Herglotz representations in our proofs.

1 Introduction

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane. Let $\mathcal{A}_n$ be the class of analytic functions of the form

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$$

which are defined in the unit disk $\mathbb{U}$, and let $\mathcal{A}_1 = \mathcal{A}$. Evidently $\mathcal{A}_{n+1} \subset \mathcal{A}_n$. The subclass of $\mathcal{A}$, consisting of functions f for which the domain $f(\mathbb{U})$ is starlike with respect to 0, is denoted by S^*. It is well-known that $f \in S^* \Leftrightarrow \mathrm{Re}\frac{zf'(z)}{f(z)} > 0,\ z \in \mathbb{U}$. A function $f \in \mathcal{A}$ for which the domain $f(\mathbb{U})$ is convex, is called convex function. The class of convex functions is denoted by K. We have $f \in K \Leftrightarrow \mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{U}$. Let $\mu \in [0,1)$. If for some function $f \in \mathcal{A}$ we have $\mathrm{Re}\frac{zf'(z)}{f(z)} > \mu,\ z \in U,\ \mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \mu,\ z \in U,\ \left|\arg\frac{zf'(z)}{f(z)}\right| < \mu\frac{\pi}{2},\ z \in U$, then we say that the function f is starlike of order μ, convex of order μ, and strongly starlike of order μ, respectively. We introduce the notations:

$$\mathcal{V}[\lambda,\gamma;f](z) \equiv \left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^{\gamma}, \tag{1}$$

2010 Mathematics Subject Classification: 30C45
Key words and phrases: convex functions, starlike functions, differential subordination

$$\mathcal{W}[\lambda,\gamma;f](z) \equiv \gamma\frac{\mathcal{F}(z)}{\mathcal{F}'(z)}\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)', \tag{2}$$

where

$$\mathcal{F}(z) = (1-\lambda)f(z) + \lambda z f'(z), \quad z \in \mathbb{U},\ \gamma \in \mathbb{C}^*,\ \lambda \in [0,1],\ f \in \mathcal{A}_n.$$

The authors proved in a recent paper [3] the following results:

Theorem 1 *If* $0 < \beta \le 1, \quad f \in \mathcal{A},$ *then*

$$\left|\mathcal{W}[\lambda,\gamma;f](z)\right| < \beta,\ z \in \mathbb{U}, \Rightarrow \left|\arg \mathcal{V}[\lambda,\gamma;f](z)\right| < \frac{\pi}{2}\beta,\ z \in \mathbb{U}.$$

Theorem 2 *If* $M \ge 1,\ z \in \mathbb{U},\ n \in \mathbb{N},\ f \in \mathcal{A}_n,$ *then*

$$\mathcal{R}e\{\mathcal{W}[\lambda,\gamma;f](z)\} < \frac{nM}{1+nM},\ z \in \mathbb{U} \Rightarrow \ \left|\mathcal{V}[\lambda,\gamma;f](z)\right| < M,\ z \in \mathbb{U}.$$

Theorem 3 *If* $0 \le \mu < 1$*,* $z \in \mathbb{U}$*,* $n \in \mathbb{N}$*,* $f \in \mathcal{A}_n$*, then*

$$\mathcal{R}e\{\mathcal{W}[\lambda,\gamma;f](z)\} > \Delta_n(\mu),\ z \in \mathbb{U} \Rightarrow \ \mathcal{R}e\mathcal{V}[\lambda,\gamma;f](z) > \mu,\ z \in \mathbb{U},$$

where

$$\Delta(\mu) = \begin{cases} \frac{n\mu}{2(\mu-1)}, & \textit{if} \quad \nu \in [0, \frac{1}{2}] \\ \frac{n(\mu-1)}{2\mu}, & \textit{if} \quad \nu \in [\frac{1}{2}, 1). \end{cases}$$

The goal of this paper is to prove the sharp version of Theorem 1 and also the sharp version of Theorem 2 and Theorem 3 in case of $n = 1$. To do this we need some preliminary results which will be exposed in the following section.

2 Preliminaries

Lemma 1 [2, p.24] *Let* f *and* g *be two analytic functions in* $\mathbb{U}$ *such that* $f(0) = g(0)$*, and* g *is univalent. If* $f \not\prec g$ *then there are two points,* $z_0 \in \mathbb{U}$ *and* $\zeta_0 \in \partial\mathbb{U}$*, and a real number* $m \in [1,\infty)$ *such that:*

$$1. \quad f(z_0) = g(\zeta_0), \qquad 2. \quad z_0 f'(z_0) = m\zeta_0 g'(\zeta_0).$$

Lemma 2 [1, p.27] *If* p *is an analytic function in* $\mathbb{U}$*, with* $p(0) = 1$ *and* $\mathcal{R}ep(z) \ge 0,\ z \in \mathbb{U}$*, then there is a probability measure* ν *on the interval* $[0, 2\pi]$*, such that* $f(z) = \int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}} d\nu(t)$.

Lemma 3 *If* $\alpha \in [1,2)$, *then the following inequality holds:* $\left|(1+z)^\alpha - 1\right| \le 2^\alpha - 1, \quad z \in \mathbb{U}$.

Proof. According to the maximum modulus principle for analytic functions, we have to prove the inequality only in case if $z = e^{i\theta}, \ \theta \in [-\pi,\pi]$. The inequality $\left|(1+e^{i\theta})^\alpha - 1\right| \le 2^\alpha - 1, \quad \theta \in [-\pi,\pi], \ \alpha \in [1,2)$ is equivalent to

$$2^{\alpha-1}\big(1-\cos^{2\alpha}\frac{\theta}{2}\big) - 1 + \cos^\alpha\frac{\theta}{2}\cos\frac{\alpha\theta}{2} \ge 0, \quad \theta \in [-\pi,\pi], \quad \alpha \in [1,2). \tag{3}$$

Let $f : [-\pi,\pi] \to \mathbb{R}$ be the function defined by $f(\theta) = 2^{\alpha-1}\big(1-\cos^{2\alpha}\frac{\theta}{2}\big) - 1 + \cos^\alpha\frac{\theta}{2}\cos\frac{\alpha\theta}{2}$. We have: $f'(\theta) = \alpha\cos^{\alpha-1}\frac{\theta}{2}\sin\frac{\theta}{2}\left[2^{\alpha-1}\cos^\alpha\frac{\theta}{2} - \frac{1}{2}\frac{\sin\frac{(\alpha+1)\theta}{2}}{\sin\frac{\theta}{2}}\right]$.
Let $\alpha \in (1,2)$, and $\theta \in [0, \frac{2\pi}{\alpha+1})$ be two fixed real numbers. We define the functions $g_1, g_2 : [0,\alpha] \to \mathbb{R}$ by $\quad g_1(x) = \big(2\cos\frac{\theta}{2}\big)^x, \quad g_2(x) = \frac{\sin\frac{(x+1)\theta}{2}}{\sin\frac{\theta}{2}}$.
It is simple to prove that g_1 is a convex and g_2 is a concave function. Thus the graphs of the two functions have at most two common points . Since $g_1(0) = g_2(0)$ and $g_1(1) = g_2(1)$, it follows that the two graphs have exactly two common points, and $g_2(x) > g_1(x), \ \ x \in (0,1)$, and $g_1(x) > g_2(x), \ \ x \in (1,\alpha]$. Thus we have $g_1(\alpha) > g_2(\alpha)$ in case of $\alpha \in (1,2)$, and $\theta \in [0, \frac{2\pi}{\alpha+1})$. The inequality $g_1(\alpha) > g_2(\alpha)$ holds in case of $\alpha \in (1,2)$ and $\theta \in [\frac{2\pi}{\alpha+1}, \pi]$ too, because in this case we have: $g_1(\alpha) > 0 \ge g_2(\alpha)$. This means that the inequality $g_1(\alpha) > g_2(\alpha)$ holds for $\alpha \in (1,2)$ and $\theta \in [0,\pi]$. It is easily seen that the inequality $g_1(\alpha) > g_2(\alpha)$ can be extended to $\alpha \in (1,2)$ and $\theta \in [-\pi,\pi]$.
Consequently, $2^{\alpha-1}\cos^\alpha\frac{\theta}{2} - \frac{1}{2}\frac{\sin\frac{(\alpha+1)\theta}{2}}{\sin\frac{\theta}{2}} \ge 0, \quad (\forall)\ \alpha \in (1,2), \quad (\forall)\ \theta \in [-\pi,\pi]$;
$f'(\theta) < 0, \ \theta \in (-\pi,0)$ and $f'(\theta) > 0, \ \theta \in (0,\pi)$.
Thus it follows that $\min_{\theta\in[-\pi,\pi]} f(\theta) = f(0) = 0$, and the inequality (3) is proved. $\square$

3 Main result

The following theorem is the sharp version of Theorem 1.

Theorem 4 *If* $0 < \beta \le 1, \quad f \in \mathcal{A}$ *then we have:*

$$\left|\mathcal{W}[\lambda,\gamma;f](z)\right| < \beta, \ z \in \mathbb{U} \Rightarrow \left|\arg\mathcal{V}[\lambda,\gamma;f](z)\right| < \beta, \ z \in \mathbb{U}.$$

Proof. Let $p(z) = \mathcal{V}[\lambda,\gamma;f](z)$. We have $\frac{zp'(z)}{p(z)} = \mathcal{W}[\lambda,\gamma;f](z)$, and consequently $\left|\frac{zp'(z)}{p(z)}\right| < \beta,\ z \in \mathbb{U}$. This inequality is equivalent to

$$\frac{zp'(z)}{p(z)} \prec h(z) = \beta z,\ z \in \mathbb{U}. \tag{4}$$

We prove the subordination $p(z) \prec q(z) = e^{\beta z},\quad z \in \mathbb{U}$. If this subordination does not hold, then according to Lemma 1, there are two points $z_0 \in \mathbb{U}$, $\zeta_0 \in \partial\mathbb{U}$ and a real number $m \in [1,\infty)$, such that $p(z_0) = q(\zeta_0)$, and $z_0 p'(z_0) = m\zeta_0 q(\zeta_0)$. Thus $\frac{z_0 p'(z_0)}{p(z_0)} = m\frac{\zeta_0 q'(\zeta_0)}{q(\zeta_0)} = mh(\zeta_0) \notin h(\mathbb{U})$.
This contradicts (4) and the contradiction implies $p(z) \prec q(z),\quad z \in \mathbb{U}$. The proved subordination implies $\left|\arg p(z)\right| \leq \max_{z\in\mathbb{U}}\{\arg(e^{\beta z})\} = \beta,\quad z \in \mathbb{U}$, and the proof is done. □

We present in the followings the sharp version of Theorem 2 and Theorem 3 in case of $n = 1$.

Theorem 5 *If* $M \geq 1,\ z \in \mathbb{U},,\ f \in \mathcal{A}$, *then*

$$\mathcal{R}e\{\mathcal{W}[\lambda,\gamma;f](z)\} < \frac{M}{1+M},\ z \in \mathbb{U} \Rightarrow \left|\mathcal{V}[\lambda,\gamma;f](z)\right| < 2^{\frac{2M}{M+1}} - 1,\ z \in \mathbb{U}.$$

Proof. The condition of the theorem can be rewritten in the following way $\mathcal{R}e\left\{1 - \frac{M+1}{M}\mathcal{W}[\lambda,\gamma;f](z)\right\} > 0,\quad z \in \mathbb{U}$. The Herglotz formula implies that there is a probability measure ν on $[0,2\pi]$ such that $1 - \frac{M+1}{M}\mathcal{W}[\lambda,\gamma;f](z) = \int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}} d\nu(t)$. This is equivalent to $\mathcal{W}[\lambda,\gamma;f](z) = -\frac{M}{M+1}\int_0^{2\pi} \frac{2ze^{-it}}{1-ze^{-it}} d\nu(t)$. On the other hand, if we denote $1 + p(z) = \mathcal{V}[\lambda,\gamma;f](z)$, we get $\frac{zp'(z)}{1+p(z)} = \mathcal{W}[\lambda,\gamma;f](z)$ and $\frac{p'(z)}{1+p(z)} = -\frac{M}{M+1}\int_0^{2\pi} \frac{2e^{-it}}{1-ze^{-it}} d\nu(t)$. This implies

$$\log(1 + p(z)) = \frac{2M}{M+1}\int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\nu(t).$$

It is easily seen that $g(z) = \log(1+z) \in K$. Thus it follows $\int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\nu(t) \in g(\mathbb{U}),\quad \forall\, z \in \mathbb{U}$, and this leads to the subordination $\int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\nu(t) \prec g(z),\ z \in \mathbb{U}$. Consequently we have $p(z) \prec \exp\left\{\frac{2M}{M+1}\log(1+z)\right\} - 1 = (1+z)^{\frac{2M}{M+1}} - 1,\ z \in \mathbb{U}$. This subordination implies

$$\left|p(z)\right| \leq \max_{z\in\mathbb{U}}\left|(1+z)^{\frac{2M}{M+1}} - 1\right|,\quad z \in \mathbb{U}.$$

Now from Lemma 3 we obtain the inequality $\left|p(z)\right| \leq 2^{\frac{2M}{M+1}} - 1, \ \ z \in \mathbb{U}$. This inequality is equivalent to $\left|\mathcal{V}[\lambda, \gamma; f](z)\right| \leq 2^{\frac{2M}{M+1}} - 1, \ \ z \in \mathbb{U}$. It is easy to show that if $M \geq 1$, then $2^{\frac{2M}{M+1}} - 1 \leq M$, so the proved result is an improvement of Theorem 2 in case $n = 1$. Moreover the proof shows that this is the best possible result in this particular case. □

Theorem 6 *Let* $0 \leq \mu < 1$, $z \in \mathbb{U}$, $f \in \mathcal{A}$. *Then:*

$$\mathcal{R}e\{\mathcal{W}[\lambda, \gamma; f](z)\} > \Delta(\mu), \ z \in \mathbb{U} \Rightarrow \ \mathcal{R}e\mathcal{V}[\lambda, \gamma; f](z) > 2^{2\Delta(\mu)}, \ z \in \mathbb{U},$$

where

$$\Delta(\mu) = \begin{cases} \frac{\mu}{2(\mu-1)}, & \textit{if } \mu \in [0, \frac{1}{2}] \\ \frac{\mu-1}{2\mu}, & \textit{if } \mu \in [\frac{1}{2}, 1). \end{cases}$$

Proof. We rewrite the condition $\mathcal{R}e\{\mathcal{W}[\lambda, \gamma; f](z)\} > \Delta(\mu), \ z \in \mathbb{U}$ in the following form: $\mathcal{R}e\frac{\Delta(\mu) - \mathcal{W}[\lambda, \gamma; f](z)}{\Delta(\mu)} > 0, \ z \in \mathbb{U}$. We use the Herglotz formula again and we get:

$$\frac{\Delta(\mu) - \mathcal{W}[\lambda, \gamma; f](z)}{\Delta(\mu)} = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\nu(t),$$

where ν is a probability measure on $[0, 2\pi]$. If we denote $p(z) = \mathcal{V}[\lambda, \gamma; f](z)$ then: $\frac{zp'(z)}{p(z)} = \mathcal{W}[\lambda, \gamma; f](z)$ and $\frac{p'(z)}{p(z)} = -\Delta(\mu) \int_0^{2\pi} \frac{2e^{-it}}{1-ze^{-it}} d\nu(t)$.
This leads to: $p(z) = \exp\left\{2\Delta(\mu) \int_0^{2\pi} \log(1 - ze^{-it}) d\nu(t)\right\}$.
Since $g(z) = \log(1 + z) \in K$, it follows the inclusion: $\int_0^{2\pi} \log(1 - ze^{-it}) d\nu(t) \in g(\mathbb{U}), \ \ z \in \mathbb{U}$, and this implies the subordination:
$\int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\nu(t) \prec g(z), \ z \in \mathbb{U}$. Thus we obtain: $p(z) \prec q(z) = (1 + z)^{2\Delta(\mu)}, \ z \in \mathbb{U}$, and consequently: $\mathcal{R}ep(z) \geq \mathcal{R}e(1 + z)^{2\Delta(\mu)}, \ z \in \mathbb{U}$. According to the definition of $\Delta(\mu)$, we have $-2\Delta(\mu) \in (0, 1)$. This implies $q \in K$. The equivalency $f(z) \in \mathbb{R} \Leftrightarrow z \in \mathbb{R}$, and the fact that the domain $q(\mathbb{U})$ is convex and symmetric with respect to the real axis, imply the inequality: $\mathcal{R}eq(z) \geq \min\{q(-1), q(1)\} = 2^{2\Delta(\mu)}, \ z \in \overline{\mathbb{U}}$. Thus it follows:

$$\mathcal{R}ep(z) \geq 2^{2\Delta(\mu)}, \ \ z \in \overline{\mathbb{U}}.$$

It is easily seen that $2^{2\Delta(\mu)} \geq \mu$, for every $0 \leq \mu < 1$, and $2^{2\Delta(\mu)}$ is the biggest value, for which the inequality

$$\mathcal{R}e\mathcal{V}[\lambda, \gamma; f](z) \geq 2^{2\Delta(\mu)}, \ \ z \in \overline{\mathbb{U}}$$

holds. According to the minimum principle, inside the unit disk we have the strict inequality: $\mathcal{R}e\mathcal{V}[\lambda,\gamma;f](z) > 2^{2\Delta(\mu)}, \quad z \in \mathbb{U}$. $\square$

By choosing suitable values of the parameters, we obtain sharp results concerning starlikeness. Theorem 4 implies in case of $\gamma = 1,\ \lambda = 0$ the following result:

Corollary 1 *If* $\beta \in (0,1]$, $f \in \mathcal{A}$, *then:*

$$\left|1 + z\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right)\right| < \beta, \quad z \in \mathbb{U} \Rightarrow \left|\arg\frac{zf'(z)}{f(z)}\right| < \beta.$$

The result is sharp, the extremal function is: $f(z) = z\exp\left(\int_0^z \frac{e^{\beta t}-1}{t}dt\right)$.

If we take $\gamma = 1,\ M = \alpha + 1,\ \lambda = 0$ then Theorem 2 implies:

Corollary 2 *If* $\alpha \in [0,1)$, $f \in \mathcal{A}$, *then:*

$$\mathcal{R}e\left[z\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right)\right] < \frac{-1}{\alpha+2},\ z \in \mathbb{U} \Rightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < 2^{\frac{2\alpha+2}{\alpha+3}} - 1,\ z \in \mathbb{U},$$

and the result is sharp. The extremal function is:
$f(z) = z\exp\left(\int_0^z \frac{(1+t)^{\frac{2\alpha+2}{\alpha+3}}-1}{t}dt\right)$. *Since* $2^{\frac{2\alpha+2}{\alpha+3}} - 1 < 1$, *it follows that* f *is a starlike function.*

Finally, for $\gamma = \lambda = 1$ Theorem 6 implies:

Corollary 3 *If* $\mu \in [0,1)$, $f \in \mathcal{A}$, *then:*

$$\mathcal{R}e\left\{z\left[\frac{(zf'(z))''}{(zf'(z))'} - \frac{(zf'(z))'}{zf'(z)}\right]\right\} > \Delta(\mu) - 1 \Rightarrow \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 2^{2\Delta(\mu)},\ z \in \mathbb{U}.$$

The result is sharp. The extremal function is:

$$f(z) = \int_0^z \exp\left(\int_0^\nu \frac{(1+t)^{2\Delta(\mu)}-1}{t}dt\right)d\nu.$$

Since $2^{2\Delta(\mu)} > \mu,\ \mu \in [0,1)$, *it follows that* f *is a convex function of order* μ.

References

[1] D. J. Hallenbeck, T. H. MacGregor, *Linear problems and convexity techniques in geometric function theory*, Pitman Advanced Publishing Program, Boston, 1984.

[2] S. S. Miller, P. T. Mocanu, *Differential subordinations theory and applications*, Marcel Dekker, New York, Basel, 2000.

[3] H. Irmak, M. Şan, Some relations between certain inequalities concerning analytic and univalent functions, *Appl. Math. Lett.* **23** (2010), 897–901.

Received: March 11, 2012

Contents
Volume 4, 2012

Acta Universitatis Sapientiae

The scientific journal of the Sapientia University publishes original papers and deep surveys in several areas of sciences written in English.
Information about the appropriate series can be found at the Internet address `http://www.acta.sapientia.ro`.

Acta Universitatis Sapientiae, Mathematica

Contact address and subscription:
Acta Universitatis Sapientiae, Mathematica
RO 400112 Cluj-Napoca
Str. Matei Corvin nr. 4.
Email: acta-math@acta.sapientia.ro

Each volume contains two issues.

Sapientia University

Scientia Publishing House

ISSN 1844-6094
http://www.acta.sapientia.ro

Information for authors

Acta Universitatis Sapientiae, Mathematica publishes original papers and surveys in all field of Mathematics. All papers will be peer reviewed.

Papers published in current and previous volumes can be found in Portable Document Format (pdf) form at the address: `http://www.acta.sapientia.ro`.

The submitted papers should not be considered for publication by other journals. The corresponding author is responsible for obtaining the permission of coauthors and of the authorities of institutes, if needed, for publication, the Editorial Board disclaims any responsibility.

Submission must be made by email (`acta-math@acta.sapientia.ro`) only, using the LaTeX style and sample file at the address: `http://www.acta.sapientia.ro`. Beside the LaTeX source a pdf format of the paper is needed too.

Prepare your paper carefully, including keywords, 2010 Mathematics Subject Classification (MSC 2010) codes (`http://www.ams.org/msc//msc2010.html`), and the reference citation should be written in brackets in the text as [3]. References should be listed alphabetically using the following examples:

For papers in journals:
A. Hajnal, V. T. Sós, Paul Erdős is seventy, *J. Graph Theory*, **7** (1983), 391–393.
For books:
D. Stanton, D. White, *Constructive combinatorics*, Springer, New York, 1986.
For papers in contributed volumes:
Z. Csörnyei, *Compilers* in *Algorithms of informatics, Vol. 1. Foundations* (ed. A. Iványi), mondAt Kiadó, Budapest, 2007, pp. 80–130.
For internet sources:
E. Ferrand, An analogue of the Thue-Morse sequence, *Electron. J. Comb.*, **14** (2007) #R30, http://www.combinatorics.org/.

Illustrations should be given in Encapsulated Postcript (eps) format.

Authors are encouraged to submit papers not exceeding 15 pages, but no more than 10 pages are preferable.

One issue is offered each author. No reprints will be available.

Printed by Gloria Printing House
Director: Péter Nagy