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On the expectation and variance of the reversal distance

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Abstract. We give a pair of well-matched lower and upper bounds for the expectation of reversal distance under the hypothesis of random gene order by investigating the expected number of cycles in the breakpoint graph of linear signed permutations. Sankoff and Haque [9] proved similar results for circular signed permutations based on approximations based on a slightly different model; while our approach is discrete. We also provide an near-tight upper bound for the variance of reversal distance, which gives information on the distribution of reversal distance.

1 Introduction

In the late 1980s, Jeffrey Palmer[6] and his colleagues compared the mitochondrial genomes of cabbage and turnip, which are very closely related. To their surprise, these genomes, which are almost identical in gene sequences, differ dramatically in gene order. This discovery and many other studies in the last decade convincingly proved that genome rearrangements represent a common mode of molecular evolution.

A framework of possible models to study genome rearrangements is to represent genomes as *signed permutations* of genes and compute their distances based on the minimum number of certain operations (evolutionary events)

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needed to transform one permutation into another. Under these models, the shorter the distance, the closer the genomes are.

In general, genes are represented as integers from 1 to n, and the genome is represented by a permutation π: $\{1, 2, \ldots n\} \mapsto \{1, 2, \ldots n\}$ by $(\pi_1\pi_2 \ldots \pi_n)$, where π_i denotes $\pi(i)$.

Permutations (more precisely, at this point, the permuted elements) may get *signed*, reflecting whether a gene or its mirror image is present. In this case each entry π_i has a positive or negative sign to model the orientation of genes. We denote the set of all permutations of size n by S_n and the set of all signed permutations by $\overline{S_n}$, respectively. Clearly $|S_n| = n!$ and $|\overline{S_n}| = 2^n n!$. We call $\pi_i\pi_{i+1} \ldots \pi_j$, where $(1 \leq i \leq j \leq n)$, a *segment* of the permutation $(\pi_1\pi_2 \ldots \pi_n)$.

An extensively studied operation on genomes is *reversal*. A reversal is an operation that reverses the order of the genes on a certain segment of the permutation of S_n - this operation is usually called *unsigned reversal*. We avoid them in this paper. Reversals are also considered acting on $\overline{S_n}$, in this case they are called *signed reversals*, and they also change the sign of the genes in the segment which was reversed. (There are other operations considered in the literature that correspond to evolutionary events, such as transposition, block interchange, transversal and translocation, see [11].) In 1995, Hannenhalli and Pevzner [4] discovered an elegant polynomial time algorithm to compute the signed reversal distance of signed permutations. However, in 1999 Caprara [2] showed that computing the unsigned reversal distance is NP-hard. Even before that, in 1996, Bafna and Pevzner [1] gave a 1.5-approximation algorithm to compute the unsigned reversal distance.

1.1 Reversals

The formal definition for a signed reversal on a signed permutation follows. A *signed permutation* is a bijection of the set $[-n, n] \setminus \{0\}$ onto itself such that $\pi(-a) = -\pi(a)$ for all $a \in [-n, n] \setminus \{0\}$ holds. It is easy to see that these bijections make a group for the composition of bijections as the group operation. This group is usually known as the group of "signed permutations" on $[n]$, or as the *hyper-octahedral group* of rank n. A signed permutation can be (uniquely) represented by the sequence of values it assigns to $1, 2, ..., n$ — this is how we described them earlier. The signed permutation in the previous section is the array of images of $1, 2, ..., n$ under the bijection. We identify the group with $\overline{S_n}$. The *identity* signed permutation assigns values $1, 2, 3, \ldots, n$ to $1, 2, 3, \ldots, n$ in this order. We denote the identity of $\overline{S_n}$ by id. For any

$0 \leq i < j \leq n$, we define a signed reversal $\rho_{i,j}$ as a signed permutation, whose values on $1, 2, ..., n$ are $1, 2, ..., (i-2), (i-1), -j, ..., -i, (j+1), (j+2), ..., n$. Observe that its action is

$$\rho_{i,j}(\pi) = \pi \circ \rho_{i,j},$$

and the values of this signed permutation on the sequence $1, 2, ..., n$ are

$$\pi_1, \ldots, \pi_{i-1}, -\pi_j, \ldots, -\pi_i, \pi_{j+1} \ldots, \pi_n.$$

For $\pi_1, \pi_2 \in \overline{S_n}$, the *reversal distance* of π_1 from π_2 is the smallest k such that

$$\pi_1 = \pi_2 \circ \rho_{i_1,j_1} \circ \rho_{i_2,j_2} \circ \cdots \circ \rho_{i_k,j_k}, \tag{1.1}$$

where the ρ's are reversals. As reversals have order two, the reversal distance is symmetric. We define the reversal distance of a signed permutation π to be the reversal distance of π and the identity $\mathtt{id}$, and denote it by $d(\pi)$.

We are interested in the expected reversal distance of two random signed permutations selected from the uniform distribution, π_1 and π_2. It follows from (1.1), that the reversal distance of π_1 and π_2 is the same as the reversal distance of $\pi_2^{-1} \circ \pi_1$ and $\mathtt{id}$. Furthermore, $\pi_2^{-1} \circ \pi_1$ is equidistributed with π_1 and π_2. Therefore it is sufficient to compute or estimate the expected reversal distance of a random signed permutation selected from the uniform distribution.

1.2 Breakpoint graph

An efficient tool, widely applied in the research of genome rearrangement is the *breakpoint graph*.

We define the breakpoint graph $G(\pi)$, together with its layout, for a signed permutation $\pi = (\pi_1\pi_2 \ldots \pi_n)$ as follows. If the entry π_i has positive sign, replace it by two vertices π_i^l, π_i^r in this order, and if the entry is negative, by π_i^r, π_i^l. Put these vertices in the order of π with two endpoints, 0^r on the extreme left and $(n+1)^l$ on the extreme right added—these vertices do not have a left (resp. right) companion. Connect any two vertices, which are consecutive in the layout (other than π_i^l and π_i^r from the same π_i) by a *black edge*, and connect i^r and $(i+1)^l$ by a *gray edge* for $0 \leq i \leq n$.

For convenience we call x the *value* of the vertex x^a, and a, which can be l or r, the *direction* of x^a. We call the vertices with the same value *conjugates*. For instance, the value of 3^r is 3, its direction is r, and it is the conjugate of 3^l, a relationship that we denote by $3^r = \overline{3^l}$. We define the *sign-function* s on $\{l, r\}$ such that $s(l) = 1$ and $s(r) = -1$. We will call the pair of vertices i^r and

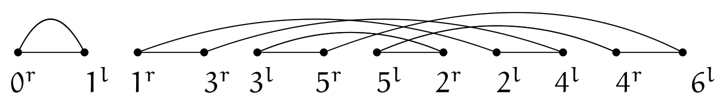


Figure 1: The breakpoint graph of π=(1,–3,–5,–2,4), straight lines are black edges and curved arcs are gray edges.

$(i+1)^l$, the *mates* of each other. For convenience, we denote the set of black edges of $G(\pi)$ by $B(\pi)$, and the vertex set $\{0^r, 1^l, 1^r, \ldots, n^l, n^r, (n+1)^l\}$ by V_n .

Each vertex is *adjacent* to exactly one black and one gray edge, so there is a *unique decomposition* of $G(\pi)$ into disjoint cycles of alternating edge colors. By the *length* of a cycle we mean the number of black edges it contains. We say that two gray edges g_1 and g_2 *cross*, if g_1 links vertices a and c, g_2 links vertices b and d, but these vertices are ordered a, b, c, d in $G(\pi)$. If g_1 and g_2 are crossing gray edges, and the cycle C_i contains the edge g_i for $i = 1, 2$, then we say that the two cycles, are *connected.* There is a finest equivalence relation on the set of cycles of $G(\pi)$, in which pairs of connected cycles fall in one class. A *component* of $G(\pi)$ is a class of this equivalence relation.

Using the breakpoint graph, Hannenhalli and Pevzner showed that the minimum number of reversals necessary to transform a signed permutation π to id is:

$$d(\pi) = n + 1 - c(\pi) + h(\pi) + fr(\pi), \tag{1.2}$$

where $c(\pi)$ is the number of cycles in the breakpoint graph, $h(\pi)$ is the number of hurdles, which are some special components, and $fr(\pi)$ takes value 1 or 0 based on whether G is a fortress. Caprara[3] showed that the probability of a random signed permutation of length n containing a hurdle is $\Theta(n^{-2})$ and the probability of a random signed permutation of length n including a fortress is $\Theta(n^{-15})$. Recently Swenson et al. [10] simplified Caprara's proof. Based on two facts above, in approximations for $d(\pi)$, the terms $h(\pi)$ and $fr(\pi)$ are often dropped. Hence we can easily find approximation for the distribution of reversal distance, if we find approximation to the distribution of the number of cycles in the breakpoint graph.

The permutations (unsigned or signed) that we discussed so far, are referred to as *linear permutations*, in order to distinguish them from the *circular permutations* which are arrangements of $\{1, 2, \ldots, n\}$ (or of $\{\pm 1, \pm 2, \ldots, \pm n\}$) along a cycle. Mathematically, the circular permutations are just the equivalence classes of linear permutations under rotation. The total number of circular unsigned permutations is $(n-1)!$ and the total number of circular signed permutations is $2^{n-1}(n-1)!$. As genomes or chromosomes can be linear or circular, the circular analogues of all concepts that we discussed so far are also relevant for bioinformatics. There exist concepts of reversals on circular permutations and breakpoint graphs of circular permutations, signed or unsigned. Hannenhalli and Pevzner [4] also computed that reversal distance of a circular permutation π of size n from the $\mathtt{id}$, an analogue of (1.2).

Sankoff and Haque [9] investigated the distribution of reversal distance among two randomly and uniformly selected circular signed permutations. They derived the expected number of cycles in the graph obtained by two random matchings on $2n$ vertices and claimed it approximately equals to the expected number of the cycles in the breakpoint graph of two random circular signed permutations. The precision of this approximation hinges on results of Kim and Wormald [5], which works with proof for sufficiently large n only (see the $\kappa < n/40$ condition in [9]). Depending this approximation, Sankoff and Haque [9] in their further calculations use continuous limit distributions to approximate the discrete probabilities in question, and put emphasis on the plotted simulation results as evidence for the result. Personal communication from Friedberg is cited in [9] as source for an asymptotic formula for the expected number of cycles in the breakpoint graph of circular (unsigned or signed) permutations.

It is expected that for linear signed permutations one would get similar results. Sankoff and Haque [9] claim that their approach extends to linear signed permutations, but do not give any specifics. The goal of this paper is to achieve such results, with rigorous and complete proofs, using a more discrete approach. We will give matching lower and upper bounds for the expected number of cycles in the breakpoint graph of a randomly and uniformly selected signed permutation.

We are not aware of any earlier results on the variance of the number of cycles in the breakpoint graph of a randomly and uniformly selected signed permutation, either linear or circular. The expectation result is analogous with the expected number of cycles in an unsigned linear permutation, which is also logarithmic, but the analogy fails for the variance, which is still logarithmic for an unsigned linear permutation [8], but jumps to $\Theta(\log^2 n)$ for signed linear permutations.

1.3 B-cycle and test graph

Definition 1 *We call an alternating edge colored cycle in black and gray on a subset of the vertex set* V_n *a* B-cycle*, if there exists a signed permutation, whose breakpoint graph has it among its cycles.*

We will denote by [] the black edges and by () the gray edges. Notice that not every alternating colored cycle is a B-cycle. For example $[2^r, 5^l](5^r, 6^l)$ $[6^l, 1^r](1^r, 2^l)[2^l, 5^l](5^l, 4^r)[4^r, 3^l](3^l, 2^r)$ is a alternating colored cycle, but not a B-cycle, because if it is a B-cycle $[2^r, 5^r]$ determined 2 is followed by -5 or 5 is followed by -2 in the permutations whose breakpoint graphs contain this cycle, while $[2^l, 5^l]$ determined -2 is followed by 5 or -5 is followed by 2. Here we define another kind of auxiliary graph.

Definition 2 *Let* $\mathcal{E}$ *be a partial matching on the vertex set* V_n*. If for any* i*,* $(i^l, i^r) \notin \mathcal{E}, 0 < i < n+1$ *and* $(0^r, (n+1)^l) \notin \mathcal{E}$*, then we call* $\mathcal{E}$ *as a* standard partial matching.

Definition 3 *Let* $\mathcal{E}$ *be a standard partial matching on the vertex set* V_n*. The* test graph *of* $\mathcal{E}$*, denoted by* $T(\mathcal{E})$ *is defined as follows:*

- *The vertex set is* V_n*.*
- *The edge set consists of all the edges in* $\mathcal{E}$ *and the edges* $\langle i^l, i^r\rangle$ *for all* $0 < i < n+1$ *and* $\langle 0^r, (n+1)^l\rangle$*.*

In a test graph, we call the edges in $\mathcal{E}$ as *real edges* denoted by [] and the edges $\langle i^l, i^r\rangle, 0 < i < n+1$ and $\langle 0^r, (n+1)^l\rangle$ as *imaginary edges.* Notice that in a test graph each vertex is incident to one imaginary edge and at most one real edge; henceforth the test graph is composed of alternating cycles and paths. These pathes begin and end with imaginary edges. By the *length* of a cycle or a path we will understand the number of real edges in the cycle or path. We describe below a condition to tell whether a standard partial matching is a subset of the set of black edges of the breakpoint graph of some signed permutation.

Theorem 1.1 *Let* $\mathcal{E}$ *be a standard partial matching on the vertex set* V_n*. Then* $\mathcal{E}$ *is a subset of the black edges of the breakpoint graph of some permutation if and only if the test graph* $T(\mathcal{E})$ *is cycle-free (consists of paths only), or is a single cycle of length* $n+1$*.*

Proof. It is obvious that the test graph $T(B(\pi))$ for the set of all black edges of the breakpoint graph of some permutation π is an alternating cycle of length $n+1$. If $\mathcal{E}$ is a subset of $B(\pi)$, then $T(\mathcal{E})$ is a subgraph of $T(B(\pi))$. Since $T(B(\pi))$ is a cycle of length $n+1$, its subgraphs have to be itself or a set of pathes.

If $T(\mathcal{E})$ is a cycle of length $n+1$, we begin to read the numbers from 0^r to $(n+1)^l$ along the longer side of this cycle, and take the positive sign to the number if i^l proceed $i^r, 0 < i < n+1$, otherwise the negative sign. Thus we obtain a signed permutation, such that $\mathcal{E}$ is exactly the black edges set of this signed permutation. If $T(\mathcal{E})$ consists of pathes, we can connect them into an $n+1$- cycle, and from that, like above, we can get a signed permutation. Clearly, $\mathcal{E}$ is a subset of the black edge set of this signed permutation. ∎

2 Expected number of cycles

Selecting a signed permutation randomly and uniformly with probability $\frac{1}{2^n n!}$, the number of cycles in its breakpoint graph, c, will be a random variable and we are interested in the expectation and variance of this random variable c.

In order to get the expected number of cycles $E[c]$, it is enough to get the expected number of cycles of fixed lengths and sum them up, according to the linearity of expectation.

Lemma 1 *Let O_t be any B-cycle of length t, where $0 < t < n+1$. The probability that a randomly and uniformly selected signed permutation contains O_t is $p_t = \frac{(n-t)!}{2^t n!}$.*

Proof. We have to count how many signed permutations contain O_t as a cycle in their breakpoint graph. Since each cycle is determined by its black edges, we only have to count the signed permutations whose breakpoint graph contains the black edges of O_t. Let $B(O_t)$ be the set of black edges of O_t. Since O_t is a B-cycle of length less than $n+1$, from Theorem 1.1, $T(B(O_t))$ consists of alternating paths. The test graph of the whole black edge set of any breakpoint graph that contains $B(O_t)$ is a cycle of length $n+1$ that contains these paths as subgraphs. Just as in the proof of Theorem 1.1, we read the permutations from these cycles and we will observe that

(A) Let $P = \langle x_1^{a_1}, \overline{x_1^{a_1}}\rangle[\overline{x_1^{a_1}}, x_2^{a_2}]\dots[\overline{x_k^{a_k}}, x_{k+1}^{a_{k+1}}]\langle x_{k+1}^{a_{k+1}}, \overline{x_{k+1}^{a_{k+1}}}\rangle$ be a path of length k which does not contain 0^r and $(n+1)^l$ in $T(B(O_t))$. Then either the segment $(s(a_1)\cdot x_1, s(a_2)\cdot x_2 \dots\ s(a_{k+1})\cdot x_{k+1})$ or the segment $(-s(a_{k+1})\cdot x_{k+1}, -s(a_k)\cdot x_k, \dots - s(a_1)\cdot x_1)$ lies in the permutations whose breakpoint graphs contain O_t depending on the direction of reading. Here notice the length of these segments is $k+1$.

For instance the path $\langle 3^l, 3^r\rangle[3^r, 5^l]\langle 5^l, 5^r\rangle[5^r, 7^r]\ \langle 7^r, 7^l\rangle[7^l, 4^r]\langle 4^r, 4^l\rangle$ is a path of length 3. A permutation whose breakpoint graph contains the black edges $[3^r, 5^l]$, $[5^r, 7^r]$ and $[7^l, 4^r]$ must contain the segment $(3, 5, -7, -4)$ or $(4, 7, -5, -3)$.

(B) Let $P = \langle x_1^{a_1}, \overline{x_1^{a_1}}\rangle[\overline{x_1^{a_1}}, x_2^{a_2}]\ldots[\overline{x_i^{a_i}}, (n+1)^l]\langle(n+1)^l, 0^r\rangle[0^r, x_{i+1}^{a_{i+1}}]\langle x_{i+1}^{a_{i+1}}, \overline{x_{i+1}^{a_{i+1}}}\rangle\ldots[\overline{x_{k-1}^{a_{k-1}}}, x_k^{a_k}]\langle x_k^{a_k}, \overline{x_k^{a_k}}\rangle$ be a path of length k which contains 0^r and $(n+1)^l$ in $T(B(O_t))$, then P determines that all the permutations whose breakpoint graphs contain O_t must begin with $s(a_{i+1})\cdot x_{i+1}, s(a_{i+2})\cdot x_{i+2}\ldots s(a_k)\cdot x_k)$ and end with $(s(a_1)\cdot x_1, s(a_2)\cdot x_2, \ldots s(a_i)\cdot x_i)$.

For example, the path $\langle 6^l, 6^r\rangle[6^r, 4^l]\langle 4^l, 4^r\rangle[4^r, 8^l]\langle 8^l, 0^r\rangle[0^r, 3^l]\langle 3^l, 3^r\rangle[3^r, 2^r]\ \langle 2^r, 2^l\rangle$ implies that the permutation whose breakpoint graph contains these black edges in the path must begin with $(3, -2)$ and end with $(6, 4)$.

Let l_i be the number of paths of length i which do not contain vertices 0^r and $(n+1)^l$ in $T(B(O_t))$. Here $0 < i < n$ for a path of length n or $n+1$ must contain 0^r and $(n+1)^l$.

Case 1. The length of the path containing 0^r and $(n+1)^l$ is 0. In this case, we have $\sum_{i=1}^{n-1} i\cdot l_i = t$. From observation (A), each path of length i determine one segment of length $i+1$ in the permutation . So the number of permutations whose breakpoint graphs contain O_t is

$$2^{n-\sum_{i=1}^{n-1}(i+1)l_i+\sum_{i=1}^{n-1} l_i}(n - \sum_{i=1}^{n-1}(i+1)l_i + \sum_{i=1}^{n-1} l_i)!,$$

which turns out to be just $2^{n-t}(n-t)!$.

Case 2. The length of the path containing 0^r and $(n+1)^l$ is $s > 0$. In this Case, we have $\sum_{i=1}^{n-1} i\cdot l_i = t - s$. From observation (B), the start and end segments of the permutations of total length s are fixed. So we only need to consider the number of permutations on the remained $n-s$ numbers. According to case 1, it should be $2^{(n-s)-(t-s)}((n-s)-(t-s))!$ which still equals to $2^{n-t}(n-t)!$.

So from Cases 1 and 2, we conclude that for any B-cycle, there are $2^{n-t}(n-t)!$ signed permutations containing it. Since $|\overline{S_n}|$ is $2^n n!$, we have $p_t = \frac{2^{n-t}(n-t)!}{2^n n!} = \frac{(n-t)!}{2^t n!}$. ∎

The following lemma is a basic fact which will be applied in the coming computation.

Lemma 2

$$\log(n+1) \leq \sum_{i=1}^{n} \frac{1}{i} \leq \log n + 1.$$

2.1 Upper bound

Lemma 3 *Let* $c_t(\pi)$ *be the number of different B-cycles of length* t *which do not contain* 0^r *and* 1^l *in the breakpoint graphs of permutation. Then*

$$c_t < \frac{2^{t-1} n!}{t(n-t)!}.$$

Proof. A B-cycle of length t without 0^r and 1^l can be written as a circular sequence of edges as

$$\{[x_t', x_1], (x_1, x_1'), [x_1', x_2], (x_2, x_2') \ldots, [x_{t-1}', x_t], (x_t, x_t')\}$$

or

$$\{[x_1, x_t'](x_t', x_t)[x_t, x_{t-1}'] \ldots, (x_2', x_2), [x_2, x_1'](x_1', x_1)\}$$

for it is undirected, where $x_i \in \{1^r, 2^l, 2^r \ldots n^l, n^r, (n+1)^l\}$ and x_i' is the mate of x_i. Observe that the vertex set of a B-cycle of length t is just t pairs of mates and it corresponds to the circular sequence of the second vertex of each black edge. The first sequence corresponds to the circular t-permutation $x_1 x_2 \ldots, x_t$ of the vertices set $V_n \setminus \{0^r, 1^l\}$, and the other corresponds to $\{x_t', x_{t-1}' \ldots, x_1'\}$. So each B-cycle corresponds to two circular t-permutations and each circular t-permutation corresponds to at most one B-cycle. Notice that if a t-permutation corresponds to a B-cycle, there is no pair of mates both in the t-permutation. Now lets count the number of such permutations. Let's select x_i's one by one to get a linear t-permutation. We have $2n$ choices for x_1, $2n-2$ choices x_2 since x_1' can not be selected $\ldots 2n - 2t + 2$ choices for x_t. Thus we have totally $\frac{2^t n!}{(n-t)!}$ such linear t- permutations i.e $\frac{2^t n!}{t(n-t)!}$ such circular t-permutation. Hence there are at most $\frac{2^{t-1} n!}{t(n-t)!}$ cycles of length t for each cycle corresponding to a pair of circular t-permutations. ∎

Theorem 2.2 *Let* $c(\pi)$ *be the random variable counting the number of cycles in the breakpoint graph of a random signed permutation* π*. Then*

$$E[c] \leq \frac{1}{2} \log n + \frac{3}{2}.$$

Proof. Since 0^r and 1^l are mates, they are contained in one cycle. Thus we have

$$E[c] \leq \sum_{i=1}^{n} c_i p_i + 1 \leq \sum_{t=1}^{n} \frac{2^{t-1} n!}{t(n-t)!} \cdot \frac{(n-t)!}{2^t n!} + 1 = \sum_{t=1}^{n} \frac{1}{2t} + 1 \leq \frac{1}{2} \log n + \frac{3}{2},$$

where the last inequality is obtained by Lemma 2. ∎

2.2 Lower bound

Definition 4 *Let O_t be a B-cycle of length t. Let O_{t+1} be a B-cycle of length $t+1$ on V_n obtained by replacing a black edge of O_t with an alternating path of two black edges and a gray edge. Then we call O_t the shadow of O_{t+1} and O_{t+1} the shade of O_t.*

Lemma 4 *For any i, $1 < i < n$, let c_i be the number of different B-cycles of length i among the breakpoint graphs of all the signed permutations, then we have*

$$(i+1)c_{i+1} \geq 2i(n-i)c_i.$$

Proof. We prove it by counting the set of ordered pairs $\mathcal{P} = \{(O_i, O_{i+1}) | O_i$ is a shadow of $O_{i+1}\}$.

For a given B-cycle of length $i+1$, we can replace an alternating path of two black edges and a gray edge by a black edge to get a new cycle which could be a shadow depending on whether the new cycle is a B-cycle. Since we have $i+1$ ways to select the alternating path of two black edges and a gray edge, each B-cycle has at most $i+1$ shadows which implies that $|\mathcal{P}| \leq (i+1)c_{i+1}$.

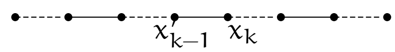
For a given B-cycle O_i of length i, its shades must lie in the set of cycles which are obtained by replacing one black edge with an alternating path of two black edges and a gray edge. We denote that set by $\mathcal{S}(O_i)$. Next let's consider how many of cycles in $\mathcal{S}(O_i)$ are B-cycles.

From Theorem1.1, we only need to count how many of the cycles in $\mathcal{S}(O_i)$ with the test graph of their black edges are cycle-free.

Assume that O_i is written as

$$\{[x_i', x_1], (x_1, x_1'), [x_1', x_2], (x_2, x_2') \ldots, [x_{i-1}', x_i], (x_i, x_i')\}.$$

we replace the black edge $[x_{k-1}', x_k]$ by the path$[x_{k-1}, y](y, y')[y', x_k]$. Clearly y can not be in O_i. What happens to the test graph $T(B(O_i))$ is the following: we delete the black edge $[x_{k-1}', x_k]$ and add two black edges $[x_{k-1}, y]$ and $[y', x_k]$



A path in $T(B(O_i))$

Figure 2: The path after deleting $[x'_{k-1}, x_k]$ and adding $[x'_{k-1}, y]$ and $[x_k, y']$

by selecting a vertex y. Let $[x'_{k-1}, x_k]$ be in a path in $T(B(O_i))$, only when y or y' is one of the two end points of this path could cause a cycle in the new test graph (see Fig. 2).

So here the "y" has at least $(2n+2)-(2i+2)$ choices. Hence $|\mathcal{P}| \geq i(2n-2i)c_i$. Thus we have

$$i(2n-2i)c_i \leq |\mathcal{P}| \leq (i+1)c_{i+1}.$$

which implies our lemma. ∎

We know that the number of B-cycles of length one is $n+1$. Recursively using Lemma 4 we get the following corollary:

Corollary 1 *Let c_t be the number of B-cycles of length t among the breakpoint graphs of all signed permutations of size n, where $1 < t < n+1$. Then*

$$c_t > \frac{2^{t-1}(n+1)(n-1)!}{t(n-t)!}.$$

Theorem 2.3 *Let $c(\pi)$ be the random variable by the number of cycles in the breakpoint graph of a random permutation π. Then*

$$E[c] \geq \frac{n+1}{2n}\log(n+1).$$

Proof.

$$E[c] \geq \sum_{i=1}^{n} c_i p_i \geq \sum_{i=1}^{n} \frac{2^{t-1}(n+1)(n-1)!}{t(n-t)!} \cdot \frac{(n-t)!}{2^t n!} \geq \frac{n+1}{2n}\log(n+1),$$

where the last inequality is obtained by Lemma 2. ∎

3 Variance of the number of cycles

Theorem 3.4 *Let* $c(\pi)$ *be the random variable counting the number of cycles in the breakpoint graph of of a randomly and uniformly selected signed permutation* π*, then*

$$E[c^2] \leq \frac{3}{4}\log^2 n + \frac{5}{2}\log n + \frac{7}{2}.$$

Proof. Let X_i be the random variable such that $X_i(\pi) = 1$ if $G(\pi)$ contains the B-cycle o_i and $X_i(\pi) = 0$ otherwise. Then we have:

- $c(\pi) = \sum_i X_i(\pi)$;
- $X_i(\pi)X_j(\pi) = 1$, if $G(\pi)$ contains the cycles o_i and o_j, $= 0$. Furthermore, if $o_i \cap o_j \neq \varnothing$, then $X_i X_j = 0$.
- $E[X_i^2(\pi)] = E[X_i(\pi)]$.

Hence

$$\begin{aligned} E[c^2] &= E[\sum_i X_i \sum_j X_j] = \sum_i E[X_i^2] + \sum_{\substack{i,j,\\ i\neq j}} E[X_i X_j] \\ &= E[c] + \sum_{\substack{i,j\\ i\neq j}} E[X_i X_j] = E[c] + \sum_{\substack{i,j\\ o_i\cap o_j=\varnothing}} E[X_i X_j] \end{aligned} \tag{3.1}$$

Let us be given two cycles A and B of length $a > 0$ and $b > 0$ with $A \cap B = \varnothing$. Then a permutation contains A and B only if it contains the $a+b$ black edges of them. From the proof of Lemma 1, we have there are $2^{n-a-b}(n-a-b)!$ permutations containing cycles A and B provided $a+b \leq n$. So the probability of a random permutation containing the cycles A and B is $\frac{2^{n-a-b}(n-a-b)!}{2^n n!}$ for $2 \leq a+b \leq n$. When $a+b = n+1$, the breakpoint graph of the permutation which contains A and B is uniquely determined by the black edges of A and B. Hence the probability of a permutation containing the cycles A and B is $\frac{1}{2^n n!}$ when $a+b = n+1$.

Now let us count how many ordered pairs of non-intersecting cycles have length $a > 0$ and $b > 0$. Since each cycle of length t is determined by a pair of circular t-permutations, the number of ordered pairs of non-intersecting cycles which have length $a > 0$ and $b > 0$ is bounded by one fourth of the ways of getting a pair of circular a-permutation and b-permutation. Just as in the proof of Lemma 3, we could select a circular a-permutation with

$$\frac{(2n+2)(2n+2-2)\dots(2n+2-(2a-2))}{a} = \frac{2^a(n+1)!}{a(n+1-a)!}$$

ways and select a circular b-permutation which is not intersected with the a-permutation with

$$\frac{(2n+2-2a)(2n+2-2a-2)\dots(2n+2-2a-(2b-2))}{b} = \frac{2^b(n+1-a)!}{(n+1-a-b)!b}$$

ways. Thus totally we have at most

$$\frac{2^a(n+1)!}{a(n+1-a)!} \cdot \frac{2^b(n+1-a)!}{(n+1-a-b)!b} = \frac{2^{a+b}(n+1)!}{(n+1-a-b)!ab}$$

such pairs. Thus,

$$\begin{aligned}
\sum_{\substack{i,j \\ o_i \cap o_j = \varnothing}} E[X_i X_j] &\leq \frac{1}{4} \sum_{a,b \geq 1}^{a+b \leq n} \frac{2^{a+b}(n+1)!}{(n+1-a-b)!ab} \frac{2^{n-a-b}(n-a-b)!}{2^n n!} \\
&+ \frac{1}{4} \sum_{a,b \geq 1}^{a+b=n+1} \frac{2^{a+b}(n+1)!}{(n+1-a-b)!ab} \frac{1}{2^n n!} \\
&= \frac{1}{4} \sum_{a,b \geq 1}^{a+b \leq n} \frac{n+1}{ab(n+1-a-b)} + \frac{1}{2} \sum_{a,b \geq 1}^{a+b=n+1} \frac{n+1}{ab} \\
&= \frac{1}{4} \sum_{t=2}^{n} \sum_{a=1}^{t-1} \frac{n+1}{a(t-a)(n+1-t)} + \frac{1}{2} \sum_{a=1}^{n} (\frac{1}{a} + \frac{1}{n+1-a})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{t=2}^{n}\sum_{a=1}^{t-1}\left(\frac{1}{a(t-a)}+\frac{1}{a(n+1-t)}+\frac{1}{(t-a)(n+1-t)}\right)+\sum_{a=1}^{n}\frac{1}{a}\\
&= \frac{1}{4}\sum_{t=2}^{n}\left((\frac{1}{t}+\frac{1}{n+1-t})\sum_{a=1}^{t-1}(\frac{1}{a}+\frac{1}{t-a})\right)+\sum_{a=1}^{n}\frac{1}{a}\\
&\le \sum_{t=2}^{n}\frac{1}{2}\left(\log(t-1)+1\right)\left(\frac{1}{t}+\frac{1}{n+1-t}\right)+(\log n+1)\\
&= \frac{1}{2}\sum_{t=2}^{n}\left(\frac{\log(t-1)}{t}+\frac{\log(t-1)}{n+1-t}\right)+\frac{1}{2}\sum_{t=2}^{n}\left(\frac{1}{t}+\frac{1}{n+1-t}\right)+(\log n+1)\\
&\le \frac{1}{2}\sum_{t=2}^{n}\left(\frac{\log(t-1)}{t-1}+\frac{\log(n-1)}{n+1-t}\right)+\frac{1}{2}(\log n+\log(n-1)+1)\\
&\quad +(\log n+1)\\
&\le \frac{1}{2}\left(\frac{\log^2 n}{2}+\log(n-1)(\log(n-1)+1)\right)\\
&\quad +\frac{1}{2}(\log n+\log(n-1)+1)+(\log n+1)\\
&\le \frac{3}{4}\log^2 n+\frac{5}{2}\log n+\frac{3}{2}.
\end{aligned}
$$

From (3.1) and Lemma 3, we have

$$
\mathsf{E}[c^2] \le \frac{1}{2}\log n+\frac{3}{2}+\frac{3}{4}\log^2 n+\frac{5}{2}\log n+\frac{3}{2}=\frac{3}{4}\log^2 n+3\log n+3.
$$

∎

Theorem 3.5 *Let* $c(\pi)$ *be the random variable that counts the number of cycles in the breakpoint graph of* π*, then*

$$
\mathsf{Var}[c] \le \frac{1}{2}\log^2 n+3\log n+3.
$$

Proof. For $\mathsf{Var}[c]=\mathsf{E}[c^2]-\mathsf{E}[c]^2$, substitute the upper bound of $\mathsf{E}[c^2]$ and the lower bound of $\mathsf{E}[c]$ to obtain

$$\begin{aligned}\mathsf{Var}[c] &\le \frac{3}{4}\log^2 n + 3\log n + 3 - (\frac{n+1}{2n}\log(n+1))^2\\ &\le \frac{3}{4}\log^2 n + 3\log n + 3 - (\frac{1}{2}\log(n))^2\\ &= \frac{1}{2}\log^2 n + 3\log n + 3.\end{aligned}$$

We leave it to reader to verify that the calculations in Theorems 3.4 and 3.5 are asymptotically tight. ∎

4 Expectation and variance of the reversal distance

Recall that $h(\pi)$ is the number of hurdles and $fr(\pi)$ is the number of fortresses in the breakpoint graph of π. For a randomly and uniformly selected signed permutation π, we have Prob($h(\pi) \geq 1) = \Theta(\frac{1}{n^2})$ according to Swenson et al. [10]. There are at most n hurdles in the breakpoint graph of any permutation. So we have $E[h] = O(\frac{1}{n})$. Swenson et al. [10] also showed that Prob($fr(\pi) = 1) = \Theta(\frac{1}{n^{15}})$, which implies that $E[fr] = \Theta(\frac{1}{n^{15}})$ for $fr(\pi)$ only takes value 1 or 0. Hence from (1.2) we have

$$E[d(\pi)] = n + 1 - E[c(\pi)] + O(\frac{1}{n}),$$

which implyies the following theorem:

Theorem 4.6 $n+1-\frac{1}{2}\log n-\frac{3}{2}+O(\frac{1}{n}) \leq E[d] \leq n+1-\frac{n+1}{2n}\log(n+1)+O(\frac{1}{n})$.

Since Prob($h(\pi) \geq 1$) and Prob($fr(\pi) = 1$) are both very small, we can drop the terms $h(\pi)$ and $fr(\pi)$ in most of the cases when we compute $d(\pi)$. Let $\widetilde{d} = n + 1 - c(\pi)$. Then we have the following result as a consequence of Theorem 3.5:

Theorem 4.7

$$\mathsf{Var}[\widetilde{d}] \leq \frac{1}{2}\log^2 n + 3\log n + 3.$$

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Anti-Ramsey numbers of spanning double stars

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Abstract. In this paper we shall prove and sharpen a conjecture of A. Bialostocki on anti-Ramsey colorings of the complete graph K_n. Assume the edges of a K_n are colored by t colors. The basic question is how many colors ensure that K_n has a spanning subtree of diameter at most d in which each edge has a different color. The surprising fact is that the answers are the same for every $d \geq 3$. Moreover, we can set the maximal degree of the spanning tree at least $n-4$ without altering the answer. This implies that in these cases there is an extremal anti-Ramsey coloring using only one color more than once. Recently Jiang showed that this is not the case for $d = 2$. We also prove a new extremal property of Moore graphs of diameter 2 (e.g. the Petersen graph), that yields a bit shorter proof of a weaker version of our main theorem.

1 Introduction

Erdős, Simonovits and Sós [10] initiated the investigation of anti-Ramsey problems for graphs. Call an edge-colored graph *totally multicolored* (TMC, for short) if any two edges have different colors. Given a family $\mathcal{L}$ of graphs, what is the maximum t for which there exist t-colorings of the edges of K_n, where every color is used at least once, without a TMC subgraph that belongs to $\mathcal{L}$? This maximum will be denoted by $R^*(n, \mathcal{L})$. When $\mathcal{L}$ consists a single graph G, we shall use $R^*(n, G)$ for $R^*(n, \{G\})$.

These questions are related to extremal graph problems. Define $\mathrm{ex}(n, \mathcal{L})$ as the maximal integer m of edges such that there is a graph with n vertices

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and m edges which does not contain a subgraph isomorphic to a graph in $\mathcal{L}$. Denote ex$(n, \{G\})$ by ex(n, G).

It is easy to see that $R^*(n, \mathcal{L}) \leq$ ex$(n, \mathcal{L})$; indeed, if we have a coloring which uses t colors and no TMC subgraph belonging to $\mathcal{L}$ is obtained, then, by choosing for every color an edge with this color arbitrarily we have a graph with t edges which does not contain subgraphs belonging to $\mathcal{L}$. On the other hand, let $\mathcal{L}^* = \{L - e : L \in \mathcal{L}, e \in E(L)\}$, and let G be a subgraph of K_n containing no member of $\mathcal{L}^*$. Then, as it was observed in [10], every coloring of K_n for which all edges not belonging to G has the same color contains no TMC member of $\mathcal{L}$. In particular, since G can be chosen to have ex$(n, \mathcal{L}^*)$ edges,

$$R^*(n, \mathcal{L}) \geq \mathrm{ex}(n, \mathcal{L}^*) + 1. \tag{1}$$

The above inequality is sharp iff there is an extremal anti-Ramsey coloring using only one color more than once. Erdős, Simonovits and Sós showed also that sometimes this is the case, and sometimes it is not. (1) is sharp if $\mathcal{L}$ consists of one clique with at least 4 vertices and n is large enough. That is,

$$R^*(n, K_k) = \mathrm{ex}(n, K_k - e) + 1 \tag{2}$$

for sufficiently large n and $k \geq 4$. (Their proofs gave $R^*(n, K_k)$ explicitly, since they proved (2) by showing

$$R^*(n, K_k) \leq \mathrm{ex}(n, K_{k-1}) + 1. \tag{3}$$

for sufficiently large n and $k \geq 4$. (3), combining with (1) and an earlier theorem of Dirac [7] stating Recently Montellano-Ballesteros and Neumann-Lara [18] and Schiermeyer [19] proved (2) for every possible n.) On the other hand, it was also shown in [10] that (1) is not sharp if $\mathcal{L}$ consists one cycle of length k and $n \geq 2k - 1$.

Calculation of the exact anti-Ramsey numbers has proven to be very hard. They have been found only for few $\mathcal{L}$ so far, and, for most of these $\mathcal{L}$s, $|\mathcal{L}| = 1$ (see [10], [18] and [19] for cliques, [10], [1] and [14] for short cycles, [20] and [17] for paths, [13] for stars, [15] for brooms, that is, for trees formed by identifying the center of a star and an endpoint of a path, and [19] for matchings).

The case $|\mathcal{L}| \geq 2$ was examined first by Bialostocki and Voxman [5] and by Jiang and West [15]. They considered the family of all trees with k edges, denoted by $\mathcal{T}_k$. $R^*(n, \mathcal{T}_k)$ makes sense iff $k \leq n-1$. Bialostocki and Voxman [5] gave $R^*(n, \mathcal{T}_{n-1})$ for every n; independently Jiang and West [15] gave $R^*(n, \mathcal{T}_k)$ for every n and k. (1) proved to be sharp for all of them. In particular, $\mathcal{T}^*_{n-1}$ obviously consists of forests of order n with exactly two components, hence for $n \geq 3$ a graph of order n is $\mathcal{T}^*_{n-1}$-free iff it has at least three components; therefore it is easy to see that

$$\mathrm{ex}(n, \mathcal{T}^*_{n-1}) = \binom{n-2}{2} \tag{4}$$

for $n \geq 3$. Bialostocki and Voxman [5] and Jiang and West [15] proved the next theorem:

Theorem 1 *For* $n \geq 3$,

$$R^*(n, \mathcal{T}_{n-1}) = \binom{n-2}{2} + 1.$$

(As we said, Jiang and West proved a much more general theorem.) Bialostocki [3] conjectured a generalization in another direction.

Let $\mathcal{T}^d_{n-1}$ be the family of the trees of size $n-1$ and diameter at most d. Bialostocki conjectured that

$$R^*(n, \mathcal{T}^4_{n-1}) = R^*(n, \mathcal{T}_{n-1}). \tag{5}$$

In other words, if the number of colors is greater than $\binom{n-2}{2} + 1$ then there is a TMC spanning tree of diameter at most 4. His motivation was a result of him and his collaborators about corresponding Ramsey-type questions. It is well-known that if a graph is disconnected, i.e., if its diameter is infinite, then the diameter of its complement is at most 2. Bialostocki, Dierker and Voxman [4] proved that the same is true for complements of graphs of diameter greater than 4.

Our main aim in this paper is to show that much more is true. First of all,

$$R^*(n, \mathcal{T}^3_{n-1}) = R^*(n, \mathcal{T}_{n-1}). \tag{6}$$

That is, if the number of colors is greater than $\binom{n-2}{2} + 1$ then there is a TMC spanning tree with an edge dominating the whole tree, i.e., every vertex is adjacent to an endpoint of this edge. We shall call such a tree a *double star*, and the edge dominating the whole tree the *central edge*. (Note that a star is also a double star, and all its edges are central.) It is obvious that in (6) the diameter 3 cannot be replaced by 2, since $\mathcal{T}^2_{n-1} = \{K_{1,n-1}\}$, $\mathcal{T}^{2^*}_{n-1} = \{K_{1,n-2}\}$

and therefore $R^*(n, K_{1,n-2}) \geq \mathrm{ex}(n, K_{1,n-2}) + 1 = \binom{n}{2} - n + 1 > R^*(n, \mathcal{T}_{n-1})$ for $n \geq 4$. Jiang [13] showed the considerably more interesting fact, that

$$R^*(n, \mathcal{T}_k^2) - \mathrm{ex}(n, \mathcal{T}_k^{2^*}) > 1, \tag{7}$$

for $k \geq n/2 + 2$. In particular,

$$R^*(n, \mathcal{T}_{n-1}^2) - \mathrm{ex}(n, \mathcal{T}_{n-1}^{2^*}) > 1$$

for $n \geq 6$. That is, for these cases the extremal anti-Ramsey colorings use at least two colors more than once. (7) disproves a conjecture of Manoussakis, Spyratos, Tuza and Voigt [16] who expected equality in (7) for every k and n. (7) is implied from the trivial facts that $\mathcal{T}_k^2 = \{K_{1,k}\}$, for $k \geq 2$, $\mathcal{T}_k^{2^*} = \{K_{1,k-1}\}$ and

$$\mathrm{ex}(n, K_{1,k-1}) = \left\lfloor \frac{n(k-2)}{2} \right\rfloor,$$

and the next theorem of Jiang [13]:

Theorem 2 *For every $n > k \geq 2$,*

$$0 \leq R^*(n, K_{1,k}) - \left(\left\lfloor \frac{n(k-2)}{2} \right\rfloor + \left\lfloor \frac{n}{n-k+2} \right\rfloor \right) \leq 1,$$

and the lower bound is sharp unless all of n, k and $\lfloor 2n/(n-k+2) \rfloor$ are odd.

(Jiang conjectures that the lower bound is the truth also in the remaining case.)

However, $R^*(n, \mathcal{L}) = R^*(n, \mathcal{T}_{n-1})$ holds even for a family $\mathcal{L}$ much smaller than $\mathcal{T}_{n-1}^3$. Let $\mathcal{DS}_{n-1}^m$ be the families of the double stars with n vertices whose maximal degree is at least $n - m$, that is, their second largest degrees are at most m. Obviously, for $n \geq 2$ the second largest degree of a double star of size $n - 1$ can be any positive integer between 1 and $n/2$, and all of these possible second largest degrees determines the double star uniquely, therefore $|\mathcal{T}_{n-1}^3| = \lfloor n/2 \rfloor$ for $n \geq 2$, and $|\mathcal{DS}_{n-1}^m| = m$ for $n \geq 2m$. In comparison, it is easy to see that $|\mathcal{T}_{n-1}^4| > P(\lfloor (n-1)/2 \rfloor)$, where P is the partition function, that is, $P(k)$ is the number of writing the integer k as a sum of positive integers without regard to order. Hardy and Ramanujan [11] showed that, for some absolute constants A and B, $e^{A\sqrt{k}} < P(k) < e^{B\sqrt{k}}$.

We shall show that

$$R^*(n, \mathcal{T}_{n-1}) = R^*(n, \mathcal{DS}_{n-1}^4). \tag{8}$$

Thus the conjectured equality (5) is true even if we replace the superpolinomially large family $\mathcal{T}^4_{n-1}$ by the family $\mathcal{DS}^4_{n-1}$ consisting four graphs.

Jiang's theorem above shows that $\mathcal{DS}^4_{n-1}$ cannot be replaced by $\mathcal{DS}^3_{n-1}$. Since all of the three members of $\mathcal{DS}^3_{n-1}$ contain $K_{1,n-3}$, $R^*(n, \mathcal{DS}^3_{n-1}) \geq R^*(n, K_{1,n-3})$. Therefore Theorem 2 implies that, for $n \geq 25$,

$$\begin{aligned} R^*(n, \mathcal{DS}^3_{n-1}) \geq \frac{n(n-5)}{2} + \left\lfloor \frac{n}{5} \right\rfloor &= \binom{n-2}{2} + \left\lfloor \frac{n}{5} \right\rfloor - 3 \\ &> \binom{n-2}{2} + 1 = R^*(n, \mathcal{T}_{n-1}). \end{aligned}$$

In the next section we prove our main result.

In the last section we present an alternative, simpler proof of statement (6), by proving another theorem that perhaps is of its own interest. A graph is called Moore graph if it has diameter d and girth $2d+1$ for some integer d. The trivial examples are complete graphs and odd cycles. Hoffman and Singleton proved that: every regular nontrivial Moore graph of diameter 2 has degree 3, 7 or 57; the unique 3-regular Moore graph of diameter 2 is the Petersen graph; there is exactly one 7-regular Moore graph (now it is called Hoffman-Singleton graph); and there is no nontrivial regular Moore graph of diameter 3. Erdős and Rényi [8] (see also [9]) found the next extremal property of regular Moore graphs of diameter 2.

Theorem 3 *Let G be a graph of order n, diameter d and maximal degree* Δ. *Then*

$$|E(G)| \geq \frac{n(n-1)(\Delta-2)}{2((\Delta-1)^d-1)},$$

and equality holds iff G is a regular Moore graph.

Later Singleton [21] showed that, perhaps surprisingly, every Moore graph is regular. Finally, Bannai and Ito [2] and, independently, Damerell [6] proved that every nontrivial Moore graph has diameter 2. It yields that equality can hold in Theorem 3 only if $d = 2$ and $\Delta \in \{2; 3; 7; 57\}$. It is still unknown if there are 57-regular Moore graphs. It is easy to see that if they exist then their order is $1 + 57 + 57 \cdot 56 = 3250$.

We will show the next extremal property of the Moore graphs of diameter 2. If the graph G has diameter 2 and minimal degree δ then its size is at least $((\delta+1)/2)n - (\delta^2+1)/2$, and equality holds iff G is a Moore graph. In case of $d = 2$, this fact can be considered as a dual of Theorem 3.

2 Proof of the main theorem

The main result of this chapter is the following.

Theorem 4 *For* $n \geq 3$,

$$R^*(n, \mathcal{DS}^4_{n-1}) = \binom{n-2}{2} + 1.$$

Theorem 4 and Theorem 1 immediately imply (8) for $n \geq 3$, and it is obvious for $n = 2$, since then (and even for $n \leq 4$) $\mathcal{T}_{n-1} = \mathcal{DS}^4_{n-1}$.

As usual, we shall denote by $N_G(x)$ the set of the vertices adjacent to x in G, and let $N_G[x]$ be the set $N_G(x) \cup \{x\}$. For any subgraph G of K_n, we define $V(G)$ as the whole $V(K_n)$ even if, for some $v \in V(K_n)$, there is no edge of G incident to v. We say that a set $D \subset V(K_n)$ is a dominating set in a subgraph G of K_n if every vertex of $\overline{D}$ is adjacent to at least one vertex of D. We shall denote the minimal degree of a graph G by $\delta(G)$ and the subgraph of G formed by the edges joining the disjoint subsets $A, B \subseteq V(G)$ by $G[A, B]$.

Before starting the proof of Theorem 4, we state and prove three lemmas. All of them are very simple, but stating them as lemmas will make the proof of Theorem 4 easier to read.

The first lemma does not concern colorings, but rather only subgraphs of K_n whose complements do not contain double stars with high maximal degree.

Lemma 1 *Let* m, n *be integers with* $m \leq n/2$. *Let* G *be a subgraph of* K_n *for which* $\overline{G}$ *does not contain a spanning double star with maximal degree at least* $n - m$, *and let* u *be a vertex of* G *of degree at most* $m - 1$. *Then* $N_G(u)$ *is a dominating set in* G.

We will use this lemma for $m = 4$ only, but its proof is essentially the same for every m.

Proof. Suppose that $N_G(u)$ is not a dominating set in G, that is, there is a vertex v whose distance from u in G is at least 3. Then $uv \in E(\overline{G})$, and $N_G(u) \cap N_G(v) = \emptyset$. Hence uv is a central edge of a spanning double star contained in $\overline{G}$ in which the degree of u is $|\overline{N_G[u]}| \geq n - m$. ■

The next two lemmas concern edge-colored K_ns and their some particular subgraphs we define as follows. A subgraph of an edge-colored K_n is called *representing* iff it has exactly one edge of every color appearing on K_n. We shall call a subgraph H *representing-complement* (RC, for short) iff $\overline{H}$ is representing. Furthermore, for a given edge-coloring of K_n, we shall call a subgraph

H *special representing-complement* (SRC, for short) iff $\delta(H)$ is minimal over all RC subgraphs.

Lemma 2 *Let a subgraph* H *of an edge-colored* K_n *be SRC, and let* v *be a vertex whose degree in* H *is* $\delta(H)$*. Then for any edge* e *of* H *incident to* v *there is an edge of* $\overline{H}$ *incident to* v *colored with the color of* e*.*

Proof. Since H is RC, there is an edge f of $\overline{H}$ colored with the color of e. Since $\overline{H}$ is representing, $\overline{H} - f + e$ is also representing, hence $H - e + f$ is RC. If f was not incident to v, then the degree of v in $H - e + f$ would be $\delta(H) - 1$, contradicting the fact that H is SRC. ■

Lemma 3 *In every SRC subgraph* H *of an edge-colored* K_n*, the vertices whose degree in* H *is* $\delta(H)$ *forms an independent set.*

Proof. Let u, v be two vertices whose degrees in H is $\delta(H)$. By Lemma 2 in $\overline{H}$ there are edges f_1, f_2 incident to u, v, respectively, colored with the color of uv. Since H is RC, only one edge of $\overline{H}$ can be colored with the color of uv, hence $f_1 = f_2$. But then f_1 is incident to u and v both, that is, $f_1 = uv$, contradicting the facts $uv \in E(H)$ and $f_1 \in E(\overline{H})$. ■

Proof.[Proof of Theorem 4] Obviously $R^*(n, \mathcal{DS}^4_{n-1}) \geq R^*(n, \mathcal{T}_{n-1})$, and, for $n \geq 3$,

$$R^*(n, \mathcal{T}_{n-1}) \geq \binom{n-2}{2} + 1 \tag{9}$$

is a part of Theorem 1. (As we saw in the introduction, (9) is the easy direction of Theorem 1, since it immediately follows from (1) and (4).) Therefore, all we need to prove is

$$R^*(n, \mathcal{DS}^4_{n-1}) \leq \binom{n-2}{2} + 1.$$

For the sake of brevity, we shall call a subgraph *good* if it is RC and for some vertex of degree at most 3, the set of its neighbors is not dominating. By Lemma 1 it suffices to prove that, for every coloring of $E(K_n)$ that uses $\binom{n-2}{2} + 2$ colors, there is a good subgraph.

Consider an arbitrary coloring of the edges of K_n that uses $\binom{n-2}{2} + 2$ colors. Let H be an arbitrary SRC subgraph. Then its size is $\binom{n}{2} - \binom{n-2}{2} - 2 = 2n - 5$.

If $\delta(H) = 0$ then there is a vertex u with $N_H(u) = \emptyset$, hence H is good.

If $\delta(H) = 1$ then let u, v be vertices such that v is the only neighbor of u in H. By Lemma 2 there is a vertex w different from v such that the colors of uv and uw are the same, hence $H - uv + uw$ is also RC. If neither H nor $H - uv + uw$ is good, then v is adjacent to every vertex in H, and w is adjacent to every vertex in $H - uv + uw$, that is, to every vertex other than u in H. Hence the degrees of v and w in H are $n - 1$ and $n - 2$, respectively, so H has at least $2n - 4$ edges, a contradiction.

Suppose that $\delta(H) = 2$. In the remainder of this proof, the graph, whose adjacency relation is considered, is always H. Let u be a vertex of degree 2 and with neighbors v_1, v_2. First we assume that uv_1 and uv_2 have the same color, say, red. By Lemma 2 there is a vertex v_3 different from v_1 and v_2 such that uv_3 is also red. For each $1 \leq i \leq 3$, $H - uv_i + uv_3$ is RC. Thus if none of them is good, then for any permutation (i, j, k) of $(1, 2, 3)$, $\{v_i, v_j\}$ is a dominating set. Therefore every element of $V(K_n) \setminus \{u, v_1, v_2, v_3\}$ has at least 2 neighbors among the v_js, and every v_i has at least 1 neighbor among the other two v_js. Hence there are at least $2(n - 4)$ edges between $\{v_1, v_2, v_3\}$ and $V(K_n) \setminus \{u, v_1, v_2, v_3\}$, and at least 2 edges inside of $\{v_1, v_2, v_3\}$. Since the degree of u is 2, we have again the contradiction $|E(H)| \geq 2(n - 4) + 2 + 2 = 2n - 4$.

We may therefore assume that uv_1 and uv_2 have different colors. By Lemma 2 there are $w_1, w_2 \in V(K_n) \setminus \{u, v_1, v_2\}$ such that uw_i have the same color as uv_i. Let $z_1, \ldots, z_{n-5}$ be the remaining vertices. If none of the RC graphs H, $H - uv_1 + uw_1$, $H - uv_2 + uw_2$, $H - uv_1 + uw_1 - uv_2 + uw_2$ is good, then any pair containing one vertex among v_1, w_1 and one vertex among v_2, w_2 form a dominating set. Hence, for every $1 \leq i \leq n - 5$, z_i is adjacent to either v_1 and w_1 or to v_2 and w_2. Therefore there are at least $2(n - 5)$ edges that are incident with the set $\{z_1, \ldots, z_{n-5}\}$.

Assume that $v_1w_1 \notin E(H)$. If H is not good, then the set $\{v_1, v_2\}$ is dominating, hence $v_2w_1 \in E(H)$. Similarly, if $H - uv_1 + uw_1$, $H - uv_2 + uw_2$ or $H - uv_1 + uw_1 - uv_2 + uw_2$ is not good then, in order, v_2v_1, w_2w_1 or w_2v_1 is an edge of H. But, since the degree of u is 2, these imply that if none of the four mentioned subgraphs are good, then there are at least $2(n-5)+4+2 = 2n-4$ edges in H. Thus $v_1w_1 \in E(H)$. Similarly, $v_2w_2 \in E(H)$, and therefore there is an edge e of H such that the $2n - 6$ edges of $H - e$ are the following: uv_1, uv_2, v_1w_1, v_2w_2, and for every $1 \leq i \leq n - 5$, either the pair v_1z_i, w_1z_i or the pair v_1z_i, w_1z_i.

If $n \geq 8$ then there is an integer l such that $1 \leq l \leq n - 5$ and e is not adjacent to z_l. Hence the only neighbors of z_l are either v_1 and w_1 or v_2 and w_2. Thus if $N(z_l)$ is dominating then there are at least 2 edges between $\{v_1, w_1\}$ and $\{v_2, w_2\}$. However, as both are outside of $H - e$, this is impossible.

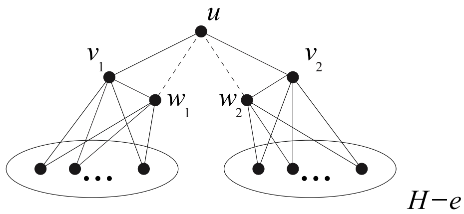

Figure 1: Case $\delta(H) = 2$ if uv_1 and uv_2 has different colors.

Therefore H is good.

On the other hand, by Lemma 3 there are no two adjacent vertices of degree 2. Hence there are at least two vertices of degree greater than 2, therefore the sum of the degrees is at least $2n + 2$. Since the size of H is $2n - 5$, we have $4n - 10 \geq 2n + 2$ and $n \geq 6$. Moreover, $n > 6$ since otherwise H would have two vertices of degree 3 and they would be adjacent to all the other four vertices, a contradiction.

Therefore the only remaining case is when $n = 7$ and $e = z_1z_2$. Then $N(w_1)$ is not dominating since v_2 and w_2 are not adjacent to any vertices in it. Clearly, the degree of w_1 is at most 3. (In fact, it is 2, since otherwise the degree of w_2 is 1 and so $\delta(H) \neq 2$.) Hence H is good.

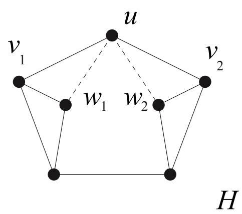

Figure 2: Subcase $n = 7$.

Finally, let $\delta(H) = 3$ and let u be a vertex of degree 3, with neighbors v_1, v_2, v_3. As in the previous argument, we can assume that there are (not necessarily different) vertices $w_1, w_2, w_3 \in V(K_n) \setminus \{u, v_1, v_2, v_3\}$ such that uw_i have the same color as uv_i. Set $W = \{v_1, v_2, v_3, w_1, w_2, w_3\}$ and let Z be the set of the remaining vertices, that is, $Z = V(K_n) \setminus (W \cup \{u\})$. As in the previous cases, every vertex of Z has at least 2 neighbors in W and every vertex of W has at least 1 neighbor in W. As we shall show, these facts lead to a lower bound on the sum of all degrees, that implies an upper bound on

n. The degree of u is 3. The sum of the degrees of the vertices in W is $|E(H[W,\{u\}])| + 2|E(H[W])| + |E(H[W,Z])|$, that is at least $3 + |W| + 2|Z|$. The sum of the degrees of the vertices in Z is at least $3|Z|$. Hence the sum of all degrees is at least $6 + |W| + 5|Z| = 6 + |W| + 5(n - |W| - 1) = 1 + 5n - 4|W|$, which is at least $5n - 23$. Since this sum is exactly $2|E(H)| = 4n - 10$, we have $5n - 23 \le 4n - 10$, that is, $n \le 13$.

On the other hand, $2|E(H)| = 4n - 10$ and $\delta(H) = 3$ imply that there are at least 10 vertices of degree 3. By Lemma 3, they form an independent set. Hence the graph has at least 30 edges. It implies that $2n - 5 \ge 30$, that is, $n \ge 18$, a contradiction. ■

3 An extremal property of the nontrivial Moore graphs

Theorem 1 and Theorem 4 imply immediately the following theorem.

Theorem 5 *For* $n \ge 3$*,*

$$R^*(n, \mathcal{T}^3_{n-1}) = \binom{n-2}{2} + 1.$$

In this section we shall prove an extremal property of the Moore graphs of diameter 2. As we saw in the introduction, the family of these graphs consists of C_5, the Petersen graph, the Hoffman-Singleton graph and perhaps some 57-regular graphs of order 3250. The fact that the Petersen graph possesses this property leads to a proof of Theorem 5 that is a bit shorter than our proof for Theorem 4.

We will prove the following theorem.

Theorem 6 *Let* G *be a graph of order* n*, diameter* 2 *and minimal degree* δ*. Then*

$$|E(G)| \ge \frac{(\delta + 1)n - \delta^2 - 1}{2},$$

and equality holds iff G *is a Moore graph.*

First, we show how can one simplify the proof of Theorem 5 by using Theorem 6.

Proof. [Proof of Theorem 5.] We repeat the proof of Theorem 4, changing the end only, which handled the case $\delta(H) = 3$. As we saw in the proof of

Lemma 1, if there are two vertices u, v of a graph G such that their distance is at least 3, than $\overline{G}$ contains a spanning double star (with central edge uv). In other words, if a graph does not contain a spanning double star, then the diameter of its complement is at most 2. Therefore, if a coloring of $E(K_n)$ does not yield any TMC spanning double star, then its RC subgraphs have diameter at most 2. If the coloring uses $\binom{n-2}{2}+2$ colors, then the size of its RC subgraphs is $\binom{n}{2}-\binom{n-2}{2}-2=2n-5$. Therefore if H is an SRC subgraph with $\delta(H)=3$, then, by Theorem 6, H is the Petersen graph. On the other hand, by Lemma 3, if an SRC subgraph of any coloring of $E(K_n)$ is not empty (that is, if the coloring uses less than $\binom{n}{2}$ colors) then it cannot be regular, thus we have the desired contradiction. ■

For proving Theorem 6 we need the following lemma.

Lemma 4 *Let n be an integer with $n \geq 2$, and let G be a graph of order n, diameter at most 2 and degree sequence $d_1, d_2, \ldots, d_n$. Then*

$$\sum_{i=1}^{n} d_i^2 \geq n^2 - n,$$

and equality holds iff either the girth of G is 5 or $G = K_{1,n-1}$.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and d_i be degree of v_i for every i. Count the walks of length 2 in G, that is, the ordered triples (v_i, v_j, v_k) of vertices with $v_iv_j, v_jv_k \in E(G)$ ($i = k$ being allowed). For a given j the number is obviously d_j^2, therefore the total number is $\sum_{i=1}^{n} d_i^2$.

Now we show that there is an injection f from the set of ordered pairs of distinct vertices to the set of these walks. For $v_iv_j \notin E(G)$, let $f(v_i, v_j) = (v_i, v_k, v_j)$ with arbitrary k such that $v_iv_k, v_kv_j \in E(G)$ (such j exists since the diameter of G is at most 2). For $v_iv_j \in E(G)$, let $f(v_i, v_j) = (v_i, v_j, v_i)$. f is an injection since for $i \neq k$, (v_i, v_j, v_k) can only be the image of (v_i, v_k), and for $i = k$, it can only be the image of (v_i, v_j).

Since the number of ordered pairs of distinct vertices is $n^2 - n$, $\sum_{i=1}^{n} d_i^2 \geq n^2 - n$. Equality holds iff f is surjective, that is, iff there is exactly one k with $v_iv_k, v_kv_j \in E(G)$ for every i, j with $v_iv_j \notin E(G)$ and there is no such k for any i, j with $v_iv_j \in E(G)$. In other words, iff G contains neither C_3 nor C_4, that is, G is either a forest or its girth is at least 5. The only forest with order n and diameter at most 2 is $K_{1,n-1}$. Since the distance of two vertices of a shortest cycle of a graph is obviously the length of the shorter arc of this cycle connecting them, a girth of a graph with diameter 2 cannot be greater than 5. ■

Proof. [Proof of Theorem 6.] Let $e = |E(G)|$. By Lemma 4, it is sufficient to show that if the degree sequence of G is $d_1, d_2, \dots, d_n$ and

$$\sum_{i=1}^{n} d_i^2 \geq n^2 - n, \tag{10}$$

then

$$e \geq \frac{(\delta+1)n - \delta^2 - 1}{2}, \tag{11}$$

and equality holds in (11) only if it holds in (10).

First assume that $n \leq \delta^2 + 1$. In this case, since obviously $e \geq \delta n/2$, (11) is trivially true. Equality holds iff $n = \delta^2 + 1$ and $d_i = \delta$ for every i. Then $\sum_{i=1}^{n} d_i^2 = n\delta^2 = n(n-1) = n^2 - n$.

Now assume $n > \delta^2+1$. Because of convexity of the function x^2, if $x_1, \dots, x_n$ and s are real numbers such that, for every i, $x_i \geq \delta$, $s \geq \delta n$ and $\sum_1^n x_i = s$, then $\sum_{i=1}^{n} x_i^2 \leq (n-1)\delta^2 + (s-(n-1)\delta)^2$, and equality holds iff there is at most one i with $x_i > \delta$. Hence if $\sum_{i=1}^{n} d_i^2 \geq n^2 - n$ then

$$(n-1)\delta^2 + (2e - (n-1)\delta)^2 \geq n^2 - n,$$

that is,

$$\begin{aligned} 4e^2 - 4\delta(n-1)e + \delta^2((n-1)^2 + n - 1) &\geq n(n-1), \\ e^2 - \delta(n-1)e + \frac{\delta^2 - 1}{4} n(n-1) &\geq 0, \end{aligned}$$

therefore

$$\begin{aligned} 2e &\geq \delta(n-1) + \sqrt{\delta^2(n-1)^2 - (\delta^2-1)n(n-1)} \\ &= \delta(n-1) + \sqrt{(n-1)(n-\delta^2)}. \end{aligned}$$

Thus, it suffices to show

$$\delta(n-1) + \sqrt{(n-1)(n-\delta^2)} > (\delta+1)n - \delta^2 - 1,$$

that is,

$$\delta^2 - \delta + 1 > n - \sqrt{(n-1)(n-\delta^2)}. \tag{12}$$

It is easy to verify that the right-hand side is strictly decreasing in n. Since equality holds in (12) for $n = \delta^2 + 1$, the inequality is valid for $n > \delta^2 + 1$. ■

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Deciding soccer scores and partial orientations of graphs

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Abstract. We show that deciding if a simple graph has a partial orientation of its edges such that all vertices have a prescribed in-, out- and undirected degree, is NP-complete even for planar graphs. We prove that two related questions are also NP-complete, one is the decision of whether a score vector of a soccer-tournament is legal or not (we know who played who so far, but do not know the outcomes), the other is about a special edge-coloring of 3-uniform hypergraphs.

1 Introduction

The problem of deciding whether we can direct a graph with each vertex having a prescribed in- and out-degree is well-known to be in P. It is another interesting question to determine the complexity of the problem where instead of a directed graph, we want to obtain a mixed graph, ie. a graph that has both directed and undirected edges, and we prescribe the in-, out- and undirected-degree of each vertex. Let us denote the problem of deciding whether this can be done or not by Partial Orientation for general graphs and Pl-PO for planar graphs. We show that both Partial Orientation and Pl-PO are NP-complete.

The Elimination Problem is to decide whether a given team can still win the tournament at some point. This was shown to be NP-complete not so long ago independently by Bernholt et al. ([1]) and Kern and Paulusma ([3]). Later it was also generalized to various other point-systems by Kern and

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Paulusma ([4]), in this paper they solve completely for which score allocation rules the problem is NP-complete, assuming that we do not require that the score vector is reachable in a valid tournament. They suspected that deciding if a score vector is reachable or not (if we know the remaining games) is a difficult problem. So let us denote the problem of deciding whether a given score vector is a possible result of a soccer-tournament or not (if we know which team played against which so far) by SCORE VECTOR. In this paper we prove that SCORE VECTOR is NP-complete (in the case when teams get $1 < p \neq 2$ points for winning, 1 for drawing and 0 for losing a game). The proof is an easy consequence of our construction given to the PARTIAL ORIENTATION problem.

Let us define the *tricoloring* of a hyperedge containing 3 vertices such that we color its vertices *red*, *green* and *blue*, using all three colors. Given a 3-uniform hypergraph and a color requirement for each vertex that prescribes how many times it has to be red, green and blue, the problem of deciding whether there is a suitable tricoloring or not, is denoted by TRICOLORING. We show that TRICOLORING is NP-complete.

2 Partial orientation of general graphs

We denote the degree of a vertex v in a simple graph by $d(v)$. In the mixed graph the in-degree is denoted by $\rho(v)$, the out-degree by $\delta(v)$ and the number of the adjacent undirected edges by $\theta(v)$. Thus $d(v) = \rho(v) + \delta(v) + \theta(v)$. When we say orientation, we mean *three* possibilities: The two directions and the undirected case. Thus in the beginning we have a graph with unoriented edges and we want to orient them.

We reduce 3-SAT to PARTIAL ORIENTATION as follows: We construct a graph for each input formula to 3-SAT. For each x_i variable the graph will have a tree that is almost binary; its root has degree two, each vertex on an odd level has degree three and each vertex on an even level has degree two. The last level is an even one, and from each leaf there is an edge connecting the tree to the rest of the graph, whose other end will be determined later. (See Figure 1.) For the root we prescribe $\rho(r_i) = \delta(r_i) = 1$. For the orientation of each edge of the tree there will be exactly two possibilities. The direction of the two edges of r_i will determine the orientation of each other edge in the tree.

For each vertex w on an odd level of the tree we prescribe $\rho(w) = \delta(w) = \theta(w) = 1$ and for each vertex v on an even level we prescribe either $\rho(v) =$

$\delta(v) = 1$ or $\rho(v) = \theta(v) = 1$ or $\theta(v) = \delta(v) = 1$. When we say that v is $\rho\delta$ (or $\rho\theta$ or $\delta\theta$), we mean that for the degree two vertex v the prescription is $\rho(v) = \delta(v) = 1$. One of the two grandchildren of a $\rho\delta$ vertex is always a $\rho\theta$, while the other is always a $\delta\theta$. Similarly, the $\rho\theta$ vertices have $\rho\delta$ and $\delta\theta$ grandchildren and $\delta\theta$ vertices have $\rho\delta$ and $\rho\theta$ grandchildren. The root has four grandchildren, both of its children have one $\rho\theta$ and one $\delta\theta$ child. This finishes the description of the tree. Note that since every edge in the tree is incident to a vertex of degree two, we have exactly two possible orientation for each edge. When we say that an edge is $\rho\delta$, we mean that its orientation cannot be undirected.

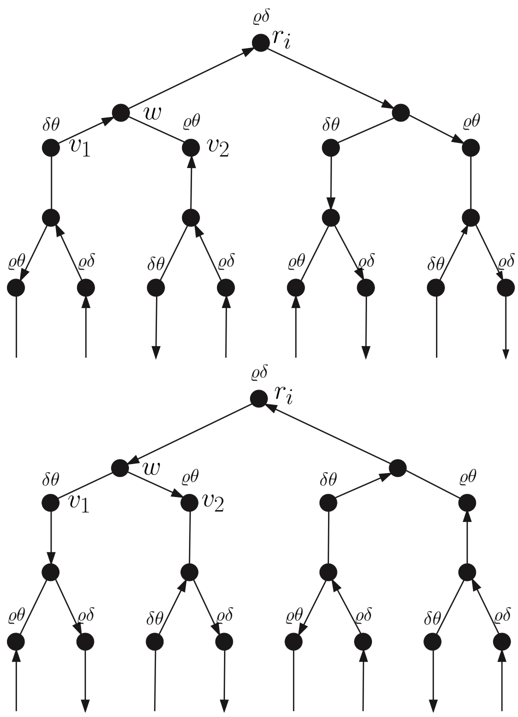

Figure 1: The two possible orientations of the tree associated with x_i.

Eg., let us take one of r_i's children, w, and both of w's children, v_1 and v_2. The edge $r_i w$ can be either directed towards w or away from w but it cannot be undirected. (See the two possibilities in Figure 1.) The edge connecting v_1 to its child can be undirected or directed away from v_1 but this is determined by the orientation of $r_i w$. The edge connecting v_2 to its child can be directed towards v_2 or be undirected and this is also determined by the orientation of $r_i w$. These edges determine the orientation of the edges under them and therefore the orientation of the whole tree depends on the choice at the root. This way we can achieve that from one decision at r_i we have an arbitrary number of edges directed to the same way from the leaves of the tree. Let us count how many.

Let us denote the number of the $\rho\delta$ edges (the ones that cannot be undirected) that are going from the $2l$th level to the $2l+1$th by $a(l)$ and the number of the other edges at the same level by $b(l)$. We have $a(0) = 2$ and $b(0) = 0$ and it is easy to see that the equations $a(l) = b(l-1)$ and $b(l) = 2a(l-1)+b(l-1)$ hold. Solving these we get $a(l+1) = b(l) = 4(2^l - (-1)^l)/3$. For each variable x_i, let us denote the (unnegated) occurrences of x_i in the clauses by u_i and the occurrences of $\overline{x_i}$ by n_i. We choose the height h_i of the tree associated with x_i to be the smallest number that satisfies $a(h_i) \geq 2\max(u_i, n_i)$. This implies that the size of each tree is at most linear. Note that half of the edges counted in $a(l)$ are directed towards the tree, and the other half away from the tree, whichever orientations we choose at r_i. We will call one of these orientations *true* and the other orientation *false*. For each clause that contains x_i, we reserve an edge that is directed away from the tree in the *true* orientation and towards the tree in the *false* orientation. Similarly, for each clause that contains $\overline{x_i}$, we reserve an edge that is directed towards the tree in the *true* orientation and away from the tree in the *false* orientation. This can be done since $a(h_i)$ is sufficiently large.

For each clause C the graph will have a vertex v_c of degree 5. The prescription for each v_C is $\rho(v_C) = 3$ and $\delta(v_C) = 2$. The three edges reserved for clause C (adjacent to the leaves of the trees associated with the variables of C) are connected to the vertex v_C. The remaining two edges are connected to the degree two $\rho\delta$ vertices v_{C1} and v_{C2}. The other neighbor of these vertices are to be determined.

Now we are done with the representation of our formula, we only need to somehow take care of the edges that have only one incident vertex so far. To this end, we add the *mirrored reflection* of everything constructed so far to the graph. This means for every vertex v that belongs to a tree or a clause, we add a v' vertex that is connected to w' if and only if v is connected to w. We also connect v and v' if and only if v has an edge that was not connected to any other vertex yet. The prescription of v' is $\rho(v') = \delta(v), \delta(v') = \rho(v)$ and $\theta(v') = \theta(v)$. This finishes our construction.

Now we have to prove that this graph has a mixed orientation fulfilling the required prescriptions if and only if the original formula had a true assignment.

First, if the formula had a true assignment, then let us orient the edges of the trees associated with the true variables in their *true* orientation and orient edges of the trees associated with the false variables in their *false* orientation. Each v_C will have at least one edge entering from a tree, we can pick the two edges connecting it to v_{C1} and v_{C2} such that $\rho(v_C) = 3$. We do the opposite with each edge in the mirrored part of the graph, this guarantees a good orientation for the vv' type edges.

Similarly, if the graph has a good orientation, then let us pick the variables associated with the trees whose orientation is *true* to be true, and the rest to be false. Since $\rho(v_C) = 3$ and only two edges can enter v_C that are not coming from a tree, therefore one of the trees associated with a variable of C must have *true* orientation, thus each clause must have a true literal.

3 Partial orientation of planar graphs

The construction will be very similar to the previous one, but now instead of 3-SAT we will reduce Pl-1-Ex3MonoSat to Pl-PO. The Pl-1-Ex3MonoSat problem is the following. The input is a CNF which consists of clauses containing exactly 3 variables, each unnegated. Furthermore, the CNF has a planar realization, ie. there is a planar, bipartite graph such that one class represents the variables, the other the clauses and there is an edge iff the variable is contained in the clause. The problem is to decide if there is an assignment such that there is exactly one true literal in every clause. Pl-1-Ex3MonoSat was shown to be NP-complete by Hunt et al. [2].

Our reduction is similar as in the case of general graphs, but the same does not work because the edges going to the mirrored part might intersect each other. So instead of the mirrored part, we have to come up with a new idea how to take care of the unneeded edges.

Each variable occurring t times in the clauses, will be represented by t copies of a tree that are connected to each other. Each copy will be a tree with three levels (seven vertices) that was defined in the previous section. These copies are connected to each other in a cycle - the other end of the edge of the rightmost leaf of the ith tree is the leftmost leaf of the $i+1$th tree (mod r). (See Figure 2.) Because of this, we either have $2r$ undirected or r pairs of directed edges (where from each pair one is directed away, the other towards the leaf) leaving the variable component. We call the first the *true* orientation.

Each clause is represented by a single vertex v_C for which we prescribe $\rho(w) = \delta(w) = \theta(w) = 2$. From each variable, that is in the clause, we connect

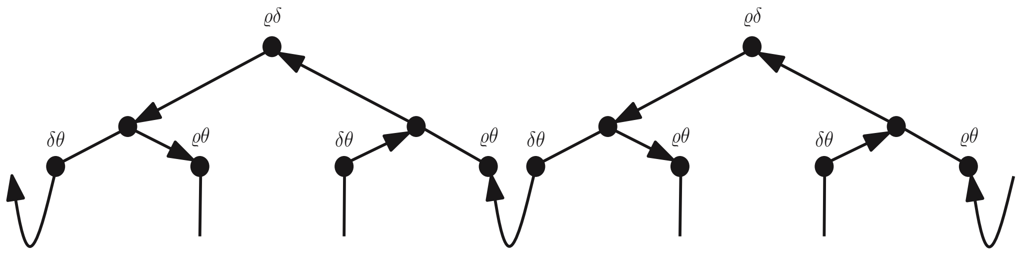

Figure 2: Copies of trees associated with x_i are connected to each other.

a pair of edges to v_C. This means that exactly one of the variables of the clause must be true. Therefore the graph has a good orientation if and only if the original formula had a true assignment.

4 Score vector problem

To prove that Score Vector is NP-complete, we associate a vertex of a graph to each of the teams. The graph is the same as in the Partial Orientation of general graphs, but instead of prescribing the degrees of a vertex v, we prescribe the score of the team associated with that vertex to be $p\delta(v) + \theta(v)$ (it would get this much if it had won $\delta(v)$, drew $\theta(v)$ and lost $\rho(v)$ games). Now we only have to notice that in our construction the score of each vertex that has degree at most three, determines the number of games that the team associated with that vertex won, drew and lost. Eg., if a vertex w has $p+1$ points and $d(w) = 3$, then this is only possible if it has won one game, drew one game and lost one game (since $1 < p \neq 2$). Since none of the vertices adjacent to the v_C's drew any of their games, the v_C's must have 3 wins and 2 losses. Therefore our construction reduces 3-SAT to Score Vector if instead of the degrees we prescribe the scores.

Note that when $p = 2$, the construction fails because one win, one draw and one losing worth the same number of points as three draws. For this $p = 2$ case the problem is in P and the proof is a folklore; just take the original simple graph, double every edge and ask whether this graph can be (completely) directed such that for every vertex v the number of edges directed away from v equals the score of v.

In a soccer tournament usually the teams have played the same number of matches at a given time, while in our construction the degrees vary. We can fix this by adding a few new vertices who have won all their matches and played some of the teams whose degree is less than the average. Also, in tournaments everyone plays with everyone else in a round, so at any point the who-played-who-so-far graph can be partitioned into perfect matchings. Our construction with a little modification can be transformed into a regular bipartite graph, that always have this property.

5 Triorientation problem

First we will modify a bit our construction given for the PARTIAL ORIENTATION of general graphs. Delete all the vertices that belong to the mirrored part (half of the vertices) and replace them with a single vertex z. The neighbors of z are all the vertices that were connected to the mirrored part. This way we obtain a bipartite graph $G = (A, B, E)$ and in one class (eg. in A) every vertex has degree two. We claim that if we let $\theta(z) = \sum\{\theta(a) : a \in A\} - \sum\{\theta(b) : b \in B \setminus \{z\}\}$, $\rho(z) = \sum\{\delta(a) : a \in A\} - \sum\{\rho(b) : b \in B \setminus \{z\}\}$, $\delta(z) = \sum\{\rho(a) : a \in A\} - \sum\{\delta(b) : b \in B \setminus \{z\}\}$, then it is NP-complete to decide if this graph has a mixed orientation. We can use the same argument as we did in Section 2, we only have to check that the degree prescriptions of z are not violated and this follows from the fact that G is bipartite; if all other requirements are satisfied, then its requirements are satisfied as well.

Now we are ready to present a 3-uniform hypergraph. The vertices of the hypergraph are the same as the vertices of G. For each vertex in A, add one hyperedge, $H = \{(a, u, v) : a \in A, \overline{au} \in E, \overline{av} \in E\}$. The color-prescriptions of the hypergraph are determined by the degree-prescriptions of G. For $b \in B : \mathtt{red}(b) = \rho(b), \mathtt{green}(b) = \delta(b), \mathtt{blue}(b) = \theta(b)$, for $a \in A : \mathtt{red}(a) = 1 - \rho(a), \mathtt{green}(a) = 1 - \delta(a), \mathtt{blue}(a) = 1 - \theta(a)$. This way, for instance an $a \in A$ vertex that is $\rho\delta$ in G, becomes $\mathtt{blue}$ in its only hyperedge. We claim that this hypergraph has a triorientation iff G has a mixed orientation.

If G has a mixed orientation, then the color of u in $(a, u, v) \in H$ is $\mathtt{red}$ if $\overline{au}$ is directed away from u, $\mathtt{green}$ if $\overline{au}$ is directed towards u and $\mathtt{blue}$ if $\overline{au}$ is undirected. It is easy to see that this is a good triorientation.

If the hypergraph has a good triorientation, then if u in $(a, u, v) \in H$ is $\mathtt{red}$, we direct $\overline{au}$ away from u, if u is $\mathtt{green}$, we direct $\overline{au}$ towards u and if u is $\mathtt{blue}$, we let $\overline{au}$ to be undirected. Since the color of a, u and v are different, this gives a good mixed orientation, satisfying all the degree-requirements.

6 Acknowledgments and concluding remarks

I would like to thank my supervisor, Zoltán Király for early discussions and suggesting the solution for the Tricoloring problem. I would also like to thank Attila Bernáth for his useful advices. He also noticed that if instead of the 3-SAT problem we use the ONE-IN-THREE-SAT problem (meaning that in a 3-CNF we want exactly one literal to be true, also NP-complete), then we do not need the v_{Ci} vertices and thus we obtain a graph with maximum degree three, which is clearly optimal. It is also possible to modify the hypergraph construction such that every vertex has degree at most three.

An interesting open question remains to determine the complexity of the problem when we only know the score (or the in-, out- and undirected degrees) of each vertex and the number of games it played (but do not know against whom) and we have to decide whether it is a possible outcome of a real tournament or not. We conjecture these problems to be in P although we could not even solve it in the case when we know that everyone played with everyone else exactly once (meaning the tournament is finished, ie. the graph is the complete graph). A similar question can be raised concerning the Elimination Problem.

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Hadamard product of GCD matrices

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Abstract. Let f be an arithmetical function. The matrix $[f(i,j)]_{n\times n}$ given by the value of f in greatest common divisor of (i,j), $f\big((i,j)\big)$ as its i, j entry is called the greatest common divisor (GCD) matrix. We consider the Hadamard product of this matrices and we calculate the Hadamard product and the determinant of Hadamard product of two GCD matrices.

1 Introduction

The classical Smith determinant introduced by H. J. Smith [6] is

$$\det[(i,j)]_{n\times n} = \begin{vmatrix} (1,1) & (1,2) & \cdots & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n) \\ \cdots & \cdots & \cdots & \cdots \\ (n,1) & (n,2) & \cdots & (n,n) \end{vmatrix} = \varphi(1)\cdot\varphi(2)\cdots\varphi(n), \tag{1}$$

where (i,j) is the greatest common divisor of i and j, and $\varphi(n)$ is Euler's totient function.

The GCD matrix with respect to f is

$$[f(i,j)]_{n\times n} = \begin{bmatrix} f((1,1)) & f((1,2)) & \cdots & f((1,n)) \\ f((2,1)) & f((2,2)) & \cdots & f((2,n)) \\ \cdots & \cdots & \cdots & \cdots \\ f((n,1)) & f((n,2)) & \cdots & f((n,n)) \end{bmatrix}$$

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If we consider the GCD matrix $[f(i,j)]_{n\times n}$ where

$$f(n)=\sum_{d|n} g(d),$$

H. J. Smith proved that

$$\det[f(i,j)]_{n\times n}=g(1)\cdot g(2)\cdots g(n).$$

For $g=\varphi$

$$f(i,j)=\sum_{d|(i,j)} \varphi(d)=(i,j).$$

this formula reduces to (1).
If we consider the GCD matrix $[f(i,j)]_{n\times n}$ where $f(n)=\sum_{d|n} g(d)$ Pólya and Szegő [5] proved that

$$[f(i,j)]_{n\times n}=G\cdot C^T, \tag{2}$$

where G and A are lower triangular matrices given by

$$g_{ij}=\begin{cases} g(j), & j\mid i\\ 0, & \text{otherwise}\end{cases}$$

and

$$c_{ij}=\begin{cases} 1, & j\mid i\\ 0, & \text{otherwise}\end{cases}$$

L. Carlitz [2] in 1960 gave a new form of (2)

$$[f(i,j)]_{n\times n}=C\ \mathrm{diag}\big(g(1),\ g(2),\dots,g(n)\big)\ C^T, \tag{3}$$

where $C=[c_{ij}]_{n\times n}$,

$$c_{ij}=\begin{cases} 1, & \text{if} \quad j\mid i\\ 0, & \text{if} \quad j\not\mid i\end{cases},$$

$D=[d_{ij}]_{n\times n}$ diagonal matrix

$$d_{ij}=\begin{cases} g(i), & \text{ha} \quad i=j\\ 0, & \text{ha} \quad i\neq j\end{cases}.$$

From (3) follows that the value of the determinant is

$$\det[f(i,j)]_{n\times n}=g(1)g(2)\cdots g(n). \tag{4}$$

Here we present some examples which are relevant in our study.

Example 1 *If*

$$g(n) = \beta(n) = \sum_{i=1}^{n} (i, n)$$

the Pillai function then

$$f(n) = \sum_{d|n} \beta(n) = n\tau(n),$$

where $\tau(n)$ *the number of divisors. The GCD matrix and determinant in this case have the following form:*

$$[(i,j)\tau(i,j)]_{n\times n} = C \ \mathit{diag}\big(\beta(1),\ \beta(2), \ldots, \beta(n)\big)\ C^T, \tag{5}$$

$$\det[(i,j)\tau(i,j)]_{n\times n} = \beta(1)\beta(2)\cdots\beta(n). \tag{6}$$

Example 2 *If* $g(n) = \dfrac{\varphi(n)}{n}$*, then*

$$f(n) = \sum_{d|n} \frac{\varphi(d)}{d} = \frac{\beta(n)}{n},$$

$$\left[\frac{\beta(i,j)}{(i,j)}\right]_{n\times n} = C \ \mathit{diag}\left(\frac{\varphi(1)}{1},\ \frac{\varphi(2)}{2}, \ldots, \frac{\varphi(n)}{n}\right)\ C^T, \tag{7}$$

$$\det\left[\frac{\beta(i,j)}{(i,j)}\right]_{n\times n} = \frac{\varphi(1)\varphi(2)\cdots\varphi(n)}{n!}.$$

For other contributions, we mention the papers of S. Beslin and S. Ligh [1], P. Haukkanen, J. Wang and J. Sillanpää [3].
We introduce the concept of Hadamard product (see F. Zhang [7]).

Definition 1 *The Hadamard product* $C = A \circ B = [c_{ij}]_{n\times n}$ *of two matrices* $A = [a_{ij}]_{n\times n}$ *and* $B = [b_{ij}]_{n\times n}$ *is simply their elementwise product,*

$$c_{ij} = a_{ij}b_{ij}, \quad i,j \in \{1,2,\ldots,b\}.$$

A. Ocal [4] establishes various results concerning GCD matrices and least common multiple (LCM) matrices. In examples 1 and 2 appears Hadamard products of special GCD matrices:

$$\det\Big[[\tau(i,j)]_{n\times n} \circ [(i,j)]_{n\times n}\Big]_{n\times n} = \beta(1)\beta(2)\cdots\beta(n),$$

$$\det\left[[\beta(i,j)]_{n\times n}\circ\left[\frac{1}{(i,j)}\right]_{n\times n}\right]_{n\times n}=\frac{\varphi(1)\varphi(2)\cdots\varphi(n)}{n!}.$$

Let f and g be two arithmetical functions. In this paper we calculate the Hadamard product and the determinant of Hadamard product of $[f(i,j)]_{n\times n}$ and $[g(i,j)]_{n\times n}$.

2 Main results

Theorem 1 *Let* h *and* g *be two arithmetical functions and* g *totally multiplicative. If*

$$f(n)=\sum_{d|n}h(d)g\left(\frac{n}{d}\right), \tag{8}$$

then

1.

$$\left[[f(i,j)]_{n\times n}\circ\left[\frac{1}{g(i,j)}\right]_{n\times n}\right]_{n\times n}=C\ \mathit{diag}\left(\frac{h(1)}{g(1)},\frac{h(2)}{g(2)},\ldots,\frac{h(n)}{g(n)}\right)\ C^T,$$

where $C=[c_{ij}]_{n\times n}$,

$$c_{ij}=\begin{cases}1, & \text{if}\quad j\mid i\\ 0, & \text{if}\quad j\not\mid i\end{cases},$$

2.

$$\det\left[[f(i,j)]_{n\times n}\circ\left[\frac{1}{g(i,j)}\right]_{n\times n}\right]_{n\times n}=\frac{h(1)}{g(1)}\frac{h(2)}{g(2)}\cdots\frac{h(n)}{g(n)}, \tag{9}$$

3. *Exists* $H(n)$ *and* $G(n)$ *arithmetical functions such that*

$$\det\left[[f(i,j)]_{n\times n}\circ\left[\frac{1}{g(i,j)}\right]_{n\times n}\right]_{n\times n}=\frac{\det[H(i,j)]}{\det[G(i,j)]}.$$

Proof. Let

$$A=[a_{ij}]_{n\times n}=\left[[f(i,j)]_{n\times n}\circ\left[\frac{1}{g(i,j)}\right]_{n\times n}\right]_{n\times n}.$$

By the definition of Hadamard product we have

$$a_{ij}=\frac{f(i,j)}{g(i,j)}.$$

If we calculate

$$B = [b_{ij}]_{n\times n} = C \text{ diag}\left(\frac{h(1)}{g(1)}, \frac{h(2)}{g(2)}, \dots, \frac{h(n)}{g(n)}\right)$$

we have

$$b_{ij} = \sum_{k|j}\sum_{k|i} \frac{h(k)}{g(k)}.$$

But taking in consideration that g is totally multiplicative and by (8) we can deduce that

$$\begin{aligned} b_{ij} &= \sum_{k|(j,i)} \frac{h(k)}{g(k)} = \sum_{k|(j,i)} \frac{h(k)g\left(\frac{(i,j)}{k}\right)}{g(k)g\left(\frac{(i,j)}{k}\right)} = \frac{1}{g\big((i,j)\big)} \sum_{k|(j,i)} h(k)g\left(\frac{(i,j)}{k}\right) = \\ &= \frac{f(i,j)}{g(i,j)} = a_{ij}, \end{aligned}$$

which means that $A = B$.
If we calculate the determinant of both parts we have (9).
Let

$$H(n) = \sum_{d|n} h(d)$$

and

$$G(n) = \sum_{d|n} g(d).$$

By (4) we have

$$\det[H(i,j)]_{n\times n} = h(1)h(2)\cdots h(n)$$

and

$$\det[G(i,j)]_{n\times n} = g(1)g(2)\cdots g(n).$$

which means that

$$\det\left[[f(i,j)]_{n\times n} \circ \left[\frac{1}{g(i,j)}\right]_{n\times n}\right]_{n\times n} = \frac{\det[H(i,j)]}{\det[G(i,j)]}.$$

■

Example 3 *If* $g(n) = n$ *then*

$$f(n) = \sum_{d|n} h(d)\frac{n}{d}.$$

and

$$\left[\frac{f(i,j)}{(i,j)}\right]_{n\times n} = \left[[f(i,j)]_{n\times n} \circ \left[\frac{1}{(i,j)}\right]_{n\times n}\right]_{n\times n} =$$

$$= C \ \mathit{diag}\left(\frac{h(1)}{1}, \frac{h(2)}{2}, \ldots, \frac{h(n)}{n}\right) \ C^T$$

$$\det\left[\frac{f(i,j)}{(i,j)}\right]_{n\times n} = \det\left[[f(i,j)]_{n\times n} \circ \left[\frac{1}{(i,j)}\right]_{n\times n}\right]_{n\times n} = \frac{h(1)h(2)\cdots h(n)}{n!}.$$

Example 4 *If* $g(n) = \frac{1}{n}$ *then*

$$f(n) = \sum_{d|n} h(d)\frac{d}{n}$$

and

$$[f(i,j)(i,j)]_{n\times n} = C \ \mathit{diag}(h(1)1, h(2)2, \ldots, h(n)n) \ C^T,$$

$$\det[f(i,j)(i,j)]_{n\times n} = \det\left[[f(i,j)]_{n\times n} \circ [(i,j)]_{n\times n}\right]_{n\times n} = h(1)\cdots h(n)n!.$$

If we want to apply this theorem to given f and g, by Mobius inversion formula we have

$$h(n) = \sum_{d|n} \mu(d)g(d)f\left(\frac{n}{d}\right)$$

where $\mu(n)$ is the usual Mobius function and we can formulate the following result.

Theorem 2 *Let* f *and* g *be two arithmetical functions and* g *totally multiplicative. We have*

$$\begin{aligned}\left[\frac{f(i,j)}{g(i,j)}\right]_{n\times n} &= \left[[f(i,j)]_{n\times n} \circ \left[\frac{1}{g(i,j)}\right]_{n\times n}\right]_{n\times n} = \\ &= C \ \mathit{diag}\left(\frac{f(1)}{g(1)}, \ldots, \frac{\sum_{d|n}\mu(d)g(d)f\left(\frac{n}{d}\right)}{g(n)}\right) \ C^T,\end{aligned}$$

and

$$\det\left[[f(i,j)]_{n\times n} \circ \left[\frac{1}{g(i,j)}\right]_{n\times n}\right]_{n\times n} = \frac{f(1)}{g(1)} \cdots \frac{\sum_{d|n}\mu(d)g(d)f\left(\frac{n}{d}\right)}{g(n)}.$$

Example 5 *If* f *power free multiplicative arithmetical function* ($f(p^\alpha) = f(p)$)

$$\det\left[f(i,j)(i,j)\right]_{n\times n} = \prod_{k=1}^{n} \varphi(k)f(k),$$

in particular if $f(n) = \gamma(n)$ *the greatest square free divisor of* n

$$\det\left[\gamma((i,j))(i,j)\right]_{n\times n} = \prod_{k=1}^{n} \varphi(k)\gamma(k).$$

Example 6 *For a power GCD matrix and determinant we have*

$$[(i,j)^s]_{n\times n} = C\ \mathit{diag}\big(J_s(1),\ J_s(2),\ldots,J_s(n)\big)\ C^T,$$

$$\det[(i,j)^s]_{n\times n} = J_s(1)J_s(2)\cdots J_s(n).$$

where $J_s(n)$ *the jordan totient function.*

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Real vector space with scalar product of quasi-triangular fuzzy numbers

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Abstract. The construction of membership function of fuzzy numbers is an important problem in vagueness modeling. Theoretically, the shape of fuzzy numbers must depend on the applied triangular space. The membership function must defined in a such a way that the change of the triangular norm modifies the shape of fuzzy number, but the calculus with them remain valid. The quasi-triangular fuzzy numbers introduced by M. Kovacs in 1992 are satisfied this requirement. The shortage that not any quasi-triangular fuzzy number has opposite (inverse) can be solved if the set of quasi-triangular fuzzy numbers is included isomorphically in an extended set and this extended set with addition forms a group. In the present paper we formulate the extended set of the quasi-triangular fuzzy numbers, being also shown that the extended set is a real vector space with scalar product.

1 Introduction

The concept of quasi-triangular fuzzy numbers generated by a continuous decreasing function was introduced first by M. Kovács in 1992. The shortage that not any quasi-triangular fuzzy number has opposite (inverse) but only the ones with spread zero, can be solved if the set of quasi-triangular fuzzy numbers is included isomorphically in an extended set and this extended set with addition forms a group. In section 3 this group is constructed and in

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section 4 it is shown that the extended set with addition and multiplication with a scalar is a real vector space. In the section 5 we construct the real vector space with scalar product of quasi-triangular fuzzy numbers.

In the study of algebraic structures for fuzzy numbers many results since the 1970s have been obtained. For example D. Dubois and H. Prade (1978) investigates the operations with fuzzy numbers and theirs properties, R. Goetschel and W. Voxman (1986) and S. Gähler (1999) continue this work and M. Kovacs and L. H. Tran (1991) constructs and studies the set of centered M-fuzzy numbers. M. Kovacs (1992) introduces a notion of quasi-triangular fuzzy number which was used in the fuzzy linear programming by Z. Mako (2006). The properties of another class of quasi-triangular fuzzy numbers were investigated by M. Mares (1992,1992/1993, 1993, 1997), J. Dombi and N. Győrbíró (2006) and D. H. Hong (2007) obtains some properties of the operations with fuzzy numbers. A. M. Bica (2007) investigates the operations over the class of fuzzy numbers.

2 Preliminaries

The fuzzy set concept was introduced in mathematics by K. Menger in 1942 and reintroduced in the system theory by L. A. Zadeh in 1965. L. A. Zadeh has introduced this notion to measure quantitatively the vague of the linguistic variable. The basic idea was: if X is a set, then all A subsets of X can be identified with its characteristic function $\chi_A : X \to \{0, 1\}$, $\chi_A(x) = 1 \Leftrightarrow x \in A$ and $\chi_A(x) = 0 \Leftrightarrow x \notin A$.

The notion of fuzzy set is another approach of the subset notion. There exist continue and transitory situations in which we have to sugest that an element belongs to a set by different level. This fact we indicate with the membership degree.

Definition 1 *Let X be a set. A mapping $\mu : X \to [0, 1]$ is called* membership function*, and the set $\bar{A} = \{(x, \mu(x)) \ / x \in X\}$ is called* fuzzy set *on X. The membership function of $\bar{A}$ is denoted by $\mu_{\bar{A}}$.*

The collection of all fuzzy subsets of X we will denote by $\mathcal{F}(X)$. We place a bar over a symbol if it represents a fuzzy set. If $\bar{A}$ is a fuzzy set of X, then $\mu_{\bar{A}}(x)$ represents the membership degree of x to X. The empty fuzzy set is denoted by $\bar{\emptyset}$, where $\mu_{\bar{\emptyset}}(x) = 0$ for all $x \in X$. The total fuzzy set is denoted by $\bar{X}$, where $\mu_{\bar{X}}(x) = 1$ for all $x \in X$.

Definition 2 *The height of* $\bar{A}$ *is defined as* $\mathrm{hgt}\left(\bar{A}\right) = \sup_{x \in X} \mu_{\bar{A}}(x)$. *The* support *of* $\bar{A}$ *is the subset of* X *given by supp*$\bar{A} = \{x \in X \ / \ \mu_{\bar{A}}(x) > 0\}$.

Definition 3 *Let* X *be a topological space. The* $\alpha-$level *of* $\bar{A}$ *is defined as*

$$\left[\bar{A}\right]^{\alpha} = \begin{cases} \{x \in X / \ \mu_{\bar{A}}(x) \geq \alpha\} & \text{if} \quad \alpha > 0, \\ \mathrm{cl}\left(\mathrm{supp}\bar{A}\right) & \text{if} \quad \alpha = 0. \end{cases}$$

where $\mathrm{cl}\left(\mathrm{supp}\bar{A}\right)$ *is closure of the support of* $\bar{A}$.

Definition 4 *A fuzzy set* $\bar{A}$ *on vector space* X *is* convex*, if all* α *-levels are convex subsets of* X*, and it is* normal *if* $\left[\bar{A}\right]^{1} \neq \emptyset$.

In many situations people are only able to characterize imprecisely numerical data. For example people use terms like: "about 100" or "near 10". These are examples of what are called fuzzy numbers.

Definition 5 *A convex, normal fuzzy set on the real line* $\mathbb{R}$ *with upper semi-continuous membership function will be called* fuzzy number*.*

Triangular norms and co-norms were introduced by K. Menger (1942) and studied first by B. Schweizer and A. Sklar (1961, 1963, 1983) to model distances in probabilistic metric spaces. In fuzzy sets theory triangular norms and co-norms are extensively used to model logical connection and and or. In the fuzzy literatures, these concepts was studied e. g. in E. Creţu (2001), J. Dombi (1982), D. Dubois and H. Prade (1985), J. Fodor (1991, 1999), S. Jenei (1998, 1999, 2000, 2001, 2004), V. Radu (1974, 1984, 1992).

Definition 6 *The function* $N : [0,1] \to [0,1]$ *is a* negation operation *if:*

(i) $N(1) = 0$ *and* $N(0) = 1$*;*

(ii) N *is continuous and strictly decreasing;*

(iii) $N(N(x)) = x$*, for all* $x \in [0,1]$.

Definition 7 *Let* N *be a negation operation. The mapping* $T : [0,1] \times [0,1] \to [0,1]$ *is a* triangular norm *(briefly* t-norm*) if satisfies the properties:*

$$\begin{aligned} \textit{Symmetry}: & \ T(x,y) = T(y,x), \quad \forall x,y \in [0,1]; \\ \textit{Associativity}: & \ T(T(x,y),z) = T(x,T(y,z)), \quad \forall x,y,z \in [0,1]; \\ \textit{Monotonicity}: & \ T(x_1,y_1) \leq T(x_2,y_2) \quad \mathrm{if} x_1 \leq x_2 \mathrm{and} y_1 \leq y_2; \\ \textit{One identity}: & \ T(x,1) = x, \ \forall x \in [0,1] \end{aligned}$$

and the mapping $S : [0,1] \times [0,1] \to [0,1]$,

$$S(x,y) = N(T(N(x), N(y)))$$

is a triangular co-norm (the dual of T *given by* N*).*

Definition 8 *The t-norm* T *is* Archimedean *if* T *is continuous and* $T(x,x) < x$*, for all* $x \in (0,1)$.

Definition 9 *The t-norm* T *is called* strict *if* T *is strictly increasing in both arguments.*

Theorem 1 ([22]) *Every Archimedean t-norm* T *is representable by a continuous and decreasing function* $g : [0,1] \to [0,+\infty]$ *with* $g(1) = 0$ *and*

$$T(x,y) = g^{[-1]}(g(x) + g(y)),$$

where

$$g^{[-1]}(x) = \begin{cases} g^{-1}(x) & \text{if } 0 \leq x < g(0), \\ 0 & \text{if } x \geq g(0). \end{cases}$$

If g_1*and* g_2 *are the generator function of* T*, then there exist* $c > 0$ *such that* $g_1 = cg_2$.

Remark 1 *If the Archimedean t-norm* T *is strict, then* $g(0) = +\infty$ *otherwise* $g(0) = p < \infty$.

Theorem 2 ([38]) *An application* $N : [0,1] \to [0,1]$ *is a negation if and only if there exist an increasing and continuous function* $e : [0,1] \to [0,1]$, *with* $e(0) = 0$, $e(1) = 1$ *such that*$N(x) = e^{-1}(1 - e(x))$, *for all* $x \in [0,1]$.

Remark 2 *The generator function of negation* $N(x) = 1 - x$ *is* $e(x) = x$. *Another negation generator function is*

$$e_\lambda(x) = \frac{\ln(1 + \lambda x)}{\ln(1 + \lambda)},$$

where $\lambda > -1$, $\lambda \neq 0$.

Remark 3 *Examples to t-norm are following:*

- *minim:* $\min(x,y) = \min\{x,y\}$;

- *product:* $P(x,y) = xy$, *the generator function is* $g(x) = -\ln x$;
- *weak:* $W(x,y) = \begin{cases} \min\{x,y\} & \textit{if} \quad \max\{x,y\} = 1, \\ 0 & \textit{otherwise.} \end{cases}$

If the negation operation is $N(x) = 1 - x$, *then the dual of these t-norms are:*

- *maxim:* $\max(x,y) = \max\{x,y\}$;
- *probability:* $S_P(x,y) = x + y - xy$;
- *strong:* $S_W(x,y) = \begin{cases} \max\{x,y\} & \textit{if} \quad \min\{x,y\} = 0, \\ 1 & \textit{otherwise.} \end{cases}$

Proposition 1 *If* T *is a t-norm and* S *is the dual of* T, *then*

$$W(x,y) \leq T(x,y) \leq \min\{x,y\},$$
$$\max\{x,y\} \leq S(x,y) \leq S_W(x,y),$$

for all $x, y \in [0,1]$.

Let X be a nonempty set, T be a t-norm, N be a negation operation and S the dual of T given by N. The intersection, union, complement and Cartesian product of fuzzy sets may be defined in the following way.

Definition 10 *The* $T-$intersection*'s membership function of fuzzy sets* $\bar{A}$ *and* $\bar{B}$ *is defined as*

$$\mu_{\bar{A}\sqcap\bar{B}}(x) = T(\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)), \quad \forall x \in X.$$

The $S-$union*'s membership function of fuzzy sets* $\bar{A}$ *and* $\bar{B}$ *is defined as*

$$\mu_{\bar{A}\sqcup\bar{B}}(x) = S(\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)), \quad \forall x \in X.$$

The $N-$complement*'s membership function of fuzzy sets* $\bar{A}$ *and* $\bar{B}$ *is defined as*

$$\mu_{\neg\bar{A}}(x) = N(\mu_{\bar{A}}(x)), \quad \forall x \in X.$$

Definition 11 *The* T-Cartesian product*'s membership function of fuzzy sets* $\bar{A}_i \in \mathcal{F}(X_i)$, $i = 1, ..., n$ *is defined as*

$$\mu_{\bar{A}}(x_1, x_2, ..., x_n) = $$
$$T\left(\mu_{\bar{A}_1}(x_1), T\left(\mu_{\bar{A}_2}(x_2), T\left(...T\left(\mu_{\bar{A}_{n-1}}(x_{n-1}), \mu_{\bar{A}_n}(x_n)\right)...\right)\right)\right),$$

for all $(x_1, x_2, ..., x_n) \in X_1 \times X_2 \times ... \times X_n$.

In order to use fuzzy sets and relations in any intelligent system we must be able to perform arithmetic operations. In fuzzy theory the extension of arithmetic operations to fuzzy sets was formulated by L.A. Zadeh in 1965. Using any t-norm the extension is possible to generalize.

Definition 12 (Generalized Zadeh's extension principle) *Let* T *be a t-norm and let* $X_1, X_2, ..., X_n$ *(*$n \geq 2$*) and* Y *be a family of sets. Assume that* $f : X_1 \times X_2 \times ... \times X_n \to Y$ *is a mapping. On the basis of the* generalized extension principle *(*sup-T extension principle*) to* f *a mapping* $F : \mathcal{F}(X_1) \times \mathcal{F}(X_2) \times ... \times \mathcal{F}(X_n) \to \mathcal{F}(Y)$ *is ordered such that for all* $\left(\bar{A}_1, \bar{A}_2, ..., \bar{A}_n\right) \in \mathcal{F}(X_1) \times \mathcal{F}(X_2) \times ... \times \mathcal{F}(X_n)$ *the membership function of* $F\left(\bar{A}_1, \bar{A}_2, ..., \bar{A}_n\right)$ *is*

$$\mu_{F(\bar{A}_1, \bar{A}_2, ..., \bar{A}_n)}(y) =$$
$$\begin{cases} \sup\limits_{(x_1, ..., x_n) \in f^{-1}(y)} \left\{ T\left(\mu_{\bar{A}_1}(x_1), T\left(...T\left(\mu_{\bar{A}_{n-1}}(x_{n-1}), \mu_{\bar{A}_n}(x_n)\right)...\right)\right)\right\} & \\ & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \\ & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

If $n = 1$, *then*

$$\mu_{F(\bar{A}_1)}(y) = \begin{cases} \sup_{x_1 \in f^{-1}(y)} \left\{\mu_{\bar{A}_1}(x_1)\right\} & \text{if } \ f^{-1}(y) \neq \emptyset, \\ 0 & \text{if } \ f^{-1}(y) = \emptyset. \end{cases}$$

If we use the generalized Zadeh's extension principle, the operations on $\mathcal{F}(X)$ are uniquely determined by T, N and the corresponding operations of X.

Definition 13 *The triplet* $(\mathcal{F}(X), T, N)$ *will be called* fuzzy triangular space.

If T is a t-norm and "$*$" is a binary operation on $\mathbb{R}$, then "$*$" can be extended to fuzzy quantities in the sense of the generalized extension principle of Zadeh.

Definition 14 *Let* $\bar{A}$ *and* $\bar{B}$ *be two fuzzy numbers. Then the membership function of fuzzy set* $\bar{A} * \bar{B} \in \mathcal{F}(\mathbb{R})$ *is*

$$\mu_{\bar{A}*\bar{B}}(y) = \sup\{T(\mu_{\bar{A}}(x_1), \mu_{\bar{B}}(x_2)) \ / \ x_1 * x_2 = y\}, \tag{1}$$

for all $y \in \mathbb{R}$.

If we replace "$*$" with operations "$+$", "$-$","$\cdot$", or "$/$", then we get the membership functions of sum, difference, product or fraction.

3 Additive group of quasi-triangular fuzzy numbers

The construction of membership function of fuzzy numbers is an important problem in vagueness modeling. Theoretically, the shape of fuzzy numbers must depend on the applied triangular space.

We noticed that, if the model constructed on the computer does not comply the requests of the given problem, then we choose another norm. The membership function must defined in a such a way that the change of the t-norm modifies the shape of fuzzy number, but the calculus with them remain valid. This desideratum is satisfied, for instance if quasi-triangular fuzzy numbers introduced by M. Kovacs [21] are used.

Let $p \in [1, +\infty]$ and $g : [0,1] \to [0,\infty]$ be a continuous, strictly decreasing function with the boundary properties $g(1) = 0$ and $\lim_{t\to 0} g(t) = g_0 \leq \infty$. The quasi-triangular fuzzy number we define in the fuzzy triangular space $(\mathcal{F}(\mathbb{R}), T_{gp}, N)$, where

$$T_{gp}(x,y) = g^{[-1]}\left((g^p(x) + g^p(y))^{\frac{1}{p}}\right) \tag{2}$$

is an Archimedean t-norm generated by g and

$$N(x) = \begin{cases} 1 - x & \text{if} \quad g_0 = +\infty, \\ g^{-1}(g_0 - g(x)) & \text{if} \quad g_0 \in \mathbb{R}. \end{cases} \tag{3}$$

is a negation operation.

Definition 15 *The* set of quasi-triangular fuzzy numbers *is*

$$\begin{aligned} \mathcal{N}_g = \Big\{ \bar{A} \in \mathcal{F}(\mathbb{R}) \ / \ & \textit{there is } a \in \mathbb{R}, d > 0 \textit{ such that} \\ & \mu_{\bar{A}}(x) = g^{[-1]}\left(\frac{|x-a|}{d}\right) \textit{ for all } x \in \mathbb{R} \Big\} \bigcup \\ & \Big\{ \bar{A} \in \mathcal{F}(\mathbb{R}) \ / \ \textit{there is } a \in \mathbb{R} \textit{ such that} \\ & \mu_{\bar{A}}(x) = \chi_{\{a\}}(x) \ \textit{ for all } x \in \mathbb{R} \Big\}, \end{aligned} \tag{4}$$

where χ_A is characteristic function of the set A. The elements of $\mathcal{N}_g$ will be called quasi-triangular fuzzy numbers *generated by g with center λ and spread d and we will denote them with* $< \lambda, d >$.

Remark 4 *The quasi-triangular fuzzy numbers $< a_1, d_1 >$ and $< a_2, d_2 >$ are equal if and only if $a_1 = a_2$ and $d_1 = d_2$.*

Remark 5 *If* $< \lambda, d > \in \mathcal{N}_g$ *and* $d > 0$, *then*

$$[< \lambda, d >]^{\alpha} = [\lambda - dg(\alpha), \lambda + dg(\alpha)]$$

and if $d = 0$, *then* $[< \lambda, d >]^{\alpha} = \{\lambda\}$, *for all* $\alpha \in [0, 1]$.

Example 1 *Let* $g : (0, 1] \to [0, \infty)$ *be a function given by* $g(t) = \sqrt{-2 \ln t}$. *Then the membership function of quasi-triangular fuzzy numbers* $< a, d >$ *is*

$$\mu(t) = e^{-\frac{(t-a)^2}{2d^2}} \quad \text{if } d > 0, \quad \text{and}$$

$$\mu(t) = \begin{cases} 1 & \text{if } t = a, \\ 0 & \text{if } t \neq a \end{cases} \quad \text{if } d = 0.$$

Suppose $\bar{A}$ and $\bar{B}$ are fuzzy sets on $\mathbb{R}$. Then using the generalized Zadeh's extension principle we get:

Definition 16 *If* $p \in [1, +\infty)$, *then the* T_{gp}-sum *of* $\bar{A}$ *and* $\bar{B}$ *is defined by*

$$\mu_{\bar{A}+\bar{B}}(z) = \sup_{x+y=z} \left[g^{[-1]} \left([g^p(\mu_{\bar{A}}(x)) + g^p(\mu_{\bar{B}}(y))]^{\frac{1}{p}} \right) \right],$$

for all $z \in \mathbb{R}$.

If $p = +\infty$, *then the* T_{gp}-sum *of* $\bar{A}$ *and* $\bar{B}$ *is defined by*

$$\mu_{\bar{A}+\bar{B}}(z) = \sup_{x+y=z} \min\{\mu_{\bar{A}}(x), \mu_{\bar{B}}(y)\},$$

for all $z \in \mathbb{R}$.

M. Kovács and T. Keresztfalvi in [19] proved the formula (5) for the T_{gp}-sum of quasi-triangular fuzzy numbers.

Theorem 3 *Let* $p \in [1, +\infty]$. *If* $\bar{A} = \langle a, d \rangle$ *and* $\bar{B} = \langle b, e \rangle$ *are quasi-triangular fuzzy numbers, then* $\bar{A} + \bar{B}$ *is quasi-triangular fuzzy number too, and*

$$\bar{A} + \bar{B} = \left\langle a + b, (d^q + e^q)^{\frac{1}{q}} \right\rangle, \tag{5}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 4 ([23]) $(\mathcal{N}_g, +)$ *is a commutative monoid with element zero* $\bar{0} =< 0, 0 >$ *and if* $p \in (1, +\infty]$, *then it possesses the simplification property.*

As follows from the theorem 4, the quasi-triangular fuzzy numbers do not form an additive group. This fact can complicate some theoretical considerations or applied procedures. This deficiency can be removed if the set of quasi-triangular fuzzy numbers is included isomorphically in an extended set and this extended set forms an additive group with T_{gp}-sum. In this section we construct this group if $p > 1$.

As follows from the definition of T_{gp}-Cartesian product, the membership function of the pair $(< a_1, d_1 >, < a_2, d_2 >)$ is

$$\mu_{(\langle a_1,d_1\rangle,\langle a_2,d_2\rangle)}(x,y) = T_{gp}\left(\mu_{\langle a_1,d_1\rangle}(x), \mu_{\langle a_2,d_2\rangle}(y)\right), \tag{6}$$

for all $(x,y) \in \mathbb{R} \times \mathbb{R}$. We denote the set of pairs by $\mathfrak{I}_{gp}$.

Definition 17 *Let* $(< a_1, d_1 >, < a_2, d_2 >)$, $(< a_3, d_3 >, < a_4, d_4 >) \in \mathfrak{I}_{gp}$. *Then we say that* $(< a_1, d_1 >, < a_2, d_2 >)$ equivalent *to* $(< a_3, d_3 >, < a_4, d_4 >)$, *and write* $(< a_1, d_1 >, < a_2, d_2 >) \sim (< a_3, d_3 >, < a_4, d_4 >)$ *if*

$$a_1 + a_4 = a_2 + a_3,$$
$$\left(d_1^q + d_4^q\right)^{1/q} = \left(d_2^q + d_3^q\right)^{1/q}.$$

It can be easily seen that "~" is an equivalence relation. This relation generates in $\mathfrak{I}_{gp}$ a division on equivalence class.

Definition 18 *The* factor *set is*

$$\mathfrak{I}_{gp}/_\sim = \left\{\overline{(< a_1, d_1 >, < a_2, d_2 >)} \ / \ < a_1, d_1 >, < a_2, d_2 > \in \mathcal{N}_g\right\},$$

where

$$\overline{(< a_1, d_1 >, < a_2, d_2 >)} = \{(< a_3, d_3 >, < a_4, d_4 >) \ / \ < a_3, d_3 >, < a_4, d_4 > \in \mathcal{N}_g \ \textit{and} \ a_1 + a_4 = a_2 + a_3, \ \left(d_1^q + d_4^q\right)^{1/q} = \left(d_2^q + d_3^q\right)^{1/q}\right\}.$$

Definition 19 *The* addition *operation in* $\mathfrak{I}_{gp}/_\sim$ *is defined by*

$$\overline{(< a_1, d_1 >, < a_2, d_2 >)} \oplus \overline{(< a_3, d_3 >, < a_4, d_4 >)} = \overline{\left(\left\langle a_1 + a_3, \left(d_1^q + d_3^q\right)^{\frac{1}{q}}\right\rangle, \left\langle a_2 + a_4, \left(d_2^q + d_4^q\right)^{\frac{1}{q}}\right\rangle\right)},$$

for all $\overline{(< a_1, d_1 >, < a_2, d_2 >)}, \overline{(< a_3, d_3 >, < a_4, d_4 >)} \in \mathfrak{I}_{gp}/_\sim$.

Because the commutative monoid $(\mathcal{N}_g, +)$ possesses simplification property if $p > 1$, it follows that:

Theorem 5 *If* $p > 1$, *then* $(\mathfrak{I}_{gp}/_\sim, \oplus)$ *is an additive commutative group.*

The *opposite* of $\overline{(< a_1, d_1 >, < a_2, d_2 >)}$ we denote by $\ominus\overline{(< a_1, d_1 >, < a_2, d_2 >)}$.

Proposition 2 *If* $< x, y >, < a, d > \in \mathcal{N}_g$, *then*

$$\overline{(< a, d > + < x, y >, < a, d >)} = \overline{(< x, y >, < 0, 0 >)}.$$

Proposition 3 ([24]) *If* $p > 1$, *then the function* $F : \mathcal{N}_g \to \mathfrak{I}_{gp}/_\sim$ *with* $F(< x, y >) = \overline{(< x, y >, < 0, 0 >)}$ *is a homomorphism.*

Theorem 6 ([24]) $(\mathcal{N}_g, +)$ *is isomorphic to* $(F(\mathcal{N}_g), \oplus)$.

The consequence of Theorem 6 $\overline{(< x, y >, < 0, 0 >)}$ is identical with $< x, y >$, if we consider this isomorphism. Using this property we introduce the following notations:

By the Theorem 6 it follows that $\overline{(< x, y >, < 0, 0 >)}$ is identical with $< x, y >$, if we consider the isomorphism in Theorem 6. Using this property we introduce the following notations:

We denote by $[x, y] = \overline{(< x, y >, < 0, 0 >)}$ the *quasi-triangular fuzzy number with center* x and spread y, and its *opposite* by $\ominus [x, y] = \overline{(< 0, 0 >, < x, y >)}$.

Definition 20 *If* $p > 1$, *then the extended set of quasi-triangular fuzzy number is* $\mho_{gp} = \mho_{gp}^{\oplus} \cup \mho_{gp}^{\ominus}$, *where*

$$\mho_{gp}^{\oplus} = \{[x, y] \ / \ < x, y > \in \mathcal{N}_g\} \text{ and } \mho_{gp}^{\ominus} = \{\ominus [x, y] \ / \ < x, y > \in \mathcal{N}_g\}.$$

Theorem 7 ([24]) *If* $p > 1$, *then* $\mathfrak{I}_{gp}/_\sim = \mho_{gp}$.

If we introduce the notation $[x_1, y_1] \ominus [x_2, y_2] = [x_1, y_1] \oplus (\ominus [x_2, y_2])$, then from Theorem 7 it follows:

Theorem 8 *If* $p > 1$, *then* $(\mho_{gp}, \oplus)$ *is an additive commutative group.*

Corollary 1 **(i)** *If* $[x, y] \in \mho_{gp}$, *then* $[0, 0] \ominus [x, y] = \ominus [x, y]$.

(ii) *If* $[x, y] \in \mho_{gp}$, *then* $\ominus (\ominus [x, y]) = [x, y]$.

(iii) *If* $[x_1, y_1], [x_2, y_2] \in \mho_{gp}$, *then*

$$(\ominus [x_1, y_1]) \oplus (\ominus [x_2, y_2]) = \ominus ([x_1, y_1] \oplus [x_2, y_2]).$$

(iv) *If* $[x_1, y_1], [x_2, y_2] \in \mho_{gp}$, *then*

$$[x_1, y_1] \ominus [x_2, y_2] = \begin{cases} \left[x_1 - x_2, \left(y_1^q - y_2^q\right)^{\frac{1}{q}}\right] & \textit{if} \quad y_1 \geq y_2, \\ \ominus \left[x_2 - x_1, \left(y_2^q - y_1^q\right)^{\frac{1}{q}}\right] & \textit{if} \quad y_2 > y_1. \end{cases}$$

(v) *If* $[x_1, y_1], [x_2, y_2], [x_3, y_3], [x_4, y_4] \in \mho_{gp}$ *and*

$$[x_1, y_1] \ominus [x_2, y_2] = [x_3, y_3] \ominus [x_4, y_4],$$

then

$$[x_1, y_1] \oplus [x_4, y4] = [x_3, y_3] \oplus [x_2, y_2].$$

4 Real vector space of quasi-triangular fuzzy numbers

In this section we construct the vector space of quasi-triangular fuzzy numbers if $p > 1$. We know that $2 \cdot [x, y] = [x, y] \oplus [x, y] = \left[2x, 2^{\frac{1}{q}} y\right]$ for all $[x, y] \in \mho_{gp}$. We generalize this property as follows.

Definition 21 *For all* $[x, y] \in \mho_{gp}^{\oplus}$ *and for all* $a \in \mathbb{R}$ *the* scalar multiplication $a [x, y]$ *is defined by*

$$a [x, y] = \begin{cases} \left[ax, a^{\frac{1}{q}} y\right] & \textit{if} \quad a \geq 0, \\ \ominus \left[-ax, (-a)^{\frac{1}{q}} y\right] & \textit{if} \quad a < 0, \end{cases}$$

and for all $\ominus [x, y] \in \mho_{gp}^{\ominus}$ *the* scalar multiplication $a (\ominus [x, y])$ *is defined by*

$$a (\ominus [x, y]) = \ominus (a [x, y]).$$

Remark 6 *For all* $\ominus [x, y] \in \mho_{gp}^{\ominus}$ *and for all* $a \in \mathbb{R}$ *we have*

$$a (\ominus [x, y]) = \begin{cases} \ominus \left[ax, a^{\frac{1}{q}} y\right] & \textit{if} \quad a \geq 0, \\ \left[-ax, (-a)^{\frac{1}{q}} y\right] & \textit{if} \quad a < 0. \end{cases}$$

Similarly, for all $[x,y] \in \mho_{gp}^{\oplus}$ *and* $a \geq 0$ *we have*

$$(-a)\,[x,y] = \ominus \left[ax, a^{\frac{1}{q}}y\right] = a\,(\ominus\,[x,y]) \quad \textit{and}$$
$$(-a)\,(\ominus\,[x,y]) = a\,[x,y]\,.$$

Theorem 9 *If* $p > 1$*, then the triple* $(\mho_{gp}, \oplus, \cdot)$ *is a real vector space.*

Proof. Since $(\mho_{gp}, \oplus)$ is an additive commutative group the following properties must be proved.

(i) If $a, b \in \mathbb{R}$ and $Z \in \mho_{gp}$, then $(a+b)\,Z = aZ \oplus bZ$.
If $Z = [x,y] \in \mho_{gp}^{\oplus}$, $a \geq 0$ and $b \geq 0$, then

$$\begin{aligned}(a+b)\,[x,y] &= \left[(a+b)\,x, (a+b)^{\frac{1}{q}}\,y\right] \\ &= \left[ax, a^{\frac{1}{q}}y\right] \oplus \left[bx, b^{\frac{1}{q}}y\right] \\ &= a\,[x,y] \oplus b\,[x,y]\,.\end{aligned}$$

If $Z = \ominus\,[x,y] \in \mho_{gp}^{\ominus}$, $a \geq 0$ and $b \geq 0$, then

$$\begin{aligned}(a+b)\,(\ominus\,[x,y]) &= \ominus\,((a+b)\,[x,y]) \\ &= \ominus\,(a\,[x,y] \oplus b\,[x,y]) \\ &= (\ominus a\,[x,y]) \oplus (\ominus b\,[x,y]) \\ &= a\,(\ominus\,[x,y]) \oplus b\,(\ominus\,[x,y])\,.\end{aligned}$$

If $Z = [x,y] \in \mho_{gp}^{\oplus}$, $a \geq 0$ $b < 0$ and $a + b \geq 0$, then

$$\begin{aligned}a\,[x,y] \oplus b\,[x,y] &= \left[ax, a^{\frac{1}{q}}y\right] \ominus \left[-bx, (-b)^{\frac{1}{q}}\,y\right] \\ &= \left[ax - (-b)\,x, (a-(-b))^{\frac{1}{q}}\,y\right] \\ &= (a+b)\,[x,y]\,.\end{aligned}$$

If $Z = \ominus\,[x,y] \in \mho_{gp}^{\ominus}$, $a \geq 0$ $b < 0$ and $a + b \geq 0$, then

$$\begin{aligned}a\,(\ominus\,[x,y]) \oplus b\,(\ominus\,[x,y]) &= (\ominus a\,[x,y]) \oplus ((-b)\,[x,y]) \\ &= \ominus a\,[x,y] \ominus b\,[x,y] \\ &= \ominus\,(a\,[x,y] \oplus b\,[x,y]\ ) \\ &= (a+b)\,(\ominus\,[x,y])\,.\end{aligned}$$

If $Z = [x, y] \in \mho_{gp}^{\oplus}$, $a \geq 0$ $b < 0$ and $a + b < 0$, then

$$\begin{aligned} a\,[x, y] \oplus b\,[x, y] &= \left[ax, a^{\frac{1}{q}} y\right] \ominus \left[-bx, (-b)^{\frac{1}{q}}\, y\right] \\ &= \ominus \left[-ax + (-b)\,x, (-b - a)^{\frac{1}{q}}\, y\right] \\ &= (a + b)\,[x, y]\,. \end{aligned}$$

If $Z = \ominus\,[x, y] \in \mho_{gp}^{\ominus}$, $a \geq 0$ $b < 0$ and $a + b < 0$, then

$$\begin{aligned} a\,(\ominus\,[x, y]) \oplus b\,(\ominus\,[x, y]) &= (\ominus a\,[x, y]) \oplus ((-b)\,[x, y]) \\ &= \ominus a\,[x, y] \ominus b\,[x, y] \\ &= \ominus\,(a\,[x, y] \oplus b\,[x, y]\ ) \\ &= (a + b)\,(\ominus\,[x, y])\,. \end{aligned}$$

If $Z = [x, y] \in \mho_{gp}^{\oplus}$, $a < 0$ and $b < 0$, then

$$\begin{aligned} a\,[x, y] \oplus b\,[x, y] &= \ominus \left[-ax, (-a)^{\frac{1}{q}}\, y\right] \ominus \left[-bx, (-b)^{\frac{1}{q}}\, y\right] \\ &= \ominus \left[-ax - bx, (-a - b)^{\frac{1}{q}}\, y\right] \\ &= (a + b)\,[x, y]\,. \end{aligned}$$

If $Z = \ominus\,[x, y] \in \mho_{gp}^{\ominus}$, $a < 0$ and $b < 0$, then

$$\begin{aligned} a\,(\ominus\,[x, y]) \oplus b\,(\ominus\,[x, y]) &= ((-a)\,[x, y]) \oplus ((-b)\,[x, y]) \\ &= (-a - b)\,[x, y] \\ &= (a + b)\,(\ominus\,[x, y])\,. \end{aligned}$$

(ii) If $Z_1, Z_2 \in \mho_{gp}$ and $a \in \mathbb{R}$, then $a\,(Z_1 \oplus Z_2) = aZ_1 \oplus aZ_2$.
If $Z_1 = [x_1, y_1] \in \mho_{gp}^{\oplus}$, $Z_2 = [x_2, y_2] \in \mho_{gp}^{\oplus}$ and $a \geq 0$, then

$$\begin{aligned} a\,(Z_1 \oplus Z_2) &= a \left[x_1 + x_2, \left(y_1^q + y_2^q\right)^{\frac{1}{q}}\right] \\ &= \left[ax_1 + ax_2, \left(ay_1^q + ay_2^q\right)^{\frac{1}{q}}\right] \\ &= a\,[x_1, y_1] \oplus a\,[x_2, y_2] \\ &= aZ_1 \oplus aZ_2. \end{aligned}$$

If $Z_1 = [x_1, y_1] \in \mho_{gp}^{\oplus}$, $Z_2 = \ominus [x_2, y_2] \in \mho_{gp}^{\ominus}$, $y_1 \geq y_2$ and $a \geq 0$, then

$$\begin{aligned} a\left(Z_1 \oplus Z_2\right) &= a\left[x_1 - x_2, \left(y_1^q - y_2^q\right)^{\frac{1}{q}}\right] \\ &= \left[ax_1 - ax_2, \left(ay_1^q - ay_2^q\right)^{\frac{1}{q}}\right] \\ &= a\left[x_1, y_1\right] \ominus a\left[x_2, y_2\right] \\ &= aZ_1 \oplus aZ_2. \end{aligned}$$

If $Z_1 = [x_1, y_1] \in \mho_{gp}^{\oplus}$, $Z_2 = \ominus [x_2, y_2] \in \mho_{gp}^{\ominus}$, $y_1 < y_2$ and $a \geq 0$, then

$$\begin{aligned} a\left(Z_1 \oplus Z_2\right) &= (-a)\left[x_2 - x_1, \left(y_2^q - y_1^q\right)^{\frac{1}{q}}\right] \\ &= \ominus\left[ax_2 - ax_1, \left(ay_2^q - ay_1^q\right)^{\frac{1}{q}}\right] \\ &= a\left[x_1, y_1\right] \ominus a\left[x_2, y_2\right] \\ &= aZ_1 \oplus aZ_2. \end{aligned}$$

If $Z_1 = \ominus [x_1, y_1]$, $Z_2 = \ominus [x_2, y_2] \in \mho_{gp}^{\ominus}$ and $a \geq 0$, then

$$\begin{aligned} a\left(Z_1 \oplus Z_2\right) &= (-a)\left[x_1 + x_2, \left(y_1^q + y_2^q\right)^{\frac{1}{q}}\right] \\ &= \ominus\left[ax_1 + ax_2, \left(ay_1^q + ay_2^q\right)^{\frac{1}{q}}\right] \\ &= a\left(\ominus\left[x_1, y_1\right]\right) \oplus a\left(\ominus\left[x_2, y_2\right]\right) \\ &= aZ_1 \oplus aZ_2. \end{aligned}$$

If $a < 0$, then

$$\begin{aligned} aZ_1 \oplus aZ_2 &= \ominus\left(\left(-a\right) Z_1\right) \ominus \left(\left(-a\right) Z_2\right) \\ &= \ominus\left(\left(-a\right) Z_1 \oplus \left(-a\right) Z_2\right) \\ &= \ominus\ \left(-a\right)\left(Z_1 \oplus Z_2\right) \\ &= a\left(Z_1 \oplus Z_2\right)\ . \end{aligned}$$

(iii) If $a, b \in \mathbb{R}$ and $Z \in \mho_{gp}$, then $(ab)\,Z = a\,(bZ)$.
If $Z = [x, y] \in \mho_{gp}^{\oplus}$, $a \geq 0$ and $b \geq 0$, then

$$\begin{aligned} (ab)\,[x, y] &= \left[(ab)\,x, (ab)^{\frac{1}{q}}\,y\right] \\ &= a\left[bx, b^{\frac{1}{q}}y\right] \\ &= a\left(b\left[x, y\right]\right). \end{aligned}$$

If $Z = [x, y] \in \mho_{gp}^{\oplus}$, $a \geq 0$ and $b < 0$, then

$$\begin{aligned} (ab)[x, y] &= \ominus \left[(-ab)x, (-ab)^{\frac{1}{q}} y\right] \\ &= a\left(\ominus \left[-bx, (-b)^{\frac{1}{q}} y\right]\right) \\ &= a(b[x, y]). \end{aligned}$$

If $Z = [x, y] \in \mho_{gp}^{\oplus}$, $a < 0$ and $b < 0$, then

$$\begin{aligned} (ab)[x, y] &= \left[(-a)(-b)x, ((-a)(-b))^{\frac{1}{q}} y\right] \\ &= (-a)((-b)[x, y]) \\ &= a(\ominus((-b)[x, y])) \\ &= a(b[x, y]). \end{aligned}$$

If $Z = \ominus [x, y] \in \mho_{gp}^{\ominus}$, then

$$\begin{aligned} (ab)(\ominus [x, y]) &= \ominus((ab)[x, y]) \\ &= \ominus(a(b[x, y])) \\ &= a(\ominus(b[x, y])) \\ &= a(b(\ominus [x, y])). \end{aligned}$$

(iv) If $Z \in \mho_{gp}$, then $1 \cdot Z = Z$.
If $Z = [x, y] \in \mho_{gp}^{\oplus}$, then $1[x, y] = [x, y]$.
If $Z = \ominus [x, y] \in \mho_{gp}^{\ominus}$, then $1(\ominus [x, y]) = \ominus [x, y]$.

■

5 Scalar product of quasi-triangular fuzzy numbers

In this section we construct the real vector space with scalar product of quasi-triangular fuzzy numbers.

Definition 22 *The* product *of the classes*

$$\overline{(< a_1, d_1 >, < a_2, d_2 >)}, \overline{(< a_3, d_3 >, < a_4, d_4 >)} \in \mathfrak{I}_{gp}/_{\sim}$$

is defined by

$$\begin{aligned} &\overline{(< a_1, d_1 >, < a_2, d_2 >)} \cdot \overline{(< a_3, d_3 >, < a_4, d_4 >)} \\ &= (a_1 - a_2)(a_3 - a_4) + \left(d_1^q - d_2^q\right)\left(d_3^q - d_4^q\right). \end{aligned} \tag{7}$$

Theorem 10 $(\mho_{gp}, \oplus, \cdot)$ *is a real vector space with scalar product given by (7).*

Proof. Let

$$(<a_5, d_5>, <a_6, d_6>) \in \overline{(<a_1, d_1>, <a_2, d_2>)} \quad \text{and}$$
$$(<a_7, d_7>, <a_8, d_8>) \in \overline{(<a_3, d_3>, <a_4, d_4>)}.$$

Since

$$a_5 - a_6 = a_1 - a_2, a_7 - a_8 = a_3 - a_4, d_5^q - d_6^q = d_1^q - d_2^q, d_7^q - d_8^q = d_3^q - d_4^q$$

follows that the (7) does not depend on choice of the elements.
Let

$$\overline{(<a_1, d_1>, <a_2, d_2>)}, \ \overline{(<a_3, d_3>, <a_4, d_4>)},$$
$$\overline{(<a_5, d_5>, <a_6, d_6>)} \in \mathfrak{I}_{gp}/_{\sim}.$$

(i) The scalar product is commutative since:

$$\overline{(<a_1, d_1>, <a_2, d_2>)} \cdot \overline{(<a_3, d_3>, <a_4, d_4>)} =$$
$$\overline{(<a_3, d_3>, <a_4, d_4>)} \cdot \overline{(<a_1, d_1>, <a_2, d_2>)}.$$

(ii) For all $\lambda \geq 0$ we have

$$\begin{aligned}
&\left(\lambda\overline{(<a_1, d_1>, <a_2, d_2>)}\right) \cdot \overline{(<a_3, d_3>, <a_4, d_4>)} \\
&= \left(\lambda\left([a_1, d_1] \ominus [a_2, d_2]\right)\right) \cdot \overline{(<a_3, d_3>, <a_4, d_4>)} \\
&= \overline{\left(<\lambda a_1, \lambda^{1/q} d_1>, <\lambda a_2, \lambda^{1/q} d_2>\right)} \cdot \overline{(<a_3, d_3>, <a_4, d_4>)} \\
&= \left(\lambda a_1 - \lambda a_2\right)\left(a_3 - a_4\right) + \left(\lambda d_1^q - \lambda d_2^q\right)\left(d_3^q - d_4^q\right) \\
&= \lambda\left[\left(a_1 - a_2\right)\left(a_3 - a_4\right) + \left(d_1^q - d_2^q\right)\left(d_3^q - d_4^q\right)\right] \\
&= \lambda\overline{(<a_1, d_1>, <a_2, d_2>)} \cdot \overline{(<a_3, d_3>, <a_4, d_4>)}.
\end{aligned}$$

For all $\lambda < 0$ we have

$$\begin{aligned}
&\left(\lambda\overline{(<a_1,d_1>,<a_2,d_2>)}\right)\cdot\overline{(<a_3,d_3>,<a_4,d_4>)}\\
&=(\lambda\left([a_1,d_1]\ominus[a_2,d_2]\right))\cdot\overline{(<a_3,d_3>,<a_4,d_4>)}\\
&=\overline{\left(<-\lambda a_2,(-\lambda)^{1/q}\,d_2>,<-\lambda a_1,(-\lambda)^{1/q}\,d_1>\right)}\cdot\\
&\quad\overline{(<a_3,d_3>,<a_4,d_4>)}\\
&=(-\lambda a_2+\lambda a_1)(a_3-a_4)+\left(-\lambda d_2^q+\lambda d_1^q\right)\left(d_3^q-d_4^q\right)\\
&=\lambda\left[(a_1-a_2)(a_3-a_4)+\left(d_1^q-d_2^q\right)\left(d_3^q-d_4^q\right)\right]\\
&=\lambda\overline{(<a_1,d_1>,<a_2,d_2>)}\cdot\overline{(<a_3,d_3>,<a_4,d_4>)}.
\end{aligned}$$

(iii) The distributivity follows by

$$\begin{aligned}
&\left(\overline{(<a_1,d_1>,<a_2,d_2>)}\oplus\overline{(<a_3,d_3>,<a_4,d_4>)}\right)\cdot\\
&\quad\overline{(<a_5,d_5>,<a_6,d_6>)}\\
&=\overline{\left(<a_1+a_3,\left(d_1^q+d_3^q\right)^{1/q}>,<a_2+a_4,\left(d_2^q+d_4^q\right)^{1/q}>\right)}\cdot\\
&\quad\overline{(<a_5,d_5>,<a_6,d_6>)}\\
&=(a_1+a_3-a_2-a_4)(a_5-a_6)+\left(d_1^q+d_3^q-d_2^q-d_4^q\right)\left(d_5^q-d_6^q\right)\\
&=(a_1-a_2)(a_5-a_6)+\left(d_1^q-d_2^q\right)\left(d_5^q-d_6^q\right)+\\
&\quad(a_3-a_4)(a_5-a_6)+\left(d_3^q-d_4^q\right)\left(d_5^q-d_6^q\right)\\
&=\left(\overline{(<a_1,d_1>,<a_2,d_2>)}\cdot\overline{(<a_5,d_5>,<a_6,d_6>)}\right)\\
&\quad\oplus\left(\overline{(<a_3,d_3>,<a_4,d_4>)}\cdot\overline{(<a_5,d_5>,<a_6,d_6>)}\right).
\end{aligned}$$

(iv) The positivity also satisfied:

$$\begin{aligned}
&\overline{(<a_1,d_1>,<a_2,d_2>)}\cdot\overline{(<a_1,d_1>,<a_2,d_2>)}\\
&=(a_1-a_2)^2+\left(d_1^q-d_2^q\right)^2\geq 0.
\end{aligned}$$

If

$$\overline{(<a_1,d_1>,<a_2,d_2>)}\cdot\overline{(<a_1,d_1>,<a_2,d_2>)}=0,$$

then $a_1=a_2$ and $d_1=d_2$. In conclusion $\overline{(\langle a_1,d_1\rangle,\langle a_2,d_2\rangle)}$ is the zero element. ∎

Proposition 4 *For all* $[a_1, d_1], [a_2, d_2] \in \mho_{gp}$ *we have*

$$\begin{aligned} \ominus [a_1, d_1] \cdot (\ominus [a_2, d_2]) &= [a_1, d_1] \cdot [a_2, d_2], \\ \ominus [a_1, d_1] \cdot [a_2, d_2] &= -[a_1, d_1] \cdot [a_2, d_2]. \end{aligned}$$

Proof. Since

$$\begin{aligned} [a_1, d_1] &= \overline{(< a_1, d_1 >, < 0, 0 >)}, \\ \ominus [a_1, d_1] &= \overline{(< 0, 0 >, < a_1, d_1 >)}, \\ [a_2, d_2] &= \overline{(< a_2, d_2 >, < 0, 0 >)}, \\ \ominus [a_2, d_2] &= \overline{(< 0, 0 >, < a_2, d_2 >)} \end{aligned}$$

it follows that

$$\begin{aligned} [a_1, d_1] \cdot [a_2, d_2] &= a_1 a_2 + d_1^q d_2^q, \\ \ominus [a_1, d_1] \cdot [a_2, d_2] &= -a_1 a_2 - d_1^q d_2^q \\ &= -[a_1, d_1] \cdot [a_2, d_2], \\ \ominus [a_1, d_1] \cdot (\ominus [a_2, d_2]) &= a_1 a_2 + d_1^q d_2^q \\ &= [a_1, d_1] \cdot [a_2, d_2]. \end{aligned}$$

■

Definition 23 *In the real vector space* $\mho_{gp}$ *the* norm *of* $[a, d] \in \mho_{gp}^{\oplus}$ *and* $\ominus [a, d] \in \mho_{gp}^{\ominus}$ *is defined by*

$$\begin{aligned} \|[a, d]\| &= \sqrt{a^2 + d^{2q}}, \\ \|\ominus [a, d]\| &= \sqrt{a^2 + d^{2q}}. \end{aligned}$$

Definition 24 *In the real vector space* $\mho_{gp}$ *the* distance *of* $C_1, C_2 \in \mho_{gp}$ *is defined by*

$$d(C_1, C_2) = \|C_1 \ominus C_2\|.$$

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Generalized perfect numbers

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Abstract. Let $\sigma(n)$ denote the sum of positive divisors of the natural number n. A natural number is perfect if $\sigma(n) = 2n$. This concept was already generalized in form of superperfect numbers $\sigma^2(n) = \sigma(\sigma(n)) = 2n$ and hyperperfect numbers $\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k}$.
In this paper some new ways of generalizing perfect numbers are investigated, numerical results are presented and some conjectures are established.

1 Introduction

For the natural number n we denote the sum of positive divisors by

$$\sigma(n) = \sum_{d|n} d.$$

Definition 1 *A positive integer n is called perfect number if it is equal to the sum of its proper divisors. Equivalently:*

$$\sigma(n) = 2n,$$

where

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Example 1 *The first few perfect numbers are:* $6, 28, 496, 8128, \ldots$ *(Sloane's A000396 [15]), since*

$$\begin{aligned} 6 &= 1 + 2 + 3 \\ 28 &= 1 + 2 + 4 + 7 + 14 \\ 496 &= 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 \end{aligned}$$

Euclid discovered that the first four perfect numbers are generated by the formula $2^{n-1}(2^n - 1)$. *He also noticed that* $2^n - 1$ *is a prime number for every instance, and in Proposition IX.36 of "Elements" gave the proof, that the discovered formula gives an even perfect number whenever* $2^n - 1$ *is prime.*
Several wrong assumptions were made, based on the four known perfect numbers:

- *Since the formula* $2^{n-1}(2^n - 1)$ *gives the first four perfect numbers for* $n = 2, 3, 5,$ *and 7 respectively, the fifth perfect number would be obtained when* $n = 11$. *However* $2^{11} - 1 = 23 \cdot 89$ *is not prime, therefore this doesn't yield a perfect number.*
- *The fifth perfect number would have five digits, since the first four had 1, 2, 3, and 4 digits respectively, but it has 8 digits. The perfect numbers would alternately end in 6 or 8.*
- *The fifth perfect number indeed ends with a 6, but the sixth also ends in a 6, therefore the alternation is disturbed.*

In order for $2^n - 1$ to be a prime, n must itself to be a prime.

Definition 2 *A* ***Mersenne prime*** *is a prime number of the form:*

$$M_n = 2^{p_n} - 1$$

where p_n *must also be a prime number.*

Perfect numbers are intimately connected with these primes, since there is a concrete one-to-one association between *even* perfect numbers and Mersenne primes. The fact that Euclid's formula gives all possible even perfect numbers was proved by Euler two millennia after the formula was discovered.
Only 46 Mersenne primes are known by now (November, 2008 [14]), which means there are 46 known even perfect numbers. There is a conjecture that there are infinitely many perfect numbers. The search for new ones is the

goal of a distributed search program via the Internet, named GIMPS (Great Internet Mersenne Prime Search) in which hundreds of volunteers use their personal computers to perform pieces of the search.
It is not known if any *odd* perfect numbers exist, although numbers up to 10^{300} (R. Brent, G. Cohen, H. J. J. te Riele [1]) have been checked without success. There is also a distributed searching system for this issue of which the goal is to increase the lower bound beyond the limit above. Despite this lack of knowledge, various results have been obtained concerning the odd perfect numbers:

- Any odd perfect number must be of the form $12m + 1$ or $36m + 9$.
- If n is an odd perfect number, it has the following form:
  $$n = q^{\alpha} p_1^{2e_1} \dots p_k^{2e_k},$$
  where $q, p_1, \dots, p_k$ are distinct primes and $q \equiv \alpha \equiv 1 \pmod 4$. (see L. E. Dickson [3])
- In the above factorization, k is at least 8, and if 3 does not divide N, then k is at least 11.
- The largest prime factor of odd perfect number n is greater than 10^8 (see T. Goto, Y. Ohno [4]), the second largest prime factor is greater than 10^4 (see D. Ianucci [6]), and the third one is greater than 10^2 (see D. Iannucci [7]).
- If any odd perfect numbers exist in form
  $$n = q^{\alpha} p_1^{2e_1} \dots p_k^{2e_k},$$
  they would have at least 75 prime factor in total, that means: $\alpha + 2\sum\limits_{i=1}^{k} e_i \geq 75$. (see K. G. Hare [5])

D. Suryanarayana introduced the notion of superperfect number in 1969 [12], here is the definition.

Definition 3 *A positive integer* n *is called* ***superperfect number*** *if*

$$\sigma(\sigma(n)) = 2n.$$

Some properties concerning superperfect numbers:

- Even superperfect numbers are 2^{p-1}, where $2^p - 1$ is a Mersenne prime.
- If any odd superperfect numbers exist, they are square numbers (G. G. Dandapat [2]) and either n or $\sigma(n)$ is divisible by at least three distinct primes. (see H. J. Kanold [8])

2 Hyperperfect numbers

Minoli and Bear [10] introduced the concept of k-hyperperfect number and they conjecture that there are k-hyperperfect numbers for every k.

Definition 4 *A positive integer n is called **k-hyperperfect number** if*

$$n = 1 + k[\sigma(n) - n - 1]$$

rearranging gives:

$$\sigma(n) = \frac{k+1}{k} n + \frac{k-1}{k}.$$

We remark that a number is perfect iff it is 1-hyperperfect. In the paper of J. S. Craine [9] all hyperperfect numbers less than 10^{11} have been computed

Example 2 *The table below shows some k-hyperperfect numbers for different k values:*

k	k-***hyperperfect*** *number*
1	*6 ,28, 496, 8128, ...*
2	*21, 2133, 19521, 176661, ...*
3	*325, ...*
4	*1950625, 1220640625, ...*
6	*301, 16513, 60110701, ...*
10	*159841, ...*
12	*697, 2041, 1570153, 62722153, ...*

Some results concerning hyperperfect numbers:

- If $k > 1$ is an odd integer and $p = (3k+1)/2$ and $q = 3k+4$ are prime numbers, then p^2q is k-hyperperfect; J. S. McCraine [9] has conjectured in 2000 that all k-hyperperfect numbers for odd $k > 1$ are of this form, but the hypothesis has not been proven so far.

- If p and q are distinct odd primes such that $k(p+q) = pq - 1$ for some integer, k then $n = pq$ is k-hyperperfect.
- If $k > 0$ and $p = k+1$ is prime, then for all $i > 1$ such that $q = p^i - p + 1$ is prime, $n = p^{i-1}q$ is k-hyperperfect (see H. J. J. te Riele [13], J. C. M. Nash [11]).

We have proposed some other forms of generalization, different from k-hyperperfect numbers, and also we have examined **super-hyperperfect numbers** ("super" in the way as super perfect):

$$\sigma(\sigma(n)) = \frac{k+1}{k}n + \frac{k-1}{k}$$

$$\sigma(n) = \frac{2k-1}{k}n + \frac{1}{k}$$

$$\sigma(\sigma(n)) = \frac{2k-1}{k}n + \frac{1}{k}$$

$$\sigma(n) = \frac{3}{2}(n+1)$$

$$\sigma(\sigma(n)) = \frac{3}{2}(n+1)$$

3 Numerical results

For finding the numerical results for the above equalities we have used the ANSI C programming language, the Maple and the Octave programs. Small programs written in C were very useful for going through the smaller numbers up to 10^7, and for the rest we used the two other programs. In this chapter the small numerical results are presented only in the cases where solutions were found.

3.1. Super-hyperperfect numbers. The table below shows the results we have reached:

k	n
1	$2, 2^2, 2^4, 2^6, 2^{12}, 2^{16}, 2^{18}$
2	$3^2, 3^6, 3^{12}$
4	5^2

3.2. $\sigma(n) = \frac{2k-1}{k}n + \frac{1}{k}$
For $k = 2$:

n	prime factorization
21	$3 \cdot 7 = 3(3^2 - 2)$
2133	$3^3 \cdot 79 = 3^3 \cdot (3^4 - 2)$
19521	$3^4 \cdot 241 = 3^4 \cdot (3^5 - 2)$
176661	$3^5 \cdot 727 = 3^5 \cdot (3^6 - 2)$

We have performed searches for $k = 3$ and $k = 5$ too, but we haven't found any solution

3.3. $\sigma(\sigma(n)) = \frac{2k-1}{k}n + \frac{1}{k}$

For $k = 2:$

k	prime factorization
9	3^2
729	3^6
531441	3^{12}

We have performed searches for $k = 3$ and $k = 5$ too, but we haven't found any solution

3.4. $\sigma(n) = \frac{3}{2}(n + 1)$

k	prime factorization
15	$3 \cdot 5$
207	$3^2 \cdot 23$
1023	$3 \cdot 11 \cdot 31$
2975	$5^2 \cdot 7 \cdot 17$
19359	$3^4 \cdot 239$
147455	$5 \cdot 7 \cdot 11 \cdot 383$
1207359	$3^3 \cdot 97 \cdot 461$
5017599	$3^3 \cdot 83 \cdot 2239$

4 Results and conjectures

Proposition 1 *If* $n = 3^{k-1}(3^k - 2)$ *where* $3^k - 2$ *is prime, then* n *is a 2-hyperperfect number.*

Proof. Since the divisor function σ is multiplicative and for a prime p and prime power we have:

$$\sigma(p) = p + 1$$

and

$$\sigma(p^{\alpha}) = \frac{p^{\alpha+1}-1}{p-1},$$

it follows that:

$$\begin{aligned}\sigma(n) &= \sigma(3^{k-1}(3^k-2)) = \sigma(3^{k-1}) \cdot \sigma(3^k-2) = \frac{3^{(k-1)+1}-1}{3-1} \cdot (3^k-2+1) = \\ &= \frac{(3^k-1)\cdot(3^k-1)}{2} = \frac{3^{2k}-2\cdot 3^k+1}{2} = \frac{3}{2}3^{k-1}(3^k-2)+\frac{1}{2}.\end{aligned}$$

■

Conjecture 2 *All 2-hyperperfect numbers are of the form* $n = 3^{k-1}(3^k-2)$, *where* 3^k-2 *is prime.*

We were looking for adequate results fulfilling the suspects, therefore we have searched for primes that can be written as 3^k-2. We have reached the following results:

#	k **for which** 3^k-2 **is prime**
1	2
2	4
3	5
4	6
5	9
6	22
7	37
8	41
9	90

#	k for which $3^k - 2$ is prime
10	102
11	105
12	317
13	520
14	541
15	561
16	648
17	780
18	786
19	957
20	1353
21	2224
22	2521
23	6184
24	7989
25	8890
26	19217
27	20746

Therefore the last result we reached is: $3^{20745}(3^{20746} - 2)$, which has 19796 digits.
If we consider the super-hiperperfect numbers in special form $\sigma(\sigma(n)) = \frac{3}{2}n + \frac{1}{2}$ we prove the following result.

Proposition 3 *If* $n = 3^{p-1}$ *where* p *and* $(3^p - 1)/2$ *are primes, then* n *is a super-hyperperfect number.*

Proof.

$$\begin{aligned} \sigma(\sigma(n)) &= \sigma(\sigma(3^{p-1})) = \sigma\left(\frac{3^p - 1}{2}\right) = \frac{3^p - 1}{2} + 1 = \\ &= \frac{3}{2} \cdot 3^{p-1} + \frac{1}{2} = \frac{3}{2}n + \frac{1}{2}. \end{aligned}$$

■

Conjecture 4 *All solutions for this generalization are* 3^{p-1}*-like numbers, where* p *and* $(3^p - 1)/2$ *are primes.*

We were looking for adequate results fulfilling the suspects, therefore we have searched for primes p for which $(3^p - 1)/2$ is also prime. We have reached the following results:

#	$p-1$ **for which** p and $(3^p-1)/2$ **are primes**
1	2
2	6
3	12
4	540
5	1090
6	1626
7	4176
8	9010
9	9550

Therefore the last result we reached is: 3^{9550}, which has 4556 digits.

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Haar measure is not approximable by balls

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Abstract. We construct a compact topological Abelian group with an invariant metric and a compact subset of Haar measure zero that cannot be covered by balls of total measure less than 1. This answers a question of Christensen.

1 Introduction

J.P.R. Christensen asked in 1979 in a conference in Oberwolfach (see [2]) the following question: *let X be a compact Abelian group with invariant metric and let* μ *be its Haar measure. Is it true that for any subset* $A \subset X$*,* $\mu(A)$ *is the infimum of* $\sum_i \mu(B_i)$*, where* B_i*'s are balls covering* A*?*

It follows from Christensen's paper, and it was also proved independently by P. Mattila (in the same proceedings [2], page 270, Remark 3.7) that this is true if we only require the balls to cover *almost all* of A. Mattila showed that this latter statement holds for any uniformly distributed measure on a metric space (i.e. for any Borel regular measure for which balls of the same size have the same, positive and finite measure). An interesting example of R.O. Davies shows (see [1]) that even if we require the balls to cover only almost all of A, it is not always possible to choose the balls to be disjoint.

The question of Christensen is equivalent to the following problem: *is it true that if* $\mu(A) = 0$*, then* A *can be covered by balls of arbitrary small total measure?* It turns out that the answer to this question is negative. In this

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note we construct a compact Abelian group X (we simply take a product of finite sets, i.e. X is homeomorphic to a Cantor set) and we choose an invariant metric on X, so that there is a compact subset $A \subset X$ of Haar measure zero for which $\sum_i \mu(B_i) \geq 1$ for any covering by balls $A \subset \bigcup_i B_i$. The main idea of our proof is to find a metric on a finite set so that all non-degenarate balls (i.e. balls of more than one element) overlap so badly that one needs many balls, of very large total measure, to cover half of the points. One can of course always use in a finite metric space the degenarate balls only, and so cover any set by balls whose total measure does not exceed the measure of the set. However, by taking the product of these finite sets, we obtain a counterexample to Christensen's problem.

2 Construction

Let $n_0 = N_0 = 1$, and choose inductively for each $k \in \mathbb{N}$ positive integers n_k, N_k for which

$$\frac{n_k}{2}\left(1-\left(\frac{n_k-1}{n_k}\right)^{N_k}\right) \geq \prod_{j<k} n_j^{N_j} \tag{1}$$

and n_k is even. Let Z_{n_k} denote the additive group mod n_k, and let $Y_k = Z_{n_k}^{N_k}$ be the product group on the product space $\{0, 1, \ldots, n_{k-1}\}^{N_k}$ equipped with the discrete topology. The Haar measure μ_k on Y_k is uniformly distributed, each element of Y_k has measure $1/n_k^{N_k}$.

We define an invariant metric on Y_k as follows. If $x = (x_1, x_2, \ldots, x_{N_k})$ and $y = (y_1, y_2, \ldots, y_{N_k})$ are two elements of Y_k, let

$$d_k(x,y) = \begin{cases} 0 & \text{if } x_j = y_j \text{ for all } j \\ 2 & \text{if } x_j \neq y_j \text{ for all } j \\ 1 & \text{otherwise.} \end{cases}$$

This is indeed a metric, since for any three distinct points x, y, z the distance between any two of them is 1 or 2, so the triangle-inequality is automatically satisfied. It is also immediate to see that d_k is invariant under the group actions.

Now let X be the direct product of the groups Y_k, $k \in \mathbb{N}$. This is a compact Abelian group on the product space $\prod_{k=1}^{\infty}\{0, 1, \ldots, n_{k-1}\}^{N_k}$ and its Haar measure is the product measure $\mu = \prod_{k=1}^{\infty} \mu_k$. For two distinct points

$x = (x_1, x_2, \dots) \in X$, $y = (y_1, y_2, \dots) \in X$ define

$$d(x, y) = \frac{d_k(x_k, y_k)}{2^k},$$

where k is the least index for which the coordinates $x_k, y_k \in Y_k$ are different. This defines a metric on X. Indeed, if $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots)$, $z = (z_1, z_2, \dots)$ are three distinct points, let k be the least index such that x_k, y_k, z_k are not all the same. If they are all different, then they satisfy the triangle-inequality since d_k satisfies it. If two of them are the same, say, $x_k = y_k \neq z_k$, then $d(x, z) = d(y, z) \in \{1/2^k, 1/2^{k-1}\}$ and $d(x, y) \in \{1/2^\ell, 1/2^{\ell-1}\}$ for some $\ell \geq k+1$, so again, x, y, z satisfy the triangle-inequality. The metric d is invariant under the group actions, since each d_k is invariant.

Since the distance between any two points of X is a power of 1/2, each (open or closed) ball in X coincides with a closed ball whose radius is $1/2^k$ for some $k \in \mathbb{N}$. For $x = (x_1, x_2, \dots) \in X$, the closed ball $B(x, 1)$ of centre x and radius 1 is the whole space X, and for $k \geq 1$, the closed ball $B(x, 1/2^k)$ of centre x and radius $1/2^k$ is

$$\{(y_1, y_2, \dots) \in X : x_j = y_j \text{ for } j < k, \text{ and } d_k(x_k, y_k) = 0 \text{ or } 1\}. \tag{2}$$

Therefore each ball is a (finite) union of cylinder sets, and each cylinder set of X is a finite union of balls. This shows that indeed the topology induced by the metric d is the product topology of X.

Let

$$A_k = \{(x_1, x_2, \dots, x_{N_k}) \in Y_k : 0 \leq x_1 < n_k/2\} \subset Y_k,$$

and let $A = \prod_{k=1}^{\infty} A_k \subset X$. Then $\mu_k(A_k) = 1/2$ for each k, hence $\mu(A) = \prod_{k=1}^{\infty} 1/2 = 0$. We show that $\sum_i \mu(B_i) \geq 1$ for any balls B_i whose union covers A.

Suppose that $A \subset \bigcup B_i$ for some balls B_i. The set A is compact since it is a product of the compact sets A_k, therefore we can assume that $A \subset \bigcup B_i$ is a finite cover. Without loss of generality we can also assume that each B_i meets A, and that none of the balls B_i is contained in any of the others. Finally, we can assume that all the balls are closed and their radius is a power of 1/2.

Take the smallest ball $B_i = B(x, 1/2^k)$. If it has radius 1, then $\mu(B_i) = \mu(X) = 1$. If its radius is less than 1, then it is of the form given by (2). Since B_i meets A, therefore $x_j \in A_j$ for all $j < k$. Consider the cylinder set

$$C = \{(y_1, y_2, \dots) \in X : x_j = y_j \text{ for } j < k\}.$$

If a ball $B(y, 1/2^\ell)$ meets C and it has radius larger than $1/2^k$, then it covers our ball B_i, so all the balls of our cover that meet C must have the same radius $1/2^k$. Consider the projections of these balls to the kth coordinate. These are closed balls $B'_i \subset Y_k$ of d_k-radius 1, and their union covers A_k. It follows from the definition of the set A_k and of the distance d_k that A_k cannot be covered by less than $n_k/2$ balls of d_k-radius 1. Indeed, for any given set $E \subset Y_k$ of less than $n_k/2$ points, one can always find a point in Y_k whose first coordinate is less than $n_k/2$, and whose d_k-distance from E is 2.

One checks easily that for each B'_i, $(n_k - 1)^{N_k}$ points of Y_k are in the complement of B'_i, hence

$$\mu_k(B'_i) = 1 - \left(\frac{n_k - 1}{n_k}\right)^{N_k}$$

and so

$$\begin{aligned} \mu(B_i) &= \left(1 - \left(\frac{n_k - 1}{n_k}\right)^{N_k}\right) / \prod_{j<k} n_j^{N_j} \\ \sum_i \mu(B_i) &\geq \frac{n_k}{2} \cdot \left(1 - \left(\frac{n_k - 1}{n_k}\right)^{N_k}\right) / \prod_{j<k} n_j^{N_j} \geq 1 \end{aligned}$$

by (1). ∎

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About a differential inequality

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Abstract.
A differential inequality concerning holomorfic function is generalised and improved. Several other differential inequalities are considered.

1 Introduction

Let $\mathcal{H}(U)$ be the set of holomorfic functions defined on the unit disc

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Y. Komatsu in [2] proved, the following implication:
If $f \in \mathcal{H}(U)$, $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ and $\mathrm{Re}\sqrt{f'(z)} > \frac{1}{2}$, $z \in U$, then $\frac{f(z)}{z} > \frac{1}{2}$, $z \in U$.
The aim of this paper is to generalize this inequality.
In the paper each multiple-valued function is taken with the principal value.

2 Preliminaries

In our study we need the following definitions and lemmas:
Let X be a locally convex linear topological space. For a subset $U \subset X$ the closed convex hull of U is defined as the intersection of all closed convex sets containing U and will be denoted by $\mathrm{co}(U)$. If $U \subset V \subset X$ then U is called an

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extremal subset of V provided that whenever $u = tx + (1-t)y$ where $u \in U$, $x, y \in V$ and $t \in (0,1)$ then $x, y \in U$.

An extremal subset of U consisting of just one point is called an extreme point of U.

The set of the extreme points of U will be denoted by EU.

Lemma 1 ([1], pp. 45) *If $J : \mathcal{H}(U) \to \mathbb{R}$ is a real-valued, continuous convex functional and $\mathcal{F}$ is a compact subset of $\mathcal{H}(U)$, then*

$$\max\{J(f) : f \in \mathrm{co}(\mathcal{F})\} = \max\{J(f) : f \in \mathcal{F}\} = \max\{J(f) : f \in E(\mathrm{co}(\mathcal{F}))\}.$$

In the particular case if J is a linear map then we also have:

$$\min\{J(f) : f \in \mathrm{co}(\mathcal{F})\} = \min\{J(f) : f \in \mathcal{F}\} = \min\{J(f) : f \in E(\mathrm{co}(\mathcal{F}))\}.$$

Suppose that $f, g \in \mathcal{H}(U)$. The function f is subordinate to g if there exists a function $\theta \in \mathcal{H}(U)$ such that $\theta(0) = 0$, $|\theta(z)| < 1$, $z \in U$ and $f(z) = g(\theta(z))$, $z \in U$.

The subordination will be denoted by $f \prec g$.

Remark 1 *Suppose that $f, g \in \mathcal{H}(U)$ and g is univalent. If $f(0) = g(0)$ and $f(U) \subset g(U)$ then $f \prec g$.*

When $F \in \mathcal{H}(U)$ we use the notation

$$s(F) = \{f \in \mathcal{H}(U) : f \prec F\}.$$

Lemma 2 ([1] pp. 51) *Suppose that F_α is defined by the equality*

$$F_\alpha(z) = \left(\frac{1+cz}{1-z}\right)^\alpha, \ |c| \le 1, \ c \ne -1.$$

If $\alpha \ge 1$ then $\mathrm{co}(s(F_\alpha))$ consists of all functions in $\mathcal{H}(U)$ represented by

$$f(z) = \int_0^{2\pi} \left(\frac{1+cze^{-it}}{1-ze^{-it}}\right)^\alpha d\mu(t)$$

where μ is a positive measure on $[0, 2\pi]$ having the property $\mu([0, 2\pi]) = 1$ and

$$E(\mathrm{co}(s(F_\alpha))) = \left\{\frac{1+cze^{-it}}{1-ze^{-it}} \mid t \in [0, 2\pi]\right\}.$$

Remark 2 *If $L : \mathcal{H}(U) \to \mathcal{H}(U)$ is an invertible linear map and $\mathcal{F} \subset \mathcal{H}(U)$ is a compact subset, then $L(\mathrm{co}(\mathcal{F})) = \mathrm{co}(L(\mathcal{F}))$ and the set $E(\mathrm{co}(\mathcal{F}))$ is in one-to-one correspondence with $EL(\mathrm{co}(\mathcal{F}))$.*

3 The main result

Theorem 1 *Let* $f \in \mathcal{H}(U)$ *be a function normalized by the conditions* $f(0) = f'(0) - 1 = 0 \quad m, p \in \mathbb{N}^*$; $a_k \in \mathbb{R}$, $k = \overline{1, p}$ *and*

$$\operatorname{Re} \sqrt[m]{f'(z) + a_1 z f''(z) + \cdots + a_p z^p f^{(p+1)}(z)} > \frac{1}{2}, \; z \in U, \tag{1}$$

then

$$1 + \inf_{\theta \in (0,2\pi)} \left(\sum_{n=1}^{\infty} \frac{C_{n+m-1}^{m-1}}{P(n+1)} \cos n\theta \right) < \operatorname{Re} \frac{f(z)}{z} < 1 +$$
$$+ \sup_{\theta \in (0,2\pi)} \left(\sum_{n=1}^{\infty} \frac{C_{n+m-1}^{m-1}}{P(n+1)} \cos n\theta \right)$$
$$1 + \inf_{\theta \in (0,2\pi)} \left(\sum_{n=1}^{\infty} \frac{(n+1) C_{n+m-1}^{m-1}}{P(n+1)} \cos n\theta \right) < \operatorname{Re} f'(z) < 1 +$$
$$+ \sup_{\theta \in (0,2\pi)} \left(\sum_{n=1}^{\infty} \frac{(n+1) C_{n+m-1}^{m-1}}{P(n+1)} \cos n\theta \right)$$

where $P(x) = x + a_1 x(x-1) + a_2 x(x-1)(x-2) + \cdots + a_p x(x-1)\ldots(x-p)$.

Proof. The condition (1) is equivalent to:

$$\sqrt[m]{f'(z) + a_1 z f''(z) + \cdots + a_p z^p f^{(p+1)}(z)} \prec \frac{1}{1-z}$$

and this can be rewritten as follows:

$$f'(z) + a_1 z f''(z) + \ldots a_p z^p f^{(p+1)}(z) \prec \frac{1}{(1-z)^m}.$$

According to the Lemma 2,

$$f'(z) + a_1 z f''(z) + \cdots + a_p z^p f^{(p+1)}(z) = \int_0^{2\pi} \frac{1}{(1 - z e^{-it})^m} d\mu(t) = h(z)$$

where $\mu([0, 2\pi]) = 1$.

Denoting

$$f(z) = z + \sum_{n=2}^{\infty} b_n z^n, \; z \in U$$

we get

$$f'(z)+a_1zf''(z)+\cdots+a_pz^pf^{(p+1)}(z)=1+\sum_{n=2}^{\infty}b_nP(n)z^{n-1},$$

on the other hand

$$\int_0^{2\pi}\frac{1}{(1-ze^{-it})^m}d\mu(t)=1+\sum_{n=2}^{\infty}C_{n+m-2}^{m-1}z^{n-1}\cdot\int_0^{2\pi}e^{-i(n-1)t}d\mu(t).$$

The above two developments in power series imply that:

$$1+\sum_{n=2}^{\infty}b_nP(n)z^{n-1}=1+\sum_{n=2}^{\infty}C_{n+m-2}^{m-1}z^{n-1}\int_0^{2\pi}e^{-i(n-1)t}d\mu(t),$$

and

$$b_n=\frac{C_{n+m-2}^{m-1}}{P(n)}\int_0^{2\pi}e^{-i(n-1)t}d\mu(t),\ n\in\mathbb{N},n\geq 2.$$

Thus

$$f(z)=z+\sum_{n=2}^{\infty}\frac{C_{n+m-2}^{m-1}}{P(n)}z^n\int_0^{2\pi}e^{-i(n-1)t}d\mu(t) \tag{2}$$

and we deduce:

$$\frac{f(z)}{z}=1+\sum_{n=1}^{\infty}\frac{C_{n+m-1}^{m-1}}{P(n+1)}z^n\int_0^{2\pi}e^{-int}d\mu(t)$$

$$f'(z)=1+\sum_{n=1}^{\infty}\frac{(n+1)C_{n+m-1}^{m-1}}{P(n+1)}z^n\int_0^{2\pi}e^{-int}d\mu(t).$$

If

$$\mathcal{B}=\left\{h\in\mathcal{H}(U)\mid h(z)=\int_0^{2\pi}\frac{1}{(1-ze^{-it})^m}d\mu(t),\ z\in U,\ \mu([0,2\pi])=1\right\},$$

$$\mathcal{C}=\left\{f\in\mathcal{H}(U)\mid \mathrm{Re}\left(\sqrt[m]{f(z)+a_1zf'(z)+\cdots+a_pz^pf^{(p)}(z)}\right)>0,\ z\in U\right\}$$

then the correspondence $L:\mathcal{B}\to\mathcal{C}$, $L(h)=f$ defines an invertible linear map and according to the Observation 2 the extreme points of the class $\mathcal{C}$ are

$$f_t(z)=z+\sum_{n=1}^{\infty}\frac{C_{n+m-1}^{m-1}}{P(n+1)}z^{n+1}e^{-int}.$$

This result, Lemma 1 and the minimum and maximum principle for harmonic functions imply the assertion of Theorem 1. ■

4 Particular cases

Let $\mathcal{A}$ be the class of analytic functions defined by the equality:

$$\mathcal{A} = \{f \in \mathcal{H} : f(0) = f'(0) - 1 = 0\}.$$

If we put $p = 2, a_1 = a_2 = m = 1$ in Theorem 1 then we get:

Corollary 1 (Komatu) [2]) *If* $f \in \mathcal{A}$ *and* $Re\sqrt{f'(z)} > \frac{1}{2}$, $z \in U$, *then* $Re\frac{f(z)}{z} > \frac{1}{2}$, $z \in U$, *and the result is sharp.*

Proof.
We apply Theorem 1 in the particular case $a_1 = 1$, $a_2 = a_3 = \ldots = a_p = 0$ i $m = 2$. We get $P(n+1) = (n+1)^2$ and

$$\mathrm{Re}\frac{f(z)}{z} > 1 + \inf_{z\in U} \mathrm{Re} \sum_{n=1}^{\infty} \frac{C_{n+1}^1}{(n+1)^2} z^n = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} = \ln 2, \ z \in U.$$

The other case can be proved as follows:

$$\mathrm{Re} f'(z) > 1 + \inf_{z\in U} \mathrm{Re} \sum_{n=1}^{\infty} \frac{(n+1)C_{n+1}^1}{(n+1)^2} z^n = 1 + \inf_{z\in U} \mathrm{Re}\frac{z}{1-z} = \frac{1}{2}, \ z \in U.$$

■

Corollary 2 *If* $f \in \mathcal{A}$ *and* $Re\sqrt{f'(z) + zf''(z)} > \frac{1}{2},$ $z \in U$ *then*
1) $Re\frac{f(z)}{z} > \ln 2, \ z \in U$
2) $Re f'(z) > \frac{1}{2}, \ z \in U$ *and the results are sharp.*

Proof. We apply again Theorem 1 in case of $a_1 = 1$, $a_2 = a_3 = \ldots = a_p = 0$ and $m = 2$. It is easily seen that $P(n+1) = (n+1)^2$ and

$$\mathrm{Re}\frac{f(z)}{z} > 1 + \inf_{z\in U} \mathrm{Re} \sum_{n=1}^{\infty} \frac{C_{n+1}^1}{(n+1)^2} z^n = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} = \ln 2, \ z \in U.$$

In the other case :

$$\mathrm{Re} f'(z) > 1 + \inf_{z\in U} \mathrm{Re} \sum_{n=1}^{\infty} \frac{(n+1)C_{n+1}^1}{(n+1)^2} z^n = 1 + \inf_{z\in U} \mathrm{Re}\frac{z}{1-z} = \frac{1}{2}, \ z \in U.$$

Corollary 3 *Let* $p \in \mathbb{N}$, $p \geq 3$. *If* $f \in \mathcal{A}$ *and* $S(p,k)$, $p \geq k$ *are the numbers of Stirling of the second kind defined by*

$$S(p,k) = \frac{1}{k!}\sum_{l=1}^{k-1}(-1)^l C_k^l (k-l)^p,\ k = \overline{1,p},$$

then the inequality

$$Re\left(\sqrt{\sum_{k=1}^{p} S(p,k) z^{k-1} f^{(k)}(z)}\right) > \frac{1}{2},\ z \in U \tag{3}$$

implies that

$$Re\frac{f(z)}{z} > \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{p-1}},\ z \in U,$$

and the result is sharp.

Proof. According to Theorem 1follows that:

$$\mathrm{Re}\frac{f(z)}{z} > 1 + \inf_{z\in U}\mathrm{Re}\left(\sum_{n=1}^{\infty}\frac{C_{n+1}^1}{P(n+1)}z^n\right) \tag{4}$$

and we have:

$$P(x) = \sum_{k=1}^{p} S(p,k)x(x-1)\dots(x-k+1) = x^p. \tag{5}$$

We have to determine:

$$\inf_{z\in U}\mathrm{Re}\left(\sum_{n=1}^{\infty}\frac{C_{n+1}^1}{P(n+1)}z^n\right) = \inf_{\theta\in(0,2\pi)}\mathrm{Re}\left(\sum_{n=1}^{\infty}\frac{e^{in\theta}}{(n+1)^{p-1}}\right).$$

We will use the following integral representation:

$$\sum_{n=1}^{\infty}\frac{e^{in\theta}}{(n+1)^{p-1}} = \underbrace{\int_0^1\int_0^1\dots\int_0^1}_{p-1}(t_1t_2\dots t_{p-1}e^{i\theta})^n dt_1dt_2\dots dt_{p-1} =$$

$$= \underbrace{\int_0^1\int_0^1\dots\int_0^1}_{p-1} t_1t_2\dots t_{p-1}\frac{e^{i\theta} - t_1t_2\dots t_{p-1}}{1 + t_1^2t_2^2\dots t_{p-1}^2 - 2t_1t_2\dots t_{p-1}\cos\theta}dt_1dt_2\dots dt_{p-1}$$

If we denote $t_1 t_2 \dots t_{p-1} = u$, then $u \in [0, 1]$ and

$$\frac{\cos\theta - u}{1 + u^2 - 2u\cos\theta} \geq \frac{-1}{1+u}, \theta \in [0, 2\pi]. \tag{6}$$

We get from (6) the inequality:

$$\int_0^1 \dots \int_0^1 t_1 \dots t_{p-1} \frac{\cos\theta - t_1 \dots t_{p-1}}{1 + t_1^2 \dots t_{p-1}^2 - 2t_1 \dots t_{p-1}\cos\theta} dt_1 \dots dt_{p-1} \geq$$
$$\geq - \int_0^1 \dots \int_0^1 \frac{t_1 \dots t_{p-1}}{1 + t_1 \dots t_{p-1}} dt_1 \dots dt_{p-1}$$

where in case of $\theta = \pi$ the equality holds. This implies that:

$$\inf_{\theta \in (0, 2\pi)} \operatorname{Re} \sum_{n=1}^{\infty} \frac{e^{in\theta}}{(n+1)^{p-1}} =$$
$$= \inf_{\theta \in (0, 2\pi)} \int_0^1 \dots \int_0^1 t_1 \dots t_{p-1} \frac{\cos\theta - t_1 \dots t_{p-1}}{1 + t_1^2 \dots t_{p-1}^2 - 2t_1 \dots t_{p-1}\cos\theta} dt_1 \dots dt_{p-1} =$$
$$= - \int_0^1 \int_0^1 \dots \int_0^1 \frac{t_1 \dots t_{p-1}}{1 + t_1 \dots t_{p-1}} dt_1 \dots dt_{p-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^{p-1}},$$

and the proof is done. ■

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