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Asymptotic behavior of the solution of quasilinear parametric variational inequalities in a beam with a thin neck

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Abstract. In this paper we study the asymptotic behavior of the solution of quasilinear parametric variational inequalities posed in a cylinder with a thin neck, and we obtain the limit problem.

1 Introduction

The aim of the paper is to study the asymptotic behavior of the solution of quasilinear variational inequalities in a beam with a thin neck. Mathematically, this notched beam is given by

$$\Omega_\epsilon = \{(x_1, x') \in \mathbb{R}^3 : -1 < x_1 < 1, |x'| < \epsilon \text{ if } |x_1| > t_\epsilon, |x'| < \epsilon r_\epsilon \text{ if } |x_1| \leq t_\epsilon\},$$

where ϵ, r_ϵ, and t_ϵ are positive parameters such that $\frac{\epsilon r_\epsilon}{t_\epsilon} \to 0$.

Previous work on domains of this type was done by Hale & Vegas [7], Jimbo [8, 9], Cabib, Freddi, Morassi, & Percivale [2], Rubinstein, Schatzman & Sternberg [13], Casado-Díaz, Luna-Laynez & Murat [3, 4] and Kohn & Slastikov [10].

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The most recent results are of Casado-Díaz, Luna-Laynez & Murat [4]. They studied the asymptotic behavior of the solution of a diffusion equation in the notched beam Ω_ϵ and obtained at the limit a one-dimensional model.

In the present article the geometrical setting is the same as in [4], but we consider quasilinear variational inequalities instead of linear variational equalities.

The paper is organized as follows. In Section 2 the geometrical setting is described, the studied problem is given, and the assumptions for our results are formulated. In Section 3 the asymptotic behavior of the solution is studied. Some results from [11] are recalled which, unfortunately, don't provide information about what happening near to the notch. Thus we need to prove some auxiliary results. In Section 4 the limit problem is obtained. To prove the results in this section, we combine the ideas from [5] with the adaptation to variational inequalities of the method used in [4].

2 Setting the problem

Let $\epsilon > 0$ be a parameter, r_ϵ ($r_\epsilon > 0$) and t_ϵ ($t_\epsilon > 0$) be two sequences of real numbers, with

$$r_\epsilon \to 0, \qquad t_\epsilon \to 0, \quad \text{when } \epsilon \to 0.$$

We assume that

$$\frac{t_\epsilon}{r_\epsilon^2} \to \mu, \quad \frac{\epsilon}{r_\epsilon} \to \nu, \qquad \text{with } 0 \le \mu < +\infty,\ 0 \le \nu < +\infty, \quad \text{when } \epsilon \to 0.$$

Let $S \subset \mathbb{R}^2$ be a bounded domain such that $0 \in S$, which is sufficiently smooth to apply the Poincaré-Wirtinger inequality.

Define the following subsets of $\mathbb{R}^3$:

$$\Omega_\epsilon^- = (-1, -t_\epsilon) \times (\epsilon S), \quad \Omega_\epsilon^0 = [-t_\epsilon, t_\epsilon] \times (\epsilon r_\epsilon S), \quad \Omega_\epsilon^+ = (t_\epsilon, 1) \times (\epsilon S),$$

$$\Omega_\epsilon = \Omega_\epsilon^- \cup \Omega_\epsilon^0 \cup \Omega_\epsilon^+, \quad \text{and} \quad \Omega_\epsilon = \Omega_\epsilon^- \cup \Omega_\epsilon^+.$$

Ω_ϵ is a notched beam, the main part of the beam is Ω_ϵ^1 and the notched part Ω_ϵ^0. A point of Ω^ϵ is denoted by $x = (x_1, x') = (x_1, x_2, x_3)$.

Denote by

$$\Gamma_\epsilon^- = \{-1\} \times (\epsilon S) \quad \text{and} \quad \Gamma_\epsilon^+ = \{1\} \times (\epsilon S)$$

the two bases of the beam, and let

$$\Gamma_\epsilon = \Gamma_\epsilon^- \cup \Gamma_\epsilon^+$$

be the union of the two bases.

Denote

$$\mathcal{V}_\epsilon = \{V \in H^1(\Omega_\epsilon), \quad V = 0 \text{ on } \Gamma_\epsilon\}.$$

We consider the following problem:
Find $U_\epsilon \in M_\epsilon$ such that, for all $V_\epsilon \in M_\epsilon$,

$$\int_{\Omega_\epsilon} [A_\epsilon \Phi_\epsilon(x, U_\epsilon, B_\epsilon)\nabla U_\epsilon, \nabla(V_\epsilon - U_\epsilon)] \; dx \geq 0 \tag{1}$$

with A_ϵ, B_ϵ, and Φ_ϵ, given functions, M_ϵ a closed, convex, nonempty cone in $\mathcal{V}_\epsilon$.

This problem has applications in Physics. Bruno [1] observed that when a ferromagnet has a thin neck, this will be preferred location for the domain wall. He also noticed that if the geometry of the neck varies rapidly enough, it can influence and even dominate the structure of the wall.

Consider problem (1). We impose the following assumptions:

(A1) The matrix A_ϵ has the following form

$$A_\epsilon(x) = \chi_{\Omega_\epsilon^1}(x) A^1\left(x_1, \frac{x'}{\epsilon}\right) + \chi_{\Omega_\epsilon^0}(x) A^0\left(\frac{x_1}{t_\epsilon}, \frac{x'}{\epsilon r_\epsilon}\right),$$

where $A^1, A^0 \in L^\infty((-1,1) \times S)^{3\times 3}$.

(A2) The matrix B_ϵ has the following form

$$B_\epsilon(x) = \chi_{\Omega_\epsilon^1}(x) B^1\left(x_1, \frac{x'}{\epsilon}\right) + \chi_{\Omega_\epsilon^0}(x) B^0\left(\frac{x_1}{t_\epsilon}, \frac{x'}{\epsilon r_\epsilon}\right),$$

where $B^1, B^0 \in L^\infty((-1,1) \times S)^{3\times 3}$.

(A3) The functions $\Phi_\epsilon : \Omega_\epsilon \times \mathbb{R} \to \mathbb{R}^{3\times 3}$ and $\Psi_\epsilon : \Omega_\epsilon \times \mathbb{R} \to \mathbb{R}^3$ are Carathéodory mappings having the following form:

$$\Phi_\epsilon(x, \eta) = \chi_{\Omega_\epsilon^1}(x) \Phi_\epsilon^1\left(x_1, \frac{x'}{\epsilon}, \eta\right) + \chi_{\Omega_\epsilon^0}(x) \Phi_\epsilon^0\left(\frac{x_1}{t_\epsilon}, \frac{x'}{\epsilon r_\epsilon}, \eta\right);$$

for a.e. $x \in \Omega_\epsilon$, for all $\eta \in \mathbb{R}$;
for all $U_\epsilon \in L^2(\Omega_\epsilon)$, $W_\epsilon \in L^2(\Omega_\epsilon)^3$, $\Phi_\epsilon^1(\cdot, U_\epsilon(\cdot))W_\epsilon(\cdot), \Phi_\epsilon^0(\cdot, U_\epsilon(\cdot))W_\epsilon(\cdot) \in L^2((-1,1) \times S)^3$.

(A4) *Coercivity condition*
There exist $C_1, C_2 > 0$ and $k_1 \in L^\infty(\Omega_\epsilon)$ such that for all $\xi \in \mathbb{R}^3$, $\eta \in \mathbb{R}$

$$[A_\epsilon(x)\Phi_\epsilon(x,\eta)B_\epsilon(x)\xi, \xi] \geq C_1\|\xi\|^2 + C_2|\eta|^{q_1} - k_1(x) \quad \text{a.e. } x \in \Omega_\epsilon \tag{2}$$

for some $1 < q_1 < 2$, for each $\epsilon > 0$.

(A5) *Growth condition*
There exist $C > 0$ and $\alpha \in L^\infty(\Omega_\epsilon)$ such that for all $\xi \in \mathbb{R}^3$, $\eta \in \mathbb{R}$

$$\|A_\epsilon(x)\Phi_\epsilon(x,\eta)\xi\| \leq C\|\xi\| + C|\eta| + \alpha(x) \quad \text{a.e. } x \in \Omega_\epsilon, \tag{3}$$

for each $\epsilon > 0$.

(A6) *Monotonicity condition*
For all $\xi, \tau \in \mathbb{R}^n$, $\eta \in \mathbb{R}$,

$$[A_\epsilon(x)\Phi_\epsilon(x,\eta)B_\epsilon(x)\xi - A_\epsilon(x)\Phi_\epsilon(x,\eta)B_\epsilon(x)\tau, \xi - \tau] \geq 0, \quad \text{a. e. } x \in \Omega_\epsilon,$$

for each $\epsilon > 0$.

(A7) If $u_\epsilon \to u$ and $w_\epsilon \rightharpoonup w$ in $L^2(Y^1)$, then

$$\Phi^1_\epsilon(\cdot, u_\epsilon(\cdot))w(\cdot) \to \Phi^1(\cdot, u(\cdot))w(\cdot) \text{ strongly in } L^2(Y^1).$$

If $u_\epsilon \to u$ and $w_\epsilon \rightharpoonup w$ in $L^2(Z)$, then

$$\Phi^0_\epsilon(\cdot, u_\epsilon(\cdot))w(\cdot) \to \Phi^0(\cdot, u(\cdot))w(\cdot) \text{ strongly in } L^2(Z).$$

3 Asymptotic behavior of the solution

To study the asymptotic behavior we use the change of variables $y = y_\epsilon(x)$ given by

$$y_1 = x_1 \quad y' = \frac{x'}{\epsilon} \tag{4}$$

which transforms the beam (except the notch) in a cylinder of fixed diameter. This change of variable is classical in the study of asymptotic behavior of variational equalities in thin cylinders or beams (see [6], [12], [14]). We denote by Y^-_ϵ, Y^0_ϵ, Y^+_ϵ, Y_ϵ, and Y^S_ϵ the images of Ω^-_ϵ, Ω^0_ϵ, Ω^+_ϵ, Ω_ϵ, and Ω^S_ϵ by the change of variables $y = y_\epsilon(x)$, i.e.

$$Y^-_\epsilon = (-1, -t_\epsilon) \times S, \quad Y^0_\epsilon = [-t_\epsilon, t_\epsilon] \times (r_\epsilon S), \quad Y^+_\epsilon = (t_\epsilon, 1) \times S,$$

$$Y_\epsilon = Y_\epsilon^- \cup Y_\epsilon^0 \cup Y_\epsilon^+, \quad Y_\epsilon^1 = Y_\epsilon^- \cup Y_\epsilon^+.$$

Denote by Y^-, Y^+, and Y^1 the "limits"of Y_ϵ^-, Y_ϵ^+, and Y_ϵ^1, i.e.

$$Y^- = (-1,0) \times S, \quad Y^+ = (0,1) \times S, \quad Y^1 = Y^- \cup Y^+.$$

Note that Y_ϵ^1 is contained in its limit Y^1.

The two bases of the beam Γ_ϵ^- and Γ_ϵ^+ are transformed to Λ^- and Λ^+, respectively, where

$$\Lambda^- = \{-1\} \times S \quad \text{and} \quad \Lambda^+ = \{1\} \times S.$$

Γ_ϵ transforms to $\Lambda = \Lambda^- \cup \Lambda^+$.

Let $U_\epsilon \in M_\epsilon$ be the solution of the variational inequality (1). Define $u_\epsilon \in K_\epsilon$ by

$$u_\epsilon(y) = U_\epsilon(y_\epsilon^{-1}(y)) \quad \text{a.e. } y \in Y_\epsilon. \tag{5}$$

K_ϵ being the image of M_ϵ. K_ϵ is a closed, convex, nonempty cone in $\mathcal{D}_\epsilon$, with $\mathcal{D}_\epsilon = \{v \in H^1(Y_\epsilon) \mid v = 0 \text{ on } \Lambda\}$. We need the following two assumptions:

(A8) There exists a nonempty, convex cone K in $H^1(Y^1)$ such that
(i) $K \cap H^1((-1,0) \cup (0,1)) \neq \emptyset$;
(ii) $\epsilon_i \to 0$, $u_{\epsilon_i} \in K_{\epsilon_i}$, $u \in H^1((-1,0) \cup (0,1))$, $u_{\epsilon_i} \rightharpoonup u$ (weakly) in $H^1(Y^1)$ imply $u \in K$.

(A9) There exists a nonempty, convex cone L in $L^2((-1,1); H^1(S))$ such that
$\epsilon_i \to 0$, $w_{\epsilon_i} \in K_{\epsilon_i}$, $w \in L^2((-1,1); H^1(S))$, $w_{\epsilon_i} \rightharpoonup w$ (weakly) in $L^2((-1,1); H^1(S))$ imply $w \in L$.

By change of variables $y = y_\epsilon(x)$ the operator ∇ transforms to

$$\nabla^\epsilon \cdot = \left(\frac{\partial \cdot}{\partial y_1}, \frac{1}{\epsilon}\frac{\partial \cdot}{\partial y_2}, \frac{1}{\epsilon}\frac{\partial \cdot}{\partial y_3} \right).$$

In the following we recall some results from [11, 4].

Lemma 1 ([11]) *Let $U_\epsilon \in M_\epsilon$ be the solution of the inequality (1) and $u_\epsilon \in K_\epsilon$ given by (5). If assumptions (A1) - (A6) are verified then the sequence U_ϵ satisfies*

$$U_\epsilon \in M_\epsilon, \quad \frac{1}{|\Omega_\epsilon|} \int_{\Omega_\epsilon} |\nabla U_\epsilon|^2 dx \leq C. \tag{6}$$

Theorem 1 ([11]) *Let* U_ϵ *be the solution of the variational inequality (1) and* $u_\epsilon \in K_\epsilon$ *defined by*

$$u_\epsilon(y) = U_\epsilon(y_\epsilon^{-1}(y)) \quad a.e.\ y \in Y_\epsilon.$$

If assumptions (A1)-(A6) and (A8)-(A9) are verified, then there exist three functions u, w, *and* σ^1 *with*

$$u \in H^1((-1,0) \cup (0,1)) \cap K, \quad u(-1) = u(1) = 0,$$

$$w \in L, \quad \sigma^1 \in L^2(Y^1)^3,$$

such that up to extraction of a subsequence

$$\chi_{Y_\epsilon^1} u_\epsilon \to u \ \text{ in } \ L^2(Y^1);$$

$$\chi_{Y_\epsilon^-} \frac{\partial u_\epsilon}{\partial y_1} \rightharpoonup \frac{\partial u}{\partial y_1} \ \text{ in } \ L^2(Y^-);$$

$$\chi_{Y_\epsilon^+} \frac{\partial u_\epsilon}{\partial y_1} \rightharpoonup \frac{\partial u}{\partial y_1} \ \text{ in } \ L^2(Y^+);$$

$$\chi_{Y_\epsilon^1} \frac{1}{\epsilon} \nabla_{y'} u_\epsilon \rightharpoonup \nabla_{y'} w \ \text{ in } \ L^2(Y^1)^2;$$

and

$$\chi_{Y_\epsilon^1} \sigma_\epsilon \rightharpoonup \sigma^1 \quad in \ L^2(Y^1)^3.$$

Theorem 2 ([11]) *Let* U_ϵ *be the solution of the variational inequality (1) and* $u \in H^1((-1,0) \cup (0,1)) \cap K$ *given in Theorem 1. If assumptions (A1)-(A6) and (A8) are verified, then there exists a subsequence of solutions* U_ϵ, *also denoted by* U_ϵ, *such that*

$$\lim_{\epsilon \to 0} \frac{1}{|\Omega_\epsilon|} \int_{\Omega_\epsilon} |U_\epsilon(x) - u(x_1)|^2 \, dx = 0. \tag{7}$$

Unfortunately, this change of variables doesn't provide information about what happening near the notch. Thus we use another change of variables, which was given in [4]. Consider the case, when

$$\mu < +\infty \quad \text{and} \quad \nu < +\infty.$$

The change of variables $z = z_\epsilon(x)$ is defined as follows

$$z_1 = \begin{cases} \begin{cases} \frac{1}{\epsilon r_\epsilon}(x_1 + t_\epsilon) - \frac{t_\epsilon}{r_\epsilon^2}, & \text{if} \quad -1 \le x_1 \le -t_\epsilon, \\ \frac{x_1}{r_\epsilon^2}, & \text{if} \quad -t_\epsilon \le x_1 \le t_\epsilon, \\ \frac{1}{\epsilon r_\epsilon}(x_1 - t_\epsilon) + \frac{t_\epsilon}{r_\epsilon^2}, & \text{if} \quad t_\epsilon \le x_1 \le 1, \end{cases} & \text{if } \mu = 0, \\ \begin{cases} \frac{\mu r_\epsilon}{\epsilon t_\epsilon}(x_1 + t_\epsilon) - \mu, & \text{if} \quad -1 \le x_1 \le -t_\epsilon, \\ \frac{\mu}{t_\epsilon} x_1, & \text{if} \quad -t_\epsilon \le x_1 \le t_\epsilon, \\ \frac{\mu r_\epsilon}{\epsilon t_\epsilon}(x_1 - t_\epsilon) + \mu, & \text{if} \quad t_\epsilon \le x_1 \le 1 \end{cases} & \text{if } \mu > 0, \end{cases} \qquad z' = \frac{x'}{\epsilon r_\epsilon}. \tag{8}$$

This change of variables transforms the notch in a cylinder of fixed diameter and length, but transforms the rest of the beam in a very large domain. But it allows to describe the behavior of the solution U_ϵ of inequality (1) when x_1 is close to zero.

We denote by Z_ϵ^-, Z_ϵ^0, Z_ϵ^+, Z_ϵ, and Z_ϵ^1 the images of Ω_ϵ^-, Ω_ϵ^0, Ω_ϵ^+, Ω_ϵ, and Ω_ϵ^1 by the change of variables $z = z_\epsilon(x)$, i.e.

$$Z_\epsilon^- = \left(-\frac{1-t_\epsilon}{\epsilon r_\epsilon} - \frac{t_\epsilon}{r_\epsilon^2}, -\frac{t_\epsilon}{r_\epsilon^2}\right) \times \left(\frac{1}{r_\epsilon} S\right), \quad Z_\epsilon^0 = \left[-\frac{t_\epsilon}{r_\epsilon^2}, \frac{t_\epsilon}{r_\epsilon^2}\right] \times S,$$

$$\text{and} \quad Z_\epsilon^+ = \left(\frac{t_\epsilon}{r_\epsilon^2}, \frac{1-t_\epsilon}{\epsilon r_\epsilon} + \frac{t_\epsilon}{r_\epsilon^2}\right) \times \left(\frac{1}{r_\epsilon} S\right)$$

if $\mu = 0$, and

$$Z_\epsilon^- = \left(-\frac{\mu r_\epsilon(1-t_\epsilon)}{\epsilon t_\epsilon} - \mu, -\mu\right) \times \left(\frac{1}{r_\epsilon} S\right), \quad Z_\epsilon^0 = [-\mu, \mu] \times S,$$

$$\text{and } Z_\epsilon^+ = \left(\mu, \frac{\mu r_\epsilon(1-t_\epsilon)}{\epsilon t_\epsilon} + \mu\right) \times \left(\frac{1}{r_\epsilon} S\right)$$

if $\mu > 0$. We set

$$Z_\epsilon = Z_\epsilon^- \cup Z_\epsilon^0 \cup Z_\epsilon^+, \quad Z_\epsilon^1 = Z_\epsilon^- \cup Z_\epsilon^+.$$

We denote by Z^-, Z^+, and Z^0 the "limits" of Z_ϵ^-, Z_ϵ^+, and Z_ϵ^0, i.e.

$$Z^- = (-\infty, -\mu) \times \mathbb{R}^2, \; Z^+ = (\mu, +\infty) \times \mathbb{R}^2, \; Z^0 = [-\mu, \mu] \times S,$$

and define

$$Z = Z^- \cup Z^0 \cup Z^+, \; Z^1 = Z^- \cup Z^+.$$

Remark 1 ([4]) *In (8) there are two definitions of z_ϵ corresponding to the cases $\mu = 0$ and $\mu > 0$. Actually when $\mu > 0$, we could define z_ϵ by the definition given for $\mu = 0$ because*

$$\mu \sim \frac{t_\epsilon}{r_\epsilon^2}, \quad \frac{\mu r_\epsilon}{\epsilon t_\epsilon} \sim \frac{1}{\epsilon r_\epsilon}, \quad \text{and} \quad \frac{\mu}{t_\epsilon} \sim \frac{1}{r_\epsilon^2}.$$

The definition (8) which distinguishes the cases $\mu = 0$ and $\mu > 0$ has the advantage that the image Z_ϵ of Ω_ϵ by the change of variables $z = z_\epsilon(x)$ is contained in its "limit" Z for every $\epsilon > 0$ and Z_ϵ^0 is fixed for $\mu > 0$; then a function defined in Z has a restriction to Z_ϵ.

Theorem 3 ([4]) *Let $(U_\epsilon)_\epsilon$ be a sequence which satisfies (6). Define $\hat{u}_\epsilon \in H^1(Z_\epsilon)$ by*

$$\hat{u}_\epsilon(z) = U_\epsilon(z_\epsilon^{-1}(z)), \quad \text{a.e. } z \in Z_\epsilon. \tag{9}$$

Then there exists a function $\hat{u}$, with

$$\hat{u} \in H^1_{loc}(Z),\ \hat{u} - u(0^-) \in L^6(Z^-),\ \hat{u} - u(0^+) \in L^6(Z^+),\ \nabla\hat{u} \in L^2(Z)^3,$$

(where u is defined in Corollary 1), such that for every $R > 0$, up to extraction of a subsequence,

$$\chi_{Z_\epsilon \cap B_3(0,R)}\hat{u}_\epsilon \to \chi_{B_3(0,R)}\hat{u} \quad \text{in } L^2(Z) \text{ strongly},$$

$$\chi_{Z_\epsilon}\nabla\hat{u}_\epsilon \rightharpoonup \nabla\hat{u} \quad \text{in } L^2(Z)^3 \text{ weakly},$$

where $B_3(0,R)$ denotes the 3-dimensional ball with center (0, 0, 0) and diameter R. Moreover, if $\mu = 0$, then $\hat{u}$ only depends on z_1 and satisfies

$$\hat{u} = u(0^-) \quad \text{in } Z^-, \quad \hat{u} = u(0^+) \quad \text{in } Z^+.$$

If $\nu = \mu = 0$, then $u(0^-) = u(0^+)$.
If $\nu = 0$ and $\mu > 0$, then there exists a function $\hat{w} \in L^2((-\mu,\mu); H^1(S))$ such that up to extraction of a subsequence,

$$\frac{r_\epsilon}{\epsilon}\nabla_{z'}\hat{u}_\epsilon \rightharpoonup \nabla_{z'}\hat{w} \quad \text{in } L^2(Z^0)^2 \text{ weakly}.$$

Let $\hat{K}_\epsilon$ be the image of M_ϵ by the change of variables $z = z_\epsilon(x)$. $\hat{K}_\epsilon$ is a closed, convex, nonempty cone in $H^1(Z_\epsilon)$. We need the following two assumptions:

(A10) There exists a nonempty subset $\hat{K}$ of $H^1_{loc}(Z)$ such that

$$\epsilon_i \to 0,\ R > 0,\ \hat{u}_{\epsilon_i} \in \hat{K}_{\epsilon_i},\ \hat{u} \in H^1_{loc}(Z),$$

$$\chi_{Z_\epsilon \cap B_3(0,R)} \hat{u}_{\epsilon_i} \to \chi_{B_3(0,R)} \hat{u} \ \text{ (strongly) in } L^2(Z),$$

and

$$\chi_{Z_\epsilon} \nabla \hat{u}_{\epsilon_i} \rightharpoonup \nabla \hat{u} \ \text{ (weakly) in } (L^2(Z))^3,$$

imply $\hat{u} \in \hat{K}$.

(A11) There exists a nonempty, convex cone $\hat{L}$ in $L^2((-\mu, \mu); H^1(S))$ such that

$\epsilon_i \to 0$, $\hat{w}_{\epsilon_i} \in K_{\epsilon_i}$, $\hat{w} \in L^2((-\mu, \mu); H^1(S))$, $\hat{w}_{\epsilon_i} \rightharpoonup \hat{w}$ (weakly) in $L^2((-\mu, \mu); H^1(S))$ imply $\hat{w} \in \hat{L}$.

Theorem 4 *Let $U_\epsilon \in M_\epsilon$ be the solution of the variational inequality (1), $u \in H^1((-1, 0) \cup (0, 1)) \cap K$ defined in Theorem 1, and $\hat{u}_\epsilon \in \hat{K}_\epsilon$ given by (9). If assumptions (A1)-(A6) and (A8)-(A11) are verified, then there exists a function $\hat{u} \in \hat{K}$, with*

$$\hat{u} - u(0^-) \in L^6(Z^-),\ \ \hat{u} - u(0^+) \in L^6(Z^+),\ \ \nabla \hat{u} \in L^2(Z)^3, \tag{10}$$

such that for every $R > 0$, up to extraction of a subsequence,

$$\chi_{Z_\epsilon \cap B_3(0,R)} \hat{u}_\epsilon \to \chi_{B_3(0,R)} \hat{u} \quad \text{in } L^2(Z) \text{ strongly,}$$

$$\chi_{Z_\epsilon} \nabla \hat{u}_\epsilon \rightharpoonup \nabla \hat{u} \quad \text{in } L^2(Z)^3 \text{ weakly.}$$

Moreover, if $\mu = 0$, then $\hat{u}$ only depends on z_1 and satisfies

$$\hat{u} = u(0^-) \ \text{ in } Z^-, \quad \hat{u} = u(0^+) \ \text{ in } Z^+.$$

If $\nu = \mu = 0$, then $u(0^-) = u(0^+)$.
If $\nu = 0$ and $\mu > 0$, then there exists a function $\hat{w} \in \hat{L}$ such that up to extraction of a subsequence,

$$\frac{r_\epsilon}{\epsilon} \nabla_{z'} \hat{u}_\epsilon \rightharpoonup \nabla_{z'} \hat{w} \quad \text{in } L^2(Z^0)^2 \text{ weakly.} \tag{11}$$

Proof. From Lemma 1 it follows that there exists a subsequence of solutions U_ϵ, also denoted by U_ϵ, such that (6) is satisfied. Thus by Theorem 3 we get that there exists a function $\hat{u} \in H^1_{loc}(Z)$ such that the statement of the theorem is true. By assumption (A10) we get that $\hat{u} \in \hat{K}$.

If $\nu = 0$ and $\mu > 0$ then, by Theorem 3, there exists a function $\hat{w} \in L^2((-\mu, \mu); H^1(S))$ such that up to extraction of a subsequence, (11) holds. Then by assumption (A11) we get that $\hat{w} \in \hat{L}$. ■

Lemma 2 *Let* u_ϵ *be one solution of the variational inequality (1),* $\hat{u}_\epsilon$ *defined by (8). Assume that (A1)-(A3) and (A5) hold. Then*

$$\left\| A^0\left(\frac{\cdot}{\mu},\cdot\right)\Phi^0_\epsilon\left(\frac{\cdot}{\mu},\cdot,\hat{u}_\epsilon(\cdot)\right)B^0\left(\frac{\cdot}{\mu},\cdot\right)\nabla\hat{u}_\epsilon(\cdot)\right\|_{L^2(Z^0)}$$

is bounded.

Proof. Taking the square of the first growth condition from (A5), multiplying by $\frac{1}{\epsilon^2}$, and integrating on Ω^0_ϵ, we obtain

$$\frac{1}{\epsilon^2}\int_{\Omega^0_\epsilon}\|A_\epsilon(x)\Phi(x,u_\epsilon(x))B_\epsilon(x)\nabla u_\epsilon(x)\|^2\,dx \leq$$
$$\leq \frac{1}{\epsilon^2}\int_{\Omega^0_\epsilon}\|\nabla u_\epsilon(x)\|^2\,dx+\frac{1}{\epsilon^2}\int_{\Omega^0_\epsilon}|u_\epsilon(x)|^2\,dx+\frac{|\Omega^0_\epsilon|}{\epsilon^2}\|\alpha\|_\infty .$$

Applying the change of variable z_ϵ and taking out $\frac{1}{r_\epsilon^2}$ from $\hat{\nabla}^\epsilon\hat{u}_\epsilon$, we get

$$\int_{Z^0}\left\|A^0\left(\frac{z_1}{\mu},z'\right)\Phi^0_\epsilon\left(\frac{z_1}{\mu},z',\hat{u}_\epsilon(z)\right)B^0\left(\frac{z_1}{\mu},z'\right)\nabla\hat{u}_\epsilon(z)\right\|^2\,dz \leq$$
$$\leq C\int_{Z^0}\left\|\left(\frac{\partial\hat{u}_\epsilon(z)}{\partial z_1},\frac{r_\epsilon}{\epsilon}\frac{\partial\hat{u}_\epsilon(z)}{\partial z_2},\frac{r_\epsilon}{\epsilon}\frac{\partial\hat{u}_\epsilon(z)}{\partial z_3}\right)\right\|^2\,dz+r_\epsilon^4C\int_{Z^0}|\hat{u}_\epsilon(z)|^2\,dz+\bar{\alpha}.$$

By Theorem 3, $\|\nabla\hat{u}_\epsilon\|_{L^2(Z^0)^3}$ and $\|\hat{u}_\epsilon\|_{L^2(Z^0)}$ are bounded, thus the statement of the lemma holds. ■

Corollary 1 *Suppose that the assumptions of Lemma 2 are verified. Then there exists* $\sigma^0\in L^2(Z^0)$ *such that*

$$A^0\left(\frac{\cdot}{\mu},\cdot\right)\Phi^0_\epsilon\left(\frac{\cdot}{\mu},\cdot,\hat{u}_\epsilon(\cdot)\right)B^0\left(\frac{\cdot}{\mu},\cdot\right)\nabla\hat{u}_\epsilon(\cdot)\rightharpoonup\sigma^0 \quad in\ L^2(Z^0).$$

4 The limit variational inequality

In this section we obtain the limit problem in two cases: when $0<\mu<+\infty$ and $\nu=0$ respectively when $\mu=+\infty$ and $0<\nu<+\infty$. In these cases

$$\frac{\epsilon r_\epsilon}{t_\epsilon}=\frac{\epsilon}{r_\epsilon}\cdot\frac{r_\epsilon^2}{t_\epsilon}\to\frac{\nu}{\mu}=0,$$

thus the beam has a thin neck.

4.1 The case $0 < \mu < \infty$ and $\nu = 0$

Theorem 5 *Let $0 < \mu < \infty$ and $\nu = 0$.*

Assume that (A1)-(A11) are verified and the following four conditions are satisfied:

(C1) $\varphi \in K$ *implies* $\chi_{Y^1_\epsilon} \varphi \in K_\epsilon$*;*

(C2) $\psi \in L$ *implies* $\chi_{Y^1_\epsilon} \psi \in K_\epsilon$*;*

(C3) $\hat{\varphi} \in \hat{K}$ *implies* $\chi_{Z^0_\epsilon} \hat{\varphi} \in \hat{K}_\epsilon$*;*

(C4) $\hat{\psi} \in \hat{L}$ *implies* $\chi_{Z^0_\epsilon} \hat{\psi} \in \hat{K}_\epsilon$.

Then the following three statements hold:

1) There exists a subsequence of the sequence U_ϵ of solutions of (1), also denoted by U_ϵ, and a function $u \in H^1((-1,0) \cup (0,1)) \cap K$ such that (7) is satisfied.

2) Let u and w be as given in Theorem 1 and $\hat{u}$ and $\hat{w}$ as in Theorem 4. Then $(u, w, \hat{u}, \hat{w})$ solves the limit variational problem:
find $u \in H^1((-1,0) \cup (0,1)) \cap K$, $u(-1) = u(1) = 0$, $w \in L$, and $\hat{u} \in \hat{K}$, $\hat{u}(-\mu) = u(0^-)$, $\hat{u}(\mu) = u(0^+)$, $\hat{w} \in \hat{L}$ such that for all $v \in H^1((-1,0) \cup (0,1)) \cap K$, $v(-1) = v(1) = 0$, $h \in L$, and $\hat{v} \in \hat{K}$, $\hat{v}(-\mu) = v(0^-)$, $\hat{v}(\mu) = v(0^+)$, $\hat{h} \in \hat{L}$,

$$\int_{Y^1} [A^1(y)\Phi^1(y, u(y_1))B^1(y)\nabla'(u, w)(y), \nabla'(v, h)(y) - \nabla'(u, w)(y)] \quad (12)$$
$$+ \int_{Z^0} \left[A^0\left(\frac{z_1}{\mu}, z'\right) \Phi^0\left(\frac{z_1}{\mu}, z', \hat{u}(z)\right) B^0\left(\frac{z_1}{\mu}, z'\right) \nabla'(\hat{u}, \hat{w})(z), \right.$$
$$\left. \nabla'(\hat{v}, \hat{h})(z) - \nabla'(\hat{u}, \hat{w})(z) \right] \, dz \geq 0.$$

3) Let σ^1 be as given in Theorem 1, σ^0 as given in Corollary 1. Then

$$\sigma^1(y) = A^1(y)\Phi^1(y, u(y))B^1(y)\nabla'(u, w)(y) \quad \textit{for a.e. } y \in Y^1,$$

$$\sigma^0(z) = A^0\left(\frac{z_1}{\mu}, z'\right) \Phi^0\left(\frac{z_1}{\mu}, z', \hat{u}(z)\right) B^0\left(\frac{z_1}{\mu}, z'\right) \nabla'\left(\hat{u}, \frac{1}{\nu}\hat{u}\right)$$

for a.e. $z \in Z^0$.

Proof. Statement 1) follows from Theorem 2.

2) Since $\nu = 0$, from Theorem 4 it follows that $\hat{u} \in \hat{K}$ only depends on z_1 with
$$\hat{u} = u(0^-) \text{ in } Z^-, \quad \hat{u} = u(0^+) \text{ in } Z^+,$$
and there exists a function $\hat{w} \in \hat{L}$ such that up to extraction of a subsequence,
$$\frac{r_\epsilon}{\epsilon}\nabla_{z'}\hat{u}_\epsilon \rightharpoonup \nabla_{z'}\hat{w} \quad \text{in } L^2(Z^0)^2 \text{ weakly.}$$
Let $\varphi^- \in H^1([-1,0])$ and $\varphi^+ \in H^1([0,1])$ and define $\varphi \in H^1((-1,0) \cup (0,1)) \cap K$ such that
$$\varphi(x_1) = \begin{cases} \varphi^-(x_1), & \text{if } x_1 \in (-1,0) \\ \varphi^+(x_1), & \text{if } x_1 \in (0,1). \end{cases}$$
Let $\psi \in L$, $\hat{\varphi} \in \hat{K}$, and $\hat{\psi} \in \hat{L}$. For ϵ small enough, the sequence V_ϵ defined by
$$V_\epsilon(x) = \chi_{\Omega_\epsilon^1}(x)\left(\varphi(x_1) + \epsilon\psi\left(x_1, \frac{x'}{\epsilon}\right)\right) + \\ + \chi_{\Omega_\epsilon^0}(x)\left(\hat{\varphi}\left(\frac{\mu x_1}{t_\epsilon}\right) + \frac{\epsilon}{r_\epsilon}\hat{\psi}\left(\frac{\mu x_1}{t_\epsilon}, \frac{x'}{\epsilon r_\epsilon}\right)\right), \quad \text{a.e. } x \in \Omega_\epsilon$$
belongs to M_ϵ.

Putting $\eta = U_\epsilon(x)$, $\xi = \nabla U_\epsilon(x)$ and
$$\tau = \tau_\epsilon(x) = \chi_{\Omega_\epsilon^1}(x)(\nabla'(\varphi,\psi) + \lambda f_1)(y_\epsilon(x)) + \\ + \chi_{\Omega_\epsilon^0}(x)\frac{1}{r_\epsilon^2}(\nabla'(\hat{\varphi},\hat{\psi}) + \lambda f_2)(z_\epsilon(x)), \quad \text{a.e. } x \in \Omega_\epsilon$$
in the monotonicity condition, we get
$$0 \le \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} [A_\epsilon(x)\Phi_\epsilon(x,U_\epsilon(x))B_\epsilon(x)\nabla U_\epsilon(x) - A_\epsilon(x)\Phi_\epsilon(x,U_\epsilon(x))B_\epsilon(x)\tau_\epsilon(x), \\ \nabla U_\epsilon(x) - \tau_\epsilon(x)] \, dx = \\ = \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} [A_\epsilon(x)\Phi_\epsilon(x,U_\epsilon(x))B_\epsilon(x)\nabla U_\epsilon(x), \nabla U_\epsilon(x)] \, dx - \\ - \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} [A_\epsilon(x)\Phi_\epsilon(x,U_\epsilon(x))B_\epsilon(x)\nabla U_\epsilon(x), \tau_\epsilon(x)] \, dx + \\ - \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} [A_\epsilon(x)\Phi_\epsilon(x,U_\epsilon(x))B_\epsilon(x)\tau_\epsilon(x), \nabla U_\epsilon(x)] \, dx - \\ + \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} [A_\epsilon(x)\Phi_\epsilon(x,U_\epsilon(x))B_\epsilon(x)\tau_\epsilon(x), \tau_\epsilon(x)] \, dx = \\ = T_1^\epsilon - T_2^\epsilon - T_3^\epsilon + T_4^\epsilon.$$

In the following we study each term separately. The first term

$$
\begin{aligned}
T_1^\epsilon &= \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} [A_\epsilon(x)\Phi_\epsilon(x,U_\epsilon(x))B_\epsilon(x)\nabla U_\epsilon(x), \nabla U_\epsilon(x)]\ dx \leq \\
&\leq \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} [A_\epsilon(x)\Phi_\epsilon(x,U_\epsilon(x))B_\epsilon(x)\nabla U_\epsilon(x), \nabla V_\epsilon(x)]\ dx \\
&= \frac{1}{\epsilon^2}\int_{\Omega_\epsilon^1} \Big[A_\epsilon^1(y_\epsilon(x))\Phi_\epsilon^1(y_\epsilon(x),U_\epsilon(x))B_\epsilon^1(y_\epsilon(x))\nabla U_\epsilon(x), \\
&\left(\frac{d\varphi(x_1)}{dx_1} + \epsilon\frac{\partial\psi(y_\epsilon(x))}{\partial x_1}, \frac{\partial\psi(y_\epsilon(x))}{\partial x_2}, \frac{\partial\psi(y_\epsilon(x))}{\partial x_3}\right)\Big]\ dx + \\
&+\frac{1}{\epsilon^2}\int_{\Omega_\epsilon^0} \Big[A_\epsilon^0(z_\epsilon(x))\Phi_\epsilon^0(z_\epsilon(x),U_\epsilon(x))B_\epsilon^0(z_\epsilon(x))\nabla U_\epsilon(x), \\
&\left(\frac{\mu}{t_\epsilon}\frac{\partial\hat{\varphi}\left(\frac{\mu x_1}{t_\epsilon}\right)}{\partial x_1} + \frac{\epsilon\mu}{r_\epsilon t_\epsilon}\frac{\partial\hat{\psi}(z_\epsilon(x))}{\partial x_1}, \frac{1}{r_\epsilon^2}\frac{\partial\hat{\psi}(z_\epsilon(x))}{\partial x_2}, \frac{1}{r_\epsilon^2}\frac{\partial\hat{\psi}(z_\epsilon(x))}{\partial x_3}\right)\Big]\ dx
\end{aligned}
$$

(using the change of variable $y = y_\epsilon(x)$ in the integral over Ω_ϵ^1 and the change of variables $z = z_\epsilon(x)$ in the integral over Ω_ϵ^0)

$$
\begin{aligned}
&= \int_{Y_\epsilon^1} \Big[A^1(y)\Phi_\epsilon^1(y,u_\epsilon(y))B^1(y)\nabla^\epsilon u_\epsilon(y), \\
&\qquad \left(\frac{d\varphi(y_1)}{dy_1} + \epsilon\frac{\partial\psi(y)}{\partial y_1}, \frac{\partial\psi(y)}{\partial y_2}, \frac{\partial\psi(y)}{\partial y_3}\right)\Big]\ dy + \\
&+ \frac{1}{\mu}t_\epsilon r_\epsilon^2\int_{Z^0} \Big[A^0\left(\frac{z_1}{\mu},z'\right)\Phi_\epsilon^0\left(\frac{z_1}{\mu},z',\hat{u}(z)\right)B^0\left(\frac{z_1}{\mu},z'\right)\cdot \\
&\qquad\qquad \cdot\left(\frac{\mu}{t_\epsilon}\frac{\partial\hat{u}_\epsilon(z)}{\partial z_1}, \frac{1}{\epsilon r_\epsilon}\frac{\partial\hat{u}_\epsilon(z)}{\partial z_2}, \frac{1}{\epsilon r_\epsilon}\frac{\partial\hat{u}_\epsilon(z)}{\partial z_3}\right), \\
&\qquad\qquad \left(\frac{\mu}{t_\epsilon}\frac{d\hat{\varphi}(z_1)}{dz_1} + \frac{\epsilon}{r_\epsilon t_\epsilon}\frac{\partial\hat{\psi}(z)}{\partial z_1}, \frac{1}{r_\epsilon^2}\frac{\partial\hat{\psi}(z)}{\partial z_2}, \frac{1}{r_\epsilon^2}\frac{\partial\hat{\psi}(z)}{\partial z_3}\right)\Big]\ dz
\end{aligned}
$$

Taking the limit, we get

$$
T_1^\epsilon \to \int_{Y^1}\Big[\sigma^1(y), \nabla'(\varphi,\psi)(y)\Big]\ dy + \int_{Z^0}\Big[\sigma^0(z), \nabla'(\hat{\varphi},\hat{\psi})(z)\Big]\ dz.
$$

The second term

$$T_2^\epsilon = \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} [A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x))B_\epsilon(x)\nabla U_\epsilon(x), \tau_\epsilon(x)]\ dx \to$$
$$\to \int_{Y^1} \left[\sigma^1(y), (\nabla'(\varphi,\psi) + \lambda f_1)(y)\right]\ dy +$$
$$+ \int_{Z^0} \left[\sigma^0(z), (\nabla'(\hat\varphi,\hat\psi) + \lambda f_2)(z)\right]\ dz,$$

when ϵ tends to zero.

The third term

$$T_3^\epsilon = \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} [A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x))B_\epsilon(x)\tau_\epsilon(x), \nabla U_\epsilon(x)]\ dx \to$$
$$\to \int_{Y^1} \left[A^1(y)\Phi^1(y, u(y))B^1(y)(\nabla'(\varphi,\psi) + \lambda f_1)(y), \nabla'(u, w)(y)\right]\ dy +$$
$$+ \int_{Z^0} \left[A^0\left(\frac{z_1}{\mu}, z'\right)\Phi^0\left(\frac{z_1}{\mu}, z', \hat u(z)\right)B^0\left(\frac{z_1}{\mu}, z'\right)(\nabla'(\hat\varphi,\hat\psi) + \lambda f_2)(z),\right.$$
$$\left.\nabla'(\hat u, \hat w)(z),\right]\ dz,$$

when ϵ tends to zero.

The last term

$$T_4^\epsilon = \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} [A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x))B_\epsilon(x)\tau_\epsilon(x), \tau_\epsilon(x)]\ dx \to$$
$$\to \int_{Y^1} \left[A^1(y)\Phi^1(y, u(y))B^1(y)(\nabla'(\varphi,\psi) + \lambda f_1)(y),\right.$$
$$\left.(\nabla'(\varphi,\psi) + \lambda f_1)(y)\right]\ dy +$$
$$+ \int_{Z^0} \left[A^0\left(\frac{z_1}{\mu}, z'\right)\Phi^0\left(\frac{z_1}{\mu}, z', \hat u(z)\right)B^0\left(\frac{z_1}{\mu}, z'\right)(\nabla'(\hat\varphi,\hat\psi) + \lambda f_2)(z),\right.$$
$$\left.(\nabla'(\hat\varphi,\hat\psi) + \lambda f_2)(z)\right]\ dz,$$

when ϵ tends to zero.

Adding the limits of T_1^ϵ, T_2^ϵ, T_3^ϵ, and T_4^ϵ, we get

$$-\int_{Y^1} [\sigma^1(y), \lambda f_1(y)]\ dy - \int_{Z^0} [\sigma^0(z), \lambda f_2(z)]\ dz + \tag{13}$$
$$+ \int_{Y^1} [A^1(y)\Phi^1(y, u(y_1))B^1(y)(\nabla'(\varphi,\psi) + \lambda f_1)(y), \nabla'(\varphi,\psi)(y) -$$
$$- \nabla'(u, w)(y) + \lambda f_1(y)] +$$

$$+\int_{Z^0}\left[A^0\left(\frac{z_1}{\mu},z'\right)\Phi^0\left(\frac{z_1}{\mu},z',\hat{u}(z)\right)B^0\left(\frac{z_1}{\mu},z'\right)(\nabla'(\hat{\varphi},\hat{\psi})+\lambda f_2)(z),\right.$$
$$\left.\nabla'(\hat{\varphi},\hat{\psi})(z)-\nabla'(\hat{u},\hat{w})(z)+\lambda f_2(z),\right]\ dz\geq 0.$$

Setting

$$\varphi-u=\theta(v-u),\quad \psi-w=\theta(h-w),\quad \hat{\varphi}=\theta\hat{v},\quad \text{and } \hat{\psi}=\theta\hat{h},$$

where $\theta>0$, dividing by θ, then letting $\theta\to 0$, we get the limit variational inequality.

Putting

$$(\varphi,u)=(\psi,w)\quad \text{and}\quad (\hat{\varphi},\hat{u})=(\hat{\psi},\hat{w}),$$

dividing by λ, and letting $\lambda\to 0$, we get

$$\int_{Y^1}[\sigma^1(y)-A^1(y)\Phi^1(y,u(y_1))B^1(y)\nabla'(u,w)(y),f_1(y)]\ dy+$$
$$+\int_{Z^0}\left[\sigma^0(z)-A^0\left(\frac{z_1}{\mu},z'\right)\Phi^0\left(\frac{z_1}{\mu},z',\hat{u}(z)\right)B^0\left(\frac{z_1}{\mu},z'\right)\nabla'\left(\hat{u},\hat{w}\right)(z),\right.$$
$$\left. f_2(z)\right]\ dz\geq 0,\quad \forall f_1\in H^1(Y^1),\forall f_2\in H^1(Z).$$

Then 3) follows. ■

4.2 The case $\mu=+\infty$ and $0<\nu<+\infty$

Theorem 6 *Let $\mu=+\infty$ and $0<\nu<+\infty$. Assume that (A1)-(A9) are verified and the following two conditions are satisfied:*

(C1) $\varphi\in K$ *implies* $\chi_{Y^1_\epsilon}\varphi\in K_\epsilon$*;*

(C2) $\psi\in L$ *implies* $\chi_{Y^1_\epsilon}\psi\in K_\epsilon$*.*

Then the following three statements hold:

1) There exists a subsequence of the sequence U_ϵ of solutions of (1), also denoted by U_ϵ, and a function $u\in H^1((-1,0)\cup(0,1))\cap K$ such that (7) is satisfied.

2) Let u and w be given as in Theorem 1. Then (u,w) solves the limit variational problem:
find $u\in H^1((-1,0)\cup(0,1))\cap K$, $u(-1)=u(1)=0$ and $w\in L$ such that for all $v\in H^1((-1,0)\cup(0,1))\cap K$, $v(-1)=v(1)=0$ and $h\in L$

$$\int_{Y^1}[A^1(y)\Phi^1(y,u(y_1))B^1(y)\nabla'(u,w)(y),\nabla'(v,h)(y)-\nabla'(u,w)(y)]\geq 0. \tag{14}$$

3) Let σ^1 *given in Theorem 1. Then*

$$\sigma^1(y) = A^1(y)\Phi^1(y, u(y))B^1(y)\nabla'(u, w)(y) \quad \textit{for a.e. } y \in Y^1.$$

Proof. Statement 1) follows from Theorem 2.

To prove statement 2), let $\varphi^- \in H^1([-1, 0])$ and $\varphi^+ \in H^1([0, 1])$ and define $\varphi \in H^1((-1, 0) \cup (0, 1)) \cap K$ such that

$$\varphi(x_1) = \begin{cases} \varphi^-(x_1), & \text{if } \ x_1 \in (-1, 0) \\ \varphi^+(x_1), & \text{if } \ x_1 \in (0, 1). \end{cases}$$

Let $\psi \in L$ and $\gamma^0 : [0, +\infty) \to \mathbb{R}$ defined by

$$\gamma^0(\tau) = \begin{cases} \tau, & \text{if } 0 \le \tau \le 1 \\ 1, & \text{if } \tau \ge 1. \end{cases}$$

and

$$V_\epsilon(x) = \varphi(x_1)\gamma^0\left(\frac{|x_1|}{t_\epsilon}\right) + \epsilon\psi\left(x_1, \frac{x'}{\epsilon}\right), \quad \text{a.e } \in \Omega_\epsilon,$$

which belongs to M_ϵ.
For ϵ small enough, by a simple calculation we obtain

$$\frac{1}{\epsilon^2}\int_{\Omega_\epsilon^1}\left|\nabla V_\epsilon - \frac{d\varphi(x_1)}{dx_1}e_1 - \nabla_{y'}\psi\left(x_1, \frac{x'}{\epsilon}\right)\right| \, dx + \frac{1}{\epsilon^2}\int_{\Omega_\epsilon^0} |\nabla V_\epsilon| \, dx \le$$
$$\le C\left(\epsilon^2 + \frac{r_\epsilon^2}{t_\epsilon}\right)$$

which tends to zero since $\mu = +\infty$.

Putting $\eta = U_\epsilon(x)$, $\xi = \nabla U_\epsilon(x)$ and

$$\tau = \tau_\epsilon(x) = \begin{cases} (\nabla'(\varphi, \psi) + \lambda f_1)(y_\epsilon(x)), & \text{if } \ x \in \Omega_\epsilon^1 \\ 0, & \text{if } \ x \in \Omega_\epsilon^0 \end{cases}$$

in the monotonicity condition, we get

$$
\begin{aligned}
0 \leq \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} & [A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x))B_\epsilon(x)\nabla U_\epsilon(x) - A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x))B_\epsilon(x)\tau_\epsilon(x), \\
& \nabla U_\epsilon(x) - \tau_\epsilon(x)] \ dx = \\
= \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} & [A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x))B_\epsilon(x)\nabla U_\epsilon(x), \nabla U_\epsilon(x)] \ dx- \\
- \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} & [A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x))B_\epsilon(x)\nabla U_\epsilon(x), \tau_\epsilon(x)] \ dx- \\
- \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} & [A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x))B_\epsilon(x)\tau_\epsilon(x), \nabla U_\epsilon(x)] \ dx+ \\
+ \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} & [A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x))B_\epsilon(x)\tau_\epsilon(x), \tau_\epsilon(x)] \ dx = \\
= T_1^\epsilon - T_2^\epsilon & - T_3^\epsilon + T_4^\epsilon.
\end{aligned}
$$

In the following we study each term separately. The first term

$$
\begin{aligned}
T_1^\epsilon &= \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} [A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x))B_\epsilon(x)\nabla U_\epsilon(x), \nabla U_\epsilon(x)] \ dx \leq \\
&\leq \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} [A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x))B_\epsilon(x)\nabla U_\epsilon(x), \nabla V_\epsilon(x)] \ dx = \\
&= \frac{1}{\epsilon^2}\int_{\Omega_\epsilon^1} [A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x))B_\epsilon(x)\nabla U_\epsilon(x), \nabla V_\epsilon(x)] \ dx+ \\
&+ \frac{1}{\epsilon^2}\int_{\Omega_\epsilon^0} [A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x))B_\epsilon(x)\nabla U_\epsilon(x), \nabla V_\epsilon(x)] \ dx,
\end{aligned}
$$

where the second term tends to zero. We use the change of variables $y = y_\epsilon(x)$ in the first term:

$$
\begin{aligned}
T_1^\epsilon \leq \int_{Y_\epsilon^1} & \Big[A^1(y)\Phi_\epsilon^1(y, u_\epsilon(y))B^1(y)\nabla^\epsilon u_\epsilon(y), \\
& \left(\frac{d\varphi(y_1)}{dy_1} + \epsilon\frac{\partial\psi(y)}{\partial y_1}, \frac{\partial\psi(y)}{\partial y_2}, \frac{\partial\psi(y)}{\partial y_3}\right)\Big] \ dy + O_\epsilon =
\end{aligned}
$$

$$= \int_{Y^1} \Big[A^1(y)\Phi^1_\epsilon(y, u_\epsilon(y))B^1(y)\nabla^\epsilon u_\epsilon(y),$$
$$\left(\frac{d\varphi(y_1)}{dy_1} + \epsilon\frac{\partial\psi(y)}{\partial y_1}, \frac{\partial\psi(y)}{\partial y_2}, \frac{\partial\psi(y)}{\partial y_3}\right)\Big]\ dy + O_\epsilon.$$

Taking the limit of both sides, we get

$$\lim_{\epsilon\to 0} T_1^\epsilon \le \int_{Y^1} \Big[\sigma^1(y), \nabla'(\varphi,\psi)(y)\Big]\ dy.$$

The third term

$$T_3^\epsilon = \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} [A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x))B_\epsilon(x)\tau_\epsilon(x), \nabla U_\epsilon(x)]\ dx =$$
$$= \frac{1}{\epsilon^2}\int_{\Omega^1_\epsilon} \Big[A^1(y_\epsilon(x))\Phi_\epsilon(y_\epsilon(x), U_\epsilon(x))B^1(y_\epsilon(x))(\nabla'(\varphi,\psi) + \lambda f_1)(y_\epsilon(x)),$$
$$\nabla U_\epsilon(x)]\ dx,$$

as the integral on Ω^0_ϵ is equal with zero because $\tau_\epsilon = 0$ on Ω^0_ϵ. Using the change of variable $y = y_\epsilon(x)$ we get

$$T_3^\epsilon = \int_{Y^1_\epsilon} \Big[A^1(y)\Phi_\epsilon(y, u_\epsilon(y))B^1(y)(\nabla'(\varphi,\psi) + \lambda f_)(y), \nabla^\epsilon u_\epsilon(y)\Big]\ dy =$$
$$= \int_{Y^1} \Big[A^1(y)\Phi_\epsilon(y, u_\epsilon(y))B^1(y)(\nabla'(\varphi,\psi) + \lambda f_1)(y), \nabla^\epsilon u_\epsilon(y)\Big]\ dy + O_\epsilon.$$

Taking the limit when $\epsilon \to 0$, we get

$$T_3^\epsilon \to \int_{Y^1} \Big[A^1(y)\Phi(y, u(y_1))B^1(y)(\nabla'(\varphi,\psi) + \lambda f_1)(y), \nabla'(u, w)(y)\Big]\ dy.$$

Similarly

$$T_2^\epsilon \to \int_{Y^1} \Big[\sigma^1(y), (\nabla'(\varphi,\psi) + \lambda f_1)(y)\Big]\ dy$$

and

$$T_4^\epsilon \to \int_{Y^1} \Big[A^1(y)\Phi(y, u(y_1))B^1(y)(\nabla'(\varphi,\psi) + \lambda f_1)(y),$$
$$(\nabla'(\varphi,\psi) + \lambda f_1)(y)\Big]\ dy,$$

when $\epsilon \to 0$.

Adding the limits of T_1^ϵ, T_2^ϵ, T_3^ϵ, and T_4^ϵ, we get

$$\int_{Y^1} [A^1(y)\Phi^1(y,u(y_1))B^1(y)(\nabla'(\varphi,\psi)+\lambda f_1)(y), \nabla'(\varphi,\psi)(y) - \\ - \nabla'(u,w)(y) + \lambda f_1(y)]\, dz - \int_{Y^1} [\sigma^1(y), \lambda f_1(y)]\, dy \geq 0. \tag{15}$$

Setting

$$\varphi - u = \theta(v-u), \quad \text{and} \quad \psi - w = \theta(h-w),$$

where $\theta > 0$, dividing by θ, then letting $\theta \to 0$, we get the limit variational inequality.

3) Putting

$$(\varphi, u) = (\psi, w),$$

dividing by λ, and letting $\lambda \to 0$, we get

$$\int_{Y^1} [\sigma^1(y) - A^1(y)\Phi^1(y,u(y_1))B^1(y)\nabla'(u,w)(y), f_1(y)]\, dy \geq 0$$
$$\forall f_1 \in H^1(Y^1).$$

Then 3) follows. ■

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Oscillation of fast growing solutions of linear differential equations in the unit disc

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Abstract. In this paper, we investigate the relationship between solutions and their derivatives of the differential equation $f^{(k)} + A(z) f = 0$, $k \geq 2$, where $A(z) \not\equiv 0$ is an analytic function with finite iterated p-order and analytic functions of finite iterated p-order in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Instead of looking at the zeros of $f^{(j)}(z) - z$ $(j = 0, .., k)$, we proceed to a slight generalization by considering zeros of $f^{(j)}(z) - \varphi(z)$ $(j = 0, .., k)$, where φ is a small analytic function relative to f such that $\varphi^{(k-j)}(z) \not\equiv 0$ $(j = 0, ..., k)$, while the solution f is of infinite iterated p-order. This paper improves some very recent results of T. B. Cao and G. Zhang, A. Chen.

1 Introduction and statement of results

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's theory on the complex plane and in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ (see [13, 21, 23, 25, 26]). Many authors have investigated the growth and oscillation of the solutions of complex linear differential equations in $\mathbb{C}$ (see [2, 3, 4, 6, 9, 12, 16, 17, 18, 20, 24]). In the unit disc, there already exist many results [7, 8, 10, 11, 14, 15, 19, 22, 28], but the

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study is more difficult than that in the complex plane, because the efficient tool of Wiman-Valiron theory which doesn't hold in the unit disc.

We need to give some definitions and discussions. Firstly, let us give two definitions about the degree of small growth order of functions in Δ as polynomials on the complex plane $\mathbb{C}$. There are many types of definitions of small growth order of functions in Δ (i.e., see [10, 11]).

Definition 1 *Let* f *be a meromorphic function in* Δ*, and*

$$D(f) = \overline{\lim_{r\to 1^-}} \frac{T(r, f)}{-\log(1-r)} = b.$$

If $b < \infty$*, we say that* f *is of finite* b *degree (or is non-admissible); if* $b = \infty$*, we say that* f *is of infinite degree (or is admissible), both defined by characteristic function* $T(r, f)$*.*

Definition 2 *Let* f *be an analytic function in* Δ*, and*

$$D_M(f) = \overline{\lim_{r\to 1^-}} \frac{\log^+ M(r, f)}{-\log(1-r)} = a < \infty \quad (\textit{or } a = \infty),$$

then we say that f *is a function of finite* a *degree (or of infinite degree) defined by maximum modulus function* $M(r, f) = \max_{|z|=r} |f(z)|$*. Moreover, for* $F \subset [0, 1)$*, the upper and lower densities of* F *are defined by*

$$\overline{dens}_\Delta F = \overline{\lim_{r\to 1^-}} \frac{m(F\cap[0,r))}{m([0,r))}, \quad \underline{dens}_\Delta F = \underline{\lim_{r\to 1^-}} \frac{m(F\cap[0,r))}{m([0,r))}$$

respectively, where $m(G) = \int_G \frac{dt}{1-t}$ *for* $G \subset [0, 1)$*.*

Now we give the definitions of iterated order and growth index to classify generally the functions of fast growth in Δ as those in $\mathbb{C}$ (see [5, 16, 17]). Let us define inductively, for $r \in [0, 1)$, $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all r sufficiently large in $(0, 1)$, $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 3 (*see* [7, 8, 15]) *Let* f *be a meromorphic function in* Δ*. Then the iterated* p*-order of* f *is defined by*

$$\rho_p(f) = \overline{\lim_{r\to 1^-}} \frac{\log_p^+ T(r, f)}{-\log(1-r)} \quad (p \geq 1 \quad \textit{is an integer}),$$

where $\log_1^+ x = \log^+ x = \max\{\log x, 0\}$, $\log_{p+1}^+ x = \log^+ \log_p^+ x$. *For* $p = 1$, *this notation is called order and for* $p = 2$ *hyper-order* [14, 19].

Remark 1 *If* f *is analytic in* Δ, *then the iterated* p-*order of* f *is defined by*

$$\rho_{M,p}(f) = \varlimsup_{r \to 1^-} \frac{\log_{p+1}^+ M(r,\ f)}{-\log\ (1-r)} \quad (p \geq 1 \textit{ is an integer}).$$

Remark 2 *It follows by M. Tsuji* [23, *p.* 205] *that if* f *is an analytic function in* Δ, *then we have the inequalities*

$$\rho_1(f) \leqslant \rho_{M,1}(f) \leqslant \rho_1(f) + 1,$$

which are the best possible in the sense that there are analytic functions g *and* h *such that* $\rho_{M,1}(g) = \rho_1(g)$ *and* $\rho_{M,1}(h) = \rho_1(h) + 1$, *see* [11]. *However, it follows by Proposition 2.2.2 in* [17] *that* $\rho_{M,p}(f) = \rho_p(f)$ *for* $p \geq 2$.

Definition 4 ([8]) *The growth index of the iterated order of a meromorphic function* $f(z)$ *in* Δ *is defined by*

$$i(f) = \begin{cases} 0, & \text{if} \quad f \text{ is non-admissible,} \\ \min\{j \in \mathbb{N} : \rho_j(f) < +\infty\}, & \text{if} \quad f \text{ is admissible,} \\ +\infty, & \text{if} \quad \rho_j(f) = +\infty \text{ for all } \ j \in \mathbb{N}. \end{cases}$$

For an analytic function f *in* Δ, *we also define*

$$i_M(f) = \begin{cases} 0, & \text{if} \quad f \text{ is non-admissible,} \\ \min, \{j \in \mathbb{N} : \rho_{M,j}(f) < +\infty\}, & \text{if} \quad f \text{ is admissible,} \\ +\infty, & \text{if} \quad \rho_{M,j}(f) = +\infty \text{ for all } \ j \in \mathbb{N}. \end{cases}$$

Remark 3 *If* $\rho_p(f) < \infty$ *or* $i(f) \leq p$, *then we say that* f *is of finite iterated* p-*order; if* $\rho_p(f) = \infty$ *or* $i(f) > p$, *then we say that* f *is of infinite iterated* p-*order. In particular, we say that* f *is of finite order if* $\rho(f) < \infty$ *or* $i(f) \leq 1$; f *is of infinite order if* $\rho(f) = \infty$ *or* $i(f) > 1$.

Definition 5 ([7, 28]) *Let* f *be a meromorphic function in* Δ. *Then the iterated exponent of convergence of the sequence of zeros of* $f(z)$ *is defined by*

$$\lambda_p(f) = \varlimsup_{r \to 1^-} \frac{\log_p^+ N\left(r, \frac{1}{f}\right)}{-\log\ (1-r)}, \quad (p \geq 1 \quad \textit{is an integer}),$$

where $N\left(r,\frac{1}{f}\right)$ *is the counting function of zeros of* $f(z)$ *in* $\{|z|<r\}$*. For* $p=1$*, this notation is called exponent of convergence of the sequence of zeros and for* $p=2$ *hyper-exponent of convergence of the sequence of zeros.*

Similarly, the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ *is defined by*

$$\overline{\lambda}_p(f)=\varlimsup_{r\to 1^-}\frac{\log_p^+\overline{N}\left(r,\frac{1}{f}\right)}{-\log(1-r)},\qquad (p\geq 1 \ \ \textit{is an integer}),$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ *is the counting function of distinct zeros of* $f(z)$ *in* $\{|z|<r\}$*. For* $p=1$*, this notation is called exponent of convergence of the sequence of distinct zeros and for* $p=2$ *hyper-exponent of convergence of the sequence of distinct zeros.*

Definition 6 ([7, 28]) *Let* f *be a meromorphic function in* Δ*. Then the iterated exponent of convergence of the sequence of fixed points of* $f(z)$ *is defined by*

$$\tau_p(f)=\lambda_p(f-z)=\varlimsup_{r\to 1^-}\frac{\log_p^+ N\left(r,\frac{1}{f-z}\right)}{-\log(1-r)}\ \ (p\geq 1 \ \textit{is an integer}).$$

For $p=1$*, this notation is called exponent of convergence of the sequence of fixed points and for* $p=2$ *hyper-exponent of convergence of the sequence of fixed points. Similarly, the iterated exponent of convergence of the sequence of distinct fixed points of* $f(z)$ *is defined by*

$$\overline{\tau}_p(f)=\overline{\lambda}_p(f-z)=\varlimsup_{r\to 1^-}\frac{\log_p^+\overline{N}\left(r,\frac{1}{f-z}\right)}{-\log(1-r)}\ \ (p\geq 1 \ \ \textit{is an integer}).$$

For $p=1$*, this notation is called exponent of convergence of the sequence of distinct fixed points and for* $p=2$ *hyper-exponent of convergence of the sequence of distinct fixed points. Thus* $\overline{\tau}_p(f)=\overline{\lambda}_p(f-z)$ *is an indication of oscillation of distinct fixed points of* $f(z)$*.*

For $k\geq 2$, we consider the linear differential equation

$$f^{(k)}+A(z)f=0, \tag{1}$$

where $A(z)\not\equiv 0$ is an analytic function in the unit disc of finite iterated p-order. Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades, see [27]. However, there are few studies on the fixed points of solutions of differential equations, specially in the unit disc. In [9], Z.-X. Chen firstly studied the problem on the fixed points and hyper-order of solutions of second order linear differential equations with entire coefficients. After that, there were some results which improve those of Z.-X. Chen, see [3, 4, 18, 20, 24]. In [7], T. B. Cao firstly investigated the fixed points of solutions of linear complex differential equations in the unit disc. Very recently in [28], G. Zhang and A. Chen extended some results of [7] to the case of higher order linear differential equations with analytic coefficients and have obtained the following results.

Theorem 1 ([28]) *Let* H *be a set of complex numbers satisfying* $\overline{\mathrm{dens}}_{\Delta}\{|z| : z \in H \subseteq \Delta\} > 0$, *and let* $A(z) \not\equiv 0$ *be an analytic function in* Δ *such that* $\rho_{M,p}(A) = \sigma < +\infty$ *and for real number* $\alpha > 0$, *we have for all* $\varepsilon > 0$ *sufficiently small,*

$$|A(z)| \geq \exp_p \left\{ \alpha \left(\frac{1}{1-|z|} \right)^{\sigma - \varepsilon} \right\} \tag{2}$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of equation (1) satisfies*

$$\tau_p\left(f^{(j)}\right) = \overline{\tau}_p\left(f^{(j)}\right) = \rho_p(f) = +\infty \ \ (j = 0, \ldots, k), \tag{3}$$

$$\tau_{p+1}\left(f^{(j)}\right) = \overline{\tau}_{p+1}\left(f^{(j)}\right) = \rho_{p+1}(f) = \rho_{M,p}(A) = \sigma \ \ (j = 0, \ldots, k). \tag{4}$$

Theorem 2 ([28]) *Let* H *be a set of complex numbers satisfying* $\overline{\mathrm{dens}}_{\Delta}\{|z| : z \in H \subseteq \Delta\} > 0$, *and let* $A(z) \not\equiv 0$ *be an analytic function in* Δ *such that* $\rho_p(A) = \sigma < +\infty$ *and for real number* $\alpha > 0$, *we have for all* $\varepsilon > 0$ *sufficiently small,*

$$T(r, A(z)) \geq \exp_{p-1} \left\{ \alpha \left(\frac{1}{1-|z|} \right)^{\sigma - \varepsilon} \right\} \tag{5}$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of equation (1) satisfies*

$$\tau_p\left(f^{(j)}\right) = \overline{\tau}_p\left(f^{(j)}\right) = \rho_p(f) = +\infty \ \ (j = 0, \ldots, k), \tag{6}$$

$$\rho_{M,p}(A) \geq \tau_{p+1}\left(f^{(j)}\right) = \overline{\tau}_{p+1}\left(f^{(j)}\right) = \rho_{p+1}(f) \geq \sigma \ \ (j = 0, \ldots, k). \tag{7}$$

In the present paper, we continue to study the oscillation of solutions of equation (1) in the unit disc. The main purpose of this paper is to study the relation between solutions and their derivatives of the differential equation (1) and analytic functions of finite iterated p-order. We obtain an extension of Theorems 1-2. In fact, we prove the following results:

Theorem 3 *Assume that the assumptions of Theorem 1 hold. If $\varphi(z)$ is an analytic function in Δ such that $\varphi^{(k-j)}(z) \not\equiv 0$ $(j = 0, ..., k)$ with finite iterated p- order $\rho_p(\varphi) < +\infty$, then every solution $f(z) \not\equiv 0$ of (1), satisfies*

$$\lambda_p\left(f^{(j)} - \varphi\right) = \overline{\lambda}_p\left(f^{(j)} - \varphi\right) = \rho_p(f) = +\infty \ \ (j = 0, \ldots, k), \tag{8}$$

$$\lambda_{p+1}\left(f^{(j)} - \varphi\right) = \overline{\lambda}_{p+1}\left(f^{(j)} - \varphi\right) = \rho_{p+1}(f) = \rho_{M,p}(A) = \sigma \ \ (j = 0, \ldots, k). \tag{9}$$

Theorem 4 *Assume that the assumptions of Theorem 2 hold. If $\varphi(z)$ is an analytic function in Δ such that $\varphi^{(k-j)}(z) \not\equiv 0$ $(j = 0, \ldots, k)$ with finite iterated p- order $\rho_p(\varphi) < +\infty$, then every solution $f(z) \not\equiv 0$ of (1), satisfies*

$$\lambda_p\left(f^{(j)} - \varphi\right) = \overline{\lambda}_p\left(f^{(j)} - \varphi\right) = \rho_p(f) = +\infty \ \ (j = 0, \ldots, k), \tag{10}$$

$$\rho_{M,p}(A) \geq \lambda_{p+1}\left(f^{(j)} - \varphi\right) = \overline{\lambda}_{p+1}\left(f^{(j)} - \varphi\right) = \rho_{p+1}(f) \geq \sigma \ \ (j = 0, ..., k). \tag{11}$$

2 Auxiliary lemmas

We need the following lemmas in the proofs of our theorems.

Lemma 1 ([8]) *If f and g are meromorphic functions in Δ, $p \geq 1$ is an integer, then we have*
(i) $\rho_p(f) = \rho_p(1/f)$, $\rho_p(a.f) = \rho_p(f)$ $(a \in \mathbb{C} - \{0\})$;
(ii) $\rho_p(f) = \rho_p\left(f'\right)$;
(iii) $\max\{\rho_p(f+g), \rho_p(fg)\} \leq \max\{\rho_p(f), \rho_p(g)\}$;
(iv) if $\rho_p(f) < \rho_p(g)$, *then* $\rho_p(f+g) = \rho_p(g)$, $\rho_p(fg) = \rho_p(g)$.

Lemma 2 ([14]) *Let f be a meromorphic function in the unit disc Δ, and let $k \geq 1$ be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f), \tag{12}$$

where $S(r,f) = O\left(\log^+ T(r,f)) + \log \frac{1}{1-r}\right)$, *possibly outside a set* $E \subset [0,1)$ *with* $\int_E \frac{dr}{1-r} < +\infty$. *If f is of finite order (namely, finite iterated* 1*-order) of growth, then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log(\frac{1}{1-r})\right). \tag{13}$$

Lemma 3 *Let* f *be a meromorphic function in the unit disc* Δ *for which* $i(f) = p \geq 1$ *and* $\rho_p(f) = \beta < +\infty$, *and let* $k \geq 1$ *be an integer. Then for any* $\varepsilon > 0$,

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right) \tag{14}$$

holds for all r *outside a set* $E \subset [0,1)$ *with* $\int_E \frac{dr}{1-r} < +\infty$.

Proof. First for $k = 1$. Since $\rho_p(f) = \beta < +\infty$, we have for all $r \to 1^-$

$$T(r,f) \leq \exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}. \tag{15}$$

By Lemma 2, we have

$$m\left(r, \frac{f'}{f}\right) = O\left(\ln^+ T(r,f) + \ln \frac{1}{1-r}\right) \tag{16}$$

holds for all r outside a set $E \subset [0,1)$ with $\int_E \frac{dr}{1-r} < +\infty$. Hence, we have

$$m\left(r, \frac{f'}{f}\right) = O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right), \quad r \notin E. \tag{17}$$

Next, we assume that we have

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right), \quad r \notin E \tag{18}$$

for some integer $k \geq 1$. Since $N\left(r, f^{(k)}\right) \leq (k+1)N(r,f)$, it holds that

$$\begin{aligned} T\left(r, f^{(k)}\right) &= m\left(r, f^{(k)}\right) + N\left(r, f^{(k)}\right) \leq \\ &\leq m\left(r, \frac{f^{(k)}}{f}\right) + m(r,f) + (k+1)N(r,f) \end{aligned}$$

$$\leq O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right)+(k+1)\,T(r,f)=O\left(\exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right). \tag{19}$$

By (16) and (19) we again obtain

$$m\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right)=O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right),\ \ r\notin E \tag{20}$$

and hence,

$$\begin{aligned} m\left(r,\frac{f^{(k+1)}}{f}\right) &\leq m\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right)+m\left(r,\frac{f^{(k)}}{f}\right)= \\ &= O\left(\exp_{p-2}\left\{\frac{1}{1-r}\right\}^{\beta+\varepsilon}\right),\ \ r\notin E. \end{aligned} \tag{21}$$

■

Lemma 4 ([1]) *Let* $g:(0,1)\to\mathbb{R}$ *and* $h:(0,1)\to\mathbb{R}$ *be monotone increasing functions such that* $g(r)\leq h(r)$ *holds outside of an exceptional set* $E\subset[0,1)$ *of finite logarithmic measure. Then there exists a* $d\in(0,1)$ *such if* $s(r)=1-d(1-r)$, *then* $g(r)\leq h(s(r))$ *for all* $r\in[0,1)$.

Lemma 5 *Let* A_0, A_1,..., A_{k-1}, $F\not\equiv 0$ *be finite iterated* p- *order analytic functions in the unit disc* Δ. *If* f *is a solution with* $\rho_p(f)=+\infty$ *and* $\rho_{p+1}(f)=\rho<+\infty$ *of the*

$$f^{(k)}+A_{k-1}f^{(k-1)}+...+A_1f'+A_0f=F, \tag{22}$$

then $\overline{\lambda}_p(f)=\lambda_p(f)=\rho_p(f)=+\infty$ *and* $\overline{\lambda}_{p+1}(f)=\lambda_{p+1}(f)=\rho_{p+1}(f)=\rho$.

Proof. Since $A_0, A_1,\ldots, A_{k-1}$, $F\not\equiv 0$ are analytic in Δ, then all solutions of (22) are analytic in Δ (see [14]). By (22), we can write

$$\frac{1}{f}=\frac{1}{F}\left(\frac{f^{(k)}}{f}+A_{k-1}\frac{f^{(k-1)}}{f}+\ldots+A_1\frac{f'}{f}+A_0\right). \tag{23}$$

If f has a zero at $z_0 \in \Delta$ of order $\gamma\,(> k)$, then F must have a zero at z_0 of order at least $\gamma - k$. Hence,

$$N\left(r, \frac{1}{f}\right) \leq k\,\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right). \tag{24}$$

By (23), we have

$$m\left(r, \frac{1}{f}\right) \leq \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + \sum_{j=0}^{k-1} m\left(r, A_j\right) + m\left(r, \frac{1}{F}\right) + O\left(1\right). \tag{25}$$

Applying the Lemma 3, we have

$$m\left(r, \frac{f^{(j)}}{f}\right) = O\left(\exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right) \quad \left(j = 1, \ldots, k\right), \tag{26}$$

where $\rho_{p+1}\left(f\right) = \rho < +\infty$, holds for all r outside a set $E \subset [0,1)$ with $\int_E \frac{dr}{1-r} < +\infty$. By (24), (25) and (26) we get

$$\begin{aligned} T\left(r, f\right) &= T\left(r, \frac{1}{f}\right) + O\left(1\right) \leq k\overline{N}\left(r, \frac{1}{f}\right) + \sum_{j=0}^{k-1} T\left(r, A_j\right) + T\left(r, F\right) + \\ &\quad + O\left(\exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right) \quad \left(|z| = r \notin E\right). \end{aligned} \tag{27}$$

Set

$$\mu = \max\left\{\rho_p\left(A_j\right) \quad \left(j = 0, \ldots, k-1\right), \rho_p\left(F\right)\right\}.$$

Then for $r \to 1^-$, we have

$$T\left(r, A_0\right) + \ldots + T\left(r, A_{k-1}\right) + T\left(r, F\right) \leq \left(k+1\right)\exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\mu+\varepsilon}. \tag{28}$$

Thus, by (27) and (28), we have for $r \to 1^-$

$$\begin{aligned} T\left(r, f\right) &\leq k\,\overline{N}\left(r, \frac{1}{f}\right) + \left(k+1\right)\exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\mu+\varepsilon} + \\ &\quad + O\left(\exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right) = \\ &= k\,\overline{N}\left(r, \frac{1}{f}\right) + O\left(\exp_{p-1}\left\{\frac{1}{1-r}\right\}^{\eta}\right), \quad \left(|z| = r \notin E\right). \end{aligned} \tag{29}$$

where $\eta < +\infty$. Hence for any f with $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho$, by Lemma 4 and (29), we have

$$\lambda_p(f) \geq \overline{\lambda}_p(f) \geq \rho_p(f) = +\infty$$

and $\lambda_{p+1}(f) \geq \overline{\lambda}_{p+1}(f) \geq \rho_{p+1}(f)$. Since $\overline{\lambda}_{p+1}(f) \leq \lambda_{p+1}(f) \leq \rho_{p+1}(f)$, we have $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) = \rho$. ■

Lemma 6 ([7]) *Let* H *be a set of complex numbers satisfying* $\overline{\mathrm{dens}}_{\Delta}\{|z| : z \in H \subseteq \Delta\} > 0$, *and let* $A(z) \not\equiv 0$ *be an analytic function in* Δ *such that* $\rho_{M,p}(A) = \sigma < +\infty$ *and for real number* $\alpha > 0$, *we have for all* $\varepsilon > 0$ *sufficiently small,*

$$|A(z)| \geq \exp_p\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}, \tag{30}$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of equation (1) satisfies* $\rho_p(f) = +\infty$ *and* $\rho_{p+1}(f) = \rho_{M,p}(A) = \sigma$.

Lemma 7 ([7]) *Let* H *be a set of complex numbers satisfying* $\overline{\mathrm{dens}}_{\Delta}\{|z| : z \in H \subseteq \Delta\} > 0$, *and let* $A(z) \not\equiv 0$ *be an analytic function in* Δ *such that* $\rho_p(A) = \sigma < +\infty$ *and for real number* $\alpha > 0$, *we have for all* $\varepsilon > 0$ *sufficiently small,*

$$T(r, A(z)) \geq \exp_{p-1}\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}, \tag{31}$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of equation (1) satisfies* $\rho_p(f) = +\infty$ *and* $\rho_{M,p}(A) \geq \rho_{p+1}(f) \geq \sigma$.

3 Proof of Theorem 3

Suppose that $f(z) \not\equiv 0$ is a solution of equation (1). Then by Lemma 6, we have $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho_{M,p}(A) = \sigma$. Set $w_j = f^{(j)} - \varphi \quad (j = 0, 1, \ldots, k)$. Since $\rho_p(\varphi) < +\infty$, then by Lemma 1, we have $\rho_p(w_j) = \rho_p(f) = +\infty$, $\rho_{p+1}(w_j) = \rho_{p+1}(f) = \rho_{M,p}(A) = \sigma$, $\lambda_p(w_j) = \lambda_p\left(f^{(j)} - \varphi\right)$, $\overline{\lambda}_p(w_j) = \overline{\lambda}_p\left(f^{(j)} - \varphi\right)$ $(j = 0, 1, \ldots, k)$. Differentiating both sides of $w_j = f^{(j)} - \varphi$ and replacing $f^{(k)}$ with $f^{(k)} = -Af$, we obtain

$$w_j^{(k-j)} = -Af - \varphi^{(k-j)} \quad (j = 0, 1, \ldots, k). \tag{32}$$

Then, we have

$$f = -\frac{w_j^{(k-j)} + \varphi^{(k-j)}}{A}. \tag{33}$$

Substituting (33) into equation (1), we get

$$\left(\frac{w_j^{(k-j)}}{A}\right)^{(k)} + w_j^{(k-j)} = -\left(\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + \varphi^{(k-j)}\right). \tag{34}$$

By (34), we can write

$$\begin{aligned} w_j^{(2k-j)} &+ \Phi_{2k-j-1} w_j^{(2k-j-1)} + \ldots + \Phi_{k-j} w_j^{(k-j)} \\ &= -A\left(\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + A\left(\frac{\varphi^{(k-j)}}{A}\right)\right), \end{aligned} \tag{35}$$

where $\Phi_{k-j}(z), \ldots, \Phi_{2k-j-1}(z)$ $(j = 0, 1, \ldots, k)$ are analytic functions with

$$\rho_{M,p}(\Phi_{k-j}) \leq \sigma, \ldots, \rho_{M,p}(\Phi_{2k-j-1}) \leq \sigma \quad (j = 0, 1, \ldots, k).$$

By Lemma 1 we have $\rho_p\left(\frac{\varphi^{(k-j)}}{A}\right) < +\infty$. Thus, by $A \not\equiv 0$ $\varphi^{9k-j)} \not\equiv 0$, $(j = 0, \ldots, k)$ and Lemma 6, we obtain

$$-A\left(\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + A\left(\frac{\varphi^{(k-j)}}{A}\right)\right) \not\equiv 0. \tag{36}$$

Hence, by Lemma 5, we have $\lambda_p(w_j) = \overline{\lambda}_p(w_j) = \rho_p(w_j) = +\infty$ and $\lambda_{p+1}(w_j) = \overline{\lambda}_{p+1}(w_j) = \rho_{p+1}(w_j) = \rho_{M,p}(A) = \sigma$ $(j = 0, 1, \ldots, k)$. Thus

$$\lambda_p\left(f^{(j)} - \varphi\right) = \overline{\lambda}_p\left(f^{(j)} - \varphi\right) = \rho_p(f) = +\infty \ \ (j = 0, 1, \ldots, k),$$

$$\lambda_{p+1}\left(f^{(j)} - \varphi\right) = \overline{\lambda}_{p+1}\left(f^{(j)} - \varphi\right) = \rho_{p+1}(f) = \rho_{M,p}(A) = \sigma \ \ (j = 0, 1, \ldots, k).$$

4 Proof of Theorem 4

Suppose that $f(z) \not\equiv 0$ is a solution of equation (1). Then by Lemma 7, we have $\rho_p(f) = +\infty$ and $\rho_{M,p}(A) \geq \rho_{p+1}(f) \geq \sigma$. By using similar reasoning as in the proof of Theorem 3, we obtain Theorem 4.

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Global dynamics of a HIV transmission model

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Abstract. In this paper a simple mathematical model will be considered describing transmission dynamics of the human immunodeficiency virus in a special situation. A unique interior equilibrium is found and its stability is investigated. Results are verified by computer simulation.

1 Introduction

Acquired immunodeficiency syndrome (AIDS) was firstly recognized in 1981 among men who have sex with men (MSM) in the United States of America (USA), and shortly thereafter in populations such as injective drug users (IDU), hemophiliacs and blood transfusion recipients and infants of women with AIDS (cf. [12]). By 1983 the viral cause of AIDS, the human immunodeficiency virus (HIV) had been discovered and the basic models of transmission established: sexual transmission, parenteral explosure to blood and blood products, and perinatal transmission (cf. [7], pp. 3–17). Because HIV is primarily a sexual transmitted disease (STD), its spread reflects the social patterning of human sexual relationships. The understanding of the long-time behaviour of STD-s will help to find whether this epidemics will die out or stay in the population and to design strategies of fighting them. Various approaches for studying epidemiology of STD have been developed from time

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to time. Since the famous Kermack-McKendrick model for a spread of disease (cf. [9]), differential equations have been widely used to study a disease transmission, to evaluate the spread of epidemics, and more importantly, to understand the mechanisms of epidemics in order to prevent them or minimise the transmission of this disease (cf. [1, 3, 5, 8]).

AIDS is a major public health problem in the USA. The epidemic in Europe has shown similar trends to those in the USA. Transmission has been the greatest among MSM and IDUs. More new cases of AIDS are reported each year among MSM than for any other group. The majority of MSM practice anal intercourse (cf. [10]) and this activity occurs within a community with a considerable prevalence of HIV infection (cf. [10, 14]). Within the MSM community there are high risk sexual zones (such as bathhouses) where MSM congregate for sexual activity (cf. [15]). Bathhouses are a feature of most major North American and European metropolitan areas, e.g. in New York City (NYC), Los Angeles.

Several models have been proposed in this area, e.g. one for treating IDUs and perinatal transmission in NYC (cf. [2]) and recently one for those with HIV transmission among MSM in a bathhouse (cf. [4]):

$$\left\{ . \begin{array}{lcl} \dot{S} & = & f(S,I,E) := \pi - \beta \dfrac{SI}{N} - \mu S \\ \dot{I} & = & g(S,I,E) := \rho - \mu I \\ \dot{E} & = & h(S,I,E) := \sigma + \beta \dfrac{SI}{N} - \mu E \end{array} \right. , \tag{1}$$

where the dot means differentiation with respect to time t. The total number of visitors in the bathhouse at any time, $N(t)$, is subdivided into three parts: susceptibles (i.e., the uninfecteds), $S(t)$, the HIV-infecteds, $I(t)$, and the HIV-exposeds $E(t)$. The biological meaning of the parameters in (1) is the following: $\pi > 0$, $\rho \geq 0$ and $\sigma \geq 0$ are the inflow rates of susceptibles, infecteds and exposeds, respectively; the average time spent by individuals in the bathhouse is $1/\mu$ with $\mu > 0$. The transfer mechanism from the class of susceptibles to the class of exposeds is guided by the fraction $\beta I/N$, where the

$$\beta = c_i\beta_i\,(I - \eta^c\psi_i^c)\,(1 - \eta^m\psi_i^m) + c_r\beta_r\,(1 - \eta^c\psi_r^c)\,(1 - \eta^m\psi_r^m)$$

denotes the probability of HIV transmission. For the detailed meaning of the parameters in β we refer to the Table 1 in [4]. We note that from the section explaining the dependence on the parameters, β seems to be I-independent, while in its definition the I-dependence is present. The scalar β would essentially simplify the system while the linear I-dependence can be considered as the fact that the probability of HIV transmission is (linearly) proportional to the size of the infected population. Thus the scalar case can be considered as a specialization of this extended version.

The aim of our study is to show that the above (extended) model is well posed and it has a unique equilibrium which is globally asymptotically stable.

2 Equilibria and their stability

We shall present some results, including the positivity and boundedness of solutions, furthermore the existence and stability of possible equilibria.

First of all, Picard–Lindelöf's Theorem guarantees that solutions of the initial value problem for system (1) exist locally and are unique.

Clearly, the interior of the positive octant of the phase space $[S, I, E]$, denoted by $\mathbb{R}^3_+$, is an invariant region. Indeed, $f(0, I, E) \equiv \pi$, $g(S, 0, E) \equiv \rho$ and $h(S, I, 0) = \sigma + \beta S/(S + I)$, thus the time derivatives of S, I and E are positive at the boundary – provided as the inflow rates ρ, σ are positive – which implies nonnegativity. Hence, for the rest of the paper we only focus on system (1) restricted to $\mathbb{R}^3_+$.

Setting the right hand sides of the three differential equations of (1) equal to zero, we find that system (1) has only one equilibrium which lies in the interior of the positive octant of the phase space $[S, I, E]$:

$$\begin{aligned} S^* &:= \frac{\pi\mu(\pi+\rho+\sigma)}{\kappa_{ri}\mu\rho + \kappa_i\rho^2 + \mu^2(\pi+\rho+\sigma)}, \\ I^* &:= \frac{\rho}{\mu}, \\ E^* &:= \frac{\sigma}{\mu} + \frac{\pi\rho(\kappa_{ri}\mu + \kappa_i\rho)}{\kappa_{ri}\mu^2\rho + \kappa_i\mu\rho^2 + \mu^3(\pi+\rho+\sigma)}, \end{aligned} \tag{2}$$

where $\kappa_{ri} := c_r\beta_r(1-\eta^c\psi^c_r)(1-\eta^m\psi^m_r) - c_i\beta_i\eta^c\psi^c_i(1-\eta^m\psi^m_i)$ and $\kappa_i := c_i\beta_i(1-\eta^m\psi^m_i)$.

Theorem 1 (S^*, I^*, E^*) *is a locally asymptotically stable equilibrium of system (1).*

Proof. If we linearize the system at this equilibrium then the characteristic polynomial turns out to be

$$p(\lambda) :\equiv \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0, \tag{3}$$

where

$$\begin{aligned}
a_0 &:= \mu\left(\mu^2 + \frac{\rho\left(\kappa_{ri}\mu + \kappa_i\rho\right)}{\pi+\rho+\sigma}\right), \\
a_1 &:= 3\mu^2 + \frac{2\rho\left(\kappa_{ri}\mu + \kappa_i\rho\right)}{\pi+\rho+\sigma}, \\
a_2 &:= 3\mu + \frac{\rho\left(\kappa_{ri}\mu + \kappa_i\rho\right)}{\mu(\pi+\rho+\sigma)},
\end{aligned}$$

which is stable by Routh-Hurwitz criterion, because it has only positive coefficients and

$$\begin{aligned}
a_2 a_1 - a_0 &= 2\left[\kappa_{ri}\rho\mu + \kappa_i\rho^2 + \mu^2(\pi+\rho+\sigma)\right] \\
&\quad \times \frac{\kappa_{ri}\rho\mu + \kappa_i\rho^2 + 4\mu^2(\pi+\rho+\sigma)}{\mu(\pi+\rho+\sigma)^2} > 0,
\end{aligned}$$

which proves the local stability. ■

Calculating the second additive compound matrix (see e.g. [13]) of the Jacobian of the right hand side of (1), we have

$$J^{[2]}(S,I,E) = \begin{bmatrix} -2\mu - a & 0 & -b \\ c & -2\mu - a - b & -c \\ -a & 0 & -2\mu - b \end{bmatrix},$$

where the parameters a, b, c are defined for arbitrary $(S,I,E) \in \mathbb{R}^3_+$ as follows:

$$\begin{aligned}
a &:= a(S,I,E) := \frac{I(\kappa_{ri} + \kappa_i I)(I+E)}{(S+I+E)^2} > 0, \\
b &:= b(S,I,E) := \frac{S(\kappa_{ri}(S+E) + \kappa_i I(2S+I+2E))}{(S+I+E)^2} > 0, \\
c &:= c(S,I,E) := \frac{SI(\kappa_{ri} + \kappa_i I)}{(S+I+E)^2} > 0.
\end{aligned}$$

The stability modulus of $J^{[2]}(S,I,E)$ is negative: $s\left(J^{[2]}(S,I,E)\right) = -2\mu$. Hence, due to a result in [11] system (1) has no Hopf bifurcation from the equilibrium point.

Now, we are going to extend our local result about stability of the unique equilibrium point to a global one. For this first we examine the boundedness of the system. Clearly, the positive octant of the $[S, I, E]$ space is positively invariant for system (1). Therefore we have to show that all solutions with positive initial conditions stay bounded in $t \in [0, +\infty)$ and there is no periodic orbit in the positive octant.

Lemma 1 *System (1) is dissipative, i.e. all solutions are bounded.*

Proof. We define the function

$$V(S, I, E) := S + I + E.$$

The time derivative along a solution of (1) is

$$\begin{aligned} \dot{V}(S, I, E) &= \dot{S} + \dot{I} + \dot{E} = \pi + \rho + \sigma - \mu(S + I + E) \\ &= \pi + \rho + \sigma - \mu V(S, I, E). \end{aligned} \tag{4}$$

Thus, for each $\epsilon \in (0, \mu)$ the sum $\dot{V} + \epsilon V$ is bounded from above, i.e. there is a $k > 0$ such that $\dot{V} + \epsilon V \leq k$. Solving this Gronwall-type inequality, we obtain the following estimate

$$0 \leq V(S, I, E) \leq \frac{k}{\epsilon} + V(S(0), I(0), E(0)) \cdot \exp(-\epsilon t) \leq \frac{k}{\epsilon} + V(S(0), I(0), E(0)),$$

which holds for all $t \geq 0$. Hence, as $t \to +\infty$ we have $0 \leq V(S, I, E) \leq \dfrac{k}{\epsilon} + \kappa$ for any $\kappa > 0$. Therefore all the trajectories initiated in $\mathbb{R}^3_+$ enter the region

$$\Omega := \left\{ (S, I, E) \in \mathbb{R}^3_+ \;\middle|\; V(S, I, E) \leq \frac{k}{\epsilon} + \kappa, \;\text{ for any }\; \kappa > 0 \right\}.$$

This completes the proof. ■

Lemma 2 *System (1) has no nontrivial periodic solutions.*

Proof. From (4) we have

$$\dot{V}(S, I, E) = -\mu \left(V(S, I, E) - \frac{\pi + \rho + \sigma}{\mu} \right),$$

which has the consequence that the simplex

$$\Gamma := \left\{ (S, I, E) \in \mathbb{R}^3_+ : \; V(S, I, E) = \frac{\pi + \rho + \sigma}{\mu} \right\}$$

is positively invariant and all solutions approach to Γ with an exponential rate. Moreover, it suffices to study the dynamics of (1) on the simplex Γ. Hence, system (1) can be reduced to a planar system

$$\left\{\begin{array}{rcl} \dot{S} &=& f(S,I) := \pi - \dfrac{\mu}{\pi+\rho+\sigma} SI\left(\kappa_i I + \kappa_{ri}\right) - \mu S, \\ \\ \dot{I} &=& g(S,I) := \rho - \mu I, \end{array}\right. \tag{5}$$

by dropping the third equation and making the substitution

$$E = (\pi + \rho + \sigma)/\mu - S - I$$

in the remaining two equations. Due to the Bendixon's Negative Criterion (see e.g. [6]) this reduced system has no nontrivial periodic solutions, because for $\mathbf{F} := (f, g)$

$$\begin{aligned} (\operatorname{div} \mathbf{F})(S,I) &= (\partial_1 f)(S,I) + (\partial_2 g)(S,I) \\ &= -2\mu - \frac{\mu}{\pi+\rho+\sigma} I\left(\kappa_i I + \kappa_{ri}\right) < 0 \quad (S > 0,\ I > 0) \end{aligned}$$

holds. ■

Thus, we can summarize our results as follows:

Theorem 2 *System* (1) *has only one steady state* (2) *which lies in the interior of the positive octant of the phase space* $[S, I, E]$ *and is globally asymptotically stable.*

Remark 1 *In [4] the situation* $\rho = \sigma = 0$ *(when there are no infected entrants) is also mentioned. In this case one has only the disease-free equilibrium* $(S_0, I_0, E_0) = (\pi/\mu, 0, 0)$ *which is because of the stability of the Jacobian matrix*

$$J(S_0, I_0, E_0) = \begin{bmatrix} -\mu & -\kappa_{ri} & 0 \\ 0 & -\mu & 0 \\ 0 & \kappa_{ri} & -\mu \end{bmatrix}$$

locally asymptotically stable. Its global asymptotical stability can be justified in the similar way as before.

Example 1 *Set* $\pi = 50.10$, $\mu = 0.50$, $\rho = 0.30$ *resp.* $\rho = 0$, $\sigma = 0.20$ *resp.* $\sigma = 0$, $\kappa_i = 0.10$ *and* $\kappa_{ri} = 0.11$. *A Mathematica* 3D *plot shows (cf. Figure 1) that the trajectories of (1) converge to the unique positive resp. boundary equilibrium* $(S^*, I^*, E^*) = (100, 0.6, 0.6)$ *resp.* $(S_0, I_0, E_0) = (100.2, 0, 0)$.

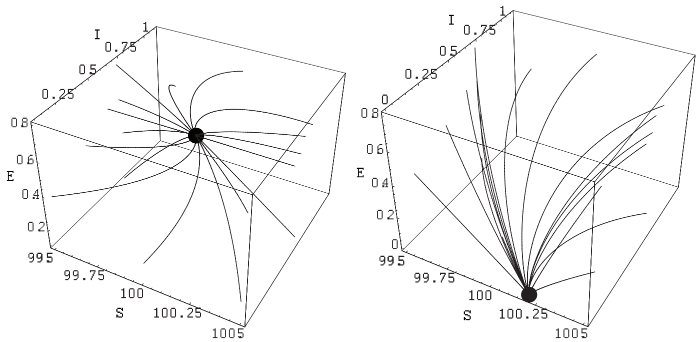

Figure 1: The unique equilibria showing their asymptotic stability.

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Reconstruction of complete interval tournaments. II.

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Abstract. Let a, b ($b \geq a$) and n ($n \geq 2$) be nonnegative integers and let $\mathcal{T}(a, b, n)$ be the set of such generalised tournaments, in which every pair of distinct players is connected at most with b, and at least with a arcs. In [40] we gave a necessary and sufficient condition to decide whether a given sequence of nonnegative integers $D = (d_1, d_2, \ldots, d_n)$ can be realized as the out-degree sequence of a $T \in \mathcal{T}(a, b, n)$. Extending the results of [40] we show that for any sequence of nonnegative integers D there exist f and g such that some element $T \in \mathcal{T}(g, f, n)$ has D as its out-degree sequence, and for any (a, b, n)-tournament T' with the same out-degree sequence D hold $a \leq g$ and $b \geq f$. We propose a $\Theta(n)$ algorithm to determine f and g and an $O(d_n n^2)$ algorithm to construct a corresponding tournament T.

1 Introduction

Let a, b ($b \geq a$) and n ($n \geq 2$) be nonnegative integers and let $\mathcal{T}(a, b, n)$ be the set of such generalised tournaments, in which every pair of distinct players is connected at most with b, and at least with a arcs. The elements of $\mathcal{T}(a, b, n)$ are called (a, b, n)-*tournaments*. The vector $D = (d_1, d_2, \ldots, d_n)$ of the out-degrees of $T \in \mathcal{T}(a, b, n)$ is called *the score vector* of T. If the elements of D are in nondecreasing order, then D is called *the score sequence* of T.

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An arbitrary vector $D = (d_1, d_2, \dots, d_n)$ of nonnegative integers is called *graphical vector,* iff there exists a loopless multigraph whose degree vector is D, and D is called *digraphical vector* (or *score vector*) iff there exists a loopless directed multigraph whose out-degree vector is D.

A nondecreasingly ordered graphical vector is called *graphical sequence*, and a nondecreasingly ordered digraphical vector is called *digraphical sequence* (or *score sequence*).

The number of arcs of T going from player P_i to player P_j is denoted by m_{ij} $(1 \le i, j \le n)$, and the matrix $\mathcal{M} = [1..n, 1..n]$ is called *point matrix* or *tournament matrix* of T.

In the last sixty years many efforts were devoted to the study of both types of vectors, resp. sequences. E.g. in the papers [8, 16, 18, 19, 20, 21, 26, 30, 32, 34, 36, 45, 68, 84, 85, 88, 90, 98] the graphical sequences, while in the papers [1, 2, 3, 7, 8, 11, 17, 27, 28, 29, 31, 33, 37, 49, 48, 50, 55, 58, 57, 60, 61, 62, 64, 65, 66, 69, 78, 79, 82, 94, 86, 87, 97, 100, 101] the score sequences were discussed.

Even in the last two years many authors investigated the conditions, when D is graphical (e.g. [4, 9, 12, 13, 22, 23, 24, 25, 38, 39, 43, 47, 51, 52, 59, 75, 81, 92, 93, 95, 96, 104]) or digraphical (e.g. [5, 35, 40, 46, 54, 56, 63, 67, 70, 71, 72, 73, 74, 83, 87, 89, 102]).

In this paper we deal only with directed graphs and usually follow the terminology used by K. B. Reid [79, 80]. If in the given context a, b and n are fixed or non important, then we speak simply on *tournaments* instead of generalised or (a, b, n)-tournaments.

We consider the loopless directed multigraphs as generalised tournaments, in which the number of arcs from vertex/player P_i to vertex/player P_j is denoted by m_{ij}, where m_{ij} means the number of points won by player P_i in the match with player P_j.

The first question: how one can characterise the set of the score sequences of the (a, b, n)-tournaments. Or, with another words, for which sequences D of nonnegative integers does exist an (a, b, n)-tournament whose out-degree sequence is D. The answer is given in Section 2.

If T is an (a, b, n)-tournament with point matrix $\mathcal{M} = [1..n, 1..n]$, then let $E(T)$, $F(T)$ and $G(T)$ be defined as follows: $E(T) = \max_{1 \le i,j \le n} m_{ij}$, $F(T) = \max_{1 \le i < j \le n}(m_{ij} + m_{ji})$, and $g(T) = \min_{1 \le i < j \le n}(m_{ij} + m_{ji})$. Let $\Delta(D)$ denote the set of all tournaments having D as out-degree sequence, and let $e(D)$, $f(D)$ and $g(D)$ be defined as follows: $e(D) = \{\min\ E(T) \mid T \in \Delta(D)\}$, $f(D) = \{\min\ F(T) \mid T \in \Delta(D)\}$, and $g(D) = \{\max\ G(T) \mid T \in \Delta(D)\}$. In the sequel we use the short notations E, F, G, e, f, g, and Δ.

Hulett et al. [39, 99], Kapoor et al. [44], and Tripathi et al. [91, 92] investigated the construction problem of a minimal size graph having a prescribed degree set [77, 103]. In a similar way we follow a mini-max approach formulating the following questions: given a sequence D of nonnegative integers,

- How to compute e and how to construct a tournament $T \in \Delta$ characterised by e? In Section 3 a formula to compute e, and an algorithm to construct a corresponding tournament are presented.
- How to compute f and g? In Section 4 an algorithm to compute f and g is described.
- How to construct a tournament $T \in \Delta$ characterised by f and g? In Section 5 an algorithm to construct a corresponding tournament is presented and analysed.

We describe the proposed algorithms in words, by examples and by the pseudocode used in [14].

Researchers of these problems often mention different applications, e.g. in biology [55], chemistry Hakimi [32], and Kim et al. in networks [47].

2 Existence of a tournament with arbitrary degree sequence

Since the numbers of points m_{ij} are not limited, it is easy to construct a $(0, d_n, n)$-tournament for any D.

Lemma 1 *If* $n \geq 2$*, then for any vector of nonnegative integers* $D = (d_1, d_2, \ldots, d_n)$ *there exists a loopless directed multigraph* T *with out-degree vector* D *so, that* $E \leq d_n$.

Proof. Let $m_{n1} = d_n$ and $m_{i,i+1} = d_i$ for $i = 1, 2, \ldots, n-1$, and let the remaining m_{ij} values be equal to zero. ■

Using weighted graphs it would be easy to extend the definition of the (a, b, n)-tournaments to allow *arbitrary real values* of a, b, and D. The following algorithm NAIVE-CONSTRUCT works without changes also for input consisting of real numbers.

We remark that Ore in 1956 [66] gave the necessary and sufficient conditions of the existence of a tournament with prescribed in-degree and out-degree vectors. Further Ford and Fulkerson [17, Theorem11.1] published in 1962 necessary and sufficient conditions of the existence of a tournament having prescribed lower and upper bounds for the in-degree and out-degree of the vertices. They results also can serve as basis of the existence of a tournament having arbitrary out-degree sequence.

2.1 Definition of a naive reconstructing algorithm

Sorting of the elements of D is not necessary.

Input. n: the number of players ($n \geq 2$);
$D = (d_1, d_2, \ldots, d_n)$: arbitrary sequence of nonnegative integer numbers.

Output. $\mathcal{M} = [1..n, 1..n]$: the point matrix of the reconstructed tournament.

Working variables. i, j: cycle variables.

Naive-Construct(n, D)

```
for i ← 1 to n
    for j ← 1 to n
        do m_ij ← 0
m_n1 ← d_n
for i ← 1 to n − 1
    do m_i,i+1 ← d_i
return M
```

The running time of this algorithm is $\Theta(n^2)$ in worst case (in best case too). Since the point matrix $\mathcal{M}$ has n^2 elements, this algorithm is asymptotically optimal.

3 Computation of e

This is also an easy question. From here we suppose that D is a nondecreasing sequence of nonnegative integers, that is $0 \leq d_1 \leq d_2 \leq \ldots \leq d_n$. Let $h = \lceil d_n/(n-1) \rceil$.

Since $\Delta(D)$ is a finite set for any finite score vector D, $e(D) = \min\{E(T) | T \in \Delta(D)\}$ exists.

Lemma 2 *If $n \geq 2$, then for any sequence $D = (d_1, d_2, \ldots, d_n)$ there exists a $(0, b, n)$-tournament T such that*

$$E \leq h \qquad \textit{and} \quad b \leq 2h, \tag{1}$$

and h *is the smallest upper bound for* e*, and* $2h$ *is the smallest possible upper bound for* b*.*

Proof. If all players gather their points in a uniform as possible manner, that is

$$\max_{1 \le j \le n} m_{ij} - \min_{1 \le j \le n,\ i \ne j} m_{ij} \le 1 \quad \text{for } i = 1,\ 2,\ \ldots,\ n, \tag{2}$$

then we get $E \le h$, that is the bound is valid. Since player P_n has to gather d_n points, the pigeonhole principle [6, 15, 42] implies $E \ge h$, that is the bound is not improvable. $E \le h$ implies $\max_{1 \le i < j \le n} m_{ij} + m_{ji} \le 2h$. The score sequence $D = (d_1, d_2, \ldots, d_n) = (2n(n-1), 2n(n-1), \ldots, 2n(n-1))$ shows, that the upper bound $b \le 2h$ is not improvable. ■

Corollary 1 *If* $n \ge 2$*, then for any sequence* $D = (d_1, d_2, \ldots, d_n)$ *holds* $e(D) = \lceil d_n/(n-1) \rceil$*.*

Proof. According to Lemma 2 $h = \lceil d_n/(n-1) \rceil$ is the smallest upper bound for e. ■

3.1 Definition of a construction algorithm

The following algorithm constructs a $(0, 2h, n)$-tournament T having $E \le h$ for any D.

Input. n: the number of players ($n \ge 2$);
$D = (d_1, d_2, \ldots, d_n)$: arbitrary sequence of nonnegative integer numbers.

Output. $\mathcal{M} = [1..n, 1..n]$: the point matrix of the tournament.

Working variables. i, j, l: cycle variables;
k: the number of the "larger parts" in the uniform distribution of the points.

PIGEONHOLE-CONSTRUCT(n, D)

```
for i ← 1 to n
    do m_ii ← 0
       k ← d_i − (n − 1)⌊d_i/(n − 1)⌋
    for j ← 1 to k
        do l ← i + j (mod n)
           m_il ← ⌈d_n/(n − 1)⌉
    for j ← k + 1 to n − 1
        do l ← i + j (mod n)
           m_il ← ⌊d_n/(n − 1)⌋
return M
```

The running time of Pigeonhole-Construct is $\Theta(n^2)$ in worst case (in best case too). Since the point matrix $\mathcal{M}$ has n^2 elements, this algorithm is asymptotically optimal.

4 Computation of f and g

Let S_i $(i = 1,\ 2,\ \ldots,\ n)$ be the sum of the first i elements of D, B_i $(i = 1,\ 2,\ \ldots,\ n)$ be the binomial coefficient $n(n-1)/2$. Then the players together can have S_n points only if $fB_n \geq S_n$. Since the score of player P_n is d_n, the pigeonhole principle implies $f \geq \lceil d_n/(n-1) \rceil$.

These observations result the following lower bound for f:

$$f \geq \max\left(\left\lceil \frac{S_n}{B_n} \right\rceil, \left\lceil \frac{d_n}{n-1} \right\rceil\right). \tag{3}$$

If every player gathers his points in a uniform as possible manner then

$$f \leq 2\left\lceil \frac{d_n}{n-1} \right\rceil. \tag{4}$$

These observations imply a useful characterisation of f.

Lemma 3 *If $n \geq 2$, then for arbitrary sequence $D = (d_1, d_2, \ldots, d_n)$ there exists a (g, f, n)-tournament having D as its out-degree sequence and the following bounds for f and g:*

$$\max\left(\left\lceil \frac{S}{B_n} \right\rceil, \left\lceil \frac{d_n}{n-1} \right\rceil\right) \leq f \leq 2\left\lceil \frac{d_n}{n-1} \right\rceil, \tag{5}$$

$$0 \leq g \leq f. \tag{6}$$

Proof. (5) follows from (3) and (4), (6) follows from the definition of f. ■

It is worth to remark, that if $d_n/(n-1)$ is integer and the scores are identical, then the lower and upper bounds in (5) coincide and so Lemma 3 gives the exact value of F.

In connection with this lemma we consider three examples. If $d_i = d_n = 2c(n-1)$ $(c > 0,\ i = 1,\ 2,\ \ldots,\ n-1)$, then $d_n/(n-1) = 2c$ and $S_n/B_n = c$, that is S_n/B_n is twice larger than $d_n/(n-1)$. In the other extremal case, when $d_i = 0$ $(i = 1,\ 2,\ \ldots, n-1)$ and $d_n = cn(n-1) > 0$, then $d_n/(n-1) = cn$, $S_n/B_n = 2c$, so $d_n/(n-1)$ is $n/2$ times larger, than S_n/B_n.

Player/Player	P_1	P_2	P_3	P_4	P_5	P_5	Score
P_1	—	0	0	0	0	0	0
P_2	0	—	0	0	0	0	0
P_3	0	0	—	0	0	0	0
P_4	10	10	10	—	5	5	40
P_5	10	10	10	5	—	5	40
P_6	10	10	10	5	5	—	40

Figure 1: Point matrix of a $(0, 10, 6)$-tournament with $f = 10$ for $D = (0, 0, 0, 40, 40, 40)$.

If $D = (0, 0, 0, 40, 40, 40)$, then Lemma 3 gives the bounds $8 \leq f \leq 16$. Elementary calculations show that Figure 1 contains the solution with minimal f, where $f = 10$.

In [40] we proved the following assertion.

Theorem 1 *For $n \geq 2$ a nondecreasing sequence $D = (d_1, d_2, \dots, d_n)$ of nonnegative integers is the score sequence of some (a, b, n)-tournament if and only if*

$$aB_k \leq \sum_{i=1}^{k} d_i \leq bB_n - L_k - (n-k)d_k \quad (1 \leq k \leq n), \tag{7}$$

where

$$L_0 = 0, \ \textit{and} \ L_k = \max\left(L_{k-1}, \ bB_k - \sum_{i=1}^{k} d_i\right) \quad (1 \leq k \leq n). \tag{8}$$

The theorem proved by Moon [61], and later by Kemnitz and Dolff [46] for (a, a, n)-tournaments is the special case $a = b$ of Theorem 1. Theorem 3.1.4 of [22] is the special case $a = b = 2$. The theorem of Landau [55] is the special case $a = b = 1$ of Theorem 1.

4.1 Definition of a testing algorithm

The following algorithm Interval-Test decides whether a given D is a score sequence of an (a, b, n)-tournament or not. This algorithm is based on Theorem 1 and returns $W =$ True if D is a score sequence, and returns $W =$ False otherwise.

Input. a: minimal number of points divided after each match;
b: maximal number of points divided after each match.

Output. W: logical variable (W = TRUE shows that D is an (a, b, n)-tournament.

Local working variables. i: cycle variable;
$L = (L_0, L_1, \ldots, L_n)$: the sequence of the values of the loss function.

Global working variables. n: the number of players ($n \geq 2$);
$D = (d_1, d_2, \ldots, d_n)$: a nondecreasing sequence of nonnegative integers;
$B = (B_0, B_1, \ldots, B_n)$: the sequence of the binomial coefficients;
$S = (S_0, S_1, \ldots, S_n)$: the sequence of the sums of the i smallest scores.

INTERVAL-TEST(a, b)

```
for i ← 1 to n
    do L_i ← max(L_{i−1}, bB_n − S_i − (n − i)d_i)
       if S_i < aB_i
          then W ← FALSE
               return W
       if S_i > bB_n − L_i − (n − i)d_i
          then W ← FALSE
               return W
return W
```

In worst case INTERVAL-TEST runs in $\Theta(n)$ time even in the general case $0 < a < b$ (n the best case the running time of INTERVAL-TEST is $\Theta(n)$). It is worth to mention, that the often referenced Havel–Hakimi algorithm [32, 36] even in the special case $a = b = 1$ decides in $\Theta(n^2)$ time whether a sequence D is digraphical or not.

4.2 Definition of an algorithm computing f and g

The following algorithm is based on the bounds of f and g given by Lemma 3 and the logarithmic search algorithm described by D. E. Knuth [53, page 410].

Input. No special input (global working variables serve as input).

Output. b: f (the minimal F);
a: g (the maximal G).

Local working variables. i: cycle variable;
l: lower bound of the interval of the possible values of F;
u: upper bound of the interval of the possible values of F.

Global working variables. n: the number of players ($n \geq 2$);
$D = (d_1, d_2, \ldots, d_n)$: a nondecreasing sequence of nonnegative integers;
$B = (B_0, B_1, \ldots, B_n)$: the sequence of the binomial coefficients;
$S = (S_0, S_1, \ldots, S_n)$: the sequence of the sums of the i smallest scores;
W: logical variable (its value is TRUE, when the investigated D is a score sequence).

MINF-MAXG

```
01 B_0 ← S_0 ← L_0 ← 0                        ▷ Initialisation
02 for i ← 1 to n
03     do B_i ← B_{i-1} + i - 1
04        S_i ← S_{i-1} + d_i
05 l ← max(⌈S_n/B_n⌉, ⌈d_n/(n - 1)⌉)
06 u ← 2 ⌈d_n/(n - 1)⌉
07 W ← TRUE                                   ▷ Computation of f
08 INTERVAL-TEST(0, l)
09 if W = TRUE
10    then b ← l
11         go to 21
12 b ← ⌈(l + u)/2⌉
13 INTERVAL-TEST(0, f)
14 if W = TRUE
15    then go to 17
16 l ← b
17 if u = l + 1
18    then b ← u
19         go to 21
20 go to 14
21 l ← 0                                      ▷ Computation of g
22 u ← f
23 INTERVAL-TEST(b, b)
24 if W = TRUE
25    then a ← f
26         go to 37
27 a ← ⌈(l + u)/2⌉
28 INTERVAL-TEST(0, a)
29 if W = TRUE
30    then l ← a
31         go to 33
```

```
u ← a
if u = l + 1
   then a ← l
        go to 37
go to 27
return a, b
```

MinF-MaxG determines f and g.

Lemma 4 *Algorithm* MinG-MaxG *computes the values* f *and* g *for arbitrary sequence* $D = (d_1, d_2, \ldots, d_n)$ *in* $O(n \log(d_n/(n))$ *time.*

Proof. According to Lemma 3 F is an element of the interval $[\lceil d_n/(n-1)\rceil, \lceil 2d_n/(n-1)\rceil]$ and g is an element of the interval $[0, f]$. Using Theorem B of [53, page 412] we get that $O(\log(d_n/n))$ calls of Interval-Test is sufficient, so the $O(n)$ run time of Interval-Test implies the required running time of MinF-MaxG. ■

4.3 Computing of f and g in linear time

Analysing Theorem 1 and the work of algorithm MinF-MaxG one can observe that the maximal value of G and the minimal value of F can be computed independently by Linear-MinF-MaxG.

Input. No special input (global working variables serve as input).
Output. b: f (the minimal F).
a: g (the maximal G).
Local working variables. i: cycle variable.
Global working variables. n: the number of players ($n \geq 2$);
$D = (d_1, d_2, \ldots, d_n)$: a nondecreasing sequence of nonnegative integers;
$B = (B_0, B_1, \ldots, B_n)$: the sequence of the binomial coefficients;
$S = (S_0, S_1, \ldots, S_n)$: the sequence of the sums of the i smallest scores.

Linear-MinF-MaxG

```
B_0 ← S_0 ← L_0 ← 0                      ▷ Initialisation
for i ← 1 to n
    do B_i ← B_{i−1} + i − 1
       S_i ← S_{i−1} + d_i
a ← 0
b ← min 2 ⌈d_n/(n − 1)⌉
for i ← 1 to n                           ▷ Computation of g
```

08 **do** $a_i \leftarrow \lceil 2S_i/(n^2 - n) \rceil$
09 **if** $a_i > a$
10 **then** $a \leftarrow a_i$
11 **for** $i \leftarrow 1$ **to** n ▷ Computation of f
12 **do** $L_i \leftarrow \max(L_{i-1}, bB_n - S_i - (n-i)d_i)$
13 $b_i \leftarrow (S_i + (n-i)d_i + L_i)/B_i$
14 **if** $b_i < b$
15 **then** $b \leftarrow b_i$
16 **return** a, b

Lemma 5 *Algorithm* Linear-MinG-MaxG *computes the values* f *and* g *for arbitrary sequence* $D = (d_1, d_2, \dots, d_n)$ *in* $\Theta(n)$ *time.*

Proof. Lines 01–03, 07, and 18 require only constant time, lines 04–06, 09–12, and 13–17 require $\Theta(n)$ time, so the total running time is $\Theta(n)$. ■

5 Tournament with f and g

The following reconstruction algorithm Score-Slicing2 is based on balancing between additional points (they are similar to ,,excess", introduced by Brauer et al. [10]) and missing points introduced in [40]. The greediness of the algorithm Havel–Hakimi [32, 36] also characterises this algorithm.

This algorithm is an extended version of the algorithm Score-Slicing proposed in [40].

5.1 Definition of the minimax reconstruction algorithm

The work of the slicing program is managed by the following program Mini-Max.

Input. No special input (global working variables serve as input).

Output. $\mathcal{M} = [1\,..\,n, 1\,..\,n]$: the point matrix of the reconstructed tournament.

Local working variables. i, j: cycle variables.

Global working variables. n: the number of players ($n \geq 2$);
$D = (d_1, d_2, \dots, d_n)$: a nondecreasing sequence of nonnegative integers;
$p = (p_0, p_1, \dots, p_n)$: provisional score sequence;
$P = (P_0, P_1, \dots, P_n)$: the partial sums of the provisional scores;
$\mathcal{M}[1\,..\,n, 1\,..\,n]$: matrix of the provisional points.

Mini-Max

```
MinF-MaxG                                  ▷ Initialisation
p_0 ← 0
for i ← 1 to n
    do for j ← 1 to i − 1
           do M[i, j] ← b
       for j ← i to n
           do M[i, j] ← 0
    p_i ← d_i
if n ≥ 3                                        ▷ Score slicing for n ≥ 3 players
   then for k ← n downto 3
            do Score-Slicing2(k, p_k, M)
if n = 2                                        ▷ Score slicing for 2 players
   then m_{1,2} ← p_1
        m_{2,1} ← p_2
return M
```

5.2 Definition of the score slicing algorithm

The key part of the reconstruction is the following algorithm Score-Slicing2 [40].

During the reconstruction process we have to take into account the following bounds:

$$a \leq m_{i,j} + m_{j,i} \leq b \quad (1 \leq i < j \leq n); \tag{9}$$

$$\text{modified scores have to satisfy (7)}; \tag{10}$$

$$m_{i,j} \leq p_i \ (1 \leq i,\ j \leq n, i \neq j); \tag{11}$$

$$\text{the monotonicity } p_1 \leq p_2 \leq \ldots \leq p_k \text{ has to be saved } \ (1 \leq k \leq n) \tag{12}$$

$$m_{ii} = 0 \quad (1 \leq i \leq n). \tag{13}$$

Input. k: the number of the actually investigated players $(k > 2)$;
$\mathbf{p}_k = (p_0, p_1, p_2, \ldots, p_k)$ $(k = 3,\ 4,\ \cdots,\ n)$: prefix of the provisional score sequence p;
$\mathcal{M}[1\ ..\ n, 1\ ..\ n]$: matrix of provisional points.

Output. $\mathcal{M}[1\ ..\ n, 1\ ..\ n]$: matrix of provisional points;
$\mathbf{p}_k = (p_0, p_1, p_2, \ldots, p_k)$ $(k = 2,\ 3,\ 4,\ \cdots,\ n-1)$: prefix of the provisional score sequence p.

Local working variables. $A = (A_1, A_2, \ldots, A_n)$: the number of the additional points;

M: missing points (the difference of the number of actual points and the number of maximal possible points of P_k);
d: difference of the maximal decreasable score and the following largest score;
y: minimal number of sliced points per player;
f: frequency of the number of maximal values among the scores p_1, p_2, $\ldots$, p_{k-1};
i, j: cycle variables;
m: maximal amount of sliceable points;
$P = (P_0, P_1, \ldots, P_n)$: the sums of the provisional scores;
x: the maximal index i with $i < k$ and $m_{i,k} < b$.

Global working variables. n: the number of players ($n \geq 2$);
$B = (B_0, B_1, B_2, \ldots, B_n)$: the sequence of the binomial coefficients;
a: minimal number of points divided after each match;
b: maximal number of points divided after each match.

Score-Slicing2$(k, \mathbf{p}_k, \mathcal{M})$

```
P_0 ← 0                                   ▷ Initialisation
for i ← 1 to k − 1
    do P_i ← P_{i−1} + p_i
       A_i ← P_i − aB_i
M ← (k − 1)b − p_k
while M > 0 and A_{k−1} > 0      ▷ There are missing and additional points
      do x ← k − 1
         while r_{x,k} = b
               do x ← x − 1
         f ← 1
         while p_{x−f+1} = p_{x−f}
               do f = f + 1
         d ← p_{x−f+1} − p_{x−f}
         m ← min(b, d, ⌈A_x/f⌉, ⌈M/f⌉)
         for i ← f downto 1
             do y ← min(b − m_{x+1−i,k}, m, M, A_{x+1−i}, p_{x+1−i})
                m_{x+1−i,k} ← m_{x+1−i,k} + y
                p_{x+1−i} ← p_{x+1−i} − y
                m_{k,x+1−i} ← m_{k,x+1−i} − m_{x+1−i,k}
                M ← M − y
         for j ← i downto 1
             A_{x+1−i} ← A_{x+1−i} − y
while M > 0 and A_{k−1} = 0               ▷ No additional points
```

24   **do for** $i \leftarrow k-1$ **downto** 1
25           $y\,\min(m_{k,i}, M, m_{k,i+m_{i,k}-a})$
26           $m_{ki} \leftarrow m_{k,i} - y$
27           $M \leftarrow M - y$
28 **return** $\mathbf{p}_k, \mathcal{M}$

Let's consider an example. Figure 2 shows the point table of a $(2,10,6)$-tournament T.

Player/Player	P_1	P_2	P_3	P_4	P_5	P_6	Score
P_1	—	1	5	1	1	1	9
P_2	1	—	4	2	0	2	9
P_3	3	3	—	5	4	4	19
P_4	8	2	5	—	2	3	20
P_5	9	9	5	7	—	2	32
P_6	8	7	5	6	8	—	34

Figure 2: The point table of a $(2,10,6)$-tournament T.

The score sequence of T is D = (9,9,19,20,32,34). In [40] the algorithm Score-Slicing2 resulted the point table represented in Figure 3.

Player/Player	P_1	P_2	P_3	P_4	P_5	P_6	Score
P_1	—	1	1	6	1	0	9
P_2	1	—	1	6	1	0	9
P_3	1	1	—	6	8	3	19
P_4	3	3	3	—	8	3	20
P_5	9	9	2	2	—	10	32
P_6	10	10	7	7	0	—	34

Figure 3: The point table of T reconstructed by Score-Slicing2.

The algorithm Mini-Max starts with the computation of f. MinF-MaxG called in line 01 begins with initialisation, including provisional setting of the elements of $\mathcal{M}$ so, that $m_{ij} = b$, if $i > j$, and $m_{ij} = 0$ otherwise. Then MinF-MaxG sets the lower bound $l = \max(9,7) = 9$ of f in line 05 and tests it in line 08 by Interval-Test. The test shows that $l = 9$ is large enough so Mini-Max sets $b = 9$ in line 12 and jumps to line 21 and begins to compute g. Interval-Test called in line 23 shows that $a = 9$ is too large, therefore

MinF-MaxG continues with the test of $a = 5$ in line 27. The result is positive, therefore comes the test of $a = 7$, then the test of $a = 8$. Now $u = l + 1$ in line 33, so $a = 8$ is fixed, and the control returns to line 02 of Mini-Max.

Lines 02–08 contain initialisation, and Mini-Max begins the reconstruction of a $(8, 9, 6)$-tournament in line 9. The basic idea is that Mini-Max successively determines the won and lost points of P_6, P_5, P_4 and P_3 by repeated calls of Score-Slicing2 in line 11, and finally it computes directly the result of the match between P_2 and P_1 in lines 12–14.

At first Mini-Max computes the results of P_6 calling Score-Slicing2 with parameter $k = 6$. The number of additional points of the first five players is $A_5 = 89 - 8 \cdot 10 = 9$ according to line 04, the number of missing points of P_6 is $M = 5 \cdot 9 - 34 = 11$ according to line 05. Then Score-Slicing2 determines the number of maximal numbers among the provisional scores $p_1, p_2, \ldots, p_5$ ($f = 1$ according to lines 10–12) and computes the difference between p_5 and p_4 ($d = 12$ according to line 13). In line 14 we get, that $m = 9$ points are sliceable, and P_5 gets these points in the match with P_6 in line 17, so the number of missing points of P_6 decreases to $M = 11 - 9 = 2$ (line 20) and the number of additional point decreases to $A_5 = 9 - 9 = 0$. Therefore the computation continues in lines 23–28 and m_{64} and m_{63} will be decreased by 1 resulting $m_{64} = 8$ and $m_{63} = 8$ as the seventh line and seventh column of Figure 4 show. The returned score sequence is $\mathbf{p}_5 = (9, 9, 19, 20, 23)$.

Player/Player	P_1	P_2	P_3	P_4	P_5	P_6	Score
P_1	—	4	4	1	0	0	9
P_2	4	—	4	1	0	0	9
P_3	4	4	—	7	4	0	19
P_4	7	7	1	—	5	0	20
P_5	8	8	4	3	—	9	32
P_6	9	9	8	8	0	—	34

Figure 4: The point table of T reconstructed by Mini-Max.

Second time Mini-Max calls Score-Slicing2 with parameter $k = 5$, and get $A_4 = 9$ and $M = 13$. At first P_4 gets 1 point, then P_3 and P_4 get both 4 points, reducing M to 4 and A_4 to 0. The computation continues in line 23 and results the further decrease of m_{54}, m_{53}, m_{52}, and m_{51} by 1, resulting $m_{54} = 3$, $m_{53} = 4$, $m_{52} = 8$, and $m_{51} = 8$ as the sixth row of Figure 4 shows. The returned score sequence is $\mathbf{p}_4 = (9, 9, 15, 15)$

Third time Mini-Max calls Score-Slicing2 with parameter $k = 4$, and get $A_3 = 11$ and $M = 11$. At first P_3 gets 6 points, then P_3 further 1 point, and P_2 and P_1 also both get 1 point, resulting $m_{34} = 7$, $m_{43} = 2$, $m_{42} = 8$, $m_{24} = 1$, $m_{14} = 1$ and $m_{14} = 8$, further $A_3 = 0$ and $M = 2$. The computation continues in lines 23–28 and results a decrease of m_{43} by 1 point resulting $m_{43} = 1$, $m_{42} = 7$, and $m_{41} = 7$, as the fifth row and fifth column of Figure 4 show. The returned score sequence is $\mathbf{p}_3 = (8, 8, 8)$.

Fourth time Mini-Max calls Score-Slicing2 with parameter $k = 3$, and gets $A_2 = 8$ and $M = 10$. At first P_1 and P_2 get 4 points, resulting $m_{13} = 4$, and $m_{23} = 4$, and $M = 2$, and $A_2 = 0$. Then Mini-Max sets in lines 23–26 $m_{31} = 4$ and $m_{32} = 4$. The returned score sequence is $\mathbf{p}_2 = (4, 4)$.

Finally Mini-Max sets $m_{12} = 4$ and $m_{21} = 4$ in lines 14–15 and returns the point matrix represented in Figure 4.

The comparison of Figures 3 and 4 shows a large difference between the simple reconstruction of Score-Slicing2 and the minimax reconstruction of Mini-Max: while in the first case the maximal value of $m_{ij} + m_{ji}$ is 10 and the minimal value is 2, in the second case the maximum equals to 9 and the minimum equals to 8, that is the result is more balanced (the given D does not allow to build a perfectly balanced (k, k, n)-tournament).

5.3 Analysis of the minimax reconstruction algorithm

The main result of this paper is the following assertion.

Theorem 2 *If $n \geq 2$ is a positive integer and $D = (d_1, d_2, \ldots, d_n)$ is a nondecreasing sequence of nonnegative integers, then there exist positive integers f and g, and a (g, f, n)-tournament T with point matrix $\mathcal{M}$ such, that*

$$f = \min(m_{ij} + m_{ji}) \leq b, \tag{14}$$

$$g = \max m_{ij} + m_{ji} \geq a \tag{15}$$

for any (a, b, n)-tournament, and algorithm Linear-MinF-MaxG *computes f and g in $\Theta(n)$ time, and algorithm* Mini-Max *generates a suitable T in $O(d_n n^2)$ time.*

Proof. The correctness of the algorithms Score-Slicing2, MinF-MaxG implies the correctness of Mini-Max.

Lines 1–46 of Mini-Max require $O(\log(d_n/n))$ uses of MinG-MaxF, and one search needs $O(n)$ steps for the testing, so the computation of f and g can be executed in $O(n \log(d_n/n))$ times.

The reconstruction part (lines 47–55) uses algorithm Score-Slicing2, which runs in $O(bn^3)$ time [40]. Mini-Max calls Score-Slicing2 $n-2$ times with $f \leq 2\lceil d_n/n \rceil$, so $n^3 d_n/n = d_n n^2$ finishes the proof. ■

The property of the tournament reconstruction problem that the extremal values of f and g can be determined independently and so there exists a tournament T having both extremal features is called linking property. This concept was introduced by Ford and Fulkerson in 1962 [17] and later extended by A. Frank in [22].

6 Summary

A nondecreasing sequence of nonnegative integers $D = (d_1, d_2, \ldots, d_n)$ is a score sequence of a $(1,1,1)$-tournament, iff the sum of the elements of D equals to B_n and the sum of the first i ($i = 1, 2, \ldots, n-1$) elements of D is at least B_i [55].

D is a score sequence of a (k,k,n)-tournament, iff the sum of the elements of D equals to kB_n, and the sum of the first i elements of D is at least kB_i [46, 60].

D is a score sequence of an (a,b,n)-tournament, iff (7) holds [40].

In all 3 cases the decision whether D is digraphical requires only linear time.

In this paper the results of [40] are extended proving that for any D there exists an optimal minimax realization T, that is a tournament having D as its out-degree sequence, and maximal G, and minimal F in the set of all realizations of D.

In a continuation [41] of this paper we construct balanced as possible tournaments in a similar way if not only the out-degree sequence but the in-degree sequence is also given.

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Fifteen problems in number theory

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Abstract. In this paper we collected problems, which was either proposed or follow directly from results in our papers.

1 Introduction

In this paper, which is based on a talk delivered at the Winter School on Explicit Methods in Number Theory, Debrecen, January 29, 2009 we collected problems, which we proposed and/or tried to solve. The problems are dealing with perfect powers in linear recursive sequences, solutions of parametrized families of Thue equations, patterns in the set of solutions of norm form equations and generalized radix representations.

In each case we give a short description of the background information, cite some relevant paper, especially papers, where the problem appeared at the first time. Sometime we present our feeling about the hardness of the problem and how one could solve it. The collection is subjective.

2 Powers in linear recursive sequences

To find perfect powers and polynomial values in linear recursive sequences is one of my favorite topics. A long standing problem was to prove that $0, 1, 8$

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and 144 are the only powers in the Fibonacci sequence. This was proved finally by Bugeaud, Mignotte and Siksek in 2006 [9].

In 1996 at The Seventh International Research Conference on Fibonacci Numbers and Their Applications I proposed the following [17]

Problem 1 *The sequence of tribonacci numbers is defined by* $T_0 = T_1 = 0, T_2 = 1$ *and* $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ *for* $n \geq 0$*. Are the only squares* $T_0 = T_1 = 0, T_2 = T_3 = 1, T_5 = 4, T_{10} = 81, T_{16} = 3136 = 56^2$ *and* $T_{18} = 10609 = 103^2$ *among the numbers* T_n*?*

By using the sieve method from [16] with the moduli $3, 7, 11, 13, 29, 41, 43, 53, 79, 101, 103, 131, 239, 97, 421, 911, 1021$ and 1123 one can show that this is true for $n \leq 2\cdot 10^6$, but known methods do not seem to be applicable for its solution.

The problem is still unsolved, although in the edited version of the second part of that talk [18] combining results of Shorey and Stewart [23] with that of Corvaja and Zannier [10] I proved

Theorem 1 *Let* G_n *be a third order LRS. For the roots* $\alpha_i, i = 1, 2, 3$ *of the characteristic polynomial of* G_n *assume that* $|\alpha_1| > |\alpha_2| \geq |\alpha_3|$ *and non of them is a root of unity. Then there are only finitely many perfect powers in* G_n*.*

As the characteristic polynomial of the tribonacci sequence $x^3 - x^2 - x - 1$ is irreducible with one dominating real root ≈ 1.839286755 it follows that there exist finitely many perfect powers in it. Unfortunately the proof of Theorem 1 is only partially effective. We have an effective bound for the exponent of the possible perfect powers, but no effective bound for the size of a fixed power, e.g., for squares.

I think that Theorem 1 can be generalized at least in the following form:

Problem 2 *Let* G_n *be an LRS such that its characteristic polynomial is irreducible and has a dominating root, then there is only finitely many perfect powers in it.*

By a result of Shorey and Stewart [23] the exponent of perfect powers can be bounded effectively. The problem is to handle the powers with bounded exponent. Combining this with the result of Corvaja and Zannier [10] and with the combinatorics of the roots, like in Pethő [18], one can probably settle this conjecture.

Like the Fibonacci sequence, we can continue the tribonacci sequence in "negative direction", and get $T_{-n} = -T_{-n+1} - T_{-n+2} + T_{-n+3}$ with initial terms $T_0 = 0, T_{-1} = 1, T_{-2} = -1$. We call this sequence n-tribonacci. One can ask again, which are the perfect powers in this sequence. After a simple search we find: $T_0 = T_{-3} = T_{-16} = 0, T_{-1} = T_{-6} = -T_{-2} = 1, T_{-7} = 2^2, T_{-8} = (-2)^3, T_{-13} = 3^2, T_{-29} = 3^4, T_{-32} = 56^2, T_{-33} = 103^2$ and $T_{-62} = 6815^2$. It is interesting to observe that $T_{10} = T_{-29}, T_{16} = T_{-32}$ and $T_{18} = T_{-33}$.

Problem 3 *Are all perfect powers of the n-tribonacci sequence listed above? Are there only finitely many perfect powers in the n-tribonacci sequence?*

The answer seems to be very difficult, because the characteristic polynomial of the n-tribonacci sequence has two conjugate complex roots of the same absolute value and its real root is less than one. Thus the result of Shorey and Stewart is not applicable.

Let $a, b \in \mathbb{Z}$ and $\delta \in \{1, -1\}$ such that $a^2 - 4(b - 2\delta) \neq 0$, $b\delta \neq 2$ and if $\delta = 1$ then $b \neq 2a - 2$. Let further the sequence $G_n = G_n(a, b, \delta)$, $n \geq 0$ defined by the initial terms $G_0 = 0$, $G_1 = 1$, $G_2 = a$, $G_3 = a^2 - b - \delta$ and by the recursion

$$G_{n+4} = aG_{n+3} - bG_{n+2} + \delta a G_{n+1} - G_n, \quad n \geq 0. \tag{1}$$

I proved in [19] that these are divisibility sequences, i.e., $G_n | G_m$, whenever $n | m$. More precisely, the roots of the characteristic polynomial of G_n can be numbered so that they are $\eta, \frac{\delta}{\eta}, \vartheta, \frac{\delta}{\vartheta}$ and

$$G_n = \frac{\eta^n - \vartheta^n}{\eta - \vartheta} \frac{1 - \left(\frac{\delta}{\eta\vartheta}\right)^n}{1 - \frac{\delta}{\eta\vartheta}}$$

Here we ask again to prove

Problem 4 *For fixed* a, b *there are only finitely many perfect powers in* G_n.

We can again bound the exponent by the result of Shorey and Stewart [23], but can not treat the equation $G_n = x^q$ for fixed $q > 1$. Especially complicated seems the case $q = 2$, because the greatest common divisor of the algebraic numbers $\frac{\eta^n - \vartheta^n}{\eta - \vartheta}$ and $\frac{1 - \left(\frac{\delta}{\eta\vartheta}\right)^n}{1 - \frac{\delta}{\eta\vartheta}}$ can be arbitrary large.

3 Thue equations

After the work of E. Thomas [24] several paper appeared about the solutions of parametrized families of Thue equations. With Halter-Koch, Lettl and Tichy we proved [13] the following:

Theorem 2 *Let* $n \geq 3$, $a_1 = 0, a_2, \ldots, a_{n-1}$ *be distinct integers and* $a_n = a$ *an integral parameter. Let* $\alpha = \alpha(a)$ *be a zero of* $P(x) = \prod_{i=1}^{n}(x - a_i) - d$ *with* $d = \pm 1$ *and suppose that the index* I *of* $\langle \alpha - a_1, \ldots, \alpha - a_{n-1} \rangle$ *in* $U_{\mathcal{O}}$, *the group of units of* $\mathcal{O}$, *is bounded by a constant* $J = J(a_1, \ldots, a_{n-1}, n)$ *for every* a *from some subset* $\Omega \subset \mathbb{Z}$. *Assume further that the Lang-Waldschmidt conjecture is true. Then for all but finitely many values* $a \in \Omega$ *the diophantine equation*

$$\prod_{i=1}^{n}(x - a_i y) - dy^n = \pm 1 \tag{2}$$

only has trivial solutions, except when $n = 3$ *and* $|a_2| = 1$, *or when* $n = 4$ *and* $(a_2, a_3) \in \{(1, -1), (\pm 1, \pm 2)\}$, *in which cases (2) has exactly one more general solution.*

The assumption on the index I is technical, the essential assumption is the Lang-Waldschmidt conjecture. In the cited paper we formulated:

Problem 5 *The last theorem is true for all large enough parameter value without further assumptions.*

A weaker version of this conjecture was formulated by E. Thomas [25]. He assumed that $a_i = p_i(a), i = 2, \ldots, n-1$ and $0 < \deg p_2 < \cdots < \deg p_{n-1}$, where p_i denotes monic polynomial with integer coefficients. This weaker conjecture was proved by C. Heuberger [14] under some technical conditions on the degree of the polynomials.

4 Progressions in the set of solutions of norm form equations

Let $\mathbb{K}$ be an algebraic number field of degree k, and let $\alpha_1, \ldots, \alpha_n$ be linearly independent elements of $\mathbb{Z}_{\mathbb{K}}$ over $\mathbb{Q}$. Let m be a non-zero integer and consider the norm form equation

$$N_{\mathbb{K}/\mathbb{Q}}(x_1\alpha_1 + \ldots + x_n\alpha_n) = m \tag{3}$$

in integer vectors $(x_1, \dots, x_n)$. Let H denote the solution set of (3) and $|H|$ the size of H. Note that if the $\mathbb{Z}$-module generated by $\alpha_1, \dots, \alpha_n$ contains a submodule, which is a full module in a subfield of $\mathbb{Q}(\alpha_1, \dots, \alpha_n)$ different from the imaginary quadratic fields and $\mathbb{Q}$, then equation (3) can have infinitely many solutions (see e.g. Schmidt [22]).

Arranging the elements of H in an $|H| \times n$ array $\mathcal{H}$, one may ask at least two natural questions about arithmetical progressions appearing in H. The "horizontal" one: do there exist infinitely many rows of $\mathcal{H}$, which form arithmetic progressions; and the "vertical" one: do there exist arbitrary long arithmetic progressions in some column of $\mathcal{H}$? Note that the first question is meaningful only if $n > 2$.

We are now presenting an example. Let $\mathbb{K} := \mathbb{Q}(\alpha)$ with $\alpha^5 = 3$. Then

$$\begin{aligned} N_{\mathbb{K}/\mathbb{Q}}(x_1 + x_2\alpha + \dots + x_5\alpha^4) = {} & 9x_3^5 + 81x_5^5 + x_1^5 + 27x_4^5 + 3x_2^5 - 135x_5^3x_4x_1 + \\ & +45x_5x_4^2x_1^2 + 135x_2x_4^2x_5^2 - 45x_2x_4^3x_1 + 45x_5^2x_3x_1^2 - 45x_2x_3^3x_4 + \\ & +135x_3^2x_5^2x_4 + 45x_1x_5^2x_2^2 - 45x_4x_2^3x_5 + 45x_4^2x_2^2x_3 + 45x_4^2x_1x_3^2 - \\ & -15x_4x_1^3x_3 + 15x_4x_1^2x_2^2 + 15x_2x_3^2x_1^2 + 45x_5x_2^2x_3^2 - 15x_5x_1^3x_2 - \\ & -135x_5x_3x_4^3 - 135x_2x_5^3x_3 - 45x_5x_3^3x_1 - 15x_2^3x_3x_1 - 45x_2x_5x_3x_4x_1. \end{aligned}$$

The next table contains a finite portion of the set of solutions of the equation

$$N_{\mathbb{K}/\mathbb{Q}}(x_1 + x_2\alpha + \dots + x_5\alpha^4) = 1.$$

x_1	x_2	x_3	x_4	x_5
4	-5	4	**-2**	0
1	2	-1	**-1**	0
4	2	0	**0**	1
1	1	0	**1**	0
1	5	1	**2**	2
-17	1	-6	**3**	8
7	**6**	**5**	**4**	**3**
-2	-1	1	1	0
-11	-5	5	6	0
-2	0	1	-1	1
-8	-8	1	6	2
28	16	4	3	8
10	12	12	4	9
...	...	...	...	...

The bold face numbers form a five term horizontal AP and a seven terms vertical AP. The "horizontal" problem was treated by Bérczes and Pethő [7] by proving that if $\alpha_i = \alpha^{i-1}$ $(i = 1, \ldots, n)$ then in general $\mathcal{H}$ contains only finitely many effectively computable "horizontal" AP's and they were able to localize the possible exceptional cases. The following question remains unanswered:

Problem 6 *Does there exist infinitely many quartic algebraic integers α such that $\frac{4\alpha^4}{\alpha^4-1} - \frac{\alpha}{\alpha-1}$ is a quadratic algebraic number.*

We were able to found only one example with defining polynomial $x^4+2x^3+5x^2+4x+2$ such that the corresponding element is a real quadratic number. It is a root of x^2-4x+2. Allowing however α not to be integral we can obtain a lot of examples.

The investigation of the "vertical" AP's is much more difficult. In this direction Bérczes, Hajdu and Pethő [6] proved

Theorem 3 *Let $(x_1^{(j)}, \ldots, x_n^{(j)})$ $(j = 1, \ldots, t)$ be a sequence of distinct elements in H such that $x_i^{(j)}$ is a non-zero arithmetic progression for some $i \in \{1, \ldots, n\}$. Then we have $t \le c_1$, where $c_1 = c_1(k, m)$ is an explicitly computable constant.*

It is interesting to note that c_1 depends only on the degree of the norm form and not on its coefficients. One can probably strengthen this result such that the upper bound for the length of the AP's depend not on m, but only on the number of its prime divisors. It is even possible that the bound depends only on k.

Earlier Pethő and Ziegler [21] as well as Dujella, Pethő and Tadić [11] investigated the AP's on Pell equations, which are quadratic norm form equations. We proved that for all but one non-constant AP of integers of length four y_1, y_2, y_3, y_4 there exist infinitely many integers d, m for which $x_i^2 - dy_i^2 = m, i = 1, 2, 3, 4$ with some integers $x_i = x_i(d, m, y_1, \ldots, y_4), i = 1, 2, 3, 4$. In contrast, five term AP's are lying on only finitely many Pell equations.

Problem 7 *Prove analogous result for norm form equations over cubic number fields. More specifically: let $y^{(i)}, i = 1, \ldots, 5$ an AP of integers. Then there exist infinitely many $m \in \mathbb{Z}$ and $\mathbb{Q}$-independent algebraic integers $\alpha_1, \alpha_2, \alpha_3$ such that $\mathbb{K} = \mathbb{Q}(\alpha_2, \alpha_3)$ has degree three and (3) holds for $(x_1^{(i)}, x_2^{(i)}, y^{(i)}), i = 1, \ldots, 5$ with some $x_1^{(i)}, x_2^{(i)} \in \mathbb{Z}$. Can 5 be replaced with a larger number?*

In the above mentioned papers we worked out a systematic method to find Pell equations having long AP's. For example the AP $-7,-5,-3,-1,1,3,5,7$ is lying on the equation $x^2-570570y^2 = 4406791$ and $-461,-295,-129,37,203,369,535$ on $x^2+1245y^2 = 375701326$.

Problem 8 *Find a systematic method to construct cubic norm form equations with long AP. Do the same for higher degree norm form equations.*

Problem 9 *Prove analogous results for geometric progressions.*

5 Polynomials

Problem 10 *Let* K *be a algebraically closed field of characteristic zero. Characterize all* $P(X) \in K[X], Q(Y) \in K[Y], R(X,Y) \in K[X,Y]$ *such that the set of zeroes of* $P(X)$ *and* $Q(Y)$ *coincide, provided* $R(X,Y) = 0$.

The case $R(X,Y) = Y - A(X)$ was solved completely by Fuchs, Pethő and Tichy [12]. They proved

Theorem 4 *Assume that* $P(X)$ *has* k *different zeroes. Then there exist* $a,b,c \in K, a,c \neq 0$ *such that:*
if $k = 1$ *then*

$$P(X) = a(X-b)^{\deg P} \text{ and } A(X) = c(X-b)^{\deg A} + b;$$

if $k \geq 2$ *then either* $A(X) = X$ *or* $A(X) = aX + b, a \neq 1$ *and in this case*

$$P(X) = c\left(X + \frac{b}{a-1}\right)^s \prod_{i=1}^{r}\prod_{j=0}^{\ell-1}\left(X - a^j x_i - b\frac{a^j-1}{a-1}\right),$$

where $x_1,\dots,x_r$ *are all different and* ℓ *is the multiplicative order of* a.

6 Shift radix systems

For $(r_1,\dots,r_d) = \mathbf{r} \in \mathbb{R}^d$ and $\mathbf{a} = (a_1,\dots,a_d) \in \mathbb{Z}^d$ let $\tau_{\mathbf{r}}(\mathbf{a}) = (a_2,\dots,a_d,-\lfloor \mathbf{ra} \rfloor)^T$, where $\mathbf{ra}$ denotes the scalar product. This nearly linear mapping was introduced by Akiyama, Borbély, Brunotte, Thuswaldner and myself [1]. We proved that it can be considered as a common generalization of canonical number systems (CNS) and β-expansions.

We also defined the sets

$$\begin{aligned}\mathcal{D}_d &= \{\mathbf{r} : \{\tau_{\mathbf{r}}^k(\mathbf{a})\}_{k=0}^{\infty} \quad \text{is bounded for all} \quad \mathbf{a} \in \mathbb{Z}^d\},\\ \mathcal{D}_d^0 &= \{\mathbf{r} : \{\tau_{\mathbf{r}}^k(\mathbf{a})\}_{k=0}^{\infty} \quad \text{is ultimately zero for all} \quad \mathbf{a} \in \mathbb{Z}^d\}\end{aligned}$$

and $\mathcal{E}_d$, which is the set of real monic polynomials, whose roots lie in the closed unit disc. We proved in the same paper that if $\mathbf{r} \in \mathcal{D}_d$ then $R(X) = X^d + r_d X^{d-1} + \cdots + r_2 X + r_1 \in \mathcal{E}_d$ and if $R(X)$ is lying in the interior of $\mathcal{E}_d$ then $\mathbf{r} \in \mathcal{D}_d$.

We called $\tau_{\mathbf{r}}$ a shift radix system (SRS), if $\mathbf{r} \in \mathcal{D}_d^0$ and gave an algorithm, which decides whether $\mathbf{r} \in \mathbb{Q}^d$ is a SRS. However this algorithm is exponential, moreover we are not able to give a polynomial time verification for $\mathbf{r} \notin \mathcal{D}_d^0 \cap \mathbb{Q}^d$. We found points $\mathbf{r} \in \mathbb{Q}^2$ such that $\mathbf{r} \notin \mathcal{D}_2^0$, but the cycles proving this can be arbitrary long. Computational experiments, see e.g. [1, 15] support the following :

Problem 11 *Prove that the SRS problem can not be solved by a polynomial time algorithm. Stronger statement is that it does not belong to the NP complexity class.*

The structure of $\mathcal{D}_d^0$, especially near to its boundary, is very complicated, see [2] for $d = 2$. On the other hand we know [1], that the closure of $\mathcal{D}_d$ is $\mathcal{E}_d$. However the investigation of the boundary points of $\mathcal{E}_d$ leads to interesting and hard problems. The case $d = 2$ was studied by Akiyama et al. in [2]. They proved that $\mathcal{D}_2$ is equal to the closed triangle with vertices $(-1, 0), (1, -2), (1, 2)$, but without the points $(1, -2), (1, 2)$, the line segment $\{(x, -x - 1) : 0 < x < 1\}$ and, possibly, some points of the line segment $\{(1, \lambda) : -2 < \lambda < 2\}$. Write in the last case $\lambda = 2\cos\alpha$ and $\omega = \cos\alpha + i \sin\alpha$. It is easy to see, that if $\lambda = 0, \pm 1$ (i.e., $\alpha = 0, \pm\pi/2$) then $(1, \lambda)$ belongs to $\mathcal{D}_2$ and we conjectured in [2] that this is true for all points of this line segment. In [4] the conjecture was proved for the golden mean, i.e., for $\lambda = \frac{1+\sqrt{5}}{2}$ and in [5] for those ω, which are quadratic algebraic numbers. The conjecture has the following nice arithmetical form:

Problem 12 *Let $|\lambda| < 2$ be a real number. If the sequence of integers $\{a_n\}$ satisfies the relation*

$$0 \le a_{n-1} + \lambda a_n + a_{n+1} < 1$$

then it is periodic.

If ω, defined above, is a root of unity then the problem may be easier as in the general case. On the other hand from the point of view of arithmetic the cases, when λ is a rational number, e.g., $\lambda = \frac{1}{2}$ seems simpler.

If the point $\mathbf{r}$ belongs to the boundary of $\mathcal{E}_d$ then either $\mathbf{r} \in \mathcal{D}_d$ or $\mathbf{r} \notin \mathcal{D}_d$. With other words this means that the sequence $\{\tau_{\mathbf{r}}(\mathbf{a})\}$ is ultimately periodic for all $\mathbf{a} \in \mathbb{Z}^d$ as well as there exists $\mathbf{a} \in \mathbb{Z}^d$ for which $\{\tau_{\mathbf{r}}(\mathbf{a})\}$ is divergent. However we do not know any general method to distinguish between these cases. Recently I gave an algorithm [20] in the special case, when $\pm 1, \pm i$ is a simple root of $X^d + r_d X^{d-1} + \cdots + r_2 X + r_1$.

Problem 13 *Is it algorithmically decidable for* $\mathbf{r} \in \mathcal{E}_d \cap \mathbb{Q}^d$ *whether* $\mathbf{r} \in \mathcal{D}_d$*?*

I am not sure that the answer is affirmative. The problem is open even for $d = 2$. In this case, by the results of [2], the status only points of the line segment $\{(1, y) : -2 < y < 2\}$ is questionable. If the answer to Problem 9 is affirmative, which I strongly believe, then $d = 2$ would be completely solved. A related, probably easier problem is:

Problem 14 *Prove that there are no elements of* $\mathcal{D}_d^0$ *on the boundary of* $\mathcal{E}_d$*.*

This is true for $d = 2$ [2], but open for $d \geq 3$.

For each $d \in \mathbb{N}$, $d \geq 1$ define the set

$$\mathcal{B}_d = \{(b_1, \ldots, b_d) \in \mathbb{Z}^d : X^d - b_1 X^{d-1} - \cdots - b_d \text{ is a Pisot or Salem polynomial}\}.$$

Further for $M \in \mathbb{N}_{>0}$ set

$$\mathcal{B}_d(M) = \left\{(b_2, \ldots, b_d) \in \mathbb{Z}^{d-1} : (M, b_2, \ldots, b_d) \in \mathcal{B}_d\right\}. \tag{4}$$

It is clear that $\mathcal{B}_d(M)$ is a finite set. In [3] we proved

Theorem 5 *Let* $d \geq 2$*. We have*

$$\left| \frac{|\mathcal{B}_d(M)|}{M^{d-1}} - \lambda_{d-1}(\mathcal{D}_{d-1}) \right| = O(M^{-1/(d-1)}), \tag{5}$$

where λ_{d-1} *denotes the* $(d-1)$*-dimensional Lebesgue measure.*

To fix the coefficient of the term X^{d-1} of a d-th degree monic polynomial is unusual. Generally the height, i.e., the maximum of the absolute values of its coefficients is used to measure polynomials. Having this in mind we define

$$\hat{\mathcal{B}}_d(M) = \left\{(b_1, b_2, \ldots, b_d) \in \mathbb{Z}^d \cap \mathcal{B}_d : \max\{|b_1|, |b_2|, \ldots, |b_d|\} \leq M\right\}.$$

and propose our last problem.

Problem 15 *Does there exist a constant* c*, such that*

$$\lim_{M\to\infty} \frac{|\hat{\mathcal{B}}_d(M)|}{M^d} = c?$$

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A note on logarithmically completely monotonic ratios of certain mean values

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Abstract. We offer a new, unitary proof of some generalizations of results from paper [2]. Our method leads to similar results for other special means, too.

1 Introduction

A function $f : (0, \infty) \to \mathbb{R}$ is said to be completely monotonic (c.m. for short), if f has derivatives of all orders and satisfies

$$(-1)^n \cdot f^{(n)}(x) \geq 0 \text{ for all } x > 0 \text{ and } n = 0, 1, 2, \ldots. \tag{1}$$

J. Dubourdieu [3] pointed out that, if a non-constant function f is c.m., then strict inequality holds in (1). It is known (and called as Bernstein theorem) that f is c.m. iff f can be represented as

$$f(x) = \int_0^\infty e^{-xt} d\mu(t), \tag{2}$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$ (see [11]).

Completely monotonic functions appear naturally in many fields, like, for example, probability theory and potential theory. The main properties of

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these functions are given in [11]. We also refer to [4, 1, 2], where detailed lists of references can be found.

Let $a, b > 0$ be two positive real numbers. The power mean of order $k \in \mathbb{R} \setminus \{0\}$ of a and b is defined by

$$A_k = A_k(a, b) = \left(\frac{a^k + b^k}{2} \right)^{1/k}.$$

Denote $A = A_1(a, b) = \dfrac{a+b}{2}$, $G = G(a, b) = A_0(a, b) = \lim\limits_{k \to \infty} A_k(a, b) = \sqrt{ab}$ the arithmetic, resp. geometric means of a and b.

The identric, resp. logarithmic means of a and b are defined by

$$I = I(a, b) = \frac{1}{e} \left(b^b / a^a \right)^{1/(b-a)} \quad \text{for } a \neq b; \quad I(a, a) = a;$$

and

$$L = L(a, b) = \frac{b - a}{\log b - \log a} \quad \text{for } a \neq b; \quad L(a, a) = a.$$

Consider also the weighted geometric mean S of a and b, the weights being $a/(a+b)$ and $b/(a+b)$:

$$S = S(a, b) = a^{a/(a+b)} \cdot b^{b/(a+b)}.$$

As one has the identity (see [6])

$$S(a, b) = \frac{I(a^2, b^2)}{I(a, b)},$$

the mean S is connected with the identric mean I.

Other means which occur in this paper are

$$H = H(a, b) = A_{-1}(a, b) = \frac{2ab}{a+b}, \quad Q = Q(a, b) = A_2(a, b) = \sqrt{\frac{a^2 + b^2}{2}},$$

as well as Seiffert's mean (see [10], [9])

$$P = P(a, b) = \frac{a - b}{2 \arcsin \left(\dfrac{a-b}{a+b} \right)} \quad \text{for } a \neq b, \quad P(a, a) = a.$$

In the paper [2] C.–P. Chen and F. Qi have considered the ratios

a) $\frac{A}{I}(x, x+1)$, b) $\frac{A}{G}(x, x+1)$, c) $\frac{A}{H}(x, x+1)$,

d) $\frac{I}{G}(x, x+1)$, e) $\frac{I}{H}(x, x+1)$, f) $\frac{G}{H}(x, x+1)$,

g) $\frac{A}{L}(x, x+1)$,

where $\frac{A}{I}(x, x+1) = \frac{A(x, x+1)}{I(x, x+1)}$ etc., and proved that the logarithms of the ratios $a) - f)$ are c.m., while the ratio from $g)$ is c.m.

In [2] the authors call a function f as logarithmically completely monotonic (l.c.m. for short) if the function $g = \log f$ is c.m. They notice that they proved earlier (in 2004) that if f is l.c.m., then it is also c.m. We note that this result has been proved already in paper [4]:

Lemma 1 *If f is l.c.m, then it is also c.m.*

The following basic property is well-known (see e.g. [4]):

Lemma 2 *If $a > 0$ and f is c.m., then $a \cdot f$ is c.m., too. The sum and the product of two c.m. functions is c.m., too.*

Corollary 1 *If k is a positive integer and f is c.m., then the function f^k is c.m., too.*

Indeed, it follows by induction from Lemma 2 that, the product of a finite number of c.m. functions is c.m., too.

Particularly, when there are k equal functions, Corollary 1 follows.

The aim of this note is to offer new proofs for more general results than in [2], and involving also the means S, P, Q.

2 Main results

First we note that, as one has the identity

$$H = \frac{G^2}{A},$$

we get immediately

$$\frac{A}{H} = \frac{A^2}{G^2}, \quad \frac{G}{H} = \frac{A}{G}$$

so that as
$$\log\frac{A}{H} = 2\log\frac{A}{G} \quad \text{and} \quad \log\frac{G}{H} = \log\frac{A}{G},$$
by Lemma 2 the ratios c) and f) may be reduced to the ratio a).

Similarly, as
$$\frac{I}{H} = \frac{A}{G}\cdot\frac{I}{G},$$
the study of ratio e) follows (based again on Lemma 2) from the ratios b) and d).

As one has
$$\frac{A}{G} = \frac{A}{I}\cdot\frac{I}{G},$$
it will be sufficient to consider the ratios a) and d).

Therefore, in Theorem 1 of [2] we should prove only that $\frac{A}{I}(x, x+1)$ and $\frac{I}{G}(x, x+1)$ are l.c.m., and $\frac{A}{L}(x, x+1)$ is c.m.

A more general result is contained in the following:

Theorem 1 *For any $a > 0$ (fixed), the ratios*
$$\frac{A}{I}(x, x+a) \text{ and } \frac{I}{G}(x, x+a)$$
are l.c.m., and the ratio
$$\frac{A}{L}(x, x+a)$$
is c.m. function.

Proof. The following series representations are well-known (see e.g. [6, 9]):
$$\log\frac{A}{G}(x,y) = \sum_{k=1}^{\infty}\frac{1}{2k}\cdot\left(\frac{y-x}{y+x}\right)^{2k}, \tag{3}$$
$$\log\frac{I}{G}(x,y) = \sum_{k=1}^{\infty}\frac{1}{2k+1}\cdot\left(\frac{y-x}{y+x}\right)^{2k}. \tag{4}$$

By substraction, from (3) and (4) we get
$$\log\frac{A}{I}(x,y) = \sum_{k=1}^{\infty}\frac{1}{2k(2k+1)}\cdot\left(\frac{y-x}{y+x}\right)^{2k}, \tag{5}$$

where $\frac{A}{G}(x,y) = \frac{A(x,y)}{G(x,y)}$, etc.

By letting $y = x + a$ in (4), we get that

$$\log \frac{I}{G}(x, x+a) = \sum_{k=1}^{\infty} \frac{a^{2k}}{2k+1} \cdot \left(\frac{1}{2x+a}\right)^{2k}. \tag{6}$$

As $\frac{1}{2x+a}$ is c.m., by Corollary 1, $g(x) = \left(\frac{1}{2x+a}\right)^{2k}$ will be c.m., too. This means that

$$(-1)^n g^{(n)}(x) \geq 0 \text{ for any } x > 0,\ n \geq 0,$$

so by n times differentiation of the series from (6), we get that $\log \frac{I}{G}(x, x+a)$ is c.m., thus $\frac{I}{G}(x, x+a)$ is l.c.m.

The similar proof for $\frac{A}{I}(x, x+a)$ follows from the series representation (5).

Finally, by the known identity (see e.g. [6], [9])

$$\log \frac{I}{G} = \frac{A}{L} - 1 \tag{7}$$

we get the last part of Theorem 1. ∎

Remark 1 *It follows from the above that* $\frac{A}{G}(x, x+a)$, $\frac{A}{H}(x, x+a)$, $\frac{I}{H}(x, x+a)$, $\frac{G}{H}(x, x+a)$ *are all l.c.m. functions.*

Theorem 2 *For any* $a > 0$, *the ratios*

$$\frac{\sqrt{2A^2+G^2}}{I\sqrt{3}}(x, x+a),\ \frac{\sqrt{2A^2+G^2}}{G\sqrt{3}}(x, x+a) \text{ and } \frac{Q}{G}(x, x+a)$$

are l.c.m. functions.

Proof. In paper [8] it is proved that

$$\log \frac{\sqrt{2A^2+G^2}}{I\sqrt{3}} = \sum_{k=1}^{\infty} \frac{1}{2k} \cdot \left(\frac{1}{2k+1} - \frac{1}{3^k}\right) \cdot \left(\frac{y-x}{y+x}\right)^{2k}, \tag{8}$$

while in [9] that

$$\log \frac{\sqrt{2A^2+G^2}}{G\sqrt{3}} = \sum_{k=1}^{\infty} \frac{1}{2k} \cdot \left(1 - \frac{1}{3^k}\right) \cdot \left(\frac{y-x}{y+x}\right)^{2k}. \tag{9}$$

Letting $y = x + a$, by the method of proof of Theorem 1, the first part of Theorem 2 follows. Finally, the identity

$$\log \frac{Q}{G} = \sum_{k=1}^{\infty} \frac{1}{2k-1} \cdot \left(\frac{y-x}{y+x}\right)^{4k-2} \tag{10}$$

appears in [9]. This leads also to the proof of l.c.m. monotonicity of the ratio $\frac{Q}{G}(x, x+a)$. ■

Theorem 3 *For any* $a > 0$, *the ratios*

$$\frac{L}{G}(x, x+a),\ -\frac{H}{L}(x, x+a) \text{ and } \frac{A}{P}(x, x+a)$$

are c.m. functions.

Proof. In [5] (see also [9] for a new proof) it is shown that

$$\frac{L}{G}(x, y) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \cdot \left(\frac{\log x - \log y}{2}\right)^{2k}. \tag{11}$$

Letting $y = x + a$ and remarking that the function $f(x) = \log(x+a) - \log x$ is c.m., by Corollary 1, and by differentiation of the series from (11), we get that $\frac{L}{G}(x, x+a)$ is c.m.

The identity

$$\log \frac{S}{I} = 1 - \frac{H}{L} \tag{12}$$

appears in [9]. Since we have the series representations (see [7], [9])

$$\log \frac{S}{G}(x, y) = \sum_{k=1}^{\infty} \frac{1}{2k-1} \cdot \left(\frac{y-x}{y+x}\right)^{2k} \tag{13}$$

and

$$\log \frac{S}{A}(x, y) = \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} \cdot \left(\frac{y-x}{y+x}\right)^{2k}, \tag{14}$$

by using relation (4), we get $\log\frac{S}{G} - \log\frac{I}{G} = \log\frac{S}{I}$, so

$$\log\frac{S}{I}(x,y) = \sum_{k=1}^{\infty} \frac{2}{4k^2-1} \cdot \left(\frac{y-x}{y+x}\right)^{2k}, \tag{15}$$

thus $\frac{S}{I}(x, x+a)$ is l.c.m., which by (12) implies that the ratio $-\frac{H}{L}$ is l.c.m. function.

Finally, Seiffert's identity (see [10], [9])

$$\log\frac{A}{P}(x,y) = \sum_{k=0}^{\infty} \frac{1}{4^k(2k+1)} \cdot \binom{2k}{k} \cdot \left(\frac{y-x}{y+x}\right)^{2k}, \tag{16}$$

implies the last part of the theorem. ■

Remark 2 *By* (13), (14) *and* (15) *we get also that* $\frac{S}{G}(x, x+a)$, $\frac{S}{A}(x, x+a)$ *and* $\frac{S}{I}(x, x+a)$ *are l.c.m. functions.*

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Sets with prescribed lower and upper weighted densities

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Abstract. It is known that we can prescribe the lower and upper asymptotic and logarithmic density of a set of positive integers. The only limitation is the inequality between asymptotic and logarithmic density. We generalize this result.

1 Introduction

Denote by $\mathbb{N}$ the set of all positive integers, let $A \subset \mathbb{N}$ and let $f : \mathbb{N} \to (0, \infty)$ be a weight function. For $A \subset \mathbb{N}$ and $n \in \mathbb{N}$ denote

$$S_f(A, n) = \sum_{\substack{a \le n \\ a \in A}} f(a), \qquad S_f(n) = \sum_{a \le n} f(a)$$

and define

$$\underline{d}_f(A) = \liminf_{n\to\infty} \frac{S_f(A, n)}{S_f(n)} \qquad \text{and} \qquad \overline{d}_f(A) = \limsup_{n\to\infty} \frac{S_f(A, n)}{S_f(n)}$$

the lower and upper f-densities of A, respectively. In the case when $\underline{d}_f(A) = \overline{d}_f(A)$ we say that A possesses f-density $d_f(A)$.

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Notice that the well-known asymptotic density corresponds to $f(n) = 1$ and the logarithmic density corresponds to $f(n) = \frac{1}{n}$. The concept of weighted densities was introduced in [7] and [1]. The continuity of densities given by the weight function n^α, $\alpha \geq -1$, was studied in [3]. Inequalities between upper and lower weighted densities for different weight functions were proved in [2].

The independence (within admissible bounds) of the asymptotic and logarithmic densities was proved in [6] and [5] showing that for any given real numbers $0 \leq \alpha \leq \beta \leq \gamma \leq \delta \leq 1$ there exists a set $A \subset \mathbb{N}$ such that

$$\underline{d}_1 = \alpha, \quad \underline{d}_{\frac{1}{n}}(A) = \beta, \quad \overline{d}_{\frac{1}{n}}(A) = \gamma, \quad \overline{d}_1(A) = \delta.$$

We generalize this result. We prove that under some assumptions on the weighted densities an analogous result holds. In [4], generalized asymptotic and logarithmic densities over an arithmetical semigroup were considered.

We call a weight function f *regular* if the corresponding weighted density fulfills the condition that for arbitrary positive integers a, b we have

$$d_f(a\mathbb{N} + b) = \frac{1}{a}$$

(f-density of the terms of arbitrary infinite arithmetical progression with the same difference are equal). Note that from this condition follows that

$$\sum_{n=1}^{\infty} f(n) = \infty, \qquad \lim_{n\to\infty} \frac{f(n)}{S_f(n)} = 0. \tag{1}$$

2 Results

The following lemma will be useful

Lemma 1 *Let* f, g *be regular weight functions. Let* B *be a subset of positive integers such that*

$$\underline{d}_f(B) = 0, \quad \overline{d}_f(B) = 1 \quad \textit{and} \quad d_g(B) = 0.$$

Then for any $0 \leq \alpha \leq \beta \leq \gamma \leq \delta \leq 1$, *rational numbers there exists a set* $E \subset \mathbb{N}$ *such that*

$$d_g(E) = \underline{d}_f(E) = \gamma \quad \textit{and} \quad \overline{d}_f(E) = \delta$$

and a set $H \subset \mathbb{N}$ *with the property*

$$d_g(H) = \overline{d}_f(H) = \beta \quad \textit{and} \quad \underline{d}_f(H) = \alpha.$$

Proof. Write γ and δ as fractions with a common denominator, let $\gamma = \frac{p}{t}$ and $\delta = \frac{q}{t}$. Define

$$E = \bigcup_{i=1}^{p}(t\mathbb{N}+i) \cup \Big(B \cap \big(\bigcup_{i=p+1}^{q}(t\mathbb{N}+i)\big)\Big).$$

As $d_g(B) = 0$, therefore

$$d_g(E) = d_g\big(\bigcup_{i=1}^{p}(t\mathbb{N}+i)\big) = \frac{p}{t} = \gamma.$$

Analogously we get $\underline{d}_f(E) = \frac{p}{t} = \gamma$.

Clearly $\overline{d}_f(E) \leq \delta = \frac{q}{t}$. The case $\overline{d}_f(E) < \frac{q}{t}$ yields a contradiction because

$$\begin{aligned} 1 &= \overline{d}_f(B \cap \mathbb{N}) = \overline{d}_f\big(\bigcup_{i=1}^{q}(B \cap (t\mathbb{N}+i))\big) + \overline{d}_f\big(\bigcup_{i=q+1}^{t}(B \cap (t\mathbb{N}+i))\big) \leq \\ &\leq \overline{d}_f(E) + \overline{d}_f\big(\bigcup_{i=q+1}^{t}(t\mathbb{N}+i)\big) < \frac{q}{t} + \frac{t-q}{t} = 1. \end{aligned}$$

In analogous way we can prove the existence of the set H with the prescribed properties. For $\alpha = \frac{r}{t}$ and $\beta = \frac{s}{t}$ let

$$H = \bigcup_{i=1}^{s}(t\mathbb{N}+i) \setminus \Big(B \cap \big(\bigcup_{i=r+1}^{s}(t\mathbb{N}+i)\big)\Big).$$

Note, from the construction of the sets E, H follows $H \subset E$. ∎

Theorem 1 *Let f, g be regular weight functions. Let B be a subset of positive integers such that*

$$\underline{d}_f(B) = 0, \quad \overline{d}_f(B) = 1 \quad \textit{and} \quad d_g(B) = 0.$$

Let $0 \leq \alpha \leq \beta \leq \gamma \leq \delta \leq 1$ be given real numbers. Then there exists a set $A \subset \mathbb{N}$ such that

$$\underline{d}_f(A) = \alpha, \quad \underline{d}_g(A) = \beta, \quad \overline{d}_g(A) = \gamma, \quad \overline{d}_f(A) = \delta.$$

Proof. On the contrary, we suppose that there exist rational numbers $0 < \alpha < \beta < \gamma < \delta < 1$ and $\varepsilon > 0$ such that at least one of the following inequalities does not hold.

$$|\underline{d}_f(A) - \alpha| < \varepsilon, \quad |\underline{d}_g(A) - \beta| < \varepsilon, \quad |\overline{d}_g(A) - \gamma| < \varepsilon, \quad |\overline{d}_f(A) - \delta| < \varepsilon$$

Using the sets H, E defined in the previous lemma we construct a set A such that

$$H \subset A \subset E.$$

Then clearly

$$\underline{d}_f(A) \geq \underline{d}_f(H), \quad \underline{d}_g(A) \geq \underline{d}_g(H)$$

and

$$\overline{d}_f(A) \leq \overline{d}_f(E), \quad \overline{d}_g(A) \leq \overline{d}_g(E).$$

Define the set A by "intertwinning" the sets E and H

$$A = H \bigcup_{k=1}^{\infty} [n_{2k}, n_{2k+1}] \cap E,$$

where $n_1 = 1$ and for $k = 1, 2, \ldots$ let n_k be sufficiently large, such that for some i, j between n_{k-1} and n_k the

$$\left|\frac{S_f(A,i)}{S_f(i)} - \alpha\right| < \varepsilon \quad \text{and} \quad \left|\frac{S_g(A,j)}{S_g(j)} - \beta\right| < \varepsilon \tag{2}$$

inequalities hold. Analogously, sufficiently large n_{2k+1} guarantees the inequalities

$$\left|\frac{S_f(A,m)}{S_f(m)} - \delta\right| < \varepsilon \quad \text{and} \quad \left|\frac{S_g(A,l)}{S_g(l)} - \gamma\right| < \varepsilon \tag{3}$$

for some m, l. From this we can deduce that (2) and (3) hold for infinitely many i, j, m, l what is a contradiction to our assumption. ∎

Roughly speaking, the proved theorem says that under some conditions to prove the existence of a set A with prescribed upper and lower weighted densities it is sufficient to consider only one, the "worst" case.

Lemma 2 *If the function* $f : \mathbb{N} \to (0, \infty)$ *satisfies the conditions*

$$\sum_{n=1}^{\infty} f(n) = \infty, \tag{4}$$

$$\lim_{n\to\infty} \frac{f(n)}{S_f(n)} = 0, \tag{5}$$

then for the function g *defined as*

$$g(n) = \frac{f(n)}{\sum_{i=1}^{n} f(i)} \tag{6}$$

we have

$$\sum_{n=1}^{\infty} g(n) = \infty \tag{7}$$

and

$$\lim_{n\to\infty} \frac{g(n)}{S_g(n)} = 0. \tag{8}$$

Proof. We prove only (7), using the fact $\lim_{n\to\infty} g(n) = 0$ together with (7) it follows immediately (8).

For arbitrary positive integers $m > n$ we have

$$\sum_{k=n}^{m} g(k) = \sum_{k=n}^{m} \frac{f(k)}{\sum_{i=1}^{k} f(i)} \geq \frac{\sum_{k=n}^{m} f(k)}{\sum_{i=1}^{m} f(i)}.$$

The proof is completed by showing that for given n and sufficiently large m

$$\sum_{k=n}^{m} g(k) \geq \frac{1}{2}.$$

From (4) we see that $\sum\limits_{i=1}^{\infty} f(i) = \infty$, therefore for arbitrary $n \in \mathbb{N}$

$$\lim_{m\to\infty} \sum_{k=n}^{m} g(k) \geq \lim_{m\to\infty} \frac{\sum\limits_{k=n}^{m} f(k)}{\sum\limits_{i=1}^{m} f(i)} = 1$$

and the lemma follows. ∎

Theorem 2 *Let the functions $f, g : \mathbb{N} \to (0, \infty)$ satisfy the assumptions (4)–(6). Then there exists a set $B \subset \mathbb{N}$ such that*

$$\underline{d}_f(B) = 0, \quad \overline{d}_f(B) = 1 \quad \textit{and} \quad d_g(B) = 0.$$

Proof. Consider

$$B = \bigcup_{k=1}^{\infty} [n_{2k}, n_{2k+1}].$$

Let $n_1 = 1$. Assume $n_1, n_2, \ldots, n_{2k-1}$ are given. We are looking for n_{2k} such that

$$\frac{f(n)}{f(1) + f(2) + \cdots + f(n)} < \frac{1}{k+1} \quad \text{for arbitrary } n \geq n_{2k}, \tag{9}$$

$$\frac{f(1)+f(2)+\cdots+f(n_{2k-1})}{f(1)+f(2)+\cdots+f(n_{2k})} < \frac{1}{k}, \tag{10}$$

$$\frac{g(1)+g(2)+\cdots+g(n_{2k-1})}{g(1)+g(2)+\cdots+g(n_{2k})} < \frac{1}{k}, \tag{11}$$

$$g(1)+g(2)+\cdots+g(n_{2k}) > k^2. \tag{12}$$

Moreover, let n_{2k+1} satisfy the inequalities

$$\frac{k-1}{k+1} < \frac{f(n_{2k})+f(n_{2k}+1)+\cdots+f(n_{2k+1})}{f(1)+f(2)+\cdots+f(n_{2k+1})} < \frac{k}{k+1}. \tag{13}$$

Inequalities (10)–(13) follow from the prescribed conditions on the functions f and g.

By (10) we have $\underline{d}_f(B) = 0$, by (11) we have $\underline{d}_g(B) = 0$ and taking into account (13) we get $\overline{d}_f(B) = 0$.

We proceed to show that $\overline{d}_g(B) = 0$. In virtue of [2], Lemma 2.1 it is sufficient to consider only the values $\frac{S_g(B,n_k)}{S_g(n_k)}$. We have

$$\frac{S_g(B,n_{2k+1})}{S_g(n_{2k+1})} \leq$$

$$\frac{g(1)+g(2)+\cdots+g(n_{2k-1})+g(n_{2k})+g(n_{2k}+1)+\cdots+g(n_{2k+1})}{g(1)+g(2)+\cdots+g(n_{2k+1})} <$$

$$\frac{1}{k} + \frac{g(n_{2k})+g(n_{2k}+1)+\cdots+g(n_{2k+1})}{g(1)+g(2)+\cdots+g(n_{2k})} \leq \frac{1}{k} + \frac{\frac{f(n_{2k})+f(n_{2k}+1)+\cdots+f(n_{2k+1})}{f(1)+f(2)+\cdots+f(n_{2k-1})}}{g(1)+g(2)+\cdots+g(n_{2k})}.$$

Using (13) we can show the inequality

$$\frac{f(n_{2k})+f(n_{2k}+1)+\cdots+f(n_{2k+1})}{f(1)+f(2)+\cdots+f(n_{2k-1})} < k.$$

Using this together with (12) we have

$$\frac{S_g(B,n_{2k+1})}{S_g(n_{2k+1})} < \frac{1}{k} + \frac{k}{k^2} = \frac{2}{k}$$

and hence $\overline{d}_g(B) = 0$ and $d_g(B) = 0$ follows. ■

It is not hard to show that if a monotone function f satisfies (4)–(5), then it is regular (see, e.g. [2], Example 2.1). If the function g defined by (6) is monotonely decreasing, then it is regular, too. We have

Corollary 1 *Let the monotone function* f *and monotone decreasing function* g *satisfy the assumptions (4)–(6). Let* $0 \le \alpha \le \beta \le \gamma \le \delta \le 1$ *be given real numbers. Then there exists a set* $A \subset \mathbb{N}$ *such that*

$$\underline{d}_f(A) = \alpha, \quad \underline{d}_g(A) = \beta, \quad \overline{d}_g(A) = \gamma, \quad \overline{d}_f(A) = \delta.$$

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An elementary proof that almost all real numbers are normal

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Abstract. A real number is called normal if every block of digits in its expansion occurs with the same frequency. A famous result of Borel is that almost every number is normal. Our paper presents an elementary proof of that fact using properties of a special class of functions.

1 Introduction

The concept of normal number was introduced by Borel. A number is called normal if in its base b expansion every block of digits occurs with the same frequency. More exact definition is

Definition 1 *A real number* $x \in (0,1)$ *is called simply normal to base* $b \geq 2$ *if its base* b *expansion is* $0.c_1c_2c_3\ldots$ *and*

$$\lim_{N\to\infty} \frac{\#\{n \leq N \mid c_n = a\}}{N} = \frac{1}{b} \quad \textit{for every } a \in \{0,\ldots,b-1\}.$$

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A number is called normal to base b *if for every block of digits* $a_1 \dots a_L$, $L \geq 1$

$$\lim_{N\to\infty} \frac{\#\{n \leq N - L \mid c_{n+1} = a_1, \dots, c_{n+L} = a_L\}}{N} = \frac{1}{b^L}\,.$$

A number is called absolutely normal if it is normal to every base $b \geq 2$.

A famous result of Borel [1] is

Theorem 1 *Almost every real number is absolutely normal.*

This theorem can be proved in many ways. Some proofs use uniform distribution [5], combinatorics [7], probability [8] or ergodic theory [2]. There are also some elementary proofs almost avoiding higher mathematics. Kac [3] proves the theorem for simply normal numbers to base 2 using Rademacher functions and Beppo Levi's Theorem. Nillsen [6] also considers binary case. He uses series of integrals of step functions and avoids usage of measure theory in the proof by defining a null set in a different way. Khoshnevisan [4] makes a survey about known results on normal numbers and their consequences in diverse areas in mathematics and computer science.

This paper presents another elementary proof of Theorem 1. Our proof is based on the fact that a bounded monotone function has finite derivative in almost all points. We also use the fact that a countable union of null sets is a null set.

Here is a sketch of the proof. We introduce a special class of functions. In Section 2 we prove elementary properties of the functions $\mathcal{F}$. We prove boundedness and monotonicity and assuming that the derivative $\mathcal{F}'(x)$ exists in point x we prove that the product (5) has finite value. We deduce that the product (5) has finite value for almost every x. In Section 3 we prove that every non-normal number belongs to some set P. We take a particular function $\mathcal{F}$. We finish the proof by showing that for elements of P the product (5) does not have finite value.

For the proof of Theorem 1 it is obviously sufficient to consider only numbers in the interval $(0, 1)$.

Definition 2 *Let* $\mathbf{b} = \{b_k\}_{k=1}^{\infty}$ *be a sequence of integers* $b_k \geq 2$. *Let* $\boldsymbol{\omega} = \{\omega_k\}_{k=1}^{\infty}$ *be a sequence of divisions of the interval* $[0, 1]$,

$$\omega_k = \{f_k(c)\}_{c=0}^{b_k}, \quad f_k(0) = 0\,, \quad f_k(c) < f_k(c+1)\,, \quad f_k(b_k) = 1\,.$$

Put

$$\Delta_k(c) := f_k(c+1) - f_k(c)\,.$$

Function $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}\colon [0,1] \to [0,\infty)$ *corresponding to* $\mathbf{b}$ *and* $\boldsymbol{\omega}$ *is defined as follows. For* $x \in [0,1)$*, let*

$$x = \sum_{n=1}^{\infty} \frac{c_n}{\prod_{k=1}^{n} b_k} \tag{1}$$

be its $\{b_k\}_{k=1}^{\infty}$*-Cantor series. Then*

$$\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(x) := \sum_{n=1}^{\infty} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k)\,.$$

We define $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(1) = 1$*.*

The reason for defining $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(1) = 1$ is that the actual range of $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}$ is $\subseteq [0,1]$. This is proved in Lemma 2.

2 Properties of the function $\mathcal{F}$

In this section we derive some basic properties of a general function $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}$.

Lemma 1 allows us to express a particular value $\mathcal{F}(x)$ in terms of values of some other function $\mathcal{F}$.

For $N \in \mathbb{N}$ define $\mathbf{b}^{(N)} := \{b_n^{(N)}\}_{n=1}^{\infty}$, $\boldsymbol{\omega}^{(N)} := \{\omega_n^{(N)}\}_{n=1}^{\infty}$ and $\{\Delta_n^{(N)}\}_{n=1}^{\infty}$ by

$$b_n^{(N)} := b_{N+n}\,, \quad \omega_n^{(N)} := \omega_{N+n}\,, \quad \Delta_n^{(N)} := \Delta_{N+n}\,.$$

Moreover, for $x = \sum_{n=1}^{\infty} \frac{c_k}{\prod_{k=1}^{n} b_k} \in (0,1)$ define

$$x^{(N)} := \sum_{n=1}^{\infty} \frac{c_{N+n}}{\prod_{k=1}^{n} b_k^{(N)}}\,.$$

Lemma 1 (Shift property) *We have*

$$\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(x) = \sum_{n=1}^{N} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k) + \prod_{k=1}^{N} \Delta_k(c_k) \cdot \mathcal{F}_{\mathbf{b}^{(N)},\boldsymbol{\omega}^{(N)}}(x^{(N)})\,.$$

Proof. An easy computation yields

$$\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(x) = \sum_{n=1}^{\infty} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k)$$

$$
\begin{aligned}
&= \sum_{n=1}^{N} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k) \\
&+ \prod_{k=1}^{N} \Delta_k(c_k) \sum_{n=1}^{\infty} f_{N+n}(c_{N+n}) \prod_{k=1}^{n-1} \Delta_{N+k}(c_{N+k}) \\
&= \sum_{n=1}^{N} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k) + \prod_{k=1}^{N} \Delta_k(c_k) \cdot \mathcal{F}_{\mathbf{b}^{(N)},\boldsymbol{\omega}^{(N)}}(x^{(N)})\,.
\end{aligned}
$$

■

Lemma 2 (Range) *For* $x \in [0,1]$ *the value* $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(x) \in [0,1]$.

Proof. First we prove that for every $\mathbf{b}, \boldsymbol{\omega}$ and every $x = \sum_{n=1}^{N} \frac{c_n}{\prod_{k=1}^{n} b_k}$

$$
\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}\Big(\sum_{n=1}^{N} \frac{c_n}{\prod_{k=1}^{n} b_k}\Big) = \sum_{n=1}^{N} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k) \le 1\,. \tag{2}
$$

We will proceed by induction on N.

For $N = 1$ we have $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}\big(\frac{c_1}{b_1}\big) = f_1(c_1) \le 1$.

For $N+1$ we use Lemma 1. By the induction assumption we have

$$
\mathcal{F}_{\mathbf{b}^{(1)},\boldsymbol{\omega}^{(1)}}(x^{(1)}) \le 1\,.
$$

Hence

$$
\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(x) = f_1(c_1) + \Delta_1(c_1)\mathcal{F}_{\mathbf{b}^{(1)},\boldsymbol{\omega}^{(1)}}(x^{(1)}) \le f_1(c_1) + \Delta_1(c_1) = f_1(c_1+1) \le 1\,.
$$

Now we use (2) and pass to the limit $N \to \infty$. For $x = \sum_{n=1}^{\infty} \frac{c_n}{\prod_{k=1}^{n} b_k}$ we have

$$
\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(x) = \lim_{N\to\infty} \sum_{n=1}^{N} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k) \le 1\,.
$$

■

Lemma 3 (Monotonicity) *The function* $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}$ *is nondecreasing.*

Proof. Let $0 \le x < y < 1$ be two numbers with

$$x = \sum_{n=1}^{\infty} \frac{c_n}{\prod_{k=1}^{n} b_k} \quad \text{and} \quad y = \sum_{n=1}^{\infty} \frac{d_n}{\prod_{k=1}^{n} b_k}\,.$$

We prove that $\mathcal{F}_{\mathfrak{b},\boldsymbol{\omega}}(x) \le \mathcal{F}_{\mathfrak{b},\boldsymbol{\omega}}(y)$.

Let N be the integer such that $c_n = d_n$ for $n \le N-1$ and $c_N < d_N$. Then Lemmas 1 and 2 imply

$$\begin{aligned}
\mathcal{F}_{\mathfrak{b},\boldsymbol{\omega}}(x) &= \sum_{n=1}^{N} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k) + \prod_{k=1}^{N} \Delta_k(c_k) \cdot \mathcal{F}_{\mathfrak{b}^{(N)},\boldsymbol{\omega}^{(N)}}(x^{(N)}) \\
&\le \sum_{n=1}^{N} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k) + \prod_{k=1}^{N} \Delta_k(c_k) \\
&= \sum_{n=1}^{N-1} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k) + f_N(c_N+1) \prod_{k=1}^{N-1} \Delta_k(c_k) \\
&\le \sum_{n=1}^{N-1} f_n(d_n) \prod_{k=1}^{n-1} \Delta_k(d_k) + f_N(d_N) \prod_{k=1}^{N-1} \Delta_k(d_k) \\
&\le \sum_{n=1}^{\infty} f_n(d_n) \prod_{k=1}^{n-1} \Delta_k(d_k) = \mathcal{F}_{\mathfrak{b},\boldsymbol{\omega}}(y)\,.
\end{aligned}$$

■

For $k \in \mathbb{N}$ and $c \in \{0, \dots, b_k\}$ define $\overline{f}_k(c) := 1 - f_k(b_k - c)$. Put $\overline{\boldsymbol{\omega}} := \{\{\overline{f}_k(c)\}_{c=0}^{b_k}\}_{k=1}^{\infty}$.

Lemma 4 (Symmetry) *For every* $x = \sum_{n=1}^{N} \frac{c_n}{\prod_{k=1}^{n} b_k}$ *we have*

$$\mathcal{F}_{\mathfrak{b},\boldsymbol{\omega}}(1-x) = 1 - \mathcal{F}_{\mathfrak{b},\overline{\boldsymbol{\omega}}}(x)\,. \tag{3}$$

Proof. We have

$$\overline{\Delta}_k(c) = \overline{f}_k(c+1) - \overline{f}_k(c) = \Delta_k(b_k - c - 1)\,.$$

Now we will proceed by induction.

For $N = 1$ we have

$$\mathcal{F}_{\mathfrak{b},\boldsymbol{\omega}}(1-x) = f_1(b_1 - c_1) = 1 - \overline{f}_1(c_1) = 1 - \mathcal{F}_{\mathfrak{b},\overline{\boldsymbol{\omega}}}\Big(\frac{c_1}{b_1}\Big) = 1 - \mathcal{F}_{\mathfrak{b},\overline{\boldsymbol{\omega}}}(x)\,.$$

Now suppose that (3) holds for N,

$$\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}\Big(1-\sum_{n=1}^{N}\frac{e_n}{\prod_{k=1}^{n}b_k}\Big)=1-\mathcal{F}_{\mathbf{b},\overline{\boldsymbol{\omega}}}\Big(\sum_{n=1}^{N}\frac{e_n}{\prod_{k=1}^{n}b_k}\Big)$$

for every possible sequence $\{e_n\}_{n=1}^{\infty}$. Then using Lemma 1 we obtain for $x=\sum_{n=1}^{N+1}\frac{c_n}{\prod_{k=1}^{n}b_k}$ that

$$\begin{aligned}
&\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(1-x)\\
&=\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}\Big(\frac{b_1-c_1-1}{b_1}+\frac{1}{b_1}\Big(\sum_{n=1}^{N-1}\frac{b_{n+1}-c_{n+1}-1}{\prod_{k=1}^{n}b_{k+1}}+\frac{b_{(N-1)+1}-c_{(N-1)+1}}{\prod_{k=1}^{N-1}b_{k+1}}\Big)\Big)\\
&=f_1(b_1-c_1-1)+\Delta_1(b_1-c_1-1)\mathcal{F}_{\mathbf{b}^{(1)},\boldsymbol{\omega}^{(1)}}(1-x^{(1)})\\
&=f_1(b_1-c_1-1)+\Delta_1(b_1-c_1-1)\big(1-\mathcal{F}_{\mathbf{b}^{(1)},\overline{\boldsymbol{\omega}}^{(1)}}(x^{(1)})\big)\\
&=1-\overline{f}_1(c_1+1)+\overline{\Delta}_1(c_1)\big(1-\mathcal{F}_{\mathbf{b}^{(1)},\overline{\boldsymbol{\omega}}^{(1)}}(x^{(1)})\big)\\
&=1-\big(\overline{f}_1(c_1)+\overline{\Delta}_1(c_1)\mathcal{F}_{\mathbf{b}^{(1)},\overline{\boldsymbol{\omega}}^{(1)}}(x^{(1)})\big)=1-\mathcal{F}_{\mathbf{b},\overline{\boldsymbol{\omega}}}(x)\,.
\end{aligned}$$

■

Remark 1 *One can prove that if* $\prod_{k=1}^{\infty}\max_{c=0,\dots,b_k-1}\Delta_k(c)=0$ *then* $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}$ *is continuous on the interval* $[0,1]$. *One can then extend Lemma 4 for every* $x\in[0,1]$.

Lemma 5 (Difference) *For every* $N\in\mathbb{N}$

$$\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}\Big(\sum_{n=1}^{N-1}\frac{c_n}{\prod_{k=1}^{n}b_k}+\frac{c_N+1}{\prod_{k=1}^{N}b_k}\Big)-\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}\Big(\sum_{n=1}^{N}\frac{c_n}{\prod_{k=1}^{n}b_k}\Big)=\prod_{k=1}^{N}\Delta_k(c_k)\,. \quad (4)$$

Proof. Denote the left-hand side of (4) by LHS. Then if $c_N\le b_N-2$ then

$$\begin{aligned}
\text{LHS}&=\prod_{k=1}^{N-1}\Delta_k(c_k)\cdot\Big(\mathcal{F}_{\mathbf{b}^{(N-1)},\boldsymbol{\omega}^{(N-1)}}\Big(\frac{c_N+1}{b_N}\Big)-\mathcal{F}_{\mathbf{b}^{(N-1)},\boldsymbol{\omega}^{(N-1)}}\Big(\frac{c_N}{b_N}\Big)\Big)\\
&=\prod_{k=1}^{N-1}\Delta_k(c_k)\cdot\big(f_N(c_N+1)-f_N(c_N)\big)=\prod_{k=1}^{N}\Delta_k(c_k)\,.
\end{aligned}$$

In the case $c_N = b_N - 1$ we apply the first case on the function $\mathcal{F}_{\mathbf{b},\overline{\boldsymbol{\omega}}}$,

$$\begin{aligned}
\text{LHS} &= \left(1 - \mathcal{F}_{\mathbf{b},\overline{\boldsymbol{\omega}}}\Big(\sum_{n=1}^{N-1} \frac{b_n - c_n - 1}{\prod_{k=1}^{n} b_k}\Big)\right) \\
&\quad - \left(1 - \mathcal{F}_{\mathbf{b},\overline{\boldsymbol{\omega}}}\Big(\sum_{n=1}^{N-1} \frac{b_n - c_n - 1}{\prod_{k=1}^{n} b_k} + \frac{1}{\prod_{k=1}^{N} b_k}\Big)\right) \\
&= \prod_{k=1}^{N} \overline{\Delta}_k(b_n - c_n - 1) = \prod_{k=1}^{N} \Delta_k(c_k)\,.
\end{aligned}$$

■

In the following text we will use the symbol

$$\Theta_k(c) := b_k \Delta_k(c)\,.$$

Lemma 6 (Derivative) *Let* $x = \sum_{n=1}^{\infty} \frac{c_n}{\prod_{k=1}^{n} b_k} \in (0,1)$. *Suppose that the derivative* $\mathcal{F}'_{\mathbf{b},\boldsymbol{\omega}}(x)$ *exists and is finite. Then*

$$\mathcal{F}'_{\mathbf{b},\boldsymbol{\omega}}(x) = \prod_{k=1}^{\infty} \Theta_k(c_k)\,. \tag{5}$$

In particular, this product has a finite value.

Proof. We have

$$\lim_{N\to\infty} \frac{\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}\Big(\sum_{n=1}^{N-1} \frac{c_n}{\prod_{k=1}^{n} b_k} + \frac{c_N + 1}{\prod_{k=1}^{N} b_k}\Big) - \mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}\Big(\sum_{n=1}^{N} \frac{c_n}{\prod_{k=1}^{n} b_k}\Big)}{\Big(\sum_{n=1}^{N-1} \frac{c_n}{\prod_{k=1}^{n} b_k} + \frac{c_N + 1}{\prod_{k=1}^{N} b_k}\Big) - \Big(\sum_{n=1}^{N} \frac{c_n}{\prod_{k=1}^{n} b_k}\Big)} \tag{6}$$

$$= \lim_{N\to\infty} \left(\frac{\sum_{n=1}^{N-1} \frac{c_n}{\prod_{k=1}^{n} b_k} + \frac{c_N + 1}{\prod_{k=1}^{N} b_k} - x}{\frac{1}{\prod_{k=1}^{N} b_k}} \right. \tag{7}$$

$$\cdot \frac{\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}\Big(\sum\limits_{n=1}^{N-1} \frac{c_n}{\prod_{k=1}^{n} b_k} + \frac{c_N+1}{\prod_{k=1}^{N} b_k}\Big) - \mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(x)}{\sum\limits_{n=1}^{N-1} \frac{c_n}{\prod_{k=1}^{n} b_k} + \frac{c_N+1}{\prod_{k=1}^{N} b_k} - x} \tag{8}$$

$$+ \frac{x - \sum\limits_{n=1}^{N} \frac{c_n}{\prod_{k=1}^{n} b_k}}{\frac{1}{\prod_{k=1}^{N} b_k}} \cdot \frac{\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(x) - \mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}\Big(\sum\limits_{n=1}^{N} \frac{c_n}{\prod_{k=1}^{n} b_k}\Big)}{x - \sum\limits_{n=1}^{N} \frac{c_n}{\prod_{k=1}^{n} b_k}} \Bigg) \tag{9}$$

$$= \mathcal{F}'_{\mathbf{b},\boldsymbol{\omega}}(x) \lim_{N\to\infty} \frac{\frac{1}{\prod_{k=1}^{N} b_k}}{\frac{1}{\prod_{k=1}^{N} b_k}} = \mathcal{F}'_{\mathbf{b},\boldsymbol{\omega}}(x)\,.$$

Existence of $\mathcal{F}'_{\mathbf{b},\boldsymbol{\omega}}(x)$ implies that limits of (8) and of the second fraction in (9) are equal to $\mathcal{F}'_{\mathbf{b},\boldsymbol{\omega}}(x)$. Hence the limit (6) exists and is equal to $\mathcal{F}'_{\mathbf{b},\boldsymbol{\omega}}(x)$. In the case that $x = \sum\limits_{n=1}^{N} \frac{c_n}{\prod_{k=1}^{n} b_k}$ we obtain that (6) $= \mathcal{F}'_{\mathbf{b},\boldsymbol{\omega}}(x)$ immediately.

On the other hand, Lemma 5 implies that

$$(6) = \lim_{N\to\infty} \frac{\prod\limits_{k=1}^{N} \Delta_k(c_k)}{\frac{1}{\prod_{k=1}^{N} b_k}} = \prod_{k=1}^{\infty} \Theta_k(c_k)\,.$$

■

Corollary 1 *For almost every $x \in [0,1]$ the derivative $\mathcal{F}'_{\mathbf{b},\boldsymbol{\omega}}(x)$ exists and is finite. In particular, for almost every $x = \sum\limits_{n=1}^{\infty} \frac{c_n}{\prod_{k=1}^{n} b_k}$ the product $\prod_{n=1}^{\infty} \Theta_k(c_k)$ exists and is finite (possibly zero).*

Proof. The function $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}$ is bounded and nondecreasing, hence in almost all points it has a finite derivative. According to Lemma 6 we obtain that the product (5) is finite. ■

3 Main result

Our main result is a proof of Theorem 1.

Proof. A number $x \in (0,1)$ is not absolutely normal if there exist $b \geq 2$, $L \in \mathbb{N}$ and $a_1, \ldots, a_L \in \{0, \ldots, b-1\}$ such that if $x = \sum_{n=1}^{\infty} \frac{c_n}{b^n}$ then

$$\liminf_{n\to\infty} \frac{\#\{n \leq N - L \mid c_{n+i} = a_i, i = 1, \ldots, L\}}{N} < \frac{1}{b^L}.$$

Then there exists $s \in \{0, \ldots, L-1\}$ such that

$$\liminf_{n\to\infty} \frac{\#\{n \leq N - L, n \equiv s \ (\mathrm{mod}\, L) \mid c_{n+i} = a_i, i = 1, \ldots, L\}}{N} < \frac{1}{Lb^L}.$$

Hence for some rational $\beta < \frac{1}{Lb^L}$

$$\liminf_{n\to\infty} \frac{\#\{n \leq N - L, n \equiv s \ (\mathrm{mod}\, L) \mid c_{n+i} = a_i, i = 1, \ldots, L\}}{N} \leq \beta. \tag{10}$$

Denote by $R_{b,L,\mathbf{a},s,\beta}$ the set of all $x = \sum_{n=1}^{\infty} \frac{c_n}{b^n}$ satisfying (10). The result of the previous paragraph is that the set of not absolutely normal numbers is a subset of

$$\bigcup_{b=2}^{\infty} \bigcup_{L=1}^{\infty} \bigcup_{a_1, \ldots, a_L = 0}^{b-1} \bigcup_{s=0}^{L-1} \bigcup_{\beta \in (0, \frac{1}{Lb^L}) \cap \mathbb{Q}} R_{b,L,\mathbf{a},s,\beta}.$$

It is sufficient to prove that every set $R_{b,L,\mathbf{a},s,\beta}$ has zero measure. Then the set of not absolutely normal numbers is a subset of a countable union of null sets, hence it is a null set.

Let $b \geq 2$, $L \in \mathbb{N}$, $a_1, \ldots, a_L \in \{0, \ldots, b-1\}$, $s \in \{0, \ldots, L-1\}$ and $\beta \in \left(0, \frac{1}{Lb^L}\right)$. Put $A = a_1 b^{L-1} + a_2 b^{L-2} + \cdots + a_L$. Let

$$x = \sum_{n=1}^{\infty} \frac{c_n}{b^n} = \sum_{n=1}^{s} \frac{d_n}{b^n} + \frac{1}{b^s} \sum_{n=1}^{\infty} \frac{d_{s+n}}{b^{Ln}} \in R_{b,L,\mathbf{a},s,\beta}.$$

Then obviously,

$$\begin{aligned}\#\big\{n \leq N - L, n \equiv s \ (\mathrm{mod}\, L) \bigm| c_{n+i} = a_i, i = 1, \ldots, L\big\}& \\ = \#\Big\{s < n \leq \Big[\frac{N-s}{L}\Big] \Bigm| d_n = A\Big\}&.\end{aligned}$$

Hence

$$\liminf_{M\to\infty} \frac{\#\{s < n \leq M \mid d_n = A\}}{M} = \liminf_{N\to\infty} \frac{\#\big\{s < n \leq \big[\frac{N-s}{L}\big] \bigm| d_n = A\big\}}{\big[\frac{N-s}{L}\big]}$$

$$
\begin{aligned}
&= \liminf_{N\to\infty} \frac{N}{\left[\frac{N-s}{L}\right]} \frac{\#\big\{n \le N-L, n \equiv s \ (\mathrm{mod} L) \ \big|\ c_{n+i} = a_i, i = 1,\dots,L\big\}}{N} \\
&\le L\beta\,.
\end{aligned}
$$

From this we obtain that $R_{b,L,\mathbf{a},s,\beta} \subseteq P$, where

$$
P = \Big\{ x = \sum_{n=1}^{s} \frac{d_n}{b^n} + \frac{1}{b^s} \sum_{n=1}^{\infty} \frac{d_{s+n}}{b^{Ln}} \ \Big|\ \liminf_{N\to\infty} \frac{\#\{s < n \le N \mid d_n = A\}}{N} \le L\beta \Big\}\,.
$$

Thus it is sufficient to prove that the set P has zero measure.

Let $\alpha \in \big(L\beta, \frac{1}{b^L}\big)$. For $t \in [0,1]$ define

$$
\varphi_\alpha(t) := t^\alpha \Big(\frac{b^L - t}{b^L - 1}\Big)^{1-\alpha}.
$$

The function φ_α is continuous with $\varphi_\alpha(0) = 0$, $\varphi_\alpha(1) = 1$ and $\varphi'_\alpha(1) = \dfrac{\alpha b^L - 1}{b^L - 1} < 0$. Hence there is $T \in (0,1)$ with $\varphi_\alpha(T) = 1$. For $u \in (0,1)$ put

$$
\psi(u) := \varphi_u(T) = T^u \Big(\frac{b^L - T}{b^L - 1}\Big)^{1-u}.
$$

The function ψ is continuous and decreasing with $\psi(\alpha) = 1$.

Consider the function $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}$ corresponding to $\mathbf{b} = \{b_k\}_{k=1}^\infty$ with

$$
b_k = \begin{cases} b, & \text{if } k \le s, \\ b^L, & \text{if } k > s, \end{cases}
$$

and $\boldsymbol{\omega} = \{\omega_k\}_{k=1}^\infty$ with

$$
\Delta_k(d) = \begin{cases} \frac{1}{b}, & \text{if } k \le s, \\ \frac{T}{b^L}, & \text{if } k > s \text{ and } d = A, \\ \frac{b^L - T}{b^L(b^L - 1)}, & \text{if } k > s \text{ and } d \ne A. \end{cases}
$$

We have

$$
\Theta_k(d) = \begin{cases} 1, & \text{if } k \le s\,, \\ T, & \text{if } k > s \text{ and } d = A, \\ \frac{b^L - T}{b^L - 1}, & \text{if } k > s \text{ and } d \ne A. \end{cases}
$$

Now Corollary 1 implies that for almost every $x = \sum_{n=1}^{s} \frac{d_n}{b^n} + \frac{1}{b^s} \sum_{n=1}^{\infty} \frac{d_{n+s}}{b^{Ls}}$ the following product exists and is finite

$$\begin{aligned}\prod_{n=1}^{\infty} \Theta_n(d_n) &= \lim_{N\to\infty} T^{\#\{s<n\le N | d_n = A\}} \Big(\frac{b^L - T}{b^L - 1}\Big)^{N - \#\{s<n\le N | d_n = A\}} \\ &= \lim_{N\to\infty} \Big(\psi\Big(\frac{\#\{s < n \le N \mid d_n = A\}}{N}\Big)\Big)^N. \end{aligned} \tag{11}$$

Now suppose that $x \in P$. Then

$$\begin{aligned}\limsup_{N\to\infty} \psi\Big(\frac{\#\{s < n \le N \mid d_n = A\}}{N}\Big) &= \psi\Big(\liminf_{N\to\infty} \frac{\#\{s < n \le N \mid d_n = A\}}{N}\Big) \\ &\ge \psi(L\beta) > \psi(\alpha) = 1,\end{aligned}$$

hence

$$\limsup_{N\to\infty} \Big(\psi\Big(\frac{\#\{s < n \le N \mid d_n = A\}}{N}\Big)\Big)^N = \infty,$$

contradicting finiteness of (11). Thus the set P has zero measure. ■

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Some characterizations of special curves in the Euclidean space E^4

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Abstract. In this work, first, we give some characterizations of helices and ccr curves in the Euclidean 4-space. Thereafter, relations among Frenet-Serret invariants of Bertrand curve of a helix are presented. Moreover, in the same space, some new characterizations of involute of a helix are presented.

1 Introduction

In the local differential geometry, we think of curves as a geometric set of points, or locus. Intuitively, we are thinking of a curve as the path traced out by a particle moving in E^4. So, investigating position vectors of the curves is a classical aim to determine behavior of the particle (curve).

Natural scientists have long held a fascination, sometimes bordering on mystical obsession for helical structures in nature. Helices arise in nanosprings, carbon nanotubes, $\alpha-$helices, DNA double and collagen triple helix, the double helix shape is commonly associated with DNA, since the double helix is

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structure of DNA [3]. This fact was published for the first time by Watson and Crick in 1953 [25]. They constructed a molecular model of DNA in which there were two complementary, antiparallel (side-by-side in opposite directions) strands of the bases guanine, adenine, thymine and cytosine, covalently linked through phosphodiesterase bonds. Each strand forms a helix and two helices are held together through hydrogen bonds, ionic forces, hydrophobic interactions and van der Waals fores forming a double helix, lipid bilayers, bacterial flagella in Salmonella and E. coli, aerial hyphae in actinomycete, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, helical staircases and sea shells (helico-spiral structures) [4, 5].

Helix is one of the most fascinating curves in science and nature. Also we can see the helix curve or helical structures in fractal geometry, for instance hyperhelices [23]. In the field of computer aided design and computer graphics, helices can be used for the tool path description, the simulation of kinematic motion or the design of highways, etc. [26]. From the view of differential geometry, a helix is a geometric curve with non-vanishing constant curvature κ and non-vanishing constant torsion τ [2]. The helix may be called a *circular helix* or *W-curve* [12, 17].

It is known that straight line ($\kappa(s) = 0$) and circle ($\tau(s) = 0$) are degenerate-helix examples [13]. In fact, circular helix is the simplest three-dimensional spirals. One of the most interesting spiral examples are k-Fibonacci spirals. These curves appear naturally from studying the k-Fibonacci numbers $\{F_{k,n}\}_{n=0}^{\infty}$ and the related hyperbolic k-Fibonacci function. Fibonacci numbers and the related Golden Mean or Golden section appear very often in theoretical physics and physics of the high energy particles [7, 8]. Three-dimensional k-Fibonacci spirals was studied from a geometric point of view in [9].

Indeed, in Euclidean 3-space E^3, a helix is a special case of the *general helix*. A curve of constant slope or general helix in Euclidean 3-space is defined by the property that the tangent makes a constant angle with a fixed straight line called the axis of the general helix. A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 (see [22] for details) says that: *A necessary and sufficient condition that a curve be a general helix is that the ratio* $\frac{\kappa}{\tau}$ *is constant along the curve, where* κ *and* τ *denote the curvature and the torsion, respectively.*

The notation of a generalized helix in E^3 can be generalized to higher dimensions in the same definition is proposed but in E^n, i.e., a generalized helix as a curve $\psi : R \to E^n$ such that its tangent vector forms a constant angle with a given direction U in E^n [20].

Two curves which, at any point, have a common principal normal vector are called Bertrand curves. The notion of Bertrand curves was discovered by J. Bertrand in 1850. Bertrand curves have been investigated in E^n and many characterizations are given in [10]. Thereafter, by theory of relativity, investigators extend some of classical differential geometry topics to Lorentzian manifolds. For instance, one can see, Bertrand curves in E_1^n [6], in E_1^3 for null curves [1], and in E_1^4 for space-like curves [27]. In the fourth section of this paper, we follow the same procedure as in [27].

In this work, first, we aim to give some new characterizations of helices and ccr curves in terms of recently obtained theorems. Thereafter, we investigate relations among Frenet-Serret invariants of Bertrand curve couples, when one of is helix, in the Euclidean 4-space. Moreover, we observe that Bertrand curve of a helix is also a helix; and cannot be a spherical curve, a general helix and a 3-type slant helix, respectively. We also express some characterizations of involute of a helix. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

2 Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E^4 are briefly presented (A more complete elementary treatment can be found in [11]).

Let $\alpha : I \subset \mathbb{R} \to E^4$ be an arbitrary curve in the Euclidean space E^4. Recall that the curve α is said to be of unit speed (or parameterized by arclength function s) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle ., . \rangle$ is the standard scalar (inner) product of E^4 given by

$$\langle \xi, \zeta \rangle = \xi_1\zeta_1 + \xi_2\zeta_2 + \xi_3\zeta_3 + \xi_4\zeta_4,$$

for each $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$, $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in E^4$. In particular, the norm of a vector $\xi \in E^4$ is given by

$$\|\xi\| = \sqrt{\langle \xi, \xi \rangle}.$$

Let $\{T(s), N(s), B(s), E(s)\}$ be the moving frame along the unit speed curve α. Then the Frenet-Serret formulas are given by [10, 21]

$$\begin{bmatrix} T' \\ N' \\ B' \\ E' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \\ E \end{bmatrix}. \tag{1}$$

Here T, N, B and E are called, respectively, the tangent, the normal, the binormal and the trinormal vector fields of the curve and the functions $\kappa(s), \tau(s)$ and $\sigma(s)$ are called, respectively, the first, the second and the third curvature of a curve in E^4. Also, the functions $H_1 = \frac{\kappa}{\tau}$ and $H_2 = \frac{H_1'}{\sigma}$ are called *Harmonic Curvatures* of the curves in E^4, where $\kappa \neq 0, \tau \neq 0$ and $\sigma \neq 0$. Let $\alpha : I \subset R \to E^4$ be a regular curve. If tangent vector field T of α forms a constant angle with unit vector U, this curve is called an inclined curve or a general helix in E^4. Recall that, A curve $\psi = \psi(s)$ is called a 3-type slant helix if the trinormal lines of α make a constant angle with a fixed direction in E^4 [24]. Recall that if a regular curve has constant Frenet curvatures ratios, (i.e., $\frac{\tau}{\kappa}$ and $\frac{\sigma}{\tau}$ are constants), then it is called a *ccr-curve* [16]. It is worth noting that: the W-curve, in Euclidean 4-space E^4, is a special case of a ccr-curve.

Let $\alpha(s)$ and $\alpha^*(s)$ be regular curves in E^4. $\alpha(s)$ and $\alpha^*(s)$ are called Bertrand Curves if for each s_0, the principal normal vector to α at $s = s_0$ is the same as the principal normal vector to $\alpha^*(s)$ at $s = s_0$. We say that $\alpha^*(s)$ is a Bertrand mate for $\alpha(s)$ if $\alpha(s)$ and $\alpha^*(s)$ are Bertrand Curves.

In [14] Magden defined in the same space, a vector product and gave a method to establish the Frenet-Serret frame for an arbitrary curve by the following definition and theorem:

Definition 1 Let $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ and $c = (c_1, c_2, c_3, c_4)$ be vectors in E^4. The vector product in E^4 is defined by the determinant

$$a \wedge b \wedge c = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}, \tag{2}$$

where e_1, e_2, e_3 and e_4 are mutually orthogonal vectors (coordinate direction vectors) satisfying equations

$$e_1 \wedge e_2 \wedge e_3 = e_4,\ e_2 \wedge e_3 \wedge e_4 = e_1,\ e_3 \wedge e_4 \wedge e_1 = e_2,\ e_4 \wedge e_1 \wedge e_2 = e_3.$$

Theorem 1 *Let $\alpha = \alpha(t)$ be an arbitrary regular curve in the Euclidean space E^4 with above Frenet-Serret equations. The Frenet apparatus of α can be written as follows:*

$$T = \frac{\alpha'}{\|\alpha'\|},$$

$$N = \frac{\|\alpha'\|^2 \alpha'' - \langle \alpha', \alpha'' \rangle \alpha'}{\left\| \|\alpha'\|^2 \alpha'' - \langle \alpha', \alpha'' \rangle \alpha' \right\|},$$

$$B = \mu\, E \wedge T \wedge N,$$
$$E = \mu\, \frac{T \wedge N \wedge \alpha'''}{\|T \wedge N \wedge \alpha'''\|},$$
$$\kappa = \frac{\left\| \|\alpha'\|^2 \alpha'' - \langle \alpha', \alpha'' \rangle \alpha' \right\|}{\|\alpha'\|^4},$$
$$\tau = \frac{\|T \wedge N \wedge \alpha'''\| \, \|\alpha'\|}{\left\| \|\alpha'\|^2 \alpha'' - \langle \alpha', \alpha'' \rangle \alpha' \right\|},$$

and

$$\sigma = \frac{\left\langle \alpha^{(IV)}, E \right\rangle}{\|T \wedge N \wedge \alpha'''\| \, \|\alpha'\|},$$

where μ *is taken* -1 *or* $+1$ *to make* $+1$ *the determinant of* $[T, N, B, E]$ *matrix.*

3 Some new results of helices and ccr curves

In this section we state some related theorems and some important results about helices and ccr curves:

Theorem 2 *Let* $\alpha = \alpha(s)$ *be a regular curve in* E^4 *parameterized by arclength with curvatures* κ, τ *and* σ*. Then* $\alpha = \alpha(s)$ *lies on the hypersphere of center* m *and radius* $r \in \Re^+$ *in* E^4 *if and only if*

$$\rho^2 + \left(\frac{1}{\tau}\frac{d\rho}{ds}\right)^2 + \frac{1}{\sigma^2}\left[\rho\tau + \frac{d}{ds}\left(\frac{1}{\tau}\frac{d\rho}{ds}\right)\right]^2 = r^2, \tag{3}$$

where $\rho = \frac{1}{\kappa}$ *[16].*

Theorem 3 *Let* $\alpha = \alpha(s)$ *be a regular curve in* E^4 *parameterized by arclength with curvatures* κ*,* τ *and* σ*. Then* α *is a generalized helix if and only if*

$$H_2' + \sigma H_1 = 0, \tag{4}$$

where $H_1 = \frac{\kappa}{\tau}$ *and* $H_2 = \frac{1}{\sigma} H_1'$ *are the Harmonic Curvatures of* α *[15].*

Theorem 4 *Let* $\alpha = \alpha(s)$ *be a regular curve in* E^4 *parameterized by arclength with curvatures* κ*,* τ *and* σ*. Then* α *is a type 3-slant helix (its second binormal vector* E *makes a constant angle with a fixed diretion* U*) if and only if*

$$\tilde{H}_2' + \sigma \tilde{H}_1 = 0, \tag{5}$$

where $\tilde{H}_1 = \frac{\sigma}{\tau}$ *and* $\tilde{H}_2 = \frac{1}{\kappa} \tilde{H}_1'$ *are the Anti-Harmonic Curvatures of* α *[18].*

With the aid of the above theorems, one can easily obtain the following important results:

Theorem 5 *Let* $\alpha = \alpha(s)$ *be a helix in* $\mathbb{E}^4$ *with non-zero curvatures.*

1. α *can not be a generalized helix*

2. α *can not be a 3-type slant helix*

3. *If* α *lies on the hypersphere* S^3, *then, the sphere's radius is equal to* $\frac{\sqrt{\tau^2+\sigma^2}}{\kappa\,\sigma}$.

Theorem 6 *Let* $\alpha = \alpha(s)$ *be a ccr curve in* $\mathbb{E}^4$ *with non-zero curvatures* $\kappa(s)$, $\tau(s) = a\,\kappa(s)$ *and* $\sigma(s) = b\,\kappa(s)$. *Then*

1. α *can not be a generalized helix*

2. α *can not be a 3-type slant helix*

3. *If* α *lies on the hypersphere* S^3, *then, if and only if, the following equation is satisfied:*

$$f^2 + \frac{f'^2}{4a^2} + \frac{f}{4a^2b^2}(2a^2 + f'')^2 = r^2, \tag{6}$$

where the function $f = f(s) = \rho^2(s) = \frac{1}{\kappa^2(s)}$.

4 Bertrand curve of a helix

In this section we investigate relations among Frenet-Serret invariants of Bertrand curve of a helix in the space $\mathbb{E}^4$.

Theorem 7 *Let* $\delta = \delta(s)$ *be a helix in* $\mathbb{E}^4$. *Moreover,* ξ *be Bertrand mate of* δ. *Frenet-Serret apparatus of* ξ, $\{T_\xi, N_\xi, B_\xi, E_\xi, \kappa_\xi, \tau_\xi, \sigma_\xi\}$, *can be formed by Frenet apparatus of* δ $\{T, N, B, E, \kappa, \tau, \sigma\}$.

Proof. Let us consider a helix (W-curve, i.e.) $\delta = \delta(s)$. We may express

$$\xi = \delta + \lambda\, N. \tag{7}$$

We know that $\lambda = c =$ constant (cf. [11]). By this way, we can write that

$$\frac{d\xi}{ds_\xi}\frac{ds_\xi}{ds} = T_\xi \frac{ds_\xi}{ds} = (1 - \lambda\,\kappa)T + \lambda\,\tau\,B.$$

So, one can have

$$T_\xi = \frac{(1-\lambda\kappa)T + \lambda\tau B}{\sqrt{(1-\lambda\kappa)^2 + (\lambda\tau)^2}}, \tag{8}$$

and

$$\frac{ds_\xi}{ds} = \left\|\xi'\right\| = \sqrt{(1-\lambda\kappa)^2 + (\lambda\tau)^2}. \tag{9}$$

In order to determine relations, we differentiate:

$$\begin{gathered}\xi'' = \left[\kappa - \lambda(\kappa^2+\tau^2)\right] N + (\lambda\,\tau\,\sigma)E,\\ \xi''' = \kappa\left[\lambda(\kappa^2+\tau^2) - \kappa\right] T + \tau[\kappa - \lambda(\kappa^2+\tau^2+\sigma^2)]B,\\ \xi^{(IV)} = l_1\,N + l_2\,E\end{gathered} \tag{10}$$

where

$$l_1 = \kappa^3(\lambda\,\kappa - 1) + \lambda\,\tau^2(2\kappa^2+\tau^2+\sigma^2),$$

and

$$l_2 = \tau\,\sigma[\kappa - \lambda(\kappa^2+\tau^2+\sigma^2)].$$

Using the above equations, we can form

$$\left\|\xi'\right\|^2\,\xi'' - \left\langle \xi', \xi''\right\rangle\,\xi' = K^2\left[[\kappa - \lambda(\kappa^2+\tau^2)]N + (\lambda\,\tau\,\sigma)E\right],$$

where

$$K = \sqrt{(1-\lambda\kappa)^2 + (\lambda\tau)^2}.$$

Therefore, we obtain the principal normal and the first curvature, respectively,

$$N_\xi = \frac{1}{L}\left[[\kappa - \lambda(\kappa^2+\tau^2)]N + (\lambda\,\tau\,\sigma)E\right], \tag{11}$$

and

$$\kappa_\xi = \frac{L}{K^2}, \tag{12}$$

where

$$L = \sqrt{[\kappa - \lambda(\kappa^2+\tau^2)]^2 + (\lambda\,\tau\,\sigma)^2}.$$

Now, we can compute the vector form $T_\xi \wedge N_\xi \wedge \xi'''$ as the following:

$$\begin{aligned} T_\xi \wedge N_\xi \wedge \xi''' &= \frac{1}{KL}\begin{vmatrix} T & N & B & E\\ 1-\lambda\kappa & 0 & \lambda\tau & 0\\ 0 & \kappa - \lambda(\kappa^2+\tau^2) & 0 & \lambda\tau\sigma\\ l_1 & 0 & l_2 & 0\end{vmatrix}\\ &= -\frac{M}{KL}\left[\lambda\,\tau\,\sigma\,N - [\kappa - \lambda(\kappa^2+\tau^2)]E\right],\end{aligned}$$

where

$$M = \tau\left[\lambda(\kappa^2+\tau^2+\sigma^2) - \kappa(1+\lambda^2\sigma^2)\right].$$

Since, we have

$$E_\xi = -\frac{1}{L}\left[\lambda\,\tau\,\sigma\,N - [\kappa - \lambda(\kappa^2+\tau^2)]E\right]. \tag{13}$$

By this way, we have the third curvature as follows:

$$\tau_\xi = \frac{M}{K^2 L}. \tag{14}$$

Besides, considering last equation of Theorem 1, one can calculate

$$\sigma_\xi = \frac{\kappa\,\sigma}{L}. \tag{15}$$

Now, to determine the third vector field of Frenet frame, we write

$$E_\xi \wedge T_\xi \wedge N_\xi = -\frac{1}{KL^2}\begin{vmatrix} T & N & B & E \\ 0 & \lambda\tau\sigma & 0 & \lambda(\kappa^2+\tau^2)-\kappa \\ 1-\lambda\kappa & 0 & \lambda\tau & 0 \\ 0 & \kappa-\lambda(\kappa^2+\tau^2) & 0 & \lambda\tau\sigma \end{vmatrix}.$$

So we obtain:

$$B_\xi = -\frac{1}{K}\left[\lambda\,\tau\,T + (1-\lambda\,\kappa)B\right]. \tag{16}$$

It is worth to note that $\mu = 1$. ■

Considering obtained equations, we get:

Theorem 8 *Let* $\delta = \delta(s)$ *be a helix in* E^4. *Moreover, let* ξ *be a Bertrand mate of* δ. *Then*

1. ξ *is also a helix.*

2. ξ *can not be a generalized helix.*

3. ξ *can not be a 3-type slant helix.*

4. *If* ξ *lies on the hypersphere* S^3, *then, the sphere's radius is equal to* $\frac{\sqrt{\tau_\xi^2+\sigma_\xi^2}}{\kappa_\xi\,\sigma_\xi} = \frac{\sqrt{\tau^2+(1-\lambda\,\kappa)^2\,\sigma^2}}{\kappa\,\sigma}$.

5 Involute-evolute curve of a helix

In this section, first we correct the computations in the paper [19] and then we obtain new results:

Theorem 9 *Let* $\xi = \xi(s)$ *be involute of* δ. *Let* δ *be a helix in* E^4. *The Frenet apparatus of* ξ, $\{T_\xi, N_\xi, B_\xi, E_\xi, \kappa_\xi, \tau_\xi, \sigma_\xi\}$, *can be formed by Frenet apparatus of* δ $\{T, N, B, E, \kappa, \tau, \sigma\}$ *and take the following form.*

$$T_\xi = N, \quad N_\xi = \frac{-\kappa T + \tau B}{\sqrt{\kappa^2 + \tau^2}}, \quad B_\xi = -E, \quad E_\xi = \frac{\tau T + \kappa B}{\sqrt{\kappa^2 + \tau^2}}, \tag{17}$$

and

$$\kappa_\xi = \frac{\sqrt{\kappa^2 + \tau^2}}{\kappa\,|c - s|}, \ \tau_\xi = \frac{\tau\,\sigma}{\kappa\,\sqrt{\kappa^2 + \tau^2}\,|c - s|}, \ \sigma_\xi = -\frac{\sigma}{\sqrt{\kappa^2 + \tau^2}\,|c - s|}, \tag{18}$$

where

$$\frac{ds_\xi}{ds} = \kappa\,|c - s|. \tag{19}$$

Proof. The proof of the above theorem is similar as the proof of the previous theorem. ■

Theorem 10 *Let* ξ *and* δ *be unit speed regular curves in* E^4. ξ *be involute of* δ. *Then*

1. ξ *cannot be a helix.*
2. ξ *is a ccr-curve.*
3. ξ *cannot be a generalized helix.*
4. ξ *cannot be a 3-type slant helix.*
5. ξ *cannot be lies on the hypersphere* S^3.

Proof. The proof of points 1, 2, 3 and 4 are obviously. In the following we will proof the point 5:

Integrating the equation (19), we have

$$|c - s| = \sqrt{\frac{2s_\xi}{\kappa}},$$

which leads to

$$\kappa_\xi = \frac{A_1}{\sqrt{s_\xi}}, \quad \tau_\xi = \frac{A_2}{\sqrt{s_\xi}}, \quad \sigma_\xi = \frac{A_3}{\sqrt{s_\xi}}, \tag{20}$$

where

$$A_1 = \sqrt{\frac{\kappa^2 + \tau^2}{2\kappa}}, \ A_2 = -\frac{\tau\,\sigma}{2\kappa(\kappa^2 + \tau^2)}, \ A_3 = -\frac{\sigma\sqrt{\kappa}}{\sqrt{2(\kappa^2 + \tau^2)}}.$$

Then if the evolute ξ lies in the hypersphere the equation (6) must be satisfied. Substituting $f = \frac{s_\xi}{A_1^2}$, $\kappa_\xi = \frac{A_1}{\sqrt{s_\xi}}$, $B_1 = \frac{\tau_\xi}{\kappa_\xi}$ and $B_2 = \frac{\sigma_\xi}{\kappa_\xi}$ in the equation (6), we have

$$\frac{s_\xi (B_1^2 + B_2^2)}{A_1^2 B_2^2} + \frac{1}{4A_1^2 B_1^2} = r^2,$$

which is contradiction because the radius r of the sphere must be constant and the coefficient of s_ξ can not be equal zero. The proof is completed. ■

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