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RADII OF k-STARLIKENESS OF ORDER α OF STRUVE AND LOMMEL FUNCTIONS

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Abstract. In the present work our main objective is to determine the radii of $k-$ starlikeness of order α of the some normalized Struve and Lommel functions of the first kind. Furthermore it has been shown that the obtained radii satisfy some functional equations. The main key tool of our proofs are the Mittag-Leffler expansions of the Struve and Lommel functions of the first kind and minimum principle for harmonic functions. Also we take advantage of some basic inequalities in the complex analysis.

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1. INTRODUCTION

It is well-known that there are numerous connections between geometric function theory and special functions. Due to these close relationships many authors studied on some geometric properties of special functions like Bessel, Struve, Lommel, Wright and Mittag-Leffler functions. Especially, the authors in the papers [3–5, 7, 14–16, 19] have investigated univalence, starlikeness, convexity and close-to convexity of the above mentioned functions. Actually, the beginning of these studies is based on the papers [6, 12, 21] written by Brown, Kreyszig and Todd and Wilf, respectively. Also the authors who studied the geometric properties of special functions have used some properties of zeros of the mentioned special functions. For comprehensive information about the zeros of these functions, we refer to the studies [17, 18, 20]. Motivated by the earlier investigations on this field our main goal is to determine the radii of k-starlikeness of the normalized Struve and Lommel functions of the first kind. Morever, we show that our obtained radii are the smallest positive roots of some functional equations. Also, for some special values of k and α we obtain some earlier results given by [1–3].

Now we would like to remind some basic concepts in geometric function theory.

Let $\mathbb{D}_r$ be the open disk $\{z \in \mathbb{C} : |z| < r\}$ with radius $r > 0$ and $\mathbb{D}_1 = \mathbb{D}$. Let $\mathcal{A}$ denote the class of analytic functions $f : \mathbb{D}_r \to \mathbb{C}$,

$$f(z) = z + \sum_{n\geq 2} a_n z^n,$$

which satisfies the normalization conditions $f(0) = f'(0) - 1 = 0$. By $\mathcal{S}$ we mean the class of functions belonging to $\mathcal{A}$ which are univalent in $\mathbb{D}_r$. The class of k-starlike functions of order α is denoted by $\mathcal{ST}(k,\alpha)$, where $k \geq 0$ and $0 \leq \alpha < 1$. This class of functions was introduced by Kanas and Wiśniowska [10, 11] which generalizes the class of uniformly convex functions introduced by Goodman in [8]. On the other hand, Kanas and Srivastava defined a linear operator and determined some conditions on the parameters for which this linear operator maps the classes of starlike and univalent functions onto the classes $k-$uniformly convex functions and $k-$starlike functions in [9]. Very recently, Srivastava gave comprehensive information about the usages of $q-$analysis in geometric function theory of complex analysis in his *survey-cum-expository* article [13]. Srivastava's work in particular inspired us to prepare this paper.

Analytic characterization of the class k-starlike functions of order α is

$$\mathcal{ST}(k,\alpha) = \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha, k \geq 0, 0 \leq \alpha < 1, z \in \mathbb{D} \right\}.$$

Also, the real number

$$r(f) = \sup\left\{ r > 0 : \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha \text{ for all } z \in \mathbb{D} \right\}$$

is called the radius of $k-$starlikeness of order α of the function f.

The Struve and Lommel functions are defined as the infinite series

$$\mathbf{H}_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\nu+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2n+\nu+1}, \ -\nu - \frac{3}{2} \notin \mathbb{N},$$

and

$$s_{\mu,\nu}(z) = \frac{(z)^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)} \sum_{n\geq 0} \frac{(-1)^n}{\left(\frac{\mu-\nu+3}{2}\right)_n\left(\frac{\mu+\nu+3}{2}\right)_n} \left(\frac{z}{2}\right)^{2n}, \ \frac{1}{2}(-\mu \pm \nu - 3) \notin \mathbb{N},$$

where $z, \mu, \nu \in \mathbb{C}$. Also, we know that the Struve and Lommel functions are the solutions of the inhomogeneous Bessel differential equations

$$zw''(z) + zw'(z) + (z^2 - \nu^2)w(z) = \frac{4\left(\frac{z}{2}\right)^{\nu+1}}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}$$

and

$$zw''(z) + zw'(z) + (z^2 - \nu^2)w(z) = z^{\mu+1},$$

respectively. One can find comprehensive information about these functions in [20].

Since the functions $\mathbf{H}_\nu$ and $s_{\mu,\nu}$ do not belong to the class $\mathcal{A}$, first we consider the following six normalized forms:

$$u_\nu(z) = \left(\sqrt{\pi}2^\nu\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(z)\right)^{\frac{1}{\nu+1}}, \quad \nu \neq -1, \tag{1.1}$$

$$v_\nu(z) = \sqrt{\pi}2^\nu z^{-\nu}\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(z), \tag{1.2}$$

$$w_\nu(z) = \sqrt{\pi}2^\nu z^{\frac{1-\nu}{2}}\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(\sqrt{z}), \tag{1.3}$$

$$f_\mu(z) = \left(\mu(\mu+1)s_{\mu-\frac{1}{2},\frac{1}{2}}(z)\right)^{\frac{1}{\mu+\frac{1}{2}}}, \quad \mu \in \left(-\frac{1}{2},1\right), \quad \mu \neq 0, \tag{1.4}$$

$$g_\mu(z) = \mu(\mu+1)z^{-\mu+\frac{1}{2}}s_{\mu-\frac{1}{2},\frac{1}{2}}(z) \tag{1.5}$$

and

$$h_\mu(z) = \mu(\mu+1)z^{\frac{3-2\mu}{4}}s_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{z}). \tag{1.6}$$

As a consequence, all functions considered above belong to the analytic functions class $\mathcal{A}$.

2. Main Results

Our first main result is related to the normalized Struve functions as follows.

Theorem 1. *Let $|\nu| \leq \frac{1}{2}$, $0 \leq \alpha < 1$ and $k \geq 0$. Then, the following assertions are true:*

i. *The radius r_u is the radius of $k-$starlikeness of order α of the normalized Struve function $z \mapsto u_\nu$ and it is the smallest positive root of the equation*

$$r(1+k)\mathbf{H}'_\nu(r) - (k+\alpha)(\nu+1)\mathbf{H}_\nu(r) = 0 \tag{2.1}$$

in $(0,h_{\nu,1})$, where $h_{\nu,1}$ is the first positive zero of Struve function $\mathbf{H}_\nu$.

ii. *The radius r_v is the radius of $k-$starlikeness of order α of the normalized Struve function $z \mapsto v_\nu$ and it is the smallest positive root of the equation*

$$r(1+k)\mathbf{H}'_\nu(r) - [\nu(1+k)+(k+\alpha)]\,\mathbf{H}_\nu(r) = 0 \tag{2.2}$$

in $(0,h_{\nu,1})$.

iii. *The radius r_w is the radius of $k-$starlikeness of order α of the normalized Struve function $z \mapsto w_\nu$ and it is the smallest positive root of the equation*

$$(1+k)\sqrt{r}\mathbf{H}'_\nu(\sqrt{r}) + (1-\nu-k-\nu k-2\alpha)\mathbf{H}_\nu(\sqrt{r}) = 0 \tag{2.3}$$

in $(0,h^2_{\nu,1})$.

Proof. We know that the zeros of the functions $\mathbf{H}_\nu(z)$ and $\mathbf{H}'_\nu(z)$ are real and simple when $|\nu| \leq \frac{1}{2}$, (see [4, 17]). Also the zeros of the function $\mathbf{H}_\nu(z)$ and its derivative interlace when $|\nu| \leq \frac{1}{2}$, according to [4]. In addition, it is known from [4] that the Struve function $\mathbf{H}_\nu(z)$ has the following infinite product representation:

$$\sqrt{\pi}2^\nu z^{-\nu-1}\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(z) = \prod_{n\geq 1}\left(1-\frac{z^2}{h_{\nu,n}^2}\right), \tag{2.4}$$

where $h_{\nu,n}$ denotes $n-$th positive zero of the Struve function $\mathbf{H}_\nu$. Using this product representation one can easily see that

$$\frac{zu'_\nu(z)}{u_\nu(z)} = 1-\frac{2}{\nu+1}\sum_{n\geq 1}\frac{z^2}{h_{\nu,n}^2-z^2}, \tag{2.5}$$

$$\frac{zv'_\nu(z)}{v_\nu(z)} = 1-2\sum_{n\geq 1}\frac{z^2}{h_{\nu,n}^2-z^2} \tag{2.6}$$

and

$$\frac{zw'_\nu(z)}{w_\nu(z)} = 1-\sum_{n\geq 1}\frac{z}{h_{\nu,n}^2-z}. \tag{2.7}$$

On the other hand, it is known from [19] that the inequality

$$\Re\left(\frac{z}{\theta-z}\right) \leq \frac{|z|}{\theta-|z|} \tag{2.8}$$

holds true for $z \in \mathbb{C}$ and $\theta \in \mathbb{R}$ such that $|z| < \theta$. Now, by using inequality (2.8) in (2.5), (2.6) and (2.7), respectively, we get

$$\begin{aligned}\Re\left(\frac{zu'_\nu(z)}{u_\nu(z)}\right) &= \Re\left(1-\frac{2}{\nu+1}\sum_{n\geq 1}\frac{z^2}{h_{\nu,n}^2-z^2}\right)\\ &\geq 1-\frac{2}{\nu+1}\sum_{n\geq 1}\frac{|z|^2}{h_{\nu,n}^2-|z|^2}\\ &= \frac{|z|\,u'_\nu(|z|)}{u_\nu(|z|)},\end{aligned} \tag{2.9}$$

$$\Re\left(\frac{zv'_\nu(z)}{v_\nu(z)}\right) = \Re\left(1-2\sum_{n\geq 1}\frac{z^2}{h_{\nu,n}^2-z^2}\right) \geq 1-2\sum_{n\geq 1}\frac{|z|^2}{h_{\nu,n}^2-|z|^2} = \frac{|z|\,v'_\nu(|z|)}{v_\nu(|z|)} \tag{2.10}$$

and

$$\Re\left(\frac{zw'_\nu(z)}{w_\nu(z)}\right) = \Re\left(1-\sum_{n\geq 1}\frac{z}{h_{\nu,n}^2-z}\right) \geq 1-\sum_{n\geq 1}\frac{|z|}{h_{\nu,n}^2-|z|} = \frac{|z|\,w'_\nu(|z|)}{w_\nu(|z|)}. \tag{2.11}$$

Also, from the reverse triangle inequality

$$|z_1-z_2|| \geq ||z_1|-|z_2||$$

we have

$$\left|\frac{zu_\nu'(z)}{u_\nu(z)}-1\right|=\left|-\frac{2}{\nu+1}\sum_{n\geq1}\frac{z^2}{h_{\nu,n}^2-z^2}\right|\leq\frac{2}{\nu+1}\sum_{n\geq1}\frac{|z|^2}{h_{\nu,n}^2-|z|^2}=1-\frac{|z|\,u_\nu'(|z|)}{u_\nu(|z|)}, \tag{2.12}$$

$$\left|\frac{zv_\nu'(z)}{v_\nu(z)}-1\right|=\left|-2\sum_{n\geq1}\frac{z^2}{h_{\nu,n}^2-z^2}\right|\leq 2\sum_{n\geq1}\frac{|z|^2}{h_{\nu,n}^2-|z|^2}=1-\frac{|z|\,v_\nu'(|z|)}{v_\nu(|z|)} \tag{2.13}$$

and

$$\left|\frac{zw_\nu'(z)}{w_\nu(z)}-1\right|=\left|-\sum_{n\geq1}\frac{z}{h_{\nu,n}^2-z}\right|\leq\sum_{n\geq1}\frac{|z|}{h_{\nu,n}^2-|z|}=1-\frac{|z|\,w_\nu'(|z|)}{w_\nu(|z|)}. \tag{2.14}$$

As a result of the above inequalities, one can easily obtain that

$$\Re\left(\frac{zu_\nu'(z)}{u_\nu(z)}\right)-k\left|\frac{zu_\nu'(z)}{u_\nu(z)}-1\right|-\alpha\geq(1+k)\frac{|z|\,u_\nu'(|z|)}{u_\nu(|z|)}-(k+\alpha), \tag{2.15}$$

$$\Re\left(\frac{zv_\nu'(z)}{v_\nu(z)}\right)-k\left|\frac{zv_\nu'(z)}{v_\nu(z)}-1\right|-\alpha\geq(1+k)\frac{|z|\,v_\nu'(|z|)}{v_\nu(|z|)}-(k+\alpha), \tag{2.16}$$

and

$$\Re\left(\frac{zw_\nu'(z)}{w_\nu(z)}\right)-k\left|\frac{zw_\nu'(z)}{w_\nu(z)}-1\right|-\alpha\geq(1+k)\frac{|z|\,w_\nu'(|z|)}{w_\nu(|z|)}-(k+\alpha). \tag{2.17}$$

It is important to emphasize here that the equalities in the last three inequalities hold true for $z=|z|=r$. If we consider the minimum principle for harmonic functions in the inequalities (2.15), (2.16) and (2.17), then we can say that these inequalities are valid if and only if $|z|<r_u$, $|z|<r_v$ and $|z|<r_w$, where r_u, r_v and r_w are the smallest positive roots of the following equations

$$(1+k)\frac{ru_\nu'(r)}{u_\nu(r)}-(k+\alpha)=0,$$

$$(1+k)\frac{rv_\nu'(r)}{v_\nu(r)}-(k+\alpha)=0$$

and

$$(1+k)\frac{rw_\nu'(r)}{w_\nu(r)}-(k+\alpha)=0,$$

respectively. Taking into account the definitions of the functions u_ν, v_ν and w_ν, it can be easily seen that the last three equations are equivalent to (2.1), (2.2) and (2.3), respectively. Now, we would like to show that equation (2.1) has an unique root on the interval $(0,h_{\nu,1})$. To show this, let us consider the function $\Psi_\nu:(0,h_{\nu,1})\mapsto\mathbb{R}$,

$$\Psi_\nu(r)=(1+k)\frac{ru_\nu'(r)}{u_\nu(r)}-(k+\alpha)=(1+k)\left(1-\frac{2}{\nu+1}\sum_{n\geq1}\frac{r^2}{h_{\nu,n}^2-r^2}\right)-(k+\alpha).$$

The function $r \mapsto \Psi_\nu(r)$ is strictly decreasing since

$$\Psi_\nu'(r) = -\frac{4r(1+k)}{\nu+1}\sum_{n\geq 1}\frac{h_{\nu,n}^2}{\left(h_{\nu,n}^2-r^2\right)^2} < 0.$$

Morever, we have

$$\lim_{r\searrow 0}(1+k)\left(1-\frac{2}{\nu+1}\sum_{n\geq 1}\frac{r^2}{h_{\nu,n}^2-r^2}\right)-(k+\alpha)=1-\alpha>0$$

and

$$\lim_{r\nearrow h_{\nu,1}}(1+k)\left(1-\frac{2}{\nu+1}\sum_{n\geq 1}\frac{r^2}{h_{\nu,n}^2-r^2}\right)-(k+\alpha)=-\infty.$$

As a result of these limit relations, we can say that equation (2.1) has an unique root in $(0,h_{\nu,1})$. Similarly, it can be shown that equations (2.2) and (2.3) have a root in $(0,h_{\nu,1})$ and $(0,h_{\nu,1}^2)$, respectively. □

The following main result is regarding the normalized Lommel functions of the first kind.

Theorem 2. *The following assertions are true:*

i. *Let $\mu \in (-\frac{1}{2},1)$ and $\mu \neq 0$. Then, the radius r_f is the radius of $k-$starlikeness of order α of the normalized Lommel function $z \mapsto f_\mu$ and it is the smallest positive root of the equation*

$$r(1+k)s'_{\mu-\frac{1}{2},\frac{1}{2}}(r)-(k+\alpha)(\mu+\frac{1}{2})s_{\mu-\frac{1}{2},\frac{1}{2}}(r)=0 \tag{2.18}$$

in $(0,l_{\mu,1})$, where $l_{\mu,1}$ is the first positive zero of Lommel function $s_{\mu-\frac{1}{2},\frac{1}{2}}$.

ii. *Let $\mu \in (-1,1)$ and $\mu \neq 0$. Then, the radius r_g is the radius of $k-$starlikeness of order α of the normalized Lommel function $z \mapsto g_\mu$ and it is the smallest positive root of the equation*

$$r(1+k)s'_{\mu-\frac{1}{2},\frac{1}{2}}(r)+\left((1+k)(\frac{1}{2}-\mu)-(k+\alpha)\right)s_{\mu-\frac{1}{2},\frac{1}{2}}(r)=0 \tag{2.19}$$

in $(0,l_{\mu,1})$.

iii. *Let $\mu \in (-1,1)$ and $\mu \neq 0$. Then, the radius r_h is the radius of $k-$starlikeness of order α of the normalized Lommel function $z \mapsto h_\mu$ and it is the smallest positive root of the equation*

$$2\sqrt{r}(1+k)s'_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{r})+((1+k)(3-2\mu)-4(k+\alpha))s_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{r})=0 \tag{2.20}$$

in $(0,l_{\mu,1}^2)$.

Proof. It is known from [4, 18] that the Lommel function $s_{\mu-\frac{1}{2},\frac{1}{2}}$ and its derivative $s'_{\mu-\frac{1}{2},\frac{1}{2}}$ have only real and simple zeros when $\mu \in (-1,1)$ and $\mu \neq 0$. Morever, the zeros of the Lommel function $s_{\mu-\frac{1}{2},\frac{1}{2}}$ and its derivative $s'_{\mu-\frac{1}{2},\frac{1}{2}}$ interlace under the same conditions, according to [4]. Also, the Lommel function $s_{\mu-\frac{1}{2},\frac{1}{2}}$ can be written as the product (see [4])

$$s_{\mu-\frac{1}{2},\frac{1}{2}}(z) = \frac{z^{\mu+\frac{1}{2}}}{\mu(\mu+1)} \prod_{n\geq 1}\left(1-\frac{z^2}{l^2_{\mu,n}}\right), \tag{2.21}$$

where $l_{\mu,n}$ denotes $n-$th positive zero of the Lommel function $s_{\mu-\frac{1}{2},\frac{1}{2}}$. Using equality (2.21), it can be easily seen that

$$\frac{zf'_\mu(z)}{f_\mu(z)} = 1-\frac{2}{1+\frac{\mu}{2}}\sum_{n\geq 1}\frac{z^2}{l^2_{\mu,n}-z^2}, \tag{2.22}$$

$$\frac{zg'_\mu(z)}{g_\mu(z)} = 1-2\sum_{n\geq 1}\frac{z^2}{l^2_{\mu,n}-z^2} \tag{2.23}$$

and

$$\frac{zh'_\mu(z)}{h_\mu(z)} = 1-\sum_{n\geq 1}\frac{z}{l^2_{\mu,n}-z}. \tag{2.24}$$

Now, if we consider inequality (2.8) in the equalities (2.22), (2.23) and (2.24), respectively, then we have that

$$\begin{aligned}\Re\left(\frac{zf'_\mu(z)}{f_\mu(z)}\right) &= \Re\left(1-\frac{2}{1+\frac{\mu}{2}}\sum_{n\geq 1}\frac{z^2}{l^2_{\mu,n}-z^2}\right)\\ &\geq 1-\frac{2}{1+\frac{\mu}{2}}\sum_{n\geq 1}\frac{|z|^2}{l^2_{\mu,n}-|z|^2}\\ &= \frac{|z|\,f'_\mu(|z|)}{f_\mu(|z|)},\end{aligned} \tag{2.25}$$

$$\Re\left(\frac{zg'_\mu(z)}{g_\mu(z)}\right) = \Re\left(1-2\sum_{n\geq 1}\frac{z^2}{l^2_{\mu,n}-z^2}\right) \geq 1-2\sum_{n\geq 1}\frac{|z|^2}{l^2_{\mu,n}-|z|^2} = \frac{|z|\,g'_\mu(|z|)}{g_\nu(|z|)} \tag{2.26}$$

and

$$\Re\left(\frac{zh'_\mu(z)}{h_\mu(z)}\right) = \Re\left(1-\sum_{n\geq 1}\frac{z}{l^2_{\nu,n}-z}\right) \geq 1-\sum_{n\geq 1}\frac{|z|}{l^2_{\mu,n}-|z|} = \frac{|z|\,h'_\mu(|z|)}{h_\mu(|z|)}. \tag{2.27}$$

By using the reverse triangle inequality again we can write that

$$\left|\frac{zf'_\mu(z)}{f_\mu(z)}-1\right| = \left|-\frac{2}{1+\frac{\mu}{2}}\sum_{n\geq 1}\frac{z^2}{l^2_{\mu,n}-z^2}\right| \leq \frac{2}{1+\frac{\mu}{2}}\sum_{n\geq 1}\frac{|z|^2}{l^2_{\mu,n}-|z|^2} = 1-\frac{|z|\,f'_\mu(|z|)}{f_\mu(|z|)}, \tag{2.28}$$

$$\left|\frac{zg'_\mu(z)}{g_\mu(z)}-1\right|=\left|-2\sum_{n\geq1}\frac{z^2}{l^2_{\mu,n}-z^2}\right|\leq 2\sum_{n\geq1}\frac{|z|^2}{l^2_{\mu,n}-|z|^2}=1-\frac{|z|\,g'_\mu(|z|)}{g_\mu(|z|)} \tag{2.29}$$

and

$$\left|\frac{zh'_\mu(z)}{h_\mu(z)}-1\right|=\left|-\sum_{n\geq1}\frac{z}{l^2_{\mu,n}-z}\right|\leq \sum_{n\geq1}\frac{|z|}{l^2_{\mu,n}-|z|}=1-\frac{|z|\,h'_\mu(|z|)}{h_\mu(|z|)}. \tag{2.30}$$

As consequences of the above inequalities, it can be easily obtained that

$$\Re\left(\frac{zf'_\mu(z)}{f_\mu(z)}\right)-k\left|\frac{zf'_\mu(z)}{f_\mu(z)}-1\right|-\alpha\geq(1+k)\frac{|z|\,f'_\mu(|z|)}{f_\mu(|z|)}-(k+\alpha), \tag{2.31}$$

$$\Re\left(\frac{zg'_\mu(z)}{g_\mu(z)}\right)-k\left|\frac{zg'_\mu(z)}{g_\mu(z)}-1\right|-\alpha\geq(1+k)\frac{|z|\,g'_\mu(|z|)}{g_\mu(|z|)}-(k+\alpha) \tag{2.32}$$

and

$$\Re\left(\frac{zh'_\mu(z)}{h_\mu(z)}\right)-k\left|\frac{zh'_\mu(z)}{h_\mu(z)}-1\right|-\alpha\geq(1+k)\frac{|z|\,h'_\mu(|z|)}{h_\mu(|z|)}-(k+\alpha). \tag{2.33}$$

It is worth mentioning that the equalities in the inequalities (2.31), (2.32) and (2.33) hold true for $z=|z|=r$. Also, if we consider the minimum principle for harmonic functions in these inequalities, then we can say that these inequalities are valid if and only if $|z|<r_f$, $|z|<r_g$ and $|z|<r_h$, where r_f, r_g and r_h are the smallest positive roots of the following equations

$$(1+k)\frac{rf'_\mu(r)}{f_\mu(r)}-(k+\alpha)=0,$$

$$(1+k)\frac{rg'_\mu(r)}{g_\mu(r)}-(k+\alpha)=0$$

and

$$(1+k)\frac{rh'_\mu(r)}{h_\mu(r)}-(k+\alpha)=0,$$

respectively. Taking into account the definitions of the functions f_μ, g_μ and h_μ, it can be easily seen that the last three equations are equivalent to (2.18), (2.19) and (2.20), respectively. In addition, we can easily show that equations (2.18) and (2.19) have one root in the interval $(0,l_{\mu,1})$, while equation (2.20) has a root in $(0,l^2_{\mu,1})$. Because the proof of these assertions are similar to the proof of the previous theorem, details are omitted. □

Remark 1. For $k=0$ and $k=\alpha=0$, Theorem 1 and Theorem 2 reduce to some earlier results given by [1–3], respectively.

Now, we would like present some applications regarding our main results. For this, we consider the following relationships between Struve and elementary trigonometric functions:

$$\mathbf{H}_{-\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\sin z \text{ and } \mathbf{H}_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}(1-\cos z).$$

Using these relationships for $\nu=-\frac{1}{2}$ and $\nu=\frac{1}{2}$, we have

$$u_{\frac{1}{2}}(z)=\left(\frac{2(1-\cos z)}{\sqrt{z}}\right)^{\frac{2}{3}},\ v_{\frac{1}{2}}(z)=\frac{2(1-\cos z)}{z},\ w_{\frac{1}{2}}(z)=2(1-\cos\sqrt{z})$$

and

$$u_{-\frac{1}{2}}(z)=\frac{\sin^2 z}{z},\ v_{-\frac{1}{2}}(z)=\sin z,\ w_{-\frac{1}{2}}(z)=\sqrt{z}\sin\sqrt{z}.$$

Corollary 1. *The following statements are true.*

i. *The radius of $k-$starlikeness of order α of the function $u_{\frac{1}{2}}(z)=\left(\frac{2(1-\cos z)}{\sqrt{z}}\right)^{\frac{2}{3}}$ is the smallest positive root of the equation*

$$2(1+k)r\sin r+(1+4k+3\alpha)(\cos r-1)=0$$

in $\left(0,h_{\frac{1}{2},1}\right)$.

ii. *The radius of $k-$starlikeness of order α of the function $v_{\frac{1}{2}}(z)=\frac{2(1-\cos z)}{z}$ is the smallest positive root of the equation*

$$(1+k)r\sin r+(1+2k+\alpha)(\cos r-1)=0$$

in $\left(0,h_{\frac{1}{2},1}\right)$.

iii. *The radius of $k-$starlikeness of order α of the function $w_{\frac{1}{2}}(z)=2(1-\cos\sqrt{z})$ is the smallest positive root of the equation*

$$(1+k)\sqrt{r}\sin\sqrt{r}+2(k+\alpha)(\cos\sqrt{r}-1)=0$$

in $\left(0,h^2_{\frac{1}{2},1}\right)$.

iv. *The radius of $k-$starlikeness of order α of the function $u_{-\frac{1}{2}}(z)=\frac{\sin^2 z}{z}$ is the smallest positive root of the equation*

$$2(1+k)r\cos r-(1+2k+\alpha)\sin r=0$$

in $\left(0,h_{-\frac{1}{2},1}\right)$.

v. *The radius of $k-$starlikeness of order α of the function $v_{-\frac{1}{2}}(z)=\sin z$ is the smallest positive root of the equation*

$$(1+k)r\cos r-(k+\alpha)\sin r=0$$

in $\left(0,h_{-\frac{1}{2},1}\right)$.

vi. *The radius of $k-$starlikeness of order α of the function $w_{-\frac{1}{2}}(z)=\sqrt{z}\sin\sqrt{z}$ is the smallest positive root of the equation*

$$(1+k)\sqrt{r}\cos\sqrt{r}-(k+2\alpha-1)\sin\sqrt{r}=0$$

in $\left(0,h^2_{-\frac{1}{2},1}\right)$.

Now, by taking $k=\alpha=0$ in Corollary 1 we get the following result.

Corollary 2. *The following assertions are true.*

- **i.** *The radius of starlikeness of the function $u_{\frac{1}{2}}(z)=\left(\frac{2(1-\cos z)}{\sqrt{z}}\right)^{\frac{2}{3}}$ is $r\cong 2.7865$ and it is the smallest positive root of the equation $2r\sin r+\cos r-1=0$.*
- **ii.** *The radius of starlikeness of the function $v_{\frac{1}{2}}(z)=\frac{2(1-\cos z)}{z}$ is $r\cong 2.33112$ and it is the smallest positive root of the equation $2r\sin r+2\cos r-1=0$.*
- **iii.** *The radius of starlikeness of the function $w_{\frac{1}{2}}(z)=2(1-\cos\sqrt{z})$ is $r\cong 9.8696$ and it is the smallest positive root of the equation $\sqrt{r}\sin\sqrt{r}=0$.*
- **iv.** *The radius of starlikeness of the function $u_{-\frac{1}{2}}(z)=\frac{\sin^2 z}{z}$ is $r\cong 1.16556$ and it is the smallest positive root of the equation $2r\cos r-\sin r=0$.*
- **v.** *The radius of starlikeness of the function $v_{-\frac{1}{2}}(z)=\sin z$ is $r\cong 1.5708$ and it is the smallest positive root of the equation $r\cos r=0$.*
- **vi.** *The radius of starlikeness of the function $w_{-\frac{1}{2}}(z)=\sqrt{z}\sin\sqrt{z}$ is $r\cong 4.11586$ and it is the smallest positive root of the equation $\sqrt{r}\cos\sqrt{r}+\sin\sqrt{r}=0$.*

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REFERENCES

[1] İ. Aktaş, Á. Baricz, and H. Orhan, "Bounds for radii of starlikeness and convexity of some special functions," *Turkish J. Math.*, vol. 42, no. 1, pp. 211–226, 2018, doi: 10.3906/mat-1610-41.

[2] İ. Aktaş, Á. Baricz, and N. Yağmur, "Bounds for the radii of univalence of some special functions," *Math. Inequal. Appl.*, vol. 20, no. 3, pp. 825–843, 2017, doi: 10.7153/mia-20-52.

[3] Á. Baricz, D. Dimitrov, H. Orhan, and N. Yağmur, "Radii of starlikeness of some special functions," *Proc. Amer. Math. Soc.*, vol. 144, no. 8, pp. 3355–3367, 2016, doi: 10.1090/proc/13120.

[4] Á. Baricz and N. Yağmur, "Geometric properties of some Lommel and Struve functions," *Ramanujan J.*, vol. 42, no. 2, pp. 325–346, 2017, doi: 10.1007/s11139-015-9724-6.

[5] N. Bohra and V. Ravichandran, "Radii problems for normalized Bessel functions of first kind," *Comput. Methods Funct. Theory*, vol. 18, no. 1, pp. 99–123, 2018, doi: 10.1007/s40315-017-0216-0.

[6] R. Brown, "Univalence of Bessel functions," *Proc. Amer. Math. Soc.*, vol. 11, no. 2, pp. 278–283, 1960.

[7] A. Gangadharan, T. N. Shanmugam, and H. M. Srivastava, "Generalized hypergeometric functions associated with $k-$uniformly convex functions," *Comput. Math. Appl.*, vol. 44, no. 12, pp. 1515–1526, 2002, doi: 10.1016/S0898-1221(02)00275-4.

[8] A. W. Goodman, "On uniformly convex functions," in *Ann. Polon. Math.*, vol. 56, no. 1, 1991, pp. 87–92.

[9] S. Kanas and H. M. Srivastava, "Linear operators associated with $k-$uniformly convex functions," *Integral Transforms Spec. Funct.*, vol. 9, no. 2, pp. 121–132, 2000, doi: 10.1080/10652460008819249.

[10] S. Kanas and A. Wisniowska, "Conic regions and $k-$uniform convexity," *J. Comput. Appl. Math.*, vol. 105, no. 1-2, pp. 327–336, 1999, doi: 10.1016/S0377-0427(99)00018-7.

[11] S. Kanas and A. Wisniowska, "Conic domains and starlike functions," *Rev. Roumaine Math. Pures Appl.*, vol. 45, no. 4, pp. 647–658, 2000.

[12] E. Kreyszig and J. Todd, "The radius of univalence of Bessel functions I," *Illinois J. Math.*, vol. 4, no. 1, pp. 143–149, 1960, doi: 10.1215/ijm/1255455740.

[13] H. M. Srivastava, "Operators of basic (or $q-$) calculus and fractional $q-$calculus and their applications in geometric function theory of complex analysis," *Iran. J. Sci. Technol. Trans. A Sci.*, vol. 44, no. 1, pp. 327–344, 2020, doi: 10.1007/s40995-019-00815-0.

[14] H. M. Srivastava and D. Bansal, "Close-to-convexity of a certain family of $q-$Mittag-Leffler functions," *J. Nonlinear Var. Anal.*, vol. 1, no. 1, pp. 61–69, 2017.

[15] H. M. Srivastava, G. Murugusundaramoorthy, and T. Janani, "Uniformly starlike functions and uniformly convex functions associated with the Struve function," *J. Appl. Computat. Math.*, vol. 3, no. 6, pp. 1–4, 2014, doi: 10.4172/2168-9679.1000180.

[16] H. M. Srivastava, G. Murugusundaramoorthy, and S. Sivasubramanian, "Hypergeometric functions in the parabolic starlike and uniformly convex domains," *Integral Transforms Spec. Funct.*, vol. 18, no. 7, pp. 511–520, 2007, doi: 10.1080/10652460701391324.

[17] J. Steinig, "The real zeros of Struve's function," *SIAM J. Math. Anal.*, vol. 1, no. 3, pp. 365–375, 1970, doi: 10.1137/0501034.

[18] J. Steinig, "The sign of Lommel's function," *Trans. Amer. Math. Soc.*, vol. 163, pp. 123–129, 1972, doi: 10.2307/1995711.

[19] R. Szász, "On starlikeness of Bessel functions of the first kind," in *Proceedings of the 8th Joint Conference on Mathematics and Computer Science, Komárno, Slovakia*, vol. 9, 2010.

[20] G. N. Watson, *A treatise on the theory of Bessel functions*. Cambridge University Press, 1922.

[21] H. S. Wilf, "The radius of univalence of certain entire functions," *Illinois J. Math.*, vol. 6, no. 2, pp. 242–244, 1962, doi: 10.1215/ijm/1255632321.

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DILATIONS, MODELS AND SPECTRAL PROBLEMS OF NON-SELF-ADJOINT SRURM-LIUVILLE OPERATORS

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Abstract. In this study, we investigate the maximal dissipative singular Sturm-Liouville operators acting in the Hilbert space $L_r^2(a,b)$ $(-\infty \leq a < b \leq \infty)$, that the extensions of a minimal symmetric operator with defect index $(2,2)$ (in limit-circle case at singular end points a and b). We examine two classes of dissipative operators with separated boundary conditions and we establish, for each case, a self-adjoint dilation of the dissipative operator as well as its incoming and outgoing spectral representations, which enables us to define the scattering matrix of the dilation. Moreover, we construct a functional model of the dissipative operator and identify its characteristic function in terms of the Weyl function of a self-adjoint operator. We present several theorems on completeness of the system of root functions of the dissipative operators and verify them.

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1. INTRODUCTION

Dissipative operators are one of the important classes of non-self-adjoint operators. It is well recognized ([1–3, 9, 13–16]), that the theory of dilations with application of functional models gives an ample approach to the spectral theory of dissipative (contractive) operators. By carrying the complete information on the spectral properties of the dissipative operator, we can say that characteristic function plays the primary role in this theory. Hence, in the incoming spectral representation of the dilation, the dissipative operator becomes the model. Completeness problem of the system of eigenvectors and associated (or root) vectors is solved through the factorization of the characteristic function. The computation of the characteristic functions of dissipative operators is preceded by the construction and investigation of the self-adjoint dilation and the corresponding scattering problem, in which the characteristic function is considered as the scattering matrix. According to the Lax-Phillips scattering theory

[10], the unitary group $\{U(s)\}$ $(s \in \mathbb{R} := (-\infty, \infty))$ has typical properties in the subspaces D^- and D^+ of the Hilbert space H, which are called respectively the incoming and outgoing subspaces. One can find the adequacy of this approach to dissipative Schrödinger and Sturm-Liouville operators, for example, in [1–3, 9, 13–15].

In this paper, we take the minimal symmetric singular Sturm-Liouville operator acting in the Hilbert space $\mathcal{L}_r^2(a,b)$ $(-\infty \leq a < b \leq \infty)$ with maximal defect index $(2,2)$ (in Weyl's limit-circle cases at singular end points a and b) into consideration. We define all maximal dissipative, maximal accumulative and self-adjoint extensions of such a symmetric operator using the boundary conditions at a and b. We investigate two classes of non-self-adjoint operators with separated boundary conditions, called 'dissipative at a' and 'dissipative at b'. In each of these two cases, we construct a self-adjoint dilation of the maximal dissipative operator together with its incoming and outgoing spectral representations so that we can determine the scattering matrix (function) of the dilation as stated in the scheme of Lax and Phillips [10]. Then, we create a functional model of the maximal dissipative operator via the incoming spectral representation and define its characteristic function in terms of the Weyl function (or scattering matrix of the dilation) of a self-adjoint operator. Finally, using the results found for characteristic functions, we prove the theorems on completeness of the system of eigenfunctions and associated functions (or root functions) of the maximal dissipative Sturm-Liouville operators. Results of the present paper are new even in the case $p = r = 1$ (in the case of the one-dimensional Schrödinger operator).

2. Extensions of a symmetric operator and self-adjoint dilations of the dissipative operators

We address the following Sturm-Liouville differential expression with two singular end points a and b:

$$\tau(x) := \frac{1}{r(t)}[-(p(t)x'(t))' + q(t)x(t)] \ (t \in J := (a,b), \ -\infty \leq a < b \leq +\infty), \quad (2.1)$$

where p, q and r are real-valued, Lebesgue measurable functions on J, and $p^{-1}, q, r \in \mathcal{L}_{loc}^1(J)$, $p \neq 0$ and $r > 0$ almost everywhere on J.

In order to pass from the differential expression to operators, we shall take the Hilbert space $\mathcal{L}_r^2(J)$ consisting of all complex-valued functions f satisfying $\int_a^b r(t)\,|f(t)|^2\,dt < \infty$, with the inner product $(f,g) = \int_a^b r(t) f(t)\overline{g(t)}dt$.

Let $\mathcal{D}_{\max}$ represent the linear set of all functions $f \in \mathcal{L}_r^2(J)$ such that f and pf' are locally absolutely continuous functions on J, and $\tau(f) \in \mathcal{L}_r^2(J)$. Let us define the *maximal operator* $T_{\max}$ on $\mathcal{D}_{\max}$ as $T_{\max} f = \tau(f)$.

For any two functions $f, g \in \mathcal{D}_{\max}$, Green's formula is given by

$$(T_{\max} f, g) - (f, T_{\max} g) = [f,g](b) - [f,g](a), \quad (2.2)$$

where

$$[f,g](t) := W_t(f,\overline{g}) := f(t)\,(p\overline{g}')\,(t) - (pf')\,(t)\,\overline{g}\,(t) \ \ (t \in J),$$

$$[f,g](a):=\lim_{t\to a^+}[f,g](t),\ [f,g](b):=\lim_{t\to b^-}[f,g](t).$$

In $\mathcal{L}_r^2(J)$, we consider the dense linear set $\mathcal{D}_{\min}$ consisting of smooth, compactly supported functions on J. Let us indicate the restriction of the operator $T_{\max}$ to $\mathcal{D}_{\min}$ by $T_{\min}$. We can conclude from (2.2) that $T_{\min}$ is symmetric. Thus, it admits closure denoted by $T_{\min}$. The *minimal operator* $T_{\min}$ is a symmetric operator with defect index $(0,0)$, $(1,1)$ or $(2,2)$, and $T_{\max}=T_{\min}^*$ ([4, 5, 12, 18, 19]). Note that the operator $T_{\min}$ is self-adjoint for defect index $(0,0)$, that is, $T_{\min}^*=T_{\min}=T_{\max}$.

Moreover, we assume that $T_{\min}$ has defect index $(2,2)$. Under this assumption, Weyl's limit-circle cases are obtained for the differential expression τ at a and b (see [4–6, 8, 11, 12, 17–19]). The domain of the operator $T_{\min}$ consists of precisely the functions $f\in\mathcal{D}_{\max}$, which satisfy the following condition

$$[f,g](b)-[f,g](a)=0,\ \forall g\in\mathcal{D}_{\max}. \tag{2.3}$$

Let $T_{\min}^-$ and $T_{\min}^+$ denote respectively the minimal symmetric operators generated by the expression τ on the intervals $(a,c]$ and $[c,b)$ for some $c\in J$, and $\mathcal{D}_{\min}^{\mp}$ represents the domain of $T_{\min}^{\mp}$. It is known ([5, 12, 18]), that the defect number $defT_{\min}$ of $T_{\min}$ can be computed using the formula $defT_{\min}=defT_{\min}^+ + defT_{\min}^- - 2$. Thus, we obtain that $defT_{\min}^+ + defT_{\min}^- = 4$, $defT_{\min}^+=2$ and $defT_{\min}^-=2$.

We denote by $\theta(t)$ and $\chi(t)$ the solutions of the equation

$$\tau(y)=0\ (t\in J) \tag{2.4}$$

satisfying the conditions

$$\theta(c)=1,\ (p\theta')(c)=0,\ \chi(c)=0,\ (p\chi')(c)=1,\ c\in J. \tag{2.5}$$

The Wronskian of the two solutions of (2.4) does not depend on t, and the two solutions of this equation are linearly independent if and only if their Wronskian is nonzero. Conditions (2.5) and the constancy of the Wronskian imply that

$$W_t(\theta,\chi)=W_c(\theta,\chi)=1\ (a\le t\le b). \tag{2.6}$$

Hence, θ and χ form a fundamental set of solutions of (2.4). Since $T_{\min}$ has defect index $(2,2)$, we have $\theta,\chi\in\mathcal{L}_r^2(J)$, and $\theta,\chi\in\mathcal{D}_{\max}$ as well.

The following equality holds for arbitrary functions $f,g\in\mathcal{D}_{\max}$ ([2])

$$[f,g](t)=[f,\theta](t)[\overline{g},\chi](t)-[f,\chi](t)[\overline{g},\theta](t)\ (a\le t\le b). \tag{2.7}$$

The domain $\mathcal{D}_{\min}$ of the operator $T_{\min}$ is composed of precisely the functions $f\in\mathcal{D}_{\max}$ satisfying the boundary conditions given as follows ([1])

$$[f,\theta](a)=[f,\chi](a)=[f,\theta](b)=[f,\chi](b)=0. \tag{2.8}$$

Recall that a linear operator $\mathbf{A}$ (with dense domain $\mathcal{D}(\mathbf{A})$) acting on some Hilbert space $\mathbf{H}$ is called *dissipative* (*accumulative*) if $\Im(\mathbf{A}y,y)\ge 0$ ($\Im(\mathbf{A}y,y)\le 0$) for all $y\in\mathcal{D}(\mathbf{A})$ and *maximal dissipative* (*maximal accumulative*) if it does not have a proper dissipative (accumulative) extension ([7], p.149).

Now, consider the linear maps of $\mathcal{D}_{\max}$ into $\mathbb{C}^2$ given by

$$\Psi_1 f = \begin{pmatrix} [f,\chi](a) \\ [f,\theta](b) \end{pmatrix}, \ \Psi_2 f = \begin{pmatrix} [f,\theta](a) \\ [f,\chi](b) \end{pmatrix}. \tag{2.9}$$

Then we get the following statement ([1]).

Theorem 1. *For any contraction $S \in \mathbb{C}^2$ the restriction of the operator $T_{\max}$ to the set of vectors $f \in \mathcal{D}_{\max}$ satisfying the boundary condition*

$$(S-I)\Psi_1 f + i(S+I)\Psi_2 f = 0 \tag{2.10}$$

or

$$(S-I)\Psi_1 f - i(S+I)\Psi_2 f = 0 \tag{2.11}$$

is, respectively, a maximal dissipative or a maximal accumulative extension of the operator $T_{\min}$. Conversely, every maximally dissipative (accumulative) extension of $T_{\min}$ is the restriction of $T_{\max}$ to the set consisting of vectors $f \in \mathcal{D}_{\max}$ satisfying (2.10) ((2.11)), and the contraction S is uniquely determined by the extension. These conditions describe a self-adjoint extension if and only if S is unitary. In the latter case, (2.10) and (2.11) are equivalent to the condition $(\cos B)\Psi_1 f - (\sin B)\Psi_2 f = 0$, where B is a self-adjoint operator (Hermitian matrix) in $\mathbb{C}^2$. The general forms of dissipative and accumulative extensions of the operator $T_{\min}$ are respectively given by the conditions

$$S(\Psi_1 f + i\Psi_2 f) = \Psi_1 f - i\Psi_2 f, \ \Psi_1 f + i\Psi_2 f \in \mathcal{D}(S), \tag{2.12}$$

$$S(\Psi_1 f - i\Psi_2 f) = \Psi_1 f + i\Psi_2 f, \ \Psi_1 f - i\Psi_2 f \in \mathcal{D}(S), \tag{2.13}$$

where S is a linear operator with $\|Sf\| \le \|f\|, f \in \mathcal{D}(S)$. For an isometric operator S in (2.12) and (2.13) we have the general forms of symmetric extensions.

Particularly, the boundary conditions ($f \in \mathcal{D}_{\max}$)

$$[f,\chi](a) - \alpha_1[f,\theta](a) = 0, \tag{2.14}$$

$$[f,\theta](b) - \alpha_2[f,\chi](b) = 0 \tag{2.15}$$

with $\Im\alpha_1 \ge 0$ or $\alpha_1 = \infty$, and $\Im\alpha_2 \ge 0$ or $\alpha_2 = \infty$ $\Im\alpha_1 \le 0$ or $\alpha_1 = \infty$, and $\Im\alpha_2 \le 0$ or $\alpha_2 = \infty$) characterize all maximal dissipative (maximal accumulative) extensions of $T_{\min}$ with separated boundary conditions. If $\Im\alpha_1 = 0$ or $\alpha_1 = \infty$, and $\Im\alpha_2 = 0$ or $\alpha_2 = \infty$ hold true, then self-adjoint extensions of $T_{\min}$ are obtained. Here for $\alpha_1 = \infty$ ($\alpha_2 = \infty$), condition (2.14) ((2.15)) should be replaced by $[f,\theta](a) = 0$ ($[f,\chi](b) = 0$).

Next, we shall consider the maximal dissipative operators $T^{\mp}_{\alpha_1\alpha_2}$ generated by (2.1) and the boundary conditions given by (2.14) and (2.15) of two different types: 'dissipative at a', i.e., either $\Im\alpha_1 > 0$ and $\Im\alpha_2 = 0$ or $\alpha_2 = \infty$; and 'dissipative at b', i.e., $\Im\alpha_1 = 0$ or $\alpha_1 = \infty$ and $\Im\alpha_2 > 0$.

In order to establish a self-adjoint dilation of the maximal dissipative operator $T^{-}_{\alpha_1\alpha_2}$ for the case 'dissipative at a' (i.e., $\Im\alpha_1 > 0$ and $\Im\alpha_2 = 0$ or $\alpha_2 = \infty$), we associate with $\mathcal{H} := \mathcal{L}^2_r(J)$ the 'incoming' and 'outgoing' channels $\mathcal{L}^2\ (\mathbb{R}_-)$ $(\mathbb{R}_- := (-\infty, 0])$ and $\mathcal{L}^2(\mathbb{R}_+)$ $(\mathbb{R}_+ := [0,\infty))$, we form the orthogonal sum $\mathfrak{H} := \mathcal{L}^2(\mathbb{R}_-) \oplus \mathcal{H} \oplus \mathcal{L}^2(\mathbb{R}_+)$. Let us call the space $\mathfrak{H}$ as the *main Hilbert space of the dilation* and consider in this space the operator $\mathfrak{T}^{-}_{\alpha_1\alpha_2}$ generated by the expression

$$\mathfrak{T}\langle u_-, y, u_+\rangle = \langle i\frac{du_-}{d\xi}, \tau(y), i\frac{du_+}{d\varsigma}\rangle \tag{2.16}$$

on the set $\mathcal{D}(\mathfrak{T}^{-}_{\alpha_1\alpha_2})$ consisting of vectors $\langle u_-, y, u_+\rangle$, where $u_- \in \mathcal{W}^1_2(\mathbb{R}_-)$, $u_+ \in \mathcal{W}^1_2(\mathbb{R}_+)$, $y \in \mathcal{D}_{\max}$ and

$$[y,\chi](a) - \alpha_1[y,\theta](a) = \gamma u_-(0),\ [y,\chi](a) - \overline{\alpha}_1[y,\theta](a) = \gamma u_+(0),$$
$$[y,\theta](b) - \alpha_2[y,\chi](b) = 0. \tag{2.17}$$

Here $\mathcal{W}^1_2(\mathbb{R}_\mp)$ denotes the Sobolev space, and $\gamma^2 := 2\Im\alpha_1,\ \gamma > 0$. Then we obtain the next assertion.

Theorem 2. *The operator $\mathfrak{T}^{-}_{\alpha_1\alpha_2}$ is self-adjoint in the space $\mathfrak{H}$and it is a self-adjoint dilation of the maximal dissipative operator $T^{-}_{\alpha_1\alpha_2}$.*

Proof. We assume that $Y, Z \in \mathcal{D}(\mathfrak{T}^{-}_{\alpha_1\alpha_2})$, $Y = \langle u_-, y, u_+\rangle$ and $Z = \langle v_-, z, v_+\rangle$. If we use integration by parts and (2.16), we find that

$$\begin{aligned}(\mathfrak{T}^{-}_{\alpha_1\alpha_2}Y, Z)_{\mathfrak{H}} &= \int_{-\infty}^{0} iu_-'\overline{v}_- d\xi + (T_{\max}y, z)_{\mathcal{H}} + \int_0^\infty iu_+'\overline{v}_+ d\varsigma \\ &= iu_-(0)\overline{v_-(0)} - iu_+(0)\overline{v_+(0)} + [y,z](b) - [y,z](a) + (Y, \mathfrak{T}^{-}_{\alpha_1\alpha_2}Z)_{\mathfrak{H}}.\end{aligned} \tag{2.18}$$

Moreover, if the boundary conditions (2.17) for the components of the vectors Y, Z and (2.7) are used, it can be seen easily that $iu_-(0)\overline{v_-(0)} - iu_+(0)\overline{v_+(0)} + [y,z](b) - [y,z](a) = 0$. Hence, we conclude that $\mathfrak{T}^{-}_{\alpha_1\alpha_2}$ is symmetric. Thus, in order to prove that $\mathfrak{T}^{-}_{\alpha_1\alpha_2}$ is self-adjoint, it is sufficient to show that $(\mathfrak{T}^{-}_{\alpha_1\alpha_2})^* \subseteq \mathfrak{T}^{-}_{\alpha_1\alpha_2}$ Take $Z = \langle v_-, z, v_+\rangle \in \mathcal{D}((\mathfrak{T}^{-}_{\alpha_1\alpha_2})^*)$. Let $(\mathfrak{T}^{-}_{\alpha_1\alpha_2})^*Z = Z^* = \langle v_-^*, z^*, v_+^*\rangle \in \mathfrak{H}$, so that

$$(\mathfrak{T}^{-}_{\alpha_1\alpha_2}Y, Z)_{\mathfrak{H}} = (Y, Z^*)_{\mathfrak{H}},\ \forall Y \in \mathcal{D}(\mathfrak{T}^{-}_{\alpha_1\alpha_2}). \tag{2.19}$$

If we choose suitable components for $Y \in \mathcal{D}(\mathfrak{T}^{-}_{\alpha_1\alpha_2})$ in (2.19), it can be shown easily that $v_- \in \mathcal{W}^1_2(\mathbb{R}_-)$, $v_+ \in \mathcal{W}^1_2(\mathbb{R}_+)$, $z \in \mathcal{D}_{\max}$ and $Z^* = \mathfrak{T}Z$, where $\mathfrak{T}$ is given by (2.16). Therefore, (2.19) takes the following form $(\mathfrak{T}Y, Z)_{\mathfrak{H}} = (Y, \mathfrak{T}Z)_{\mathfrak{H}},\ \forall Y \in \mathcal{D}(\mathfrak{T}^{-}_{\alpha_1\alpha_2})$. Hence, the sum of the integrated terms in the bilinear form $(\mathfrak{T}Y, Z)_{\mathfrak{H}}$ must be zero:

$$iu_-(0)\overline{v_-(0)} - iu_+(0)\overline{v_+(0)} + [y,z](b) - [y,z](a) = 0 \tag{2.20}$$

for all $Y = \langle u_-, y, u_+\rangle \in \mathcal{D}(\mathfrak{T}^-_{\alpha_1\alpha_2})$. Additionally, after the boundary conditions (2.17) for $[y,\theta](a)$ and $[y,\chi](a)$ are solved, it is found that

$$[y,\theta](a) = -\frac{i}{\gamma}(u_+(0) - u_-(0)),\ [y,\chi](a) = \gamma u_-(0) - \frac{i\alpha_1}{\gamma}(u_+(0) - u_-(0)). \quad (2.21)$$

Therefore, (2.7) and (2.21) imply that (2.20) is equivalent to the equality given as follows

$$\begin{aligned} &iu_-(0)\overline{v_-(0)} - iu_+(0)\overline{v_+(0)} = [y,z](a) - [y,z](b) \\ &= -\frac{i}{\gamma}(u_+(0) - u_-(0))\,[\bar z,\chi]\,(a) - \gamma[u_-(0) - \frac{i\alpha_1}{\gamma^2}(u_+(0) - u_-(0))]\,[\bar z,\theta]\,(a) \\ &\quad - [y,\theta](b)\,[\bar z,\chi]\,(b) + [y,\chi](b)\,[\bar z,\theta]\,(b) \\ &= -\frac{i}{\gamma}(u_+(0) - u_-(0))\,[\bar z,\chi]\,(a) - \gamma[u_-(0) - \frac{i\alpha_1}{\gamma^2}(u_+(0) - u_-(0))]\,[\bar z,\theta]\,(a) \\ &\quad + ([\bar z,\theta]\,(b) - \alpha_2\,[\bar z,\chi]\,(b))[y,\chi](b). \end{aligned}$$

Note that $u_\pm(0)$ can be arbitrary complex numbers. If we compare the coefficients of $u_\pm(0)$ on the left and right sides of the last equality, we see that the vector $Z = \langle v_-, z, v_+\rangle$ satisfies the boundary conditions $[z,\chi](a) - \alpha_1[z,\theta](a) = \gamma v_-(0)$, $[z,\chi](a) - \overline{\alpha}_1[z,\theta](a) = \gamma v_+(0)$, $[z,\theta](b) - \alpha_2[z,\chi](b) = 0$. Consequently, the inclusion $(\mathfrak{T}^-_{\alpha_1\alpha_2})^* \subseteq \mathfrak{T}^-_{\alpha_1\alpha_2}$ is fulfilled. This proves that $\mathfrak{T}^-_{\alpha_1\alpha_2} = (\mathfrak{T}^-_{\alpha_1\alpha_2})^*$.

In the space $\mathfrak{H}$, self-adjoint operator $\mathfrak{T}^-_{\alpha_1\alpha_2}$ generates a unitary group $\mathfrak{U}^-(s) := \exp[i\mathfrak{T}^-_{\alpha_1\alpha_2}s]$ $(s \in \mathbb{R})$. Denote by $\mathcal{P} : \mathfrak{H} \to \mathcal{H}$ and $\mathcal{P}_1 : \mathcal{H} \to \mathfrak{H}$ the mappings acting in keeping with the formulas $\mathcal{P} : \langle u_-, y, u_+\rangle \to y$ and $\mathcal{P}_1 : y \to \langle 0, y, 0\rangle$. Set $\mathcal{V}(s) = \mathcal{P}\mathfrak{U}^-(s)\mathcal{P}_1$ $(s \geq 0)$. The family $\{\mathcal{V}(s)\}$ $(s \geq 0)$ of operators is a strongly continuous semigroup of completely non-unitary contractions on $\mathcal{H}$. Let A represent the generator of this semigroup, i.e, $Az = \lim_{s\to+0}[(is)^{-1}(\mathcal{V}(s)z - z)]$. All vectors for which this limit exists belong to the domain of A. The operator A is maximal dissipative and the operator $\mathfrak{T}^-_{\alpha_1\alpha_2}$ is called the *self-adjoint dilation* of A ([13–15]). We aim to show that $A = T^-_{\alpha_1\alpha_2}$, which implies in turn that $\mathfrak{T}^-_{\alpha_1\alpha_2}$ is a self-adjoint dilation of $T^-_{\alpha_1\alpha_2}$. To achieve this goal, we first verify the following equality ([13–15])

$$\mathcal{P}(\mathfrak{T}^-_{\alpha_1\alpha_2} - \lambda I)^{-1}\mathcal{P}_1 y = (T^-_{\alpha_1\alpha_2} - \lambda I)^{-1}y,\ y \in \mathcal{H},\ \Im\lambda < 0. \quad (2.22)$$

Let $(\mathfrak{T}^-_{\alpha_1\alpha_2} - \lambda I)^{-1}\mathcal{P}_1 y = Z = \langle v_-, z, v_+\rangle$. Then $(\mathfrak{T}^-_{\alpha_1\alpha_2} - \lambda I)Z = \mathcal{P}_1 y$, and so, $T_{\max}z - \lambda z = y$, $v_-(\xi) = v_-(0)e^{-i\lambda\xi}$ and $v_+(\varsigma) = v_+(0)e^{-i\lambda\varsigma}$. Since $Z \in \mathcal{D}(\mathfrak{T}^-_{\alpha_1\alpha_2})$ and hence, $v_- \in \mathcal{L}^2(\mathbb{R}_-)$; we have $v_-(0) = 0$, and consequently, z satisfies the boundary conditions $[z,\chi](a) - \alpha_1[z,\theta](a) = 0$, $[z,\theta](b) - \alpha_2[z,\chi](b) = 0$. Therefore, $z \in \mathcal{D}(T^-_{\alpha_1\alpha_2})$, and since a dissipative operator cannot have an eigenvalue λ with $\Im\lambda < 0$, we conclude that $z = (T^-_{\alpha_1\alpha_2} - \lambda I)^{-1}y$. Here, we evaluate $v_+(0)$ using the formula $v_+(0) = \gamma^{-1}([z,\chi](a) - \overline{\alpha}_1[z,\theta](a))$. Then

$$(\mathfrak{T}^-_{\alpha_1\alpha_2} - \lambda I)^{-1}\mathcal{P}_1 y = \left\langle 0, (T^-_{\alpha_1\alpha_2} - \lambda I)^{-1}y, \gamma^{-1}([z,\chi](a) - \overline{\alpha}_1[z,\theta](a))\,e^{-i\lambda\varsigma}\right\rangle$$

for $y \in \mathcal{H}$ and $\Im\lambda < 0$. Applying $\mathcal{P}$, we get the desired equality (2.22).

Now, it is not difficult to show that $A = T^{-}_{\alpha_1\alpha_2}$. In fact, it follows from (2.22) that

$$\begin{aligned}(T^{-}_{\alpha_1\alpha_2} - \lambda I)^{-1} &= \mathcal{P}(\mathfrak{T}^{-}_{\alpha_1\alpha_2} - \lambda I)^{-1}\mathcal{P}_1 = -i\mathcal{P}\int_0^\infty \mathfrak{U}^{-}(s)e^{-i\lambda s}ds\mathcal{P}_1 \\ &= -i\int_0^\infty \mathcal{V}(s)e^{-i\lambda s}ds = (A-\lambda I)^{-1},\ \Im\lambda < 0,\end{aligned}$$

and thus we have $T^{-}_{\alpha_1\alpha_2} = A$ proving Theorem 2. $\square$

In order to construct a self-adjoint dilation of the maximal dissipative operator $T^{+}_{\alpha_1\alpha_2}$ in the case 'dissipative at b' (i.e., $\Im\alpha_1 = 0$ or $\alpha_1 = 0$ and $\Im\alpha_2 > 0$) in $\mathfrak{H}$, we consider the operator $\mathfrak{T}^{+}_{\alpha_1\alpha_2}$ generated by the expression (2.16) on the set $\mathcal{D}(\mathfrak{T}^{+}_{\alpha_1\alpha_2})$ of vectors $\langle u_-, y, u_+\rangle$ satisfying the conditions: $u_- \in \mathcal{W}_2^1(\mathbb{R}_-)$, $u_+ \in \mathcal{W}_2^1(\mathbb{R}_+)$, $y \in \mathcal{D}_{\max}$ and

$$\begin{aligned}[y,\chi](a) - \alpha_1[y,\theta](a) = 0,\ \ [y,\theta](b) - \alpha_2[y,\chi](b) = \beta u_-(0), \\ [y,\theta](b) - \overline{\alpha}_2[y,\chi](b) = \beta u_+(0),\end{aligned} \tag{2.23}$$

where $\beta^2 := 2\Im\alpha_2$, $\beta > 0$.

Since the proof of the next theorem is similar to that of Theorem 2, we omit it here.

Theorem 3. *The operator $\mathfrak{T}^{+}_{\alpha_1\alpha_2}$ is self-adjoint in $\mathfrak{H}$ and it is a self-adjoint dilation on the maximal dissipative operator $T^{+}_{\alpha_1\alpha_2}$.*

3. Scattering theory of the dilations, functional models and completeness of root functions of the dissipative operators

The unitary group $\mathfrak{U}^{\pm}(s) = \exp[i\mathfrak{T}^{\pm}_{\alpha_1\alpha_2}s]$ $(s \in \mathbb{R})$ possesses a crucial feature through which we can apply to it the Lax-Phillips scheme ([10]). Namely, it has incoming and outgoing subspaces $\mathfrak{D}^- := \langle \mathcal{L}^2(\mathbb{R}_-), 0, 0\rangle$ and $\mathfrak{D}^+ := \langle 0, 0, \mathcal{L}^2(\mathbb{R}_+)\rangle$ satisfying the following properties:

(1) $\mathfrak{U}^{\pm}(s)\mathfrak{D}^- \subset \mathfrak{D}^-$, $s \le 0$ and $\mathfrak{U}^{\pm}(s)\mathfrak{D}^+ \subset \mathfrak{D}^+$, $s \ge 0$;

(2) $\bigcap\limits_{s\le 0}\mathfrak{U}^{\pm}(s)\mathfrak{D}^- = \bigcap\limits_{s\ge 0}\mathfrak{U}^{\pm}(s)\mathfrak{D}^+ = \{0\}$;

(3) $\overline{\bigcup\limits_{s\ge 0}\mathfrak{U}^{\pm}(s)\mathfrak{D}^-} = \overline{\bigcup\limits_{s\le 0}\mathfrak{U}^{\pm}(s)\mathfrak{D}^+} = \mathfrak{H}$;

(4) $\mathfrak{D}^- \perp \mathfrak{D}^+$.

It is evident that property **(4)** holds true. Let us prove property **(1)** for $\mathfrak{D}^+$ (the proof for $\mathfrak{D}^-$ is similar). For this end, we define $\mathcal{R}^{\pm}_\lambda = (\mathfrak{T}^{\pm}_{\alpha_1\alpha_2} - \lambda I)^{-1}$ for all λ with $\Im\lambda < 0$. Then, for any $Y = \langle 0, 0, u_+\rangle \in \mathfrak{D}^+$, we get

$$\mathcal{R}^{\pm}_\lambda Y = \langle 0, 0, -ie^{-i\lambda\varsigma}\int_0^\varsigma e^{-i\lambda\xi}u_+(\xi)d\xi\rangle.$$

Therefore, we see that $\mathcal{R}_\lambda Y \in \mathfrak{D}^+$. Further, if $Z \perp \mathfrak{D}^+$, then

$$0 = \left(\mathcal{R}^{\pm}_\lambda Y, Z\right)_{\mathfrak{H}} = -i\int_0^\infty e^{-i\lambda s}\left(\mathfrak{U}^{\pm}(s)Y, Z\right)_{\mathfrak{H}} ds,\ \Im\lambda < 0,$$

which implies that $(\mathfrak{U}^{\pm}(s)Y,Z)_{\mathfrak{H}}=0$ for all $s\geq 0$. So, we obtain $\mathfrak{U}^{\pm}(s)\mathfrak{D}^{+}\subset\mathfrak{D}^{+}$ for $s\geq 0$, proving property (**1**).

To prove property (**2**) for $\mathfrak{D}^{+}$ (the proof for $\mathfrak{D}^{-}$ is similar), we denote by $P^{+}:\mathfrak{H}\to\mathcal{L}^2(\mathbb{R}_+)$ and $P_1^{+}:\mathcal{L}^2(\mathbb{R}_+)\to\mathfrak{D}^{+}$ the mappings acting according to the formulae $P^{+}:\langle u_-,y,u_+\rangle\to u_+$ and $P_1^{+}:u\to\langle 0,0,u\rangle$, respectively. The semigroup of isometries $\mathcal{X}(s)=P^{+}\mathfrak{U}^{-}(s)P_1^{+}$, $s\geq 0$ is a one-sided shift in $\mathcal{L}^2(\mathbb{R}_+)$. In fact, the generator of the semigroup of the one-sided shift $\mathcal{Y}(s)$ in $\mathcal{L}^2(\mathbb{R}_+)$ is the differential operator $i\frac{d}{d\xi}$ satisfying the boundary condition $u(0)=0$. On the other hand, the generator $\mathcal{B}$ of the semigroup of isometries $\mathcal{X}(s)$, $s\geq 0$, is the operator defined by

$$\mathcal{B}u=P^{+}\mathfrak{T}_{\alpha_1\alpha_2}^{-}P_1^{+}Y=P^{+}\mathfrak{T}_{\alpha_1\alpha_2}^{-}\langle 0,0,u\rangle=P^{+}\langle 0,0,i\frac{du}{d\xi}\rangle=i\frac{du}{d\xi},$$

where $u\in\mathcal{W}_2^1(\mathbb{R}_+)$ and $u(0)=0$. However, since a semigroup is uniquely determined by its generator, we have $\mathcal{X}(s)=\mathcal{Y}(s)$, and thus,

$$\bigcap_{s\geq 0}\mathfrak{U}^{-}(s)\mathfrak{D}^{+}=\langle 0,0,\bigcap_{s\geq 0}\mathcal{Y}(s)\mathcal{L}^2(\mathbb{R}_+)\rangle=\{0\},$$

(the proof for $\mathfrak{U}^{+}(s)$ is similar) verifying that property (**2**) is valid.

As stated in the scheme of the Lax-Phillips scattering theory, the scattering matrix is defined using the spectral representations theory. Now, we shall continue with their construction. During this process, we shall also have proved property (**3**) of the incoming and outgoing subspaces.

Recall that the linear operator **A** (with domain $\mathcal{D}(\mathbf{A})$) acting in the Hilbert space **H** is called *completely non-self-adjoint* (or *pure*) if invariant subspace $\mathbf{M}\subseteq\mathcal{D}(\mathbf{A})$ ($\mathbf{M}\neq\{0\}$) of the operator **A** whose restriction on **M** is self-adjoint, does not exist.

Lemma 1. *The operator $T_{\alpha_1\alpha_2}^{\pm}$ is completely non-self-adjoint (pure).*

Proof. Let $\mathcal{H}'\subset\mathcal{H}$ be a non-trivial subspace in which the operator $T_{\alpha_1\alpha_2}^{-}$ (the proof for $T_{\alpha_1\alpha_2}^{+}$ is similar) induces a self-adjoint operator T' with domain $\mathcal{D}(T')=\mathcal{H}'\cap\mathcal{D}(T_{\alpha_1\alpha_2}^{-})$. If $z\in\mathcal{D}(T')$, then we have $z\in\mathcal{D}(T'^{*})$ and $[z,\chi](a)-\alpha_1[z,\theta](a)=0$, $[z,\chi](a)-\overline{\alpha}_1[z,\theta](a)=0$, $[z,\theta](b)-\alpha_2[z,\chi](b)=0$. Hence, we have $[z,\theta](a)=0$ for the eigenfunctions $z(t,\lambda)$ of the operator $T_{\alpha_1\alpha_2}^{-}$ that lie in $\mathcal{H}'$ and are eigenfunctions of T'. Since $[z,\chi](a)-\alpha_1[z,\theta](a)=0$, we derive that $[z,\chi](a)=0$ and $z(t,\lambda)\equiv 0$. Since all solutions of $\tau(z)=\lambda z$ $(t\in J)$ lie in $\mathcal{L}_r^2(J)$, we can see that the resolvent $R_\lambda(T_{\alpha_1\alpha_2}^{-})$ of the operator $T_{\alpha_1\alpha_2}^{-}$ is a Hilbert-Schmidt operator, and thus the spectrum of $T_{\alpha_1\alpha_2}^{-}$ is purely discrete. Hence, the theorem on the expansion of the self-adjoint operator T' in eigenfunctions implies that $\mathcal{H}'=\{0\}$, that is, $T_{\alpha_1\alpha_2}^{-}$ is pure. This completes the proof. $\square$

In order to prove third property, we set

$$\mathfrak{H}_{-}^{\pm}=\overline{\bigcup_{s\geq 0}\mathfrak{U}^{\pm}(s)\mathfrak{D}^{-}},\ \mathfrak{H}_{+}^{\pm}=\overline{\bigcup_{s\leq 0}\mathfrak{U}^{\pm}(s)\mathfrak{D}^{+}}$$

and first prove the next result.

Lemma 2. *The equality $\mathfrak{H}_-^{\pm}+\mathfrak{H}_+^{\pm}=\mathfrak{H}$ is fulfilled.*

Proof. By means of the property (**1**) of the subspace $\mathfrak{D}_{\pm}$, it can be shown that the subspace $\mathfrak{H}'_{\pm}=\mathfrak{H}\ominus\left(\mathfrak{H}_-^{\pm}+\mathfrak{H}_+^{\pm}\right)$ is invariant with respect to the group $\{\mathfrak{U}^{\pm}(s)\}$ and it can be described as $\mathfrak{H}'_{\pm}=\left\langle 0,\mathcal{H}'_{\pm},0\right\rangle$, where $\mathcal{H}'_{\pm}$ is a subspace in $\mathcal{H}$. Therefore, if the subspace $\mathfrak{H}'_{\pm}$ (and hence also $\mathcal{H}'_{\pm}$) were non-trivial, then the unitary group $\{\mathfrak{U}^{\pm\prime}(s)\}$, restricted to this subspace, would be a unitary part of the group $\{\mathfrak{U}^{\pm}(s)\}$, and thus the restriction $T^{\pm\prime}_{\alpha_1\alpha_2}$ of $T^{\pm}_{\alpha_1\alpha_2}$ to $\mathcal{H}'_{\pm}$ would be a self-adjoint operator in $\mathcal{H}'_{\pm}$. Since the operator $T^{\pm}_{\alpha_1\alpha_2}$ is pure, we conclude that $\mathcal{H}'_{\pm}=\{0\}$, i.e., $\mathfrak{H}'_{\pm}=\{0\}$. Hence, the lemma is proved. □

Let $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ be the solutions of the equation $\tau(y)=\lambda y$ $(t\in J)$ satisfying the conditions given by

$$[\varphi,\theta](a)=-1,\ [\varphi,\chi](a)=0,\ [\psi,\theta](a)=0,\ [\psi,\chi](a)=1. \tag{3.1}$$

The *Weyl function* $m_{\infty\alpha_2}(\lambda)$ of the self-adjoint operator $T^{-}_{\infty\alpha_2}$ is determined by the condition

$$[\psi+m_{\infty\alpha_2}\varphi,\theta](b)-\alpha_2[\psi+m_{\infty\alpha_2}\varphi,\chi](b)=0,$$

which implies in turn that

$$m_{\infty\alpha_2}(\lambda)=-\frac{[\psi,\theta](b)-\alpha_2[\psi,\chi](b)}{[\varphi,\theta](b)-\alpha_2[\varphi,\chi](b)}. \tag{3.2}$$

It follows from (3.2) that $m_{\infty\alpha_2}(\lambda)$ is a meromorphic function on the complex plane $\mathbb{C}$ with a countable number of poles on the real axis. We note that these poles coincide with the eigenvalues of the self-adjoint operator $T^{-}_{\infty\alpha_2}$. Furthermore, we can show that the function $m_{\infty\alpha_2}(\lambda)$ has the following properties: $\Im\lambda\Im m_{\infty\alpha_2}(\lambda)>0$ for $\Im\lambda\neq 0$ and $m_{\infty\alpha_2}(\bar\lambda)=\overline{m_{\infty\alpha_2}(\lambda)}$ for complex λ, except the real poles of $m_{\infty\alpha_2}(\lambda)$.

For convenience, we adopt the following notations:

$$\omega(t,\lambda)=\psi(t,\lambda)+m_{\infty\alpha_2}(\lambda)\varphi(t,\lambda),$$

$$\Theta^{-}_{\alpha_1\alpha_2}(\lambda)=\frac{m_{\infty\alpha_2}(\lambda)-\alpha_1}{m_{\infty\alpha_2}(\lambda)-\overline{\alpha}_1}. \tag{3.3}$$

Set

$$\mathcal{V}^{-}_{\lambda}(t,\xi,\varsigma)=\langle e^{-i\lambda\xi},(m_{\infty\alpha_2}(\lambda)-\alpha_1)^{-1}\gamma\omega(t,\lambda),\overline{\Theta}^{-}_{\alpha_1\alpha_2}(\lambda)e^{-i\lambda\varsigma}\rangle.$$

By means of the vector $\mathcal{V}^{-}_{\lambda}(t,\xi,\varsigma)$, we consider the transformation $\Phi_-:Y\to\tilde{Y}_-(\lambda)$ by $(\Phi_-Y)(\lambda):=\tilde{Y}_-(\lambda):=\frac{1}{\sqrt{2\pi}}(Y,\mathcal{V}^{-}_{\lambda})_{\mathfrak{H}}$ on the vector $Y=\langle u_-,y,u_+\rangle$, where u_-,u_+, and y are smooth, compactly supported functions.

Lemma 3. *The transformation Φ_- maps $\mathfrak{H}_-^-$ onto $\mathcal{L}^2(\mathbb{R})$ isometrically. For all vectors $Y, Z \in \mathfrak{H}_-^-$ the Parseval equality and the inversion formula hold:*

$$(Y,Z)_{\mathfrak{H}} = (\tilde{Y}_-, \tilde{Z}_-)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \tilde{Y}_-(\lambda)\overline{\tilde{Z}_-(\lambda)}d\lambda, \ Y = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{Y}_-(\lambda)\mathcal{V}_\lambda^- d\lambda,$$

where $\tilde{Y}_-(\lambda) := (\Phi_- Y)(\lambda)$ and $\tilde{Z}_-(\lambda) := (\Phi_- Z)(\lambda)$.

Proof. For $Y, Z \in \mathfrak{D}^-$, $Y = \langle u_-, 0, 0\rangle$, $Z = \langle v_-, 0, 0\rangle$, we get

$$\tilde{Y}_-(\lambda) : = \frac{1}{\sqrt{2\pi}}(Y, \mathcal{V}_\lambda^-)_{\mathfrak{H}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{9} u_-(\xi)e^{i\lambda\xi}d\xi \in \mathcal{H}_-^2$$

and

$$(Y,Z)_{\mathfrak{H}} = \int_{-\infty}^{0} u_-(\xi)\overline{v_-(\xi)}d\xi = \int_{-\infty}^{\infty} \tilde{Y}_-(\lambda)\overline{\tilde{Z}_-(\lambda)}d\lambda = (\Phi_- Y, \Phi_- Z)_{\mathcal{L}^2}$$

in view of the usual Parseval equality for Fourier integrals. Here and below, $\mathcal{H}_\pm^2$ denote the Hardy classes in $\mathcal{L}^2(\mathbb{R})$ consisting of the functions analytically extendable to the upper and lower half-planes, respectively.

We aim to extend the Parseval equality to the whole of $\mathfrak{H}_-^-$. In this context, we consider in $\mathfrak{H}_-^-$ the dense set $\mathfrak{H}'_-$ of vectors acquired from the smooth, compactly supported functions in $\mathfrak{D}^- : Y \in \mathfrak{H}'_-$ if $Y = \mathfrak{U}^-(s)Y_0$, $Y_0 = \langle u_-, 0, 0\rangle$, $u_- \in C_0^\infty(\mathbb{R}_-)$, where $s = s_Y$ is a non-negative number depending on Y. If $Y, Z \in \mathfrak{H}'_-$, then for $s > s_Y$ and $s > s_Z$ we have $\mathfrak{U}^-(-s)Y$, $\mathfrak{U}^-(-s)Z \in \mathfrak{D}^-$ and, moreover, the first components of these vectors lie in $C_0^\infty(\mathbb{R}_-)$. Then, as the operators $\mathfrak{U}^-(s)$ $(s \in \mathbb{R})$ are unitary, it follows from the equality

$$\Phi_-\mathfrak{U}^-(-s)Y = (\mathfrak{U}^-(-s)Y, \mathcal{V}_\lambda^-)_{\mathfrak{H}} = e^{-i\lambda s}(Y, \mathcal{V}_\lambda^-)_{\mathfrak{H}} = e^{-i\lambda s}\Phi_- Y,$$

that

$$\begin{aligned}(Y,Z)_{\mathfrak{H}} &= (\mathfrak{U}^-(-s)Y, \mathfrak{U}^-(-s)Z)_{\mathfrak{H}} = (\Phi_-\mathfrak{U}^-(-s)Y, \Phi_-\mathfrak{U}^-(-s)Z)_{\mathcal{L}^2} \\ &= (e^{-i\lambda s}\Phi_- Y, e^{-i\lambda s}\Phi_- Z)_{\mathcal{L}^2} = (\Phi_- Y, \Phi_- Z)_{\mathcal{L}^2}. \qquad (3.4)\end{aligned}$$

If we take the closure in (3.4), we find the Parseval equality for the entire space $\mathfrak{H}_-^-$. If all integrals in the Parseval equality are considered as limits in the mean of integrals over finite intervals, we get the inversion formula. In conclusion, we have

$$\Phi_-\mathfrak{H}_-^- = \overline{\bigcup_{s\geq 0}\Phi_-\mathfrak{U}^-(s)\mathfrak{D}^-} = \overline{\bigcup_{s\geq 0} e^{-i\lambda s}\mathcal{H}_-^2} = \mathcal{L}^2(\mathbb{R}),$$

i.e., Φ_- maps $\mathfrak{H}_-^-$ onto whole $\mathcal{L}^2(\mathbb{R})$, proving the lemma. □

Let us set

$$\mathcal{V}_\lambda^+(t,\xi,\varsigma) = \langle \Theta_{\alpha_1\alpha_2}^-(\lambda)e^{-i\lambda\xi}, (m_{\infty\alpha_2}(\lambda) - \overline{\alpha}_1)^{-1}\gamma\omega(t,\lambda), e^{-i\lambda\varsigma}\rangle.$$

By using the vectors $\mathcal{V}_\lambda^+(t,\xi,\varsigma)$, we define the map $\Phi_+ : Y \to \tilde{Y}_+(\lambda)$ on vectors $Y = \langle u_-, y, u_+\rangle$ in which u_-, u_+, and y are smooth, compactly supported functions by setting $(\Phi_+ Y)(\lambda) := \tilde{Y}_+(\lambda) := \frac{1}{\sqrt{2\pi}}(Y, \mathcal{V}_\lambda^+)_{\mathfrak{H}}$. The next result can be proved by following the procedure used in the proof of Lemma 3.

Lemma 4. *The transformation Φ_+ isometrically maps $\mathfrak{H}_+^-$ onto $\mathcal{L}^2(\mathbb{R})$ and besides, the Parseval equality and the inversion formula hold for all vectors $Y, Z \in \mathfrak{H}_+^-$ as follows:*

$$(Y,Z)_{\mathfrak{H}} = (\tilde{Y}_+, \tilde{Z}_+)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \tilde{Y}_+(\lambda)\overline{\tilde{Z}_+(\lambda)}d\lambda, \; Y = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{Y}_+(\lambda)\mathcal{V}_\lambda^+ d\lambda,$$

where $\tilde{Y}_+(\lambda) := (\Phi_+ Y)(\lambda)$ and $\tilde{Z}_+(\lambda) := (\Phi_+ Z)(\lambda)$.

Equality given by (3.3) implies that $\Theta_{\alpha_1\alpha_2}^-(\lambda)$ satisfies $\left|\Theta_{\alpha_1\alpha_2}^-(\lambda)\right| = 1$ for all $\lambda \in \mathbb{R}$. Then, we conclude from the explicit formula for the vectors $\mathcal{V}_\lambda^+$ and $\mathcal{V}_\lambda^-$ that

$$\mathcal{V}_\lambda^- = \overline{\Theta_{\alpha_1\alpha_2}^-}(\lambda)\mathcal{V}_\lambda^+ \; (\lambda \in \mathbb{R}). \tag{3.5}$$

Lemmas 3 and 4 imply that $\mathfrak{H}_-^- = \mathfrak{H}_+^-$. This, together with Lemma 2, verifies that $\mathfrak{H} = \mathfrak{H}_-^- = \mathfrak{H}_+^-$ and property (**3**) for $\mathfrak{U}^-(s)$ above has been established for the incoming and outgoing subspaces.

Hence, Φ_- isometrically maps onto $\mathcal{L}^2(\mathbb{R})$ with the subspace $\mathfrak{D}^-$ mapped onto $\mathcal{H}_-^2$, and the operators $\mathfrak{U}^-(s)$ are transformed by the operators of multiplication by $e^{i\lambda s}$. This means that Φ_- (Φ_+) is the incoming (outgoing) spectral representation for the group $\{\mathfrak{U}^-(s)\}$. Using (3.5), we can pass from the Φ_+-representation of a vector $Y \in \mathfrak{H}$ to its Φ_--representation by multiplication of the function $\Theta_{\alpha_1\alpha_2}^-(\lambda)$: $\tilde{Y}_-(\lambda) = \Theta_{\alpha_1\alpha_2}^-(\lambda)\tilde{Y}_+(\lambda)$. Based on [10], the scattering function (matrix) of the group $\{\mathfrak{U}^-(s)\}$ with respect to the subspaces $\mathfrak{D}^-$ and $\mathfrak{D}^+$, is the coefficient by which the Φ_--representation of a vector $Y \in \mathfrak{H}$ must be multiplied in order to get the corresponding Φ_+-representation: $\tilde{Y}_+(\lambda) = \overline{\Theta_{\alpha_1\alpha_2}^-}(\lambda)\tilde{Y}_-(\lambda)$ and thus we have proved the following statement.

Theorem 4. *The function $\overline{\Theta_{\alpha_1\alpha_2}^-}(\lambda)$ is the scattering function (matrix) of the group $\{\mathfrak{U}^-(s)\}$ or of the self-adjoint operator $\mathfrak{T}_{\alpha_1\alpha_2}^-$).*

Let $\mathcal{S}(\lambda)$ be an arbitrary non-constant inner function ([16]) defined on the upper half-plane (we recall that a function $\mathcal{S}(\lambda)$ analytic in the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $|\mathcal{S}(\lambda)| \leq 1$ for $\lambda \in \mathbb{C}_+$, and $|\mathcal{S}(\lambda)| = 1$ for almost all $\lambda \in \mathbb{R}$). Setting $\mathcal{K} = \mathcal{H}_+^2 \ominus \mathcal{S}\mathcal{H}_+^2$, we can see that $\mathcal{K} \neq \{0\}$ is a subspace of the Hilbert space $\mathcal{H}_+^2$. We deal with the semigroup of the operators $\mathcal{X}(s)$ $(s \geq 0)$ acting in $\mathcal{K}$ according to the formula $\mathcal{X}(s)u = P\left[e^{i\lambda s}u\right]$, $u := u(\lambda) \in \mathcal{K}$, where P is the orthogonal projection from $\mathcal{H}_+^2$ onto $\mathcal{K}$. The generator of the semigroup $\{\mathcal{X}(s)\}$ is represented as $\mathcal{B} : \mathcal{B}u = \lim_{s \to +0}[(is)^{-1}(\mathcal{X}(s)u - u)]$. $\mathcal{B}$ is a maximal dissipative operator acting in $\mathcal{K}$ and its domain $\mathcal{D}(\mathcal{B})$ consisting of all functions $u \in \mathcal{K}$, for which the limit given above exists. The operator $\mathcal{B}$ is called a *model dissipative operator* (we remark that this model dissipative operator, which is associated with the names of Lax and Phillips

[10], is a special case of a more general model dissipative operator constructed by Sz.-Nagy and Foiaş [16]). It is the basic assertion that $\mathcal{S}(\lambda)$ is the *characteristic function* of the operator $\mathcal{B}$.

If we set $\mathfrak{N}=\langle 0,\mathcal{H},0\rangle$, then it is obtained that $\mathfrak{H}=\mathfrak{D}^{-}\oplus\mathfrak{N}\oplus\mathfrak{D}^{+}$. From the explicit form of the unitary transformation Φ_{-} that under the mapping Φ_{-}, we have

$$\mathfrak{H}\to\mathcal{L}^{2}(\mathbb{R}),\ Y\to\tilde{Y}_{-}(\lambda)=(\Phi_{-}Y)(\lambda),\ \mathfrak{D}^{-}\to\mathcal{H}_{-}^{2},$$
$$\mathfrak{D}^{+}\to\Theta_{\alpha_1\alpha_2}^{-}\mathcal{H}_{+}^{2},\ \mathfrak{N}\to\mathcal{H}_{+}^{2}\ominus\Theta_{\alpha_1\alpha_2}^{-}\mathcal{H}_{+}^{2},$$
$$\mathfrak{U}^{-}(s)Y\to(\Phi_{-}\mathfrak{U}^{-}(s)\Phi_{-}^{-1}\tilde{Y}_{-})(\lambda)=e^{i\lambda s}\tilde{Y}_{-}(\lambda). \tag{3.6}$$

The formulas in (3.6) imply that our operator $T_{\alpha_1\alpha_2}^{-}$ is unitary equivalent to the model dissipative operator with the characteristic function $\Theta_{\alpha_1\alpha_2}^{-}(\lambda)$. The fact that characteristic functions of unitary equivalent dissipative operators coincide ([13–16]) leads us the following theorem.

Theorem 5. *The characteristic function of the maximal dissipative operator $T_{\alpha_1\alpha_2}^{-}$ coincides with the function $\Theta_{\alpha_1\alpha_2}^{-}(\lambda)$given by* (3.3).

Weyl function of the self-adjoint operator $T_{\alpha_1\infty}^{+}$, denoted by $m_{\alpha_1\infty}(\lambda)$, can be expressed in terms of the Wronskians of the solutions:

$$m_{\alpha_1\infty}(\lambda)=-\frac{[\vartheta,\chi](b)}{[\phi,\chi](b)},$$

where $\phi(t,\lambda)$ and $\vartheta(t,\lambda)$ are solutions of $\tau(y)=\lambda y\,(t\in J)$ and satisfying the conditions

$$[\phi,\theta](a)=-\frac{1}{\sqrt{1+\alpha_1^2}},\ [\phi,\chi](a)=-\frac{\alpha_1}{\sqrt{1+\alpha_1^2}},$$
$$[\vartheta,\theta](a)=\frac{\alpha_1}{\sqrt{1+\alpha_1^2}},\ [\vartheta,\chi](a)=\frac{1}{\sqrt{1+\alpha_1^2}}.$$

Let us adopt the following notations:

$$k(\lambda):=\frac{[\phi,\theta](b)}{[\vartheta,\chi](b)},\ m(\lambda):=m_{\alpha_1\infty}(\lambda),$$
$$\Theta^{+}(\lambda):=\Theta_{\alpha_1\alpha_2}^{+}(\lambda):=\frac{m(\lambda)k(\lambda)-\alpha_2}{m(\lambda)k(\lambda)-\overline{\alpha}_2}. \tag{3.7}$$

Let

$$\mathcal{W}_{\lambda}^{-}(t,\xi,\varsigma)=\langle e^{-i\lambda\xi},\beta m(\lambda)[(m(\lambda)k(\lambda)-\alpha_2)[[\vartheta,v](b)]^{-1}\phi(t,\lambda),\overline{\Theta}^{+}(\lambda)e^{-i\lambda\varsigma}\rangle.$$

By means of the vector $\mathcal{W}_{\lambda}^{-}$, we set the transformation $\Upsilon_{-}:Y\to\tilde{Y}_{-}(\lambda)$ given by $(\Upsilon_{-}Y)(\lambda):=\tilde{Y}_{-}(\lambda):=\frac{1}{\sqrt{2\pi}}(Y,\mathcal{W}_{\lambda}^{-})_{\mathfrak{H}}$ on the vector $Y=\langle u_{-},y,u_{+}\rangle$ in which u_{-},u_{+}, and y are smooth, compactly supported functions. The proof of the next result is similar to that of Lemma 3.

Lemma 5. *The transformation Υ_- isometrically maps $\mathfrak{H}_-^+$ onto $\mathcal{L}^2(\mathbb{R})$. For all vectors $Y, Z \in \mathfrak{H}_-^+$, we obtain the Parseval equality and the inversion formula given by:*

$$(Y,Z)_{\mathfrak{H}} = (\tilde{Y}_-, \tilde{Z}_-)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \tilde{Y}_-(\lambda)\overline{\tilde{Z}_-(\lambda)}d\lambda,\ Y = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{Y}_-(\lambda)\mathcal{W}_\lambda^- d\lambda,$$

where $\tilde{Y}_-(\lambda) = (\Upsilon_- Y)(\lambda)$ and $\tilde{Z}_-(\lambda) = (\Upsilon_- Z)(\lambda)$.

Let

$$\mathcal{W}_\lambda^+(t,\xi,\varsigma) = \langle \Theta^+(\lambda)e^{-i\lambda\xi}, \beta m(\lambda)[(m(\lambda)k(\lambda) - \overline{\alpha}_2)[\vartheta,\chi](b)]^{-1}\phi(t,\lambda), e^{-i\lambda\varsigma}\rangle.$$

With the help of the vector $\mathcal{W}_\lambda^+(t,\xi,\varsigma)$, define the transformation $\Upsilon_+ : Y \to \tilde{Y}_+(\lambda)$ on vectors $Y = \langle u_-, y, u_+\rangle$ by setting $(\Upsilon_+ Y)(\lambda) := \tilde{Y}_+(\lambda) := \frac{1}{\sqrt{2\pi}}(Y, \mathcal{W}_\lambda^+)_{\mathfrak{H}}$. Here, we consider u_-, u_+, and y as smooth, compactly supported functions.

Lemma 6. *The transformation Υ_+ isometrically maps $\mathfrak{H}_+^+$ onto $\mathcal{L}^2(\mathbb{R})$, and for all vectors $Y, Z \in \mathfrak{H}_+^+$, the Parseval equality and the inversion formula hold:*

$$(Y,Z)_{\mathfrak{H}} = (\tilde{Y}_+, \tilde{Z}_+)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \tilde{Y}_-(\lambda)\overline{\tilde{Z}_-(\lambda)}d\lambda,\ Y = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{Y}_+(\lambda)\mathcal{W}_\lambda^+ d\lambda,$$

where $\tilde{Y}_+(\lambda) := (\Upsilon_+ Y)(\lambda)$and $\tilde{Z}_+(\lambda) := (\Upsilon_+ Z)(\lambda)$

It follows from (3.7) that the function $\Theta_{\alpha_1\alpha_2}^+(\lambda)$ satisfies $\left|\Theta_{\alpha_1\alpha_2}^+(\lambda)\right| = 1$ for $\lambda \in \mathbb{R}$. Then, the explicit formula for the vectors $\mathcal{W}_\lambda^+$ and $\mathcal{W}_\lambda^-$ implies that

$$\mathcal{W}_\lambda^- = \overline{\Theta}_{\alpha_1\alpha_2}^+(\lambda)\mathcal{W}_\lambda^+,\ \lambda \in \mathbb{R}. \tag{3.8}$$

Lemmas 5 and 6 result in $\mathfrak{H}_-^+ = \mathfrak{H}_+^+$. By means of Lemma 2, we can conclude that $\mathfrak{H} = \mathfrak{H}_-^+ = \mathfrak{H}_+^+$. According to the formula (3.8), we can see that the passage from the Υ_--representation of a vector $Y \in \mathfrak{H}$ to its Υ_+-representation is achieved as follows: $\tilde{Y}_+(\lambda) = \overline{\Theta}_{\alpha_1\alpha_2}^+(\lambda)\tilde{Y}_-(\lambda)$. Hence, according to [10], the following theorem follows.

Theorem 6. *The function $\overline{\Theta}_{\alpha_1\alpha_2}^+(\lambda)$ is the scattering matrix of the group $\{\mathfrak{U}^+(s)\}$ of the self-adjoint operator $\mathfrak{T}_{\alpha_1\alpha_2}^+$).*

We derive from the explicit form of the unitary transformation Φ_- that

$$\mathfrak{H} \to \mathcal{L}^2(\mathbb{R}),\ \ Y \to \tilde{Y}_-(\lambda) = (\Upsilon_- Y)(\lambda),\ \ \mathfrak{D}^- \to \mathcal{H}_-^2,$$
$$\mathfrak{D}^+ \to \Theta_{\alpha_1\alpha_2}^+\mathcal{H}_+^2,\ \ \mathfrak{N} \to \mathcal{H}_+^2 \ominus \Theta_{\alpha_1\alpha_2}^+\mathcal{H}_+^2,$$
$$\mathfrak{U}^+(s)Y \to (\Upsilon_-\mathfrak{U}^+(s)\Upsilon_-^{-1}\tilde{Y}_-)(\lambda) = e^{i\lambda s}\tilde{Y}_-(\lambda). \tag{3.9}$$

The formulas given by (3.9) state that the operator $T_{\alpha_1\alpha_2}^+$ is a unitary equivalent to the model dissipative operator with characteristic function $\Theta_{\alpha_1\alpha_2}^+(\lambda)$. We have thus proved the next assertion.

Theorem 7. *The characteristic function of the maximal dissipative operator $T^{+}_{\alpha_1\alpha_2}$ coincides with the function $\Theta^{+}_{\alpha_1\alpha_2}(\lambda)$defined by (3.7).*

Let $\mathbf{S}$ represent the linear operator acting in the Hilbert space $\mathbf{H}$ with the domain $\mathcal{D}(\mathbf{S})$. We know that a complex number λ_0 is called an *eigenvalue* of an operator $\mathbf{S}$ if there exists a non-zero vector $z_0 \in \mathcal{D}(\mathbf{S})$ satisfying the equation $\mathbf{S}z_0 = \lambda_0 z_0$; here, z_0 is called an *eigenvector* of $\mathbf{S}$ for λ_0. The eigenvector for λ_0 spans a subspace of $\mathcal{D}(\mathbf{S})$, called the *eigenspace* for λ_0 and the *geometric multiplicity* of λ_0 is the dimension of its eigenspace. The vectors $z_1, z_2, ..., z_k$ are called the *associated vectors* of the eigenvector z_0 if they belong to $\mathcal{D}(\mathbf{S})$ and $\mathbf{S}z_j = \lambda_0 z_j + z_{j-1},\ j = 1, 2, ..., k$. The non-zero vector $z \in \mathcal{D}(\mathbf{S})$ is called a *root vector* of the operator $\mathbf{S}$ corresponding to the eigenvalue λ_0, if all powers of $\mathbf{S}$ are defined on this element and $(\mathbf{S} - \lambda_0 I)^m z = 0$ for some integer m. The set of all root vectors of $\mathbf{S}$ corresponding to the same eigenvalue λ_0 with the vector $z = 0$ forms a linear set $\mathbf{M}_{\lambda_0}$ and is called the root lineal. The dimension of the lineal $\mathbf{M}_{\lambda_0}$ is called the *algebraic multiplicity* of the eigenvalue λ_0. The root lineal $\mathbf{M}_{\lambda_0}$ coincides with the linear span of all eigenvectors and associated vectors of $\mathbf{S}$ corresponding to the eigenvalue λ_0. As a result, the completeness of the system of all eigenvectors and associated vectors of $\mathbf{S}$ is equivalent to the completeness of the system of all root vectors of this operator.

Characteristic function of a maximal dissipative operator $T^{\pm}_{\alpha_1\alpha_2}$ carries complete information about the spectral properties of this operator ([9, 13–16]). For example, when a singular factor $\theta^{\pm}(\lambda)$ of the characteristic function $\Theta^{\pm}_{\alpha_1\alpha_2}(\lambda)$ in the factorization $\Theta^{\pm}_{\alpha_1\alpha_2}(\lambda) = \theta^{\pm}(\lambda)B^{\pm}(\lambda)$ (where $B^{\pm}(\lambda)$ is a Blaschke product) is absent, we are sure that system of eigenfunctions and associated functions (or root functions) of the maximal dissipative Sturm-Liouville operator $T^{\pm}_{\alpha_1\alpha_2}$ is complete.

Theorem 8. *For all values of $\alpha_1$where $\Im\alpha_1 > 0$, with the possible exception of a single value $\alpha_1 = \alpha_1^0$, and for a fixed α_2 ($\Im\alpha_2 = 0$ or $\alpha_2 = 0$), the characteristic function $\Theta^{-}_{\alpha_1\alpha_2}(\lambda)$ of the maximal dissipative operator $T^{-}_{\alpha_1\alpha_2}$ is a Blaschke product, and the spectrum of $T^{-}_{\alpha_1\alpha_2}$ is purely discrete, and lies in the open upper half plane. The operator $T^{-}_{\alpha_1\alpha_2}$ $(\alpha_1 \neq \alpha_1^0)$ has a countable number of isolated eigenvalues having finite multiplicity and limit points at infinity, and the system of all eigenfunctions and associated functions (or all root functions) of this operator is complete in the space $\mathcal{L}^2_r(J)$.*

Proof. It can be seen from the explicit formula (3.3) that $\Theta^{-}_{\alpha_1\alpha_2}(\lambda)$ is an inner function in the upper half-plane and, besides, it is meromorphic in the whole λ-plane. Therefore, we can factorize it in the following way

$$\Theta^{-}_{\alpha_1\alpha_2}(\lambda) = e^{i\lambda l(\alpha_1)} B_{\alpha_1\alpha_2}(\lambda),\ l(\alpha_1) \geq 0, \tag{3.10}$$

where $B_{\alpha_1\alpha_2}(\lambda)$ is a Blaschke product. Using (3.10), we find that

$$\left|\Theta^{-}_{\alpha_1\alpha_2}(\lambda)\right| \leq e^{-l(\alpha_1)\Im\lambda},\ \Im\lambda \geq 0. \tag{3.11}$$

Additionally, if we express $m_{\infty\alpha_2}(\lambda)$ in terms of $\Theta^-_{\alpha_1\alpha_2}(\lambda)$ and use (3.3), we get

$$m_{\infty\alpha_2}(\lambda) = \frac{\overline{\alpha}_1\Theta^-_{\alpha_1\alpha_2}(\lambda) - \alpha_1}{\Theta^-_{\alpha_1\alpha_2}(\lambda) - 1}. \tag{3.12}$$

For a given value α_1 ($\Im\alpha_1 > 0$), if $l(\alpha_1) > 0$ then we have $\lim_{s\to+\infty}\Theta^-_{\alpha_1\alpha_2}(is) = 0$ by (3.11). This, together with (3.12), results in $\lim_{s\to+\infty} m_{\infty\alpha_2}(is) = \alpha_1$. Since $m_{\infty\alpha_2}(\lambda)$ is independent of α_1, $l(\alpha_1)$ can be non-zero at not more than a single point $\alpha_1 = \alpha_1^0$ (and, further, $\alpha_1^0 = \lim_{s\to+\infty} m_{\infty\alpha_2}(is)$). Then, the theorem is proved. □

The next result can be proved in a similar manner in the proof of Theorem 8.

Theorem 9. *For all values of α_2 with $\Im\alpha_2 > 0$, with the possible exception of a single value $\alpha_2 = \alpha_2^0$, and for a fixed α_1 ($\Im\alpha_1 = 0$ or $\alpha_1 = \infty$), the characteristic function $\Theta^+_{\alpha_1\alpha_2}(\lambda)$ of the maximal dissipative operator $T^+_{\alpha_1\alpha_2}$ is a Blaschke product, and the spectrum of $T^+_{\alpha_1\alpha_2}$ is purely discrete, and lies in the open upper half-plane. The operator $T^+_{\alpha_1\alpha_2}$ ($\alpha_2 \neq \alpha_2^0$) has a countable number of isolated eigenvalues having finite multiplicity and limit points at infinity, and the system of all eigenfunctions and associated functions (or all root functions) of this operator is complete in the space $\mathcal{L}^2_r(J)$.*

REFERENCES

[1] B. P. Allahverdiev, "Dissipative Sturm-Liouville operators with nonseparated boundary conditions." *Monatsh. Math.*, vol. 140, no. 1, pp. 1–17, 2003, doi: 10.1007/s00605-003-0035-4.

[2] B. P. Allahverdiev, "Nonself adjoint singular Sturm Lioville operators in limit-circle case." *Taiwan. J. Math.*, vol. 16, no. 6, pp. 2035–2052, 2012, doi: 10.11650/twjm/1500406837.

[3] B. P. Allahverdiev, "Extensions, dilations and spectral analysis of singular Sturm-Liouville operators." *Math. Reports*, vol. 19(69), no. 2, pp. 225–243, 2017.

[4] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations.* New York: McGraw-Hill, 1955.

[5] N. Dunford and J. T. Schwartz, *Linear Operators. Part II.* New York: Interscience, 1964.

[6] W. N. Everitt, I. W. Knowles, and T. T. Reed, "Limit-point and limit-circle criteria for Sturm-Liouville equations with intermittently negative principal coefficient." *Proc. Royal Soc. Edinburgh, Sect. A Math.*, vol. 103, pp. 215–228, 1986, doi: 10.1017/S0308210500018874.

[7] V. I. Gorbachuk and M. L. Gorbachuk, *Boundary Value Problems for Operator Differential Equations.* Dordrecht: Kluwer, 1991.

[8] B. J. Harris, "Limit-circle criteria for second order differential expression." *Q. J. Math. Oxford2 Ser.*, vol. 35, pp. 415–427, 1984, doi: 10.1093/qmath/35.4.415.

[9] A. Kuzhel, *Characteristic Functions and Models of Nonself-adjoint Operators.* Boston and London: Kluwer, 1996.

[10] P. D. lax and R. S. Phillips, *Scattering Theory.* New York: Academic Press, 1967.

[11] B. M. Levitan and I. S. Sargsjan, *Sturm-liouville and Dirac Operators.* Dordrecht: Kluwer, 1990.

[12] M. A. Naimark, *Lineer Differential Operators. Parts 1,2.* New York: Ungar, 1967, 1968.

[13] B. S. Pavlov, "On conditions for separation of the spectral components of a dissipative operator," *Math. USSR Izvestiya*, vol. 9, pp. 113–1337, 1976.

[14] B. S. Pavlov, "Dilation theory and spectral analysis of nonselfadjoint differential operators," *Transl. II. Ser., Amer. Math. Soc.*, vol. 115, pp. 103–142, 1981.

[15] B. S. Pavlov, "Spectral analysis of a dissipative singular Schrödinger operator in terms of a functional model." *Partial Differential Equations 8, Encyc. Math. Sci.*, vol. 65, pp. 87–163, 1996.
[16] B. Sz.-Nagy and C. Foiaş, *Analyse Harmonique des Operateurs de L′espace de Hilbert*. Paris and Budapest: Masson, Akad Kiadó, 1967.
[17] E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second Order Differential Equations. Part1, 2nd edn.* Oxford: University Press, 1962.
[18] J. Weidmann, *Spectral Theory of Ordinary Differential Operators.* Berlin: Springer-Verlag, 1987.
[19] A. Zettl, *Sturm-Liouville Theory.* Providence, RI: American Mathematical Society, 2005.

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ERRATUM: SIMPSON TYPE QUANTUM INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. We have shown that the results of [4] were wrong. Additionally, correct results concerning the Simpson type quantum integral inequalities are proved.

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Keywords: q-integral inequalities, q-derivative, Convex functions, Simpson's inequalities.

1. Introduction

In 2018 Tunç et al. [4] obtained Simpson's type quantum integral inequalities. Unfortunately, there are many mistakes in the proofs. Many q-integrals are calculated incorrectly. Besides, the results of lemma and theorems are also wrong. In this paper, we show the errors in the [4].

2. Preliminaries and Definitions of q-Calculus

Throughout this paper, let $a < b$ and $0 < q < 1$ be a constant. The following definitions and theorems for q- derivative and q- integral of a function f on $[a,b]$ are given in [2, 3].

Definition 1. For a continuous function $f : [a,b] \to \mathbb{R}$ then q- derivative of f at $x \in [a,b]$ is characterized by the expression

$$ {}_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \; x \neq a. \tag{2.1}$$

Since $f : [a,b] \to \mathbb{R}$ is a continuous function, thus we have ${}_aD_q f(a) = \lim_{x \to a} {}_aD_q f(x)$. The function f is said to be q- differentiable on $[a,b]$ if ${}_aD_q f(t)$ exists for all $x \in [a,b]$. If $a = 0$ in (2.1), then ${}_0D_q f(x) = D_q f(x)$, where $D_q f(x)$ is familiar q-derivative of f at $x \in [a,b]$ defined by the expression (see [1])

$$ D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \qquad x \neq 0. \tag{2.2}$$

Definition 2. Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Then the q-definite integral on $[a,b]$ is delineated as

$$\int_a^x f(t)_a\, d_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \tag{2.3}$$

for $x\in[a,b]$.

If $a=0$ in (2.3), then $\int_0^x f(t)_0\, d_q t = \int_0^x f(t)\, d_q t$, where $\int_0^x f(t)\, d_q t$ is familiar q-definite integral on $[0,x]$ defined by the expression (see [1])

$$\int_0^x f(t)_0\, d_q t = \int_0^x f(t)\, d_q t = (1-q)\, x\sum_{n=0}^{\infty} q^n f(q^n x). \tag{2.4}$$

If $c\in(a,x)$, then the q- definite integral on $[c,x]$ is expressed as

$$\int_c^x f(t)_a\, d_q t = \int_a^x f(t)_a\, d_q t - \int_a^c f(t)_a\, d_q t. \tag{2.5}$$

$[n]_q$ notation

$$[n]_q = \frac{q^n-1}{q-1}$$

Lemma 1. *[3] For $\alpha\in\mathbb{R}\backslash\{-1\}$, the following formula holds:*

$$\int_a^x (t-a)_a^{\alpha}\, d_q t = \frac{(x-a)^{\alpha+1}}{[\alpha+1]_q}. \tag{2.6}$$

3. Erratum: Simpson type quantum integral inequalities for convex functions

Here, we will show the errors we mentioned above. For example, in Lemma 4 the followin equality is not correct:

$$\begin{aligned}\int_0^{\frac{1}{2}} (1-t)\left|qt-\frac{1}{6}\right|_0 d_q t &= \int_0^{\frac{1}{2}} \left|qt-\frac{1}{6}\right|_0 d_q t - \int_0^{\frac{1}{2}} t\left|qt-\frac{1}{6}\right|_0 d_q t \\ &= \int_0^{\frac{1}{6q}} \left(qt-\frac{1}{6}\right)_0 d_q t + \int_{\frac{1}{6q}}^{\frac{1}{2}} \left(\frac{1}{6}-qt\right)_0 d_q t\end{aligned}$$

$$-\left(\int_{0}^{\frac{1}{6q}} t\left(qt-\frac{1}{6}\right)_{0} d_{q}t+\int_{\frac{1}{6q}}^{\frac{1}{2}} t\left(\frac{1}{6}-qt\right)_{0} d_{q}t\right).$$

Here, for $q\in(0,1)$, $\frac{1}{6q}\nleq\frac{1}{2}$. For instance, $q=\frac{1}{6}\to 1\nleq\frac{1}{2}$. So, the proof of Lemma 4 is not correct. Lemma 5 also have the same errors. On the other hand, since Lemma 4 and Lemma 5 are used in proof of Theorem 1, there are errors in this theorem. Moreover, Theorem 2 and 3 have the same mistakes. For instance, because of (2.6), the following equalities are also not true:

$$\int_{0}^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|_{0}^{p} d_{q}t=\frac{\left(1+(3q-1)^{p+1}\right)(1-q)}{6^{p+1}q\left(1-q^{p+1}\right)},$$

$$\int_{\frac{1}{2}}^{1}\left|qt-\frac{5}{6}\right|_{0}^{p} d_{q}t=\frac{\left[(5-3q)^{p+1}+(6q-5)^{p+1}\right](1-q)}{6^{p+1}q\left(1-q^{p+1}\right)}.$$

The integral boundaries that cause all these errors are chosen independently of q.

Now, let show the following Theorem 1 in [4] is not correct. For this, we give an example.

Theorem 1. *Suppose that $f:[a,b]\to\mathbb{R}$ is a q-differentiable function on (a,b) and $0<q<1$. If $|{}_aD_qf|$ is convex and integrable function on $[a,b]$, then we possess the inequality*

$$\frac{1}{6}\left|f(a)+4f\left(\frac{a+b}{2}\right)+f(b)-\frac{1}{(b-a)}\int_{a}^{b} f(t)_{a} d_{q}t\right| \tag{3.1}$$
$$\leq\frac{(b-a)}{12}\left\{\frac{2q^2+2q+1}{q^3+2q^2+2q+1}\left|{}_aD_qf(b)\right|+\frac{1}{3}\frac{6q^3+4q^2+4q+1}{q^3+2q^2+2q+1}\left|{}_aD_qf(a)\right|\right\}.$$

Example 1. Let choose $f(t)=1-t$ on $[0,1]$ and $f(t)$ satisfies the conditions of Theorem 1. On the other hand, $|{}_aD_qf|=|{}_aD_q(1-t)|=1$ is convex and integrable on $[0,1]$. Then we have

$$\frac{1}{6}\left|f(a)+4f\left(\frac{a+b}{2}\right)+f(b)-\frac{1}{(b-a)}\int_{a}^{b} f(t)_{a} d_{q}t\right| \tag{3.2}$$
$$=\frac{1}{6}\left|1+2+0-\int_{0}^{1}(1-t)_{0} d_{q}t\right|$$

$$=\frac{1}{6}\left|3-\left(t-\frac{t^2}{1+q}\right)\Big|_0^1\right|=\frac{3+2q}{6(1+q)}.$$

Also,

$$\begin{aligned}
&\frac{(b-a)}{12}\left\{\frac{2q^2+2q+1}{q^3+2q^2+2q+1}\left|{}_aD_qf(b)\right|+\frac{1}{3}\frac{6q^3+4q^2+4q+1}{q^3+2q^2+2q+1}\left|{}_aD_qf(a)\right|\right\}\\
&\qquad=\frac{1}{12}\left\{\frac{2q^2+2q+1}{q^3+2q^2+2q+1}+\frac{1}{3}\frac{6q^3+4q^2+4q+1}{q^3+2q^2+2q+1}\right\}\\
&\qquad=\frac{1}{36}\frac{6q^2+6q+3+6q^3+4q^2+4q+1}{q^3+2q^2+2q+1}\\
&\qquad=\frac{1}{36}\frac{6q^3+10q^2+10q+4}{q^3+2q^2+2q+1}\\
&\qquad=\frac{1}{18}\frac{3q^3+5q^2+5q+2}{q^3+2q^2+2q+1}.
\end{aligned}\tag{3.3}$$

As we seen, from (3.2) and (3.3) and for $q\in(0,1)$ we write

$$\frac{3+2q}{6(1+q)}\nleq\frac{1}{18}\frac{3q^3+5q^2+5q+2}{q^3+2q^2+2q+1}.$$

For instance, choosing $q=\frac{1}{2}$ we have

$$\frac{4}{9}\nleq\frac{7}{54}.$$

Therefore, Inequality (3.1) is not correct.

Similarly, other theorems can be shown to be false.

References

[1] V. Kac and P. Cheung, *Quantum calculus*. Springer-Verlag New York, 2001. doi: 10.1007/978-1-4613-0071-7.

[2] J. Tariboon and S. K. Ntouyas, "Quantum calculus on finite intervals and applications to impulsive difference equations," *Advances in Difference Equations*, vol. 2013, no. 1, p. 282, 2013, doi: 10.1186/1687-1847-2013-282.

[3] J. Tariboon and S. K. Ntouyas, "Quantum integral inequalities on finite intervals," *Journal of Inequalities and Applications*, vol. 2014, no. 1, p. 121, 2014, doi: 10.1186/1029-242X-2014-121.

[4] M. Tunç, E. Gov, and S. Balgecti, "Simpson type quantum integral inequalities for convex functions," *Miskolc Math. Notes*, vol. 19, no. 1, pp. 649–664, 2018, doi: 10.18514/MMN.2018.1661.

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A NUMERICAL METHOD FOR A SECOND ORDER SINGULARLY PERTURBED FREDHOLM INTEGRO-DIFFERENTIAL EQUATION

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Abstract. The boundary-value problem for a second order singularly perturbed Fredholm integro-differential equation was considered in this paper. For the numerical solution of this problem, we use an exponentially fitted difference scheme on a uniform mesh which is succeeded by the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form. Also, the method is first order convergent in the discrete maximum norm. Numerical example shows that recommended method has a good approximation characteristic.

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Keywords: Fredholm integro-differential equation, singular perturbation, finite difference, uniform convergence

1. Introduction

Fredholm integro-differential equations (FIDEs) have in large quantities applications in every branches of science. FIDEs arise from the mathematical modeling of many scientific phenomena, such as the study of fluid, physics, chemistry, biology, mechanics, astronomy, potential theory, electrostatics, control theory of industrial mathematics and chemical kinetics [13, 14, 18]. On the other hand, FIDEs are quite difficult to find exact solutions. For this reason, numerical methods play a significant role in this problems, for example, in [5, 8–11](see, also references therein).

Below, the boundary-value problem for a singularly perturbed Fredholm integro-differential equation(SPFIDE) is considered:

$$\begin{aligned} Lu :=& -\varepsilon u'' + a(x)u + \lambda \int_0^l K(x,s)u(s)ds = f(x), \qquad x \in (0,l), \\ u(0) =& A, \qquad u(l) = B, \end{aligned} \tag{1.1}$$

where $\varepsilon \in (0,1]$ is a perturbation parameter, λ is real parameter. We assume that $a(x) \geq \alpha > 0$, $f(x)$ and $K(x,s)$ are the sufficiently smooth functions satisfying certain regularity conditions to be specified. The solution $u(x)$ of (1.1) has in general a boundary layer near $x = 0$ and $x = l$.

Singularly perturbed differential equations are typically characterized by a small parameter ε multiplying some or all of the highest order terms in the differential equation. This problem undergo rapid changes within very thin layers near the boundary or inside the problem domain, so most of the conventional methods fail when this small parameter approaches to zero. These singularly perturbed differential equations arise in the modeling of various modern complicated processes, such as reaction-diffusion processes, epidemic dynamics, high Reynold's number flow in the fluid dynamics, heat transport problem. For more details on singular perturbation, one can refer the books [4, 15–17, 19] and the references therein. Survey of some existence and uniqueness results of singularly perturbed equations can be found in [7, 15–17].

In recent years, there has been a growing interest in the numerical solution of integral equations. The Adomian decomposition method for solving linear second-order FIDEs is presented in [10]. Qing Xue et al. [20] studied on an improved reproducing kernel method to find the numerical solution of FIDE type boundary value problems. Emamzadeh and Kajani [6] used a numerical method for solving the nonlinear Fredholm integral equation. Jackiewicz et al. [9] proposed several approaches to the numerical solution of a new FIDEs modelling neural networks. Gegele et al. [8] presented some approximation methods to solve higher order linear FIDEs. Karimi and Jozi [11] proposed a new numerical method for solving system of linear Fredholm integral equations of the second kind.

The above mentioned papers, related to FIDEs were concerned only with the regular cases. Also, current studies for the numerical solution of SPFIDEs have not widespread yet. Various difference schemes for singularly perturbed integro-differential equations and problems with integral boundary condition were investigated in [3, 12].

In this paper, we present fitted type difference scheme on an uniform mesh for the numerical solution of the problem (1.1). The difference scheme is constructed by the method of integral identities with the use exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form [1]. To approximate the integral part of (1.1), the composite right-side rectangle rule with the remainder term in integral form is being used.

The organization of the paper is as follows. In Section 2, we state some significant properties of the exact solution. In Section 3, we describe the finite difference discretization and appropriate mesh. The error analysis for the approximate solution is presented in Section 4. Uniform convergence is proved in the discrete maximum norm. Numerical results are given in Section 5 to support the predicted theory. The paper ends with a summary of the main conclusions.

Notation 1. Throughout the paper, C will denote a generic positive constant independent of ε and the mesh parameter and $\|g\|_\infty$ is the continuous maximum norm on the corresponding closed interval for any continuous function $g(x)$.

2. The Continuous Problem

Lemma 1. *If* $a, f \in C^1[0,l]$, $\frac{\partial^s K}{\partial x^s} \in C[0,l]^2$, $(s = 0,1)$ *and*

$$|\lambda| < \frac{\alpha}{\max\limits_{0\le x\le l} \int\limits_0^l |K(x,s)|ds}, \tag{2.1}$$

then for the solution $u(x)$ *of the problem (1.1) hold the following estimates*

$$\|u\|_\infty \le C, \tag{2.2}$$

$$\left|u'(x)\right| \le C\left\{1 + \frac{1}{\sqrt{\varepsilon}}\left(e^{-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}} + e^{-\frac{\sqrt{\alpha}(l-x)}{\sqrt{\varepsilon}}}\right)\right\}, \qquad x \in [0,l]. \tag{2.3}$$

Proof. Using the maximum principle for the operator $L_0 u = -\varepsilon u'' + a(x)u$, we obtain the estimate

$$||u||_\infty \le |A| + |B| + \alpha^{-1}||f||_\infty + \alpha^{-1}|\lambda| \max_{0\le x\le l} \int\limits_0^l |K(x,s)|\,|u(s)|\,ds$$

which after taking into account (2.1), leads to (2.2).
Next, we prove the estimate (2.3). Using (2.2) on (1.1) we have

$$|u''(x)| = \frac{1}{\varepsilon}\left| f(x) - a(x)u(x) - \lambda \int\limits_0^l K(x,s)u(s)ds \right| \le \frac{C}{\varepsilon}, \qquad 0 \le x \le l.$$

Moreover, we now proceed with the estimation of $|u'(0)|$, $|u'(l)|$. Here we use the following relation which holds for any function $g \in C^2[0,l]$:

$$g'(x) = g[\alpha_0, \alpha_1] - \int\limits_{\alpha_0}^{\alpha_1} K_0(\xi, x) g''(\xi) d\xi, \qquad \alpha_0 < \alpha_1, \tag{2.4}$$

where

$$g(\alpha_0; \alpha_1) = \frac{g(\alpha_1) - g(\alpha_0)}{\alpha_1 - \alpha_0},$$
$$K_0(\xi, x) = T_0(\xi - x) - (\alpha_1 - \alpha_0)^{-1}(\xi - \alpha_0)$$

and

$$T_0(\lambda) = \begin{cases} 1, & \lambda \ge 0 \\ 0, & \lambda < 0. \end{cases}$$

Equality (2.4) with the values $g(x)=u(x)$, $x=0$, $\alpha_0=0$, and $\alpha_1=\sqrt{\varepsilon}$ yields

$$|u'(0)| \le \frac{u(\sqrt{\varepsilon})-u(0)}{\sqrt{\varepsilon}} - \int_0^{\sqrt{\varepsilon}} K_0(\xi,0)u''(\xi)d\xi \le \frac{C}{\sqrt{\varepsilon}}. \tag{2.5}$$

Similarly, using (2.4) for $g(x)=u(x)$, $x=l$, $\alpha_0=l-\sqrt{\varepsilon}$, and $\alpha_1=l$ we confirm that

$$|u'(l)| \le \frac{u(l)-u(\sqrt{\varepsilon})}{\sqrt{\varepsilon}} - \int_{l-\sqrt{\varepsilon}}^{l} K_0(\xi,l)u''(\xi)d\xi \le \frac{C}{\sqrt{\varepsilon}}. \tag{2.6}$$

Next, differentiating (1.1), according to (2.5) and (2.6), we get

$$-\varepsilon v''+a(x)v=F(x), \qquad v(0)=O\left(\frac{1}{\sqrt{\varepsilon}}\right), \qquad v(l)=O\left(\frac{1}{\sqrt{\varepsilon}}\right) \tag{2.7}$$

with

$$v(x)=u'(x), \qquad F(x)=f'(x)-a'(x)u(x)-\lambda\int_0^l \frac{\partial}{\partial x}K(x,s)u(s)ds.$$

By virtue of (2.2) evidently

$$|F(x)|\le C. \tag{2.8}$$

In order to estimate the solution of the problem (2.7), we present it in the form

$$v(x)=v_1(x)+v_2(x),$$

where the functions $v_1(x)$ and $v_2(x)$ are the solutions of the following problems respectively:

$$\begin{aligned} &-\varepsilon v_1''+a(x)v_1=F(x),\\ &v_1(0)=v_1(l)=0, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &-\varepsilon v_2''+a(x)v_2=0,\\ &v_2(0)=O\left(\frac{1}{\sqrt{\varepsilon}}\right), \qquad v_2(l)=O\left(\frac{1}{\sqrt{\varepsilon}}\right). \end{aligned} \tag{2.10}$$

For the solution of the problem (2.9), using the maximum principle and (2.8), we have

$$|v_1(x)|\le \alpha^{-1}\|F\|_\infty \le C, \qquad 0\le x\le l. \tag{2.11}$$

According to the maximum principle, from the problem (2.10), we also conclude that

$$|v_2(x)|\le w(x), \tag{2.12}$$

where the function $w(x)$ is the solution of the following problem:

$$\begin{gathered} -\varepsilon w'' + \alpha w = 0, \\ w(0) = |v_2(0)|, \quad w(l) = |v_2(l)|. \end{gathered} \tag{2.13}$$

The solution of problem (2.13) is given by

$$w(x) = \frac{1}{\sinh\left(\frac{\sqrt{\alpha}l}{\sqrt{\varepsilon}}\right)} \left\{ w(0)\sinh\left(\frac{\sqrt{\alpha}(l-x)}{\sqrt{\varepsilon}}\right) + w(l)\sinh\left(\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}\right) \right\}.$$

Hence, taking into consideration (2.10) we obtain

$$w(x) \leq \frac{C}{\sqrt{\varepsilon}} \left\{ e^{-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}} + e^{-\frac{\sqrt{\alpha}(l-x)}{\sqrt{\varepsilon}}} \right\}. \tag{2.14}$$

Finally, the use bounds (2.11), (2.12) and (2.14) in the inequality

$$|u'(x)| \leq |v_1(x)| + |v_2(x)|$$

immediately leads to (2.3). $\square$

3. The Mesh and Discretization

Let ω_N be an uniform mesh on $[0,l]$:

$$\omega_N = \left\{ x_i = ih,\, i = 1,2,...,N-1,\, h = \frac{l}{N} \right\}$$

and

$$\bar{\omega}_N = \omega_N \cup \{x = 0, x_N = l\}.$$

To construct the difference scheme for the problem (1.1), we start with the following identity

$$\chi_i^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}} Lu(x)\varphi_i(x)dx = \chi_i^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)dx, i = 1,2,..,N-1, \tag{3.1}$$

with the basis functions

$$\varphi(x) = \begin{cases} \varphi_i^{(1)}(x) \equiv \frac{\sinh\gamma_i(x-x_i)}{\sinh\gamma_i h}, & x_{i-1} < x < x_i, \\ \varphi_i^{(2)}(x) \equiv \frac{\sinh\gamma_i(x_{i+1}-x)}{\sinh\gamma_i h}, & x_i < x < x_{i+1}, \\ 0, & x \notin (x_{i-1}, x_{i+1}), \end{cases}$$

where

$$\gamma_i = \sqrt{\frac{a(x_i)}{\varepsilon}}, \qquad \chi_i = h^{-1}\int_{x_{i-1}}^{x_{i+1}} \varphi_i(x)dx = \frac{2tanh(\gamma_i h/2)}{\gamma_i h}.$$

We note that the functions $\varphi_i^{(1)}$ and $\varphi_i^{(2)}$ are the solutions of the following problems respectively:

$$\begin{aligned} -\varepsilon\varphi'' + a_i(x)\varphi(x) &= 0, & x_{i-1} < x < x_i \\ \varphi(x_{i-1}) &= 0, & \varphi(x_i) = 1, \\ -\varepsilon\varphi'' + a_i(x)\varphi(x) &= 0, & x_i < x < x_{i+1} \\ \varphi(x_i) &= 1, & \varphi(x_{i+1}) = 0. \end{aligned}$$

By using the method of exact difference schemes (see e.g. [1–3]), it follows that

$$\begin{aligned} &-\chi_i^{-1}h^{-1}\varepsilon\int_{x_{i-1}}^{x_{i+1}}\varphi_i(x)u''(x)dx + \chi_i^{-1}h^{-1}a_i\int_{x_{i-1}}^{x_{i+1}}\varphi_i(x)u(x)dx = \\ &\qquad -\varepsilon\chi_i^{-1}\left\{1 + a_i\varepsilon^{-1}\int_{x_{i-1}}^{x_i}\varphi_i^{(1)}(x)(x-x_i)dx\right\}u_{\bar{x}x,i} \\ &\qquad + a_i\chi_i^{-1}\left\{h^{-1}\int_{x_{i-1}}^{x_i}\varphi_i^{(1)}dx + h^{-1}\int_{x_i}^{x_{i+1}}\varphi_i^{(2)}dx\right\}u_i = -\varepsilon\theta_i u_{\bar{x}x,i} + a_i u_i \end{aligned}$$

with

$$\theta_i = \frac{a_i\rho^2}{4sinh^2\left(\sqrt{a_i}\rho/2\right)}, \qquad \rho = \frac{h}{\sqrt{\varepsilon}}.$$

Thereby

$$\chi_i^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}}\left[\varepsilon u''(x) + a(x)u(x)\right]\varphi_i(x)dx = -\varepsilon\theta_i u_{\bar{x}x,i} + a_i u_i + R_i^{(1)} \tag{3.2}$$

with remainder term

$$R_i^{(1)} = \chi_i^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}}\left[a(x) - a(x_i)\right]u(x)\varphi_i(x)dx. \tag{3.3}$$

Further for the right-side in (3.1) we have

$$\chi_i^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)dx = f_i + R_i^{(2)} \tag{3.4}$$

with remainder term

$$R_i^{(2)} = \chi_i^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}}\left[f(x) - f(x_i)\right]\varphi_i(x)dx. \tag{3.5}$$

For integral term involving kernel function, we have from (3.1)

$$\chi_i^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}} dx\varphi_i(x)\int_0^l K(x,s)u(s)ds = \int_0^l K(x_i,s)u(s)ds$$
$$+\chi_i^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}} dx\varphi_i(x)\int_0^l \left[K(x,s)-K(x_i,s)\right]u(s)ds.$$

Further using the composite right side rectangle rule, we obtain

$$\int_0^l K(x_i,s)u(s)ds = h\sum_{j=1}^{N} K_{ij}u_j - \sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(\xi-x_{j-1})\frac{\partial}{\partial\xi}\left[K(x_i,\xi)u(\xi)\right]d\xi.$$

Therefore we get

$$\chi_i^{-1}h^{-1}\int_{x_{i-1}}^{x_{i+1}} dx\varphi_i(x)\int_0^l K(x,s)u(s)ds = h\sum_{j=1}^{N} K_{ij}u_j + R_i^{(3)} \tag{3.6}$$

with remainder term

$$\begin{aligned} R_i^{(3)} =& \chi_i^{-1}h^{-1}\lambda\int_{x_{i-1}}^{x_{i+1}} dx\varphi_i(x)\int_0^l \left[K(x,s)-K(x_i,s)\right]u(s)ds \\ &-\lambda\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(\xi-x_{j-1})\frac{\partial}{\partial\xi}\left[K(x_i,\xi)u(\xi)\right]d\xi. \end{aligned} \tag{3.7}$$

The relations (3.2), (3.4) and (3.6) yield the following exact relation for $u(x_i)$

$$L_N u_i := -\varepsilon\theta_i u_{\bar{x}x,i} + a_i u_i + \lambda h\sum_{j=1}^{N} K_{ij}u_j + R_i = f_i, \qquad 1\le i\le N-1 \tag{3.8}$$

with remainder term

$$R_i = R_i^{(1)} + R_i^{(2)} + R_i^{(3)}, \tag{3.9}$$

where $R_i^{(k)};(k=1,2,3)$ are defined by (3.3), (3.5) and (3.7) respectively. Based on (3.8) we propose the following difference scheme for approximating (1.1).

$$\begin{aligned} L_N y_i :=& -\varepsilon\theta_i y_{\bar{x}x,i} + a_i y_i + \lambda h\sum_{j=1}^{N} K_{ij}y_j = f_i, \qquad 1\le i\le N-1, \\ y_0 =& A, \qquad y_N = B. \end{aligned} \tag{3.10}$$

4. Error Analysis

From (3.8) and (3.10) for the error of the approximate solution $z_i = y_i - u_i$ we have

$$\begin{aligned} L_N z_i :=& -\varepsilon\theta_i z_{\bar{x}x,i} + a_i z_i + \lambda h \sum_{j=1}^{N} K_{ij} z_j = R_i, \qquad 1 \le i \le N-1, \\ z_0 =& 0, \qquad z_N = 0. \end{aligned} \tag{4.1}$$

where R_i are defined by (3.9).

Theorem 1. *Under the conditions of Lemma (2.1) and*

$$|\lambda| < \frac{\alpha}{\max\limits_{1\le i\le N} \sum\limits_{j=1}^{N} h|K_{ij}|},$$

the solution of (3.10) converges ε-uniformly to the solution of (1.1). For the error of approximate solution the following estimate hols

$$||y-u||_{\infty,\bar{\omega}_N} \le Ch.$$

Proof. Applying the maximum principle, from (4.1) we have

$$\begin{aligned} ||z||_{\infty,\bar{\omega}_N} &\le \alpha^{-1}||R - \lambda h \sum_{j=1}^{N} K_{ij} z_j||_{\infty,\omega_N} \\ &\le \alpha^{-1}||R||_{\infty,\omega_N} + |\lambda|\alpha^{-1} \max_{1\le i\le N} \sum_{j=1}^{N} h|K_{ij}|||z||_{\infty,\bar{\omega}_N}, \end{aligned}$$

hence

$$||z||_{\infty,\bar{\omega}_N} \le \frac{\alpha^{-1}||R||_{\infty,\omega_N}}{1 - |\lambda|\alpha^{-1} \max\limits_{1\le i\le N} \sum\limits_{j=1}^{N} h|K_{ij}|}$$

which implies of

$$||z||_{\infty,\bar{\omega}_N} \le C||R||_{\infty,\omega_N}. \tag{4.2}$$

Further we estimate $R_i^{(1)}$, $R_i^{(2)}$ and $R_i^{(3)}$ seperately. For $a(x)$, by the mean value theorem, we have

$$|a(x) - a(x_i)| \le |a'(\xi_i)||x - x_i| \le Ch, \qquad x_i \le \xi_i \le x.$$

Thereby for $R_i^{(1)}$, by $a \in C^1[0,l]$ and (2.2) we get

$$\begin{aligned} \left|R_i^{(1)}\right| &\leq \chi_i^{-1} h^{-1} \left| \int_{x_{i-1}}^{x_{i+1}} [a(x) - a(x_i)]\, u(x) \varphi_i(x) dx \right| \\ &\leq C h \chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_{i+1}} \varphi_i(x) dx = Ch. \end{aligned} \tag{4.3}$$

Similarly, for $R_i^{(2)}$ we get

$$\left|R_i^{(2)}\right| \leq Ch. \tag{4.4}$$

Finally for $R_i^{(3)}$, taking into account the boundedness of $\frac{\partial K}{\partial x}$ and (2.3) it follows that

$$\begin{aligned} \left|R_i^{(3)}\right| \leq & \chi_i^{-1} h^{-1} |\lambda| \int_{x_{i-1}}^{x_{i+1}} dx \varphi_i(x) \int_0^l |K(x_i,s) - K(x,s)|\, |u(s)|\, ds \\ & + |\lambda| \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} (\xi - x_{j-1}) \left| \frac{\partial}{\partial \xi} [K(x_i,\xi) u(\xi)] \right| d\xi \\ & \leq \chi_i^{-1} h^{-1} |\lambda| \int_{x_{i-1}}^{x_{i+1}} dx \varphi_i(x) \int_0^l (x - x_i) \left| \frac{\partial}{\partial \xi} K(\xi,s) u(s) \right| ds \\ & + |\lambda|\, h \int_0^l \left| \frac{\partial}{\partial \xi} [K(x_i,\xi) u(\xi)] \right| d\xi \\ & \leq Ch|\lambda| + |\lambda|\, h \int_0^l \left\{ \left| \frac{\partial K(x_i,\xi)}{\partial \xi} \right| |u(\xi)| + |K(x_i,\xi)|\, |u'(\xi)| \right\} d\xi \\ & \leq C \left\{ h + h \int_0^l \left(1 + \frac{1}{\sqrt{\varepsilon}} \left(e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}} + e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}} \right) \right) d\xi \right\} \leq Ch. \end{aligned} \tag{4.5}$$

Thus from (4.3)-(4.5) we see easily the estimate

$$||R||_{\infty} \leq Ch. \tag{4.6}$$

The bound (4.6) together with (4.2) completes the proof. $\square$

5. Numerical Results

Consider the particular problem with

$$a(x)=1,\qquad K(x,s)=x,\qquad f(x)=x-\varepsilon+\varepsilon e^{-\frac{x}{\varepsilon}},\qquad x\in(0,1),$$
$$\lambda=\frac{1}{2},\qquad A=1,\qquad B=2-\varepsilon+\varepsilon e^{-\frac{1}{\varepsilon}},$$

The exact solution is given by

$$u(x)=\frac{e^{-\frac{x}{\sqrt{\varepsilon}}}+e^{\frac{x-1}{\sqrt{\varepsilon}}}-e^{\frac{x-2}{\sqrt{\varepsilon}}}-e^{-\frac{x+1}{\sqrt{\varepsilon}}}}{1-e^{-\frac{2}{\sqrt{\varepsilon}}}}+x-\varepsilon+\varepsilon e^{-\frac{x}{\sqrt{\varepsilon}}}.$$

We define the exact error e_ε^h and the computed ε-uniform maximum pointwise error e^h as follows:

$$e_\varepsilon^h=\|y-u\|_\infty,\qquad e^h=\max_\varepsilon e_\varepsilon^h.$$

We also define the computed parameter-uniform rate of convergence to be

$$p^h=\ln\left(e^h/e^{h/2}\right)/\ln 2.$$

The resulting errors e^h and the corresponding numbers p^h for various values ε and h are listed in Table 1.

Table 1 Exact errors e_ε^h, computed ε-uniform errors e^h and convergence rates p^h on ω_N.

ε	$h=1/32$	$h=1/64$	$h=1/128$	$h=1/256$	$h=1/512$	$h=1/1024$
1	0.00343868	0.00198874	0.00110332	0.00060368	0.00030394	0.00015197
	0.79	0.85	0.87	0.99	1.00	
2^{-4}	0.01032126	0.00605257	0.00338123	0.00185003	0.00094445	0.00047551
	0.77	0.84	0.87	0.97	0.99	
2^{-8}	0.01125894	0.00660244	0.00368841	0.0020181	0.00103025	0.00051871
	0.77	0.84	0.87	0.97	0.99	
2^{-12}	0.011200979	0.00656845	0.00366942	0.00200771	0.00102495	0.00051604
	0.77	0.84	0.87	0.97	0.99	
2^{-16}	0.0112049	0.00657075	0.00367071	0.00200842	0.00102531	0.00051622
	0.77	0.84	0.87	0.97	0.99	
e^h	0.01125894	0.00660244	0.00368841	0.0020181	0.00103025	0.00051622
p^h	0.77	0.84	0.87	0.97	0.99	

The obtained results show that the convergence rate of difference scheme is essentially in accord with the theoretical analysis.

6. Conclusion

A boundary-value problem for a second order singularly perturbed Fredholm integro-differential equation has been considered. For the numerical solution of this problem, we proposed a fitted finite difference scheme on a uniform type mesh. The difference scheme is constructed by the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form. It is shown that the method displays uniform convergence independently of the perturbation parameter in the discrete maximum norm. We have implemented the present method on a example. Numerical results were carried out to show the efficiency and accuracy of the method.

References

[1] G. M. Amiraliyev and Y. D. Mamedov, "Difference schemes on the uniform mesh for singularly perturbed pseudo-parabolic equations." *Turkish J. Math.*, vol. 19, pp. 207–222, 1995.

[2] G. M. Amiraliyev and Ö. Yapman, "On the Volterra delay-integro-differential equation with layer behavior and its numerical solution." *Miskolc Math. Notes*, vol. 20, no. 1, pp. 75–87, 2019, doi: 10.18514/mmn.2019.2424.

[3] G. M. Amiraliyev and B. Yilmaz, "Fitted difference method for a singularly perturbed initial value problem." *Int. J. Math. Comput.*, vol. 22, no. 1, pp. 1–10, 2014.

[4] R. Bellman and K. L. Cooke, *Differential difference equations.*, New York, 1963.

[5] M. C. Bonis and C. Laurita, "Numerical treatment of second kind Fredholm integral equations systems on bounded intervals." *J. Comput. Appl. Math.*, vol. 217, no. 1, pp. 64–87, 2008, doi: 10.1016/j.cam.2007.06.014.

[6] M. J. Emamzadeh and M. T. Kajani, "Nonlinear Fredholm integral equation of the second kind with quadrature methods." *Journal of Mathematical Extension*, vol. 4, no. 2, pp. 51–58, 2010.

[7] M. Feckan, "Singularly perturbed higher order boundary value problems." *J. Differential Equations*, vol. 111, pp. 79–102, 1994, doi: 10.1006/jdeq.1994.1076.

[8] O. A. Gegele, O. P. Evans, and D. Akoh, "Numerical solution of higher order linear Fredholm-integro-differential equations." *American Journal of Engineering Research*, vol. 3, no. 8, pp. 243–247, 2014.

[9] Z. Jackiewicz, M. Rahman, and B. D. Welfert, "Numerical solution of a Fredholm integro-differential equation modelling θ neural networks." *Appl. Math. Comput.*, vol. 195, no. 2, pp. 523–536, 2008, doi: 10.1016/j.amc.2007.05.031.

[10] M. F. Karim, M. Mohamad, M. S. Rusiman, N. Che–him, R. Roslan, and K. Khalid, "ADM for solving linear second-order Fredholm integro-differential equations." *Journal of Physics: Conference Series*, vol. 995, p. 012009, 2018, doi: 10.1088/1742-6596/995/1/012009.

[11] S. Karimi and M. Jozi, "Numerical solution of the system of linear Fredholm integral equations based on degenerating kernels." *J. Pure Appl. Math.*, vol. 6, no. 1, pp. 111–119, 2015.

[12] M. Kudu, I. Amirali, and G. M. Amiraliyev, "A finite-difference method for a singularly perturbed delay integro-differential equation." *J. Comput. Appl. Math.*, vol. 308, pp. 379–390, 2016, doi: 10.1016/j.cam.2016.06.018.

[13] N. Kurt and M. Sezer, "Polynomial solution of high-order Fredholm integro-differential equations with constant coeffcients." *J. Franklin Inst.*, vol. 345, no. 8, pp. 839–850, 2008, doi: 10.1016/j.jfranklin.2008.04.016.

[14] P. Kythe and P. Puri, *Computational methods for linear integral equations.* Boston: Birkhäuser Basel, 2002. doi: 10.1007/978-1-4612-0101-4.

[15] J. J. H. Miller, E. O'Riordan, and G. I. Shishkin, *Fitted Numerical methods for singular perturbation problems.* Singapore: World Scientific, 1996.
[16] A. H. Nayfeh, *Introduction to perturbation techniques.* New York: Wiley, 1993.
[17] R. E. O'Malley, *Singular perturbation methods for ordinary differential equations.* New York: Springer-Verlag, 1991.
[18] A. D. Polyanin and A. V. Manzhirov, *Handbook of integral equations.* Boca Raton, Fla, USA: Chapman and Hall/CRC, 2008.
[19] H. G. Roos, M. Stynes, and L. Tobiska, *Numerical methods for singularly perturbed differential equations.* Berlin: Springer-Verlag, 1996.
[20] Q. Xue, J. Niu, D. Yu, and C. Ran, "An improved reproducing kernel method for Fredholm integro-differential type two-point boundary value problems." *Int. J. Comput. Math.*, vol. 95, no. 5, pp. 1015–1023, 2018, doi: 10.1080/00207160.2017.1322201.

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OPTIMALITY CONDITIONS AND DUALITY RESULTS FOR A NEW CLASS OF NONCONVEX NONSMOOTH VECTOR OPTIMIZATION PROBLEMS

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Abstract. In this paper, a new class of nonconvex nonsmooth multiobjective programming problems with both inequality and equality constraints defined in a real Banach space is considered. Under the nondifferentiable vectorial $(\Phi,\rho)^w$-invexity notion introduced in the paper, optimality conditions and duality results in Mond-Weir sense are established for the considered nonsmooth vector optimization problem. It turns out that the results developed here under $(\Phi,\rho)^w$-invexity are applicable for a larger class of nonconvex nondifferentiable multiobjective programming problems than under several generalized convexity notions existing in the literature.

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Keywords: nonsmooth mutiobjective programming, efficient solution, nonsmooth $(\Phi,\rho)^w$-invexity, optimality conditions, Mond-Weir duality

1. INTRODUCTION

The term multiobjective programming (or vector optimization) is used to denote a type of optimization problems where two or more objectives are to be minimized subject to certain constraints. Investigation on sufficiency of (weak) Pareto optimality and duality has been one of the most attraction topics in the theory of multi-objective problems. This is a consequence of the fact that vector optimization problems are useful mathematical models of most real-life problems in economics, physics, mechanics, decision making, game theory, engineering, optimal control, etc. It is well known that the concept of convexity and its various generalizations play an important role in deriving sufficient optimality conditions and duality results for multiobjective programming problems. In recent years, therefore, multiobjective programming has grown remarkably in different directions in the settings of optimality conditions and duality theory. It has been enriched by the applications of various types of generalizations of convexity theory, with and without differentiability assumptions (see, for example, [1, 2, 4, 5, 7, 8, 11, 13–16, 18, 20, 21] and others).

The aim of the present work is to introduce a new concept of nondifferentiable generalized invexity notion and to use it to prove optimality and duality results for a new class of nonsmooth multiobjective programming problems defined in a real Banach space. By taking the motivation from Antczak and Stasiak [6] and Stefanescu and Stefanescu [22], we introduce the concept of nondifferentiable $(\Phi,\rho)^w$-invexity for a nonsmooth multiobjective programming problem in which every component of functions involved is a locally Lipschitz function. However, the central purpose of this paper is to discuss application of the introduced vectorial nondifferentiable $(\Phi,\rho)^w$-invexity notion in proving the optimality results for a new class of nonconvex nondifferentiable multiobjective programming problems. Namely, we prove Karush-Kuhn-Tucker necessary optimality conditions for a (weak) Pareto optimal solution in the considered nondifferentiable multiobjective programming problem in which constraint functions are $(\Phi,\rho)^w$-invex. Sufficiency of these necessary optimality conditions for both weak Pareto and Pareto solutions is established for the class of constrained vector optimization problems with nondifferentiable $(\Phi,\rho)^w$-invex functions, not necessarily, with respect to the same ρ. Further, under $(\Phi,\rho)^w$-invexity hypotheses, several duality results are established between the considered nonsmooth multiobjective programming problem and its nondifferentiable vector dual problem in the sense of Mond-Weir. The optimality results proved in the paper are illustrated by an example of a nonconvex nonsmooth vector optimization problem involving nondifferentiable $(\Phi,\rho)^w$-invex functions.

2. Preliminaries

Throughout this paper, we use the following conventions for vectors $x = (x_1,x_2,...,x_n)^T$, $y = (y_1,y_2,...,y_n)^T$ in the Euclidean space R^n:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1,2,...,n$;
(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1,2,...,n$;
(iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1,2,...,n$;
(iv) $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

In this section, we provide some definitions and some results that we shall use in the sequel. Throughout this paper, we denote a real Banach space by X, the (continuous) dual of X by X^*, and the value of the function ξ in X^* at $v \in X$ by $\langle \xi, v\rangle$.

Definition 1 ([9])**.** The Clarke generalized directional derivative of a locally Lipschitz function $f : X \to R$ at $x \in X$ in the direction $v \in X$, denoted by $f^0(x;v)$, is given by

$$f^0(x;v) = \limsup_{\substack{y\to x\\ \lambda\downarrow 0}} \frac{f(y+\lambda v)-f(y)}{\lambda}.$$

Definition 2 ([9])**.** The Clarke generalized subgradient of a locally Lipschitz function $f : X \to R$ at $x \in X$, denoted by $\partial f(x)$, is defined as follows

$$\partial f(x) = \left\{\xi \in X^* : f^0(x;v) \geq \langle \xi, v\rangle \text{ for all } v \in X\right\}.$$

Let S be a nonempty convex subset of X.

Definition 3. The function $\Phi : S \to R$ is said to be quasi-convex if, for each $\alpha \in R$, the level set $\{x \in S : \Phi(x) \leqq \alpha\}$ is convex, or equivalently, if $\Phi(\lambda y + (1-\lambda)x) \leqq \max\{\Phi(y), \Phi(x)\}$ for every $y, x \in S$ and $\lambda \in [0,1]$.

A stronger property is also considered as follows:

Definition 4. The function $\Phi : S \to R$ is said to be strictly quasi-convex if it is quasi-convex and $\Phi(\lambda y + (1-\lambda)x) < 0$, whenever $\Phi(y) < 0$, $\Phi(x) \leqq 0$ and $\lambda \in (0,1)$.

Proposition 1. *If $\Phi : S \to R$ is a strictly quasi-convex function and there are $x^1, ..., x^k \in S$ such that $\Phi\left(x^i\right) \leqq 0$, $i = 1, ..., k$ and $\Phi\left(x^{i^*}\right) < 0$ for at least one $i^* \in \{1, ..., k\}$, then $\Phi\left(\sum_{i=1}^{k} \lambda_i x_i\right) < 0$ for every $\lambda = (\lambda_1, ..., \lambda_k) \geq 0$ such that $\sum_{i=1}^{k} \lambda_i = 1$ and $\lambda_{i^*} > 0$.*

In [22], Stefanescu and Stefanescu introduced the definition of a differentiable $(\Phi, \rho)^w$-invex function. Now, in the natural way, we generalize this definition to the nondifferentiable vectorial case.

Definition 5. Let $f = (f_1, ..., f_k) : X \to R^k$ be defined on X, every its component f_i, $i = 1, ..., k$, be a locally Lipschitz function on X and $u \in X$. If there exist a function $\Phi : X \times X \times X^* \times R \to R$, where $\Phi(x, u, (\cdot, \cdot))$ is strictly quasi-convex on $X^* \times R$, $\Phi(x, u, (0, a)) \geqq 0$ for all $x \in X$ and each $a \in R_+$ and $\rho = (\rho_1, ..., \rho_k) \in R^k$, where ρ_i, $i = 1, ..., k$, are real numbers such that the following inequalities

$$f_i(x) - f_i(u) \geqq \Phi(x, u, (\xi_i, \rho_i)), \; i = 1, ..., k \quad (>) \tag{2.1}$$

hold for all $x \in X$ $(x \neq u)$ and each $\xi_i \in \partial f_i(u)$, $i = 1, ..., k$, then f is said to be a nonsmooth vector $(\Phi, \rho)^w$-invex at u on X. If inequalities (2.1) are satisfied at any point u, then f is said to be a nonsmooth (strictly) vector $(\Phi, \rho)^w$-invex function on X.

In order to define an analogous class of nondifferentiable vector (strictly) $(\Phi, \rho)^w$-incave functions, the direction of the inequality in the definition of these functions should be changed to the opposite one.

Remark 1. Let $X \subseteq R^n$ and the functional $\Phi(x, u, (\cdot, \cdot))$ be convex on $R \times R$. From Definition 5, there are the following special cases:

a) If $\Phi(x, u, (\xi_i, \rho_i)) = \langle \xi_i, x - u \rangle$, where $\xi_i \in \partial f_i(u)$, $i = 1, ..., k$, then we obtain the definition of a nondifferentiable convex function.
b) If $\Phi(x, u, (\xi_i, \rho_i)) = \langle \xi_i, \eta(x, u) \rangle$ for a certain mapping $\eta : X \times X \to R^n$, where $\xi_i \in \partial f_i(u)$, $i = 1, ..., k$, then we obtain the definition of a locally Lipschitz invex function (with respect to the function η) (see Lee [16] and Kim and Schaible [14] in a nonsmooth vectorial case).
c) If $\Phi(x, u, (\xi_i, \rho_i)) = \frac{1}{b_i(x,u)} \langle \xi_i, \eta(x, u) \rangle$, where $b_i : X \times X \to R_+ \backslash \{0\}$ and $\eta : X \times X \to R^n$, then we obtain the definition of a nondifferentiable b-invex

function (with respect to the function η) (see, Li et al. [17] in a nondifferentiable scalar case).

d) If $\Phi(x,u,(\xi_i,\rho_i)) = \langle \xi_i, x-u\rangle + \rho_i \|x-u\|^2$, then $(\Phi,\rho)^w$-invexity reduces to the definition of a nonsmooth ρ-convex function (see Zalmai [23]).

e) If $\Phi(x,u,(\xi_i,\rho_i)) = \langle \xi_i, \eta(x,u)\rangle + \rho_i \|\theta(x,u)\|^2$ for a certain mapping $\eta : X\times X \to R^n$, where $\theta : X\times X \to R^n$, $\theta(x,u)\neq 0$, whenever $x\neq u$, then $(\Phi,\rho)^w$-invexity reduces to the definition of a nonsmooth ρ-invex function (with respect to η and θ) introduced by Jeyakumar [12] in a scalar case.

f) If $\Phi(x,u,(\xi_i,\rho_i)) = \alpha_i(x,u)\langle \xi_i, \eta(x,u)\rangle$, where $\alpha_i : X\times X \to R_+\backslash\{0\}$, $\theta : X\times X \to R^n$, $\theta(x,u)\neq 0$, whenever $x\neq u$, then $(\Phi,\rho)^w$-invexity reduces to the definition of a V-invex function (with respect to η) introduced by Jeyakumar and Mond [13] in a differentiable case and Mishra and Mukherjee [19] in a nonsmooth case.

g) If $\Phi(x,u,(\xi_i,\rho_i)) = \alpha_i(x,u)\langle \xi_i, \eta(x,u)\rangle + \rho_i \|\theta(x,u)\|^2$, where $\alpha_i : X\times X \to R_+\backslash\{0\}$, $\theta : X\times X \to R^n$, $\theta(x,u)\neq 0$, whenever $x\neq u$, then $(\Phi,\rho)^w$-invexity reduces to the definition of a nonsmooth V-ρ-invex function (with respect to η and θ) introduced by Kuk et al. [15].

h) If $\Phi(x,u,(\xi_i,\rho_i)) = F(x,u,\xi_i)$, where $F(x,u,\cdot)$ is a sublinear functional with respect to the third component, then the definition of a $(\Phi,\rho)^w$-invex function reduces to the definition F-convexity introduced by Hanson and Mond [10].

i) If $\Phi(x,u,(\xi_i,\rho_i)) = F(x,u,\xi_i) + \rho_i d^2(x,u)$, where $F(x,u,\cdot)$ is a sublinear functional with respect to the third component and $d : X\times X \to R$ is a pseudometric on X, then the definition of a $(\Phi,\rho)^w$-invex function reduces to the definition (F,ρ)-convexity introduced in a nondifferentiable case by Bhatia and Jain [8] .

j) If the functional $\Phi(x,u,(\cdot,\cdot))$ is convex on $X^*\times R$, then we obtain the definition of a nondifferentiable (Φ,ρ)-invex function (see Antczak and Stasiak [6] in a scalar case).

3. Optimality under nonsmooth $(\Phi,\rho)^w$-invexity

In the paper, we consider the following nonsmooth vector optimization problem:

$$\begin{aligned} & f(x) := (f_1(x),...,f_k(x)) \to V\text{-min} \\ & \text{subject to} \quad g_j(x) \leqq 0,\ \ j\in J = \{1,...m\}, \\ & h_t(x) = 0,\ t\in T = \{1,...,q\}, \\ & x \in X, \end{aligned} \tag{VP}$$

where $f_i : X\to R$, $i\in I = \{1,...,k\}$, $g_j : X\to R$, $j\in J$, and $h_t : X\to R$, $t\in T$ are locally Lipschitz functions on X. For the purpose of simplifying our presentation, we will next introduce some notations which will be used frequently throughout this paper. Let $D := \{x\in X : g_j(x)\leqq 0,\ \ j\in J,\ h_t(x) = 0,\ t\in T\}$ be the set of all feasible solutions in problem (VP), and $J(\overline{x})$ be a set of active inequality constraints at point $\overline{x} \in D$, that is, $J(\overline{x}) = \{ j \in J : g_j(\overline{x}) = 0 \}$.

Definition 6. A feasible point $\overline{x}$ is said to be a weak Pareto solution (weakly efficient solution, weak minimum) for (VP) if and only if there exists no other $x \in D$ such that $f(x) < f(\overline{x})$.

Definition 7. A feasible point $\overline{x}$ is said to be a Pareto solution (efficient solution) for (VP) if and only if there exists no other $x \in D$ such that $f(x) \leq f(\overline{x})$.

Theorem 1 (Generalized F. John necessary optimality conditions, [9]). *Let $\overline{x} \in D$ be a (weakly) efficient solution of the considered nonsmooth multiobjective programming problem (VP). Then, there exist $\overline{\lambda} \in R^k$, $\overline{\mu} \in R^m$ and $\overline{\vartheta} \in R^q$ such that*

$$0 \in \sum_{i=1}^{k} \overline{\lambda}_i \partial f_i(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j \partial g_j(\overline{x}) + \sum_{t=1}^{q} \overline{\vartheta}_t \partial h_t(\overline{x}), \tag{3.1}$$

$$\overline{\mu}_j g_j(\overline{x}) = 0, \ j \in J, \tag{3.2}$$

$$\left(\overline{\lambda}, \overline{\mu}\right) \geq 0. \tag{3.3}$$

Now, we prove the so-called Generalized Karush-Kuhn-Tucker necessary optimality conditions for a nonsmooth vector optimization problem with locally Lipschitz $(\Phi, \rho)^w$-invex constraint functions.

Theorem 2 (Generalized Karush-Kuhn-Tucker necessary optimality conditions). *Let $\overline{x} \in D$ be a (weakly) efficient solution of the considered multiobjective programming problem (VP) and Generalized F. John necessary optimality conditions (3.1)-(3.3) be satisfied at $\overline{x}$ with Lagrange multipliers $\overline{\lambda} \in R^k$, $\overline{\mu} \in R^m$ and $\overline{\vartheta} \in R^q$. Further, assume that there exists a feasible solution $\widetilde{x}$ such that $g_j(\widetilde{x}) < 0$, $j \in J(\overline{x})$ and, moreover, g_j, $j \in J(\overline{x})$, is locally Lipschitz $\left(\Phi, \rho_{g_j}\right)^w$-invex at $\overline{x}$ on D, h_t, $t \in T^+(\overline{x}) := \{ t \in T : \overline{\vartheta}_t > 0 \}$, is locally Lipschitz $(\Phi, \rho_{h_t})^w$-invex at $\overline{x}$ on D $-h_t$, $t \in T^-(\overline{x}) := \{ t \in T : \overline{\vartheta}_t < 0 \}$, is locally Lipschitz $(\Phi, \rho_{h_t})^w$-invex at $\overline{x}$ on D and $\sum_{j=1}^{m} \overline{\mu}_j \rho_{g_j} + \sum_{t \in T^+(\overline{x})} \overline{\vartheta}_t \rho_{h_t}^+ - \sum_{t \in T^-(\overline{x})} \overline{\vartheta}_t \rho_{h_t}^- \geqq 0$. Then $\overline{\lambda} \neq 0$.*

Proof. Let $\overline{x} \in D$ be an efficient (weakly efficient) solution of the considered multiobjective programming problem (VP). Then, the necessary optimality conditions of F. John type (3.1)-(3.3) are fulfilled with the Lagrange multipliers $\overline{\lambda} \in R^k$, $\overline{\mu} \in R^m$ and $\overline{\vartheta} \in R^q$ (see, for example, [9]). We prove that $\overline{\lambda} \neq 0$. Suppose, contrary to the result, that $\overline{\lambda} = 0$. Hence, as it follows from the necessary optimality conditions of F. John type (3.3), we have $\left(\overline{\lambda}, \overline{\mu}\right) \geq 0$. Since $\overline{\lambda} = 0$, the above relation implies that $\overline{\mu} \neq 0$. Using $\overline{\lambda} = 0$ together with the necessary optimality conditions of F. John type (3.1), we get

$$0 \in \sum_{j=1}^{m} \overline{\mu}_j \partial g_j(\overline{x}) + \sum_{t=1}^{q} \overline{\vartheta}_t \partial h_t(\overline{x}). \tag{3.4}$$

By (3.4), there exist $\zeta_j \in \partial g_j(\overline{x})$, $j \in J$ and $\varsigma_t \in \partial h_t(\overline{x})$, $t \in T$ such that

$$0 = \sum_{j=1}^{m} \overline{\mu}_j \zeta_j + \sum_{t=1}^{q} \overline{\vartheta}_t \varsigma_t. \tag{3.5}$$

Since $\overline{\mu} \neq 0$, we have that $A = \sum_{j=1}^{m} \overline{\mu}_j + \sum_{t \in T^+(\overline{x})} \overline{\vartheta}_t - \sum_{t \in T^-(\overline{x})} \overline{\vartheta}_t > 0$. Let us denote

$$\widehat{\beta}_j = \frac{\overline{\mu}_j}{\sum_{j=1}^{m} \overline{\mu}_j + \sum_{t \in T^+(\overline{x})} \overline{\vartheta}_t - \sum_{t \in T^-(\overline{x})} \overline{\vartheta}_t}, \ j \in J(\overline{x}), \tag{3.6}$$

$$\widehat{\gamma}_t^+ = \frac{\overline{\vartheta}_t}{\sum_{j=1}^{m} \overline{\mu}_j + \sum_{t \in T^+(\overline{x})} \overline{\vartheta}_t - \sum_{t \in T^-(\overline{x})} \overline{\vartheta}_t}, \ t \in T^+(\overline{x}), \tag{3.7}$$

$$\widehat{\gamma}_t^- = \frac{-\vartheta_t}{\sum_{j=1}^{m} \overline{\mu}_j + \sum_{t \in T^+(\overline{x})} \overline{\vartheta}_t - \sum_{t \in T^-(\overline{x})} \overline{\vartheta}_t}, \ t \in T^-(\overline{x}). \tag{3.8}$$

By (3.6)-(3.8), it follows that $\beta = \left(\widehat{\beta}_1, \ldots, \widehat{\beta}_m\right) \geq 0$, $0 \leqq \widehat{\beta}_j \leqq 1$, $j \in J(\overline{x})$, $0 \leqq \widehat{\gamma}_t^+ \leqq 1$, $t \in T^+(\overline{x})$, $0 \leqq \widehat{\gamma}_t^- \leqq 1$, $t \in T^-(\overline{x})$, and, moreover, $\sum_{j \in J(\overline{x})} \widehat{\beta}_j + \sum_{t \in T^+(\overline{x})} \widehat{\gamma}_t^+ + \sum_{t \in T^-(\overline{x})} \widehat{\gamma}_t^- = 1$. By assumption, g_j, $j \in J(\overline{x})$, is $\left(\Phi, \rho_{g_j}\right)^w$-invex at $\overline{x}$ on D, h_t, $t \in T^+(\overline{x}) := \left\{t \in T : \overline{\vartheta}_t > 0\right\}$, is locally Lipschitz $\left(\Phi, \rho_{h_t}^+\right)^w$-invex at $\overline{x}$ on D, $-h_t$, $t \in T^-(\overline{x}) := \left\{t \in T : \overline{\vartheta}_t < 0\right\}$, is locally Lipschitz $\left(\Phi, \rho_{h_t}^-\right)^w$-invex at $\overline{x}$ on D. Further, as it follows from the assumption, there exists $\widetilde{x} \in D$ such that $g_j(\widetilde{x}) < 0$, $j \in J(\overline{x})$. Hence, by Definition 5, it follows that the following inequalities

$$g_j(\widetilde{x}) - g_j(\overline{x}) \geqq \Phi\left(\widetilde{x}, \overline{x}, \left(\zeta_j, \rho_{g_j}\right)\right), \quad j \in J(\overline{x}), \tag{3.9}$$

$$h_t(\widetilde{x}) - h_t(\overline{x}) \geqq \Phi\left(\widetilde{x}, \overline{x}, \left(\varsigma_t, \rho_{h_t}^+\right)\right), \quad t \in T^+(\overline{x}), \tag{3.10}$$

$$-h_t(\widetilde{x}) + h_t(\overline{x}) \geqq \Phi\left(\widetilde{x}, \overline{x}, \left(-\varsigma_t, \rho_{h_t}^-\right)\right), \quad t \in T^-(\overline{x}). \tag{3.11}$$

hold for each $\xi_i \in \partial f_i(\overline{x})$, $i \in I(\overline{x})$, $\zeta_j \in \partial g_j(\overline{x})$, $j \in J(\overline{x})$, $\varsigma_t \in \partial h_t(\overline{x})$, $t \in T^+(\overline{x}) \cup T^-(\overline{x})$. Combining $g_j(\widetilde{x}) < 0$, $j \in J(\overline{x})$ and (3.9), we get

$$\Phi\left(\widetilde{x}, \overline{x}, \left(\zeta_j, \rho_{g_j}\right)\right) < 0, \quad j \in J(\overline{x}). \tag{3.12}$$

By $\widetilde{x} \in D$, $\overline{x} \in D$, inequalities (3.10) and (3.11) yield, respectively

$$\Phi\left(\widetilde{x}, \overline{x}, \left(\varsigma_t, \rho_{h_t}^+\right)\right) \leqq 0, \quad t \in T^+(\overline{x}), \tag{3.13}$$

$$\Phi\left(\widetilde{x}, \overline{x}, \left(-\varsigma_t, \rho_{h_t}^-\right)\right) \leqq 0, \quad t \in T^-(\overline{x}). \tag{3.14}$$

By Definition 5, we have that $\Phi(\widetilde{x}, \overline{x}, (\cdot, \cdot))$ is a strictly quasi-convex function on R^{n+1}. Since (3.12)-(3.14) are satisfied, by Proposition 1, it follows that

$$\Phi\left(\widetilde{x}, \overline{x}, \left(\sum_{j \in J(\overline{x})} \widehat{\beta}_j \zeta_j + \sum_{t \in T^+(\overline{x})} \widehat{\gamma}_t^+ \varsigma_t + \sum_{t \in T^-(\overline{x})} \widehat{\gamma}_t^- (-\varsigma_t)\right.\right., \tag{3.15}$$

$$\sum_{j\in J(\bar{x})}\widehat{\beta}_j\rho_{g_j}+\sum_{t\in T^+(\bar{x})}\widehat{\gamma}_t^+\rho_{h_t}^+ +\sum_{t\in T^-(\bar{x})}\widehat{\gamma}_t^-\rho_{h_t}^-\Bigg)\Bigg)<0.$$

Taking into account (3.6)-(3.8) and Lagrange multipliers equal to 0 in (3.15), we get

$$\Phi\left(\widetilde{x},\bar{x},\frac{1}{A}\left(\sum_{j=1}^{m}\bar{\mu}_j\zeta_j+\sum_{t=1}^{q}\overline{\vartheta}_t\varsigma_t,\sum_{j=1}^{m}\bar{\mu}_j\rho_{g_j}+\sum_{t\in T^+(\bar{x})}\overline{\vartheta}_t\rho_{h_t}^+-\sum_{t\in T^-(\bar{x})}\overline{\vartheta}_t\rho_{h_t}^-\right)\right)<0. \tag{3.16}$$

By the necessary optimality condition of F. John type (3.1), it follows that

$$\Phi\left(\widetilde{x},\bar{x},\frac{1}{A}\left(0,\sum_{j=1}^{m}\bar{\mu}_j\rho_{g_j}+\sum_{t\in T^+(\bar{x})}\overline{\vartheta}_t\rho_{h_t}^+-\sum_{t\in T^-(\bar{x})}\overline{\vartheta}_t\rho_{h_t}^-\right)\right)<0. \tag{3.17}$$

By assumption, $\sum_{j=1}^{m}\bar{\mu}_j\rho_{g_j}+\sum_{t\in T^+(\bar{x})}\overline{\vartheta}_t\rho_{h_t}^+-\sum_{t\in T^-(\bar{x})}\overline{\vartheta}_t\rho_{h_t}^-\geqq 0$. Thus, by Definition 5, the following inequality

$$\Phi\left(\widetilde{x},\bar{x},\frac{1}{A}\left(0,\sum_{j=1}^{m}\bar{\mu}_j\rho_{g_j}+\sum_{t\in T^+(\bar{x})}\overline{\vartheta}_t\rho_{h_t}^+-\sum_{t\in T^-(\bar{x})}\overline{\vartheta}_t\rho_{h_t}^-\right)\right)\geqq 0$$

holds, contradicts (3.17). This means that $\overline{\lambda}\neq 0$. If $\sum_{i=1}^{k}\overline{\lambda}_i\neq 1$, then it is sufficient to normalize the Lagrange multipliers $\overline{\lambda}_i$, $i\in I$. This completes the proof of this theorem. □

Remark 2. Theorem 2 can also be proved if hypotheses that each function h_t, $t\in T^+(\bar{x}):=\{t\in T:\overline{\vartheta}_t>0\}$, is locally Lipschitz $(\Phi,\rho_{h_t})^w$-invex at $\bar{x}$ on D and each function $-h_t$, $t\in T^-(\bar{x}):=\{t\in T:\overline{\vartheta}_t<0\}$, is locally Lipschitz $(\Phi,\rho_{h_t})^w$-invex at $\bar{x}$ on D are replaced by, in general, a weaker hypothesis that $\sum_{t=1}^{q}\overline{\vartheta}_t h_t$ is locally Lipschitz $(\Phi,\rho_h)^w$-invex at $\bar{x}$ on D.

Definition 8. The point $\left(\bar{x},\overline{\lambda},\bar{\mu},\overline{\vartheta}\right)\in D\times R^k\times R^m\times R^q$ is said to be a Karush-Kuhn-Tucker point of the considered vector optimization problem (VP) if the necessary optimality conditions (3.1)-(3.2) and, in place of (3.3), the conditions $\overline{\lambda}\geq 0$, $\sum_{i=1}^{k}\overline{\lambda}_i=1$, $\bar{\mu}\geqq 0$ are satisfied at $\bar{x}$ with Lagrange multipliers $\overline{\lambda}$, $\bar{\mu}$ and $\overline{\vartheta}$.

Now, we prove the sufficient optimality conditions for weak efficiency of a feasible solution in the considered nonsmooth multiobjective programming problem (VP) under nonsmooth $(\Phi,\rho)^w$-invexity.

Theorem 3. *Let $\left(\bar{x},\overline{\lambda},\bar{\mu},\overline{\vartheta}\right)\in D\times R^k\times R^m\times R^q$ be a Karush-Kuhn-Tucker point of the considered nonsmooth multiobjective programming problem (VP). Further, assume that f_i, $i\in I$, is locally Lipschitz $(\Phi,\rho_{f_i})^w$-invex at $\bar{x}$ on D, g_j, $j\in J(\bar{x})$, is locally Lipschitz $\left(\Phi,\rho_{g_j}\right)^w$-invex at $\bar{x}$ on D, h_t, $t\in T^+(\bar{x}):=\{t\in T:\overline{\vartheta}>0\}$, is locally Lipschitz $\left(\Phi,\rho_{h_t}^+\right)^w$-invex at $\bar{x}$ on D, $-h_t$, $t\in T^-(\bar{x}):=\{t\in T:\overline{\vartheta}<0\}$, is locally*

Lipschitz $\left(\Phi,\rho_{h_t}^-\right)^w$*-invex at* $\overline{x}$ *on* D*. If* $\sum_{i=1}^k \overline{\lambda}_i\rho_{f_i} + \sum_{j\in J(\overline{x})}\overline{\mu}_j\rho_{g_j} + \sum_{t\in T^+(\overline{x})}\overline{\vartheta}_t\rho_{h_t}^+ - \sum_{t\in T^-(\overline{x})}\overline{\vartheta}_t\rho_{h_t}^- \geqq 0$, *then* $\overline{x}$ *is a weak Pareto optimal solution of the problem (VP).*

Proof. Suppose, contrary to the result, that $\overline{x}$ is not a weak Pareto optimal solution of the problem (VP). Then, by Definition 6, there exists a feasible solution $\widetilde{x}$ such that

$$f(\widetilde{x}) < f(\overline{x}). \tag{3.18}$$

By assumption, $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\vartheta}\right) \in D\times R^k\times R^m\times R^q$ is a Karush-Kuhn-Tucker point of the considered nonsmooth multiobjective programming problem (VP). Then, by Definition 8, the necessary optimality conditions (3.1)-(3.2) hold with $\overline{\lambda}\geq 0$, $\sum_{i=1}^k\overline{\lambda}_i = 1$. By (3.1), there exist $\xi_i\in\partial f_i(\overline{x})$, $i\in I$, $\zeta_j\in\partial g_j(\overline{x})$, $j\in J$, $\varsigma_t\in\partial h_t(\overline{x})$, $t\in T$, such that

$$\sum_{i=1}^k\overline{\lambda}_i\xi_i + \sum_{j\in J(\overline{x})}\overline{\mu}_j\zeta_j + \sum_{t\in T^+(\overline{x})}\overline{\vartheta}_t\varsigma_t + \sum_{t\in T^-(\overline{x})}\overline{\vartheta}_t\varsigma_t = 0. \tag{3.19}$$

As it follows from Definition 5, the following inequality $\Phi(\widetilde{x},\overline{x},(0,a))\geqq 0$ holds for each $a\geqq 0$. Hence, by (3.19), hypothesis $\sum_{i\in I(\overline{x})}\overline{\lambda}_i\rho_{f_i} + \sum_{j\in J(\overline{x})}\overline{\mu}_j\rho_{g_j} + \sum_{t\in T^+(\overline{x})}\overline{\vartheta}_t\rho_{h_t}^+ - \sum_{t\in T^-(\overline{x})}\overline{\vartheta}_t\rho_{h_t}^- \geqq 0$ implies

$$\Phi\left(\widetilde{x},\overline{x},\frac{1}{\overline{A}}\left(\sum_{i=1}^k\overline{\lambda}_i\xi_i + \sum_{j\in J(\overline{x})}\overline{\mu}_j\zeta_j + \sum_{t\in T^+(\overline{x})}\overline{\vartheta}_t\varsigma_t + \sum_{t\in T^-(\overline{x})}\overline{\vartheta}_t\varsigma_t,\right.\right. \tag{3.20}$$
$$\left.\left.\sum_{i=1}^k\overline{\lambda}_i\rho_{f_i} + \sum_{j\in J(\overline{x})}\overline{\mu}_j\rho_{g_j} + \sum_{t\in T^+(\overline{x})}\overline{\vartheta}_t\rho_{h_t}^+ - \sum_{t\in T^-(\overline{x})}\overline{\vartheta}_t\rho_{h_t}^-\right)\right)\geqq 0,$$

where

$$\overline{A} = \sum_{i=1}^k\overline{\lambda}_i + \sum_{j=1}^m\overline{\mu}_j + \sum_{t\in T^+(\overline{x})}\overline{\vartheta}_t - \sum_{t\in T^-(\overline{x})}\overline{\vartheta}_t > 0. \tag{3.21}$$

Let us denote

$$\overline{\alpha}_i = \frac{\overline{\lambda}_i}{\overline{A}},\ i\in I(\overline{x}),\ \overline{\beta}_j = \frac{\overline{\mu}_j}{\overline{A}},\ j\in J(\overline{x}), \tag{3.22}$$

$$\overline{\gamma}_t^+ = \frac{\overline{\vartheta}_t}{\overline{A}},\ t\in T^+(\overline{x}),\ \overline{\gamma}_t^- = \frac{-\overline{\vartheta}_t}{\overline{A}},\ t\in T^-(\overline{x}). \tag{3.23}$$

Then, by $\overline{\lambda}\geq 0$, $\sum_{i=1}^k\overline{\lambda}_i = 1$, it follows that $\overline{\alpha} := (\overline{\alpha}_1,\dots,\overline{\alpha}_k)\geq 0$, $0\leqq\overline{\alpha}_i\leqq 1$, $i\in I$, $0<\overline{\alpha}_i\leqq 1$ for at least one $i\in I$, $\overline{\beta} = \left(\overline{\beta}_1,\dots,\overline{\beta}_m\right)\geqq 0$, $0\leqq\overline{\beta}_j\leqq 1$, $j\in J$, $0\leqq\overline{\gamma}_t^+\leqq 1$, $t\in T^+(\overline{x})$, $0\leqq\overline{\gamma}_t^-\leqq 1$, $t\in T^-(\overline{x})$, and, moreover,

$$\sum_{i=1}^k\overline{\alpha}_i + \sum_{j\in J(\overline{x})}\overline{\beta}_j + \sum_{t\in T^+(\overline{x})}\overline{\gamma}_t^+ + \sum_{t\in T^-(\overline{x})}\overline{\gamma}_t^- = 1. \tag{3.24}$$

Since f_i, $i \in I$, is locally Lipschitz $(\Phi, \rho_{f_i})^w$-invex at $\overline{x}$ on D, g_j, $j \in J(\overline{x})$, is $(\Phi, \rho_{g_i})^w$-invex at $\overline{x}$ on D, h_t, $t \in T^+(\overline{x}) := \{t \in T : \overline{\vartheta} > 0\}$, is locally Lipschitz $\left(\Phi, \rho_{h_t}^+\right)^w$-invex at $\overline{x}$ on D, $-h_t$, $t \in T^-(\overline{x}) := \{t \in T : \overline{\vartheta} < 0\}$, is locally Lipschitz $\left(\Phi, \rho_{h_t}^-\right)^w$-invex at $\overline{x}$ on D, by Definition 5, it follows that the following inequalities

$$f_i(\widetilde{x}) - f_i(\overline{x}) \geqq \Phi(\widetilde{x}, \overline{x}, (\xi_i, \rho_{f_i})), \quad i \in I, \tag{3.25}$$

$$g_j(\widetilde{x}) - g_j(\overline{x}) \geqq \Phi\left(\widetilde{x}, \overline{x}, \left(\zeta_j, \rho_{g_j}\right)\right), \quad j \in J(\overline{x}), \tag{3.26}$$

$$h_t(\widetilde{x}) - h_t(\overline{x}) \geqq \Phi\left(\widetilde{x}, \overline{x}, \left(\varsigma_t, \rho_{h_t}^+\right)\right), \quad t \in T^+(\overline{x}), \tag{3.27}$$

$$-h_t(\widetilde{x}) + h_t(\overline{x}) \geqq \Phi\left(\widetilde{x}, \overline{x}, \left(-\varsigma_t, \rho_{h_t}^-\right)\right), \quad t \in T^-(\overline{x}) \tag{3.28}$$

hold for each $\xi_i \in \partial f_i(\overline{x})$, $i \in I$, $\zeta_j \in \partial g_j(\overline{x})$, $j \in J(\overline{x})$, $\varsigma_t \in \partial h_t(\overline{x})$, $t \in T^+(\overline{x}) \cup T^-(\overline{x})$, respectively. By (3.18), inequality (3.25) implies

$$\Phi(\widetilde{x}, \overline{x}, (\xi_i, \rho_{f_i})) < 0, \quad i \in I. \tag{3.29}$$

By $\widetilde{x} \in D$, $\overline{x} \in D$, inequalities (3.26)-(3.28) yield, respectively,

$$\Phi\left(\widetilde{x}, \overline{x}, \left(\zeta_j, \rho_{g_j}\right)\right) \leqq 0, \quad j \in J(\overline{x}), \tag{3.30}$$

$$\Phi\left(\widetilde{x}, \overline{x}, \left(\varsigma_t, \rho_{h_t}^+\right)\right) \leqq 0, \quad t \in T^+(\overline{x}), \tag{3.31}$$

$$\Phi\left(\widetilde{x}, \overline{x}, \left(-\varsigma_t, \rho_{h_t}^-\right)\right) \leqq 0, \quad t \in T^-(\overline{x}). \tag{3.32}$$

As it follows from Definition 5, $\Phi(\widetilde{x}, \overline{x}, \cdot)$ is a strictly quasi-convex function on R^{n+1}. Since (3.24) is satisfied, by inequalities (3.29)-(3.32), Proposition 1 implies

$$\Phi\left(\widetilde{x}, \overline{x}, \left(\sum_{i \in I(\overline{x})} \overline{\alpha}_i \xi_i + \sum_{j \in J(\overline{x})} \overline{\beta}_j \zeta_j + \sum_{t \in T^+(\overline{x})} \overline{\gamma}_t^+ \varsigma_t + \sum_{t \in T^-(\overline{x})} \overline{\gamma}_t^- (-\varsigma_t), \right.\right. \tag{3.33}$$

$$\left.\left. \sum_{i \in I(\overline{x})} \overline{\alpha}_i \rho_{f_i} + \sum_{j \in J(\overline{x})} \overline{\beta}_j \rho_{g_j} + \sum_{t \in T^+(\overline{x})} \overline{\gamma}_t^+ \rho_{h_t}^+ + \sum_{t \in T^-(\overline{x})} \overline{\gamma}_t^- \rho_{h_t}^- \right)\right) < 0.$$

Taking into account (3.22)-(3.23) in (3.33), we obtain that the following inequality

$$\Phi\left(\widetilde{x}, \overline{x}, \frac{1}{A}\left(\sum_{i=1}^{k} \overline{\lambda}_i \xi_i + \sum_{j \in J(\overline{x})} \overline{\mu}_j \zeta_j + \sum_{t \in T^+(\overline{x})} \overline{\vartheta}_t \varsigma_t + \sum_{t \in T^-(\overline{x})} \overline{\vartheta}_t \varsigma_t, \right.\right.$$

$$\left.\left. \sum_{i \in I(\overline{x})} \overline{\lambda}_i \rho_{f_i} + \sum_{j \in J(\overline{x})} \overline{\mu}_j \rho_{g_j} + \sum_{t \in T^+(\overline{x})} \overline{\vartheta}_t \rho_{h_t}^+ - \sum_{t \in T^-(\overline{x})} \overline{\vartheta}_t \rho_{h_t}^- \right)\right) < 0$$

holds, contradicting (3.20). This means that $\overline{x}$ is a weakly efficient solution of (VP) and completes the proof of this theorem. □

In order to prove the sufficient optimality conditions for a Pareto optimal solution of the nonsmooth multiobjective programming problem (VP) with nonsmooth (Φ,ρ)-invex functions, some stronger hypotheses should be assumed.

Theorem 4. *Let* $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\vartheta}\right) \in D \times R^k \times R^m \times R^q$ *be a Karush-Kuhn-Tucker point of the considered nonsmooth multiobjective programming problem (VP). Further, assume that any one of the following hypotheses is satisfied:*

i) *the objective function* f_i, $i \in I$, *is locally Lipschitz strictly* $(\Phi,\rho_{f_i})^w$*-invex at* $\overline{x}$ *on* D, *the constraint function* g_j, $j \in J(\overline{x})$, *is locally Lipschitz* $\left(\Phi,\rho_{g_j}\right)^w$*-invex at* $\overline{x}$ *on* D, h_t, $t \in T^+(\overline{x}) := \left\{t \in T : \overline{\vartheta}_t > 0\right\}$, *is locally Lipschitz* $\left(\Phi,\rho^+_{h_t}\right)^w$*-invex at* $\overline{x}$ *on* D, $-h_t$, $t \in T^-(\overline{x}) := \left\{t \in T : \overline{\vartheta}_t < 0\right\}$, *is locally Lipschitz* $\left(\Phi,\rho^-_{h_t}\right)^w$*-invex at* $\overline{x}$ *on* D *and, moreover,* $\sum_{i=1}^k \overline{\lambda}_i\rho_{f_i} + \sum_{j\in J(\overline{x})}\overline{\mu}_j\rho_{g_j} + \sum_{t\in T^+(\overline{x})}\overline{\vartheta}_t\rho^+_{h_t} - \sum_{t\in T^-(\overline{x})}\overline{\vartheta}_t\rho^-_{h_t} \geqq 0$,

ii) *the Lagrange multipliers* $\overline{\lambda}_i > 0$, $i \in I$, *the objective function* f_i, $i \in I$, *is locally Lipschitz* $(\Phi,\rho_{f_i})^w$*-invex at* $\overline{x}$ *on* D, *the constraint function* g_j, $j \in J(\overline{x})$, *is locally Lipschitz* $\left(\Phi,\rho_{g_j}\right)$*-invex at* $\overline{x}$ *on* D, h_t, $t \in T^+(\overline{x}) := \left\{t \in T : \overline{\vartheta}_t > 0\right\}$, *is locally Lipschitz* $\left(\Phi,\rho^+_{h_t}\right)^w$*-invex at* $\overline{x}$ *on* D, $-h_t$, $t \in T^-(\overline{x}) := \left\{t \in T : \overline{\vartheta}_t < 0\right\}$, *is locally Lipschitz* $\left(\Phi,\rho^-_{h_t}\right)^w$*-invex at* $\overline{x}$ *on* D *and* $\sum_{i=1}^k \overline{\lambda}_i\rho_{f_i} + \sum_{j\in J(\overline{x})}\overline{\mu}_j\rho_{g_j} + \sum_{t\in T^+(\overline{x})}\overline{\vartheta}_t\rho^+_{h_t} - \sum_{t\in T^-(\overline{x})}\overline{\vartheta}_t\rho^-_{h_t} \geqq 0$.

Then $\overline{x}$ *is an efficient solution of the problem (VP).*

Proof. The proof of this theorem is similar to the proof of Theorem 3. □

In order to illustrate the sufficient optimality results established in this section, we consider the following example of a nondifferentiable multiobjective programming problem with $(\Phi,\rho)^w$-invex functions, not necessarily, with respect to the same ρ.

Example 1. Consider the following nondifferentiable multiobjective programming problem defined as follows

$$f(x) = \left(\arctan(|x_1|) + \arctan(|x_2|), x_1^2 + x_2^2 + \arctan(|x_1x_2|)\right) \to V\text{-}\min$$
$$g(x) = |x_1| + |x_2| - 2\arctan(|x_1x_2|) \leqq 0, \qquad \text{(VP1)}$$
$$X = R^2.$$

It is not difficult to see that $D = \left\{(x_1,x_2) \in R^2 : |x_1| + |x_2| - 2\arctan(|x_1x_2|) \leqq 0\right\}$ and $\overline{x} = (0,0)$ is such a feasible point at which the Generalized Karush-Kuhn-Tucker necessary optimality conditions are satisfied. It can be established, by Definition 5, that the objective function f_i, $i = 1,2$, is locally Lipschitz strictly $(\Phi,\rho_{f_i})^w$-invex at $\overline{x}$ on D and the constraint function g is $(\Phi,\rho_g)^w$-invex at $\overline{x}$ on D, where

$$\Phi(x,\overline{x},(\varsigma,\rho)) = \arctan(\varsigma_1|x_1|) + \arctan(\varsigma_2|x_2|) + \arctan(\rho)(\arctan|x_1x_2| - \arctan|\overline{x}_1\overline{x}_2|)$$

$$\rho_{f_1} = 0,\ \rho_{f_2} = \tan(1),\ \rho_g = \tan(-2),$$

and $\varsigma \in \partial k(\bar{x})$, where k denotes f_1 or f_2 or g, respectively, and ρ is equal to ρ_{f_1}, ρ_{f_2} or ρ_g, respectively.

Since all hypotheses of Theorem 4 are satisfied, $\bar{x}$ is an efficient solution of the considered nonsmooth multiobjective programming problem. Note that we are not in a position to prove efficiency of $\bar{x}$ in the considered nonconvex nonsmooth multiobjective programming problem (VP1) under other generalized convexity notions existing in the literature, that is, invexity [14,16], b-invexity [17], F-convexity [10], r-invexity [3], V-invexity [13], G-invexity [4], V-r-invexity [5], univexity [18]. This follows from the fact that not every stationary point of the functions constituting problem (VP1) is a global minimum of such a function. Whereas one of the main property of the concepts generalized convexity notions mentioned above is that a stationary point of every function belonging to the aforesaid classes of generalized convex functions is its global minimizer. Further, we can't use also the sufficient optimality conditions under nondifferentiable (Φ,ρ)-invexity since the functional $\Phi(x,\bar{x},\cdot)$ is not convex for all $x \in D$ as it follows from the definition of this concept of generalized convexity (see [6]). As it follows even from this example, the introduced concept of nondifferentiable $(\Phi,\rho)^w$-invexity is useful to prove the sufficiency of Generalized Karush-Kuhn-Tucker necessary optimality conditions for a larger class of nonconvex nondifferentiable vector optimization problems in comparison to other generalized convexity notions, earlier defined in the literature.

4. Mond-Weir duality

In this section, for the considered nonsmooth multiobjective programming problem (VP), we define a vector dual problem in the Mond-Weir sense. Then, we prove several duality results between the primal multiobjective programming problem and its Mond-Weir dual problem under $(\Phi,\rho)^w$-invexity hypotheses.

Now, for the considered nonsmooth multiobjective programming problem (VP), we state the following vector Mond-Weir dual problems as follows:

$$\begin{gathered} f(y) \to V\text{-max} \\ \text{s.t. } 0 \in \textstyle\sum_{i=1}^k \lambda_i \partial f_i(y) + \sum_{j=1}^m \mu_j \partial g_j(y) + \sum_{t=1}^q \vartheta_t \partial h_t(y), \\ \textstyle\sum_{j=1}^m \mu_j g_j(y) \geqq 0, \quad \sum_{t=1}^q \vartheta_t h_t(y) \geqq 0, \\ \lambda \in R^k,\ \lambda \geq 0,\ \textstyle\sum_{i=1}^k \lambda_i = 1,\ \mu \in R^m,\ \mu \geqq 0,\ \vartheta \in R^q. \end{gathered} \tag{VD}$$

We denote by Ω the set of all feasible solutions in the vector Mond-Weir dual problem (VD) and, moreover, let Y be the projection of the set Ω on X, that is, $Y = \{y \in X : (y,\lambda,\mu,\vartheta) \in \Omega\}$.

Theorem 5 (Weak duality)**.** *Let x and $(y,\lambda,\mu,\vartheta)$ be any feasible solutions for the problems (VP) and (VD), respectively. Further, assume that f_i, $i \in I$, is locally Lipschitz $(\Phi,\rho_f)^w$-invex at y on $D\cup Y$, $\sum_{j=1}^m \mu_j g_j$ is locally Lipschitz $(\Phi,\rho_g)^w$-invex at y on $D\cup Y$, $\sum_{t=1}^q \vartheta_t h_t$ is locally Lipschitz $(\Phi,\rho_h)^w$-invex at y on $D\cup Y$. If $\sum_{i=1}^k \lambda_i \rho_{f_i} + \rho_g + \rho_h \geqq 0$, then $f(x) \not< f(y)$.*

Proof. We proceed by contradiction. Suppose, contrary to the result, that there exist $x \in D$ and $(y,\lambda,\mu,\vartheta) \in \Omega$ such that

$$f(x) < f(y). \tag{4.1}$$

By assumption, f_i, $i \in I$, is locally Lipschitz $(\Phi,\rho_f)^w$-invex at y on $D\cup Y$. Hence, by Definition 5, the following inequalities

$$f_i(z) - f_i(y) \geqq \Phi\left(z,y,\left(\xi_i,\rho_{f_i}\right)\right),\, i \in I \tag{4.2}$$

hold for all $z \in D\cup Y$ and for each $\xi_i \in \partial f_i(y)$. Therefore, they are also satisfied for $z = x \in D$. Thus, inequalities (4.2) yield

$$f_i(x) - f_i(y) \geqq \Phi\left(x,y,\left(\xi_i,\rho_{f_i}\right)\right),\, i \in I. \tag{4.3}$$

Combining (4.1) and (4.3), we have

$$\Phi\left(x,y,\left(\xi_i,\rho_{f_i}\right)\right) < 0,\, i \in I. \tag{4.4}$$

By assumptions, $\sum_{j=1}^m \mu_j g_j$ is locally Lipschitz $(\Phi,\rho_g)^w$-invex at y on $D\cup Y$, $\sum_{t=1}^q \vartheta_t h_t$ is locally Lipschitz $(\Phi,\rho_h)^w$-invex at y on $D\cup Y$. Hence, by Definition 5, the following inequalities

$$\sum_{j=1}^m \mu_j g_j(x) - \sum_{j=1}^m \mu_j g_j(y) \geqq \Phi\left(x,y,\left(\sum_{j=1}^m \mu_j \zeta_j, \rho_g\right)\right), \tag{4.5}$$

$$\sum_{t=1}^q \vartheta_t h_t(x) - \sum_{t=1}^q \vartheta_t h_t(y) \geqq \Phi\left(x,y,\left(\sum_{t=1}^q \vartheta_t \varsigma_t, \rho_h\right)\right) \tag{4.6}$$

hold for each $\zeta_j \in \partial g_j(y)$, $j \in J$ and $\varsigma_t \in \partial h_t(y)$, $t \in T$, respectively. By $x \in D$ and $(y,\lambda,\mu,\vartheta) \in \Omega$, (4.5) and (4.6) yield, respectively,

$$\Phi\left(x,y,\left(\sum_{j=1}^m \mu_j \zeta_j, \rho_g\right)\right) \leqq 0, \tag{4.7}$$

$$\Phi\left(x,y,\left(\sum_{t=1}^q \vartheta_t \varsigma_t, \rho_h\right)\right) \leqq 0. \tag{4.8}$$

By Definition 5, $\Phi(x,y,\cdot)$ is strictly quasi-convex on R^{n+1}. Then, by Proposition 1, inequalities (4.4), (4.7) and (4.8) imply

$$\Phi\left(x,y,\left(\sum_{i=1}^k \frac{\lambda_i}{3}\xi_i + \frac{1}{3}\sum_{j=1}^m \mu_j \zeta_j + \frac{1}{3}\sum_{t=1}^q \vartheta_t \varsigma_t\, ,\right.\right. \tag{4.9}$$

$$\sum_{i=1}^{k} \frac{\lambda_i}{3} \rho_{f_i} + \frac{1}{3}\rho_g + \frac{1}{3}\rho_h \Bigg) \Bigg) < 0.$$

Thus, (4.9) gives

$$\Phi\left(x, y, \frac{1}{3}\left(\sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \zeta_j + \sum_{t=1}^{q} \vartheta_t \varsigma_t, \sum_{i=1}^{k} \lambda_i \rho_{f_i} + \rho_g + \rho_h\right)\right) < 0. \tag{4.10}$$

Using $(y, \lambda, \mu, \vartheta) \in \Omega$ again, the first constraint of dual problem (VD) gives

$$\Phi\left(x, y, \frac{1}{3}\left(0\,, \sum_{i=1}^{k} \lambda_i \rho_{f_i} + \rho_g + \rho_h\right)\right) < 0. \tag{4.11}$$

By Definition 5, it follows that $\Phi(x, y, (0, a)) \geqq 0$ for any $a \in R_+$. Therefore, hypothesis $\sum_{i=1}^{k} \lambda_i \rho_{f_i} + \rho_g + \rho_{h_t} \geqq 0$ implies that the following inequality

$$\Phi\left(x, y, \frac{1}{3}\left(0\,, \sum_{i=1}^{k} \lambda_i \rho_{f_i} + \rho_g + \rho_h\right)\right) \geqq 0$$

holds, contradicting (4.11). Hence, the proof of this theorem is completed. □

If a stronger $(\Phi, \rho_f)^w$-invexity assumption is imposed on the objective functions constituting considered vector optimization problems, then the following stronger result can be established.

Theorem 6 (Weak duality). *Let x and $(y, \lambda, \mu, \vartheta)$ be feasible solutions for the problems (VP) and (VD), respectively. Further, assume that f_i, $i \in I$, is locally Lipschitz strictly $(\Phi, \rho_f)^w$-invex at y on $D \cup Y$, $\sum_{j=1}^{m} \mu_j g_j$ is locally Lipschitz $(\Phi, \rho_g)^w$-invex at y on $D \cup Y$, $\sum_{t=1}^{q} \vartheta_t h_t$ is locally Lipschitz $(\Phi, \rho_h)^w$-invex at y on $D \cup Y$. If $\sum_{i=1}^{k} \lambda_i \rho_{f_i} + \rho_g + \rho_h \geqq 0$, then $f(x) \nleq f(y)$.*

Theorem 7 (Strong duality). *Let $\overline{x}$ be a weak Pareto solution (a Pareto solution) of the primal multiobjective programming (VP) and all hypotheses of Theorem 2 be satisfied at $\overline{x}$. Then there exist $\overline{\lambda} \in R^k$, $\overline{\mu} \in R^m$ and $\overline{\vartheta} \in R^q$ such that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\vartheta}\right)$ is feasible in (VD) and the objective functions of (VP) and (VD) are equal at these points. Further, if all hypotheses of the weak duality theorem (Theorem 5) are satisfied, then $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\vartheta}\right)$ is a weakly efficient solution of a maximum type in (VD). If $\overline{\lambda} > 0$ and all hypotheses of the weak duality theorem (Theorem 6) are satisfied, then $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\vartheta}\right)$ is an efficient solution of a maximum type for the vector Mond-Weir dual problem (VD).*

Proof. By assumption, $\overline{x} \in D$ is a weak Pareto optimal solution (a Pareto solution) of the problem (VP) and the constraint qualification is satisfied at $\overline{x}$. Then there exist the Lagrange multipliers $\overline{\lambda} \in R^k$, $\overline{\mu} \in R^m$ and $\overline{\vartheta} \in R^q$ such that the Karush-Kuhn-Tucker necessary optimality conditions (3.3)-(3.5) are satisfied at $\overline{x}$. Thus, the feasibility of $\left(\bar{x},\bar{\lambda},\bar{\mu},\bar{\vartheta}\right)$ in (VD) follows from these necessary optimality conditions and, moreover, $\bar{x} \in D$. Therefore, the objective functions of the problems (VP) and (VD) are equal at $\bar{x}$ and $\left(\bar{x},\bar{\lambda},\bar{\mu},\bar{\vartheta}\right)$. Hence, weakly efficiency of a maximum type of $\left(\bar{x},\bar{\lambda},\bar{\mu},\bar{\vartheta}\right)$ for (VD) follows directly from weak duality (Theorem 5), whereas efficiency of a maximum type of $\left(\bar{x},\bar{\lambda},\bar{\mu},\bar{\vartheta}\right)$ follows from Theorem 6. □

Theorem 8 (Converse duality)**.** *Let $\left(\bar{y},\bar{\lambda},\bar{\mu},\bar{\vartheta}\right)$ be an efficient solution of a maximum type (a weakly efficient solution of a maximum type) for the vector mixed dual problem (VD) such that $\bar{y} \in D$. Further, assume that f_i, $i \in I$, is (locally Lipschitz $(\Phi,\rho_f)^w$-invex) locally Lipschitz strictly $(\Phi,\rho_f)^w$-invex at $\bar{y}$ on $D \cup Y$, $\sum_{j=1}^{m} \bar{\mu}_j g_j$ is locally Lipschitz $(\Phi,\rho_g)^w$-invex at $\bar{y}$ on $D \cup Y$, $\sum_{t=1}^{q} \bar{\vartheta}_t h_t$ is locally Lipschitz $(\Phi,\rho_h)^w$-invex at $\bar{y}$ on $D \cup Y$. If $\sum_{i=1}^{k} \bar{\lambda}_i \rho_{f_i} + \rho_g + \rho_h \geqq 0$. Then $\bar{y}$ is an efficient solution (a weakly efficient solution) of the considered multiobjective programming problem (VP).*

Proof. The proof of the theorem follows directly from weak duality (Theorem 5 or Theorem 6, respectively). □

5. Conclusions

In the paper, a new class of nonconvex nonsmooth multiobjective programming problems is considered in which every component of functions involved is locally Lipschitz $(\Phi,\rho)^w$-invex. Hence, the sufficient optimality conditions for weak efficiency and efficiency and duality results in the sense of Mond-Weir have been established for the considered nonconvex nonsmooth multiobjective programming problem under the concept of nondifferentiable $(\Phi,\rho)^w$-invexity introduced in the paper. Note that the definition of nondifferentiable $(\Phi,\rho)^w$-invexity unifies many generalized convex notions earlier introduced in the literature (see Remark 1). In order to illustrate the results established in the paper, some example of a nonconvex nonsmooth multiobjective programming problem with nondifferentiable $(\Phi,\rho)^w$-invex functions has been presented. It is interesting that not all functions constituting the considered nonsmooth vector optimization problem have the fundamental property of the most classes of generalized convex functions, namely that a stationary point of such a function is also its global minimum. Thus, we have also shown that many generalized convexity notions existing in the literature (that is, invexity [14], b-invexity [17], F-convexity [10], univexity [18], r-invexity [3], V-invexity [19], V-r-invexity [5], G-invexity [4]) may fail in proving the sufficiency of the Karush-Kuhn-Tucker necessary optimality conditions and Mond-Weir duality results for the considered nonconvex nonsmooth vector optimization problem. Thus, the concept of nondifferentiable $(\Phi,\rho)^w$-invexity extend the class of nonconvex nonsmooth multiobjective programming problems for which it is possible to prove the sufficiency of the Generalized Karush-Kuhn-Tucker necessary optimality conditions and several duality theorems in the sense of Mond-Weir in comparison to similarly results proved under other generalized convexity notions.

REFERENCES

[1] R. P. Agarwal, I. Ahmad, Z. Husain, and A. Jayswal, "Optimality and duality in nonsmooth multiobjective optimization involving V-type I invex functions." *J. Inequal. Appl.*, vol. 2010, p. 898626, 2010, doi: 10.1155/2010/898626.

[2] G. An and X. Gao, "Sufficiency and Wolfe type duality for nonsmooth multiobjective programming problems." *Adv. Pure Math.*, vol. 8, pp. 755–763, 2018, doi: 10.4236/apm.2018.88045.

[3] T. Antczak, "Lipschitz r-invex functions and nonsmooth programming." *Numer. Funct. Anal. Optim.*, vol. 23, pp. 265–284, 2002, doi: 10.1081/NFA-120006693.

[4] T. Antczak, "On G -invex multiobjective programming. Part I. Optimality." *J. Global Optim.*, vol. 43, pp. 97–109, 2009, doi: 10.1007/s10898-008-9299-5.

[5] T. Antczak, "Optimality and duality for nonsmooth multiobjective programming problems with V-r-invexity." *J.Global Optim.*, vol. 45, pp. 319–334, 2009, doi: 10.1007/s10898-008-9377-8.

[6] T. Antczak and A. Stasiak, "(Φ,ρ) - invexity in nonsmooth optimization." *Numer. Funct. Anal. Optim.*, vol. 32, pp. 1–25, 2011, doi: 10.1080/01630563.2010.526408.

[7] M. Arana-Jimenez, A. Rufian-Lizana, R. Osuna-Gómez, and G. Ruiz-Garzón, "Pseudoinvexity, optimality conditions and efficiency in multiobjective problems: duality." *Nonlinear Anal.*, vol. 68, pp. 24–34, 2008, doi: 10.1016/j.na.2006.10.028.

[8] D. Bhatia and P. Jain, "Generalized (F,ρ) -convexity and duality for non smooth multi-objective programs." *Optimization*, vol. 31, pp. 153–164, 1994, doi: 10.1080/02331939408844012.

[9] F. H. Clarke, *Optimization and nonsmooth analysis.* United States of America: A Wiley-Interscience Publication, John Wiley&Sons, Inc., 1983.

[10] M. A. Hanson and B. Mond, "Further generalization of convexity in mathematical programming." *J. Inform. Optim. Sci.*, vol. 3, pp. 25–32, 1982, doi: 10.1080/02522667.1982.10698716.

[11] A. Jaddar and K. El Moutaouakil, "On weak Pareto optimality for pseudoconvex nonsmooth multiobjective optimization problems." *Internat. J. Math. Anal.*, vol. 7, pp. 2995–3003, 2013, doi: 10.12988/ijma.2013.311276.

[12] V. Jeyakumar, "Equivalence of saddle-points and optima, and duality for a class of nonsmooth non-convex problems." *J. Math. Anal. Appl.*, vol. 130, pp. 334–343, 1988, doi: 10.1016/0022-247X(88)90309-5.

[13] V. Jeyakumar and B. Mond, "On generalized convex mathematical programming." *J. Austral. Math. Soc. Ser.B*, vol. 34, pp. 43–53, 1992, doi: 10.1017/S0334270000007372.

[14] D. S. Kim and S. Schaible, "Optimality and duality for invex nonsmooth multiobjective programming problems." *Optimization*, vol. 53, pp. 165–176, 2004, doi: 10.1080/0233193042000209435.

[15] H. Kuk, G. M. Lee, and D. S. Kim, "Nonsmooth multiobjective programs with V-ρ - invexity." *Indian J. Pure Appl. Math.*, vol. 29, pp. 405–412, 1998.

[16] G. M. Lee, "Nonsmooth invexity in multiobjective programming." *J. Inform. Optim. Sci.*, vol. 15, pp. 127–136, 1994, doi: 10.1080/02522667.1994.10699173.

[17] X. F. Li and Q. H. L. J. L. Dong, "Lipschitz B - vex functions and nonsmooth programming." *J. Optim. Theory Appl.*, vol. 93, pp. 557–574, 1997, doi: 10.1023/A:1022643129733.

[18] X. Long, "Sufficiency and duality for nonsmooth multiobjective programming problems involving generalized univex functions." *J. Syst. Sci. Complex.*, vol. 26, pp. 1002–1018, 2013, doi: 10.1080/02522667.1994.10699173.

[19] S. K. Mishra and R. N. Mukherjee, “On generalised convex multi-objective nonsmooth programming.” *J. Austral. Math. Soc. Ser.B*, vol. 38, pp. 140–148, 1996, doi: 10.1017/S0334270000000515.

[20] C. Nahak and R. N. Mohapatra, “Nonsmooth $\rho-(\eta,\theta)$ - invexity in multiobjective programming problems.” *Optim. Lett.*, vol. 6, pp. 253–260, 2012, doi: 10.1007/s11590-010-0239-1.

[21] S. Nobakhtian, “Sufficiency in nonsmooth multiobjective programming involving generalized (f,ρ) - convexity,” *J. Optim. Theory Appl.*, vol. 130, pp. 359–365, 2006, doi: 10.1007/s10957-006-9105-9.

[22] M. V. Stefanescu and A. Stefanescu, “On semi-infinite minmax programming with generalized invexity.” *Optimization*, vol. 61, pp. 1307–1319, 2012, doi: 10.1080/02331934.2011.563304.

[23] G. J. Zalmai, “Optimality conditions and duality models for generalized fractional programming problems containing locally subdifferentiable and ρ - convex functions.” *Optimization*, vol. 32, pp. 95–124, 1995, doi: 10.1080/02331939508844040.

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ON SOME DIOPHANTINE EQUATIONS

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Abstract. In this paper we deal with some Diophantine equations and present infinitely many positive integer solutions for each one of them.

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1. INTRODUCTION

The Diophantine equations

$$x^n + y^n = u^n + v^n, \qquad n = 2,3,4$$

have been considered by many mathematicians. The case $n = 2$ was presented in [13, 14] while Euler [8] and Binet [3] considered the case $n = 3$. Parametric solutions of the above equation for $n = 4$ can be found in [4, 10]. Some researches considered more general Diophantine equations with more variables or with integer coefficients that are not all equal to 1 [5–7, 11, 12].

In this paper, we deal with the equations

$$\begin{aligned} X^3 - Y^2 &= X^2 - Z^3, \\ X^3 \pm Y^3 &= U^4 - V^4 \end{aligned} \tag{1.1}$$

and obtain infinitely many positive integer solutions for each one of them. We consider the equation of the form

$$Am^4 + Bn^4 + Cp^4 + Dq^4 = Ar^4 + Bs^4 + Ct^4 + Du^4, \qquad A,B,C,D \in \mathbb{Z},$$

introduce some linear transformations and set some special conditions on its coefficients. Some recent papers deal with the similar problems. In [9] the authors investigate Diophantine equations of the form

$$T^2 = G(\overline{X}), \qquad \overline{X} = (X_1, X_2, \cdots, X_m),$$

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where $m=3$ or $m=4$ and G is a specific homogenous quintic form. The equations

$$a(X_1'^5+X_2'^5)+\sum_{i=0}^{n}a_1X_i^5=b(Y_1'^3+Y_2'^3)+\sum_{i=0}^{n}b_iY_i^3, \tag{1.2}$$

where $m,n\in\mathbb{N}\cup\{0\}$ and $a,b\neq 0$, a_i,b_i are fixed arbitrary rational numbers are studied in [2]. The theory of elliptic curves is used in order to solve (1.2) which is transformed to a cubic or a quartic elliptic curve with a positive rank. In [1, Main Theorem 2] authors prove that

$$\sum_{i=1}^{n}p_i\cdot x_i^{a_i}=\sum_{j=1}^{m}q_j\cdot y_j^{b_j},$$

$m,n,a_i,b_j\in\mathbb{N},\ p_i,q_j\in\mathbb{Z},\ i=1,\dots,n,\ j=1,\dots,m$ has a parametric solution and infinitely many solutions in nonzero integers if there exists an i such that $p_i=1$ and $(a_i,a_1a_2\cdots a_{i-1}a_{i+1}\cdots a_nb_1b_2\cdots b_m)=1$ or there exists a j such that $q_j=1$ and $(b_j,a_1\cdots a_nb_1\cdots b_{j-1}b_{j+1}\cdots b_m)=1$. In this article, even though linear transformations are also used, we introduce a different approach and some different conditions on the integer coefficients in order to solve (1.1).

2. Equation $X^3-Y^2=X^2-Z^3$

For start, we deal with the Diophantine equation

$$X^3-Y^2=X^2-Z^3. \tag{2.1}$$

It is easily shown that equation (2.1) has infinitely many solutions $(X,Y,Z)=(1,n^3,n^2)$, $n\in\mathbb{N}$. The main task of our work in this section is to discover whether there are more positive integer solutions of (2.1). We set

$$c=x^2,\qquad\qquad d=y^2$$

and obtain

$$\begin{aligned}c^3+d^3=x^6+y^6&=(x^3-(yi)^3)(x^3+(yi)^3)\\&=(x-yi)(x^2+xyi-y^2)(x+yi)(x^2-xyi-y^2).\end{aligned}$$

If we define

$$a-bi=(x+yi)(x^2+xyi-y^2)\qquad\text{and}\qquad a+bi=(x-yi)(x^2-xyi-y^2),$$

we get

$$c^3+d^3=a^2+b^2 \tag{2.2}$$

for $a=x^3-2y^2x$ and $b=y^3-2x^2y$. From (2.2) we obtain the equation

$$(x^2)^3+(y^2)^3=(x^3-2y^2x)^2+(y^3-2x^2y)^2.$$

First we deal with the case

$$y^2=y^3-2x^2y,\qquad\qquad y\neq 0$$

and get

$$(2y-1)^2 - 8x^2 = 1. \tag{2.3}$$

After introducing $y' = 2y-1$ in (2.3) a Pell equation

$$y'^2 - 8x^2 = 1$$

is obtained. Some of its solutions are $(y',x) = \{(3,1),(17,6),(99,35),\dots\}$. Finally, solutions of equation (2.3) are

$$(x,y) = \{(1,2),(6,9),(35,50),\dots\}.$$

The inequality $x < y$ implies $x^3 - 2y^2x < 0$, so $Y < 0$. Because we deal with equation of the form (2.1), we take $|Y|$ and get infinitely many positive integer solutions

$$(X,Y,Z) = (y^2,\ x^3-2y^2x,\ x^2) = \{(4,7,1),\ (81,756,36),\ (50^2,132125,35^2),\ \dots\}$$

of (2.1).

Alternatively, we get

$$8y^2 = \frac{8x^3}{2x+1}$$

for $y^2 = x^3 - 2y^2x$. Obviously, $\frac{8x^3}{2x+1} \in \mathbb{N}$ if and only if $(2x+1)|1$ which happens for only $2x+1 = 1$ which implies $x = 0,\ y = 0$. This case is not considered. Cases $x^2 = x^3 - 2y^2x$ and $x^2 = y^3 - 2x^2y$ do not provide us with new solutions. Consequently, (2.1) has infinitely many positive integer solutions of the form

$$(X,Y,Z) = (y^2,\ x^3-2y^2x,\ x^2),$$

where (x,y) are solutions of equation (2.3).

Remark 1. It can be noticed that for $X = Z$ equation (2.1) becomes

$$X^2(2X-1) = Y^2,$$

so $X = Z = 2k^2+2k+1,\ k \in \mathbb{N}$ will provide a solution. This approach can be generalized by taking $X = mZ,\ Y = nZ$ for $m,n \in \mathbb{N}$. We get

$$Z = \frac{m^2+n^2}{m^3+1}$$

and by fixing m we may yield some solutions. For example, if $m = 4$, then $n \equiv 7,32,33,58 \pmod{65}$ will provide solutions. Similarily, for $m = 9$, we obtain that $n \equiv 97,243,487,630 \pmod{730}$ will provide solutions. This approach works if m is a square, however we also have solutions for $m = 28$. Therefore, it may be difficult to completely classify all the solutions here.

3. Equation $X^3+Y^3=U^4-V^4$

In this section we deal with the equation

$$X^3+Y^3=U^4-V^4. \tag{3.1}$$

It is easy to notice that $(X,Y,U,V)=(m,-m,n,n)$ is a trivial solution of (3.1) for $m,n\in\mathbb{N}$, while the smallest nontrivial solution of (3.1) is $(X,Y,U,V)=(4,1,3,2)$.

Two different linear transformations are considered and for each one of them we give a different class of infinitely many positive integer solutions of equation (3.1).

3.1. *The First Method*

Let

$$X=px+u, \qquad Y=qx-u, \qquad U=x+v, \qquad V=px-v, \tag{3.2}$$

$p,q,u,v\in\mathbb{N}$. Introducing (3.2) into the initial equation (3.1), we get

$$\alpha x^4+\beta x^3+\gamma x^2+\delta x=0, \tag{3.3}$$

where

$$\begin{aligned} &\alpha=p^4-1, && \beta=p^3+q^3-4v-4p^3v,\\ &\gamma=3p^2u-3q^2u-6v^2+6p^2v^2, && \delta=3pu^2+3qu^2-4v^3-4pv^3. \end{aligned} \tag{3.4}$$

For $\delta=0$ in (3.4), we obtain

$$3(p+q)u^2=4(p+1)v^3.$$

Additionally, we set $u=t^3$, $v=t^2$ and get $p=3q-4$. Finally, the following is obtained

$$\alpha=81q^4-432q^3+864q^2-768q+255,$$

$$\beta=432q^2t^2-576t^2q-108q^2+252t^2-108q^3t^2+28q^3+144q-64,$$

$$\gamma=24q^2t^3+54q^2t^4-72qt^3-144qt^4+90t^4+48t^3.$$

Let

$$\gamma=6t^3(q-1)(4q+9qt-15t-8)=0.$$

In that case, we have $q=\frac{15t+8}{9t+4}$ and therefore (3.3) becomes

$$\frac{11664t^3+23328t^2+16128t+3840}{(9t+4)^4}x^4+$$
$$+\frac{-52488t^6-128304t^5-87480t^4+27216t^3+61632t^2+27648t+4096}{(9t+4)^4}x^3=0.$$

We get

$$x=\frac{(9t+4)(243t^4+324t^3-27t^2-192t-64)}{6(81t^2+108t+40)}$$

and

$$\begin{aligned} X &= \frac{2673t^5 + 5508t^4 + 2589t^3 - 1944t^2 - 2112t - 512}{6(81t^2 + 108t + 40)}, \\ Y &= \frac{3159t^5 + 6156t^4 + 1947t^3 - 3096t^2 - 2496t - 512}{6(81t^2 + 108t + 40)}, \\ U &= \frac{2187t^5 + 4374t^4 + 1701t^3 - 1596t^2 - 1344t - 256}{6(81t^2 + 108t + 40)}, \\ V &= \frac{2187t^5 + 4374t^4 + 1701t^3 - 2184t^2 - 2112t - 512}{6(81t^2 + 108t + 40)}. \end{aligned} \tag{3.5}$$

After eliminating denominators in (3.5), we have

$$X = 216(81t^2 + 108t + 40)^3(2673t^5 + 5508t^4 + 2589t^3 - 1944t^2 - 2112t - 512),$$

$$Y = 216(81t^2 + 108t + 40)^3(3159t^5 + 6156t^4 + 1947t^3 - 3096t^2 - 2496t - 512),$$

$$U = 36(81t^2 + 108t + 40)^2(9t + 8)(243t^4 + 270t^3 - 51t^2 - 132t - 32),$$

$$V = 36(81t^2 + 108t + 40)^2(2187t^5 + 4374t^4 + 1701t^3 - 2184t^2 - 2112t - 512).$$

For $t = 1$, the solutions of (3.1) are

$$(X,Y,U,V) = (16087625952048,\ 13379550896592,\ 9563979816,\ 65207230704),$$

while $t = 2$ leads us to

$$\begin{aligned} (X,Y,U,V) = (&7664511333888000, 8313869044224000, \\ &1746900979200, 1696715481600). \end{aligned}$$

We get a positive integer solution (X,Y,U,V) of equation (3.1) for every $t \in \mathbb{N}$. So, the presented method generates infinitely many positive integer solutions of the initial equation (3.1).

3.2. *The Second Method*

Again, we deal with (3.1) and start this new method by introducing a different linear transformation in order to reach more (different) positive integer solutions. Let

$$X = u, \qquad Y = qx - u, \qquad U = x + u, \qquad V = px + u, \tag{3.6}$$

$p, q, u \in \mathbb{N}$. Like in the previous subsection, introducing these linear transformations into (3.3), leads us to the expression of the form

$$Ax^4 + Bx^3 + Cx^2 + Dx = 0, \tag{3.7}$$

where

$$\begin{aligned} A &= p^4-1, & B &= q^3+4p^3u-4u, \\ C &= 6p^2u^2-6u^2-3q^2u, & D &= 4u^3p-4u^3+3u^2q. \end{aligned} \tag{3.8}$$

We obtain $u = \frac{3q}{4(1-p)}$, $p \neq 1$ for $D=0$. After introducing the latter expression into (3.8) and canceling the denominators we get

$$A = 8p^5-8p^4-8p+8, \quad B = 8q^3p-24p^3q-8q^3+24q, \quad C = 27q^2p+18q^3+27q^2.$$

We obtain $q = -\frac{3}{2}(p+1)$ for $C=0$. Under those conditions we get

$$A = p^4-1, \qquad B = \frac{9}{8}(p+1)(p-1)^2$$

and (3.7) becomes $(p^4-1)x^4 + \frac{9}{8}(p+1)(p-1)^2x^3 = 0$ where

$$x = \frac{9(1-p)}{8(p^2+1)}.$$

After plugging all of these results into (3.6) and canceling the denominators, we finally get

$$\begin{aligned} X &= 18(p-1)^3(p+1)(p^2+1)^4, \\ Y &= 9(p+1)(p^2-6p+1)(p-1)^3(p^2+1)^3, \\ U &= 9p(p^2+3)(p-1)^2(p^2+1)^2, \\ V &= 9(3p^2+1)(p^2+1)^2(p-1)^2. \end{aligned} \tag{3.9}$$

Remark 2. According to (3.9), $X > 0$ is satisfied for $p \in \mathbb{Z}\backslash\{0,1\}$. Also, $Y > 0$ is satisfied for $p \in \mathbb{Z}\backslash\{-1,0,1,2,3,4,5\}$. Therefore, by the introduced method we are again able to generate infinitely many positive integer solutions of (3.1).

4. The equation $X^3 - Y^3 = U^4 - V^4$

In this section we deal with the equation

$$X^3 - Y^3 = U^4 - V^4. \tag{4.1}$$

If we introduce $Y \to -Y$ or $X \to -X$ in (3.1), it is clear that one gets solutions of (4.1), but, as we mentioned earlier, we are interested only in positive integer solutions. Therefore, we consider (3.1) and (4.1) as two different equations.

Clearly, $(X,Y,U,V) = (1,n^4,1,n^3)$, $n \in \mathbb{N}$ are trivial solutions of (4.1).

After introducing linear transformations

$$X = px+u, \qquad Y = qx+u, \qquad U = rx+u, \qquad V = u,$$

$p,q,r,u \in \mathbb{N}$ into (4.1), we get the equation

$$Mx^4 + Nx^3 + Px^2 + Qx = 0,$$

where

$$\begin{aligned} M &= -r^4, & N &= p^3 - q^3 - 4r^3u, \\ P &= 3p^2u - 3q^2u - 6r^2u^2, & Q &= 3u^2p - 3u^2q - 4u^3r. \end{aligned} \tag{4.2}$$

If we set $Q = 0$, it is easily obtained $u = \frac{3(p-q)}{4r}$. Plugging this new form of u into (4.2), we get

$$M = -r^4, \quad N = (p-q)(p^2+pq-3r^2+q^2), \quad P = \frac{9(p-q)^2(2p+2q-3r)}{8r}. \tag{4.3}$$

We obtain $r = \frac{2}{3}(p+q)$ for $2p+2q-3r = 0$. So, introducing the above expressions into (4.3), we get

$$-\frac{16}{81}(p+q)^4x^4 - \frac{1}{3}(p-q)(p^2+5pq+q^2)x^3 = 0,$$

$$x = -\frac{27}{16}\frac{(p-q)(p^2+5pq+q^2)}{(p+q)^4}.$$

Finally, we define $p^2+5pq+q^2-16 = 0$ which is a quadratic equation in p. So,

$$p = \frac{-5q \pm \sqrt{21q^2+64}}{2}.$$

Let $21q^2+64 = r^2$, $r \in \mathbb{Z}$. Solution $(q,r) = (0,8)$ is a trivial solution for this equation. Therefore, considering $r = mq+8$ and $21q^2+64 = r^2$ leads us to $q = \frac{16m}{21-m^2}$ and $p = \frac{4(m^2+10m+21)}{m^2-21}$ or $p = \frac{4(m^2-10m+21)}{21-m^2}$. We get

$$x = \frac{-27(m^2-21)^3(m^2+14m+21)}{64(m^2+6m+21)^4}$$

for $p = \frac{4(m^2+10m+21)}{m^2-21}$. After canceling the denominators, we obtain

$$\begin{aligned} X &= -9(m^2+14m+21)(m^6-6m^5-405m^4-3204m^3-8505m^2-2646m+9261), \\ Y &= 18(m^2+14m+21)(m^6+24m^5+171m^4+720m^3+3591m^2+10584m+9261), \\ U &= 27\cdot 2^2\cdot m(m^2+14m+21)(m^2+10m+21), \\ V &= 9(m^2+6m+21)^2(m^2+14m+21). \end{aligned} \tag{4.4}$$

Some elementary analysis leads us to conclusion that $X, Y > 0$ for every $m \in \mathbb{N}$. The described method generates infinitely many positive integer solutions (X,Y,U,V) for the initial equation (4.1). Some of them are introduced in the following table.

m	X	Y	U	V
1	1783296	15780096	124416	254016
2	29712807	51631434	515160	653013
3	126531072	146686464	1399680	1492992
4	375132519	380970594	3093552	3114477
5	911771136	917584128	6013440	6030144

Remark 3. If we apply the previous process by taking $p = \frac{4(m^2-10m+21)}{21-m^2}$, no new solutions are obtained.

Some basic calculations give us $X, Y > 0$ for $m \le -1$ and these solutions are already obtained for $m > 0$ in (4.4). This is shown in the following table. It is useful to notice that, even though we get integer solutions in this case, we do not consider negative ones because U, V are introduced with even powers in (4.1).

m	X	Y	U	V
-1	1783296	15780096	-124416	254016
-2	29712807	51631434	-515160	653013
-3	126531072	146686464	-1399680	1492992
-4	375132519	380970594	-3093552	3114477
-5	911771136	917584128	-6013440	6030144

5. The equation $Am^4 + Bn^4 + Cp^4 + Dq^4 = Ar^4 + Bs^4 + Ct^4 + Du^4$

We deal with the equation of the form

$$Am^4 + Bn^4 + Cp^4 + Dq^4 = Ar^4 + Bs^4 + Ct^4 + Du^4, \qquad A, B, C, D \in \mathbb{Z}. \tag{5.1}$$

Let

$$\begin{aligned} m &= lx+1, & n &= 5x+k, & p &= 4x+1, & q &= 2x, \\ r &= lx-1, & s &= 4x-k, & t &= 5x-1, & u &= x, \end{aligned}$$

for $k, l \in \mathbb{N}$. Introducing these linear transformations into (5.1), one gets the equation of the form

$$ax^4 + bx^3 + cx^2 + dx = 0, \tag{5.2}$$

where

$$\begin{aligned} a &= -396C + 15E + 369B, & b &= 756Bk + 8Al^3 + 756C, \\ c &= -54C + 54Bk^2, & d &= 8Al + 36Bk^3 + 36C. \end{aligned}$$

We set the conditions $c = d = 0$ and get

$$C = Bk^2 \qquad \text{and} \qquad l = -\frac{9}{2} \cdot \frac{Bk^3 + C}{A}. \tag{5.3}$$

Introducing (5.3) into (5.2) leads us to

$$\begin{aligned} &-\frac{3(-123BA^2 + 123Bk^2A^2 - 5EA^2)x^4}{A^2} \\ &-\frac{3(729B^3k^7 + 243B^3k^6 + 243B^3k^9 + 729B^3k^8 - 252Bk^2A^2 - 252BA^2k)x^3}{A^2} = 0 \end{aligned}$$

and

$$x = \frac{9Bk(-27B^2k^8 - 81B^2k^7 - 81B^2k^6 - 27B^2k^5 + 28A^2 + 28kA^2)}{A^2(123Bk^2 - 5E - 123B)}.$$

After plugging all results into (5.3) and cancelling the denominators, we get

$$\begin{aligned}
m =& 2187B^4k^{12} + 8748B^4k^{11} + 13122B^4k^{10} + 8748B^4k^9 \\
& - 4536B^2k^4A^2 - 2268B^2k^5A^2 + 2187B^4k^8 - 2268B^2k^3A^2 \\
& + 246A^3Bk^2 - 10A^3E - 246A^3B, \\
n =& 2Ak\left(-1215B^3k^8 - 3645B^3k^7 - 3645B^3k^6 - 1215B^3k^5\right. \\
& \left.+1137BA^2 + 1260BkA^2 + 123A^2Bk^2 - 5A^2E\right), \\
p =& 2A\left(-972B^3k^9 - 2916B^3k^8 - 2916B^3k^7 - 972B^3k^6\right. \\
& \left.+1008BkA^2 + 1131A^2Bk^2 - 5A^2E - 123BA^2\right), \\
q =& 36ABk(-27B^2k^8 - 81B^2k^7 - 81B^2k^6 - 27B^2k^5 + 28A^2 + 28kA^2), \\
r =& -2187B^4k^{12} - 8748B^4k^{11} - 13122B^4k^{10} - 8748B^4k^9 \\
& + 4536B^2k^4A^2 + 2268B^2k^5A^2 - 2187B^4k^8 + 2268B^2k^3A^2 \\
& + 246A^3Bk^2 - 10A^3E - 246A^3B, \\
s =& 2Ak\left(972B^3k^8 + 2916B^3k^7 + 2916B^3k^6 + 972B^3k^5 - 1131BA^2\right. \\
& \left.-1008BkA^2 + 123A^2Bk^2 - 5A^2E\right), \\
t =& 2A\left(-1215B^3k^9 - 1645B^2k^8 - 1645B^3k^7 - 1215B^3k^6 + 1260BkA^2\right. \\
& \left.+1137A^2Bk^2 + 5A^2E + 123BA^2\right), \\
u =& 18ABk(-27B^2k^8 - 81B^2k^7 - 81B^2k^6 - 27B^2k^5 + 28A^2 + 28kA^2).
\end{aligned} \tag{5.4}$$

5.1. *Numerical examples*

We solve equation (5.1) of the form

$$m^4 + 2n^4 + 2p^4 + 5q^4 = r^4 + 2s^4 + 2t^4 + 5u^4$$

where $A = 1,\ B = 2,\ C = 2,\ D = 5$, so $k = 1$. According to (5.4), we obtain

$$\begin{aligned}
&m = 523534, && n = 145490, && p = 116402, && q = 58176, \\
&r = 523634, && s = 116302, && t = 145390, && u = 39088
\end{aligned}$$

or

$$\begin{aligned}
&m = 261767, && n = 72745, && p = 58201, && q = 29088, \\
&r = 261817, && s = 58151, && t = 72695, && u = 14544.
\end{aligned}$$

Let

$$3m^4 + 2n^4 + 8p^4 + 5q^4 = 3r^4 + 2s^4 + 8t^4 + 5u^4.$$

We have

$$\begin{array}{llll} m = 719753958, & n = 99883476, & p = 79929882 & q = 39984192, \\ r = 719676954, & s = 80045388, & t = 99998982, & u = 19992096 \end{array}$$

or

$$\begin{array}{llll} m = 13328777, & n = 1849694, & p = 1480183, & q = 740448, \\ r = 13327351, & s = 1482322, & t = 1851833, & u = 370224. \end{array}$$

Remark 4. Even though the introduced method in Section 2 provides us with the primitive solution of (2.1), methods for solving equations (3.1) and (4.1) do not provide us with primitive solutions. In these particular cases, we can find some of them using a few simple computer algorithms. For example,

$$\begin{aligned}(X,Y,U,V) =&(4,15,10,9),(4,16,9,7),(14,23,16,15),\\ &(20,31,14,5),(25,71,37,35),\dots\end{aligned}$$

are primitive integer solutions of (3.1) and

$$\begin{aligned}(X,Y,U,V) =&(9,22,3,10),(10,25,2,11),(16,81,8,27),\\ &(26,73,20,27),(57,58,3,10),(62,87,21,28),\\ &(70,71,15,16),(79,92,18,25),(148,177,10,39),\dots.\end{aligned}$$

are primitive integer solutions of (4.1). So, introducing a slightly different approach could be a good starting point for our further research.

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References

[1] M. Baghalaghdam and F. Izadi, "A Note On The High Power Diophantine Equations," *Proc. Math. Sci.*, vol. 129, no. 14, 2019, doi: 10.1007/s12044-018-0457-5.

[2] M. Baghalaghdam and F. Izadi, "On The Diophantine Equation in The Form That A Sum Of Cubes Equals a Sum of Quantics," *Math. J. Okayama Univ*, vol. 61, pp. 75–84, 2019. [Online]. Available: https://www.math.okayama-u.ac.jp/mjou/mjou61/_04_Izadi.pdf

[3] J. P. M. Binet, "Notes Sur Une Question Relative àla Thérie Des Nombres," *Comptes Rendus Paris*, vol. 12, pp. 248–250, 1841.

[4] A. Choudhry, "The Diophantine Equation $A^4+B^4=C^4+D^4$," *Indian J. Pure Appl. Math.*, vol. 22, no. 1, pp. 9–11, 1991.

[5] A. Choudhry, "On The Diophantine Equation $A^4+hB^4=C^4+hD^4$," *Indian J. Pure Appl. Math.*, vol. 26, no. 11, pp. 1057–1061, 1995.

[6] A. Choudhry, "The Diophantine Equation $ax^5+by^5+cz^5=au^5+bv^5+cw^5$," *Rocky Mountain J. Math*, vol. 29, no. 2, pp. 459–462, 1999, doi: 10.1216/rmjm/1181071645.

[7] N. Elkies, "On $A^4+B^4+C^4=D^4$," *Math. Comput.*, vol. 51, no. 184, pp. 825–835, 1988, doi: 10.1090/S0025-5718-1988-0930224-9.
[8] L. Euler, *Novi Comm. Acad. Petrop., 6,1756-7, 155; Comm. Aerith., I, 193; Op. Om., (1), II, 428.*, vol. 6, 1756-7.
[9] M. Gawron and M. Ulas, "On Primitive Integer Solutions Of The Diophantine Equation $t^2 = G(x,y,z)$ and Related Results," *Journal of Number Theory*, vol. 159, pp. 101–122, 2016, doi: 10.1016/j.jnt.2015.07.012.
[10] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*. London: Oxford University Press, 1960.
[11] F. Izadi and K. Nabardi, "Diophantine Equation $X^4+Y^4=2(U^4+V^4)$," *Math. Slovaca*, vol. 66, no. 3, pp. 557–560, 2016, doi: 10.1515/ms-2015-0157.
[12] A. S. Janfada and K. Nabardi, "On the Diophantine Equation $x^4+y^4=n(u^4+v^4)$," *Math. Slovaca*, vol. 69, no. 6.
[13] J. Kersey, *The Elements of Algebra*, London, 1674.
[14] P. Pasternak, "Zeitschr," *Math. Naturw. Unterrieht*, vol. 37, pp. 33–35, 1906.

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ON CENTRAL FUBINI-LIKE NUMBERS AND POLYNOMIALS

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Abstract. We introduce *the central Fubini-like* numbers and polynomials using Rota approach. Several identities and properties are established as generating functions, recurrences, explicit formulas, parity, asymptotics and determinantal representation.

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1. Introduction

We start by giving some definitions that will be used throughout this paper.
For $n \geq 1$, the falling factorial denoted $x^{\underline{n}}$ is defined by

$$x^{\underline{n}} = x(x-1)\cdots(x-n+1),$$

and the central factorial $x^{[n]}$, see [4, 9], is defined by

$$x^{[n]} = x(x+n/2-1)(x+n/2-2)\cdots(x-n/2+1).$$

We use the convention, $x^{\underline{0}} = x^{[0]} = 1$.

It is well-known that, for all non-negative integers n and k ($k \leq n$), Stirling numbers of the second kind are defined as the coefficients $S(n,k)$ in the expansion

$$x^n = \sum_{k=0}^{n} S(n,k) x^{\underline{k}}. \tag{1.1}$$

Riordan, in his book [15], shows that, for all non-negative integers n and k ($k \leq n$), the central factorial numbers of the second kind are the coefficients $T(n,k)$ in the expansion

$$x^n = \sum_{k=0}^{n} T(n,k) x^{[k]}. \tag{1.2}$$

In combinatorics, the number of ways to partition a set of n elements into k nonempty subsets are counted by Stirling numbers $S(n,k)$, and the central factorial numbers $T(2n, 2n-2k)$ count the number of ways to place k rooks on a 3D-triangle board of size $(n-1)$, see [11].

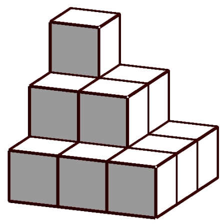

FIGURE 1. 3D-triangle board of size 3.

The coefficients $S(n,k)$ and $T(n,k)$ satisfy, respectively, the triangular recurrences

$$S(n,k) = kS(n-1,k) + S(n-1,k-1) \quad (1 \le k \le n) \tag{1.3}$$

and

$$T(n,k) = \left(\frac{k}{2}\right)^2 T(n-2,k) + T(n-2,k-2) \quad (2 \le k \le n), \tag{1.4}$$

where

$$S(n,k) = T(n,k) = 0 \text{ for } k > n,\ S(0,0) = T(0,0) = T(1,1) = 1 \text{ and } T(1,0) = 0.$$

$S(n,k)$ and $T(n,k)$ admit also the explicit expressions

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n, \tag{1.5}$$

$$T(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\frac{k}{2}-j\right)^n. \tag{1.6}$$

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	1	1				
3	0	1	3	1			
4	0	1	7	6	1		
5	0	1	15	25	10	1	
6	0	1	31	90	65	15	1

TABLE 1. The first few values of $S(n,k)$.

	0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	0	1				
3	0	$\frac{1}{4}$	0	1			
4	0	0	1	0	1		
5	0	$\frac{1}{16}$	0	$\frac{5}{2}$	0	1	
6	0	0	1	0	5	0	1

TABLE 2. The first few values of $T(n,k)$.

The usual difference operator Δ, the shift operator E^a and the central difference operator δ are given respectively by

$$\Delta f(x) = f(x+1) - f(x),$$
$$\mathsf{E}^a f(x) = f(x+a)$$

and

$$\delta f(x) = f(x+1/2) - f(x-1/2).$$

Riordan, [15], mentioned that the central factorial operator δ satisfies the following property

$$\delta f_n(x) = n f_{n-1}(x), \tag{1.7}$$

where $(f_n(x))_{n\geq 0}$ is a sequence of polynomials with $f_0(x) = 1$.

We can also express δ by means of both Δ and E^a, see [9, 15], as follows:

$$\delta f(x) = \Delta \mathsf{E}^{-1/2} f(x). \tag{1.8}$$

For more details about difference operators, we refer the reader to [9].

2. Central Fubini-like numbers and polynomials

In 1975, Tanny [17], introduced the Fubini polynomials (or ordered Bell polynomials) $F_n(x)$ by applying a linear transformation $\mathcal{L}$ defined as

$$\mathcal{L}(x^{\underline{n}}) := n! x^n.$$

The polynomials $F_n(x)$ are given by

$$F_n(x) := \sum_{k\geq 0}^{n} k! S(n,k) x^k, \tag{2.1}$$

according to,

$$F_n(x) := \mathcal{L}(x^n) = \mathcal{L}\left(\sum_{k=0}^{n} S(n,k) x^{\underline{k}}\right) = \sum_{k=0}^{n} S(n,k) \mathcal{L}(x^{\underline{k}}) = \sum_{k=0}^{n} k! S(n,k) x^k.$$

Putting $x = 1$ in (2) we get

$$F_n := F_n(1) = \sum_{k=0}^{n} k! S(n,k), \tag{2.2}$$

which is the n-th Fubini number.

The Fubini polynomial $F_n(x)$ has the exponential generating function given by, see [17],

$$\sum_{n=0} F_n(x) \frac{t^n}{n!} = \frac{1}{1 - x(e^t - 1)}. \tag{2.3}$$

For more details concerning Fubini numbers and polynomials, see [3, 6, 8, 12, 17, 18, 20] and papers cited therein.

Now, we introduce the linear transformation $\mathcal{Z}$ as follows.

Definition 1. For $n \geq 0$, we define the transformation

$$\mathcal{Z}(x^{[n]}) = n!x^n. \tag{2.4}$$

Then, we have

$$\mathcal{Z}(x^n) = \mathcal{Z}\left(\sum_{k=0}^{n} T(n,k)x^{[k]}\right) = \sum_{k=0}^{n} T(n,k)\mathcal{Z}(x^{[k]}) = \sum_{k=0}^{n} k!T(n,k)x^k. \tag{2.5}$$

And due to Formula (1.6), we are now able to introduce the main notion of the present paper.

Definition 2. The *n-th central Fubini-like* polynomial is given by

$$\mathfrak{C}_n(x) := \sum_{k=0}^{n} k!T(n,k)x^k. \tag{2.6}$$

Setting $x = 1$, we obtain the *central Fubini-like numbers*,

$$\mathfrak{C}_n = \mathfrak{C}_n(1) := \sum_{k=0}^{n} k!T(n,k). \tag{2.7}$$

The first central polynomials $\mathfrak{C}_n(x)$ are given in Table 3.

n	$\mathfrak{C}_{2n}(x)$	$2^{2n}\mathfrak{C}_{2n+1}(x)$
0	1	x
1	$2x^2$	$x+24x^3$
2	$2x^2+24x^4$	$x+240x^3+1920x^5$
3	$2x^2+120x^4+720x^6$	$x+2184x^3+67200x^5+322560x^7$
4	$2x^2+504x^4+10080x^6+40320x^8$	$x+19680x^3+1854720x^5+27095040x^7+92897280x^9$

TABLE 3. First value of $\mathfrak{C}_n(x)$.

The first few central Fubini-like numbers are

$(\mathfrak{C}_{2n})_{n\geq 0}$: $1, 2, 26, 842, 50906, 4946282, 704888186, 138502957322, \ldots$

$(2^{2n}\mathfrak{C}_{2n+1})_{n\geq 0}$: $1, 25, 2161, 391945, 121866721, 57890223865, 38999338931281, \ldots$

2.1. *Exponential generating function*

We begin by establishing the exponential generating function of the central Fubini-like polynomials.

Theorem 1. *The polynomials $\mathfrak{C}_n(x)$ have the following exponential generating function*

$$G(x;t) := \sum_{n=0} \mathfrak{C}_n(x)\frac{t^n}{n!} = \frac{1}{1-2x\sinh(t/2)}. \tag{2.8}$$

Proof. We have

$$\sum_{n=0} \mathfrak{C}_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} k!T(n,k)x^k\frac{t^n}{n!} = \sum_{k=0} k!x^k\sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!},$$

from [15, p. 214] , we have

$$\sum_{n=0} T(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left(2\sinh(t/2)\right)^k,$$

therefore

$$\sum_{n=0} \mathfrak{C}_n(x)\frac{t^n}{n!} = \sum_{k=0}^{\infty}\left(2\sinh(t/2)\right)^k x^k = \frac{1}{1-2x\sinh(t/2)}.$$

□

Corollary 1. *The sequence* $(\mathfrak{C}_n)_{n\geq 0}$ *has the following exponential generating function*

$$\sum_{k=0}^{n} \mathfrak{C}_n\frac{t^n}{n!} = \frac{1}{1-2\sinh(t/2)}. \tag{2.9}$$

2.2. *Explicit representations*

In this subsection we propose some explicit formulas for the central Fubini-like polynomials, we start by the derivative representation.

Proposition 1. *The polynomials* $(\mathfrak{C}_n(x))_{n\geq 0}$ *correspond to the higher derivative expression*

$$\mathfrak{C}_n(x) = \sum_{k=0}^{\infty}\frac{\partial^n}{\partial^n t}\left(2x\sinh(t/2)\right)^k\Big|_{t=0}.$$

Proof. Let

$$\frac{\partial^n}{\partial^n t}\left(\sum_{m=0}^{\infty}\mathfrak{C}_m(x)\frac{t^m}{m!}\right)\Bigg|_{t=0} = \sum_{m=n}^{\infty}\mathfrak{C}_m(x)\frac{t^{m-n}}{(m-n)!}\Bigg|_{t=0} = \sum_{m=0}^{\infty}\mathfrak{C}_{n+m}(x)\frac{t^m}{m!}\Bigg|_{t=0} = \mathfrak{C}_n(x).$$

Thus from Theorem 1 we get the result. □

From Formula (1.6), it is clear that the following proposition holds.

Proposition 2. *The central Fubini-like polynomials satisfy the following explicit formula*

$$\mathfrak{C}_n(x) = \sum_{k=0}^{n} x^k\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k/2-j)^n.$$

Proof. It suffices to replace $T(n,k)$ in Equation (2.6) by its explicit formula (Equation (1.6)),

$$\mathfrak{C}_n(x) = \sum_{k=0}^{n} k!T(n,k)x^k = \sum_{k=0}^{n} x^k\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k/2-j)^n.$$

□

Theorem 2. *For non-negative n, the following explicit representation holds true.*

$$\mathfrak{C}_n(x) = x\sum_{k=0}^{n-1}\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{-1}{2}\right)^{k-j}\mathfrak{C}_j(x) = x\sum_{j=0}^{n-1}\binom{n}{j}\delta[0^{n-j}]\mathfrak{C}_j(x), \tag{2.10}$$

where $\delta[0^{n-j}] = (1/2)^{n-j} - (-1/2)^{n-j}$.

The proof will depend on Lemma 1, Lemma 2 and Relation (1.8).

Lemma 1. *For all polynomials $p_n(x)$ the following relation holds true.*

$$\mathcal{Z}(p_n(x)) = x\mathcal{Z}(\delta p_n(x)).$$

Proof. We have

$$\mathcal{Z}(x^{[n]}) = n!x^n = xn(n-1)!x^{n-1} = x\mathcal{Z}(nx^{[n-1]}) = x\mathcal{Z}(\delta x^{[n]}),$$

as any polynomial can be written as sums of central factorials $x^{[n]}$. Thus, we have the result. □

Lemma 2 (Tanny [17])**.** *For all polynomials $p_n(x)$ we have*

$$\Delta p_n(x) = \sum_{k=0}^{n-1}\binom{n}{k}p_k(x). \tag{2.11}$$

Now we give the proof of Theorem 2,

Proof of Theorem 2. Using Lemma 1, Lemma 2 and setting $p_n(x) = x^n$, we get

$$\begin{aligned}\mathcal{Z}(x^n) &= x\mathcal{Z}(\delta x^n) = x\mathcal{Z}\left(\Delta E^{-1/2}x^n\right) = x\mathcal{Z}\left(\Delta\left(x-\frac{1}{2}\right)^n\right)\\ &= x\mathcal{Z}\left(\sum_{k=0}^{n-1}\binom{n}{k}\left(x-\frac{1}{2}\right)^k\right) = x\mathcal{Z}\left(\sum_{k=0}^{n-1}\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{-1}{2}\right)^{k-j}x^j\right)\\ &= x\sum_{k=0}^{n-1}\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{-1}{2}\right)^{k-j}\mathfrak{C}_j(x).\end{aligned}$$

Using binomial product identity $\binom{n}{k}\binom{k}{j} = \binom{n-j}{k-j}\binom{n}{j}$, we get the result. □

Corollary 2. *The central Fubini-like numbers satisfy*

$$\mathfrak{C}_n = \sum_{j=0}^{n-1}\binom{n}{j}\delta[0^{n-j}]\mathfrak{C}_j. \tag{2.12}$$

Now we give an explicit formula connecting the central Fubini-like polynomials with Stirling numbers of the second kind $S(n,k)$,

Theorem 3. *The central Fubini-like polynomials $\mathfrak{C}_n(x)$ satisfy*

$$\mathfrak{C}_n(x) = \sum_{k=0}^{n} k!x^k \sum_{j=0}^{n} \binom{n}{j} \left(\frac{-k}{2}\right)^j S(n-j,k). \tag{2.13}$$

Proof. From Theorem 1, we have

$$\sum_{n=0} \mathfrak{C}_n(x)\frac{t^n}{n} = \frac{1}{1-2x\sinh(t/2)}.$$

Using the exponential form of $2x\sinh(t/2)$ we get

$$\sum_{n=0} \mathfrak{C}_n(x)\frac{t^n}{n} = \frac{1}{1-xe^{(-t/2)}(e^t-1)} = \sum_{k=0} x^k e^{(-kt/2)}(e^t-1)^k.$$

It is also known that

$$\sum_{n=0} S(n,k)\frac{t^n}{n!} = \frac{(e^t-1)^k}{k!}.$$

Therefore

$$\sum_{n=0} \mathfrak{C}_n(x)\frac{t^n}{n} = \sum_{k=0} x^k k! \sum_{n=0} \left(\frac{-k}{2}\right)^n \frac{t^n}{n!} \sum_{n=0} S(n,k)\frac{t^n}{n!}.$$

Then Cauchy's product implies the identity. □

Corollary 3. *The central Fubini-like numbers $\mathfrak{C}_n$ satisfy*

$$\mathfrak{C}_n = \sum_{k=0}^{n} k! \sum_{j=0}^{n} \binom{n}{j} \left(\frac{-k}{2}\right)^j S(n-j,k). \tag{2.14}$$

2.3. *Umbral representation*

Umbral (or Blissard or symbolic) calculus originated as a method for discovering and proving combinatorial identities in which subscripts are treated as powers. Bell in [1] gave a postulational bases of this calculus. In this section we use the following property given by Riordan [16]. As specified by the author in [16], *"A sequence $a_0, a_1, \ldots$ may be replaced by $a^0, a^1, \ldots$ with the exponents are treated as powers during all formal operations, and only restored as indexes when operations are completed"*. Then when we have

$$a_n = \sum_{k=0} \binom{n}{k} b_k c_{n-k}$$

we can write it as

$$a_n = (b+c)^n,$$

where $b^n \equiv b_n$ and $c^n \equiv c_n$. We note that b^0 and c^0 is not necessary equal to 1.

In the following theorem we use the umbral notation $\mathfrak{C}_k(x) \equiv \mathfrak{C}^k(x)$ and $\mathfrak{C}_k \equiv \mathfrak{C}^k$.

Theorem 4. *Let n be a non-negative integer, for all real x we have*

$$\mathfrak{C}_n(x) = x\left[(\mathfrak{C}(x)+1/2)^n - (\mathfrak{C}(x)-1/2)^n\right].$$

Proof. From Theorem 2 and using the umbral notation, a simple calculation gives the umbral representation result. □

Corollary 4. *For non-negative integer n, we have*

$$\mathfrak{C}_n = (\mathfrak{C} + 1/2)^n - (\mathfrak{C} - 1/2)^n.$$

2.4. *Parity*

A function $f(x)$ is said to be even when $f(x) = f(-x)$ for all x and it is said to be odd when $f(x) = -f(-x)$.

Theorem 5. *For all non-negative n and real variable x we have*

$$\mathfrak{C}_n(x) = (-1)^n \mathfrak{C}_n(-x).$$

Proof. Using the fact that the function $f : t \mapsto \sinh(t)$ is odd, this gives $G(x;t) = G(-x;-t)$, then comparing the coefficients of $t^n/n!$ in $G(x;t)$ and $G(-x;-t)$ the theorem follows. □

Corollary 5. *The polynomials $\mathfrak{C}_n(x)$ are odd if and only if n is odd.*

Proof. Using Theorem 5, it suffices to replace n by $2k+1$ (resp. $2k$) and establish the property. □

2.5. *Recurrences and derivatives of higher order*

Now we are interested to derive some recurrences for $\mathfrak{C}_n(x)$ in terms of their derivatives.

First, we deal with a recurrence of second order.

Theorem 6. *For $n \geq 2$, the polynomials $\mathfrak{C}_n(x)$ satisfy the following recurrence relation*

$$\mathfrak{C}_n(x) = 2x^2 \mathfrak{C}_{n-2}(x) + \left(\frac{x}{4} + 4x^3\right) \mathfrak{C}'_{n-2}(x) + \left(\frac{x^2}{4} + x^4\right) \mathfrak{C}''_{n-2}(x).$$

Here $\mathfrak{C}'_n(x)$ and $\mathfrak{C}''_n(x)$ are respectively the first and second derivative of $\mathfrak{C}_n(x)$.

Proof. From Equation (1.4) we have

$$\begin{aligned}
\mathfrak{C}_n(x) &= \sum_{k=0}^{n} k!T(n,k)x^k \\
&= \sum_{k=2}^{n} k!T(n-2,k-2)x^k + \frac{1}{4}\sum_{k=0}^{n} k^2 k!T(n-2,k)x^k \\
&= \sum_{k=0}^{n} (k+2)!T(n-2,k)x^{k+2} + \frac{x}{4}\left(\sum_{k=0}^{n} kk!T(n-2,k)x^k\right)'
\end{aligned}$$

$$
\begin{aligned}
&= x^2\left(x^2\sum_{k=0}^{n} k!T(n-2,k)x^k\right)'' + \frac{x}{4}\left(x\left(\sum_{k=0}^{n} k!T(n-2,k)x^k\right)'\right)' \\
&= x^2\left(x^2\mathfrak{C}_{n-2}(x)\right)'' + \frac{x}{4}\left(x\mathfrak{C}'_{n-2}(x)\right)' \\
&= 2x^2\mathfrak{C}_{n-2}(x) + \left(\frac{x}{4}+4x^3\right)\mathfrak{C}'_{n-2}(x) + \left(\frac{x^2}{4}+x^4\right)\mathfrak{C}''_{n-2}(x),
\end{aligned}
$$

this concludes the proof. □

In the next theorem we give a recurrence formula for the r-th derivative of $\mathfrak{C}_n(x)$.

Proposition 3. *The r-th derivative of $G(x;t)$, defined in (2.8), is given by*

$$\frac{\partial^r}{\partial^r x}G(x;t) = \frac{r!}{x^r}G(x;t)(G(x;t)-1)^r.$$

Proof. Induction on r implies the equality. □

Theorem 7. *Let $\mathfrak{C}_n^{(r)}(x)$ be the r-th derivative of $\mathfrak{C}_n(x)$. Then $\mathfrak{C}_n^{(r)}(x)$ is given by*

$$\mathfrak{C}_n^{(r)}(x) = \frac{r!}{x^r}\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\sum_{j_0+j_1+\cdots+j_k=n}\binom{n}{j_0,j_1,\ldots,j_k}\mathfrak{C}_{j_0}(x)\mathfrak{C}_{j_1}(x)\cdots\mathfrak{C}_{j_k}(x).$$

Proof. Using Proposition 3, by applying Cauchy product and comparing the coefficients of $t^n/n!$, we get the result. □

Corollary 6. *The following equality holds for any real x:*

$$x\mathfrak{C}'_n(x) = \sum_{k=0}^{n-1}\binom{n}{k}\mathfrak{C}_k(x)\mathfrak{C}_{n-k}(x).$$

Proof. Setting $r=1$ in Proposition 3, we get the first derivative of $G(x;t)$ as

$$
\begin{aligned}
\frac{\partial}{\partial x}G(x;t) &= \frac{2\sinh\left(\frac{t}{2}\right)}{\left(1-2x\sinh(\frac{t}{2})\right)^2} = \frac{G(x;t)}{x}\left(G(x;t)-1\right), \\
x\frac{\partial}{\partial x}G(x;t) &= G(x;t)^2 - G(x;t), \\
x\sum_{n=0}\mathfrak{C}'_n(x)\frac{t^n}{n!} &= \left(\sum_{n=0}\mathfrak{C}_n(x)\frac{t^n}{n!}\right)^2 - \sum_{n=0}\mathfrak{C}_n(x)\frac{t^n}{n!},
\end{aligned}
$$

then applying the Cauchy product in the right hand side and comparing the coefficients of $t^n/n!$ we get the result. □

2.6. *Integral representation*

Integral representation is a fundamental property in analytic combinatorics. The central Fubini-like polynomials can be represented as well.

Theorem 8. *The polynomials $\mathfrak{C}_n(x)$ satisfy*

$$\mathfrak{C}_n(x) = \frac{2n!}{\pi}\mathbf{Im}\int_0^{\pi} \frac{\sin(n\theta)}{1-2x\sinh\left(e^{i\theta}/2\right)}\partial\theta.$$

Proof. We will use here the known identity, see [5],

$$k^n = \frac{2n!}{\pi}\mathbf{Im}\int_0^{\pi} \exp\left(ke^{i\theta}\right)\sin(n\theta)\partial\theta.$$

We have

$$\begin{aligned}
\mathfrak{C}_n(x) &= \sum_{k=0}^{\infty} k!T(n,k)x^k \\
&= \sum_{k=0}^{\infty} x^k \sum_{j=0}^{k} (-1)^j \binom{k}{j}\left(\frac{k}{2}-j\right)^n \\
&= \sum_{k=0}^{\infty} x^k \sum_{j=0}^{k} (-1)^j \binom{k}{j}\frac{2n!}{\pi}\mathbf{Im}\int_0^{\pi} \exp\left((k/2-j)e^{i\theta}\right)\sin(n\theta)\partial\theta \\
&= \frac{2n!}{\pi}\mathbf{Im}\int_0^{\pi} \sin(n\theta)\sum_{k=0}^{\infty} x^k \exp\left(-\frac{k}{2}e^{i\theta}\right)\left(\exp\left(e^{i\theta}\right)-1\right)^k\partial\theta \\
&= \frac{2n!}{\pi}\mathbf{Im}\int_0^{\pi} \frac{\sin(n\theta)}{1-2x\sinh\left(e^{i\theta}/2\right)}\partial\theta.
\end{aligned}$$

□

2.7. *Determinantal representation*

Several papers have been published on determinantal representations of many sequences as Bernoulli numbers, Euler numbers, ordered Bell numbers (or Fubini numbers), etc.

Komatsu and Ramírez in a recent paper gives the following theorem.

Theorem 9 (Komatsu & Ramírez [10])**.** *Let $(R(j))_{j\geq 0}$ be a sequence, and let α_n be defined by the following determinantal expression for all $n \geq 1$:*

$$\alpha_n = \begin{vmatrix} R(1) & 1 & & & \\ R(2) & R(1) & & & \\ \vdots & \vdots & \ddots & 1 & \\ R(n-1) & R(n-2) & \cdots & R(1) & 1 \\ R(n) & R(n-1) & \cdots & R(2) & R(1) \end{vmatrix}. \qquad (2.15)$$

Then we have

$$\alpha_n = \sum_{j=1}^{n} (-1)^{j-1} R(j) \alpha_{n-j} \qquad (n \geq 1). \tag{2.16}$$

We set $\alpha_0 = 1$.

By applying the previous theorem we get

Theorem 10. *For* $n \geq 1$, *we have*

$$\frac{\mathfrak{C}_n(x)}{n!} = \begin{vmatrix} R(1) & 1 & & & \\ R(2) & R(1) & & & \\ \vdots & \vdots & \ddots & 1 & \\ R(n-1) & R(n-2) & \cdots & R(1) & 1 \\ R(n) & R(n-1) & \cdots & R(2) & R(1) \end{vmatrix}, \tag{2.17}$$

where

$$R(j) = x\frac{(-1)^{j-1}}{j!}\delta[0^j] = x\frac{(-1)^{j-1}}{j!}\left(\left(\frac{1}{2}\right)^j - \left(-\frac{1}{2}\right)^j\right).$$

Proof. From Theorem 2 we have,

$$\mathfrak{C}_n(x) = x\sum_{j=0}^{n-1}\binom{n}{j}\delta[0^{n-j}]\mathfrak{C}_j(x) = x\sum_{j=1}^{n}\binom{n}{j}\delta[0^j]\mathfrak{C}_{n-j}(x)$$

$$\frac{\mathfrak{C}_n(x)}{n!} = \sum_{j=1}^{n}\frac{x}{j!}\delta[0^j]\frac{\mathfrak{C}_{n-j}(x)}{(n-j)!}.$$

It suffices to set $\alpha_n = \frac{\mathfrak{C}_n(x)}{n!}$ and $R(j) = x\frac{(-1)^{j-1}}{j!}\delta[0^j]$ to get the result. □

Remark 1. The function $R(j) = 0$ for j even.

Using Remark 1, we establish the following binomial convolution for the polynomials $\mathfrak{C}_n(x)$.

Theorem 11. *For* $n \geq 0$ *we have*

$$\mathfrak{C}_{n+1}(x) = x\sum_{k=0}^{\lfloor n/2 \rfloor} 4^{-k}\binom{n+1}{2k+1}\mathfrak{C}_{n-2k}(x). \tag{2.18}$$

Proof. From Remark 1 and using Formula (2.16) with $\alpha_n = \mathfrak{C}_n(x)/n!$ and $R(j) = x\frac{(-1)^{j-1}}{j!}\left(\left(\frac{1}{2}\right)^j - \left(-\frac{1}{2}\right)^j\right)$ we get the result. □

Remark 2. Formula (2.18) is better than result of Theorem 2 from a computational point of view.

2.8. *Asymptotic result with respect to $\mathfrak{C}_n$*

Find an asymptotic behaviour of a sequence $(a_n)_{n\geq 0}$ means to find a second function depending on n simple than the expression of a_n which gives a good approximation to the values of a_n when n is large.

In this subsection, we are interested to obtaining the asymptotic behaviour of the central Fubini-like numbers.

Let $(a_n)_{n\geq 0}$ be a sequence of non-negative real numbers, the asymptotic behaviour a_n is closely tied to the poles in $G(z)$, where $G(z)$ is the generating function of a_n,

$$G(z) = \sum_{n=0} a_n z^n.$$

Wilf, in his book [19] and Flajolet et al. in [7] gave a method to determine the asymptotic behaviour a_n which can be summarized in the following steps:

(1) Find the poles $z_0, z_1, \ldots, z_s$ in $G(z)$.
(2) Calculate the principal parts $P(G(z), z_i)$ at the dominant singularities z_i (which have the smallest modulus R) as

$$P(G(z), z_i) = \frac{Res(G(z), z_i)}{(z - z_i)},$$

where $Res(G(z), z_i)$ is the residue of $G(z)$ at the pole z_i.
(3) Set $H(z) = \sum_{i=0}^{s} P(G(z), z_i)$ then write $H(z)$ as the expansion below,

$$H(z) = \sum_{n=0} b_n z^n.$$

(4) The sequence $(b_n)_{n=0}$ is the asymptotic behaviour of a_n when n is big enough,

$$a_n \sim b_n + O\left(\left(\frac{1}{R'} + \varepsilon\right)^n\right), \quad n \longmapsto \infty.$$

where R' is the next smallest modulus of the poles.

For more details about singularities analysis method we refer to [7].

Remark 3. Poles $z_0, z_1, \ldots, z_s$ are considered as simple poles (has a multiplicity equal to 1).

Analytic methods of determining the asymptotic behavior of a sequence $(a_n)_n$ are widely discussed on [2, 7, 13, 14, 19].

Theorem 12. *The asymptotic behaviour of the $\mathfrak{C}_n$ is given by*

$$\mathfrak{C}_n \sim \frac{n!}{2^n \sqrt{5} \log^{n+1}(\phi)} + O\left((0.15732 + \varepsilon)^n\right), \quad n \longmapsto \infty$$

where ϕ is the Golden ratio.

Proof. Applying the previous steps in the generating function $G(z) = \frac{1}{1 - 2\sinh(z/2)}$ gives

(1) The poles of $G(z)$ are

$$z_0 = -2\log\left(\frac{1+\sqrt{5}}{2}\right) + 2i\pi + 4i\pi k \text{ and } z_1 = 2\log\left(\frac{1+\sqrt{5}}{2}\right) + 4i\pi k,$$

with $k \in \mathbb{Z}$.

(2) By setting $k = 0$, the dominant singularity is $z_1 = 2\log(\phi)$ (the modulus $R = 0.96$), then,

$$P(G(z), z_1) = -\frac{2}{\sqrt{5}(z - 2\log(\phi))}.$$

(3) Set $H(z) = -\frac{2}{\sqrt{5}(z-2\log(\phi))}$, if we write $H(z)$ as the expansion we get

$$H(z) = \sum_{n=0} \frac{1}{2^n\sqrt{5}\log^{n+1}(\phi)} z^n.$$

(4) The the next smallest modulus of the poles $R' = 6.356\ldots$, then the asymptotic behaviour of $\mathfrak{C}_n$ when n is big enough is,

$$\mathfrak{C}_n \sim \frac{n!}{2^n\sqrt{5}\log^{n+1}(\phi)} + O\left((0.15732+\varepsilon)^n\right), \quad n \longmapsto \infty.$$

□

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References

[1] E. T. Bell, "Postulational bases for the umbral calculus," *American Journal of Mathematics*, vol. 62, no. 1, pp. 717–724, 1940, doi: 10.2307/2371481.

[2] E. A. Bender, "Asymptotic methods in enumeration," *SIAM review*, vol. 16, no. 4, pp. 485–515, 1974, doi: 10.1137/1016082.

[3] K. N. Boyadzhiev, "A series transformation formula and related polynomials," *International Journal of Mathematics and Mathematical Sciences*, vol. 2005, no. 23, pp. 3849–3866, 2005, doi: 10.1155/IJMMS.2005.3849.

[4] P. L. Butzer, K. Schmidt, E. Stark, and L. Vogt, "Central factorial numbers; their main properties and some applications." *Numerical Functional Analysis and Optimization*, vol. 10, no. 5-6, pp. 419–488, 1989, doi: 10.1080/01630568908816313.

[5] D. Callan, "Cesaro's integral formula for the Bell numbers (corrected)," *arXiv preprint arXiv:0708.3301*, 2007.

[6] A. Dil and V. Kurt, "Investigating geometric and exponential polynomials with Euler-Seidel matrices," *J. Integer Seq*, vol. 14, no. 4, 2011.

[7] P. Flajolet and R. Sedgewick, *Analytic combinatorics*. Cambridge University Press, 2009.

[8] O. A. Gross, "Preferential arrangements," *The American Mathematical Monthly*, vol. 69, no. 1, pp. 4–8, 1962, doi: 10.1080/00029890.1962.11989826.

[9] C. Jordan and K. Jordán, *Calculus of finite differences*. American Mathematical Soc., 1965, vol. 33.
[10] T. Komatsu and J. L. Ramírez, "Some determinants involving incomplete Fubini numbers," *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. 26, no.3*, 2018, doi: 10.2478/auom-2018-0038.
[11] N. Krzywonos and F. Alayont, "Rook polynomials in three and higher dimensions," *Involve*, vol. 6, no. 1, pp. 35–52, 2013, doi: 10.2140/involve.2013.6.35.
[12] I. Mező, "Periodicity of the last digits of some combinatorial sequences," *J. Integer Seq*, vol. 17, pp. 1–18, 2014.
[13] A. M. Odlyzko, "Asymptotic enumeration methods," *Handbook of combinatorics*, vol. 2, no. 1063, p. 1229, 1995.
[14] J. Plotkin and J. Rosenthal, "Some asymptotic methods in combinatorics," *Journal of the Australian Mathematical Society*, vol. 28, no. 4, pp. 452–460, 1979, doi: 10.1017/S1446788700012593.
[15] J. Riordan, *Combinatorial identities*. Wiley New York, 1968, vol. 6.
[16] J. Riordan, *Introduction to combinatorial analysis*. Courier Corporation, 2012.
[17] S. M. Tanny, "On some numbers related to the Bell numbers," *Canadian Mathematical Bulletin*, vol. 17, no. 5, pp. 733–738, 1975, doi: 10.4153/CMB-1974-132-8.
[18] W. A. Whitworth, *Choice and chance: with 1000 exercises*. D. Bell and Company;[etc., etc.,], 1901.
[19] H. S. Wilf, *generatingfunctionology*. AK Peters/CRC Press, 2005.
[20] D. Zeitlin, "Remarks on a formula for preferential arrangements," *The American Mathematical Monthly*, vol. 70, no. 2, pp. 183–187, 1963, doi: 10.2307/2312890.

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FUZZY SUB-HOOPS BASED ON FUZZY POINTS

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Abstract. Using the *belongs to* relation $(\in)$ and *quasi-coincident with* relation (q) between fuzzy points and fuzzy sets, the notions of an $(\in,\in)$-fuzzy sub-hoop, an $(\in,\in\vee q)$-fuzzy sub-hoop and a $(q,\in\vee q)$-fuzzy sub-hoop are introduced, and several properties are investigated. Characterizations of an $(\in,\in)$-fuzzy sub-hoop and an $(\in,\in\vee q)$-fuzzy sub-hoop are displayed. Relations between an $(\in,\in)$-fuzzy sub-hoop, an $(\in,\in\vee q)$-fuzzy sub-hoop and a $(q,\in\vee q)$-fuzzy sub-hoop are discussed. Conditions for a fuzzy set to be a $(q,\in\vee q)$-fuzzy sub-hoop are considered, and condition for an $(\in,\in\vee q)$-fuzzy sub-hoop to be a $(q,\in\vee q)$-fuzzy sub-hoop are provided.

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1. INTRODUCTION

After the introduction of the concept of a fuzzy set by Zadeh [18], several researches were conducted on the generalizations of the concept of a fuzzy set. One of the least satisfactory areas in the early development of fuzzy topology has been that surrounding the concept of fuzzy point. In the original classical theory, where values are taken in the closed unit interval I, it soon became apparent that, in order to build up a reasonable theory, points should be defined as fuzzy singletons while membership requires strict inequality. So crisp points, taking value 1, are excluded, and fuzzy topology would seem not to include general topology. This disturbing state of affairs was to some extent overcome by [15] who replaced membership by quasi-coincidence (not belonging to the complement, where belonging is taken as $\leq$), thus reinstating crisp points. More recently [11] has drawn attention to a duality between quasi-coincidence and strict inequality membership. The duality, however, is only partial [17].

Hoop, which is introduced by B. Bosbach in [9], is naturally ordered commutative residuated integral monoids. Several properties of hoops are displayed in [3–5, 8, 10, 13, 16, 19]. For example, Blok [3, 4], investigated structure of hoops and their applicational reducts. Borzooei and Aaly Kologani in [5] defined (implicative, positive implicative, fantastic) filters in a hoop and discussed their relations and properties. Using filter, they considered a congruence relation on a hoop, and induced the quotient structure which is a hoop. They also provided conditions for the quotient structure to be Brouwerian semilattice, Heyting algebra and Wajesberg hoop. After that in [2], they studied these notions in pseudo-hoops. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [15], played a vital role to generate some different types of fuzzy subalgebras in of BCK/BCI-algebras. On (α,β)-fuzzy subalgerbas of BCK/BCI-algebras, introduced by Jun [12]. In particular, $(\in,\in\vee q)$-fuzzy subalgebra is an important and useful generalization of a fuzzy subalgebra in BCK/BCI-algebras. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures.

In this paper, we introduce the notions of an $(\in,\in)$-fuzzy sub-hoop, an $(\in,\in\vee q)$-fuzzy sub-hoop and a $(q,\in\vee q)$-fuzzy sub-hoop, and investigate several properties. We discuss characterizations of an $(\in,\in)$-fuzzy sub-hoop and an $(\in,\in\vee q)$-fuzzy sub-hoop. We find relations between an $(\in,\in)$-fuzzy sub-hoop, an $(\in,\in\vee q)$-fuzzy sub-hoop and a $(q,\in\vee q)$-fuzzy sub-hoop. We consider conditions for a fuzzy set to be a $(q,\in\vee q)$-fuzzy sub-hoop of H. We provide a condition for an $(\in,\in\vee q)$-fuzzy sub-hoop to be a $(q,\in\vee q)$-fuzzy sub-hoop.

2. Preliminaries

By a *hoop* we mean an algebra $(H,\odot,\to,1)$ in which $(H,\odot,1)$ is a commutative monoid and the following assertions are valid.

(H1) $(\forall x\in H)(x\to x=1)$,
(H2) $(\forall x,y\in H)(x\odot(x\to y)=y\odot(y\to x))$,
(H3) $(\forall x,y,z\in H)(x\to(y\to z)=(x\odot y)\to z)$.

By a *sub-hoop* of a hoop H we mean a subset S of H which satisfies the condition:

$$(\forall x,y\in H)(x,y\in S\ \Rightarrow\ x\odot y\in S,\ x\to y\in S). \tag{2.1}$$

Note that every non-empty sub-hoop contains the element 1.

Every hoop H satisfies the following conditions (see [9]).

$$(\forall x,y\in H)(x\odot y\le z\ \Leftrightarrow\ x\le y\to z). \tag{2.2}$$

$$(\forall x,y\in H)(x\odot y\le x,y). \tag{2.3}$$

$$(\forall x,y\in H)(x\le y\to x). \tag{2.4}$$

$$(\forall x\in H)(x\to 1=1). \tag{2.5}$$

$$(\forall x\in H)(1\to x=x). \tag{2.6}$$

A fuzzy set λ in a set X of the form

$$\lambda(y):=\begin{cases} t\in(0,1] & \text{if } y=x,\\ 0 & \text{if } y\neq x,\end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set λ in a set X, Pu and Liu [15] gave meaning to the symbol $x_t\alpha\lambda$, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$.

To say that $x_t \in \lambda$ (resp. $x_t q\lambda$) means that $\lambda(x) \geq t$ (resp. $\lambda(x)+t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set λ.

To say that $x_t \in\vee q\,\lambda$ (resp. $x_t \in\wedge q\,\lambda$) means that $x_t \in \lambda$ or $x_t q\lambda$ (resp. $x_t \in \lambda$ and $x_t q\lambda$).

3. (α,β)-FUZZY SUB-HOOPS FOR $(\alpha,\beta) \in \{(\in,\in),(\in,\in\vee q),(q,\in\vee q)\}$

In what follows, let H be a hoop unless otherwise specified.

Definition 1. A fuzzy set λ in H is called an $(\in,\in)$-*fuzzy sub-hoop* of H if the following assertion is valid.

$$(\forall x,y \in H)(\forall t,k \in (0,1])\left(x_t \in \lambda,\ y_k \in \lambda \ \Rightarrow \left\{ \begin{array}{l} (x\odot y)_{\min\{t,k\}} \in \lambda \\ (x \to y)_{\min\{t,k\}} \in \lambda) \end{array} \right. \right). \tag{3.1}$$

Example 1. Let $H = \{0,a,b,c,d,1\}$ be a set with binary operations $\odot$ and $\to$ in Table 1 and Table 2, respectively.

TABLE 1. Cayley table for the binary operation "$\odot$"

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	d	0	d	a
b	0	d	c	c	0	b
c	0	0	c	c	0	c
d	0	d	0	0	0	d
1	0	a	b	c	d	1

TABLE 2. Cayley table for the binary operation "$\to$"

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Then $(H,\odot,\rightarrow,1)$ is a hoop. Define a fuzzy set λ in H as follows:

$$\lambda : H \rightarrow [0,1],\ x \mapsto \begin{cases} 0.5 & \text{if } x = 0, \\ 0.7 & \text{if } x = a, \\ 0.3 & \text{if } x = b, \\ 0.5 & \text{if } x = c, \\ 0.3 & \text{if } x = d, \\ 0.8 & \text{if } x = 1 \end{cases}$$

It is routine to verify that λ is an $(\in,\in)$-fuzzy sub-hoop of H.

We consider characterizations of an $(\in,\in)$-fuzzy sub-hoop.

Theorem 1. *A fuzzy set λ in H is an $(\in,\in)$-fuzzy sub-hoop of H if and only if the following assertion is valid.*

$$(\forall x,y \in H)\left(\begin{array}{l} \lambda(x\odot y) \geq \min\{\lambda(x),\lambda(y)\} \\ \lambda(x\rightarrow y) \geq \min\{\lambda(x),\lambda(y)\} \end{array}\right). \tag{3.2}$$

Proof. Assume that λ is an $(\in,\in)$-fuzzy sub-hoop of H. Note that $x_{\lambda(x)} \in \lambda$ and $y_{\lambda(y)} \in \lambda$ for all $x,y \in H$. It follows from (3.1) that $(x\odot y)_{\min\{\lambda(x),\lambda(y)\}} \in \lambda$ and $(x \rightarrow y)_{\min\{\lambda(x),\lambda(y)\}} \in \lambda$. Hence

$$\lambda(x\odot y) \geq \min\{\lambda(x),\lambda(y)\}$$

and

$$\lambda(x\rightarrow y) \geq \min\{\lambda(x),\lambda(y)\}$$

for all $x,y \in H$.

Conversely, suppose that λ satisfies the condition (3.2). Let $x,y \in H$ and $t,k \in (0,1]$ such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$, which implies from (3.2) that

$$\lambda(x\odot y) \geq \min\{\lambda(x),\lambda(y)\} \geq \min\{t,k\}$$

and

$$\lambda(x\rightarrow y) \geq \min\{\lambda(x),\lambda(y)\} \geq \min\{t,k\}$$

for all $x,y \in H$. Hence $(x\odot y)_{\min\{t,k\}} \in \lambda$ and $(x \rightarrow y)_{\min\{t,k\}} \in \lambda$. Therefore λ is an $(\in,\in)$-fuzzy sub-hoop of H. □

Given a fuzzy set λ in H, we consider the set

$$U(\lambda;t) := \{x \in H \mid \lambda(x) \geq t\},$$

which is called an $\in$-*level set* of λ (related to t).

Theorem 2. *A fuzzy set λ in H is an $(\in,\in)$-fuzzy sub-hoop of H if and only if the non-empty $\in$-level set $U(\lambda;t)$ of λ is a sub-hoop of H for all $t \in [0,1]$.*

Proof. Let λ be a fuzzy set in H such that $U(\lambda;t)$ is a non-empty sub-hoop of H for all $t \in [0,1]$. Let $x,y \in H$ and $t,k \in (0,1]$ such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$, and so $x,y \in U(\lambda;\min\{t,k\})$. By hypothesis, we have $x \odot y \in U(\lambda;\min\{t,k\})$ and $x \to y \in U(\lambda;\min\{t,k\})$. Hence $(x \odot y)_{\min\{t,k\}} \in \lambda$ and $(x \to y)_{\min\{t,k\}} \in \lambda$. Therefore λ is an $(\in,\in)$-fuzzy sub-hoop of H.

Conversely, assume that λ is an $(\in,\in)$-fuzzy sub-hoop of H. Let $x,y \in U(\lambda;t)$ for all $t \in [0,1]$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq t$, that is, $x_t \in \lambda$ and $y_t \in \lambda$. It follows from (3.1) that $(x \odot y)_t \in \lambda$ and $(x \to y)_t \in \lambda$, that is, $x \odot y \in U(\lambda;t)$ and $x \to y \in U(\lambda;t)$. Therefore $U(\lambda;t)$ of λ is a sub-hoop of H for all $t \in [0,1]$. □

Theorem 3. *Let λ be an $(\in,\in)$-fuzzy sub-hoop of H such that $|\mathrm{Im}(\lambda)| \geq 3$. Then λ can be expressed as the union of two fuzzy sets μ and ν where μ and ν are $(\in,\in)$-fuzzy sub-hoops of H such that*

(1) $\mathrm{Im}(\mu)$ *and* $\mathrm{Im}(\nu)$ *have at least two elements.*
(2) μ *and* ν *have no same family of* $\in$*-level sub-hoops.*

Proof. Let λ be an $(\in,\in)$-fuzzy sub-hoop of H with $\mathrm{Im}(\lambda) = \{t_0, t_1, \ldots, t_n\}$ where $t_0 > t_1 > \cdots > t_n$ and $n \geq 2$. Then

$$U(\lambda;t_0) \subseteq U(\lambda;t_1) \subseteq \cdots \subseteq U(\lambda;t_n) = H$$

is a chain of $\in$-level sub-hoops of λ. Define two fuzzy sets μ and ν in H by

$$\mu(x) = \begin{cases} k_1 & \text{if } x \in U(\lambda;t_1), \\ t_r & \text{if } x \in U(\lambda;t_r) \setminus U(\lambda;t_{r-1}) \text{ for } r = 2,3,\cdots,n, \end{cases}$$

and

$$\nu(x) = \begin{cases} t_0 & \text{if } x \in U(\lambda;t_0), \\ t_1 & \text{if } x \in U(\lambda;t_1) \setminus U(\lambda;t_0), \\ k_2 & \text{if } x \in U(\lambda;t_3) \setminus U(\lambda;t_1), \\ t_r & \text{if } x \in U(\lambda;t_r) \setminus U(\lambda;t_{r-1}) \text{ for } r = 4,5\cdots,n, \end{cases}$$

respectively, where $k_1 \in (t_2,t_1)$ and $k_2 \in (t_4,t_2)$. Then μ and ν are $(\in,\in)$-fuzzy sub-hoops of H, and their $\in$-level sub-hoops are chains as follows:

$$U(\mu;t_1) \subseteq U(\mu;t_2) \subseteq \cdots \subseteq U(\mu;t_n) = H$$

and

$$U(\nu;t_0) \subseteq U(\nu;t_1) \subseteq U(\nu;t_3) \subseteq \cdots \subseteq U(\mu;t_n) = H$$

It is clear that $\mu \subseteq \lambda$, $\nu \subseteq \lambda$ and $\mu \cup \nu = \lambda$. This completes the proof. □

Definition 2. A fuzzy set λ in H is called an $(\in,\in\vee q)$-*fuzzy sub-hoop* of H if the following assertion is valid.

$$(\forall x,y \in H)(\forall t,k \in (0,1])\left(x_t \in \lambda,\ y_k \in \lambda \Rightarrow \begin{cases} (x \odot y)_{\min\{t,k\}} \in \vee q\, \lambda \\ (x \to y)_{\min\{t,k\}} \in \vee q\, \lambda) \end{cases} \right). \tag{3.3}$$

Example 2. Consider the hoop $(H,\odot,\to,1)$ which is described in Example 1.
(1) Define a fuzzy set λ in H as follows:

$$\lambda : H \to [0,1],\ x \mapsto \begin{cases} 0.5 & \text{if } x=1, \\ 0.3 & \text{if } x=c, \\ 0.2 & \text{if } x=b, \\ 0.1 & \text{if } x\in\{0,a,d\}. \end{cases}$$

It is routine to verify that λ is an $(\in,\in\vee q)$-fuzzy sub-hoop of H.
(2) Define a fuzzy set μ in H as follows:

$$\mu : H \to [0,1],\ x \mapsto \begin{cases} 0.8 & \text{if } x=0, \\ 0.7 & \text{if } x=a, \\ 0.3 & \text{if } x=b, \\ 0.4 & \text{if } x=c, \\ 0.3 & \text{if } x=d, \\ 0.5 & \text{if } x=1 \end{cases}$$

It is routine to verify that μ is an $(\in,\in\vee q)$-fuzzy sub-hoop of H.

We consider characterizations of $(\in,\in\vee q)$-fuzzy sub-hoop.

Theorem 4. *A fuzzy set λ in H is an $(\in,\in\vee q)$-fuzzy sub-hoop of H if and only if the following assertion is valid.*

$$(\forall x,y\in H)\left(\begin{array}{l} \lambda(x\odot y)\geq\min\{\lambda(x),\lambda(y),0.5\} \\ \lambda(x\to y)\geq\min\{\lambda(x),\lambda(y),0.5\} \end{array}\right). \tag{3.4}$$

Proof. Assume that λ is an $(\in,\in\vee q)$-fuzzy sub-hoop of H and let $x,y\in H$. Suppose that $\min\{\lambda(x),\lambda(y)\}<0.5$.

If $\lambda(x\odot y)<\min\{\lambda(x),\lambda(y)\}$ or $\lambda(x\to y)<\min\{\lambda(x),\lambda(y)\}$, then $\lambda(x\odot y)<t\leq\min\{\lambda(x),\lambda(y)\}$ or $\lambda(x\to y)<k\leq\min\{\lambda(x),\lambda(y)\}$ for some $t,k\in(0,1]$. It follows that

$$x_t\in\lambda \text{ and } y_t\in\lambda$$

or

$$x_k\in\lambda \text{ and } y_k\in\lambda.$$

But $(x\odot y)_{\min\{t,t\}}=(x\odot y)_t\overline{\in\vee q}\lambda$ or $(x\to y)_{\min\{k,k\}}=(x\to y)_k\overline{\in\vee q}\lambda$. This is a contradiction, and so $\lambda(x\odot y)\geq\min\{\lambda(x),\lambda(y)\}$ and $\lambda(x\to y)\geq\min\{\lambda(x),\lambda(y)\}$ whenever $\min\{\lambda(x),\lambda(y)\}<0.5$.

Assume that $\min\{\lambda(x),\lambda(y)\}\geq 0.5$. Then $x_{0.5}\in\lambda$ and $y_{0.5}\in\lambda$. It follows from (3.3) that $(x\odot y)_{0.5}=(x\odot y)_{\min\{0.5,0.5\}}\in\vee q\,\lambda$ and $(x\to y)_{0.5}=(x\to y)_{\min\{0.5,0.5\}}\in\vee q\,\lambda$. Thus $\lambda(x\odot y)\geq 0.5$ and $\lambda(x\to y)\geq 0.5$. Consequently, $\lambda(x\odot y)\geq\min\{\lambda(x),\lambda(y),0.5\}$ and $\lambda(x\to y)\geq\min\{\lambda(x),\lambda(y),0.5\}$.

Conversely, suppose that λ satisfies the condition (3.4). Let $x,y\in H$ and $t,k\in(0,1]$ such that $x_t\in\lambda$ and $y_k\in\lambda$. Then $\lambda(x)\geq t$ and $\lambda(y)\geq k$. If $\lambda(x\odot y)<\min\{t,k\}$, then

$\min\{\lambda(x),\lambda(y)\} \geq 0.5$ because if $\min\{\lambda(x),\lambda(y)\} < 0.5$, then

$$\lambda(x\odot y) \geq \min\{\lambda(x),\lambda(y),0.5\} \geq \min\{\lambda(x),\lambda(y)\} \geq \min\{t,k\}$$

which is a contradiction. Similarly, if $\lambda(x \to y) < \min\{t,k\}$, then $\min\{\lambda(x),\lambda(y)\} \geq 0.5$. It follows that

$$\lambda(x\odot y) + \min\{t,k\} > 2\lambda(x\odot y) \geq 2\min\{\lambda(x),\lambda(y),0.5\} = 1$$

and

$$\lambda(x \to y) + \min\{t,k\} > 2\lambda(x \to y) \geq 2\min\{\lambda(x),\lambda(y),0.5\} = 1.$$

Hence $(x\odot y)_{\min\{t,k\}} q\lambda$ and $(x \to y)_{\min\{t,k\}} q\lambda$, and so $(x\odot y)_{\min\{t,k\}} \in\vee q\lambda$ and $(x \to y)_{\min\{t,k\}} \in\vee q\lambda$. Therefore λ is an $(\in,\in\vee q)$-fuzzy sub-hoop of H. □

Theorem 5. *A fuzzy set λ in H is an $(\in,\in\vee q)$-fuzzy sub-hoop of H if and only if the non-empty $\in$-level set $U(\lambda;t)$ of λ is a sub-hoop of H for all $t \in (0,0.5]$.*

Proof. Assume that λ is an $(\in,\in\vee q)$-fuzzy sub-hoop of H. Let $x,y \in U(\lambda;t)$ for $t \in (0,0.5]$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq t$.

It follows from Theorem 4 that $\lambda(x\odot y) \geq \min\{\lambda(x),\lambda(y),0.5\} \geq \min\{t,0.5\} = t$ and

$\lambda(x \to y) \geq \min\{\lambda(x),\lambda(y),0.5\} \geq \min\{t,0.5\} = t$. Hence $x\odot y \in U(\lambda;t)$ and $x \to y \in U(\lambda;t)$. Therefore $U(\lambda;t)$ is a sub-hoop of H.

Conversely, suppose that the non-empty $\in$-level set $U(\lambda;t)$ of λ is a sub-hoop of H for all $t \in (0,0.5]$. If there exists $x,y \in H$ such that $\lambda(x\odot y) < \min\{\lambda(x),\lambda(y),0.5\}$ or $\lambda(x \to y) < \min\{\lambda(x),\lambda(y),0.5\}$, then $\lambda(x\odot y) < t \leq \min\{\lambda(x),\lambda(y),0.5\}$ or $\lambda(x \to y) < t \leq \min\{\lambda(x),\lambda(y),0.5\}$ for some $t \in (0,1]$. Hence $t \leq 0.5$ and $x,y \in U(\lambda;t)$, and so $x\odot y \in U(\lambda;t)$ and $x \to y \in U(\lambda;t)$. This is a contradiction, and therefore $\lambda(x\odot y) \geq \min\{\lambda(x),\lambda(y),0.5\}$ and $\lambda(x \to y) \geq \min\{\lambda(x),\lambda(y),0.5\}$. Using Theorem 4, we conclude that λ is an $(\in,\in\vee q)$-fuzzy sub-hoop of H. □

Theorem 6. *Every $(\in,\in)$-fuzzy sub-hoop is an $(\in,\in\vee q)$-fuzzy sub-hoop.*

Proof. Straightforward. □

The converse of Theorem 6 is not true in general as seen in the following example.

Example 3. The $(\in,\in\vee q)$-fuzzy sub-hoop μ in Example 2(2) is not an $(\in,\in)$-fuzzy sub-hoop of H since $a_{0.55} \in \mu$ and $0_{0.75} \in \mu$, but $(a \to 0)_{\min\{0.55,0.75\}} \overline{\in} \mu$.

We provide a condition for an $(\in,\in\vee q)$-fuzzy sub-hoop to be an $(\in,\in)$-fuzzy sub-hoop.

Theorem 7. *If an $(\in,\in\vee q)$-fuzzy sub-hoop λ of H satisfies the condition*

$$(\forall x \in H)(\lambda(x) < 0.5), \tag{3.5}$$

then λ is an $(\in,\in)$-fuzzy sub-hoop of H.

Proof. Let $x, y \in H$ and $t, k \in (0,1]$ such that $x_t \in \lambda$ and $y_k \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq k$. Using (3.5) and Theorem 4, we have

$$\lambda(x \odot y) \geq \min\{\lambda(x), \lambda(y), 0.5\} = \min\{\lambda(x), \lambda(y)\} \geq \min\{t, k\}$$

and

$$\lambda(x \to y) \geq \min\{\lambda(x), \lambda(y), 0.5\} = \min\{\lambda(x), \lambda(y)\} \geq \min\{t, k\}.$$

Hence $(x \odot y)_{\min\{t,k\}} \in \lambda$ and $(x \to y)_{\min\{t,k\}} \in \lambda$. Therefore λ is an $(\in, \in)$-fuzzy sub-hoop of H. □

Proposition 1. *If λ is a non-zero $(\in, \in \vee q)$-fuzzy sub-hoop of H, then $\lambda(1) > 0$.*

Proof. Assume that $\lambda(1) = 0$. Since λ is non-zero, there exists $x \in H$ such that $\lambda(x) = t \neq 0$, and so $x_t \in \lambda$. Then $\lambda(x \to x) = \lambda(1) = 0$ and $\lambda(x \to x) + t = \lambda(1) + t = t \leq 1$, that is, $(x \to x)_t \overline{\in} \lambda$ and $(x \to x)_t \overline{q} \lambda$. Thus $(x \to x)_t \overline{\in \vee q} \lambda$, which is a contradiction. Therefore $\lambda(1) > 0$. □

Corollary 1. *If λ is a non-zero $(\in, \in)$-fuzzy sub-hoop of H, then $\lambda(1) > 0$.*

Theorem 8. *If λ is a non-zero $(\in, \in)$-fuzzy sub-hoop of H, then the set*

$$H_0 := \{x \in H \mid \lambda(x) \neq 0\} \tag{3.6}$$

is a sub-hoop of H.

Proof. Let $x, y \in H_0$. Then $\lambda(x) > 0$ and $\lambda(y) > 0$. Note that $x_{\lambda(x)} \in \lambda$ and $y_{\lambda(y)} \in \lambda$. If $\lambda(x \odot y) = 0$ or $\lambda(x \to y) = 0$, then $\lambda(x \odot y) = 0 < \min\{\lambda(x), \lambda(y)\}$ or $\lambda(x \to y) = 0 < \min\{\lambda(x), \lambda(y)\}$, that is, $(x \odot y)_{\min\{\lambda(x),\lambda(y)\}} \overline{\in} \lambda$ or $(x \to y)_{\min\{\lambda(x),\lambda(y)\}} \overline{\in} \lambda$. This is a contradiction, and so $\lambda(x \odot y) \neq 0$ and $\lambda(x \to y) \neq 0$. Hence $x \odot y \in H_0$ and $x \to y \in H_0$. Therefore H_0 is a sub-hoop of H. □

Theorem 9. *For any sub-hoop S of H and $t \in (0, 0.5]$, there exists an $(\in, \in \vee q)$-fuzzy sub-hoop λ of H such that $U(\lambda; t) = S$.*

Proof. Let λ be a fuzzy set in H defined by

$$\lambda : H \to [0,1],\ x \mapsto \begin{cases} t & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases} \tag{3.7}$$

where $t \in (0, 0.5]$. It is clear that $U(\lambda; t) = S$. Suppose that $\lambda(x \odot y) < \min\{\lambda(x), \lambda(y), 0.5\}$ or $\lambda(x \to y) < \min\{\lambda(x), \lambda(y), 0.5\}$ for some $x, y \in H$. Since $|\mathrm{Im}(\lambda)| = 2$, it follows that $\lambda(x \odot y) = 0$ or $\lambda(x \to y) = 0$, and $\min\{\lambda(x), \lambda(y), 0.5\} = t$. Since $t \leq 0.5$, we have $\lambda(x) = t = \lambda(y)$ and so $x, y \in S$. Then $x \odot y \in S$ and $x \to y \in S$, which imply that $\lambda(x \odot y) = t$ and $\lambda(x \to y) = t$. This is a contradiction, and so $\lambda(x \odot y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ and $\lambda(x \to y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$. Using Theorem 4, we know that λ is an $(\in, \in \vee q)$-fuzzy sub-hoop of H. □

For any fuzzy set λ in H and $t \in (0,1]$, we consider the following sets so called *q-set* and $\in\vee q$*-set*, respectively.

$$\lambda_q^t := \{x \in H \mid x_t\, q\, \lambda\} \text{ and } \lambda_{\in\vee q}^t := \{x \in H \mid x_t \in\vee q \lambda\}$$

It is clear that $\lambda_{\in\vee q}^t = U(\lambda;t) \cup \lambda_q^t$.

Theorem 10. *A fuzzy set λ in H is an $(\in,\in\vee q)$-fuzzy sub-hoop of H if and only if $\lambda_{\in\vee q}^t$ is a sub-hoop of H for all $t \in (0,1]$.*

We call $\lambda_{\in\vee q}^t$ an $\in\vee q$*-level sub-hoop* of λ.

Proof. Assume that λ is an $(\in,\in\vee q)$-fuzzy sub-hoop of H. Let $x, y \in \lambda_{\in\vee q}^t$ for $t \in (0,1]$. Then $x_t \in\vee q\lambda$ and $y_t \in\vee q\lambda$, i.e., $\lambda(x) \geq t$ or $\lambda(x) + t > 1$, and $\lambda(y) \geq t$ or $\lambda(y) + t > 1$. It follows from (3.4) that $\lambda(x \odot y) \geq \min\{t, 0.5\}$ and $\lambda(x \to y) \geq \min\{t, 0.5\}$. In fact, if $\lambda(x \odot y) < \min\{t, 0.5\}$ or $\lambda(x \to y) < \min\{t, 0.5\}$, then $x_t \overline{\in\vee q}\lambda$ or $y_t \overline{\in\vee q}\lambda$, a contradiction.

If $t \leq 0.5$, then $\lambda(x \odot y) \geq \min\{t, 0.5\} = t$ and $\lambda(x \to y) \geq \min\{t, 0.5\} = t$. Hence $x \odot y \in U(\lambda;t) \subseteq \lambda_{\in\vee q}^t$ and $x \to y \in U(\lambda;t) \subseteq \lambda_{\in\vee q}^t$.

If $t > 0.5$, then $\lambda(x \odot y) \geq \min\{t, 0.5\} = 0.5$ and $\lambda(x \to y) \geq \min\{t, 0.5\} = 0.5$. Hence $\lambda(x \odot y) + t > 0.5 + 0.5 = 1$ and $\lambda(x \to y) + t > 0.5 + 0.5 = 1$, that is, $(x \odot y)_t\, q\, \lambda$ and $(x \to y)_t\, q\, \lambda$. It follows that $x \odot y \in \lambda_q^t \subseteq \lambda_{\in\vee q}^t$ and $x \to y \in \lambda_q^t \subseteq \lambda_{\in\vee q}^t$. Therefore $\lambda_{\in\vee q}^t$ is a sub-hoop of H for all $t \in (0,1]$.

Conversely, let λ be a fuzzy set in H and $t \in (0,1]$ such that $\lambda_{\in\vee q}^t$ is a sub-hoop of H. Suppose that $\lambda(x \odot y) < \min\{\lambda(x), \lambda(y), 0.5\}$ or $\lambda(x \to y) < \min\{\lambda(x), \lambda(y), 0.5\}$ for some $x, y \in H$. Then $\lambda(x \odot y) < t < \min\{\lambda(x), \lambda(y), 0.5\}$ or $\lambda(x \to y) < t < \min\{\lambda(x), \lambda(y), 0.5\}$ for some $t \in (0, 0.5)$. Hence $x, y \in U(\lambda;t) \subseteq \lambda_{\in\vee q}^t$, and so $x \odot y \in \lambda_{\in\vee q}^t$ and $x \to y \in \lambda_{\in\vee q}^t$. Thus $\lambda(x \odot y) \geq t$ or $\lambda(x \odot y) + t > 1$, and $\lambda(x \to y) \geq t$ or $\lambda(x \to y) + t > 1$. This is a contradiction, and therefore $\lambda(x \odot y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ and $\lambda(x \to y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ for all $x, y \in H$. Consequently, λ is an $(\in,\in\vee q)$-fuzzy sub-hoop of H by Theorem 4. $\square$

Theorem 11. *If λ is an $(\in,\in\vee q)$-fuzzy sub-hoop of H, then the q-set λ_q^t is a sub-hoop of H for all $t \in (0.5,1]$.*

Proof. Let $x, y \in \lambda_q^t$ for $t \in (0.5,1]$. Then $\lambda(x) + t > 1$ and $\lambda(y) + t > 1$, which imply from Theorem 4 that

$$\begin{aligned}\lambda(x \odot y) + t &\geq \min\{\lambda(x), \lambda(y), 0.5\} + t \\ &= \min\{\lambda(x) + t, \lambda(y) + t, 0.5 + t\} > 1,\end{aligned}$$

and

$$\begin{aligned}\lambda(x \to y) + t &\geq \min\{\lambda(x), \lambda(y), 0.5\} + t \\ &= \min\{\lambda(x) + t, \lambda(y) + t, 0.5 + t\} > 1,\end{aligned}$$

that is, $(x\odot y)_t\, q\,\lambda$ and $(x\to y)_t\, q\,\lambda$. Hence $x\odot y\in\lambda_q^t$ and $x\to y\in\lambda_q^t$. Therefore λ_q^t is a sub-hoop of H for all $t\in(0.5,1]$. $\square$

Theorem 12. *Let $f: H\to K$ be a homomorphism of hoops. If λ and μ are $(\in,\in\vee q)$-fuzzy sub-hoops of H and K, respectively, then*

(1) *$f^{-1}(\mu)$ is an $(\in,\in\vee q)$-fuzzy sub-hoop of H.*

(2) *If f is onto and λ satisfies the condition*

$$(\forall T\subseteq H)(\exists x_0\in T)\left(\lambda(x_0)=\sup_{x\in T}\lambda(x)\right), \tag{3.8}$$

then $f(\lambda)$ is an $(\in,\in\vee q)$-fuzzy sub-hoop of K.

Proof. (1) Let $x,y\in H$ and $t,k\in(0,1]$ such that $x_t\in f^{-1}(\mu)$ and $y_k\in f^{-1}(\mu)$. Then $(f(x))_t\in\mu$ and $(f(y))_k\in\mu$. Since μ is an $(\in,\in\vee q)$-fuzzy sub-hoop of K, we have

$$(f(x\odot y))_{\min\{t,k\}}=(f(x)\odot f(y))_{\min\{t,k\}}\in\vee\, q\,\mu$$

and

$$(f(x\to y))_{\min\{t,k\}}=(f(x)\to f(y))_{\min\{t,k\}}\in\vee\, q\,\mu.$$

Hence $(x\odot y)_{\min\{t,k\}}\in\vee\, q\, f^{-1}(\mu)$ and $(x\to y)_{\min\{t,k\}}\in\vee\, q\, f^{-1}(\mu)$. Therefore $f^{-1}(\mu)$ is an $(\in,\in\vee q)$-fuzzy sub-hoop of H.

(2) Let $a,b\in K$ and $t,k\in(0,1]$ such that $a_t\in f(\lambda)$ and $b_k\in f(\lambda)$. Then $(f(\lambda))(a)\geq t$ and $(f(\lambda))(b)\geq k$. Using the condition (3.8), there exist $x\in f^{-1}(a)$ and $y\in f^{-1}(b)$ such that

$$\lambda(x)=\sup_{z\in f^{-1}(a)}\lambda(z)\text{ and }\lambda(y)=\sup_{w\in f^{-1}(b)}\lambda(w).$$

Then $x_t\in\lambda$ and $y_k\in\lambda$, which imply that $(x\odot y)_{\min\{t,k\}}\in\vee\, q\,\lambda$ and $(x\to y)_{\min\{t,k\}}\in\vee\, q\,\lambda$ since λ is an $(\in,\in\vee q)$-fuzzy sub-hoop of H. Now $x\odot y\in f^{-1}(a\odot b)$ and $x\to y\in f^{-1}(a\to b)$, and so $(f(\lambda))(a\odot b)\geq\lambda(x\odot y)$ and $(f(\lambda))(a\to b)\geq\lambda(x\to y)$. Hence

$$(f(\lambda))(a\odot b)\geq\min\{t,k\}\text{ or }(f(\lambda))(a\odot b)+\min\{t,k\}>1$$

and

$$(f(\lambda))(a\to b)\geq\min\{t,k\}\text{ or }(f(\lambda))(a\to b)+\min\{t,k\}>1,$$

that is, $(a\odot b)_{\min\{t,k\}}\in\vee\, q\, f(\lambda)$ and $(a\to b)_{\min\{t,k\}}\in\vee\, q\, f(\lambda)$. Therefore $f(\lambda)$ is an $(\in,\in\vee q)$-fuzzy sub-hoop of K. $\square$

Theorem 13. *Let λ be an $(\in,\in\vee q)$-fuzzy sub-hoop of H such that $|\{\lambda(x)\mid\lambda(x)<0.5\}|\geq 2$. Then there exist two $(\in,\in\vee q)$-fuzzy sub-hoops μ and ν of H such that*

(1) *$\lambda=\mu\cup\nu$.*

(2) *$\mathrm{Im}(\mu)$ and $\mathrm{Im}(\nu)$ have at least two elements.*

(3) *μ and ν have no the same family of $\in\vee q$-level sub-hoops.*

Proof. Let $\{\lambda(x) \mid \lambda(x) < 0.5\} = \{t_1, t_2, \ldots, t_r\}$ where $t_1 > t_2 > \cdots > t_r$ and $r \geq 2$. Then the chain of $\in\vee q$-level sub-hoops of λ is

$$\lambda^{0.5}_{\in\vee q} \subseteq \lambda^{t_1}_{\in\vee q} \subseteq \lambda^{t_2}_{\in\vee q} \subseteq \cdots \subseteq \lambda^{t_r}_{\in\vee q} = H.$$

Define two fuzzy sets μ and ν in H by

$$\mu(x) = \begin{cases} t_1 & \text{if } x \in \lambda^{t_1}_{\in\vee q}, \\ t_n & \text{if } x \in \lambda^{t_n}_{\in\vee q} \setminus \lambda^{t_{n-1}}_{\in\vee q} \text{ for } n = 2, 3, \cdots, r, \end{cases}$$

and

$$\nu(x) = \begin{cases} \lambda(x) & \text{if } x \in \lambda^{0.5}_{\in\vee q}, \\ k & \text{if } x \in \lambda^{t_2}_{\in\vee q} \setminus \lambda^{0.5}_{\in\vee q}, \\ t_n & \text{if } x \in \lambda^{t_n}_{\in\vee q} \setminus \lambda^{t_{n-1}}_{\in\vee q} \text{ for } n = 3, 4, \cdots, r, \end{cases}$$

respectively, where $k \in (t_3, t_2)$. Then μ and ν are $(\in, \in\vee q)$-fuzzy sub-hoops of H, and $\mu \subseteq \lambda$ and $\nu \subseteq \lambda$. The chains of $\in\vee q$-level sub-hoops of μ and ν are given by

$$\mu^{t_1}_{\in\vee q} \subseteq \mu^{t_2}_{\in\vee q} \subseteq \cdots \subseteq \mu^{t_r}_{\in\vee q} \text{ and } \nu^{0.5}_{\in\vee q} \subseteq \nu^{t_2}_{\in\vee q} \subseteq \cdots \subseteq \nu^{t_r}_{\in\vee q},$$

respectively. It is clear that $\mu \cup \nu = \lambda$. This completes the proof. □

Definition 3. A fuzzy set λ in H is called a $(q, \in\vee q)$-*fuzzy sub-hoop* of H if the following assertion is valid.

$$(\forall x, y \in H)(\forall t, k \in (0,1]) \left(x_t \, q\lambda,\ y_k \, q\lambda \Rightarrow \begin{cases} (x \odot y)_{\min\{t,k\}} \in\vee q\lambda \\ (x \to y)_{\min\{t,k\}} \in\vee q\lambda) \end{cases} \right). \quad (3.9)$$

Example 4. Let $H = \{0, a, b, 1\}$ be a set with binary operations $\odot$ and $\to$ in Table 3 and Table 4, respectively.

TABLE 3. Cayley table for the binary operation "$\odot$"

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

TABLE 4. Cayley table for the binary operation "$\to$"

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Define a fuzzy set λ in H as follows:

$$\lambda : H \to [0,1],\ x \mapsto \begin{cases} 0.8 \text{ if } x = 1, \\ 0.6 \text{ if } x = b, \\ 0.55 \text{ if } x = a, \\ 0.7 \text{ if } x = 0. \end{cases}$$

It is routine to verify that λ is a $(q, \in\vee q)$-fuzzy sub-hoop of H.

Question 1. Let λ be a fuzzy set in H such that

(1) $0 \neq \lambda(a) \leq 0.5$ for some $a \in H$,
(2) $(\forall x \in H)\ (x \neq a \ \Rightarrow\ \lambda(x) \geq 0.5)$.

Then is λ a $(q, \in\vee q)$-fuzzy sub-hoop of H?

The answer to this question is negative as seen in the following example.

Example 5. Consider the hoop $(H, \odot, \to, 1)$ which is described in Example 4. Let λ be a fuzzy set in H defined by $\lambda(0) = 0.6$, $\lambda(a) = 0.4$, $\lambda(b) = 0.55$ and $\lambda(1) = 0.8$. Then λ is not a $(q, \in\vee q)$-fuzzy sub-hoop of H since $a_{0.7}\, q\, \lambda$ and $b_{0.46}\, q\, \lambda$, but $(a \odot b)_{\min\{0.7,0.46\}}\, \overline{\in\vee q}\, \lambda$ and/or $(b \to a)_{\min\{0.7,0.46\}}\, \overline{\in\vee q}\, \lambda$.

We consider conditions for a fuzzy set to be a $(q, \in\vee q)$-fuzzy sub-hoop of H.

Theorem 14. *Let S be a sub-hoop of H and let λ be a fuzzy set in H such that*

$$(\forall x \in H) \begin{pmatrix} \lambda(x) = 0 \text{ if } x \notin S \\ \lambda(x) \geq 0.5 \text{ if } x \in S \end{pmatrix}. \tag{3.10}$$

Then λ is a $(q, \in\vee q)$-fuzzy sub-hoop of H.

Proof. Let $x, y \in H$ and $t, k \in (0,1]$ such that $x_t\, q\, \lambda$ and $y_k\, q\, \lambda$, that is, $\lambda(x) + t > 1$ and $\lambda(y) + k > 1$. Then $x \odot y \in S$ and $x \to y \in S$ because if $x \odot y \notin S$, then $x \in H \setminus S$ or $y \in H \setminus S$. Thus $\lambda(x) = 0$ or $\lambda(y) = 0$, and so $t > 1$ or $k > 1$. This is contradiction. Similarly, if $x \to y \notin S$, then we arrive at a contradiction. If $\min\{t,k\} > 0.5$, then $\lambda(x \odot y) + \min\{t,k\} > 1$ and $\lambda(x \to y) + \min\{t,k\} > 1$, and so $(x \odot y)_{\min\{t,k\}}\, q\, \lambda$ and $(x \to y)_{\min\{t,k\}}\, q\, \lambda$. If $\min\{t,k\} \leq 0.5$, then $\lambda(x \odot y) \geq 0.5 \geq \min\{t,k\}$ and $\lambda(x \to y) \geq 0.5 \geq \min\{t,k\}$. Thus $(x \odot y)_{\min\{t,k\}} \in \lambda$ and $(x \to y)_{\min\{t,k\}} \in \lambda$. Therefore $(x \odot y)_{\min\{t,k\}} \in\vee q \lambda$ and $(x \to y)_{\min\{t,k\}} \in\vee q \lambda$. Consequently, λ is a $(q, \in\vee q)$-fuzzy sub-hoop of H. □

Corollary 2. *If a fuzzy set λ in H satisfies $\lambda(x) \geq 0.5$ for all $x \in H$, then λ is a $(q, \in\vee q)$-fuzzy sub-hoop of H.*

Theorem 15. *If λ is a $(q, \in\vee q)$-fuzzy sub-hoop of H such that λ is not constant on H_0, then there exists $x \in H$ such that $\lambda(x) \geq 0.5$. Moreover $\lambda(x) \geq 0.5$ for all $x \in H_0$.*

Proof. If $\lambda(x) < 0.5$ for all $x \in H$, then there exists $a \in H_0$ such that $t_a = \lambda(a) \neq \lambda(1) = t_1$ since λ is not constant on H_0. Then $t_a < t_1$ or $t_a > t_1$. If $t_1 < t_a$, then we can take $\delta > 0.5$ such that $t_1 + \delta < 1 < t_a + \delta$. It follows that $a_\delta \, q \, \lambda$, $\lambda(a \to a) = \lambda(1) = t_1 < \delta = \min\{\delta, \delta\}$ and $\lambda(a \to a) + \min\{\delta, \delta\} = \lambda(1) + \delta = t_1 + \delta < 1$. Hence $(a \to a)_{\min\{\delta,\delta\}} \overline{\in \vee q} \lambda$, which is a contradiction. If $t_1 > t_a$, then $t_a + \delta < 1 < t_1 + \delta$ for some $\delta > 0.5$. It follows that $1_\delta \, q \, \lambda$ and $a_1 \, q \, \lambda$, but $(1 \to a)_{\min\{1,\delta\}} = a_\delta \overline{\in \vee q} \lambda$ since $\lambda(a) < 0.5 < \delta$ and $\lambda(a) + \delta = t_a + \delta < 1$. This leads a contradiction, and therefore $\lambda(x) \geq 0.5$ for some $x \in H$. We now prove that $\lambda(1) \geq 0.5$. Suppose that $\lambda(1) = t_1 < 0.5$. Since $\lambda(x) = t_x \geq 0.5$ for some $x \in H$, it follows that $t_1 < t_x$. Choose $t_0 > t_1$ such that $t_1 + t_0 < 1 < t_x + t_0$. Then $\lambda(x) + t_0 = t_x + t_0 > 1$, i.e., $x_{t_0} \, q \, \lambda$. Also we have

$$\lambda(x \to x) = \lambda(1) = t_1 < t_0 = \min\{t_0, t_0\}$$

and

$$\lambda(x \to x) + \min\{t_0, t_0\} = \lambda(1) + t_0 = t_1 + t_0 < 1.$$

Thus $(x \to x)_{\min\{t_0,t_0\}} \overline{\in \vee q} \lambda$, a contradiction. Hence $\lambda(1) \geq 0.5$. Finally, assume that $t_a = \lambda(a) < 0.5$ for some $a \in H_0$. Take $t \in (0, 1]$ such that $t_a + t < 0.5$. Then $\lambda(a) + 1 = t_a + 1 > 1$ and $\lambda(1) + (0.5 + t) > 1$, which imply that $a_1 \, q \, \lambda$ and $1_{0.5+t} \, q \, \lambda$. But $(1 \to a)_{\min\{1,0.5+t\}} = a_{\min\{1,0.5+t\}} \overline{\in \vee q} \lambda$ since $\lambda(1 \to a) = \lambda(a) < 0.5 + t < \min\{1, 0.5 + t\}$ and

$$\lambda(1 \to a) + \min\{1, 0.5 + t\} = \lambda(a) + 0.5 + t = t_a + 0.5 + t < 0.5 + 0.5 = 1.$$

This is a contradiction. Therefore $\lambda(x) \geq 0.5$ for all $x \in H_0$. This completes the proof. □

Theorem 16. *If λ is a $(q, \in \vee q)$-fuzzy sub-hoop of H, then the set H_0 in (3.6) is a sub-hoop of H.*

Proof. Let $x, y \in H_0$. Then $\lambda(x) + 1 > 1$ and $\lambda(y) + 1 > 1$, that is, $x_1 \, q \, \lambda$ and $y_1 \, q \, \lambda$. Assume that $\lambda(x \odot y) = 0$ or $\lambda(x \to y) = 0$. Then

$$\lambda(x \odot y) < 1 = \min\{1, 1\} \text{ and } \lambda(x \odot y) + \min\{1, 1\} = 1$$

or

$$\lambda(x \to y) < 1 = \min\{1, 1\} \text{ and } \lambda(x \to y) + \min\{1, 1\} = 1,$$

that is, $(x \odot y)_{\min\{1,1\}} \overline{\in \vee q} \lambda$ or $(x \to y)_{\min\{1,1\}} \overline{\in \vee q} \lambda$. This is a contradiction, and so $\lambda(x \odot y) \neq 0$ and $\lambda(x \to y) \neq 0$, i.e., $x \odot y \in H_0$ and $x \to y \in H_0$. Consequently, H_0 is a sub-hoop of H. □

Theorem 17. *If λ is a $(q, \in \vee q)$-fuzzy sub-hoop of H, then the q-set λ_q^t is a sub-hoop of H for all $t \in (0.5, 1]$.*

Proof. Let $x, y \in \lambda_q^t$ for $t \in (0.5, 1]$. Then $x_t \, q \, \lambda$ and $y_t \, q \, \lambda$. Since λ is a $(q, \in \vee q)$-fuzzy sub-hoop of H, we have $(x \odot y)_t \in \vee q \lambda$ and $(x \to y)_t \in \vee q \lambda$. If $(x \odot y)_t \, q \, \lambda$ (and $(x \to y)_t \, q \, \lambda$), then $x \odot y \in \lambda_q^t$ (and $x \to y \in \lambda_q^t$). If $(x \odot y)_t \in \lambda$ (and $(x \to y)_t \in \lambda$), then $\lambda(x \odot y) \geq t > 1 - t$ (and $\lambda(x \to y) \geq t > 1 - t$) since $t > 0.5$. Thus $(x \odot y)_t \, q \, \lambda$ (and $(x \to y)_t \, q \, \lambda$), that is, $x \odot y \in \lambda_q^t$ (and $x \to y \in \lambda_q^t$). Therefore λ_q^t is a sub-hoop of H for all $t \in (0.5, 1]$. □

We consider relations between $(\in, \in\vee q)$-fuzzy sub-hoop and $(q, \in\vee q)$-fuzzy sub-hoop.

Theorem 18. *Every $(q, \in\vee q)$-fuzzy sub-hoop is an $(\in, \in\vee q)$-fuzzy sub-hoop.*

Proof. Let λ be a $(q, \in\vee q)$-fuzzy sub-hoop of H. Let $x, y \in H$ and $t, k \in (0, 1]$ such that $x_t \in \lambda$ and $y_k \in \lambda$. Suppose that $(x \odot y)_{\min\{t,k\}} \overline{\in\vee q}\lambda$ or $(x \to y)_{\min\{t,k\}} \overline{\in\vee q}\lambda$. Then

$$\lambda(x \odot y) < \min\{t, k\} \text{ and } \lambda(x \odot y) + \min\{t, k\} \le 1 \tag{3.11}$$

or

$$\lambda(x \to y) < \min\{t, k\} \text{ and } \lambda(x \to y) + \min\{t, k\} \le 1. \tag{3.12}$$

It follows that $\lambda(x \odot y) < \min\{t, k, 0.5\}$ or $\lambda(x \to y) < \min\{t, k, 0.5\}$. Hence

$$\begin{aligned} 1 - \lambda(x \odot y) > 1 - \min\{t, k, 0.5\} &= \max\{1 - t, 1 - k, 0.5\} \\ &\ge \max\{1 - \lambda(x), 1 - \lambda(y), 0.5\} \end{aligned}$$

or

$$\begin{aligned} 1 - \lambda(x \to y) > 1 - \min\{t, k, 0.5\} &= \max\{1 - t, 1 - k, 0.5\} \\ &\ge \max\{1 - \lambda(x), 1 - \lambda(y), 0.5\}. \end{aligned}$$

Therefore there exist $\delta_1, \delta_2 \in (0, 1]$ such that

$$1 - \lambda(x \odot y) \ge \delta_1 > \max\{1 - \lambda(x), 1 - \lambda(y), 0.5\} \tag{3.13}$$

or

$$1 - \lambda(x \to y) \ge \delta_2 > \max\{1 - \lambda(x), 1 - \lambda(y), 0.5\}. \tag{3.14}$$

From the right inequalities in (3.13) and (3.14), we have

$$\lambda(x) + \delta_1 > 1 \text{ and } \lambda(y) + \delta_1 > 1, \text{ i.e., } x_{\delta_1} \, q \, \lambda \text{ and } y_{\delta_1} \, q \, \lambda$$

or

$$\lambda(x) + \delta_2 > 1 \text{ and } \lambda(y) + \delta_2 > 1, \text{ i.e., } x_{\delta_2} \, q \, \lambda \text{ and } y_{\delta_2} \, q \, \lambda.$$

Since λ is a $(q, \in\vee q)$-fuzzy sub-hoop of H, it follows that $(x \odot y)_{\delta_1} \in\vee q \lambda$ or $(x \to y)_{\delta_2} \in\vee q \lambda$. From the left inequalities in (3.13) and (3.14), we have $\lambda(x \odot y) + \delta_1 \le 1$ or $\lambda(x \to y) + \delta_2 \le 1$, that is, $(x \odot y)_{\delta_1} \overline{q} \lambda$ or $(x \to y)_{\delta_2} \overline{q} \lambda$. Also $\lambda(x \odot y) \le 1 - \delta_1 < 1 - 0.5 = 0.5 < \delta_1$ or $\lambda(x \to y) \le 1 - \delta_2 < 1 - 0.5 = 0.5 < \delta_2$. Hence $(x \odot y)_{\delta_1} \overline{\in\vee q}\lambda$ or $(x \to y)_{\delta_2} \overline{\in\vee q}\lambda$. This is a contradiction, and so $(x \odot y)_{\min\{t,k\}} \in\vee q \lambda$ and $(x \to y)_{\min\{t,k\}} \in\vee q \lambda$. Therefore λ is an $(\in, \in\vee q)$-fuzzy sub-hoop of H. □

The following example shows that any $(\in, \in\vee q)$-fuzzy sub-hoop may not be a $(q, \in\vee q)$-fuzzy sub-hoop.

Example 6. In Example 1, the fuzzy set λ is an $(\in, \in\vee q)$-fuzzy sub-hoop of H. But it is not a $(q, \in\vee q)$-fuzzy sub-hoop of H since $a_{0.4} \, q \, \lambda$ and $b_{0.8} \, q \, \lambda$. But $(a \odot b)_{\min\{0.4, 0.8\}} \overline{\in\vee q}\lambda$ and/or $(a \to b)_{\min\{0.4, 0.8\}} \overline{\in\vee q}\lambda$.

We provide a condition for an $(\in, \in\vee q)$-fuzzy sub-hoop to be a $(q, \in\vee q)$-fuzzy sub-hoop.

Theorem 19. *Let λ be an $(\in, \in\vee q)$-fuzzy sub-hoop of H. If every fuzzy point has the value in $(0, 0.5]$, then λ is a $(q, \in\vee q)$-fuzzy sub-hoop of H.*

Proof. Let $x, y \in H$ and $t, k \in (0, 0.5]$ such that $x_t\, q\, \lambda$ and $y_k\, q\, \lambda$. Then $\lambda(x) > 1 - t \geq t$ and $\lambda(y) > 1 - k \geq k$, that is, $x_t \in \lambda$ and $y_k \in \lambda$. Since λ is an $(\in, \in\vee q)$-fuzzy sub-hoop of H, it follows that $(x \odot y)_{\min\{t,k\}} \in\vee q\, \lambda$ and $(x \to y)_{\min\{t,k\}} \in\vee q\, \lambda$. Therefore λ is a $(q, \in\vee q)$-fuzzy sub-hoop of H. □

4. Conclusion

Our aim was to define the concepts of an $(\in, \in)$-fuzzy sub-hoop, an $(\in, \in\vee q)$-fuzzy sub-hoop and a $(q, \in\vee q)$-fuzzy sub-hoop, and we discussed some properties and found some equivalent definitions of them. Then, we discussed characterizations of an $(\in, \in)$-fuzzy sub-hoop and an $(\in, \in\vee q)$-fuzzy sub-hoop. Also, we found relations between an $(\in, \in)$-fuzzy sub-hoop, an $(\in, \in\vee q)$-fuzzy sub-hoop and a $(q, \in\vee q)$-fuzzy sub-hoop and considered conditions for a fuzzy set to be a $(q, \in\vee q)$-fuzzy sub-hoop of H, and provided a condition for an $(\in, \in\vee q)$-fuzzy sub-hoop to be a $(q, \in\vee q)$-fuzzy sub-hoop. By [1, 6, 7, 14] we defined the concept of $(\in, \in)$-fuzzy filters (fuzzy implicative filters, fuzzy positive implicative filters, fuzzy fantastic filters) of hoop and $(\in, \in\vee q)$-fuzzy filters (fuzzy implicative filters, fuzzy positive implicative filters, fuzzy fantastic filters) of hoop and have investigated some equivalent definitions and properties of them.

References

[1] M. Aaly Kologani, M. Mohseni Takallo, and H. Kim, "Fuzzy filters of hoops based on fuzzy points," *Mathematics*, vol. 7, no. 5, p. 430, 2019, doi: 10.3390/math7050430.

[2] S. Z. Alavia, R. A. Borzooei, and M. Aaly Kologani, "Fuzzy filters in pseudo hoops." *J. Intell. Fuzzy Syst*, vol. 3, no. 32, pp. 1997–2007, 2017, doi: 10.3233/JIFS-161586.

[3] W. Blok and I. Ferreirim, " Hoops and their implicational reducts," *Logic in Computer Science*, vol. 1, no. 28, pp. 219–230, 1993.

[4] W. Blok and I. Ferreirim, "On the structure of hoops," *Algebra Universalis*, no. 43, pp. 233–257, 2000.

[5] R. Borzooei and M. Aaly Kologani, " Filter theory of hoop-algebras," *J. Adv. Res. Pure Math*, vol. 6, no. 4, pp. 72–86, 2014.

[6] R. Borzooei, M. Aaly Kologani, Y. B. Jun, and M. Mohseni Takallo, "Fuzzy fantastic filters of hoops based on fuzzy points," *Submitted to Fuzzy Information and Engineering*.

[7] R. Borzooei, M. Aaly Kologani, M. Sabetkish, and Y. B. Jun, "Fuzzy positive implicative filters of hoops based on fuzzy points," *Mathematics*, vol. 7, no. 6, p. 566, 2019, doi: 10.3390/math7060566.

[8] R. Borzooei, H. R. Varasteh, and K. Borna, " Fundamental hoop-algebras," *Ratio Mathematica*, vol. 29, pp. 25–40, 2015.

[9] B. Bosbach, " Komplementäre halbgruppen, axiomatik und arithmetik," *Fundamenta Mathematicae*, vol. 64, no. -, pp. 257–287, 1969.

[10] S. Ghorbani, "Localization of hoop algebras ," *J. Adv. Res. Pure Math.*, vol. 5, no. 3, pp. 1–13, 2013.
[11] J. M. Hu, " Fuzzy topological spaces," *J. Math. Anal. Appl.*, vol. 110, pp. 141–178, 1985.
[12] Y. B. Jun, "On (α,β)-fuzzy subalgerbas of BCK/BCI-algebras," *Bull. Korean Math. Soc*, vol. 42, no. 4, pp. 703–711, 2005.
[13] C. Luo, X. Xin, and P. He, "n-fold (positive) implicative filters of hoops," *Ital. J. Pure Appl. Math.*, vol. 38, pp. 631–642, 2017.
[14] M. Mohseni Takallo, M. Aaly Kologani, X. Xin, Y. Jun, and R. Borzooei, "Fuzzy implicative filters of hoops based on fuzzy points," *Submitted*.
[15] P. M. Pu and Y. M. Liu, "Fuzzy topology I, Neighborhood structure of a fuzzy point and Moore-Smith convergence," *J. Math. Anal. Appl.* , vol. 76, pp. 571–599, 1980.
[16] R. Tayebi Khorami and A. Borumand Saeid, "Some unitary operators on hoop-algebras," *Fuzzy Inf. Eng.*, vol. 9, no. 2, pp. 205–223, 2017.
[17] M. W. Warner, "A note on the relation between G- and Q-theories for first countability and sequentiality," *Fuzzy Sets and Systems*, vol. 33, pp. 321–327, 1989.
[18] L. A. Zadeh, " Fuzzy sets ," *Inform. Control*, vol. 8, pp. 338–353, 1965.
[19] M. A. Zander, "Decomposability of the finitely generated free hoop residuation algebra," *Studia Logica* , vol. 88, no. 2, pp. 233–246, 2008.

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SOME NEW INEQUALITIES FOR DIFFERENTIABLE h-CONVEX FUNCTIONS AND APPLICATIONS

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Abstract. In this paper, the authors established a new identity for differentiable functions, afterwards they obtained some new inequalities for functions whose first derivatives in absolute value at certain powers are h-convex by using the identity. Also they give some applications for special means for arbitrary positive numbers.

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1. INTRODUCTION

1.1. *Definitions*

A function $f : I \to R$, $I \subseteq R$ is an interval, is said to be a convex function on I if

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y) \tag{1.1}$$

holds for all $x,y \in I$ and $t \in [0,1]$. If the reversed inequality in (1.1) holds, then f is concave.

We say that $f : I \to \mathbb{R}$ is Godunova-Levin function or that f belongs to the class $Q(I)$ if f is nonnegative and for all $x,y \in I$ and $t \in (0,1)$ we have

$$f(tx+(1-t)y) \leq \frac{f(x)}{t}+\frac{f(y)}{1-t} \tag{1.2}$$

[13, Godunova and Levin, 1985].

Let $s \in (0,1]$. A function $f : (0,\infty] \to [0,\infty]$ is said to be s-convex in the second sense if

$$f(tx+(1-t)y) \leq t^s f(x)+(1-t)^s f(y), \tag{1.3}$$

for all $x,y \in (0,\infty]$ and $t \in [0,1]$. This class of s-convex functions is usually denoted by K_s^2 [14, Hudzik and Maligranda, 1994].

In 1978, Breckner introduced s-convex functions as a generalization of convex functions in [6]. Also, in that work Breckner proved the important fact that the set valued map is s-convex only if the associated support function is s-convex function in

[7]. A number of properties and connections with s-convex in the first sense and its generalizations are discussed in the papers [9, 10, 14]. Of course, s-convexity means just convexity when $s = 1$.

We say that $f : I \to \mathbb{R}$ is a P-function or that f belongs to the class $P(I)$ if f is nonnegative and for all $x, y \in I$ and $t \in [0,1]$, we have

$$f(tx + (1-t)y) \leq f(x) + f(y) \tag{1.4}$$

[12, Dragomir, Pečarić and Persson, 1995].

Let $h : J \to \mathbb{R}$ be a nonnegative function, $h \not\equiv 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h,I)$, if f is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ we have

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y). \tag{1.5}$$

If inequality (1.5) is reversed, then f is said to be h-concave, i.e. $f \in SV(h,I)$. Obviously, if $h(t) = t$, then all nonnegative convex functions belong to $SX(h,I)$ and all nonnegative concave functions belong to $SV(h,I)$; if $h(t) = \frac{1}{t}$, then $SX(h,I) = Q(I)$; if $h(t) = 1$, then $SX(h,I) \supseteq P(I)$; and if $h(t) = t^s$, where $s \in (0,1)$, then $SX(h,I) \supseteq K_s^2$ [22, Varošanec, 2007].

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class of $MT(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ satisfies the inequality;

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y) \tag{1.6}$$

[21, Tunç and Yıldırım, 2012]. Definition of MT-convex function may be regarded as a special case of h-convex function. And in (1.6), if we take $t = 1/2$, inequality (1.6) reduces to Jensen convex.

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a nonnegative function. We say that $f : I \to \mathbb{R}$ is $tgs-$convex function on I if the inequality

$$f(tx + (1-t)y) \leq t(1-t)[f(x) + f(y)] \tag{1.7}$$

holds for all $x, y \in I$ and $t \in (0,1)$. We say that f is $tgs-$concave if $(-f)$ is $tgs-$convex [20]. In (1.5), if we take $h(t) = t(1-t)$, inequality (1.5) reduces to inequality (1.7).

1.2. *Theorems*

If f is integrable on $[a,b]$, then the average value of f on $[a,b]$ is

$$\frac{1}{b-a}\int_a^b f(x)dx.$$

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.8}$$

is known as **Hermite-Hadamard inequality** for convex mappings. For particular choice of the function f in (1.8) yields some classical inequalities of means. Both inequalities in (1.8) hold in reversed direction if f is concave. The refinement of the second inequality in (1.8) is due to Bullen as follows:

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2} \tag{1.9}$$

where f is as above. This (1.9) integral inequality is well known in the literature as **Bullen Inequality** [18, Pečarić, Proschan and Tong, 1991]. For some recent results in connection with Hermite-Hadamard inequality and its applications we refer to [1–5, 12, 15, 16, 21, 22] where further references are given.

The following inequality is well known in the literature as **Simpson's inequality** [11, Dragomir, Agarwal, and Cerone, 2000];

$$\int_a^b f(x)\,dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right] \leq \frac{1}{1280}\left\|f^{(4)}\right\|_\infty (b-a)^5,$$

where the mapping $f:[a,b]\to\mathbb{R}$ is assumed to be four times continuously differentiable on the interval and $f^{(4)}$ to be bounded on (a,b), that is,

$$\left\|f^{(4)}\right\|_\infty = \sup_{t\in(a,b)}\left|f^{(4)}(t)\right| < \infty.$$

In [19], M. Z. Sarıkaya, A. Sağlam and H. Yıldırım established the following Hadamard type inequality for h-convex functions:

Let $f \in SX(h,I)$, $a,b\in I$ and $f\in L_1([a,b])$, then

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq [f(a)+f(b)]\int_0^1 h(t)\,dt. \tag{1.10}$$

For recent results and generalizations concerning h-convex functions see [5, 8, 17, 19, 22] and references therein.

In this paper, firstly we will derive a new general inequality for functions whose first derivatives in absolute value are h-convex, which not only provides a generalization of the previous theorems but also gives some other interesting special results. Then we give some corollaries and remarks for different type convex functions. Finally, applications to some special means of real numbers are considered.

2. Main Results

Lemma 1. *Let $f: I\subset R\to R$ be a differentiable function on I° such that $f'\in L^1[a,b]$, where $a,b\in I$ with $a<b$. Then, for any $\varepsilon\in[0,1]$, the following equality holds:*

$$(1-2\varepsilon)f\left(\frac{a+b}{2}\right)+\varepsilon[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(x)\,dx \tag{2.1}$$

$$= \frac{b-a}{4}\left\{\int_0^1 (t-2\varepsilon) f'\left(t\frac{a+b}{2}+(1-t)a\right)dt + \int_0^1 (2\varepsilon - t) f'\left(t\frac{a+b}{2}+(1-t)b\right)dt\right\}$$

Proof. Integrating by parts, we have the following identity:

$$\begin{aligned} I_1 &= \int_0^1 (t-2\varepsilon) f'\left(t\frac{a+b}{2}+(1-t)a\right)dt \\ &= (t-2\varepsilon)\frac{2}{b-a} f\left(t\frac{a+b}{2}+(1-t)a\right)\Big|_0^1 - \frac{2}{b-a}\int_0^1 f\left(t\frac{a+b}{2}+(1-t)a\right)dt \\ &= \frac{2(1-2\varepsilon)}{b-a} f\left(\frac{a+b}{2}\right) + \frac{4\varepsilon}{b-a} f(a) - \frac{2}{b-a}\int_0^1 f\left(t\frac{a+b}{2}+(1-t)a\right)dt. \end{aligned} \tag{2.2}$$

Using the change of variable $x = t\frac{a+b}{2}+(1-t)a$ for $t \in [0,1]$ and multiplying both sides of (2.2) by $\frac{b-a}{4}$, we obtain

$$\begin{aligned} &\frac{b-a}{4}\int_0^1 (t-2\varepsilon) f'\left(t\frac{a+b}{2}+(1-t)a\right)dt \\ &\qquad = \frac{1-2\varepsilon}{2} f\left(\frac{a+b}{2}\right) + \varepsilon f(a) - \frac{1}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx. \end{aligned} \tag{2.3}$$

Similarly, we observe that

$$\begin{aligned} I_2 &= \int_0^1 (2\varepsilon - t) f'\left(t\frac{a+b}{2}+(1-t)b\right)dt \\ &= \frac{2(2\varepsilon-1)}{a-b} f\left(\frac{a+b}{2}\right) - \frac{4\varepsilon}{a-b} f(b) + \frac{2}{a-b}\int_0^1 f\left(t\frac{a+b}{2}+(1-t)b\right)dt. \end{aligned} \tag{2.4}$$

Using the change of variable $x = t\frac{a+b}{2}+(1-t)b$ for $t \in [0,1]$ and multiplying both sides of (2.4) by $\frac{b-a}{4}$, we obtain

$$\begin{aligned} &\frac{b-a}{4}\int_0^1 (2\varepsilon - t) f'\left(t\frac{a+b}{2}+(1-t)b\right)dt \\ &\qquad = \frac{1-2\varepsilon}{2} f\left(\frac{a+b}{2}\right) + \varepsilon f(b) - \frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} f(x)\,dx. \end{aligned} \tag{2.5}$$

Thus, adding (2.3) and (2.5), we get the required identity (2.1). □

Theorem 1. *Let $I \subset [0,\infty)$, $f : I \to R$ be a differentiable function on I° such that $f' \in L^1[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is h-convex on $[a,b]$ for some fixed*

$t \in (0,1)$ and $q \geq 1$, then the following inequalities hold

$$\left|(1-2\varepsilon) f\left(\frac{a+b}{2}\right)+\varepsilon[f(a)+f(b)]-\frac{1}{b-a} \int_a^b f(x)\, dx\right| \tag{2.6}$$
$$\leq \frac{b-a}{4}\left(4\varepsilon^2-2\varepsilon+\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q \int_0^1|2\varepsilon-t| h(t)\, dt+\left|f'(a)\right|^q \int_0^1|2\varepsilon-t| h(1-t)\, dt\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q \int_0^1|2\varepsilon-t| h(t)\, dt+\left|f'(b)\right|^q \int_0^1|2\varepsilon-t| h(1-t)\, dt\right\}^{\frac{1}{q}}\right]$$

for $0 \leq \varepsilon \leq \frac{1}{2}$, and

$$\left|(1-2\varepsilon) f\left(\frac{a+b}{2}\right)+\varepsilon[f(a)+f(b)]-\frac{1}{b-a} \int_a^b f(x)\, dx\right| \tag{2.7}$$
$$\leq \frac{b-a}{4}\left(2\varepsilon-\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q \int_0^1|2\varepsilon-t| h(t)\, dt+\left|f'(a)\right|^q \int_0^1|2\varepsilon-t| h(1-t)\, dt\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q \int_0^1|2\varepsilon-t| h(t)\, dt+\left|f'(b)\right|^q \int_0^1|2\varepsilon-t| h(1-t)\, dt\right\}^{\frac{1}{q}}\right]$$

for $\frac{1}{2} \leq \varepsilon \leq 1$.

Proof. In case $0 \leq \varepsilon \leq \frac{1}{2}$, by Lemma 1 and using the Hölder inequality, we have

$$\left|(1-2\varepsilon) f\left(\frac{a+b}{2}\right)+\varepsilon[f(a)+f(b)]-\frac{1}{b-a} \int_a^b f(x)\, dx\right|$$
$$\leq \frac{b-a}{4}\left\{\left(\int_0^1|t-2\varepsilon|\, dt\right)^{1-\frac{1}{q}}\left(\int_0^1|t-2\varepsilon|\left|f'\left(t \frac{a+b}{2}+(1-t) a\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1|2\varepsilon-t|\, dt\right)^{1-\frac{1}{q}}\left(\int_0^1|2\varepsilon-t|\left|f'\left(t \frac{a+b}{2}+(1-t) b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$
$$\leq \frac{b-a}{4}\left(4\varepsilon^2-2\varepsilon+\frac{1}{2}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\int_{0}^{1}|2\varepsilon-t|\,h(t)\,dt+\left|f'(a)\right|^{q}\int_{0}^{1}|2\varepsilon-t|\,h(1-t)\,dt\right\}^{\frac{1}{q}}\right.$$

$$\left.+\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\int_{0}^{1}|2\varepsilon-t|\,h(t)\,dt+\left|f'(b)\right|^{q}\int_{0}^{1}|2\varepsilon-t|\,h(1-t)\,dt\right\}^{\frac{1}{q}}\right],$$

where

$$\int_{0}^{1}|t-2\varepsilon|\,dt=4\varepsilon^{2}-2\varepsilon+\frac{1}{2}.$$

In case $\frac{1}{2}\leq\varepsilon\leq1$, by Lemma 1 and using the Hölder inequality, we have

$$\left|(1-2\varepsilon)f\left(\frac{a+b}{2}\right)+\varepsilon\left[f(a)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right|$$

$$\leq\frac{b-a}{4}\left[\left(\int_{0}^{1}|t-2\varepsilon|\,dt\right)^{1-\frac{1}{q}}\left\{\int_{0}^{1}|t-2\varepsilon|\left|f'\left(t\frac{a+b}{2}+(1-t)\,a\right)\right|^{q}dt\right\}^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{0}^{1}|2\varepsilon-t|\,dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}|2\varepsilon-t|\left|f'\left(t\frac{a+b}{2}+(1-t)\,b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right]$$

$$\leq\frac{b-a}{4}\left(2\varepsilon-\frac{1}{2}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\int_{0}^{1}|2\varepsilon-t|\,h(t)\,dt+\left|f'(a)\right|^{q}\int_{0}^{1}|2\varepsilon-t|\,h(1-t)\,dt\right\}^{\frac{1}{q}}\right.$$

$$\left.+\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\int_{0}^{1}|2\varepsilon-t|\,h(t)\,dt+\left|f'(b)\right|^{q}\int_{0}^{1}|2\varepsilon-t|\,h(1-t)\,dt\right\}^{\frac{1}{q}}\right],$$

where

$$\int_{0}^{1}|t-2\varepsilon|\,dt=2\varepsilon-\frac{1}{2}.$$

Thus, the proof is completed. □

Corollary 1. *Let $I\subset[0,\infty)$, $f:I\to R$ be a differentiable function on I° such that $f'\in L^{1}[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^{q}$ is h-convex on $[a,b]$ for some fixed $t\in(0,1)$ and $q=1$, then the following inequalities hold*

$$\left|(1-2\varepsilon)f\left(\frac{a+b}{2}\right)+\varepsilon\left[f(a)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right| \tag{2.8}$$

$$\leq\frac{b-a}{4}\left[2\left|f'\left(\frac{a+b}{2}\right)\right|\int_{0}^{1}|2\varepsilon-t|\,h(t)\,dt+\left(\left|f'(b)\right|+\left|f'(a)\right|\right)\int_{0}^{1}|2\varepsilon-t|\,h(1-t)\,dt\right]$$

for $0\leq\varepsilon\leq1$.

Proof. Inequalities (2.8) is immediate by setting $q=1$ in (2.6) and (2.7) of Theorem 1. □

Remark 1. If we take $\varepsilon=0$ in (2.8) then we get a midpoint type inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \\ \le \frac{b-a}{4}\left[2\left|f'\left(\frac{a+b}{2}\right)\right|\int_0^1 th(t)\,dt + \left(\left|f'(a)\right|+\left|f'(b)\right|\right)\int_0^1 th(1-t)\,dt\right].$$

If we take $\varepsilon=\frac{1}{2}$ in (2.8), then we get a trapezoid type inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \le \frac{b-a}{4}\left[2\left|f'\left(\frac{a+b}{2}\right)\right|\left(\int_0^1 |t-1|\,h(t)\,dt\right)\right. \\ \left. + \left(\left|f'(a)\right|+\left|f'(b)\right|\right)\left(\int_0^1 |t-1|\,h(1-t)\,dt\right)\right].$$

If we take $\varepsilon=\frac{1}{4}$ in (2.8), then we get a Bullen type inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\ \le \frac{b-a}{4}\left\{2\left|f'\left(\frac{a+b}{2}\right)\right|\left(\int_0^1 \left|t-\frac{1}{2}\right| h(t)\,dt\right)\right. \\ \left. + \left(\left|f'(a)\right|+\left|f'(b)\right|\right)\left(\int_0^1 \left|t-\frac{1}{2}\right| h(1-t)\,dt\right)\right\}.$$

If we take $\varepsilon=\frac{1}{6}$ in (2.8), then we get a Simpson type inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{6}\left[f(a)+4\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\ \le \frac{b-a}{4}\left\{2\left|f'\left(\frac{a+b}{2}\right)\right|\left(\int_0^1 \left|t-\frac{1}{3}\right| h(t)\,dt\right)\right. \\ \left. + \left(\left|f'(a)\right|+\left|f'(b)\right|\right)\left(\int_0^1 \left|t-\frac{1}{3}\right| h(1-t)\,dt\right)\right\}.$$

Corollary 2. *Under the assumption of Theorem 1, if $|f'|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$ and $q\ge 1$, then the following inequalities hold:*

$$\begin{aligned}
&\left|(1-2\varepsilon) f\left(\frac{a+b}{2}\right)+\varepsilon[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\leq \frac{b-a}{4}\left(4\varepsilon^2-2\varepsilon+\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q P(s,\varepsilon)+\left|f'(a)\right|^q Q(s,\varepsilon)\right)^{\frac{1}{q}}\right. \\
&\qquad\qquad \left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q P(s,\varepsilon)+\left|f'(b)\right|^q Q(s,\varepsilon)\right)^{\frac{1}{q}}\right], \quad (2.9)
\end{aligned}$$

for $0\leq\varepsilon\leq\frac{1}{2}$, *where* $P(s,\varepsilon)=\frac{s-4\varepsilon-2s\varepsilon+2(2\varepsilon)^{s+2}+1}{(s+1)(s+2)}$, $Q(s,\varepsilon)=\frac{2(1-2\varepsilon)^{s+2}+4\varepsilon+2s\varepsilon-1}{(s+1)(s+2)}$, *and*

$$\begin{aligned}
&\left|(1-2\varepsilon) f\left(\frac{a+b}{2}\right)+\varepsilon[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\leq \frac{b-a}{4}\left(2\varepsilon-\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q U(s,\varepsilon)+\left|f'(a)\right|^q V(s,\varepsilon)\right)^{\frac{1}{q}}\right. \\
&\qquad\qquad \left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q U(s,\varepsilon)+\left|f'(b)\right|^q V(s,\varepsilon)\right)^{\frac{1}{q}}\right], \quad (2.10)
\end{aligned}$$

for $\frac{1}{2}\leq\varepsilon\leq 1$ *where* $U(s,\varepsilon)=\frac{2\varepsilon(s+2)-(s+1)}{(s+1)(s+2)}$, $V(s,\varepsilon)=\frac{4\varepsilon+2s\varepsilon-1}{(s+1)(s+2)}$.

Proof. In case $0\leq\varepsilon\leq\frac{1}{2}$, by Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|(1-2\varepsilon) f\left(\frac{a+b}{2}\right)+\varepsilon[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{b-a}{4}\left[\left(\int_0^1|t-2\varepsilon|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|t-2\varepsilon|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\int_0^1|2\varepsilon-t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|2\varepsilon-t|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\leq \frac{b-a}{4}\left(4\varepsilon^2-2\varepsilon+\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1|2\varepsilon-t|\left(t^s\left|f'\left(\frac{a+b}{2}\right)\right|^q+(1-t)^s\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\int_0^1|2\varepsilon-t|\left(t^s\left|f'\left(\frac{a+b}{2}\right)\right|^q+(1-t)^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right] \\
&= \frac{b-a}{4}\left(4\varepsilon^2-2\varepsilon+\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left\{\int_0^{2\varepsilon}(2\varepsilon-t)\left(t^s\left|f'\left(\frac{a+b}{2}\right)\right|^q+(1-t)^s\left|f'(a)\right|^q\right)dt\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_{2\varepsilon}^{1}(t-2\varepsilon)\left(t^{s}\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+(1-t)^{s}\left|f'(a)\right|^{q}\right)dt\Bigg\}^{\frac{1}{q}}\\
&+\Bigg\{\int_{0}^{2\varepsilon}(2\varepsilon-t)\left(t^{s}\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+(1-t)^{s}\left|f'(b)\right|^{q}\right)dt\\
&+\int_{2\varepsilon}^{1}(t-2\varepsilon)\left(t^{s}\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+(1-t)^{s}\left|f'(b)\right|^{q}\right)dt\Bigg\}^{\frac{1}{q}}\Bigg]\\
=&\frac{b-a}{4}\left(4\varepsilon^{2}-2\varepsilon+\frac{1}{2}\right)^{1-\frac{1}{q}}\\
&\times\Bigg[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\frac{s-4\varepsilon-2s\varepsilon+2(2\varepsilon)^{s+2}+1}{(s+1)(s+2)}+\left|f'(a)\right|^{q}\frac{2(1-2\varepsilon)^{s+2}+4\varepsilon+2s\varepsilon-1}{(s+1)(s+2)}\right)^{\frac{1}{q}}\\
&+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\frac{s-4\varepsilon-2s\varepsilon+2(2\varepsilon)^{s+2}+1}{(s+1)(s+2)}+\left|f'(b)\right|^{q}\frac{2(1-2\varepsilon)^{s+2}+4\varepsilon+2s\varepsilon-1}{(s+1)(s+2)}\right)^{\frac{1}{q}}\Bigg]
\end{aligned}
$$

where

$$
\begin{aligned}
\int_{0}^{1}|t-2\varepsilon|\,dt&=\int_{0}^{2\varepsilon}(2\varepsilon-t)\,dt+\int_{2\varepsilon}^{1}(t-2\varepsilon)\,dt=4\varepsilon^{2}-2\varepsilon+\frac{1}{2}\\
\int_{0}^{2\varepsilon}t^{s}(2\varepsilon-t)\,dt&=\frac{(2\varepsilon)^{s+2}}{(s+1)(s+2)}\\
\int_{0}^{2\varepsilon}(2\varepsilon-t)(1-t)^{s}\,dt&=\frac{(1-2\varepsilon)^{s+2}+4\varepsilon+2s\varepsilon-1}{(s+1)(s+2)}\\
\int_{2\varepsilon}^{1}t^{s}(t-2\varepsilon)\,dt&=\frac{s-4\varepsilon-2s\varepsilon+(2\varepsilon)^{s+2}+1}{(s+1)(s+2)}\\
\int_{2\varepsilon}^{1}(t-2\varepsilon)(1-t)^{s}\,dt&=\frac{(1-2\varepsilon)^{s+2}}{(s+1)(s+2)}.
\end{aligned}
$$

In case $\frac{1}{2}\leq\varepsilon\leq 1$, by Lemma 1 and using the Hölder inequality, we have

$$
\begin{aligned}
&\left|(1-2\varepsilon)f\left(\frac{a+b}{2}\right)+\varepsilon\left[f(a)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right|\\
\leq&\frac{b-a}{4}\Bigg[\left(\int_{0}^{1}|t-2\varepsilon|\,dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}|t-2\varepsilon|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}}\\
&+\left(\int_{0}^{1}|2\varepsilon-t|\,dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}|2\varepsilon-t|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|^{q}dt\right)^{\frac{1}{q}}\Bigg]
\end{aligned}
$$

$$\leq \frac{b-a}{4}\left(2\varepsilon-\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 |2\varepsilon-t|\left(t^s\left|f'\left(\frac{a+b}{2}\right)\right|^q+(1-t)^s\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 |2\varepsilon-t|\left(t^s\left|f'\left(\frac{a+b}{2}\right)\right|^q+(1-t)^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right]$$
$$=\frac{b-a}{4}\left(2\varepsilon-\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q\int_0^1 (2\varepsilon-t)\,t^s dt+\left|f'(a)\right|^q\int_0^1 (2\varepsilon-t)(1-t)^s dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q\int_0^1 (2\varepsilon-t)\,t^s dt+\left|f'(b)\right|^q\int_0^1 (2\varepsilon-t)(1-t)^s dt\right)^{\frac{1}{q}}\right]$$
$$=\frac{b-a}{4}\left(2\varepsilon-\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q\frac{2\varepsilon(s+2)-(s+1)}{(s+1)(s+2)}+\left|f'(a)\right|^q\frac{4\varepsilon+2s\varepsilon-1}{(s+1)(s+2)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q\frac{2\varepsilon(s+2)-(s+1)}{(s+1)(s+2)}+\left|f'(b)\right|^q\frac{4\varepsilon+2s\varepsilon-1}{(s+1)(s+2)}\right)^{\frac{1}{q}}\right].$$

The proof is completed. □

Corollary 3. *Let $I\subset[0,\infty)$, $f:I\to R$ be a differentiable function on I° such that $f'\in L^1[a,b]$, where $a,b\in I$ with $a<b$. If $\left|f'\right|$ is s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$, then the following inequalities hold:*

$$\left|(1-2\varepsilon)f\left(\frac{a+b}{2}\right)+\varepsilon[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{b-a}{4}\left(2\left|f'\left(\frac{a+b}{2}\right)\right|P+\left(\left|f'(a)\right|+\left|f'(b)\right|\right)Q\right)\quad(2.11)$$

for $0\leq\varepsilon\leq\frac{1}{2}$, *where* $P=\frac{s-4\varepsilon-2s\varepsilon+2(2\varepsilon)^{s+2}+1}{(s+1)(s+2)}$, $Q=\frac{2(1-2\varepsilon)^{s+2}+4\varepsilon+2s\varepsilon-1}{(s+1)(s+2)}$, *and*

$$\left|(1-2\varepsilon)f\left(\frac{a+b}{2}\right)+\varepsilon[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{b-a}{4}2\left(\left|f'\left(\frac{a+b}{2}\right)\right|U+\left(\left|f'(a)\right|+\left|f'(b)\right|\right)V\right)\quad(2.12)$$

for $\frac{1}{2}\leq\varepsilon\leq 1$ *where* $U=\frac{2\varepsilon(s+2)-(s+1)}{(s+1)(s+2)}$, $V=\frac{4\varepsilon+2s\varepsilon-1}{(s+1)(s+2)}$.

Proof. Inequalities (2.11) and (2.12) are immediate by setting $q=1$ in (2.9) and (2.10) of Corollary 2. □

Remark 2. If we take $\varepsilon = 0$ in (2.11), then we get a midpoint type inequality

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{4} \left(2 \left| f'\left(\frac{a+b}{2}\right) \right| \frac{1}{s+2} + \frac{|f'(a)| + |f'(b)|}{(s+1)(s+2)} \right). \tag{2.13}$$

If we take $\varepsilon = \frac{1}{2}$ in (2.11) or (2.12), then we get a trapezoid type inequality

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2} \right| \\ &\qquad \leq \frac{b-a}{4} \left(2 \left| f'\left(\frac{a+b}{2}\right) \right| \frac{1}{(s+1)(s+2)} + \frac{(|f'(a)| + |f'(b)|)}{(s+2)} \right). \end{aligned} \tag{2.14}$$

If we take $\varepsilon = \frac{1}{4}$ in (2.11), then we get a Bullen type inequality

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \\ &\leq \frac{b-a}{4} \left(\left| f'\left(\frac{a+b}{2}\right) \right| \frac{s+2^{-s}}{(s+1)(s+2)} + \left(\left|f'(a)\right| + \left|f'(b)\right| \right) \frac{s+2^{-s}}{2(s+1)(s+2)} \right). \end{aligned} \tag{2.15}$$

If we take $\varepsilon = \frac{1}{6}$ in (2.11), then we get a Simpson type inequality

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{1}{6}\left[f(a) + 4\left(\frac{a+b}{2}\right) + f(b) \right] \right| \\ &\leq \frac{b-a}{4} \left(2 \left| f'\left(\frac{a+b}{2}\right) \right| \frac{2s+2\times 3^{s+1}+1}{3(s+1)(s+2)} + \left(\left|f'(a)\right| + \left|f'(b)\right| \right) \frac{s+2^{s+3}\times 3^{-s-1}-1}{3(s+1)(s+2)} \right). \end{aligned} \tag{2.16}$$

Remark 3. If we put $M = \sup_{x\in[a,b]} |f'|$ in (2.13)-(2.16), then we have

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{2} \frac{M}{s+1} \tag{2.17}$$

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2} \right| \leq \frac{b-a}{2} \frac{M}{s+1} \tag{2.18}$$

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \\ &\qquad\qquad \leq \frac{b-a}{2} \left(M \frac{s+2^{-s}}{(s+1)(s+2)} \right) \end{aligned} \tag{2.19}$$

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{6}\left[f(a)+4\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$
$$\leq M\frac{b-a}{6}\left(\frac{3s+2\times 3^{s+1}}{(s+1)(s+2)}+\frac{2^{s+3}\times 3^{-s-1}}{(s+1)(s+2)}\right). \quad (2.20)$$

Remark 4. If we further take $s=1$ in (2.17)-(2.20) for functions f with convex $|f'|$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{M(b-a)}{4} \quad (2.21)$$

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{f(a)+f(b)}{2}\right|\leq\frac{M(b-a)}{4} \quad (2.22)$$

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\leq\frac{M(b-a)}{8} \quad (2.23)$$

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{6}\left[f(a)+4\left(\frac{a+b}{2}\right)+f(b)\right]\right|\leq\frac{205M(b-a)}{324}. \quad (2.24)$$

Corollary 4. *Under the assumption of Theorem 1, if* $|f'|^q$ *is* $P(I)$*, then the following inequality holds:*

$$\left|(1-2\varepsilon)f\left(\frac{a+b}{2}\right)+\varepsilon[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \quad (2.25)$$
$$\leq\frac{b-a}{4}\left(4\varepsilon^2-2\varepsilon+\frac{1}{2}\right)\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]$$

for $0\leq\varepsilon\leq\frac{1}{2}$,

$$\left|(1-2\varepsilon)f\left(\frac{a+b}{2}\right)+\varepsilon[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \quad (2.26)$$
$$\leq\frac{b-a}{4}\left(2\varepsilon-\frac{1}{2}\right)\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]$$

for $\frac{1}{2}\leq\varepsilon\leq 1$.

Proof. Proof of inequalities (2.25) and (2.26) is explicit by choosing $h(t)=1$ in (2.6) and (2.7) of Theorem 1. □

Corollary 5. *Under the assumption of Theorem 1, if $|f'|^q$ is tgs-convex, then the following inequality holds:*

$$\left|(1-2\varepsilon)f\left(\frac{a+b}{2}\right)+\varepsilon\left[f(a)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{4}\left(4\varepsilon^2-2\varepsilon+\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{(1-4\varepsilon)}{12}+\frac{8\varepsilon^3(1-\varepsilon)}{3}\right)^{\frac{1}{q}}$$
$$\times\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right], \tag{2.27}$$

for $0\leq\varepsilon\leq\frac{1}{2}$, *and*

$$\left|(1-2\varepsilon)f\left(\frac{a+b}{2}\right)+\varepsilon\left[f(a)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.28}$$
$$\leq \frac{b-a}{8\times 6^{1/q}}(4\varepsilon-1)\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'(b)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right],$$

for $\frac{1}{2}\leq\varepsilon\leq 1$.

Proof. Proof of inequalities (2.27) and (2.28) is explicit by taking $h(t)=t(1-t)$ in (2.6) and (2.7) of Theorem 1. ☐

3. Applications

We consider the means for arbitrary positive numbers a,b $(a\neq b)$ as follows:
The arithmetic mean:

$$A(a,b)=\frac{a+b}{2},$$

the generalized log-mean:

$$L_p(a,b)=\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}},\ p\in\mathbb{R}\setminus\{-1,0\}.$$

Now, by using the result of the second section, we give some applications to special means of real numbers.

Proposition 1. *Let $0<a<b$, $s\in(0,1)$. Then the following inequalities hold:*

$$\left|L_s^s(a,b)-A^s(a,b)\right|\leq\frac{s(b-a)}{2}\left(\frac{A^s(a,b)}{s+2}+\frac{A(a^s,b^s)}{(s+1)(s+2)}\right) \tag{3.1}$$
$$\left|L_s^s(a,b)-A(a^s,b^s)\right|\leq\frac{s(b-a)}{2}\left(\frac{A^s(a,b)}{(s+1)(s+2)}+\frac{A(a^s,b^s)}{s+2}\right) \tag{3.2}$$

$$\left|L_s^s(a,b)-\frac{A^s(a,b)+A(a^s,b^s)}{2}\right|\leq\frac{s(b-a)}{4}\frac{s+2^{-s}}{(s+1)(s+2)}\left(A^s(a,b)+2A(a^s,b^s)\right) \tag{3.3}$$

$$\begin{aligned}&\left|L_s^s(a,b)-\frac{2A^s(a,b)+A(a^s,b^s)}{3}\right|\\&\leq\frac{s(b-a)}{6}\left(A^s(a,b)\frac{2s+2\times3^{s+1}+1}{(s+1)(s+2)}+A(a^s,b^s)\frac{s+2^{s+3}\times3^{-s-1}-1}{(s+1)(s+2)}\right).\end{aligned} \tag{3.4}$$

Proof. The inequalities are derived from (2.13)-(2.16) applied to the s-convex functions $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^s$, $s\in(0,1)$, $x\in[a,b]$ and $f'(x)=sx^{s-1}$, $s\in(0,1)$, $x\in[a,b]$. The details are disregarded. □

Proposition 2. *Let* $0<a<b$, $s\in(0,1)$. *Then the following inequalities hold:*

$$|L_s^s(a,b)-A^s(a,b)|\leq\frac{(b-a)(s+2)}{2a^{1-s}} \tag{3.5}$$

$$|L_s^s(a,b)-A(a^s,b^s)|\leq\frac{(b-a)(s+2)}{2a^{1-s}} \tag{3.6}$$

$$\left|L_s^s(a,b)-\frac{A^s(a,b)+A(a^s,b^s)}{2}\right|\leq\frac{b-a}{2}\frac{s+2^{-s}}{a^{1-s}} \tag{3.7}$$

$$\left|L_s^s(a,b)-\frac{2A^s(a,b)+A(a^s,b^s)}{3}\right|\leq\frac{b-a}{6a^{1-s}}\left(3s+2\times3^{s+1}+2^{s+3}\times3^{-s-1}\right). \tag{3.8}$$

Proof. The inequalities are derived from (2.17)-(2.20) applied to the s-convex functions $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^s$, $s\in(0,1)$, $x\in[a,b]$ and $f'(x)=sx^{s-1}$, $s\in(0,1)$, $x\in[a,b]$ and we might take $M=\frac{(s+1)(s+2)}{a^{1-s}}$. The details are disregarded. □

REFERENCES

[1] H. Alzer, "A superadditive property of Hadamard's gamma function," *Abh. Math. Semin. Univ. Hambg.*, vol. 79, pp. 11–23, 2009, doi: 10.1007/s12188-008-0009-5.

[2] M. Bessenyei, "The Hermite-Hadamard inequality in Beckenbach's setting." *J. Math. Anal. Appl..*, vol. 364, no. 2, pp. 366–383, 2010, doi: 10.1016/j.jmaa.2009.11.015.

[3] M. Bessenyei and Z. Páles, "Higher-order generalizations of Hadamard's inequality." *Publ. Math. Debrecen.*, vol. 61, no. 34, pp. 623–643, 2002.

[4] M. Bessenyei and Z. Páles, "Characterizations of convexity via Hadamard's inequality." *Math. Inequal. Appl.*, vol. 9, no. 1, pp. 53–62, 2006, doi: 10.7153/mia-09-06.

[5] M. Bombardelli and S. Varošanec, " Properties of $h-$convex functions related to the Hermite-Hadamard-Fejér inequalities." *Comput. Math. Appl.*, vol. 58, pp. 1869–1877, 2009, doi: 10.1016/j.camwa.2009.07.073.

[6] W. W. Breckner, "Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Raumen." *Pupl. Inst. Math.*, vol. 23, pp. 13–20, 1978.

[7] W. W. Breckner, "Continuity of generalized convex and generalized concave set-valued functions." *Rev Anal. Numer. Thkor. Approx.*, vol. 22, pp. 39–51, 1993.

[8] P. Burai and A. Házy, "On approximately $h-$convex functions." *J. Convex Anal.*, vol. 18, no. 2, pp. 447–454, 2011.
[9] P. Burai, A. Házy, and T. Juhász, "Bernstein–Doetsch type results for $s-$convex functions." *Publ. Math. Debrecen.*, vol. 75, no. 1-2, pp. 23–31, 2009.
[10] P. Burai, A. Házy, and T. Juhász, "On approximately Breckner $s-$convex functions." *Control Cybernet.*, vol. 40, no. 1, pp. 91–99, 2011.
[11] S. S. Dragomir, R. P. Agarwal, and P. Cerone, "On Simpson's inequality and applications." *J. Inequal. Appl.*, vol. 5, no. 6, pp. 533–579, 2000, doi: 10.1155/S102558340000031X.
[12] S. S. Dragomir, J. Pečarić, and L. E. Persson, "Some inequalities of Hadamard type." *Soochow J.Math.*, vol. 21, pp. 235–241, 1995.
[13] E. K. Godunova and V. I. Levin, "Neravenstva dlja funkcii sirokogo klassa, soderzascego vypuklye, monotonnye i nekotorye drugie vidy funkii." *Vycislitel. Mat. i. Fiz. Mezvuzov. Sb. Nauc. Trudov, MGPI. Moskva*, pp. 138–142, 1985.
[14] H. Hudzik and L. Maligranda, "Some remarks on $s-$convex functions." *Aequationes Math.*, vol. 48, pp. 100–111, 1994, doi: 10.1007/BF01837981.
[15] J. Makó and Z. Páles, " Implications between approximate convexity properties and approximate Hermite-Hadamard inequalities," *Cent. Eur. J. Math.*, vol. 10, no. 3, pp. 1017–1041, 2012, doi: 10.2478/s11533-012-0027-5.
[16] J. Makó and Z. Páles, "Approximate Hermite-Hadamard type inequalities for approximately convex functions," *Math. Inequal. Appl.*, vol. 16, no. 2, pp. 507–526, 2013, doi: 10.7153/mia-16-37.
[17] M. E. Özdemir, M. Gürbüz, and A. O. Akdemir, "Inequalities for $h-$convex functions via further properties." *RGMIA Research Report Collection.*, vol. 14, no. 22, 2011.
[18] J. Pečarić, F. Proschan, and Y. Tong, *Convex Functions, Partial Orderings and Statistical Applications*. Academic Press, 1991.
[19] M. Z. Sarikaya, A. Saglam, and H.Yildirim, "On some Hadamard-type inequalities for $h-$convex functions." *J. Math. Inequal.*, vol. 2, no. 3, pp. 335–341, 2008, doi: 10.7153/jmi-02-30.
[20] M. Tunc, E. Gov, and U. Sanal, "On tgs-convex functions and their inequalities," *Facta Universitatis Ser. Math. Inform.*, vol. 30, no. 5, pp. 679–691, 2015.
[21] M. Tunc and H. Yildirim, "On MT-convexity," *http://arxiv.org/pdf/1205.5453.pdf, preprint*.
[22] S. Varošanec, "On $h-$convexity." *J. Math. Anal. Appl.*, vol. 326, no. 1, pp. 303–311, 2007, doi: 10.1016/j.jmaa.2006.02.086.

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ON THIRD-ORDER JACOBSTHAL POLYNOMIALS AND THEIR PROPERTIES

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Abstract. Third-order Jacobsthal polynomial sequence is defined in this study. Some properties involving this polynomial, including the Binet-style formula and the generating function are also presented. Furthermore, we present the modified third-order Jacobsthal polynomials, and derive adaptations for some well-known identities of third-order Jacobsthal and modified third-order Jacobsthal numbers.

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1. INTRODUCTION

The Jacobsthal numbers have many interesting properties and applications in many fields of science (see, [1]). The Jacobsthal numbers $(J_n)_{n\geq 0}$ are defined by the recurrence relation

$$J_0=0,\ J_1=1,\quad J_{n+2}=J_{n+1}+2J_n,\ n\geq 0. \tag{1.1}$$

Another important sequence is the Jacobsthal–Lucas sequence. This sequence is defined by the recurrence relation $j_{n+2}=j_{n+1}+2j_n$, where $j_0=2$ and $j_1=1$.

In Cook and Bacon's work [5] the Jacobsthal recurrence relation is extended to higher order recurrence relations and the basic list of identities provided by A. F. Horadam [9] is expanded and extended to several identities for some of the higher order cases. In fact, the third-order Jacobsthal numbers, $\{J_n^{(3)}\}_{n\geq 0}$, and third-order Jacobsthal–Lucas numbers, $\{j_n^{(3)}\}_{n\geq 0}$, are defined by

$$J_{n+3}^{(3)}=J_{n+2}^{(3)}+J_{n+1}^{(3)}+2J_n^{(3)},\ J_0^{(3)}=0,\ J_1^{(3)}=J_2^{(3)}=1,\ n\geq 0, \tag{1.2}$$

and

$$j_{n+3}^{(3)}=j_{n+2}^{(3)}+j_{n+1}^{(3)}+2j_n^{(3)},\ j_0^{(3)}=2,\ j_1^{(3)}=1,\ j_2^{(3)}=5,\ n\geq 0, \tag{1.3}$$

respectively.

Some of the following properties given for third-order Jacobsthal numbers and third-order Jacobsthal–Lucas numbers are used in this paper (for more details, see

[2–5]). Note that Eqs. (1.7) and (1.11) have been corrected in [3], since they have been wrongly described in [5]. Then, we have

$$3J_n^{(3)} + j_n^{(3)} = 2^{n+1}, \tag{1.4}$$

$$j_n^{(3)} - 3J_n^{(3)} = 2j_{n-3}^{(3)},\ n \geq 3, \tag{1.5}$$

$$J_{n+2}^{(3)} - 4J_n^{(3)} = \begin{cases} -2 & \text{if} \quad n \equiv 1 \pmod 3 \\ 1 & \text{if} \quad n \not\equiv 1 \pmod 3 \end{cases}, \tag{1.6}$$

$$j_{n+1}^{(3)} + j_n^{(3)} = 3J_{n+2}^{(3)}, \tag{1.7}$$

$$j_n^{(3)} - J_{n+2}^{(3)} = \begin{cases} 1 & \text{if} \quad n \equiv 0 \pmod 3 \\ -1 & \text{if} \quad n \equiv 1 \pmod 3 \\ 0 & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}, \tag{1.8}$$

$$\left(j_{n-3}^{(3)}\right)^2 + 3J_n^{(3)} j_n^{(3)} = 4^n, \tag{1.9}$$

$$\sum_{k=0}^{n} J_k^{(3)} = \begin{cases} J_{n+1}^{(3)} & \text{if} \quad n \not\equiv 0 \pmod 3 \\ J_{n+1}^{(3)} - 1 & \text{if} \quad n \equiv 0 \pmod 3 \end{cases} \tag{1.10}$$

and

$$\left(j_n^{(3)}\right)^2 - 9\left(J_n^{(3)}\right)^2 = 2^{n+2} j_{n-3}^{(3)},\ n \geq 3. \tag{1.11}$$

Using standard techniques for solving recurrence relations, the auxiliary equation, and its roots are given by

$$x^3 - x^2 - x - 2 = 0;\ x = 2,\ \text{and}\ x = \frac{-1 \pm i\sqrt{3}}{2}.$$

Note that the latter two are the complex conjugate cube roots of unity. Call them ω_1 and ω_2, respectively. Thus the Binet formulas can be written as

$$J_n^{(3)} = \frac{2}{7}2^n - \left(\frac{3+2i\sqrt{3}}{21}\right)\omega_1^n - \left(\frac{3-2i\sqrt{3}}{21}\right)\omega_2^n \tag{1.12}$$

and

$$j_n^{(3)} = \frac{8}{7}2^n + \left(\frac{3+2i\sqrt{3}}{7}\right)\omega_1^n + \left(\frac{3-2i\sqrt{3}}{7}\right)\omega_2^n, \tag{1.13}$$

respectively. Now, we use the notation

$$Z_n = \frac{A\omega_1^n - B\omega_2^n}{\omega_1 - \omega_2} = \begin{cases} 2 & \text{if} \quad n \equiv 0 \pmod 3 \\ -3 & \text{if} \quad n \equiv 1 \pmod 3 \\ 1 & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}, \tag{1.14}$$

where $A = -3 - 2\omega_2$ and $B = -3 - 2\omega_1$. Furthermore, note that for all $n \geq 0$ we have

$$Z_{n+2} = -Z_{n+1} - Z_n, \quad Z_0 = 2,\ Z_1 = -3. \tag{1.15}$$

From the Binet formulas (1.12), (1.13) and Eq. (1.14), we have

$$J_n^{(3)} = \frac{1}{7}\left(2^{n+1} - Z_n\right) \text{ and } j_n^{(3)} = \frac{1}{7}\left(2^{n+3} + 3Z_n\right). \tag{1.16}$$

A systematic investigation of the incomplete generalized Jacobsthal numbers and the incomplete generalized Jacobsthal–Lucas numbers was featured in [6]. In [7], Djordjević and Srivastava introduced the generalized incomplete Fibonacci polynomials and the generalized incomplete Lucas polynomials. In [8], the authors investigated some properties and relations involving generalizations of the Fibonacci numbers. In [10], Raina and Srivastava investigated the a new class of numbers associated with the Lucas numbers. Moreover they gave several interesting properties of these numbers.

In this paper, we introduce the third-order Jacobsthal polynomials and we give some properties, including the Binet-style formula and the generating functions for these sequences. Some identities involving these polynomials are also provided.

2. The Third-order Jacobsthal polynomial, Binet's formula and the generating function

The principal goals of this section will be to define the third-order Jacobsthal polynomial and to present some elementary results involving it.

For any variable quantity x such that $x^3 \neq 1$. We define the third-order Jacobsthal polynomial, denoted by $\{J_n^{(3)}(x)\}_{n\geq 0}$. This sequence is defined recursively by

$$J_{n+3}^{(3)}(x) = (x-1)J_{n+2}^{(3)}(x) + (x-1)J_{n+1}^{(3)}(x) + xJ_n^{(3)}(x), \quad n \geq 0, \tag{2.1}$$

with initial conditions $J_0^{(3)}(x) = 0$, $J_1^{(3)}(x) = 1$ and $J_2^{(3)}(x) = x-1$.

In order to find the generating function for the third-order Jacobsthal polynomial, we shall write the sequence as a power series where each term of the sequence correspond to coefficients of the series. As a consequence of the definition of generating function of a sequence, the generating function associated to $\{J_n^{(3)}(x)\}_{n\geq 0}$, denoted by $\{j(t)\}$, is defined by

$$j(t) = \sum_{n\geq 0} J_n^{(3)}(x)t^n.$$

Consequently, we obtain the following result:

Theorem 1. *The generating function for the third-order Jacobsthal polynomials* $\{J_n^{(3)}(x)\}_{n\geq 0}$ *is* $j(t) = \frac{t}{1-(x-1)t-(x-1)t^2-xt^3}$.

Proof. Using the definition of generating function, we have

$$j(t) = J_0^{(3)}(x) + J_1^{(3)}(x)t + J_2^{(3)}(x)t^2 + \cdots + J_n^{(3)}(x)t^n + \cdots.$$

Multiplying both sides of this identity by $-(x-1)t$, $-(x-1)t^2$ and by $-xt^3$, and then from Eq. (2.1), we have

$$(1-(x-1)t-(x-1)t^2-xt^3)j(t)$$
$$=J_0^{(3)}(x)+(J_1^{(3)}(x)-(x-1)J_0^{(3)}(x))t+(J_2^{(3)}(x)-(x-1)J_1^{(3)}-(x-1)J_0^{(3)}(x))t^2 \tag{2.2}$$

and the result follows. □

The following result gives the Binet-style formula for $J_n^{(3)}(x)$.

Theorem 2. *For $n \geq 0$, we have*

$$J_n^{(3)}(x)=\frac{x^{n+1}}{x^2+x+1}-\frac{\omega_1^{n+1}}{(x-\omega_1)(\omega_1-\omega_2)}+\frac{\omega_2^{n+1}}{(x-\omega_2)(\omega_1-\omega_2)},$$

where ω_1, ω_2 are the roots of the characteristic equation associated with the respective recurrence relations $\lambda^2+\lambda+1=0$.

Proof. Since the characteristic equation has three distinct roots, the sequence $J_n^{(3)}(x)=a(x)x^n+b(x)\omega_1^n+c(x)\omega_2^n$ is the solution of the Eq. (2.1). Considering $n=0,1,2$ in this identity and solving this system of linear equations, we obtain a unique value for $a(x)$, $b(x)$ and $c(x)$, which are, in this case, $(x^2+x+1)a(x)=x$, $(x-\omega_1)(\omega_1-\omega_2)b(x)=-\omega_1$ and $(x-\omega_2)(\omega_1-\omega_2)c(x)=\omega_2$. So, using these values in the expression of $J_n^{(3)}(x)$ stated before, we get the required result. □

We define the modified third-order Jacobsthal polynomial sequence, denoted by $\{K_n^{(3)}(x)\}_{n\geq 0}$. This sequence is defined recursively by

$$K_{n+3}^{(3)}(x)=(x-1)K_{n+2}^{(3)}(x)+(x-1)K_{n+1}^{(3)}(x)+xK_n^{(3)}(x), \tag{2.3}$$

with initial conditions $K_0^{(3)}(x)=3$, $K_1^{(3)}(x)=x-1$ and $K_2^{(3)}(x)=x^2-1$.

We give their versions for the third-order Jacobsthal and modified third-order Jacobsthal polynomials.

For simplicity of notation, let

$$\begin{aligned} Z_n(x)&=\frac{1}{\omega_1-\omega_2}\left((x-\omega_2)\omega_1^{n+1}-(x-\omega_1)\omega_2^{n+1}\right),\\ Y_n&=\omega_1^n+\omega_2^n. \end{aligned} \tag{2.4}$$

Then, we can write

$$J_n^{(3)}(x)=\frac{1}{x^2+x+1}\left(x^{n+1}-Z_n(x)\right)$$

and

$$K_n^{(3)}(x)=x^n+Y_n.$$

Then, $Z_n(x)=-Z_{n-1}(x)-Z_{n-2}(x)$, $Z_0(x)=x$ and $Z_1(x)=-(x+1)$.

Furthermore, we easily obtain the identities stated in the following result:

Proposition 1. *For a natural number n and m, if $J_n^{(3)}(x)$ and $K_n^{(3)}(x)$ are, respectively, the n-th third-order Jacobsthal and modified third-order Jacobsthal polynomials, then the following identities are true:*

$$K_n^{(3)}(x) = (x-1)J_n^{(3)}(x) + 2(x-1)J_{n-1}^{(3)}(x) + 3xJ_{n-2}^{(3)}(x),\ n \geq 2, \tag{2.5}$$

$$J_n^{(3)}(x)J_m^{(3)}(x) + J_{n+1}^{(3)}(x)J_{m+1}^{(3)}(x) + J_{n+2}^{(3)}(x)J_{m+2}^{(3)}(x) = \frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c} (1+x^2+x^4)\cdot x^{n+m+2} \\ -x^{n+1}\left((1-x^2)Z_m(x)+x(1-x)Z_{m+1}(x)\right) \\ -x^{m+1}\left((1-x^2)Z_n(x)+x(1-x)Z_{n+1}(x)\right) \\ +(x^2+x+1)(\omega_1^n\omega_2^m+\omega_1^m\omega_2^n) \end{array}\right\}, \tag{2.6}$$

$$\left(J_n^{(3)}(x)\right)^2 + \left(J_{n+1}^{(3)}(x)\right)^2 + \left(J_{n+2}^{(3)}(x)\right)^2 = \frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c} (1+x^2+x^4)\cdot x^{2n+2} \\ -2x^{n+1}\left((1-x^2)Z_n(x)+x(1-x)Z_{n+1}(x)\right) \\ +2(x^2+x+1) \end{array}\right\}, \tag{2.7}$$

and $Z_n(x)$ as in Eq. (2.4).

Proof. (2.5): To prove Eq. (2.5), we use induction on n. Let $n=2$, we get

$$\begin{aligned} (x-1)J_2^{(3)}(x) + 2(x-1)J_1^{(3)}(x) + 3xJ_0^{(3)}(x) &= (x-1)(x-1)+2(x-1) \\ &= x^2-1 \\ &= K_2^{(3)}(x). \end{aligned}$$

Let us assume that $K_m^{(3)}(x) = (x-1)J_m^{(3)}(x) + 2(x-1)J_{m-1}^{(3)}(x) + 3xJ_{m-2}^{(3)}(x)$ is true for all values m less than or equal $n \geq 2$. Then,

$$\begin{aligned} K_{m+1}^{(3)}(x) &= (x-1)K_m^{(3)}(x) + (x-1)K_{m-1}^{(3)}(x) + xK_{m-2}^{(3)}(x) \\ &= (x-1)\left((x-1)J_m^{(3)}(x) + 2(x-1)J_{m-1}^{(3)}(x) + 3xJ_{m-2}^{(3)}(x)\right) \\ &\quad + (x-1)\left((x-1)J_{m-1}^{(3)}(x) + 2(x-1)J_{m-2}^{(3)}(x) + 3xJ_{m-3}^{(3)}(x)\right) \\ &\quad + x\left((x-1)J_{m-2}^{(3)}(x) + 2(x-1)J_{m-3}^{(3)}(x) + 3xJ_{m-4}^{(3)}(x)\right) \\ &= (x-1)J_{m+1}^{(3)}(x) + 2(x-1)J_m^{(3)}(x) + 3xJ_{m-1}^{(3)}(x). \end{aligned}$$

(2.6): Using the Binet formula of $J_n^{(3)}(x)$ in Theorem 2, we have

$$J_n^{(3)}(x)J_m^{(3)}(x) + J_{n+1}^{(3)}(x)J_{m+1}^{(3)}(x) + J_{n+2}^{(3)}(x)J_{m+2}^{(3)}(x) = \frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c} \left(x^{n+1}-Z_n(x)\right)\left(x^{m+1}-Z_m(x)\right) \\ +\left(x^{n+2}-Z_{n+1}(x)\right)\left(x^{m+2}-Z_{m+1}(x)\right) \\ +\left(x^{n+3}-Z_{n+2}(x)\right)\left(x^{m+3}-Z_{m+2}(x)\right) \end{array}\right\}.$$

Then, we obtain

$$\begin{aligned}
&J_n^{(3)}(x)J_m^{(3)}(x)+J_{n+1}^{(3)}(x)J_{m+1}^{(3)}(x)+J_{n+2}^{(3)}(x)J_{m+2}^{(3)}(x)\\
&=\frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c}(1+x^2+x^4)\cdot x^{n+m+2}\\ -x^{n+1}\left(Z_m(x)+xZ_{m+1}(x)+x^2Z_{m+2}(x)\right)\\ -x^{m+1}\left(Z_n(x)+xZ_{n+1}(x)+x^2Z_{n+2}(x)\right)\\ +Z_n(x)Z_m(x)+Z_{n+1}(x)Z_{m+1}(x)+Z_{n+2}(x)Z_{m+2}(x)\end{array}\right\}\\
&=\frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c}(1+x^2+x^4)\cdot x^{n+m+2}\\ -x^{n+1}\left((1-x^2)Z_m(x)+x(1-x)Z_{m+1}(x)\right)\\ -x^{m+1}\left((1-x^2)Z_n(x)+x(1-x)Z_{n+1}(x)\right)\\ +(x^2+x+1)(\omega_1^n\omega_2^m+\omega_1^m\omega_2^n)\end{array}\right\}.
\end{aligned}$$

Then, we obtain the Eq. (2.7) if $m=n$ in Eq. (2.6). □

3. Some identities involving this type of polynomials

In this section, we state some identities related with these type of third-order polynomials. As a consequence of the Binet formula of Theorem 2, we get for this sequence the following interesting identities.

Proposition 2 (Catalan-like identity)**.** *For a natural numbers n, s, with $n\geq s$, if $J_n^{(3)}(x)$ is the n-th third-order Jacobsthal polynomials, then the following identity*

$$\begin{aligned}
&J_{n+s}^{(3)}(x)J_{n-s}^{(3)}(x)-\left(J_n^{(3)}(x)\right)^2\\
&=\frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c}x^{n+1}(x^s-x^{-s})X_sZ_{n+1}(x)\\ -x^{n+1}(2+x^sX_{s+1}-x^{-s}X_{s-1})Z_n(x)\\ -(x^2+x+1)X_s^2\end{array}\right\}
\end{aligned}$$

is true, where $Z_n(x)$ as in Eq. (2.4), $X_n=\frac{\omega_1^n-\omega_2^n}{\omega_1-\omega_2}$ and ω_1, ω_2 are the roots of the characteristic equation associated with the recurrence relation $x^2+x+1=0$.

Proof. Using the Eq. (2.4) and the Binet formula of $J_n^{(3)}(x)$ in Theorem 2, we have

$$\begin{aligned}
&J_{n+s}^{(3)}(x)J_{n-s}^{(3)}(x)-\left(J_n^{(3)}(x)\right)^2\\
&=\frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c}\left(x^{n+s+1}-Z_{n+s}(x)\right)\left(x^{n-s+1}-Z_{n-s}(x)\right)\\ -\left(x^{n+1}-Z_n(x)\right)^2\end{array}\right\}\\
&=\frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c}-x^{n+1}\left(x^sZ_{n-s}(x)+x^{-s}Z_{n+s}(x)-2Z_n(x)\right)\\ +Z_{n+s}(x)Z_{n-s}(x)-(Z_n(x))^2\end{array}\right\}.
\end{aligned}$$

Using the following identity for the sequence $Z_n(x)$:

$$Z_{n+s}(x)=X_sZ_{n+1}(x)-X_{s-1}Z_n(x),$$

where $X_s = \frac{\omega_1^s - \omega_2^s}{\omega_1 - \omega_2}$ and $X_{-s} = -X_s$. Then, we obtain

$$J_{n+s}^{(3)}(x)J_{n-s}^{(3)}(x) - \left(J_n^{(3)}(x)\right)^2$$
$$= \frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c} x^{n+1}\left(x^s - x^{-s}\right)X_s Z_{n+1}(x) \\ -x^{n+1}\left(x^s X_{s+1} - x^{-s}X_{s-1} - 2\right)Z_n(x) \\ -(x^2+x+1)X_s^2 \end{array}\right\}.$$

Hence the result holds. □

Note that for $s = 1$ in the Catalan-like identity obtained, we get the Cassini-like identity for the third-order Jacobsthal polynomial. Furthermore, for $s = 1$, the identity stated in Proposition 2, yields

$$J_{n+1}^{(3)}(x)J_{n-1}^{(3)}(x) - \left(J_n^{(3)}(x)\right)^2$$
$$= \frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c} x^{n+1}\left(x^1 - x^{-1}\right)X_1 Z_{n+1}(x) \\ -x^{n+1}\left(x^1 X_{1+1} - x^{-1}X_{1-1} - 2\right)Z_n(x) \\ -(x^2+x+1) \end{array}\right\}.$$

and using $X_0 = 0$ and $X_1 = 1$ in Proposition 2, we obtain the following result.

Proposition 3 (Cassini-like identity)**.** *For a natural numbers n, if $K_n^{(3)}$ is the n-th third-order Jacobsthal numbers, then the identity*

$$J_{n+1}^{(3)}(x)J_{n-1}^{(3)}(x) - \left(J_n^{(3)}(x)\right)^2$$
$$= \frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c} x^n\left((x^2-1)Z_{n+1}(x) + x(x+2)Z_n(x)\right) \\ -(x^2+x+1) \end{array}\right\}.$$

is true.

The d'Ocagne-like identity can also be obtained using the Binet formula and in this case we obtain

Proposition 4 (d'Ocagne-like identity)**.** *For a natural numbers m, n, with $m \geq n$ and $J_n^{(3)}(x)$ is the n-th third-order Jacobsthal polynomial, then the following identity*

$$J_{m+1}^{(3)}(x)J_n^{(3)}(x) - J_m^{(3)}(x)J_{n+1}^{(3)}(x)$$
$$= \frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c} x^{m+1}\left(Z_{n+1}(x) - xZ_n(x)\right) \\ -x^{n+1}\left(Z_{m+1}(x) - xZ_m(x)\right) + (x^2+x+1)X_{m-n} \end{array}\right\}$$

is true.

Proof. Using the Eq. (2.4) and the Theorem 2, we get the required result. □

In addition, some formulae involving sums of terms of the third-order Jacobsthal polynomial sequence will be provided in the following proposition.

Proposition 5. *For a natural numbers m, n, with $n \geq m$, if $J_n^{(3)}(x)$ and $K_n^{(3)}(x)$ are, respectively, the n-th third-order Jacobsthal and modified third-order Jacobsthal polynomials, then the following identities are true:*

$$\sum_{s=m}^{n} J_s^{(3)}(x) = \frac{1}{3(x-1)} \left\{ \begin{array}{c} (3x-2)J_n^{(3)}(x) + (2x-1)J_{n-1}^{(3)}(x) \\ +xJ_{n-2}^{(3)}(x) - J_{m+2}^{(3)}(x) \\ +(x-2)J_{m+1}^{(3)}(x) + (2x-3)J_m^{(3)}(x) \end{array} \right\}, \tag{3.1}$$

$$\sum_{s=0}^{n} K_s^{(3)}(x) = \frac{1}{x-1} \left\{ \begin{array}{ccc} x^{n+1} + 2x - 3 & if & n \equiv 0 \pmod 3 \\ x^{n+1} + x - 2 & if & n \equiv 1 \pmod 3 \\ x^{n+1} - 1 & if & n \equiv 2 \pmod 3 \end{array} \right. . \tag{3.2}$$

Proof. (3.1): Using Eq. (2.1), we obtain

$$\begin{aligned} \sum_{s=m}^{n} J_s^{(3)}(x) &= J_m^{(3)}(x) + J_{m+1}^{(3)}(x) + J_{m+2}^{(3)}(x) + \sum_{s=m+3}^{n} J_s^{(3)}(x) \\ &= J_m^{(3)}(x) + J_{m+1}^{(3)}(x) + J_{m+2}^{(3)}(x) + (x-1) \sum_{s=m+2}^{n-1} J_s^{(3)}(x) \\ &\quad + (x-1) \sum_{s=m+1}^{n-2} J_s^{(3)}(x) + x \sum_{s=m}^{n-3} J_s^{(3)}(x) \end{aligned}$$

Then,

$$\begin{aligned} \sum_{s=m}^{n} J_s^{(3)}(x) &= (3x-2) \sum_{s=m}^{n} J_s^{(3)}(x) + J_{m+2}^{(3)}(x) - (x-2)J_{m+1}^{(3)}(x) - (2x-3)J_m^{(3)}(x) \\ &\quad - (3x-2)J_n^{(3)}(x) - (2x-1)J_{n-1}^{(3)}(x) - xJ_{n-2}^{(3)}(x). \end{aligned}$$

Finally, the result in Eq. (3.1) is completed.

(3.2): As a consequence of the Eq. (2.4) of Theorem 2 and

$$\begin{aligned} \sum_{s=0}^{n} Y_s &= \sum_{s=0}^{n} (\omega_1^s + \omega_2^s) \\ &= \frac{\omega_1^{n+1} - 1}{\omega_1 - 1} + \frac{\omega_2^{n+1} - 1}{\omega_2 - 1} \\ &= \frac{1}{3}(Y_n - Y_{n+1}) + 1, \end{aligned}$$

we have

$$\begin{aligned} \sum_{s=0}^{n} K_s^{(3)}(x) &= \sum_{s=0}^{n} x^s + \sum_{s=0}^{n} Y_s \\ &= \frac{x^{n+1} - 1}{x - 1} + \frac{1}{3}(Y_n - Y_{n+1}) + 1 \end{aligned}$$

$$= \frac{1}{x-1} \begin{cases} x^{n+1} + 2x - 3 & \text{if} \quad n \equiv 0 \pmod 3 \\ x^{n+1} + x - 2 & \text{if} \quad n \equiv 1 \pmod 3 \\ x^{n+1} - 1 & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}.$$

Hence, we obtain the result. □

For example, if $n \equiv 0 (\text{mod } 3)$ we have that $x^{n+1} + 2x - 3$ is divisible by $x - 1$.

For negative subscripts terms of the sequence of modified third-order Jacobsthal polynomial we can establish the following result:

Proposition 6. *For a natural number n and $x^3 \neq 0$ the following identities are true:*

$$K_{-n}^{(3)}(x) = K_n^{(3)}(x) + x^{-n} - x^n, \tag{3.3}$$

$$\sum_{s=0}^{3n} K_{-s}^{(3)}(x) = \frac{1}{x-1}\left(3x - 2 - x^{-3n}\right). \tag{3.4}$$

Proof. (3.3): Since $Y_{-n} = Y_n$, using the Binet formula stated in Theorem 2 and the fact that $\omega_1 \omega_2 = 1$, all the results of this Proposition follow. In fact,

$$\begin{aligned} K_{-n}^{(3)}(x) &= x^{-n} + Y_{-n} \\ &= x^{-n} + x^n + Y_n - x^n \\ &= K_n^{(3)}(x) + x^{-n} - x^n. \end{aligned}$$

So, the proof is completed.

(3.4): The proof is similar to the proof of Eq. (3.1) using Eq. (3.3). □

4. Conclusion

Sequences of polynomials have been studied over several years, including the well-known Tribonacci polynomial and, consequently, on the Tribonacci-Lucas polynomial. In this paper, we have also contributed for the study of third-order Jacobsthal and modified third-order Jacobsthal polynomials, deducing some formulae for the sums of such polynomials, presenting the generating functions and their Binet-style formula. It is our intention to continue the study of this type of sequences, exploring some their applications in the science domain. For example, a new type of sequences in the quaternion algebra with the use of these polynomials and their combinatorial properties.

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REFERENCES

[1] P. Barry, "Triangle geometry and Jacobsthal numbers," *Irish Math. Soc. Bulletin*, vol. 51, no. 1, pp. 45–57, 2003.

[2] G. Cerda-Morales, "Identities for third order Jacobsthal quaternions," *Adv. Appl. Clifford Algebr.*, vol. 27, no. 2, pp. 1043–1053, 2017, doi: 10.1007/s00006-016-0654-1.

[3] G. Cerda-Morales, "Dual third-order Jacobsthal quaternions," *Proyecciones Journal of Mathematics*, vol. 37, no. 4, pp. 731–747, 2018.

[4] G. Cerda-Morales, "On the third-order Jacobsthal and third-order Jacobsthal–Lucas sequences and their matrix representations," *Mediterr. J. Math.*, vol. 16, no. 2, pp. 1–12, 2019, doi: 10.1007/s00009-019-1319-9.

[5] C. K. Cook and M. R. Bacon, "Some identities for Jacobsthal and Jacobsthal–Lucas numbers satisfying higher order recurrence relations," *Ann. Math. Inform*, vol. 41, no. 1, pp. 27–39, 2013.

[6] G. B. Djordjević and H. M. Srivastava, "Incomplete generalized Jacobsthal and Jacobsthal–Lucas numbers," *Mathl. Comput. Modelling*, vol. 42, pp. 1049–1056, 2005, doi: 10.1016/j.mcm.2004.10.026.

[7] G. B. Djordjević and H. M. Srivastava, "Some generalizations of the incomplete Fibonacci and the incomplete Lucas polynomials," *Adv. Stud. Contemp. Math.*, vol. 11, pp. 11–32, 2005.

[8] G. B. Djordjević and H. M. Srivastava, "Some generalizations of certain sequences associated with the Fibonacci numbers," *J. Indonesian Math. Soc.*, vol. 12, pp. 99–112, 2006.

[9] A. F.Horadam, "Jacobsthal representation numbers," *Fibonacci Q.*, vol. 34, no. 1, pp. 40–54, 1996.

[10] R. K. Raina and H. M. Srivastava, "A class of numbers associated with the Lucas numbers," *Mathl. Comput. Modelling*, vol. 25, no. 7, pp. 15–22, 1997, doi: 10.1016/S0895-7177(97)00045-9.

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LINEAR EQUATIONS WITH ONE CONSTRAINT AND THEIR CONNECTION TO NONLINEAR EQUATIONS OF THE FOURTH ORDER

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Abstract. The purpose of this paper is to present several new results concerning relations between linear differential equations of the fourth order with one constraint and nonlinear differential equations of the fourth order. We consider linear differential equations of the second, the third and the fourth order and nonlinear fourth order differential equations related via the Schwarzian derivative. The method is based on the use of the Schwarzian derivative, which is defined as the ratio of two linearly independent solutions of the linear differential equations of the second or third and fourth order. As a result we obtain new relations between the solutions of these linear and nonlinear equations. To illustrate theorems and our constructive approach we give two examples. The given method may be generalized to differential equations of higher orders.

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1. INTRODUCTION

The Schwarzian derivative is a differential operator that is invariant under all linear fractional transformations, see [1, 8, 10]. It plays a significant role in the theory of modular forms, hypergeometric functions, univalent functions and conformal mappings [1, 8]. It is defined by

$$(Sf)(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

The well-known relation between a second-order linear differential equation of the form

$$y''(z) + Q(z)y(z) = 0$$

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and the Schwarzian derivative of the ratio of two linearly independent solutions y_1, y_2 of the linear equation above is as follows:

$$(S\xi)(z) = 2Q(z),$$

where $\xi = y_1/y_2$ and z is, in general, a complex variable. See [1] and [8] for more details.

If we have a general second order differential equation

$$y''(z) + p(z)y'(z) + q(z)y(z) = 0, \tag{1.1}$$

then substituting $y(z) = \xi(z)y_1(z)$ with the condition that y_1 is also a solution of (1.1), we get an expression for ξ, y_1 and their derivatives (up to order 2 and 1 respectively). Differentiating again and eliminating y_1, y_1', we get that the function

$$w(z) = (S\xi)(z)$$

satisfies

$$w(z) = \frac{1}{2}(4q(z) - p(z)^2 - 2p'(z)). \tag{1.2}$$

We call expression (1.2) the invariant for the second order linear differential equation (1.1).

On extension of this approach for linear differential equation of the third order see in [10–12]. The generalization of the method for the linear differential equations of the fourth order is given in [6]. In papers [2, 4, 5] special classes of the fourth order linear differential equations and the nonlinear fourth order differential equations related via the Schwarzian derivative are considered and general solutions of both differential equations are found. In paper [3] the generalization of the method for a special type of linear differential equations of the fifth order is given along with a computer realization of this method in Mathematica (www.wolfram.com).

Several questions arise. What happens if the second, the third and the fourth order linear differential equations are related? What happens if we modify the function to be the ratio of solutions of two different equations? These questions were answered during the studies of linear differential equation of the third order in [9].

Similar questions were resolved for the fours order ordinary differential equations with coefficients satisfying a system of two first order differential equations [7].

The main objective of this paper is to answer these questions for linear differential equation of the fourth order with coefficients that satisfy the differential equation of the first order. The proofs of statements are computational, that is the results can be verified by using any computer algebra system.

2. Main Results

In this section we shall present 5 main results concerning the relations between linear

$$y''''(z) + p(z)y'''(z) + q(z)y''(z) + r(z)y'(z) + s(z)y(z) = 0, \tag{2.1}$$

where

$$p' = \frac{1}{12}\left(8q - 3p^2\right) \tag{2.2}$$

and nonlinear differential equations.

Theorem 1. *Let y be a solution of the fourth order linear differential equation (2.1) and y_1 be a solution of another fourth order linear differential equation*

$$y''''(z) + p_1(z)y'''(z) + q_1(z)y''(z) + r_1(z)y'(z) + s_1(z)y(z) = 0, \tag{2.3}$$

where

$$p_1' = \frac{1}{12}(8q_1 - 3p_1^2).$$

If the function $w(z) = (S\xi)(z)$ with $\xi = y/y_1$ solves a nonlinear differential equation

$$\sum_{k=0}^{1}\sum_{j=0}^{2}\sum_{i=0}^{3}\alpha_{ijk}(z)w^i(w')^j(w'')^k w'''' + \sum_{l=0}^{2}\sum_{k=0}^{3}\sum_{j=0}^{4}\sum_{i=0}^{6}\beta_{ijkl}(z)w^i(w')^j(w'')^k(w''')^l = 0, \tag{2.4}$$

which is explicitly given by

$$\begin{aligned}
&(240ww'' - 300w'^2 - 40\psi w' + 160w^3 + (80q'' + \frac{40}{3}q^2 - 10p^2q + 60pr - 480s - 20\phi)w \\
&\quad - \psi^2)w'''' - 280ww'''^2 + ((56\psi + 840w')w'' - (1120w^2 - 110\phi + 55p\psi - 20\psi')w' \\
&\quad - \frac{448}{3}\psi w^2 + (10pq^2 - \frac{5}{2}p^3q - \frac{45}{4}p^2\psi - 90\phi' + 45p\psi' - \frac{20}{3}q(3r - 4\psi - 3p'') - 20\psi'')w \\
&\quad + 4\phi\psi - 2p\psi^2 + 2\psi\psi')w''' - 504w''^3 + (192w^2 - 153\phi + \frac{153}{2}p\psi - 66\psi')w''^2 \\
&\quad + (2040ww'^2 + 448w^4 - 7\phi^2 - \frac{5}{16}p^2\psi^2 - \frac{23}{6}q\psi^2 + \frac{23}{2}\psi\phi' + \phi(7p\psi - 18\psi') \\
&\quad + \frac{13}{4}p\psi\psi' - 2\psi'^2 + (42p\psi - 84\phi + 152\psi')w^2 \\
&\quad + (\frac{55}{3}\psi^2 - \frac{15}{4}p^3\psi - 30r\psi + 15p^2\psi' - 40q\psi' + 60\phi'' + 15p(q\psi - 2\psi''))w \\
&\quad + \psi\psi'' + (\frac{135}{8}p^2\psi - 45q\psi + 528\psi w + 135\phi' - \frac{135}{2}p\psi' + 30\psi'')w')w'' \\
&\quad - 1275w'^4 - 520\psi w'^3 + (-560w^3 + \frac{15}{2}(38\phi - 19p\psi - 4\psi')w + \frac{75}{16}p^3\psi \\
&\quad + \frac{75}{2}r\psi - \frac{611}{12}\psi^2 - \frac{75}{4}p^2\psi' + 50q\psi' - 75\phi'' - \frac{75}{4}p(q\psi - 2\psi''))w'^2 \\
&\quad + (-128\psi w^3 + \frac{35}{32}p^2\phi\psi - \frac{35}{12}q\phi\psi + \frac{5}{64}p^3\psi^2 - \frac{25}{24}pq\psi^2 + 5r\psi^2 - \frac{7}{6}\psi^3 \\
&\quad + \frac{35}{4}\phi\phi' + \frac{5}{16}p^2\psi\psi' + 5q\psi\psi' + 5\phi'\psi' + (32\phi\psi - 16p\psi^2 + 16\psi\psi')w \\
&\quad - 10\psi\phi'' - \frac{5}{8}p(\psi'(7\phi + 4\psi') + \psi(7\phi' - 2\psi''))
\end{aligned}$$

$$
\begin{aligned}
&-(\frac{45}{2}p^2\psi-60q\psi+180\phi'-90p\psi'+40\psi'')w^2+\frac{15}{2}\phi\psi'')w'+64w^6\\
&-(8\phi-4p\psi-\frac{272}{3}\psi')w^4-\Big(\frac{5}{2}p^3\psi+20r\psi+\frac{82}{9}\psi^2-10p^2\psi'+\frac{80}{3}q\psi'-40\phi''\\
&-10p(q\psi-2\psi'')\Big)\,w^3+(344p\psi\psi'-84\phi^2-105p^2\psi^2+4\phi(21p\psi-4\psi')\\
&+16(14q\psi^2-42\psi\phi'+31\psi'^2-28\psi\psi''))\frac{w^2}{48}+(\frac{13}{24}\phi\psi^2-\frac{5}{32}p^3\phi\psi-\frac{5}{9}q^2\psi^2\\
&+\frac{10}{3}q\psi\phi'-5\phi'^2-\frac{5}{3}q\phi\psi'+\frac{5}{12}\psi^2\psi'-\frac{10}{3}q\psi'^2-\frac{5}{4}r\psi(\phi+2\psi')\\
&+\frac{5}{48}p^2(q\psi^2-12\psi\phi'+6\phi\psi')+\frac{5}{2}\phi\phi''+5\psi'\phi''+\frac{5}{3}q\psi\psi''-5\phi'\psi''\\
&+\frac{1}{48}p(30r\psi^2-13\psi^3+240\phi'\psi'+10q\psi(3\phi+2\psi')-60\psi\phi''-60\phi\psi''))w\\
&+\frac{1}{32}(3\phi^2(p\psi-4\psi')-2\phi^3+2\psi(2r\psi^2-3p\psi\phi'+8\phi'\psi'-4\psi\phi'')\\
&+2\phi(\psi(6\phi'+3p\psi'+4\psi'')-2q\psi^2-8\psi'^2))=0,
\end{aligned}
$$

then conditions

$$p_1(z)=p(z),\ q_1(z)=q(z),\ r_1(z)=r(z),\ s_1(z)=s(z)$$

and

$$\phi=pr-16s+4r',\quad \psi=pq-6r+4q' \tag{2.5}$$

hold.

Proof. We substitute $w(z)=(S\xi)(z)$ into equation (2.4) with unknown coefficients and then replace ξ by the ratio of y and y_1. Replacing the fourth and higher order derivatives of y and y_1 by using the linear equations, we collect the coefficients of y, y_1 and their derivatives up to order 3. In the result we obtain a system of equations on the coefficients of linear and nonlinear equations, from which we get the desired result. □

Theorem 2. *Let y be a solution of equation (2.1) and y_1 be a solution of the third order linear differential equation of the form*

$$y'''(z)+q_1(z)y''(z)+r_1(z)y'(z)+s_1(z)y(z)=0. \tag{2.6}$$

If the function $w(z)=(S\xi)(z)$ with $\xi=y/y_1$ solves the nonlinear differential equation (2.4), then we have conditions (2.5) for (2.4) and three additional conditions on the coefficients of the linear equation (2.6)

$$q_1'=q_1^2+q-p\,q_1-r_1,\ r_1'=r-pr_1+q_1r_1-s_1,\ s_1'=s_1(q_1-p)+s. \tag{2.7}$$

Proof. We substitute $w(z)=(S\xi)(z)$ into equation (2.4), (2.5) with unknown coefficients and then replace ξ by the ratio of y and y_1. Replacing the fourth and higher order derivatives of y and the third and higher order derivatives of y_1 by using the linear equations, we collect the coefficients of y, y_1 and their derivatives up to order

3, order 2 and order 1 respectively. In the result we obtain a system of equations on the coefficients of linear and nonlinear equations, from which we get the desired result. □

Example 1. Assume that we know a particular solution y and coefficient p of equation (2.1)

$$y(z) = z, \;\; p(z) = b, \tag{2.8}$$

where b is a constant. We substitute functions (2.8) into equations (2.1), (2.2). Solving the obtained equations we find

$$q(z) = \frac{3b^2}{8}, \;\; s(z) = -\frac{r(z)}{z}. \tag{2.9}$$

Assume that we know a particular solution y_1 and coefficient q_1 of the equation (2.6)

$$y_1(z) = z, \;\; q_1(z) = b. \tag{2.10}$$

Then from equation (2.6) we find

$$s_1(z) = -\frac{r_1(z)}{z}. \tag{2.11}$$

We substitute functions (2.8) -(2.11) into equations (2.7) . Solving the resulting equations we find

$$r_1(z) = \frac{3b^2}{8}, \;\; r(z) = -\frac{3b^2}{8z}. \tag{2.12}$$

We substitute functions (2.10), (2.11) into equation (2.6) and obtain

$$8zy''' + 8bzy'' + 3b^2zy' - 3b^2y = 0. \tag{2.13}$$

The substitution

$$y = z\int v(z)dz \tag{2.14}$$

reduces equation (2.13) to the second order differential equation

$$8zv'' + 8(bz+3)v' + b(3bz+16)v = 0. \tag{2.15}$$

The general solution of equation (2.15) is of the form

$$v = e^{-\frac{1}{4}i(\sqrt{2}-2i)bz}\left(C_1U_1 + C_2L^2_{-\frac{3}{2}-\frac{i}{\sqrt{2}}}\left(\frac{biz}{\sqrt{2}}\right)\right), \tag{2.16}$$

where $U_1 = U\left(\frac{3}{2}+\frac{i}{\sqrt{2}}, 3, \frac{biz}{\sqrt{2}}\right)$ is the confluent hypergeometric function and has the integral representation

$$U(a,b,z) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^{a-1}(1+t)^{b-a-1}dt; \;\; L^2_{-\frac{3}{2}-\frac{i}{\sqrt{2}}}\left(\frac{biz}{\sqrt{2}}\right)$$

is the Laguerre polynomial, that satisfy the differential equation

$$zv'' + (3-z)v' - \left(\frac{3}{2}+\frac{i}{\sqrt{2}}\right)v = 0,$$

and C_1, C_2 are arbitrary constants.

The general solution of equation (2.13) is of the form (2.14), (2.16). We choose, for example, the values of the arbitrary constants equal to $C_1 = 1$, $C_2 = 0$. Then the particular solution is

$$y_1 = z\int e^{-\frac{1}{4}i(\sqrt{2}-2i)bz}U\left(\frac{3}{2}+\frac{i}{\sqrt{2}},3,\frac{biz}{\sqrt{2}}\right)dz$$

and

$$\xi = \left(\int e^{-\frac{1}{4}i(\sqrt{2}-2i)bz}U\left(\frac{3}{2}+\frac{i}{\sqrt{2}},3,\frac{biz}{\sqrt{2}}\right)dz\right)^{-1}.$$

Then

$$\begin{aligned} w &= \frac{\xi'''}{\xi'} - \frac{3}{2}\left(\frac{\xi''}{\xi'}\right)^2 \\ &= \frac{b^2}{16}\left(U^{-2}(3(7+6\sqrt{2}i)U_2^2 + 2U_1(5-2i\sqrt{2})U_2 - (13+8i\sqrt{2})U_3)) - 2i\sqrt{2} - 1\right), \end{aligned} \tag{2.17}$$

where

$$U_2 = U\left(\frac{5}{2}+\frac{i}{\sqrt{2}},4,\frac{biz}{\sqrt{2}}\right), \quad U_3 = U\left(\frac{7}{2}+\frac{i}{\sqrt{2}},5,\frac{biz}{\sqrt{2}}\right).$$

The differential equation (2.4), (2.5) has coefficients (2.8), (2.9), (2.12) and

$$\phi(z) = -\frac{3b^2(12+bz)}{8z^2}, \quad \psi(z) = \frac{3b^2(6+bz)}{8z}.$$

Then according to Theorem 2 this nonlinear differential equation has a solution (2.17) which can be easily verified by substitution.

Theorem 3. *Let y be a general solution of the third order linear differential equation (2.6), (2.7). Then this solution is a three parameter family of solutions of the fourth order linear differential equation (2.1), (2.2).*

Proof. The proof is computational. □

Theorem 4. *Let y be a solution of equation (2.1) and y_1 be a solution of the second order linear differential equation of the form*

$$y''(z) + r_1(z)y'(z) + s_1(z)y(z) = 0. \tag{2.18}$$

If the function $w(z) = (S\xi)(z)$ with $\xi = y/y_1$ solves a nonlinear differential equation (2.4), then we have condition (2.5) for (2.4) and two additional conditions on the coefficients of the linear equation (2.18)

$$\begin{aligned} r_1'' &= -pr_1' + pr_1^2 - ps_1 - qr_1 + r + 3r_1r_1' + 2r_1s_1 - r_1^3 - 2s_1', \\ s_1'' &= pr_1s_1 - ps_1' - qs_1 + 2s_1r_1' + r_1s_1' - r_1^2s_1 + s + s_1^2. \end{aligned} \tag{2.19}$$

Proof. We substitute $w(z) = (S\xi)(z)$ into equation (2.4), (2.5) with unknown coefficients and then replace ξ by the ratio of y and y_1. Replacing the fourth and higher order derivatives of y and the second and higher order derivatives of y_1 by using the linear equations, we collect the coefficients of y, y_1 and their derivatives up to order 3 and order 1 respectively. In the result we obtain a system of equations on the coefficients of linear and nonlinear equations, from which we get the desired result. □

Example 2. Let

$$y(z) = \sqrt{z}, \quad p(z) = \frac{2}{z}, \quad r(z) = \frac{3}{2z^3}. \tag{2.20}$$

We substitute functions (2.20) into equations (2.1), (2.2). Solving the resulting equations we find

$$q(z) = -\frac{3}{2z^2}, \quad s(z) = -\frac{15}{16z^4}. \tag{2.21}$$

Let

$$r_1(z) = -\frac{3}{z}. \tag{2.22}$$

We substitute the relations (2.20)-(2.22) into (2.19). After simplifications we obtain the system

$$s_1''(z) = s_1^2(z) - \frac{5}{z}s_1'(z) - \frac{15}{2z^2}s_1(z) - \frac{5}{16z^4}, \; 4z^3 s_1'(z) + 16z^2 s_1(z) = 30. \tag{2.23}$$

We find the following solution of system (2.23):

$$s_1(z) = \frac{15}{4z^2}. \tag{2.24}$$

We substitute the relations (2.22), (2.24) into equation (2.18) and integrate it. We write the general solution in the form

$$y_1 = C_1 z^{3/2} + C_2 z^{5/2}, \tag{2.25}$$

where C_1, C_2 are arbitrary constants. We choose, for example, the values of arbitrary constants equal to one. Then we obtain $y_1 = z^{3/2} + z^{5/2}$ and $\xi = \frac{1}{z^2+z}$. Then we find the solution

$$w = \frac{\xi'''}{\xi'} - \frac{3}{2}\left(\frac{\xi''}{\xi'}\right)^2 = -\frac{6}{(2z+1)^2}. \tag{2.26}$$

Differential equation (2.4), (2.5) for coefficients (2.20), (2.21) has the form

$$\begin{gathered} 20(8w^3 + 12ww'' - 15w'^2)w'''' - 280ww'''^2 - 280w'(4w^2 - 3w'')w''' - 504w''^3 \\ + 192w^2w''^2 + 8(56w^4 + 255ww'^2)w'' - 1275w'^4 - 560w^3w'^2 + 64w^6 = 0. \end{gathered} \tag{2.27}$$

According to Theorem 4 equation (2.27) has the solution (2.26) that can be easily verified by the direct substitution.

If we choose the function (2.25), then the corresponding solution of the equation (2.27) takes the form

$$w = -\frac{6{C_2}^2}{(2C_2 z + C_1)^2}.$$

It is a one-parameter family of solutions. This is easily seen by introducing the substitution $c = C_2/C_1$.

Theorem 5. *Let y be a general solution of the second order linear differential equation (2.18), (2.19). Then this solution is a two parameter family of solutions of the fourth order linear differential equation (2.1), (2.2).*

Proof. The proof is computational. □

3. Conclusions

One research direction is to replace linear differential equations with nonlinear equations of second and higher order and to consider the Schwarzian derivative of the ratio of 2 solutions. This might give a new insight into the theory of some nonlinear special functions.

Taking into account the obtained results for the known solutions of the fourth and the second order linear equations, we can formulate the corresponding theorems for the known solutions of the fourth order linear equation and the Riccati equation, to which the second order linear equation reduces. Here it seems appropriate to use the results of [9] and the method of V. Orlov [13,14] for the study of the Riccati equation and nonlinear differential equations of the second order.

From the point of view of programming algorithms for solving the considered problems, the opportunities of Wolfram Research technologies described in [15] are essential. They significantly complement the set of tools for creating, maintaining and distributing dynamic content when constructing and studying solutions of differential equations.

References

[1] L. Ahlfors, *Mobius transformations in several dimensions.* Minneapolis: Lecture notes at the University of Minnesota, 1981.

[2] A. Chichurin, "The computer method of construction of the general solution of the nonlinear differential equation of the fourth order." *Studia i Materialy EWSIE*, vol. 7, pp. 39–47, 2014.

[3] A. Chichurin, *Recent Developments in Mathematics and Informatics, Contemporary Mathematics and Computer Science*. Lublin: KUL, 2016, vol. 1, ch. 2, pp. 19–36.

[4] A. Chichurin and G. Stepaniuk, " The computer method of construction of the general solution of the linear differential equation of the third order." *Studia i Materialy EWSIE*, vol. 8, pp. 17–27, 2014.

[5] A. Chichurin and G. Stepaniuk, "General solution to the fourth order linear differential equation with coefficients satisfying the system of three the first order differential equations." *Bulletin of Taras Shevchenko National University of Kyiv, Series: Physics and Mathematics.*, no. 31, pp. 7–10, 2014.

[6] A. V. Chichurin, *The Chazy equation and linear equations of the Fuchs class.* Moscow: RUDN, 2003, (in Russian).
[7] A. Chichurin and G. Filipuk, "The properties of certain linear and nonlinear differential equations of the fourth order." *preprint*, 2019.
[8] V. Dobrovolsky, *Essays on the development of the analytic theory of differential equations.* Kiev: Vishcha shkola, 1974, (in Russian).
[9] G. Filipuk and A. Chichurin, *Advances in Mechanics and Mathematics*. Cham: Springer, 2019, vol. 41, ch. The Properties of Certain Linear and Nonlinear Differential Equations, pp. 193–200, doi: 10.1007/978-3-030-02487-1-11.
[10] N. Lukashevich, "On third-order linear equations." *Differ. Equ.*, vol. 35, no. 10, pp. 1384–1390, 1999.
[11] N. Lukashevich and A. Chichurin, *Differential equations of the first order.* Minsk: BSU, 1999, (in Russian).
[12] N. Lukashevich and I. Martynov, *Proc. of the International Scientific Conference "Differential Equations and Their Applications"*. Grodno: GrSU, 1998, pp. 78–85.
[13] V. Orlov and O. Kovalchuk, "Research of one class of nonlinear differential equations of third order for mathematical modeling the complex structures." *IOP Conf. Series: Materials Science and Engineering, FORM 2018*, vol. 365, no. 042045, pp. 1–8, 2018, doi: 10.1088 / 1757-899X / 365/4/042045.
[14] V. N. Orlov, *Approximate method for solving scalar and matrix Riccati differential equations.* Cheboksary: Chuvash State Pedagogical University, 2015, (in Russian).
[15] V. Taranchuk and M. Zhuravkov, " Development of interactive teaching materials for computer mechanics." *Vest. BSU. Ser. 1, Fiz. Mat. Inform.*, no. 3, pp. 97–107, 2016.

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OPTIMIZATION THROUGH BEST PROXIMITY POINTS FOR MULTIVALUED F-CONTRACTIONS

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Abstract. Best proximity point theorems ensure the existence of an approximate optimal solution to the equations of the type $f(x) = x$ when f is not a self-map and a solution of the same does not necessarily exist. Best proximity points theorems, therefore, serve as a powerful tool in the theory of optimization and approximation. The aim of this article is to consider a global optimization problem in the context of best proximity points in a complete metric space. We establish an existence of best proximity result for multivalued mappings using Wardowski's technique.

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1. INTRODUCTION AND PRELIMINARIES

Nadler [9] defined a Hausdorff concept by considering the distance between two arbitrary sets as follows.

Let (Ω, η) be a complete metric space (in short, MS) and let $CB(\Omega)$ be the family of all nonempty closed and bounded subsets of the nonempty set Ω. For $\mathcal{M}, \mathcal{N} \in CB(\Omega)$, define the map $\mathcal{H} : CB(\Omega) \times CB(\Omega) \to [0, \infty)$ by

$$\mathcal{H}(\mathcal{M}, \mathcal{N}) = \max\{\sup_{\xi \in \mathcal{N}} \Delta(\xi, \mathcal{M}), \sup_{\delta \in \mathcal{M}} \Delta(\delta, \mathcal{N})\},$$

where $\Delta(\delta, \mathcal{N}) = \inf_{\xi \in \mathcal{N}} \eta(\delta, \xi)$. Then $(CB(\Omega), \mathcal{H})$ is an MS induced by η.

Let $\mathcal{M}, \mathcal{N}$ be any two nonempty subsets of the MS (Ω, η). The following notations will be used throughout:

$$\mathcal{M}_0 = \{\mu \in \mathcal{M} : \eta(\mu, \nu) = \eta(\mathcal{M}, \mathcal{N}) \text{ for some } \nu \in \mathcal{N}\},$$
$$\mathcal{N}_0 = \{\nu \in \mathcal{N} : \eta(\mu, \nu) = \eta(\mathcal{M}, \mathcal{N}) \text{ for some } \mu \in \mathcal{M}\},$$

where $\eta(\mathcal{M}, \mathcal{N}) = \inf\{\eta(\mu, \nu) : \mu \in \mathcal{M}, \nu \in \mathcal{N}\}$.

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For $\mathcal{M},\mathcal{N} \in CB(\Omega)$, we have

$$\eta(\mathcal{M},\mathcal{N}) \leq \mathcal{H}(\mathcal{M},\mathcal{N}).$$

We say that $\mu \in \mathcal{M}$ is a best proximity point (in short, BPP) of the multivalued map $\Gamma : \mathcal{M} \to CB(\mathcal{N})$ if $\Delta(\mu,\Gamma\mu) = \eta(\mathcal{M},\mathcal{N})$. $\upsilon \in \Omega$ is said to be a fixed point of the multivalued map $\Gamma : \Omega \to CB(\Omega)$ if $\upsilon \in \Gamma\upsilon$.

Remark 1.

(1) In the MS $(CB(\Omega),\mathcal{H})$, $\upsilon \in \Omega$ is a fixed point of Γ if and only if $\Delta(\upsilon,\Gamma\upsilon) = 0$.
(2) If $\eta(\mathcal{M},\mathcal{N}) = 0$, then a fixed point and a BPP are identical.
(3) The metric function $\eta : \Omega \times \Omega \to [0,\infty)$ is continuous in the sense that if $\{\upsilon_n\},\{\xi_n\}$ are two sequences in Ω with $(\upsilon_n,\xi_n) \to (\upsilon,\xi)$ for some $\upsilon,\xi \in \Omega$, as $n \to \infty$, then $\eta(\upsilon_n,\xi_n) \to \eta(\upsilon,\xi)$ as $n \to \infty$. The function Δ is continuous in the sense that if $\upsilon_n \to \upsilon$ as $n \to \infty$, then $\Delta(\upsilon_n,\mathcal{M}) \to \Delta(\upsilon,\mathcal{M})$ as $n \to \infty$ for any $\mathcal{M} \subseteq \Omega$.

The following Lemmas are noteworthy.

Lemma 1 ([2,4])**.** *Let (Ω,η) be an MS and $\mathcal{M},\mathcal{N} \in CB(\Omega)$. Then*

(1) *$\Delta(\mu,\mathcal{N}) \leq \eta(\mu,\gamma)$ for any $\gamma \in \mathcal{N}$ and $\mu \in \Omega$;*
(2) *$\Delta(\mu,\mathcal{N}) \leq \mathcal{H}(\mathcal{M},\mathcal{N})$ for any $\mu \in \mathcal{M}$.*

Lemma 2 ([9])**.** *Let $\mathcal{M},\mathcal{N} \in CB(\Omega)$ and let $\upsilon \in \mathcal{M}$, then for any $r > 0$, there exists $\xi \in \mathcal{N}$ such that*

$$\eta(\upsilon,\xi) \leq \mathcal{H}(\mathcal{M},\mathcal{N}) + r.$$

But we may not have any $\xi \in \mathcal{N}$ such that

$$\eta(\upsilon,\xi) \leq \mathcal{H}(\mathcal{M},\mathcal{N}).$$

Further, when $\mathcal{N}$ is compact, there exists $\xi \in \Omega$ such that $\eta(\upsilon,\xi) \leq \mathcal{H}(\mathcal{M},\mathcal{N})$.

The concept of $\mathcal{H}$-continuity for multivalued maps is listed next.

Definition 1 ([5])**.** Let (Ω,η) be an MS. We say that a multivalued map $\Gamma : \Omega \to CB(\Omega)$ is $\mathcal{H}$-continuous at a point μ_0, if for each sequence $\{\mu_n\} \subset \Omega$, such that $\lim_{n\to\infty} \eta(\mu_n,\mu_0) = 0$, we have $\lim_{n\to\infty} \mathcal{H}(\Gamma\mu_n,\Gamma\mu_0) = 0$ (i.e., if $\mu_n \to \mu_0$, then $\Gamma\mu_n \to \Gamma\mu_0$ as $n \to \infty$).

Definition 2 ([9])**.** Let $\Gamma : \Omega \to CB(\Omega)$ be a multivalued map. We say that Γ is a multivalued contraction if $\mathcal{H}(\Gamma\mu,\Gamma\nu) \leq \lambda\eta(\mu,\nu)$ for all $\mu,\nu \in \Omega$, where $\lambda \in [0,1)$.

Remark 2.

(1) If Γ is $\mathcal{H}$-continuous on every point of $\mathcal{M} \subseteq \Omega$, then it is said to be continuous on $\mathcal{M}$.
(2) A multivalued contraction Γ is $\mathcal{H}$-continuous.

In 2012, Wardowski [16] defined the concept of F-contraction as follows.

Definition 3. Let $F:(0,+\infty)\to(-\infty,+\infty)$ be a function which satisfies the following:

(F1) F is strictly increasing;

(F2) For each sequence $\{u_n\}_{n\in\mathbb{N}}\subset(0,+\infty)$,
$$\lim_{n\to+\infty} u_n = 0 \text{ if and only if } \lim_{n\to+\infty} F(u_n) = -\infty;$$

(F3) There is $t\in(0,1)$ such that $\lim_{u\to 0^+} u^t F(u) = 0$.

Let $\mathcal{F}$ denote the class of all such functions F. If (Ω,η) is an MS, then a self-map $T:\Omega\to\Omega$ is said to be an $F-$contraction if there exist $\tau>0$, $F\in\mathcal{F}$, such that for all $\mu,\nu\in\Omega$,
$$\eta(T\mu,T\nu)>0\Rightarrow \tau+F(\eta(T\mu,T\nu))\le F(\eta(\mu,\nu)).$$

Multivalued F-contractions were defined by Altun et al. [1] as follows.

Definition 4 ([1])**.** Let (Ω,η) be an MS. A multivalued map $\Gamma:\Omega\to CB(\Omega)$ is said to be a multivalued F-contraction (MVFC, in short) if there exist $\tau>0$ and $F\in\mathcal{F}$ such that
$$\tau+F(\mathcal{H}(\Gamma\mu,\Gamma\nu))\le F(\eta(\mu,\nu)) \tag{1.1}$$
for all $\mu,\nu\in\Omega$ with $\Gamma\mu\neq\Gamma\nu$.

Remark 3. An MVFC is $\mathcal{H}$-continuous.

We can find the concept of P-property in [12], whereas the notion of weak P property was defined by Zhang et al. [18].

Definition 5 ([12])**.** Let (Ω,η) be an MS and $\mathcal{M},\mathcal{N}$ be two non-empty subsets of Ω such that $\mathcal{M}_0\neq\phi$. The pair $(\mathcal{M},\mathcal{N})$ is said to have the P-property if and only if $\eta(\mu_1,\nu_1)=\eta(\mathcal{M},\mathcal{N})=\eta(\mu_2,\nu_2)$ implies $\eta(\mu_1,\mu_2)=\eta(\nu_1,\nu_2)$, where $\mu_1,\mu_2\in\mathcal{M}_0$ and $\nu_1,\nu_2\in\mathcal{N}$.

Definition 6 ([18])**.** Let (Ω,η) be an MS and $\mathcal{M},\mathcal{N}$ be two non-empty subsets of Ω such that $\mathcal{M}_0\neq\phi$. The pair $(\mathcal{M},\mathcal{N})$ is said to have the weak P-property if and only if $\eta(\mu_1,\nu_1)=\eta(\mathcal{M},\mathcal{N})=\eta(\mu_2,\nu_2)$ implies $\eta(\mu_1,\mu_2)\le\eta(\nu_1,\nu_2)$, where $\mu_1,\mu_2\in\mathcal{M}_0$ and $\nu_1,\nu_2\in\mathcal{N}_0$.

BPP theorems for F-contractive non-self mappings were studied by Omidvari et al. [11] with the help of P-property. Later, Nazari [10] investigated BPPs for a particular type of generalized multivalued contractions by using the weak P-property.

Srivastava et al. [13, 14] presented Krasnosel'skii type hybrid fixed point theorems and found their very interesting applications to fractional integral equations. Xu et al. [17] proved Schwarz lemma that involves boundary fixed point. Very recently, Debnath and Srivastava [6] investigated common BPPs for multivalued contractive pairs of mappings in connection with global optimization. Debnath and Srivastava [7] also proved new extensions of Kannan's and Reich's theorems in the context of multivalued mappings using Wardowski's technique. Further, a very significant application of fixed points of $F(\psi,\varphi)$-contractions to fractional differential equations was recently provided by Srivastava et al. [15].

In this paper, we introduce a best proximity result for multivalued mappings with the help of F-contraction and the weak P property. Also we provide an example where the P-property is not satisfied but the weak P-property holds.

2. Best proximity point for MVFC

In this section, with the help of the notion of F-contraction, we show that an MVFC satisfying certain conditions admits a BPP.

Theorem 1. *Let (Ω,η) be a complete MS and $\mathcal{M},\mathcal{N}$ be two non-empty closed subsets of Ω such that $\mathcal{M}_0 \neq \phi$ and that the pair $(\mathcal{M},\mathcal{N})$ has the weak P-property. Suppose $\Gamma : \mathcal{M} \to CB(\mathcal{N})$ be a MVFC such that $\Gamma\mu$ is compact for each $\mu \in \mathcal{M}$ and $\Gamma\mu \subseteq \mathcal{N}_0$ for all $\mu \in \mathcal{M}_0$. Then Γ has a BPP.*

Proof. Fix $\mu_0 \in \mathcal{M}_0$ and choose $\nu_0 \in \Gamma\mu_0 \subseteq \mathcal{N}_0$. By the definition of $\mathcal{N}_0$, we can select $\mu_1 \in \mathcal{M}_0$ such that

$$\eta(\mu_1,\nu_0) = \eta(\mathcal{M},\mathcal{N}). \tag{2.1}$$

If $\nu_0 \in \Gamma\mu_1$, then

$$\eta(\mathcal{M},\mathcal{N}) \leq \Delta(\mu_1,\Gamma\mu_1) \leq \eta(\mu_1,\nu_0) = \eta(\mathcal{M},\mathcal{N}).$$

Thus $\eta(\mathcal{M},\mathcal{N}) = \Delta(\mu_1,\Gamma\mu_1)$, i.e., μ_1 is a BPP of Γ. Therefore, assume that $\nu_0 \notin \Gamma\mu_1$. Since $\Gamma\mu_1$ is compact, by Lemma 2, there exists $\nu_1 \in \Gamma\mu_1$ such that

$$0 < \eta(\nu_0,\nu_1) \leq \mathcal{H}(\Gamma\mu_0,\Gamma\mu_1).$$

Since F is strictly increasing, the last inequality implies that

$$\begin{aligned} F(\eta(\nu_0,\nu_1)) &\leq F(\mathcal{H}(\Gamma\mu_0,\Gamma\mu_1)) \\ &\leq F(\eta(\mu_0,\mu_1)) - \tau. \end{aligned} \tag{2.2}$$

Since $\nu_1 \in \Gamma\mu_1 \subseteq \mathcal{N}_0$, there exists $\mu_2 \in \mathcal{M}_0$ such that

$$\eta(\mu_2,\nu_1) = \eta(\mathcal{M},\mathcal{N}). \tag{2.3}$$

From (2.1) and (2.3) and using weak $P-$property , we have that

$$\eta(\mu_1,\mu_2) \leq \eta(\nu_0,\nu_1). \tag{2.4}$$

From (2.2) and (2.4), we have

$$F(\eta(\mu_1,\mu_2)) \leq F(\eta(\nu_0,\nu_1)) \leq F(\eta(\mu_0,\mu_1)) - \tau. \tag{2.5}$$

If $\nu_1 \in \Gamma\mu_2$, then

$$\eta(\mathcal{M},\mathcal{N}) \leq \Delta(\mu_2,\Gamma\mu_2) \leq \eta(\mu_2,\nu_1) = \eta(\mathcal{M},\mathcal{N}).$$

Thus $\eta(\mathcal{M},\mathcal{N}) = \Delta(\mu_2,\Gamma\mu_2)$, i.e., μ_1 is a BPP of Γ. So, assume that $\nu_1 \notin \Gamma\mu_2$.

Since $\Gamma\mu_2$ is compact, by Lemma 2, there exists $\nu_2 \in \Gamma\mu_2$ such that

$$0 < \eta(\nu_1, \nu_2) \leq \mathcal{H}(\Gamma\mu_1, \Gamma\mu_2).$$

Using the fact that F is strictly increasing, we have that

$$\begin{aligned} F(\eta(\nu_1,\nu_2)) &\leq F(\mathcal{H}(\Gamma\mu_1,\Gamma\mu_2)) \\ &\leq F(\eta(\mu_1,\mu_2)) - \tau \\ &\leq F(\eta(\mu_0,\mu_1)) - 2\tau \text{ (using 2.5)}. \end{aligned}$$

Since $\nu_2 \in \Gamma\mu_2 \subseteq \mathcal{N}_0$, there exists $\mu_3 \in \mathcal{M}_0$ such that

$$\eta(\mu_3, \nu_2) = \eta(\mathcal{M}, \mathcal{N}). \tag{2.6}$$

From (2.5) and (2.6) and using weak property P, we have that

$$\eta(\mu_2, \mu_3) \leq \eta(\nu_1, \nu_2). \tag{2.7}$$

From (2.6) and (2.7), we have

$$F(\eta(\mu_2,\mu_3)) \leq F(\eta(\nu_1,\nu_2)) \leq F(\eta(\mu_0,\mu_1)) - 2\tau. \tag{2.8}$$

Continuing in this manner, we obtain two sequences $\{\mu_n\}$ and $\{\nu_n\}$ in $\mathcal{M}_0$ and $\mathcal{N}_0$ respectively, satisfying

(B1) $\nu_n \in \Gamma\mu_n \subseteq \mathcal{N}_0$,
(B2) $\eta(\mu_{n+1}, \nu_n) = \eta(\mathcal{M}, \mathcal{N})$,
(B3) $F(\eta(\mu_n,\mu_{n+1})) \leq F(\eta(\nu_{n-1},\nu_n)) \leq F(\eta(\mu_0,\mu_1)) - n\tau$,

for each $n = 0, 1, 2, \ldots$.

Put $\alpha_n = \eta(\mu_n, \mu_{n+1})$ for each $n = 0, 1, 2, \ldots$. Taking limit on both sides of **(B3)** as $n \to \infty$, we have

$$\lim_{n\to\infty} F(\alpha_n) = -\infty.$$

Using **(F2)**, we obtain

$$\lim_{n\to\infty} \alpha_n = 0. \tag{2.9}$$

Using **(F3)**, there exists $k \in (0,1)$ such that

$$\alpha_n^k F(\alpha_n) \to 0 \text{ as } n \to \infty. \tag{2.10}$$

From **(B3)**, for each $n \in \mathbb{N}$, we have that

$$F(\alpha_n) - F(\alpha_0) \leq -n\tau.$$

This implies

$$\alpha_n^k F(\alpha_n) - \alpha_n^k F(\alpha_0) \leq -n\alpha_n^k \tau \leq 0. \tag{2.11}$$

Letting $n \to \infty$ in (2.11) and using (2.9), (2.10), we obtain

$$\lim_{n\to\infty} n\alpha_n^k = 0.$$

Thus there exists $n_0 \in \mathbb{N}$ such that $n\alpha_n^k \leq 1$ for all $n \geq n_0$, i.e., $\alpha_n \leq \frac{1}{n^{\frac{1}{k}}}$ for all $n \geq n_0$.

Let $m,n \in \mathbb{N}$ with $m > n \geq n_0$. Then

$$\begin{aligned}\eta(\mu_m,\mu_n) &\leq \sum_{i=n}^{m-1}\eta(\mu_i,\mu_{i+1}) = \sum_{i=n}^{m-1}\alpha_i\\ &\leq \sum_{i=n}^{\infty}\alpha_i \leq \sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}.\end{aligned}$$

Since the series $\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}$ is convergent for $k \in (0,1)$, we have $\eta(\mu_m,\mu_n) \to 0$ as $m,n \to \infty$. Hence $\{\mu_n\}$ is Cauchy in $\mathcal{M}_0 \subseteq \mathcal{M}$. Since (Ω,η) is complete and $\mathcal{M}$ is closed, we have $\lim_{n\to\infty}\mu_n = \theta$ for some $\theta \in \mathcal{M}$.
Since Γ is $\mathcal{H}$-continuous (for it is an MVFC), we have

$$\lim_{n\to\infty}\mathcal{H}(\Gamma\mu_n,\Gamma\theta) = 0. \tag{2.12}$$

Exactly in the similar manner as above, using **(B3)**, we can prove that $\{\nu_n\}$ is Cauchy in $\mathcal{N}$ and since $\mathcal{N}$ is closed, there exists $\xi \in B$ such that $\lim_{n\to\infty}\nu_n = \xi$.
Since $\eta(\mu_{n+1},\nu_n) = \eta(\mathcal{M},\mathcal{N})$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty}\eta(\mu_{n+1},\nu_n) = \eta(\theta,\xi) = \eta(\mathcal{M},\mathcal{N}).$$

We claim that $\xi \in \Gamma\theta$. Indeed, since $\nu_n \in \Gamma\mu_n$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty}\Delta(\nu_n,\Gamma\theta) \leq \lim_{n\to\infty}\mathcal{H}(\Gamma\mu_n,\Gamma\theta) = 0.$$

Therefore, $\Delta(\xi,\Gamma\theta) = 0$. Since $\Gamma\theta$ is closed, we have $\xi \in \Gamma\theta$.
Now,

$$\eta(\mathcal{M},\mathcal{N}) \leq \Delta(\theta,\Gamma\theta) \leq \eta(\theta,\xi) = \eta(\mathcal{M},\mathcal{N}).$$

Hence $\Delta(\theta,\Gamma\theta) = \eta(\mathcal{M},\mathcal{N})$, i.e., θ is a BPP of Γ. □

A Geraghty type [8] result follows as a consequence of our previous theorem. Let $\mathcal{G}$ be the class of functions $g : [0,\infty) \to [0,1)$ satisfying the condition: $g(\xi_n) \to 1$ implies $\xi_n \to 0$. An example of such a map is $g(\xi) = (1+\xi)^{-1}$ for all $\xi > 0$ and $g(0) \in [0,1)$.

Definition 7. Let $\mathcal{M},\mathcal{N}$ be two non-empty subsets of a MS (Ω,η). A multivalued map $\Gamma : \mathcal{M} \to CB(\mathcal{N})$ is said to be a multivalued Geraghty-type F-contraction (MVGFC, in short) if there exist $\tau > 0$, $F \in \mathcal{F}$ and $g \in \mathcal{G}$ such that

$$\tau + F(\mathcal{H}(\Gamma\mu,\Gamma\nu)) \leq g(\eta(\mu,\nu)) \cdot F(\eta(\mu,\nu)) \tag{2.13}$$

for all $\mu,\nu \in \Omega$ with $\Gamma\mu \neq \Gamma\nu$.

Corollary 1. *Let (Ω,η) be a complete MS and $\mathcal{M},\mathcal{N}$ be two non-empty closed subsets of Ω such that $\mathcal{M}_0 \neq \phi$ and that the pair $(\mathcal{M},\mathcal{N})$ satisfies the weak P-property. Suppose $\Gamma : \mathcal{M} \to CB(\mathcal{N})$ be a MVGFC such that $\Gamma\mu$ is compact for each $\mu \in \mathcal{M}$ and $\Gamma\mu \subseteq \mathcal{N}_0$ for all $\mu \in \mathcal{M}_0$. Then Γ has a BPP.*

Proof. Since $g(t) \in [0,1)$ for all $t \in [0,\infty)$, from (2.13), we have that

$$\tau + F(\mathcal{H}(\Gamma\mu,\Gamma\nu)) \leq F(\eta(\mu,\nu)) \tag{2.14}$$

for all $\mu,\nu \in \mathcal{M}$ with $\Gamma\mu \neq \Gamma\nu$. Thus, Γ is an MVFC and hence from Theorem 1 it follows that Γ has a BPP. $\square$

Remark 4. Corollary 1 extends the results due to Caballero et al. [3] and Zhang et al. [18] to their multivalued analogues using F-contraction.

Next, we provide some examples in support of our main result.

Example 1. Consider $\Omega = \mathbb{R}$ with usual metric $\eta(\mu,\nu) = |\mu - \nu|$ for all $\mu,\nu \in \Omega$. Let $\mathcal{M} = [5,6]$ and $\mathcal{N} = [-6,-5]$. Then $\eta(\mathcal{M},\mathcal{N}) = 10$ and $\mathcal{M}_0 = \{5\}$, $\mathcal{N}_0 = \{-5\}$. Define the multivalued map $\Gamma : \mathcal{M} \to CB(\mathcal{N})$ such that

$$\Gamma\mu = [\frac{-\mu-5}{2}, -5] \text{ for all } \mu \in [5,6].$$

Therefore $\Gamma(5) = \{-5\}$ (i.e., $\Gamma\mu \subseteq \mathcal{N}_0$ for all $\mu \in \mathcal{M}_0$).
We claim that Γ is a MVFC. Let $\mathcal{H}(\Gamma\mu,\Gamma\nu) > 0$. Then we have

$$\begin{aligned}
\mathcal{H}(\Gamma\mu,\Gamma\nu) &= \mathcal{H}([\frac{-\mu-5}{2},-5],[\frac{-\nu-5}{2},-5]) \\
&= |(\frac{-\mu-5}{2}) - (\frac{-\nu-5}{2})| \\
&= \frac{|\nu-\mu|}{2} \\
&= \frac{\eta(\mu,\nu)}{2} \\
&< \eta(\mu,\nu).
\end{aligned}$$

From the last inequality, we have that $\ln(\mathcal{H}(\Gamma\mu,\Gamma\nu)) < \ln(\eta(\mu,\nu))$, and further, $\tau + \ln(\mathcal{H}(\Gamma\mu,\Gamma\nu)) \leq \ln(\eta(\mu,\nu))$, for any $\tau \in (0,\ln 2]$. Therefore, we have that $\tau + F(\mathcal{H}(\Gamma\mu,\Gamma\nu)) \leq F(\eta(\mu,\nu))$, for any $\tau \in (0,\ln 2]$, where $F(t) = \ln t, t > 0$.

Finally, it is easy to check that $(\mathcal{M},\mathcal{N})$ satisfies weak P-property. Thus, all conditions of Theorem 1 are satisfied and we observe that $\mu = 5$ is a BPP of Γ.

In fact, in Example 1, the pair $(\mathcal{M},\mathcal{N})$ satisfies P-property (and hence the weak P-property as well). Next, we present an example in which the pair $(\mathcal{M},\mathcal{N})$ satisfies only the weak P-property but not the P-property.

Example 2. Consider $\Omega = \mathbb{R}^2$ with the Euclidean metric η.
Let $\mathcal{M} = \{(-5,0),(0,1),(5,0)\}$ and $\mathcal{N} = \{(\mu,\nu) : \nu = 2 + \sqrt{2-\mu^2}, \mu \in [-\sqrt{2},\sqrt{2}]\}$.
Then $\eta(\mathcal{M},\mathcal{N}) = \sqrt{3}$ and $\mathcal{M}_0 = \{(0,1)\}$, $\mathcal{N}_0 = \{(\sqrt{2},2),(-\sqrt{2},2)\}$.
Define the multivalued map $\Gamma : \mathcal{M} \to CB(\mathcal{N})$ such that

$$\Gamma(-5,0) = \{(-\sqrt{2},2),(-1,3)\},\ \Gamma(0,1) = \{(\sqrt{2},2)\},\ \Gamma(5,0) = \{(\sqrt{2},2),(1,3)\}.$$

It is easy to check that Γ is a MVFC with $\tau = \ln 2$ and $F(t) = \ln t, t > 0$.

Finally, we observe that

$$\eta((0,1),(\sqrt{2},2)) = \eta((0,1),(-\sqrt{2},2)) = \sqrt{3} = \eta(\mathcal{M},\mathcal{N}),$$

but

$$\eta((0,1),(0,1)) = 0 < \eta((\sqrt{2},2),(-\sqrt{2},2)) = 2\sqrt{2}.$$

Thus, $(\mathcal{M},\mathcal{N})$ satisfies weak P-property, but not the P-property. Therefore, all conditions of Theorem 1 are satisfied and since $\Delta((0,1),\Gamma(0,1)) = \sqrt{3} = \eta(\mathcal{M},\mathcal{N})$, we conclude that $(0,1)$ is a BPP of Γ.

3. CONCLUSION

We have proved our main result with a strong condition that images of the MVFC are compact sets. Relaxation of this compactness criterion is a suggested future work. We have shown the non-triviality of the assumption of the weak P-property by presenting an example which does not satisfy the P-property but satisfies only the weak P-property. The results due to Caballero et al. [3] and Zhang et al. [18] are also extended to their multivalued analogues as a consequence of our results.

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REFERENCES

[1] I. Altun, G. Minak, and H. Dag, "Multivalued F-contractions on complete metric spaces," *J. Nonlinear Convex Anal.*, vol. 16, no. 4, pp. 659–666, 2015.

[2] M. Boriceanu, A. Petrusel, and I. Rus, "Fixed point theorems for some multivalued generalized contractions in b-metric spaces," *Internat. J. Math. Statistics*, vol. 6, pp. 65–76, 2010.

[3] J. Caballero, J. Harjani, and K. Sadarangani, "A best proximity point theorem for Geraghty-contractions," *Fixed Point Theory Appl.*, vol. 2012, no. 231, pp. 1–9, 2012, doi: 10.1186/1687-1812-2012-231.

[4] S. Czerwik, "Nonlinear set-valued contraction mappings in b-metric spaces," *Atti Sem. Mat. Univ. Modena*, vol. 46, pp. 263–276, 1998.

[5] P. Debnath and M. de La Sen, "Fixed points of eventually Δ-restrictive and $\Delta(\varepsilon)$-restrictive set-valued maps in metric spaces," *Symmetry*, vol. 12, no. 1, pp. 1–7, 2020, doi: 10.3390/sym12010127.

[6] P. Debnath and H. M. Srivastava, "Global optimization and common best proximity points for some multivalued contractive pairs of mappings," *Axioms*, vol. 9, no. 3, pp. 1–8, 2020, doi: 10.3390/axioms9030102.

[7] P. Debnath and H. M. Srivastava, "New extensions of Kannan's and Reich's fixed point theorems for multivalued maps using Wardowski's technique with application to integral equations," *Symmetry*, vol. 12, no. 7, pp. 1–9, 2020, doi: 10.3390/sym12071090.

[8] M. A. Geraghty, "On contractive mappings," *Proc. Amer. Math. Soc.*, vol. 40, pp. 604–608, 1973.

[9] S. B. Nadler, "Multi-valued contraction mappings," *Pac. J. Math.*, vol. 30, no. 2, pp. 475–488, 1969, doi: 10.2140/pjm.1969.30.475.

[10] E. Nazari, "Best proximity point theorems for generalized multivalued contractions in metric spaces," *Miskolc Math. Notes*, vol. 16, no. 2, pp. 1055–1062, 2015, doi: 10.18514/MMN.2015.1329.

[11] M. Omidvari, S. M. Vaezpour, and R. Saadati, "Best proximity point theorems for F-contractive non-self mappings," *Miskolc Math. Notes*, vol. 15, no. 2, pp. 615–623, 2014, doi: 10.18514/MMN.2014.1011.

[12] V. Sankar Raj, "A best proximity point theorem for weakly contractive non-self-mappings," *Nonlinear Anal.*, vol. 74, no. 14, pp. 4804–4808, 2011, doi: 10.1016/j.na.2011.04.052.

[13] H. M. Srivastava, S. V. Bedre, S. M. Khairnar, and B. S. Desale, "Krasnosel'skii type hybrid fixed point theorems and their applications to fractional integral equations," *Abstr. Appl. Anal.*, vol. 2014, no. Article ID: 710746, pp. 1–9, 2014, doi: 10.1155/2014/710746.

[14] H. M. Srivastava, S. V. Bedre, S. M. Khairnar, and B. S. Desale, "Corrigendum to "Krasnosel'skii type hybrid fixed point theorems and their applications to fractional integral equations"," *Abstr. Appl. Anal.*, vol. 2015, no. Article ID: 467569, pp. 1–2, 2015, doi: 10.1155/2015/467569.

[15] H. M. Srivastava, A. Shehata, and S. I. Moustafa, "Some fixed point theorems for $f(\psi,\varphi)$-contractions and their application to fractional differential equations," *Russian J. Math. Phys.*, vol. 27, pp. 385–398, 2020, doi: 10.1134/S1061920820030103.

[16] D. Wardowski, "Fixed points of a new type of contractive mappings in complete metric space," *Fixed Point Theory Appl.*, vol. 2012, no. 94, pp. 1–6, 2012, doi: 10.1186/1687-1812-2012-94.

[17] Q. Xu, Y. Tang, T. Yang, and H. M. Srivastava, "Schwarz lemma involving the boundary fixed point," *Fixed Point Theory Appl.*, vol. 2016, no. Article ID: 84, pp. 1–8, 2016, doi: 10.1186/s13663-016-0574-8.

[18] J. Zhang, Y. Su, and Q. Cheng, "A note on 'A best proximity point theorem for Geraghty-contractions'," *Fixed Point Theory Appl.*, vol. 2013, no. 99, pp. 1–4, 2013, doi: : 10.1186/1687-1812-2013-83.

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FIXED POINT THEOREMS IN COMPLEX VALUED FUZZY *b*-METRIC SPACES WITH APPLICATION TO INTEGRAL EQUATIONS

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Abstract. In this paper, firstly, we introduce the concept of a complex valued fuzzy b-metric space, which is inspired by the work of Shukla et al. [24]. Also, we investigate some of its topological properties which strengthen this concept. Next, we establish some fixed point theorems in the context of complex valued fuzzy b-metric spaces and give suitable examples to illustrate the usability of the obtained main results. These results extend and generalize the corresponding results given in the existing literature. Moreover, we provide some applications on the existence and uniqueness of solutions for a certain type of nonlinear integral equations.

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1. INTRODUCTION

Fixed point theory plays a fundamental role in mathematics and applied sciences, such as optimization, mathematical models and economic theories. Also, this theory have been applied to show the existence and uniqueness of the solutions of differential equations, integral equations and many other branches of mathematics [6, 18, 19]. A basic result in fixed point theory is the Banach contraction principle. Since the appearance of this principle, there has been a lot of activity in this area.

In 2011, Azam et al. [4] defined the notion of a complex valued metric space which is more general than the well-known metric space and obtained some fixed point results for a pair of mappings satisfying a rational inequality. In this line, Rouzkard et al. [21] studied some common fixed point theorems in this space to generalize the result of [4]. Ahmad et al. [2] investigated some common fixed point results for the mappings satisfying rational expressions on a closed ball in such space. Later, Rao et al. [20] gave a common fixed point theorem in complex valued b-metric spaces, generalizing both the b-metric spaces introduced by Czerwik [5] and the complex valued metric spaces. After the establishment of this new idea, Mukhemier [14] presented common fixed point results of two self-mappings satisfying a rational inequality in complex valued b-metric spaces. Verma [26] studied common fixed point theorems using property (CLCS) in these spaces. In recent years, there has been a considerable literature on fixed point theory in complex valued metric spaces [1, 15, 16, 25].

In 1965, Zadeh [28] introduced the concept of a fuzzy set theory to deal with the unclear or inexplicit situations in daily life. Using this theory, Kramosil and Michalek [12] defined the concept of a fuzzy metric space. Grabiec [8] gave contractive mappings on a fuzzy metric space and extended fixed point theorems of Banach and Edelstein in such space. Successively, George and Veeramani [7] slightly modified the notion of a fuzzy metric space introduced by Kramosil and Michalek [12] and then obtained a Hausdorff topology and a first countable topology on it. In the light of the results given in [7], Sapena [22] gave some examples and properties of fuzzy metric spaces. Also, Shukla et al. [24] extended the concept of fuzzy metric space to complex valued fuzzy metric space and obtained some fixed point results in this space. In recent years, many researchers have improved and generalized fixed point results for various contractive mappings in fuzzy metric spaces [3, 9, 10, 13, 17, 23, 27].

In this paper, we introduce the concept of a complex valued fuzzy b-metric space, generalizing both the notion of a complex valued fuzzy metric space introduced by Shukla et al. [24] and the notion of a b-metric space. Then, we give the topology induced by this space and also study some properties about this topology such as Hausdorffness. Moreover, we present some fixed point theorems for contraction mappings in this more general class of fuzzy metric spaces. Finally, we investigate the applicability of the obtained results to integral equations and show a concrete example which illustrate the application part.

2. Preliminaries

Consistent with Shukla, Rodriguez-Lopez and Abbas [24], the following definitions and results will be needed in what follows.

$\mathbb{C}$ denotes the complex number system over the field of real numbers. We set $P = \{(a,b) : 0 \le a < \infty, 0 \le b < \infty\} \subset \mathbb{C}$. The elements $(0,0), (1,1) \in P$ are denoted by θ and ℓ, respectively.

Define a partial ordering $\preceq$ on $\mathbb{C}$ by $c_1 \preceq c_2$ (or, equivalently, $c_2 \succeq c_1$) if and only if $c_2 - c_1 \in P$. We write $c_1 \prec c_2$ (or, equivalently, $c_2 \succ c_1$) to indicate $Re(c_1) < Re(c_2)$ and $Im(c_1) < Im(c_2)$ (see, also, [4]). The sequence $\{c_n\}$ in $\mathbb{C}$ is said to be monotonic with respect to $\preceq$ if either $c_n \preceq c_{n+1}$ for all $n \in \mathbb{N}$ or $c_{n+1} \preceq c_n$ for all $n \in \mathbb{N}$.

We define the closed unit complex interval by $I = \{(a,b) : 0 \le a \le 1, 0 \le b \le 1\}$, and the open unit complex interval by $I_\theta = \{(a,b) : 0 < a < 1, 0 < b < 1\}$. P_θ denotes the set $\{(a,b) : 0 < a < \infty, 0 < b < \infty\}$. It is obvious that for $c_1, c_2 \in \mathbb{C}$, $c_1 \prec c_2$ if and only if $c_2 - c_1 \in P_\theta$.

For $A \subset \mathbb{C}$, if there exists an element $\inf A \in \mathbb{C}$ such that it is a lower bound of A, that is, $\inf A \preceq a$ for all $a \in A$ and $u \preceq \inf A$ for every lower bound $u \in \mathbb{C}$ of A, then $\inf A$ is called the greatest lower bound or infimum of A. Similarly, we define $\sup A$, the least upper bound or supremum of A, in usual manner.

Remark 1 ([24]). Let $c_n \in P$ for all $n \in \mathbb{N}$. Then,

(i) If the sequence $\{c_n\}$ is monotonic with respect to $\preceq$ and there exists $\alpha, \beta \in P$ such that $\alpha \preceq c_n \preceq \beta$, for all $n \in \mathbb{N}$, then there exists a $c \in P$ such that $\lim_{n\to\infty} c_n = c$.

(ii) Although the partial ordering $\preceq$ is not a linear (total) order on $\mathbb{C}$, the pair $(\mathbb{C}, \preceq)$ is a lattice.

(iii) If $S \subset \mathbb{C}$ is such that there exist $\alpha, \beta \in \mathbb{C}$ with $\alpha \preceq s \preceq \beta$ for all $s \in S$, then $\inf S$ and $\sup S$ both exist.

Remark 2 ([24]). Let $c_n, c_n', z \in P$, for all $n \in \mathbb{N}$. Then,

(i) If $c_n \preceq c_n' \preceq \ell$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} c_n = \ell$, then $\lim_{n\to\infty} c_n' = \ell$.

(ii) If $c_n \preceq z$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} c_n = c \in P$, then $c \preceq z$.

(iii) If $z \preceq c_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} c_n = c \in P$, then $z \preceq c$.

Definition 1 ([24])**.** Let $\{c_n\}$ be a sequence in P. Then, the sequence $\{c_n\}$ is said to diverge to ∞ as $n \to \infty$, and we write $\lim_{n\to\infty} c_n = \infty$, if for all $c \in P$ there exists an $n_0 \in \mathbb{N}$ such that $c \preceq c_n$ for all $n > n_0$.

Definition 2 ([24])**.** Let X be a nonempty set. A complex fuzzy set M is characterized by a mapping with domain X and values in the closed unit complex interval I.

Definition 3 ([24])**.** A binary operation $* : I \times I \to I$ is called a complex valued t-norm if:

(n_1) $c_1 * c_2 = c_2 * c_1$;
(n_2) $c_1 * c_2 \preceq c_3 * c_4$ whenever $c_1 \preceq c_3$, $c_2 \preceq c_4$;
(n_3) $c_1 * (c_2 * c_3) = (c_1 * c_2) * c_3$;
(n_4) $c * \theta = \theta$, $c * \ell = c$

for all $c, c_1, c_2, c_3, c_4 \in I$.

Example 1 ([24]). Let the binary operations $*_1, *_2, *_3 : I \times I \to I$ be defined, respectively, by

(1) $c_1 *_1 c_2 = (a_1a_2, b_1b_2)$, for all $c_1 = (a_1, b_1), c_2 = (a_2, b_2) \in I$;
(2) $c_1 *_2 c_2 = (\min\{a_1, a_2\}, \min\{b_1, b_2\})$, for all $c_1 = (a_1, b_1), c_2 = (a_2, b_2) \in I$;
(3) $c_1 *_3 c_2 = (\max\{a_1 + a_2 - 1, 0\}, \max\{b_1 + b_2 - 1, 0\})$

for all $c_1 = (a_1, b_1)$, $c_2 = (a_2, b_2) \in I$.

Then, $*_1, *_2$ and $*_3$ are complex valued t-norms.

Example 2 ([24]). Define $*_4 : I \times I \to I$ as follows:

$$c_1 *_4 c_2 = \begin{cases} (a_1, b_1), & \text{if } (a_2, b_2) = \ell; \\ (a_2, b_2), & \text{if } (a_1, b_1) = \ell; \\ \theta, & \text{otherwise,} \end{cases}$$

for all $c_1 = (a_1, b_1)$, $c_2 = (a_2, b_2) \in I$. Then, $*_4$ is a complex valued t-norm.

Definition 4 ([24])**.** Let X be a nonempty set, $*$ a continuous complex valued t-norm and M a complex fuzzy set on $X^2 \times P_\theta$ satisfying the following conditions:

(M_1) $\theta \prec M(x,y,c)$;
(M_2) $M(x,y,c) = \ell$ for every $c \in P_\theta$ if and only if $x = y$;
(M_3) $M(x,y,c) = M(y,x,c)$;
(M_4) $M(x,y,c) * M(y,z,c') \preceq M(x,z,c+c')$;
(M_5) $M(x,y,\cdot) : P_\theta \to I$ is continuous

for all $x,y,z \in X$ and $c,c' \in P_\theta$.

Then, the triplet $(X,M,*)$ is called a complex valued fuzzy metric space and M is called a complex valued fuzzy metric on X. A complex valued fuzzy metric can be thought of as the degree of nearness between two points of X with respect to a complex parameter $c \in P_\theta$.

3. On complex valued fuzzy b-metric spaces

In this section, we present the notion of a complex valued fuzzy b-metric space and study some of its topological aspects which strengthen this concept.

Definition 5. Let X be a nonempty set, $s \geq 1$ a given real number, $*$ a continuous complex valued t-norm and M a complex fuzzy set on $X^2 \times P_\theta$ satisfying the following conditions:

(bM_1) $\theta \prec M(x,y,c)$;
(bM_2) $M(x,y,c) = \ell$ for every $c \in P_\theta$ if and only if $x = y$;
(bM_3) $M(x,y,c) = M(y,x,c)$;
(bM_4) $M(x,y,c) * M(y,z,c') \preceq M(x,z,s(c+c'))$;
(bM_5) $M(x,y,\cdot) : P_\theta \to I$ is continuous

for all $x,y,z \in X$ and $c,c' \in P_\theta$.

Then, the quadruple $(X,M,*,s)$ is called a complex valued fuzzy b-metric space and M is called a complex valued fuzzy b-metric on X.

It is seen that the above definition coincides with that of the complex valued fuzzy metric when $s = 1$. Thus, the class of the complex valued fuzzy b-metric spaces is larger than that of the complex valued fuzzy metric spaces, that is, every complex valued fuzzy metric space is a complex valued fuzzy b-metric space.

Now, we shall give the examples of complex valued fuzzy b-metric spaces induced by the b-metric spaces.

Example 3. Let (X,d,s) be a b-metric space. Let us consider a complex fuzzy set $M : X^2 \times P_\theta \to I$ such that

$$M(x,y,c) = \frac{a.b}{ab + d(x,y)}\ell,$$

where $c = (a,b) \in P_\theta$. Then, $(X,M,*_2,s)$ is a complex valued fuzzy b-metric space.

Example 4. Let (X,d,s) be a b-metric space. Define the mapping $M : X^2 \times P_\theta \to I$ by

$$M(x,y,c) = e^{-\frac{d(x,y)}{a+b}}\ell,$$

where $c = (a,b) \in P_\theta$. Then, $(X,M,*_2,2s)$ is a complex valued fuzzy b-metric space.

As shown in the following examples, every complex valued fuzzy b-metric space may not be induced by a b-metric space.

Example 5. Let $X = (3,+\infty)$ and let $M : X^2 \times P_\theta \to I$ be defined by

$$M(x,y,c) = \begin{cases} \ell, & \text{if } x = y; \\ (\frac{1}{x}+\frac{1}{y})\ell, & \text{if } x \neq y. \end{cases}$$

Then, it is easy to see that $(X,M,*_3,s)$ is a complex valued fuzzy b-metric space. Moreover, there is not a b-metric d on X inducing the given complex valued fuzzy b-metric.

Example 6. Let $X = (0,+\infty)$ be endowed with the mapping $M : X^2 \times P_\theta \to I$ given by

$$M(x,y,c) = \begin{cases} (\frac{x}{y})^a\ell, & \text{if } x \leq y; \\ (\frac{y}{x})^a\ell, & \text{if } y \leq x, \end{cases}$$

where $a > 0$. Then, $(X,M,*_1,s)$ is a complex valued fuzzy b-metric space. Also, there is not a b-metric d on X inducing the given complex valued fuzzy b-metric.

Lemma 1. *Let $(X,M,*,s)$ be a complex valued fuzzy b-metric space and $c_1,c_2 \in \mathbb{C}$. If $c_1 \prec c_2$, then $M(x,y,c_1) \preceq M(x,y,sc_2)$ for all $x,y \in X$.*

Proof. Let us take $c_1,c_2 \in P_\theta$ such that $c_1 \prec c_2$. Therefore, $c_2 - c_1 \in P_\theta$ and so we have that for all $x,y \in X$

$$M(x,y,c_1) = \ell * M(x,y,c_1) = M(x,x,c_2-c_1) * M(x,y,c_1) \preceq M(x,y,sc_2).$$

□

Let $(X,M,*,s)$ be a complex valued fuzzy b-metric space. An open ball $B_M(x,r,c)$ with center $x \in X$ and radius $r \in I_\theta$, $c \in P_\theta$ is defined by

$$B_M(x,r,c) = \{y \in X : \ell - r \prec M(x,y,c)\}.$$

Definition 6. Let $(X,M,*,s)$ be a complex valued fuzzy b-metric space. Then, $(X,M,*,s)$ is called a Hausdorff space if for any two distinct points $x,y \in X$, there exist two open balls $B(x,r_1,c_1)$ and $B(y,r_2,c_2)$ such that $B(x,r_1,c_1) \cap B(y,r_2,c_2) = \varnothing$.

Theorem 1. *Every complex valued fuzzy b-metric space is a Hausdorff space.*

Proof. Let $(X,M,*,s)$ be a complex valued fuzzy b-metric space and $x,y \in X$ with $x \neq y$. Then, we have $\theta \prec M(x,y,c) \prec \ell$. Taking $M(x,y,c) = r$, we obtain an $r_1 \in I_\theta$ such that $r \prec r_1 \prec \ell$. Therefore, there exists an $r_2 \in I_\theta$ satisfying $r_2 * r_2 \succ r_1$. It is clear that $x \in B(x,\ell - r_2,\frac{c}{2s})$ and $y \in B(y,\ell - r_2,\frac{c}{2s})$. Also, we verify that $B(x,\ell - r_2,\frac{c}{2s}) \cap$

$B(y, \ell - r_2, \frac{c}{2s}) = \varnothing$. Suppose instead that there is a $z \in B(x, \ell - r_2, \frac{c}{2s}) \cap B(y, \ell - r_2, \frac{c}{2s})$. Hence,

$$r \prec r_1 \prec r_2 * r_2 \prec M(x, z, \frac{c}{2s}) * M(y, z, \frac{c}{2s}) \preceq M(x, y, c) = r$$

and so we get a contradiction. □

Theorem 2. *Let $(X, M, *, s)$ be a complex valued fuzzy b-metric space. Then, the family*

$$\tau_M = \{G \subseteq X : \text{ for all } x \in G, \text{ there exist } r \in I_\theta \text{ and } c \in P_\theta \text{ such that } B_M(x, r, c) \subseteq G\}$$

is a topology on X.

Proof. It is enough to show that if $G_1, G_2 \in \tau_M$, then $G_1 \cap G_2 \in \tau_M$, since the other axioms are readily verified. Let $x \in G_1 \cap G_2$. Then, there exist $r_1 = (a_1, b_1), r_2 = (a_2, b_2) \in I_\theta$ and $c_1 = (m_1, n_1), c_2 = (m_2, n_2) \in P_\theta$ such that $B_M(x, r_1, c_1) \subseteq G_1$ and $B_M(x, r_2, c_2) \subseteq G_2$. Take

$$r = (\min\{a_1, a_2\}, \min\{b_1, b_2\}) \text{ and } c = (\min\{\frac{m_1}{s}, \frac{m_2}{s}\}, \min\{\frac{n_1}{s}, \frac{n_2}{s}\}).$$

It is clear that $r \in I_\theta$ and $c \in P_\theta$. Therefore, by applying Lemma 1, we get $B(x, r, c) \subseteq B(x, r, c_1)$ and $B(x, r, c) \subseteq B(x, r, c_2)$. Thus, we obtain $B(x, r, c) \subseteq G_1 \cap G_2$, completing the proof. □

Then, (X, τ_M) is called the topological space induced by the complex valued fuzzy b-metric space $(X, M, *, s)$.

Example 7. (i) The complex valued fuzzy b-metric space defined in Example 5 induces the discrete topological space on X since for $x \in X$, $B_M(x, r, c) = \{x\}$ whenever $r_1 = r_2 < \frac{1}{3} - \frac{1}{x}$.
(ii) The complex valued fuzzy b-metric space defined in Example 6 induces the usual topological space on $X \subset \mathbb{R}$ because

$$B_M(x, r, c) = \big(\max\{x(1 - r_1)^{\frac{1}{a}}, x(1 - r_2)^{\frac{1}{a}}\}, \min\big\{\frac{x}{(1 - r_1)^{\frac{1}{a}}}, \frac{x}{(1 - r_2)^{\frac{1}{a}}}\big\}\big)$$

for $x \in X$, $r \in I_\theta$ and $c \in P_\theta$.

Proposition 1. *Let $(X, M_1, *, s)$ and $(X, M_2, *, s)$ be two complex valued fuzzy b-metric spaces. Define the mappings $M : X^2 \times P_\theta \to I$ and $N : X^2 \times P_\theta \to I$ by*

$$M(x, y, c) = M_1(x, y, c) * M_2(x, y, c),$$

and

$$N(x, y, c) = \big(\min\{Re(M_1(x, y, c)), Re(M_2(x, y, c))\}, \min\{Im(M_1(x, y, c)), Im(M_2(x, y, c))\}\big).$$

Then, the following results hold:

(i) *$(X, M, *, s)$ is a complex valued fuzzy b-metric space if $p * q \neq \theta$ with $p, q \neq \theta$.*
(ii) *$(X, N, *, s)$ is a complex valued fuzzy b-metric space.*
(iii) *$\tau_M = \tau_N$.*

Proof. (i) and (ii) are obvious.

(iii) Let $G \in \tau_M$. Then, for all $x \in G$, there exist an $r \in I_\theta$ and a $c \in P_\theta$ such that $B_M(x,r,c) \subseteq G$. Now, take an $r' \in I_\theta$ with $(\ell - r') * (\ell - r') \succ \ell - r$. If $z \in B_N(x,r',c)$, then we have

$$\ell - r' \prec \big(\min\{Re(M_1(x,z,c)), Re(M_2(x,z,c))\}, \min\{Im(M_1(x,z,c)), Im(M_2(x,z,c))\}\big).$$

Therefore, from the fact that

$$\ell - r \prec (\ell - r') * (\ell - r') \prec M_1(x,z,c) * M_2(x,z,c) = M(x,z,c)$$

it follows that $z \in B_M(x,r,c)$. Thus, we infer that $G \in \tau_N$.

Conversely, let $G \in \tau_N$. Then, for all $x \in G$, there exist an $r \in I_\theta$ and a $c \in P_\theta$ such that $B_N(x,r,c) \subseteq G$. If $z \in B_M(x,r,c)$, then we have

$$\ell - r \prec M(x,z,c) = M_1(x,z,c) * M_2(x,z,c).$$

Therefore, since $M_1(x,z,c) * M_2(x,z,c) \preceq M_1(x,z,c)$ and $M_1(x,z,c) * M_2(x,z,c) \preceq M_2(x,z,c)$, we get $\ell - r \prec N(x,z,c)$. Thus, $z \in B_N(x,r,c)$ and this implies that $G \in \tau_M$. □

Let (X,d,s) be a b-metric space and τ_d be a topology induced by the b-metric d on X. Then, we shall show that the topology τ_d coincides with the topology τ_M, where $(X,M,*,s)$ is deduced from the b-metric d.

Example 8. Consider Example 3. Then, we have $\tau_M = \tau_d$. Indeed, let $G \in \tau_M$. Then, for all $x \in G$, there exist an $r = (r_1,r_2) \in I_\theta$ and a $c = (a,b) \in P_\theta$ such that $B_M(x,r,c) \subseteq G$. Let us choose a positive number $h = \min\{\frac{abr_1}{1-r_1}, \frac{abr_2}{1-r_2}\}$. Therefore, we obtain $B_d(x,h) \subseteq G$, where $B_d(x,h)$ is an open ball with centre x and radius h for the b-metric d and thus $G \in \tau_d$.

On the other hand, let $G \in \tau_d$. Then, for all $x \in G$, there exists a positive number h such that $B_d(x,h) \subseteq G$. Let us now take an arbitrary $c = (a,b) \in P_\theta$ and an $r = (r_1,r_2) = (\frac{h}{ab+h}, \frac{h}{ab+h}) \in I_\theta$. Hence, we get $B_M(x,r,c) \subseteq G$ and so that $G \in \tau_M$.

Example 9. Let $(X,M,*_2,2s)$ be a complex valued fuzzy b-metric space defined in Example 4. Then, $\tau_M = \tau_d$. Indeed, if $G \in \tau_M$, then, for all $x \in G$, there exist an $r = (r_1,r_2) \in I_\theta$ and a $c = (a,b) \in P_\theta$ such that $B_M(x,r,c) \subseteq G$. Take a positive number $h = \min\{-(a+b)In(1-r_1), -(a+b)In(1-r_2)\}$. Clearly, $B_d(x,h) \subseteq G$ and this shows that $G \in \tau_d$.

For the reverse inclusion, let $G \in \tau_d$. Then, for all $x \in G$, there exists a positive number h such that $B_d(x,h) \subseteq G$. Let us consider an arbitrary $c = (a,b) \in P_\theta$ and an $r = (r_1,r_2) = (1 - e^{\frac{-h}{a+b}}, 1 - e^{\frac{-h}{a+b}}) \in I_\theta$. Thus, it follows from $B_M(x,r,c) \subseteq G$ that $G \in \tau_M$.

Definition 7. Let $(X,M,*,s)$ be a complex valued fuzzy b-metric space.

(i) A sequence $\{x_n\}$ in X converges to $x \in X$ if for every $r \in I_\theta$ and every $c \in P_\theta$, there exists an $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, $\ell - r \prec M(x_n, x, c)$. We denote this by $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence in $(X, M, *, s)$ if for every $c \in P_\theta$, $\lim_{n\to\infty} \inf_{m>n} M(x_n, x_m, c) = \ell$.

(iii) $(X, M, *, s)$ is said to be a complete complex valued fuzzy b-metric space if for every Cauchy sequence $\{x_n\}$ in $(X, M, *, s)$, there exists an $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

The proofs of the following lemmas follow along similar lines as in [24] and are therefore omitted.

Lemma 2. *Let $(X, M, *, s)$ be a complex valued fuzzy b-metric space. A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if* $\lim_{n\to\infty} M(x_n, x, c) = \ell$ *holds for all $c \in P_\theta$.*

Lemma 3. *Let $(X, M, *, s)$ be a complex valued fuzzy b-metric space. A sequence $\{x_n\}$ in X is a Cauchy sequence if and only if for every $r \in I_\theta$ and every $c \in P_\theta$, there exists an $n_0 \in \mathbb{N}$ such that, for all $m, n > n_0$, $\ell - r \prec M(x_n, x_m, c)$.*

Example 10. Let $X = [0,1] \times \{0\} \cup \{0\} \times [0,1]$ and let $d : X \times X \to \mathbb{C}$ be the mapping defined by

$$d((x,0),(y,0)) = (x-y)^2(\alpha, 1)$$
$$d((0,x),(0,y)) = (x-y)^2(1, \beta)$$
$$d((x,0),(0,y)) = d((0,y),(x,0)) = (\alpha x^2 + y^2, x^2 + \beta y^2)$$

where α, β are fixed nonnegative real constants satisfying $\alpha \neq \frac{1}{\beta}$. Then, (X, d, s) is a complete complex valued b-metric space with $s \geq 2$. Moreover, we define

$$M(u,v,c) = \frac{ab}{ab + |d(u,v)|}\ell$$

for all $u, v \in X$, $c = (a,b) \in P_\theta$. Thus, one can check that $(X, M, *_2, s)$ is a complete complex valued fuzzy b-metric space.

It follows from the above example that a complete complex valued fuzzy b-metric space can be induced by a complete complex valued b-metric space.

Definition 8. Let $(X, M, *, s)$ be a complex valued fuzzy b-metric space, $f : X \to X$ be a mapping and $x \in X$. Then, the mapping f is continuous at x if for any sequence $\{x_n\}$ in X, $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} f x_n = f x$.

If f is continuous at each point $x \in X$, then we say that f is continuous on X.

4. Main Results

Firstly, we prove the Banach Contraction Theorem in the setting of complex valued fuzzy b-metric space.

Theorem 3. *Let $(X, M, *, s)$ be a complete complex valued fuzzy b-metric space such that, for every sequence $\{c_n\}$ in P_θ with $\lim_{n\to\infty} c_n = \infty$, we have*

$$\lim_{n\to\infty} \inf_{y\in X} M(x,y,c_n) = \ell$$

for all $x \in X$. Let $f : X \to X$ be a mapping satisfying

$$M(fx, fy, \frac{\lambda c}{s}) \succeq M(x,y,c) \tag{4.1}$$

for all $x, y \in X$ and $c \in P_\theta$, where $\lambda \in (0,1)$. Then, f has a unique fixed point in X.

Proof. We start by an arbitrary $x_0 \in X$ and generate a sequence $\{x_n\}$ in X by the iterative process

$$x_n = fx_{n-1} \text{ for all } n \in \mathbb{N}.$$

If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of f. Consequently, assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. Now, we will prove that $\{x_n\}$ is a Cauchy sequence in X. Define

$$B_n = \{M(x_n, x_m, c) : m > n\}$$

for all $n \in \mathbb{N}$ and $c \in P_\theta$. Due to $\theta \prec M(x_n, x_m, c) \preceq \ell$, for all $m \in \mathbb{N}$ with $m > n$ and from Remark 1(iii), $inf B_n = \beta_n$ exists for all $n \in \mathbb{N}$. Applying Lemma 1 and (4.1), we get

$$M(x_n, x_m, c) \preceq M(x_n, x_m, \frac{sc}{\lambda}) \preceq M(fx_n, fx_m, c) = M(x_{n+1}, x_{m+1}, c), \tag{4.2}$$

for $c \in P_\theta$ and $m, n \in \mathbb{N}$ with $m > n$. So, from the fact that

$$\theta \preceq \beta_n \preceq \beta_{n+1} \preceq \ell \text{ for all } n \in \mathbb{N}$$

it follows that $\{\beta_n\}$ is a monotonic sequence in P. Therefore, utilizing Remark 1(i), we have an $\ell_0 \in P$ satisfying

$$\lim_{n\to\infty} \beta_n = \ell_0. \tag{4.3}$$

Now, by successive application of the contractive condition (4.1), we have

$$\begin{aligned} M(x_{n+1}, x_{m+1}, c) &\succeq M(x_n, x_m, \frac{sc}{\lambda}) = M(fx_{n-1}, fx_{m-1}, \frac{sc}{\lambda}) \\ &\succeq M(x_{n-1}, x_{m-1}, \frac{s^2c}{\lambda^2}) = M(fx_{n-2}, fx_{m-2}, \frac{s^2c}{\lambda^2}) \\ &\succeq M(x_{n-2}, x_{m-2}, \frac{s^3c}{\lambda^3}) \\ &\succeq \cdots \succeq M(x_0, x_{m-n}, \frac{s^{n+1}c}{\lambda^{n+1}}), \end{aligned}$$

for $c \in P_\theta$ and $m, n \in \mathbb{N}$ with $m > n$. Thus,

$$\beta_{n+1} = \inf_{m>n} M(x_{n+1}, x_{m+1}, c) \succeq \inf_{m>n} M(x_0, x_{m-n}, \frac{s^{n+1}c}{\lambda^{n+1}}) \succeq \inf_{y\in X} M(x_0, y, \frac{s^{n+1}c}{\lambda^{n+1}}).$$

Since $\lim_{n\to\infty} \frac{s^{n+1}c}{\lambda^{n+1}} = \infty$, by using the hypothesis along with (4.3), we obtain

$$\ell_0 \succeq \lim_{n\to\infty} \inf_{y\in X} M(x_0, y, \frac{s^{n+1}c}{\lambda^{n+1}}) = \ell,$$

which implies that $\ell_0 = \ell$. Thus, $\{x_n\}$ is a Cauchy sequence in X.
Since $(X, M, *, s)$ is a complete complex valued fuzzy b-metric space, by Lemma 2, there exists a $p \in X$ such that for all $c \in P_\theta$,

$$\lim_{n\to\infty} M(x_n, p, c) = \ell. \tag{4.4}$$

Next, we will show that p is the fixed point of f. Due to (bM_4) and the contractive condition (4.1), we have

$$\begin{aligned} M(p, fp, c) &\succeq M(p, x_{n+1}, \frac{c}{2s}) * M(x_{n+1}, fp, \frac{c}{2s}) \\ &= M(p, x_{n+1}, \frac{c}{2s}) * M(fx_n, fp, \frac{c}{2s}) \\ &\succeq M(p, x_{n+1}, \frac{c}{2s}) * M(x_n, p, \frac{c}{2\lambda}), \end{aligned}$$

for any $c \in P_\theta$. Letting the limit as $n \to \infty$, by (4.4) and Remark 2(ii), we get $M(p, fp, c) = \ell$ for all $c \in P_\theta$, which gives $fp = p$.

To prove the uniqueness of the fixed point p, let q be another fixed point of f, that is, there is a $c \in P_\theta$ with $M(p, q, c) \neq \ell$. From (4.1), we obtain that

$$\begin{aligned} M(p, q, c) = M(fp, fq, c) &\succeq M(p, q, \frac{sc}{\lambda}) = M(fp, fq, \frac{sc}{\lambda}) \\ &\succeq M(p, q, \frac{s^2c}{\lambda^2}) \\ &\vdots \\ &\succeq M(p, q, \frac{s^nc}{\lambda^n}) \\ &\succeq \inf_{y\in X} M(p, y, \frac{s^nc}{\lambda^n}), \end{aligned}$$

for all $n \in \mathbb{N}$. Hence, since $\lim_{n\to\infty} \frac{s^nc}{\lambda^n} = \infty$, the above inequality turns into

$$M(p, q, c) \succeq \ell,$$

which gives a contradiction. Thus, we conclude that the fixed point of f is unique. □

Now, we present an example which shows the superiority of our assertion.

Example 11. Let $X = [0, 1]$ and let $M : X^2 \times P_\theta \to I$ be defined by

$$M(x, y, c) = \frac{ab}{ab + (x - y)^2}\ell$$

where $c = (a,b) \in P_\theta$. Then, one can readily verify that $(X, M, *_1, s)$ is a complete complex valued fuzzy b-metric space with $s = 2$. Moreover, following the same procedure as in Example 3.10 of [24], we conclude that for any sequence $\{c_n\}$ in P_θ with $\lim_{n\to\infty} c_n = \infty$, we have $\lim_{n\to\infty} \inf_{y\in X} M(x,y,c_n) = \ell$ for all $x \in X$.

Now, we define a mapping $f : X \to X$ such that $fx = \alpha x^2$, where $0 < \alpha < \frac{1}{4}$. By a routine calculation, we see that

$$M(fx, fy, \frac{\lambda c}{2}) \succeq M(x,y,c)$$

for all $x, y \in X$ and $c \in P_\theta$, where $\lambda = 4\alpha \in (0,1)$. Hence, all the conditions of Theorem 3 are satisfied and 0 is the unique fixed point of f.

Next, we establish the following fixed point theorem that extends the Jungck's Theorem [11] to the setting of complex valued fuzzy b-metric spaces.

Theorem 4. *Let $(X, M, *, s)$ be a complete complex valued fuzzy b-metric space such that, for every sequence $\{c_n\}$ in P_θ with $\lim_{n\to\infty} c_n = \infty$, we have*

$$\lim_{n\to\infty} \inf_{y\in X} M(x,y,c_n) = \ell$$

for all $x \in X$ and $f, g : X \to X$ be two mappings satisfying the following conditions:

(i) *$g(X) \subseteq f(X)$,*
(ii) *f and g commute on X,*
(iii) *f is continuous on X,*
(iv) *$M(gx, gy, \frac{\lambda c}{s}) \succeq M(fx, fy, c)$ for all $x, y \in X$ and $c \in P_\theta$, where $\lambda \in (0,1)$.*

Then, f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Due to $g(X) \subseteq f(X)$, we can choose an $x_1 \in X$ such that $gx_0 = fx_1$. Continuing this process, we can choose an $x_n \in X$ such that $fx_n = gx_{n-1}$. Now, we shall show that the sequence $\{fx_n\}$ is a Cauchy sequence. For all $n \in \mathbb{N}$ and $c \in P_\theta$, we define

$$B_n = \{M(fx_n, fx_m, c) : m > n\}.$$

Since $\theta \prec M(fx_n, fx_m, c) \preceq \ell$, for all $m \in \mathbb{N}$ with $m > n$ and from Remark 1(iii) it follows that $inf B_n = \beta_n$ exists for all $n \in \mathbb{N}$. For $c \in P_\theta$ and $m, n \in \mathbb{N}$ with $m > n$, we obtain, by Lemma 1 and the condition (iv),

$$M(fx_n, fx_m, c) \preceq M(fx_n, fx_m, \frac{sc}{\lambda}) \preceq M(gx_n, gx_m, c) = M(fx_{n+1}, fx_{m+1}, c).$$

Therefore, due to

$$\theta \preceq \beta_n \preceq \beta_{n+1} \preceq \ell \text{ for all } n \in \mathbb{N},$$

$\{\beta_n\}$ is a monotonic sequence in P. So, using Remark 1(i), there exists an $\ell_0 \in P$ such that

$$\lim_{n\to\infty} \beta_n = \ell_0. \tag{4.5}$$

For $c \in P_\theta$ and $m,n \in \mathbb{N}$ with $m > n$, by utilizing the condition (iv), we have

$$\begin{aligned} M(fx_{n+1}, fx_{m+1}, c) &= M(gx_n, gx_m, c) \\ &\succeq M(fx_n, fx_m, \frac{sc}{\lambda}) = M(gx_{n-1}, gx_{m-1}, \frac{sc}{\lambda}) \\ &\succeq M(fx_{n-1}, fx_{m-1}, \frac{s^2c}{\lambda^2}) = M(gx_{n-2}, gx_{m-2}, \frac{s^2c}{\lambda^2}) \\ &\succeq M(fx_{n-2}, fx_{m-2}, \frac{s^3c}{\lambda^3}) \\ &\succeq \cdots \succeq M(fx_0, fx_{m-n}, \frac{s^{n+1}c}{\lambda^{n+1}}), \end{aligned}$$

which gives

$$\begin{aligned} \beta_{n+1} &= \inf_{m>n} M(fx_{n+1}, fx_{m+1}, c) \\ &\succeq \inf_{m>n} M(fx_0, fx_{m-n}, \frac{s^{n+1}c}{\lambda^{n+1}}) \\ &\succeq \inf_{y\in X} M(fx_0, y, \frac{s^{n+1}c}{\lambda^{n+1}}). \end{aligned}$$

Since $\lim_{n\to\infty} \frac{s^{n+1}c}{\lambda^{n+1}} = \infty$ and from the hypothesis along with (4.5) it follows that

$$\ell_0 \succeq \lim_{n\to\infty} \inf_{y\in X} M(fx_0, y, \frac{s^{n+1}c}{\lambda^{n+1}}) = \ell,$$

which yields $\ell_0 = \ell$. Thus, $\{fx_n\}$ is a Cauchy sequence in X.

By completeness of X and Lemma 2, there exists a $p \in X$ such that

$$\lim_{n\to\infty} fx_n = p.$$

By the condition (iv), one can easily verify that continuity of f implies continuity of g. Therefore, $\lim_{n\to\infty} gfx_n = gp$. Since f and g commute on X, we have $\lim_{n\to\infty} fgx_n = gp$. Moreover, we know that $\lim_{n\to\infty} gx_{n-1} = p$ and so we obtain $\lim_{n\to\infty} fgx_{n-1} = fp$. According to the uniqueness of limit, we get $fp = gp$ and therefore $fgp = ggp$.

Now, repeated use of the condition (iv) gives

$$\begin{aligned} M(gp, ggp, c) &\succeq M(fp, fgp, \frac{sc}{\lambda}) = M(gp, ggp, \frac{sc}{\lambda}) \\ &\succeq \cdots \succeq M(gp, ggp, \frac{s^nc}{\lambda^n}) = M(gp, fgp, \frac{s^nc}{\lambda^n}) \\ &\succeq \inf_{y\in X} M(gp, y, \frac{s^nc}{\lambda^n}). \end{aligned}$$

On taking the limit $n \to \infty$ and applying the hypothesis we deduce $M(gp, ggp, c) = \ell$, which in turn implies that

$$ggp = fgp = gp.$$

That is, gp is a common fixed point of f and g.

Finally, we will investigate that such a point is unique. Let gp and q be two distinct common fixed points of f and g. On using the condition (iv) with $x = gp$ and $y = q$, we find

$$\begin{aligned}
\ell &\succeq M(gp, q, c) = M(ggp, gq, c) \\
&\succeq M(fgp, fq, \frac{sc}{\lambda}) = M(gp, q, \frac{sc}{\lambda}) \\
&\vdots \\
&\succeq M(gp, q, \frac{s^n c}{\lambda^n}) \\
&\succeq \inf_{y \in X} M(gp, y, \frac{s^n c}{\lambda^n}).
\end{aligned}$$

Hence, taking into account $\lim_{n\to\infty} \frac{s^n c}{\lambda^n} = \infty$, we conclude that $M(gp, q, c) = \ell$. Thus, $gp = q$, which completes the proof. □

Now, we give the following example to illustrate the validity of Theorem 4.

Example 12. Let $X = [0, 1]$. Define $M : X^2 \times P_\theta \to I$ as follows:

$$M(x, y, c) = e^{-\frac{(x-y)^2}{a+b}} \ell,$$

where $c = (a, b) \in P_\theta$. Clearly, $(X, M, *_2, s)$ is a complete complex valued fuzzy b-metric space with $s = 4$.

On the other hand, let $\lim_{n\to\infty} c_n = \infty$ for any sequence $\{c_n\}$ in P_θ, where $c_n = (a_n, b_n)$. From the fact that $(x - y)^2 \leq 1$ for all $x, y \in X$ it follows that

$$\inf_{y \in X} M(x, y, c_n) = \inf_{y \in X} e^{-\frac{(x-y)^2}{a_n+b_n}} \ell = e^{-\frac{\sup_{y \in X} (x-y)^2}{a_n+b_n}} \ell \succeq e^{-\frac{1}{a_n+b_n}} \ell.$$

Therefore, we have

$$\lim_{n\to\infty} \inf_{y \in X} M(x, y, c_n) \succeq \lim_{n\to\infty} e^{-\frac{1}{a_n+b_n}} \ell = \ell.$$

Consider the mappings $f, g : X \to X$ given by

$$f(x) = x \text{ and } g(x) = \frac{x}{4}.$$

One can readily verify that $g(X) \subseteq f(X)$ and f is continuous on X. Besides, f and g commute on X. Furthermore, it is easy to find that the condition (iv) holds for all $x, y \in [0, 1]$ with $\lambda = \frac{1}{4} \in (0, 1)$.

Thus, all of the assumptions of Theorem 4 are fulfilled and $0 \in X$ is the unique common fixed point of the involved mappings f and g.

Let $(X, M, *, s)$ be a complete complex valued fuzzy b-metric space. The contraction condition for the mapping $f : X \to X$ can be changed as follows:

$$\ell - M(fx, fy, c) \preceq \lambda[\ell - M(x, y, c)] \tag{4.6}$$

for all $x, y \in X$ and $c \in P_\theta$, where $\lambda \in [0, 1)$.

Then, we demonstrate a fixed point result for this class of contraction, which is a new generalization of the Banach contraction principle.

Theorem 5. *Let $(X, M, *, s)$ be a complete complex valued fuzzy b-metric space and $f : X \to X$ be a mapping satisfying the contraction condition (4.6). Then, f has a unique fixed point in X.*

Proof. Let x_0 be an arbitrary element of X. By induction, we can construct a sequence $\{x_n\}$ in X such that $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. Following the proof of Theorem 3.1 in [24], we observe that the sequence $\{x_n\}$ is a Cauchy sequence in X and converges to some $p \in X$. We shall show that p is a fixed point of f. By the contractive condition (4.6), we have

$$\ell - M(fx_n, fp, c) \preceq \lambda[\ell - M(x_n, p, c)]$$

for all $n \in \mathbb{N}$ and $c \in P_\theta$. The above inequality shows that

$$\ell(1 - \lambda) + \lambda M(x_n, p, c) \preceq M(fx_n, fp, c) \tag{4.7}$$

for all $n \in \mathbb{N}$ and $c \in P_\theta$. Therefore,

$$\begin{aligned} M(p, fp, c) &\succeq M(p, x_{n+1}, \frac{c}{2s}) * M(x_{n+1}, fp, \frac{c}{2s}) \\ &= M(p, x_{n+1}, \frac{c}{2s}) * M(fx_n, fp, \frac{c}{2s}), \end{aligned}$$

for any $c \in P_\theta$. Making the limit as $n \to \infty$, from (4.7) and Remark 2(ii), we deduce that $M(p, fp, c) = \ell$ for all $c \in P_\theta$, which yields $fp = p$.

To investigate the uniqueness of the fixed point of f, suppose that there exists another $q \in X$ such that $f(q) = q$. Then, there is a $c \in P_\theta$ satisfying $M(p, q, c) \neq \ell$. For this c, by virtue of (4.6), we have

$$\ell - M(p, q, c) = \ell - M(fp, fq, c) \preceq \lambda[\ell - M(p, q, c)].$$

Since $M(p, q, c) \neq \ell$, we obtain $Re(M(p, q, c)) \neq 1$ or $Im(M(p, q, c)) \neq 1$. Let $Re(M(p, q, c)) \neq 1$. Therefore, we get

$$1 - Re(M(p, q, c)) \leq \lambda(1 - Re(M(p, q, c))) < 1 - Re(M(p, q, c)),$$

which leads to a contradiction. The other case is similar to this one and so we skip the details. Thus, $M(p, q, c) = \ell$ for all $c \in P_\theta$ and the proof is concluded. □

The following example validates the aforesaid theorem.

Example 13. Let $X = [0,1]$ and let $M : X^2 \times P_\theta \to I$ be given by the rule

$$M(x,y,c) = \ell - \frac{(x-y)^2}{1+ab}\ell,$$

where $c = (a,b) \in P_\theta$. Then, $(X, M, *_4, s)$ is a complete complex valued fuzzy b-metric space. Define the mapping

$$f : X \to X,\ fx = \frac{x^2}{4}.$$

Therefore, we have

$$\frac{(fx-fy)^2}{1+ab}\ell \preceq \lambda\left(\frac{(x-y)^2}{1+ab}\ell\right),$$

where $\lambda \in [\frac{1}{4}, 1)$. Hence, we conclude that (4.6) holds, so all the required hypotheses of Theorem 5 are satisfied, and thus we deduce the existence and uniqueness of the fixed point of f. Here, 0 is the unique fixed point of f.

Corollary 1. *Let $(X, M, *, s)$ be a complete complex valued fuzzy b-metric space and let $f : X \to X$ be a mapping satisfying*

$$\ell - M(f^n x, f^n y, c) \preceq \lambda[\ell - M(x,y,c)]$$

for all $x, y \in X$ and $c \in P_\theta$, where $\lambda \in [0,1)$. Then, f has a unique fixed point in X (Here, f^n is the nth iterate of f).

Proof. By Theorem 5, we get a unique $x \in X$ such that $f^n x = x$. From the fact that $f^n fx = ff^n x = fx$ and from uniqueness, it follows that $fx = x$. This shows that f has a unique fixed point in X. □

5. Applications to existence of solutions of integral equations

In this section, we study the existence theorem for a solution of the following integral equation by using our main results in the previous section:

$$x(t) = \vartheta(t) + \beta \int_0^1 \xi(t,s)\varphi(s, x(s))ds,\ t \in [0,1], \tag{5.1}$$

where

(i) $\vartheta : [0,1] \to \mathbb{R}$ is continuous;

(ii) $\varphi : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $\varphi(t,x) \geq 0$ and there exists a $\lambda \in [0,1)$ such that

$$|\varphi(t,x) - \varphi(t,y)| \leq \lambda|x-y|,$$

for all $x, y \in \mathbb{R}$;

(iii) $\xi : [0,1] \times [0,1] \to \mathbb{R}$ is continuous at $t \in [0,1]$ for all $s \in [0,1]$ and measurable at $s \in [0,1]$ for all $t \in [0,1]$. Also, $\xi(t,s) \geq 0$ and $\int_0^1 \xi(t,s)ds \leq L$;

(iv) $\lambda^2 L^2 \beta^2 \leq \frac{1}{2}$.

Now, we prove the following result.

Theorem 6. *Suppose that the conditions* (i)-(iv) *hold. Then, the integral equation (5.1) has one and only one solution in $C([0,1],\mathbb{R})$, where $C([0,1],\mathbb{R})$ is the set of all continuous real valued functions on* $[0,1]$.

Proof. Let $X = C([0,1],\mathbb{R})$ and let us define a mapping $f : X \to X$ by

$$fx(t) = \vartheta(t) + \beta \int_0^1 \xi(t,s)\varphi(s,x(s))ds$$

for all $x \in X$ and for all $t \in [0,1]$. Now, we have to show that the mapping f satisfies all conditions of Theorem 5. Define a mapping $M : X^2 \times P_\theta \to I$ by

$$M(x,y,c) = \ell - \sup_{t\in[0,1]} \frac{\left(x(t)-y(t)\right)^2}{e^{ab}}\ell$$

where $c = (a,b) \in P_\theta$. Clearly, $(X,M,*_4,s)$ is a complete complex valued fuzzy b-metric space.

Moreover, for all $x,y \in X$ and $t \in [0,1]$, we have

$$\begin{aligned} |fx(t)-fy(t)| &= \beta\left|\int_0^1 \xi(t,s)\varphi\big(s,x(s)\big) - \xi(t,s)\varphi\big(s,y(s)\big)ds\right| \\ &\le \beta\int_0^1 \xi(t,s)\big|\varphi\big(s,x(s)\big) - \varphi\big(s,y(s)\big)\big|ds \\ &\le \beta\int_0^1 \xi(t,s)\lambda|x(s)-y(s)|ds \\ &\le \beta L\lambda \sup_{t\in[0,1]} |x(t)-y(t)|. \end{aligned}$$

From the fact that

$$\sup_{t\in[0,1]} |fx(t)-fy(t)| \le \beta L\lambda \sup_{t\in[0,1]} |x(t)-y(t)|$$

it follows that

$$\begin{aligned} \sup_{t\in[0,1]} \frac{|fx(t)-fy(t)|^2}{e^{ab}} &\le \beta^2 L^2 \lambda^2 \sup_{t\in[0,1]} \frac{|x(t)-y(t)|^2}{e^{ab}} \\ &\le \frac{1}{2} \sup_{t\in[0,1]} \frac{|x(t)-y(t)|^2}{e^{ab}}. \end{aligned}$$

This proves that the mapping f satisfy the contractive condition (4.6) appearing in Theorem 5, and hence f has a unique fixed point in $C([0,1],\mathbb{R})$, that is, the integral equation (5.1) has a unique solution in $C([0,1],\mathbb{R})$. $\square$

Next, we give an example of an integral equation and establish the existence of its solutions by using Theorem 6.

Example 14. Consider the following integral equation

$$x(t)=\frac{1}{1+t}+2\int_0^1\frac{s^2}{t^2+2}.\frac{|\cos x(s)|}{5e^s}ds,\ t\in[0,1]. \tag{5.2}$$

It is seen that the above equation is of the form (5.1), for

$$\beta=2,\ \vartheta(t)=\frac{1}{1+t},\ \xi(t,s)=\frac{s^2}{t^2+2},\ \varphi(t,x)=\frac{|\cos x|}{5e^t}.$$

Clearly, the mapping φ is continuous on $[0,1]\times\mathbb{R}$ and we get

$$\begin{aligned}|\varphi(t,x)-\varphi(t,y)|&=\frac{1}{5e^t}||\cos x|-|\cos y||\\&\leq\frac{1}{5e^t}|\cos x-\cos y|\\&\leq\frac{1}{5}|\cos x-\cos y|\\&\leq\frac{1}{5}|x-y|\end{aligned}$$

for all $x,y\in\mathbb{R}$. Therefore, φ satisfies the condition (ii) of the integral equation (5.1) with $\lambda=\frac{1}{5}$. One can readily check that the mapping ϑ is continuous and in view of

$$\int_0^1\xi(t,s)ds=\int_0^1\frac{s^2}{t^2+2}ds=\frac{1}{t^2+2}.\frac{1}{3}\leq\frac{1}{6}=L,$$

the mapping ξ satisfies the condition (iii). Also, we have

$$\lambda^2\beta^2L^2\leq\frac{1}{2}.$$

So, all the hypotheses (i)-(iv) are fulfilled. Thus, applying the Theorem 6, we conclude that the integral equation (5.2) has a unique solution in $C([0,1],\mathbb{R})$.

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REFERENCES

[1] M. Abbas, M. Arshad, and A. Azam, "Fixed points of asymptotically regular mappings in complex-valued metric spaces," *Georgian Math. J.*, vol. 20, no. 2, pp. 213–221, 2013, doi: 10.1515/gmj-2013-0013.

[2] J. Ahmad, A. Azam, and S. Saejung, "Common fixed point results for contractive mappings in complex valued metric spaces," *Fixed Point Theory Appl.*, vol. 2014, no. 1, p. 67, 2014, doi: 10.1186/1687-1812-2014-67.

[3] I. Altun and D. Turkoglu, "Some fixed point theorems on fuzzy metric spaces with implicit relations," *Commun. Korean Math. Soc.*, vol. 23, no. 1, pp. 111–124, 2008.

[4] A. Azam, B. Fisher, and M. Khan, "Common fixed point theorems in complex valued metric spaces," *Numer. Funct. Anal. Optim.*, vol. 32, no. 3, pp. 243–253, 2011, doi: 10.1080/01630563.2011.533046.

[5] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta mathematica et informatica universitatis ostraviensis*, vol. 1, no. 1, pp. 5–11, 1993.

[6] B. C. Dhage, "Condensing mappings and applications to existence theorems for common solution of differential equations," *Bull. Korean Math. Soc.*, vol. 36, no. 3, pp. 565–578, 1999.

[7] A. George and P. Veeramani, "On some results of analysis for fuzzy metric spaces," *Fuzzy Sets Syst.*, vol. 90, no. 3, pp. 365–368, 1997, doi: 10.1016/S0165-0114(96)00207-2.

[8] M. Grabiec, "Fixed points in fuzzy metric spaces," *Fuzzy Sets Syst.*, vol. 27, no. 3, pp. 385–389, 1988, doi: 10.1016/0165-0114(88)90064-4.

[9] V. Gregori, J.-J. Miñana, and D. Miravet, "Extended fuzzy metrics and fixed point theorems," *Mathematics*, vol. 7, no. 3, p. 303, 2019, doi: 10.3390/math7030303.

[10] N. Hussain, P. Salimi, and V. Parvaneh, "Fixed point results for various contractions in parametric and fuzzy b-metric spaces," *J. Nonlinear Sci. Appl.*, vol. 8, no. 5, pp. 719–739, 2015.

[11] G. Jungck, "Commuting maps and fixed points," *Amer. Math. Monthly*, vol. 83, pp. 261–263, 1976.

[12] I. Kramosil and J. Michálek, "Fuzzy metrics and statistical metric spaces," *Kybernetika*, vol. 11, no. 5, pp. 336–344, 1975.

[13] F. Mehmood, R. Ali, C. Ionescu, and T. Kamran, "Extended fuzzy b-metric spaces," *J. Math. Anal.*, vol. 8, no. 6, pp. 124–131, 2017.

[14] A. A. Mukheimer, "Some common fixed point theorems in complex valued-metric spaces," *Sci. World J.*, vol. 2014, 2014, doi: 10.1155/2014/587825.

[15] P. Murthy, B. Fisher, and R. Kewat, "Periodic points of rational inequality in a complex valued metric space," *Filomat*, vol. 31, no. 7, pp. 2143–2150, 2017, doi: 10.2298/FIL1707143M.

[16] H. Nashine, M. Imdad, and M. Hasan, "Common fixed point theorems under rational contractions in complex valued metric spaces," *J. Nonlinear Sci. Appl.*, vol. 7, pp. 42–50, 2014.

[17] T. Oner, M. B. Kandemir, and B. Tanay, "Fuzzy cone metric spaces," *J. Nonlinear Sci. Appl.*, vol. 8, no. 5, pp. 610–616, 2015.

[18] H. Pathak and B. Fisher, "Common fixed point theorems with applications in dynamic programming," *Glas. Mat.*, vol. 31, pp. 321–328, 1996.

[19] H. Pathak, S. Mishra, and A. Kalinde, "Common fixed point theorems with applications to nonlinear integral equations," *Demonstratio Math.*, vol. 32, no. 3, pp. 547–564, 1999.

[20] K. Rao, P. R. Swamy, and J. R. Prasad, "A common fixed point theorem in complex valued b-metric spaces," *Bull. Math. Statist. Res.*, vol. 1, no. 1, pp. 1–8, 2013.

[21] F. Rouzkard and M. Imdad, "Some common fixed point theorems on complex valued metric spaces," *Comput. Math. Appl.*, vol. 64, no. 6, pp. 1866–1874, 2012, doi: 10.1016/j.camwa.2012.02.063.

[22] A. Sapena, "A contribution to the study of fuzzy metric spaces," *Appl. Gen. Topol.*, vol. 2, no. 1, pp. 63–75, 2001.

[23] S. Shukla, D. Gopal, and W. Sintunavarat, "A new class of fuzzy contractive mappings and fixed point theorems," *Fuzzy Sets Syst.*, vol. 350, pp. 85–94, 2018, doi: 10.1016/j.fss.2018.02.010.

[24] S. Shukla, R. Rodriguez-Lopez, and M. Abbas, "Fixed point results for contractive mappings in complex valued fuzzy metric spaces," *Fixed Point Theory*, vol. 19, no. 2, pp. 751–774, 2018, doi: 10.24193/fpt-ro.2018.2.56.

[25] W. Sintunavarat, Y. J. Cho, and P. Kumam, "Urysohn integral equations approach by common fixed points in complex-valued metric spaces," *Adv. Difference Equ.*, vol. 2013, no. 1, p. 49, 2013, doi: 10.1186/1687-1847-2013-49.

[26] R. K. Verma, “Fixed point theorems using (clcs) property in complex valued b-metric spaces,” *Facta Univ. Ser. Math. Inform.*, vol. 32, no. 3, pp. 269–292, 2017, doi: 10.22190/FUMI1703269V.

[27] D. Wardowski, “Fuzzy contractive mappings and fixed points in fuzzy metric spaces,” *Fuzzy Sets Syst.*, vol. 222, pp. 108–114, 2013, doi: 10.1016/j.fss.2013.01.012.

[28] L. A. Zadeh, “Fuzzy sets,” *Inf. Control*, vol. 8, no. 3, pp. 338–353, 1965.

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INVERSE PROBLEM FOR SINGULAR DIFFUSION OPERATOR

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Abstract. In this study, singular diffusion operator with jump conditions is considered. Integral representations have been derived for solutions that satisfy boundary conditions and jump conditions. Some properties of eigenvalues and eigenfunctions are investigated. Asymtotic representation of eigenvalues and eigenfunctions have been obtained. Reconstruction of the singular diffusion operator have been shown by the Weyl function.

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1. INTRODUCTION

Let's define the following boundary value problem which will be denoted by L in the sequel all the paper

$$l(y) := -y'' + [2\lambda p(x) + q(x)]\, y = \lambda^2 \delta(x)\, y,\ x \in [0,\pi] / \{p_1, p_2\} \tag{1.1}$$

with the boundary conditions

$$y'(0) = 0, \qquad\qquad y(\pi) = 0 \tag{1.2}$$

and the jump conditions

$$y(p_1+0) = \alpha_1 y(p_1-0), \tag{1.3}$$

$$y'(p_1+0) = \beta_1 y'(p_1-0) + i\lambda\gamma_1 y(p_1-0), \tag{1.4}$$

$$y(p_2+0) = \alpha_2 y(p_2-0), \tag{1.5}$$

$$y'(p_2+0) = \beta_2 y'(p_2-0) + i\lambda\gamma_2 y(p_2-0), \tag{1.6}$$

where λ is a spectral parameter, $q(x) \in L_2[0,\pi]$, $p(x) \in W_2^1[0,\pi]$, $p_1, p_2 \in (0,\pi)$, $p_1 < p_2$, $|\alpha_1 - 1|^2 + \gamma_1^2 \neq 0$, $|\alpha_2 - 1|^2 + \gamma_2 \neq 0$, $\left(\beta_i = \frac{1}{\alpha_i}\,(i = 1,2)\right)$ and $\delta(x) = \begin{cases} 1 & x \in (0,p_1); \\ \alpha^2 & x \in (p_1,p_2); \\ \beta^2 & x \in (p_2,\pi); \end{cases}$ to be $\alpha > 0$, $\alpha \neq 1$, $\beta > 0$, $\beta \neq 1$ real numbers.

Direct and inverse problems are important in mathematics, physics and engineering. The inverse problem is called the reconstruction of the operator whose spectral characteristics are given in sequences. For example; to learn the distribution of density in the nonhomogeneous arc according to the wave lengths in mechanics and finding the field potentials according to scattering data in the quantum physics are examples of inverse problems. The first study on inverse problems for differential equations was made by Ambartsumyan in [25]. A significant study in the spectral theory of the singular differential operators was carried out by Levitan in [4]. An important method in the solution of inverse problems is the transformation operators. In [14], Guseinov studied the regular differential equation and the direct spectral problem of the operator under certain initial conditions. In recent years, Weyl function has frequently been used to solve inverse problems. The Weyl function was introduced by H. Weyl in 1910 in the literature. Many studies have been made on direct or inverse problems [1–28]. The solution of discontinuous boundary value problem can be given as an example of concrete problem of mathematical physics. Boundary value problems with discontinuous coefficients are important for applied mathematics and applied sciences.

In [17], Koyunbakan and Panakhov proved that the potential function can be determined on $\left[\frac{\pi}{2},\pi\right]$ while it is known on $\left[0,\frac{\pi}{2}\right]$ by single spectrum in [12]. In [26], Yang showed that can be determined uniquely diffusion operator from nodal data.

2. Preliminaries

Let $\phi(x,\lambda)$, $\psi(x,\lambda)$ be solutions of (1.1) respectively under the boundary conditions

$$\begin{aligned} \phi(0,\lambda) &= 1, & \phi'(0,\lambda) &= 0 \\ \psi(\pi,\lambda) &= 0, & \psi'(\pi,\lambda) &= 1 \end{aligned}$$

and discontinuity conditions $(1.3)-(1.6)$, where $Q(t)=2\lambda p(t)+q(t)$.

It is obvious that the function $\phi(x,\lambda)$ is similar to [8] satisfies the following integral equations if $0\le x<p_1$:

$$\phi(x,\lambda)=e^{i\lambda x}+\frac{1}{\lambda}\int_0^x \sin\lambda(x-t)\,Q(t)\,y(t,\lambda)\,dt, \tag{2.1}$$

if $p_1<x<p_2$:

$$\begin{aligned} \phi(x,\lambda)=&\beta_1^+e^{i\lambda\varsigma^+(x)}+\beta_1^-e^{i\lambda\varsigma^-(x)}+\frac{\gamma_1}{2\alpha}e^{i\lambda\varsigma^+(x)}-\frac{\gamma_1}{2\alpha}e^{i\lambda\varsigma^-(x)} \\ &+\beta_1^+\int_0^{p_1}\frac{\sin\lambda(\varsigma^+(x)-t)}{\lambda}J(t)\,y(t,\lambda)\,dt \\ &+\beta_1^-\int_0^{p_1}\frac{\sin\lambda(\varsigma^-(x)-t)}{\lambda}J(t)\,y(t,\lambda)\,dt \end{aligned} \tag{2.2}$$

$$
\begin{aligned}
&-i\frac{\gamma_1}{2\alpha}\int_0^{p_1}\frac{\cos\lambda\left(\varsigma^+(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+i\frac{\gamma_1}{2\alpha}\int_0^{p_1}\frac{\cos\lambda\left(\varsigma^-(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+\int_{p_1}^{x}\frac{\sin\lambda(x-t)}{\lambda}J(t)\,y(t,\lambda)\,dt,
\end{aligned}
$$

if $p_2 < x \leq \pi$:

$$
\begin{aligned}
\phi(x,\lambda) =&\, \xi^+e^{i\lambda b^+(x)}+\xi^-e^{i\lambda b^-(x)}+\vartheta^+e^{i\lambda s^+(x)}+\vartheta^-e^{i\lambda s^-(x)}\\
&+\left(\beta_1^+\beta_2^+ +\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\int_0^{p_1}\frac{\sin\lambda\left(b^+(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+\left(\beta_1^+\beta_2^- -\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\int_0^{p_1}\frac{\sin\lambda\left(s^+(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+\left(\beta_1^-\beta_2^- -\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\int_0^{p_1}\frac{\sin\lambda\left(b^-(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+\left(\beta_1^-\beta_2^+ +\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\int_{p_1}^{p_2}\frac{\sin\lambda\left(s^-(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&-i\left(\frac{\gamma_1\beta_2^+}{2\alpha}+\frac{\gamma_2\beta_1^+}{2\beta}\right)\int_0^{p_1}\frac{\cos\lambda\left(b^+(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&-i\left(\frac{\gamma_1\beta_2^-}{2\alpha}-\frac{\gamma_2\beta_1^+}{2\beta}\right)\int_0^{p_1}\frac{\cos\lambda\left(s^+(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+i\left(\frac{\gamma_1\beta_2^-}{2\alpha}-\frac{\gamma_2\beta_1^-}{2\beta}\right)\int_0^{p_1}\frac{\cos\lambda\left(b^-(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+i\left(\frac{\gamma_1\beta_2^+}{2\alpha}+\frac{\gamma_2\beta_1^-}{2\beta}\right)\int_{p_1}^{p_2}\frac{\cos\lambda\left(s^-(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+\beta_2^+\int_{p_1}^{p_2}\frac{\sin\lambda\left(\beta x-\beta p_2+\alpha p_2-\alpha t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&-\beta_2^-\int_{p_1}^{p_2}\frac{\sin\lambda\left(\beta x-\beta p_2-\alpha p_2+\alpha t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&-i\frac{\gamma_2}{2\beta}\int_{p_1}^{p_2}\frac{\cos\lambda\left(\beta x-\beta p_2+\alpha p_2-\alpha t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+i\frac{\gamma_2}{2\beta}\int_{p_1}^{p_2}\frac{\cos\lambda\left(\beta x-\beta p_2-\alpha p_2+\alpha t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+\int_{p_2}^{x}\frac{\sin\lambda(x-t)}{\lambda}J(t)\,y(t,\lambda)\,dt,
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
\phi(x,\lambda) =&\xi^{+}e^{i\lambda b^{+}(x)}+\xi^{-}e^{i\lambda b^{-}(x)}+\vartheta^{+}e^{i\lambda s^{+}(x)}+\vartheta^{-}e^{i\lambda s^{-}(x)}\\
&+\left(\beta_1^{+}\beta_2^{+}+\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\int_0^{p_1}\frac{\sin\lambda\left(b^{+}(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+\left(\beta_1^{+}\beta_2^{-}-\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\int_0^{p_1}\frac{\sin\lambda\left(s^{+}(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+\left(\beta_1^{-}\beta_2^{-}-\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\int_0^{p_1}\frac{\sin\lambda\left(b^{-}(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+\left(\beta_1^{-}\beta_2^{+}+\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\int_{p_1}^{p_2}\frac{\sin\lambda\left(s^{-}(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&-i\left(\frac{\gamma_1\beta_2^{+}}{2\alpha}+\frac{\gamma_2\beta_1^{+}}{2\beta}\right)\int_0^{p_1}\frac{\cos\lambda\left(b^{+}(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&-i\left(\frac{\gamma_1\beta_2^{-}}{2\alpha}-\frac{\gamma_2\beta_1^{+}}{2\beta}\right)\int_0^{p_1}\frac{\cos\lambda\left(s^{+}(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+i\left(\frac{\gamma_1\beta_2^{-}}{2\alpha}-\frac{\gamma_2\beta_1^{-}}{2\beta}\right)\int_0^{p_1}\frac{\cos\lambda\left(b^{-}(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+i\left(\frac{\gamma_1\beta_2^{+}}{2\alpha}+\frac{\gamma_2\beta_1^{-}}{2\beta}\right)\int_{p_1}^{p_2}\frac{\cos\lambda\left(s^{-}(x)-t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+\beta_2^{+}\int_{p_1}^{p_2}\frac{\sin\lambda\left(\beta x-\beta p_2+\alpha p_2-\alpha t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&-\beta_2^{-}\int_{p_1}^{p_2}\frac{\sin\lambda\left(\beta x-\beta p_2-\alpha p_2+\alpha t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&-i\frac{\gamma_2}{2\beta}\int_{p_1}^{p_2}\frac{\cos\lambda\left(\beta x-\beta p_2+\alpha p_2-\alpha t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+i\frac{\gamma_2}{2\beta}\int_{p_1}^{p_2}\frac{\cos\lambda\left(\beta x-\beta p_2-\alpha p_2+\alpha t\right)}{\lambda}J(t)\,y(t,\lambda)\,dt\\
&+\int_{p_2}^{x}\frac{\sin\lambda(x-t)}{\lambda}J(t)\,y(t,\lambda)\,dt,
\end{aligned}
\tag{2.4}
$$

and it is obvious that the function $\psi(x,\lambda)$ satisfies the following integral equations if $p_2 < x \le \pi$:

$$
\psi(x,\lambda)=\frac{\sin\lambda\beta(x-\pi)}{\lambda\beta}+\int_x^{\pi}\frac{\sin\lambda\beta(x-t)}{\lambda\beta}Q(t)\,y(t,\lambda)\,dt, \tag{2.5}
$$

if $p_1 < x < p_2$:

$$
\psi(x,\lambda)=\left(\frac{\alpha\beta_2-\gamma_2}{2\alpha\beta_2\lambda\alpha_2\beta}-\frac{1}{2\alpha\beta_2\lambda}\right)e^{-i\lambda(\beta(p_2-\pi)+\alpha(p_2-x))}
$$

$$
\begin{aligned}
&+\left(\frac{\alpha\beta_2+\gamma_2}{2\alpha\beta_2\lambda\alpha_2\beta}+\frac{1}{2\alpha\beta_2\lambda}\right)e^{-i\lambda(\beta(p_2-\pi)-\alpha(p_2-x))}\\
&-\left(\frac{\alpha\beta_2-\gamma_2}{2\alpha\beta_2}-\frac{1}{2}\right)\int_{p_1}^{p_2}\frac{\sin\lambda\left(x-p_2+\alpha t-\alpha p_2\right)}{\lambda\alpha}Q(t)\,y(t,\lambda)\,dt\\
&+\left(\frac{\alpha\beta_2-\gamma_2}{2\alpha\beta_2}+\frac{1}{2}\right)\int_{p_1}^{p_2}\frac{\sin\lambda\left(x-p_2-\alpha t+\alpha p_2\right)}{\lambda\alpha}Q(t)\,y(t,\lambda)\,dt \qquad (2.6)\\
&+\frac{1}{2}\left(\frac{\alpha\beta_2-\gamma_2}{\alpha\beta_2\alpha_2\beta}-\frac{1}{\alpha\beta_2}\right)\int_{p_2}^{\pi}\frac{\sin\lambda\left(x-p_2+\beta\left(t-p_2\right)\right)}{\lambda\beta}Q(t)\,y(t,\lambda)\,dt\\
&-\frac{1}{2}\left(\frac{\alpha\beta_2-\gamma_2}{\alpha\beta_2\alpha_2\beta}-\frac{1}{\alpha\beta_2}\right)\int_{p_2}^{\pi}\frac{\sin\lambda\left(x-p_2-\beta\left(t-p_2\right)\right)}{\lambda\beta}Q(t)\,y(t,\lambda)\,dt\\
&+\frac{\gamma_2}{2\alpha\beta_2\lambda}\int_{p_1}^{p_2}\frac{\cos\lambda\left(x-p_2+\alpha t-\alpha p_2\right)}{\lambda\alpha}Q(t)\,y(t,\lambda)\,dt\\
&-\frac{\gamma_2}{2\alpha\beta_2\lambda}\int_{p_1}^{p_2}\frac{\cos\lambda\left(x-p_2-\alpha t+\alpha p_2\right)}{\lambda\alpha}Q(t)\,y(t,\lambda)\,dt\\
&+\int_{p_1}^{x}\frac{\sin\lambda\alpha\left(x-t\right)}{\lambda\alpha}Q(t)\,y(t,\lambda)\,dt,
\end{aligned}
$$

if $0\leq x<p_1$:

$$
\begin{aligned}
\psi(x,\lambda)=&\left(\xi^{+}+\frac{\alpha}{2\beta_1}\right)\eta^{-}e^{-i\lambda(b^{-}(\pi)+x)}+\left(\xi^{-}-\frac{\alpha}{2\beta_1}\right)\eta^{+}e^{-i\lambda(b^{+}(\pi)+x)}\\
&+\left(\xi^{-}+\frac{\alpha}{2\beta_1}\right)\eta^{-}e^{-i\lambda(s^{+}(\pi)+x)}+\left(\xi^{-}-\frac{\alpha}{2\beta_1}\right)\eta^{+}e^{-i\lambda(s^{-}(\pi)+x)}\\
&+\left(\frac{1}{2\alpha_1}-\frac{\mu^{+}}{4\beta_1}\right)\int_{a_2}^{\pi}\frac{\sin\lambda\left(x-p_2-\beta t+\beta p_2\right)}{\lambda}Q(t)\,y(t,\lambda)\,dt\\
&-\left(\frac{1}{2\alpha_1}+\frac{\mu^{+}}{4\beta_1}\right)\int_{p_2}^{\pi}\frac{\sin\lambda\left(x-2p_1+p_2+\beta t-\beta p_2\right)}{\lambda}Q(t)\,y(t,\lambda)\,dt\\
&+\left(\frac{1}{2\alpha_1}+\frac{\mu^{-}}{4\beta_1}\right)\int_{p_2}^{\pi}\frac{\sin\lambda\left(x-p_2-\beta t+\beta p_2\right)}{\lambda}Q(t)\,y(t,\lambda)\,dt \qquad (2.7)\\
&-\left(\frac{1}{2\alpha_1}-\frac{\mu^{-}}{4\beta_1}\right)\int_{p_2}^{\pi}\frac{\sin\lambda\left(x-2p_1+p_2-\beta t+\beta p_2\right)}{\lambda}Q(t)\,y(t,\lambda)\,dt\\
&+\frac{i\gamma_1}{2\alpha_1\beta_1}\int_{p_2}^{\pi}\frac{\cos\lambda\left(x-p_2+\beta t-\beta p_2\right)}{\lambda}Q(t)\,y(t,\lambda)\,dt\\
&-\frac{i\gamma_1}{2\alpha_1\beta_1}\int_{p_2}^{\pi}\frac{\cos\lambda\left(x-2p_1+p_2-\beta t+\beta p_2\right)}{\lambda}Q(t)\,y(t,\lambda)\,dt\\
&-\frac{i\gamma_1}{2\alpha_1\beta_1}\int_{p_2}^{\pi}\frac{\cos\lambda\left(x-p_2-\beta t+\beta p_2\right)}{\lambda}Q(t)\,y(t,\lambda)\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{i\gamma_1}{2\alpha_1\beta_1}\int_{p_2}^{\pi}\frac{\cos\lambda\left(x-2p_1+p_2+\beta t-\beta p_2\right)}{\lambda}Q(t)\,y(t,\lambda)\,dt\\
&+A\int_{p_1}^{p_2}\frac{\sin\lambda\left(x-p_2+\alpha t-\alpha p_2\right)}{\lambda\alpha}Q(t)\,y(t,\lambda)\,dt\\
&+A\int_{p_1}^{p_2}\frac{\sin\lambda\left(x-2p_1+p_2-\alpha t+\alpha p_2\right)}{\lambda\alpha}Q(t)\,y(t,\lambda)\,dt\\
&+B\int_{p_1}^{p_2}\frac{\cos\lambda\left(x-p_2+\alpha t-\alpha p_2\right)}{\lambda\alpha}Q(t)\,y(t,\lambda)\,dt\\
&+B\int_{p_1}^{p_2}\frac{\cos\lambda\left(x-2p_1+p_2-\alpha t+\alpha p_2\right)}{\lambda\alpha}\\
&+C\int_{p_1}^{p_2}\frac{\sin\lambda\left(x-p_2-\alpha t+\alpha p_2\right)}{\lambda\alpha}Q(t)\,y(t,\lambda)\,dt\\
&+C\int_{p_1}^{p_2}\frac{\sin\lambda\left(x-2p_1+p_2+\alpha t-\alpha p_2\right)}{\lambda\alpha}Q(t)\,y(t,\lambda)\,dt\\
&+D\int_{p_1}^{p_2}\frac{\cos\lambda\left(x-p_2-\alpha t+\alpha p_2\right)}{\lambda\alpha}Q(t)\,y(t,\lambda)\,dt\\
&+D\int_{p_1}^{p_2}\frac{\cos\lambda\left(x-2p_1+p_2+\alpha t-\alpha p_2\right)}{\lambda\alpha}Q(t)\,y(t,\lambda)\,dt\\
&+\int_{0}^{x}\frac{\sin\lambda\left(x-t\right)}{\lambda}Q(t)\,y(t,\lambda)\,dt,
\end{aligned}
$$

where

$$
\begin{aligned}
&\varsigma^{\pm}(x)=\pm\alpha x\mp\alpha p_1+p_1, && \beta_1^{\pm}=\frac{1}{2}\left(\alpha_1\pm\frac{\beta_1}{\alpha}\right),\\
&b^{\pm}(x)=\beta x-\beta p_2+\mu^{\pm}(p_2), && s^{\pm}(x)=-\beta x+\beta p_2+\mu^{\pm}(p_2),\\
&\beta_2^{\mp}=\frac{1}{2}\left(\alpha_2\mp\frac{\alpha\beta_2}{\beta}\right), && \xi^{\mp}=\frac{1}{2}\left(\beta_1^{\mp}\mp\frac{\gamma_1}{2\alpha}\right)\left(\alpha_2\mp\frac{\alpha\beta_2}{\beta}+\frac{\gamma_2}{\beta}\right),\\
&\vartheta^{\mp}=\frac{1}{2}\left(\beta_1^{\mp}\mp\frac{\gamma_1}{2\alpha}\right)\left(\alpha_2\pm\frac{\alpha\beta_2}{\beta}-\frac{\gamma_2}{\beta}\right), && \mu^{\pm}=\left(\frac{\alpha\beta_2\pm\gamma_2}{2\alpha\beta_2\lambda\alpha_2\beta}\pm\frac{1}{2\alpha\beta_2\lambda}\right),
\end{aligned}
$$

$$
\begin{aligned}
A&=\left[\left(\frac{i\gamma_1\gamma_2}{4\lambda\alpha\alpha_1\beta_1\beta_2}+\left(\frac{-1}{2\alpha_1}-\frac{1}{4\beta_1}\right)\left(\frac{\alpha\beta_2-\gamma_2}{2\alpha\beta_2}-\frac{1}{2}\right)\right)\right],\\
B&=\left[\frac{-i\gamma_1}{2\alpha_1\beta_1}\left(\frac{\alpha\beta_2-\gamma_2}{2\alpha\beta_2}-\frac{1}{2}\right)+\frac{1}{2\alpha_1}\frac{\gamma_2}{2\alpha\beta_2\lambda}\right],\\
C&=\left[\left(\frac{1}{2\alpha_1}+\frac{1}{2\beta_1}\right)\left(\frac{\alpha\beta_2-\gamma_2}{2\alpha\beta_2}+\frac{1}{2}\right)+\frac{i\gamma_1\gamma_2}{4\lambda\alpha\alpha_1\beta_1\beta_2}\right],
\end{aligned}
$$

$$D=\left[\frac{i\gamma_1}{2\alpha_1\beta_1}\left(\frac{\alpha\beta_2-\gamma_2}{2\alpha\beta_2}+\frac{1}{2}\right)-\frac{\gamma_2\left(1-\alpha^2\right)}{4\alpha_1\alpha\beta_2\lambda}\right].$$

Theorem 1. *If $p(x)\in W_2^1(0,\pi)$ and $q(x)\in L_2(0,\pi)$; $y_\upsilon(x,\lambda)$ be solutions of (1.1), that satisfies conditions (1.2) – (1.6), has the form*

$$y_\upsilon(x,\lambda)=y_{0\upsilon}(x,\lambda)+\int_{-x}^{x}K_\upsilon(x,t)\,e^{i\lambda t}dt \qquad \left(\upsilon=\overline{1,3}\right)$$

where

$$y_{0\upsilon}(x,\lambda)=\begin{cases} R_0(x)\,e^{i\lambda x} & 0\le x<p_1;\\ R_1(x)\,e^{i\lambda\varsigma^+(x)}+R_2(x)\,e^{i\lambda\varsigma^-(x)} & p_1<x<p_2;\\ R_3(x)\,e^{i\lambda b^+(x)}+R_4(x)\,e^{i\lambda b^-(x)} & \\ +R_5(x)\,e^{i\lambda s^+(x)}+R_6(x)\,e^{i\lambda s^-(x)} & p_2<x\le\pi;\end{cases}$$

$$R_0(x)=e^{-i\int_0^x p(x)dx},\qquad R_1(x)=\left(\beta_1^++\frac{\gamma_1}{2\alpha}\right)R_0(p_1)\,e^{-\frac{i}{\alpha}\int_{p_1}^x p(t)dt},$$

$$R_2(x)=\left(\beta_1^--\frac{\gamma_1}{2\alpha}\right)R_0(p_1)\,e^{\frac{i}{\alpha}\int_{p_1}^x p(t)dt},\qquad R_3(x)=\left(\beta_2^++\frac{\gamma_2}{2\beta}\right)R_1(p_2)\,e^{-\frac{i}{\beta}\int_{p_2}^x p(t)dt},$$

$$R_4(x)=\left(\beta_2^-+\frac{\gamma_2}{2\beta}\right)R_2(p_2)\,e^{-\frac{i}{\beta}\int_{p_2}^x p(t)dt},\qquad R_5(x)=\left(\beta_2^--\frac{\gamma_2}{2\beta}\right)R_1(p_2)\,e^{\frac{i}{\beta}\int_{p_2}^x p(t)dt},$$

$$R_6(x)=\left(\beta_2^+-\frac{\gamma_2}{2\beta}\right)R_2(p_2)\,e^{\frac{i}{\beta}\int_{p_2}^x p(t)dt}.$$

and $\varpi(x)=\int_0^x(2|p(t)|+(x-t)|q(t)|)\,dt$ and the functions $K_\upsilon(x,t)$ satisfies the inequality

$$\int_{-x}^{x}|K_\upsilon(x,\lambda)|\,dt\le e^{c_\upsilon\varpi(x)}-1$$

with

$$c_1=1,\qquad c_2=\left(\beta_1^++|\beta_1^-|+\frac{\gamma_1}{\alpha}+\frac{2}{\alpha}\right),$$

$$c_3=\left(\alpha_2\left(\beta_1^++|\beta_1^-|\right)+\frac{1}{\alpha}\left(\beta_2^++|\beta_2^-|\right)+\frac{\beta^+}{\beta}+\frac{\gamma_2}{\beta}\right),$$

where

$$\varsigma^\pm(x)=\pm\alpha x\mp\alpha p_1+p_1,\qquad \beta_1^\pm=\frac{1}{2}\left(\alpha_1\pm\frac{\beta_1}{\alpha}\right),$$

$$b^\pm(x)=\beta x-\beta p_2+\varsigma^\pm(p_2),\qquad s^\pm(x)=-\beta x+\beta p_2+\varsigma^\pm(p_2),$$

$$\beta_2^\mp=\frac{1}{2}\left(\alpha_2\mp\frac{\alpha\beta_2}{\beta}\right),\qquad \xi^\mp=\frac{1}{2}\left(\beta_1^\mp\mp\frac{\gamma_1}{2\alpha}\right)\left(\alpha_2\mp\frac{\alpha\beta_2}{\beta}+\frac{\gamma_2}{\beta}\right),$$

$$\vartheta^\mp=\frac{1}{2}\left(\beta_1^\mp\mp\frac{\gamma_1}{2\alpha}\right)\left(\alpha_2\pm\frac{\alpha\beta_2}{\beta}-\frac{\gamma_2}{\beta}\right),\qquad \beta^\pm=\frac{1}{2}\left(1\pm\frac{1}{\beta}\right).$$

The proof is done as in [8].

Theorem 2. *Let $p(x) \in W_2^1(0,\pi)$ and $q(x) \in L_2(0,\pi)$. The functions $A(x,t)$, $B(x,t)$, whose first order partial derivatives, are summable on $[0,\pi]$, for each $x \in [0,\pi]$ such that representation*

$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + \int_0^x A(x,t)\cos\lambda t\,dt + \int_0^x B(x,t)\sin\lambda t\,dt$$

is satisfied.

If $p_1 < x < p_2$:

$$\begin{aligned}\varphi(x,\lambda) = &\left(\beta_1^+ + \frac{\gamma_1}{2\alpha}\right) R_0(p_1)\cos\left[\lambda\varsigma^+(x) - \frac{1}{\alpha}\int_{p_1}^x p(t)\,dt\right] \\ &+ \left(\beta_1^- - \frac{\gamma_1}{2\alpha}\right) R_0(p_1)\cos\left[\lambda\varsigma^-(x) + \frac{1}{\alpha}\int_{p_1}^x p(t)\,dt\right] \\ &+ \int_0^{\varsigma^+(x)} A(x,t)\cos\lambda t\,dt + \int_0^{\varsigma^+(x)} B(x,t)\sin\lambda t\,dt,\end{aligned} \tag{2.8}$$

where $\beta_1^{\pm} = \frac{1}{2}\left(\alpha_1 \pm \frac{\beta_1}{\alpha}\right)$. If $p_2 < x \leq \pi$:

$$\begin{aligned}\varphi(x,\lambda) = &\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right) R_1(p_2)\cos\left[\lambda b^+(x) - \frac{1}{\beta}\int_{p_2}^x p(t)\,dt\right] \\ &+ \left(\beta_2^- + \frac{\gamma_2}{2\beta}\right) R_2(p_2)\cos\left[\lambda b^-(x) - \frac{1}{\beta}\int_{p_2}^x p(t)\,dt\right] \\ &+ \left(\beta_2^- - \frac{\gamma_2}{2\beta}\right) R_1(p_2)\cos\left[\lambda s^+(x) + \frac{1}{\beta}\int_{p_2}^x p(t)\,dt\right] \\ &+ \left(\beta_2^+ - \frac{\gamma_2}{2\beta}\right) R_2(p_2)\cos\left[\lambda s^-(x) + \frac{1}{\beta}\int_{p_2}^x p(t)\,dt\right] \\ &+ \int_{p_2}^x A(x,t)\cos\lambda t\,dt + \int_{p_2}^x B(x,t)\sin\lambda t\,dt,\end{aligned} \tag{2.9}$$

where $\beta_2^{\mp} = \frac{1}{2}\left(\alpha_2 \mp \frac{\alpha\beta_2}{\beta}\right)$. Moreover, the equations

$$\begin{aligned}&A\left(x,\varsigma^+(x)\right)\cos\frac{\beta(x)}{\alpha} + B\left(x,\varsigma^+(x)\right)\sin\frac{\beta(x)}{\alpha} \\ &\qquad = \left(\beta_1^+ + \frac{\gamma_1}{2\alpha}\right)\frac{R_0(p_1)}{2\alpha}\int_0^x\left(q(t) + \frac{p^2(t)}{\alpha^2}\right)dt\end{aligned} \tag{2.10}$$

$$\begin{aligned}&A\left(x,\varsigma^+(x)\right)\sin\frac{\beta(x)}{\alpha} - B\left(x,\varsigma^+(x)\right)\cos\frac{\beta(x)}{\alpha} \\ &\qquad = \left(\beta_1^+ + \frac{\gamma_1}{2\alpha}\right)\frac{R_0(p_1)}{2\alpha^2}\left(p(x) - p(0)\right)\end{aligned} \tag{2.11}$$

$$A\left(x,\varsigma^{-}(x)+0\right)-A\left(x,\varsigma^{-}(x)-0\right)=\left(\beta_1^{-}-\frac{\gamma_1}{2\alpha}\right)\frac{R_0(p_1)}{2\alpha^2}\sin\frac{\beta(x)}{\alpha}\left(p(x)-p(0)\right)$$
$$+\left(\beta_1^{-}-\frac{\gamma_1}{2\alpha}\right)\frac{R_0(p_1)}{2\alpha}\cos\frac{\beta(x)}{\alpha}\int_0^x\left(q(t)+\frac{p^2(t)}{\alpha^2}\right)dt \quad (2.12)$$

$$B\left(x,\varsigma^{-}(x)+0\right)-B\left(x,\varsigma^{-}(x)-0\right)=\left(\beta_1^{-}-\frac{\gamma_1}{2\alpha}\right)\frac{R_0(p_1)}{2\alpha^2}\cos\frac{\beta(x)}{\alpha}\left(p(x)-p(0)\right)$$
$$-\left(\beta_1^{-}-\frac{\gamma_1}{2\alpha}\right)\frac{R_0(p_1)}{2\alpha}\sin\frac{\beta(x)}{\alpha}\int_0^x\left(q(t)+\frac{p^2(t)}{\alpha^2}\right)dt \quad (2.13)$$

$$B(x,0)=\left.\frac{\partial A(x,t)}{\partial t}\right|_{t=0}=0 \quad (2.14)$$

$$A\left(x,s^{-}(x)+0\right)-A\left(x,s^{-}(x)-0\right)=-\left(\beta_2^{-}-\frac{\gamma_2}{2\beta}\right)\frac{R_2(p_2)}{2\beta^2}\left(p(x)-p(0)\right)\sin\frac{\omega(x)}{\beta}$$
$$-\left(\beta_2^{-}-\frac{\gamma_2}{2\beta}\right)\frac{R_2(p_2)}{2\beta}\int_0^x\left(q(t)+\frac{p^2(t)}{\beta^2}\right)dt\cos\frac{\omega(x)}{\beta} \quad (2.15)$$

$$B\left(x,s^{-}(x)+0\right)-B\left(x,s^{-}(x)-0\right)=-\left(\beta_2^{-}-\frac{\gamma_2}{2\beta}\right)\frac{R_2(p_2)}{2\beta^2}\left(p(x)-p(0)\right)\cos\frac{\omega(x)}{\beta}$$
$$+\left(\beta_2^{-}-\frac{\gamma_2}{2\beta}\right)\frac{R_2(p_2)}{2\beta}\int_0^x\left(q(t)+\frac{p^2(t)}{\beta^2}\right)dt\sin\frac{\omega(x)}{\beta} \quad (2.16)$$

$$A\left(x,s^{+}(x)+0\right)-A\left(x,s^{+}(x)-0\right)=-\left(\beta_2^{-}-\frac{\gamma_2}{2\beta}\right)\frac{R_1(p_2)}{2\beta^2}\left(p(x)-p(0)\right)\sin\frac{\omega(x)}{\beta}$$
$$-\left(\beta_2^{-}-\frac{\gamma_2}{2\beta}\right)\frac{R_1(p_2)}{2\beta}\int_0^x\left(q(t)+\frac{p^2(t)}{\beta^2}\right)dt\cos\frac{\omega(x)}{\beta} \quad (2.17)$$

$$B\left(x,s^{+}(x)+0\right)-B\left(x,s^{+}(x)-0\right)=-\left(\beta_2^{-}-\frac{\gamma_2}{2\beta}\right)\frac{R_1(p_2)}{2\beta^2}\left(p(x)-p(0)\right)\cos\frac{\omega(x)}{\beta}$$
$$+\left(\beta_2^{-}-\frac{\gamma_2}{2\beta}\right)\frac{R_1(p_2)}{2\beta}\int_0^x\left(q(t)+\frac{p^2(t)}{\beta^2}\right)dt\sin\frac{\omega(x)}{\beta} \quad (2.18)$$

$$A\left(x,b^{-}(x)+0\right)-A\left(x,b^{-}(x)-0\right)=-\left(\beta_2^{-}+\frac{\gamma_2}{2\beta}\right)\frac{R_2(p_2)}{2\beta^2}\left(p(x)-p(0)\right)\sin\frac{\omega(x)}{\beta}$$
$$-\left(\beta_2^{-}-\frac{\gamma_2}{2\beta}\right)\frac{R_2(p_2)}{2\beta}\int_0^x\left(q(t)+\frac{p^2(t)}{\beta^2}\right)dt\cos\frac{\omega(x)}{\beta} \quad (2.19)$$

$$B\left(x,b^{-}(x)+0\right)-B\left(x,b^{-}(x)-0\right)=\left(\beta_2^{-}+\frac{\gamma_2}{2\beta}\right)\frac{R_2(p_2)}{2\beta^2}\left(p(x)-p(0)\right)\cos\frac{\omega(x)}{\beta}$$
$$-\left(\beta_2^{-}-\frac{\gamma_2}{2\beta}\right)\frac{R_2(p_2)}{2\beta}\int_0^x\left(q(t)+\frac{p^2(t)}{\beta^2}\right)dt\,\sin\frac{\omega(x)}{\beta} \tag{2.20}$$

$$A\left(x,b^{+}(x)+0\right)-A\left(x,b^{+}(x)-0\right)=-\left(\beta_2^{+}+\frac{\gamma_2}{2\beta}\right)\frac{R_1(p_2)}{2\beta^2}\left(p(x)-p(0)\right)\sin\frac{\omega(x)}{\beta}$$
$$-\left(\beta_2^{+}+\frac{\gamma_2}{2\beta}\right)\frac{R_1(p_2)}{2\beta}\int_0^x\left(q(t)+\frac{p^2(t)}{\beta^2}\right)dt\,\cos\frac{\omega(x)}{\beta} \tag{2.21}$$

$$B\left(x,b^{+}(x)+0\right)-B\left(x,b^{+}(x)-0\right)=\left(\beta_2^{+}+\frac{\gamma_2}{2\beta}\right)\frac{R_1(p_2)}{2\beta^2}\left(p(x)-p(0)\right)\cos\frac{\omega(x)}{\beta}$$
$$-\left(\beta_2^{+}+\frac{\gamma_2}{2\beta}\right)\frac{R_1(p_2)}{2\beta}\int_0^x\left(q(t)+\frac{p^2(t)}{\beta^2}\right)dt\,\sin\frac{\omega(x)}{\beta} \tag{2.22}$$

are held. If in addition we suppose that $p(x)\in W_2^2(0,\pi)$, $q(x)\in W_2^1(0,\pi)$, the functions $A(x,t)$, $B(x,t)$ the following system are provided.

$$\begin{cases}\dfrac{\partial^2 A(x,t)}{\partial x^2}-q(x)A(x,t)-2p(x)\dfrac{\partial B(x,t)}{\partial t}=\eta\dfrac{\partial^2 A(x,t)}{\partial t^2}\\[2mm] \dfrac{\partial^2 B(x,t)}{\partial x^2}-q(x)B(x,t)+2p(x)\dfrac{\partial A(x,t)}{\partial t}=\eta\dfrac{\partial^2 B(x,t)}{\partial t^2}\end{cases} \tag{2.23}$$

where

$$\eta=\begin{cases}\alpha^2 & p_1<x<p_2;\\ \beta^2 & p_2<x<\pi.\end{cases}$$

The proof is done as in [7].

Conversely, if the second order derivatives of functions $A(x,t)$, $B(x,t)$ are summable on $[0,\pi]$ and $A(x,t)$, $B(x,t)$ provides (2.23) system and equations (2.10) – (2.22), then the function $\varphi(x,\lambda)$ which is defined by (1.3) – (1.6) is a solution of (1.1) satisfying boundary conditions (1.2).

Definition 1. If there is a nontrivial solution $y_0(x)$ that provides the (1.2) conditions for the (1.1) problem, then λ_0 is called eigenvalue. Additionally, $y_0(x)$ is called the eigenfunction of the problem corresponding to the eigenvalue λ_0.

Let us assume that $q(x)$ satisfies the following conditions.

$$\int_0^\pi\left\{\left|y'(x)\right|^2+q(x)\left|y(x)\right|^2\right\}dx>0. \tag{2.24}$$

For all $y(x)\in W_2^2[0,\pi]$ such that $y(x)\neq 0$ and $y'(0)\cdot\overline{y(0)}-y'(\pi)\cdot\overline{y(\pi)}=0$.

Lemma 1. *The eigenvalues of the boundary value problem L are real.*

Proof. We set $l(y) := [-y'' + q(x) y]$. Integration by part yields

$$(l(y), y) = \int_0^{\pi} l(y) \cdot \overline{y(x)} dx = \int_0^{\pi} \left\{ |y'(x)|^2 + q(x) |y(x)|^2 \right\} dx. \tag{2.25}$$

Since condition (2.24) holds, it follow that $(l(y), y) > 0$. □

Lemma 2. *Eigenfunction corresponding to different eigenvalues of problem L are orthogonal in the sense of the equality*

$$(\lambda_n + \lambda_k) \int_0^{\pi} \delta(x) y(x, \lambda_n) y(x, \lambda_k) dx - 2 \int_0^{\pi} p(x) y(x, \lambda_n) y(x, \lambda_k) dx = 0. \tag{2.26}$$

The proof of Lemma 2 carried out as claim [14].

3. Properties of the spectrum

Let $\psi(x, \lambda)$ and $\varphi(x, \lambda)$ be any two solutions of equation (1.1),

$$W[\psi(x, \lambda), \varphi(x, \lambda)] = \psi(x, \lambda) \varphi'(x, \lambda) - \psi'(x, \lambda) \varphi(x, \lambda),$$

Wronskian dosen't depend on x. In this case, it depends only on the λ parameter. Although it is shown as $W[\psi, \varphi] = \Delta(\lambda)$. $\Delta(\lambda)$ is called the characteristic function of L . Clearly, the function $\Delta(\lambda)$ is entire in λ. It follows that, $\Delta(\lambda)$ has at most a countable set of zeros $\{\lambda_n\}$.

Lemma 3. *The zeros $\{\lambda_n\}$ of the characteristic function $\Delta(\lambda)$ coincide with the eigenvalues of the boundary value problem L . The functions $\psi(x, \lambda_0)$ and $\varphi(x, \lambda_0)$ are eigenfunctions corresponding to the eigenvalue λ_n, and there exist a sequence (β_n) such that*

$$\psi(x, \lambda_n) = \beta_n \varphi(x, \lambda_n), \qquad \beta_n \neq 0. \tag{3.1}$$

The proof of the Lemma 3 is done as in [27].

Let use denote

$$\alpha_n = \int_0^{\pi} \delta(x) \varphi^2(x, \lambda_n) dx - \frac{1}{\lambda_n} \int_0^{\pi} p(x) \varphi^2(x, \lambda_n) dx, \qquad n = 1, 2, 3, \ldots. \tag{3.2}$$

The numbers $\{\alpha_n\}$ are called normalized numbers of the problem L.

Lemma 4. *The equality $\overset{\bullet}{\Delta}(\lambda_n) = 2\lambda_n \beta_n \alpha_n$ is obtained. Here $\overset{\bullet}{\Delta} = \frac{d\Delta}{d\lambda}$.*

Proof. Since $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ are the solutions of (1.1),

$$-\varphi''(x, \lambda) + [2\lambda p(x) + q(x)] \varphi(x, \lambda) = \lambda^2 \delta(x) \varphi(x, \lambda),$$
$$-\psi''(x, \lambda) + [2\lambda p(x) + q(x)] \psi(x, \lambda) = \lambda^2 \delta(x) \psi(x, \lambda)$$

equations are provided. Hence, we differentiate the equalities with respect to

$$-\overset{\bullet}{\varphi}{}''(x, \lambda) + [2\lambda p(x) + q(x)] \overset{\bullet}{\varphi}(x, \lambda) = \lambda^2 \delta(x) \overset{\bullet}{\varphi}(x, \lambda) + [2\lambda \delta(x) - 2p(x)] \varphi(x, \lambda),$$

$$-\overset{\bullet}{\psi}''(x,\lambda)+[2\lambda p(x)+q(x)]\overset{\bullet}{\psi}(x,\lambda)=\lambda^2\delta(x)\overset{\bullet}{\psi}(x,\lambda)+[2\lambda\delta(x)-2p(x)]\psi(x,\lambda).$$

Thanks to these equations

$$\frac{d}{dx}\left\{\varphi(x,\lambda)\cdot\overset{\bullet}{\psi}'(x,\lambda)-\varphi'(x,\lambda)\cdot\overset{\bullet}{\psi}(x,\lambda)\right\}=-[2\lambda\delta(x)-2p(x)]\varphi(x,\lambda)\psi(x,\lambda),$$

$$\frac{d}{dx}\left\{\overset{\bullet}{\varphi}(x,\lambda)\cdot\psi'(x,\lambda)-\overset{\bullet}{\varphi}'(x,\lambda)\cdot\psi(x,\lambda)\right\}=[2\lambda\delta(x)-2p(x)]\varphi(x,\lambda)\psi(x,\lambda).$$

If the last equations are integrated from x to π and from 0 to x, respectively, by the discontinuity conditions, we obtain

$$-\left\{\varphi(\xi,\lambda)\cdot\overset{\bullet}{\psi}'(\xi,\lambda)-\varphi'(\xi,\lambda)\cdot\overset{\bullet}{\psi}(\xi,\lambda)\right\}\Big|_x^\pi =\int_x^\pi[2\lambda\delta(\xi)-2p(\xi)]\varphi(\xi,\lambda)\psi(\xi,\lambda)\,d\xi$$

and

$$\left\{\overset{\bullet}{\varphi}(\xi,\lambda)\cdot\psi'(\xi,\lambda)-\overset{\bullet}{\varphi}'(\xi,\lambda)\cdot\psi(\xi,\lambda)\right\}\Big|_0^x=\int_0^x[2\lambda\delta(\xi)-2p(\xi)]\varphi(\xi,\lambda)\psi(\xi,\lambda)\,d\xi.$$

If we add the last equalities side by side, we get

$$W\left[\varphi(\xi,\lambda),\dot{\psi}(\xi,\lambda)\right]+W\left[\dot{\varphi}(\xi,\lambda),\psi(\xi,\lambda)\right]=-\dot{\Delta}(\lambda) =\int_0^\pi[2\lambda\delta(\xi)-2p(\xi)]\varphi(\xi,\lambda)\psi(\xi,\lambda)\,d\xi$$

for $\lambda\to\lambda_n$, this yields

$$\overset{\bullet}{\Delta}(\lambda_n)=-\int_0^\pi[2\lambda_n\delta(\xi)-2p(\xi)]\beta_n\varphi^2(\xi,\lambda_n)\,d\xi$$
$$=2\lambda_n\beta_n\left\{\int_0^\pi\delta(\xi)\varphi^2(\xi,\lambda_n)\,d\xi-\frac{1}{\lambda_n}\int_0^\pi p(\xi)\varphi^2(\xi,\lambda_n)\,d\xi\right\}=2\lambda_n\beta_n\alpha_n.$$

Denote,

$$\Gamma_n=\left\{\lambda:|\lambda|=\left|\lambda_n^0\right|+\delta,\delta>0,n=0,1,2,\dots\right\},$$
$$G_n=\left\{\lambda:\left|\lambda-\lambda_n^0\right|\geq\delta,\delta>0,n=0,1,2,\dots\right\},$$

where δ is sufficiently small positive number. For sufficiently large values of n, one has

$$|\Delta(\lambda)-\Delta_0(\lambda)|<\frac{C_\delta}{2}e^{|\tau|(\beta\pi-\beta a_2+\alpha p_2-\alpha p_1+p_1)},\qquad\lambda\in\Gamma_n. \tag{3.3}$$

As it is shown in [19], $|\Delta_0(\lambda)|\geq C_\delta e^{|Im\lambda|\pi}$ for all $\lambda\in\bar{G}_\delta$, where $C_\delta>0$

$$\lim_{|\lambda|\to\infty}e^{-|Im\lambda|\pi}(\Delta(\lambda)-\Delta_0(\lambda))$$

$$= \lim_{|\lambda|\to\infty} e^{-|Im\lambda|\pi} \left(\int_0^\pi \tilde{A}(\pi,t)\cos\lambda t dt + \int_0^\pi \tilde{B}(\pi,t)\sin\lambda t dt \right) = 0$$

is constant. On the other hand, since for sufficiently large values of n (see[23]) we get (3.3). The Lemma 4 is proved. □

Lemma 5. *The problem $L(\alpha, p_1, p_2)$ has countable set of eigenvalues. If one denotes by $\lambda_1, \lambda_2, \ldots$ the positive eigenvalues arranged in increasing order and by $\lambda_{-1}, \lambda_{-2}, \ldots$ the negative eigenvalues arranged in decreasing order, then eigenvalues of the problem $L(\alpha, p_1, p_2)$ have the asymptotic behavior*

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{k_n}{\lambda_n^0} \qquad n \to \infty,$$

where $k_n \in l_2$, d_n is a bounded sequence and

$$\lambda_n^0 = \frac{n\pi}{\beta\pi - \beta p_2 + \alpha p_2 - \alpha p_1 + p_1} + \psi_1(n); \qquad \sup_n |\psi_1(n)| = c < +\infty.$$

Proof. According to previous lemma, if n is a sufficiently large natural number and $\lambda \in \Gamma_n$, we have $|\Delta_0(\lambda)| \geq C_\delta e^{|Im\lambda|\pi} > \frac{C_\delta}{2} e^{|Im\lambda|\pi} > |\Delta(\lambda) - \Delta_0(\lambda)|$. Applying Rouche's theorem, we conclude that for sufficiently large n inside the contour Γ_n the functions $\Delta_0(\lambda)$ and $\Delta_0(\lambda) + \{\Delta(\lambda) - \Delta_0(\lambda)\} = \Delta(\lambda)$ have the same number of zeros. That is, there are exactly $(n+1)$ zeros $\lambda_1, \lambda_2, \ldots, \lambda_n$. Analogously, it is shown by Rouche's theorem that, for sufficiently large values of n, the function $\Delta(\lambda)$ has a unique zero inside each circle $\left|\lambda - \lambda_n^0\right| < \delta$. Since $\delta > 0$ is a arbitrary, it follows that $\lambda_n = \lambda_n^0 + \varepsilon_n$, where $\lim_{n\to\infty} \varepsilon_n = 0$. If $\Delta(\lambda_n) = 0$, we have

$$\Delta_0\left(\lambda_n^0 + \varepsilon_n\right) + \int_0^\pi A(\pi,t)\cos\left(\lambda_n^0 + \varepsilon_n\right) t dt + \int_0^\pi B(\pi,t)\sin\left(\lambda_n^0 + \varepsilon_n\right) t dt = 0, \quad (3.4)$$

$$\begin{aligned}\Delta_0\left(\lambda_n^0 + \varepsilon_n\right) = &\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right) R_1(p_2)\cos\left[\left(\lambda_n^0 + \varepsilon_n\right) b^+(\pi) - \frac{1}{\beta}\int_{p_2}^\pi p(t)\,dt\right] \\ &+ \left(\beta_2^- + \frac{\gamma_2}{2\beta}\right) R_2(p_2)\cos\left[\left(\lambda_n^0 + \varepsilon_n\right) b^-(\pi) - \frac{1}{\beta}\int_{p_2}^\pi p(t)\,dt\right] \\ &+ \left(\beta_2^- - \frac{\gamma_2}{2\beta}\right) R_1(p_2)\cos\left[\left(\lambda_n^0 + \varepsilon_n\right) s^+(\pi) + \frac{1}{\beta}\int_{p_2}^\pi p(t)\,dt\right] \\ &+ \left(\beta_2^+ - \frac{\gamma_2}{2\beta}\right) R_2(p_2)\cos\left[\left(\lambda_n^0 + \varepsilon_n\right) s^-(\pi) + \frac{1}{\beta}\int_{p_2}^\pi p(t)\,dt\right].\end{aligned} \quad (3.5)$$

Since $\Delta_0(\lambda)$ is an analytical function,

$$\Delta_0\left(\lambda_n^0 + \varepsilon_n\right) = \Delta_0\left(\lambda_n^0\right)\varepsilon_n + \dot{\underset{0}{\Delta}}\left(\lambda_n^0\right)\varepsilon_n + \frac{\ddot{\underset{0}{\Delta}}\left(\lambda_n^0\right)}{2!}\varepsilon_n^2 + \ldots, \qquad \lim_{n\to\infty} \varepsilon_n = 0.$$

λ_n^0 is the roots of the $\Delta_0(\lambda)=0$ equation $\Delta_0\left(\lambda_n^0+\varepsilon_n\right)=\left[\underset{0}{\dot{\Delta}}\left(\lambda_n^0\right)+o(1)\right]\varepsilon_n$, $n\to\infty$ is provided.

$$\begin{aligned}
&\left[\underset{0}{\dot{\Delta}}\left(\lambda_n^0\right)+o(1)\right]\varepsilon_n+\int_{p_2}^{s^-(x)-0}A(\pi,t)\cos\left(\lambda_n^0+\varepsilon_n\right)tdt\\
&\quad+\int_{s^-(x)+0}^{s^+(x)-0}A(\pi,t)\cos\left(\lambda_n^0+\varepsilon_n\right)tdt+\int_{s^+(x)+0}^{b^-(x)-0}A(\pi,t)\cos\left(\lambda_n^0+\varepsilon_n\right)tdt\\
&\quad+\int_{b^-(x)+0}^{b^+(x)-0}A(\pi,t)\cos\left(\lambda_n^0+\varepsilon_n\right)tdt+\int_{b^+(x)+0}^{x}A(\pi,t)\cos\left(\lambda_n^0+\varepsilon_n\right)tdt\\
&\quad+\int_{p_2}^{s^-(x)-0}B(\pi,t)\sin\left(\lambda_n^0+\varepsilon_n\right)tdt+\int_{s^-(x)+0}^{s^+(x)-0}B(\pi,t)\sin\left(\lambda_n^0+\varepsilon_n\right)tdt\\
&\quad+\int_{s^+(x)+0}^{b^-(x)-0}B(\pi,t)\sin\left(\lambda_n^0+\varepsilon_n\right)tdt+\int_{b^-(x)+0}^{b^+(x)-0}B(\pi,t)\sin\left(\lambda_n^0+\varepsilon_n\right)tdt\\
&\qquad\qquad+\int_{b^+(x)+0}^{x}B(\pi,t)\sin\left(\lambda_n^0+\varepsilon_n\right)tdt=0
\end{aligned}$$

It is easy to see that the function $\Delta_0(\lambda)=0$ is type of [16], so there is a $\eta_\delta>0$ such that $\left|\underset{0}{\dot{\Delta}}\left(\lambda_n^0\right)\right|\geq\eta_\delta>0$ is satisfied for all n. We also have

$$\lambda_n^0=\frac{n\pi}{\beta\pi-\beta p_2+\alpha p_2-\alpha p_1+p_1}+\psi_1(n), \tag{3.6}$$

where $\sup\limits_n|\psi_1(n)|<M$ is for some constant $M>0$ [18]. Further, substituting (3.6) into (3.5) after certain transformations, we reach $\varepsilon_n\in l_2$.

Since $\left(\int_0^\pi A_t(\pi,t)\sin\left(\lambda_n^0+\varepsilon_n\right)tdt\right)\in l_2$ and $\left(\int_0^\pi B_t(\pi,t)\cos\left(\lambda_n^0+\varepsilon_n\right)tdt\right)\in l_2$, we have

$$\begin{aligned}
\varepsilon_n=&\frac{1}{2\lambda_n^0\Delta_0\left(\lambda_n^0\right)}\left\{\left[\left(\beta_2^--\frac{\gamma_2}{2\beta}\right)\frac{R_2(p_2)}{2\beta}\sin\left[\lambda_n^0s^-(\pi)+\frac{\omega(x)}{\beta}\right]\right.\right.\\
&+\left(\beta_2^--\frac{\gamma_2}{2\beta}\right)\frac{R_1(p_2)}{2\beta}\sin\left[\lambda_n^0s^+(\pi)+\frac{\omega(x)}{\beta}\right]\\
&+\left(\beta_2^--\frac{\gamma_2}{2\beta}\right)\frac{R_2(p_2)}{2\beta}\sin\left[\lambda_n^0b^-(\pi)-\frac{\omega(x)}{\beta}\right]\\
&+\left.\left(\beta_2^++\frac{\gamma_2}{2\beta}\right)\frac{R_1(p_2)}{2\beta}\sin\left[\lambda_n^0b^+(\pi)-\frac{\omega(x)}{\beta}\right]\right]\int_0^\pi\left(q(t)+p^2(t)\right)dt\\
&+\left[-\left(\beta_2^--\frac{\gamma_2}{2\beta}\right)\frac{R_2(p_2)}{2\beta^2}\cos\left[\lambda_n^0s^-(\pi)+\frac{\omega(x)}{\beta}\right]\right.
\end{aligned}$$

$$-\left(\beta_2^- - \frac{\gamma_2}{2\beta}\right)\frac{R_1(p_2)}{2\beta^2}\cos\left[\lambda_n^0 s^+(\pi)+\frac{\omega(x)}{\beta}\right]$$
$$+\left(\beta_2^- + \frac{\gamma_2}{2\beta}\right)\frac{R_2(p_2)}{2\beta^2}\cos\left[\lambda_n^0 b^-(\pi)-\frac{\omega(x)}{\beta}\right]$$
$$+\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)\frac{R_1(p_2)}{2\beta^2}\cos\left[\lambda_n^0 b^+(\pi)-\frac{\omega(x)}{\beta}\right]\Bigg][p(\pi)-p(0)]\Bigg\}+\frac{k_n}{\lambda_n^0},$$

where

$$\begin{aligned}
d_n =&\frac{1}{2\Delta_0(\lambda_n^0)}\left\{\left[\left(\beta_2^- - \frac{\gamma_2}{2\beta}\right)\frac{R_2(p_2)}{2\beta}\sin\left[\lambda_n^0 s^-(\pi)+\frac{\omega(x)}{\beta}\right]\right.\right.\\
&+\left(\beta_2^- - \frac{\gamma_2}{2\beta}\right)\frac{R_1(p_2)}{2\beta}\sin\left[\lambda_n^0 s^+(\pi)+\frac{\omega(x)}{\beta}\right]\\
&+\left(\beta_2^- - \frac{\gamma_2}{2\beta}\right)\frac{R_2(p_2)}{2\beta}\sin\left[\lambda_n^0 b^-(\pi)-\frac{\omega(x)}{\beta}\right]\\
&\left.+\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)\frac{R_1(p_2)}{2\beta}\sin\left[\lambda_n^0 b^+(\pi)-\frac{\omega(x)}{\beta}\right]\right]\int_0^\pi\left(q(t)+p^2(t)\right)dt\\
&+\left[-\left(\beta_2^- - \frac{\gamma_2}{2\beta}\right)\frac{R_2(p_2)}{2\beta^2}\cos\left[\lambda_n^0 s^-(\pi)+\frac{\omega(x)}{\beta}\right]\right.\\
&-\left(\beta_2^- - \frac{\gamma_2}{2\beta}\right)\frac{R_1(p_2)}{2\beta^2}\cos\left[\lambda_n^0 s^+(\pi)+\frac{\omega(x)}{\beta}\right]\\
&+\left(\beta_2^- + \frac{\gamma_2}{2\beta}\right)\frac{R_2(p_2)}{2\beta^2}\cos\left[\lambda_n^0 b^-(\pi)-\frac{\omega(x)}{\beta}\right]\\
&\left.\left.+\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)\frac{R_1(p_2)}{2\beta^2}\cos\left[\lambda_n^0 b^+(\pi)-\frac{\omega(x)}{\beta}\right]\right][p(\pi)-p(0)]\right\}
\end{aligned}$$

is bounded sequence. The proof is completed. □

The $\varphi(x,\lambda)$ function is $|\lambda|\to\infty$ in the region $D=\{\lambda : \arg\lambda\in[\varepsilon,\pi-\varepsilon]\}$ for $x>p_2$,

$$\varphi(x,\lambda)=\frac{1}{2}\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)\exp\left(-i\left(\lambda b^+(x)-w(x)\right)\right)\left(1+O\left(\frac{1}{\lambda}\right)\right)\qquad |\lambda|\to\infty$$

it has an asymptotic representation where $w(x)=\int_{p_2}^x p(t)\,dt$ and $\beta_2^\mp=\frac{1}{2}\left(\alpha_2\mp\frac{\alpha\beta_2}{\beta}\right)$.

4. Inverse Problem

Let us consider the boundary value problem $\tilde{L}$:

$$\tilde{L} := \begin{cases} l(y) := -y'' + [2\lambda \tilde{p}(x) + \tilde{q}(x)]\, y = \lambda^2 \tilde{\delta}(x)\, y,\ x \in (0,\pi) \\ y'(0) = 0, y(\pi) = 0 \\ y(\tilde{p}_1 + 0) = \tilde{\alpha}_1 y(\tilde{p}_1 - 0) \\ y'(\tilde{p}_1 + 0) = \tilde{\beta}_1 y'(\tilde{p}_1 - 0) + i\lambda\tilde{\gamma}_1 y(\tilde{p}_1 - 0) \\ y(\tilde{p}_2 + 0) = \tilde{\alpha}_2 y(\tilde{p}_2 - 0) \\ y'(\tilde{p}_2 + 0) = \tilde{\beta}_2 y'(\tilde{p}_2 - 0) + i\lambda\tilde{\gamma}_2 y(\tilde{p}_2 - 0) \end{cases}$$

Let the function $\Phi(x,\lambda)$ denote solution of (1.1) that satisfy the conditions $\Phi'(0) = 1$, $\Phi(\pi) = 0$ respectively and jump conditions $(1.3) - (1.6)$. Lets define it as $M(\lambda) := \Phi(0,\lambda)$. The $\Phi(x,\lambda)$ and $M(\lambda)$ functions are called the Weyl solution and the Weyl function, respectively.

$$\Phi(x,\lambda) = M(\lambda)\,.\varphi(x,\lambda) + S(x,\lambda) \qquad \lambda \neq \lambda_n, \qquad n = 1,2,3,\ldots$$

is true. Because of $W[\varphi,S]|_{x=0} = \varphi(0,\lambda)\, S'(0,\lambda) - \varphi'(0,\lambda)\, S(0,\lambda) = 1 \neq 0$, $\varphi(x,\lambda)$ and $S(x,\lambda)$ solutions are linear independent. When $\psi(x,\lambda)$ is solution (1.1),

$$\begin{aligned} \psi(x,\lambda) &= A(\lambda)\,\varphi(x,\lambda) + B(\lambda)\, S(x,\lambda), \\ \psi'(x,\lambda) &= A(\lambda)\,\varphi'(x,\lambda) + B(\lambda)\, S'(x,\lambda). \end{aligned}$$

Due to boundary conditions, $A(\lambda) = \psi(0,\lambda)$, $B(\lambda) = \psi'(0,\lambda) = -\Delta(\lambda)$. Then $\psi(x,\lambda) = \psi(0,\lambda)\,\varphi(x,\lambda) - \Delta(\lambda)\, S(x,\lambda)$ is obtained. Hence,

$$\Phi(x,\lambda) := -\frac{\psi(x,\lambda)}{\Delta(\lambda)} = S(x,\lambda) + M(\lambda)\,\varphi(x,\lambda), \qquad M(\lambda) = -\frac{\psi(0,\lambda)}{\Delta(\lambda)}.$$

The $M(\lambda)$ function is a meromorphic function.

Theorem 3. *If $M(\lambda) = \tilde{M}(\lambda)$, then $L = \tilde{L}$.*

Proof. Let us define the matrix $P(x,\lambda) = [P_{j,k}(x,\lambda)]$, $(j,k = 1,2)$ by the formula

$$P(x,\lambda) \cdot \begin{pmatrix} \widetilde{\varphi}(x,\lambda)\widetilde{\Phi}(x,\lambda) \\ \widetilde{\varphi}'(x,\lambda)\widetilde{\Phi}'(x,\lambda) \end{pmatrix} = \begin{pmatrix} \varphi(x,\lambda)\Phi(x,\lambda) \\ \varphi'(x,\lambda)\Phi'(x,\lambda) \end{pmatrix}.$$

In this case

$$\begin{aligned} P_{11}(x,\lambda) &= -\varphi(x,\lambda)\,\frac{\widetilde{\psi}'(x,\lambda)}{\widetilde{\Delta}(\lambda)} + \widetilde{\varphi}'(x,\lambda)\,\frac{\psi(x,\lambda)}{\Delta(\lambda)}, \\ P_{12}(x,\lambda) &= -\widetilde{\varphi}(x,\lambda)\,\frac{\psi(x,\lambda)}{\Delta(\lambda)} + \varphi(x,\lambda)\,\frac{\widetilde{\psi}(x,\lambda)}{\widetilde{\Delta}(\lambda)}, \end{aligned}$$

$$P_{21}(x,\lambda) = -\varphi'(x,\lambda)\frac{\widetilde{\psi}'(x,\lambda)}{\widetilde{\Delta}(\lambda)} - \widetilde{\varphi}'(x,\lambda)\frac{\psi'(x,\lambda)}{\Delta(\lambda)},$$

$$P_{22}(x,\lambda) = -\widetilde{\varphi}(x,\lambda)\frac{\psi'(x,\lambda)}{\Delta(\lambda)} + \varphi'(x,\lambda)\frac{\widetilde{\psi}(x,\lambda)}{\widetilde{\Delta}(\lambda)}.$$

Hence,

$$\begin{aligned}
P_{11}(x,\lambda) =& \varphi(x,\lambda)\left[\widetilde{S}'(x,\lambda) + \widetilde{M}(\lambda)\cdot\widetilde{\varphi}'(x,\lambda)\right] - \widetilde{\varphi}'(x,\lambda)\left[S(x,\lambda) + M(\lambda)\cdot\varphi(x,\lambda)\right] \\
=& \varphi(x,\lambda)\widetilde{S}'(x,\lambda) - \widetilde{\varphi}'(x,\lambda)S(x,\lambda) + \left[\widetilde{M}(\lambda) - M(\lambda)\right]\varphi(x,\lambda)\widetilde{\varphi}'(x,\lambda), \\
P_{12}(x,\lambda) =& \widetilde{\varphi}(x,\lambda)\left[S(x,\lambda) + M(\lambda)\cdot\varphi(x,\lambda)\right] - \varphi(x,\lambda)\left[\widetilde{S}(x,\lambda) + \widetilde{M}(\lambda)\cdot\widetilde{\varphi}(x,\lambda)\right] \\
=& \widetilde{\varphi}(x,\lambda)S(x,\lambda) - \varphi(x,\lambda)\widetilde{S}(x,\lambda) + \left[M(\lambda) - \widetilde{M}(\lambda)\right]\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda), \\
P_{21}(x,\lambda) =& \varphi'(x,\lambda)\left[\widetilde{S}'(x,\lambda) + \widetilde{M}(\lambda)\cdot\widetilde{\varphi}'(x,\lambda)\right] - \widetilde{\varphi}'(x,\lambda)\left[S'(x,\lambda) + M(\lambda)\cdot\varphi'(x,\lambda)\right] \\
=& \varphi'(x,\lambda)\widetilde{S}'(x,\lambda) - \widetilde{\varphi}'(x,\lambda)S'(x,\lambda) + \left[\widetilde{M}(\lambda) - M(\lambda)\right]\varphi'(x,\lambda)\widetilde{\varphi}'(x,\lambda), \\
P_{22}(x,\lambda) =& \widetilde{\varphi}(x,\lambda)\left[S'(x,\lambda) + M(\lambda)\cdot\varphi'(x,\lambda)\right] + \varphi'(x,\lambda)\left[\widetilde{S}(x,\lambda) + \widetilde{M}(\lambda)\cdot\widetilde{\varphi}(x,\lambda)\right] \\
=& \varphi'(x,\lambda)S'(x,\lambda) - \varphi'(x,\lambda)\widetilde{S}(x,\lambda) + \left[M(\lambda) - \widetilde{M}(\lambda)\right]\varphi'(x,\lambda)\widetilde{\varphi}(x,\lambda).
\end{aligned}$$

from $M(\lambda) \equiv \widetilde{M}(\lambda)$:

$$\begin{aligned}
P_{11}(x,\lambda) &= \varphi(x,\lambda)\widetilde{S}'(x,\lambda) - \widetilde{\varphi}'(x,\lambda)S(x,\lambda), \\
P_{12}(x,\lambda) &= \widetilde{\varphi}(x,\lambda)S(x,\lambda) - \varphi(x,\lambda)\widetilde{S}(x,\lambda), \\
P_{21}(x,\lambda) &= \varphi'(x,\lambda)\widetilde{S}'(x,\lambda) - \widetilde{\varphi}'(x,\lambda)S'(x,\lambda), \\
P_{22}(x,\lambda) &= \varphi'(x,\lambda)S'(x,\lambda) - \varphi'(x,\lambda)\widetilde{S}(x,\lambda)
\end{aligned}$$

are obtained. When $M(\lambda) \equiv \widetilde{M}(\lambda)$, it is clear that the $P_{j,k}(x,\lambda), (j,k = 1,2)$ functions are full functions according to λ. From (3.3); for $\forall x \in [0,\pi]$, c_δ , C_δ constants that provide $|P_{11}(x,\lambda)| \le c_\delta$ and $|P_{12}(x,\lambda)| \le C_\delta$ inequalities can be shown. From the Liouville theorem $P_{11}(x,\lambda) \equiv A(x)$ and $P_{12}(x,\lambda) \equiv 0$. From

$$\varphi(x,\lambda)\cdot\Phi'(x,\lambda) - \widetilde{\varphi}'(x,\lambda)\cdot\Phi(x,\lambda) = A(x),$$

$$\widetilde{\varphi}(x,\lambda)\cdot\Phi(x,\lambda) - \varphi(x,\lambda)\cdot\widetilde{\Phi}(x,\lambda) = 0$$

$$\varphi(x,\lambda) = \widetilde{\varphi}(x,\lambda)\cdot A(x), \qquad \Phi(x,\lambda) = \widetilde{\Phi}(x,\lambda)\cdot A(x) \tag{4.1}$$

are obtained and

$$W[\varphi,\Phi] = W\left[\varphi(x,\lambda), -\frac{\psi(x,\lambda)}{\Delta(\lambda)}\right]$$

$$=\frac{1}{\Delta(\lambda)}W\left[\varphi(x,\lambda),-\psi(0,\lambda)\varphi(x,\lambda)+\Delta(\lambda)S(x,\lambda)\right]$$
$$=-\frac{\psi(0,\lambda)}{\Delta(\lambda)}W\left[\varphi(x,\lambda),\varphi(x,\lambda)\right]+W\left[\varphi(x,\lambda),S(x,\lambda)\right]=1.$$

And similarly $W\left[\widetilde{\varphi},\widetilde{\Phi}\right]=1$ is obtained. If this equation is written in place of (4.1),

$$1=W\left[\varphi(x,\lambda),\Phi(x,\lambda)\right]=W\left[A(x)\widetilde{\varphi}(x,\lambda),A(x)\widetilde{\Phi}(x,\lambda)\right]$$
$$=A^2(x)W\left[\widetilde{\varphi}(x,\lambda),\widetilde{\Phi}(x,\lambda)\right]=A^2(x)$$

is obtained.

Therefore, $\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)\neq 1;\ p_1=\tilde{p}_1,\ p_2=\tilde{p}_2$. We have $A(x)=1$ from (4.1) $\varphi(x,\lambda)\equiv\tilde{\varphi}(x,\lambda)$ and $\Phi(x,\lambda)\equiv\tilde{\Phi}(x,\lambda)$.

When $\varphi(x,\lambda)\equiv\tilde{\varphi}(x,\lambda)$,

$$-\varphi''+\left[2\lambda p(x)+q(x)\right]\varphi=\lambda^2\delta(x)\varphi,$$
$$-\varphi''+\left[2\lambda p(x)+q(x)\right]\varphi=\lambda^2\tilde{\delta}(x)\varphi$$

are obtained.

$$\left\{\lambda^2\left(\delta(x)-\tilde{\delta}(x)\right)+2\lambda\left(p(x)-\tilde{p}(x)\right)+\left(q(x)-\tilde{q}(x)\right)\right\}\varphi\equiv 0 \qquad (\text{ for } \forall\,\lambda\ )$$

$\delta(x)=\tilde{\delta}(x)$, $p(x)=\tilde{p}(x)$ and $q(x)=\tilde{q}(x)$ a.e. For every λ in discontinuity conditions,

$$(\alpha_1-\tilde{\alpha}_1)\varphi(p_1-0,\lambda)=0$$
$$\left(\beta_1-\tilde{\beta}_1\right)\varphi'(p_1-0,\lambda)+(\gamma_1-\tilde{\gamma}_1)\varphi(p_1-0,\lambda)=0$$
$$(\alpha_2-\tilde{\alpha}_2)\varphi(p_2-0,\lambda)=0$$
$$\left(\beta_2-\tilde{\beta}_2\right)\varphi'(p_2-0,\lambda)+(\gamma_2-\tilde{\gamma}_2)\varphi(p_2-0,\lambda)=0$$

$\alpha_1=\tilde{\alpha}_1,\beta_1=\tilde{\beta}_1,\gamma_1=\tilde{\gamma}_1$ and $\alpha_2=\tilde{\alpha}_2,\beta_2=\tilde{\beta}_2,\gamma_2=\tilde{\gamma}_2$.
Consequently $L=\tilde{L}$. The proof is completed. □

REFERENCES

[1] V. Ambarzumian, "Uber eine frage der eigenwerttheorie," *Zeitschrift fur Physik*, vol. 53, no. 9-10, pp. 690–695, sep 1929, doi: 10.1007/bf01330827.
[2] R. K. Amirov and A. A. Nabiev, "Inverse problems for the quadratic pencil of the Sturm-Liouville equations with impulse," *Abstract and Applied Analysis*, vol. 2013, pp. 1–10, 2013, doi: 10.1155/2013/361989.
[3] L. Anderson, "Inverse eigenvalue problems with discontiuous coefficients," *Inverse Problems*, vol. 4, pp. 353–397, 1998.

[4] G. Borg, "Eine umkehrung der sturm-liouvilleschen eigenwertaufgabe," *Acta Mathematica*, vol. 78, pp. 1–96, 1946.

[5] R. Carlson, "An inverse spectral problem for Sturm-Liouville operators with discontinuous coefficients," *Proceedings of the American Mathematical Society*, vol. 120, no. 2, pp. 475–475, feb 1994, doi: 10.1090/s0002-9939-1994-1197532-5.

[6] A. Ergün, "A half-inverse problem for the singular diffusion operator with jump conditions," *Miskolc Mathematical Notes*, to appear.

[7] A. Ergün and R. Amirov, "Direct and inverse problem for diffusion operator with discontinuoity points," *TWMS J. App. Eng. Math.*, vol. 9, pp. 9–21, 2019.

[8] A. Ergün and R. Amirov, "Properties of the spectrum for the singular diffusion operator," *Asian Journal of Mathematics and Computer Research*, vol. 26, pp. 74–87, 2019.

[9] G. Freiling and V. Yurko, "Inverse spectral problems for singular non-selfadjoint differential operators with discontinuities in an interior point," *Inverse Problems*, vol. 18, no. 3, pp. 757–773, apr 2002, doi: 10.1088/0266-5611/18/3/316.

[10] S. Gala, Q. Liu, and M. Ragusa, "A new regularity criterion for the nematic liquid crystal flows," *Applicable Analysis*, vol. 91, no. 9, pp. 1741–1747, 2012.

[11] S. Gala and M. Ragusa, "Logarithmically improved regularity criterion for the boussinesq equations in besov spaces with negative indices," *Applicable Analysis*, vol. 95, no. 6, pp. 1271–1279, 2016.

[12] M. Gasymov and G. Guseinov, "Determination of a diffusion operator from spectral data," *Akad. Nauk Azerbaidzhanskoi SSR*, vol. 37, pp. 19–23, 1981.

[13] G. Guseinov, "On the spectral analysis of a quadratic pencil of Sturm-Liouville operators," *Dodlady Akad. Nauk SSSR*, vol. 285, pp. 1292–1296, 1985.

[14] G. Guseinov, "Inverse spectral problems for a quadratic pencil of Sturm-Liouville operators on a finite interval," *Spectral Theory of Operators and Its Applications*, pp. 51–101, 1986.

[15] O. Hald, "Discontinuous inverse eigenvalue problems," *Comm. on Pure and Appl. Math*, vol. 37, pp. 539–577, 1984.

[16] B. Jdanovich, "Formulae for the zeros of drichlet polynomials and quasi-polynomials," *Doklady Akad. Nauk SSSR*, vol. 135, pp. 1046–1049, 1960.

[17] H. Koyunbakan and E. S. Panakhov, "Half-inverse problem for diffusion operators on the finite interval," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 1024–1030, feb 2007, doi: 10.1016/j.jmaa.2006.03.068.

[18] M. Krein and B. Y. Levin, "On entire almost periodic functions of exponential type," *Doklady Akad. Nauk SSSR.*, vol. 64, pp. 285–287, 1949.

[19] B. I. Levin, *Entire Functions*. MGV, 1971.

[20] B. Levitan, *Inverse Sturm-Liouville problems*. Netherlands: VNU Science Press., 1987.

[21] B. Levitan and M. Gasymov, "Determination of a differantial equation by two spectra," *Uspekhi Mathematicheskikh Nauk.*, vol. 19, pp. 3–63, 1964.

[22] B. Levitan and I. Sargsyan, *Introduction to Spectral Theory*. Amer. Math. Soc., 1975.

[23] V. Marchenko, *Sturm-Liouville Operators and Applications*. AMS Chelsea Publishing, 1986.

[24] E. Maris and S. Goktas, "A study of the uniform convergence of spectral expansions for the continuous functions on a sturm-liouville problem," *Miskolc Mathematical Notes*, vol. 20, no. 2, pp. 1063–1086, 2019.

[25] M. Nabiev, "The inverse spectral problem for the diffusion operator on an interval," *Mathematicheskaya Fizika, Analiz, Geometriya*, vol. 11, pp. 302–313, 2004.

[26] C.-F. Yang, "Reconstruction of the diffusion operator from nodal data," *Zeitschrift für Naturforschung A*, vol. 65, no. 1-2, pp. 100–106, jan 2010, doi: 10.1515/zna-2010-1-211.

[27] V. A. Yurko, *Inverse Spectral Problems for Linear Differential Operators and Their Applications*. CRC Press, 2000. [Online]. Available: https://www.ebook.de/de/product/4351615/v_a_yurko_inverse_spectral_problems_for_linear_differential_operators_and_their_applications.html
[28] V. Yurko, *Introduction to the Theory of Inverse Spectral Problems*. Fizmatlit. Moscow, 2007.

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ON THE GENERALIZED k–FIBONACCI NUMBERS

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Abstract. In this paper, new types of k–Fibonacci numbers are described, with respect to the definition of the distance between numbers using a recurrence relation. However, these sequences differ not only by the value of the natural number k but also according to the value of a new parameter r which is used in defining this distance. Furthermore, various properties of these new numbers are discussed.

In the second part of this paper, we apply the binomial transform to these generalized k–Fibonacci sequences.

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1. INTRODUCTION

Classical Fibonacci numbers have been generalized in different ways [8–11]. One such generalization which has recently increased interest among researchers mathematical terms pertains to the k–Fibonacci numbers [4, 5].

k–Fibonacci numbers are defined by the recurrence relation $F_{k,n+1} = k F_{k,n} + F_{k,n-1}$ for $n \geq 1$ with the initial conditions $F_{k,0} = 0;\ F_{k,1} = 1$. If $F_k = \{F_{k,n}\}$, then F_1 is the classical Fibonacci sequence while F_2 is the Pell sequence.

2. GENERALIZED k–FIBONACCI NUMBERS

In this section we apply the definition of r–distance to k–Fibonacci numbers in a generalized approach to previous results [2, 12]. The formulas used to calculate the general term of the sequences generated by the above definition are very interesting and they allow us to find the sum of n first terms.

Definition 1. With respect to natural numbers $k \geq 1,\ n \geq 0,\ r \geq 1$, we define the generalized (k,r)–Fibonacci numbers $F_{k,n}(r)$ by the recurrence relation

$$F_{k,n}(r) = k\, F_{k,n-r}(r) + F_{k,n-2}(r) \quad \text{for} \quad n \geq r, \tag{2.1}$$

with initial conditions $F_{k,n}(r) = 1,\ n = 0, 1, 2, \ldots r-1$, except $F_{k,1}(1) = k$.

The following proposition shows the formulae used to calculate the general term of the sequence $F_k(r)$, where $r \geq 2$ is odd or even (for $r = 1$ see [4,5]).

Theorem 1 (Main formula)**.** (1) *If r is even, $r = 2p$:*

$$F_{k,2n}(2p) = F_{k,2n+1}(2p) = \sum_{j=0}^{n/p} \binom{n-(p-1)j}{j} k^j. \tag{2.2}$$

(2) *If r is odd, $r = 2p+1 \geq 3$:*

$$F_{k,2n}(2p+1) = \sum_{j=0} \left[\binom{n-(2p-1)j}{2j} k^{2j} + \binom{n-p-(2p-1)j}{2j+1} k^{2j+1} \right], \tag{2.3}$$

$$F_{k,2n+1}(2p+1) = \sum_{j=0} \left[\binom{n-(2p-1)j}{2j} k^{2j} + \binom{n-(p-1)-(2p-1)j}{2j+1} k^{2j+1} \right]. \tag{2.4}$$

Proof. By induction.

Formula (2.2). Let $r = 2p$.

For $n = 0$, by definition, let $F_{k,0}(2p) = 1$, and the right hand side (RHS) of (2.2) is

$$F_{k,0}(2p) = \sum_{0}^{0} \binom{(1-p)j}{j} k^j = 1.$$

For $n = 1$, by definition, let $F_{k,2}(2) = 1+k$, and $F_{k,2}(r) = 1$ for $r > 2$. In formula (2.2) we have

$$F_{k,2}(2p) = \sum_{0}^{2} \binom{1-(p-1)j}{j} k^j = 1 + \binom{1-(p-1)j}{1} k.$$

Then, $F_{k,2}(2) = 1+k$ and $F_{k,2}(2p) = 1$ for $2p = 4, 6, 8, \ldots$

Suppose this formula holds for n. Then

$$\begin{aligned}
F_{k,2n+2}(2p) &= kF_{k,2n+2-2p}(2p) + F_{k,2n}(2p) \\
&= \sum_{j=0} \binom{n-(p-1)j}{j} k^j + kF_{k,2(n+1-p)}(2p) \\
&= \sum_{j=0} \binom{n-(p-1)j}{j} k^j + k \sum_{j=0} \binom{n+1-p-(p-1)j}{j} k^j \\
&= 1 + \sum_{j=1} \binom{n-(p-1)j}{j} k^j + \sum_{j=0} \binom{n-(p-1)(j+1)}{j} k^{j+1} \\
&= 1 + \sum_{j=0} \left[\binom{n-(p-1)(j+1)}{j+1} k^{j+1} + \binom{n-(p-1)(j+1)}{j} k^{j+1} \right] \\
&= 1 + \sum_{j=0} \binom{n-(p-1)(j+1)+1}{j+1} k^{j+1}
\end{aligned}$$

$$= \sum_{j=0} \binom{n+1-(p-1)j}{j} k^j = F_{k,2n+2}(2p)$$

as we proposed to prove.

We will prove formulae (2.3) and (2.4) together.

For $n=0$ we have $F_{k,0}(2p+1)=1$ and the RHS of (2.3) is

$$\sum_0^0 \left[\binom{-(2p-1)j}{2j} k^{2j} + \binom{-p-(2p-1)j}{2j+1} k^{2j+1} \right] = 1.$$

In the same manner, we have $F_{k,1}(2p+1)=1$ and the RHS of (2.4) is

$$\sum_0^0 \left[\binom{-(2p-1)j}{2j} k^{2j} + \binom{1-p-(2p-1)j}{2j+1} k^{2j+1} \right] = 1 \text{ because } 1-p<0.$$

For $n=1$ we have $F_{k,2}(2p+1)=1$ and the RHS of (2.3) is

$$\sum_0^0 \left[\binom{1-(2p-1)j}{2j} k^{2j} + \binom{1-p-(2p-1)j}{2j+1} k^{2j+1} \right] = 1,$$

because the condition $p \geq 1$ involves $1-p-(2p-1)j<0$.

For formula (2.4), if $p=1$, the left hand side (LHS) is $F_{k,3}(2p+1)=1+k$, while the RHS is

$$\sum_0^0 \left[\binom{1-j}{2j} k^{2j} + \binom{1-j}{2j+1} k^{2j+1} \right] = 1+k.$$

If $p>1$, the LHS and the RHS of (2.4) are equal to 1.

Suppose the formula holds for $2n$ and $2n+1$. Then

$$\begin{aligned}
F_{k,2n+2}(2p+1) &= kF_{k,2n+1-2p}(2p+1) + F_{k,2n}(2p+1) \\
&= kF_{k,2(n-p)+1}(2p+1) + F_{k,2n}(2p+1) \\
&= k\sum_{j=0} \left[\binom{n-p-(2p-1)j}{2j} k^{2j} + \binom{n-(2p-1)-(2p-1)j}{2j+1} k^{2j+1} \right] \\
&\quad + \sum_{j=0} \left[\binom{n-(2p-1)j}{2j} k^{2j} + \binom{n-p-(2p-1)j}{2j+1} k^{2j+1} \right] \\
&= \sum_{j=0} \left[\binom{n-p-(2p-1)j}{2j} + \binom{n-p-(2p-1)j}{2j+1} \right] k^{2j+1} \\
&\quad + \sum_{j=0} \binom{n-(2p-1)(j+1)}{2j+1} k^{2j+2} + 1 + \sum_{j=1} \binom{n-(2p-1)j}{2j} k^{2j} \\
&= \sum_{j=0} \binom{n+1-p-(2p-1)j}{2j+1} k^{2j+1}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}\binom{n-(2p-1)j}{2j-1}k^{2j}+1+\sum_{j=1}\binom{n-(2p-1)j}{2j}k^{2j}\\
&=\sum_{j=0}\binom{n+1-p-(2p-1)j}{2j+1}k^{2j+1}+\sum_{j=1}\binom{n+1-(2p-1)j}{2j}k^{2j}+1\\
&=\sum_{j=0}\binom{n+1-p-(2p-1)j}{2j+1}k^{2j+1}+\sum_{j=0}\binom{n+1-(2p-1)j}{2j}k^{2j}\\
&=F_{k,2n+2}(2p+1).
\end{aligned}
$$

A similar development shows the formula for $F_{k,2n+3}(2p+1)$. □

In [3] the following formulas are proven:

(1) Sum: $S_{k,n}(r)=\frac{1}{k}\Big(F_{k,n+r-1}(r)+F_{k,n+r}(r)-2\Big)$.

(2) Generating function: $f_k(r,x)=\dfrac{1+x}{1-x^2-k\,x^r}$.

3. Binomial Transform of the Generalized k–Fibonacci Numbers

In this section we will apply the binomial transform to the preceding sequences and will obtain new integer sequences [7].

Definition 2. Binomial transform of the generalized k–Fibonacci sequence is defined in the classical form as

$$BF_{k,n}(r)=\sum_{j=0}^{n}\binom{n}{j}F_{k,j}(r). \tag{3.1}$$

For $r=1$, see [6].

So, for $r=2,3,\ldots$ the sequences obtained by applying this transformation are:

$$
\begin{aligned}
BF_k(2)=\{&1,2,4+k,8+4k,16+12k+k^2,32+32k+6k^2,\\
&64+80k+24k^2+k^3,128+192k+80k^2+8k^3,\ldots\}\\
BF_k(3)=\{&1,2,4,8+k,16+5k,32+17k,64+49k+k^2,128+129k+8k^2,256+321k\\
&+39k^2,512+769k+150k^2+k^3,1024+1793k+501k^2+11k^3,\ldots\}\\
BF_k(4)=\{&1,2,4,8,16+k,32+6k,64+23k,128+72k,256+201k+k^2,\\
&512+522k+10k^2,1024+1291k+58k^2,2048+3084k+256k^2,\ldots\}
\end{aligned}
$$

Curiously, $BF_1(2)$ is the classical Pell sequence.

3.1. *Generating Function of the Sequences $BF_k(r)$*

[1] shows that if $A(x)$ is the generating function of the sequence $\{a_n\}$, then $S(x)=\dfrac{1}{1-x}A\left(\dfrac{x}{1-x}\right)$ is the generating function of the sequence $\{b_n\}$ with $b_n=\sum_j\binom{n}{j}a_n$.

So, we can deduce the generating function of the $F_k(r)$ sequence, for $r = 2,3,\ldots$ is $g_r(x) = \frac{1}{1-x} f_r\left(\frac{x}{1-x}\right)$, that is:

$$g_r(x) = \frac{(1-x)^{r-2}}{(1-2x)(1-x)^{r-2} - kx^r}. \tag{3.2}$$

3.2. *Recurrence Relation of the Sequences $BF_k(r)$*

Taking into account $g_r(x)$ is the generating function of the sequences $BF_k(r)$, the coefficients of the denominator of this function shows the recurrence relation of the sequences $BF_k(r)$.

For clarity, we indicate as $b_n(r)$ the elements of the sequence $BF_k(r)$. The coefficients of the polynomial $D(x) = (1-2x)(1-x)^{r-2} - kx^r$, for $r = 2,3\ldots$, show the recurrence relation of the sequences $\{b_n(r)\}$, and consequently:

$$r = 2 \to \{1,-2,-k\} \to b_n(2) = 2b_{n-1}(2) + kb_{n-2}(2) \tag{3.3}$$

$$r = 3 \to \{1,-3,2,-k\} \to b_n(3) = 3b_{n-1}(3) - 2b_{n-2}(3) + kb_{n-3}(3) \tag{3.4}$$

$$\begin{aligned} r = 4 \to \{1,-4,5,-2,-k\} \to b_n(4) \\ = 4b_{n-1}(4) - 5b_{n-2}(4) + 2b_{n-3}(4) + kb_{n-4}(4) \end{aligned} \tag{3.5}$$

$$\begin{aligned} r = 5 \to \{1,-5,9,-7,2,-k\} \to b_n(5) \\ = 5b_{n-1}(5) - 9b_{n-2}(5) + 7b_{n-3}(5) - 2b_{n-4}(5) + kb_{n-5}(5) \end{aligned} \tag{3.6}$$

with the initial conditions $b_n(r) = 2^n$ for $n = 0,1,2\ldots r-1$.

3.3. *Sums of the Sequences $BF_k(r)$*

In the sequel we will prove the formulas for the sums of the sequences $BF_k(2)$ and $BF_k(3)$ and show for $BF_k(4)$.
Let $b_n(2) = BF_{k,n}(2)$ be and we will indicate as b_n.
From (3.3) it is $b_n = 2b_{n-1} + kb_{n-2}$. Then

$$\begin{aligned} S_n(2) &= \sum_{j=0}^{n} b_j = b_0 + b_1 + \sum_{2}^{n} b_j = 1 + 2 + 2\sum_{2}^{n} b_{j-1} + k\sum_{2}^{n} b_{j-2} \\ &= 3 + 2(S_n - b_0 - b_n) + k(S_n - b_{n-1} - b_n) \\ &= 1 + 2S_n + kS_n - (2+k)b_n - kb_{n-1} \to \\ S_n(2) &= \frac{1}{1+k}\Big((2+k)b_n(2) + kb_{n-1}(2) - 1\Big). \end{aligned}$$

Let $b_n(3) = BF_{k,n}(3)$ be and we will idicate as b_n.
From (3.4) it is $b_n = 3b_{n-1} - 2b_{n-2} + kb_{n-3}$. Then

$$S_n(3) = \sum_{j=0}^{n} b_j = b_0 + b_1 + b_2 + \sum_{3}^{n} b_j$$

$$
\begin{aligned}
&= 1+2+4+3\sum_3^n b_{j-1} - 2\sum_3^n b_{j-2} + k\sum_3^n b_{j-3}\\
&= 7+3(S_n-b_0-b_1-b_n) - 2(S_n-b_0-b_{n-1}-b_n) + k(S_n-b_{n-2}-b_{n-1}-b_n)\\
&= S_n + kS_n - (1+k)S_n - (k-2)b_{n-1} - kb_{n-2} \to\\
S_n(3) &= \frac{1}{k}\left((1+k)b_n + (k-2)b_{n-1} + kb_{n-2}\right)\\
&= \frac{1}{k}(b_n - 2b_{n-1} + kb_{n-2} + (b_n + b_{n-1} + b_{n-2})\\
&= \frac{1}{k}(b_{n+1}-2b_n) + S_n - S_{n-3} \to S_{n-3}(3) = \frac{1}{k}(b_{n+1}-2b_n) \to\\
S_n(3) &= \frac{1}{k}\Big(b_{n+4}(3) - 2b_{n+3}(3)\Big).
\end{aligned}
$$

And so,

$$
\begin{aligned}
S_n(4) &= \frac{1}{k}\Big(b_{n+4}(4) - 3b_{n+3}(4) + 2b_{n+2}(4)\Big),\\
S_n(5) &= \frac{1}{k}\Big(b_{n+4}(5) - 4b_{n+3}(5) + 5b_{n+2}(5) - 2b_{n+1}(5)\Big).
\end{aligned}
$$

And we see that the coefficients of the sum $S_n(r)$ are the constant coefficients of the polynomial $D(x)$ for $r = n-1$. So, the following sum must be

$$S_n(6) = \frac{1}{k}\Big(b_{n+4}(6) - 5B_{n+3}(6) + 9b_{n+2}(6) - 7b_{n+1}(6) + 2b_n(6)\Big).$$

And rightly so!

CONCLUSIONS

We have generalized the r–distance Fibonacci numbers to the case of k–Fibonacci numbers, getting more general formulas that previously found.

These formulas include which allows to find the general term of a sequence of this type according to r is even or odd.

This paper also shows the formula to find the sum of the terms of the generalized (k,r)–Fibonacci sequences.

We apply the binomial transform to these sequences and find its generating function. Later, we found both the recurrence relation for the sequences of the binomial transforms of this new type of k–Fibonacci numbers and the formula for the sum of the terms of these sequences.

REFERENCES

[1] P. Barry, "On integer-sequence-based constructions of generalized Pascal triangles," *Journal of Integer Sequences*, vol. 9, no. Article 06.2.4, 2006.

[2] D. Bród, K. Piejko, and I. Wloch, "Distance Fibonacci numbers, distance Lucas numbers and their applications," *Ars Combinatoria*, vol. CXII, pp. 397–410, 2013.

[3] S. Falcon, "Generalized (k,r)–Fibonacci numbers," *General Mathematics Notes*, vol. 25, no. 2, pp. 148–158, 2014.

[4] S. Falcon and A. Plaza, "On the Fibonacci k–numbers," *Chaos, Solitons & Fractals*, vol. 32, no. 5, pp. 1615–1624, 2006, doi: 10.1016/j.chaos.2006.09.022.

[5] S. Falcon and A. Plaza, "The k–Fibonacci sequence and the Pascal 2–triangle," *Chaos, Solitons & Fractals*, vol. 33, no. 1, pp. 38–49, 2007, doi: 10.1016/j.chaos.2006.10.022.

[6] S. Falcon and A. Plaza, "Binomial transforms of the k–Fibonacci sequence," *International Journal of Nonlinear Sciences & Numerical Simulation*, vol. 10, no. 11–12, pp. 1527 – 1538, 2009, doi: 10.1515/IJNSNS.2009.10.11-12.1527.

[7] R. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*. Addison Wesley, 1998.

[8] V. E. Hoggatt, *Fibonacci and Lucas numbers*. Houghton Mifflin, 1969.

[9] A. F. Horadam, "A generalized Fibonacci sequence," *Math. Mag.*, vol. 68, pp. 455–459, 1961.

[10] A. G. Shanon, "Generalized Fibonacci triples," *Amer. Math. Monthly*, vol. 80, pp. 187–190, 1973.

[11] V. W. Spinadel, "The family of metallic means," *VisMath*, vol. 1, no. 3, 1999. [Online]. Available: http://members.tripod.com/vismath/

[12] I. Wloch, U. Bednarz, D. Bród, A. Wloch, and M. Wolowiecz-Musial, "On a new type of distance Fibonacci numbers," *Discrete Applied Mathematics*, vol. 161, no. 16-17, pp. 2695–2701, 2013, doi: 10.1016/j.dam.2013.05.029.

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ON THE PRIME GRAPH OF A FINITE GROUP

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Abstract. Let G be a finite group. We define the prime graph $\Gamma(G)$ of G as follows: The vertices of $\Gamma(G)$ are the primes dividing the order of G and two distinct vertices p, q are joined by an edge, denoted by $p \sim q$, if there is an element in G of order pq. We denote by $\pi(G)$, the set of all prime divisors of $|G|$. The degree $deg(p)$ of a vertex p of $\Gamma(G)$ is the number of edges incident with p. If $\pi(G) = \{p_1, p_2, ..., p_k\}$ where $p_1 < p_2 < ... < p_k$, then we define $D(G) = (deg(p_1), deg(p_2), ..., deg(p_k))$, which is called the degree pattern of G. Given a finite group M, if the number of non-isomorphic groups G such that $|G| = |M|$ and $D(G) = D(M)$ is equal to r, then M is called r-fold OD-characterizable. Also a 1-fold OD-characterizable group is simply called OD-characterizable. In this paper we give some results on characterization of finite groups by prime graphs and OD-characterizability of finite groups. In particular we apply our results to show that the simple groups $G_2(7)$, $B_3(5)$, $\mathbb{A}_{11}$, and $\mathbb{A}_{19}$ are OD-characterizable.

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1. Introduction

Throughout this paper, groups under consideration are finite. For any group G, we denote by $\pi(G)$ the set of prime divisors of $|G|$. We denote the set of elements of G by $\pi_e(G)$. We associate to $\pi_e(G)$ a graph called prime graph of G, denoted by $\Gamma(G)$. The vertex set of this graph is $\pi(G)$ and two distinct vertices p, q are joined by an edge, denote by $p \sim q$, if $pq \in \pi_e(G)$. The connected components of $\Gamma(G)$ is denoted by π_1, π_2, ..., $\pi_{t(G)}$, where $t(G)$ is the number of connected components of $\Gamma(G)$. If the order of G is even, the notation is chosen so that $2 \in \pi_1$. Clearly the order of G can be expressed as the product of m_1, m_2, ..., $m_{t(G)}$, where $\pi(m_i) = \pi_i$, $1 \leq i \leq t(G)$.

The degree $deg(p)$ of a vertex p of $\Gamma(G)$ is the number of edges incident with p. If $\pi(G) = \{p_1, p_2, ..., p_k\}$ with $p_1 < p_2 < ... < p_k$, then we define

$$D(G) = (deg(p_1), deg(p_2), ..., deg(p_k)),$$

which is called the degree pattern of G. Given a finite group M, if the number of non-isomorphic groups G such that $|G| = |M|$ and $D(G) = D(M)$ is equal to r, then M is called r-fold OD-characterizable. Also a 1-fold OD-characterizable group is simply called OD-characterizable.

We call a directed graph strongly connected if there is a directed path from each vertex in the graph to every other vertex. Given an integer a and a positive integer n with $(a,n)=1$, the multiplicative order of a modulo n is the smallest positive integer k such that $a^k \equiv 1(modn)$. We denote the order of a modulo n by $Ord_n(a)$. It is easy to see that if $a^l \equiv 1(modn)$, then $Ord_n(a)\big| l$. Let G be a finite group with $|G| = p_1{}^{\alpha_1} p_2{}^{\alpha_2} \ldots p_k{}^{\alpha_k}$, where $p_1 < p_2 < \ldots < p_k$ are prime numbers. We define a directed graph $\gamma(G)$ as follows: the vertex set is $\pi(G)$ and two distinct vertices p_i, p_j are joined by an edge, denote by $p_i \sim p_j$, whenever $p_i \nsim p_j$ in $\Gamma(G)$ and $Ord_{p_j{}^{\alpha_j}}(p_i) > \alpha_i$.

The problem of OD-characterizability of simple groups was raised in [2] for the first time. Then many researchers paid attention to characterize finite simple groups by orders and degree patterns of their prime graphs, to mention a few references we will quote [8] and [7].

In this paper we consider the prime graph of a finite group G and prove results which will be used to prove the OD-characterizability of the simple groups $G_2(7)$, $B_3(5)$, $\mathbb{A}_{11}$, and $\mathbb{A}_{19}$. Of course there are many other simple groups whose OD-characterizability can be proved using the results of this paper.

If m and l are natural numbers and p is a prime number, the notation $p^m \parallel n$ means that $p^m \big| n$ and $p^{m+1} \nmid n$. For a prime number r and a positive integer n, n_r denotes the r-part of n, i.e. type n_r is a power of r and $n = mn_r$, where $(m,r)=1$.

2. Preliminaries

Lemma 1. *Let $a > 1$ and n be natural numbers and r be a prime number. If $2 \neq r^n \parallel a-1$, then $r^{n+1} \parallel (a^r - 1)$.*

Proof. See [3], 3.2. □

Lemma 2. *Let p_i and p_j be two distinct prime numbers, $p_j \neq 2$, $Ord_{p_j}(p_i) = m$ and $p_j \parallel p_i{}^m - 1$, then $Ord_{p_j{}^d}(p_i) = mp_j{}^{d-1}$, where d is a positive integer.*

Proof. By Lemma 1 and induction on t we see that

$$p_j{}^t \parallel p_i{}^{mp_j{}^{t-1}} - 1, \tag{2.1}$$

where t is an arbitrary natural number. Now we prove the lemma by induction on d. If $d = 1$, then clearly the lemma holds.

Suppose that $Ord_{p_j{}^k}(p_i) = mp_j{}^{k-1}$. Set $s = Ord_{p_j{}^{k+1}}(p_i)$. Thus $p_j{}^{k+1}\big|p_i{}^s - 1$ and so $p_j{}^k\big|p_i{}^s - 1$. Hence $mp_j{}^{k-1}\big|s$, because $Ord_{p_j{}^k}(p_i) = mp_j{}^{k-1}$. On the other hand by (2.1) we have $p_j{}^{k+1}\big|p_i{}^{mp_j{}^k} - 1$ and since $Ord_{p_j{}^{k+1}}(p_i) = s$, $s\big|mp_j{}^k$. It follows that $mp_j{}^{k-1}\big|\ s\big|mp_j{}^k$. This means that $s = mp_j{}^{k-1}$ or $s = mp_j{}^k$. If $s = mp_j{}^{k-1}$, then we have $p_j{}^{k+1}\big|p_i{}^{mp_j{}^{k-1}} - 1$. But by (2.1) $p_j{}^k \parallel p_i{}^{mp_j{}^{k-1}} - 1$. This contradiction shows that $Ord_{p_j{}^{k+1}}(p_i) = s = mp_j{}^k$. Therefore $Ord_{p_j{}^{k+1}}(p_i) = mp_j{}^k$ and the lemma is proved. □

Lemma 3. *Let G be a finite group with $t(G) \geq 2$. If $N \trianglelefteq G$ is a π_i-group, then* $(\prod_{j=1, j\neq i}^{t(G)} m_j)\big| |N|-1$.

Proof. See Lemma 8 of [1]. $\square$

Lemma 4. *Let G be a finite group with $|G| = {p_1}^{\alpha_1}{p_2}^{\alpha_2}...{p_n}^{\alpha_n}$, $p_1 < p_2 < ... < p_n$ where p_i is a prime number, $1 \leq i \leq n$. Also assume that M is an arbitrary normal subgroup of G. Then the following holds:*

1) *If p_i, $p_j \in \pi(G)$ and $p_i \sim p_j$ in $\gamma(G)$, then $p_i \big| |M|$ implies that $p_j \big| |M|$, where p_i, p_j are distinct prime numbers.*
2) *Let p_i, $p_j \in \pi(M)$, $p_i \nsim p_j$ in $\Gamma(G)$ and ${p_j}^{\alpha_j} \nmid \prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i} - {p_i}^{k-1})$. If $[\prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j} \big| {p_j}^{\alpha_j}$, then $\frac{{p_j}^{\alpha_j}}{[\prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j}} \big| |M|$.*
3) *If p_i, $p_j \in \pi(M)$, $p_i \nsim p_j$ in $\Gamma(G)$ and $Ord_{{p_j}^d}(p_i) > \alpha_i$ for some integer $1 \leq d \leq \alpha_j$, then ${p_j}^{\alpha_j+1-d} \big| |M|$.*

Proof. 1) Since $p_i \sim p_j$ in $\gamma(G)$, we conclude that $p_i \nsim p_j$ in $\Gamma(G)$ and $Ord_{{p_j}^{\alpha_j}}(p_i) > \alpha_i$. We suppose that $p_i \big| |M|$. By Frattini argument $N_G(M_{p_i})M = G$, where M_{p_i} is a Sylow p_i-subgroup of M. If $p_j \nmid |M|$, then since ${p_j}^{\alpha_j} \big| |G|$, we have ${p_j}^{\alpha_j} \big| |N_G(M_{p_i})|$ and so $N_G(M_{p_i})$ has a subgroup, say L of order ${p_j}^{\alpha_j}$. $M_{p_i} \trianglelefteq N_G(M_{p_i})$ implies that $LM_{p_i} \leq N_G(M_{p_i})$. On the other hand there is an positive integer $\beta \leq \alpha_i$ such that $|LM_{p_i}| = {p_j}^{\alpha_j}{p_i}^{\beta}$ and since $p_i \nsim p_j$ in $\Gamma(G)$, the prime graph of LM_{p_i} is not connected. Also $M_{p_i} \trianglelefteq LM_{p_i}$. Thus ${p_j}^{\alpha_j} \big| {p_i}^{\beta} - 1$ by Lemma 3. Hence $Ord_{{p_j}^{\alpha_j}}(p_i) \big| \beta$. In particular we have $Ord_{{p_j}^{\alpha_j}}(p_i) \leq \alpha_i$ and this is a contradiction and so $p_j \big| |M|$.

2) We have $N_G(M_{p_i})M = G$. Thus $\frac{{p_j}^{\alpha_j}}{|N_G(M_{p_i})|_{p_j}} \big| |M|$. Moreover if N is a minimal normal subgroup of $N_G(M_{p_i})$ such that $N \leq M_{p_i}$, then N is isomorphic to a direct product of cyclic groups $\mathbb{Z}_{p_i}$. Assume that N is isomorphic to a direct product of r cyclic group $\mathbb{Z}_{p_i}$. $(N \cong \mathbb{Z}_{p_i} \times ... \times \mathbb{Z}_{p_i})$. Since $\frac{N_G(M_{p_i})}{C_{N_G(M_{p_i})}(N)} \hookrightarrow Aut(N)$, we have $\frac{|N_G(M_{p_i})|}{|C_{N_G(M_{p_i})}(N)|} \big| |Aut(N)| = |Aut({\mathbb{Z}_{p_i}}^r)| = |Gl_r(p_i)| = \prod_{k=1}^{r}({p_i}^r - {p_i}^{k-1})$. This implies that $|N_G(M_{p_i})| \big| |C_{N_G(M_{p_i})}(N)| \prod_{k=1}^{r}({p_i}^r - {p_i}^{k-1})$. But since $p_i \nsim p_j$ in $\Gamma(G)$, $p_j \nmid C_{N_G(M_{p_i})}(N)|$. (Note that N is a p_i-group).

Thus $|N_G(M_{p_i})|_{p_j} \big| [\prod_{k=1}^{r}({p_i}^r - {p_i}^{k-1})]_{p_j}$. Also since $r \leq \alpha_i$,

$$[\prod_{k=1}^{r}({p_i}^r - {p_i}^{k-1})]_{p_j} \big| [\prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j}.$$

Therefore $|N_G(M_{p_i})|_{p_j} \big| [\prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j}$.

Now from $N_G(M_{p_i})M = G$, we conclude that $|G| = |N_G(M_{p_i})M| = \frac{|N_G(M_{p_i})||M|}{|N_G(M_{p_i})\cap M|}$ and so $|G|\big||N_G(M_{p_i})||M|$. Thus ${p_j}^{\alpha_j} = |G|_{p_j}\big||N_G(M_{p_i})|_{p_j}|M|_{p_j}$ and since

$$|N_G(M_{p_i})|_{p_j}\big|[\prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j},\ {p_j}^{\alpha_j}\big|[\prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j}|M|_{p_j}.$$

By assumption $[\prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j}\big|{p_j}^{\alpha_j}$ and so $\frac{{p_j}^{\alpha_j}}{[\prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i}-{p_i}^{k-1})]_{p_j}}\big||M|_{p_j}\big||M|$.

3) We will prove that ${p_j}^d \nmid |N_G(M_{p_i})|$.

If ${p_j}^d\big||N_G(M_{p_i})|$, then $N_G(M_{p_i})$ has a subgroup, say J of order ${p_j}^d$.

Since $M_{p_i} \trianglelefteq JM_{p_i}$ and the prime graph of JM_{p_i} is not connected ($p_i \nsim p_j$ in $\Gamma(G)$) by Lemma 3, we have ${p_j}^d\big|{p_i}^e - 1$ for a positive integer $e \leq \alpha_i$. It means that ${p_i}^e \equiv 1 (mod\, {p_j}^d)$. It follows that $Ord_{{p_j}^d}(p_i) \leq \alpha_i$, which is a contradiction. Thus ${p_j}^d \nmid |N_G(M_{p_i})|$ and so $|N_G(M_{p_i})|_{p_j}\big|{p_j}^{d-1}$. But since $N_G(M_{p_i})M = G$, we conclude that $|G|\big||N_G(M_{p_i})||M|$, which implies that ${p_j}^{\alpha_j} = |G|_{p_j}\big||N_G(M_{p_i})|_{p_j}|M|_{p_j}|{p_j}^{d-1}|M|_{p_j}$ and so ${p_j}^{\alpha_j+1-d} = {p_j}^{\alpha_j-(d-1)}\big||M|$. The proof is completed. □

3. CHARACTERIZATION OF FINITE GROUPS BY PRIME GRAPH AND ORDER OF THE GROUP

Theorem 1. *Let G be a finite group with $|G| = {p_1}^{\alpha_1}{p_2}^{\alpha_2}...{p_n}^{\alpha_n}$, $p_1 < p_2 < \cdots < p_n$ where p_i is a prime number, $1 \leq i \leq n$. If the directed graph $\gamma(G)$ is strongly connected, then the following assertions hold.*

1) *There is a simple group S such that $S \trianglelefteq G \leq Aut(S)$ and $\pi(S) = \pi(G)$. Also if $p_i \nsim p_j$ in $\Gamma(G)$, then $p_i \nsim p_j$ in $\Gamma(S)$ too and if $p_i \sim p_j$ in $\Gamma(G)$, then $p_i \sim p_j$ in $\Gamma(Aut(S))$ too.*
2) *Let p_i, $p_j \in \pi(G)$, $p_i \nsim p_j$ in $\Gamma(G)$ and ${p_j}^{\alpha_j} \nmid \prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i} - {p_i}^{k-1})$. If $[\prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j}\big|{p_j}^{\alpha_j}$, then $\frac{{p_j}^{\alpha_j}}{[\prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i}-{p_i}^{k-1})]_{p_j}}\big||S|$.*
3) *If p_i, $p_j \in \pi(G)$, $p_i \nsim p_j$ in $\Gamma(G)$ and for some integer $1 \leq d \leq \alpha_j$, $Ord_{{p_j}^d}(p_i) > \alpha_i$, then ${p_j}^{\alpha_j+1-d}\big||S|$.*

Proof. Assume that L is a minimal normal subgroup of G. Thus $L \neq 1$ and so there is a prime number $p_i \in \pi(G)$ such that $p_i\big||L|$. Since $\gamma(G)$ is strongly connected, for all $p_j \in \pi(G)$ there exists a directed path from p_i to p_j. So by Lemma 4 and induction on the length of path we can easily see that $p_j\big||L|$ for all $p_j \in \pi(G)$. Therefore $\pi(L) = \pi(G)$ and since $\gamma(G)$ is strongly connected, clearly $\Gamma^c(G)$ is connected, where $\Gamma^c(G)$ denotes the complement of the graph $\Gamma(G)$. Now if L is a direct product of more than one isomorphic simple groups, then since $\pi(L) = \pi(G)$, $\Gamma(G)$ is a complete graph and so $\Gamma(G)^c$ is not connected, a contradiction. Hence L is a simple group. On the other hand if for some $q \in \pi(G)$, $q\big||C_G(L)|$, then $q \sim t$ in $\Gamma(G)$ for all $t \in \pi(G) - \{q\}$ and so $\Gamma^c(G)$ is not connected, which is contradiction. Thus $C_G(L) = 1$ and since

$\frac{G}{C_G(L)} \hookrightarrow Aut(L)$, we conclude that $G \hookrightarrow Aut(L)$. So the proof of Part 1 is completed. We conclude Part 2 and 3 of the Theorem from Lemma 4. $\square$

Theorem 2. *Let G be a finite group, $|G| = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\ldots{p_n}^{\alpha_n}$, where $p_1 < p_2 < \cdots < p_n$ and p_i is a prime number, $1 \le i \le n$. If γ_1 is a strongly connected directed subgraph of the graph $\gamma(G)$ and V_1 is the vertex set of γ_1, then the following assertions hold.*

1) *There is a simple group S such that $S \trianglelefteq \frac{G}{O_{\pi(G)-V_1}(G)} \le Aut(S)$, $V_1 \subseteq \pi(S) \subseteq \pi(G)$ and if $p_i,\ p_j \in V_1$ and $p_i \nsim p_j$ in $\Gamma(G)$, then $p_i \nsim p_j$ in $\Gamma(S)$ and if $p_i \sim p_j$ in $\Gamma(G)$, then $p_i \sim p_j$ in $\Gamma(Aut(S))$ ($O_{\pi(G)-V_1}(G)$ is the largest normal subgroup N with $\pi(N) = \pi(G) - V_1$).*
2) *Let $p_i,\ p_j \in V_1$, $p_i \nsim p_j$ in $\Gamma(G)$ and ${p_j}^{\alpha_j} \nmid \prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i} - {p_i}^{k-1})$. If $[\prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j} \big| {p_j}^{\alpha_j}$, then $\frac{{p_j}^{\alpha_j}}{[\prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i}-{p_i}^{k-1})]_{p_j}} \big| |S|$.*
3) *If $p_i,\ p_j \in V_1$ and $p_i \nsim p_j$ in $\Gamma(G)$ and for some integer $1 \le d \le \alpha_j$, $Ord_{{p_j}^d}(p_i) > \alpha_i$, then ${p_j}^{\alpha_j+1-d} \big| |S|$.*

Proof. Set $L = O_{\pi(G)-V_1}(G)$ and $\bar{G} = \frac{G}{L}$. Suppose that S is a minimal normal subgroup of $\bar{G}$. Thus for a normal subgroup of G, say M_1, we have $S = \frac{M_1}{L}$, where $L \le M_1$. It is obvious that there is a prime number $q \in V_1$, such that $q \big| |M_1|$. But there exists a path between q and t for all $t \in V_1 - \{q\}$. Therefore by Lemma 4 and induction on length we see that $V_1 \subseteq \pi(M_1)$. It follows that $V_1 \subseteq \pi(S) \subseteq \pi(G)$. Since γ_1 is a strongly connected subgraph of $\gamma(G)$, for all $p_i \in V_1$, there exists $p_j \in V_1$ such that $p_i \nsim p_j$ in $\Gamma(G)$ and so S is not a direct product of more than one isomorphic simple groups. Hence S is a simple group. Now we prove that $C_{\bar{G}}(S) = 1$. Assume that $C_{\bar{G}}(S) \ne 1$. Thus there is a subgroup of G, say K such that $C_{\bar{G}}(S) = \frac{K}{L}$, $L \ne K$. It follows that there is a prime number $r \in V_1$ such that $r \big| |K|$. It means that $r \big| |C_{\bar{G}}(S)|$. Moreover since $V_1 \subseteq \pi(S)$, we conclude that $r \sim t$ in $\Gamma(\bar{G})$ for all $t \in V_1 - \{r\}$. It is easy to see that $r \sim t$ in $\Gamma(G)$ for all $t \in V_1 - \{r\}$ and so $r \nsim t$ in $\gamma(G)$, in particular in γ_1 for all $t \in V_1 - \{r\}$, but this is a contradiction with γ_1 being a strongly connected graph and thus $C_{\bar{G}}(S) = 1$. Hence $S \trianglelefteq \bar{G} = \frac{\bar{G}}{1} = \frac{\bar{G}}{C_{\bar{G}}(S)} \le Aut(S)$.

Now we assume that $p_i,\ p_j \in V_1$ and $p_i \nsim p_j$ in $\Gamma(G)$ and ${p_j}^{\alpha_j} \nmid \prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i} - {p_i}^{k-1})$. Also suppose that $[\prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i} - {p_i}^{k-1})]_{p_j} \big| {p_j}^{\alpha_j}$. By using Part 2 of Lemma 4 for $M_1 \trianglelefteq G$, we conclude that $\frac{{p_j}^{\alpha_j}}{[\prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i}-{p_i}^{k-1})]_{p_j}} \big| |M_1|$ and since $p_j \in V_1$, $p_j \nmid |L|$ and so $\frac{{p_j}^{\alpha_j}}{[\prod_{k=1}^{\alpha_i}({p_i}^{\alpha_i}-{p_i}^{k-1})]_{p_j}} \big| |\frac{M_1}{L}| = |S|$, thus the proof of Part 2 is completed.

Similar arguments prove Part 3.

If $p_i,\ p_j \in V_1$, $p_i \sim p_j$ in $\Gamma(S)$, then clearly $p_i \sim p_j$ in $\Gamma(\bar{G})$ and so in $\Gamma(G)$. Thus if $p_i \nsim p_j$ in $\Gamma(G)$, then $p_i \nsim p_j$ in $\Gamma(S)$. Also if $p_i,\ p_j \in V_1$ and $p_i \sim p_j$ in $\Gamma(G)$, then there is an element $g \in G$, such that $g^{p_ip_j} = 1$ and $o(g) = p_ip_j$. Thus $g^{p_ip_j} \in L$. Since $o(g) = p_ip_j \nmid |L|$, $g \notin L$. If $g^{p_i} \in L$, then since $\pi(L) \subseteq \pi(G) - V_1$ and $p_i,\ p_j \in V_1$, we conclude that there is a positive integer m such that $(p_ip_j, m) = 1$

and $(g^{p_i})^m = g^{p_i m} = 1$. This implies that $p_i p_j \big| p_i m$, because $o(g) = p_i p_j$, thus $p_j \big| m$, a contradiction. Therefore $g^{p_i} \notin L$. Similarly $g^{p_j} \notin L$ and so $o(gL) = p_i p_j$. Thus $p_i \sim p_j$ in $\Gamma(\bar{G})$. But since $\bar{G} \leq Aut(S)$, $p_i \sim p_j$ in $\Gamma(Aut(S))$ and the proof is completed. $\square$

4. OD-CHARACTERIZABILITY OF FINITE GROUPS

Let G be a finite group and $|G| = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \ldots {p_n}^{\alpha_n}$, where $p_1 < p_2 < \cdots < p_n$ and p_i is a prime number, $1 \leq i \leq n$. For $i = 1, 2, \ldots, n$, set $R(p_i) = |\{p_j \in \pi(G) | p_i \neq p_j$, $Ord_{{p_j}^{\alpha_j}}(p_i) > \alpha_i$ and $Ord_{{p_i}^{\alpha_i}}(p_j) > \alpha_j\}|$. We have the following three propositions.

Proposition 1. *Let G be a finite group with $|G| = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \ldots {p_n}^{\alpha_n}$, $p_1 < p_2 < \cdots < p_n$, p_i is a prime number, $1 \leq i \leq n$. Assume that there is $p_m \in \pi(G)$ such that $deg(p_m) = 0$ in $\Gamma(G)$ and $R(p_m) = n-1$. Then the following assertions hold.*

1) *There is a simple group S such that $S \trianglelefteq G \leq Aut(S)$, $\pi(S) = \pi(G)$. Also we have $deg_{\Gamma(S)}(p_i) \leq deg_{\Gamma(G)}(p_i) \leq deg_{\Gamma(Aut(S))}(p_i)$, $1 \leq i \leq n$.*
2) *If $p_l \in \pi(G)$, ${p_l}^{\alpha_l} \nmid \prod_{k=1}^{\alpha_m} ({p_m}^{\alpha_m} - {p_m}^{k-1})$ and $[\prod_{k=1}^{\alpha_m} ({p_m}^{\alpha_m} - {p_m}^{k-1})]_{p_l} \big| {p_l}^{\alpha_l}$, then $\frac{{p_l}^{\alpha_l}}{[\prod_{k=1}^{\alpha_m} ({p_m}^{\alpha_m} - {p_m}^{k-1})]_{p_l}} \big| |S|$.*
3) *If $p_l \in \pi(G)$, ${p_m}^{\alpha_m} \nmid \prod_{k=1}^{\alpha_l} ({p_l}^{\alpha_l} - {p_l}^{k-1})$ and $[\prod_{k=1}^{\alpha_l} ({p_l}^{\alpha_l} - {p_l}^{k-1})]_{p_m} \big| {p_m}^{\alpha_m}$, then $\frac{{p_m}^{\alpha_m}}{[\prod_{k=1}^{\alpha_l} ({p_l}^{\alpha_l} - {p_l}^{k-1})]_{p_m}} \big| |S|$.*
4) *If $p_l \in \pi(G)$ and $Ord_{{p_l}^d}(p_m) > \alpha_m$ for some integer $1 \leq d \leq \alpha_l$, then ${p_l}^{\alpha_l + 1 - d} \big| |S|$.*
5) *If $p_l \in \pi(G)$ and $Ord_{{p_m}^d}(p_l) > \alpha_l$. for some integer $1 \leq d \leq \alpha_m$, then ${p_m}^{\alpha_m + 1 - d} \big| |S|$.*

Proof. By Theorem 1 it is sufficient to prove that $\gamma(G)$ is strongly connected. Since $deg(p_m) = 0$ in $\Gamma(G)$, $p_i \nsim p_m$ in $\Gamma(G)$ for all $i \neq m$, $1 \leq i \leq n$ and since $R(p_m) = n-1$, $Ord_{{p_m}^{\alpha_m}}(p_i) > \alpha_i$ and $Ord_{{p_i}^{\alpha_i}}(p_m) > \alpha_m$ for all $i \neq m$, $1 \leq i \leq n$. Hence there is a directed edge from p_i to p_m and from p_m to p_i for all $i \neq m$, $1 \leq i \leq n$.

Now assume that p_a, p_b are two arbitrary vertices in $\gamma(G)$. Then by above discussion there is a directed edge from p_a to p_m and from p_m to p_a in $\gamma(G)$. Also there is a directed edge from p_m to p_b and from p_b to p_m. Thus there is a directed path from p_a to p_b. Therefore $\gamma(G)$ is strongly connected.

Since for all $q \in \pi(G) - \{p_m\}$, $q \nsim p_m$ in $\Gamma(G)$, 2, 3, 4 and 5 are concluded from Theorem 1 Part 2 and 3. $\square$

Proposition 2. *Let G be a finite group with $|G| = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \ldots {p_n}^{\alpha_n}$, $p_1 < p_2 < \cdots < p_n$, p_i is a prime number, $1 \leq i \leq n$. Assume that there exists $p_m \in \pi(G)$ such that $deg(p_m) = 1$ in $\Gamma(G)$ and $R(p_m) = n-1$. Then the following assertions hold.*

1) *There exists a simple group S and a prime number $p_r \in \pi(G) - \{p_m\}$ such that $S \trianglelefteq \frac{G}{O_{p_r}(G)} \leq Aut(S)$ and $\pi(G) - \{p_r\} \subseteq \pi(S) \subseteq \pi(G)$. ($O_{p_r}(G)$ is the largest normal subgroup N of G with $\pi(N) = \{p_r\}$).*

2) a) *If* $p_s \in \pi(G)$ *and* $deg(p_s) = n-1$ *in* $\Gamma(G)$*, then there is a simple group* S *such that* $S \trianglelefteq \frac{G}{O_{p_s}(G)} \leq Aut(S)$ *and* $\pi(G) - \{p_s\} \subseteq \pi(S) \subseteq \pi(G)$*.*

b) *If* $p_t \in \pi(G) - \{p_s, p_m\}$*,* $p_t^{\alpha_t} \nmid \prod_{k=1}^{\alpha_m}(p_m{}^{\alpha_m} - p_m{}^{k-1})$ *and* $[\prod_{k=1}^{\alpha_m}(p_m{}^{\alpha_m} - p_m{}^{k-1})]_{p_t} | p_t^{\alpha_t}$*, then* $\frac{p_t^{\alpha_t}}{[\prod_{k=1}^{\alpha_m}(p_m{}^{\alpha_m} - p_m{}^{k-1})]_{p_t}} \Big| |S|$*.*

c) *If* $p_t \in \pi(G) - \{p_s, p_m\}$*,* $p_m{}^{\alpha_m} \nmid \prod_{k=1}^{\alpha_t}(p_t{}^{\alpha_t} - p_t{}^{k-1})$ *and* $[\prod_{k=1}^{\alpha_t}(p_t{}^{\alpha_t} - p_t{}^{k-1})]_{p_m} | p_m{}^{\alpha_m}$*, then* $\frac{p_m{}^{\alpha_m}}{[\prod_{k=1}^{\alpha_t}(p_t{}^{\alpha_t} - p_t{}^{k-1})]_{pm}} \Big| |S|$*.*

d) *If* $p_t \in \pi(G) - \{p_s, p_m\}$ *and* $Ord_{p_t{}^d}(p_m) > \alpha_m$ *for some integer* $1 \leq d \leq \alpha_t$*, then* $p_t{}^{\alpha_t+1-d} \Big| |S|$*.*

e) *If* $p_t \in \pi(G) - \{p_s, p_m\}$ *and* $Ord_{p_m{}^d}(p_t) > \alpha_t$ *for some integer* $1 \leq d \leq \alpha_m$*, then* $p_m{}^{\alpha_m+1-d} \Big| |S|$*.*

Proof. 1) By Theorem 2 it is sufficient to prove that $\gamma(G)$ has a strongly connected subgraph with $n-1$ vertices. Since $deg(p_m) = 1$ in $\Gamma(G)$, there exists $p_r \in \pi(G)$ such that $p_r \sim p_m$ in $\Gamma(G)$. If p_i is an arbitrary vertex of the directed graph $\gamma(G)$ such that $p_i \neq p_r, p_m$, then since $R(p_m) = n-1$, we conclude that $Ord_{p_i{}^{\alpha_i}}(p_m) > \alpha_m$ and $Ord_{p_m{}^{\alpha_m}}(p_i) > \alpha_i$. On the other hand $p_i \nsim p_m$ in $\Gamma(G)$ and so there is an edge from p_i to p_m and from p_m to p_i in $\gamma(G)$.

Now if p_a, p_b are two arbitrary vertices of $\gamma(G)$ such that $p_a, p_b \neq p_r, p_m$, then there is an edge from p_a to p_m and from p_m to p_a, also from p_b to p_m and from p_m to p_b in $\gamma(G)$. Thus there is a path from p_a to p_b. Hence there is a strongly connected subgraph of $\gamma(G)$ such that its vertex set is equal to $\pi(G) - \{p_r\}$. Therefore by Theorem 2, there is a simple group S such that $S \trianglelefteq \frac{G}{O_{p_r}(G)} \leq Aut(S)$ and $\pi(G) - \{p_r\} \subseteq \pi(S) \subseteq \pi(G)$.

2) Assume that there exists $p_s \in \pi(G)$ such that $deg(p_s) = n-1$ in $\Gamma(G)$. So p_s is joint to all vertices in $\Gamma(G)$. In particular $p_s \sim p_m$ in $\Gamma(G)$.

By similar argument as in Part 1 we can see that $\gamma(G)$ has a strongly connected subgraph such that its vertex set is equal to $\pi(G) - \{p_s\}$. Thus by Theorem 2, there is a simple group S such that $S \trianglelefteq \frac{G}{O_{p_s}(G)} \leq Aut(S)$ and $\pi(G) - \{p_s\} \subseteq \pi(S) \subseteq \pi(G)$. Also b, c, d, e are concluded from Theorem 2 Part 2 and 3. $\square$

We define $(m)^*$ for all $m \in \mathbb{Z}$ by $(m)^* = \begin{cases} m & \text{for} \quad m > 0 \\ 0 & \text{for} \quad m \leq 0. \end{cases}$

Proposition 3. *Let* G *be a finite group with* $|G| = p_1{}^{\alpha_1} p_2{}^{\alpha_2} \ldots p_n{}^{\alpha_n}$*,* $p_1 < p_2 < \cdots < p_n$*,* p_i *is a prime number,* $1 \leq i \leq n$*. We set* $M = max\{(R(p_i) - deg(p_i))^* | 1 \leq i \leq n\}$ *and* $m = min\{(R(p_i) - deg(p_i))^* | 1 \leq i \leq n\}$*. If* $M + m \geq n-1$*, then there is a simple group* S *such that* $S \trianglelefteq G \leq Aut(S)$ *and* $\pi(S) = \pi(G)$*. Also* $deg_{\Gamma(S)}(q) \leq deg_{\Gamma(G)}(q) \leq deg_{\Gamma(Aut(S))}(q)$ *for all* $q \in \pi(G)$*.*

Proof. By Theorem 1 it is sufficient to prove that $\gamma(G)$ is strongly connected. So assume that p_d is an arbitrary vertex of $\gamma(G)$. We define A_d and B_d as follows:

$$A_d = \{p_i \in \pi(G) | p_i \neq p_d,\ p_i \nsim p_d \text{ in } \Gamma(G)\},$$

$$B_d = \{p_j \in \pi(G) | p_j \neq p_d,\ Ord_{p_j^{\alpha_j}}(p_d) > \alpha_d \text{ and } Ord_{p_d^{\alpha_d}}(p_j) > \alpha_j\}.$$

Thus $|A_d| = n-1-deg(p_d)$, $|B_d| = R(p_d)$, where $deg(p_d)$ is the degree of p_d in $\Gamma(G)$.

Moreover $A_d \cap B_d$ is equal to set of all vertices in $\gamma(G)$ that are joined to p_d and also p_d is joined to them by an edge. Since $p_d \notin A_d \cup B_d$, $A_d \cup B_d \subseteq \pi(G) - \{p_d\}$ and so $|A_d \cup B_d| \leq n-1$. Therefore we have $|A_d \cup B_d| = n-1-deg(p_d)+R(p_d)-|A_d \cap B_d| \leq n-1$. Hence $|A_d \cap B_d| \geq R(p_d)-deg(p_d)$. Since $|A_d \cap B_d| \geq 0$, we have $|A_d \cap B_d| \geq (R(p_d)-deg(p_d))^*$.

But $(R(p_d)-deg(p_d))^* \geq m$, which implies that $|A_d \cap B_d| \geq m$. Thus there exist m vertices in $\gamma(G)$ that are joined to p_d and also p_d is joined to them by an edge, where p_d is an arbitrary vertex of $\gamma(G)$. Denote the set of all these m vertices by E_d.

Now we assume that $p_c \in \pi(G)$ and $M = (R(p_c)-deg(p_c))^*$. Then by a similar argument we see that there exist M vertices in $\gamma(G)$ that are joined to p_c and also p_c is joined to them by an edge. Denote the set of all these M vertices by F_c.

We will show that if $p_u \in \pi(G)$ is different from p_c, then there is a directed path from p_u to p_c and from p_c to p_u. Since $p_u \neq p_c$, $p_u \nsim p_c$ and $p_c \nsim p_u$ in $\gamma(G)$. We know that $p_u \sim q$ and $q \sim p_u$ in $\gamma(G)$ for all $q \in E_u$. If $E_u \cap F_c = \varnothing$, then since $\{p_u\} \cup E_u \cup \{p_c\} \cup F_c \subseteq \pi(G)$, $p_u \neq p_c$, $p_u \nsim p_c$ and $p_c \nsim p_u$ in $\gamma(G)$, we have $|\{p_u\} \cup E_u \cup \{p_c\} \cup F_c| = 1+m+1+M \leq n$, which is a contradiction with assumption, $(M+m \geq n-1\ )$. Thus $E_u \cap F_c \neq \varnothing$. Suppose that $p_v \in E_u \cap F_c$. It follows that $p_u \sim p_v$, $p_v \sim p_u$, $p_c \sim p_v$ and $p_v \sim p_c$. Hence $p_u \to p_v \to p_c$ is a directed path from p_u to p_c and $p_c \to p_v \to p_u$ is a directed path from p_c to p_u. So we proved that for all $p_u \in \pi(G)$ there exists a directed path from p_u to p_c and there is a directed path from p_c to p_u.

Now we assume that p_a, p_b are two arbitrary vertices of $\gamma(G)$. Thus by the above discussion there is a path from p_c to p_a and from p_a to p_c, also there is a path from p_c to p_b and from p_b to p_c. Therefore there is a path from p_a to p_b and so $\gamma(G)$ is strongly connected and the proof is completed. □

5. APPLICATIONS

We give some examples of characterization of finite groups by prime graph and OD-characterization of them.

We note that the following examples are proved in [4] and a few more papers. But our proofs are based on Theorems 1 and 2 and Propositions 1, 2 and 3. The prime graphs of all groups considered are obtained by [6].

Example 1. We consider the simple group $C_2(7)$. We know that $|C_2(7)| = 2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$ and $2 \sim 3$, $2 \sim 7$, $3 \sim 7$, $5 \nsim 2$, $5 \nsim 7$ and $5 \nsim 3$ in $\Gamma(C_2(7))$. Since $Ord_{5^2}(2) > 8$, $Ord_{5^2}(3) > 2$, $Ord_{2^8}(5) > 2$, $Ord_{3^2}(5) > 2$, we deduce that $E_{\gamma(C_2(7))} \supseteq \{(2,5),(5,2),(3,5),(5,3)\}$, where $E_{\gamma(C_2(7))}$ is the edge set of $\gamma(G)$. Hence there exists a strongly connected subgraph of $\gamma(C_2(7))$ that its vertex set is $\{2,3,5\}$.

Now if G is a finite group with $|G| = |C_2(7)|$ and $\Gamma(G) = \Gamma(C_2(7))$, then $\gamma(G) = \gamma(C_2(7))$ and so there exists a strongly connected subgraph of $\gamma(G)$ that its vertex set is $\{2,3,5\}$. Thus by Theorem 2 there is a simple group S such that $S \trianglelefteq \frac{G}{O_7(G)} \leq Aut(S)$ and $\{2,3,5\} \subseteq \pi(S) \subseteq \pi(G) = \{2,3,5,7\}$. Since $3 \nsim 5$ in $\Gamma(C_2(7)) = \Gamma(G)$ and $Ord_5(3) > 2$ by Part 3 of Theorem 2, we conclude that $5^{2+1-1} = 5^2 \big| |S|$. Similarly since $2 \nsim 5$ in $\Gamma(G)$ and $Ord_{2^4}(5) > 2$, $2^{8+1-4} = 2^5 \big| |S|$. Now by Table 4 of [5] we see that $S \cong B_2(7)$ or $S \cong C_2(7)$ and since $S \leq \frac{G}{O_7(G)}$ and $\frac{|G|}{|O_7(G)|} \big| |G| = |C_2(7)|$, we conclude that $O_7(G) = 1$ and $G = S$ and so $G \cong B_2(7)$ or $G \cong C_2(7)$.

Hence if $\Gamma(G) = \Gamma(C_2(7))$ and $|G| = |C_2(7)|$, then $G \cong C_2(7)$ or $G \cong B_2(7)$.

Example 2. We consider the simple group $B_3(5)$. We know that $|B_3(5)| = 2^9 \cdot 3^4 \cdot 5^9 \cdot 7 \cdot 13 \cdot 31$ and $2 \sim 3$, $2 \sim 5$, $2 \sim 13$ $2 \sim 31$, $3 \sim 5$, $3 \sim 7$, $3 \sim 13$ and $5 \sim 13$ in $\Gamma(B_3(5))$ and $7 \nsim i$, $31 \nsim j$ for $i \in \{2,5,13,31\}$ and $j \in \{3,5,7,13\}$ in $\Gamma(G)$. We have $Ord_{31}(3) > 4$, $Ord_{3^4}(31) > 1$, $Ord_7(31) > 1$, $Ord_{31}(7) > 1$, $Ord_{31}(13) > 1$ and $Ord_{13}(31) > 1$. Thus $E_{\gamma(B_3(5))} \supseteq \{(31,3),(3,31),(31,7),(7,31),(31,13),(13,31)\}$, where $E_{\gamma(B_3(5))}$ is the edge set of $\gamma(B_3(5))$. Therefore there exists a strongly connected subgraph of $\gamma(B_3(5))$ that its vertex set is $\{3,7,13,31\}$. Now if G is a finite group with $|G| = |B_3(5)|$ and $\Gamma(G) = \Gamma(B_3(5))$, then $\gamma(G) = \gamma(B_3(5))$ and so there exists a strongly connected subgraph of $\gamma(G)$ that its vertex set is $\{3,7,13,31\}$ Thus by Theorem 2, there is a simple group S such that $S \trianglelefteq \frac{G}{O_{\{2,5\}}(G)} \leq Aut(S)$ and $\{3,7,13,31\} \subseteq \pi(S) \subseteq \{2,3,5,7,13,31\}$. But since $3 \nsim 31$ in $\Gamma(G) = \Gamma(B_3(5))$ and $3^4 \nmid 31-1$ by Theorem 2 Part 2 we have $\frac{3^4}{[31-1]_3} = 3^3 \big| |S|$ and so $3^3 \cdot 7 \cdot 13 \cdot 31 \big| |S|$. Now by Table 4 of [5], we conclude that $S \cong B_3(5)$ or $S \cong C_3(5)$. Thus $O_{\{2,5\}}(G) = 1$ and since $|G| = |B_3(5)|$, we conclude that $G \cong B_3(5)$ or $G \cong C_3(5)$.

Hence if $\Gamma(G) = \Gamma(B_3(5))$ and $|G| = |B_3(5)|$, then $G \cong B_3(5)$ or $G \cong C_3(5)$.

Example 3. We consider the simple group $\mathbb{A}_{11}$. We know that $|\mathbb{A}_{11}| = 2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$. We can easily see that $deg(11) = 0$ in $\Gamma(\mathbb{A}_{11})$. Assume that G is a finite group with $D(G) = D(\mathbb{A}_{11})$ and $|G| = |\mathbb{A}_{11}|$.

Since $Ord_{11}(2) > 7$, $Ord_{11}(3) > 4$, $Ord_{11}(5) > 2$, $Ord_{11}(7) > 1$, $Ord_{2^7}(11) > 1$, $Ord_{3^4}(11) > 1$, $Ord_{5^2}(11) > 1$ and $Ord_7(11) > 1$, we conclude that $R(11) = 4$ and since $|\pi(G)| = 5$ by Proposition 1 there is a simple group S such that $S \trianglelefteq G \leq Aut(S)$ and $\pi(S) = \pi(G) = \{2,3,5,7,11\}$. Since $2^7 \nmid 11-1 = 10$ and $[10]_2 \big| 2^7$ by Part 2 of Proposition 1 we have $\frac{2^7}{[10]_2} = 2^6 \big| |S|$. Similarly $3^4 \big| |S|$. Thus $|S| = 2^a \cdot 3^4 \cdot 5^b \cdot 7 \cdot 11$, where $6 \leq a \leq 7$, $1 \leq b \leq 2$ and so by Table 4 of [5] S is isomorphic to $\mathbb{A}_{11}$ and since $S \leq G$, $|G| = |\mathbb{A}_{11}|$, we conclude that $G \cong \mathbb{A}_{11}$.

Hence $\mathbb{A}_{11}$ is OD-characterizable.

Example 4. We consider the simple group $\mathbb{A}_{19}$. We know that $|\mathbb{A}_{19}| = 2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$. Obviously $deg(19) = 0$ in $\Gamma(\mathbb{A}_{19})$. Now assume that G is a finite group with $D(G) = D(\mathbb{A}_{19})$ and $|G| = |\mathbb{A}_{19}|$.

We have $Ord_{19}(2) > 15$, $Ord_{19}(3) > 8$, $Ord_{19}(5) > 3$, $Ord_{19}(7) > 2$, $Ord_{19}(11) > 1$, $Ord_{19}(13) > 1$, $Ord_{19}(17) > 1$, $Ord_{2^{15}}(19) > 1$, $Ord_{3^8}(19) > 1$, $Ord_{5^3}(19) > 1$, $Ord_{7^2}(19) > 1$, $Ord_{11}(19) > 1$, $Ord_{13}(19) > 1$ and $Ord_{17}(19) > 1$, thus $R(19) = 7$ and since $|\pi(G)| = 8$ by Proposition 1 there is a simple group S such that $S \trianglelefteq G \leq Aut(S)$ and $\pi(S) = \pi(G) = \{2,3,5,7,11,13,17,19\}$. Since $2^{15} \nmid 19-1 = 18$, $[18]_2 \big| 2^{15}$ by Part 2 of Proposition 1 we have $\frac{2^{15}}{[18]_2} = 2^{14} \big| |S|$. Similarly $\frac{3^8}{[18]_3} = 3^6 \big| |S|$, $5^3 \big| |S|$ and $7^2 \big| |S|$. Thus $|S| = 2^a \cdot 3^b \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$, where $14 \leq a \leq 15$, $6 \leq b \leq 8$ and so by Table 4 of [5] $S \cong \mathbb{A}_{19}$ and since $S \leq G$, $|G| = |\mathbb{A}_{19}|$, we conclude that $G \cong \mathbb{A}_{19}$.

Hence $\mathbb{A}_{19}$ is OD-characterizable.

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REFERENCES

[1] G. Chen, "Further reflections on Thompson's conjecture," *Journal of Algebra - J ALGEBRA*, vol. 218, pp. 276–285, 1999.

[2] M. Darafsheh, A. Moghaddamfar, and A. Zokayi, "A characterization of finite simple groups by the degrees of vertices of their prime graphs," *Algebra Colloquium*, vol. 12, pp. 431–442, 2005.

[3] C. Hering, "Transitive linear groups and linear groups which contain irreducible subgroups of prime order," *Geometriae Dedicata*, vol. 2, no. 4, pp. 425–460, 1974, doi: 10.1007/BF00147570.

[4] A. Iranmanesh and B. Khosravi, "A characterization of $C_2(q)$ where $q > 5$," *Commentationes Mathematicae Universitatis Carolinae*, vol. 43, pp. 9–21, 2002.

[5] A. Moghaddamfar and S. , "More on the od-characterizability of a finite group," *Algebra Colloquium*, vol. 18, no. 4, pp. 663–674, 2012.

[6] A. Vasiliev and E. Vdovin, "An adjacency criterion for the prime graph of a finite simple group," *Algebra i Logika*, vol. 6, pp. 381–406, 2005.

[7] L. Zhang and W. Shi, "Od-characterization of the projective special linear groups $L_2(q)$," *Algebra Colloquium*, vol. 19, no. 03, pp. 509–524, 2012, doi: 10.1142/S1005386712000375.

[8] L. Zhang, W. Shi, C. Shao, and L. Wang, "Od-characterization of the simple group $L_3(9)$," *Journal of Guangxi University. Natural Science Edition*, vol. 34, pp. 120–122, 2009.

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S^p-ALMOST PERIODIC AND S^p-ALMOST AUTOMORPHIC SOLUTIONS OF AN INTEGRAL EQUATION WITH INFINITE DELAY

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Abstract. We state sufficient conditions for the existence and uniqueness of Stepanov-like pseudo almost periodic and Stepanov-like pseudo almost automorphic solutions for a class of nonlinear Volterra integral with infinite delay of the form

$$x(t) = f(t,x(t),x(t-r(t))) - \int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s.$$

Our approach is based on Bochner's transform, some analytic techniques, and a Banach fixed point theorem. Then we apply these results to a nonlinear differential equation when the delay is time-dependent and the force function is continuous

$$x'(t) = ax(t) + \alpha x'(t-r(t)) - q(t,x(t),x(t-r(t))) + h(t).$$

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1. Introduction

The existence and stability of almost periodic solutions of some models are among the most attractive topics in the qualitative theory of differential and integral equations due to their applications in physical science, mathematical biology, population growth... Hence, in the literature, several studies have been conducted on Bohr's almost periodicity and Bochner's almost automorphic to establish sufficient conditions for the existence and uniqueness of various types of differential and integral equations. For instance, one can see [1–3, 6, 20] and the references therein. In particular, it can be noted that several qualitative studies of various differential and integral equations have been carried out in recently published articles [7, 17–19, 25]. The notion of pseudo almost periodicity functions which is the central issue in this paper is a new concept introduced a few years ago by Zhang [27] as a generalization of the classical notion of Bohr's almost periodicity. Also, the notion of almost automorphic (Stepanov) was then defined firstly by N'G uérékata and Pankov [24] as an extension of the classical and well-known almost automorphic concept. It should be mentioned that the study of the existence of almost periodic solutions of the integral equation with a discrete delay was initiated in [16], where Fink and Gatica established the existence of a positive almost periodic solution to the following equation

$$x(t) = \int_{t-\tau}^{t} f(s,x(s))\mathrm{d}s, \tag{1.1}$$

which arises in models for the spread of epidemics. Since then, many works related to the sufficient conditions on the delay and the function f in order to establish the existence of almost periodic solutions to equation (1.1).

In 1997, Ait Dads and Ezzinbi [11] studied the existence of positive almost periodic solutions for the following neutral integral equation

$$x(t) = \gamma x(t-\tau) + (1-\gamma)\int_{t-\tau}^{t} f(s,x(s))\mathrm{d}s. \tag{1.2}$$

Later, Ait Dads et al. [10] established the existence of positive pseudo almost periodic solutions in the case of infinite delay for the equation

$$x(t) = \int_{-\infty}^{t} a(t,t-s)f(s,x(s))\mathrm{d}s, \quad t \in \mathbb{R}. \tag{1.3}$$

Afterwards, Ding et al. [15] developed the above results to the following integral equation with neutral delay

$$x(t) = \alpha x(t-\beta) + \int_{-\infty}^{t} a(t,t-s)f(s,x(s))\mathrm{d}s + h(t,x(t)), \quad t \in \mathbb{R}. \tag{1.4}$$

Equations similar to (1.4) arise in the study of [28] where the authors established the existence and uniqueness of almost periodic and pseudo almost periodic solutions of the integral equation given by

$$x(t) = \alpha(t)x(t-\sigma(t)) + \int_{-\infty}^{t} \beta(t,t-s)f(s,x(s),x'(s))\mathrm{d}s, \quad t \in \mathbb{R}, \tag{1.5}$$

where $\sigma(t)$ is almost periodic (respectively pseudo almost periodic). Recently, the authors in [12] consider two variants of Eq. (1.5), a variant where the delay $\sigma(t)$ is compact almost automorphic in time and another variant where the delay is state-dependent. Also, the existence and uniqueness of periodic solutions of a more general model were established via three fixed point theorems by Islam [21].

Hence, one of the still topical subjects in the study of integral equations and/or differential equations is that if the force functions and/or the coefficients possess a specific property, are we going to find the same characteristics in the solution? Roughly speaking, if the considered functions are Stepanov-like pseudo almost periodic, will the expected solutions of the differential or integral equation be of the same type? The aim of this work is to study the existence and uniqueness of Stepanov-like pseudo almost periodic and Stepanov-like pseudo almost automorphic solutions for the following integral equation

$$x(t) = f(t, x(t), x(t - r(t))) - \int_t^{+\infty} c(t,s) g(s, x(s), x(s - r(s)))\mathrm{d}s, \tag{1.6}$$

where $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $c : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions, $r(\cdot)$ is a time-dependent delay. To the best of our knowledge, there are no papers published on the S^p-pseudo almost periodic solutions and/or S^p-pseudo almost automorphic solutions of this class of Volterra equation.

Our main contributions in this paper are:

(1) The existence and uniqueness of Stepanov-like pseudo almost periodic solution for system (1.6) are proved.
(2) A new proof for the composition theorem in the space $PAPS^p$ is given based mainly on the Banach's transform.
(3) The existence and uniqueness of Stepanov-like pseudo almost automorphic solution for system (1.6) are proved.
(4) The existence and uniqueness of Stepanov-like pseudo almost periodic solutions and Stepanov-like pseudo almost automorphic of a class of logistic differential equation are established.

The organization of this work is as follows. In Section 2, we present some definitions and lemmas that will be used later. In Section 3, we state our main results. More precisely, we give sufficient conditions for the existence and the uniqueness of S^p-pseudo almost automorphic and S^p-pseudo almost periodic solutions of the integral equation (1.6). Our approach is based mainly on Bochner's transform, using analytic techniques and Banach's fixed point theorem. Finally, in Section 4, we study the validity of our theoretical result, therefore we give an illustrating application. It should be mentioned that the main results of this paper include Theorems 1, 2, 3 and 4.

2. Preliminaries: spaces of functions

Throughout this article $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ and $(\mathbb{F}, \|\cdot\|_{\mathbb{F}})$ denote Banach spaces and $C(\mathbb{E}, \mathbb{F})$ the Banach space of continuous functions from $\mathbb{E}$ to $\mathbb{F}$. We denote by $BC(\mathbb{R}, \mathbb{E})$ the Banach space of bounded and continuous defined functions on $\mathbb{R}$ with the sup norm defined by

$$\|f\| = \sup_{t \in \mathbb{R}} \|f(t)\|. \tag{2.1}$$

Definition 1 ([4])**.** A set D of real numbers is said to be relatively dense if there exists a number $\ell > 0$ such that any interval of length ℓ contains at least one number of D.

Definition 2 ([4])**.** A function $f \in C(\mathbb{R}, \mathbb{E})$ is called (Bohr) almost periodic if for each $\varepsilon > 0$ the set $T(f, \varepsilon) = \{\tau : f(t + \tau) - f(t)\}$ is relatively dense, i.e. for any $\varepsilon > 0$

there exists $l = l(\varepsilon) > 0$ such that every interval of length l contains a number τ with the property that

$$\|f(t+\tau) - f(t)\| < \varepsilon, \quad t \in \mathbb{R}.$$

The collection of all such functions will be denoted by $AP(\mathbb{R},\mathbb{E})$.

Definition 3 ([26]). A function $f \in C(\mathbb{R}\times\mathbb{E},\mathbb{F})$ is called (Bohr) almost periodic in $t \in \mathbb{R}$ uniformly in $y \in K$ where $K \subset \mathbb{E}$ is any compact subset if for each $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that every interval of length l contains a number τ with the property that

$$\|f(t+\tau,y) - f(t,y)\| < \varepsilon, \quad t \in \mathbb{R},\ y \in K.$$

The collection of such functions will be denoted by $AP(\mathbb{R}\times\mathbb{E},\mathbb{F})$.

Lemma 1 ([5]). *Let $f \in AP(\mathbb{R}\times\mathbb{E},\mathbb{F})$ and $\phi \in AP(\mathbb{R},\mathbb{E})$ then the function* $[t \longmapsto F(t,\phi(t))] \in AP(\mathbb{R},\mathbb{F})$.

Definition 4 ([27]). A continuous function $f : \mathbb{R} \longrightarrow \mathbb{E}$ is called pseudo almost periodic if it can be written as $f = h + \phi$ where $h \in AP(\mathbb{R},\mathbb{E})$ and $\phi \in PAP_0(\mathbb{R},\mathbb{E})$ where the space $PAP_0(\mathbb{R},\mathbb{E})$ is defined by

$$PAP_0(\mathbb{R},\mathbb{E}) = \left\{ f \in BC(\mathbb{R},\mathbb{E}), \mathcal{M}(\|f\|) = \lim_{T\longrightarrow\infty} \frac{1}{2T}\int_{-T}^{T} \|f(t)\|\mathrm{d}t = 0 \right\}.$$

The functions h and ϕ in above definition are respectively called the almost periodic components and the ergodic perturbation of the pseudo-almost periodic function f. The collection of all pseudo almost periodic functions which map from $\mathbb{R}$ to $\mathbb{E}$ will be denoted by $PAP(\mathbb{R},\mathbb{E})$.

Definition 5 ([13]). The Bochner transform $f^b(t,s)$ with $t \in \mathbb{R}, s \in [0,1]$ of a function $f : \mathbb{R} \mapsto \mathbb{E}$ is defined by $f^b(t,s) := f(t+s)$.

Definition 6 ([13]). The Bochner transform $F^b : \mathbb{R}\times[0,1]\times\mathbb{E} \mapsto \mathbb{E}$ of a function $F : \mathbb{R}\times\mathbb{E} \mapsto \mathbb{E}$ is defined by $F^b(t,s,u) := F(t+s,u)$ for each $t \in \mathbb{R},\ s \in [0,1]$, and $u \in \mathbb{E}$.

Definition 7 ([13]). Let $p \in [1,\infty[$. The space $BS^p(\mathbb{R},\mathbb{E})$ of all Stepanov-like bounded functions, with exponent p, consists of all measurable functions $f : \mathbb{R} \mapsto \mathbb{E}$ such that $f^b \in L^\infty(\mathbb{R}, L^p((0,1),\mathbb{E}))$. This is a Banach space with the norm

$$\|f\|_{BS^p(\mathbb{R},\mathbb{E})} := \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p \mathrm{d}\tau \right)^{1/p}.$$

Definition 8 ([14]). A function $f \in BS^p(\mathbb{R},\mathbb{E})$ is called Stepanov-like almost periodic if $f^b \in AP(\mathbb{R}, L^p((0,1),\mathbb{E}))$. The collection of these functions will be denoted by $APS^p(\mathbb{R},\mathbb{E})$.

Definition 9 ([14]). A function $f:\mathbb{R}\times\mathbb{E}\to\mathbb{F},(t,u)\mapsto f(t,u)$ with $f(\cdot,u)\in BS^p(\mathbb{R},)$, for each $u\in\mathbb{E}$, is called Stepanov almost periodic function in $t\in\mathbb{R}$ uniformly for $u\in\mathbb{E}$ if for each $\varepsilon>0$ and each compact set $K\subset\mathbb{E}$ there exists a relatively dense set $P=P(\varepsilon,f,K)\subset\mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}}\left(\int_0^1\|f(t+s+\tau,u)-f(t+s,u)\|\mathrm{d}s\right)^{1/p}<\varepsilon, \tag{2.2}$$

for each $\tau\in P$, $u\in K$. We denote by $APS^p(\mathbb{R}\times\mathbb{E},\mathbb{F})$ the set of such functions.

Definition 10 ([13]). Let $p\geq 1$. A function $f\in BS^p(\mathbb{R},\mathbb{E})$ is called S^p-pseudo almost periodic (or Stepanov-like pseudo almost periodic) if it can be expressed as

$$f=h+\phi, \tag{2.3}$$

where $h^b\in AP(L^p((0,1),\mathbb{E}))$ and $\phi^b\in PAP_0(L^p((0,1),\mathbb{E}))$. In other words, a function $f\in L^p(\mathbb{R},\mathbb{E})$ is said to be S^p-pseudo almost periodic if its Bochner transform $f^b:\mathbb{R}\longrightarrow L^p((0,1),\mathbb{E})$ is pseudo almost periodic in the sense that there exist two functions $h,\phi:\mathbb{R}\to\mathbb{E}$ such that $f=h+\phi$, where $h^b\in AP(L^p((0,1),\mathbb{E}))$ and $\phi^b\in PAP_0(L^p((0,1),\mathbb{E}))$ that is,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\|\varphi(\sigma)\|^p\mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t=0. \tag{2.4}$$

The collection of such functions will be denoted by $PAPS^p(\mathbb{R},\mathbb{E})$.

Definition 11 ([23]). A continuous function $f:\mathbb{R}\longrightarrow\mathbb{E}$ is almost automorphic if for every sequence of real numbers $(s_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t)=\lim_{n\to+\infty}f(t+s_n) \tag{2.5}$$

is well defined for each $t\in\mathbb{R}$, and

$$\lim_{n\to+\infty}g(t-s_n)=f(t) \tag{2.6}$$

for each $t\in\mathbb{R}$. The collection of all almost automorphic functions which map from $\mathbb{R}$ to $\mathbb{E}$ is denoted by $AA(\mathbb{R},\mathbb{E})$.

Definition 12 ([5]). A function $f:\mathbb{R}\times\mathbb{E}\longrightarrow\mathbb{E}$ $(t,x)\longmapsto f(t,x)$ is said to be almost automorphic in $t\in\mathbb{R}$ for each $u\in\mathbb{E}$ when it satisfies the two following conditions:

(1) For all $x\in\mathbb{E}$, the function $f(\cdot,x)\in AA(\mathbb{R},\mathbb{E})$.
(2) For all subset compact K of $\mathbb{E}$, for all $\varepsilon>0$ there exists $\delta=\delta(k,\varepsilon)>0$ such that, for all $x,z\in k$, if $d(x,z)\leq\delta$ then we have $d(f(x,t),f(z,t))\leq\varepsilon$ for all $t\in\mathbb{R}$.

The collection of such functions will be denoted by $AA(\mathbb{E}\times\mathbb{R},\mathbb{F})$.

Lemma 2 ([5]). *Let $f \in AA(\mathbb{R}\times\mathbb{E},\mathbb{F})$ and $u \in AA(\mathbb{R},\mathbb{E})$, then we have*

$$[t \longmapsto f(t,u(t))] \in AA(\mathbb{R},\mathbb{F}).$$

Lemma 3 ([9]). *If the functions $x(\cdot) \in PAPS^p(\mathbb{R},\mathbb{R})$ and $r(\cdot) \in APS^p(\mathbb{R},\mathbb{R})$ then we have $x(\cdot - r(\cdot)) \in PAPS^p(\mathbb{R},\mathbb{R})$.*

Definition 13 ([14]). A function $f \in BS^p(\mathbb{R},\mathbb{E})$ is called S^p-almost automorphic if $f^b \in AA(L^p((0,1),\mathbb{E}))$. The collection of such functions will be denoted by $AAS^p(\mathbb{R},\mathbb{E})$.

Definition 14 ([14]). A function $f \in BS^p(\mathbb{R}\times\mathbb{E},\mathbb{F})$, $(t,u) \mapsto F(t,u)$ where $F(\cdot,u) \in L^p(\mathbb{R},\mathbb{E})$ for each $u \in \mathbb{E}$, is called S^p-pseudo almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{E}$ if $t \mapsto F(t,u)$ is S^p-pseudo automorphic for each $u \in K$ where $K \subset \mathbb{E}$ is a bounded subset. The collection of such functions will be denoted by $PAAS^p(\mathbb{R}\times\mathbb{E},\mathbb{F})$.

3. Main results

3.1. *Stepanov-like pseudo almost periodic solutions*

In this section, we consider the following integral equation

$$x(t) = f(t,x(t),x(t-r(t))) - \int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s, \tag{3.1}$$

where $f : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $c, r : \mathbb{R} \to \mathbb{R}$ are continuous functions. We give sufficient conditions which guarantee the existence of S^p-pseudo almost periodic solutions for equation (3.1).

(H1) $f : \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is S^p-pseudo almost periodic, i.e. $f^b = h^b + \phi^b$, where $h^b \in AP(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and $\phi^b \in PAP_0(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \left(\int_t^{t+1} |\phi(\sigma,u)|^p \,\mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t = 0, \tag{3.2}$$

uniformly in $u \in \mathbb{R}^2$.

(H2) f is Lipschitz i.e. $\exists L_f^1, L_f^2 > 0$ such that $\forall x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le L_f^1|x_1-y_1| + L_f^2|x_2-y_2|. \tag{3.3}$$

(H3) $g : \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is S^p-pseudo almost periodic, i.e. $g^b = g_1^b + g_2^b$, where $g_1^b \in AP(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and $g_2^b \in PAP_0(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \left(\int_t^{t+1} |g_2(\sigma,u)|^p \,\mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t = 0, \tag{3.4}$$

uniformly for all $u \in \mathbb{R}^2$.

(H4) g is Lipschitz i.e. $\exists L_g^1, L_g^2 > 0$ such that $\forall x_1, x_2, y_1, y_2 \in \mathbb{R}$

$$|g(t,x_1,x_2) - g(t,y_1,y_2)| \leq L_g^1 |x_1 - y_1| + L_g^2 |x_2 - y_2|. \tag{3.5}$$

(H5) There exists a constant $\lambda > 0$ such that $c(t,s) \leq \mathrm{e}^{\lambda(t-s)}$, for all $s \geq t$.

(H6) The function $t \mapsto r(t) \in APS^p(\mathbb{R},\mathbb{R}) \cap C^1(\mathbb{R},\mathbb{R})$ with

$$0 \leq r(t) \leq \bar{r},\ r(t) \leq r^* < 1, \quad \text{for all } t \in \mathbb{R}. \tag{3.6}$$

Lemma 4. *Assume that (H1)-(H3) hold, if $x(\cdot) \in PAPS^p(\mathbb{R},\mathbb{R})$, then the function $\beta : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\beta(\cdot) = f(\cdot, x(\cdot), x(\cdot - r(\cdot)))$ belongs to $PAPS^p(\mathbb{R},\mathbb{R})$.*

Proof. Let $f = h + \phi$ where $h^b \in AP(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and the function $\phi^b \in PAP_0(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$. Similarly, let $x^b(\cdot) = x_1^b(\cdot) + x_2^b(\cdot)$ where the function $x_1^b \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}))$ and $x_2^b \in PAP_0(\mathbb{R}, L^p((0,1),\mathbb{R}))$ that is

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_t^{t+1} |x_2(\sigma)|^p \,\mathrm{d}\sigma \right)^{\frac{1}{p}} \mathrm{d}t = 0, \tag{3.7}$$

for all $t \in \mathbb{R}$. By Lemma 3 we get $x(\cdot - r(\cdot)) \in PAPS^p(\mathbb{R},\mathbb{R})$, then

$$x^b(\cdot - r(\cdot)) = x_1^b(\cdot - r^b(\cdot)) + x_2^b(\cdot - r^b(\cdot)), \tag{3.8}$$

where $x_1^b(\cdot - r^b(\cdot)) \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}))$ and $x_2^b(\cdot - r^b(\cdot)) \in PAP_0(\mathbb{R}, L^p((0,1),\mathbb{R}))$ that is

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_t^{t+1} |x_2(\sigma - r(\sigma))|^p \,\mathrm{d}\sigma \right)^{\frac{1}{p}} \mathrm{d}t = 0. \tag{3.9}$$

Since $f^b : \mathbb{R} \longrightarrow L^p((0,1),\mathbb{R})$ we decompose f^b as follows

$$\begin{aligned}
&f^b(\cdot, x^b(\cdot), x^b(\cdot - r(t))) \\
&= h^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r(t))) + f^b(\cdot, x^b(\cdot), x^b(\cdot - r(t))) - h^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r(t))) \\
&= h^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r(t))) + f^b(\cdot, x^b(\cdot), x^b(\cdot - r(t))) - f^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r(t))) \\
&\quad + \phi^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r(t))).
\end{aligned}$$

Let us prove that $h^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r(\cdot))) \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}))$. First, the function $x_1^b(\cdot) \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}))$ and $x_1^b(\cdot - r^b(\cdot)) \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}))$. Then the function $u_1^b(\cdot) = (x_1^b(\cdot), x_1^b(\cdot - r(t))) \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}^2))$. Indeed, $\forall \varepsilon > 0$, $\exists \ell > 0$, $\forall a \in \mathbb{R}$, $\exists \tau \in [a, a+\ell]$ such that

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \left(\int_0^1 \left\| u_1^b(t+\tau) - u_1^b(t) \right\|_\infty^p \mathrm{d}s \right)^{\frac{1}{p}} \\
&= \sup_{t\in\mathbb{R}} \left(\int_0^1 \left(\max\left(|x_1^b(t+\tau) - x_1^b(t)|, |x_1^b(t+\tau - r^b(t)) - x_1^b(t - r^b(t))| \right) \right)^p \mathrm{d}s \right)^{\frac{1}{p}} \\
&\leq \varepsilon.
\end{aligned}$$

Since the function $h^b \in AP(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and $u_1^b(\cdot) \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}^2))$ then, we can apply the composition theorem of almost periodic functions 1, thus $h^b(\cdot, u_1^b(\cdot)) \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}))$. Now, set

$$G^b(\cdot) = f^b(\cdot, x^b(\cdot), x^b(\cdot - r^b(\cdot))) - f^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r^b(\cdot))). \tag{3.10}$$

$G^b(\cdot) \in PAP_0(\mathbb{R}, (L^p((0,1),\mathbb{R}))$. Indeed, let $T>0$, we have

$$\begin{aligned}
&\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T}\left(\int_0^1 \left|G^b(\sigma)\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t \\
&\le \lim_{T\to+\infty} \frac{L_f^1}{2T}\int_{-T}^{T}\left(\int_0^1 \left|x^b(\sigma)-x_1^b(\sigma)\right|^p d\sigma\right)^{\frac{1}{p}} \mathrm{d}t \\
&\quad + \lim_{T\to+\infty} \frac{L_f^2}{2T}\int_{-T}^{T}\left(\int_0^1 \left|x^b(\sigma-r^b(\sigma))-x_1^b(\sigma-r^b(\sigma))\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t \\
&\le \lim_{T\to+\infty} \frac{L_f^1}{2T}\int_{-T}^{T}\left(\int_0^1 \left|x_2^b(\sigma)\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t \\
&\quad + \lim_{T\to+\infty} \frac{L_f^2}{2T}\int_{-T}^{T}\left(\int_0^1 \left|x_2^b(\sigma-r^b(\sigma))\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t.
\end{aligned}$$

Using (3.7) and (3.9) we get $\frac{1}{2T}\int_{-T}^{T}\left(\int_0^1 \left|G_2^b(\sigma)\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t = 0.$

Moreover, using the composition theorem of ergodic functions (cf. [22]) we have $\phi^b(\cdot, u_1^b(\cdot)) \in PAP_0(\mathbb{R}, L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left(\int_0^1 \left|\phi^b(\sigma, u_1^b(\sigma))\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t = 0. \tag{3.11}$$

□

Lemma 5. *Assume that (H3)-(H5) hold. If $x(\cdot) \in PAPS^p(\mathbb{R},\mathbb{R})$, then the function $\Theta : t \longmapsto \int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s \in PAPS^p(\mathbb{R},\mathbb{R})$ for all $s\in\mathbb{R}$.*

Proof. Using Lemma 4 and the hypothesis (H5), we obtain that the integral is convergent and consequently $t \longmapsto \int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s$ is well defined. Otherwise, since $[s \longmapsto g(s,x(s),x(s-r(s))] \in PAPS^p(\mathbb{R},\mathbb{R})$, one can write

$$g = g_1 + g_2 \tag{3.12}$$

with $g_1 \in APS^p(\mathbb{R},\mathbb{R})$ i.e. for each $\varepsilon' > 0$, there exists $\ell > 0$ such that every interval of length ℓ contains a τ such that $\|g_1(t+\tau) - g_1(t)\|_{S^p} < \varepsilon'$ and the function

$g_2 \in PAP_0(\mathbb{R}, L^p((t,t+1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}|g_2(\sigma)|^p\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t = 0. \tag{3.13}$$

Then

$$\Theta(t) = \int_t^{+\infty} c(t,s)g_1(s)\mathrm{d}s + \int_t^{+\infty} c(t,s)g_2(s)\mathrm{d}s = \Theta_1(t)+\Theta_2(t).$$

Now, we shall study the S^p-almost periodicity of $\Theta_1(\cdot)$. We have

$$\begin{aligned}|\Theta_1(t+\tau)-\Theta_1(t)| &\le \left|\int_{t+\tau}^{+\infty} \mathrm{e}^{\lambda(t+\tau-s)}g_1(s)\mathrm{d}s - \int_t^{+\infty}\mathrm{e}^{\lambda(t-s)}g_1(s)\mathrm{d}s\right| \\ &= \left|\int_t^{+\infty}\mathrm{e}^{\lambda(t-\xi)}g_1(\xi+\tau)\mathrm{d}\xi - \int_t^{+\infty}\mathrm{e}^{\lambda(t-s)}g_2(s)\mathrm{d}s\right| \\ &\le \int_t^{+\infty}\mathrm{e}^{\lambda(t-s)}|g_1(s+\tau)-g_1(s)|\mathrm{d}s.\end{aligned}$$

According to Hölder inequality $\left(\dfrac{1}{p}+\dfrac{1}{q}=1\right)$ one has for all $\tau\in\mathbb{R}$,

$$\begin{aligned}|\Theta_1(t+\tau)-\Theta_1(t)| &\le \int_0^{+\infty}\mathrm{e}^{-\lambda s}|g_1(s+t+\tau)-g_1(s+t)|\mathrm{d}s \\ &\le \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\int_0^{+\infty}\mathrm{e}^{\frac{-\lambda ps}{2}}|g_1(s+t+\tau)-g_1(s+t)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}.\end{aligned}$$

Using Fubini's theorem, we get for all $\tau\in\mathbb{R}$

$$\begin{aligned}&\sup_{x\in\mathbb{R}}\left(\int_x^{x+1}|\Theta_1(t+\tau)-\Theta_1(t)|^p\mathrm{d}t\right)^{\frac{1}{p}} \\ &\le \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\sup_{x\in\mathbb{R}}\left(\int_x^{x+1}\int_0^{+\infty}\mathrm{e}^{\frac{-\lambda ps}{2}}|g_1(s+t+\tau)-g_1(s+t)|^p\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{p}} \\ &\le \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\int_0^{+\infty}\mathrm{e}^{\frac{-\lambda ps}{2}}\sup_{x\in\mathbb{R}}\int_x^{x+1}|g_1(s+t+\tau)-g_1(s+t)|^p\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{p}}.\end{aligned}$$

As g_1 is S^p almost periodic, for $\varepsilon' = C\varepsilon > 0$ we have

$$\sup_{x\in\mathbb{R}}\left(\int_x^{x+1}|g_1(t++s+\tau)-g_1(t+s)|^p\mathrm{d}t\right)^{\frac{1}{p}} < \varepsilon' = C\varepsilon, \tag{3.14}$$

where $C = \left(\dfrac{\lambda p}{2}\right)^{\frac{1}{p}}\left(\dfrac{\lambda q}{2}\right)^{\frac{1}{q}}$. Then $\sup\limits_{x\in\mathbb{R}}\left(\displaystyle\int_x^{x+1}|\Theta_1(t+\tau)-\Theta_1(t)|^p\mathrm{d}t\right)^{\frac{1}{p}} \le \varepsilon$. This proves the S^p-almost periodicity of Θ_1. Now let's show the ergodicity of $\Theta_2(\cdot)$. Since

$\Theta_2(\cdot) \in BC(\mathbb{R},\mathbb{R})$, it remains to show that

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1} |\Theta_2(\sigma)|^p \,\mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t = 0. \tag{3.15}$$

Let $q \in [1,\infty[$ such that $\dfrac{1}{p}+\dfrac{1}{q}=1$ then, by Hölder's inequality and Fubini's theorem we obtain

$$\begin{aligned}
&\int_{-T}^{T}\left(\int_t^{t+1} |\Theta_2(\sigma)|^p \,\mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t \\
&\le (2T)^{\frac{1}{q}}\left[\int_{-T}^{T}\left(\int_t^{t+1} |\Theta_2(\sigma)|^p \,\mathrm{d}\sigma\right)\mathrm{d}t\right]^{\frac{1}{p}} \\
&\le |\Theta_2|_\infty^{\frac{1}{q}}\,(2T)\left[\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\left|\int_\sigma^{+\infty} \mathrm{e}^{\lambda(\sigma-s)} g_2(s)\mathrm{d}s\right|\mathrm{d}\sigma\right)\mathrm{d}t\right]^{\frac{1}{p}},
\end{aligned}$$

which gives

$$\begin{aligned}
&\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1} |\Theta_2(\sigma)|^p \,\mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t \\
&\qquad \le \lim_{T\to+\infty} |\Theta_2|_\infty^{\frac{1}{q}}\left[\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\left|\int_\sigma^{+\infty} \mathrm{e}^{\lambda(\sigma-s)} g_2(s)\mathrm{d}s\right|\mathrm{d}\sigma\right)\mathrm{d}t\right]^{\frac{1}{p}}.
\end{aligned}$$

On the other hand, we get

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\left|\int_\sigma^{+\infty} \mathrm{e}^{\lambda(\sigma-s)} g_2(s)\mathrm{d}s\right|\mathrm{d}\sigma\right)\mathrm{d}t \le I+J, \tag{3.16}$$

with

$$I = \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\left|\int_\sigma^{T} \mathrm{e}^{\lambda(\sigma-s)} g_2(s)\mathrm{d}s\right|\mathrm{d}\sigma\right)\mathrm{d}t \tag{3.17}$$

and

$$J = \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\left|\int_T^{+\infty} \mathrm{e}^{\lambda(\sigma-s)} g_2(s)\mathrm{d}s\right|\mathrm{d}\sigma\right)\mathrm{d}t. \tag{3.18}$$

Further

$$I \le \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1} \mathrm{e}^{\lambda\sigma}\int_\sigma^{T} \mathrm{e}^{-\lambda s}|g_2(s)|\,\mathrm{d}s\mathrm{d}\sigma\right)\mathrm{d}t.$$

Now, we have $[s \longmapsto \mathrm{e}^{-\lambda s}]$ and $[s \longmapsto |g_2(s)|]$ are two continuous functions on $[\sigma, T]$, furthermore $[s \longmapsto |g_2(s)|]$ keep a constant sign, so $\exists\, \xi \in]\sigma, T[$ such that,

$$\begin{aligned}
I &= \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}\left(\mathrm{e}^{\lambda\sigma}|g_2(\xi)|\int_\sigma^{T} \mathrm{e}^{-\lambda s}\mathrm{d}s\right)\mathrm{d}\sigma\right)\mathrm{d}t \\
&= \lim_{T\to+\infty}\frac{1}{2T\lambda}\int_{-T}^{T}\left(\int_t^{t+1}\left(|g_2(\xi)|\left[1-\mathrm{e}^{-\lambda(T-\sigma)}\right]\right)\mathrm{d}\sigma\right)\mathrm{d}t.
\end{aligned}$$

Now as $1-\mathrm{e}^{-\lambda(T-\sigma)} \le 1$, one has $I \le \lim_{T\to+\infty} \frac{1}{2T\lambda}\int_{-T}^{T}\left(\int_{t}^{t+1}|g_2(\xi)|\,\mathrm{d}\sigma\right)\mathrm{d}t$. Since $\xi \in]\sigma,T[$, then $\xi=(1-\alpha)\sigma+\alpha T$, where $\alpha\in]0,1[$. Hence,

$$I=\lim_{T\to+\infty}\frac{1}{2T\lambda}\int_{-T}^{T}\left(\int_{t}^{t+1}(|g_2((1-\alpha)\sigma+\alpha T)|)\,\mathrm{d}\sigma\right)\mathrm{d}t. \tag{3.19}$$

On the other hand, since $t\le\sigma\le t+1$, we get

$$(1-\alpha)t+\alpha T\le(1-\alpha)\sigma+\alpha T\le(1-\alpha)t+(1-\alpha)+\alpha T.$$

Besides, $1-\alpha<1$, thus

$$(1-\alpha)t+\alpha T\le(1-\alpha)\sigma+\alpha T\le(1-\alpha)t+\alpha T+1.$$

Set $z=(1-\alpha)\sigma+\alpha T$ and $u=(1-\alpha)\sigma+\alpha T$. We obtain,

$$I\le\lim_{T\to+\infty}\frac{1}{2T\lambda}\int_{-T}^{T}\left(\int_{z}^{z+1}|g_2(u)|\,du\right)\mathrm{d}t. \tag{3.20}$$

According to the hypothesis $g_2\in PAP_0(\mathbb{R},L^p((t,t+1),\mathbb{R}))$ we conclude that $I=0$. Meanwhile, by applying Fubini's theorem we obtain

$$\begin{aligned}J&=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\left|\int_{T}^{+\infty}\mathrm{e}^{\lambda(\alpha-s)}g_2(s)\mathrm{d}s\right|\mathrm{d}\alpha\right)\mathrm{d}t\\&=\lim_{T\to+\infty}\frac{1}{2T\lambda}\int_{-T}^{T}\left(\int_{T}^{+\infty}|g_2(s)|\,\mathrm{e}^{-s\lambda}\left[\mathrm{e}^{\lambda(t+1)}-\mathrm{e}^{\lambda t}\right]\mathrm{d}s\right)\mathrm{d}t.\end{aligned}$$

Thus, $J\le J_1+J_2$ with

$$J_1=\lim_{T\to+\infty}\frac{\mathrm{e}^{\lambda}}{2T\lambda}\int_{-T}^{T}\int_{T}^{+\infty}|g_2(s)|\,\mathrm{e}^{\lambda(t-s)}\mathrm{d}s\mathrm{d}t, \tag{3.21}$$

and

$$J_2=\lim_{T\to+\infty}\frac{1}{2T\lambda}\int_{-T}^{T}\int_{T}^{+\infty}|g_2(s)|\,\mathrm{e}^{\lambda(t-s)}\mathrm{d}s\mathrm{d}t. \tag{3.22}$$

Let $\xi=s-t$, then

$$J_1\le\lim_{T\to+\infty}\frac{\mathrm{e}^{\lambda}\,|g_2|_\infty}{2T\lambda}\int_{-T}^{T}\int_{T-t}^{+\infty}\mathrm{e}^{-\lambda\xi}\mathrm{d}\xi\mathrm{d}t=\lim_{T\to+\infty}\frac{\mathrm{e}^{\lambda}\,|g_2|_\infty}{2T\lambda^3}\left[1-\mathrm{e}^{-2\lambda T}\right]=0.$$

Similarly, it is easy to see that $J_2=0$ which implies that $J=0$. Then we have

$$\lim_{T\to+\infty}|\Theta_2|_\infty^{\frac{1}{q}}\left[\frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}\left|\int_{\sigma}^{+\infty}\mathrm{e}^{\lambda(\sigma-s)}g_2(s)\mathrm{d}s\right|\mathrm{d}\sigma\right)\mathrm{d}t\right]^{\frac{1}{p}}=0. \tag{3.23}$$

Consequently,

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_{t}^{t+1}|\Theta_2(\sigma)|^p\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t=0. \tag{3.24}$$

Therefore the function $\Theta : t \longmapsto \int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s$ belongs to $PAPS^p(\mathbb{R},\mathbb{R})$. $\square$

Now, we are able to establish the existence and uniqueness of the Stepanov-like pseudo almost periodic solutions of (3.1).

Theorem 1. *We assume (H1)-(H5) hold. If $m<1$ then, (3.1) has a unique S^p-pseudo almost periodic solution with*

$$m=\max\left(L_f^1, L_f^2(1-r^*)^{-\frac{1}{p}}, L_g^1\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}, L_g^2\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}(1-r^*)^{-\frac{1}{p}}\right).$$

Proof. Define the operator on $PAPS^p(\mathbb{R},\mathbb{R})$ by

$$\Gamma(x)(t)=f(t,x(t),x(t-r(t)))-\int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s, \quad t\in\mathbb{R}. \tag{3.25}$$

Using Lemma 3 and (H1) we get that the function $t\mapsto f(t,x(t),x(t-r(t)))$ is continuous. Furthermore, by Lemma 4 and the hypothesis (H5) we get that the integral defined by $t\longmapsto \int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s$ exists. Thus, Γx is well defined. Moreover, from Lemmas 4 and 5 we deduce that

$$\Gamma : PAPS^p(\mathbb{R},\mathbb{R}) \longrightarrow PAPS^p(\mathbb{R},\mathbb{R}).$$

Let $x,\, y\in PAPS^p(\mathbb{R},\mathbb{R})$, according to Hölder's inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, we get

$$\begin{aligned}
|\Gamma x(t)-\Gamma y(t)| &\le |f(t,x(t),x(t-r(t)))-f(t,y(t),y(t-r(t)))|\\
&\quad+\left|\int_t^{+\infty} c(t,s)\Big(g(s,y(s),y(s-r(s)))\mathrm{d}s-g(s,x(s),x(s-r(s)))\Big)\mathrm{d}s\right|\\
&= |f(t,x(t),x(t-r(t)))-f(t,y(t),y(t-r(t)))|\\
&\quad+\left(\frac{2}{q\lambda}\right)^{\frac{1}{q}}\left(\int_0^{+\infty} \mathrm{e}^{\frac{-\lambda ps}{2}}|g(s+t,y(s+t),y(s+t-r(t)))\right.\\
&\quad\left.-\ g(s+t,x(s+t),x(s+t-r(t)))|^p\,\mathrm{d}s\right)^{\frac{1}{p}}.
\end{aligned}$$

Then using Fubini's theorem and Minkowski's inequality, we get

$$\begin{aligned}
&\sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}|\Gamma x(t)-\Gamma y(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\\
&\le \sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}\left(L_f^1|x(t)-y(t)|+L_f^2|x(t-r(t))-y(t-r(t))|\right)^p\mathrm{d}t\right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\int_0^\infty \mathrm{e}^{\frac{-\lambda ps}{2}}\sup_{\xi\in\mathbb{R}}\int_\xi^{\xi+1}\left(L_g^1|y(t+s)-x(t+s)|\right.\right.\\
&\left.\left.+\ L_g^2|y(t+s-r(s))-x(t+s-r(s))|\right)^p \mathrm{d}t\ \mathrm{d}s\right)^{\frac{1}{p}}\\
&\le \sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}\left(L_f^1|x(t)-y(t)|+L_f^2|x(t-r(t))-y(t-r(t))|\right)^p \mathrm{d}t\right)^{\frac{1}{p}}\\
&+\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\times\left(\int_0^\infty \mathrm{e}^{\frac{-\lambda ps}{2}}\sup_{\xi'\in\mathbb{R}}\int_{\xi'}^{\xi'+1}\left(L_g^1|y(t')-x(t')|\right.\right.\\
&\left.\left.+\ L_g^2|y(t'-r(t))-x(t'-r(t))|\right)^p \mathrm{d}t'\mathrm{d}s\right)^{\frac{1}{p}}\\
&\le \sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}\left(L_f^1|x(t)-y(t)|+L_f^2|x(t-r(t))-y(t-r(t))|\right)^p \mathrm{d}t\right)^{\frac{1}{p}}\\
&+\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}\left(\sup_{\xi'\in\mathbb{R}}\int_{\xi'}^{\xi'+1}\left(L_g^1|y(t')-x(t')|\right.\right.\\
&\left.\left.+\ L_g^2|y(t'-r(t'))-x(t'-r(t'))|\right)^p \mathrm{d}t'\right)^{\frac{1}{p}}\\
&\le L_f^1\|x-y\|_{S^p}+L_g^1\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}\|x-y\|_{S^p}\\
&+L_f^2\left(1-r'(t)\right)^{-\frac{1}{p}}\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-r(\xi)}^{\xi+1-r(\xi+1)}|x(\rho)-y(\rho)|^p\,d\rho\right)^{\frac{1}{p}}\\
&+L_g^2\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}\left(1-r'(t)\right)^{-\frac{1}{p}}\times\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-r(\xi)}^{\xi+1-r(\xi+1)}|x(\rho)-y(\rho)|^p\,d\rho\right)^{\frac{1}{p}}\\
&\le L_f^1\|x-y\|_{S^p}+L_f^2(1-r^*)^{-\frac{1}{p}}\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-\bar{r}}^{\xi+1-\bar{r}}|x(\rho)-y(\rho)|^p\,d\rho\right)^{\frac{1}{p}}\\
&+L_g^1\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}\|x-y\|_{S^p}\\
&+L_g^2\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}(1-r^*)^{-\frac{1}{p}}\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-\bar{r}}^{\xi+1-\bar{r}}|x(\rho)-y(\rho)|^p\,d\rho\right)^{\frac{1}{p}}\\
&\le m\|x-y\|_{S^p},
\end{aligned}
$$

where

$$m = \max\left(L_f^1, L_f^2(1-r^*)^{-\frac{1}{p}}, L_g^1\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}, L_g^2\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}(1-r^*)^{-\frac{1}{p}}\right).$$

Since $m < 1$, the operator $\Gamma : (PAPS^p(\mathbb{R},\mathbb{R}), \|\cdot\|_{S^p}) \longrightarrow (PAPS^p(\mathbb{R},\mathbb{R}), \|\cdot\|_{S^p})$ is a contraction. Therefore, by applying the Banach fixed point theorem, there is a unique $x_* \in PAPS^p(\mathbb{R},\mathbb{R})$ such that $\Gamma(x_*) = x_*$, which corresponds to the unique S^p-almost periodic pseudo solution of equation (3.1). $\square$

3.2. *Stepanov like (pseudo) almost automorphic solutions*

In this section, we establish the existence of pseudo almost automorphic solutions of equation (3.1). For this study, we make the following assumptions:

(H1) $f : \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is S^p-pseudo almost automorphic, i.e. $f^b = h^b + \phi^b$, where the function $h^b \in AA(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and the function $\phi^b \in PAP_0((\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}|\phi(\sigma,u)|^p\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t = 0, \tag{3.26}$$

uniformly for all $u \in \mathbb{R}^2$.

(H2) f is a Lipschitz function, i.e. $\exists L_f^1, L_f^2 > 0$ such that $\forall x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le L_f^1|x_1-y_1| + L_f^2|x_2-y_2|. \tag{3.27}$$

(H3) $g : \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is S^p-pseudo almost automorphic, i.e. $g^b = g_1^b + g_2^b$, where $g_1^b \in AA(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and $g_2^b \in PAP_0(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}|g_2(\sigma,u)|^p\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t = 0, \tag{3.28}$$

uniformly for all $u \in \mathbb{R}^2$.

(H4) g is Lipschitz , i.e. $\exists L_g^1, L_g^2 > 0$ such that $\forall x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$|g(t,x_1,x_2) - g(t,y_1,y_2)| \le L_g^1|x_1-y_1| + L_g^2|x_2-y_2|. \tag{3.29}$$

(H5) There exists a constant $\lambda > 0$ such that $c(t,s) \le \mathrm{e}^{\lambda(t-s)}$, for all $s \ge t$.

(H6) The function $t \mapsto r(t) \in C^1(\mathbb{R},\mathbb{R})$ with

$$0 \le r(t) \le \bar{r}, \quad r(t) \le r^* < 1. \tag{3.30}$$

Lemma 6. *Assume that (H6) holds. If $x(\cdot) \in PAAS^p(\mathbb{R},\mathbb{R})$ then $x(\cdot - r(\cdot)) \in PAAS^p(\mathbb{R})$.*

Proof. Since $x(\cdot) \in PAAS^p(\mathbb{R},\mathbb{R})$, then $x(\cdot)$ can be written as $x = x_1 + x_2$, where $x_1^b(\cdot) \in AA(\mathbb{R}, L^p([0,1],\mathbb{R}))$ and $x_2^b(\cdot) \in PAP_0(\mathbb{R}, L^p([0,1],\mathbb{R}))$, such that

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_t^{t+1} |x_2(\sigma)|^p \, \mathrm{d}\sigma \right)^{\frac{1}{p}} \mathrm{d}t = 0. \tag{3.31}$$

Let

$$x(\cdot - r(\cdot)) = x_1(\cdot - r(\cdot)) + x(\cdot - r(\cdot)) - x_1(\cdot - r(\cdot)) = \Psi_1(\cdot) + \Psi_2(\cdot), \tag{3.32}$$

where $\Psi_1(\cdot) = x_1(\cdot - r(\cdot))$ and $\Psi_2(\cdot) = x(\cdot - r(\cdot)) - x_1(\cdot - r(\cdot))$. Note that the function $\Psi_2^b(\cdot) \in PAP_0(\mathbb{R}, L^p((0,1),\mathbb{R}))$ (cf. [9]). Hence, it only remains to show that $x_1^b(\cdot - r^b(\cdot)) \in AA(\mathbb{R}, L^p((0,1),\mathbb{R}))$. Since $x_1^b(\cdot) \in AA(\mathbb{R}, L^p((0,1),\mathbb{R}))$ then for any sequence $(s_n')_{n\in\mathbb{R}}$ there exists a subsequence $(s_n)_{n\in\mathbb{R}}$ and a function $g \in L^p_{loc}(\mathbb{R},\mathbb{R})$ such that

$$\left(\int_t^{t+1} |x_1(s+s_n) - g(s)|^p \, \mathrm{d}s \right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0 \tag{3.33}$$

and

$$\left(\int_t^{t+1} |g(s-s_n) - x_1(s)|^p \, \mathrm{d}s \right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.34}$$

Thus we could find

$$\begin{aligned}
&\left(\int_t^{t+1} |x_1(s+s_n - r(s+s_n)) - g(s-r(s))|^p \, \mathrm{d}s \right)^{\frac{1}{p}} \\
&\leq \left(\int_t^{t+1} |x_1(s+s_n - r(s+s_n)) - g(s - r(s+s_n)|^p \, \mathrm{d}s \right)^{\frac{1}{p}} \\
&\quad + \left(\int_t^{t+1} |g(s - r(s+s_n) - g(s-r(s))|^p \, \mathrm{d}s \right)^{\frac{1}{p}}.
\end{aligned}$$

Let $(g_n)_{n\in\mathbb{N}}$ be a sequence such that $g_n \to g$ as $n\to\infty$ in $BS^p(\mathbb{R},\mathbb{R})$ which is dominated by some integrable function w, then

$$\left(\int_t^{t+1} |g(s - r(s+s_n) - g(s-r(s))|^p \, \mathrm{d}s \right)^{\frac{1}{p}} \leq I + J + K,$$

where

$$I = \left(\int_t^{t+1} |g(s - r(s+s_n) - g_n(s - r(s+s_n)|^p \, \mathrm{d}s \right)^{\frac{1}{p}}, \tag{3.35}$$

$$J = \left(\int_t^{t+1} |g_n(s - r(s+s_n) - g_n(s-r(s))|^p \, \mathrm{d}s \right)^{\frac{1}{p}} \tag{3.36}$$

and

$$K = \left(\int_t^{t+1} |g_n(s-r(s)) - g(s-r(s))|^p \, \mathrm{d}s \right)^{\frac{1}{p}}. \tag{3.37}$$

Let us show that $I=0$. For that, letting $s'=s-r(s+s_n)$ one obtains

$$\begin{aligned} I &= \big(1-r'(s+s_n)\big)^{-\frac{1}{p}} \left(\int_{t-r(t+s_n)}^{t+1-r(t+1-s_n)} \big|g(s')-g_n(s')\big|^p \,\mathrm{d}s'\right)^{\frac{1}{p}} \\ &\le (1-r^*))^{-\frac{1}{p}} \left(\int_{t-\bar{r}}^{t+1-\bar{r}} \big|g(s')-g_n(s')\big|^p \,\mathrm{d}s'\right)^{\frac{1}{p}} \\ &\le (1-r^*))^{-\frac{1}{p}} \left(\int_{t}^{t+1} \big|g(s')-g_n(s')\big|^p \,\mathrm{d}s'\right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

In addition, by applying the dominated convergence theorem we get $J=0$. Moreover, let $s'=s-r(s)$, then

$$\begin{aligned} K &= \big(1-r'(s)\big)^{-\frac{1}{p}} \left(\int_{t-r(t)}^{t+1-r(t+1)} \big|g(s')-g_n(s')\big|^p \,\mathrm{d}s'\right)^{\frac{1}{p}} \\ &\le (1-r^*))^{-\frac{1}{p}} \left(\int_{t-\bar{r}}^{t+1-\bar{r}} \big|g(s')-g_n(s')\big|^p \,\mathrm{d}s'\right)^{\frac{1}{p}} \\ &\le (1-r^*))^{-\frac{1}{p}} \left(\int_{t}^{t+1} \big|g(s')-g_n(s')\big|^p \,\mathrm{d}s'\right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

What is left to show that

$$\left(\int_t^{t+1} |x_1(s+s_n-r(s+s_n))-g(s-r(s+s_n)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.38}$$

For this purpose, we set $s-r(s+s_n)=s'$, then

$$\begin{aligned} &\left(\int_t^{t+1} |x_1(s+s_n-r(s+s_n))-g(s-r(s+s_n)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \\ &= \big(1-r'(s+s_n)\big)^{-\frac{1}{p}} \left(\int_{t-r(t+s_n)}^{t+1-r(t+1-s_n)} \big|x_1(s'+s_n)-g(s')\big|^p \,\mathrm{d}s'\right)^{\frac{1}{p}} \\ &\le (1-r^*))^{-\frac{1}{p}} \left(\int_{t-\bar{r}}^{t+1-\bar{r}} \big|x_1(s'+s_n)-g(s')\big|^p \,\mathrm{d}s'\right)^{\frac{1}{p}} \\ &\le (1-r^*))^{-\frac{1}{p}} \left(\int_{t}^{t+1} \big|x_1(s'+s_n)-g(s')\big|^p \,\mathrm{d}s'\right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Similarly, we can get

$$\left(\int_t^{t+1} |g(s-s_n+r(s-s_n))-x_1(s-r(s))|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.39}$$

Consequently,

$$x_1^b(\cdot-r^b(\cdot))\in AA(\mathbb{R},L^p([0,1],\mathbb{R})).$$

Therefore the function

$$x(\cdot - r(\cdot)) \in PAAS^p(\mathbb{R}).$$

□

Lemma 7. *We assume that (H1)-(H2) hold. If $x(\cdot) \in PAAS^p(\mathbb{R},\mathbb{R})$, then the function $\beta : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\beta(\cdot) = f(\cdot, x(\cdot), x(\cdot - r(\cdot)))$ belongs to $PAAS^p(\mathbb{R},\mathbb{R})$.*

Proof. By (H1), we have $f^b = h^b + \phi^b$ where $h^b \in AA(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and $\phi^b \in PAP_0(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \left(\int_t^{t+1} |\phi(\sigma,u)|^p \,\mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t = 0, \tag{3.40}$$

uniformly for all $u \in \mathbb{R}^2$. Similarly, $x^b = x_1^b + x_2^b$ where $x_1^b(\cdot) \in AA(\mathbb{R}, L^p((0,1),\mathbb{R}))$ and $x_2^b(\cdot) \in PAP_0(\mathbb{R}, L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \left(\int_t^{t+1} |x_2(\sigma)|^p \,d\sigma\right)^{\frac{1}{p}} \mathrm{d}t = 0, \tag{3.41}$$

for all $t \in \mathbb{R}$. Then, by Lemma 6, we get $x(\cdot - r(\cdot)) \in PAAS^p(\mathbb{R},\mathbb{R})$, i.e.

$$x^b(\cdot - r(\cdot)) = x_1^b(\cdot - r^b(\cdot)) + x_2^b(\cdot - r^b(\cdot)),$$

where $x_1^b(\cdot - r^b(\cdot)) \in AA(\mathbb{R}, L^p((0,1),\mathbb{R}))$ and $x_2^b(\cdot - r^b(\cdot)) \in PAP_0(\mathbb{R}, L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \left(\int_t^{t+1} |x_2(\sigma - r(\sigma))|^p \,d\sigma\right)^{\frac{1}{p}} \mathrm{d}t = 0, \tag{3.42}$$

for all $t \in \mathbb{R}$. In addition, $f^b : \mathbb{R} \longrightarrow L^p((0,1),\mathbb{R})$. Now decompose f^b as follows

$$\begin{aligned}
&f^b(\cdot, x^b(\cdot), x^b(\cdot - r^b(\cdot)))\\
&= h^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r^b(\cdot))) + f^b(\cdot, x^b(\cdot), x^b(\cdot - r^b(\cdot))) - h^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r^b(\cdot)))\\
&= h^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r^b(\cdot))) + f^b(\cdot, x^b(\cdot), x^b(\cdot - r^b(\cdot))) - f^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r^b(\cdot)))\\
&\quad + \phi^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r^b(\cdot))).
\end{aligned}$$

We start by demonstrating that $h^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r^b(\cdot))) \in AA(\mathbb{R}, L^p((0,1),\mathbb{R}))$. Since $x_1^b(\cdot) \in AA(\mathbb{R}, L^p((0,1),\mathbb{R}))$, i.e. there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $y_1^b(\cdot) \in L^p_{loc}(\mathbb{R},\mathbb{E})$ such that

$$\left(\int_t^{t+1} |x_1(s+s_n) - y_1(s)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0, \tag{3.43}$$

and

$$\left(\int_t^{t+1} |y_1(s-s_n) - x_1(s)|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.44}$$

Similarly, $x_1^b(\cdot - r^b(\cdot)) \in AA(\mathbb{R}, L^p((0,1),\mathbb{R}))$, i.e. there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $y_1^b(\cdot - r^b(\cdot)) \in L^p_{loc}(\mathbb{R},\mathbb{E})$ such that

$$\left(\int_t^{t+1} |x_1(s+s_n-r(s+s_n)) - y_1(s-r(s))|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0, \tag{3.45}$$

and

$$\left(\int_t^{t+1} |y_1(s-s_n-r(s-s_n)) - x_1(s-r(s))|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.46}$$

Then $u_1^b(\cdot) = (x_1^b(\cdot), x_1^b(\cdot - r^b(\cdot))) \in AA(R, L^p((0,1),\mathbb{R}^2))$. Indeed, let $(s'_n)_{n'\in\mathbb{N}}$ a sequence has a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $w_1^b(\cdot) = (y_1^b(\cdot), y_1^b(\cdot - r^b(\cdot))) \in L^p_{loc}(\mathbb{R},\mathbb{R}^2)$ such that

$$\begin{aligned}
&\left(\int_t^{t+1} \|u_1(s+s_n) - w_1(s)\|^p \,\mathrm{d}s\right)^{\frac{1}{p}} \\
\Leftrightarrow &\left(\int_t^{t+1} \|(x_1(s+s_n), x_1(s+s_n-r(s+s_n))) - (y_1(s), y_1(s-r(s)))\|_\infty^p \,\mathrm{d}s\right)^{\frac{1}{p}} \\
\Leftrightarrow &\left(\int_t^{t+1} \max\left(|x_1(s+s_n) - y_1(s)|^p, |x_1(s+s_n-r(s+s_n) - y_1(s-r(s))|^p\right) \mathrm{d}s\right)^{\frac{1}{p}}.
\end{aligned}$$

We deduce from (3.43) and (3.45) that

$$\lim_{n\to\infty}\left(\int_t^{t+1} \max\left(|x_1(s+s_n) - y_1(s)|^p, |x_1(s+s_n-r(s+s_n) - y_1(s-r(s))|^p\right) \mathrm{d}s\right)^{\frac{1}{p}} = 0.$$

Moreover, since $h^b \in AA(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and by applying Lemma 2, it is easy to see that the function $h^b(.,u_1^b(.)) \in AA(\mathbb{R}, L^p([0,1],\mathbb{R}))$. Now set

$$G^b(.) = f^b(\cdot, x^b(\cdot), x^b(\cdot - rs + s_n(\cdot))) - f^b(\cdot, x_1^b(\cdot), x_1^b(\cdot - r^b(\cdot)). \tag{3.47}$$

$G^b(\cdot) \in PAP_0(\mathbb{R}, L^p((0,1),\mathbb{R}))$. Indeed, $T>0$ using the fact that f is Lipschitz and Minkowski's inequality we get

$$\begin{aligned}
&\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_0^1 \left|G^b(\sigma)\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t \\
&\leq \lim_{T\to+\infty}\frac{L_f^1}{2T}\int_{-T}^{T}\left(\int_0^1 \left|x^b(\sigma) - x_1^b(\sigma)\right|^p d\sigma\right)^{\frac{1}{p}} \mathrm{d}t \\
&\quad + \lim_{T\to+\infty}\frac{L_f^2}{2T}\int_{-T}^{T}\left(\int_0^1 \left|x^b(\sigma - r^b(\sigma)) - x_1^b(\sigma - r^b(\sigma))\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t \\
&\leq \lim_{T\to+\infty}\frac{L_f^1}{2T}\int_{-T}^{T}\left(\int_0^1 \left|x_2^b(\sigma)\right|^p \mathrm{d}\sigma\right)^{\frac{1}{p}} \mathrm{d}t
\end{aligned}$$

$$+\lim_{T\to+\infty}\frac{L_f^2}{2T}\int_{-T}^{T}\left(\int_0^1\left|x_2^b(\sigma-r^b(\sigma))\right|^p\mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t.$$

By (3.41) and (3.42) we obtain

$$\frac{1}{2T}\int_{-T}^{T}\left(\int_0^1\left|G_2^b(\sigma)\right|^p d\sigma\right)^{\frac{1}{p}}\mathrm{d}t=0. \tag{3.48}$$

It remains to show that $\phi^b(\cdot,x_1^b(\cdot),x_1^b(\cdot-r^b(\cdot)))\in PAP_0(L^p((0,1),\mathbb{R}))$. We have already shown that the function $u_1^b(\cdot)=(x_1^b(\cdot),x_1^b(\cdot-r^b(\cdot)))\in AP(R,L^p((0,1),\mathbb{R}^2))$. Since the function $\phi^b\in PAP_0(\mathbb{R}\times\mathbb{R}^2,L^p((0,1),\mathbb{R}))$, hence by applying the composition theorem of ergodic functions (cf. [22]), we get $\phi^b(\cdot,u_1^b(\cdot))\in PAP_0(L^p([0,1],\mathbb{R})$ such that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_0^1\left|\phi^b(\sigma,u_1^b(\sigma))\right|^p d\sigma\right)^{\frac{1}{p}}\mathrm{d}t=0. \tag{3.49}$$

□

Lemma 8. *Suppose that assumptions (H3)-(H5) hold. If $x(\cdot)\in PAAS^p(\mathbb{R},\mathbb{R})$ then the function Θ defined by $\Theta: t\longmapsto\int_t^{+\infty}c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s$ belongs to $PAAS^p(\mathbb{R},\mathbb{R})$.*

Proof. Using Lemma 7 and hypothesis (H5) one can easily check that the integral $t\longmapsto\int_t^{+\infty}c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s$ is well defined. Since the function $[\,s\longmapsto g(s,x(s),x(s-r(s))]\in PAAS^p(\mathbb{R},\mathbb{R})$, we can write $g=g_1+g_2$ where $g_1^b\in AA(\mathbb{R},L^p((0,1),\mathbb{R}))$ and $g_2^b\in PAP_0(\mathbb{R},L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}|g_2(\sigma)|^p d\sigma\right)^{\frac{1}{p}}\mathrm{d}t=0. \tag{3.50}$$

Then

$$\Theta(t)=\int_t^{+\infty}c(t,s)g_1(s)\mathrm{d}s+\int_t^{+\infty}c(t,s)g_2(s)\mathrm{d}s=\Theta_1(t)+\Theta_2(t). \tag{3.51}$$

We have already shown that $[t\longmapsto\theta_2(t)]\in PAP_0(\mathbb{R},L^p([0,1],\mathbb{R}))$ (see Lemma 5). So to prove the S^p-pseudo almost periodicity of the function $\theta(\cdot)$, it suffices to show that $[t\longmapsto\theta_1(t)]\in SAA(\mathbb{R},\mathbb{R})$. Let $(s_n')_{n\in\mathbb{N}}$ a sequence of real numbers and g_1 is S^p-almost automorphic function then there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $g_1^*\in L^p_{loc}(\mathbb{R},\mathbb{E})$ such that

$$\left(\int_0^1|g_1(s_n+s+t)-g_1^*(s+t)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\underset{n\to\infty}{\longrightarrow}0, \tag{3.52}$$

and

$$\left(\int_0^1 |g_1^*(t+s-s_n)-g_1(t+s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.53}$$

Set

$$\theta_1^*(t)=\int_t^{+\infty} c(t,s)g_1^*(s)\mathrm{d}s. \tag{3.54}$$

Then we have

$$\begin{aligned}
|\theta_1(u+s_n)-\theta_1^*(u)| &= \left|\int_{u+s_n}^{+\infty} c(u+s_n,s)g_1(s)\mathrm{d}s-\int_u^{+\infty} c(u,s)g_1^*(s)\mathrm{d}s\right| \\
&= \left|\int_u^{+\infty} \mathrm{e}^{\lambda(u-s)}g_1(s_n+s)ds-\int_u^{+\infty} \mathrm{e}^{\lambda(u-s)}g_1^*(s)\mathrm{d}s\right| \\
&= \int_0^{+\infty} \mathrm{e}^{-s\lambda}|g_1(s_n+s+u)-g_1^*(s+u)|\mathrm{d}s.
\end{aligned}$$

Using Hölder's inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$

$$|\theta_1(u+s_n)-\theta_1^*(u)| \le \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}} \left(\int_0^{\infty} \mathrm{e}^{\frac{-s\lambda p}{2}}|g_1(s_n+s+u)-g_1^*(s+u)|^p\mathrm{d}s\right)^{\frac{1}{p}}.$$

Then

$$\begin{aligned}
&\left(\int_t^{t+1} |\theta_1(u+s_n)-\theta_1^*(u)|^p\,\mathrm{d}u\right)^{\frac{1}{p}} \\
&\le \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}} \left(\int_t^{t+1}\int_0^{\infty} \mathrm{e}^{\frac{-\lambda ps}{2}}\,|g_1(s_n+s+u)-g_1^*(s+u)|^p\,\mathrm{d}s\mathrm{d}u\right)^{\frac{1}{p}} \\
&\le \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}} \left(\frac{2}{\lambda p}\right)^{\frac{1}{p}} \left(\int_{t'}^{t'+1} |g_1(s_n+u')-g_1^*(u')|^p\mathrm{d}u'\right)^{\frac{1}{p}}.
\end{aligned}$$

Since

$$\lim_{n\to+\infty}\left(\int_{t'}^{t'+1} |g_1(s_n+u)-g_1^*(u)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}=0, \tag{3.55}$$

we obtain

$$\lim_{n\to+\infty}\left(\int_t^{t+1} |\theta_1(u+s_n)-\theta_1^*(u)|^p\,\mathrm{d}u\right)^{\frac{1}{p}}=0. \tag{3.56}$$

On the other hand, we have

$$\begin{aligned}
|\theta_1^*(u-s_n)-\theta_1(u)| &\le \left|\int_u^{+\infty} \mathrm{e}^{\lambda(u-s)}g_1^*(s-s_n)\mathrm{d}s-\int_u^{+\infty} \mathrm{e}^{\lambda(u-s)}g_1(s)\mathrm{d}s\right| \\
&\le \int_u^{+\infty} \mathrm{e}^{\lambda(u-s)}|g_1^*(s-s_n)-g_1(s)|\mathrm{d}s
\end{aligned}$$

$$= \int_0^{+\infty} \mathrm{e}^{-\lambda s} |g_1^*(s+u-s_n) - g_1(s+u)| \mathrm{d}s.$$

Using Hölder's inequality $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ we get

$$|\theta_1^*(u-s_n) - \theta_1(u)| \le \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}} \left(\int_0^{+\infty} \mathrm{e}^{\frac{-s\lambda p}{2}} |g_1(s-s_n+u) - g_1^*(s+u)|^p \mathrm{d}s\right)^{\frac{1}{p}}.$$

Now using Fubini theorem we get

$$\begin{aligned}
&\left(\int_t^{t+1} |\theta_1^*(u-s_n) - \theta_1(u)|^p \mathrm{d}u\right)^{\frac{1}{p}} \\
&\le \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}} \left(\int_t^{t+1}\int_0^{\infty} \mathrm{e}^{\frac{-\lambda ps}{2}} |g_1^*(s-s_n+u) - g_1(s+u)|^p \mathrm{d}s\mathrm{d}u\right)^{\frac{1}{p}} \\
&= \left(\frac{2}{\lambda q}\right)^{\frac{1}{q}} \left(\frac{2}{\lambda p}\right)^{\frac{1}{p}} \left(\int_{t'}^{t'+1} |g_1^*(s_n+u') - g_1(u')|^p \mathrm{d}u'\right)^{\frac{1}{p}}.
\end{aligned}$$

Using the fact that

$$\lim_{n\to+\infty} \left(\int_{t'}^{t'+1} \left|g_1^*(u'-s_n) - g_1(u')\right|^p \mathrm{d}u'\right)^{\frac{1}{p}} = 0, \tag{3.57}$$

we get

$$\lim_{n\to+\infty} \left(\int_{t'}^{t'+1} |\theta_1^*(u-s_n) - \theta_1(u)|^p \mathrm{d}u\right)^{\frac{1}{p}} = 0. \tag{3.58}$$

Therefore, $\Theta : t \longmapsto \int_t^{+\infty} c(t,s)\, g(s,x(s),x(s-r(s)))\, \mathrm{d}s \in PAAS^p(\mathbb{R},\mathbb{R})$. □

Theorem 2. *Assume that (H1)-(H5) hold. If $m<1$, then (3.1) has a unique S^p-pseudo almost automorphic solution with*

$$m = \max\left(L_f^1, L_f^2(1-r^*)^{-\frac{1}{p}}, L_g^1\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}, L_g^2\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}(1-r^*)^{-\frac{1}{p}}\right).$$

Proof. Let us consider the operator Γ defined on $PAAS^p(\mathbb{R},\mathbb{R})$ by

$$\Gamma(x)(t) = f(t,x(t),x(t-r(t))) - \int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s, \quad t\in\mathbb{R}.$$

By Lemma 6 and (H1) we obtain that the function $t \longmapsto f(t,x(t),x(t-r(t)))$ is continuous. Furthermore, using Lemma 7 and (H5), we get that the integral defined by $t \longmapsto \int_t^{+\infty} c(t,s)g(s,x(s),x(s-r(s)))\mathrm{d}s$ exists. Thus, Γx is well defined. Moreover, from Lemmas 7 and 8 we deduce that

$$\Gamma : PAAS^p(\mathbb{R},\mathbb{R}) \to PAAS^p(\mathbb{R},\mathbb{R}).$$

Let $x, y \in PAAS^p(\mathbb{R},\mathbb{R})$, by making a change of variables and according to Hölder's inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ we get

$$\begin{aligned}
&\sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}|\Gamma x(t)-\Gamma y(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\\
&\leq \sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}\left(L_f^1|x(t)-y(t)|+L_f^2|x(t-r(t))-y(t-r(t))|\right)^p\,\mathrm{d}t\right)^{\frac{1}{p}}\\
&\quad+\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\int_0^{\infty}\mathrm{e}^{\frac{-\lambda ps}{2}}\sup_{\xi\in\mathbb{R}}\int_{\xi}^{\xi+1}\left(L_g^1|y(t+s)-x(t+s)|\right.\right.\\
&\quad\left.\left.+\ L_g^2|y(t+s-r(s))-x(t+s-r(s))|\right)^p\,\mathrm{d}t\ \mathrm{d}s\right)^{\frac{1}{p}}\\
&\leq \sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}\left(L_f^1|x(t)-y(t)|+L_f^2|x(t-r(t))-y(t-r(t))|\right)^p\,\mathrm{d}t\right)^{\frac{1}{p}}\\
&\quad+\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\times\left(\int_0^{\infty}\mathrm{e}^{\frac{-\lambda ps}{2}}\sup_{\xi'\in\mathbb{R}}\int_{\xi'}^{\xi'+1}\left(L_g^1|y(t')-x(t')|\right.\right.\\
&\quad\left.\left.+\ L_g^2|y(t'-r(t))-x(t'-r(t))|\right)^p\,\mathrm{d}t'\mathrm{d}s\right)^{\frac{1}{p}}\\
&\leq \sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}\left(L_f^1|x(t)-y(t)|+L_f^2|x(t-r(t))-y(t-r(t))|\right)^p\,\mathrm{d}t\right)^{\frac{1}{p}}\\
&\quad+\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}\left(\sup_{\xi'\in\mathbb{R}}\int_{\xi'}^{\xi'+1}\left(L_g^1|y(t')-x(t')|\right.\right.\\
&\quad\left.\left.+\ L_g^2|y(t'-r(t'))-x(t'-r(t'))|\right)^p\,\mathrm{d}t'\right)^{\frac{1}{p}}\\
&\leq L_f^1\|x-y\|_{S^p}+L_g^1\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}\|x-y\|_{S^p}\\
&\quad+L_f^2\left(1-r'(t)\right)^{-\frac{1}{p}}\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-r(\xi)}^{\xi+1-r(\xi+1)}|x(\rho)-y(\rho)|^p\,d\rho\right)^{\frac{1}{p}}\\
&\quad+L_g^2\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}\left(1-r'(t)\right)^{-\frac{1}{p}}\times\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-r(\xi)}^{\xi+1-r(\xi+1)}|x(\rho)-y(\rho)|^p\,d\rho\right)^{\frac{1}{p}}
\end{aligned}$$

$$\leq L_f^1\|x-y\|_{S^p}+L_f^2(1-r^*)^{-\frac{1}{p}}\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-\bar r}^{\xi+1-\bar r}|x(\rho)-y(\rho)|^p\,d\rho\right)^{\frac{1}{p}}$$
$$+L_g^1\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}\|x-y\|_{S^p}$$
$$+L_g^2\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}(1-r^*)^{-\frac{1}{p}}\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-\bar r}^{\xi+1-\bar r}|x(\rho)-y(\rho)|^p\,d\rho\right)^{\frac{1}{p}}$$
$$\leq m\|x-y\|_{S^p},$$

where

$$m=\max\left(L_f^1,L_f^2(1-r^*)^{-\frac{1}{p}},L_g^1\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}},L_g^2\left(\frac{2}{\lambda q}\right)^{\frac{1}{q}}\left(\frac{2}{\lambda p}\right)^{\frac{1}{p}}(1-r^*)^{-\frac{1}{p}}\right).$$

Since $m<1$, the operator $\Gamma:(PAAS^p(\mathbb{R},\mathbb{R}),\|\cdot\|_{S^p})\longrightarrow(PAAS^p(\mathbb{R},\mathbb{R}),\|\cdot\|_{S^p})$ is a contraction. Therefore, by applying the Banach fixed point theorem there is a unique $x_*\in PAAS^p(\mathbb{R},\mathbb{R})$ such that $\Gamma(x_*)=x_*$, which corresponds to the unique S^p-almost periodic pseudo solution of the equation (3.1). □

4. Application

The purpose of this section is to show the existence and uniqueness of the S^p-pseudo almost periodic and S^p-pseudo almost automorphic solutions of the following logistic differential equation

$$x'(t)=ax(t)+\alpha x'(t-r(t))-q(t,x(t),x(t-r(t)))+h(t). \tag{4.1}$$

But rather than dealing with equation (4.1) we will study the existence and uniqueness of S^p-pseudo almost periodic and S^p-pseudo almost automorphic solutions of the following integral equation

$$x(t)=\alpha x(t-r(t))-\int_t^{+\infty}[q(s,x(s),x(s-r(s)))-a\alpha x(s-r(s))]\,e^{a(t-s)}ds+p(t), \tag{4.2}$$

where $h:\mathbb{R}\longrightarrow\mathbb{R}$, $q:\mathbb{R}\times\mathbb{R}^2\longrightarrow\mathbb{R}$ are continuous functions, $p:\mathbb{R}\longrightarrow\mathbb{R}$ a differentiable function, $a>0$, $0\leq|\alpha|<1$ are respectively constants and $r(\cdot)$ is a time-dependent delay. Indeed, let x a solution of (4.2) then

$$x'(t)=\alpha x'(t-r(t))-q(t,x(t),x(t-r(t)))+a\alpha x(t-r(t))$$
$$-a\int_t^{\infty}[q(s,x(s),x(s-r(s)))-a\alpha x(s-r(s))]e^{a(t-s)}ds+p'(t)$$
$$=a\left[\alpha x(t-r(t))-\int_t^{\infty}[q(s,x(s),x(s-r(s)))-a\alpha x(s-r(s))]\,e^{a(t-s)}ds\right]$$
$$+\alpha x'(t-r(t))-q(t,x(t),x(t-h))+p'(t)$$

$$= ax(t) + \alpha x'(t - r(t)) - q(t, x(t), x(t - r(t))) + h(t),$$

where $h(t) = p'(t)$. Then, the solutions of equation (4.1) are exactly those of the integral equation (4.2).

4.1. *Stepanov-like pseudo almost periodic solutions*

We will study the S^p-pseudo almost periodic solutions of (4.2). For this study, we formulate the following assumptions

(H1) $q : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is S^p-pseudo almost periodic function, i.e.

$$q^b = q_1^b + q_2^b, \tag{4.3}$$

with $q_1^b \in AP(\mathbb{R} \times \mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and $q_2^b \in PAP_0(\mathbb{R} \times \mathbb{R}^2, L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_t^{t+1} |q_2(\sigma, u)|^p \, d\sigma \right)^{\frac{1}{p}} dt = 0, \tag{4.4}$$

uniformly for all $u \in \mathbb{R}^2$.

(H2) q is Lipschitz, i.e. $\exists L_q^1, L_q^2 > 0$ such that $\forall x_1, x_2, y_1, y_2 \in \mathbb{R}$

$$|q(t, x_1, x_2) - q(t, y_1, y_2)| \leq L_q^1 |x_1 - y_1| + L_q^2 |x_2 - y_2|. \tag{4.5}$$

(H3) $p : \mathbb{R} \longrightarrow \mathbb{R}$ is S^p-pseudo almost periodic function, i.e.

$$p^b = p_1^b + p_2^b \tag{4.6}$$

where $p_1^b \in AP(\mathbb{R}, L^p((0,1),\mathbb{R}))$ and $p_2^b \in PAP_0(\mathbb{R}, L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_t^{t+1} |p_2(\sigma)|^p \, d\sigma \right)^{\frac{1}{p}} dt = 0. \tag{4.7}$$

(H4) The function $t \mapsto r(t) \in APS^p(\mathbb{R},\mathbb{R}) \cap C^1(\mathbb{R},\mathbb{R})$, with

$$0 \leq r(t) \leq \bar{r}, \quad r(t) \leq r^* < 1. \tag{4.8}$$

Theorem 3. *Assume that (H1)-(H3) hold. If $m_1 < 1$ then (4.2) has a unique S^p-pseudo almost perodic solution, where*

$$m_1 = \max\left(\alpha(1 - r^*), L_q^1 \left(\frac{2}{aq}\right)^{\frac{1}{q}} \left(\frac{2}{ap}\right)^{\frac{1}{p}}, (1 - r^*)\left(L_q^2 - a\alpha\right) \left(\frac{2}{aq}\right)^{\frac{1}{q}} \left(\frac{2}{ap}\right)^{\frac{1}{p}} \right).$$

Proof. Let the operator Λ defined on $PAPS^p(\mathbb{R},\mathbb{R})$ by

$$\Lambda(x)(t) = \alpha x(t - r(t)) - \int_t^{+\infty} [q(s, x(s), x(s - r(s))) - a\alpha x(s - r(s))] \, e^{a(t-s)} ds + p(t). \tag{4.9}$$

$\Lambda x \in PAPS^p(\mathbb{R},\mathbb{R})$. In fact, pose

$$f(\cdot, x(\cdot), x(\cdot - r(\cdot))) = \alpha x(\cdot - r(\cdot)) + p(\cdot), \tag{4.10}$$

and

$$g(\cdot,x(\cdot),x(\cdot-r(\cdot))) = [q(\cdot,x(\cdot),x(\cdot-r(\cdot))) - a\alpha x(\cdot-r(\cdot))]. \tag{4.11}$$

Since, $[t \mapsto x(t)] \in PAAS^p(\mathbb{R},\mathbb{R})$, Lemma 3 implies that

$$[t \mapsto x(t-r(t))] \in PAPS^p(\mathbb{R},\mathbb{R}),$$

for all $t \in \mathbb{R}$. Thus,

$$[t \longmapsto \alpha x(t-r(t))] \in PAAS^p(\mathbb{R},\mathbb{R}),$$

for all $t \in \mathbb{R}$. In accordance with Lemma 4, the function

$$[t \mapsto f(t,x(t),x(t-r(t)))] \in PAPS^p(\mathbb{R},\mathbb{R}).$$

Moreover, under (H1)-(H2) and using Lemma 4, we obtain that the function

$$[s \to q(s,x(s),x(s-r(s)))] \in PAPS^p(\mathbb{R},\mathbb{R}).$$

As previously we show that the function $[s \longmapsto -a\alpha x(s-r(s))] \in PAPS^p(\mathbb{R},\mathbb{R})$, for all $t \in \mathbb{R}$. Then, $[s \longmapsto g(s,x(s),x(s-r(s)))] \in PAPS^p(\mathbb{R},\mathbb{R})$, as being the sum of two S^p-pseudo almost periodic functions. It follows from Lemma 5 that

$$\left[t \longmapsto \int_t^{+\infty} [q(s,x(s),x(s-r(s))) - a\alpha x(s-r(s))]\,\mathrm{e}^{a(t-s)}\mathrm{d}s\right] \in PAPS^p(\mathbb{R},\mathbb{R}).$$

Therefore we deduce that $\Lambda x \in PAPS^p(\mathbb{R},\mathbb{R})$.

Let $x,\, y \in PAPS^p(\mathbb{R},\mathbb{R})$, then

$$\begin{aligned}
&|\Lambda x(t) - \Lambda y(t)| \\
&\leq |\alpha x(t-r(t)) - \alpha y(t-r(t))| \\
&\quad + \int_t^{+\infty} \mathrm{e}^{a(t-s)}\, |q(s,y(s),y(s-r(s))) - q(s,x(s),x(s-r(s))) \\
&\quad + a\alpha x(s-r(s)) - a\alpha y(s-r(s))|\,\mathrm{d}s \\
&\leq |\alpha x(t-r(t)) - \alpha y(t-r(t))| + \int_0^{+\infty} \mathrm{e}^{-as}\, |q(s+t,y(s+t),y(s+t-r(s+t))) \\
&\quad - q(s+t,x(s+t),x(s+t-r(s+t))) \\
&\quad + a\alpha x(s+t-r(s+t)) - a\alpha y(s+t-r(s+t))|\,\mathrm{d}s \\
&\leq |\alpha x(t-r(t)) - \alpha y(t-r(t))| + \left(\frac{2}{qa}\right)^{\frac{1}{q}} \\
&\quad \times \int_0^{\infty} \mathrm{e}^{\frac{-aps}{2}}\, |q(s+t,y(s+t),y(s+t-r(s+t))) + a\alpha x(s+t-r(s+t)) \\
&\quad - q(s+t,x(s+t),x(s+t-r(s+t))) - a\alpha y(s+t-r(s+t))|^p\,\mathrm{d}s.
\end{aligned}$$

So, using Fubini's theorem and Minkowski's inequality we have

$$\sup_{\xi\in\mathbb{R}} \left(\int_\xi^{\xi+1} |\Lambda x(t) - \Lambda y(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}$$

$$\leq \sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}|\alpha x(t-r(t))-\alpha y(t-r(t))|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}$$

$$\times\left(\int_0^{+\infty}\mathrm{e}^{\frac{-aps}{2}}\sup_{\xi\in\mathbb{R}}\int_{\xi}^{\xi+1}|a\alpha x(s+t-r(s+t))+q(s+t,y(s+t),y(s+t-r(s+t)))\right.$$

$$\left.-q(s+t,x(s+t),x(s+t-r(s+t)))-a\alpha y(s+t-r(s+t))|^p\,\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{p}}$$

$$\leq \sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}|\alpha x(t-r(t))-\alpha y(t-r(t))|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}$$

$$\left(\int_0^{+\infty}\mathrm{e}^{\frac{-aps}{2}}\sup_{\xi'\in\mathbb{R}}\int_{\xi'}^{\xi'+1}\left|a\alpha x(t'-r(t'))\right.\right.$$

$$\left.\left.+q(t',y(t'),y(t'-r(t')))-q(t',x(t'),x(t'-r(t')))-a\alpha y(t'-r(t'))\right|^p\mathrm{d}t'\mathrm{d}s\right)^{\frac{1}{p}}$$

$$\leq \sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}|\alpha x(t-r(t))-\alpha y(t-r(t))|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}$$

$$\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'}^{\xi'+1}\left|a\alpha x(t'-r(t'))+q(t',y(t'),y(t'-r(t')))\right.\right.$$

$$\left.\left.-\ q(t',x(t'),x(t'-r(t')))-a\alpha y(t'-r(t'))\right|^p\mathrm{d}t'\right)^{\frac{1}{p}}$$

$$\leq \sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}|\alpha x(t-r(t))-\alpha y(t-r(t))|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}$$

$$\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'}^{\xi'+1}\left(L_q^1\left|y(t')-x(t')\right|+(L_q^2-a\alpha)\left|y(t'-r(t'))-x(t'-r(t'))\right|\right)^p\mathrm{d}t'\right)^{\frac{1}{p}}$$

$$\leq \alpha\sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}|x(t-r(t))-y(t-r(t))|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}L_q^1$$

$$\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'}^{\xi'+1}\left|y(t')-x(t')\right|^p\mathrm{d}t'\right)^{\frac{1}{p}}$$

$$+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}(L_q^2-a\alpha)\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'}^{\xi'+1}\left|y(t'-r(t'))-x(t'-r(t'))\right|^p\mathrm{d}t'\right)^{\frac{1}{p}}$$

$$
\begin{aligned}
&\leq \alpha\left(1-r'(t)\right)\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-r(\xi)}^{\xi+1-r(\xi+1)}\left|x(t')-y(t')\right|^p \mathrm{d}t'\right)^{\frac{1}{p}}\\
&\quad+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}L_q^1\|x-y\|_{S^p}\\
&\quad+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}(L_q^2-a\alpha)\left(1-r(t')\right)\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'-r(\xi')}^{\xi'+1-r(\xi'+1)}|y(t)-x(t)|^p\,\mathrm{d}t'\right)^{\frac{1}{p}}\\
&\leq \alpha(1-r^*)\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-\bar{r}}^{\xi+1-\bar{r}}\left|x(t')-y(t')\right|^p \mathrm{d}t'\right)^{\frac{1}{p}}\\
&\quad+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}L_q^1\|x-y\|_{S^p}\\
&\quad+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}(L_q^2-a\alpha)(1-r^*)\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'-\bar{r}}^{\xi'+1-\bar{r}}|y(t)-x(t)|^p\,\mathrm{d}t'\right)^{\frac{1}{p}}\\
&\leq m_1\|x-y\|_{S^p},
\end{aligned}
$$

with

$$
m_1=\max\left(\alpha(1-r^*),L_q^1\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}},(1-r^*)\left(L_q^2-a\alpha\right)\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}\right).
$$

As $m_1<1$, the operator $\Lambda:(PAPS^p(\mathbb{R},\mathbb{R}),\|\cdot\|_{S^p})\longrightarrow(PAPS^p(\mathbb{R},\mathbb{R}),\|\cdot\|_{S^p})$ is a contraction. □

4.2. *Example 1*

Let us consider the following logistic differential equation

$$
x'(t)=3x(t)+\frac{x'(t-r(t))}{2}-q(t,x(t),x(t-r(t)))+h(t), \tag{4.12}
$$

where $q:\mathbb{R}\times\mathbb{R}^2\longrightarrow\mathbb{R}$ defined by:

$$
\begin{aligned}
q(t,\sin(t),\sin(t-\cos(t)))&=\left(\sin(t)+\sin(\sqrt{2}t)\right)[\sin(t)+\sin(t-\cos(t))]\\
&\quad+\frac{[\sin(t)+\sin(t-\cos(t))]}{1+t^2}\\
&=q_1(t,\sin(t),\sin(t-\cot(t)))+q_2(t,\sin(t),\sin(t-\cos(t))).
\end{aligned}
$$

$h:\mathbb{R}\longrightarrow\mathbb{R}$ defined by $h(t)=h_1(t)+h_2(t)$, with

$$
h_1(t)=\begin{cases}-\sin(t) & \text{for} t\neq k\pi\\ 0 & \text{for } t=k\pi\end{cases} \tag{4.13}
$$

and

$$h_2(t) = \arctan(t). \tag{4.14}$$

Solving (4.12) returns to work out the following integral equation

$$x(t) = \frac{x(t-r(t))}{2} - \int_t^{+\infty} \left[q(s,x(s),x(s-r(s))) - \frac{3x(s-r(s))}{2} \right] \mathrm{e}^{3(t-s)} \mathrm{d}s + p(t), \tag{4.15}$$

where the function $p : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $p(t) = p_1(t) + p_2(t)$, with

$$p_1(t) = \begin{cases} \cos(t) & \text{for } t \neq k\pi \\ k & \text{for } t = k\pi \end{cases} \tag{4.16}$$

and

$$p_2(t) = \frac{1}{1+t^2}. \tag{4.17}$$

Firstly, the function $q_1^b(\cdot) \in AP(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$. In addition, the function $q_2^b \in PAP_0(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$. Then, the function $q : \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is S^p-pseudo almost periodic, thus hypothesis (H1) holds. Secondly, the function q is Lipschitz. Then hypothesis (H2) holds. Meanwhile, it is well-known that $[t \longmapsto p_1(t)] \in SAP(\mathbb{R},\mathbb{R})$ (cf. [8]). Moreover, $p_2^b(\cdot) \in PAP_0(\mathbb{R}, L^p((0,1),\mathbb{R}))$. Then $p(\cdot) \in PAPS^p(\mathbb{R},\mathbb{R})$, which implies that (H3) holds. Now, by virtue of Theorem 3 equation (4.15) admits a unique S^p-pseudo almost periodic solution when $m_1 < 1$, with

$$m_1 = \max\left(\frac{1}{2}, 3\left(\frac{2}{3q}\right)^{\frac{1}{q}} \left(\frac{2}{3p}\right)^{\frac{1}{p}}, \frac{3}{2}\left(\frac{2}{3q}\right)^{\frac{1}{q}} \left(\frac{2}{3p}\right)^{\frac{1}{p}} \right).$$

4.3. *Stepanov-like pseudo almost automorphic solutions*

We will study the S^p-pseudo almost automorphic solutions of (4.2). For this study, we formulate the following assumptions

(H1) $q : \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is S^p-pseudo almost automorphic, i.e.

$$q^b = q_1^b + q_2^b, \tag{4.18}$$

where $q_1^b \in AA(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ and $q_2^b \in PAP_0(\mathbb{R}\times\mathbb{R}^2, L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_t^{t+1} |q_2(\sigma,u)|^p \,\mathrm{d}\sigma \right)^{\frac{1}{p}} \mathrm{d}t = 0, \tag{4.19}$$

uniformly in $u \in \mathbb{R}^2$.

(H2) q is Lipschitz, i.e. $\exists L_q^1, L_q^2 > 0$ such that $\forall x_1, x_2, y_1, y_2 \in \mathbb{R}$

$$|q(t,x_1,x_2) - q(t,y_1,y_2)| \leq L_q^1|x_1-y_1| + L_q^2|x_2-y_2|. \tag{4.20}$$

(H3) $p:\mathbb{R}\longrightarrow\mathbb{R}$ is S^p-pseudo almost automorphic, i.e. $p^b=p_1^b+p_2^b$ where $p_1^b\in AP(\mathbb{R},L^p((0,1),\mathbb{R}))$ and $p_2^b\in PAP_0(\mathbb{R},L^p((0,1),\mathbb{R}))$ such that

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_t^{t+1}|p_2(\sigma)|^p\,\mathrm{d}\sigma\right)^{\frac{1}{p}}\mathrm{d}t=0. \tag{4.21}$$

(H6) The function $t\mapsto r(t)\in C^1(\mathbb{R},\mathbb{R})$ with

$$0\le r(t)\le\bar{r},\quad r(t)\le r^*<1. \tag{4.22}$$

Theorem 4. *Assume that (H1)-(H3) hold. If $m_2<1$ then (4.2) has a unique S^p-pseudo almost automorphic solution where*

$$m_2=\max\left(\alpha\,(1-r^*)\,,L_q^1\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}},(1-r^*)\,(L_q^2-a\alpha)\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}\right).$$

Proof. Let the operator Λ_2 defined on $PAAS^p(\mathbb{R},\mathbb{R})$ by

$$\Lambda_2(x)(t)=\alpha x(t-r(t))-\int_t^{+\infty}[q(s,x(s),x(s-r(s)))-a\alpha x(s-r(s))]\,\mathrm{e}^{a(t-s)}\mathrm{d}s+p(t).$$

Now, showing that $\Lambda_2(x)\in PAAS^p(\mathbb{R},\mathbb{R})$, set the functions

$$f(\cdot,x(\cdot),x(\cdot-r(t)))=\alpha x(\cdot-r(t))+p(\cdot), \tag{4.23}$$

and

$$g(\cdot,x(\cdot),x(\cdot-r(t)))=[q(\cdot,x(\cdot),x(\cdot-r(t)))-a\alpha x(\cdot-r(t)).] \tag{4.24}$$

Since, $x(\cdot)\in PAAS^p(\mathbb{R},\mathbb{R})$, Lemma 6 implies that

$$[t\longmapsto x(t-r(t))]\in PAAS^p(\mathbb{R},\mathbb{R}).$$

Then

$$[t\mapsto\alpha x(t-r(t))]\in PAAS^p(\mathbb{R},\mathbb{R}).$$

By Lemma 7 we obtain that

$$[t\longmapsto f(t,x(t),x(t-r(t)))]\in PAAS^p(\mathbb{R},\mathbb{R}).$$

Assumptions (H1)-(H2) and Lemma 7, yield that

$$[s\to q(s,x(s),x(s-r(s)))]\in PAAS^p(\mathbb{R},\mathbb{R}).$$

Moreover, as previously we show, $[s\longmapsto -a\alpha x(s-r(s))]\in PAAS^p(\mathbb{R},\mathbb{R})$. Hence, the function $[s\longmapsto g(s,x(s),x(s-r(s)))]\in PAAS^p(\mathbb{R},\mathbb{R})$, as the sum of two S^p-pseudo almost periodic functions. Then, it follows from Lemma 8 that

$$\left[t\longmapsto\int_t^{+\infty}[q(s,x(s),x(s-r(s)))-a\alpha x(s-r(s))]\,\mathrm{e}^{a(t-s)}\mathrm{d}s\right]\in PAAS^p(\mathbb{R},\mathbb{R}).$$

We deduce that $\Lambda_2x\in PAAS^p(\mathbb{R},\mathbb{R})$. It remains to show that Λ_2 admits a unique fixed point. Let $x,\,y\in PAAS^p(\mathbb{R},\mathbb{R})$

$$|\Lambda_2x(t)-\Lambda_2y(t)|\le|\alpha x(t-r(t))-\alpha y(t-r(t))|$$

$$
\begin{aligned}
&+\int_t^{+\infty} \mathrm{e}^{a(t-s)}\,|q(s,y(s),y(s-r(s)))-q(s,x(s),x(s-r(s)))\\
&+a\alpha x(s-r(s))-a\alpha y(s-r(s))|\,\mathrm{d}s\\
\le{}& |\alpha x(t-r(t))-\alpha y(t-r(t))|+\int_0^{+\infty}\mathrm{e}^{-as}\,|q(s+t,y(s+t),y(s+t-r(s+t)))\\
&-q(s+t,x(s+t),x(s+t-r(s+t)))\\
&+a\alpha x(s+t-r(s+t))-a\alpha y(s+t-r(s+t))|\,\mathrm{d}s\\
\le{}& |\alpha x(t-r(t))-\alpha y(t-r(t))|+\left(\frac{2}{qa}\right)^{\frac{1}{q}}\\
&\times\int_0^{\infty}\mathrm{e}^{\frac{-aps}{2}}\,|q(s+t,y(s+t),y(s+t-r(s+t)))+a\alpha x(s+t-r(s+t))\\
&-q(s+t,x(s+t),x(s+t-r(s+t)))-a\alpha y(s+t-r(s+t))|^p\,\mathrm{d}s.
\end{aligned}
$$

So, using Fubini's theorem and Minkowski's inequality we have

$$
\begin{aligned}
&\sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}|\Lambda_2 x(t)-\Lambda_2 y(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\\
\le{}&\sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}|\alpha x(t-r(t))-\alpha y(t-r(t))|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\\
\times&\Bigg(\int_0^{+\infty}\mathrm{e}^{\frac{-aps}{2}}\sup_{\xi\in\mathbb{R}}\int_\xi^{\xi+1}|a\alpha x(s+t-r(s+t))+q(s+t,y(s+t),y(s+t-r(s+t)))\\
&-q(s+t,x(s+t),x(s+t-r(s+t)))-a\alpha y(s+t-r(s+t))|^p\,\mathrm{d}t\mathrm{d}s\Bigg)^{\frac{1}{p}}\\
\le{}&\sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}|\alpha x(t-r(t))-\alpha y(t-r(t))|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\\
&\times\Bigg(\int_0^{+\infty}\mathrm{e}^{\frac{-aps}{2}}\sup_{\xi'\in\mathbb{R}}\int_{\xi'}^{\xi'+1}\big|a\alpha x(t'-r(t'))\\
&+q(t',y(t'),y(t'-r(t')))-q(t',x(t'),x(t'-r(t')))-a\alpha y(t'-r(t'))\big|^p\,\mathrm{d}t'\mathrm{d}s\Bigg)^{\frac{1}{p}}\\
\le{}&\sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}|\alpha x(t-r(t))-\alpha y(t-r(t))|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'}^{\xi'+1}\left|a\alpha x(t'-r(t'))+q(t',y(t'),y(t'-r(t')))\right.\right.\\
&\left.\left.-\;q(t',x(t'),x(t'-r(t')))-a\alpha y(t'-r(t'))\right|^p \mathrm{d}t'\right)^{\frac{1}{p}}\\
&\leq \sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}|\alpha x(t-r(t))-\alpha y(t-r(t))|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}\\
&\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'}^{\xi'+1}\left(L_q^1\left|y(t')-x(t')\right|+(L_q^2-a\alpha)\left|y(t'-r(t'))-x(t'-r(t'))\right|\right)^p\mathrm{d}t'\right)^{\frac{1}{p}}\\
&\leq \alpha\sup_{\xi\in\mathbb{R}}\left(\int_{\xi}^{\xi+1}|x(t-r(t))-y(t-r(t))|^p\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}L_q^1\\
&\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'}^{\xi'+1}\left|y(t')-x(t')\right|^p\mathrm{d}t'\right)^{\frac{1}{p}}\\
&+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}(L_q^2-a\alpha)\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'}^{\xi'+1}\left|y(t'-r(t'))-x(t'-r(t'))\right|^p\mathrm{d}t'\right)^{\frac{1}{p}}\\
&\leq \alpha\left(1-r'(t)\right)\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-r(\xi)}^{\xi+1-r(\xi+1)}\left|x(t')-y(t')\right|^p\mathrm{d}t'\right)^{\frac{1}{p}}\\
&+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}L_q^1\|x-y\|_{S^p}\\
&+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}(L_q^2-a\alpha)\left(1-r(t')\right)\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'-r(\xi')}^{\xi'+1-r(\xi'+1)}\left|y(t')-x(t')\right|^p\mathrm{d}t'\right)^{\frac{1}{p}}\\
&\leq \alpha(1-r^*)\sup_{\xi\in\mathbb{R}}\left(\int_{\xi-\bar{r}}^{\xi+1-\bar{r}}\left|x(t')-y(t')\right|^p\mathrm{d}t'\right)^{\frac{1}{p}}+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}L_q^1\|x-y\|_{S^p}\\
&+\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}(L_q^2-a\alpha)(1-r^*)\sup_{\xi'\in\mathbb{R}}\left(\int_{\xi'-\bar{r}}^{\xi'+1-\bar{r}}\left|y(t')-x(t')\right|^p\mathrm{d}t'\right)^{\frac{1}{p}}\\
&\leq m_2\|x-y\|_{S^p},
\end{aligned}
$$

with

$$
m_2=\max\left(\alpha(1-r^*),L_q^1\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}},(1-r^*)\left(L_q^2-a\alpha\right)\left(\frac{2}{aq}\right)^{\frac{1}{q}}\left(\frac{2}{ap}\right)^{\frac{1}{p}}\right).
$$

As $m_2 < 1$, the operator $\Lambda_2 : (PAAS^p(\mathbb{R},\mathbb{R}), \|\cdot\|_{S^p}) \longrightarrow (PAAS^p(\mathbb{R},\mathbb{R}), \|\cdot\|_{S^p})$ is a contraction and the result holds by Banach's fixed point theorem. $\square$

4.4. *Example 2*

In order to illustrate Theorem 4, we consider the following logistic differential equation

$$x'(t) = 2x(t) + \frac{x'(t-r(t))}{4} - q(t,x(t),x(t-r(t))) + h(t), \tag{4.25}$$

where $q : \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by:

$$\begin{aligned} q(t,\sin(t),\sin(t-4)) &= \frac{[\sin(t)+\sin(t-\cos(t))]}{2+\cos(t)+\cos(\sqrt{2}t)} + \frac{[\sin(t)+\sin(t-\cos(t))]}{1+t^2} \\ &= q_1(t,\sin(t),\sin(t-\cos(t))) + q_2(t,\sin(t),\sin(t-\cos(t))), \end{aligned}$$

and $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(t) = h_1(t) + h_2(t)$ where

$$h_1(t) = \frac{\sin(t)+\sqrt{2}\sin(\sqrt{2}t)}{(2+\cos(t)+\cos(\sqrt{2}t))^2}\cos\left(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\right)$$

and

$$h_2(t) = \arctan(t). \tag{4.26}$$

Rather than dealing with (4.25) we will study the following integral equation

$$x(t) = \frac{x(t-r(t))}{2} - \int_t^{+\infty}\left[q(s,x(s),x(s-r(s))) - \frac{3x(s-r(s))}{2}\right]e^{3(t-s)}ds + p(t), \tag{4.27}$$

the function q is S^p-pseudo almost automorphic and lipschtizian. In addition, the function $p : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $p(t) = p_1(t) + p_2(t)$ where

$$p_1(t) = \sin\left(\frac{1}{2+\cos(t)+\sin(\sqrt{2}t)}\right) \tag{4.28}$$

and

$$p_2(t) = \frac{1}{1+t^2}, \tag{4.29}$$

belongs to $PAAS^p(\mathbb{R},\mathbb{R})$. Hence, one can deduce that all the assumptions (H1), (H2) and (H3) of Theorem 4 are satisfied and thus equation (4.25) has a unique S^p-pseudo almost automorphic solution.

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REFERENCES

[1] M. Amdouni and F. Chérif, "The pseudo almost periodic solutions of the new class of Lotka–Volterra recurrent neural networks with mixed delays," *Chaos, Solitons & Fractals*, vol. 113, pp. 79–88, aug 2018, doi: 10.1016/j.chaos.2018.05.004.

[2] B. Ammar, F. Chérif, and A. M. Alimi, "Existence and uniqueness of pseudo almost-periodic solutions of recurrent neural networks with time-varying coefficients and mixed delays," *IEEE Trans. Neural Netw. Learning Syst.*, vol. 23, no. 1, pp. 109–118, 2012, doi: 10.1109/TNNLS.2011.2178444.

[3] C. Aouiti , M. S. M'hamdi, F. Chérif, and A. M. Alimi, "Impulsive generalised high-order Recurrent Neural Networks with mixed delays: Stability and periodicity," *Neurocomputing*, nov 2017, doi: 10.1016/j.neucom.2017.11.037.

[4] A. S. Besicovitch, *Almost periodic functions*. Fellow of Trinity College Cambridg, 1954.

[5] J. Blot, P. Cieutat, G. M. N'Guérékata, and D. Pennequin, "Superposition operators between various almost periodic function spaces and applications." *Communications in Mathematical Analysis*, vol. 6, no. 1, pp. 42–70, 2009. [Online]. Available: https://hal.archives-ouvertes.fr/hal-00662577

[6] Z. Cai, J. Huang, and L. Huang, "Periodic orbit analysis for the delayed Filippov system," *Proceedings of the American Mathematical Society*, vol. 146, no. 11, pp. 4667–4682, aug 2018, doi: 10.1090/proc/13883.

[7] Z. Cai, L. Huang, and Z. Wang, "Mono/multi-periodicity generated by impulses control in time-delayed memristor-based neural networks," *Nonlinear Analysis: Hybrid Systems*, vol. 36, p. 100861, may 2020, doi: 10.1016/j.nahs.2020.100861.

[8] F. Chérif, "A various types of almost periodic functions on Banach spaces: I." *Int. Math. Forum*, vol. 6, no. 17-20, pp. 921–952, 2011.

[9] F. Chérif and M. Abdelaziz, "Stepanov-like pseudo almost periodic solution of quaternion-valued for fuzzy recurrent neural networks with mixed delays," *Neural Processing Letters*, jan 2020, doi: 10.1007/s11063-020-10193-z.

[10] E. Dads, P. Cieutat, and L. Lhachimi, "Existence of positive almost periodic or ergodic solutions for some neutral nonlinear integral equations," *Differential and Integral Equations*, vol. 22, 11 2009.

[11] E. A. Dads and K. Ezzinbi, "Almost periodic solution for some neutral nonlinear integral equation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 28, no. 9, pp. 1479–1489, may 1997, doi: 10.1016/s0362-546x(96)00012-0.

[12] E. H. A. Dads, F. Boudchich, and B. Es-sebbar, "Compact almost automorphic solutions for some nonlinear integral equations with time-dependent and state-dependent delay," *Advances in Difference Equations*, vol. 2017, no. 1, oct 2017, doi: 10.1186/s13662-017-1364-2.

[13] T. Diagana, "Stepanov-like pseudo almost periodic functions and their applications to differential equations," *Communications in Mathematical Analysis*, vol. 3, no. 1, 01 2007.

[14] T. Diagana, *Almost automorphic type and almost periodic type functions in abstract spaces*. Springer, 08 2013.

[15] H.-S. Ding, Y.-Y. Chen, and G. M. N'Guérékata, "Existence of positive pseudo almost periodic solutions to a class of neutral integral equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 18, pp. 7356–7364, dec 2011, doi: 10.1016/j.na.2011.07.054.

[16] A. Fink and J. Gatica, "Positive almost periodic solutions of some delay integral equations," *Journal of Differential Equations*, vol. 83, no. 1, pp. 166–178, jan 1990, doi: 10.1016/0022-0396(90)90073-x.

[17] H. Hu, T. Yi, and X. Zou, "On spatial-temporal dynamics of a Fisher-KPP equation with a shifting environment," *Proceedings of the American Mathematical Society*, vol. 148, no. 1, pp. 213–221, jul 2019, doi: 10.1090/proc/14659.

[18] H. Hu, X. Yuan, L. Huang, and C. Huang, "Global dynamics of an SIRS model with demographics and transfer from infectious to susceptible on heterogeneous networks," *Mathematical Biosciences and Engineering*, vol. 16, no. 5, pp. 5729–5749, 2019, doi: 10.3934/mbe.2019286.

[19] C. Huang, X. Long, L. Huang, and S. Fu, "Periodicity on Nicholson's blowflies systems involving patch structure and mortality terms," *Canadian Mathematical Bulletin*, pp. 1–18, sep 2019, doi: 10.4153/s0008439519000511.

[20] C. Huang, H. Zhang, and L. Huang, "Almost periodicity analysis for a delayed Nicholson's blowflies model with nonlinear density-dependent mortality term," *Communications on Pure & Applied Analysis*, vol. 18, no. 6, pp. 3337–3349, 2019, doi: 10.3934/cpaa.2019150.

[21] M. N. Islam, "Periodic solutions of Volterra type integral equations with finite delay," *Communications in Applied Analysis*, vol. 15, no. 1, p. 57, 01 2011.

[22] H. X. Li, F. L. Huang, and J. Y. Li, "Composition of pseudo almost-periodic functions and semilinear sifferential qquations," *Journal of Mathematical Analysis and Applications*, vol. 255, no. 2, pp. 436–446, mar 2001, doi: 10.1006/jmaa.2000.7225.

[23] G. M. N'Guerekata, *Almost automorphic and almost periodic functions in abstract Spaces*. Springer US, 2001. doi: 10.1007/978-1-4757-4482-8.

[24] G. M. N'guerekata and A. Pankov, "Stepanov-like almost automorphic functions and monotone evolution equations," *Nonlinear Analysis-theory Methods & Applications*, vol. 68, no. 9, pp. 2658–2667, 2008.

[25] J. Wang, X. Chen, and L. Huang, "The number and stability of limit cycles for planar piecewise linear systems of node–saddle type," *Journal of Mathematical Analysis and Applications*, vol. 469, no. 1, pp. 405–427, jan 2019, doi: 10.1016/j.jmaa.2018.09.024.

[26] T. Yoshizawa, "Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions," *Springer-Verlag New York*, vol. 9, no. 5, pp. 314–314, 1975, doi: 10.1109/tsmc.1979.4310210.

[27] C. Zhang, *Almost periodic type functions and ergodicity*. Springer Netherlands, 2003. doi: 10.1007/978-94-007-1073-3.

[28] Q.-F. Zou and H.-S. Ding, "Almost periodic solutions for a nonlinear integro-differential equation with neutral delay," *Journal of Nonlinear Sciences and Applications*, vol. 09, no. 06, pp. 4500–4508, jun 2016, doi: 10.22436/jnsa.009.06.88.

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INEQUALITIES OF HERMITE–HADAMARD TYPE FOR EXTENDED HARMONICALLY (s,m)-CONVEX FUNCTIONS

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Abstract. In the paper, the authors introduce a new notion "extended harmonically (s,m)-convex function" and establish some integral inequalities of the Hermite–Hadamard type for extended harmonically (s,m)-convex functions in terms of hypergeometric functions.

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1. INTRODUCTION

We recall some definitions on diverse convex functions in the literature.

Definition 1. A function $f : I \subseteq \mathbb{R} = (-\infty,\infty) \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)$$

holds for all $x,y \in I$ and $t \in [0,1]$.

Definition 2 ([2,11]). Let $s \in (0,1]$ be a real number. A function $f : \mathbb{R}_0 = [0,\infty) \to \mathbb{R}_0$ is said to be s-convex in the second sense if the inequality

$$f(tx+(1-t)y) \le t^s f(x)+(1-t)^s f(y)$$

holds for all $x,y \in I$ and $t \in [0,1]$.

Definition 3 ([25]). For $f : [0,b] \to \mathbb{R}$, $b>0$, and $m \in (0,1]$, if the inequality

$$f(tx+m(1-t)y) \le tf(x)+m(1-t)f(y)$$

is valid for all $x,y \in [0,b]$ and $t \in [0,1]$, then we say that f is an m-convex function on $[0,b]$.

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Definition 4 ([28]). A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be extended s-convex if the inequality

$$f(tx+(1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in (0,1)$ and for some fixed $s \in [-1,1]$.

Definition 5 ([13]). Let $I \subseteq \mathbb{R} \setminus \{0\}$ be a real interval. A function $I \to \mathbb{R}$ is said to be harmonically convex if the inequality

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le t f(x) + (1-t) f(y) \tag{1.1}$$

holds for all $x, y \in$I and $t \in [0,1]$. If the inequality in (1.1) is reversed, then f is said to be harmonically concave.

Definition 6 ([17, Definition 2.6]). A function $I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is said to be harmonically s-convex if the inequality

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le t^s f(x) + (1-t)^s f(y)$$

is valid for $x, y \in I$, $t \in (0,1)$, and $s \in [-1,1]$.

Definition 7 ([29]). Let $f : (0,b] \to \mathbb{R}$ and let $m \in (0,1]$ be a constant. If the inequality

$$f\left(\left(\frac{t}{x} + m\frac{1-t}{y}\right)^{-1}\right) \le t f(x) + m(1-t) f(y)$$

is valid for all $x, y \in (0,b]$ and $t \in [0,1]$, then f is said to be an m-harmonic-arithmetically convex function or, simply speaking, an m-HA-convex function.

Definition 8 ([9]). Let $f : (0,b] \to \mathbb{R}$ and let $\alpha, m \in (0,1]$ be constants. If the inequality

$$f\left(\left(\frac{t}{x} + m\frac{1-t}{y}\right)^{-1}\right) \le t^{\alpha} f(x) + m(1-t^{\alpha}) f(y)$$

is valid for all $x, y \in (0,b]$ and $t \in [0,1]$, then f is said to be an (α,m)-harmonic-arithmetically convex function or, simply speaking, an (α,m)-HA-convex function.

In recent decades, establishing integral inequalities of the Hermite–Hadamard type for diverse convex functions has been being an active direction in mathematics. Some of these results can be reformulated as follows.

Theorem 1 ([5, Theorem 2.2]). *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^{\circ}$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

Theorem 2 ([20, Theorems 1 and 2]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ and $q \geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}.$$

Theorem 3 ([6]). *Let $m \in (0,1]$ and $f : \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ be m-convex. If $f \in L_1([a,b])$ for $0 \leq a < b < \infty$, then*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \min\left\{\frac{f(a)+mf(b/m)}{2}, \frac{mf(a/m)+f(b)}{2}\right\}.$$

Theorem 4 ([14]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'|^q$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $q \geq 1$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\qquad \leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\left[\frac{2+1/2^s}{(s+1)(s+2)}\right]^{1/q}\left[|f'(a)|^q + |f'(b)|^q\right]^{1/q}.\end{aligned}$$

Theorem 5 ([12]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $q > 1$, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}\left(\frac{1}{2}\right)^{1/p} \\ &\quad \times \left\{\left[|f'(a)|^q + (s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right. \\ &\quad \left. + \left[|f'(b)|^q + (s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right\},\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 6 ([24]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|$ is s-convex on $[a,b]$ for some $s \in (0,1]$ and $p > 1$, then*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\quad \leq \frac{(s-4)6^{s+1} + 2\times 5^{s+2} - 2\times 3^{s+2} + 2}{6^{s+2}(s+1)(s+2)}(b-a)\left(|f'(a)| + |f'(b)|\right),\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

For more information developed in recent decades on this topic, please refer to the papers [1, 3, 4, 7, 8, 10, 15, 16, 18, 19, 21–23, 26, 27, 30] and closely related references.

In this paper, we will introduce a new notion "extended harmonically (s,m)-convex function" and establish some new integral inequalities of the Hermite–Hadamard type for extended harmonically (s,m)-convex functions.

2. A DEFINITION AND A LEMMA

Now we introduce the notion "extended harmonically (s,m)-convex function".

Definition 9. For $m \in (0,1]$ and $s \in [-1,1]$, a function $f:(0,b] \to \mathbb{R}$ is said to be extended harmonically (s,m)-convex on $(0,b]$ if the inequality

$$f\left(\left(\frac{t}{x}+m\frac{1-t}{y}\right)^{-1}\right) \le t^s f(x)+m(1-t)^s f(y)$$

holds for all $x,y \in (0,b]$ and $t \in (0,1)$.

Example 1. Let $s \in [-1,1]$ and $f(x)=\frac{1}{x^r}$ for $x \in \mathbb{R}_+$ and $r \ge 1$. Since

$$f\left(\left(\frac{t}{x}+\frac{m(1-t)}{y}\right)^{-1}\right) \le \frac{ty^r+(1-t)(mx)^r}{(xy)^r} \le t^s f(x)+m(1-t)^s f(y)$$

for all $x,y \in \mathbb{R}_+$ and $t \in (0,1)$, the function $f(x)=\frac{1}{x^r}$ is extended harmonically (s,m)-convex on $\mathbb{R}_+$.

To establish some new integral inequalities of the Hermite–Hadamard type for extended harmonically (s,m)-convex functions, we need the following lemma.

Lemma 1. *Let $f: I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I° and $a,b \in I^\circ$ with $a<b$. If $f' \in L_1([a,b])$ and $0 \le \lambda,\mu \le 1$, then*

$$\begin{aligned}\frac{\lambda f(a)+\mu f(b)}{2}&+\frac{2-\lambda-\mu}{2}f(H(a,b))-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \qquad (2.1)\\ &=\frac{b-a}{4ab}\int_0^1\left[(1-\lambda-t)\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2} f'\left(\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-1}\right)\right.\\ &\quad\left.+(\mu-1+t)\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2} f'\left(\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-1}\right)\right]dt.\end{aligned}$$

In particular, if $\lambda=\mu=0$, then

$$\begin{aligned}f(H(a,b))&-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \qquad (2.2)\\ &=\frac{b-a}{4ab}\int_0^1\left[(1-t)\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2} f'\left(\left(\frac{1+t}{2a}+\frac{1-t}{2b}\right)^{-1}\right)\right.\end{aligned}$$

$$-(1-t)\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2} f'\left(\left(\frac{1+t}{2b}+\frac{1-t}{2a}\right)^{-1}\right)\Bigg] dt,$$

where $H(a,b)=\frac{2ab}{a+b}$.

Proof. Putting $x=\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-1}$ for $t\in[0,1]$ gives

$$\int_0^1 (1-\lambda-t)\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2} f'\left(\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-1}\right) dt \tag{2.3}$$

$$=\frac{2ab}{b-a}[\lambda f(a)+(1-\lambda)f(H(a,b))]-\left(\frac{2ab}{b-a}\right)^2\int_a^{H(a,b)}\frac{f(x)}{x^2}dx.$$

Similarly, letting $x=\left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-1}$ for $t\in[0,1]$ results in

$$\int_0^1 (\mu-1+t)\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2} f'\left(\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-1}\right) dt \tag{2.4}$$

$$=\frac{2ab}{b-a}[\mu f(b)+(1-\mu)f([H(a,b)])]-\left(\frac{2ab}{b-a}\right)^2\int_{H(a,b)}^{b}\frac{f(x)}{x^2}dx.$$

Adding the equalities (2.3) and (2.4) leads to the equality (2.1). The proof of Lemma 1 is thus complete. □

3. Integral inequalities of Hermite–Hadamard type

Now we start out to establish some new integral inequalities of the Hermite–Hadamard type for extended harmonically (s,m)-convex functions.

Theorem 7. *Let* $f:(0,d]\to\mathbb{R}$ *be differentiable,* $a,b\in(0,d]$ *with* $a<b$*,* $f'\in L_1([a,b])$*, and* $0\le\lambda,\mu\le 1$*. If* $|f'|^q$ *for* $q\ge 1$ *is extended harmonically* (s,m)*-convex on* $(0,d]$ *for some fixed* $m\in(0,1]$ *and* $s\in[-1,1]$*, then*

(1) *when* $-1<s\le 1$,

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \tag{3.1}$$

$$\le\frac{(b-a)^{(2-q)/q}}{2^{(q+1)/q}(ab)^{1/q}}\Big\{[T(a,b,\lambda)]^{1-1/q}\left(mK(a,H(a,b),s,\lambda)|f'(mH(a,b))|^q\right.$$

$$\left.+K(H(a,b),a,s,1-\lambda)|f'(a)|^q\right)^{1/q}+[T(b,a,\mu)]^{1-1/q}\left(mK(b,H(a,b),s,\mu)\right.$$

$$\left.\times|f'(mH(a,b))|^q+K(H(a,b),b,s,1-\mu)|f'(b)|^q\right)^{1/q}\Big\};$$

(2) *when* $s=-1$,

$$\left|f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\le\frac{(ab)^{(q-2)/q}(a+b)^{(1-q)/q}}{2^{1/q}(b-a)^{(q-2)/q}}\{[(a+b)(\ln a$$

$$-\ln H(a,b)) + (b-a)]^{1-1/q}\left[\left(2b^2(\ln(2a) - \ln H(a,b)) - bH(a,b)\right)|f'(a)|^q\right.$$
$$\left.+maH(a,b)|f'(mb)|^q\right]^{1/q} + \left[m(2a^2(\ln(2b) - \ln H(a,b)) - aH(a,b))|f'(mb)|^q\right.$$
$$\left.+bH(a,b)|f'(a)|^q\right]^{1/q}[(a+b)(\ln b - \ln H(a,b)) - (b-a)]^{1-1/q}\Big\},$$

where

$$T(a,b,\lambda) = 2ab(\ln a + \ln H(a,b) - 2\ln[(1-\lambda)H(a,b) + \lambda a]) + (b-a)[(1-\lambda)H(a,b) - \lambda a]$$

and

$$K(a,u,s,\lambda) = \frac{2\lambda^{s+2}a^2}{(s+1)(s+2)}{}_2F_1\left(2,s+1,s+3,\frac{\lambda(u-a)}{u}\right) - \frac{\lambda a^2}{s+1}{}_2F_1\left(2,s+1,s+2,\frac{u-a}{u}\right) + \frac{a^2}{s+2}{}_2F_1\left(2,s+2,s+3,\frac{u-a}{u}\right)$$

with the hypergeometric function

$$ {}_2F_1(c,d,e;z) = \frac{\Gamma(e)}{\Gamma(d)\Gamma(e-d)}\int_0^1 t^{d-1}(1-t)^{e-d-1}(1-zt)^{-c}\,dt \tag{3.2}$$

for $e > d > 0$, $|z| < 1$, $c \in \mathbb{R}$, *and* $u > 0$.

Proof. When $-1 < s \le 1$, by virtue of Lemma 1 and the extended harmonic (s,m)-convexity of $|f'|^q$, we obtain

$$\left|\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2}f(H(a,b)) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \tag{3.3}$$
$$\le \frac{b-a}{4ab}\left[\left(\int_0^1 |1-\lambda-t|\left(\frac{t}{a} + \frac{1-t}{H(a,b)}\right)^{-2}dt\right)^{1-1/q}\right.$$
$$\times\left(\int_0^1 |1-\lambda-t|\left(\frac{t}{a} + \frac{1-t}{H(a,b)}\right)^{-2}\left|f'\left(\left(\frac{t}{a} + \frac{1-t}{H(a,b)}\right)^{-1}\right)\right|^q dt\right)^{1/q}$$
$$+\left(\int_0^1 |1-\mu-t|\left(\frac{t}{b} + \frac{1-t}{H(a,b)}\right)^{-2}dt\right)^{1-1/q}$$
$$\left.\times\left(\int_0^1 |1-\mu-t|\left(\frac{t}{b} + \frac{1-t}{H(a,b)}\right)^{-2}\left|f'\left(\left(\frac{t}{b} + \frac{1-t}{H(a,b)}\right)^{-1}\right)\right|^q dt\right)^{1/q}\right]$$
$$\le \frac{b-a}{4ab}\left[\left(\int_0^1 |1-\lambda-t|\left(\frac{t}{a} + \frac{1-t}{H(a,b)}\right)^{-2}dt\right)^{1-1/q}\left(\int_0^1 |1-\lambda-t|\right.\right.$$

$$
\begin{aligned}
&\times\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2}\left[t^s|f'(a)|^q+m(1-t)^s|f'(mH(a,b))|^q\right]dt\Bigg)^{1/q}\\
&+\left(\int_0^1|1-\mu-t|\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2}dt\right)^{1-1/q}\Bigg(\int_0^1|1-\mu-t|\\
&\times\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2}\left[t^s|f'(b)|^q+m(1-t)^s|f'(mH(a,b))|^q\right]dt\Bigg)^{1/q}\Bigg],
\end{aligned}
$$

where we used the facts

$$
\int_0^1|1-\lambda-t|\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2}dt=\frac{2ab}{(b-a)^2}T(a,b,\lambda),
$$

$$
\int_0^1|1-\lambda-t|t^s\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2}dt=K(H(a,b),a,s,1-\lambda),
$$

and

$$
\int_0^1|1-\lambda-t|(1-t)^s\left[\frac{t}{a}+\frac{1-t}{H(a,b)}\right]^{-2}dt=K(a,H(a,b),s,\lambda).
$$

The inequality (3.1) is thus proved.

When $s=-1$, by the identity (2.2) and the extended harmonic (s,m)-convexity of $|f'|^q$, we have

$$
\begin{aligned}
&\left|f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\\
&\le\frac{b-a}{2^{2-1/q}ab}\Bigg[\left(\int_0^1(1-t)\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2}dt\right)^{1-1/q}\\
&\times\Bigg(\int_0^1(1-t)\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2}\left[(1+t)^{-1}|f'(a)|^q\right.\\
&\left.+m(1-t)^{-1}|f'(mb)|^q\right]dt\Bigg)^{1/q}+\left(\int_0^1(1-t)\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2}dt\right)^{1-1/q}\\
&\times\Bigg(\int_0^1(1-t)\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2}\left[(1-t)^{-1}|f'(a)|^q\right.\\
&\left.+m(1+t)^{-1}|f'(mb)|^q\right]dt\Bigg)^{1/q}\Bigg] \qquad (3.4)\\
&=\frac{(ab)^{(q-2)/q}(a+b)^{(1-q)/q}}{2^{1/q}(b-a)^{(q-2)/q}}\Big\{[(a+b)(\ln a-\ln H(a,b))+(b-a)]^{1-1/q}
\end{aligned}
$$

$$\times\left[\left(2b^2(\ln(2a)-\ln H(a,b))-bH(a,b)\right)|f'(a)|^q+maH(a,b)|f'(mb)|^q\right]^{1/q}$$
$$+\left[m\left(2a^2(\ln(2b)-\ln H(a,b))-aH(a,b)\right)|f'(mb)|^q+bH(a,b)|f'(a)|^q\right]^{1/q}$$
$$\times[(a+b)(\ln b-\ln H(a,b))-(b-a)]^{1-1/q}\Big\}.$$

The proof of Theorem 7 is thus complete. □

Corollary 1. *Under conditions of Theorem 7, when $q=1$,*

(1) *if $-1<s\le 1$, then*

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$
$$\le\frac{b-a}{4ab}\left\{K(H(a,b),a,s,1-\lambda)|f'(a)|+K(H(a,b),b,s,1-\mu)|f'(b)|\right.$$
$$\left.+m[K(a,H(a,b),s,\lambda)+K(b,H(a,b),s,\mu)]|f'(mH(a,b))|\right\};$$

(2) *if $s=-1$, then*

$$\left|f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$
$$\le\frac{b-a}{ab}\left(b^2[\ln(2a)-\ln H(a,b)]|f'(a)|+ma^2[\ln(2b)-\ln H(a,b)]|f'(mb)|\right).$$

Corollary 2. *Under conditions of Theorem 7, when $q=s=1$ and $\lambda=\mu=\frac{1}{2}$, we have*

$$\left|\frac{f(a)+2f(H(a,b))+f(b)}{4}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dt\right|\le\frac{1}{4(b-a)^2}$$
$$\times\left\{\left[(20ab^2+12a^2b)\left(\ln\sqrt{aH(a,b)}-\ln H(a,H(a,b))\right)-a(b-a)^2\right]|f'(a)|\right.$$
$$+4mab\left[(7b+a)\left(\ln H(a,H(a,b))-\ln\sqrt{aH(a,b)}\right)\right.$$
$$\left.+(7a+b)\left(\ln\sqrt{bH(a,b)}-\ln H(b,H(a,b))\right)\right]|f'(mH(a,b))|$$
$$\left.+\left[\left(20a^2b+12ab^2\right)\left(\ln H(b,H(a,b))-\ln\sqrt{bH(a,b)}\right)+b(b-a)^2\right]|f'(b)|\right\}.$$

Theorem 8. *Let $f:(0,d]\to\mathbb{R}$ be differentiable, $a,b\in(0,d]$ with $a<b$, $f'\in L_1([a,b])$, and $0\le\lambda,\mu\le 1$. If $|f'|^q$ for $q>1$ is extended harmonically (s,m)-convex on $(0,d]$ for some fixed $s\in[-1,1]$ and $0<m\le 1$, then*

(1) *when $-1<s\le 1$,*

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \tag{3.5}$$
$$\le\frac{b-a}{4ab[(s+1)(s+2)]^{1/q}}\left\{Q^{1-1/q}(a,H(a,b),\lambda)\left[(2(1-\lambda)^{s+2}+\lambda(s+2)\right.\right.$$

$$-1)\,|f'(a)|^q+m\left(2\lambda^{s+2}+(1-\lambda)(s+2)-1\right)|f'(mH(a,b))|^q\big]^{1/q}$$
$$+Q^{1-1/q}(b,H(a,b),\mu)\left[\left(2(1-\mu)^{s+2}+\mu(s+2)-1\right)|f'(b)|^q\right.$$
$$\left.+m\left(2\mu^{s+2}+(1-\mu)(s+2)-1\right)|f'(mH(a,b))|^q\right]^{1/q}\Big\};$$

(2) *when* $s=-1$,

$$\left|f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\le\frac{b-a}{2^{(2q-1)/q}ab} \tag{3.6}$$
$$\times\left\{\left[(2\ln 2-1)|f'(a)|^q+m|f'(mb)|^q\right]^{1/q}Q^{1-1/q}(a,H(a,b),0)\right.$$
$$\left.+\left[(2\ln 2-1)|f'(b)|^q+m|f'(ma)|^q\right]^{1/q}Q^{1-1/q}(b,H(a,b),0)\right\},$$

where, for $u>0$ *and* $u\neq a$

$$Q(a,u,\lambda)=\frac{(q-1)(au)^{2q/(q-1)}}{(q+1)(u-a)}\left\{(1-\lambda)a^{-(q+1)/(q-1)}-\lambda u^{-(q+1)/(q-1)}\right.$$
$$\left.-\frac{q-1}{2(u-a)}\left[a^{-2/(q-1)}-2[(1-\lambda)(u-a)+a]^{-2/(q-1)}+u^{-2/(q-1)}\right]\right\}.$$

Proof. If $-1<s\le 1$, by the inequality (3.3) and the extended harmonic (s,m)-convexity of $|f'|^q$, we derive

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \tag{3.7}$$
$$\le\frac{b-a}{4ab}\left[\left(\int_0^1|1-\lambda-t|\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2q/(q-1)}dt\right)^{1-1/q}\right.$$
$$\times\left(\int_0^1|1-\lambda-t|\left[t^s|f'(a)|^q+m(1-t)^s|f'(mH(a,b))|^q\right]dt\right)^{1/q}$$
$$+\left(\int_0^1|1-\mu-t|\left(\frac{t}{b}+\frac{1-t}{H(a,b)}\right)^{-2q/(q-1)}dt\right)^{1-1/q}$$
$$\left.\times\left(\int_0^1|1-\mu-t|\left[t^s|f'(b)|^q+m(1-t)^s|f'(mH(a,b))|^q\right]dt\right)^{1/q}\right],$$

where

$$\int_0^1|1-\lambda-t|\left(\frac{t}{a}+\frac{1-t}{H(a,b)}\right)^{-2q/(q-1)}dt=Q(a,H(a,b),\lambda),$$
$$\int_0^1|1-\lambda-t|t^s dt=\frac{2(1-\lambda)^{s+2}+\lambda(s+2)-1}{(s+1)(s+2)},$$

and

$$\int_0^1 |1-\lambda-t|(1-t)^s dt = \frac{2\lambda^{s+2}+(1-\lambda)(s+2)-1}{(s+1)(s+2)}. \tag{3.8}$$

Combining (3.7) with (3.8) gives the required inequality (3.5).

Similarly, by the inequality (3.4), we can prove the inequality (3.6). The proof of Theorem 8 is complete. □

Corollary 3. *Under assumptions of Theorem 8,*

(1) *if* $-1 < s \le 1$, *then*

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(H(a,b))-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\
&\le \frac{b-a}{24ab}\left[\frac{6}{(s+1)(s+2)}\right]^{1/q}\Big\{\Big[\big(2(1-\lambda)^{s+2}+\lambda(s+2)-1\big)|f'(a)|^q \\
&\quad +m\left(2\lambda^{s+2}+(1-\lambda)(s+2)-1\right)|f'(mH(a,b))|^q\Big]^{1/q}\Big[\big(2(1-\lambda)^3 \\
&\quad +3\lambda-1\big)a^{2q/(q-1)}+(2\lambda^3-3\lambda+2)H^{2q/(q-1)}(a,b)\Big]^{1-1/q}+\Big[\big(2\,(1-\mu)^{s+2} \\
&\quad +\mu(s+2)-1\big)|f'(b)|^q+m\left(2\mu^{s+2}+(1-\mu)(s+2)-1\right)|f'(mH(a,b))|^q\Big]^{1/q} \\
&\quad \times\Big[(2(1-\mu)^3+3\mu-1)b^{2q/(q-1)}+\left(2\mu^3-3\mu+2\right)H^{2q/(q-1)}(a,b)\Big]^{1-1/q}\Big\};
\end{aligned}$$

(2) *if* $s=-1$, *then*

$$\begin{aligned}
&\left|f(H(a,b))-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{b-a}{2\times 12^{(q-1)/q}ab}\Big\{\big[(2\ln 2-1)|f'(a)|^q \\
&\quad +m|f'(mb)|^q\big]^{1/q}\left[a^{2q/(q-1)}+2H^{2q/(q-1)}(a,b)\right]^{1-1/q}+\big[|f'(a)|^q \\
&\quad +m(2\ln 2-1)|f'(mb)|^q\big]^{1/q}\left[b^{2q/(q-1)}+2H^{2q/(q-1)}(a,b)\right]^{1-1/q}\Big\}.
\end{aligned}$$

Proof. Substituting

$$Q(a,H(a,b),\lambda) \le \frac{2(1-\lambda)^3+3\lambda-1}{6}a^{2q/(q-1)}+\frac{2\lambda^3-3\lambda+2}{6}H^{2q/(q-1)}(a,b)$$

into (3.7) yields Corollary 3. □

Theorem 9. *Let* $f:(0,d]\to\mathbb{R}$ *is extended harmonically* (s,m)*-convex for some fixed* $m\in(0,1]$, $s\in(-1,1]$, *and* $a,b\in(0,d]$ *with* $a<b$. *If* $f\in L_1([a,b])$, *then*

$$2^s f(H(a,b)) \le \frac{ab}{b-a}\int_a^b \frac{f(x)+mf(mx)}{x^2}dx \tag{3.9}$$

and

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \min\left\{\frac{f(a)+mf(mb)}{s+1},\frac{mf(ma)+f(b)}{s+1}\right\}.$$

In particular, when $m = 1$, we have

$$2^{s-1} f(H(a,b)) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{f(a)+f(b)}{s+1}. \tag{3.10}$$

Remark 1. The inequality (3.10) appeared in [17].

Proof. From the extended harmonic (s,m)-convexity f, it follows that

$$\begin{aligned} f(H(a,b)) &= \int_0^1 f\left(\frac{2}{ta^{-1}+(1-t)b^{-1}+tb^{-1}+(1-t)a^{-1}}\right)dt \\ &\le \frac{1}{2^s}\int_0^1 \left[f\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)+mf\left(m\left(\frac{t}{b}+\frac{1-t}{a}\right)^{-1}\right)\right]dt. \end{aligned}$$

Let $x = \left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}$ for $t \in [0,1]$. Then

$$\int_0^1 f\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)dt = \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx. \tag{3.11}$$

Similarly, we have

$$\int_0^1 f\left(m\left(\frac{t}{b}+\frac{1-t}{a}\right)^{-1}\right)dt = \frac{ab}{b-a}\int_a^b \frac{f(mx)}{x^2}dx. \tag{3.12}$$

From (3.11) and (3.12), the inequality (3.9) follows immediately.

Let $x = \left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}$ for $t \in [0,1]$. Then

$$\begin{aligned} \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx &= \int_0^1 f\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)dt \\ &\le \int_0^1 [t^s f(a)+m(1-t)^s f(mb)]\,dt = \frac{f(a)+mf(mb)}{s+1}. \end{aligned}$$

The proof of Theorem 9 is complete. □

Corollary 4. *Under assumptions of Theorem 9, if $s = m = 1$, then*

$$f(H(a,b)) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{f(a)+f(b)}{2}.$$

Theorem 10. *For $m \in (0,1]$ and $s \in (-1,1]$, let $f : (0,d] \to \mathbb{R}$ is extended harmonically (s,m)-convex and $a,b \in (0,d]$ with $a < b$. If $f \in L_1([a,b])$, then*

$$2^{s-2}H^2(a,b)f(H(a,b)) \le \frac{ab}{b-a}\int_a^b [f(x)+mf(mx)]dx$$

and

$$\frac{ab}{b-a}\int_a^b f(x)dx \le \frac{a^2}{s+1}\left[m \times {}_2F_1\left(2,1,s+2,1-ab^{-1}\right)f(ma)\right.$$

$$+{}_2F_1\left(2, s+1, s+2, 1-ab^{-1}\right) f(b)\right].$$

where ${}_2F_1(a,b;c;z)$ *is the hypergeometric function defined by* (3.2).

Proof. By the extended harmonic (s,m)-convexity of f, we have

$$\begin{aligned}
H(a,b)]^2 f(H(a,b)) &\le 2^{2-s}\int_0^1 \left[\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-2}\right] f\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right) dt \\
&\quad + 2^{2-s} m \int_0^1 \left[\left(\tfrac{t}{b}+\tfrac{1-t}{a}\right)^{-2}\right] f\left(m\left(\tfrac{t}{b}+\tfrac{1-t}{a}\right)^{-1}\right) dt \\
&\le \frac{ab}{b-a}\int_a^b \frac{f(x)+mf(mx)}{2^{s-2}} dx
\end{aligned}$$

and

$$\begin{aligned}
\frac{ab}{b-a}\int_a^b f(x)dx &= \int_0^1 \left(\frac{1-t}{a}+\frac{t}{b}\right)^{-2} f\left(\left(\frac{1-t}{a}+\frac{t}{b}\right)^{-1}\right) dt \\
&\le \int_0^1 \left(\frac{1-t}{a}+\frac{t}{b}\right)^{-2} [m(1-t)^s f(ma) + t^s f(b)] dt \\
&= \frac{a^2}{s+1}\left[m \times {}_2F_1(2,1,s+2,1-ab^{-1}) f(ma)\right. \\
&\quad \left. + {}_2F_1(2,s+1,s+2,1-ab^{-1}) f(b)\right].
\end{aligned}$$

The proof of Theorem 10 is thus complete. □

Corollary 5. *Under assumptions of Theorem 10, if* $s = m = 1$*, then*

$$\begin{aligned}
\frac{[H(a,b)]^2 f(H(a,b))}{4} &\le \frac{ab}{b-a}\int_a^b f(x)dx \\
&\le \frac{a^2 b[b\ln(a^{-1}b)-(b-a)]}{(b-a)^2} f(a) + \frac{ab^2[(b-a)-a\ln(a^{-1}b)]}{(b-a)^2} f(b).
\end{aligned}$$

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REFERENCES

[1] R.-F. Bai, F. Qi, and B.-Y. Xi, "Hermite-Hadamard type inequalities for the m- and (α,m)-logarithmically convex functions," *Filomat*, vol. 27, no. 1, pp. 1–7, 2013, doi: 10.2298/FIL1301001B.

[2] W. W. Breckner, "Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen," *Publ. Inst. Math. (Beograd) (N.S.)*, vol. 23(37), pp. 13–20, 1978.

[3] P. Burai, A. Házy, and T. Juhász, "Bernstein-Doetsch type results for s-convex functions," *Publ. Math. Debrecen*, vol. 75, no. 1-2, pp. 23–31, 2009.

[4] S. S. Dragomir, "On some new inequalities of Hermite-Hadamard type for m-convex functions," *Tamkang J. Math.*, vol. 33, no. 1, pp. 55–65, 2002, doi: 10.5556/j.tkjm.33.2002.45-56.

[5] S. S. Dragomir and R. P. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Appl. Math. Lett.*, vol. 11, no. 5, pp. 91–95, 1998, doi: 10.1016/S0893-9659(98)00086-X.

[6] S. S. Dragomir and G. Toader, "Some inequalities for m-convex functions," *Studia Univ. Babeş-Bolyai Math.*, vol. 38, no. 1, pp. 21–28, 1993.

[7] A. Házy, "Bernstein-Doetsch type results for h-convex functions," *Math. Inequal. Appl.*, vol. 14, no. 3, pp. 499–508, 2011, doi: 10.7153/mia-14-42.

[8] A. Házy, "Bernstein-Doetsch type results for (k,h)-convex functions," *Miskolc Math. Notes*, vol. 13, no. 2, pp. 325–336, 2012, doi: 10.18514/mmn.2012.538.

[9] C.-Y. He, Y. Wang, B.-Y. Xi, and F. Qi, "Hermite-Hadamard type inequalities for (α,m)-HA and strongly (α,m)-HA convex functions," *J. Nonlinear Sci. Appl.*, vol. 10, no. 1, pp. 205–214, 2017, doi: 10.22436/jnsa.010.01.20.

[10] J. Hua, B.-Y. Xi, and F. Qi, "Inequalities of Hermite-Hadamard type involving an s-convex function with applications," *Appl. Math. Comput.*, vol. 246, pp. 752–760, 2014, doi: 10.1016/j.amc.2014.08.042.

[11] H. Hudzik and L. Maligranda, "Some remarks on s-convex functions," *Aequationes Math.*, vol. 48, no. 1, pp. 100–111, 1994, doi: 10.1007/BF01837981.

[12] S. Hussain, M. I. Bhatti, and M. Iqbal, "Hadamard-type inequalities for s-convex functions. I," *Punjab Univ. J. Math. (Lahore)*, vol. 41, pp. 51–60, 2009.

[13] I. İşcan, "Hermite-Hadamard type inequalities for harmonically convex functions," *Hacet. J. Math. Stat.*, vol. 43, no. 6, pp. 935–942, 2014.

[14] U. S. Kirmaci, M. Klaričić Bakula, M. E. Özdemir, and J. Pečarić, "Hadamard-type inequalities for s-convex functions," *Appl. Math. Comput.*, vol. 193, no. 1, pp. 26–35, 2007, doi: 10.1016/j.amc.2007.03.030.

[15] M. Klaričić Bakula, M. E. Özdemir, and J. Pečarić, "Hadamard type inequalities for m-convex and (α,m)-convex functions," *J. Inequal. Pure Appl. Math.*, vol. 9, no. 4, pp. Article 96, 12, 2008. [Online]. Available: http://www.emis.de/journals/JIPAM/article1032.html

[16] M. A. Latif, S. S. Dragomir, and E. Momoniat, "Fejér type inequalities for harmonically-convex functions with applications," *J. Appl. Anal. Comput.*, vol. 7, no. 3, pp. 795–813, 2017.

[17] M. A. Noor, K. I. Noor, M. U. Awan, and S. Costache, "Some integral inequalities for harmonically h-convex functions," *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, vol. 77, no. 1, pp. 5–16, 2015.

[18] M. A. Noor, K. I. Noor, and M. U. Awan, "Some characterizations of harmonically log-convex functions," *Proc. Jangjeon Math. Soc.*, vol. 17, no. 1, pp. 51–61, 2014.

[19] M. A. Noor, K. I. Noor, M. U. Awan, and S. Khan, "Fractional Hermite-Hadamard inequalities for some new classes of Godunova-Levin functions," *Appl. Math. Inf. Sci.*, vol. 8, no. 6, pp. 2865–2872, 2014, doi: 10.12785/amis/080623.

[20] C. E. M. Pearce and J. Pečarić, "Inequalities for differentiable mappings with application to special means and quadrature formulæ," *Appl. Math. Lett.*, vol. 13, no. 2, pp. 51–55, 2000, doi: 10.1016/S0893-9659(99)00164-0.
[21] B. T. Polyak, "Existence theorems and convergence of minimizing sequences in extremum problems with restictions," *Soviet Math. Dokl.*, vol. 35, no. 7, pp. 72–75, 1966.
[22] F. Qi, T.-Y. Zhang, and B.-Y. Xi, "Hermite-Hadamard-type integral inequalities for functions whose first derivatives are convex," *Ukrainian Math. J.*, vol. 67, no. 4, pp. 625–640, 2015, reprint of Ukraïn. Mat. Zh. **6**7 (2015), no. 4, 555–567, doi: 10.1007/s11253-015-1103-3.
[23] F. Qi and B.-Y. Xi, "Some Hermite-Hadamard type inequalities for geometrically quasi-convex functions," *Proc. Indian Acad. Sci. Math. Sci.*, vol. 124, no. 3, pp. 333–342, 2014, doi: 10.1007/s12044-014-0182-7.
[24] M. Z. Sarikaya, E. Set, and M. E. Ozdemir, "On new inequalities of Simpson's type for s-convex functions," *Comput. Math. Appl.*, vol. 60, no. 8, pp. 2191–2199, 2010, doi: 10.1016/j.camwa.2010.07.033.
[25] G. Toader, "Some generalizations of the convexity," in *Proceedings of the colloquium on approximation and optimization (Cluj-Napoca, 1985).* Univ. Cluj-Napoca, Cluj-Napoca, 1985, pp. 329–338.
[26] B.-Y. Xi, R.-F. Bai, and F. Qi, "Hermite-Hadamard type inequalities for the m- and (α,m)-geometrically convex functions," *Aequationes Math.*, vol. 84, no. 3, pp. 261–269, 2012, doi: 10.1007/s00010-011-0114-x.
[27] B.-Y. Xi and F. Qi, "Hermite-Hadamard type inequalities for geometrically r-convex functions," *Studia Sci. Math. Hungar.*, vol. 51, no. 4, pp. 530–546, 2014, doi: 10.1556/SScMath.51.2014.4.1294.
[28] B.-Y. Xi and F. Qi, "Inequalities of Hermite-Hadamard type for extended s-convex functions and applications to means," *J. Nonlinear Convex Anal.*, vol. 16, no. 5, pp. 873–890, 2015.
[29] B.-Y. Xi, F. Qi, and T.-Y. Zhang, "Some inequalities of Hermite–Hadamard type for m-harmonic-arithmetically convex functions," *ScienceAsia*, vol. 41, no. 5, pp. 357–361, 2015, doi: 10.2306/scienceasia1513-1874.2015.41.357.
[30] B.-Y. Xi, S.-H. Wang, and F. Qi, "Some inequalities for (h,m)-convex functions," *J. Inequal. Appl.*, pp. 2014:100, 12, 2014, doi: 10.1186/1029-242X-2014-100.

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VARIANTS OF R-WEAKLY COMMUTING MAPPINGS SATISFYING A WEAK CONTRACTION

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Abstract. In this paper, first we prove a common fixed point theorem for pairs of weakly compatible mappings satisfying a generalized ϕ-weak contraction condition that involves cubic terms of metric functions. Secondly, we prove some results using different variants of R-weakly commuting mappings. At the end, we give an application in support of our results.

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1. Introduction and preliminaries

The Banach Contraction Principle is a basic tool to study fixed point theory, which ensures the existence and uniqueness of a fixed point under appropriate conditions. It is most widely applied to understand fixed point results in many branches of mathematics because it requires the structure of complete metric spaces. Generalizations of Banach Contraction Principle gave new direction to researchers in the field of fixed point theory. In 1969, Boyd and Wong [4] replaced the constant k in Banach Contraction Principle by a control function ψ as follows:

Let (X,d) be a complete metric space and $\psi : [0,\infty) \to [0,\infty)$ be an upper semi continuous from the right such that $0 \leq \psi(t) < t$ for all $t > 0$. If $T : X \to X$ satisfies $d(T(x),T(y)) \leq \psi(d(x,y))$ for all $x,y \in X$, then it has a unique fixed point.

In 1994, Pant [13] introduced the notion of R-weakly commuting mappings in metric spaces. In 1997, Pathak *et al.* [14] improved the notion of R-weakly commuting mappings to the notion of R-weakly commuting mappings of type (A_g) and R-weakly commuting mappings of type (A_f). In fact, the main application of R-weakly commuting mappings of type (A_f) or type (A_g) is to study common fixed

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points for noncompatible mappings. In 1998, Jungck and Rhoades [9] introduced the notion of weakly compatible mappings. In 2006, Imdad and Ali [5] introduced R-weakly commuting mappings of type (P) in fuzzy metric spaces. In 2009, Kumar and Garg [12] introduced the concept of R-weakly commuting mappings of type (P) in metric spaces analogue to the notion in fuzzy metric spaces given in [5]. In 1997, Alber and Guerre-Delabriere [2] introduced the concept of a weak contraction and further Rhoades [15] showed that the results of Alber and Gueree-Delabriere are also valid in complete metric spaces. A mapping $T : X \to X$ is said to be a *weak contraction* if for all $x, y \in X$, there exists a function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(t) > 0$ and $\phi(0) = 0$ such that

$$d(Tx,Ty) \leq d(x,y) - \phi(d(x,y)).$$

In 2017, Jain *et al.* [6] introduced a new type of inequality having cubic terms of $d(x,y)$ that extended and generalized the results of Alber and Gueree-Delabriere [2] and others cited in the literature of fixed point theory. See [1, 3, 7, 10, 11] for more information on fixed point theory.

In this paper, we extend and generalize the result of Jain *et al.* [6] for two pairs of R-weakly commuting mappings and its variants satisfying the generalized ϕ-weak contractive condition involving various combinations of the metric functions.

Our improvement in this paper is four-fold:

(i) to relax the continuity requirement of mappings completely;
(ii) to derogate the commutativity requirement of mappings to the point of coincidence;
(iii) to soften the completeness requirement of the space;
(iv) to engage a more general contraction condition in proving our results.

2. Basic properties

In this section, we give some basic definitions and results that are useful for proving our main results.

Definition 1 ([8])**.** Two self-mappings f and g of a metric space (X,d) are said to be *commuting* if $fgx = gfx$ for all $x \in X$.

The notion of weak commutativity as an improvement over the notion of commutativity was introduced by Sessa [16] in 1982 as a sharpener tool to obtain fixed point.

Definition 2 ([16])**.** Two self-mappings f and g of a metric space (X,d) are said to be *weakly commuting* if $d(fgx, gfx) \leq d(gx, fx)$ for all $x \in X$.

Remark 1. Commutative mappings must be weak commutative mappings, but the converse is not true.

Definition 3 ([9])**.** Two self-mappings f and g of a metric space (X,d) are called *weakly compatible* if they commute at their coincidence point.

Definition 4 ([13])**.** Two self-mappings f and g of a metric space (X,d) are said to be *R-weakly commuting* if there exists some $R \geq 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$ for all $x \in X$.

Remark 2. Notice that weak commutativity of a pair of self-mappings implies R-weak commutativity and the converse is true only when $R \leq 1$.

Example 1. Let $X = [1, \infty)$ be endowed with the usual metric. Define $f, g : X \to X$ by $f(x) = 2x - 1$ and $g(x) = x^2$ for all $x \in X$. Then $d(fgx, gfx) = 2d(fx, gx)$. Thus f and g are R-weakly commuting ($R = 2$) but are not weakly commuting.

Definition 5 ([14])**.** Two self-mappings f and g of a metric space (X,d) are said to be *R-weakly commuting of type* (A_f) if there exists a positive real number R such that $d(fgx, ggx) \leq Rd(fx, gx)$ for all $x \in X$.

Definition 6 ([14])**.** Two self-mappings f and g of a metric space (X,d) are said to be *R-weakly commuting of type* (A_g) if there exists a positive real number R such that $d(gfx, ffx) \leq Rd(fx, gx)$ for all $x \in X$.

It may be observed that Definition 6 can be obtained from Definition 5 by interchanging the role of f and g. Further, R-weakly commuting pair of self-mappings is independent of R-weakly commuting of type (A_f) or type (A_g). In Example 1, we note that $d(fgx, ggx) > Rd(fx, gx)$ for all $x > 1$ and some $R > 0$. Thus f and g are R-weakly commuting but not R-weakly commuting of type (A_f).

Definition 7 ([5, 12])**.** Two self-mappings f and g of a metric space (X,d) are said to be *R-weakly commuting mapping of type* (P) if there exists some $R >0$ such that $d(ffx, ggx) \leq Rd(fx, gx)$ for all $x \in X$.

Remark 3. If f and g are R-weakly commuting or R-weakly commuting (A_f) or R-weakly commuting of type (A_g) or R-weakly commuting (P) and if z is a coincidence point, i.e., $fz = gz$, then we get $ffz = fgz = gfz = ggz$. Thus at a coincidence point, all the analogous notions of R-weak commutativity including R-weak commutativity are equivalent to each other and imply their commutativity.

3. Main results

Let S, T, A and B be four self-mappings of a metric space (X,d) satisfying the following conditions:

(C1) $S(X) \subset B(X), \quad T(X) \subset A(X)$;

(C2) $(1 + pd(Ax, By))d(Sx, Ty)^2$

$$\leq p \cdot \max\{\frac{1}{2}(d(Ax, Sx)^2 d(By, Ty) + d(Ax, Sx)d(By, Ty)^2),$$
$$d(Ax, Sx)d(Ax, Ty)d(By, Sx), d(Ax, Ty)d(By, Sx)d(By, Ty)\}$$
$$+ m(Ax, By) - \phi(m(Ax, By))$$

for all $x, y \in X$, where

$$m(Ax,By) = \max\{d(Ax,By)^2, d(Ax,Sx)d(By,Ty), d(Ax,Ty)d(By,Sx), \frac{1}{2}[d(Ax,Sx)d(Ax,Ty)+d(By,Sx)d(By,Ty)]\},$$

$p \geq 0$ is a real number and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) > t$ for all $t > 0$.

From (C1), for any arbitrary point $x_0 \in X$, we can find an x_1 such that $S(x_0) = B(x_1) = y_0$ and for this x_1 one can find an $x_2 \in X$ such that $T(x_1) = A(x_2) = y_1$. Continuing in this way one can construct a sequence $\{y_n\}$ such that

$$y_{2n} = S(x_{2n}) = B(x_{2n+1}), \quad y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2}) \tag{3.1}$$

for each $n \geq 0$.

Lemma 1 ([6]). *Let S, T, A and B be four self-mappings of a metric space (X,d) satisfying the conditions (C1) and (C2). Then the sequence $\{y_n\}$ defined by (3.1) is a Cauchy sequence in X.*

For the convenience of the reader, we give the following proof of Lemma 1.

Proof. For brevity, we write $\alpha_{2n} = d(y_{2n}, y_{2n+1})$.
First, we prove that $\{\alpha_{2n}\}$ is a nonincreasing sequence and converges to zero.
Case I: Suppose that n is even. Taking $x = x_{2n}$ and $y = x_{2n+1}$ in (C2), we get

$$\begin{aligned}
&[1+pd(Ax_{2n},Bx_{2n+1})]d(Sx_{2n},Tx_{2n+1})^2 \\
&\leq p\cdot\max\{\frac{1}{2}(d(Ax_{2n},Sx_{2n})^2 d(Bx_{2n+1},Tx_{2n+1}) + d(Ax_{2n},Sx_{2n})d(Bx_{2n+1},Tx_{2n+1})^2), \\
&\qquad d(Ax_{2n},Sx_{2n})d(Ax_{2n},Tx_{2n+1})d(Bx_{2n+1},Sx_{2n}), \\
&\qquad d(Ax_{2n},Tx_{2n+1})d(Bx_{2n+1},Sx_{2n})d(Bx_{2n+1},Tx_{2n+1})\} \\
&\quad + m(Ax_{2n},Bx_{2n+1}) - \phi(m(Ax_{2n},Bx_{2n+1})),
\end{aligned}$$

where

$$\begin{aligned}
&m(Ax_{2n},Bx_{2n+1}) \\
&= \max\{d(Ax_{2n},Bx_{2n+1})^2, d(Ax_{2n},Sx_{2n})d(Bx_{2n+1},Tx_{2n+1}), d(Ax_{2n},Tx_{2n+1})d(Bx_{2n+1},Sx_{2n}), \\
&\qquad \frac{1}{2}(d(Ax_{2n},Sx_{2n})d(Ax_{2n},Tx_{2n+1}) + d(Bx_{2n+1},Sx_{2n})d(Bx_{2n+1},Tx_{2n+1}))\}.
\end{aligned}$$

Using $\alpha_{2n} = d(y_{2n}, y_{2n+1})$ in (3.1), we have

$$\begin{aligned}
&[1+p\alpha_{2n-1}]\alpha_{2n}^2 \\
&\leq p\max\{\frac{1}{2}[\alpha_{2n-1}^2\alpha_{2n} + \alpha_{2n-1}\alpha_{2n}^2], 0, 0)\} + m(y_{2n-1},y_{2n}) - \phi(m(y_{2n-1},y_{2n})),
\end{aligned} \tag{3.2}$$

where

$$m(y_{2n-1},y_{2n}) = \max\{\alpha_{2n-1}^2, \alpha_{2n-1}\alpha_{2n}, 0, \frac{1}{2}[\alpha_{2n-1}d(y_{2n-1},y_{2n+1})+0]\}.$$

By the triangular inequality, we get

$$d(y_{2n-1},y_{2n+1}) \le d(y_{2n-1},y_{2n}) + d(y_{2n},y_{2n+1}) = \alpha_{2n-1}+\alpha_{2n},$$

$$m(y_{2n-1},y_{2n}) \le \max\{\alpha_{2n-1}^2, \alpha_{2n-1}\alpha_{2n}, 0, \frac{1}{2}[\alpha_{2n-1}(\alpha_{2n-1}+\alpha_{2n}),0]\}.$$

If $\alpha_{2n-1} < \alpha_{2n}$, then (3.2) reduces to $p\alpha_{2n}^2 \le p\alpha_{2n}^2 - \phi(\alpha_{2n}^2)$, which is a contradiction. Thus $\alpha_{2n} \le \alpha_{2n-1}$.

In a similar way, if n is odd, then we can obtain $\alpha_{2n+1} \le \alpha_{2n}$. It follows that the sequence $\{\alpha_{2n}\}$ is decreasing.

Let $\lim_{n\to\infty}\alpha_{2n} = r$ for some $r \ge 0$. Then from the inequality (C2), we have

$$\begin{aligned}
&[1+pd(Ax_{2n},Bx_{2n+1})]d(Sx_{2n},Tx_{2n+1})^2\\
&\le p\cdot\max\{\frac{1}{2}(d(Ax_{2n},Sx_{2n})^2d(Bx_{2n+1},Tx_{2n+1}) + d(Ax_{2n},Sx_{2n})d(Bx_{2n+1},Tx_{2n+1})^2),\\
&\qquad d(Ax_{2n},Sx_{2n})d(Ax_{2n},Tx_{2n+1})d(Bx_{2n+1},Sx_{2n}),\\
&\qquad d(Ax_{2n},Tx_{2n+1})d(Bx_{2n+1},Sx_{2n})d(Bx_{2n+1},Tx_{2n+1})\}\\
&\quad + m(Ax_{2n},Bx_{2n+1}) - \phi(m(Ax_{2n},Bx_{2n+1})),
\end{aligned}$$

where

$$\begin{aligned}
&m(Ax_{2n},Bx_{2n+1})\\
&= \max\{d(Ax_{2n},Bx_{2n+1})^2, d(Ax_{2n},Sx_{2n})d(Bx_{2n+1},Tx_{2n+1}), d(Ax_{2n},Tx_{2n+1})d(Bx_{2n+1},Sx_{2n}),\\
&\qquad \frac{1}{2}(d(Ax_{2n},Sx_{2n})d(Ax_{2n},Tx_{2n+1}) + d(Bx_{2n+1},Sx_{2n})d(Bx_{2n+1},Tx_{2n+1}))\}.
\end{aligned}$$

Now using (3.2), the property of ϕ and passing to the limit as $n\to\infty$, we get

$$[1+pr]r^2 \le pr^3 + r^2 - \phi(r^2).$$

So $\phi(r^2) \le 0$. Since r is positive, by the property of ϕ, we get $r = 0$. Therefore, we conclude that

$$\lim_{n\to\infty}\alpha_{2n} = \lim_{n\to\infty} d(y_{2n},y_{2n-1}) = r = 0. \tag{3.3}$$

Now we show that $\{y_n\}$ is a Cauchy sequence. Assume that $\{y_n\}$ is not a Cauchy sequence. For given $\varepsilon > 0$, we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$

$$d(y_{m(k)},y_{n(k)}) \ge \varepsilon, \qquad d(y_{m(k)},y_{n(k)-1}) < \varepsilon. \tag{3.4}$$

Thus $\varepsilon \le d(y_{m(k)},y_{n(k)}) \le d(y_{m(k)},y_{n(k)-1}) + d(y_{n(k)-1},y_{n(k)})$. Taking the limit as $k\to\infty$, we get $\lim_{k\to\infty} d(y_{m(k)},y_{n(k)}) = \varepsilon$.

Now using the triangular inequality, we have

$$|d(y_{n(k)},y_{m(k)+1})-d(y_{m(k)},y_{n(k)})|\leq d(y_{m(k)},y_{m(k)+1}).$$

Taking the limit as $k\to\infty$ and using (3.3) and (3.4), we have

$$\lim_{k\to\infty} d(y_{n(k)},y_{m(k)+1})=\varepsilon.$$

Again from the triangular inequality, we have

$$|d(y_{m(k)},y_{n(k)+1})-d(y_{m(k)},y_{n(k)})|\leq d(y_{n(k)},y_{n(k)+1}).$$

Taking the limit as $k\to\infty$ and using (3.3) and (3.4), we have

$$\lim_{k\to\infty} d(y_{m(k)},y_{n(k)+1})=\varepsilon.$$

Similarly, we have

$$|d(y_{m(k)+1},y_{n(k)+1})-d(y_{m(k)},y_{n(k)})|\leq d(y_{m(k)},y_{m(k)+1})+d(y_{n(k)},y_{n(k)+1}).$$

Taking the limit as $k\to\infty$ in the above inequality and using (3.3) and (3.4), we have

$$\lim_{k\to\infty} d(y_{n(k)+1},y_{m(k)+1})=\varepsilon.$$

Putting $x=x_{m(k)}$ and $y=x_{n(k)}$ in (C2), we get

$$\begin{aligned}
&[1+pd(Ax_{m(k)},Bx_{n(k)})]d(Sx_{m(k)},Tx_{n(k)})^2\\
&\leq p\cdot\max\{\frac{1}{2}(d(Ax_{m(k)},Sx_{m(k)})^2d(Bx_{n(k)},Tx_{n(k)})+d(Ax_{m(k)},Sx_{m(k)})d(Bx_{n(k)},Tx_{n(k)})^2),\\
&\qquad d(Ax_{m(k)},Sx_{m(k)})d(Ax_{m(k)},Tx_{n(k)})d(Bx_{n(k)},Sx_{m(k)}),\\
&\qquad d(Ax_{m(k)},Tx_{n(k)})d(Bx_{n(k)},Sx_{m(k)})d(Bx_{n(k)},Tx_{n(k)})\}\\
&\quad +m(Ax_{m(k)},Bx_{n(k)})-\phi(m(Ax_{m(k)},Bx_{n(k)})),
\end{aligned}$$

where

$$\begin{aligned}
m(Ax_{m(k)},Bx_{n(k)})=\max\{&d(Ax_{m(k)},Bx_{n(k)})^2,d(Ax_{m(k)},Sx_{m(k)})d(Bx_{n(k)},Tx_{n(k)}),\\
&d(Ax_{m(k)},Tx_{n(k)})d(Bx_{n(k)},Sx_{m(k)}),\\
&\frac{1}{2}(d(Ax_{m(k)},Sx_{m(k)})d(Ax_{m(k)},Tx_{n(k)})\\
&+d(Bx_{n(k)},Sx_{m(k)})d(Bx_{n(k)},Tx_{n(k)}))\}.
\end{aligned}$$

Using (3.1), we obtain

$$\begin{aligned}
&[1+pd(y_{m(k)-1},y_{n(k)-1})]d(y_{m(k)},y_{n(k)})^2\\
&\leq p\cdot\max\{\frac{1}{2}(d(y_{m(k)-1},y_{m(k)})^2d(y_{n(k)-1},y_{n(k)})+d(y_{m(k)-1},y_{m(k)})d(y_{n(k)-1},y_{n(k)})^2),\\
&\qquad d(y_{m(k)-1},y_{m(k)})d(y_{m(k)-1},y_{n(k)})d(y_{n(k)-1},y_{m(k)}),\\
&\qquad d(y_{m(k)-1},y_{n(k)})d(y_{n(k)-1},y_{m(k)})d(y_{n(k)-1},y_{n(k)})\}\\
&\quad +m(Ax_{m(k)},Bx_{n(k)})-\phi(m(Ax_{m(k)},Bx_{n(k)})),
\end{aligned}$$

where

$$\begin{aligned} m(Ax_{m(k)}, Bx_{n(k)}) = \max\{& d(y_{m(k)-1}, y_{n(k)-1})^2, d(y_{m(k)-1}, y_{m(k)})d(y_{n(k)-1}, y_{n(k)}), \\ & d(y_{m(k)-1}, y_{n(k)})d(y_{n(k)-1}, y_{m(k)}), \\ & \frac{1}{2}(d(y_{m(k)-1}, y_{m(k)})d(y_{m(k)-1}, y_{n(k)}) \\ & + d(y_{n(k)-1}, y_{m(k)})d(y_{n(k)-1}, y_{n(k)}))\}. \end{aligned}$$

Taking the limit as $k \to \infty$, we get

$$[1 + p\varepsilon]\varepsilon^2 \le p\max\{\frac{1}{2}[0+0], 0, 0\} + \varepsilon^2 - \phi(\varepsilon^2) = \varepsilon^2 - \phi(\varepsilon^2),$$

which is a contradiction. Thus $\{y_n\}$ is a Cauchy sequence in X. □

Now we prove our main results as follows:

Theorem 1. *Let S, T, A and B be four self-mappings of a metric space (X, d) satisfying the conditions (C1) and (C2) and one of the subspaces AX, BX, SX and TX be complete. Then*

(i) *A and S have a point of coincidence;*
(ii) *B and T have a point of coincidence.*

Moreover, if the pairs (A, S) and (B, T) are weakly compatible, then S, T, A and B have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. From (C1), we can find an x_1 such that $S(x_0) = B(x_1) = y_0$ and for this x_1 one can find an $x_2 \in X$ such that $T(x_1) = A(x_2) = y_1$. Continuing in this way, one can construct a sequence such that

$$y_{2n} = S(x_{2n}) = B(x_{2n+1}), \quad y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2})$$

for all $n \ge 0$ and $\{y_n\}$ is a Cauchy sequence in X.

Now suppose that AX is a complete subspace of X. Then there exists $z \in X$ such that

$$y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2}) \to z$$

as $n \to \infty$. Consequently, we can find $w \in X$ such that $Aw = z$. Further, a Cauchy sequence $\{y_n\}$ has a convergent subsequence $\{y_{2n+1}\}$ and so the sequence $\{y_n\}$ converges and hence a subsequence $\{y_{2n}\}$ also converges. Thus we have $y_{2n} = S(x_{2n}) = B(x_{2n+1}) \to z$ as $n \to \infty$. Letting $x = w$ and $y = z$ in (C2), we get

$$\begin{aligned} & [1 + pd(Aw, Bz)]d(Sw, Tz)^2 \\ & \le p \cdot \max\{\frac{1}{2}[d(Aw, Sw)^2 d(Bz, Tz) + d(Aw, Sw)d(Bz, Tz)^2], \\ & \qquad d(Aw, Sw)d(Aw, Tz)d(Bz, Sw), d(Aw, Tz)d(Bz, Sw)d(Bz, Tz)\} \\ & \quad + m(Aw, Bz) - \phi(m(Aw, Bz)), \end{aligned}$$

where

$$m(Aw,Bz) = \max\{d(Aw,Bz)^2, d(Aw,Sw)d(Bz,Tz), d(Aw,Tz)d(Bz,Sw),$$
$$\frac{1}{2}[d(Aw,Sw)d(Aw,Tz) + d(Bz,Sw)d(Bz,Tz)]\}.$$

Since

$$m(Aw,Bz) = \max\{d(z,z)^2, d(z,Sw)d(Tz,Tz), d(z,z)d(z,Sw),$$
$$\frac{1}{2}[d(z,Sw)d(z,z) + d(z,Sw)d(Tz,Tz)]\} = 0,$$

$$[1+pd(z,z)]d(Sw,z)^2 \leq p\cdot\max\{\frac{1}{2}[d(z,Sw)^2d(z,z) + d(z,Sw)d(z,z)^2],$$
$$d(z,Sw)d(z,z)d(z,Sw), d(z,z)d(z,Sw)d(z,z)\} + 0 - \phi(0).$$

This implies that $Sw = z$ and hence $Sw = Aw = z$. Therefore, w is a coincidence point of A and S. Since $z = Sw \in SX \subset BX$, there exists $v \in X$ such that $z = Bv$.

Next, we claim that $Tv = z$. Now letting $x = x_{2n}$ and $y = v$ in $(C2)$, we get

$$[1+pd(Ax_{2n},Bv)]d(Sx_2n,Tv)^2$$
$$\leq p\cdot\max\{\frac{1}{2}[d(Ax_{2n},Sx_{2n})^2d(Bv,Tv) + d(Ax_{2n},Sx_{2n})d(Bv,Tv)^2],$$
$$d(Ax_{2n},Sx_{2n})d(Ax_{2n},Tv)d(Bz,Sx_{2n}), d(Ax_{2n},Tv)d(Bv,Sx_{2n})d(Bv,Tv)\}$$
$$+ m(Ax_{2n},Bv) - \phi(m(Ax_{2n},Bv)),$$

where

$$m(Ax_{2n},Bv) = \max\{d(Ax_{2n},Bv)^2, d(Ax_{2n},Sx_{2n})d(Bv,Tv), d(Ax_{2n},Tv)d(Bv,Sx_{2n}),$$
$$\frac{1}{2}[d(Ax_{2n},Sx_{2n})d(Ax_{2n},Tv) + d(Bv,Sx_{2n})d(Bv,Tv)]\} = 0.$$

Therefore,

$$[1+pd(z,z)]d(z,Tv)^2 \leq p\cdot\max\{\frac{1}{2}[0+0],0,0\} + 0 - \phi(0).$$

This gives $z = Tv$ and hence $z = Tv = Bv$. Therefore, v is a coincidence point of B and T. Since the pairs (A,S) and (B,T) are weakly compatible, we have

$$Sz = S(Aw) = A(Sw) = Az, \qquad Tz = T(Bv) = B(Tv) = Bz.$$

Now, we show that $Sz = z$. For this, letting $x = z$ and $y = x_{2n+1}$ in (C2), we get

$$[1+pd(Az,Bx_{2n+1})]d(Sz,Tx_{2n+1})^2$$
$$\leq p\cdot\max\{\frac{1}{2}[d(Az,Sz)^2d(z,z) + d(Az,Sz)d(z,z)^2],$$
$$d(Az,Sz)d(Az,z)d(z,Sz), d(Az,z)d(z,Sz)d(z,z)\} + m(Az,z) - \phi(m(Az,z)),$$

where

$$m(Az,z) = \max\{d(Az,z)^2, d(Az,Sz)d(z,z), d(Az,z)d(z,Sz), \frac{1}{2}[d(Az,Sz)d(Az,z) + d(z,Sz)d(z,z)]\} = d(Sz,z)^2.$$

Therefore, we get

$$[1 + pd(Sz,z)]d(Sz,z)^2 \le p \cdot \max\{\frac{1}{2}[0+0], 0, 0\} + d(Sz,z)^2 - \phi(d(Sz,z)^2).$$

Thus we get $d(Sz,z)^2 = 0$. This implies that $Sz = z$. Hence $Sz = Az = z$.

Next, we claim that $Tz = z$. Now letting $x = x_{2n}$ and $y = z$ in (C2), we get

$$\begin{aligned}
&[1 + pd(Ax_{2n}, Bz)]d(Sx_2n, Tz)^2 \\
&\le p \cdot \max\{\frac{1}{2}[d(Ax_{2n}, Sx_{2n})^2 d(Bz,Tz) + d(Ax_{2n}, Sx_{2n})d(Bz,Tz)^2], \\
&\qquad d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tz)d(Bz, Sx_{2n}), d(Ax_{2n}, Tz)d(Bz, Sx_{2n})d(Bz, Tz)\} \\
&\quad + m(Ax_{2n}, Bz) - \phi(m(Ax_{2n}, Bz)),
\end{aligned}$$

where

$$m(Ax_{2n}, Bz) = \max\{d(Ax_{2n}, Bz)^2, d(Ax_{2n}, Sx_{2n})d(Bz,Tz), d(Ax_{2n}, Tz)d(Bz, Sx_{2n}), \frac{1}{2}[d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tz) + d(Bz, Sx_{2n})d(Bz,Tz)]\} = d(z,Tz)^2.$$

Hence we get

$$[1 + pd(z,Tz)]d(z,Tz)^2 \le p \cdot \max\{\frac{1}{2}[0+0], 0, 0\} + d(z,Tz)^2 - \phi(d(z,Tz)^2).$$

This gives $z = Tz$ and hence $z = Tz = Bz$. Therefore, z is a common fixed point of A, B, S and T.

Similarly, we can complete the proofs for the cases that BX or SX or TX is complete.

Now, we prove the uniqueness. Suppose z and w are two common fixed points of S, T, A and B with $z \ne w$. Letting $x = z$ and $y = w$ in (3.2), we get

$$[1 + pd(Az,Bw)]d(Sz,Tw)^2 \le p \cdot \max\{0,0,0\} + m(Az,Bw) - \phi(m(Az,Bw)),$$

$$[1 + pd(Az,Bw)]d(Sz,Tw)^2 \le p \cdot \max\{0,0,0\} + d(Sz,Tw)^2 - \phi(d(Sz,Tw)^2),$$

which implies that $d(z,w)^2 = 0$. Hence $z = w$. This completes the proof. □

Theorem 2. *If a 'weakly compatible' property in the statement of Theorem 1 is replaced by one (retaining the rest of hypotheses) of the following:*

(i) *R-weakly commuting property;*
(ii) *R-weakly commuting mappings of type (A_f);*
(iii) *R-weakly commuting mappings of type (A_g);*
(iv) *R-weakly commuting mappings of type (P);*

(v) *weakly commuting,*

then Theorem 1 remains true.

Proof. Since all the conditions of Theorem 1 are satisfied, the existence of coincidence points for both the pairs is insured. Let w be an arbitrary point of coincidence for the pair (A,S). Then using R-weak commutativity, one gets

$$d(ASw,SAw) \le Rd(Aw,Sw),$$

which implies $ASw = SAw$. Thus the pair (A,S) is coincidentally commuting. Similarly, (B,T) commutes at all of its coincidence points. Now applying Theorem 1, one concludes that S,T,A and B have a unique common fixed point.

If (A,S) are R-weakly commuting mappings of type (A_f), then

$$d(ASw,SSw) \le Rd(Aw,Sw),$$

which implies that $ASw = SSw$. Since

$$d(ASw,SAw) \le d(ASw,SSw) + d(SSw,SAw) = 0+0 = 0,$$

which implies that $ASw = SAw$.

Similarly, if (A,S) are R-weakly commuting mappings of type (A_g) or of type (P) or weakly commuting, then (A,S) also commute at their points of coincidence.

Similarly, one can show that the pair (B,T) is also coincidentally commuting. Now in view of Theorem 1, for all four cases, A,B,S and T have a unique common fixed point. This completes the proof. □

As an application of Theorem 1, we prove a common fixed point theorem for four finite families of mappings.

Theorem 3. *Let $\{A_1,A_2,\cdots,A_m\}$, $\{B_1,B_2,\cdots,B_n\}$, $\{S_1,S_2,\cdots,S_p\}$ and $\{T_1,T_2,\cdots,T_q\}$ be four finite families of self-mappings of a metric space (X,d) such that $A = A_1A_2\cdots A_m$, $B = B_1B_2\cdots B_n$, $S = S_1S_2\cdots S_p$ and $T = T_1T_2\cdots T_q$ satisfy the conditions $(C1)$, $(C2)$ and one of the mappings $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a complete subspace of X. Then*

(i) *A and S have a point of coincidence,*

(ii) *B and T have a point of coincidence.*

Moreover, if $A_iA_j = A_jA_i$, $B_kB_l = B_lB_k$, $S_rS_s = S_sS_r$, $T_tT_u = T_uT_t$, $A_iS_r = S_rA_i$ and $B_kT_t = T_tB_k$ for all $i,j \in I_1 = \{1,2,\cdots,m\}$, $k,l \in I_2 = \{1,2,\cdots,n\}$, $r,s \in I_3 = \{1,2,\cdots,p\}$ and $t,u \in I_4 = \{1,2,\cdots,q\}$, then (for all $i \in I_1$, $k \in I_2$, $r \in I_3$ and $t \in I_4$) A_i,S_r,B_k and T_t have a common fixed point.

Proof. The conclusions (i) and (ii) are immediate since A,S,B and T satisfy all the conditions of Theorem 1. Now appealing to component wise commutativity of various pairs, one can immediately prove that $AS = SA$ and $BT = TB$ and hence, obviously, both pairs (A,S) and (B,T) are weakly compatible. Note that all the conditions of Theorem 1 (for mappings A,S,B and T) are satisfied to ensure the existence of a unique common fixed point, say, z. Now one needs to show that z remains the fixed point of all the component mappings. For this, consider

$$\begin{aligned}S(S_rz) &= ((S_1S_2\cdots S_p)S_r)z = (S_1S_2\cdots S_{p-1})((S_pS_r)z)\\ &= (S_1S_2\cdots S_{p-1})(S_rS_pz) = (S_1S_2\cdots S_{p-2})(S_{p-1}S_r(S_pz))\\ &= (S_1S_2\cdots S_{p-2})(S_rS_{p-1}(S_pz)) = \cdots\\ &= S_1S_r(S_2S_3S_4\cdots S_pz) = S_rS_1(S_2S_3\cdots S_pz) = S_r(Sz) = S_rz.\end{aligned}$$

Similarly, one can show that

$$\begin{aligned}A(S_rz) &= S_r(Az) = S_rz, A(A_iz) = A_i(Az) = A_iz,\\ S(A_iz) &= A_i(Sz) = A_iz, B(B_kz) = B_k(Bz) = B_kz,\\ B(T_tz) &= T_t(Bz) = T_tz, T(T_tz) = T_t(Tz) = T_tz,\\ T(B_kz) &= B_k(Tz) = B_kz,\end{aligned}$$

which implies that (for all i, r, k and t) A_iz and S_rz are other fixed points of the pair (A,S), whereas B_kz and T_tz are other fixed points of the pair (B,T).

Now appealing to the uniqueness of common fixed points of both pairs, separately, we get

$$z = A_iz = S_rz = B_kz = T_tz,$$

which shows that z is a common fixed point of A_i, S_r, B_k and T_t for all i, r, k and t. □

Setting $A = A_1 = A_2 = \cdots = A_m$, $B = B_1 = B_2 = \cdots = B_n$, $S = S_1 = S_2 = \cdots = S_p$ and $T = T_1 = T_2 = \cdots = T_q$, one can deduce the following result for certain iterates of mappings.

Corollary 1. *Let A, B, S and T be four self-mappings of a metric space (X,d) such that A_m, B_n, S_p and T_q satisfy the conditions $(C1)$ and $(C2)$. If one of the mappings $A_m(X), B_n(X), S_p(X)$ and $T_q(X)$ is a complete subspace of X, then A, B, S and T have a unique common fixed point provided (A,S) and (B,T) commute.*

Theorem 4. *Let S, T, A, B be four mappings of a complete metric space (X,d) into itself satisfying all the conditions of Theorem 1 except $(C2)$, where $(C2)$ is replaced by $(C3)$*

$$\int_0^{M(x,y)} \gamma(t)dt \le p\int_0^{N(x,y)} \gamma(t)dt. \tag{C3}$$

Here

$$M(x,y) = (1 + pd(Ax,By))d(Sx,Ty)^2,$$

$$\begin{aligned}N(x,y) = \max\{&\frac{1}{2}(d(Ax,Sx)^2d(By,Ty) + d(Ax,Sx)d(By,Ty)^2),\\ &d(Ax,Sx)d(Ax,Ty)d(By,Sx), d(Ax,Ty)d(By,Sx)d(By,Ty)\}\\ &+ m(Ax,By) - \phi(m(Ax,By)),\end{aligned}$$

$p \geq 0$ is a real number, $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) > t$ for all $t > 0$ and $\gamma : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable function which is summable on each compact subset of $[0,\infty)$ such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \gamma(t)dt > 0$. Then S,T,A,B have a unique common fixed point.

Proof. Letting $\gamma(t) = c$ in Theorem 1, we obtain the required results. □

Example 2. Let $X = [2,20]$ and d be a usual metric. Define self-mappings A,B,S and T on X by

$$Ax = \begin{cases} 12 & \text{if } 2 < x \leq 5 \\ x-3 & \text{if } x > 5 \\ 2 & \text{if } x = 2, \end{cases} \qquad Bx = \begin{cases} 2 & \text{if } x = 2 \\ 6 & \text{if } x > 2, \end{cases}$$

$$Sx = \begin{cases} 6 & \text{if } 2 < x \leq 5 \\ x & \text{if } x = 2 \\ 2 & \text{if } x > 5, \end{cases} \qquad Tx = \begin{cases} x & \text{if } x = 2 \\ 3 & \text{if } x > 2. \end{cases}$$

Let us consider a sequence $\{x_n\}$ with $x_n = 2$. It is easy to verify that all the conditions of Theorem 1 are satisfied. In fact, 2 is the unique common fixed point of S,T,A and B.

Conclusion

In this paper, we have proved a common fixed point theorem for pairs of weakly compatible mappings satisfying a generalized ϕ-weak contraction condition that involves cubic terms of metric functions. Next, we have proved some results using different variants of R-weakly commuting mappings. Finally, we have given an application in support of our results.

Competing interests

The authors declare that they have no competing interests.

References

[1] O. Acar, "A fixed point theorem for multivalued almost F-δ-contraction," *Results Math.*, vol. 2, no. 3, pp. 1545–1553, 2017, doi: 10.1007/s00025-017-0705-5.

[2] Y. I. Alber and S. Guerre-Delabriere, "Principle of weakly contractive maps in Hilbert spaces," *New Results Operator Theory Adv. Appl.* , vol. 98, no. 1, pp. 7–22, 1997, doi: 10.1137/050641867.

[3] B. Boonsri and S. Saejung, "Fixed point theorems for contractions of Reich type on a metric space with a graph," *J. Fixed Point Theory Appl.*, vol. 20, no. 2, pp. 1–17, 2018, doi: 10.1007/s11784-018-0565-y.

[4] D. W. Boyd and J. S. W. Wong, "On nonlinear contractions," *Proc. Am. Math. Soc.*, vol. 20, no. 2, pp. 458–464, 1969.

[5] M. Imdad and J. Ali, "Some common fixed point theorems in fuzzy metric spaces," *Math. Commun.*, vol. 11, no. 2, pp. 153–163, 2006.

[6] D. Jain, S. Kumar, S. Kang, and C. Jung, "Weak contraction condition for compatible mappings involving cubic terms of the metric function," *Far Esat J. Math. Sci.*, vol. 103, no. 4, pp. 799–818, 2018, doi: 10.17654/MS103040799.

[7] G. Jiang and S. Kang, "Common fixed point theorems with an application in dynamic programming," *Miskolc Math. Notes*, vol. 19, no. 2, pp. 923–930, 2018, doi: 10.18514/MMN.2018.2312.

[8] G. Jungck, "Commuting mappings and fixed points," *Am. Math. Monthly*, vol. 83, pp. 261–263, 1976.

[9] G. Jungck and B. E. Rhoades, "Fixed points for set valued functions without continuity," *Indian J. Pure Appl. Math.*, vol. 29, no. 3, pp. 227–238, 1998.

[10] S. Kajántó and A. Lukács, "On the conditions of fixed-point theorems concerning F-contractions," *Results Math.*, vol. 73, no. 2, pp. 1–10, 2018, doi: 10.1007/s00025-018-0846-1.

[11] K. Kim, "Convergence and stability of generalized φ-weak contraction mapping in $CAT(0)$ spaces," *Open Math.*, vol. 15, no. 1, pp. 1063–1074, 2017, doi: 10.1515/math-2017-0089.

[12] S. Kumar and S. K. Garg, "Expansion mapping theorems in metric spaces," *Int. J. Contemp. Math. Sci.*, vol. 4, no. 36, pp. 1749–1758, 2009.

[13] R. P. Pant, "Common fixed point for non commuting mappings," *J. Math. Anal. Appl.*, vol. 188, no. 2, pp. 436–440, 1994.

[14] H. K. Pathak, Y. Cho, and S. Kang, "Remarks on R-weakly commuting mappings and common fixed point theorems," *Bull. Korean Math. Soc.*, vol. 34, no. 2, pp. 247–257, 1997.

[15] B. E. Rhoades, "Some theorems on weakly contractive maps. Proceedings of the Third World Congress of Nonlinear Analysis, Part 4 (Catania, 2000)," *Nolinear Anal.*, vol. 47, no. 4, pp. 2683–2693, 2001, doi: /10.1016/S0362-546X(01)00388-1.

[16] S. Sessa, "On a weak commutativity conditions of mappings in fixed point consideration," *Publ. Inst. Math. (Beograd)*, vol. 32, pp. 146–153, 1982.

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STATISTICAL CONVERGENCE OF MARTINGALE DIFFERENCE SEQUENCE VIA DEFERRED WEIGHTED MEAN AND KOROVKIN-TYPE THEOREMS

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Abstract. In the present paper, we introduce and study the concepts of statistical convergence and statistical summability for martingale difference sequences of random variables via deferred weighted summability mean. We then establish an inclusion theorem concerning the relation between these two beautiful concepts. Also, based upon our proposed notions, we state and prove new Korovkin-type approximation theorems with algebraic test functions for a martingale difference sequence over a Banach space and demonstrate that our theorems effectively extend and improves most (if not all) of the previously existing results (in statistical and classical versions). Finally, we present an illustrative example by using the generalized Bernstein polynomial of a martingale difference sequence in order to demonstrate that our established theorems are stronger than its traditional and statistical versions.

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1. INTRODUCTION AND MOTIVATION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability measurable space and suppose that (Y_n) be a difference random variable such that $Y_n = X_n - X_{n-1}$ defined over this space, where (X_n) and X_{n-1} are also random variables belongs to this space. Also, let $\mathcal{F}_n \subseteq \mathcal{F}$ $(n \in \mathbb{N})$ be a monotonically increasing sequence of σ-fields of measurable sets. Now, considering the random variable (Y_n) and the measurable functions $(\mathcal{F}_n)$, we adopt a stochastic sequence $(Y_n, \mathcal{F}_n; n \in \mathbb{N})$.

A given stochastic sequence $(Y_n, \mathcal{F}_n; n \in \mathbb{N})$ is said to be a martingale difference sequence if

(i) $\mathbb{E}|Y_n| < \infty$,
(ii) $\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = 0$ almost surely (a.s.) and
(iii) Y_n is a measurable function of $\mathcal{F}_1$, $\mathcal{F}_2$, $\cdots$, $\mathcal{F}_n$,

where $\mathbb{E}$ is the mathematical expectation.

Next, we discuss about the above properties of martingale difference sequence of random variables.

Suppose (X_n) is a martingale sequence with respect to $\mathcal{F}_n$. Also, let

$$Y_n = X_n - X_{n-1}, \quad n = 2, 3, \cdots.$$

Now,

$$\mathbb{E}|Y_n| \leqq \mathbb{E}|X_n| + \mathbb{E}|X_{n-1}| < \infty.$$

Next,

$$\begin{aligned}\mathbb{E}(Y_{n+1}|\mathcal{F}_n) &= \mathbb{E}(X_{n+1} - X_n|\mathcal{F}_n)\\ &= \mathbb{E}(X_{n+1}|\mathcal{F}_n) - X_n \quad (\because X_n \text{ is a constant on } \mathcal{F}_n)\\ &= 0 \quad (\because \mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n).\end{aligned}$$

Since, (X_n) and (X_{n-1}) are measurable, therefore (Y_n) is measurable.

We now recall the definition for convergence of martingale difference sequences of random variables.

Definition 1. A martingale difference sequence $(Y_n, \mathcal{F}_n; n \in \mathbb{N})$ with $\mathbb{E}|Y_n|$ is bounded and $\text{Prob}(Y_n) = 1$ (that is, with probability 1) is said to be convergent to a martingale $(Y_0, \mathcal{F}_0)$, if

$$\lim_{n\to\infty} (Y_n, \mathcal{F}_n) \longrightarrow (Y_0, \mathcal{F}_0) \qquad (\mathbb{E}|Y_0| < \infty).$$

The notion of statistical convergence has been one of the beautiful aspects of the sequence space theory and such an interesting notion was introduced by Fast [5]. Subsequently, the notion of probability convergence for sequences of random variables was introduced and such a notion is more general than the statistical convergence as well as of the usual convergence. Using both the concepts with different settings, various researchers developed many interesting results in several fields of pure and applied mathematics such as summability theory, Fourier series, approximation theory, probability theory, measure theory and so on, see [2,3,7–9,12,15,18,19] and [23].

Let $\mathfrak{X} \subseteq \mathbb{N}$, and also let $\mathfrak{X}_n = \{j : j \leqq n \quad \text{and} \quad j \in \mathfrak{X}\}$. Then the natural density $d(\mathfrak{X})$ of $\mathfrak{X}$ is defined by

$$d(\mathfrak{X}) = \lim_{n\to\infty} \frac{|\mathfrak{X}_n|}{n} = \chi,$$

where χ is real and a finite number, and $|\mathfrak{X}_n|$ is the cardinality of $\mathfrak{X}_n$.

We now recall the definition of statistical convergence for real sequence.

Definition 2 (see [5])**.** A given sequence (u_n) is statistically convergent to κ if, for each $\varepsilon > 0$,

$$\mathfrak{X}_\varepsilon = \{j : j \in \mathbb{N} \quad \text{and} \quad |u_j - \kappa| \geqq \varepsilon\}$$

has zero natural density. Thus, for each $\varepsilon > 0$, we have

$$d(\mathfrak{X}_\varepsilon) = \lim_{n\to\infty} \frac{|\mathfrak{X}_\varepsilon|}{n} = 0.$$

Here, we write

$$\text{stat} \lim_{n\to\infty} u_n = \kappa.$$

We now introduce the definition of statistical convergence of martingale difference sequence for random variables.

Definition 3. A bounded martingale difference sequence $(Y_n, \mathcal{F}_n; n \in \mathbb{N})$ having probability 1 is said to be statistically convergent to a martingale $(Y_0, \mathcal{F}_0)$ with $\mathbb{E}|Y_0| < \infty$ if, for all $\varepsilon > 0$,

$$\mathfrak{R}_\varepsilon = \{ j : j \leqq n \quad \text{and} \quad |(Y_j, \mathcal{F}_j) - (Y_0, \mathcal{F}_0)| \geqq \varepsilon \}$$

has zero natural density. That is, for every $\varepsilon > 0$, we have

$$d(\mathfrak{R}_\varepsilon) = \lim_{n\to\infty} \frac{|\mathfrak{R}_\varepsilon|}{n} = 0.$$

Here, we write

$$\text{stat}_{\text{MD}} \lim_{n\to\infty} (Y_n, \mathcal{F}_n) = (Y_0, \mathcal{F}_0).$$

Now we present an example illustrating that every martingale difference convergent sequence is statistically convergent, but not conversely.

Example 1. Let $(\mathcal{F}_n; n \in \mathbb{N})$ be a monotonically increasing sequence of 0-mean independent random variables over σ-fields and suppose (X_n) is a sequence of n th partial sum of $(\mathcal{F}_n; n \in \mathbb{N})$ such that $X_n - X_{n-1} = Y_n$. Consider the sequence of random variables (X_n) as

$$X_n = \begin{cases} 1 & (n = m^2; m \in \mathbb{N}) \\ \\ 0 & (\text{otherwise}). \end{cases}$$

It is easy to see that, the martingale difference sequence $(Y_n, \mathcal{F}_n; n \in \mathbb{N})$ is statistically convergent to zero but not simply martingale difference convergent.

Based on our proposed definition, we establish a theorem concerning a relation between ordinary and statistical versions of convergence of martingale difference sequences.

Theorem 1. *If a martingale difference sequence $(Y_n, \mathcal{F}_n; n \in \mathbb{N})$ is convergent to a martingale $(Y_0, \mathcal{F}_0)$ with $\mathbb{E}|Y_0| < \infty$, then it is statistically convergent to the same martingale.*

Proof. Let the martingale difference sequence $(Y_n, \mathcal{F}_n; n \in \mathbb{N})$ be bounded and converges with probability 1, then there exists a martingale $(Y_0, \mathcal{F}_0)$ with $\mathbb{E}|Y_0| < \infty$, that is

$$\lim_{n\to\infty} (Y_n, \mathcal{F}_n) \longrightarrow (Y_0, \mathcal{F}_0).$$

As the given martingale sequence $(Y_n, \mathcal{F}_n; n \in \mathbb{N})$ is bounded with probability 1, then for every $\varepsilon > 0$, we have

$$\frac{1}{n}\{j : j \leqq n \quad \text{and} \quad |(Y_j, \mathcal{F}_j) - (Y_0, \mathcal{F}_0)| \geqq \varepsilon\} \subseteq \lim_{n\to\infty} |(Y_n, \mathcal{F}_n) - (Y_0, \mathcal{F}_0)| < \varepsilon.$$

Consequently, by Definition 3, we obtain

$$\frac{1}{n}\{j : j \leqq n \quad \text{and} \quad |(Y_j, \mathcal{F}_j) - (Y_0, \mathcal{F}_0)| \geqq \varepsilon\} = 0.$$

□

Motivated essentially by the above mentioned investigations, we introduce and study the concepts of statistical convergence and statistical summability for martingale difference sequences of random variables via deferred weighted summability mean. We then establish an inclusion theorem concerning the relation between these two beautiful concepts. Also, based upon our proposed notions, we state and prove new Korovkin-type approximation theorems with algebraic test functions for a martingale difference sequence over a Banach space and demonstrate that our theorems effectively extend and improves most (if not all) of the previously existing results (in statistical and classical versions). Finally, we present an illustrative example by using the generalized Bernstein polynomial of a martingale difference sequence in order to demonstrate that our established theorems are stronger than its traditional and statistical versions.

2. Deferred Weighted Martingale Difference Sequence

Let (a_n) and (b_n) be sequences of non-negative integers such that $a_n < b_n$ and $\lim_{n\to\infty} b_n = +\infty$, and let (p_i) be a sequence of non-negative numbers such that

$$P_n = \sum_{i=a_n+1}^{b_n} p_i.$$

Then the deferred weighted mean for the martingale difference sequence $(Y_n, \mathcal{F}_n; n \in \mathbb{N})$ of random variables is defined by

$$\mathfrak{W}(Y_n, \mathcal{F}_n) = \frac{1}{P_n} \sum_{i=a_n+1}^{b_n} p_i (Y_i, \mathcal{F}_i).$$

It will be interesting to see that, for $p_i = 1$, $\mathfrak{W}(Y_n, \mathcal{F}_n)$ reduces to deferred Cesàro mean $\{\mathcal{D}(X_n, \mathcal{F}_n) : X_n = \sum_{i=1}^{n} Y_i\}$ which has been recently introduced by Srivastava *et al.* [17]. Moreover, recalling another result of Srivastava *et al.* [20] via deferred Nörlund mean $D_a^b(N,p,q)$ for real sequence given by

$$t_n = \frac{1}{\mathcal{R}_n} \sum_{m=a_n+1}^{b_n} p_{b_n-m} q_m x_m,$$

one can also extend the same for the martingale difference sequence.

We now present the definitions of deferred weighted statistical convergence and statistically deferred weighted summability of martingale difference sequences of random variables.

Definition 4. Let (a_n) and (b_n) be sequences of non-negative integers, and let (p_n) be a sequence of non-negative numbers. A bounded martingale difference sequence $(Y_n,\mathcal{F}_n;n\in\mathbb{N})$ of random variables having probability 1 is deferred weighted statistically convergent to a martingale $(Y_0,\mathcal{F}_0)$ with $\mathbb{E}|Y_0|<\infty$ if, for all $\varepsilon>0$,

$$\mathfrak{Y}_\varepsilon = \{j : j \leqq P_n \quad \text{and} \quad p_j|(Y_j,\mathcal{F}_j)-(Y_0,\mathcal{F}_0)| \geqq \varepsilon\}$$

has zero natural density. That is, for every $\varepsilon>0$, we have

$$\lim_{n\to\infty} \frac{1}{P_n} |\{j : j \leqq P_n \quad \text{and} \quad p_j|(Y_j,\mathcal{F}_j)-(Y_0,\mathcal{F}_0)| \geqq \varepsilon\}| = 0.$$

We write

$$\mathrm{DWMD_{stat}} \lim_{n\to\infty} (Y_n,\mathcal{F}_n) = (Y_0,\mathcal{F}_0).$$

Definition 5. Let (a_n) and (b_n) be sequences of non-negative integers, and let (p_n) be a sequence of non-negative numbers. A bounded martingale difference sequence $(Y_n,\mathcal{F}_n;n\in\mathbb{N})$ of random variables having probability 1 is statistically deferred weighted summable to a martingale $(Y_0,\mathcal{F}_0)$ with $\mathbb{E}|Y_0|<\infty$ if, for all $\varepsilon>0$,

$$\mathfrak{Z}_\varepsilon = \{j : a_n < j \leqq b_n \quad \text{and} \quad |\mathfrak{W}(Y_j,\mathcal{F}_j)-(Y_0,\mathcal{F}_0)| \geqq \varepsilon\}$$

has zero natural density. That is, for every $\varepsilon>0$, we have

$$\lim_{n\to\infty} \frac{|\{j : a_n < j \leqq b_n \quad \text{and} \quad |\mathfrak{W}(Y_j,\mathcal{F}_j)-(Y_0,\mathcal{F}_0)| \geqq \varepsilon\}|}{b_n-a_n} = 0.$$

We write

$$\mathrm{stat_{DWMD}} \lim_{n\to\infty} \mathfrak{W}(Y_j,\mathcal{F}_j) = (Y_0,\mathcal{F}_0).$$

Now we establish an inclusion theorem concerning the above mentioned two new interesting definitions.

Theorem 2. *If a given martingale difference sequence $(Y_n,\mathcal{F}_n;n\in\mathbb{N})$ of random variables is deferred weighted statistically convergent to a martingale $(Y_0,\mathcal{F}_0)$ with $\mathbb{E}|Y_0|<\infty$, then it is statistically deferred weighted summable to the same martingale, but not conversely.*

Proof. Suppose the given martingale sequence $(Y_n,\mathcal{F}_n;n\in\mathbb{N})$ of random variables is deferred weighted statistically convergent to a martingale $(Y_0,\mathcal{F}_0)$ with $\mathbb{E}|Y_0|<\infty$, then by Definition 4, we have

$$\lim_{n\to\infty}\frac{1}{P_n}|\{j: j\leqq P_n \quad \text{and} \quad p_j|(Y_j,\mathcal{F}_j)-(Y_0,\mathcal{F}_0)|\geqq\varepsilon\}|=0.$$

Now assuming two sets as follows:

$$\mathcal{W}_\varepsilon=\{j: j\leqq P_n \quad \text{and} \quad p_j|(Y_j,\mathcal{F}_j)-(Y_0,\mathcal{F}_0)|\geqq\varepsilon\}$$

and

$$\mathcal{W}_\varepsilon^c=\{j: j\leqq P_n \quad \text{and} \quad p_j|(Y_j,\mathcal{F}_j)-(Y_0,\mathcal{F}_0)|<\varepsilon\},$$

we have

$$\begin{aligned}
|\mathfrak{W}(Y_n,\mathcal{F}_n)-(Y_0,\mathcal{F}_0)| &= \left|\frac{1}{P_n}\sum_{i=a_n+1}^{b_n} p_i(Y_i,\mathcal{F}_i)-(Y_0,\mathcal{F}_0)\right| \\
&\leqq \left|\frac{1}{P_n}\sum_{i=a_n+1}^{b_n} p_i\left[(Y_i,\mathcal{F}_i)-(Y_0,\mathcal{F}_0)\right]\right| \\
&\quad + \left|\frac{1}{P_n}\sum_{i=a_n+1}^{b_n} p_i(Y_0,\mathcal{F}_0)-(Y_0,\mathcal{F}_0)\right| \\
&\leqq \frac{1}{P_n}\sum_{\substack{i=a_n+1\\(j\in\mathcal{W}_\varepsilon)}}^{b_n} |p_i(Y_i,\mathcal{F}_i)-(Y_0,\mathcal{F}_0)| \\
&\quad + \frac{1}{P_n}\sum_{\substack{i=a_n+1\\(j\in\mathcal{W}_\varepsilon^c)}}^{b_n} |p_i(Y_i,\mathcal{F}_i)-(Y_0,\mathcal{F}_0)| \\
&\quad + |(Y_0,\mathcal{F}_0)|\left|\frac{1}{P_n}\sum_{i=a_n+1}^{b_n} p_i-1\right| \\
&\leqq \frac{1}{P_n}\left|\mathcal{W}_\varepsilon\right|+\frac{1}{P_n}|\mathcal{W}_\varepsilon^c|=0.
\end{aligned}$$

Clearly, we obtain

$$|\mathfrak{W}(Y_n,\mathcal{F}_n)-(Y_0,\mathcal{F}_0)|<\varepsilon.$$

Thus, the martingale difference sequence $(Y_n,\mathcal{F}_n;n\in\mathbb{N})$ of random variables is statistically deferred weighted summable to the martingale $(Y_0,\mathcal{F}_0)$ with $\mathbb{E}|Y_0|<\infty$.

Next, in support of the non-validity of the converse statement, we present here an example demonstrating that a statistically deferred weighted summable martingale difference sequence of random variables is not necessarily deferred weighted statistically convergent.

Example 2. Suppose that $a_n = 2n$, $b_n = 4n$ and $p_n = n$, and let $(\mathcal{F}_n; n \in \mathbb{N})$ be a monotonically increasing sequence of 0-mean independent random variables of σ-fields and suppose that (X_n) is a sequence of n th partial sum of $(\mathcal{F}_n; n \in \mathbb{N})$ such that $X_n - X_{n-1} = Y_n$. Consider the sequence of random variables (X_n) as

$$X_n = \begin{cases} 1 & (n = \text{even}) \\ \\ -1 & (n = \text{odd}). \end{cases}$$

It is easy to see that, the martingale difference sequence $(Y_n, \mathcal{F}_n; n \in \mathbb{N})$ is neither convergent nor deferred weighted statistically convergent; however, it is deferred weighted summable to 0. Therefore, it is statistically deferred weighted summable to 0.

□

3. A Korovkin-type Theorem for Martingale Difference Sequence

Quite recently, a few researchers worked toward extending (or generalizing) the approximation of Korovkin-type theorems in different fields of mathematics such as sequence space, Banach space, Probability space, Measurable space, etc. This concept is extremely valuable in Real Analysis, Functional Analysis, Harmonic Analysis, and so on. Here, we like to refer the interested readers to the recent works [4, 13, 18, 20, 21] and [26].

In fact, we establish here the statistical versions of new Korovin-type approximation theorems for martingale difference sequences of positive linear operators via deferred weighted summability mean.

Let $\mathcal{C}([0,1])$ be the space of all real valued continuous functions defined on $[0,1]$ under the norm $\|.\|_\infty$. Also, let $\mathcal{C}[0,1]$ be a Banach space. Then for $f \in \mathcal{C}[0,1]$, the norm of f denoted by $\|f\|$ is given by

$$\|f\|_\infty = \sup_{x \in [0,1]} \{|f(x)|\}.$$

We say that, an operator $\mathfrak{A}$ is a martingale difference sequence of positive linear operators provided that

$$\mathfrak{A}(f;x) \geqq 0 \quad \text{whenever} \quad f \geqq 0, \text{ with } \mathfrak{A}(f;x) < \infty \text{ and } \operatorname{Prob}(\mathfrak{A}(f;x)) = 1.$$

Theorem 3. *Let*

$$\mathfrak{A}_m : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$$

be a martingale difference sequence of positive linear operators. Then, for all $f \in \mathcal{C}[0,1]$,

$$\mathrm{DWMD_{stat}} \lim_{m \to \infty} \|\mathfrak{A}_m(f;x) - f(x)\|_\infty = 0 \tag{3.1}$$

if and only if

$$\mathrm{DWMD_{stat}}\lim_{m\to\infty}\|\mathfrak{A}_m(1;x)-1\|_\infty=0, \tag{3.2}$$

$$\mathrm{DWMD_{stat}}\lim_{m\to\infty}\|\mathfrak{A}_m(2x;x)-2x\|_\infty=0 \tag{3.3}$$

and

$$\mathrm{DWMD_{stat}}\lim_{m\to\infty}\|\mathfrak{A}_m(3x^2;x)-3x^2\|_\infty=0. \tag{3.4}$$

Proof. Since each of the following functions

$$f_0(x)=1,\quad f_1(x)=2x\quad \text{and}\quad f_2(x)=3x^2$$

belong to $\mathcal{C}[0,1]$ and are continuous, the implication given by (3.1) implies (3.2) to (3.4) is obvious.

In order to complete the proof of the Theorem 3, we first assume that the conditions (3.2) to (3.4) hold true. If $f\in\mathcal{C}[0,1]$, then there exists a constant $\mathcal{N}>0$ such that

$$|f(x)|\leqq\mathcal{N}\quad(\forall\, x\in[0,1]).$$

We thus find that

$$|f(r)-f(x)|\leqq 2\mathcal{N}\quad(r,x\in[0,1]). \tag{3.5}$$

Clearly, for given $\varepsilon>0$, there exists $\delta>0$ such that

$$|f(r)-f(x)|<\varepsilon \tag{3.6}$$

whenever

$$|r-x|<\delta,\quad\text{for all}\quad r,x\in[0,1].$$

Let us choose

$$\varphi_1=\varphi_1(r,x)=(2r-2x)^2.$$

If $|r-x|\geqq\delta$, then we obtain

$$|f(r)-f(x)|<\frac{2\mathcal{N}}{\delta^2}\varphi_1(r,x). \tag{3.7}$$

From equation (3.6) and (3.7), we get

$$|f(r)-f(x)|<\varepsilon+\frac{2\mathcal{N}}{\delta^2}\varphi_1(r,x),$$

which implies that

$$-\varepsilon-\frac{2\mathcal{N}}{\delta^2}\varphi_1(r,x)\leqq f(r)-f(x)\leqq\varepsilon+\frac{2\mathcal{N}}{\delta^2}\varphi_1(r,x). \tag{3.8}$$

Now, since $\mathfrak{A}_m(1;x)$ is monotone and linear, by applying the operator $\mathfrak{A}_m(1;x)$ to this inequality, we have

$$\mathfrak{A}_m(1;x)\left(-\varepsilon-\frac{2\mathcal{N}}{\delta^2}\varphi_1(r,x)\right)\leqq\mathfrak{A}_m(1;x)(f(r)-f(x))$$

$$\leqq \mathfrak{A}_m(1;x)\left(\varepsilon + \frac{2\mathcal{N}}{\delta^2}\varphi_1(r,x)\right).$$

We note that x is fixed and so $f(x)$ is a constant number. Therefore, we have

$$\begin{aligned}-\varepsilon\mathfrak{A}_m(1;x) - \frac{2\mathcal{N}}{\delta^2}\mathfrak{A}_m(\varphi_1;x) &\leqq \mathfrak{A}_m(f;x) - f(x)\mathfrak{A}_m(1;x)\\ &\leqq \varepsilon\mathfrak{A}_m(1;x) + \frac{2\mathcal{N}}{\delta^2}\mathfrak{A}_m(\varphi_1;x).\end{aligned} \tag{3.9}$$

Also, we know that

$$\mathfrak{A}_m(f;x) - f(x) = [\mathfrak{A}_m(f;x) - f(x)\mathfrak{A}_m(1;x)] + f(x)[\mathfrak{A}_m(1;x) - 1]. \tag{3.10}$$

Using (3.9) and (3.10), we have

$$\mathfrak{A}_m(f;x) - f(x) < \varepsilon\mathfrak{A}_m(1;x) + \frac{2\mathcal{A}}{\delta^2}\mathfrak{A}_m(\varphi_1;x) + f(x)[\mathfrak{A}_m(1;x) - 1]. \tag{3.11}$$

We now estimate $\mathfrak{A}_m(\varphi_1;x)$ as follows:

$$\begin{aligned}\mathfrak{A}_m(\varphi_1;x) &= \mathfrak{A}_m((2r-2x)^2;x) = \mathfrak{A}_m(2r^2 - 8xr + 4x^2;x)\\ &= \mathfrak{A}_m(4r^2;x) - 8x\mathfrak{A}_m(r;x) + 4x^2\mathfrak{A}_m(1;x)\\ &= 4[\mathfrak{A}_m(r^2;x) - x^2] - 8x[\mathfrak{A}_m(r;x) - x]\\ &\quad + 4x^2[\mathfrak{A}_m(1;x) - 1].\end{aligned}$$

Using (3.11), we obtain

$$\begin{aligned}\mathfrak{A}_m(f;x) - f(x) &< \varepsilon\mathfrak{A}_m(1;x) + \frac{2\mathcal{N}}{\delta^2}\{4[\mathfrak{A}_m(r^2;x) - x^2]\\ &\quad - 8x[\mathfrak{A}_m(r;x) - x] + 4x^2[\mathfrak{A}_m(1;x) - 1]\}\\ &\quad + f(x)[\mathfrak{A}_m(1;x) - 1].\\ &= \varepsilon[\mathfrak{A}_m(1;x) - 1] + \varepsilon + \frac{2\mathcal{N}}{\delta^2}\{4[\mathfrak{A}_m(r^2;x) - x^2]\\ &\quad - 8x[\mathfrak{A}_m(r;x) - x] + 4x^2[\mathfrak{A}_m(1;x) - 1]\}\\ &\quad + f(x)[\mathfrak{A}_m(1;x) - 1].\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we can write

$$\begin{aligned}|\mathfrak{A}_m(f;x) - f(x)| &\leqq \varepsilon + \left(\varepsilon + \frac{8\mathcal{N}}{\delta^2} + \mathcal{N}\right)|\mathfrak{A}_m(1;x) - 1|\\ &\quad + \frac{16\mathcal{N}}{\delta^2}|\mathfrak{A}_m(r;x) - x| + \frac{8\mathcal{N}}{\delta^2}|\mathfrak{A}_m(r^2;x) - x^2|\\ &\leqq \mathcal{E}(|\mathfrak{A}_m(1;x) - 1| + |\mathfrak{A}_m(r;x) - x|\\ &\quad + |\mathfrak{A}_m(r^2;x) - x^2|),\end{aligned} \tag{3.12}$$

where

$$\mathcal{E} = \max\left(\varepsilon + \frac{8\mathcal{N}}{\delta^2} + \mathcal{N}, \frac{16\mathcal{N}}{\delta^2}, \frac{8\mathcal{N}}{\delta^2}\right).$$

Now, for a given $\mu > 0$, there exists $\varepsilon > 0$ $(\varepsilon < \mu)$, we get

$$\mathfrak{G}_m(x;\mu) = \{m : m \leqq P_n \quad \text{and} \quad p_m |\mathfrak{A}_m(f;x) - f(x)| \geqq \mu\}.$$

Furthermore, for $k = 0,1,2$, we have

$$\mathfrak{G}_{k,m}(x;\mu) = \left\{m : m \leqq P_n \quad \text{and} \quad p_m |\mathfrak{A}_m(f;x) - f_k(x)| \geqq \frac{\mu - \varepsilon}{3\mathcal{E}}\right\},$$

so that,

$$\mathfrak{G}_m(x;\mu) \leqq \sum_{k=0}^{2} \mathfrak{G}_{k,m}(x;\mu).$$

Clearly, we obtain

$$\frac{\|\mathfrak{G}_m(x;\mu)\|_{C[0,1]}}{P_n} \leqq \sum_{k=0}^{2} \frac{\|\mathfrak{G}_{k,m}(x;\mu)\|_{C[0,1]}}{P_n}. \tag{3.13}$$

Now, using the above assumption about the implications in (3.2) to (3.4) and by Definition 4, the right-hand side of (3.13) is seen to tend to zero as $n \to \infty$. Consequently, we get

$$\lim_{n\to\infty} \frac{\|\mathfrak{G}_m(x;\mu)\|_{C[0,1]}}{P_n} = 0 \ (\delta, \mu > 0).$$

Therefore, implication (3.1) holds true. This completes the proof of Theorem 3. □

Next, by using Definition 5, we present the following theorem.

Theorem 4. *Let $\mathfrak{A}_m : C[0,1] \to C[0,1]$ be a martingale difference sequence of positive linear operators and let $f \in C[0,1]$. Then*

$$\text{stat}_{\text{DWMD}} \lim_{m\to\infty} \|\mathfrak{A}_m(f;x) - f(x)\|_\infty = 0 \tag{3.14}$$

if and only if

$$\text{stat}_{\text{DWMD}} \lim_{m\to\infty} \|\mathfrak{A}_m(1;x) - 1\|_\infty = 0, \tag{3.15}$$

$$\text{stat}_{\text{DWMD}} \lim_{m\to\infty} \|\mathfrak{A}_m(2x;x) - 2x\|_\infty = 0 \tag{3.16}$$

and

$$\text{stat}_{\text{DMD}} \lim_{m\to\infty} \|\mathfrak{A}_m(3x^2;x) - 3x^2\|_\infty = 0. \tag{3.17}$$

Proof. The proof of Theorem 4 is similar to the proof of Theorem 3. We, therefore, choose to skip the details involved. □

We present below an illustrative example for the martingale difference sequence of positive linear operators that does not satisfy the conditions of the weighted statistical convergence versions of Korovkin-type approximation Theorem 3 and also the results of Srivastava *et al.* [22], and Paikray *et al.* [11], but it satisfies the conditions of statistical weighted summability versions of our Korovkin-type approximation Theorem 4. Thus, our Theorem 4 is stronger than the results asserted by Theorem 3 and also, the results of Srivastava *et al.* [22] and Paikray *et al.* [11].

We now recall the operator

$$\vartheta(1+\vartheta D) \quad \left(D=\frac{d}{d\vartheta}\right),$$

which was used by Al-Salam [1] and, more recently, by Viskov and Srivastava [25] (see [14] and the monograph by Srivastava and Manocha [24] for various general families of operators and polynomials of this kind). Here, in our Example 3 below, we use this operator in conjunction with the Bernstein polynomial.

Example 3. We consider *Bernstein polynomial* $\mathfrak{B}_m(f;\vartheta)$ on $C[0,1]$ given by

$$\mathfrak{B}_m(f;\vartheta)=\sum_{m=0}^{n} f\left(\frac{m}{n}\right)\binom{n}{m}\vartheta^m(1-\vartheta)^{n-m} \quad (\vartheta\in[0,1]). \tag{3.18}$$

Next, we present the martingale difference sequences of positive linear operators on $C[0,1]$ defined as follows:

$$\mathfrak{A}_m(f;\vartheta)=[1+(Y_n,\mathcal{F}_n)]\vartheta(1+\vartheta D)\mathfrak{B}_m(f;\vartheta) \quad (\forall\, f\in C[0,1]), \tag{3.19}$$

where $(Y_n,\mathcal{F}_n)$ is already mentioned in Example 2.

Now, we calculate the values of the functions 1, 2ϑ and $3\vartheta^2$ by using our proposed operators (3.19),

$$\begin{aligned}\mathfrak{A}_m(1;\vartheta)&=[1+(Y_m,\mathcal{F}_m)]\vartheta(1+\vartheta D)1=[1+(Y_m,\mathcal{F}_m)]\vartheta,\\ \mathfrak{A}_m(2\vartheta;\vartheta)&=[1+(X_m,\mathcal{F}_m)]\vartheta(1+\vartheta D)2\vartheta=[1+(Y_m,\mathcal{F}_m)]\vartheta(1+2\vartheta),\end{aligned}$$

and

$$\begin{aligned}\mathfrak{A}_m(3\vartheta^2;\vartheta)&=[1+(Y_m,\mathcal{F}_m)]\vartheta(1+\vartheta D)3\left\{\vartheta^2+\frac{\vartheta(1-\vartheta)}{m}\right\}\\ &=[1+(Y_m,\mathcal{F}_m)]\left\{\vartheta^2\left(6-\frac{9\vartheta}{m}\right)\right\},\end{aligned}$$

so that we have

$$\begin{aligned}\mathrm{stat}_{\mathrm{DWMD}}\lim_{m\to\infty}\|\mathfrak{A}_m(1;\vartheta)-1\|_\infty&=0,\\ \mathrm{stat}_{\mathrm{DWMD}}\lim_{m\to\infty}\|\mathfrak{A}_m(2\vartheta;\vartheta)-2\vartheta\|_\infty&=0\end{aligned}$$

and

$$\mathrm{stat}_{\mathrm{DWMD}}\lim_{m\to\infty}\|\mathfrak{A}_m(3\vartheta^2;\vartheta)-3\vartheta^2\|_\infty=0.$$

Consequently, the sequence $\mathfrak{A}_m(f;\vartheta)$ satisfies the conditions (3.15) to (3.17). Therefore, by Theorem 4, we have

$$\mathrm{stat}_{\mathrm{DWMD}} \lim_{m\to\infty} \|\mathfrak{A}_m(f;\vartheta)-f\|_\infty = 0.$$

Here, the given martingale difference sequence $(Y_m,\mathcal{F}_m)$ of functions in Example 2 is statistically deferred weighted summable but not deferred weighted statistically convergent. Thus, martingale difference operators defined by (3.19) satisfy the Theorem 4; however, it is not satisfying Theorem 3.

Moreover, if one considers the positive linear operators of the types Baskakov and Szász-Mirakyan [6], and Beta Szász-Mirakjan [16] in place of *Bernstein polynomial* $\mathfrak{B}_m(f;\vartheta)$ in Example 3, then with the same algebraic test functions it will also satisfy the conclusion of Korovkin-type approximation theorem via our purposed mean for martingale difference sequences of random variables. Consequently, these operators are also valid for Theorem 4; however, it will not satisfy Theorem 3.

4. Concluding Remarks and Observations

In this concluding section of our investigation, we present several further remarks and observations concerning to various results which we have proved here.

Remark 1. Let $(Y_n,\mathcal{F}_n;n\in\mathbb{N})$ be a martingale difference sequence given in Example 2. Then, since

$$\mathrm{stat}_{\mathrm{DWMD}} \lim_{m\to\infty} Y_m = 0 \ \text{ on } \ [0,1],$$

we have

$$\mathrm{stat}_{\mathrm{DWMD}} \lim_{m\to\infty} \|\mathfrak{A}_m(f_k;x)-f_k(x)\|_\infty = 0 \quad (k=0,1,2). \tag{4.1}$$

Thus, by Theorem 4, we can write

$$\mathrm{stat}_{\mathrm{DWMD}} \lim_{m\to\infty} \|\mathfrak{A}_m(f;x)-f(x)\|_\infty = 0, \tag{4.2}$$

where

$$f_0(x)=1, \quad f_1(x)=2x \quad \text{and} \quad f_2(x)=3x^2.$$

Here, the martingale difference sequence $(Y_n,\mathcal{F}_n;n\in\mathbb{N})$ is neither statistically convergent nor converges uniformly in the ordinary sense; thus, the classical and statistical versions of Korovkin-type theorems do not work here for the operators defined by (3.19). Hence, this application indicates that our Theorem 4 is a non-trivial generalization of the classical as well as statistical versions of Korovkin-type theorems (see [5] and [10]).

Remark 2. Let $(Y_n,\mathcal{F}_n;n\in\mathbb{N})$ be a martingale difference sequence given already in Example 2. Then, since

$$\mathrm{stat}_{\mathrm{DWMD}} \lim_{m\to\infty} Y_m = 0 \text{ on } [0,1],$$

so (4.1) holds true. Now, by applying (4.1) and Theorem 4, condition (4.2) also holds true. However, since the martingale difference sequence $(Y_n, \mathcal{F}_n; n \in \mathbb{N})$ is not deferred weighted statistically convergent but it is statistically deferred weighted summable. Thus, Theorem 4 is certainly a non-trivial extension of Theorem 3. Therefore, Theorem 4 is stronger than Theorem 3.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

REFERENCES

[1] W. A. Al-Salam, "Operational representations for the Laguerre and other polynomials." *Duke Math. J.*, vol. 31, pp. 127–142, 1964, doi: 10.1215/S0012-7094-64-03113-8.

[2] N. L. Braha, V. Loku, and H. M. Srivastava, "Λ^2-Weighted statistical convergence and Korovkin and Voronovskaya type theorems." *Appl. Math. Comput.*, vol. 266, pp. 675–686, 2015.

[3] N. L. Braha, H. M. Srivastava, and S. A. Mohiuddine, "A Korovkin-type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée Poussin mean." *Appl. Math. Comput.*, vol. 228, pp. 162–169, 2014.

[4] A. A. Das, S. K. Paikray, T. Pradhan, and H. Dutta, "Statistical $(C,1)(E,\mu)$-summablity and associated fuzzy approximation theorems with statistical fuzzy rates." *Soft Comput.*, vol. 24, pp. 10 883–10 892, 2020, doi: 10.1007/s00500-019-04591-2.

[5] H. Fast, "Sur la convergence statistique." *Colloq. Math.*, vol. 2, pp. 241–244, 1951.

[6] V. Gupta, A. M. Acu, and H. M. Srivastava, "Difference of some positive linear approximation operators for higher-order derivatives." *Symmetry*, vol. 12, pp. 1–19, 2020, doi: 10.3390/sym12060915.

[7] B. B. Jena and S. K. Paikray, "Product of statistical probability convergence and its applications to Korovkin-type theorem." *Miskolc Math. Notes*, vol. 20, pp. 969–984, 2019, doi: 10.18514/MMN.2019.3014.

[8] B. B. Jena, S. K. Paikray, and H. Dutta, "On various new concepts of statistical convergence for sequences of random variables via deferred Cesàro mean." *J. Math. Anal. Appl.*, vol. 487, pp. 1–18, 2020, doi: 10.1016/j.jmaa.2020.123950.

[9] B. B. Jena, S. K. Paikray, S. A. Mohiuddine, and V. N. Mishra, "Relatively equi-statistical convergence via deferred Nörlund mean based on difference operator of fractional-order and related approximation theorems." *AIMS Math.*, vol. 5, pp. 650–672, 2020, doi: 10.3934/math.2020044.

[10] P. P. Korovkin, "Convergence of linear positive operators in the spaces of continuous functions (in Russian)." *Doklady Akad. Nauk. SSSR (New Ser.)*, vol. 90, pp. 961–964, 1953.

[11] S. K. Paikray and H. Dutta, *Applied Mathematical Analysis: Theory, Methods, and Applications: On statistical deferred weighted $\mathcal{B}$-convergence.* Springer Nature Switzerland AG: Springer, 2019. doi: 10.1007/978-3-319-99918-0_20.

[12] P. Parida, S. K. Paikray, and B. B. Jena, "Generalized deferred Cesàro equi-statistical convergence and analogous approximation theorems," *Proyecciones J. Math.*, vol. 39, pp. 307–331, 2020, doi: 10.22199.

[13] T. Pradhan, S. K. Paikray, B. B. Jena, and H. Dutta, "Statistical deferred weighted $\mathcal{B}$-summability and its applications to associated approximation theorems." *J. Inequal. Appl.*, vol. 2018, pp. 1–21, 2018, doi: 10.1186/s13660-018-1650-x.

[14] H. M. Srivastava, "A note on certain operational representations for the Laguerre polynomials." *J. Math. Anal. Appl.*, vol. 138, pp. 209–213, 1989, doi: 10.1016/0022-247X(89)90331-4.

[15] H. M. Srivastava and M. Et, "Lacunary statistical convergence and strongly lacunary summable functions of order α." *Filomat*, vol. 31, pp. 1573–1582, 2017.

[16] H. M. Srivastava, G. İçöz, and B. Çekim, "Approximation properties of an extended family of the Szász-Mirakjan Beta-type operators." *Axioms*, vol. 8, pp. 1–13, 2019, doi: 10.3390/axioms8040111.

[17] H. M. Srivastava, B. B. Jena, and S. K. Paikray, "Deferred Cesàro statistical convergence of martingale sequence and Korovkin-type approximation theorems." *Miskolc Math. Notes, to be published.*

[18] H. M. Srivastava, B. B. Jena, and S. K. Paikray, "Deferred Cesàro statistical probability convergence and its applications to approximation theorems." *J. Nonlinear Convex Anal.*, vol. 20, pp. 1777–1792, 2019.

[19] H. M. Srivastava, B. B. Jena, and S. K. Paikray, "A certain class of statistical probability convergence and its applications to approximation theorems." *Appl. Anal. Discrete Math.*, vol. 14, pp. 579–598, 2020, doi: doi.org/10.2298/AADM190220039S.

[20] H. M. Srivastava, B. B. Jena, and S. K. Paikray, "Statistical deferred Nörlund summability and Korovkin-type approximation theorem." *Mathematics*, vol. 8, pp. 1–11, 2020, doi: 10.3390/math8040636.

[21] H. M. Srivastava, B. B. Jena, and S. K. Paikray, "Statistical probability convergence via the deferred Nörlund mean and its applications to approximation theorems." *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. (RACSAM)*, vol. 114, pp. 1–14, 2020, doi: 10.1007/s13398-020-00875-7.

[22] H. M. Srivastava, B. B. Jena, S. K. Paikray, and U. K. Misra, "Generalized equi-statistical convergence of the deferred Nörlund summability and its applications to associated approximation theorems." *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. (RACSAM)*, vol. 112, pp. 1487–1501, 2018, doi: 10.1007/s13398-017-0442-3.

[23] H. M. Srivastava, B. B. Jena, S. K. Paikray, and U. K. Misra, "Statistically and relatively modular deferred-weighted summability and Korovkin-type approximation theorems." *Symmetry*, vol. 11, pp. 1–20, 2019, doi: 10.3390/sym11040448.

[24] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions.* New York, Chichester, Brisbane and Toronto: John Wiley and Sons, 1984.

[25] O. V. Viskov and H. M. Srivastava, "New approaches to certain identities involving differential operators." *J. Math. Anal. Appl.*, vol. 186, pp. 1–10, 1994, doi: 10.1006/jmaa.1994.1281.

[26] A. Zraiqat, S. K. Paikray, and H. Dutta, "A certain class of deferred weighted statistical B-summability involving (p,q)-integers and analogous approximation theorems." *Filomat*, vol. 33, pp. 1425–1444, 2019.

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HERMITE-HADAMARD TYPE INEQUALITIES FOR SOME CONVEX DOMINATED FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. In this study, we derive some new inequalities of H-H type for $(g,m)-$ and (g,h)-convex dominated functions related fractional integral. Our obtained results are extensions of earlier works.

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1. INTRODUCTION

Let's start our study by introducing the concept of convex function and the famous inequality obtained for the mean value of a convex function (see [6], [15]).

Definition 1. Let $I \subseteq \mathbb{R}$ be an interval. Then a real-valued function $f : I \to \mathbb{R}$ is said to be convex (concave) on the interval I if the inequality

$$f(\lambda x + (1-\lambda)y) \leq (\geq) \lambda f(x) + (1-\lambda) f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

With the help of convex function and inequalities we will give below, we observe that many applications take place in pure and applied mathematics.

The following double inequality is called Hermite-Hadamard inequality in the literature. If $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

The inequalities in (1.2) hold in the reversed direction if f is concave.

Different convex function types and different Hermite-Hadamard type inequalities which are considered basic for each definition and for each inequality have been obtained through the Definition 1 and Hermite-Hadamard Inequality. It is observed that studies in this direction take a large place in the literature.

In [22], G. Toader defined $m-$convexity as the following:

Definition 2. The function $f:[0,b]\to\mathbb{R}, b>0$ is said to be $m-$convex where $m\in[0,1]$, if we have

$$f(\lambda x+m(1-\lambda)y)\leq \lambda f(x)+m(1-\lambda)f(y)$$

for all $x,y\in[0,b]$ and $\lambda\in[0,1]$. We say that f is $m-$concave if $(-f)$ is $m-$convex.

In [4], Dragomir obtained the following Theorem for $m-$ convex functions.

Theorem 1. *Let $f:[0,\infty)\to\mathbb{R}$ be a $m-$ convex function with $m\in(0,1]$ and $0\leq a<b$. If $f\in L_1[a,b]$, then one has the inequalities*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b \frac{f(x)+mf(\frac{x}{m})}{2}dx \\ &\leq \frac{1}{2}\left[\frac{f(a)+mf(\frac{a}{m})}{2}+m\frac{f(\frac{b}{m})+mf(\frac{b}{m^2})}{2}\right]. \end{aligned} \tag{1.3}$$

Let us remind the definition of $h-$ convex function [11, 23]:

Definition 3. Let $h-$ be a positive function. We say that $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is $h-$convex function or that belongs to the class $SX(h,I)$, if f is non-negative and for all $x,y\in I$ and $\lambda\in(0,1)$, we have

$$f(\lambda x+(1-\lambda)y)\leq h(\lambda)f(x)+h(1-\lambda)f(y). \tag{1.4}$$

If the inequality in (1.4) is reserved, then f is said to be $h-$concave, i.e. $SV(h,I)$. Obviously, if $f(\lambda)=\lambda; f(\lambda)=\frac{1}{\lambda}; f(\lambda)=1; f(\lambda)=\lambda^s$ where $s\in(0,1)$, then all non-negative convex function belong to $SX(h,I)$ and all nonnegative concave functions belong to $SV(h,I)$; $SX(h,I)=Q(I)$; $SX(h,I)\supseteq P(I)$; $SX(h,I)\supseteq K_s^2$, respectively.

The classical H-H inequality for $h-$ convex functions was obtained by Sarikaya et. al. in [18] is as follows:

Theorem 2. *Let $f\in SX(h,I)$, $a,b\in I$ with $a<b, f\in L_1[a,b]$. Then*

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq[f(a)+f(b)]\int_0^1 h(\lambda)d\lambda. \tag{1.5}$$

In [5], Dragomir and Ionescu introduced the following class of functions and proved some inequalities.

Definition 4. Let $g:I\to\mathbb{R}$ be a convex function on the interval I. The function $g:I\to R$ is called $g-$convex dominated on I if the following condition is satisfied:

$$|\lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y)|\leq\lambda g(x)+(1-\lambda)g(y)-g(\lambda x+(1-\lambda)y)$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Theorem 3. *(See [7]) Let $g : I \to \mathbb{R}$ be a convex function and $f : I \to R$ be a $g-$convex dominated mapping. Then, for all $a, b \in I, a < b$,*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{1}{b-a}\int_a^b g(x)dx - g\left(\frac{a+b}{2}\right) \tag{1.6}$$

and

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(x)dx. \tag{1.7}$$

$(g,m)-$ dominated convex function and interested theorem have been given as the following (see [12]):

Definition 5. Let $g : [0,b] \to \mathbb{R}$ be a given $m-$ convex function on the interval $[0,b]$. The real function $f : [0,b] \to \mathbb{R}$ is called $(g,m)-$convex dominated on $[0,b]$ if the following condition is satisfied

$$\begin{aligned} &|\lambda f(x) + m(1-\lambda)f(y) - f(\lambda x + m(1-\lambda)y)| \\ &\quad \leq \lambda g(x) + m(1-\lambda)g(y) - g(\lambda x + m(1-\lambda)y) \end{aligned} \tag{1.8}$$

for all $x, y \in [0,b]$, $m, \lambda \in [0,1]$.

Theorem 4. *Let $g : [0,\infty) \to \mathbb{R}$ be an $m-$convex function with $m \in (0,1]$. $f : [0,\infty) \to \mathbb{R}$ is $(g,m)-$ convex dominated mapping and $0 \leq a < b$. If $f \in L_1[a,b]$, then one has the inequalities:*

$$\begin{aligned} &\left| \frac{1}{b-a}\int_a^b \frac{f(x)+mf(\frac{x}{m})}{2}dx - f\left(\frac{a+b}{2}\right) \right| \\ &\quad \leq \frac{1}{b-a}\int_a^b \frac{g(x)+mg(\frac{x}{m})}{2}dx - g\left(\frac{a+b}{2}\right) \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} &\left| \frac{1}{2}\left[\frac{f(a)+mf(\frac{a}{m})}{2} + m\frac{f(\frac{b}{m})+mf(\frac{b}{m^2})}{2} \right] - \frac{1}{b-a}\int_a^b \frac{f(x)+mf(\frac{x}{m})}{2}dx \right| \\ &\quad \leq \frac{1}{2}\left[\frac{g(a)+mg(\frac{a}{m})}{2} + m\frac{g(\frac{b}{m})+mg(\frac{b}{m^2})}{2} \right] - \frac{1}{b-a}\int_a^b \frac{g(x)+mg(\frac{x}{m})}{2}dx. \end{aligned} \tag{1.10}$$

Definition 6. (See [12]) Let $h \neq 0, h : J \to \mathbb{R}$ be a nonnegative function, $g : I \to \mathbb{R}$ be an $h-$convex function. The real function $f : I \to \mathbb{R}$ is called $(g,h)-$convex dominated on I if the following condition is satisfied:

$$\begin{aligned} &|h(\lambda)f(x) + h(1-\lambda)f(y) - f(\lambda x + (1-\lambda)y)| \\ &\quad \leq h(\lambda)g(x) + h(1-\lambda)g(y) - g(\lambda x + (1-\lambda)y) \end{aligned}$$

for all $x, y \in I$ and $\lambda \in (0,1)$.

Theorem 5. *Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$, $g : I \to \mathbb{R}$, be an $h-$ convex function and the real function $f : I \to \mathbb{R}$ be $(g,h)-$ convex dominated on I. Then one has the inequalities:*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - \frac{1}{2h(\frac{1}{2})} f\Big(\frac{a+b}{2}\Big) \right| \leq \frac{1}{b-a} \int_a^b g(x)dx - \frac{1}{2h(\frac{1}{2})} g\Big(\frac{a+b}{2}\Big)$$

and

$$\left| [f(a)+f(b)] \int_0^1 h(\lambda)d\lambda - \frac{1}{b-a} \int_a^b f(x)dx \right| \tag{1.11}$$

$$\leq [g(a)+g(b)] \int_0^1 h(\lambda)d\lambda - \frac{1}{b-a} \int_a^b g(x)dx \tag{1.12}$$

for all $x, y \in I$ and $\lambda \in (0,1]$.

Many authors study integral inequalities involving various fractional operators like Erdelyi-Kober, Riemann-Liouville, conformable fractional integral operators, Katugampola, etc. in last years. The most studied in them is Riemann-Liouville fractional integral operators. In [16], Liouville and Riemann introduced the fractional calculus at last of the nineteenth century. Now, we remind the definition of Riemann-Liouville fractional integrals.

Definition 7. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a^+} f$ and $J^{\alpha}_{b^-} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \, x > a \tag{1.13}$$

and

$$J^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \, x < b \tag{1.14}$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} u^{\alpha-1} du$. Here $J^0_{a^+} f(x) = J^0_{b^-} f(x) = f(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

One can find the interested properties and inequalities the references [1–3, 8–10, 13, 16, 17, 19–21, 24, 25].

In [19], Sarikaya et. al. obtained the Hermite-Hadamard type inequality for fractional calculus as following:

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J^{\alpha}_{a^+} f(b) + J^{\alpha}_{b^-} f(a)] \leq \frac{f(a)+f(b)}{2} \tag{1.15}$$

with $\alpha \geq 0$.

In [14], another Hermite-Hadamard type inequality via fractional calculus have been presented by Özdemir and Önalan.

Theorem 7. *Let* $f,g:[a,b]\to\mathbb{R}$ *be positive functions with* $0\leq a<b$ *and* $f,g\in L_1[a,b]$. *If* g *is a convex function on* $[a,b]$ *and* f *is a* $g-$ *convex dominated function, then the following inequalities for fractional integrals hold:*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)]-f\left(\frac{a+b}{2}\right)\right|\\&\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}g(b)+J^{\alpha}_{b^-}g(a)]-g\left(\frac{a+b}{2}\right)\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)]\right|\\&\leq\frac{g(a)+g(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}g(b)+J^{\alpha}_{b^-}g(a)].\end{aligned}$$

In [14], Özdemir and Önalan present Hermite-Hadamard type inequalities for fractional calculus as following:

Theorem 8. *Let* $f:[0,\infty)\to\mathbb{R}$ *be a positive function with* $0\leq a<b$ *and* $f\in L_1[a,\frac{b}{m}]$. *If* f *is an m-convex function on* $[0,\infty)$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b)+m^{\alpha+1}J^{\alpha}_{\frac{b}{m}^-}f(\frac{a}{m})\right]\\&\leq\frac{1}{2}\left[\frac{\alpha f(a)+mf(\frac{a}{m})}{\alpha+1}+m\frac{f(\frac{b}{m})+m\alpha f(\frac{b}{m^2})}{\alpha+1}\right]\end{aligned}$$

with $\alpha>0$ *ve* $m\in(0,1]$.

Yildiz et. al give Hermite-Hadamard type inequality for fractional calculus in [25].

Theorem 9. *Let* $f:I\subseteq R\to\mathbb{R}$ *be a real function with* $a<b,a,b\in I^{\star}$ *and* $f\in L[a,b]$. *If* f *belongs to the SX* (h,I), *we give*

$$\begin{aligned}\frac{1}{\alpha h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)&\leq\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f\left(b\right)+J^{\alpha}_{b^-}f\left(a\right)\right]\\&\leq\left[f\left[a\right]+f\left[b\right]\right]\int_0^1 t^{\alpha-1}\left[h\left(t\right)+h\left(1-t\right)\right]dt\end{aligned}$$

with $\alpha>0$.

Now, we give new Hermite-Hadamard type inequalities for $(g,h)-$convex dominated functions and $(g,m)-$convex dominated functions by using fractional calculus.

2. The Results

Theorem 10. *Let $f,g:[a,b]\to\mathbb{R}$ be positive functions with $0\le a<b$ and $f,g\in L_1[a,b]$. If g is an $h-$ convex function on $[a,b]$ and f is a $(g,h)-$ convex dominated function, then the following inequalities for fractional integrals hold:*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]-\frac{f\left(\frac{a+b}{2}\right)}{\alpha h\left(\frac{1}{2}\right)}\right|\\&\qquad\le\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\frac{g\left(\frac{a+b}{2}\right)}{\alpha h\left(\frac{1}{2}\right)}\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}&[f(a)+f(b)]\left[\int_0^1 t^{\alpha-1}(h(t)+h(1-t))\,dt\right]\\&\qquad-\frac{\Gamma(\alpha)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\\&\le[g(a)+g(b)]\left[\int_0^1 t^{\alpha-1}(h(t)+h(1-t))\,dt\right]\\&\qquad-\frac{\Gamma(\alpha)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]\end{aligned}\tag{2.2}$$

with $\alpha\ge 0$.

Proof. In Definition 6, if we choose $\lambda=\frac{1}{2}$, we get

$$\left|h\left(\frac{1}{2}\right)[f(x)+f(y)]-f\left(\frac{x+y}{2}\right)\right|\le h\left(\frac{1}{2}\right)[g(x)+g(y)]-g\left(\frac{x+y}{2}\right)$$

for all $x,y\in[a,b]$. Then if we take $x=ta+(1-t)b$ and $y=(1-t)a+tb$, we get

$$\begin{aligned}&\left|h\left(\frac{1}{2}\right)[f(ta+(1-t)b)+f((1-t)a+tb)]-f\left(\frac{a+b}{2}\right)\right|\\&\quad\le h\left(\frac{1}{2}\right)[g(ta+(1-t)b)+g((1-t)a+tb)]-g\left(\frac{a+b}{2}\right)\end{aligned}\tag{2.3}$$

for $t\in[0,1]$. Multiplying (2.3) by $t^{\alpha-1}$, then integrating the deduced inequality with respect to t over $[0,1]$, we obtain;

$$\begin{aligned}&\left|h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\alpha-1}(f(ta+(1-t)b))\,dt+\int_0^1 t^{\alpha-1}(f((1-t)a+tb))\,dt\right]\right.\\&\qquad\left.-f\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}dt\right|\end{aligned}$$

$$\leq h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\alpha-1}\left(g\left(ta+(1-t)b\right)\right)dt+\int_0^1 t^{\alpha-1}\left(g\left((1-t)a+tb\right)\right)dt\right]$$
$$-g\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}dt$$

If we correct the above inequality, we get the first part of requested inequality.

$$\left|\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]-\frac{f\left(\frac{a+b}{2}\right)}{\alpha h\left(\frac{1}{2}\right)}\right|$$
$$\leq\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\frac{g\left(\frac{a+b}{2}\right)}{\alpha h\left(\frac{1}{2}\right)}.$$

To get second part of Theorem 10, let's use Definition 6. Then,

$$\left|h(t)f(a)+h(1-t)f(b)-f(ta+(1-t)b)\right|$$
$$\leq h(t)g(a)+h(1-t)g(b)-g(ta+(1-t)b)$$

and

$$\left|h(1-t)f(a)+h(t)f(b)-f((1-t)a+tb)\right|$$
$$\leq h(1-t)g(a)+h(t)g(b)-g((1-t)a+tb).$$

Obtained last two inequalities if add side by side, we get

$$[f(a)+f(b)][h(t)+h(1-t)]-f(ta+(1-t)b)-f((1-t)a+tb)$$
$$\leq[g(a)+g(b)][h(t)+h(1-t)]-g(ta+(1-t)b)-g((1-t)a+tb).$$

Multiplying the last inequality by $t^{\alpha-1}$, then integrating with respect to t over $[0,1]$, we obtain;

$$[f(a)+f(b)]\left[\int_0^1 t^{\alpha-1}(h(t)+h(1-t))dt\right]$$
$$-\int_0^1 t^{\alpha-1}f(ta+(1-t)b)dt-\int_0^1 t^{\alpha-1}f((1-t)a+tb)dt$$
$$\leq[g(a)+g(b)]\left[\int_0^1 t^{\alpha-1}(h(t)+h(1-t))dt\right]$$
$$-\int_0^1 t^{\alpha-1}g(ta+(1-t)b)dt-\int_0^1 t^{\alpha-1}g((1-t)a+tb)dt.$$

If we correct the obtained inequality, we get the desired inequality,

$$[f(a)+f(b)]\left[\int_0^1 t^{\alpha-1}(h(t)+h(1-t))dt\right]$$
$$-\frac{\Gamma(\alpha)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]$$

$$\leq [g(a)+g(b)]\left[\int_0^1 t^{\alpha-1}(h(t)+h(1-t))\,dt\right]$$
$$-\frac{\Gamma(\alpha)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right].$$

So the proof is completed. □

Corollary 1. *If we choose $h(t)=t$ and $\alpha=1$ in Theorem 10, we get the results of Theorem 3.*

Corollary 2. *If we choose $h(t)=t$ in Theorem 10, we get the result of Theorem 7. Also, if we choose $\alpha=1$, we obtain Theorem 5.*

Theorem 11. *Let $f,g:[0,\infty)\to\mathbb{R}$ be positive functions with $0\leq a<b$ and $f,g\in L_1[a,\frac{b}{m}]$. If g is an $m-$ convex function on $[0,\infty)$ and f is a $(g,m)-$ convex dominated function, then the following inequalities for fractional integrals hold:*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+m^{\alpha+1}J_{\frac{b}{m}^-}^{\alpha}f\left(\frac{a}{m}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}g(b)+m^{\alpha+1}J_{\frac{b}{m}^-}^{\alpha}g\left(\frac{a}{m}\right)\right]-g\left(\frac{a+b}{2}\right)$$

and

$$\left|\frac{1}{2}\left[\frac{\alpha f(a)+mf\left(\frac{a}{m}\right)}{\alpha+1}+m\frac{f\left(\frac{b}{m}\right)+m\alpha f\left(\frac{b}{m^2}\right)}{\alpha+1}\right]\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+m^{\alpha+1}J_{\frac{b}{m}^-}^{\alpha}f\left(\frac{a}{m}\right)\right]\right|$$
$$\leq\frac{1}{2}\left[\frac{\alpha g(a)+mg\left(\frac{a}{m}\right)}{\alpha+1}+m\frac{g\left(\frac{b}{m}\right)+m\alpha g\left(\frac{b}{m^2}\right)}{\alpha+1}\right]$$
$$-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}g(b)+m^{\alpha+1}J_{\frac{b}{m}^-}^{\alpha}g\left(\frac{a}{m}\right)\right]$$

with $\alpha>0$ and $m\in(0,1]$.

Proof. In Definition 5, if we choose $\lambda=\frac{1}{2}$ and $x=ta+(1-t)b, y=(1-t)\frac{a}{m}+t\frac{b}{m}$, we get

$$\left|\frac{f(ta+(1-t)b)+mf\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)}{2}-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{g(ta+(1-t)b)+mg\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)}{2}-g\left(\frac{a+b}{2}\right).$$

Multiplying the last inequality by $t^{\alpha-1}$, then integrating with respect to t over $[0,1]$, we obtain;

$$\begin{aligned}&\left|\frac{1}{2}\left[\int_0^1 t^{\alpha-1}\left(f\left(ta+(1-t)b\right)\right)dt+m\int_0^1 t^{\alpha-1}f((1-t)\frac{a}{m}+t\frac{b}{m}dt\right]\right.\\&\quad\left.-f\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}dt\right|\\&\quad\leq\frac{1}{2}\left[\int_0^1 t^{\alpha-1}\left(g\left(ta+(1-t)b\right)\right)dt+m\int_0^1 t^{\alpha-1}g\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)dt\right]\\&\quad-g\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}dt.\end{aligned}$$

If the necessary calculations are made and the resulting expression is edited, the first part of Theorem 11 is obtained. To get second part of Theorem 11, we can use Definition 5. Then, if we choose $x=a$ and $y=\frac{b}{m}$, we get;

$$\begin{aligned}&\left|tf(a)+m(1-t)f\left(\frac{b}{m}\right)-f\left(ta+m(1-t)\right)\frac{b}{m}\right|\\&\quad\leq tg(a)+m(1-t)g\left(\frac{b}{m}\right)-g\left(ta+m(1-t)\frac{b}{m}\right).\end{aligned}$$

Also, in Definition 5 if we choose $x=\frac{a}{m}$ and $y=\frac{b}{m^2}$, then multiplying the obtained inequality with m, the following inequality is obtained.

$$\begin{aligned}&\left|mtf\left(\frac{a}{m}\right)+m^2(1-t)f\left(\frac{b}{m^2}\right)-mf\left(t\frac{a}{m}+m(1-t)\frac{b}{m^2}\right)\right|\\&\quad\leq mtg\left(\frac{a}{m}\right)+m^2(1-t)g\left(\frac{b}{m^2}\right)-mg\left(t\frac{a}{m}+m(1-t)\frac{b}{m^2}\right).\end{aligned}$$

Let's multiply both of the last two inequalities we got above by $t^{\alpha-1}$,then integrate the obtained inequality with respect to t over $[0,1]$,

$$\begin{aligned}&\left|\frac{f(a)}{\alpha+1}+m\frac{f\left(\frac{b}{m}\right)}{\alpha(\alpha+1)}-\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}J^{\alpha}_{a^+}f(b)\right|\\&\quad\leq\frac{g(a)}{\alpha+1}+m\frac{g\left(\frac{b}{m}\right)}{\alpha(\alpha+1)}-\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}J^{\alpha}_{a^+}g(b)\end{aligned}$$

and

$$\left|m\frac{f\left(\frac{a}{m}\right)}{\alpha(\alpha+1)}+m^2\frac{f\left(\frac{b}{m^2}\right)}{\alpha+1}-m^{\alpha+1}\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}J^{\alpha}_{\frac{b}{m}^-}f\left(\frac{a}{m}\right)\right|$$

$$\leq m\frac{g\left(\frac{a}{m}\right)}{\alpha(\alpha+1)}+m^{2}\frac{g\left(\frac{b}{m^{2}}\right)}{\alpha+1}-m^{\alpha+1}\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}J^{\alpha}_{\frac{b}{m}-}g\left(\frac{a}{m}\right).$$

Finally, the second part of the theorem is proved when we arrange the inequalities we obtained using the properties of absolute value. Thus, the proof is completed. □

Corollary 3. *If we choose $m=1$ and $\alpha=1$ in Theorem 11, we get the results of Theorem 3.*

Corollary 4. *If we choose $\alpha=1$ in Theorem 11, we get the results of Theorem 4. Also, if we choose $m=1$ in Theorem 11, we get the result of Theorem 7.*

REFERENCES

[1] S. Belarbi and Z. Dahmani, "On some new fractional integral inequalities," *J. Ineq. Pure Appl. Math.*, vol. 10, p. Art. 86, 2009. [Online]. Available: https://www.emis.de/journals/JIPAM/images/139_09_JIPAM/139_09.pdf

[2] Z. Dahmani, "New inequalities in fractional integrals," *Int. J. Nonlinear Sci.*, vol. 9, pp. 493–497, 2010.

[3] M. Dokuyucu, "Caputo and Atangana-Baleanu-Caputo fractional derivative applied to garden equation," *Turkish Journal of Science*, vol. 5, pp. 1–7, 2020. [Online]. Available: https://dergipark.org.tr/en/download/article-file/1024808

[4] S. S. Dragomir, "On some new inequalities of Hermite-Hadamard type for m-convex functions," *Tamkang Journal of Mathematics*, vol. 33, no. 1, pp. 55–65, 2002, doi: 10.5556/j.tkjm.33.2002.304.

[5] S. S. Dragomir and N. M. Ionescu, "On some inequalities for convex-dominated functions," *Anal. Num. Theor. Approx.*, vol. 19, pp. 21–28, 1990.

[6] S. S. Dragomir and C. E. M. Pearce, "Selected topics on Hermite–Hadamard inequalities and applications," *RGMIA Monographs*, 2000.

[7] S. Dragomir, C. Pearce, and J. Pecaric, "Means, g-convex dominated and Hadamard- type inequalities," *Tamsui Oxford Journal of Mathematical Sciences*, vol. 18, pp. 161–173, 2002.

[8] A. Ekinci and M. E. Ozdemir, "Some New Integral Inequalities Via Riemann Liouville Integral Operators," *Applied and Computational Mathematics*, vol. 3, pp. 288–295, 2019.

[9] R. Gorenflo and F. Mainardi, *Fractional calculus: integral and differential equations of fractional order*. Wien: Springer, 1997.

[10] M. Gürbüz and M. Özdemir, "On some inequalities for product of different kinds of convex functions," *Turkish Journal of Science*, vol. 5, pp. 23–27.

[11] A. Házy, "Bernstein-Doetsch type results for h-convex functions," *Math. Inequal. Appl.*, vol. 14, no. 3, pp. 499–508, 2011, doi: 10.7153/mia-14-42.

[12] H. Kavurmaci, "Bazı farklı türden konveks fonksiyonlar icin Ostrowski ve Hermite-Hadamard tipli esitsizlikler," *Ataturk University, Graduate School of Natural and Applied Sciences*, 2012.

[13] K. S. Miller and B. Ross, *An introduction to the Fractional Calculus and Fractional Differential Equations*. John Wiley and Sons, 1993.

[14] M. E. Ozdemir and H. K. Onalan, "Hermite-Hadamard's inequalities for different kinds of convexity via fractional integrals," *Submitted*.

[15] J. E. Pecaric, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*. Boston, MA etc.: Academic Press, 1992.

[16] I. Podlubny, *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[17] M. Z. Sarikaya and N. Alp, "On Hermite-Hadamard-Fejer type integral inequalities for generalized convex functions via local fractional integrals," *Open Journal of Mathematical Sciences*, vol. 3, pp. 273–284, 2019, doi: 10.30538/oms2019.0070.

[18] M. Z. Sarikaya, A. Saglam, and H. Yildirim, "On some Hadamard-type inequalities for h-convex functions," *J. Math. Inequal.*, vol. 2, pp. 335–341, 2008, doi: 10.7153/jmi-02-30.

[19] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, "Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, pp. 2403–2407, 2013, doi: 10.1016/j.mcm.2011.12.048.

[20] E. Set, A. Akdemir, and E. Alan, "Hermite-Hadamard and Hermite-Hadamard-Fejer type inequalities involving fractional integral operators," *Filomat*, vol. 33, no. 8, pp. 2367–2380, 2019, doi: 10.2298/FIL1908367S.

[21] E. Set, A. Akdemir, and M. Özdemir, "Simpson type integral inequalities for convex functions via Riemann-Liouville integrals," *Filomat*, vol. 31, no. 14, pp. 4415–4420, 2017, doi: 10.2298/FIL1714415S.

[22] G. Toader, "Some generalizations of the convexity," *Proc. Colloq. Approx. Optim. Univ*, pp. 329–338, 1984.

[23] S. Varosanec, "On h-convexity," *J. Math. Anal. Appl.*, vol. 326, pp. 303–311, 2007, doi: 10.1016/j.jmaa.2006.02.086.

[24] F. Yalçın and N. Bekar, "Two-dimensional operator harmonically convex functions and related generalized inequalities," *Turkish Journal of Science*, vol. 4, pp. 30–38.

[25] C. Yildiz, M. E. Ozdemir, and H. K. Onalan, "Fractional integral inequalities for different functions," *New Trends in Mathematical Sciences*, vol. 3, pp. 110–117, 2015. [Online]. Available: https://www.ntmsci.com/AjaxTool/GetArticleByPublishedArticleId?PublishedArticleId=66

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FUZZY SOFT POSITIVE IMPLICATIVE HYPER *BCK*-IDEALS OF SEVERAL TYPES

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Abstract. Fuzzy soft positive implicative hyper *BCK*-ideal of types $(\ll, \subseteq, \subseteq)$, $(\ll, \ll, \subseteq)$ and $(\subseteq, \ll, \subseteq)$ are introduced, and their relations are investigated. Relations between fuzzy soft strong hyper *BCK*-ideal and fuzzy soft positive implicative hyper *BCK*-ideal of types $(\ll, \subseteq, \subseteq)$ and $(\ll, \ll, \subseteq)$ are discussed. We prove that the level set of fuzzy soft positive implicative hyper *BCK*-ideal of types $(\ll, \subseteq, \subseteq)$, $(\ll, \ll, \subseteq)$ and $(\subseteq, \ll, \subseteq)$ are positive implicative hyper *BCK*-ideal of types $(\ll, \subseteq, \subseteq)$, $(\ll, \ll, \subseteq)$ and $(\subseteq, \ll, \subseteq)$, respectively. Conditions for a fuzzy soft set to be a fuzzy soft positive implicative hyper *BCK*-ideal of types $(\ll, \subseteq, \subseteq)$, $(\ll, \ll, \subseteq)$ and $(\subseteq, \ll, \subseteq)$, respectively, are founded, and conditions for a fuzzy soft set to be a fuzzy soft weak hyper *BCK*-ideal are considered.

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Keywords: hyper *BCK*-algebra, fuzzy soft (weak, strong) hyper *BCK*-ideal, fuzzy soft positive implicative hyper *BCK*-ideal of types $(\ll, \subseteq, \subseteq)$, $(\ll, \ll, \subseteq)$ and $(\subseteq, \ll, \subseteq)$

1. INTRODUCTION

Algebraic hyperstructures represent a natural extension of classical algebraic structures and they were introduced in 1934 by the French mathematician F. Marty [13] when Marty defined hypergroups, began to analyze their properties, and applied them to groups and relational algebraic functions (see [13]). Since then, many papers and several books have been written on this topic. Nowadays, hyperstructures have a lot of applications in several branches of mathematics and computer sciences etc. (see [1,4,11,12]). In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. In [9], Jun et al. applied the hyperstructures to *BCK*-algebras, and introduced the concept of a hyper *BCK*-algebra which is a generalization of a *BCK*-algebra. Sine then, Jun et al. studied more notions and results in [5], and [8]. Dealing with uncertainties is a major problem in many areas such as economics, engineering, environmental science, medical science and social science etc. These problems cannot be dealt with by classical methods, because classical methods have inherent difficulties. To overcome these difficulties, Molodtsov [14] proposed a new approach, which was called soft set theory, for modeling uncertainty. Jun applied the notion of soft sets to the theory of BCK/BCI-algebras, and Jun et al. [5] studied ideal theory of BCK/BCI-algebras based on soft set theory. Maji et al. [15] extended the study of soft sets to fuzzy soft sets. They introduced the concept of fuzzy soft sets as a generalization of the standard soft sets, and presented an application of fuzzy soft sets in a decision making problem. Jun et al. applied fuzzy soft set to BCK/BCI-algebras. Khademan et al. [10] applied the notion of fuzzy soft sets by Maji et al. to the theory of hyper BCK-algebras. They introduced the notion of fuzzy soft positive implicative hyper BCK-ideal, and investigated several properties. They discussed the relation between fuzzy soft positive implicative hyper BCK-ideal and fuzzy soft hyper BCK-ideal, and provided characterizations of fuzzy soft positive implicative hyper BCK-ideal. Using the notion of positive implicative hyper BCK-ideal, they established a fuzzy soft weak (strong) hyper BCK-ideal.

In this paper, we introduce the notion of fuzzy soft positive implicative hyper BCK-ideal of types $(\ll,\subseteq,\subseteq)$, $(\ll,\ll,\subseteq)$ and $(\subseteq,\ll,\subseteq)$, and investigate their relations and properties. We discuss relations between fuzzy soft strong hyper BCK-ideal and fuzzy soft positive implicative hyper BCK-ideal of types $(\ll,\subseteq,\subseteq)$ and $(\ll,\ll,\subseteq)$. We prove that the level set of fuzzy soft positive implicative hyper BCK-ideal of types $(\ll,\subseteq,\subseteq)$, $(\ll,\ll,\subseteq)$ and $(\subseteq,\ll,\subseteq)$ are positive implicative hyper BCK-ideal of types $(\ll,\subseteq,\subseteq)$, $(\ll,\ll,\subseteq)$ and $(\subseteq,\ll,\subseteq)$, respectively. We find conditions for a fuzzy soft set to be a fuzzy soft positive implicative hyper BCK-ideal of types $(\ll,\subseteq,\subseteq)$, $(\ll,\ll,\subseteq)$ and $(\subseteq,\ll,\subseteq)$, respectively. We also consider conditions for a fuzzy soft set to be a fuzzy soft weak hyper BCK-ideal.

2. PRELIMINARIES

Let H be a nonempty set endowed with a hyper operation "$\circ$", that is, "$\circ$" is a function from $H\times H$ to $\mathcal{P}^*(H)=\mathcal{P}(H)\setminus\{\varnothing\}$. For two subsets A and B of H, denote by $A\circ B$ the set $\cup\{a\circ b\mid a\in A, b\in B\}$. We shall use $x\circ y$ instead of $x\circ\{y\}$, $\{x\}\circ y$, or $\{x\}\circ\{y\}$.

By a hyper BCK-algebra (see [9]) we mean a nonempty set H endowed with a hyper operation "$\circ$" and a constant 0 satisfying the following axioms:

(H1) $(x\circ z)\circ(y\circ z)\ll x\circ y$,
(H2) $(x\circ y)\circ z=(x\circ z)\circ y$,
(H3) $x\circ H\ll\{x\}$,
(H4) $x\ll y$ and $y\ll x$ imply $x=y$,

for all $x,y,z\in H$, where $x\ll y$ is defined by $0\in x\circ y$ and for every $A,B\subseteq H$, $A\ll B$ is defined by $\forall a\in A,\exists b\in B$ such that $a\ll b$.

In a hyper BCK-algebra H, the condition (H3) is equivalent to the condition:

$$x\circ y\ll\{x\}. \tag{2.1}$$

In any hyper BCK-algebra H, the following hold (see [9]):

$$x \circ 0 \ll \{x\},\ 0 \circ x \ll \{0\},\ 0 \circ 0 \ll \{0\}, \tag{2.2}$$
$$(A \circ B) \circ C = (A \circ C) \circ B,\ A \circ B \ll A,\ 0 \circ A \ll \{0\}, \tag{2.3}$$
$$0 \circ 0 = \{0\}, \tag{2.4}$$
$$0 \ll x,\ x \ll x,\ A \ll A, \tag{2.5}$$
$$A \subseteq B \text{ implies } A \ll B, \tag{2.6}$$
$$0 \circ x = \{0\},\ 0 \circ A = \{0\}, \tag{2.7}$$
$$A \ll \{0\} \text{ implies } A = \{0\}, \tag{2.8}$$
$$x \in x \circ 0, \tag{2.9}$$
$$x \circ 0 = \{x\},\ A \circ 0 = A, \tag{2.10}$$

for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H.
A subset I of a hyper BCK-algebra H is called a *hyper BCK-ideal* of H (see [9]) if it satisfies

$$0 \in I \tag{2.11}$$
$$(\forall x, y \in H)\ (x \circ y \ll I,\ y \in I \Rightarrow x \in I) \tag{2.12}$$

A subset I of a hyper BCK-algebra H, is called a *strong hyper BCK-ideal* of H (see [8]) if it satisfies (2.11) and

$$(\forall x, y \in H)\ ((x \circ y) \cap I \neq \varnothing,\ y \in I \Rightarrow x \in I). \tag{2.13}$$

Recall that every strong hyper BCK-ideal is a hyper BCK-ideal, but the converse may not be true (see [8]). A subset I of a hyper BCK-algebra H is called a *weak hyper BCK-ideal* of H (see [9]) if it satisfies (2.11) and

$$(\forall x, y \in H)\ (x \circ y \subseteq I,\ y \in I \Rightarrow x \in I) \tag{2.14}$$

Every hyper BCK-ideal is a weak hyper BCK-ideal, but the converse may not be true.
A subset I of a hyper BCK-algebra H is said to be

- *reflexive* if $(x \circ x) \subseteq I$ for all $x \in H$,
- *closed* if the following assertion is valid.

$$(\forall x \in H)(\forall y \in I)(x \ll y\ \Rightarrow\ x \in I).$$

Given a subset I of H and $x, y, z \in H$, we consider the following conditions:

$$(x \circ y) \circ z \subseteq I,\ y \circ z \subseteq I \Rightarrow x \circ z \subseteq I \tag{2.15}$$
$$(x \circ y) \circ z \subseteq I,\ y \circ z \ll I \Rightarrow x \circ z \subseteq I \tag{2.16}$$
$$(x \circ y) \circ z \ll I,\ y \circ z \subseteq I \Rightarrow x \circ z \subseteq I \tag{2.17}$$
$$(x \circ y) \circ z \ll I,\ y \circ z \ll I \Rightarrow x \circ z \subseteq I \tag{2.18}$$

Definition 1 ([3, 6])**.** Let I be a nonempty subset of a hyper BCK-algebra H and $0 \in I$. If it satisfies (2.15) (resp. (2.16), (2.17) and (2.18)), then we say that I is a positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ (resp. $(\subseteq, \ll, \subseteq)$, $(\ll, \subseteq, \subseteq)$ and $(\ll, \ll, \subseteq)$) for all $x, y, z \in H$.

Molodtsov ([14]) defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $\mathscr{P}(U)$ denote the power set of U and $A \subseteq E$.

Definition 2 ([14])**.** A pair (λ, A) is called a *soft set* over U, where λ is a mapping given by

$$\lambda : A \to \mathscr{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\lambda(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (λ, A). Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [14].

Definition 3 ([15])**.** Let U be an initial universe set and E be a set of parameters. Let $\mathcal{F}(U)$ denote the set of all fuzzy sets in U. Then a pair $(\tilde{\lambda}, A)$ is called a *fuzzy soft set* over U where $A \subseteq E$ and $\tilde{\lambda}$ is a mapping given by $\tilde{\lambda} : A \to \mathcal{F}(U)$.

In general, for every parameter u in A, $\tilde{\lambda}[u]$ is a fuzzy set in U and it is called *fuzzy value set* of parameter u.

Given a fuzzy set μ in a hyper BCK-algebra H and a subset T of H, by $\mu_*(T)$ and $\mu^*(T)$ we mean

$$\mu_*(T) = \inf_{a \in T} \mu(a) \text{ and } \mu^*(T) = \sup_{a \in T} \mu(a). \tag{2.19}$$

Definition 4 ([2])**.** A fuzzy soft set $(\tilde{\lambda}, A)$ over a hyper BCK-algebra H is called

- a *fuzzy soft hyper BCK-ideal* based on a paramenter $u \in A$ over H (briefly, u-fuzzy soft hyper BCK-ideal of H) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0,1]$ of u satisfies the following conditions:

$$(\forall x, y \in H) \left(x \ll y \Rightarrow \tilde{\lambda}[u](x) \geq \tilde{\lambda}[u](y) \right), \tag{2.20}$$

$$(\forall x, y \in H) \left(\tilde{\lambda}[u](x) \geq \min\{\tilde{\lambda}[u]_*(x \circ y), \tilde{\lambda}[u](y)\} \right). \tag{2.21}$$

- a *fuzzy soft weak hyper BCK-ideal* based on a paramenter $u \in A$ over H (briefly, *u-fuzzy soft weak hyper BCK-ideal* of H) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0,1]$ of u satisfies condition (2.21) and

$$(\forall x \in H) \left(\tilde{\lambda}[u](0) \geq \tilde{\lambda}[u](x) \right). \tag{2.22}$$

- a *fuzzy soft strong hyper BCK-ideal* over H based on a paramenter u in A (briefly, u-fuzzy soft strong hyper BCK-ideal of H) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0,1]$ of u satisfies the following conditions:

$$(\forall x, y \in H)\left(\tilde{\lambda}[u](x) \geq \min\{\tilde{\lambda}[u]^*(x \circ y), \tilde{\lambda}[u](y)\}\right), \tag{2.23}$$

$$(\forall x \in H)\left(\tilde{\lambda}[u]_*(x \circ x) \geq \tilde{\lambda}[u](x)\right). \tag{2.24}$$

If $(\tilde{\lambda}, A)$ is a fuzzy soft (weak, strong) hyper BCK-ideal based on a paramenter u over H for all $u \in A$, we say that $(\tilde{\lambda}, A)$ is a *fuzzy soft (weak, strong) hyper BCK-ideal* of H.

3. Fuzzy soft positive implicative hyper *BCK*-ideals

In what follows, let H be a hyper BCK-algebra unless otherwise specified.

Definition 5. Let $(\tilde{\lambda}, A)$ be a fuzzy soft set over H. Then $(\tilde{\lambda}, A)$ is called

- a *fuzzy soft positive implicative hyper BCK-ideal* of type $(\subseteq, \subseteq, \subseteq)$ based on a parameter $u \in A$ over H (briefly, u-fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0,1]$ of u satisfies the following conditions:

$$(\forall x, y \in H)\ (x \ll y \Rightarrow \tilde{\lambda}[u](x) \geq \tilde{\lambda}[u](y)), \tag{3.1}$$

$$(\forall x, y, z \in H)(\tilde{\lambda}[u]_*(x \circ z) \geq \min\{\tilde{\lambda}[u]_*((x \circ y) \circ z), \tilde{\lambda}[u]_*(y \circ z)\}). \tag{3.2}$$

- a *fuzzy soft positive implicative hyper BCK-ideal* of type $(\subseteq, \ll, \subseteq)$ based on a parameter $u \in A$ over H (briefly, u-fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0,1]$ of u satisfies (3.1) and

$$(\forall x, y, z \in H)\ (\tilde{\lambda}[u]_*(x \circ z) \geq \min\{\tilde{\lambda}[u]_*((x \circ y) \circ z), \tilde{\lambda}[u]^*(y \circ z)\}). \tag{3.3}$$

- a *fuzzy soft positive implicative hyper BCK-ideal* of type $(\ll, \subseteq, \subseteq)$ based on a parameter $u \in A$ over H (briefly, u-fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0,1]$ of u satisfies (3.1) and

$$(\forall x, y, z \in H)\ (\tilde{\lambda}[u]_*(x \circ z) \geq \min\{\tilde{\lambda}[u]^*((x \circ y) \circ z), \tilde{\lambda}[u]_*(y \circ z)\}). \tag{3.4}$$

- a *fuzzy soft positive implicative hyper BCK-ideal* of type $(\ll, \ll, \subseteq)$ based on a parameter $u \in A$ over H (briefly, u-fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0,1]$ of u satisfies (3.1) and

$$(\forall x, y, z \in H)\ (\tilde{\lambda}[u]_*(x \circ z) \geq \min\{\tilde{\lambda}[u]^*((x \circ y) \circ z), \tilde{\lambda}[u]^*(y \circ z)\}). \tag{3.5}$$

Theorem 1. *Let $(\tilde{\lambda}, A)$ be a fuzzy soft set over H.*

(1) *If* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive implicative hyper* BCK*-ideal of type* $(\ll, \subseteq, \subseteq)$ *or type* $(\subseteq, \ll, \subseteq)$*, then* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive implicative hyper* BCK*-ideal of type* $(\subseteq, \subseteq, \subseteq)$*.*

(2) *If* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive implicative hyper* BCK*-ideal of type* $(\ll, \ll, \subseteq)$*, then* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive implicative hyper* BCK*-ideal of type* $(\ll, \subseteq, \subseteq)$ *and* $(\subseteq, \ll, \subseteq)$*.*

Proof. (1) Assume that $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ or type $(\subseteq, \ll, \subseteq)$. Then

$$\begin{aligned}\tilde{\lambda}[u]_*(x \circ z) &\geq \min\{\tilde{\lambda}[u]^*((x \circ y) \circ z), \tilde{\lambda}[u]_*(y \circ z)\} \\ &\geq \min\{\tilde{\lambda}[u]_*((x \circ y) \circ z), \tilde{\lambda}[u]_*(y \circ z)\}\end{aligned}$$

or

$$\begin{aligned}\tilde{\lambda}[u]_*(x \circ z) &\geq \min\{\tilde{\lambda}[u]_*((x \circ y) \circ z), \tilde{\lambda}[u]^*(y \circ z)\} \\ &\geq \min\{\tilde{\lambda}[u]_*((x \circ y) \circ z), \tilde{\lambda}[u]_*(y \circ z)\},\end{aligned}$$

respectively. Thus $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$.

(2) Suppose that $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$. Then

$$\begin{aligned}\tilde{\lambda}[u]_*(x \circ z) &\geq \min\{\tilde{\lambda}[u]^*((x \circ y) \circ z), \tilde{\lambda}[u]^*(y \circ z)\} \\ &\geq \min\{\tilde{\lambda}[u]^*((x \circ y) \circ z), \tilde{\lambda}[u]_*(y \circ z)\}\end{aligned}$$

and

$$\begin{aligned}\tilde{\lambda}[u]_*(x \circ z) &\geq \min\{\tilde{\lambda}[u]^*((x \circ y) \circ z), \tilde{\lambda}[u]^*(y \circ z)\} \\ &\geq \min\{\tilde{\lambda}[u]_*((x \circ y) \circ z), \tilde{\lambda}[u]^*(y \circ z)\}.\end{aligned}$$

Therefore $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ and $(\subseteq, \ll, \subseteq)$. □

Corollary 1. *If* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive implicative hyper* BCK*-ideal of type* $(\ll, \ll, \subseteq)$*, then* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive implicative hyper* BCK*-ideal of type* $(\subseteq, \subseteq, \subseteq)$*.*

The following example shows that any fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ is not a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.

Example 1. Consider a hyper BCK-algebra $H = \{0, a, b, c\}$ with the hyper operation "$\circ$" in Table 1.
Given a set $A = \{x, y\}$ of parameters, we define a fuzzy soft set $(\tilde{\lambda}, A)$ by Table 2. Then $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$.

TABLE 1. Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{b,c\}$	$\{0,b,c\}$

TABLE 2. Tabular representation of $(\tilde{\lambda},A)$

$\tilde{\lambda}$	0	a	b	c
x	0.9	0.8	0.5	0.3
y	0.9	0.7	0.6	0.4

Since

$$\tilde{\lambda}[x]_*(c\circ 0)=0.3<0.5=\min\left\{\tilde{\lambda}[x]^*((c\circ b)\circ 0),\tilde{\lambda}[x]_*(b\circ 0)\right\},$$

it is not an x-fuzzy soft positive implicative hyper *BCK*-ideal of type $(\ll,\subseteq,\subseteq)$, and thus it is not a fuzzy soft positive implicative hyper *BCK*-ideal of type $(\ll,\subseteq,\subseteq)$.

Question.

> Is a fuzzy soft positive implicative hyper *BCK*-ideal of type $(\subseteq,\subseteq,\subseteq)$ a fuzzy soft positive implicative hyper *BCK*-ideal of type $(\subseteq,\ll,\subseteq)$?

The following example shows that any fuzzy soft positive implicative hyper *BCK*-ideal of type $(\subseteq,\ll,\subseteq)$ is not a fuzzy soft positive implicative hyper *BCK*-ideal of type $(\ll,\subseteq,\subseteq)$ or $(\ll,\ll,\subseteq)$.

Example 2. Consider a hyper *BCK*-algebra $H=\{0,a,b\}$ with the hyper operation "$\circ$" in Table 3.

TABLE 3. Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a,b\}$

Given a set $A=\{x,y\}$ of parameters, we define a fuzzy soft set $(\tilde{\lambda},A)$ by Table 4.

TABLE 4. Tabular representation of $(\tilde{\lambda}, A)$

$\tilde{\lambda}$	0	a	b
x	0.9	0.5	0.3
y	0.8	0.7	0.1

Then $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$. Since

$$\tilde{\lambda}[x]_*(b \circ b) = 0.3 < 0.9 = \min\left\{\tilde{\lambda}[x]^*((b \circ a) \circ b), \tilde{\lambda}[x]^*(a \circ b)\right\},$$

it is not an x-fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ and so not a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$. Also, since

$$\tilde{\lambda}[y]_*(b \circ b) = 0.1 < 0.8 = \min\left\{\tilde{\lambda}[y]^*((b \circ 0) \circ b), \tilde{\lambda}[y]^*(0 \circ b)\right\},$$

it is not a y-fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$ and so not a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$.

Question.

> Is a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$ or $(\ll, \ll, \subseteq)$?

Lemma 1 ([10]). *Every fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ is a fuzzy soft hyper BCK-ideal.*

The converse of Lemma 1 is not true (see [10, Example 3.6]). Using Theorems 1 and Lemma 1, we have the following corollary.

Corollary 2. *Every fuzzy soft positive implicative hyper BCK-ideal $(\tilde{\lambda}, A)$ of types $(\ll, \subseteq, \subseteq)$, $(\subseteq, \ll, \subseteq)$ or $(\ll, \ll, \subseteq)$ is a fuzzy soft hyper BCK-ideal.*

We can check that the fuzzy soft set $(\tilde{\lambda}, A)$ in Example 1 is a fuzzy soft hyper BCK-ideal of H, but it is not a fuzzy soft positive implicative hyper BCK-ideal of types $(\ll, \subseteq, \subseteq)$. This shows that any fuzzy soft hyper BCK-ideal may not be a fuzzy soft positive implicative hyper BCK-ideal of types $(\ll, \subseteq, \subseteq)$. Also, we know that the fuzzy soft set $(\tilde{\lambda}, A)$ in Example 2 is a fuzzy soft hyper BCK-ideal of H, but it is a fuzzy soft hyper BCK-ideal of type $(\ll, \ll, \subseteq)$. Thus any fuzzy soft hyper BCK-ideal may not be a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$. Let $(\tilde{\lambda}, A)$ be a fuzzy soft hyper BCK-ideal of H. If $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal $(\tilde{\lambda}, A)$ of type $(\subseteq, \ll, \subseteq)$, then it is a fuzzy soft positive implicative hyper BCK-ideal $(\tilde{\lambda}, A)$ of type $(\subseteq, \subseteq, \subseteq)$ by Theorem 1(1). Hence every fuzzy soft hyper BCK-ideal of H is a fuzzy soft positive implicative hyper BCK-ideal $(\tilde{\lambda}, A)$ of type $(\subseteq, \subseteq, \subseteq)$. But this is contradictory to [10, Example 3.6]. Therefore we know that any fuzzy soft hyper BCK-ideal may not be a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$.

We consider relation between a fuzzy soft positive implicative hyper BCK-ideal of any type and a fuzzy soft strong hyper BCK-ideal.

Theorem 2. *Every fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ is a fuzzy soft strong hyper BCK-ideal of H.*

Proof. Let $(\tilde{\lambda}, A)$ be a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ and let u be any parameter in A. Since $x \circ x \ll x$ for all $x \in H$, it follows from (3.1) that

$$\tilde{\lambda}[u]_*(x \circ x) \geq \tilde{\lambda}[u]_*(x) = \tilde{\lambda}[u](x).$$

Taking $z = 0$ in (3.4) and using (2.10) imply that

$$\begin{aligned}\tilde{\lambda}[u](x) &= \tilde{\lambda}[u]_*(x \circ 0)\\ &\geq \min\{\tilde{\lambda}[u]^*((x \circ y) \circ 0), \tilde{\lambda}[u]_*(y \circ 0)\}\\ &= \min\{\tilde{\lambda}[u]^*(x \circ y), \tilde{\lambda}[u](y)\}.\end{aligned}$$

Therefore $(\tilde{\lambda}, A)$ is a fuzzy soft strong hyper BCK-ideal of H. □

Corollary 3. *Every fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$ is a fuzzy soft strong hyper BCK-ideal of H.*

The following example shows that the converse of Theorem 2 and Corollary 3 is not true in general.

Example 3. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" which is given in Table 5. Given a set $A = \{x, y\}$ of parameters, we define a fuzzy

TABLE 5. Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

soft set $(\tilde{\lambda}, A)$ by Table 6.

Then $(\tilde{\lambda}, A)$ is a fuzzy soft strong hyper BCK-ideal of H. Since

$$\tilde{\lambda}[x]_*(b \circ b) = 0.5 < 0.9 = \min\left\{\tilde{\lambda}[x]^*((b \circ 0) \circ b), \tilde{\lambda}[x]_*(0 \circ b)\right\},$$

TABLE 6. Tabular representation of $(\tilde{\lambda}, A)$

$\tilde{\lambda}$	0	a	b
x	0.9	0.1	0.5
y	0.7	0.2	0.6

we know that $(\tilde{\lambda}, A)$ is not an x-fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ and so it is not a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$. Also

$$\tilde{\lambda}[y]_*(b \circ b) = 0.6 < 0.7 = \min\left\{\tilde{\lambda}[y]^*((b \circ b) \circ b), \tilde{\lambda}[y]^*(b \circ b)\right\},$$

and so $(\tilde{\lambda}, A)$ it is not a y-fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$. Thus it is not a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$. Therefore any fuzzy soft strong hyper BCK-ideal of H may not be a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ or $(\ll, \ll, \subseteq)$.

Consider the hyper BCK-algebra $H = \{0, a, b, c\}$ in Example 1 and a set $A = \{x, y\}$ of parameters. We define a fuzzy soft set $(\tilde{\lambda}, A)$ by Table 2 in Example 1. Then $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ and $(\subseteq, \ll, \subseteq)$. But $(\tilde{\lambda}, A)$ is not a fuzzy soft strong hyper BCK-ideal of H since

$$\tilde{\lambda}[y](c) = 0.4 < 0.6 = \min\left\{\tilde{\lambda}[y]^*(c \circ b), \tilde{\lambda}[y](b)\right\}.$$

Hence we know that any fuzzy soft positive implicative hyper BCK-ideal of types $(\subseteq, \subseteq, \subseteq)$ and $(\subseteq, \ll, \subseteq)$ is not a fuzzy soft strong hyper BCK-ideal of H.

Given a fuzzy soft set $(\tilde{\lambda}, A)$ over H and $t \in [0, 1]$, we consider the following set

$$U(\tilde{\lambda}[u]; t) := \left\{x \in H \mid \tilde{\lambda}[u](x) \geq t\right\} \tag{3.6}$$

where u is a parameter in A, which is called *level set* of $(\tilde{\lambda}, A)$.

Lemma 2. *If a fuzzy soft set $(\tilde{\lambda}, A)$ over H satisfies the condition (3.1), then $0 \in U(\tilde{\lambda}[u]; t)$ for all $t \in [0, 1]$ and any parameter u in A with $U(\tilde{\lambda}[u]; t) \neq \varnothing$.*

Proof. Let $(\tilde{\lambda}, A)$ be a fuzzy soft set over H which satisfies the condition (3.1). For any $t \in [0, 1]$ and any parameter u in A, assume that $U(\tilde{\lambda}[u]; t) \neq \varnothing$. Since $0 \ll x$ for all $x \in H$, it follows from (3.1) that $\tilde{\lambda}[u](0) \geq \tilde{\lambda}[u](x)$ for all $x \in H$. Hence $\tilde{\lambda}[u](0) \geq \tilde{\lambda}[u](x)$ for all $x \in U(\tilde{\lambda}[u]; t)$, and so $\tilde{\lambda}[u](0) \geq t$. Thus $0 \in U(\tilde{\lambda}[u]; t)$. □

Lemma 3 ([2]). *A fuzzy soft set $(\tilde{\lambda}, A)$ over H is a fuzzy soft hyper BCK-ideal of H if and only if the set $U(\tilde{\lambda}[u]; t)$ in (3.6) is a hyper BCK-ideal of H for all $t \in [0, 1]$ and any parameter u in A with $U(\tilde{\lambda}[u]; t) \neq \varnothing$.*

Theorem 3. *If a fuzzy soft set* $(\tilde{\lambda}, A)$ *over* H *is a fuzzy soft positive implicative hyper BCK-ideal of type* $(\subseteq, \ll, \subseteq)$*, then the set* $U(\tilde{\lambda}[u];t)$ *in* (3.6) *is a positive implicative hyper BCK-ideal of type* $(\subseteq, \ll, \subseteq)$ *for all* $t \in [0,1]$ *and any parameter* u *in* A *with* $U(\tilde{\lambda}[u];t) \neq \varnothing$.

Proof. Assume that a fuzzy soft set $(\tilde{\lambda}, A)$ over H is a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$. Then $0 \in U(\tilde{\lambda}[u];t)$ by Lemma 2. Let $x, y, z \in H$ be such that $(x \circ y) \circ z \subseteq U(\tilde{\lambda}[u];t)$ and $y \circ z \ll U(\tilde{\lambda}[u];t)$. Then

$$\tilde{\lambda}[u](a) \geq t \text{ for all } a \in (x \circ y) \circ z \tag{3.7}$$

and

$$(\forall b \in y \circ z)(\exists c \in U(\tilde{\lambda}[u];t))(b \ll c). \tag{3.8}$$

The condition (3.7) implies $\tilde{\lambda}[u]_*((x \circ y) \circ z) \geq t$, and the condition (3.8) implies from (3.1) that $\tilde{\lambda}[u](b) \geq \tilde{\lambda}[u](c) \geq t$ for all $b \in y \circ z$. Let $d \in x \circ z$. Using (3.3), we have

$$\tilde{\lambda}[u](d) \geq \tilde{\lambda}[u]_*(x \circ z) \geq \min\left\{\tilde{\lambda}[u]_*((x \circ y) \circ z), \tilde{\lambda}[u]^*(y \circ z)\right\} \geq t.$$

Thus $d \in U(\tilde{\lambda}[u];t)$, and so $x \circ z \subseteq U(\tilde{\lambda}[u];t)$. Therefore $U(\tilde{\lambda}[u];t)$ is a positive implicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$. □

The following example shows that the converse of Theorem 3 is not true in general.

Example 4. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" in Table 7.

TABLE 7. Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Given a set $A = \{x, y\}$ of parameters, we define a fuzzy soft set $(\tilde{\lambda}, A)$ by Table 8.

TABLE 8. Tabular representation of $(\tilde{\lambda}, A)$

$\tilde{\lambda}$	0	a	b
x	0.9	0.5	0.8
y	0.8	0.3	0.6

Then

$$U(\tilde{\lambda}[x];t) = \begin{cases} \varnothing & \text{if } t \in (0.9, 1], \\ \{0\} & \text{if } t \in (0.8, 0.9], \\ \{0, b\} & \text{if } t \in (0.5, 0.8], \\ H & \text{if } t \in [0, 0.5] \end{cases}$$

and

$$U(\tilde{\lambda}[y];t) = \begin{cases} \varnothing & \text{if } t \in (0.8, 1], \\ \{0\} & \text{if } t \in (0.6, 0.8], \\ \{0, b\} & \text{if } t \in (0.3, 0.6], \\ H & \text{if } t \in [0, 0.3], \end{cases}$$

which are positive implicative hyper BCK-ideals of type $(\subseteq, \ll, \subseteq)$. Note that $a \ll b$ and $\tilde{\lambda}[u](a) < \tilde{\lambda}[u](b)$ for all $u \in A$. Thus $(\tilde{\lambda}, A)$ is not a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$.

Lemma 4 ([3]). *Every positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ is a weak hyper BCK-ideal of H.*

Lemma 5 ([8]). *Let I be a reflexive hyper BCK-ideal of H. Then*

$$(\forall x, y \in H)((x \circ y) \cap I \neq \varnothing \ \Rightarrow \ x \circ y \ll I). \tag{3.9}$$

Lemma 6. *If any subset I of H is closed and satisfies the condition (2.14), then the condition (2.12) is valid.*

Proof. Assume that $x \circ y \ll I$ and $y \in I$ for all $x, y \in H$. Let $a \in x \circ y$. Then there exists $b \in I$ such that $a \ll b$. Since I is closed, we have $a \in I$ and thus $x \circ y \subseteq I$. It follows from (2.14) that $x \in I$. □

Theorem 4. *Let A be a fuzzy soft set over H satisfying the condition (3.1) and*

$$(\forall T \in \mathcal{P}(H))(\exists x_0 \in T)\left(\tilde{\lambda}[u](x_0) = \tilde{\lambda}[u]^*(T)\right). \tag{3.10}$$

If the set $U(\tilde{\lambda}[u];t)$ in (3.6) is a reflexive positive implicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$ for all $t \in [0, 1]$ and any parameter u in A with $U(\tilde{\lambda}[u];t) \neq \varnothing$, then $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$.

Proof. For any $x, y, z \in H$ let

$$t := \min\{\tilde{\lambda}[u]_*((x \circ y) \circ z), \tilde{\lambda}[u]^*(y \circ z)\}.$$

Then $\tilde{\lambda}[u]_*((x \circ y) \circ z) \geq t$ and so $\tilde{\lambda}[u](a) \geq t$ for all $a \in (x \circ y) \circ z$. Since $\tilde{\lambda}[u]^*(y \circ z) \geq t$, it follows from (3.10) that $\tilde{\lambda}[u](b_0) = \tilde{\lambda}[u]^*(y \circ z) \geq t$ for some $b_0 \in y \circ z$. Hence $b_0 \in U(\tilde{\lambda}[u];t)$, and thus $U(\tilde{\lambda}[u];t) \cap (y \circ z) \neq \varnothing$. Since $U(\tilde{\lambda}[u];t)$ is a positive implicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$ and hence of type $(\subseteq, \subseteq, \subseteq)$, $U(\tilde{\lambda}[u];t)$ is a weak hyper BCK-ideal of H by Lemma 4. Let $x, \in H$ be such that $x \ll y$. If $y \in U(\tilde{\lambda}[u];t)$, then $\tilde{\lambda}[u](x) \geq \tilde{\lambda}[u](y) \geq t$ by (3.1) and so $x \in U(\tilde{\lambda}[u];t)$, that is, $U(\tilde{\lambda}[u];t)$ is closed. Hence $U(\tilde{\lambda}[u];t)$ is a hyper BCK-ideal of H by Lemma 6.

Since $U(\tilde{\lambda}[u];t)$ is reflexive, it follows from Lemma 5 that $y\circ z \ll U(\tilde{\lambda}[u];t)$. Hence $x\circ z \subseteq U(\tilde{\lambda}[u];t)$ since $U(\tilde{\lambda}[u];t)$ is a positive implicative hyper *BCK*-ideal of type $(\subseteq,\ll,\subseteq)$. Hence

$$\tilde{\lambda}[u](a) \geq t = \min\{\tilde{\lambda}[u]_*((x\circ y)\circ z), \tilde{\lambda}[u]^*(y\circ z)\}$$

for all $a \in x\circ z$, and thus

$$\tilde{\lambda}[u]_*(x\circ z) \geq \min\{\tilde{\lambda}[u]_*((x\circ y)\circ z), \tilde{\lambda}[u]^*(y\circ z)\}$$

for all $x,y,z \in H$. Therefore $(\tilde{\lambda},A)$ is a fuzzy soft positive implicative hyper *BCK*-ideal of type $(\subseteq, \ll, \subseteq)$. □

Corollary 4. *Let A be a fuzzy soft set over H satisfying the condition* (3.1) *and* (3.10). *For any $t \in [0,1]$ and any parameter u in A, assume that $U(\tilde{\lambda}[u];t)$ is nonempty and reflexive. Then $(\tilde{\lambda},A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq,\ll,\subseteq)$ if and only if $U(\tilde{\lambda}[u];t)$ is a positive implicative hyper BCK-ideal of type $(\subseteq,\ll,\subseteq)$.*

Theorem 5. *If a fuzzy soft set $(\tilde{\lambda},A)$ over H is a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll,\subseteq,\subseteq)$, then the set $U(\tilde{\lambda}[u];t)$ in* (3.6) *is a positive implicative hyper BCK-ideal of type $(\ll,\subseteq,\subseteq)$ for all $t \in [0,1]$ and any parameter u in A with $U(\tilde{\lambda}[u];t) \neq \varnothing$.*

Proof. Let $(\tilde{\lambda},A)$ be a fuzzy soft positive implicative hyper *BCK*-ideal of type $(\ll,\subseteq,\subseteq)$. Then $0 \in U(\tilde{\lambda}[u];t)$ by Lemma 2. Let $x,y,z \in H$ be such that $(x\circ y)\circ z \ll U(\tilde{\lambda}[u];t)$ and $y\circ z \subseteq U(\tilde{\lambda}[u];t)$. Then

$$(\forall a \in (x\circ y)\circ z)(\exists b \in U(\tilde{\lambda}[u];t))(a \ll b), \tag{3.11}$$

which implies from (3.1) that $\tilde{\lambda}[u](a) \geq \tilde{\lambda}[u](b)$ for all $a \in (x\circ y)\circ z$. Since $y\circ z \subseteq U(\tilde{\lambda}[u];t)$, we have

$$\tilde{\lambda}[u](a) \geq t \text{ for all } a \in y\circ z. \tag{3.12}$$

Let $c \in x\circ z$. Then

$$\tilde{\lambda}[u](c) \geq \tilde{\lambda}[u]_*(x\circ z) \geq \min\{\tilde{\lambda}[u]^*((x\circ y)\circ z), \tilde{\lambda}[u]_*(y\circ z)\} \geq t$$

for all $x,y,z \in H$ by (3.4), and thus $c \in U(\tilde{\lambda}[u];t)$. Hence $x\circ z \subseteq U(\tilde{\lambda}[u];t)$. Therefore $U(\tilde{\lambda}[u];t)$ is a positive implicative hyper *BCK*-ideal of type $(\ll,\subseteq,\subseteq)$. □

The converse of Theorem 5 is not true as seen in the following example.

Example 5. Consider the hyper BCK-algebra $H = \{0, a, b\}$ and the fuzzy soft set $(\tilde{\lambda}, A)$ in Example 2. Then

$$U(\tilde{\lambda}[x];t) = \begin{cases} \varnothing & \text{if } t \in (0.9, 1], \\ \{0\} & \text{if } t \in (0.5, 0.9], \\ \{0, a\} & \text{if } t \in (0.3, 0.5], \\ H & \text{if } t \in [0, 0.3] \end{cases}$$

and

$$U(\tilde{\lambda}[y];t) = \begin{cases} \varnothing & \text{if } t \in (0.8, 1], \\ \{0\} & \text{if } t \in (0.7, 0.8], \\ \{0, a\} & \text{if } t \in (0.1, 0.7], \\ H & \text{if } t \in [0, 0.1], \end{cases}$$

which are positive implicative hyper BCK-ideals of type $(\ll, \subseteq, \subseteq)$. But we know $(\tilde{\lambda}, A)$ is not a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.

Lemma 7 ([8]). *Every reflexive hyper BCK-ideal I of H satisfies the following implication:*

$$(\forall x, y \in H)\ ((x \circ y) \cap I \neq \varnothing \Rightarrow x \circ y \subseteq I)$$

Lemma 8 ([7]). *Every positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ is a hyper BCK-ideal.*

We provide conditions for a fuzzy soft set to be a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.

Theorem 6. *Let A be a fuzzy soft set over H satisfying the condition* (3.10). *If the set $U(\tilde{\lambda}[u];t)$ in* (3.6) *is a reflexive positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ for all $t \in [0, 1]$ and any parameter u in A with $U(\tilde{\lambda}[u];t) \neq \varnothing$, then $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.*

Proof. Assume that $U(\tilde{\lambda}[u];t) \neq \varnothing$ for all $t \in [0, 1]$ and any parameter u in A. Suppose that $U(\tilde{\lambda}[u];t)$ is a positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$. Then $U(\tilde{\lambda}[u];t)$ is a hyper BCK-ideal of H by Lemma (8). It follows from Lemma (3) that $(\tilde{\lambda}, A)$ is a fuzzy soft hyper BCK-ideal of H. Thus the condition (3.1) is valid. Now let $t = \min\left\{\tilde{\lambda}[u]^*((x \circ y) \circ z), \tilde{\lambda}[u]_*(y \circ z)\right\}$ for $x, y, z \in H$. Since $(\tilde{\lambda}, A)$ satisfies the condition (3.10), there exists $x_0 \in (x \circ y) \circ z$ such that $\tilde{\lambda}[u](x_0) = \tilde{\lambda}[u]^*((x \circ y) \circ z) \geq t$ and so $x_0 \in U(\tilde{\lambda}[u];t)$. Hence $((x \circ y) \circ z) \cap U(\tilde{\lambda}[u];t) \neq \varnothing$ and so $(x \circ y) \circ z \ll U(\tilde{\lambda}[u];t)$ by Lemma 7 and (2.6). Moreover $\tilde{\lambda}[u](c) \geq \tilde{\lambda}[u]_*(y \circ z) \geq t$ for all $c \in y \circ z$, and hence $c \in U(\tilde{\lambda}[u];t)$ which shows that $y \circ z \subseteq U(\tilde{\lambda}[u];t)$. Since $U(\tilde{\lambda}[u];t)$ is a positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$, it follows that $x \circ z \subseteq U(\tilde{\lambda}[u];t)$. Thus $\tilde{\lambda}[u](a) \geq t$ for all $a \in x \circ z$, and so

$$\tilde{\lambda}[u]_*(x \circ z) \geq t = \min\left\{\tilde{\lambda}[u]^*((x \circ y) \circ z), \tilde{\lambda}[u]_*(y \circ z)\right\}.$$

Consequently, $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$. □

Corollary 5. *Let A be a fuzzy soft set over H satisfying the condition (3.10). For any $t \in [0,1]$ and any parameter u in A, assume that $U(\tilde{\lambda}[u];t)$ is nonempty and reflexive. Then $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ if and only if $U(\tilde{\lambda}[u];t)$ is a positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.*

Using a positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ (resp., $(\subseteq, \ll, \subseteq)$, $(\ll, \subseteq, \subseteq)$ and $(\ll, \ll, \subseteq)$), we establish a fuzzy soft weak hyper BCK-ideal.

Theorem 7. *Let I be a positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ (resp., $(\subseteq, \ll, \subseteq)$, $(\ll, \subseteq, \subseteq)$ and $(\ll, \ll, \subseteq)$) and let $z \in H$. For a fuzzy soft set $(\tilde{\lambda}, A)$ over H and any parameter u in A, if we define the fuzzy value set $\tilde{\lambda}[u]$ by*

$$\tilde{\lambda}[u] : H \to [0,1],\ x \mapsto \begin{cases} t & \text{if } x \in I_z, \\ s & \text{otherwise,} \end{cases} \tag{3.13}$$

where $t > s$ in $[0,1]$ and $I_z := \{y \in H \mid y \circ z \subseteq I\}$, then $(\tilde{\lambda}, A)$ is a u-fuzzy soft weak hyper BCK-ideal of H.

Proof. It is clear that $\tilde{\lambda}[u](0) \geq \tilde{\lambda}[u](x)$ for all $x \in H$. Let $x, y \in H$. If $y \notin I_z$, then $\tilde{\lambda}[u](y) = s$ and so

$$\tilde{\lambda}[u](x) \geq s = \min\left\{\tilde{\lambda}[u](y), \tilde{\lambda}[u]_*(x \circ y)\right\}. \tag{3.14}$$

If $x \circ y \nsubseteq I_z$, then there exists $a \in x \circ y \setminus I_z$, and thus $\tilde{\lambda}[u](a) = s$. Hence

$$\min\left\{\tilde{\lambda}[u](y), \tilde{\lambda}[u]_*(x \circ y)\right\} = s \leq \tilde{\lambda}[u](x). \tag{3.15}$$

Assume that $x \circ y \subseteq I_z$ and $y \in I_z$. Then

$$(x \circ y) \circ z \subseteq I \text{ and } y \circ z \subseteq I. \tag{3.16}$$

If I is of type $(\subseteq, \subseteq, \subseteq)$, then $x \circ z \subseteq I$, i.e., $x \in I_z$. Thus

$$\tilde{\lambda}[u](x) = t \geq \min\left\{\tilde{\lambda}[u](y), \tilde{\lambda}[u]_*(x \circ y)\right\}. \tag{3.17}$$

The condition (3.16) implies that $(x \circ y) \circ z \ll I$ and $y \circ z \ll I$ by (2.6). Hence, if I is of type $(\ll, \ll, \subseteq)$, then $x \circ z \subseteq I$, i.e., $x \in I_z$. Therefore we have (3.17). From the condition (3.16), we have $(x \circ y) \circ z \subseteq I$ and $y \circ z \ll I$. If I is of type $(\subseteq, \ll, \subseteq)$, then $x \circ z \subseteq I$, i.e., $x \in I_z$. Therefore we have (3.17). From the condition (3.16), we have $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$. If I is of type $(\ll, \subseteq, \subseteq)$, then $x \circ z \subseteq I$, i.e., $x \in I_z$. Therefore we have (3.17). Therefore $(\tilde{\lambda}, A)$ is a u-fuzzy soft weak hyper BCK-ideal of H. □

Theorem 8. *Let $(\tilde{\lambda}, A)$ be a fuzzy soft set over H in which the nonempty level set $U(\tilde{\lambda}[u];t)$ of $(\tilde{\lambda}, A)$ is reflexive for all $t \in [0,1]$. If $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of H of type $(\ll, \subseteq, \subseteq)$, then the set*

$$\tilde{\lambda}[u]_z := \{x \in H \mid x \circ z \subseteq U(\tilde{\lambda}[u];t)\} \tag{3.18}$$

is a (weak) hyper BCK-ideal of H for all $z \in H$.

Proof. Assume that $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of H of type $(\ll, \subseteq, \subseteq)$. Obviously $0 \in \tilde{\lambda}[u]_z$. Then $(\tilde{\lambda}, A)$ is a fuzzy soft hyper BCK-ideal of H, and so $U(\tilde{\lambda}[u];t)$ is a hyper BCK-ideal of H. Let $x, y \in H$ be such that $x \circ y \subseteq \tilde{\lambda}[u]_z$ and $y \in \tilde{\lambda}[u]_z$. Then $(x \circ y) \circ z \subseteq U(\tilde{\lambda}[u];t)$ and $y \circ z \subseteq U(\tilde{\lambda}[u];t)$ for all $t \in [0,1]$. Using (2.6), we know that $(x \circ y) \circ z \ll U(\tilde{\lambda}[u];t)$. Since $U(\tilde{\lambda}[u];t)$ is a positive implicative hyper BCK-ideal of H of type $(\ll, \subseteq, \subseteq)$, it follows from (2.17) that $x \circ z \subseteq U(\tilde{\lambda}[u];t)$, that is, $x \in \tilde{\lambda}[u]_z$. This shows that $\tilde{\lambda}[u]_z$ is a weak hyper BCK-ideal of H. Let $x, y \in H$ be such that $x \circ y \ll \tilde{\lambda}[u]_z$ and $y \in \tilde{\lambda}[u]_z$, and let $a \in x \circ y$. Then there exists $b \in \tilde{\lambda}[u]_z$ such that $a \ll b$, that is, $0 \in a \circ b$. Thus $(a \circ b) \cap U(\tilde{\lambda}[u];t) \neq \varnothing$. Since $U(\tilde{\lambda}[u];t)$ is a reflexive hyper BCK-ideal of H, it follows from (H1) and Lemma 7 that $(a \circ z) \circ (b \circ z) \ll a \circ b \subseteq U(\tilde{\lambda}[u];t)$ and so that $a \circ z \subseteq U(\tilde{\lambda}[u];t)$ since $b \circ z \subseteq U(\tilde{\lambda}[u];t)$. Hence $a \in \tilde{\lambda}[u]_z$, and so $x \circ y \subseteq \tilde{\lambda}[u]_z$. Since $\tilde{\lambda}[u]_z$ is a weak hyper BCK-ideal of H, we get $x \in \tilde{\lambda}[u]_z$. Consequently $\tilde{\lambda}[u]_z$ is a hyper BCK-ideal of H. □

Corollary 6. *Let $(\tilde{\lambda}, A)$ be a fuzzy soft set over H in which the nonempty level set $U(\tilde{\lambda}[u];t)$ of $(\tilde{\lambda}, A)$ is reflexive for all $t \in [0,1]$. If $(\tilde{\lambda}, A)$ is a fuzzy soft positive implicative hyper BCK-ideal of H of type $(\ll, \ll, \subseteq)$, then the set*

$$\tilde{\lambda}[u]_z := \{x \in H \mid x \circ z \subseteq U(\tilde{\lambda}[u];t)\} \tag{3.19}$$

is a (weak) hyper BCK-ideal of H for all $z \in H$.

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References

[1] R. Ameri and M. Zahedi, "Hyperalgebraic systems," *Italian Journal of Pure and Applied Mathematics*, vol. 6, pp. 21–32, 1999.

[2] H. Bordbar, S. Z. Song, M. R. Bordbar, and Y. B. Jun, "Fuzzy soft set theory with applications in hyper *BCK*-algebras," *Journal of Intelligent and Fuzzy Systems*, pp. 1–9, 2019, doi: 10.3233/JIFS-190103.

[3] R. A. Borzooei and M. Bakhshi, "On positive implicative hyper *BCK*-ideals," *Scientiae Mathematicae Japonicae Online*, vol. 9, pp. 303–314, 2003.

[4] P. Corsini and V. Leoreanu, *Applications of Hyperstructure Theory*. Dordrecht: Kluwer, 2003.

[5] Y. B. Jun and C. H. Park, " Applications of soft sets in ideal theory of BCK/BCI-algebras,," *Information Sciences*, vol. 178, pp. 2466–2475., 2008, doi: 10.1016/j.ins.2008.01.017.

[6] Y. B. Jun and W. H. Shim, " Some types of positive implicative hyper *BCK*-ideals," *Scientiae Mathematicae Japonicae* , vol. 56, no. 1, pp. 63–68, 2002.

[7] Y. B. Jun and X. L. Xin, "Positive implicative hyper *BCK*-algebras ," *Scientiae Mathematicae Japonicae*, vol. 55, pp. 97–106, 2002.

[8] Y. B. Jun, X. L. Xin, M. M. Zahedi, and E. H. Roh, " Strong hyper *BCK*-ideals of hyper *BCK*-algebras," *Mathematica Japonica* , vol. 51, no. 3, pp. 493–498, 2000.

[9] Y. B. Jun, M. M. Zahedi, X. L. Xin, and R. A. Borzoei, "On hyper *BCK*-algebras ," *Italian Journal of Pure and Applied Mathematics* , vol. 8, pp. 127–136, 2000.

[10] S. Khademan, M. M. Zahedi, Y. B. Jun, and R. A. Borzooei, " Fuzzy soft positive implicative hyper *BCK*-ideals in hyper *BCK*-algebras ," *Journal of Intelligent and Fuzzy Systems* , vol. 36, pp. 2605–2613, 2019, doi: 10.3233/JIFS-181755.

[11] V. Leoreanu-Fotea and B. Davvaz, " Join *n*-spaces and lattices,," *Multiple Valued Logic Soft Computing*, vol. 5-6, pp. 421–432., 2008.

[12] V. Leoreanu-Fotea and B. Davvaz, " *n*-hypergroups and binary relations," *European Journal of Combinatorics*, vol. 29, pp. 1207–1218., 2008.

[13] F. Marty, *Sur une generalization de la notion de groupe*. 8th Congress Mathematics Scandenaves, Stockholm, 1934.

[14] D. Molodtsov, " Soft set theory - First results," *Computers & Mathematics with Applications*, pp. 19–31., 1999, doi: 10.1016/S0898-1221(99)00056-5.

[15] R. B. P. K. Maji and A. R. Roy, " Fuzzy soft sets,," *The Journal of Fuzzy Mathematics*, vol. 9, no. 3, pp. 589–602., 2001.

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A NEW CLASS OF GENERALIZED POLYNOMIALS ASSOCIATED WITH HERMITE AND POLY-BERNOULLI POLYNOMIALS

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Abstract. In this paper, we introduce a new class of generalized polynomials associated with the modified Milne-Thomson's polynomials $\Phi_n^{(\alpha)}(x,\nu)$ of degree n and order α introduced by Dere and Simsek. The concepts of poly-Bernoulli numbers, poly-Bernoulli polynomials, Hermite-Bernoulli polynomials and generalized Hermite-Bernoulli polynomials are generalized to polynomials of three positive real parameters. Numerous properties of these polynomials and some relations are established. Some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions. These results extend some known summations and identities of generalized poly-Bernoulli numbers and polynomials.

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1. Introduction

Kaneko [6] introduced and studied poly-Bernoulli numbers which generalize the classical Bernoulli numbers. poly-Bernoulli numbers $B_n^{(k)}$ with $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, appear in the following power series:

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \tag{1.1}$$

where

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}, |z| < 1,$$

and

$$\mathrm{Li}_1(z) = -\ln(1-z), \;\; \mathrm{Li}_0(z) = \frac{z}{1-z}, \;\; \mathrm{Li}_{-1}(z) = \frac{z}{(1-z)^2}, \ldots$$

Moreover when $k \geq 1$, the left hand side of (1.1) can be written in the form of iterated integrals

$$e^t \frac{1}{e^t-1}\int_0^t \frac{1}{e^t-1}\cdots\int_0^t \frac{1}{e^t-1}\int_0^t \frac{t}{e^t-1}dtdt\cdots dt = \sum_{n=0}^{\infty} B_n^{(k)}\frac{t^n}{n!}.$$

Obviously

$$B_n^{(1)} = B_n.$$

Recently, Jolany et al. [4, 5] generalized the concept of poly-Bernoulli polynomials defined as follows.

Let $a,b,c > 0$ and $a \neq b$. The generalized poly-Bernoulli numbers $B_n^{(k)}(a,b)$, the generalized poly-Bernoulli polynomials $B_n^{(k)}(x;a,b)$ and the generalized polynomials $B_n^{(k)}(x;a,b,c)$ are appeared in the following series respectively.

$$\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)}(a,b)\frac{t^n}{n!}, |t| < \frac{2\pi}{|\ln a + \ln b|}, \tag{1.2}$$

$$\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x,a,b)\frac{t^n}{n!}, |t| < \frac{2\pi}{|\ln a + \ln b|}, \tag{1.3}$$

$$\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x,a,b,c)\frac{t^n}{n!}, |t| < \frac{2\pi}{|\ln a + \ln b|}. \tag{1.4}$$

Dere and Simsek [3] modified the Milne-Thomson's polynomials $\Phi_n^{(\alpha)}(x)$ (see for detail [11]) as $\Phi_n^{(\alpha)}(x,\nu)$ of degree n and order α by the means of the following generating function:

$$g_1(t,x;\alpha,\nu) = f(t,\alpha)e^{xt+h(t,\nu)} = \sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x,\nu)\frac{t^n}{n!}, \tag{1.5}$$

where $f(t,\alpha)$ is a function of t and integer α. Note that $\Phi_n^{(\alpha)}(x,0) = \Phi_n^{(\alpha)}(x)$ (c.f. [11]).

On setting $f(t,\alpha) = \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}$ in (1.5), we obtain the following polynomials given by the generating function:

$$g_2(t,x;k,\nu) = \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{xt+h(t,\nu)} = \sum_{n=0}^{\infty} \frac{B_n^{(k)}(x,\nu)t^n}{n!}. \tag{1.6}$$

Observe that the polynomials $B_n^{(k)}(x,\nu)$ are related to not only Bernoulli polynomials but also the Hermite polynomials. For example, if $h(t,0) = 0$ in (1.6), we have

$$B_n^{(k)}(x,0) = B_n^{(k)}(x),$$

where $B_n^{(k)}(x)$ denotes the poly-Bernoulli polynomials of higher order which is defined by means of the following generating function:

$$F_B(t,x;k) = \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}. \tag{1.7}$$

One can easily see that

$$B_n^{(k)}(0,0) = B_n^{(k)}, B_n^{(k)}(x) = 1+x$$

and

$$B_n^{(k)}(x) = B_n^{(k)}(e^{x+1}, e^x), \tag{1.8}$$

where $B_n^{(k)}$ are generalized poly-Bernoulli numbers. For more information about poly-Bernoulli numbers and poly-Bernoulli polynomials, we refer to [4–8].

In [10], Luo et al. gave the following definition of the generalized Bernoulli polynomials, which generalize the concepts stated above.
Let $a,b>0$ and $a \neq b$. The generalized Bernoulli polynomials $B_n(x;a,b,c)$ for non-negative integer n are defined by

$$\Phi(x,t;a,b,c) = \frac{t}{a^t-b^t}c^{xt} = \sum_{n=0}^{\infty} B_n(x;a,b,c)\frac{t^n}{n!}, \quad |\, t\,| < 2\pi \tag{1.9}$$

Let $x,y \in \mathbb{R}$, the generalized Hermite-Bernoulli polynomials of two variables given by means of the following generating function (see [14]):

$$\left(\frac{t}{a^t-b^t}\right)^{\alpha} c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(\alpha)}(x,y;a,b,c)\frac{t^n}{n!}, \quad |t| < \frac{2\pi}{|\ln a + \ln b|}, \tag{1.10}$$

which is essentially a generalization of Bernoulli numbers, Bernoulli polynomials, Hermite polynomials and Hermite-Bernoulli polynomials ${}_HB_n(x,y)$, generalized Hermite-Bernoulli polynomials ${}_HB_n^{(\alpha)}(x,y)$ introduced by Pathan and Khan [15] and Dattoli et al. [2, p.386(1.6)] in the form:

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(\alpha)}(x,y)\frac{t^n}{n!}, \tag{1.11}$$

and

$$\left(\frac{t}{e^t-1}\right) e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n(x,y)\frac{t^n}{n!}. \tag{1.12}$$

Let $c>0$. The generalized 2-variable 1-parameter Hermite Kampé de Fériet $H_n(x,y,c)$ polynomials for nonnegative integer n are defined by

$$c^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y,c)\frac{t^n}{n!}. \tag{1.13}$$

This is an extended 2-variable Hermite Kamp'e de Feriet polynomials $H_n(x,y)$ (see [1]) defined by

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}. \tag{1.14}$$

Note that

$$H_n(x,y,e) = H_n(x,y),$$

and the definition (1.13) yields the relationship

$$H_n(x,y,c) = \sum_{j=0}^{[\frac{n}{2}]} \binom{n}{j} (\ln c)^{n-j} x^{n-2j} y^j. \tag{1.15}$$

In this note, we first give definitions of the generalized poly-Bernoulli polynomials $B_n^{(k)}(x;a,b.c)$, which generalize the concepts stated above and then research their basic properties and relationships with poly-Bernoulli numbers $B_n^{(k)}(a,b)$, poly-Bernoulli polynomials $B_n^{(k)}(x)$ and the generalized poly-Bernoulli polynomials $B_n^{(k)}(x;a,b,c)$ of Joalny et al., Hermite-Bernoulli polynomials ${}_HB_n(x,y)$ of Dattoli et al. and ${}_HB_n^{(\alpha)}(x,y)$ of Pathan and Khan. The remainder of this paper is organized as follows. We modify generating functions for the Milne-Thomson's polynomials and derive some identities related to Hermite polynomials, poly-Bernoulli polynomials and power sums. Some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions. These results extend some known summations and identities of generalized Hermite-poly-Bernoulli polynomials, degenerate Hermite poly-Bernoulli studied by Khan [7–9].

2. DEFINITION AND PROPERTIES OF THE GENERALIZED HERMITE POLY-BERNOULLI POLYNOMIALS ${}_HB_n^{(k)}(x,y;a,b,c)$

In the modified Milne Thomson's polynomials due to Dere and Simsek [3, 11] defined by (1.5), if we set $f(t,\alpha) = \frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}$, we obtain the following generalized polynomials $B_n^{(k)}(x,\nu;a,b,c)$.

Definition 1. Let $a,b,c>0$ and $a \neq b$. The generalized poly-Bernoulli polynomials $B_n^{(k)}(x,\nu;a,b,c)$ are defined by

$$G_1(t,x;\alpha,a,b,\nu) = \frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}} c^{xt+h(t,\nu)} = \sum_{n=0}^{\infty} B_n^{(k)}(x,\nu;a,b,c)\frac{t^n}{n!}, \tag{2.1}$$

$$(|\,t\,| < 2\pi/(|\ln a + \ln b\,|), x \in \mathbb{R}).$$

On setting $h(t,\nu) = h(t,y) = yt^2$, (2.1) reduces to

Definition 2. Let $a,b,c>0$ and $a\neq b$. The generalized Hermite poly-Bernoulli polynomials ${}_HB_n^{(k)}(x,y;a,b,c)$ are defined by

$$G_2(t,x,y;k,a,b,c)=\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HB_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!}, \tag{2.2}$$

$$(|\,t\,|<2\pi/(|\ln a+\ln b\,|),x,y\in\mathbb{R}),$$

whereas for $x=0$ gives

$${}_HB_n^{(k)}(0,y;a,b,c)=\sum_{m=0}^{[\frac{n}{2}]}\frac{n!}{m!(n-2m)!}(\ln c)^m B_{n-2m}^{(k)}(a,b)y^m. \tag{2.3}$$

Another special case of (2.2), for $y=0$ leads to the extension of the generalized poly-Bernoulli numbers $B_n^{(k)}(a,b)$ defined by (1.2) in the form.

Definition 3. Let $a,b,c>0$ and $a\neq b$. The generalized poly-Bernoulli polynomials $B_n^{(k)}(x;a,b,c)$ are defined by

$$\Phi(t;k,a,b)=\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt}=\sum_{n=0}^{\infty}B_n^{(k)}(x;a,b,c)\frac{t^n}{n!}, \tag{2.4}$$

$$(|\,t\,|<2\pi/(|\ln a+\ln b\,|),x\in\mathbb{R}).$$

Letting $c=e$, equation (2.2) reduces to

Definition 4. Let $a,b>0$ and $a\neq b$. The generalized Hermite poly-Bernoulli polynomials ${}_HB_n^{(k)}(x,y;a,b,e)$ are defined by

$$G_3(t,x,y;k,a,b,e)=\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HB_n^{(k)}(x,y;a,b,e)\frac{t^n}{n!}, \tag{2.5}$$

$$(|\,t\,|<2\pi/(|\ln a+\ln b\,|),x,y\in\mathbb{R}).$$

The generalized Hermite poly-Bernoulli polynomials ${}_HB_n^{(k)}(x,y;a,b,c)$ defined by (2.2) have the following properties which are stated as theorems below.

Theorem 1. *Let $a,b,c>0$ and $a\neq b$. For $x,y\in\mathbb{R}$ and $n\geq 0$, we have*

$$\begin{gathered}{}_HB_n^{(k)}(x,y;e,1,e)={}_HB_n^{(k)}(x,y),{}_HB_n^{(k)}(0,0;a,b,1)=B_n^{(k)}(a,b)\\ {}_HB_n^{(k)}(0,0;e,1,1)=B_n^{(k)},{}_HB_n^{(k)}(0,0;a,b,e)={}_HB_n^{(k)}(a,b),\end{gathered} \tag{2.6}$$

$${}_HB_n^{(k)}(x+y,z+u;a,b,c)=\sum_{m=0}^{n}\binom{n}{m}H_m(y,z;c)\,{}_HB_{n-m}^{(k)}(x,u;a,b,c), \tag{2.7}$$

$${}_HB_n^{(k)}(x+z,y;a,b,c)=\sum_{m=0}^{n}\binom{n}{m}B_{n-m}^{(k)}(x;a,b,c)H_m(y,z;c). \tag{2.8}$$

Proof. The formula in (2.6) are obvious. Applying definition (2.2), we have

$$\sum_{n=0}^{\infty} {}_HB_n^{(k)}(x+y,z+u;a,b,c)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_HB_n^{(k)}(x,u;a,b,c)\frac{t^n}{n!}\sum_{m=0}^{\infty} H_m(y,z;c)\frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} H_m(y,z;c){}_HB_{n-m}^{(k)}(x,u;a,b,c)\frac{t^n}{(n-m)!m!}.$$

Now equating the coefficients of the like powers of t in the above equation, we get the result (2.7).
Again, by using (2.2) of generalized Hermite poly-Bernoulli polynomials, we have

$$\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t - a^{-t}}c^{(x+z)t+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(k)}(x+z,y;a,b,c)\frac{t^n}{n!}, \tag{2.9}$$

which can be written as

$$\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t - a^{-t}}c^{xt}c^{zt+yt^2} = \sum_{n=0}^{\infty} B_n^{(k)}(x;a,b,c)\frac{t^n}{n!}\sum_{m=0}^{\infty} H_m(y,z;c)\frac{t^m}{m!}. \tag{2.10}$$

Replacing n by $n-m$ in (2.10) and comparing with (2.9) and equating their coefficients of t^n leads to formula (2.8). $\square$

3. Implicit summation formuale involving generalized Hermite poly-Bernoulli polynomials

For the derivation of implicit formulae involving generalized poly-Bernoulli polynomials $B_n^{(k)}(x;a,b,c)$ and generalized Hermite poly-Bernoulli polynomials ${}_HB_n^{(k)}$ $(x,y;a,b,c)$ the same considerations as developed for the ordinary Hermite and related polynomials in Khan [7–9] and Hermite-Bernoulli polynomials in Pathan and Khan [12–17] holds as well. First we prove the following results involving generalized Hermite poly-Bernoulli polynomials ${}_HB_n^{(k)}(x,y;a,b,c)$.

Theorem 2. *Let $a,b,c>0$ and $a\neq b$. For $x,y\in\mathbb{R}$ and $n\geq 0$, the following implicit summation formulae for generalized Hermite poly-Bernoulli polynomials ${}_HB_n^{(k)}(x,y;a,b,c)$ holds true:*

$$\begin{aligned} &{}_HB_{l+p}^{(k)}(z,y;a,b,c) \\ &\quad = \sum_{m,n=0}^{l,p}\binom{l}{m}\binom{p}{n}(z-x)^{m+n}(\ln c)^{m+n}{}_HB_{l+p-m-n}^{(k)}(x,y;a,b,c). \end{aligned} \tag{3.1}$$

Proof. We replace t by $t+u$ and rewrite the generating function (2.2) as

$$\frac{\mathrm{Li}_k(1-(ab)^{-(t+u)})}{b^{t+u} - a^{-(t+u)}}c^{y(t+u)^2} = c^{-x(t+u)}\sum_{l,p=0}^{\infty} {}_HB_{l+p}^{(k)}(x,y;a,b,c)\frac{t^l}{l!}\frac{u^p}{p!}. \tag{3.2}$$

Replacing x by z and equating the resulting equation to the above equation, we get

$$c^{(z-x)(t+u)}\sum_{l,p=0}^{\infty} {}_HB_{l+p}^{(k)}(x,y;a,b,c)\frac{t^l}{l!}\frac{u^p}{p!} = \sum_{l,p=0}^{\infty} {}_HB_{l+p}^{(k)}(z,y;a,b,c)\frac{t^l}{l!}\frac{u^p}{p!}. \tag{3.3}$$

On expanding exponential function (3.3) gives

$$\sum_{N=0}^{\infty}\frac{[(z-x)(t+u)]^N}{N!}\sum_{l,p=0}^{\infty} {}_HB_{l+p}^{(k)}(x,y;a,b,c)\frac{t^l}{l!}\frac{u^p}{p!}$$
$$= \sum_{l,p=0}^{\infty} {}_HB_{l+p}^{(k)}(z,y;a,b,c)\frac{t^l}{l!}\frac{u^p}{p!}, \tag{3.4}$$

which on using formula [18, p.52(2)]:

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}, \tag{3.5}$$

in the left hand side becomes

$$\sum_{m,n=0}^{\infty}\frac{(z-x)^{m+n}(\ln c)^{m+n}t^m u^n}{m!n!}\sum_{l,p=0}^{\infty} {}_HB_{l+p}^{(k)}(x,y;a,b,c)\frac{t^l}{l!}\frac{u^p}{p!} \tag{3.6}$$
$$= \sum_{l,p=0}^{\infty} {}_HB_{l+p}^{(k)}(z,y;a,b,c)\frac{t^l}{l!}\frac{u^p}{p!}. \tag{3.7}$$

Now replacing l by $l-m$, p by $p-n$ and using the lemma [18, p.100(1)] in the left hand side of (3.7), we get

$$\sum_{m,n=0}^{\infty}\sum_{l,p=0}^{\infty}\frac{(z-x)^{m+n}(\ln c)^{m+n}}{m!n!} {}_HB_{l+p-m-n}^{(k)}(x,y;a,b,c)\frac{t^l}{(l-m)!}\frac{u^p}{(p-n)!}$$
$$= \sum_{l,p=0}^{\infty} {}_HB_{l+p}^{(k)}(z,y;a,b,c)\frac{t^l}{l!}\frac{u^p}{p!}. \tag{3.8}$$

Finally, on equating the coefficients of the like powers of t and u in the above equation, we get the required result. □

Remark 1. On setting $l = 0$ in Theorem 3.1, we immediately deduce the following result.

Corollary 1. *The following implicit summation formula for Hermite poly-Bernoulli polynomials* ${}_HB_n^{(k)}(z,y;a,b,c)$ *holds true:*

$${}_HB_p^{(k)}(z,y;a,b,c) = \sum_{n=0}^{p}\binom{p}{n}(z-x)^n(\ln c)^n {}_HB_{p-n}^{(k)}(x,y;a,b,c). \tag{3.9}$$

Remark 2. Replacing z by $z+x$ and setting $y=0$ in Theorem 3.1, we obtain the following result involving generalized poly-Bernoulli polynomials of one variable

$$B_{l+p}^{(k)}(z+x;a,b,c)=\sum_{m,n=0}^{l,p}\binom{l}{m}\binom{p}{n}(z)^{m+n}(\ln c)^{m+n}B_{l+p-m-n}^{(k)}(x;a,b,c), \tag{3.10}$$

whereas by setting $z=0$ in Theorem 3.1, we obtain another result involving generalized poly-Bernoulli polynomials of one and two variables

$$B_{l+p}^{(k)}(y;a,b,c)=\sum_{m,n=0}^{l,p}\binom{l}{m}\binom{p}{n}(-x)^{m+n}(\ln c)^{m+n}{}_HB_{l+p-m-n}^{(k)}(x,y;a,b,c). \tag{3.11}$$

Remark 3. Along with the above results we will exploit extended forms of generalized poly-Bernoulli polynomials $B_{l+p}^{(k)}(z;a,b,c)$ by setting $y=0$ in the Theorem 3.1 to get

$$B_{l+p}^{(k)}(z;a,b,c)=\sum_{m,n=0}^{l,p}\binom{l}{m}\binom{p}{n}(z-x)^{n+m}(\ln c)^{m+n}B_{l+p-m-n}^{(k)}(x;a,b,c). \tag{3.12}$$

Theorem 3. *Let $a,b,c>0$ and $a\neq b$. Then $x\in\mathbb{R}$ and $n\geq 0$, we have*

$$B_n^{(k)}(x+1;a,b,c)=B_n^{(k)}\left(x;ac,\frac{b}{c},c\right). \tag{3.13}$$

Proof. From (2.3), we have

$$\sum_{n=0}^{\infty}B_n^{(k)}(x+1;a,b,c)\frac{t^n}{n!}=\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{(x+1)t}=\frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt}c^t$$

$$\sum_{n=0}^{\infty}B_n^{(k)}(x+1;a,b,c)\frac{t^n}{n!}=\frac{\mathrm{Li}_k(1-(ab)^{-t})}{(\frac{b}{c})^t-(ac)^{-t}}c^{xt}=\sum_{n=0}^{\infty}B_n^{(k)}(x;ac,\frac{b}{c},c)\frac{t^n}{n!}. \tag{3.14}$$

Equating the coefficients of t^n on both sides, we get (3.13). □

Theorem 4. *Let $a,b,c>0$ and $a\neq b$. Then for $x,y\in\mathbb{R}$ and $n\geq 0$, we have*

$${}_HB_n^{(k)}(x+1,y;a,b,c)=\sum_{j=0}^{[\frac{n}{2}]}\binom{n}{2j}y^j(\ln c)^jB_{n-2j}^{(k)}\left(x;ac,\frac{b}{c},c\right). \tag{3.15}$$

Proof. Since

$${}_HB_n^{(\alpha)}(x+1,y;a,b,c)\frac{t^n}{n!}=\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{(x+1)t+yt^2}=\frac{\mathrm{Li}_k(1-(ab)^{-t})}{(\frac{b}{c})^t-(ac)^{-t}}c^{xt}c^{yt^2}$$

$$=\left(\sum_{n=0}^{\infty}B_n^{(k)}(x;ac,\frac{b}{c},c)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}y^j(\ln c)^j\frac{t^2j}{j!}\right).$$

Now replacing n by $n-2j$ and comparing the coefficients of t^n, we obtain the result (3.15). □

Theorem 5. *Let $a,b,c>0$ and $a\neq b$. Then for $x,y\in\mathbb{R}$ and $n\geq 0$, we have*

$$ {}_HB_n^{(k)}(x,y;a,b,c)=\sum_{m=0}^{n}\binom{n}{m}B_{n-m}^{(k)}(a,b)H_m(x,y,c). \tag{3.16}$$

Proof. By using equations (2.2) and (1.2), we have

$$\begin{aligned}\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt+yt^2}&=\sum_{n=0}^{\infty}{}_HB_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!}\\&=\left(\sum_{n=0}^{\infty}B_n^{(k)}(a,b)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}H_m(x,y;c)\frac{t^m}{m!}\right).\end{aligned}$$

Replacing n by $n-m$ and comparing the coefficients of t^n, we required at the desired result (3.16). □

Remark 4. For $c=e$, (3.16) yields

$$ {}_HB_n^{(k)}(x,y;a,b,e)=\sum_{m=0}^{n}\binom{n}{m}B_{n-m}^{k}(a,b)H_m(x,y).$$

Theorem 6. *Let $a,b,c>0$ and $a\neq b$. Then for $x,y\in\mathbb{R}$ and $n\geq 0$, we have*

$$ {}_HB_n^{(k)}(x,y;a,b,c)=\sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]}y^jx^{n-m-2j}(\ln c)^{n-m-j}B_m^{(k)}(a,b)\frac{n!}{m!j!(n-2j-m)!}. \tag{3.17}$$

Proof. Applying the definition (2.2) to the term $\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}$ and expanding the exponential function c^{xt+yt^2} at $t=0$ yields

$$\begin{aligned}\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt+yt^2}&=\left(\sum_{m=0}^{\infty}B_m^{(k)}(a,b)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}x^n(\ln c)^n\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}y^j(\ln c)^j\frac{t^{2j}}{j!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}(\ln c)^{n-m}B_m^{(k)}(a,b)x^{n-m}\right)\frac{t^n}{n!}\left(\sum_{j=0}^{\infty}y^j(\ln c)^j\frac{t^{2j}}{j!}\right).\end{aligned}$$

Replacing n by $n-2j$ in the L.H.S. of above equation, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}{}_HB_n^{(k)}(x,y;a,b)\frac{t^n}{n!} \qquad (3.18)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]}\binom{n-2j}{m}(\ln c)^{n-m-j}B_m^{(k)}(a,b)x^{n-m-2j}y^j\right)\frac{t^n}{(n-2j)!j!}.\end{aligned}$$

Combining (3.18) and (2.2) and equating their coefficients of t^n produce the formula (3.17). □

Theorem 7. *Let $a,b,c>0$ and $a\neq b$. Then for $x,y\in\mathbb{R}$ and $n\geq 0$, we have*

$$ {}_HB_n^{(k)}(x+1,y;a,b,c)=\sum_{j=0}^{[\frac{n}{2}]}\sum_{m=0}^{n-2j}\binom{n-2j}{m}y^j(\ln c)^{n-m-j}B_m^{(k)}(x;a,b,c). \tag{3.19}$$

Proof. By using the definition of generalized Hermite poly-Bernoulli polynomials, we have

$$\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{(x+1)t+yt^2}=\sum_{n=0}^{\infty}{}_HB_n^{(k)}(x+1,y;a,b,c)\frac{t^n}{n!}, \tag{3.20}$$

$$=\left(\sum_{m=0}^{\infty}B_m^{(k)}(x;a,b,c)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}(\ln c)^n\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}y^j(\ln c)^j\frac{t^{2j}}{j!}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}(\ln c)^{n-m}B_m^{(k)}(x;a,b,c)\frac{t^n}{n!}\left(\sum_{j=0}^{\infty}y^j(\ln c)^j\frac{t^{2j}}{j!}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}y^j(\ln c)^{n-m+j}B_m^{(k)}(x;a,b,c)\frac{t^{n+2j}}{n!\,j!}.$$

Replacing n by $n-2j$ in the L.H.S. of above equation, we have

$$\sum_{n=0}^{\infty}{}_HB_n^{(k)}(x+1,y;a,b,c)\frac{t^n}{n!} \tag{3.21}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{[\frac{n}{2}]}\sum_{m=0}^{n-2j}\binom{n-2j}{m}y^j(\ln c)^{n-m-j}B_m^{(k)}(x;a,b,c)\right)\frac{t^n}{n!}.$$

Combining (3.20) and (3.21) and equating their coefficients of t^n leads to formula (3.19). □

Theorem 8. *Let $a,b,c>0$ and $a\neq b$. Then for $x,y\in\mathbb{R}$ and $n\geq 0$, we have*

$$ {}_HB_n^{(k)}(x+1,y;a,b,c)=\sum_{m=0}^{n}\binom{n}{m}(\ln c)^{n-m}{}_HB_m^{(k)}(x,y;a,b,c). \tag{3.22}$$

Proof. From (2.2), we have

$$\sum_{n=0}^{\infty}{}_HB_n^{(k)}(x+1,y;a,b,c)\frac{t^n}{n!}-\sum_{n=0}^{\infty}{}_HB_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!}$$

$$=\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt+yt^2}(c^t-1)$$

$$=\left(\sum_{m=0}^{\infty}{}_HB_m^{(k)}(x,y;a,b,c)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}(\ln c)^n\frac{t^n}{n!}\right)-\sum_{n=0}^{\infty}{}_HB_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}(\ln c)^{n-m}{}_HB_m^{(k)}(x,y;a,b,c)\frac{t^n}{(n-m)!}-\sum_{n=0}^{\infty}{}_HB_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!}.$$

Finally, equating the coefficients of the like powers of t^n, we get (3.22). □

Theorem 9. *Let $a,b,c>0$ and $a\neq b$. Then for $x,y\in\mathbb{R}$ and $n\geq 0$, we have*

$$\sum_{m=0}^{n}\binom{n}{m}(\ln ab)^m {}_HB_{n-m}^{(k)}(-x,y;a,b,c)=(-1)^n {}_HB_n^{(k)}(x,y;a,b,c). \tag{3.23}$$

Proof. We replace t by $-t$ in (2.2) and then subtract the result from (2.2) itself finding

$$c^{yt^2}\left[\frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}}(c^{xt}-(ab)^t c^{-xt})\right]=\sum_{n=0}^{\infty}[1-(-1)^n]{}_HB_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!},$$

which is equivalent to

$$\sum_{n=0}^{\infty}{}_HB_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!}-\left(\sum_{m=0}^{\infty}(\ln ab)^m\frac{t^m}{m!}\right)\sum_{n=0}^{\infty}{}_HB_n^{(k)}(-x,y;a,b,c)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}[1-(-1)^n]{}_HB_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!}$$
$$\sum_{n=0}^{\infty}{}_HB_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!}-\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}(\ln ab)^m\right){}_HB_{n-m}^{(k)}(-x,y;a,b,c)\frac{t^n}{(n-m)!m!}$$
$$=\sum_{n=0}^{\infty}[1-(-1)^n]{}_HB_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!},$$

and thus by equating coefficients of like powers of t^n, we get (3.23). □

4. Symmetry identities

In this section, we establish general symmetry identities for the generalized poly-Bernoulli polynomials $B_n^{(k)}(x;a,b,c)$ and the generalized Hermite-poly-Bernoulli polynomials ${}_HB_n^{(k)}(x,y;a,b,c)$ by applying the generating function (1.4) and (2.2). The results extend some known identities of Khan [7–9], Pathan and Khan [12–17].

Theorem 10. *Let $a,b,c>0$ and $a\neq b$. Then for $x,y\in\mathbb{R}$ and $n\geq 0$, the following identity holds true:*

$$\sum_{m=0}^{n}\binom{n}{m}b^m a^{n-m}{}_HB_{n-m}^{(k)}(bx,b^2y;A,B,c){}_HB_m^{(k)}(ax,a^2y;A,B,c)$$
$$=\sum_{m=0}^{n}\binom{n}{m}a^m b^{n-m}{}_HB_{n-m}^{(k)}(ax,a^2y;A,B,c){}_HB_m^{(k)}(bx,b^2y;A,B,c). \tag{4.1}$$

Proof. Start with

$$g(t)=\left(\frac{(\mathrm{Li}_k(1-(ab)^{-t}))^2}{(B^{at}-A^{-at})(B^{bt}-A^{-bt})}\right)c^{abxt+a^2b^2yt^2}. \tag{4.2}$$

Then the expression for $g(t)$ is symmetric in a and b and we can expand $g(t)$ into series in two ways to obtain:

$$g(t)=\sum_{n=0}^{\infty}{}_HB_n^{(k)}(bx,b^2y;A,B,c)\frac{(at)^n}{n!}\sum_{m=0}^{\infty}{}_HB_m^{(k)}(ax,a^2y;A,B,c)\frac{(bt)^m}{m!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^m{}_HB_{n-m}^{(k)}(bx,b^2y;A,B,c){}_HB_m^{(k)}(ax,a^2y;A,B,c)\right)\frac{t^n}{n!}$$

On the similar lines, we can show that

$$g(t)=\sum_{n=0}^{\infty}{}_HB_n^{(k)}(ax,a^2y;A,B,c)\frac{(bt)^n}{n!}\sum_{m=0}^{\infty}{}_HB_m^{(k)}(bx,b^2y;A,B,c)\frac{(at)^m}{m!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}a^{m}b^{n-m}{}_HB_{n-m}^{(k)}(ax,a^2y;A,B,c){}_HB_m^{(k)}(bx,b^2y;A,B,c)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result. □

Remark 5. For $c = e$ in Theorem 4.1, we get

$$\sum_{m=0}^{n}\binom{n}{m}b^ma^{n-m}{}_HB_{n-m}^{(k)}(bx,b^2y;A,B,e){}_HB_m^{(k)}(ax,a^2y;A,B,e)$$
$$=\sum_{m=0}^{n}\binom{n}{m}a^mb^{n-m}{}_HB_{n-m}^{(k)}(ax,a^2y;A,B,e){}_HB_m^{(k)}(bx,b^2y;A,B,e). \tag{4.3}$$

Remark 6. By setting $b = 1$ in Theorem 4.1, the following result reduces to

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}{}_HB_{n-m}^{(k)}(x,y;A,B,c){}_HB_m^{(k)}(ax,a^2y;A,B,c) \tag{4.4}$$
$$=\sum_{m=0}^{n}\binom{n}{m}a^{m}{}_HB_{n-m}^{(k)}(ax,a^2y;A,B,c){}_HB_m^{(k)}(x,y;A,B,c). \tag{4.5}$$

Theorem 11. *Let $a,b,c>0$ and $a\neq b$. Then for $x,y\in\mathbb{R}$ and $n\geq 0$, the following identity holds true:*

$$\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}{}_HB_{n-m}^{(k)}\left(bx+\frac{b}{a}i+j,b^2z;A,B,c\right)B_m^{(k)}(ay;A,B,c)b^ma^{n-m}$$
$$=\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}{}_HB_{n-m}^{(k)}\left(ax+\frac{a}{b}i+j,a^2z;A,B,c\right)B_m^{(k)}(by;A,B,c)a^mb^{n-m}. \tag{4.6}$$

Proof. Let

$$h(t)=\left(\frac{(\mathrm{Li}_k(1-(ab)^{-t}))^2}{(B^{at}-A^{-at})(B^{bt}-A^{-bt})}\right)\frac{(c^{abt}-1)^2c^{ab(x+y)t+a^2b^2zt^2}}{(c^{at}-1)(c^{bt}-1)}.$$

$$
\begin{aligned}
h(t) &= \left(\frac{\mathrm{Li}_k(1-(ab)^{-t})}{(B^{at}-A^{-at}}\right) c^{abxt+a^2b^2zt^2} \left(\frac{c^{abt}-1}{c^{bt}-1}\right) \\
&\quad \times \left(\frac{\mathrm{Li}_k(1-(ab)^{-t})}{B^{bt}-A^{-bt}}\right) c^{abyt} \left(\frac{c^{abt}-1}{c^{at}-1}\right) \\
&= \left(\frac{\mathrm{Li}_k(1-(ab)^{-t})}{(B^{at}-A^{-at}}\right) c^{abxt+a^2b^2zt^2} \sum_{i=0}^{a-1} c^{bti} \left(\frac{\mathrm{Li}_k(1-(ab)^{-t})}{B^{bt}-A^{-bt}}\right) c^{abyt} \sum_{j=0}^{b-1} c^{atj} \\
&= \left(\frac{\mathrm{Li}_k(1-(ab)^{-t})}{B^{at}-A^{-at}}\right) c^{a^2b^2zt^2} \sum_{i=0}^{a-1}\sum_{j=0}^{b-1} c^{(bx+\frac{b}{a}i+j)at} \sum_{m=0}^{\infty} B_m^{(k)}(ay;A,B,c)\frac{(bt)^m}{m!} \\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1} {}_HB_n^{(k)}\left(bx+\frac{b}{a}i+j,b^2z;A,B,c\right)\frac{(at)^n}{n!}\sum_{m=0}^{\infty} B_m^{(k)}(ay;A,B,c)\frac{(bt)^m}{(m)!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1} {}_HB_{n-m}^{(k)}\left(bx+\frac{b}{a}i+j,b^2z;A,B,c\right)\right. \\
&\quad \left. \times B_m^{(k)}(ay;A,B,c)b^m a^{n-m}\right)\frac{t^n}{n!}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
h(t) = \sum_{n=0}^{\infty}&\left(\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1} {}_HB_{n-m}^{(k)}\left(ax+\frac{a}{b}i+j,a^2z;A,B,c\right)\right. \\
&\left. \times B_m^{(k)}(by;A,B,c)a^m b^{n-m}\right)\frac{t^n}{n!}.
\end{aligned}
$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at the desired result. □

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REFERENCES

[1] E. T. Bell, "Exponential polynomials," *Annals of Mathematics*, pp. 258–277, 1934.

[2] G. Dattoli, S. Lorenzutta, and C. Cesarano, "Finite sums and generalized forms of bernoulli polynomials," *Rendiconti di Matematica*, vol. 19, pp. 385–391, 1999.

[3] R. Dere and Y. Simsek, "Bernoulli type polynomials on umbral algebra," *Russian Journal of Mathematical Physics*, vol. 22, pp. 1–5, 2015, doi: 10.1134/S106192081501001X.

[4] H. Jolany and R. Corcino, "Explicit formula for generalization of poly-bernoulli numbers and polynomials with a,b,c parameters," *Journal of Classical Analysis*, vol. 6, pp. 119–135, 2015, doi: 10.7153/jca-06-10.

[5] H. Jolany, M. Darafsheh, and R. E. Alikelaye, "Generalizations of poly-bernoulli numbers and polynomials," *Int. J. Math. Comb*, vol. 2, pp. 7–14, 2010. [Online]. Available: http://fs.unm.edu/IJMC/GeneralizationsOfPolyBernoulliNumbers.pdf

[6] M. Kaneko, "Poly-bernoulli numbers," *Journal de théorie des nombres de Bordeaux*, vol. 9, no. 1, pp. 221–228, 1997.

[7] W. A. Khan, "A note on degenerate hermite poly-bernoulli numbers and polynomials," *J. Class. Anal*, vol. 8, no. 1, pp. 65–76, 2016, doi: 10.7153/jca-08-06.

[8] W. A. Khan, "A note on hermite-based poly-euler and multi poly-euler polynomials," *Palestine J. Math*, vol. 5, no. 1, pp. 17–26, 2016. [Online]. Available: https://pjm.ppu.edu/sites/default/files/papers/PJM_JUNE_2017_14.pdf

[9] W. A. Khan, "Some properties of the generalized apostol type hermite-based polynomials," *Kyungpook Math. J*, vol. 55, pp. 597–614, 2015, doi: 10.5666/KMJ.2015.55.3.597.

[10] Q.-M. Luo, B.-N. Guo, F. Qi, and L. Debnath, "Generalizations of bernoulli numbers and polynomials," *International Journal of Mathematics and Mathematical Sciences*, vol. 2003, 2003, doi: 10.1155/S0161171203112070.

[11] L. Milne-Thomson, "Two classes of generalized polynomials," *Proceedings of the London Mathematical Society*, vol. 2, no. 1, pp. 514–522, 1933.

[12] M. A. Pathan, "Some implicit summation formulas and symmetric identities for the generalized hermite-euler polynomials," *East West Math*, vol. 16, no. 01, pp. 92–109, 2014. [Online]. Available: http://fces.ntt.edu.vn/index.php/ewm/article/download/64/64

[13] M. A. Pathan and W. A. Khan, "Some implicit summation formulas and symmetric identities for the generalized hermite based-polynomials," *Acta Universitatis Apulensis*, vol. 39, pp. 113–136, 2014, doi: 10.17114/j.aua.2014.39.11.

[14] M. A. Pathan and W. A. Khan, "A new class of generalized polynomials associated with hermite and bernoulli polynomials," *Le Matematiche*, vol. 70, no. 1, pp. 53–70, 2015. [Online]. Available: https://lematematiche.dmi.unict.it/index.php/lematematiche/article/view/1107/933

[15] M. A. Pathan and W. A. Khan, "Some implicit summation formulas and symmetric identities for the generalized hermite–bernoulli polynomials," *Mediterranean Journal of Mathematics*, vol. 12, no. 3, pp. 679–695, 2015, doi: 10.1007/s00009-014-0423-0.

[16] M. A. Pathan and W. A. Khan, "Some new classes of generalized hermite-based apostol-euler and apostol-genocchi polynomials," *Fasciculi Mathematici*, vol. 55, no. 1, pp. 153–170, 2015, doi: 10.1515/fascmath-2015-0020.

[17] M. A. Pathan and W. A. Khan, "A new class of generalized polynomials associated with hermite and euler polynomials," *Mediterranean Journal of Mathematics*, vol. 13, no. 3, pp. 913–928, 2016, doi: 10.1007/s00009-015-0551-1.

[18] H. M. Srivastava and H. L. Manocha, "*Treatise on generating functions.*" John Wiley & Sons, Inc., 605 Third Ave., New York, NY 10158, USA, 1984, 500*, 1984.*

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ON THE SOLUTIONS OF A SYSTEM OF $(2p+1)$ DIFFERENCE EQUATIONS OF HIGHER ORDER

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Abstract. In this paper we represent the well-defined solutions of the system of the higher-order rational difference equations

$$x_{n+1}^{(j)}=\frac{1+2x_{n-k}^{(j+1)mod(2p+1)}}{3+x_{n-k}^{(j+1)mod(2p+1)}},\quad n,k,p\in\mathbb{N}_0$$

in terms of Fibonacci and Lucas sequences, where the initial values $x_{-k}^{(j)},x_{-k+1}^{(j)},\ldots,x_{-1}^{(j)}$ and $x_0^{(j)}$, $j=1,2,\ldots,2p+1$, do not equal -3. Some theoretical explanations related to the representation for the general solution are also given.

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Keywords: Fibonacci sequence, Lucas sequence, system of difference equations, representation of solutions

1. INTRODUCTION

The seniority, richness and the appreciable flexibility of use, have allowed difference equations to be an attractive subject in recent times among researchers and scientists from different disciplines. Difference equations and system of difference equations have been applied in diverse mathematical models in biology, economics, genetics, population dynamics, medicine, and other fields (see [4, 8, 17]).

Solving system of difference equations in closed-form has attracted the attention of many authors, (see, for example [1–3, 5–7, 9–16, 18, 21–23] and the references therein).

It is a well-known fact that the Fibonacci sequence defined as follows

$$F_{n+1}=F_n+F_{n-1},\quad n\in\mathbb{N},\tag{1.1}$$

where $F_0=0$ and $F_1=1$. The solution of equation (1.1) is given by the formula

$$F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta},\tag{1.2}$$

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which is called the Binet formula of the Fibonacci numbers, where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad (\text{the so}-\text{called golden number}), \quad \beta = \frac{1-\sqrt{5}}{2}. \tag{1.3}$$

One can easily verify that

$$\lim_{n\to+\infty} \frac{F_{n+r}}{F_n} = \alpha^r, \quad n, r \in \mathbb{N}. \tag{1.4}$$

Also, the Lucas sequence has the same recursive relationship as the Fibonacci sequence,

$$L_{n+1} = L_n + L_{n-1}, \quad n \in \mathbb{N}, \tag{1.5}$$

but with different initial conditions, $L_0 = 2$ and $L_1 = 1$. The first few terms of the recurrence sequence are $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$. The Binet's formula for this recurrence sequence can easily be obtained and is given by

$$L_n = \alpha^n + \beta^n, \tag{1.6}$$

where α and β are the two numbers mentioned in (1.3), and we have also

$$\lim_{n\to+\infty} \frac{L_{n+r}}{L_n} = \alpha^r, \quad n, r \in \mathbb{N}. \tag{1.7}$$

Khelifa et al. in [19] gave some theoretical explanations related to the representation for the general solution of the system of three higher-order rational difference equations

$$x_{n+1} = \frac{1+2y_{n-k}}{3+y_{n-k}}, \quad y_{n+1} = \frac{1+2z_{n-k}}{3+z_{n-k}}, \quad z_{n+1} = \frac{1+2x_{n-k}}{3+x_{n-k}}, \quad n, k \in \mathbb{N}_0. \tag{1.8}$$

Motivated by the paper [19], we represents the well-defined solutions of the system of $(2p+1)$ higher-order rational difference equations

$$x_{n+1}^{(j)} = \frac{1+2x_{n-k}^{(j+1) mod (2p+1)}}{3+x_{n-k}^{(j+1) mod (2p+1)}}, \quad n, k, p \in \mathbb{N}_0, j = 1, 2, \ldots, 2p+1. \tag{1.9}$$

Clearly if take $p = 1$ in the system (1.9) we get the system (1.8). So our results generalizes the results obtained in [19].

2. On the system of first order difference equations (2.1)

In this section, to give a closed form for the well defined solutions of the system (1.9) we consider the system of $2p+1$ difference equations of first order

$$x_{n+1}^{(1)} = \frac{1+2x_n^{(2)}}{3+x_n^{(2)}}, \quad x_{n+1}^{(2)} = \frac{1+2x_n^{(3)}}{3+x_n^{(3)}}, \quad \ldots, \quad x_{n+1}^{(2p+1)} = \frac{1+2x_n^{(1)}}{3+x_n^{(1)}}, \quad n, p \in \mathbb{N}_0. \tag{2.1}$$

We replace $x_{n+1}^{(2p+1)}$ in the equation $x_{n+1}^{(2p)} = \dfrac{1+2x_n^{(2p+1)}}{3+x_n^{(2p+1)}}$, we get

$$x_{n+1}^{(2p)} = \frac{F_2+F_1x_{n-1}^{(1)}}{F_3+F_2x_{n-1}^{(1)}}, \quad n \geq 1.$$

Similarly, we replace $x_{n+1}^{(2p)}$ in the equation $x_{n+1}^{(2p-1)} = \dfrac{1+2x_n^{(2p)}}{3+x_n^{(2p)}}$, we get

$$x_{n+1}^{(2p-1)} = \frac{L_3+L_2x_{n-2}^{(1)}}{L_4+L_3x_{n-2}^{(1)}} \quad n \geq 2.$$

By induction we get

$$x_{n+1}^{(2)} = \frac{F_{2p}+F_{2p-1}x_{n-2p+1}^{(1)}}{F_{2p+1}+F_{2p}x_{n-2p+1}^{(1)}}, \quad n \geq (2p-1),$$

$$x_{n+1}^{(1)} = \frac{L_{2p+1}+L_{2p}x_{n-2p}^{(1)}}{L_{2p+2}+L_{2p+1}x_{n-2p}^{(1)}}, \quad n \geq 2p.$$

So, the system (2.1) can be written as the following equation

$$x_{n+1} = \frac{L_{2p+1}+L_{2p}x_{n-2p}}{L_{2p+2}+L_{2p+1}x_{n-2p}} \quad n \geq 2p. \tag{2.2}$$

Let

$$^{(j)}x_n = x_{(2p+1)n+j}, \quad n \in \mathbb{N}_0 \tag{2.3}$$

where $j \in \{0,1,2,\cdots,2p\}$.

Using notation (2.3), we can write (2.2) as

$$^{(j)}x_{n+1} = \frac{L_{2p+1}+L_{2p}\,{}^{(j)}x_n}{L_{2p+2}+L_{2p+1}\,{}^{(j)}x_n}, \quad n \in \mathbb{N} \tag{2.4}$$

for each $j \in \{0,1,2,\cdots,2p\}$.

Now consider the equation

$$y_{n+1} = \frac{L_{2p+1}+L_{2p}y_n}{L_{2p+2}+L_{2p+1}y_n} \quad n \in \mathbb{N}_0. \tag{2.5}$$

Using the change of variables

$$y_n = \frac{1}{L_{2p+1}}\left(w_n - L_{2p+2}\right), \quad n \in \mathbb{N}_0 \tag{2.6}$$

we can write (2.5) as

$$w_{n+1} = \frac{(L_{2p}+L_{2p+2})w_n - 5}{w_n}, \quad n \in \mathbb{N}_0. \tag{2.7}$$

In the following result, we solve in a closed form the equation (2.8) in terms of the sequences $(F_n)_{n=0}^{+\infty}$ and $(L_n)_{n=0}^{+\infty}$. The obtained formula will be very useful to get the formula of the solutions of system (1.9).

Lemma 1. *Consider the linear difference equation*

$$z_{n+1}-5F_{2p+1}z_n+5z_{n-1}=0, \quad n\in\mathbb{N}_0, \tag{2.8}$$

with initial conditions $z_{-1},z_0\in\mathbb{R}$. Then all solutions of equation (2.8) will be written under the form

$$z_n=\left(\frac{\sqrt{5}^n}{L_{2p+1}}\right)\left[\sqrt{5}z_{-1}N_{(2p+1)n}-z_0N_{(2p+1)(n+1)}\right], \tag{2.9}$$

where

$$N_{(2p+1)n}=\left(\alpha^{(2p+1)n}-(-1)^n\beta^{(2p+1)n}\right), \quad \textit{with} \quad \alpha=\frac{1+\sqrt{5}}{2}, \beta=\frac{1-\sqrt{5}}{2}.$$

So,

$$N_{(2p+1)n}=\begin{cases}\sqrt{5}F_{(2p+1)n}, & \textit{if} \quad n \textit{ even},\\ L_{(2p+1)n}, & \textit{if} \quad n \textit{ odd},\end{cases} \tag{2.10}$$

where $(F_n)_{n=0}^{+\infty}$is the Fibonacci sequence and $(L_n)_{n=0}^{+\infty}$ is the Lucas sequence.

Proof. As it is well-known, the equation

$$z_{n+1}-5F_{2p+1}z_n+5z_{n-1}=0, \quad n\in\mathbb{N}_0,$$

(the homogeneous linear second order difference equation with constant coefficients), where $z_0,z_{-1}\in\mathbb{R}$, is usually solved by using the characteristic roots λ_1 and λ_2 of the characteristic polynomial $\lambda^2-5F_{2p+1}\lambda+5$. So

$$\lambda_1=\frac{5F_{2p+1}+\sqrt{5}L_{2p+1}}{2}, \quad \lambda_2=\frac{5F_{2p+1}-\sqrt{5}L_{2p+1}}{2}$$

and the formula of general solution is

$$x_n=c_1\lambda_1^n+c_2\lambda_2^n.$$

The characteristic roots λ_1 and λ_2 check the following relationships

$$\lambda_1=\frac{5F_{2p+1}+\sqrt{5}L_{2p+1}}{2}=\sqrt{5}\left(\frac{L_{2p+1}+\sqrt{5}F_{2p+1}}{2}\right)=\sqrt{5}\alpha^{2p+1},$$

$$\lambda_2=\frac{5F_{2p+1}-\sqrt{5}L_{2p+1}}{2}=-\sqrt{5}\left(\frac{L_{2p+1}-\sqrt{5}F_{2p+1}}{2}\right)=-\sqrt{5}\beta^{2p+1}.$$

Using the initial conditions z_0 and z_{-1} with some calculations we get

$$c_1=-\frac{\sqrt{5}}{L_{2p+1}}\left(z_{-1}-\frac{z_0}{5}\lambda_1\right),$$

$$c_2 = -\frac{\sqrt{5}}{L_{2p+1}}\left(\frac{z_0}{5}\lambda_2 - z_{-1}\right).$$

So,

$$\begin{aligned}
z_n &= \left(-\frac{\sqrt{5}}{L_{2p+1}}\left(z_{-1} - \frac{z_0}{5}\lambda_1\right)\right)\lambda_1^n + \left(-\frac{\sqrt{5}}{L_{2p+1}}\left(\frac{z_0}{5}\lambda_2 - z_{-1}\right)\right)\lambda_2^n \\
&= -\frac{\sqrt{5}}{L_{2p+1}}\left(z_{-1}\left[\lambda_1^n - \lambda_2^n\right] - \frac{z_0}{5}\left[\lambda_1^{n+1} - \lambda_2^{n+1}\right]\right) \\
&= -\frac{\sqrt{5}}{L_{2p+1}}\left(z_{-1}(\sqrt{5})^n\left[\alpha^{(2p+1)n} - (-1)^n\beta^{(2p+1)n}\right]\right. \\
&\quad \left. - \frac{z_0(\sqrt{5})^{n+1}}{(\sqrt{5})^2}\left[\alpha^{(2p+1)(n+1)} - (-1)^{n+1}\beta^{(2p+1)(n+1)}\right]\right),
\end{aligned}$$

putting

$$N_{(2p+1)n} = \left(\alpha^{(2p+1)n} - (-1)^n\beta^{(2p+1)n}\right),$$

it is obtained that the general solution of equation (2.8) is

$$z_n = -\frac{(\sqrt{5})^n}{L_{2p+1}}\left[z_{-1}\sqrt{5}N_(2p+1)n - z_0 N_{(2p+1)(n+1)}\right]. \tag{2.11}$$

The lemma is proved. □

Through an analytical approach we put

$$w_n = \frac{z_n}{z_{n-1}}, \tag{2.12}$$

which reduces equation (2.7) to the following one

$$z_{n+1} = 5F_{2p+1}z_n - 5z_{n-1}. \tag{2.13}$$

So, from Lemma (1) we get

$$z_n = \left(\frac{\sqrt{5}^n}{L_{2p+1}}\right)[\sqrt{5}z_{-1}N_{(2p+1)n} - z_0 N_{(2p+1)(n+1)}],$$

with

$$N_{(2p+1)n} = \begin{cases} \sqrt{5}F_{(2p+1)n}, & \text{if} \quad n \text{ even}, \\ L_{(2p+1)n}, & \text{if} \quad n \text{ odd}, \end{cases} \tag{2.14}$$

where $(F_n)_{n=0}^{+\infty}$ is the Fibonacci sequence and $(L_n)_{n=0}^{+\infty}$ is the Lucas sequence.

By formulas (2.12) and (2.14), it follows that the general solution of equation (2.7) is

$$\begin{cases} w_{2n} = \dfrac{5F_{2(2p+1)n} - w_0 L_{(2p+1)(2n+1)}}{L_{(2p+1)(2n-1)} - w_0 F_{2(2p+1)n}}, \\ w_{2n+1} = \dfrac{5L_{(2p+1)(2n+1)} - 5w_0 F_{2(2p+1)(n+1)}}{5F_{2(2p+1)n} - w_0 L_{(2p+1)(2n+1)}}. \end{cases}$$

From all above mentioned the following theorem holds.

Theorem 1. *Let $\{y_n\}_{n\geq 0}$ b a well-defined solution of the equation (2.5). Then, for $n = 2,3,\ldots,$*

$$\begin{cases} y_{2n} = \dfrac{F_{2(2p+1)n} + F_{2(2p+1)n-1}y_0}{F_{2(2p+1)n+1} + F_{2(2p+1)n}y_0}, \\ y_{2n+1} = \dfrac{L_{2(2p+1)n+(2p+1)} + L_{2(2p+1)n+2p}y_0}{L_{2(2p+1)n+(2p+2)} + L_{2(2p+1)n+(2p+1)}y_0}, \end{cases} \tag{2.15}$$

where $(L_n)_{n=0}^{+\infty}$ is the Lucas sequence and $(F_n)_{n=0}^{+\infty}$ is the Fibonacci sequence.

Proof. According to the change of variable (2.6), and using the following equalities (see [20])

$$\begin{aligned} L_{2p+1}F_{2(2n+1)n+1} &= L_{2p+2}F_{2(2p+1)n-1} - L_{2(2p+1)n-(2p+2)}, \\ L_{2p+1}L_{2(2p+1)n+(2p+2)} &= L_{2p+1}L_{2(2p+1)n+(2p+1)} - 5F_{2(2p+1)n}, \\ L_{2p+1}F_{2(2p+1)n-1} &= L_{2(2p+1)n+(2p+1)} - L_{2p+2}F_{2(2p+1)n}, \\ L_{2p+1}L_{2(2p+1)n-(2p+2)} &= 5F_{2(2p+1)n} - L_{2p+2}L_{2(2p+1)n-(2p+1)}, \end{aligned}$$

we obtain

$$\begin{aligned} y_{2n} &= \frac{1}{L_{2p+1}}\left(w_{2n} - L_{2p+2}\right) \\ &= \frac{1}{L_{2p+1}}\left(\frac{(5F_{2(2p+1)n} - L_{2p+2}L_{2(2p+1)n-(2p+1)})}{L_{2(2n+1)n-(2n+1)} - w_0F_{2(2n+1)n}}\right) \\ &\quad + \frac{1}{L_{2p+1}}\left(\frac{+w_0(L_{2p+2}F_{2(2p+1)n} - L_{2(2p+1)n+(2p+1)})}{L_{2(2n+1)n-(2n+1)} - w_0F_{2(2n+1)n}}\right) \\ &= \frac{1}{L_{2p+1}}\left(\frac{L_{2p+1}L_{2(2p+1)n-(2p+2)} - L_{2p+1}w_0F_{2(2p+1)n-1}}{L_{2(2p+1)n-(2p+1)} - w_0F_{2(2p+1)n}}\right) \\ &= \frac{(L_{2(2p+1)n-(2p+2)} - L_{2p+2}F_{2(2p+1)n-1}) - L_{2p+1}y_0F_{2(2p+1)n-1}}{(L_{2(2p+1)n-(2p+1)} - L_{2p+2}F_{2(2p+1)n}) - L_{2p+1}y_0F_{2(2p+1)n}} \\ &= \frac{-L_{2p+1}F_{2(2p+1)n} - L_{2p+1}y_0F_{2(2p+1)n-1}}{-L_{2p+1}F_{2(2p+1)n+1} - L_{2p+1}y_0F_{2(2p+1)n}}. \end{aligned}$$

So
$$y_{2n} = \frac{F_{2(2p+1)n} + y_0 F_{2(2p+1)n-1}}{F_{2(2p+1)n+1} + y_0 F_{2(2p+1)n}}.$$
Similarly
$$\begin{aligned}
y_{2n+1} &= \frac{1}{L_{2p+1}}\left(w_{2n+1} - L_{2p+2}\right)\\
&= \frac{1}{L_{2p+1}}\left(\frac{5(L_{2(2p+1)n+(2p+1)} - L_{2p+2}F_{2(2p+1)n})}{5F_{2(2p+1)n} - w_0 L_{2(2p+1)n+(2p+1)}}\right)\\
&\quad + \frac{1}{L_{2p+1}}\left(\frac{-w_0(5F_{2(2p+1)n+(2p+1)} - 7L_{2(2p+1)n+2(2p+1)})}{5F_{2(2p+1)n} - w_0 L_{2(2p+1)n+(2p+1)}}\right)\\
&= \frac{L_{2p+1}}{L_{2p+1}}\left(\frac{5F_{2(2p+1)n-1} - w_0 L_{2(2p+1)(n+1)-(2p+2)}}{5F_{2(2p+1)n} - w_0 L_{2(2p+1)n+(2p+1)}}\right)\\
&= \frac{(5F_{2(2p+1)n-1} - L_{2p+2}L_{2(2p+1)n+2p}) - L_{2p+1}y_0 L_{2(2p+1)n+2p}}{(5F_{2(2p+1)n} - L_{2p+1}L_{2(2p+1)n+(2p+1)}) - L_{2p+1}y_0 L_{2(2p+1)n+(2p+1)}}\\
&= \frac{-L_{2p+1}}{-L_{2p+1}}\left(\frac{L_{2(2p+1)n+(2p+1)} + y_0 L_{2(2p+1)n+2p}}{L_{2(2p+1)n+(2p+2)} + y_0 L_{2(2p+1)n+(2p+1)}}\right).
\end{aligned}$$
So
$$y_{2n+1} = \frac{L_{2(2p+1)n+(2p+1)} + y_0 L_{2(2p+1)n+2p}}{L_{2(2p+1)n+(2p+2)} + y_0 L_{2(2p+1)n+(2p+1)}}.$$
□

From Theorem (1), the solution of equation (2.4) given by
$$\begin{cases} {}^{(j)}x_{2n} = \dfrac{F_{2(2p+1)n} + F_{2(2p+1)n-1}\,{}^{(j)}x_0}{F_{2(2p+1)n+1} + F_{2(2p+1)n}\,{}^{(j)}x_0},\\[2ex] {}^{(j)}x_{2n+1} = \dfrac{L_{2(2p+1)n+(2p+1)} + L_{2(2p+1)n+2p}\,{}^{(j)}x_0}{L_{2(2p+1)n+(2p+2)} + L_{2(2p+1)n+(2p+1)}\,{}^{(j)}x_0}. \end{cases} \tag{2.16}$$
By using (2.3) the following corollary is easily obtained from Theorem (1).

Corollary 1. *Let $\{x_n\}_{n\geq 0}$ be a well-defined solution of (2.2). Then, for, $n \geq 2p$*
$$\begin{cases} x_{(2p+1)(2n)+j} = \dfrac{F_{2(2p+1)n} + F_{2(2p+1)n-1}x_j}{F_{2(2p+1)n+1} + F_{2(2p+1)n}x_j},\\[2ex] x_{(2p+1)(2n+1)+j} = \dfrac{L_{2(2p+1)n+(2p+1)} + L_{2(2p+1)n+2p}x_j}{L_{2(2p+1)n+(2p+2)} + L_{2(2p+1)n+(2p+1)}x_j}, \end{cases}$$
where $j \in \{0, 1, \ldots, 2p\}$, $(L_n)_{n=0}^{+\infty}$ is the Lucas sequence and $(F_n)_{n=0}^{+\infty}$ is the Fibonacci sequence.

Corollary 2. *Let* $\left\{x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(2p+1)}\right\}_{n\geq 0}$ *be a well-defined solution of* (2.1). *Then, for* $n \geq 2p$

$$\begin{cases} x^{(q)}_{2(2p+1)n+j} = \dfrac{F_{2(2p+1)n+j} + x_0^{(q+j)mod(2p+1)} F_{2(2p+1)n+(j-1)}}{F_{2(2p+1)n+(j+1)} + x_0^{(q+j)mod(2p+1)} F_{2(2p+1)n+j}}, \\ x^{(q)}_{(2p+1)(2n+1)+j} = \dfrac{L_{(2p+1)(2n+1)+j} + x_0^{(q+j)mod(2p+1)} L_{(2p+1)(2n+1)+(j-1)}}{L_{(2p+1)(2n+1)+(j+1)} + x_0^{(q+j)mod(2p+1)} L_{(2p+1)(2n+1)+j}}, \end{cases}$$

with $j \in \{0, 2, \ldots, 2p\}$.

$$\begin{cases} x^{(q)}_{2(2p+1)n+j} = \dfrac{L_{2(2p+1)n+(j+1)} + x_0^{(q+j)mod(2p+1)} L_{2(2p+1)n+j}}{L_{2(2p+1)n+(j+2)} + x_0^{(q+j)mod(2p+1)} L_{2(2p+1)n+(j+1)}}, \\ x^{(q)}_{(2p+1)(2n+1)+j} = \dfrac{F_{(2p+1)(2n+1)+(j+1)} + x_0^{(q+j)mod(2p+1)} F_{(2p+1)(2n+1)+j}}{F_{(2p+1)(2n+1)+(j+2)} + x_0^{(q+j)mod(2p+1)} F_{(2p+1)(2n+1)+(j+1)}}. \end{cases}$$

with $j \in \{1, 3, \ldots, 2p+1\}$, $q \in \{1, 2, \ldots, 2p+1\}$, $(L_n)_{n=0}^{+\infty}$ *is the Lucas sequence and* $(F_n)_{n=0}^{+\infty}$ *is the Fibonacci sequence.*

Proof. Let $\left\{x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(2p+1)}\right\}_{n\geq 0}$ be a well-defined solution of system (2.1), so $\{x_n^{(1)}\}_{n\geq 0}$ is a solution of equation (2.2). Then,

$$x^{(1)}_{(2p+1)(2n)+j} = \frac{F_{2(2p+1)n} + F_{2(2p+1)n-1} x_j^{(1)}}{F_{2(2p+1)n+1} + F_{2(2p+1)n} x_j^{(1)}}, \tag{2.17}$$

$$x^{(1)}_{(2p+1)(2n+1)+j} = \frac{L_{(2p+1)(2n+1)} + L_{(2p+1)(2n+1)-1} x_j^{(1)}}{L_{(2p+1)(2n+1)+1} + L_{(2p+1)(2n+1)} x_j^{(1)}}, \tag{2.18}$$

$n \geq 2p, j \in \{0, 1, \ldots, 2p\}$.

On the other hand, if j is even, we have

$$x_j^{(1)} = \frac{F_j + F_{j-1}{}^1 x_0^{(1+j)}}{F_{j+1} + F_j{}^1 x_0^{(1+j)}}. \tag{2.19}$$

From (2.17) we get

$$x^{(1)}_{(2p+1)(2n)+j} = \frac{F_{2(2p+1)n} + F_{2(2p+1)n-1} x_j^{(1)}}{F_{2(2p+1)n+1} + F_{2(2p+1)n} x_j^{(1)}}.$$

Using (2.19) and the equalities

$$F_m = F_{j+1} F_{m-j} + F_j F_{m-(j+1)}, \quad j \in 2\mathbb{N}, m \in \mathbb{N}, \tag{2.20}$$

we obtain

$$x^{(1)}_{2(2p+1)n+j} = \frac{(F_{j+1}+F_j x_0^{(1+j)})F_{2(2p+1)n}+(F_j+F_{j-1}x_0^{(1+j)})F_{2(2p+1)n-1}}{(F_{j+1}+F_j x_0^{(1+j)})F_{2(2p+1)n+1}+(F_j+F_{j-1}x_0^{(1+j)})F_{2(2p+1)n}}$$
$$= \frac{F_{2(2p+1)n+j}+x_0^{(1+j)}F_{2(2p+1)n+(j-1)}}{F_{2(2p+1)n+(j+1)}+x_0^{(1+j)}F_{2(2p+1)n+j}}.$$

Similarly, from (2.17) we have

$$x^{(1)}_{(2p+1)(2n+1)-j} = \frac{L_{(2p+1)(2n+1)}+L_{(2p+1)(2n+1)-1}x_j^{(1)}}{L_{(2p+1)(2n+1)+1}+L_{(2p+1)(2n+1)}x_j^{(1)}}.$$

Using (2.19) and the (2.20) we obtain

$$x^{(1)}_{(2p+1)(2n+1)+j} = \frac{(F_{j+1}+F_j x_0^{(1+j)})L_{(2p+1)(2n+1)}+(F_j+F_{j-1}x_0^{(1+j)})L_{(2p+1)(2n+1)-1}}{(F_{j+1}+F_j x_0^{(1+j)})L_{(2p+1)(2n+1)+1}+(F_j+F_{j-1}x_0^{(1+j)})L_{(2p+1)(2n+1)}}$$
$$= \frac{L_{(2p+1)(2n+1)+j}+x_0^{(1+j)}L_{(2p+1)(2n+1)+(j-1)}}{L_{(2p+1)(2n+1)+(j+1)}+x_0^{(1+j)}L_{(2p+1)(2n+1)+j}}.$$

If j is odd, we have

$$x_j^{(1)} = \frac{F_j+F_{j-1}x_1^{(j)}}{F_{j+1}+F_j x_1^{(j)}}. \tag{2.21}$$

From (2.17) we have

$$x^{(1)}_{(2p+1)(2n)+j} = \frac{F_{2(2p+1)n}+F_{2(2p+1)n-1}x_j^{(1)}}{F_{2(2p+1)n+1}+F_{2(2p+1)n}x_j^{(1)}}.$$

From (2.20) and (2.21), we get

$$x^{(1)}_{2(2p+1)n+j} = \frac{(F_{j+1}+F_j x_1^{(j)})F_{2(2p+1)n}+(F_j+F_{j-1}x_1^{(j)})F_{2(2p+1)n-1}}{(F_{j+1}+F_j x_1^{(j)})F_{2(2p+1)n+1}+(F_j+F_{j-1}x_1^{(j)})F_{2(2p+1)n}}$$
$$= \frac{F_{2(2p+1)n+j}+x_1^{(j)}F_{2(2p+1)n+(j-1)}}{F_{2(2p+1)n+(j+1)}+x_1^{(j)}F_{2(2p+1)n+j}}.$$

So,

$$x^{(1)}_{(2p+1)(2n+1)+j} = \frac{L_{(2p+1)(2n+1)}+L_{(2p+1)(2n+1)-1}x_j^{(1)}}{L_{(2p+1)(2n+1)+1}+L_{(2p+1)(2n+1)}x_j^{(1)}}.$$

From (2.21), we have

$$x^{(1)}_{(2p+1)(2n+1)+j} = \frac{(F_{j+1}+F_j x_1^{(j)})L_{(2p+1)(2n+1)} + (F_j + F_{j-1}x_1^{(j)})L_{(2p+1)(2n+1)-1}}{(F_{j+1}+F_j x_1^{(j)})L_{(2p+1)(2n+1)+1} + (F_j + F_{j-1}x_1^{(j)})L_{(2p+1)(2n+1)}}$$
$$= \frac{L_{(2p+1)(2n+1)+j} + x_1^{(j)} L_{(2p+1)(2n+1)+(j-1)}}{L_{(2p+1)(2n+1)+(j+1)} + x_1^{(j)} L_{(2p+1)(2n+1)+j}}.$$

So

$$\begin{cases} x^{(1)}_{2(2p+1)n+j} = \dfrac{F_{2(2p+1)n+j} + x_1^{(j)} F_{2(2p+1)n+(j-1)}}{F_{2(2p+1)n+(j+1)} + x_1^{(j)} F_{2(2p+1)n+j}}, \\ x^{(1)}_{(2p+1)(2n+1)-j} = \dfrac{L_{(2p+1)(2n+1)+j} + x_1^{(j)} L_{(2p+1)(2n+1)+(j-1)}}{L_{(2p+1)(2n+1)+(j+1)} + x_1^{(j)} L_{(2p+1)(2n+1)+j}}. \end{cases} \tag{2.22}$$

Since we have

$$x_1^{(j)} = \frac{1+2x_0^{(j+1)}}{3+x_0^{(j+1)}}, \tag{2.23}$$

we get

$$\begin{cases} x^{(1)}_{2(2p+1)n+j} = \dfrac{F_{2(2p+1)n+j} + \left(\frac{1+2x_0^{(j+1)}}{3+x_0^{(j+1)}}\right) F_{2(2p+1)n+(j-1)}}{F_{2(2p+1)n+(j+1)} + \left(\frac{1+2x_0^{(j+1)}}{3+x_0^{(j+1)}}\right) F_{2(2p+1)n+j}}, \\ x^{(1)}_{(2p+1)(2n+1)-j} = \dfrac{L_{(2p+1)(2n+1)+j} + \left(\frac{1+2x_0^{(j+1)}}{3+x_0^{(j+1)}}\right) L_{(2p+1)(2n+1)+(j-1)}}{L_{(2p+1)(2n+1)+(j+1)} + \left(\frac{1+2x_0^{(j+1)}}{3+x_0^{(j+1)}}\right) L_{(2p+1)(2n+1)+j}}. \end{cases}$$

So

$$\begin{cases} x^{(1)}_{2(2p+1)n+j} = \frac{(3F_{2(2p+1)n+j}+F_{2(2p+1)n+(j-1)})+x_0^{(j+1)}(F_{2(2p+1)n+j}+2F_{2(2p+1)n+(j-1)})}{(3F_{2(2p+1)n+(j+1)}+F_{2(2p+1)n+j})+x_0^{(j+1)}(F_{2(2p+1)n+(j+1)}+2F_{2(2p+1)n+j})}, \\ x^{(1)}_{(2p+1)(2n+1)+j} \\ \quad = \frac{(3L_{(2p+1)(2n+1)+j}+L_{(2p+1)(2n+1)+(j-1)})+x_0^{(j+1)}(2L_{(2p+1)(2n+1)+(j-1)}+L_{(2p+1)(2n+1)+j})}{(3L_{(2p+1)(2n+1)+(j+1)}+L_{(2p+1)(2n+1)+j})+x_0^{(j+1)}(2L_{(2p+1)(2n+1)+j}+L_{(2p+1)(2n+1)+(j+1)})}. \end{cases}$$

Finally we get

$$\begin{cases} x^{(1)}_{2(2p+1)n+j} = \dfrac{L_{2(2p+1)n+(j+1)} + x_0^{(j+1)} L_{2(2p+1)n+j}}{L_{2(2p+1)n+(j+2)} + x_0^{(j+1)} L_{2(2p+1)n+(j+1)}}, \\ x^{(1)}_{(2p+1)(2n+1)+j} = \dfrac{F_{(2p+1)(2n+1)+(j+1)} + x_0^{(j+1)} F_{(2p+1)(2n+1)+j}}{F_{(2p+1)(2n+1)+(j+2)} + x_0^{(j+1)} F_{(2p+1)(2n+1)+(j+1)}}. \end{cases}$$

Hence

$$\begin{cases} x^{(1)}_{2(2p+1)n+j} = \dfrac{F_{2(2p+1)n+j} + x_0^{(1+j)} F_{2(2p+1)n+(j-1)}}{F_{2(2p+1)n+(j+1)} + x_0^{(1+j)} F_{2(2p+1)n+j}}, \\ x^{(1)}_{(2p+1)(2n+1)+j} = \dfrac{L_{(2p+1)(2n+1)+j} + x_0^{(1+j)} L_{(2p+1)(2n+1)+(j-1)}}{L_{(2p+1)(2n+1)+(j+1)} + x_0^{(1+j)} L_{(2p+1)(2n+1)+j}}, \end{cases}$$

with $j \in \{0, 2, \ldots, 2p\}$.

$$\begin{cases} x^{(1)}_{2(2p+1)n+j} = \dfrac{L_{2(2p+1)n+(j+1)} + x_0^{(j+1)} L_{2(2p+1)n+j}}{L_{2(2p+1)n+(j+2)} + x_0^{(j+1)} L_{2(2p+1)n+(j+1)}}, \\ x^{(1)}_{(2p+1)(2n+1)+j} = \dfrac{F_{(2p+1)(2n+1)+(j+1)} + x_0^{(j+1)} F_{(2p+1)(2n+1)+j}}{F_{(2p+1)(2n+1)+(j+2)} + x_0^{(j+1)} F_{(2p+1)(2n+1)+(j+1)}}, \end{cases}$$

with $j \in \{1, 3, \ldots, 2p+1\}$.

In the same way, after some calculations and using the fact that

$$x_n^{(2p+1)} = \frac{1 + 2x_{n-1}^{(1)}}{3 + x_{n-1}^{(1)}}, \quad x_n^{(i)} = \frac{1 + 2x_{n-1}^{(i+1)}}{3 + x_{n-1}^{(i+1)}}, \quad i = 2, 3, \ldots, 2p,$$

we obtain

$$\begin{cases} x^{(q)}_{2(2p+1)n+j} = \dfrac{F_{2(2p+1)n+j} + x_0^{(q+j) mod(2p+1)} F_{2(2p+1)n+(j-1)}}{F_{2(2p+1)n+(j+1)} + x_0^{(q+j) mod(2p+1)} F_{2(2p+1)n+j}}, \\ x^{(q)}_{(2p+1)(2n+1)+j} = \dfrac{L_{(2p+1)(2n+1)+j} + x_0^{(q+j) mod(2p+1)} L_{(2p+1)(2n+1)+(j-1)}}{L_{(2p+1)(2n+1)+(j+1)} + x_0^{(q+j) mod(2p+1)} L_{(2p+1)(2n+1)+j}}, \end{cases}$$

with $j \in \{0, 2, \ldots, 2p\}$.

$$\begin{cases} x^{(q)}_{2(2p+1)n+j} = \dfrac{L_{2(2p+1)n+(j+1)} + x_0^{(q+j) mod(2p+1)} L_{2(2p+1)n+j}}{L_{2(2p+1)n+(j+2)} + x_0^{(q+j) mod(2p+1)} L_{2(2p+1)n+(j+1)}}, \\ x^{(q)}_{(2p+1)(2n+1)+j} = \dfrac{F_{(2p+1)(2n+1)+(j+1)} + x_0^{(q+j) mod(2p+1)} F_{(2p+1)(2n+1)+j}}{F_{(2p+1)(2n+1)+(j+2)} + x_0^{(q+j) mod(2p+1)} F_{(2p+1)(2n+1)+(j+1)}}, \end{cases}$$

with $j \in \{1, 3, \ldots, 2p+1\}$. □

3. On the system of higher order difference equations (1.9)

In this section, we discuss the form of system (1.9) which generalizes (2.1) in a graceful way. We establish the solution of the system (1.9) by using an appropriate transformation reducing this system to the system of first-order difference equations (2.1).

3.1. *Analysis of the form of system* (1.9)

The initial values with the smallest indexes are $x_{-k}^{(1)}, x_{-k}^{(2)}, \dots, x_{-k}^{(2p)}$ and $x_{-k}^{(2p+1)}$. By using (1.9) with $n = 0$, we obtain the values of $x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(2p)}$ and $x_1^{(2p+1)}$ as follows

$$x_1^{(1)} = \frac{1+2x_{-k}^{(1)}}{3+x_{-k}^{(1)}}, \quad x_1^{(2)} = \frac{1+2x_{-k}^{(3)}}{3+x_{-k}^{(3)}}, \dots, \quad x_1^{(2p+1)} = \frac{1+2x_{-k}^{(1)}}{3+x_{-k}^{(1)}}.$$

After known the values of $x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(2p)}$ and $x_1^{(2p+1)}$, by using (1.9) with $n = k+1$ we get the values of $x_{k+2}^{(1)}, x_{k+2}^{(2)}, \dots, x_{k+2}^{(2p)}$ and $x_{k+2}^{(2p+1)}$. We have

$$x_{k+2}^{(1)} = \frac{1+2x_1^{(1)}}{3+x_1^{(1)}}, \quad x_{k+2}^{(2)} = \frac{1+2x_1^{(3)}}{3+x_1^{(3)}}, \dots, \quad x_{k+2}^{(2p+1)} = \frac{1+2x_1^{(1)}}{3+x_1^{(1)}}.$$

The values of $x_{k+2}^{(1)}, x_{k+2}^{(2)}, \dots, x_{k+2}^{(2p)}$ and $x_{k+2}^{(2p+1)}$, by using (1.9) with $n = 2k+2$, leads us to obtain the values of $x_{2k+3}^{(1)}, x_{2k+3}^{(2)}, \dots, x_{2k+3}^{(2p)}$ and $x_{2k+3}^{(2p+1)}$. We have

$$x_{2k+3}^{(1)} = \frac{1+2x_{k+2}^{(1)}}{3+x_{k+2}^{(1)}}, \quad x_{2k+3}^{(2)} = \frac{1+2x_{k+2}^{(3)}}{3+x_{k+2}^{(3)}}, \dots, \quad x_{2k+3}^{(2p+1)} = \frac{1+2x_{k+2}^{(1)}}{3+x_{k+2}^{(1)}}.$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\begin{cases} x_{(k+1)m+1}^{(1)} &= \frac{1+2x_{(k+1)m-k}^{(1)}}{3+x_{(k+1)m-k}^{(1)}}, \\ x_{(k+1)m+1}^{(2)} &= \frac{1+2x_{(k+1)m-k}^{(3)}}{3+x_{(k+1)m-k}^{(3)}}, \\ & \vdots \\ x_{(k+1)m+1}^{(2p+1)} &= \frac{1+2x_{(k+1)m-k}^{(1)}}{3+x_{(k+1)m-k}^{(1)}}. \end{cases} \tag{3.1}$$

In the same way, it is shown that the initial values $x_{-r}^{(1)}, x_{-r}^{(2)}, \dots, x_{-r}^{(2p)}$ and $x_{-r}^{(2p+1)}$, for a fixed $r \in \{0, 1, \dots, k\}$, determine all the values of the sequences $(x_{(k+1)(m+1)-r}^{(1)})_m$,

$(x^{(2)}_{(k+1)(m+1)-r})_m, \ldots, (x^{(2p)}_{(k+1)(m+1)-r})_m$ and $(x^{(2p+1)}_{(k+1)(m+1)-r})_m$. Also we have

$$\begin{cases} x^{(1)}_{(k+1)(m+1)-r} & = & \dfrac{1+2x^{(1)}_{(k+1)m-r}}{3+x^{(1)}_{(k+1)m-r}}, \\ x^{(2)}_{(k+1)(m+1)-r} & = & \dfrac{1+2x^{(3)}_{(k+1)m-r}}{3+x^{(3)}_{(k+1)m-r}}, \\ & \vdots & \\ x^{(2p+1)}_{(k+1)(m+1)-r} & = & \dfrac{1+2x^{(1)}_{(k+1)m-r}}{3+x^{(1)}_{(k+1)m-r}}. \end{cases} \tag{3.2}$$

3.2. *A representation of the general solution to system* (1.9)

Now we are going to apply the previous analysis. Let

$$^{(r)}x^{(q)}_n = x_{(k+1)n-r}, \tag{3.3}$$

where $r \in \{0,1,\ldots,k\}$. and $q \in \{1,2,\ldots,(2p+1)\}$.
Using notation (3.3), we can write (1.9) as

$$^{(r)}x^{(1)}_{n+1} = \frac{1+2^{(r)}x^{(1)}_n}{3+{}^{(r)}x^{(1)}_n}, \quad ^{(r)}x^{(2)}_{n+1} = \frac{1+2^{(r)}x^{(3)}_n}{3+{}^{(r)}x^{(3)}_n}, \cdots, \quad ^{(r)}x^{(2p+1)}_{n+1} = \frac{1+2^{(r)}x^{(1)}_n}{3+{}^{(r)}x^{(1)}_n}, \tag{3.4}$$

for each $r \in \{0,1,\ldots,k\}$.

It signifies that the sequences $\left({}^{(r)}x^{(1)}_n\right)_{n\in\mathbb{N}_0}, \left({}^{(r)}x^{(2)}_n\right)_{n\in\mathbb{N}_0}, \ldots, \left({}^{(r)}x^{(2p)}_n\right)_{n\in\mathbb{N}_0}$ and $\left({}^{(r)}x^{(2p+1)}_n\right)_{n\in\mathbb{N}_0}$, $r=\overline{0,k}$, are $(2p+1)(k+1)$ solutions to system (2.1) with the initial values ${}^{(r)}x^{(1)}_0, {}^{(r)}x^{(2)}_0, \ldots, {}^{(r)}x^{(2p)}_0$ and ${}^{(r)}x^{(2p+1)}_0$, $r=\overline{0,k}$, respectively.

Using Corollary (2) to the sequences $\left({}^{(r)}x^{(1)}_n\right)_{n\in\mathbb{N}_0}, \left({}^{(r)}x^{(2)}_n\right)_{n\in\mathbb{N}_0}, \ldots, \left({}^{(r)}x^{(2p)}_n\right)_{n\in\mathbb{N}_0}$ and $\left({}^{(r)}x^{(2p+1)}_n\right)_{n\in\mathbb{N}_0}$, $r=\overline{0,k}$, we show that the following representation holds

$$\begin{cases} {}^{(r)}x^{(q)}_{2(2p+1)n+j} = \dfrac{F_{2(2p+1)n+j}{}^{(r)} + x_0^{(q+j)mod(2p+1)}F_{2(2p+1)n+(j-1)}}{F_{2(2p+1)n+(j+1)} + {}^{(r)}x_0^{(q+j)mod(2p+1)}F_{2(2p+1)n+j}}, \\ {}^{(r)}x^{(q)}_{(2p+1)(2n+1)+j} = \dfrac{L_{(2p+1)(2n+1)+j} + {}^{(r)}x_0^{(q+j)mod(2p+1)}L_{(2p+1)(2n+1)+(j-1)}}{L_{(2p+1)(2n+1)+(j+1)} + {}^{(r)}x_0^{(q+j)mod(2p+1)}L_{(2p+1)(2n+1)+j}}, \end{cases}$$

with $j \in \{0,2,\dots,2p\}$.

$$\begin{cases} {}^{(r)}x^{(q)}_{2(2p+1)n+j} = \dfrac{L_{2(2p+1)n+(j+1)} + {}^{(r)}x_0^{(q+j)mod(2p+1)} L_{2(2p+1)n+j}}{L_{2(2p+1)n+(j+2)} + {}^{(r)}x_0^{(q+j)mod(2p+1)} L_{2(2p+1)n+(j+1)}}, \\ {}^{(r)}x^{(q)}_{(2p+1)(2n+1)+j} = \dfrac{F_{(2p+1)(2n+1)+(j+1)} + {}^{(r)}x_0^{(q+j)mod(2p+1)} F_{(2p+1)(2n+1)+j}}{F_{(2p+1)(2n+1)+(j+2)} + {}^{(r)}x_0^{(q+j)mod(2p+1)} F_{(2p+1)(2n+1)+(j+1)}}, \end{cases}$$

with $j \in \{1,3,\dots,2p+1\}$.

For each $q \in \{1,2,\dots,2p+1\}$, $r \in \{1,2,\dots,k\}$, $(L_n)_{n=0}^{+\infty}$ is the Lucas sequence and $(F_n)_{n=0}^{+\infty}$ is the Fibonacci sequence.

Coming back to the original notation, from (3.3), it follows that the following result holds.

Corollary 3. *Let* $\left\{x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(2p+1)}\right\}_{n\geq -k}$ *be a solution of* (1.9). *Then, for* $n = 2,3,\dots,$

$$\begin{cases} x^{(q)}_{(k+1)(2(2p+1)n+j)-r} = \dfrac{F_{2(2p+1)n+j} + x_{-r}^{(q+j)mod(2p+1)} F_{2(2p+1)n+(j-1)}}{F_{2(2p+1)n+(j+1)} + x_{-r}^{(q+j)mod(2p+1)} F_{2(2p+1)n+j}}, \\ x^{(q)}_{(k+1)((2p+1)(2n+1)+j)-r} = \dfrac{L_{(2p+1)(2n+1)+j} + x_{-r}^{(q+j)mod(2p+1)} L_{(2p+1)(2n+1)+(j-1)}}{L_{(2p+1)(2n+1)+(j+1)} + x_{-r}^{(q+j)mod(2p+1)} L_{(2p+1)(2n+1)+j}}, \end{cases}$$

with $j \in \{0,2,\dots,2p\}$.

$$\begin{cases} x^{(q)}_{(k+1)(2(2p+1)n+j)-r} = \dfrac{L_{2(2p+1)n+(j+1)} + x_{-r}^{(q+j)mod(2p+1)} L_{2(2p+1)n+j}}{L_{2(2p+1)n+(j+2)} + x_{-r}^{(q+j)mod(2p+1)} L_{2(2p+1)n+(j+1)}}, \\ x^{(q)}_{(k+1)((2p+1)(2n+1)+j)-r} = \dfrac{F_{(2p+1)(2n+1)+(j+1)} + x_{-r}^{(q+j)mod(2p+1)} F_{(2p+1)(2n+1)+j}}{F_{(2p+1)(2n+1)+(j+2)} + x_{-r}^{(q+j)mod(2p+1)} F_{(2p+1)(2n+1)+(j+1)}}, \end{cases}$$

where $j \in \{1,3,\dots,2p+1\}$, $q \in \{1,2,\dots,2p+1\}$, $r \in \{1,2,\dots,k\}$, $(L_n)_{n=0}^{+\infty}$ *is the Lucas sequence and* $(F_n)_{n=0}^{+\infty}$ *is the Fibonacci sequence.*

4. Global stability of positive solutions of (1.9)

In this section we study the global stability character of the solutions of system (1.9). It is easy to show that (1.9) has a unique real positive equilibrium point given by

$$E = \left(\overline{x^{(1)}}, \overline{x^{(2)}}, \dots, \overline{x^{(2p+1)}}\right) = (-\beta, -\beta, \dots, -\beta),$$

where β is the number defined in (1.3).

Let $I_i(0,+\infty)$ and consider the functions

$$f_i : I_1^{k+1} \times I_2^{k+1} \times \dots \times I_{2p+1}^{k+1} \longrightarrow I_i,$$

defined by

$$f_i\left(u_0^{(1)},u_1^{(1)},\ldots,u_k^{(1)},u_0^{(2)},u_1^{(2)},\ldots,u_k^{(2)},\ldots,u_0^{(2p+1)},u_1^{(2p+1)},\ldots,u_k^{(2p+1)}\right) = \frac{1+2u_k^{(i+1)mod(2p+1)}}{3+u_k^{(i+1)mod(2p+1)}},$$

with $i \in \{1,2,\ldots,2p+1\}$.

Theorem 2. *The equilibrium point E is locally asymptotically stable.*

Proof. The linearized system about the equilibrium point

$$E = (-\beta,\ldots,-\beta,-\beta,\ldots,-\beta) \in I_1^{k+1} \times I_2^{k+1} \times \ldots \times I_{2p+1}^{k+1}$$

is given by

$$X_{n+1} = AX_n, \tag{4.1}$$

$$X_n = \left(x_n^{(1)},x_{n-1}^{(1)},\ldots,x_{n-k}^{(1)},x_n^{(2)},x_{n-1}^{(2)},\ldots,x_{n-k}^{(2)},\ldots,x_n^{(2p+1)},x_{n-1}^{(2p+1)},\ldots,x_{n-k}^{(2p+1)}\right)^t \tag{4.2}$$

and

$$A = \left(\begin{array}{cccccccccccccc}
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & \frac{5}{(3-\beta)^2} & \ldots & \ldots & 0 & 0 & \ldots & 0 \\
1 & 0 & & & 0 & & & & & & & & & 0 \\
0 & 1 & \ddots & & & & & & & & & & & \vdots \\
0 & & \ddots & 0 & \ldots & \ldots & 0 & 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 \\
 & & & 1 & 0 & & & & & & & & & \vdots \\
\vdots & & \vdots & & \vdots & & & \vdots & & & \vdots & & & \vdots \\
 & & & & & 1 & & & & & & & & 0 \\
 & & & & & & \ddots & 0 & \ldots & \ldots & 0 & 0 & \ldots & \frac{5}{(3-\beta)^2} \\
\vdots & & & \vdots & & 0 & & 1 & & & & & & 0 \\
\vdots & & & & & & & & & & & & & \vdots \\
0 & \ldots & 0 & \frac{5}{(3-\beta)^2} & 0 & 0 & & & \ldots & \ldots & 0 & & & \\
0 & & & 0 & & 0 & & & & 0 & \ddots & 0 & \vdots & \vdots \\
\vdots & & & \vdots & & \vdots & & & & & & \ddots & 0 & 0 \\
0 & 0 & \ldots & 0 & & 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 & 1 & 0
\end{array}\right).$$

So, after some elementary calculations, we get

$$P(\lambda) = (-\lambda)^{(2p+1)(k+1)} + (-1)^k \left(\frac{5}{(3-\beta)^2}\right)^{2p+1}.$$

Now, consider the two functions defined by

$$\varphi(\lambda)=\lambda^{(2p+1)(k+1)},\quad \phi(\lambda)=\left(\frac{5}{(3-\beta)^2}\right)^{2p+1}.$$

We have

$$|\phi(\lambda)|<|\varphi(\lambda)|,\forall\lambda:|\lambda|=1$$

So, according to Rouche's Theorem φ and $P=\varphi+\phi$ have the same number of zeros in the unit disc $|\lambda|<1$, and since φ admits as root $\lambda=0$ of multiplicity $(2p+1)(k+1)$, then all the roots of P are in the disc $|\lambda|<1$. Thus, the equilibrium point is locally asymptotically stable. □

Theorem 3. *For every well defined solution of system* (1.9)*, we have*

$$\lim_{n\to+\infty} x_n^{(q)}=-\beta,$$

for each $q\in\{1,2,\ldots,2p+1\}$.

Proof. From Corollary (3), we have

$$\begin{aligned}
&\lim_{n\to+\infty} x^{(q)}_{(k+1)(2(2p+1)n+2(2p+1)+j)-r}\\
&\quad=\lim_{n\to+\infty}\frac{F_{2(2p+1)n+2(2p+1)+j}+x_{-r}^{(q+j)}F_{2(2p+1)n+2(2p+1)+(j-1)}}{F_{2(2p+1)n+2(2p+1)+(j+1)-r}+x_{-r}^{(q+j)}F_{2(2p+1)n+2(2p+1)+j}}\\
&\quad=\lim_{n\to+\infty}\frac{1+x_{-r}^{(q+j)}\frac{F_{2(2p+1)n+2(2p+1)+(j-1)}}{F_{2(2p+1)n+2(2p+1)+j}}}{\frac{F_{2(2p+1)n+2(2p+1)+(j+1)}}{F_{2(2p+1)n+2(2p+1)+j}}+x_{-r}^{(q+j)}}.
\end{aligned}$$

Using the limit (1.4), we get

$$\lim_{n\to+\infty} x^{(q)}_{(k+1)(2(2p+1)n+2(2p+1)+j)-r}=\frac{1+x_{-r}^{(q+j)}\frac{1}{\alpha}}{\alpha+x_{-r}^{(q+j)}}.$$

Hence

$$\lim_{n\to+\infty} x^{(q)}_{(k+1)(2(2p+1)n+2(2p+1)+j)-r}=-\beta.$$

Also,

$$\begin{aligned}
&\lim_{n\to+\infty} x^{(q)}_{(k+1)((2p+1)(2n+1)+j)-r}\\
&\quad=\lim_{n\to+\infty}\frac{L_{2(2p+1)n+(2p+1)+j}+x_{-r}^{(q+j)}L_{2(2p+1)n+(2p+1)+(j-1)}}{L_{2(2p+1)n+(2p+1)+(j+1)}+x_{-r}^{(q+j)}L_{2(2p+1)n+(2p+1)+j}}\\
&\quad=\lim_{n\to+\infty}\frac{1+x_{-r}^{(q+j)}\frac{L_{2(2p+1)n+(2p+1)+(j-1)}}{L_{2(2p+1)n+(2p+1)+j}}}{\frac{L_{2(2p+1)n+(2p+1)+(j+1)}}{L_{2(2p+1)n+(2p+1)+j}}+x_{-r}^{(q+j)}}.
\end{aligned}$$

Using the limit (1.7), we get

$$\lim_{n\to+\infty} x^{(q)}_{(k+1)((2p+1)(2n+1)+j)-r} = \frac{1+x^{(q+j)}_{-r}\frac{1}{\alpha}}{\alpha+x^{(q+j)}_{-r}}.$$

Hence

$$\lim_{n\to+\infty} x^{(q)}_{(k+1)((2p+1)(2n+1)+j)-r} = -\beta.$$

Similarly, we find

$$\lim_{n\to+\infty} x^{(q)}_{(k+1)(2(2p+1)n+2(2p+1)+j)-r} = \lim_{n\to+\infty} x^{(q)}_{(k+1)((2p+1)(2n+1)+j)-r} = -\beta.$$

So, we have

$$\lim_{n\to+\infty} x^{(q)}_{n} = -\beta.$$

□

The following result is a direct consequence of Theorems (2) and (3).

Corollary 4. *The equilibrium point E is globally asymptotically stable.*

4.1. *Numerical confirmation*

In order to verify our theoretical results we consider several interesting numerical examples in this section. These examples represent different types of qualitative behaviour of solutions of the system (1.9). All plots in this section are drawn with Matlab.

Example 1. Let $k=1$ and $p=2$ in system (1.9), then we obtain the system

$$\begin{cases} x^{(1)}_{n+1} = \frac{1+2x^{(2)}_{n-1}}{3+x^{(2)}_{n-1}}, & x^{(2)}_{n+1} = \frac{1+2x^{3)}_{n-1}}{3+x^{(3)}_{n-1}}, \quad x^{(3)}_{n+1} = \frac{1+2x^{(4)}_{n-1}}{3+x^{(4)}_{n-1}}, \\ x^{(4)}_{n+1} = \frac{1+2x^{(5)}_{n-1}}{3+x^{(5)}_{n-1}}, & x^{(5)}_{n+1} = \frac{1+2x^{(1)}_{n-1}}{3+x^{(1)}_{n-k}}, \quad n\in N_0. \end{cases} \tag{4.3}$$

Assume $x^{(1)}_{-1}=1,\ x^{(1)}_{0}=7,\ x^{(2)}_{-1}=1.3,\ x^{(2)}_{0}=0.3,\ x^{(3)}_{-1}=3,\ x^{(3)}_{0}=1.5,\ x^{(4)}_{-1}=14,$ $x^{(4)}_{0}=2,\ x^{(5)}_{-1}=3$ and $x^{(5)}_{0}=0.1$. (See Figure (1)).

Example 2. Let $k=3$ and $p=3$ in system (1.9), then we obtain the system

$$\begin{cases} x^{(1)}_{n+1} = \frac{1+2x^{(2)}_{n-3}}{3+x^{(2)}_{n-3}}, \quad x^{(2)}_{n+1} = \frac{1+2x^{3)}_{n-3}}{3+x^{(3)}_{n-3}}, \quad x^{(3)}_{n+1} = \frac{1+2x^{(4)}_{n-3}}{3+x^{(4)}_{n-3}}, \quad x^{(4)}_{n+1} = \frac{1+2x^{(5)}_{n-3}}{3+x^{(5)}_{n-3}}, \\ x^{(5)}_{n+1} = \frac{1+2x^{(6)}_{n-3}}{3+x^{(6)}_{n-3}}, \quad x^{(6)}_{n+1} = \frac{1+2x^{(7)}_{n-3}}{3+x^{(7)}_{n-3}}, \quad x^{(7)}_{n+1} = \frac{1+2x^{(1)}_{n-3}}{3+x^{(1)}_{n-3}}, \quad n\in N_0. \end{cases} \tag{4.4}$$

Assume $x^{(1)}_{-3}=1, x^{(1)}_{-2}=0.2, x^{(1)}_{-1}=6, x^{(1)}_{0}=7, x^{(2)}_{-3}=1.3, x^{(2)}_{-2}=5, x^{(2)}_{-1}=0.7, x^{(2)}_{0}=9, x^{(3)}_{-3}=0.1, x^{(3)}_{-2}=3, x^{(3)}_{-1}=6, x^{(3)}_{0}=1.5, x^{(4)}_{-3}=7, x^{(4)}_{-2}=9.3, x^{(4)}_{-1}=5.3, x^{(4)}_{0}=5.3, x^{(5)}_{-3}=2.2, x^{(5)}_{-2}=2.2, x^{(5)}_{-1}=14.3, x^{(5)}_{0}=0.8, x^{(6)}_{-3}=3.3, x^{(6)}_{-2}=6, x^{(6)}_{-1}=8, x^{(6)}_{0}=1.9, x^{(7)}_{-3}=4, x^{(7)}_{-2}=7.2, x^{(7)}_{-1}=1.6$ and $x^{(7)}_{0}=8$. (See Figure (2)).

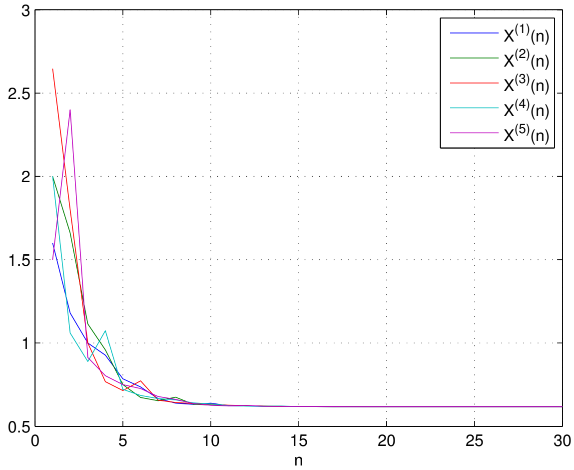

FIGURE 1. The plot of system (4.3)

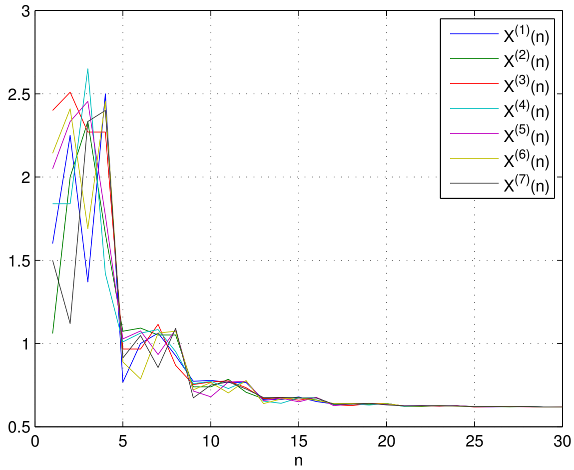

FIGURE 2. The plot of system (4.4)

5. CONCLUSIONS

In the paper, we represented the well-defined solutions of the system (1.9) composed by $2p+1$ rational difference equations. More exactly, We gave general solutions of system (1.9) in terms of Fibonacci and Lucas sequences. Also, we presented some results about the general behavior of solutions of system (1.9) and some numerical examples are carried out to support the analysis results. Our system generalized the systems studied in [18] and [19].

The results in this paper can be extended to the following system of difference equations

$$x_{n+1}^{(j)} = \frac{L_{m+2} + L_{m+1} x_{n-k}^{((j+1) mod(p))}}{L_{m+3} + L_{m+2} x_{n-k}^{((j+1) mod(p))}}, \quad n, m, p, k \in N_0, j = \overline{1, p},$$

where $(L_n)_{n=0}^{+\infty}$ is the Lucas sequence.

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REFERENCES

[1] R. Abo-Zeid, "Global behavior of a higher order difference equation," *Math. Slovaca*, vol. 64, no. 4, pp. 931–940, 2014, doi: 10.2478/s12175-014-0249-z.

[2] R. Abo-Zeid, "Forbidden sets and stability in some rational difference equations," *J. Difference Equ. Appl.*, vol. 24, no. 2, pp. 220–239, 2018, doi: 10.1080/10236198.2017.1400023.

[3] Y. Akrour, N. Touafek, and Y. Halim, "On a system of difference equations of second order solved in closed-form." *Miskolc Math. Notes*, vol. 20, no. 2, pp. 701–717, 2019, doi: 10.18514/MMN.2019.2923.

[4] S. Elaydi, *An introduction to difference equations*. Springer Science & Business Media, 2005. doi: 10.1007/0-387-27602-5.

[5] E. M. Elsayed, "Solutions of rational difference systems of order two," *Math. Comput. Modelling*, vol. 55, no. 1, pp. 378–384, 2012, doi: 10.1016/j.mcm.2011.08.012.

[6] E. M. Elsayed, "Solution for systems of difference equations of rational form of order two," *Comp. Appl. Math.*, vol. 33, no. 3, pp. 751–765, 2014, doi: 10.1007/s40314-013-0092-9.

[7] E. M. Elsayed and T. F. Ibrahim, "Periodicity and solutions for some systems of nonlinear rational difference equations," *Hacet. J. Math. Stat.*, vol. 44, no. 6, pp. 1361–1390, 2015, doi: 10.15672/HJMS.2015449653.

[8] E. A. Grove and G. Ladas, *Periodicities in nonlinear difference equations*. CRC Press, 2004, vol. 4.

[9] Y. Halim, "Form and periodicity of solutions of some systems of higher-order difference equations," *Math. Sci. Lett.*, vol. 5, no. 1, pp. 79–84, 2016, doi: 10.18576/msl/050111.

[10] Y. Halim, "A system of difference equations with solutions associated to Fibonacci numbers," *Int. J. Difference. Equ.*, vol. 11, no. 1, pp. 65–77, 2016.

[11] Y. Halim, "Global character of systems of rational difference equations," *Electron. J. Math. Analysis. Appl.*, vol. 3, no. 1, pp. 204–214, 2018.

[12] Y. Halim and M. Bayram, "On the solutions of a higher-order difference equation in terms of generalized Fibonacci sequences," *Math. Methods Appl. Sci.*, vol. 39, no. 1, pp. 2974–2982, 2016, doi: 10.1002/mma.3745.

[13] Y. Halim, A. Khelifa, and M. Berkal, "Representation of solutions of a two-dimensional System of difference equations," *Miskolc Math. Notes*, vol. 21, no. 1, pp. 203–2018, 2020, doi: 10.18514/MMN.2020.3204.

[14] Y. Halim, A. Khelifa, and A. Boussaha, "Representation of solutions of a second-order system of difference equations in terms of Padovan sequence," *Dyn. Contin. Discrete Impuls. Syst., Ser. B, Appl. Algorithms*, vol. 27, no. 3, pp. 113–131, 2020.

[15] Y. Halim and J. F. T. Rabago, "On the solutions of a second-order difference equation in terms of generalized Padovan sequences," *Math. Slovaca*, vol. 68, no. 3, pp. 625–638, 2018, doi: 10.1515/ms-2017-0130.

[16] M. Kara and Y. Yazlik, "Solvability of a system of nonlinear difference equations of higher order," *Turk. J. Math.*, vol. 43, no. 3, pp. 1533–1565, 2019, doi: 10.3906/mat-1902-24.

[17] W. G. Kelley and A. C. Peterson, *Difference equations: an introduction with applications*. Academic press, 2001.

[18] A. Khelifa, Y. Halim, and M. Berkal, "Solutions of a system of two higher-order difference equations in terms of Lucas sequence," *Univers. J. Math. Appl.*, vol. 2, no. 4, pp. 202–211, 2019, doi: 10.32323/ujma.610399.

[19] A. Khelifa, Y. Halim, A. Bouchair, and M. Berkal, "On a system of three difference equations of higher order solved in terms of lucas and fibonacci numbers," *Math. Slovaca*, vol. 70, no. 3, pp. 641–656, 2020, doi: 10.1515/ms-2017-0378.

[20] T. Koshy, *Fibonacci and Lucas numbers with applications*. John Wiley & Sons, 2019. doi: 10.1002/9781118033067.

[21] D. T. Tollu, Y. Yazlik, and N. Taskara, "On the solutions of two special types of Riccati difference equation via Fibonacci numbers," *Adv. Difference Equ.*, vol. 174, no. 1, pp. 1–7, 2013, doi: 10.1186/1687-1847-2013-174.

[22] N. Touafek, "On some fractional systems of difference equations," *Iran. J. Math. Sci. Inform.*, vol. 9, no. 2, pp. 73–86, 2014, doi: 10.7508/ijmsi.2014.02.007.

[23] N. Touafek and E. Elsayed, "On the solutions of systems of rational difference equations," *Math. Comput. Modelling*, vol. 55, no. 7-8, pp. 1987–1997, 2012, doi: 10.1016/j.mcm.2011.11.058.

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EXISTENCE RESULTS FOR A KIRCHHOFF-TYPE PROBLEM WITH SINGULARITY

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Abstract. In this work, using an infinitely many critical points theorem we establish the existence of a sequence of weak solutions for a Kirchhoff-type problem with singular term. This approach is based on variational methods and critical point theory.

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Keywords: singularity, Kirchhoff type problems, variational methods, critical point

1. INTRODUCTION

In 1883, the stationary problem

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0,$$

was proposed by Kirchhoff [14] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. In recent years, the study of elliptic problems involving Kirchhoff type operators have been studied in many works, we refer to [1, 3, 5–7, 16–18, 20, 22]. For instance, in [17], Molica Bisci and Pizzimenti considered the following problem

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^p dx\right)\Delta_p u+\alpha(x)|u|^{p-2}u=\lambda h(x)f(u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases}$$

They obtained the existence of infinitely many weak solutions by using variational methods. Also, in [5], the authors studied the non-local problem

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p dx\right)\Delta_p u=f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases}$$

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By using Browder Theorem, the writers proved the existence and uniqueness of solutions. On the other hand, singular elliptic problems have been intensively studied in the last decads. Among others, we mention the works [8–12, 15, 19, 21]. Recently, motivated by this large interest, Ferrara and Molica Bisci in [8] studied the existence of at least one non-trivial weak solution for the following elliptic Dirichlet problem

$$\begin{cases} -\Delta_p u = \mu \dfrac{|u|^{p-2}u}{|x|^p} + \lambda f(x,u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $\lambda > 0$ and $\mu \geq 0$ are two real parameters, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ containing the origin and with smooth boundary $\partial\Omega$, $1 < p < N$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying a suitable subcritical growth condition.

The aim of this paper is to investigate the existence of infinitely many weak solutions for the following problem

$$\begin{cases} -M\left(\int\limits_\Omega |\nabla u|^p dx\right) \Delta_p u + \frac{|u|^{q-2}u}{|x|^q} = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplace operator, Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ containing the origin and with smooth boundary $\partial\Omega$, $1 < q < N < p$, $M : [0, +\infty) \to R$ is a continuous function satisfying

(f_1) there are two positive constants m_0 , m_1, such that

$$m_0 \leq M(t) \leq m_1, \qquad \forall t \geq 0,$$

and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function.

Recall that a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be an L^1-Carathéodory function, if

(C_1) the function $x \mapsto f(x,t)$ is measurable for every $t \in \mathbb{R}$;
(C_2) the function $t \mapsto f(x,t)$ is continuous for a.e. $x \in \Omega$;
(C_3) for every $\rho > 0$ there exists a function $l_\rho \in L^1(\Omega)$ such that

$$\sup_{|t| \leq \rho} |f(x,t)| \leq l_\rho(x),$$

for a.e. $x \in \Omega$.

A special case of our main result is the following theorem.

Theorem 1. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a non-negative continuous function such that*

$$\liminf_{\xi \to +\infty} \frac{f(\xi)}{\xi^{p-1}} = 0 \qquad \textit{and} \qquad \limsup_{\xi \to +\infty} \frac{\int_0^\xi f(t)dt}{\xi^p} = +\infty.$$

Then, the problem

$$\begin{cases} -\left(\frac{1+2\left(\int_\Omega |\nabla u|^p dx\right)^2}{1+\left(\int_\Omega |\nabla u|^p dx\right)^2}\right)\Delta_p u + \frac{|u|^{q-2}u}{|x|^q} = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits a sequence of weak solutions which is unbounded in X.

1.1. *Preliminary considerations*

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 2)$ containing the origin and with smooth boundary $\partial\Omega$. Further, denote by X the space $W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p \, dx\right)^{1/p}.$$

Also, let $\|\cdot\|_1$ denotes the usual norm of $L^1(\Omega)$; i.e.,

$$\|u\|_1 := \int_\Omega |u(x)|dx.$$

We recall classical Hardy's inequality, which says that

$$\int_\Omega \frac{|u(x)|^q}{|x|^q} dx \leq \frac{1}{H}\int_\Omega |\nabla u(x)|^q dx, \qquad (\forall u \in X), \tag{1.2}$$

where $H := (\frac{N-q}{q})^q$; see, for instance, the paper [2].

Let us define $F(x,\xi) := \int_0^\xi f(x,t)dt$, for every (x,ξ) in $\Omega \times \mathbb{R}$. Moreover we introduce the functional $I_\lambda : X \to \mathbb{R}$ associated with (1.1),

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u),$$

for every $u \in X$, where

$$\Phi(u) := \frac{1}{p}\hat{M}(\|u\|^p) + \frac{1}{q}\int_\Omega \frac{|u(x)|^q}{|x|^q} dx,$$

and

$$\Psi(u) := \int_\Omega F(x, u(x))dx,$$

for every $u \in X$, where $\hat{M}(t) := \int\limits_0^t M(s)ds, \;\; t \geq 0$. By standard arguments, one has that Φ is well defined (by Hardy's inequality), Gâteaux differentiable and sequentially weakly lower semicontinuous, and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$ given by

$$\Phi'(u)(v) = M\left(\int_\Omega |\nabla u|^p dx\right)\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx + \int_\Omega \frac{|u(x)|^{q-2}}{|x|^q}u(x)v(x)dx,$$

for every $v \in X$ and clearly Φ is coercive. It is easy to prove that Φ is strongly continuous. On the other hand, standard arguments show that Ψ is well defined and continuously Gâteaux differentiable functional whose Gâteaux derivative

$$\Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)dx,$$

for every $v \in X$, is a compact operator from X to the dual X^*.

Fixing the real parameter λ, a function $u : \Omega \to \mathbb{R}$ is said to be a weak solution of (1.1) if $u \in X$ and

$$M\left(\int_\Omega |\nabla u|^p dx\right) \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx + \int_\Omega \frac{|u(x)|^{q-2}}{|x|^q} u(x)v(x)dx - \lambda \int_\Omega f(x,u(x))v(x)dx = 0,$$

for every $v \in X$. Hence, the critical points of I_λ are exactly the weak solutions of (1.1).

Our main tool to investigate the existence of infinitely many solutions for the problem (1.1) is a smooth version of [4, Theorem 2.1] which is a more precise version of Ricceri's variational principle [19, Theorem 2.5], which we now recall.

Theorem 2. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, put*

$$\varphi(r) := \inf_{\Phi(u)<r} \frac{\left(\sup_{\Phi(v)<r} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \qquad \textit{and} \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in]0, 1/\varphi(r)[$, the restriction of the functional*

$$I_\lambda := \Phi - \lambda\Psi$$

to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) *If $\gamma < +\infty$, then for each $\lambda \in]0, 1/\gamma[$, the following alternative holds: either*

  (b_1) *I_λ possesses a global minimum, or*

  (b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$, then for each $\lambda \in]0, 1/\delta[$, the following alternative holds: either*

(c_1) *there is a global minimum of* Φ *which is a local minimum of* I_λ, *or*

(c_2) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points* (*local minima*) *of* I_λ *which weakly converges to a global minimum of* Φ, *with*

$$\lim_{n\to+\infty}\Phi(u_n)=\inf_{u\in X}\Phi(u).$$

2. Main results

Put

$$k:=\sup_{u\in X,u\neq 0}\left(\frac{\max_{x\in\bar{\Omega}}|u(x)|}{\|u\|}\right). \tag{2.1}$$

Since the embedding $X\hookrightarrow C(\bar{\Omega})$ is compact, one has $k<+\infty$. Fix $x_0\in\Omega$ and $D>0$ such that $B(x_0,D)\subset\Omega$ and $\overline{B(x_0,D)}$ not containing the origin, where $B(x_0,D)$ denotes the ball with center at x_0 and radius D.

Put

$$\omega:=\frac{m_1}{p}\left[\left(\frac{2}{D}\right)^p m\left(D^N-\left(\frac{D}{2}\right)^N\right)\right], \tag{2.2}$$

$$\alpha:=\int_{B(x_0,\frac{D}{2})}\frac{1}{|x|^q}dx\quad,\qquad \beta:=\left(\frac{2}{D}\right)^q\int_{B(x_0,D)\setminus B(x_0,\frac{D}{2})}\frac{(D-|x-x_0|)^q}{|x|^q}dx, \tag{2.3}$$

where $m:=\frac{\pi^{N/2}}{\Gamma(1+\frac{N}{2})}$. Here Γ is the Gamma function defined by

$$\Gamma(t):=\int_0^{+\infty}z^{t-1}e^{-z}dz\qquad(\forall t>0).$$

Put

$$A:=\liminf_{\xi\to+\infty}\frac{\|l_\xi\|_1}{\xi^{p-1}},$$

and

$$B:=\limsup_{\xi\to+\infty}\frac{\int_{B(x_0,\frac{D}{2})}F(x,\xi)dx}{\xi^p},$$

where $l_\xi\in L^1(\Omega)$ satisfies condition (C_3) on $f(x,t)$ for every $\xi>0$.

Our main result is the following.

Theorem 3. *Assume that* $M:[0,+\infty[\to R$ *is a continuous function satisfying* (f_1). *Also let* $f:\Omega\times\mathbb{R}\to\mathbb{R}$ *be an* L^1-*Carathéodory function such that*

(i) $F(x,t)\geq 0$ *for every* $(x,t)\in\Omega\times\mathbb{R}^+$,

(ii) $A<\frac{1}{p\omega k^p}B$, *where* k *and* ω *are given by* (2.1) *and* (2.2), *respectively.*

Then, for every $\lambda\in\Lambda:=\left]\frac{\omega}{B},\frac{1}{pk^pA}\right[$, *the problem* (1.1) *admits a sequence of weak solutions which is unbounded in* X.

Proof. Fix $\lambda \in \left]\frac{\omega}{B}, \frac{1}{pk^pA}\right[$. Our aim is to apply Theorem 2 part (b) with $X := W_0^{1,p}(\Omega)$ and where Φ and Ψ are the functionals introduced in Section 2. As seen before, the functionals Φ and Ψ satisfy the regularity assumptions requested in Theorem 2. Now, we look on the existence of critical points of the functional $I_\lambda := \Phi - \lambda\Psi$ in X. To this end, we take $\{\xi_n\} \subset \mathbb{R}^+$ such that $\lim_{n\to+\infty} \xi_n = +\infty$, and

$$\lim_{n\to+\infty} \frac{\|l_{\xi_n}\|_1}{\xi_n^{p-1}} = A.$$

Set $r_n := \frac{m_0\xi_n^p}{pk^p}$ for all $n \in \mathbb{N}$. From (2.1) we get

$$\max_{x\in\bar{\Omega}} |u(x)| \leq k\|u\|, \tag{2.4}$$

for every $u \in X$. Then, for each $u \in X$ with $\Phi(u) < r_n$, we have

$$\max_{x\in\bar{\Omega}} |u(x)| \leq k(\frac{p}{m_0}\Phi(u))^{1/p} < k(\frac{p}{m_0}r_n)^{1/p} = \xi_n.$$

Then, since $\Phi(0) = \Psi(0) = 0$, we have

$$\begin{aligned} \varphi(r_n) &= \inf_{\Phi(v)<r_n} \frac{\left(\sup_{\Phi(u)<r_n} \Psi(u)\right) - \Psi(v)}{r_n - \Phi(u)} \\ &\leq \frac{\sup_{\Phi(u)<r_n} \int_\Omega F(x,u(x))dx}{r_n} \\ &\leq \frac{\xi_n\|l_{\xi_n}\|_1}{m_0\frac{\xi_n^p}{pk^p}}. \end{aligned}$$

Hence, it follows that

$$\gamma \leq \liminf_{n\to+\infty} \varphi(r_n) \leq \frac{p}{m_0}k^p \liminf_{n\to+\infty} \frac{\|l_{\xi_n}\|_1}{\xi_n^{p-1}} = \frac{p}{m_0}k^pA < +\infty,$$

since condition (ii) yields $A < +\infty$. Now, we claim that the functional I_λ is unbounded from below. Let $\{d_n\}$ be a real sequence such that $\lim_{n\to+\infty} d_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_{B(x_0,\frac{D}{2})} F(x,d_n)dx}{d_n^p} = B. \tag{2.5}$$

Further, for each $n \geq 1$, define $v_n \in X$ given by

$$v_n(x) := \begin{cases} 0, & x \in \Omega \setminus B(x_0,D), \\ d_n, & x \in B(x_0,\frac{D}{2}), \\ \dfrac{2d_n}{D}(D - |x-x_0|), & x \in B(x_0,D) \setminus B(x_0,\frac{D}{2}). \end{cases} \tag{2.6}$$

By using condition (i), we infer

$$\Psi(v_n) = \int_\Omega F(x, v_n(x))dx \geq \int_{B(x_0,\frac{D}{2})} F(x, d_n)dx,$$

for every $n \geq 1$. Then, we have

$$I_\lambda(v_n) \leq \omega d_n^p + \frac{\alpha+\beta}{q} d_n^q - \lambda \int_{B(x_0,\frac{D}{2})} F(x, d_n)dx.$$

If $B < +\infty$, let

$$\delta \in \left] \frac{\omega}{\lambda B}, 1 \right[.$$

By (2.5), there exists N_δ such that

$$\int_{B(x_0,\frac{D}{2})} F(x, d_n)dx > \delta B d_n^p, \qquad (\forall n > N_\delta).$$

Consequently, one has

$$\begin{aligned} I_\lambda(v_n) &< \omega d_n^p + \frac{\alpha+\beta}{q} d_n^q - \lambda\delta B d_n^p \\ &= (\omega - \lambda\delta B) d_n^p + \frac{\alpha+\beta}{q} d_n^q, \end{aligned}$$

for every $n > N_\delta$. Then, it follows that

$$\lim_{n\to+\infty} I_\lambda(v_n) = -\infty,$$

since $q < p$.

If B = $+\infty$, let us consider $L > \frac{\omega}{\lambda}$. By (2.5), there exists N_L such that

$$\int_{B(x_0,\frac{D}{2})} F(x, d_n)dx > L d_n^p, \qquad (\forall n > N_L).$$

So, we have

$$\begin{aligned} I_\lambda(v_n) &< \omega d_n^p + \frac{\alpha+\beta}{q} d_n^q - \lambda L d_n^p \\ &= (\omega - \lambda L) d_n^p + \frac{\alpha+\beta}{q} d_n^q \qquad (\forall n > N_L). \end{aligned}$$

Taking into account the choice of L, also in this case, one has

$$\lim_{n\to+\infty} I_\lambda(v_n) = -\infty,$$

since $q < p$. Therefore owing to Theorem 2(b), the functional I_λ admits an unbounded sequence $\{u_n\} \subset X$ of critical points. Then the problem (1.1) admits a sequence of weak solutions which is unbounded in X. □

Among the consequences of Theorem 3, we point out the following result.

Corollary 1. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Assume that condition* (i) *of Theorem 3 holds. Further, require that*

(iii) $A < \frac{1}{pk^p}$ *and* $B > \omega$, *where k and ω are given by* (2.1) *and* (2.2), *respectively.*

Then the following problem

$$\begin{cases} -M\left(\int\limits_{\Omega} |\nabla u|^p dx\right) \Delta_p u + \dfrac{|u|^{q-2}u}{|x|^q} = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

admits a sequences of weak solutions which is unbounded in X.

Remark 1. We note that assumption (ii) in Theorem 3 could be replaced by the following more general hypothesis:

(ii′) There exists two positive sequences $\{a_n\}$ and $\{b_n\}$ such that

$$\frac{1}{p}\int\limits_0^{\frac{\omega}{m_1}{a_n}^p} M(s)ds + \frac{(\alpha+\beta)}{q}a_n^q < \frac{m_0 b_n^p}{pk^p}, \qquad (\forall n \geq 1)$$

and $\lim_{n\to+\infty} b_n = +\infty$ such that

$$\widetilde{A} < \frac{B}{\omega},$$

where k, ω and α, β are given by (2.1), (2.2) and (2.3), respectively, and

$$\widetilde{A} := \lim_{n\to+\infty} \frac{b_n \|l_{b_n}\|_1 - \int_{B(x_0,\frac{D}{2})} F(x,a_n)dx}{\frac{m_0 b_n^p}{pk^p} - \frac{1}{p}\int\limits_0^{\frac{\omega}{m_1}{a_n}^p} M(s)ds - \frac{(\alpha+\beta)}{q}a_n^q}.$$

Then, for every

$$\lambda \in \left]\frac{\omega}{B}, \frac{1}{\widetilde{A}}\right[,$$

the problem (1.1) admits a sequence of weak solutions which is unbounded in X.

Indeed, from (ii′) we obtain (ii), by choosing $a_n = 0$ for all $n \in \mathbb{N}$. Moreover, if we assume (ii′) instead of (ii) and set $r_n := \frac{m_0 b_n^p}{pk^p}$ for every $n \geq 1$, one has

$$\begin{aligned} \varphi(r_n) &:= \inf_{\Phi(v)<r_n} \frac{\left(\sup_{\Phi(u)<r_n} \int_\Omega F(x,u(x))dx\right) - \int_\Omega F(x,v(x))dx}{r_n - \Phi(v)} \\ &\leq \frac{b_n\|l_{b_n}\|_1 - \int_\Omega F(x,v_n(x))dx}{\frac{m_0 b_n^p}{pk^p} - \Phi(v_n)} \end{aligned}$$

$$\leq \frac{b_n\|l_{b_n}\|_1 - \int_{B(x_0,\frac{D}{2})} F(x,a_n)dx}{\frac{m_0 b_n^p}{pk^p} - \frac{1}{p}\int\limits_0^{\frac{\omega}{m_1}a_n{}^p} M(s)ds - \frac{(\alpha+\beta)}{q}a_n^q},$$

by choosing

$$v_n(x) := \begin{cases} 0, & x \in \Omega \setminus B(x_0,D), \\ a_n, & x \in B(x_0,\frac{D}{2}), \\ \dfrac{2a_n}{D}(D-|x-x_0|), & x \in B(x_0,D) \setminus B(x_0,\frac{D}{2}), \end{cases}$$

for every $n \geq 1$. Therefore, since by assumption (ii$'$) one has $\widetilde{A} < +\infty$, we obtain

$$\gamma \leq \liminf_{n\to+\infty} \varphi(r_n) \leq \widetilde{A} < +\infty.$$

From now on, arguing as in the proof of Theorem 3, the conclusion follows.

Now, we present the other main result. First, put

$$A' := \liminf_{\xi\to 0^+} \frac{\|l_\xi\|_1}{\xi^{p-1}}, \qquad B' := \limsup_{\xi\to 0^+} \frac{\int_{B(x_0,\frac{D}{2})} F(x,\xi)dx}{\xi^p}.$$

Arguing as in the proof of Theorem 3 but using conclusion (c) of Theorem 2 instead of (b), the following result holds.

Theorem 4. *Assume that $f : \Omega\times\mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function such that hypothesis* (i) *in Theorem 3 holds, and*

(iv) $A' < \frac{m_0}{p\omega k^p} B'$.

Then, for every $\lambda \in \Lambda' := \left]\frac{\omega}{B'}, \frac{m_0}{pk^pA'}\right[$, the problem (1.1) *has a sequence of weak solutions, which strongly converges to zero in X.*

Proof. Fix $\lambda \in \Lambda'$. We take Φ, Ψ and I_λ as in Section 2. Now, as it has been pointed out before, the functionals Φ and Ψ satisfy the regularity assumptions reqired in Theorem 2. As first step, we will prove that $\lambda < 1/\delta$. Then, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \xi_n = 0$ and

$$\lim_{n\to+\infty} \frac{\|l_{\xi_n}\|_1}{\xi_n^p} = A'.$$

By the fact that $\inf_X \Phi = 0$ and the definition of δ, we have $\delta := \liminf_{r\to 0^+} \varphi(r)$. Put $r_n := \frac{m_0 \xi_n^p}{pk^p}$ for all $n \in \mathbb{N}$. Then, for all $u \in X$ with $\Phi(u) < r_n$, taking (2.4) into account, one has $\|u\|_\infty < \xi_n$. Thus, for all $n \in \mathbb{N}$,

$$\varphi(r_n) \leq \frac{\sup_{\Phi(u)<r_n} \Psi(u)}{r_n} \leq \frac{pk^p}{m_0}\frac{\|l_{\xi_n}\|_1}{\xi_n^{p-1}}.$$

Hence,

$$\delta \leq \liminf_{n\to+\infty} \varphi(r_n) \leq \frac{pk^p}{m_0} \liminf_{n\to+\infty} \frac{\|l_{\xi_n}\|_1}{\xi_n^{p-1}} = \frac{pk^pA'}{m_0} < +\infty,$$

and therefore $\Lambda' \subset]0, 1/\delta[$.

Let λ be fixed. We claim that the functional I_λ does not have a local minimum at zero. Let $\{d_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} d_n = 0$ and

$$\lim_{n\to+\infty} \frac{\int_{B(x_0,\frac{D}{2})} F(x,d_n)dx}{d_n^p} = B'.$$

For all $n \in \mathbb{N}$, let $v_n \in X$ defined by (2.6) with the above d_n. Now, with the same argument as in the proof of Theorem 3, we achieve $I_\lambda(v_n) < 0$ for every $n \in \mathbb{N}$ large enough. Then, since $\lim_{n\to+\infty} I_\lambda(v_n) = I_\lambda(0) = 0$, we see that zero is not a local minimum of I_λ. This, together with the fact that zero is the only global minimum of Φ, we deduce that the energy functional I_λ does not have a local minimum at the unique global minimum of Φ. Therefore, by Theorem 2(c), there exists a sequence $\{u_n\}$ of critical points of I_λ which converges weakly to zero. In view of the fact that the embedding $X \hookrightarrow C(\bar{\Omega})$ is compact, we know that the critical points converge strongly to zero, and the proof is complete. □

We end this paper with the following example to illustrate our results.

Example 1. Let $r > 0$ be a real number and $\{t_n\}$, $\{s_n\}$ be two strictly increasing sequences of real numbers that defined by induction

$$t_1 = r, \quad s_1 = 2r$$

and for $n \geq 1$,

$$t_{2n} = \left(2^{2n+1} - 1\right) t_{2n-1}, \quad t_{2n+1} = \left(2 - \frac{1}{2^{2n+1}}\right) t_{2n},$$

$$s_{2n} = \frac{t_{2n}}{2^n} = \left(2 - \frac{1}{2^{2n}}\right) s_{2n-1}, \quad s_{2n+1} = 2^{n+1} t_{2n+1} = \left(2^{2n+2} - 1\right) s_{2n}.$$

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x,t) := \begin{cases} 2\varphi(x)t, & (x,t) \in \Omega \times [0,t_1], \\ \varphi(x)\left(s_{n-1} + \frac{s_n - s_{n-1}}{t_n - t_{n-1}}(t - t_{n-1})\right), & (x,t) \in \Omega \times [t_{n-1}, t_n] \text{ for some } n > 1, \end{cases}$$

where $\varphi : \Omega \to \mathbb{R}$ is a positive continuous function with $0 < m \leq \varphi(x) \leq M$. Then f is an L^1-Carathéodory function and since $f(x,t)$ is strictly increasing with respect to t argument at every $x \in \Omega$, the function $l_\xi(x) := f(x,\xi)$ satisfies in condition (C_3) on f; i.e.,

$$\sup_{|t|\leq\xi} |f(x,t)| \leq l_\xi(x), \quad \text{for a.e. } x \in \Omega.$$

Arguing as in [13], we have

$$\limsup_{\xi\to+\infty}\frac{\int_{B(x_0,\frac{D}{2})}F(x,\xi)dx}{\xi^{\frac{7}{3}}}=+\infty,\qquad \liminf_{\xi\to+\infty}\frac{\|l_\xi\|_1}{\xi^{\frac{4}{3}}}=0,$$

for every $x_0\in\Omega$ and $D>0$ such that $B(x_0,D)\subset\Omega$ and $\overline{B(x_0,D)}$ not containing the origin, where Ω is a bounded domain in $\mathbb{R}^2$ containing the origin and with smooth boundary $\partial\Omega$. Hence, by Theorem 3, for every $\lambda\in]0,+\infty[$, the following problem

$$\begin{cases} -M\left(\int\limits_{\Omega}|\nabla u|^{\frac{7}{3}}dx\right)\Delta_{\frac{7}{3}}u+\dfrac{|u|^{\frac{-1}{2}}u}{|x|^{\frac{3}{2}}}=\lambda f(x,u), & \text{in }\Omega,\\ u=0, & \text{on }\partial\Omega,\end{cases}$$

possesses an unbounded sequence of weak solutions in $W_0^{1,\frac{7}{3}}(\Omega)$.

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REFERENCES

[1] C. O. Alves, F. J. S. A. Correa, and T. M. Ma, "Positive Solution for a quasi-linear elliptic equation of Kirchhoff type." *Comput Math Appl.*, vol. 49, no. 1, pp. 85–93, 2005, doi: 10.1016/j.camwa.2005.01.008.

[2] J. P. G. Azorero and I. P. Alonso, "Hardy inequalities and some critical elliptic and parabolic problems," *J Differ Equ.*, vol. 144, no. 2, pp. 441–476, 1998, doi: 10.1006/jdeq.1997.3375.

[3] A. Bensedik and M. Bouchekif, "On an elliptic equation of Kirchhoff type with a Potential asympt otically linear at infinity." *Math. Comput. Modelling.*, vol. 49, no. 5-6, pp. 1089–1096, 2009, doi: 10.1016/j.mcm.2008.07.032.

[4] G. Bonanno and G. Molica Bisci, "Infinitely many solutions for a boundary value problem with discontinuous nonlinearities," *Bound. Value Probl.*, vol. 2009, pp. 1–20, 2009, doi: 10.1155/2009/670675.

[5] T. Bouali and R. Guefaifia, "Existence and uniqueness of weak solution for a non-local problem involving the p-Laplacian." *Int. J.Pure Appl. Math.*, vol. 98, no. 1, pp. 11–21, 2015, doi: 10.12732/ijpam.v98i1.2.

[6] N. T. Chung, "Multiple Solutions for a $p(x)$-Kirchhoff type equation with Sing-Charging nonlinearities," *Complex Var. Elliptic Equ.*

[7] N. T. Chung, "Multiplicity results for a class of $p(x)$-Kirchhoff type equations with combined nonlinearities," *Electron. J. Qual. Theory Differ. Equ.*, vol. 2012, no. 42, pp. 1–13, 2012, doi: 10.14232/ejqtde.2012.1.42.

[8] M. Ferrara and G. M. Bisci, "Existence results for elliptic problems with Hardy potential," *Bull. Sci. Math.*, vol. 138, no. 7, pp. 846–859, 2014, doi: 10.1016/j.bulsci.2014.02.002.

[9] M. Ghergu and V. D. Rădulescu, "Sublinear singular elliptic problems with two parameters," *J Differ Equ.*, vol. 195, no. 2, pp. 520–536, 2003, doi: 10.1016/S0022-0396(03)00105-0.

[10] M. Ghergu and V. D. Rădulescu, "Multiparameter bifurcation and asymptotics for the singular Lane-Emden-Fowler equation with a convection term," *Proc. Roy.Soc. Edinburgh Sect. A.*, vol. 135, no. 1, pp. 61–83, 2005, doi: 10.1017/S0308210500003760.

[11] M. Ghergu and V. D. Rădulescu, "Ground state solutions for the singular Lane-Emden-Fowler equation with sublinear convection term," *J. Math. Anal. Appl.*, vol. 333, no. 1, pp. 265–273, 2007, doi: 10.1016/j.jmaa.2006.09.074.

[12] M. Ghergu and V. D. Rădulescu, *Singular Elliptic Problems: Bifurcation and Asymptotic Analysis.* Oxford: Oxford: Oxford University Press, 2008.

[13] S. Khalkhali, S. Heidarkhani, and AbdolrahmanRazani, "Infinitely many weak solutions for a forth-order boundary-value problem," *Electron. J. Diff. Equ.*, vol. 2012, no. 164, pp. 1–14, 2012.

[14] G. Kirchhoff, *Vorlesungen über mathematische Physik. Mechanik.* 2^{te} *Aufl.* Leipzig. Teubner (1877)., 1877.

[15] A. Kristály and C. Varga, "Multiple solutions for elliptic problems with singular and sublinear potentials," *Proc. Am. Math. Soc.*, vol. 135, no. 7, pp. 2121–2126, 2007, doi: 10.1090/S0002-9939-07-08715-1.

[16] T. Ma, "Remarks on an elliptic equation of Kirchhoff type," *Nonlinear Anal Theory Methods Appl*, vol. 63, no. 5-7, pp. 1967–1977, 2005, doi: 10.1016/j.na.2005.03.021.

[17] G. Molica Bisci and P. F. Pizzimenti, "Sequences of weak solutions for non-local elliptic problems with Dirichlet boundary condition." *Proc. Edinb. Math. Soc., II. Ser.*, vol. 57, no. 3, pp. 779–809, 2014, doi: 10.1017/S0013091513000722.

[18] K. Perera and Z. Zhang, "Nontrivial solutions of Kirchhoff-type problems via the Yang index," *J. Differ. Equations*, vol. 221, no. 1, pp. 246–255, 2006, doi: 10.1016/j.jde.2005.03.006.

[19] B. Ricceri, "A general variational principle and some of its applications," *J. Comput. Appl. Math.*, vol. 113, no. 1-2, pp. 401–410, 2000, doi: 10.1016/S0377-0427(99)00269-1.

[20] B. Ricceri, "On an elliptic Kirchhoff-type problem depending on two parameters," *J. Glob. Optim.*, vol. 46, no. 4, pp. 543–549, 2010, doi: 10.1007/s10898-009-9438-7.

[21] V. D. Rădulescu, "Combined effects in nonlinear singular elliptic problems with convenction," *Rev. Roumaine Math. Pures Appl*, vol. 53, no. 5, pp. 543–553, 2008.

[22] J. J. Sun and C. L. Tang, "Existence and multiplicity of solutions for Kirchhoff type equations," *Nonlinear Anal Theory Methods Appl*, vol. 74, no. 4, pp. 1212–1222, 2011, doi: 10.1016/j.na.2010.09.061.

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NORMAL DIRECTION CURVES AND APPLICATIONS

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Abstract. In this study, we define a new type of associated curves in the Euclidean 3-space such as normal-direction curve and normal-donor curve. We obtain characterizations for these curves. Moreover, we give applications of normal-direction curves to some special curves such as helix, slant helix, plane curve or normal-direction (ND)-normal curves in E^3. And, we show that slant helices and rectifying curves can be constructed by using normal-direction curves.

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1. Introduction

In the curve theory of Euclidean space, the most important subject is to obtain a characterization for a regular curve, since these characterizations allow to classify curves according to some relations. These characterizations can be given for a single curve or for a curve pair. Helix, slant helix, plane curve, spherical curve, etc. are the well-known examples of single special curves [1,10,12,17,20] and these curves, especially the helices, are used in many applications [2,7,9,16]. Moreover, special curves can be defined by considering Frenet planes. If the position vector of a space curve always lies on its rectifying, osculating or normal planes, then the curve is called rectifying curve, osculating curve or normal curve, respectively [4]. In the Euclidean space E^3, rectifying, normal and osculating curves satisfy Cesaro's fixed point condition, i.e., Frenet planes of such curves always contain a particular point [8, 15]. In particular, there exists a simple relationship between rectifying curves and Darboux vectors (centrodes), which play some important roles in mechanics, kinematics as well as in differential geometry in defining the curves of constant precession [4].

Moreover, special curve pairs are characterized by some relationships between their Frenet vectors or curvatures. Involute-evolute curves, Bertrand curves, Mannheim curves are the well-known examples of curve pairs and studied by some mathematicians [3, 11, 14, 19, 20].

Recently, a new curve pair in the Euclidean 3-space E^3 has been defined by Choi and Kim [6]. They have considered an integral curve γ of a unit vector field X defined in the Frenet basis of a Frenet curve α and they have given the definitions and characterizations of principal-directional curve and principal-donor curve in E^3. They also gave some applications of these curves to some special curves.

In the present paper, we consider a new type of associated curves and define a new curve pair such as normal-direction curve and normal-donor curve in E^3. We obtain some characterizations for these curves and show that normal-direction curve is a space evolute of normal-donor curve. Moreover, we give some applications of normal-direction curve to some special curves such as helix, slant helix or plane curve.

2. PRELIMINARIES

This section includes a brief summary of space curves and definitions of general helix and slant helix in the Euclidean 3-space E^3.

A unit speed curve $\alpha : I \to E^3$ is called a general helix if there is a constant vector u, so that $\langle T, u\rangle = \cos\theta$ is constant along the curve, where $\theta \neq \pi/2$ and $T(s) = \alpha'(s)$ is unit tangent vector of α at s. The curvature (or first curvature) of α is defined by $\kappa(s) = \|\alpha''(s)\|$. Then, the curve α is called Frenet curve, if $\kappa(s) \neq 0$, and the unit principal normal vector $N(s)$ of the curve α at s is given by $\alpha''(s) = \kappa(s)N(s)$. The unit vector $B(s) = T(s) \times N(s)$ is called the unit binormal vector of α at s. Then $\{T, N, B\}$ is called the Frenet frame of α. For the derivatives of the Frenet frame, the following Frenet-Serret formulae hold:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.1}$$

where $\tau(s)$ is the torsion (or second curvature) of α at s. It is well-known that the curve α is a general helix if and only if $\frac{\tau}{\kappa}(s) = \text{constant}$ [17, 18]. If both $\kappa(s) \neq 0$ and $\tau(s)$ are constants, we call α as a circular helix. A curve α with $\kappa(s) \neq 0$ is called a slant helix if the principal normal lines of α make a constant angle with a fixed direction. Also, a slant helix α in E^3 is characterized by the differential equation of its curvature κ and its torsion τ given by

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)' = \text{constant}.$$

(See [12]).

Now, we give the definitions of some associated curves defined by Choi and Kim [6]. Let $I \subset \mathbb{R}$ be an open interval. For a Frenet curve $\alpha : I \to E^3$, consider a vector field X given by

$$X(s) = u(s)T(s) + v(s)N(s) + w(s)B(s), \tag{2.2}$$

where u,v and w are arbitrary differentiable functions of s which is the arc length parameter of α. Let

$$u^2(s)+v^2(s)+w^2(s)=1, \tag{2.3}$$

holds. Then the definitions of X-direction curve and X-donor curve in E^3 are given as follows.

Definition 1. (Definition 2.1. in [6]) Let α be a Frenet curve in Euclidean 3-space E^3 and X be a unit vector field satisfying the equations (2.2) and (2.3). The integral curve $\beta : I \to E^3$ of X is called an X-direction curve of α. The curve α whose X-direction curve is β is called the X-donor curve of β in E^3.

Definition 2. (Definition 2.2. in [6]) An integral curve of principal normal vector $N(s)$ (resp. binormal vector $B(s)$) of α in (2.2) is called the principal-direction curve (resp. binormal-direction curve) of α in E^3.

Remark 1. (Remark 2.3. in [6]) A principal-direction (resp. the binormal-direction) curve is an integral curve of $X(s)$ with $u(s)=w(s)=0$, $v(s)=1$ (resp. $u(s)=v(s)=0$, $w(s)=1$) for all s in (2.2).

3. Normal-direction curve and normal-donor curve in E^3

In this section, we will give definitions of normal-direction curve and normal donor curve in E^3. We obtain some theorems and results characterizing these curves. First, we give the following definition.

Definition 3. Let α be a Frenet curve in E^3 and X be a unit vector field lying on the normal plane of α and defined by

$$X(s)=v(s)N(s)+w(s)B(s), \qquad v(s)\neq 0, \qquad w(s)\neq 0, \tag{3.1}$$

and satisfying that the vectors $X'(s)$ and $T(s)$ are linearly dependent. The integral curve $\gamma : I \to E^3$ of $X(s)$ is called a normal-direction curve of α. The curve α whose normal -direction curve is γ is called the normal-donor curve in E^3.

The Frenet frame is a rotation-minimizing with respect to the principal normal N [8]. If we consider a new frame given by $\{T,X,M\}$ where $M=T\times X$, we have that this new frame is rotation-minimizing with respect to T, i.e., the unit vector X belongs to a rotation-minimizing frame.

Since, $X(s)$ is a unit vector and $\gamma : I \to E^3$ is an integral curve of $X(s)$, without loss of generality we can take s as the arc length parameter of γ and we can give the following characterizations in the view of these information.

Theorem 1. *Let $\alpha : I \to E^3$ be a Frenet curve and an integral curve of $X(s)=v(s)N(s)+w(s)B(s)$ be the curve $\gamma : I \to E^3$. Then, γ is a normal-direction curve of*

α *if and only if the following equalities hold,*

$$v(s) = \sin\left(\int \tau ds\right) \neq 0, \qquad w(s) = \cos\left(\int \tau ds\right) \neq 0. \tag{3.2}$$

Proof. Since γ is a normal-direction curve of α, from Definition 3, we have

$$X(s) = v(s)N(s) + w(s)B(s), \tag{3.3}$$

and

$$v^2(s) + w^2(s) = 1. \tag{3.4}$$

Differentiating (3.3) with respect to s and by using the Frenet formulas, it follows

$$X'(s) = -v\kappa T + (v' - w\tau)N + (w' + v\tau)B. \tag{3.5}$$

Since we have that X' and T are linearly dependent. Then from (3.5) we can write

$$\begin{cases} -v\kappa \neq 0, \\ v' - w\tau = 0, \\ w' + v\tau = 0. \end{cases} \tag{3.6}$$

The solutions of second and third differential equations are

$$v(s) = \sin\left(\int \tau ds\right) \neq 0, \qquad w(s) = \cos\left(\int \tau ds\right) \neq 0,$$

respectively, which completes the proof. $\square$

Theorem 2. *Let $\alpha : I \to E^3$ be a Frenet curve. If γ is the normal-direction curve of α, then γ is a space evolute of α.*

Proof. Since γ is an integral curve of X, we have $\gamma' = X$. Denote the Frenet frame of γ by $\{\bar{T}, \bar{N}, \bar{B}\}$. Differentiating $\gamma' = X$ with respect to s and by using Frenet formulas we get

$$X' = \bar{T}' = \bar{\kappa}\bar{N}. \tag{3.7}$$

Furthermore, we know that X' and T are linearly dependent. Then from (3.7) we get $\bar{N}$ and T are linearly dependent, i.e, γ is a space evolute of α. $\square$

Theorem 3. *Let $\alpha : I \to E^3$ be a Frenet curve. If γ is the normal direction curve of α, then the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$ of γ are given as follows,*

$$\bar{\kappa} = \kappa\left|\sin\left(\int \tau ds\right)\right|, \qquad \bar{\tau} = \kappa\cos\left(\int \tau ds\right).$$

Proof. From (3.5), (3.6) and (3.7), we have

$$\bar{\kappa}\bar{N} = -v\kappa T. \tag{3.8}$$

By considering (3.8) and (3.2) we obtain

$$\bar{\kappa}\bar{N} = -\kappa \sin\left(\int \tau ds\right) T, \tag{3.9}$$

which gives us

$$\bar{\kappa} = \kappa \left| \sin\left(\int \tau ds \right) \right|. \tag{3.10}$$

Moreover, from (3.9) and (3.10), we can write

$$\bar{N} = T. \tag{3.11}$$

Then, we have

$$\bar{B} = \bar{T} \times \bar{N} = \cos\left(\int \tau ds \right) N - \sin\left(\int \tau ds \right) B. \tag{3.12}$$

Differentiating (3.12) with respect to s gives

$$\bar{B}' = -\kappa \cos\left(\int \tau ds \right) T. \tag{3.13}$$

Since $\bar{\tau} = -\langle \bar{B}', \bar{N} \rangle = -\langle \bar{B}', T \rangle$, from (3.13) it follows

$$\bar{\tau} = \kappa \cos\left(\int \tau ds \right), \tag{3.14}$$

that finishes the proof. □

Corollary 1. *Let γ be a normal-direction curve of the curve α. Then the relationships between the Frenet frames of curves are given as follows,*

$$\begin{aligned} X = \bar{T} &= \sin\left(\int \tau ds \right) N + \cos\left(\int \tau ds \right) B, \\ \bar{N} &= T, \\ \bar{B} &= \cos\left(\int \tau ds \right) N - \sin\left(\int \tau ds \right) B. \end{aligned}$$

Proof. The proof is clear from Theorem 3. □

Theorem 4. *Let γ be a normal-direction curve of α with curvature $\bar{\kappa}$ and torsion $\bar{\tau}$. Then curvature κ and torsion τ of α are given by*

$$\kappa = \sqrt{\bar{\kappa}^2 + \bar{\tau}^2}, \qquad \tau = \frac{\bar{\tau}^2}{\bar{\kappa}^2 + \bar{\tau}^2} \left(\frac{\bar{\kappa}}{\bar{\tau}} \right)'.$$

Proof. From (3.10) and (3.14), we easily get

$$\kappa = \sqrt{\bar{\kappa}^2 + \bar{\tau}^2}. \tag{3.15}$$

Substituting (3.15) into (3.10) and (3.14), it follows

$$\left| \sin\left(\int \tau ds \right) \right| = \frac{\bar{\kappa}}{\sqrt{\bar{\kappa}^2 + \bar{\tau}^2}}, \tag{3.16}$$

$$\cos\left(\int \tau ds\right) = \frac{\bar{\tau}}{\sqrt{\bar{\kappa}^2+\bar{\tau}^2}}, \tag{3.17}$$

respectively. Differentiating (3.16) with respect to s, we have

$$\tau\cos\left(\int \tau ds\right) = \frac{\bar{\tau}(\bar{\kappa}'\bar{\tau}-\bar{\kappa}\bar{\tau}')}{(\bar{\kappa}^2+\bar{\tau}^2)^{3/2}}. \tag{3.18}$$

From (3.17) and (3.18), it follows

$$\tau = \frac{\bar{\kappa}'\bar{\tau}-\bar{\kappa}\bar{\tau}'}{\bar{\kappa}^2+\bar{\tau}^2},$$

or equivalently,

$$\tau = \frac{\bar{\tau}^2}{\bar{\kappa}^2+\bar{\tau}^2}\left(\frac{\bar{\kappa}}{\bar{\tau}}\right)'. \tag{3.19}$$

□

Theorem 4 leads us to give the following corollary whose proof is clear.

Corollary 2. *Let γ with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$ be a normal-direction curve of α. Then*

$$\frac{\tau}{\kappa} = -\frac{\bar{\kappa}^2}{(\bar{\kappa}^2+\bar{\tau}^2)^{3/2}}\left(\frac{\bar{\tau}}{\bar{\kappa}}\right)', \tag{3.20}$$

is satisfied, where κ and τ are curvature and torsion of α, respectively.

4. Applications of normal-direction curves

In this section, we focus on relations between normal-direction curves and some special curves such as general helix, slant helix, plane curve or rectifying curve in E^3.

4.1. *General helices, slant helices and plane curves*

Considering Corollary 2, we have the following theorems which gives a way to construct the examples of slant helices by using general helices.

Theorem 5. *Let $\alpha : I \to E^3$ be a Frenet curve in $E^3$and γ be a normal-direction curve of α. Then the followings are equivalent,*

(i) A Frenet curve α is a general helix in E^3.
(ii) α is a normal-donor curve of a slant helix.
(iii) A normal-direction curve of α is a slant helix.

Theorem 6. *Let $\alpha : I \to E^3$ be a Frenet curve in $E^3$and γ be a normal-direction curve of α. Then the followings are equivalent,*

(i) A Frenet curve α is a plane curve in E^3.
(ii) α is a normal-donor curve of a general helix.

(iii) A normal-direction curve of α *is a general helix.*

Example 1. Let consider the general helix given by the parametrization $\alpha(s) = \left(\cos\frac{s}{\sqrt{2}},\ \sin\frac{s}{\sqrt{2}},\ \frac{s}{\sqrt{2}}\right)$ in E^3 (Fig 1a). The Frenet vectors and curvatures of α are obtained as follows,

$$\begin{aligned}
T(s) &= \left(-\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right),\\
N(s) &= \left(-\cos\frac{s}{\sqrt{2}},\ \sin\frac{s}{\sqrt{2}},\ 0\right),\\
B(s) &= \left(\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right),\\
\kappa &= \tau = \frac{1}{2}.
\end{aligned}$$

Then we have $X(s) = (x_1(s),\ x_2(s),\ x_3(s))$ where

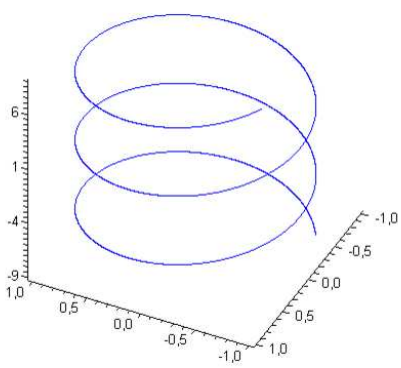
$$\begin{aligned}
x_1(s) &= -\sin\left(\frac{s}{2}+c\right)\cos\frac{s}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right)\sin\frac{s}{\sqrt{2}},\\
x_2(s) &= \sin\left(\frac{s}{2}+c\right)\sin\frac{s}{\sqrt{2}} - \frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right)\cos\frac{s}{\sqrt{2}},\\
x_3(s) &= \frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right).
\end{aligned}$$

and c is integration constant. Now, we can construct a slant helix γ which is also a normal-direction curve of α (Fig 1b):

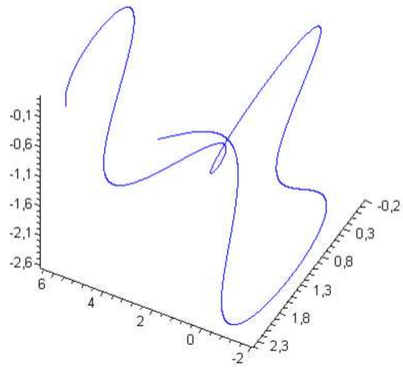
$$\gamma = \int_0^s \gamma'(s)ds = \int_0^s X(s)ds = (\gamma_1(s),\ \gamma_2(s), \gamma_3(s)),$$

where

$$\begin{aligned}
\gamma_1(s) &= \int_0^s \left[-\sin\left(\frac{s}{2}+c\right)\cos\frac{s}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right)\sin\frac{s}{\sqrt{2}}\right] ds,\\
\gamma_2(s) &= \int_0^s \left[\sin\left(\frac{s}{2}+c\right)\sin\frac{s}{\sqrt{2}} - \frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right)\cos\frac{s}{\sqrt{2}}\right] ds,\\
\gamma_3(s) &= \int_0^s \frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right) ds.
\end{aligned}$$



(A) General helix α.



(B) Slant helix γ.

FIGURE 1. Slant helix γ constructed by α.

4.2. *ND-normal Curves*

In this subsection we define normal-direction (ND)-normal curves in E^3 and give the relationships between normal-direction curves and ND-normal curves.

A space curve whose position vector always lies in its normal plane is called normal curve [5]. Moreover, if the Frenet frame and curvatures of a space curve are given by $\{T,N,B\}$ and κ, τ, respectively, then the vector $\tilde{D}(s) = \frac{\tau}{\kappa}(s)T(s) + B(s)$ is called modified Darboux vector of the curve [12, 13].

Let now α be a Frenet curve with Frenet frame $\{T,N,B\}$ and γ a normal-direction curve of α. The curve γ is called normal-direction normal curve (or ND-normal curve) of α, if the position vector of γ always lies on the normal plane of its normal-donor curve α.

The definition of ND-normal curve allows us to write the following equality,

$$\gamma(s) = m(s)N(s) + n(s)B(s), \tag{4.1}$$

where $m(s)$, $n(s)$ are non-zero differentiable functions of s. Since γ is normal-direction curve of α, from Corollary 1, we have

$$\begin{cases} N = \sin\left(\int \tau ds\right)\bar{T} + \cos\left(\int \tau ds\right)\bar{B}, \\ B = \cos\left(\int \tau ds\right)\bar{T} - \sin\left(\int \tau ds\right)\bar{B}. \end{cases} \tag{4.2}$$

Substituting (4.2) in (4.1) gives

$$\begin{aligned} \gamma(s) = &\left[m\sin\left(\int \tau ds\right) + n\cos\left(\int \tau ds\right)\right]\bar{T} \\ &+ \left[m\cos\left(\int \tau ds\right) - n\sin\left(\int \tau ds\right)\right]\bar{B}. \end{aligned} \tag{4.3}$$

Writing

$$\begin{cases} \rho(s) = m\sin\left(\int \tau ds\right) + n\cos\left(\int \tau ds\right), \\ \sigma(s) = m\cos\left(\int \tau ds\right) - n\sin\left(\int \tau ds\right), \end{cases} \tag{4.4}$$

in (4.3) and differentiating the obtained equality we obtain

$$\bar{T} = \rho'\bar{T} + (\rho\bar{\kappa} - \sigma\bar{\tau})\bar{N} + \sigma'\bar{B}. \tag{4.5}$$

Then we have

$$\sigma = a = \text{constant}, \qquad \rho = s + b = \frac{\bar{\tau}}{\bar{\kappa}}a, \tag{4.6}$$

where a, b are non-zero integration constants. From (4.6), it follows that

$$\gamma(s) = a\left(\frac{\bar{\tau}}{\bar{\kappa}}\bar{T} + \bar{B}\right)(s) = a\tilde{\bar{D}}(s), \tag{4.7}$$

where $\tilde{\bar{D}}$ is the modified Darboux vector of γ.

Now we can give the followings which characterize ND-normal curves.

Theorem 7. *Let $\alpha : I \to E^3$ be a Frenet curve in $E^3$and γ be a normal-direction curve of α. If γ is a ND-normal curve in E^3, then we have the followings,*

(i) γ is a rectifying curve in E^3 whose curvatures satisfy $\frac{\bar{\tau}}{\bar{\kappa}} = \frac{s+b}{a}$ where a, b are non-zero constants .

(ii) The position vector and modified Darboux vector $\tilde{\bar{D}}$ of γ are linearly dependent.

Theorem 7 gives a way to construct a rectifying curve by using normal-donor curve as follows:

Corollary 3. *Let $\alpha : I \to E^3$ be a Frenet curve in $E^3$and γ a ND-normal curve of α in E^3. Then the position vector of γ is obtained as follows,*

$$\begin{aligned} \gamma(s) = &\left[(s+b)\sin\left(\int \tau ds\right) + a\cos\left(\int \tau ds\right)\right] N(s) \\ &+ \left[(s+b)\cos\left(\int \tau ds\right) - a\sin\left(\int \tau ds\right)\right] B(s) \end{aligned} \tag{4.8}$$

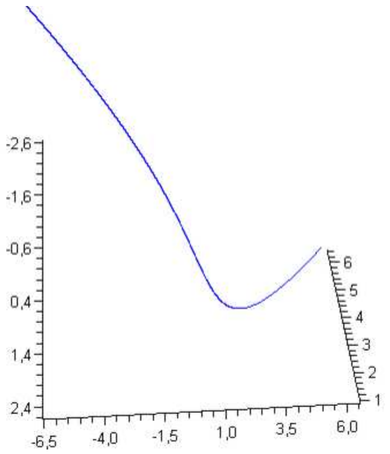
where a, b are non-zero integration constants.

Proof. The proof is clear from (4.1), (4.4) and (4.6). □

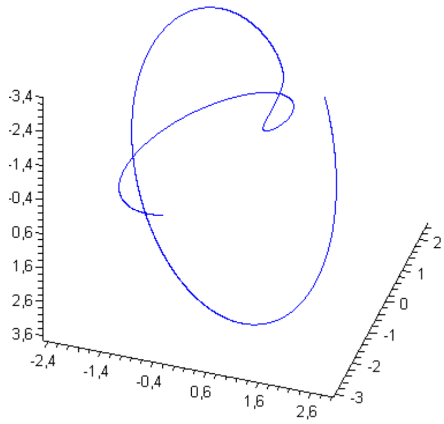
Example 2. Let consider the general helix given by the parametrization

$$\alpha(s) = \left(\sqrt{1+s^2}, s, \ln(s + \sqrt{1+s^2})\right),$$

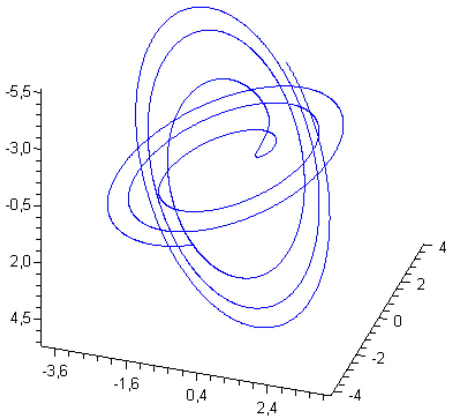
and drawn in Fig 2a.



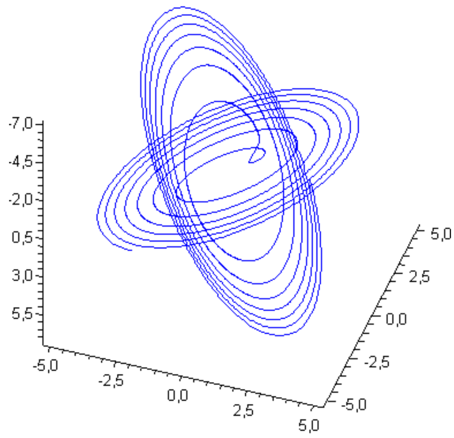
(A) General helix α.



(B) ND-normal curve γ for $-\pi \leq s \leq \pi$.



(C) ND-normal curve γ for $\frac{-3\pi}{2} \leq s \leq \frac{3\pi}{2}$.



(D) ND-normal curve γ for $-2\pi \leq s \leq 2\pi$.

FIGURE 2. ND-normal curve γ constructed by α.

Frenet vectors and curvatures of the curve are

$$T(s) = \frac{1}{\sqrt{2}\sqrt{1+s^2}}\left(s, \sqrt{1+s^2}, 1\right),$$
$$N(s) = \frac{1}{\sqrt{1+s^2}}(1, 0, -s),$$
$$B(s) = \frac{1}{\sqrt{2}\sqrt{1+s^2}}\left(-s, \sqrt{1+s^2}, -1\right),$$
$$\kappa = \tau = \frac{1+s^2}{2},$$

respectively. Then from Corollary 3, a ND-normal curve γ is obtained as follows,

$$\begin{aligned}\gamma(s) = \Bigg(& \frac{1}{\sqrt{1+s^2}}\left[(s+b)\sin\left(\frac{s}{2}+\frac{s^3}{6}+c\right)+a\cos\left(\frac{s}{2}+\frac{s^3}{6}+c\right)\right] \\ & -\frac{s}{\sqrt{2(1+s^2)}}\left[(s+b)\cos\left(\frac{s}{2}+\frac{s^3}{6}+c\right)-a\sin\left(\frac{s}{2}+\frac{s^3}{6}+c\right)\right], \\ & -\frac{1}{\sqrt{2}}\left[(s+b)\cos\left(\frac{s}{2}+\frac{s^3}{6}+c\right)-a\sin\left(\frac{s}{2}+\frac{s^3}{6}+c\right)\right], \\ & -\frac{s}{\sqrt{1+s^2}}\left[(s+b)\sin\left(\frac{s}{2}+\frac{s^3}{6}+c\right)+a\cos\left(\frac{s}{2}+\frac{s^3}{6}+c\right)\right] \\ & -\frac{1}{\sqrt{2(1+s^2)}}\left[(s+b)\cos\left(\frac{s}{2}+\frac{s^3}{6}+c\right)-a\sin\left(\frac{s}{2}+\frac{s^3}{6}+c\right)\right]\Bigg)\end{aligned}$$

which is also a rectifying curve in the view of Theroem 7 and drawn in Figures 2b, 2c, 2d by choosing $a = b = 1,\ c = 0$.

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REFERENCES

[1] M. Barros, "General helices and a theorem of Lancret." *Proc. Amer. Math. Soc.*, vol. 125, no. 5, pp. 1503–1509, 1997, doi: 10.1090/S0002-9939-97-03692-7.

[2] J. Beltran and J. Monterde, "A characterization of quintic helices." *J. Comput. Appl. Math.*, vol. 206, no. 1, pp. 116–121, 2007, doi: 10.1016/j.cam.2006.06.001.

[3] J. Burke, "Bertrand curves associated with a pair of curves." *Mathematics Magazine.*, vol. 34, no. 1, pp. 60–62, 1960, doi: 10.1080/0025570X.1960.11975181.

[4] B. Y. Chen, "When does the position vector of a space curve always lie in its normal plane?" *Amer. Math. Monthly*, vol. 110, pp. 147–152, 2003, doi: 10.2307/3647775.

[5] B. Y. Chen and F. Dillen, "Rectifying curves as centrodes and extremal curves." *Bull. Inst. Math. Academia Sinica.*, vol. 33, no. 2, pp. 77–90, 2005.

[6] J. H. Choi and Y. H. Kim, "Associated curves of a Frenet curve and their applications," *Applied Mathematics and Computation*, vol. 218, no. 18, pp. 9116–9124, 2012, doi: 10.1016/j.amc.2012.02.064.

[7] A. F. Da Fonseca and C. P. Malta, "Lancret helices," *arXiv:physics/0507105v1*. [Online]. Available: https://arxiv.org/pdf/physics/0507105.pdf

[8] R. Farouki, G. Gianelli, M. Sampoli, and A. Sestini, "Rotation-minimizing osculating frames." *Computer Aided Geometric Design.*, vol. 31, no. 1, pp. 27–42, 2014, doi: 10.1016/j.cagd.2013.11.003.

[9] R. Farouki, C. Y. Han, C. Manni, and A. Sestini, "Characterization and construction of helical polynomial space curves." *J. Comput. Appl. Math.*, vol. 162, no. 2, pp. 365–392, 2004, doi: 10.1016/j.cam.2003.08.030.

[10] H. A. Hayden, "On a generalized helix in a Riemannian n-space." *Proc. Lond. Math. Soc.*, vol. 32, pp. 337–345, 1931, doi: 10.1112/plms/s2-32.1.337.

[11] S. Izumiya and N. Takeuchi, "Generic properties of helices and Bertrand curves." *Journal of Geometry.*, vol. 74, pp. 97–109, 2002, doi: 10.1007/PL00012543.
[12] S. Izumiya and N. Takeuchi, "New special curves and developable surfaces." *Turk. J. Math.*, vol. 28, pp. 153–163, 2004.
[13] T. Kahraman, M. Önder, M. Kazaz, and H. H. Uğurlu, "Some Characterizations of Mannheim Partner Curves in Minkowski 3-space." *Proceedings of the Estonian Academy of Sciences.*, vol. 60, no. 4, pp. 210–220, 2011, doi: 10.3176/proc.2011.4.02.
[14] H. Liu and F. Wang, "Mannheim partner curves in 3-space." *Journal of Geometry.*, vol. 88, no. 1–2, pp. 120–126, 2008, doi: 10.1007/s00022-007-1949-0.
[15] T. Otsuki, *Differential Geometry.* Tokyo: Asakura Publishing Co.Ltd., 1961.
[16] J. Puig-Pey, A. Gálvez, and A. Iglesias, "Helical curves on surfaces for computer-aided geometric design and manufacturing." *Computational Science and its Applications ICCSA.*, vol. 3044, no. 2, pp. 771–778, 2004, doi: 10.1007/978-3-540-24709-8_81.
[17] D. J. Struik, *Lectures on Classical Differential Geometry.*, ser. 2. Dover: Addison Wesley., 1988.
[18] F. Wang and H. Liu, "Mannheim partner curves in 3-Euclidean space." *Mathematics in Practice and Theory.*, vol. 37, no. 1, pp. 141–143, 2007.
[19] J. K. Whittemore, "Bertrand curves and helices." *Duke Math. J.*, vol. 6, no. 1, pp. 235–245, 1950.
[20] Y. C. Wong, "On the generalized helices of Hayden and Syptak in an N-space." *Proc. Cambridge Philos. Soc.*, vol. 37, pp. 229–243, 1941, doi: 10.1017/S0305004100021721.

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THIRD HANKEL DETERMINANT FOR CERTAIN SUBCLASS OF p-VALENT ANALYTIC FUNCTIONS

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This paper is dedicated to Professor T. RAMREDDY on his 72^{nd} birthday.

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Abstract. The objective of this paper is to obtain an upper bound to the third Hankel determinant for certain subclass of p-valent functions, using Toeplitz determinants.

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Keywords: p-valent analytic function, upper bound, Hankel determinant, positive real function, Toeplitz determinants

1. Introduction

Let A_p denote the class of functions f of the form

$$f(z) = z^p + a_{p+1}z^{p+1} + \cdots, \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$ with $p \in \mathbb{N} = \{1,2,3,\ldots\}$. Let S be the subclass of $A_1 = A$, consisting of univalent functions. In 1985, Louis de Branges de Bourcia proved the Bieberbach conjecture, i.e., for a univalent function its n^{th}- coefficient is bounded by n (see [3]). The bounds for the coefficients of these functions give information about their geometric properties. In particular, the growth and distortion properties of a normalized univalent function are determined by the bound of its second coefficient. The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ (when $p = 1$) was defined by Pommerenke [10] as follows and has been extensively studied.

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.2}$$

One can easily observe that the Fekete-Szegő functional is $H_2(1) = a_3 - a_2^2$. Fekete and Szegő then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in S$. Further, sharp upper bounds for the functional $H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2$, the Hankel determinant in the case of $q = 2$ and $n = 2$, known as the second Hankel determinant (functional), were obtained for various subclasses of univalent and multivalent analytic functions. Janteng et al. [6] have considered the functional $|a_2a_4-a_3^2|$ and found a sharp upper bound for the function f in the subclass $\mathcal{R}$ of S, consisting of functions whose derivative has a positive real part (also called bounded turning functions) studied by MacGregor [9] and have showed that $|H_2(2)| \le \frac{4}{9}$. For our discussion in this paper, we consider the Hankel determinant in the case of $q=3$ and $n=p$, denoted by $H_3(p)$, given by

$$H_3(p)=\begin{vmatrix} a_p & a_{p+1} & a_{p+2} \\ a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+3} & a_{p+4} \end{vmatrix}. \tag{1.3}$$

For $f \in A_p$, $a_p=1$, so that, we have

$$H_3(p)=a_{p+2}(a_{p+1}a_{p+3}-a_{p+2}^2)-a_{p+3}(a_{p+3}-a_{p+1}a_{p+2})+a_{p+4}(a_{p+2}-a_{p+1}^2)$$

and by applying the triangle inequality, we obtain

$$|H_3(p)| \le |a_{p+2}||a_{p+1}a_{p+3}-a_{p+2}^2|+|a_{p+3}||a_{p+1}a_{p+2}-a_{p+3}|+|a_{p+4}||a_{p+2}-a_{p+1}^2|. \tag{1.4}$$

Incidentally, all of the functionals on the right hand side of the inequality (1.4) have known (and sharp) upper bounds except $|a_{p+1}a_{p+2}-a_{p+3}|$. It was known that if $f \in \mathcal{R}_p$, the class of p-valent bounded turning functions, then $|a_k| \le \frac{2p}{k}$, where $k \in \{p+1,\ p+2,\ ...\}$ and $|a_{p+2}-a_{p+1}^2| \le \frac{2p}{p+2}$, with $p \in \mathbb{N}$.

Motivated by the result obtained by Babalola [1] in finding the sharp upper bound to the Hankel determinant $|H_3(1)|$ for the class $\mathcal{R}$, in this paper we obtain an upper bound to the functional $|a_{p+1}a_{p+2}-a_{p+3}|$ and hence for $|H_3(p)|$, for the function f given in (1.1), belonging to certain subclass of p-valent analytic functions, as follows.

Definition 1 ([13]). A function $f \in A_p$ is said to be in the class $I_p(\beta)$(β is real), if it satisfies the condition

$$\operatorname{Re}\left\{(1-\beta)\frac{f(z)}{z^p}+\beta\frac{f'(z)}{pz^{p-1}}\right\}>0, \quad z \in E. \tag{1.5}$$

(1) Choosing $\beta=1$ and $p=1$, we obtain $I_1(1)=\mathcal{R}$.
(2) Selecting $\beta=1$, we get $I_p(1)=\mathcal{R}_p$.

2. Preliminary Results

In this section some preliminary lemmas are stated which are required for proving our results.

Let $\mathcal{P}$ denote the class of functions consisting of p, such that

$$p(z)=1+c_1z+c_2z^2+c_3z^3+...=1+\sum_{n=1}^{\infty}c_nz^n, \tag{2.1}$$

which are analytic in the open unit disc E and satisfy $\operatorname{Re}p(z)>0$ for any $z \in E$. Here $p(z)$ is called Carathéodory function [4].

Lemma 1 ([11, 12]). *If $p \in \mathcal{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$.*

Lemma 2 ([5]). *The power series for $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ given in (2.1) converges in the open unit disc E to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \; n = 1, 2, 3 \ldots.$$

and $c_{-k} = \bar{c}_k$, are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z)$, with $\sum_{k=1}^{m} \rho_k = 1$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \frac{1+z}{1-z}$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

We may assume without restriction that $c_1 \geq 0$. Using Lemma 2, for $n = 2$ and $n = 3$, for some complex values x and z with $|x| \leq 1$ and $|z| \leq 1$ respectively, we have

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.2}$$

$$\text{and } \; 4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z. \tag{2.3}$$

To obtain our results, we refer to the classical method devised by Libera and Zlotkiewicz [8], which is used by many authors in the literature.

3. Main Results

Theorem 1. *If $f \in I_p(\beta)$ $(0 < \beta \leq 1)$ with $p \in \mathbb{N}$, then*

$$|a_{p+1}a_{p+2} - a_{p+3}| \leq \frac{2p}{p + 3\beta}.$$

Proof. For $f = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in I_p(\beta)$, by virtue of Definition 1, there exists an analytic function $p \in \mathcal{P}$ in the open unit disc E with $p(0) = 1$ and $\mathrm{Re}\, p(z) > 0$ such that

$$(1 - \beta)\frac{f(z)}{z^p} + \beta \frac{f'(z)}{pz^{p-1}} = p(z) \Leftrightarrow (1 - \beta)pf(z) + \beta f'(z) = pz^p p(z). \tag{3.1}$$

$$(1-\beta)p\left\{z^p + \sum_{n=p+1}^{\infty} a_n z^n\right\} + \beta\left\{pz^{p-1} + \sum_{n=p+1}^{\infty} n a_n z^{n-1}\right\} = pz^p\left\{1 + \sum_{n=1}^{\infty} c_n z^n\right\}.$$

Upon simplification, we obtain

$$\begin{aligned}(p+\beta)a_{p+1}z^{p+1} + (p+2\beta)a_{p+2}z^{p+2} + (p+3\beta)a_{p+3}z^{p+3} + (p+4\beta)a_{p+4}z^{p+4} + \ldots \\ = pc_1 z^{p+1} + pc_2 z^{p+2} + pc_3 z^{p+3} + pc_4 z^{p+4} + \ldots\end{aligned} \tag{3.2}$$

Equating the coefficients of z^{p+1}, z^{p+2}, z^{p+3} and z^{p+4} respectively in 3.2, we have

$$a_{p+1} = \frac{pc_1}{p+\beta};\ a_{p+2} = \frac{pc_2}{p+2\beta};\ a_{p+3} = \frac{pc_3}{p+3\beta}\ \text{ and }\ a_{p+4} = \frac{pc_4}{p+4\beta}. \tag{3.3}$$

Substituting the values of a_{p+1}, a_{p+2} and a_{p+3} from (3.3) in the functional $|a_{p+1}a_{p+2} - a_{p+3}|$, after simplifying, we get

$$|a_{p+1}a_{p+2} - a_{p+3}| = \frac{p}{(p+\beta)(p+2\beta)(p+3\beta)} \left| p(p+3\beta)c_1c_2 - (p+\beta)(p+2\beta)c_3 \right|.$$

The above expression is equivalent to

$$|a_{p+1}a_{p+2} - a_{p+3}| = \frac{p}{(p+\beta)(p+2\beta)(p+3\beta)} \left| d_1c_1c_2 + d_2c_3 \right|, \tag{3.4}$$

$$\text{where}\quad d_1 = p(p+3\beta);\ \ d_2 = -(p+\beta)(p+2\beta). \tag{3.5}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.3) respectively from Lemma 2 on the right-hand side of (3.4), we have

$$\begin{aligned} |d_1c_1c_2 + d_2c_3| = |d_1c_1 \times \frac{1}{2}\{c_1^2 + x(4-c_1^2)\} + d_2 \\ \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\}|. \end{aligned}$$

Using the facts $|z| \le 1$ and $|xa+yb| \le |x||a| + |y||b|$, where x, y, a and b are real numbers, which simplifies to

$$\begin{aligned} 4\,|d_1c_1c_2 + d_2c_3| \le [|(2d_1+d_2)||c_1|^3 + 2|d_2||(4-c_1^2)| + 2|(d_1+d_2)||c_1||(4-c_1^2)||x| \\ + |d_2||(c_1+2)||(4-c_1^2)||x|^2]. \end{aligned} \tag{3.6}$$

From (3.5), we can write

$$2d_1 + d_2 = p^2 + 3p\beta - 2\beta^2;\ d_1 + d_2 = -2\beta^2. \tag{3.7}$$

Substituting the calculated values from (3.7) along with (3.5) on the right-hand side of (3.6), we have

$$\begin{aligned} 4|d_1c_1c_2 + d_2c_3| \le [(p^2+3p\beta-2\beta^2)c_1^3 + 2(p+\beta)(p+2\beta)(4-c_1^2) \\ + 4\beta^2c_1(4-c_1^2)|x| + (c_1+2)(p+\beta)(p+2\beta)(4-c_1^2)|x|^2]. \end{aligned}$$

Since $c_1 = c \in [0,2]$, noting that $c_1 - a \le c_1 + a$, where $a \ge 0$ and replacing $|x|$ by μ on the right-hand side of the above inequality, we get

$$\begin{aligned} 4|d_1c_1c_2 + d_2c_3| \le [(p^2+3p\beta-2\beta^2)c^3 + 2(p+\beta)(p+2\beta)(4-c^2) + 4\beta^2c(4-c^2)\mu \\ + (c-2)(p+\beta)(p+2\beta)(4-c^2)\mu^2] = F(c,\mu), \end{aligned} \tag{3.8}$$

for $0 \le \mu = |x| \le 1$ and $0 \le c \le 2$, where

$$\begin{aligned} F(c,\mu) = (p^2+3p\beta-2\beta^2)c^3 + 2(p+\beta)(p+2\beta)(4-c^2) + 4\beta^2c(4-c^2)\mu \\ + (c-2)(p+\beta)(p+2\beta)(4-c^2)\mu^2. \end{aligned} \tag{3.9}$$

Next, we need to find the maximum value of the function $F(c,\mu)$ on the closed region $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ given in (3.9) partially with respect to μ and c respectively, we obtain

$$\frac{\partial F}{\partial \mu} = 4\beta^2 c(4-c^2) + 2(p+\beta)(p+2\beta)(4c-c^3-8+2c^2)\mu. \tag{3.10}$$

and

$$\begin{aligned}\frac{\partial F}{\partial c} &= 3(p^2+3p\beta-2\beta^2)c^2 - 4c(p+\beta)(p+2\beta) + 16\beta^2\mu - 12\beta^2c^2\mu \\ &\quad + (p+\beta)(p+2\beta)(4-3c^2+4c)\mu^2. \end{aligned}\tag{3.11}$$

For the extreme values of $F(c,\mu)$, consider

$$\frac{\partial F}{\partial \mu} = 0 \quad \text{and} \quad \frac{\partial F}{\partial c} = 0. \tag{3.12}$$

In view of (3.12), on solving the equations in (3.10) and (3.11), we obtain the only critical point for the function $F(c,\mu)$ which lies in the closed region $[0,2]\times[0,1]$ is $(0,0)$. At the critical point $(0,0)$, we observe that

$$\begin{aligned}\frac{\partial^2 F}{\partial \mu^2} &= -4(p+\beta)(p+2\beta) < 0;\\ \frac{\partial^2 F}{\partial c^2} &= -16(p+\beta)(p+2\beta) < 0;\\ \frac{\partial^2 F}{\partial c\partial \mu} &= 16\beta^2;\\ \left[\left(\frac{\partial^2 F}{\partial \mu^2}\right)\left(\frac{\partial^2 F}{\partial c^2}\right) - \left(\frac{\partial^2 F}{\partial c\partial \mu}\right)^2\right] &= 64[(p+\beta)^2(p+2\beta)^2 - 4\beta^4] > 0,\end{aligned}$$

with $p\in\mathbb{N}$ and $0<\beta\le 1$.

Therefore, the function $F(c,\mu)$ has maximum value at the point $(0,0)$, from (3.9), it is given by

$$G_{max} = F(0,0) = 8(p+\beta)(p+2\beta). \tag{3.13}$$

Simplifying the expressions (3.4) and (3.8) together with (3.13), we obtain

$$|a_{p+1}a_{p+2} - a_{p+3}| \le \frac{2p}{p+3\beta}. \tag{3.14}$$

This completes the proof of our theorem. □

Remark 1. Choosing $p=1$ and $\beta=1$ in (3.14), we obtain $|a_2a_3-a_4|\le\frac{1}{2}$, this inequality is sharp and coincides with the result of Bansal et al. [2].

Theorem 2. *If $f \in I_p(\beta)$ $(0 < \beta \leq 1)$ with $p \in \mathbb{N}$ then*

$$|a_{p+2} - a_{p+1}^2| \leq \frac{2p}{p+2\beta}$$

and the inequality is sharp for the values $c_1 = c = 0,\ c_2 = 2$ and $x = 1$.

Proof. On substituting the values of a_{p+1} and a_{p+2} from (3.3) in the functional $|a_{p+2} - a_{p+1}^2|$, which simplifies to

$$|a_{p+2} - a_{p+1}^2| = \frac{p}{(p+\beta)^2(p+2\beta)} \left|(p+\beta)^2 c_2 - p(p+2\beta)c_1^2\right|. \tag{3.15}$$

The above expression is equivalent to

$$|a_{p+2} - a_{p+1}^2| = \frac{p}{(p+\beta)^2(p+2\beta)} \left|d_1 c_2 + d_2 c_1^2\right|, \tag{3.16}$$

$$\text{where} \quad d_1 = (p+\beta)^2 \quad \text{and} \quad d_2 = -p(p+2\beta). \tag{3.17}$$

Substituting the value of c_2 from (2.2) of Lemma 2, applying the triangle inequality on the right-hand side of (3.16), after simplifying, we get

$$2\left|d_1 c_2 + d_2 c_1^2\right| \leq \left[|(d_1 + 2d_2)||c_1|^2 + |d_1|\left(4 - c_1^2\right)|x|\right]. \tag{3.18}$$

From (3.17), we can write

$$d_1 + 2d_2 = -(p^2 + 2p\beta - \beta^2);\ d_1 = (p+\beta)^2. \tag{3.19}$$

Substituting the calculated values from (3.19), taking $c_1 = c \in [0,2]$, replacing $|x|$ by μ on the right-hand side of (3.18), we obtain

$$\begin{aligned} 2\left|d_1 c_2 + d_2 c_1^2\right| &\leq \left[(p^2 + 2p\beta - \beta^2)c^2 + (p+\beta)^2\left(4 - c^2\right)\mu\right] \\ &= F(c,\mu)\ ,\ \ 0 \leq \mu = |x| \leq 1 \ \text{ and } \ 0 \leq c \leq 2, \end{aligned} \tag{3.20}$$

$$\text{where} \ \ F(c,\mu) = (p^2 + 2p\beta - \beta^2)c^2 + (p+\beta)^2\left(4 - c^2\right)\mu. \tag{3.21}$$

Now, we maximize the function $F(c,\mu)$ on the closed region $[0,2] \times [0,1]$. Let us suppose that there exists a maximum value for $F(c,\mu)$ at any point in the interior of the closed region $[0,2] \times [0,1]$. Differentiating $F(c,\mu)$ given in (3.21) partially with respect to μ, we obtain

$$\frac{\partial F}{\partial \mu} = (p+\beta)^2\left(4 - c^2\right) \tag{3.22}$$

For $0 < \beta \leq 1$, for fixed values of c with $0 < c < 2$ and $p \in \mathbb{N}$, from (3.22), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ which is independent of μ becomes an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed region $[0,2] \times [0,1]$. The maximum value of $F(c,\mu)$ occurs only on the boundary i.e., when $\mu = 1$. Therefore, for fixed $c \in [0,2]$, we have

$$\max_{0 \leq \mu \leq 1} F(c,\mu) = F(c,1) = G(c). \tag{3.23}$$

In view of (3.23), replacing μ by 1 in (3.21), it simplifies to

$$G(c) = -2\beta^2 c^2 + 4(p+\beta)^2, \tag{3.24}$$

$$G'(c) = -4\beta^2 c. \tag{3.25}$$

From the expression (3.25), we observe that $G'(c) \leq 0$ for each $c \in [0,2]$ and for every β with $0 < \beta \leq 1$. Therefore, $G(c)$ becomes a decreasing function of c, whose maximum value occurs at $c = 0$ only and from (3.24), it is given by

$$G_{max} = G(0) = 4(p+\beta)^2. \tag{3.26}$$

Simplifying the expressions (3.16), (3.20) along with (3.26), we obtain

$$|a_{p+2} - a_{p+1}^2| \leq \frac{2p}{p+2\beta}. \tag{3.27}$$

This completes the proof of our theorem. □

Remark 2. If $p = 1$ and $\beta = 1$ in (3.27) then $|a_3 - a_2^2| \leq \frac{2}{3}$, this result coincides with that of Babalola [1].

Theorem 3. *If $f \in I_p(\beta)$ $(0 < \beta \leq 1)$ then*

$$|a_{p+k}| \leq \frac{2p}{p+k\beta}, \quad \text{for} \ \ p, \ \ k \in \mathbb{N}. \tag{3.28}$$

Proof. Using the fact that $|c_n| \leq 2$, for $n \in \mathbb{N}$, with the help of c_2 and c_3 values given in (2.2) and (2.3) respectively, together with the values obtained in (3.3), we get $|a_{p+k}| \leq \frac{2p}{p+k\beta}$, with $p\ ,k \in \mathbb{N}$. This completes the proof of our theorem. □

Substituting the results of Theorems 1, 2, 3 together with the known inequality $|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \left[\frac{2p}{p+2\beta}\right]^2$ (see [7]) in the inequality given in (1.4), we obtain the following Corollary.

Corollary 1. *If $f \in I_p(\beta)$ $(0 < \beta \leq 1)$ with $p \in \mathbb{N}$ then*

$$|H_3(p)| \leq 4p^2 \left[\frac{2p}{(p+2\beta)^3} + \frac{1}{(p+3\beta)^2} + \frac{1}{(p+2\beta)(p+4\beta)}\right]. \tag{3.29}$$

Remark 3. In particular for the values $p = 1$ and $\beta = 1$ in (3.29), which simplifies to $|H_3(1)| \leq \frac{439}{540}$. This result coincides with that of Bansal et al. [2].

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REFERENCES

[1] K. O. Babalola, "On H3(1) Hankel determinant for some classes of univalent functions," *Inequal. Theory Appl. (In: Dragomir, S.S., Cho, J.Y. (eds.))*, vol. 6, no. 1, pp. 1–7, 2010, doi: 10.1I55/2008/153280.

[2] D. Bansal, S. Maharana, and J. K. Prajapat, "Third Hankel determinant for certain univalent functions," *J. Korean Math. Soc.*, vol. 52, no. 6, pp. 1139–1148, 2015, doi: 10.4134/JKMS.2015.52.6.1139.

[3] L. D. Branges, "A proof of Bieberbach conjecture," *Acta Mathematica*, vol. 154, no. 1, pp. 137–152, 1985, doi: 10.1007/BF02392821.

[4] P. L. Duren, *Univalent Functions*. USA: Springer-Verlag New York Inc., 1983, vol. 259, no. 1.

[5] U. Grenander and G. Szegő, *Toeplitz Forms and Their Applications; 2nd Edition*. New York: AMS Chelsea Publishing Company, 1984.

[6] A. Janteng, S. A. Halim, and M. Darus, "Coefficient inequality for a function whose derivative has a positive real part," *J. Inequal. and Appl.*, vol. 7, no. 2, pp. 1–5, 2006.

[7] D. V. Krishna and T. RamReddy, "Coefficient inequality for certain subclass of p-valent functions," *Palestine J. Math.*, vol. 4, no. 1, pp. 223–228, 2015.

[8] R. J. Libera and E. J. Zlotkiewicz, "Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$," *Proc. Am. Math. Sc.*, vol. 87, no. 2, pp. 251–257, 1983.

[9] T. H. MacGregor, "Functions whose derivative has a positive real part," *Trans. Am. Math. Sc.*, vol. 104, no. 3, pp. 532–537, 1962, doi: 10.1090/S0002-9947-1962-0140674-7.

[10] C. Pommerenke, "On the coefficients and Hankel determinants of univalent functions," *J. Lond. Math. Sc.*, vol. 41, no. 1, pp. 111–122, 1966, doi: 10.1112/jlms/s1-41.1.111.

[11] C. Pommerenke, *Univalent Functions with a Chapter on Quadratic Differentials by Gerd Jensen*. Studia Mathematica/Mathematische Lehrbücher. Band XXV.Vandenhoeck and Ruprecht GmbH and Co KG; 1st edition (December 1974), 1975.

[12] B. Simon, *Orthogonal polynomials on the unit circle. Part 1*. American Mathematical Society Colloquium Publications, Vol. 54, American Mathematical Society, Providence, RI, 2005. Classical theory: American mathematical society colloquium publications., 2005.

[13] Öznur Ozkan Kilic, "Sufficient conditions for subordination of multivalent functions," *J. Inequa. Math. Sc.*, vol. 2008, no. 3, pp. 1–8, 2008, doi: 10.1155/2008/374356.

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HYPERSTABILITY OF RASSIAS-RAVI RECIPROCAL FUNCTIONAL EQUATION

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Abstract. The investigation of stabilities of various types of equations is an interesting and evolving research area in the field of mathematical analysis. Recently, there are many research papers published on this topic, especially mixed type and multiplicative inverse functional equations. We propose a new functional equation in this study which is quite different from the functional equations already dealt in the literature. The main feature of the equation dealt in this study is that it has two different solutions, namely additive and multiplicative inverse functions. We also prove that the hyperstability results hold good for this equation.

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Keywords: additive function, multiplicative inverse function, Ulam-Hyers stability, hyperstability

1. INTRODUCTION

The analysis of stability of Functional Equation (FE) is due to the celebrated query presented in [32]. An excellent response was presented in [15]. This method of proving stability result of FE is termed as Hyers-Ulam (H-U) stability which involves a small positive constant as upper bound. Later, this result influenced many researchers to solve stability problems via different directions in [14, 20, 21] and these results are respectively called as generalized Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias (U-G-R) stability and Hyers-Ulam-Rassias (H-U-R) stability.

For the first time, the hyperstability results associated with the ring homomorphisms were obtained in [5]. Also, the hyperstability of a class of linear functional equations were dealt in [18]. There are a number of published papers associated with hyperstability results and stability results via fixed point technique of various FEs, one may refer to [1–3, 6–10, 13, 17, 19, 25].

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On the other hand, for the first time in this theory, a rational FE of the form

$$r(x+y)=\frac{r(x)r(y)}{r(x)+r(y)} \tag{1.1}$$

was introduced and studied its stability results in [23], where $r:\mathbb{R}^{\star}\longrightarrow\mathbb{R}$ is a mapping. It is interesting to note that the rational function $f(x)=\frac{c}{x}$, c being a constant, is a solution of equation (1.1). Motivated by the equation (1.1), there are numerous papers published on the problems of solving stability of various multiplicative inverse FEs of the type reciprocal-quadratic, reciprocal-cubic, reciprocal-quintic, etc., and radical functional equation. The detailed information regarding these results are available in [4, 11, 12, 16, 22, 24, 26–31].

In an algebraic polynomial equation $g(x)=0$, if we replace the variable x by $\frac{1}{x}$ and if we get the same equation, then it is called a reciprocal equation. Also, if α is a root of $g(x)=0$, then $\frac{1}{\alpha}$ is also a root of $g(x)=0$.

These concepts together with the results of equation (1.2) instigated us to deal with a new FE of the form

$$h\left(\sum_{j=1}^{m}u_j\right)+h\left(\frac{\prod_{j=1}^{m}u_j}{\sum_{j=1}^{m}\prod_{k=1,k\neq j}^{m}u_k}\right)=\frac{\prod_{j=1}^{m}h(u_j)}{\sum_{j=1}^{m}\prod_{k=1,k\neq j}^{m}h(u_k)}+\sum_{j=1}^{m}h(u_j). \tag{1.2}$$

One can easily verify that the functions $h(u)=u$ and $h(u)=\frac{1}{u}$ are solutions of equation (1.2). We present preliminaries and some basic results connected with (1.2). We also establish hyperstability results of (1.2) in the setting of real numbers.

Thoughout this paper, let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{N}_{m_0}$, $\mathbb{R}$ and $\mathbb{R}^{\star}$ denote the set of all natural numbers, the set of all nonnegative integers, the set of all integers greater than or equal to m_0, the set of all real numbers and the set of all non-zero real numbers, respectively.

2. Preliminaries

Here we recall some significant concepts related with hyperstability and fixed point theorem [8] which are useful to prove our main results of this investigation. The ensuing three propositions are significant in obtaining the hyperstability results.

(P1) Let A and B be a nonempty set and a Banach space, respectively.
Let $h_1,h_2,\ldots,h_k:A\longrightarrow A$ and $Q_1,Q_2,\ldots,Q_k:A\longrightarrow\mathbb{R}_+$ be given mappings.

(P2) Let an operator $\eta:B^A\longrightarrow B^A$ satisfies the inequality

$$\|\eta\alpha(u)-\eta\beta(u)\|\leq\sum_{i=1}^{k}Q_i(u)\,\|\alpha(h_i(u))-\beta(h_i(u))\| \tag{2.1}$$

for all $\alpha,\beta\in B^A$, $u\in A$.

(P3) The mapping $\Gamma : \mathbb{R}_+^A \longrightarrow \mathbb{R}_+^A$ be defined by

$$\Gamma\Delta(u) = \sum_{i=1}^{k} Q_i(u)\Delta(h_i(u)), \quad \Delta \in \mathbb{R}_+^A,\ u \in A. \tag{2.2}$$

The subsequent theorem is employed in our investigation to claim the persistence of the distinct fixed point operator $\eta : B^A \longrightarrow B^A$.

Theorem 1. *Let the propositions* (P1)–(P3) *be substantial. Suppose the mappings* $\psi : A \longrightarrow \mathbb{R}_+$ *and let* $\phi : A \longrightarrow B$ *satisfy the ensuing two conditions:*

$$\|\eta\phi(u) - \phi(u)\| \leq \psi(u), \quad u \in A, \tag{2.3}$$

$$\psi^\star(u) = \sum_{n=0}^{\infty} \Gamma^n \psi(u) < \infty, \quad u \in A. \tag{2.4}$$

Then, there exists a unique fixed point χ *of* η *such that*

$$\|\psi(u) - \chi(u)\| \leq \psi^\star(u), \quad u \in A. \tag{2.5}$$

Furthermore,

$$\chi(u) = \lim_{n\to\infty} \eta^n \psi(u) \tag{2.6}$$

3. Basic Significant Results connected with equation (1.2)

The following definition will be useful to prove our main results.

Definition 1. A function $h : \mathbb{R}^\star \longrightarrow \mathbb{R}$ is said to be a Rassias-Ravi reciprocal function if it satisfies the following general FE:

$$h(mu) + h\left(\frac{u}{m}\right) = \frac{m^2+1}{m} h(u) \tag{3.1}$$

for all $u \in \mathbb{R}^\star$ and any integer m.

Remark 1. From the above definition, it is clear that (1.2) satisfies (3.1) by plugging $u_k = u$, for $k = 1, 2, \ldots, m$ in (1.2). Hence (1.2) is said to be Rassias-Ravi reciprocal functional equation.

Remark 2. When $m = 2$, (1.2) produces the following equation in two variables:

$$h(u_1+u_2) + h\left(\frac{u_1u_2}{u_1+u_2}\right) = \frac{h(u_1)h(u_2)}{h(u_1)+h(u_2)} + [h(u_1)+h(u_2)].$$

When $m = 3$, (1.2) induces the ensuing equation in three variables:

$$h(u_1+u_2+u_3) + h\left(\frac{u_1u_2u_3}{u_1u_2+u_1u_3+u_2u_3}\right) = \frac{h(u_1)h(u_2)h(u_3)}{h(u_1)h(u_2)+h(u_1)h(u_3)+h(u_2)h(u_3)} + [h(u_1)+h(u_2)+h(u_3)].$$

Remark 3. In this investigation, we assume that

$$\sum_{j=1}^{m} u_j,\ \sum_{j=1}^{m}\prod_{k=1,k\neq j}^{m} u_k,\ \sum_{j=1}^{m}\prod_{k=1,k\neq j}^{m} h(u_k)\neq 0,$$

for all $u_j \in \mathbb{R}^\star$, $j = 1,2,\dots,m$.

Theorem 2. *A mapping $h:\mathbb{R}^\star \longrightarrow \mathbb{R}$ satisfying* (1.2) *also satisfies*

$$h(m^p u)+h\left(\frac{u}{m^p}\right)=\frac{m^{2p}+1}{m^p}h(u) \tag{3.2}$$

for all $u \in \mathbb{R}^\star$ and $p > 0$ is an integer.

Proof. Firstly, let us consider u for every u_j, $j = 1,2,\dots,m$ in (1.2) to obtain

$$h(mu)+h\left(\frac{u}{m}\right)=\frac{m^2+1}{m}h(u) \tag{3.3}$$

for all $u \in \mathbb{R}^\star$. Next, reinstating u by mu in 3.3 and then multiplying by $\frac{m^2+1}{m}$ on its both sides, we get

$$\frac{m^2+1}{m}h(m^2u)+\frac{m^2+1}{m}h(u)=\frac{(m^2+1)^2}{m^2}h(mu) \tag{3.4}$$

for all $u \in \mathbb{R}^\star$. On the other hand, replacing u by $\frac{u}{m}$ in (3.3) and then multiplying by $\frac{m^2+1}{m}$ on its both sides, we obtain

$$\frac{m^2+1}{m}h(u)+\frac{m^2+1}{m}h\left(\frac{u}{m^2}\right)=\frac{(m^2+1)^2}{m^2}h\left(\frac{u}{m}\right) \tag{3.5}$$

for all $u \in \mathbb{R}^\star$. Now, adding (3.4) with (3.5) and simplifying further, we arrive at

$$h(m^2u)+h\left(\frac{u}{m^2}\right)=\frac{m^4+1}{m^2}h(u) \tag{3.6}$$

for all $u \in \mathbb{R}^\star$. Again, plugging u by mu in (3.6), we get

$$h(m^3u)+h\left(\frac{u}{m}\right)=\frac{m^4+1}{m^2}h(mu) \tag{3.7}$$

for all $u \in \mathbb{R}^\star$. Also, substituting u by $\frac{u}{m}$ in (3.6), we obtain

$$h(mu)+h\left(\frac{u}{m^3}\right)-\frac{m^4+1}{m^2}h\left(\frac{u}{m}\right) \tag{3.8}$$

for all $u \in \mathbb{R}^\star$. Now, summing (3.7) and (3.8) and simplifying further to arrive at

$$h(m^3u)+h\left(\frac{u}{m^3}\right)=\frac{m^6+1}{m^3}h(u)$$

for all $u \in \mathbb{R}^\star$. Proceeding with similar arguments and employing mathematical induction, one can find for any $p > 0$ integer,

$$h(m^p u)+h\left(\frac{u}{m^p}\right)=\frac{m^{2p}+1}{m^p}h(u)$$

for all $u \in \mathbb{R}^\star$. This completes the proof. $\square$

4. Hyperstability of equation (1.2)

In this section, by employing the notions and fixed point theorem proposed in [8], we establish the hyperstability of (1.2). For the sake of of convenience, let us define the difference operator $Dh(u_1,\dots,u_m) : \underbrace{\mathbb{R}^* \times \cdots \times \mathbb{R}^*}_{(m \text{ times})} \longrightarrow \mathbb{R}$ as follows:

$$\begin{aligned} &Dh(u_1,u_2,\dots,u_m)\\ &\quad = h\left(\sum_{j=1}^{m} u_j\right) + h\left(\frac{\prod_{j=1}^{m} u_j}{\sum_{j=1}^{m}\prod_{k=1,k\neq j}^{m} u_k}\right) - \frac{\prod_{j=1}^{m} h(u_j)}{\sum_{j=1}^{m}\prod_{k=1,k\neq j}^{m} h(u_k)} - \sum_{j=1}^{m} h(u_j) \end{aligned}$$

for all $u_1,\dots,u_m \in \mathbb{R}^\star$.

Theorem 3. *Let $k>0$ and $p<0$ be fixed constants. Let there exists $n_0 \in \mathbb{N}$ with $nu \in \mathbb{R}^\star$ for $u \in \mathbb{R}^\star$, $n \in \mathbb{N}_{n_0}$. Suppose a mapping $h : \mathbb{R}^\star \longrightarrow \mathbb{R}$ satisfies the inequality*

$$|Dh(u_1,u_2\dots,u_m)| \leq k\sum_{j=1}^{m} |u_j|^p \tag{4.1}$$

for all $u_1,\dots,u_m \in \mathbb{R}^\star$. Then there exists a unique Rassias-Ravi reciprocal function $H : \mathbb{R}^\star \longrightarrow \mathbb{R}$ satisfying (1.2) and

$$|H(u)-h(u)| \leq \frac{m^2k}{1-m^{1+p}-m^{1-p}+m^2}\,|u|^p \tag{4.2}$$

for all $u \in \mathbb{R}^\star$.

Proof. Firstly, let us plug $u_j = u$, for $j = 1,2,\dots,m$ in (4.1) and then multiply by $\frac{m}{m^2+1}$ on its both sides to get

$$\left|\frac{m}{m^2+1}h(mu) + \frac{m}{m^2+1}h\left(\frac{u}{m}\right) - h(u)\right| \leq \frac{m^2k}{m^2+1}\,|u|^p \tag{4.3}$$

for all $u \in \mathbb{R}^\star$. We can find that there exists an $m_0 \in \mathbb{N}_{m_0}$ such that

$$\frac{m^p}{(m^2+1)^p} < 1 \quad \text{for } \; m \geq m_0. \tag{4.4}$$

Let $m \in \mathbb{N}_{m_0}$ be fixed integer. Let us denote

$$\eta\alpha(u) = \frac{m}{m^2+1}\alpha(mu) + \frac{m}{m^2+1}\alpha\left(\frac{u}{m}\right), \quad u \in \mathbb{R},\;\; \alpha \in \mathbb{R}^{\mathbb{R}^\star}, \tag{4.5}$$

$$\psi(u) = \frac{m^2k}{m^2+1}\,|u|^p, \quad u \in \mathbb{R}^\star. \tag{4.6}$$

Using (4.5) and (4.6), inequality (4.3) can be written as

$$|\eta h(u) - h(u)| \leq \psi(u), \quad u \in \mathbb{R}^\star. \tag{4.7}$$

The operator is defined by the following:

$$\Gamma\rho(u) = \frac{m}{m^2+1}\rho(mu) + \frac{m}{m^2+1}\rho\left(\frac{u}{m}\right), \quad \rho \in \mathbb{R}_+^{\mathbb{R}^\star}, \ u \in \mathbb{R}^\star \tag{4.8}$$

has the form which is defined in (P3) with $k = 2$ and $h_1(u) = mu$, $h_2(u) = \frac{u}{m}$ and $Q_1(u) = Q_2(u) = \frac{m}{m^2+1}$ for $u \in \mathbb{R}^\star$. Also, for each $\alpha, \beta \in \mathbb{R}^{\mathbb{R}^\star}$, $u \in \mathbb{R}^\star$,

$$\begin{aligned}
|\eta\alpha(u) - \eta\beta(u)| &= \left| \frac{m}{m^2+1}\alpha(mu) + \frac{m}{m^2+1}\alpha\left(\frac{u}{m}\right) - \frac{m}{m^2+1}\beta(mu) - \frac{m}{m^2+1}\beta\left(\frac{u}{m}\right) \right| \\
&\le \frac{m}{m^2+1}\left|(\alpha-\beta)(mu)\right| + \frac{m}{m^2+1}\left|(\alpha-\beta)\left(\frac{u}{m}\right)\right| \\
&\le \sum_{i=1}^{2} Q_i(u)\left|(\alpha-\beta)h_i(u)\right|.
\end{aligned} \tag{4.9}$$

Since $\frac{m}{m^2+1}\left(\frac{m^{2p}+1}{m^p}\right) < 1$, we have

$$\begin{aligned}
\psi^\star(u) = \sum_{n=0}^{\infty}\Gamma^n\psi(u) &= \sum_{n=0}^{\infty}\frac{m^2k}{m^2+1}\left(\frac{m}{m^2+1}\left(\frac{m^{2p}+1}{m^p}\right)\right)^n |u|^p \\
&= \frac{m^2k}{1-m^{1+p}-m^{1-p}+m^2}|u|^p.
\end{aligned} \tag{4.10}$$

Owing to Theorem 1, there exists a unique solution $H : \mathbb{R}^\star \longrightarrow \mathbb{R}$ of the equation

$$H(u) = \frac{m}{m^2+1}h(mu) + \frac{m}{m^2+1}h\left(\frac{u}{m}\right) \tag{4.11}$$

such that the inequality (4.2) holds. Moreover,

$$H(u) = \lim_{n\to\infty}\eta^n h(u). \tag{4.12}$$

In order to show that H satisfies (1.2), we find that

$$|\eta^n Dh(u_1,u_2,\ldots,u_m)| \le k\left(\frac{m}{m^2+1}\right)^n\left(\frac{m^{2p}+1}{m^p}\right)^n\sum_{j=1}^{m}|u|^p \tag{4.13}$$

for all $u_1,\ldots,u_m \in \mathbb{R}^\star$, and $n \in \mathbb{N}_0$. Suppose $n = 0$, then (4.13) becomes (4.1). So, let us fix $n \in \mathbb{N}_0$ and suppose that (4.13) holds for n and $u_1,\ldots,u_m \in \mathbb{R}^\star$. Then

$$\begin{aligned}
&\left|\eta^{n+1}Dh(u_1,u_2,\ldots,u_m)\right| \\
&= \left|\frac{m}{m^2+1}\eta^n h\left(\sum_{j=1}^{m}mu_j\right) + \frac{m}{m^2+1}\eta^n h\left(\sum_{j=1}^{m}\frac{u_j}{m}\right)\right. \\
&\quad + \frac{m}{m^2+1}\eta^n h\left(\frac{\prod_{j=1}^{m}mu_j}{\sum_{j=1}^{m}\prod_{k=1,k\neq j}^{m}mu_k}\right) + \frac{m}{m^2+1}\eta^n h\left(\frac{\prod_{j=1}^{m}\frac{u_j}{m}}{\sum_{j=1}^{m}\prod_{k=1,k\neq j}^{m}\frac{u_k}{m}}\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{m}{m^2+1}\eta^n \left(\frac{\prod_{j=1}^{m} h(mu_j)}{\sum_{j=1}^{m}\prod_{k=1,k\neq j}^{m} h(mu_k)} \right) - \frac{m}{m^2+1}\eta^n \left(\frac{\prod_{j=1}^{m} h\left(\frac{u_j}{m}\right)}{\sum_{j=1}^{m}\prod_{k=1,k\neq j}^{m} h\left(\frac{u_k}{m}\right)} \right) \\
&- \frac{m}{m^2+1}\eta^n \sum_{j=1}^{m} h(mu_j) - \frac{m}{m^2+1}\eta^n \sum_{j=1}^{m} h\left(\frac{u_j}{m}\right) \Bigg| \\
&\le k\left(\frac{m}{m^2+1}\right)^n \left(\frac{m^{2p}+1}{m^p}\right)^n \left[\frac{m}{m^2+1}\sum_{j=1}^{m} |u_j|^p + \frac{m}{m^2+1}\sum_{j=1}^{m} \left|\frac{u_j}{m}\right|^p \right] \\
&\le k\left(\frac{m}{m^2+1}\right)^{n+1} \left(\frac{m^{2p}+1}{m^p}\right)^{n+1} \sum_{j=1}^{m} |u_j|^p . \qquad (4.14)
\end{aligned}
$$

Hence through induction method, the above inequality (4.14) implies that (4.13) holds good for all $u_j \in \mathbb{R}^\star$, for $j = 1,2,\ldots,m$. By letting n to ∞ in (4.13), we can find that H satisfies (1.2). This completes the proof. □

In the sequel, we provide two examples for the non-stability of equation (1.2).

Example 1. Let $A = [-1,1]\backslash\{0\}$ and let $h : A \longrightarrow \mathbb{R}$ be defined by $h(u) = u$, $u \in A$. Then for $u_j \in A$, $j = 1,2,\ldots,m$ such that

$$
\sum_{j=1}^{m} u_j, \sum_{j=1}^{m} \frac{u_j}{m}, \prod_{j=1}^{m} mu_j, \prod_{j=1}^{m} \frac{u_j}{m}, \frac{\prod_{j=1}^{m} mu_j}{\sum_{j=1}^{m}\prod_{k=1,k\neq j}^{m} mu_k}, \frac{\prod_{j=1}^{m} \frac{u_j}{m}}{\sum_{j=1}^{m}\prod_{k=1,k\neq j}^{m} \frac{u_k}{m}} \in A,
$$

$$
|Dh(u_1,u_2,\ldots,u_m)| \le \sum_{j=1}^{m} |u_j|^p ,
$$

with $p < 0$, but h does not satisfy (1.2).

The following theorem contains the hyperstability involving product of different powers of norms. The proof is obtained by similar arguments as in Theorem 3. Hence we omit the proof and provide only the statement.

Theorem 4. *Let $k > 0$ be a fixed constant. Let $p_j \in \mathbb{R}$, $j = 1,2,\ldots,m$ such that $p = \sum_{j=1}^{m} p_i < 0$. Let $h : \mathbb{R}^\star \longrightarrow \mathbb{R}$ satisfy the following inequality*

$$
|Dh(u_1,u_2,\ldots,u_m)| \le k\prod_{j=1}^{m} |u_j|^{p_j}
$$

for all $u_j \in \mathbb{R}^\star$, $j = 1,2,\ldots,m$. Then there exists a unique Rassias-Ravi reciprocal function $H : \mathbb{R}^\star \longrightarrow \mathbb{R}$ satisfying (1.2) and

$$
|H(u) - h(u)| \le \frac{mk}{1 - m^{1+p} - m^{1-p} + m^2} |u|^p
$$

for all $u \in \mathbb{R}^\star$.

5. Conclusion

So far various forms of additive FEs and multiplicative inverse FEs are considered in this research field to obtain their stability results through different methods. For the first time, a new FE with additive function and multiplicative inverse function is proposed in this paper and its hyperstability results are proved via fixed point method.

References

[1] L. Aiemsomboon and W. Sintunavarat, "On new hyperstability results for the generalized p-radical functional equation in quasi-Banach spaces with the illustrative example." *Results in Fixed Point Theory and Applications*, vol. 2019, no. 2019012, pp. 1–14, 2019.

[2] Y. Aribou, H. Dimou, and S. Kabbaj, "Generalized hyperstability of the cubic functional equation in ultrametric spaces." *Linear and Topological Algebra*, vol. 8, no. 2, pp. 97–104, 2019.

[3] A. Bahyrycz and M. Piszczek, "Hyperstability of the Jensen functional equation." *Acta Math. Hungarica*, vol. 142, no. 2, pp. 353–365, 2014, doi: 10.1007/s10474-013-0347-3.

[4] A. Bodaghi and B. V. Senthil Kumar, "Estimation of inexact reciprocal-quintic and reciprocal-sextic functional equations." *Mathematica*, vol. 49, no. 82, pp. 3–14, 2017.

[5] D. G. Bourgin, "Approximately isometric and multiplicative transformations on continuous function rings." *Duke Math. J.*, vol. 16, pp. 385–397, 1949.

[6] J. Brzdek, "Hyperstability of the Cauchy equation on restricted domains." *Acta Math. Hungarica*, vol. 141, no. 1-2, pp. 58–67, 2013, doi: 10.1007/s10474-013-0302-3.

[7] J. Brzdek, "Remarks on hyperstability of the Cauchy functional equations." *Aequ. Math.*, vol. 86, pp. 255–267, 2013, doi: 10.1007/s00010-012-0168-4.

[8] J. Brzdek, J. Chudziak, and Z. Pales, "A fixed point approach to stability of functional equations." *Nonlinear Anal.*, vol. 74, no. 17, pp. 6728–6732, 2011, doi: 10.1016/j.na.2011.06.052.

[9] J. Brzdek and K. Cieplinski, "A fixed point approach to the stability of functional equations in non-Archimedean metric spaces." *Nonlinear Anal.*, vol. 74, no. 18, pp. 6861–6867, 2011, doi: 10.1016/j.na.2011.06.050.

[10] L. Cadariu, L. Gavruta, and P. Gavruta, "Fixed points and generalized Hyers-Ulam stability." *Abst. Appl. Anal.*, vol. 2012, no. 712743, pp. 1–12, 2012, doi: 10.1155/2012/712743.

[11] H. Dutta and B. V. Senthil Kumar, "Geometrical elucidations and approximation of some functional equations in numerous variables." *Proc. Indian Natn. Sc. Acad.*, vol. 85, no. 3, pp. 603–611, 2019, doi: 10.16943/ptinsa/2019/49584.

[12] A. Ebadian, S. Zolfaghari, S. Ostadbashi, and C. Park, "Approximation on the reciprocal functional equation in several variables in matrix non-Archimedean random normed spaces." *Adv. Difference. Equ.*, vol. 314, pp. 1–13, 2015, doi: 10.1186/s13662-015-0656-7.

[13] E. Gselmann, "Hyperstability of a functional equation." *Acta Math. Hungarica*, vol. 124, no. 1-2, pp. 179–188, 2009, doi: 10.1007/s10474-009-8174-2.

[14] P. Găvruta, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mapppings." *J. Math. Anal. Appl.*, vol. 184, pp. 431–436, 1994, doi: 10.1006/jmaa.1994.1211.

[15] D. H. Hyers, "On the stability of the linear functional equation." *Proc. Nat. Acad. Sci. U.S.A.*, vol. 27, pp. 222–224, 1941, doi: 10.1073/pnas.27.4.222.

[16] S. O. Kim, B. V. Senthil Kumar, and A. Bodaghi, "Stability and non-stability of the reciprocal-cubic and reciprocal-quartic functional equations in non-Archimedean fields." *Adv. Difference Equ.*, vol. 2017, no. 77, pp. 1–12, 2017, doi: 10.1186/s13662-017-1128-z.

[17] Y. W. Lee and G. H. Kim, "Hyperstability and stability of a logarithmic-type functional equation." *Math. Stat.*, vol. 7, no. 1, pp. 25–32, 2019, doi: 10.13189/ms.2019.070104.

[18] G. Maksa and Z. Pales, “Hyperstability of a class of linear functional equations.” *Acta Mathematica*, vol. 17, no. 2, pp. 107–112, 2001.

[19] M. Piszczek, “Remark on hyperstability of the general linear equation.” *Aequ. Math.*, vol. 88, pp. 163–168, 2014, doi: 10.1007/s00010-013-0214-x.

[20] J. M. Rassias, “On approximately of approximately linear mappings by linear mappings.” *J. Funct. Anal., USA.*, vol. 46, pp. 126–130, 1982, doi: 10.1016/0022-1236(82)90048-9.

[21] T. M. Rassias, “On the stability of the linear mapping in Banach spaces.” *Proc. Amer. Math. Soc.*, vol. 72, pp. 297–300, 1978, doi: 10.1090/S0002-9939-1978-0507327-1.

[22] K. Ravi, J. M. Rassias, and B. V. Senthil Kumar, “Ulam stability of reciprocal difference and adjoint functional equations.” *Aust. J. Math. Anal. Appl.*, vol. 8, no. 1, pp. 1–18, 2011.

[23] K. Ravi and B. V. Senthil Kumar, “Ulam-Gavruta-Rassias stability of Rassias reciprocal functional equation.” *Global J. Appl. Math. Math. Sci.*, vol. 3, no. 1-2, pp. 57–79, 2010.

[24] K. Ravi, E. Thandapani, and B. V. Senthil Kumar, “Stability of reciprocal type functional equations.” *PanAmerican Math. J.*, vol. 21, no. 1, pp. 59–70, 2011.

[25] S. Saejung and J. Senasukh, “On stability and hyperstability of additive equations on a commutative semigroup.” *Acta Math. Hungarica*, vol. 159, pp. 358–373, 2019, doi: 10.1007/s10474-019-00998-3.

[26] B. V. Senthil Kumar, Ashish Kumar, and P. Narasimman, “Ulam stability of radical functional equation in the sense of Hyers, Rassias and Gavruta.” *Int. J. Stat. Appl. Math.*, vol. 1, no. 4, pp. 6–12, 2016.

[27] B. V. Senthil Kumar and H. Dutta, “Non-Archimedean stability of a generalized reciprocal-quadratic functional equation in several variables by direct and fixed point methods.” *Filomat*, vol. 32, no. 9, pp. 3199–3209, 2018, doi: 10.2298/FIL1809199.

[28] B. V. Senthil Kumar and H. Dutta, “Approximation of multiplicative inverse undecic and duodecic functional equations.” *Math. Meth. Appl. Sci.*, vol. 42, pp. 1073–1081, 2019, doi: 10.1002/mma.5413.

[29] B. V. Senthil Kumar and H. Dutta, “Fuzzy stability of a rational functional equation and its relevance to system design.” *Int. J. General Syst.*, vol. 48, no. 2, pp. 157–169, 2019, doi: 10.1080/03081079.2018.1523904.

[30] B. V. Senthil Kumar and H. Dutta, “Fundamental stabilities of various forms of complex valued functional equations.” *in: Dutta H., Peters J. (eds) Applied Mathematical Analysis: Theory, Methods, and Applications. Studies in Systems, Decision and Control, Springer, Cham*, vol. 177, pp. 29–59, 2020.

[31] B. V. Senthil Kumar, H. Dutta, and S. Sabarinathan, “Approximation of a system of rational functional equations of three variables.” *Int. J. Appl. Comput. Math.*, vol. 5, no. 3, pp. 1–16, 2019, doi: 10.1007/s40819-019-0626-y.

[32] S. M. Ulam, *Problems in Modern Mathematics, Chapter VI.* New York: Wiley-Interscience, 1964.

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AN EXTENSION OF THE TOPSIS FOR MULTI-ATTRIBUTE GROUP DECISION MAKING UNDER NEUTROSOPHIC ENVIRONMENT

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Abstract. Neutrosophic set theory as a generalization of the fuzzy set theory and intuitionistic fuzzy set theory is an effective tool to deal with inconsistent, imprecise, and vague information. TOPSIS is a multiple attribute method to identify solutions from a finite set of alternatives based upon simultaneous minimization of distance from an ideal point and maximization of distance from a nadir point. In this paper, we first develop a new Hamming distance between single-valued neutrosophic numbers and then present an extension of the TOPSIS method for multi-attribute group decision-making (MAGDM) based on single-valued neutrosophic sets, where the inform ation about attribute values and attribute weights are expressed by decision-makers based on neutrosophic numbers.

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1. SECTION HEAD

Multi-attribute decision making (MADM) as a component of decision science is a substantial and essential part of daily life which can be applied in various areas, such as society, economics, management, military, and engineering technology. In most cases, it is intricate for decision-makers to accurately reveal a preference when solving MADM problems with imprecise, vague or incomplete information. Under these conditions, in the past few decades, various types of sets, such as fuzzy sets [32], interval-valued fuzzy sets [33], intuitionistic fuzzy sets [1, 3], interval-valued intuitionistic fuzzy sets [2], type 2 fuzzy sets [8, 11], type n fuzzy sets [8], hesitant fuzzy sets [29] and neutrosophic set theory [26], have been introduced and widely used in the solution of significant decision-making problems. The neutrosophic set theory which is an extension of the intuitionistic fuzzy set provides a practical tool to deal with indeterminate and inconsistent information that exist commonly in the real conditions. A given neutrosophic set such as N has three independent components,

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namely the truth membership $T_N(x)$, the indeterminacy membership $I_N(x)$ and falsity-membership $F_N(x)$.

The technique for order performance by similarity to ideal solution (TOPSIS) was first developed by Hwang and Yoon [11] for solving a MADM problem. It bases upon the concept that the chosen alternative should have the shortest distance from the positive ideal solution (PIS) and the farthest from the negative ideal solution (NIS). In the process of TOPSIS, the performance ratings and the weights of the criteria are given as crisp values. In recent years a lot of MADM methods [4, 14–16, 30, 31] and multi-attribute group decision making (MAGDM) methods [6, 17, 19] based on the extension of the TOPSIS method have been proposed.

In order to evaluate human resources, Jin et al. [14] introduced an extended TOPSIS method for MADM based on intuitionistic fuzzy sets where the attribute values given by decision-makers are the intuitionistic fuzzy numbers. Wei and Liu [31] presented an extended TOPSIS method based on uncertain linguistic variables to manage high technological risks. In order to resolve MADM problems, Liu in [16] presents an extension of the TOPSIS method where the weights and decision values of the alternatives are considered as interval vague values. Liu and Su [15] proposed an extended TOPSIS based on trapezoid fuzzy linguistic numbers and present a method for determining attribute weights. Rădulescu. C. and Rădulescu. I. [24] by modifying the variable ρ in the Minkowski distance measure proposed an extended TOPSIS method for ranking cloud service providers. Verma et al. [30] proposed an interval-valued intuitionistic fuzzy TOPSIS method for solving a facility location problem. Balin [4] proposed an extension of TOPSIS based on interval-valued spherical fuzzy sets to select the most effective stabilizing system for naval ships. In [6] Chen proposed a symmetric approach to extend the TOPSIS to the fuzzy environment for MAGDM problems in which the weights of various attributes and ratings of alternatives in regard to the different attributes indicated by linguistic variables. By defining a distance formula of generalized interval-valued fuzzy numbers in [17] Liu proposed an extended TOPSIS method for MAGDM problems where the attribute values and weights given by different decision-makers are all generalized interval-valued fuzzy numbers. In this respect, to choose adequate security mechanisms in e-business processes, Mohammadi et al. [19] proposed a fuzzy TOPSIS method based on group recommendation.

In this research, we first develop a distance measure to calculate the distance between single-valued neutrosophic numbers and then present an extended TOPSIS method for MAGDM under the neutrosophic environment where the attribute values and weights given by decision-makers (DMs) are represented by single-valued neutrosophic numbers (SVNNs). The key of our proposed method is that the different neutrosophic decision matrices presented by different decision-makers are converted into a single matrix and create an aggregated group decision matrix. The remaining of this research is marshaled as follows: in the next section, we will briefly review the basic concepts of neutrosophic sets, the operation rules of single-valued neutrosophic sets, and the distance between them. Section 3 presents a distance measure to calculate the distance between SVNNs and describes the steps of the proposed method to rank the alternatives. Section 4 gives a numerical example to explain the validity of the proposed method. The study is concluded in Section 5.

2. Preliminaries

This section provides a brief review of particular preliminaries regarding neutrosophic sets, the distance between neutrosophic sets (NSs) and some other important concepts.

Definition 1 ([22, 27]). A neutrosophic set (NS) N in a domain X(finite universe of objectives) can be represented by $T_N : X \to]0^-, 1^+[$, $I_N : X \to]0^-, 1^+[$ and $F_N : X \to]0^-, 1^+[$ that satisfy the condition $0^- \leq T_N(x) + I_N(x) + F_N(x) \leq 3^+ \forall x \in X$. Where $T_N(x)$, $I_N(x)$ and $F_N(x)$ denote the truth, indeterminacy and falsity membership functions, respectively.

Definition 2 ([20, 21]). A neutrosophic set N is contained in another neutrosophic set M, if and only if:

$$\begin{aligned} Inf\, T_N(x) &\leq Inf\, T_M(x), \\ Sup\, T_N(x) &\leq Sup\, T_M(x), \\ Inf\, I_N(x) &\geq Inf\, I_M(x), \\ Sup\, I_N(x) &\geq Sup\, I_M(x), \\ Inf\, F_N(x) &\geq Inf\, F_M(x), \\ Sup\, F_N(x) &\geq Sup\, F_M(x), \end{aligned} \tag{2.1}$$

for all $x \in X$.

Definition 3 ([25]). The complement of a neutrosophic set N is denoted by N^c and can be defined as $T_N^c(x) = \{1\} \ominus T_N(x)$, $I_N^c(x) = \{1\} \ominus I_N(x)$ and $F_N^c(x) = \{1\} \ominus F_N(x)$ for all $x \in X$.

Definition 4 ([18, 28]). Let X be a domain. A single-valued neutrosophic set (SVNS) N in the domain X can be denoted as $N = \{x, T_N(x), I_N(x), F_N(x); x \in X\}$, where $T_N : X \to [0,1]$, $I_N : X \to [0,1]$ and $F_N : X \to [0,1]$ are three maps in X that satisfy the condition $0 \leq T_N(x) + F_N(x) + I_N(x) \leq 3\ \forall x \in X$. The numbers $T_N(x)$, $F_N(x)$ and $I_N(x)$ are the degree of truth, falsity and indeterminacy membership of element x to N, respectively.

Remark 1. For a SVNS N, the trinary $(T_N(x), I_N(x), F_N(x))$ is called a single-valued neutrosophic number (SVNN). For convenience, the trinary $(T_N(x), I_N(x), F_N(x))$ is often denoted by (T, I, F).

Definition 5 ([9, 18])**.** Let $x = (T_1, I_1, F_1)$ and $y = (T_2, I_2, F_2)$ be two SVNNs. The mathematical operations between x and y are defined as follows:

$$I. \quad x \oplus y = (T_1 + T_2 - T_1T_2, I_1I_2, F_1F_2), \tag{2.2}$$

$$II. \quad x \otimes y = (T_1T_2, I_1 + I_2 - I_1I_2, F_1 + F_2 - F_1F_2), \tag{2.3}$$

$$III. \quad \lambda x = \left(1 - (1 - T_1)^{\lambda}, I_1^{\lambda}, F_1^{\lambda}\right), \lambda > 0, \tag{2.4}$$

$$IV. \quad x^{\lambda} = \left(T_1^{\lambda}, 1 - (1 - I_1)^{\lambda}, 1 - (1 - F_1)^{\lambda}\right), \lambda > 0. \tag{2.5}$$

Definition 6 ([10, 13])**.** The complement of a SVNS N is denoted by N^c and is defined as $T_N^c(x) = F_N(x), I_N^c(x) = 1 - I(x)$ and $F_N^c(x) = T_N(x)$ for all $x \in X$. Therefore $[N^c = \{x, F_N(x), 1 - I_N(x), T_N(x); x \in X\}.]$

Definition 7 ([7, 12])**.** Let $N = \{x, T_N(x), I_N(x), F_N(x); x \in X\}$ and $M = \{x, T_M(x), I_M(x), F_M(x); x \in X\}$ be two single-valued neutrosophic sets, the Hamming distance between N and M is defined as follow:

$$d_H(x,y) = \frac{1}{6}\left(|T_N(x) - T_M(x)| + |I_N(x) - I_M(x)| + |F_N(x) - F_M(x)|\right), \tag{2.6}$$

also, the Euclidian distance between N and M is defined as follow:

$$d_E(N,M) = \sqrt{\frac{1}{6}\left((T_N(x) - T_M(x))^2 + (I_N(x) - I_M(x))^2 + (F_N(x) - F_M(x))^2\right)}. \tag{2.7}$$

Definition 8 ([5, 23])**.** $d(N,M)$ is said to be a distance measure between neutrosophic sets if it satisfies the following properties:

P1: $d(N,M) \geq 0$.
P2: $d(N,M) = 0$ if and only if $N = M$ for all $N, M \in$ NSs.
P3: $d(N,M) = d(M,N)$.
P4: If $N \subseteq M \subseteq O$ where $O \in$ NSs in X then: $d(N,O) \geq d(N,M)$ and $d(N,O) \geq d(M,O)$.

3. The proposed method

In this section, we first propose a new Hamming distance based on the Hausdorff metric between single-valued neutrosophic numbers. Then we will use this distance to present a new multi-attribute group decision-making method (MAGDM) based on the combination of neutrosophic sets and extended TOPSIS method.

3.1. *Extended Hausdorff distance*

Let $X = \{x_1, x_2, ..., x_n\}$ be a finite universe of objectives. Consider two neutrosophic sets N and M in X where $N = \{x_i, T_N(x_i), I_N(x_i), F_N(x_i); x_i \in X\}$ and $M = \{x_i, T_M(x_i), I_M(x_i), F_M(x_i); x_i \in X\}$. Then denote

$$d(N,M) = \frac{1}{n}\sum_{i=1}^{n}[\frac{(|T_N(x_i) - T_M(x_i)| + |I_N(x_i) - I_M(x_i)| + |F_N(x_i) - F_M(x_i)|)}{6}$$

$$+\frac{\max\left(\left|T_N(x_i)-T_M(x_i)\right|,\left|I_N(x_i)-I_M(x_i)\right|,\left|F_N(x_i)-F_M(x_i)\right|\right)}{3}]. \quad (3.1)$$

Theorem 1. *$d(N,M)$ is a distance between two neutrosophic sets N and M in X.*

Proof. It is obvious $d(N,M)$ satisfies P1-P3 of Definition 8. Therefore we only need to prove $d(N,M)$ satisfies P4. To this aim let $O=\{x,T_O(x),I_O(x),F_O(x);x\in X\}$ be another neutrosophic set. In this case, if $N\subseteq M\subseteq O$ then we have:

$$d(N,M)=\frac{1}{n}\sum_{i=1}^{n}[\frac{(|T_N(x_i)-T_M(x_i)|+|I_N(x_i)-I_M(x_i)|+|F_N(x_i)-F_M(x_i)|)}{6}$$
$$+\frac{\max\left(\left|T_N(x_i)-T_M(x_i)\right|,\left|I_N(x_i)-I_M(x_i)\right|,\left|F_N(x_i)-F_M(x_i)\right|\right)}{3}],$$

and

$$d(N,O)=\frac{1}{n}\sum_{i=1}^{n}[\frac{(|T_N(x_i)-T_O(x_i)|+|I_N(x_i)-I_O(x_i)|+|F_N(x_i)-F_O(x_i)|)}{6}$$
$$+\frac{\max\left(\left|T_N(x_i)-T_O(x_i)\right|,\left|I_N(x_i)-I_O(x_i)\right|,\left|F_N(x_i)-F_O(x_i)\right|\right)}{3}].$$

It's easy to see

$$\begin{aligned}|T_N(x_i)-T_O(x_i)|&\geq|T_N(x_i)-T_M(x_i)|,\\|I_N(x_i)-I_O(x_i)|&\geq|I_N(x_i)-I_M(x_i)|,\\|F_N(x_i)-F_O(x_i)|&\geq|F_N(x_i)-F_M(x_i)|,\end{aligned}$$

so we have:

$$\begin{aligned}&\frac{(|T_N(x_i)-T_O(x_i)|+|I_N(x_i)-I_O(x_i)|+|F_N(x_i)-F_O(x_i)|)}{6}\\&\quad+\frac{\max\left(\left|T_N(x_i)-T_O(x_i)\right|,\left|I_N(x_i)-I_O(x_i)\right|,\left|F_N(x_i)-F_O(x_i)\right|\right)}{3}\\&\geq\frac{(|T_N(x_i)-T_M(x_i)|+|I_N(x_i)-I_M(x_i)|+|F_N(x_i)-F_M(x_i)|)}{6}\\&\quad+\frac{\max\left(\left|T_N(x_i)-T_M(x_i)\right|,\left|I_N(x_i)-I_M(x_i)\right|,\left|F_N(x_i)-F_M(x_i)\right|\right)}{3}.\end{aligned}$$

Therefore we can get the inequality $d(N,O)\geq d(N,M)$. By the same reason we can get $d(N,O)\geq d(M,O)$. So $d(N,M)$ satisfies P4 of Definition 8. That is to say, $d(N,M)$ is a distance measure between neutrosophic sets N and M. □

3.2. *The extended TOPSIS method for multi-attribute group decision-making*

Suppose that $A=\{A_1,A_2,\ldots,A_n\}$ be a set of alternatives, $B=\{C_1,C_2,\ldots,C_m\}$ be a set of attributes and $D=\{D_1,D_2,\ldots D_k\}$ be a set of decision-makers (DMs). Let $\bar{w}_p=\left[\bar{w}_1^p,\bar{w}_2^p,\ldots,\bar{w}_m^p\right]$ be a vector of weights for attributes determined by DM D_p

where $\bar{w}_j^p$ is a single-valued neutrosophic number denoting the weight of attribute C_j given by decision-maker D_p. $1 \le j \le m$ and $1 \le p \le k$.

Assume that W_p represents the weight of DM D_p. If a decision group has k members then $W_P = \frac{1}{k}$, where $W_p \in [0,1]$ and $\sum_{p=1}^{k} W_p = 1$.

Let $X_p = [x_{ij}]_{m\times n}$ be a decision matrix of the n alternatives in regard to the m attributes characterized by decision-maker D_p, shown as follows:

$$X_p = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_m \end{array} \\ \begin{bmatrix} x_{11}^p & x_{12}^p & \dots & x_{1m}^p \\ x_{21}^p & x_{22}^p & \dots & x_{2m}^p \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}^p & x_{n2}^p & \dots & x_{nm}^p \end{bmatrix} \end{array}, \tag{3.2}$$

where $x_{ij} = (T_{ij}, I_{ij}, F_{ij})$ is a single value neutrosophic number for the alternative A_i in regard to the attribute C_j.

The procedure of our proposed method can be summarized as follows:

Step 1. According to the weighting vector $\bar{w}_p$, the decision matrix X_P and the multiplication operator of SVNSs presented in (2.3) calculate the weighted decision matrix (WDM) EV_P as follows:

$$EV_p = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_m \end{array} \\ \begin{bmatrix} x_{11}^p \otimes \bar{w}_1^P & x_{12}^p \otimes \bar{w}_2^P & \dots & x_{1m}^p \otimes \bar{w}_m^P \\ x_{21}^p \otimes \bar{w}_1^P & x_{22}^p \otimes \bar{w}_2^P & \dots & x_{2m}^p \otimes \bar{w}_m^P \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}^p \otimes \bar{w}_1^P & x_{n2}^p \otimes \bar{w}_2^P & \dots & x_{nm}^p \otimes \bar{w}_m^P \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_m \end{array} \\ \begin{bmatrix} y_{11}^p & y_{12}^p & \dots & y_{1m}^p \\ y_{21}^p & y_{22}^p & \dots & y_{2m}^p \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}^p & y_{n2}^p & \dots & y_{nm}^p \end{bmatrix} \end{array}. \tag{3.3}$$

Step 2. Based on the obtained WDMs and the weight of decision-makers we can get the aggregated group decision matrix AG of all decision-makers $D_1, D_2, \ldots, D_k$ as follows:

$$AG = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & \cdots & D_k \end{array} \\ \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1k} \\ G_{21} & G_{22} & \cdots & G_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n1} & G_{n2} & \cdots & G_{nk} \end{bmatrix} \end{array}, \tag{3.4}$$

where G_{ip} is a neutrosophic value, representing the sum of alternatives in regard to DM D_p, and can be calculated as follows:

$$G_{ip} = W_P[y_{i1}^p \oplus y_{i2}^p \oplus \cdots \oplus y_{im}^p],$$

where W_P is the weight of decision-maker D_P and $\oplus$ is the addition operator presented in (2.2).

Step 3. Based on the obtained aggregated group decision matrix we know that the elements G_{ip} are SVNNs. The absolute neutrosophic positive ideal solution (NPIS)

$P^+$and the neutrosophic negative ideal solution (NNIS) P^-can be defined as follows:

$$\begin{array}{l} P^+ = (G_1^+, G_2^+, ..., G_k^+), \\ P^- = (G_1^-, G_2^-, ..., G_k^-), \end{array} \tag{3.5}$$

where $G_j^+ = (1,0,0)$ and $G_j^- = (0,1,1)$, $j = 1,2,..,k$. Also we can select the virtual positive ideal solution and negative ideal solution by selecting the best values for each attribute from all alternatives as follows:

$$\begin{cases} G_j^+ = (\max_i T_{ij}, \min_i I_{ij}, \min_i F_{ij}) = (T_j^+, I_j^+, F_j^+), \\ G_j^- = (\min_i T_{ij}, \max_i I_{ij}, \max_i F_{ij}) = (T_j^-, I_j^-, F_j^-), \end{cases} \quad 1 \leq j \leq k. \tag{3.6}$$

Step 4. Based on the proposed distance measure in (3.1). calculate the distance between alternative A_i and the elements in the obtained positive ideal solution P^+ as follows:

$$\begin{aligned} d_i^+ = \sum_{j=1}^{n} G_{ij} - G_j^+ = \frac{1}{n} \sum_{j=1}^{n} [& \frac{\left(\left|T_{ij} - T_j^+\right| + \left|I_{ij} - I_j^+\right| + \left|F_{ij} - F_j^+\right|\right)}{6} \\ & + \frac{\max\left(\left|T_{ij} - T_j^+\right|, \left|I_{ij} - I_j^+\right|, \left|F_{ij} - F_j^+\right|\right)}{3}], \end{aligned} \tag{3.7}$$

also, the degree of distance between the alternative A_i and the elements in the obtained negative ideal solution P^- can be calculated as follows:

$$\begin{aligned} d_i^- = \sum_{j=1}^{n} G_{ij} - G_j^- = \frac{1}{n} \sum_{j=1}^{n} [& \frac{\left(\left|T_{ij} - T_j^-\right| + \left|I_{ij} - I_j^-\right| + \left|F_{ij} - F_j^-\right|\right)}{6} \\ & + \frac{\max\left(\left|T_{ij} - T_j^-\right|, \left|I_{ij} - I_j^-\right|, \left|F_{ij} - F_j^-\right|\right)}{3}], \end{aligned} \tag{3.8}$$

where $1 \leq \mathrm{i} \leq n, 1 \leq \mathrm{j} \leq k$.

Step 5. Compute the relative closeness coefficient to choose the most appropriate and efficient decision by ranking the alternatives as follows:

$$R_i^* = \frac{d_i^-}{d_i^+ + d_i^-}, 1, ..., n. \tag{3.9}$$

Step 6. Utilize the relative closeness coefficients to sort the alternatives. The bigger R_i^* is, the better alternative A_i is.

4. ILLUSTRATIVE EXAMPLE

In this section, an example based on TOPSIS method for MAGDM under the neutrosophic environment is used as a demonstration of the applications and the effectiveness of the proposed decision-making method.

Suppose that there is a panel to compare four car companies A_1, A_2, A_3 and A_4 as the alternatives. Also assume that three attributes such as "Quality (C_1)", "Design (C_2)" and "Price (C_3)". A committee of three decision-makers D_1, D_2 and D_3 has been formed to rank the alternatives and choose the best company. Assume that the decision values of company alternatives A_1, A_2, A_3 and A_4 in regard to the attributes "Quality", "Design" and "Price" given by the decision-makers D_1, D_2 and D_3 based on single-valued neutrosophic numbers, as shown in Table 1, Table 2 and Table 3, respectively.

TABLE 1. The decision values given by D_1

	Quality	Design	Price
A_1	(0.1771,0.5573,0.5013)	(0.1079,0.3390,0.4857)	(0.1932,0.6289,0.9274)
A_2	(0.8296,0.7725,0.4317)	(0.1822,0.2101,0.8944)	(0.8959,0.1015,0.9175)
A_3	(0.7669,0.3119,0.9976)	(0.0991,0.5102,0.1375)	(0.0991,0.3909,0.7136)
A_4	(0.9345,0.1790,0.8116)	(0.4898,0.9064,0.3900)	(0.0442,0.0546,0.6183)

TABLE 2. The decision values given by D_2

	Quality	Design	Price
A_1	(0.3433,0.5493,0.9542)	(0.6465,0.7565,0.2815)	(0.8352,0.9727,0.5906)
A_2	(0.9360,0.3304,0.0319)	(0.8332,0.4139,0.2304)	(0.3225,0.3278,0.6604)
A_3	(0.1248,0.6195,0.3369)	(0.3983,0.4923,0.7111)	(0.5523,0.8378,0.0476)
A_4	(0.7306,0.3606,0.6627)	(0.7498,0.6947,0.6246)	(0.9791,0.7391,0.3488)

TABLE 3. The decision values given by D_3

	Quality	Design	Price
A_1	(0.4513,0.5038,0.3610)	(0.2815,0.4494,0.0839)	(0.1386,0.1892,0.4035)
A_2	(0.2409,0.4896,0.6203)	(0.7311,0.9635,0.9748)	(0.5882,0.6671,0.1220)
A_3	(0.2409,0.8770,0.8112)	(0.1378,0.0423,0.6513)	(0.3662,0.5864,0.2684)
A_4	(0.8562,0.3531,0.0193)	(0.8367,0.9730,0.2312)	(0.8068,0.6751,0.2578)

Therefore the corresponding decision matrices X_1, X_2 and X_3 can be shown as follows, respectively:

$$X_1 = \begin{array}{c} \begin{array}{ccc} \quad C_1 \quad & \quad\quad C_2 \quad\quad & \quad C_3 \quad \end{array} \\ \left[\begin{array}{c} (0.1771, 0.5573, 0.5013), (0.1079, 0.3390, 0.4857), (0.1932, 0.6289, 0.9274) \\ (0.8296, 0.7725, 0.4317), (0.1822, 0.2101, 0.8944), (0.8959, 0.1015, 0.9175) \\ (0.7669, 0.3119, 0.9976), (0.0991, 0.5102, 0.1375), (0.0991, 0.3909, 0.7136) \\ (0.9345, 0.1790, 0.8116), (0.4898, 0.9064, 0.3900), (0.0442, 0.0546, 0.6183) \end{array} \right] \end{array},$$

$$X_2 = \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \left[\begin{array}{l} (0.3433,0.5493,0.9542),(0.6465,0.7565,0.2815),(0.8352,0.9727,0.5906) \\ (0.9360,0.3304,0.0319),(0.8332,0.4139,0.2304),(0.3225,0.3278,0.6604) \\ (0.1248,0.6195,0.3369),(0.3983,0.4923,0.7111),(0.5523,0.8378,0.0476) \\ (0.7306,0.3606,0.6627),(0.7498,0.6947,0.6246),(0.9791,0.7391,0.3488) \end{array} \right] \end{array},$$

$$X_3 = \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \left[\begin{array}{l} (0.4513,0.5038,0.3610),(0.2815,0.4494,0.0839),(0.1386,0.1892,0.4035) \\ (0.2409,0.4896,0.6203),(0.7311,0.9635,0.9748),(0.5882,0.6671,0.1220) \\ (0.2409,0.8770,0.8112),(0.1378,0.0423,0.6513),(0.3662,0.5864,0.2684) \\ (0.8562,0.3531,0.0193),(0.8367,0.9730,0.2312),(0.8068,0.6751,0.2578) \end{array} \right] \end{array}.$$

Suppose that the attribute weights given by three DMs D_1, D_2 and D_3 are shown as follows:

$$W_1 = \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ [(0.2834,\ 0.3900,\ 0.8344),(0.8962,\ 0.4979,\ 0.6096),(0.8266,\ 0.6948,\ 0.5747)] \end{array},$$

$$W_2 = \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ [(0.3260,\ 0.8844,\ 0.6748),(0.4564,\ 0.7209,\ 0.4385),(0.7138,\ 0.0186,\ 0.4378)] \end{array},$$

$$W_3 = \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ [(0.1170,\ 0.2462,\ 0.5466),(0.8147,\ 0.3427,\ 0.5619),(0.3249,\ 0.3757,\ 0.3958)] \end{array}.$$

Furthermore, because the decision group in this example has three members we can consider $W_1 = W_2 = W_3 = \frac{1}{3}$.
The proposed method is currently applied to solve this problem and the computational procedure is summarized as follows:

Step 1. Construct the weighted decision matrices EV_1, EV_2 and EV_3 as follows:

$$EV_1 = \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \left[\begin{array}{l} (0.0502,0.7300,0.9174),(0.0967,0.6681,0.7992),(0.1597,0.8867,0.9691) \\ (0.2351,0.8612,0.9059),(0.1633,0.6034,0.9588),(0.7405,0.7258,0.9649) \\ (0.2173,0.5803,0.9996),(0.0888,0.7540,0.6633),(0.0819,0.8141,0.8782) \\ (0.2648,0.4992,0.9688),(0.4389,0.9530,0.7619),(0.0365,0.7115,0.8377) \end{array} \right] \end{array},$$

$$EV_2 = \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \left[\begin{array}{l} (0.1119,0.9479,0.9851),(0.2951,0.9320,0.5966),(0.5962,0.9732,0.7698) \\ (0.3052,0.9226,0.6852),(0.3803,0.8364,0.5679),(0.2302,0.3403,0.8091) \\ (0.0407,0.9650,0.7908),(0.1818,0.8583,0.8378),(0.3942,0.8408,0.4646) \\ (0.2382,0.9261,0.8903),(0.3422,0.9148,0.7892),(0.6989,0.7439,0.6339) \end{array} \right] \end{array},$$

$$EV_3 = \begin{array}{c} \begin{array}{ccc} \qquad C_1 \qquad & \qquad C_2 \qquad & \qquad C_3 \qquad \end{array} \\ \begin{bmatrix} (0.0528,0.6260,0.7103), & (0.2293,0.6381,0.5987), & (0.0450,0.4938,0.6396) \\ (0.0282,0.6153,0.8278), & (0.5956,0.9760,0.9890), & (0.1911,0.7922,0.4695) \\ (0.0837,0.9073,0.9144), & (0.1122,0.3705,0.8473), & (0.1189,0.7418,0.5580) \\ (0.1002,0.5124,0.5553), & (0.6817,0.9822,0.6632), & (0.2621,0.7972,0.5516) \end{bmatrix} \end{array}.$$

Step 2. Based on the obtained WDMs EV_1, EV_2, EV_3 and addition operator of SVNN shown in (2.2) construct the aggregated group decision matrix of all decision-makers as follows:

$$AG = \begin{array}{c} \begin{array}{ccc} \qquad D_1 \qquad & \qquad D_2 \qquad & \qquad D_3 \qquad \end{array} \\ \begin{bmatrix} (0.0930,0.1442,0.2368), & (0.2491,0.2866,0.1508), & (0.1010,0.0658,0.0907) \\ (0.2780,0.1257,0.2794), & (0.2228,0.0875,0.1049), & (0.2274,0.1586,0.1281) \\ (0.1151,0.1187,0.1941), & (0.1748,0.2300,0.1026), & (0.0944,0.0831,0.1441) \\ (0.2009,0.1128,0.2061), & (0.2830,0.2101,0.1485), & (0.2629,0.1337,0.0677) \end{bmatrix} \end{array}.$$

Step 3. Determine the virtual NPIS and NNIS as:

$$G_j^+ = \left[\ (0.2780,0.1128,0.1941),(0.2830,0.0875,0.1026),(0.2629,0.0658,0.0677)\ \right],$$

$$G_j^- = \left[\ (0.0930,0.1442,0.2794),(0.1748,0.2866,0.1508),(0.0944,0.1586,0.1441)\ \right].$$

Step 4. Calculate the distance of each alternative from NPIS and NNIS, respectively, as follows:

$$\begin{array}{l} d_1^+ = 0.3028,\ d_2^+ = 0.1377,\ d_3^+ = 0.2715,\ d_4^+ = 0.1435, \\ d_1^- = 0.1148,\ d_2^- = 0.2799,\ d_3^- = 0.1246,\ d_4^- = 0.2397. \end{array}$$

Step5. Based on (3.9), the relative closeness coefficient of each candidate can be calculated as follows:

$$R_1^* = 0.2748,\ R_2^* = 0.6703,\ R_3^* = 0.3146,\ R_4^* = 0.6255.$$

Therefore, the ranking order of the four alternatives is A_2, A_4, A_3 and A_1. Obviously, the best selection is A_2.

Remark 2. In recent years, a lot of extended TOPSIS methods have been presented to deal with MAGDM problems that only consider crisp or incomplete information on their calculation. But until now there hasn't been any TOPSIS method to consider and handle indeterminate and inconsistent information that exists commonly in real decision-making problems. In order to overcome this drawback, this paper for the first time presents an extended TOPSIS method for MAGDM problems based on a single-valued neutrosophic set. Although by using the neutrosophic sets we are faced with a large class of problems the proposed method has less calculation and is more flexible for decision making in the real world.

5. CONCLUSION

In general, decision-making problems are included uncertain and imprecise information, and neutrosophic sets can depict this kind of information easier and better. Because TOPSIS is an important decision-making method, and the neutrosophic sets can handle the incomplete, indeterminate and inconsistent data, it is important to establish an extended TOPSIS method based on NSs. In this paper, we first develop a distance measures which is an effective and simple tool to measure the distance between two single-valued neutrosophic numbers and then present an extended TOPSIS method to deal with multi-attribute group decision-making (MAGDM) under neutrosophic environment, where decision-makers express the attribute weights and attribute values for alternatives by using neutrosophic numbers. Although the proposed method presented in this paper is illustrated by a personal selection problem, however, it can also be applied to problems such as information project selection, material selection and many other areas of management decision problems.

REFERENCES

[1] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy sets and systems*, vol. 1, no. 20, pp. 87–96, 1986, doi: 10.1016/S0165-0114(86)80034-3.

[2] K. T. Atanassov, "More on intuitionistic fuzzy sets," *Fuzzy sets and systems*, vol. 33, no. 1, pp. 37–45, 1989, doi: 10.1016/0165-0114(89)90215-7.

[3] K. T. Atanassov, "Intuitionistic fuzzy sets," in *Intuitionistic fuzzy sets*. Springer, 1999, pp. 1–137, doi: 10.1007/978-3-7908-1870-3-1.

[4] A. Balin, "A novel fuzzy multi-criteria decision-making methodology based upon the spherical fuzzy sets with a real case study," *Iranian Journal of Fuzzy Systems*, 2020, doi: 10.22111/ijfs.2020.5234.

[5] S. Benbernou, S. Gala, and M. A. Ragusa, "On the regularity criteria for the 3d magnetohydrodynamic equations via two components in terms of bmo space," *Mathematical Methods in the Applied Sciences*, vol. 37, no. 15, pp. 2320–2325, 2014, doi: 10.1002/mma.2981.

[6] C.-T. Chen, "Extensions of the TOPSIS for group decision-making under fuzzy environment," *Fuzzy sets and systems*, vol. 114, no. 1, pp. 1–9, 2000, doi: 10.1016/S0165-0114(97)00377-1.

[7] P. Chi and P. Liu, "An extended TOPSIS method for the multiple attribute decision making problems based on interval neutrosophic set," *Neutrosophic Sets and Systems*, vol. 1, no. 1, pp. 63–70, 2013, doi: 10.5281/zenodo.571231.

[8] D. Dubois and H. Prade, *Fuzzy sets and systems: theory and applications*. Academic press, 1980, vol. 144, doi: 10.1057/jors.1982.38.

[9] S. Gala, Q. Liu, and M. A. Ragusa, "A new regularity criterion for the nematic liquid crystal flows," *Applicable Analysis*, vol. 91, no. 9, pp. 1741–1747, 2012, doi: 10.1080/00036811.2011.581233.

[10] S. Gala and M. A. Ragusa, "Logarithmically improved regularity criterion for the boussinesq equations in Besov spaces with negative indices," *Applicable Analysis*, vol. 95, no. 6, pp. 1271–1279, 2016, doi: 10.1080/00036811.2015.1061122.

[11] C. Hwang and K. Yoon, *Multiple Attribute Decision Making*. Springer, New York, 1981, vol. 186, doi: 10.1007/978-3-642-48318-9.

[12] A. Javadpour, "Improving resources management in network virtualization by utilizing a software-based network," *Wireless Personal Communications*, vol. 106, no. 2, pp. 505–519, 2019, doi: 10.1007/s11277-019-06176-6.

[13] A. Javadpour, “An optimize-aware target tracking method combining mac layer and active nodes in wireless sensor networks,” *Wireless Personal Communications*, vol. 108, no. 2, pp. 711–728, 2019, doi: 10.1007/s11277-019-06425-8.

[14] F. Jin, L. PD, and X. Zhang, “Evaluation study of human resources based on intuitionistic fuzzy set and TOPSIS method,” *Journal of Information and Computational Science*, vol. 4, no. 3, pp. 1023–1028, 2007.

[15] P. Liu and Y. Su, “The extended TOPSIS based on trapezoid fuzzy linguistic variables,” *Journal of Convergence Information Technology*, vol. 5, no. 4, pp. 38–53, 2010, doi: 10.4156/jcit.vol5.issue4.5.

[16] P. Liu, “Multi-attribute decision-making method research based on interval vague set and TOPSIS method,” *Technological and economic development of economy*, vol. 15, no. 3, pp. 453–463, 2009, doi: 10.3846/1392-8619.2009.15.453-463.

[17] P. Liu, “An extended TOPSIS method for multiple attribute group decision making based on generalized interval-valued trapezoidal fuzzy numbers,” *Informatica*, vol. 35, no. 2, 2011.

[18] P. Liu and Y. Wang, “Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean,” *Neural Computing and Applications*, vol. 25, no. 7-8, pp. 2001–2010, 2014, doi: 10.1007/s00521-014-1688-8.

[19] S. Mohammadi, S. Golara, and N. Mousavi, “Selecting adequate security mechanisms in e-business processes using fuzzy TOPSIS,” *International Journal of Fuzzy System Applications (IJFSA)*, vol. 2, no. 1, pp. 35–53, 2012, doi: 10.4018/ijfsa.2012010103.

[20] A. H. Nafei, S. M. E. P. Azizi, and R. A. Ghasempour, “An approach in solving data envelopment analysis with stochastic data,” pp. 154–162, 2016, doi: 10.1007/978-3-319-66514-6-17.

[21] A. H. Nafei, W. Yuan, and H. Nasseri, “Group multi-attribute decision making based on interval neutrosophic sets,” *Studies in Informatics and Control*, vol. 28, no. 3, pp. 309–316, 2019, doi: 10.24846/v28i3y201907.

[22] A. Nafei, Y. Wenjun, and H. NASSERI, “A new method for solving interval neutrosophic linear programming problems,” *Gazi University Journal of Science*, vol. 33, no. 4, pp. 796–808, 2020, doi: 10.35378/gujs.689125.

[23] M. Olfati, W. Yuan, and S. Nasseri, “An integrated model of fuzzy multi-criteria decision making and stochastic programming for the evaluating and ranking of advanced manufacturing technologies,” *Iranian Journal of Fuzzy Systems*, vol. 17, no. 5, pp. 183–196, 2020, doi: 10.22111/ijfs.2020.5524.

[24] C. Z. Rădulescu and I. C. Rădulescu, “An extended TOPSIS approach for ranking cloud service providers,” *Studies in Informatics and Control*, vol. 26, no. 2, pp. 183–192, 2017, doi: 10.24846/v26i2y201706.

[25] R. Şahin, “Multi-criteria neutrosophic decision making method based on score and accuracy functions under neutrosophic environment,” *arXiv preprint arXiv:1412.5202*, 2014, doi: 10.5281/zenodo.22994.

[26] F. Smarandache, “A unifying field in logics. neutrosophy: Neutrosophic probability, set and logic,” 1999.

[27] F. Smarandache, *A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability: Neutrsophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability*. American Research Press, Rehoboth, 2005.

[28] F. Smarandache, *Multispace & Multistructure. Neutrosophic Transdisciplinarity (100 Collected Papers of Science)*. North-European Scientific Publishers, 2010, vol. 4.

[29] V. Torra and Y. Narukawa, “On hesitant fuzzy sets and decision,” pp. 1378–1382, 2009, doi: 10.1109/FUZZY.2009.5276884.

[30] A. Verma, R. Verma, and N. Mahanti, “Facility location selection: an interval valued intuitionistic fuzzy TOPSIS approach,” *Journal of Modern Mathematics and Statistics*, vol. 4, no. 2, pp. 68–72, 2010, doi: 10.3923/jmmstat.2010.68.72.

[31] Y. Wei and P. Liu, “Risk evaluation method of high-technology based on uncertain linguistic variable and TOPSIS method.” *JCP*, vol. 4, no. 3, pp. 276–282, 2009, doi: 10.4304/jcp.4.3.276-282.

[32] L. A. Zadeh, “Fuzzy sets,” *Information and control*, vol. 3, no. 8, pp. 338–353, 1965, doi: 10.1016/S0019-9958(65)90241-X.

[33] L. A. Zadeh, “The concept of a linguistic variable and its application to approximate reasoning-III,” *Information sciences*, vol. 9, no. 1, pp. 43–80, 1975, doi: 10.1016/0020-0255(75)90017-1.

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CENTRALIZERS OF *BCI*-ALGEBRAS

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Abstract. In this paper, the concept of the centralizer of a subset (an element) of a BCI-algebra by using commutators is given. The connection between commutative ideals with commutators are considered. Also the pseudo center of a BCI-algebra is defined and the relationships between center and pseudo center in BCI-algebras are discussed. Following the concept of the centralizer, we introduce C-closed subalgebras of a BCI-algebra and discuss some characteristics of these subalgebras. Finally, we define central ideal and derived ideal of a BCI-algebra and the relationship between central ideal, derived ideal and other ideals of BCI-algebras are investigated.

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Keywords: BCI-algebras, centralizer, (pseudo) center, C-closed subalgebras, central (derived) ideal

1. INTRODUCTION

K. Iséki introduced the concept of BCI-algebra in 1966 [13]. From then some mathematicians studied and developed many concepts in this algebraic structure, for instance, T. Lei and C. Xi [20] showed that each p-semisimple BCI-algebra can be converted to an Abelian group and conversely each Abelian group is converted to a BCI-algebra. S.A. Bhatti and M.A. Chaudhry introduced the concept of the center of a BCI algebra based on the center of a group [1] and showed that the center of a p-semisimple BCI-algebra is itself. Unlike Abelian group, this is not true in the case of BCI-algebras. For example the center of a BCK-algebra X is $\{0\}$. This motivates us to define pseudo center of a BCI-algebra which not only covers the mentioned deficiencies but offers a number of advantages with respect to the center. The map $\phi(x) = 0 * x$, was formally introduced in [8] for BCH-algebras, but earlier it was used in [4] and [3] to investigate some classes of BCI-algebras connected with groups. In BCI-algebras which are quasigroups, that is, BCI-algebras isotopic to commutative groups [1], any finite subset of such BCI-algebra is an ideal if and only if it is a subgroup of the corresponding group. Any group G in which the square of every element is the identity (i.e. a Boolean group) is a BCI-algebra. In [10] is proved that a BCI-algebra $(X, *, 0)$ is a Boolean group if it has a neutral element (i.e. if

$0*x = x*0 = x$ for all $x \in X$) or if it is associative. Also every para-associative BCI-algebra is a Boolean group [2,6].

Since centralizer and center are two important notions, we extend these two notions to these BCI-algebras and discuss further properties of these concepts. We use the notions of pseudo center and centralizer in BCI-algebras to develop other new concepts such as idealizer and normalizer in these structures. One of the main motivations for defining pseudo center in BCI-algebras is proving similar Lagrange and Sylow theorems if possible. The C-dimension theory, a new and interesting concept has been of interest to many mathematicians recently. The theory has been developed using centralizers in some algebraic structures including groups and rings. The concept of C-dimension in these structures is defined as the length of the longest nested chain of the centralizers, which is closely related to the general theory of groups and rings. This means that if two groups have the same general theory, especially if they are elementary equivalent, they have the same C-dimension. The converse is also correct under certain conditions. Investigation of the concept of C-dimension in BCI-algebras using centralization could be an interesting subject for further studies.

In this paper, we present a definition for the centralizer of an arbitrary element in BCI-algebras on based commutators. We define also the notion of the centralizer of a subset of a BCI-algebra, give several characterizations of it and prove that the class of C-closed subalgebras of a BCI-algebra X is a commutative monoid and a lower semi-lattice. We illustrate also these notions by some examples. Finally, we present the concepts of central ideal and derived ideal of a BCI-algebra and some properties of these notions are investigated. We verify some useful properties of these ideals in BCI-algebras such as relation between central ideal and derived ideal with radical of X.

2. Preliminaries

By a BCI-algebra we mean an algebra $(X,*,0)$ of type $(2,0)$ satisfying the following axioms: for all $x,y,z \in X$,

(BCI1) $((x*y)*(x*z))*(z*y) = 0$,
(BCI2) $(x*(x*y))*y = 0$,
(BCI3) $x*x = 0$,
(BCI4) $x*y = y*x = 0$ implies $x = y$.

A partial ordering $\leq$ on X can be defined by $x \leq y$ if and only if $x*y = 0$. In any BCI-algebra X for all $x,y \in X$, the following hold:

(1) $(x*y)*z = (x*z)*y$,
(2) $x*(x*(x*y)) = x*y$,
(3) $x*0 = x$,
(4) $x \leq y$ imply that $x*z \leq y*z$ and $z*y \leq z*x$,
(5) $(x*z)*(y*z) \leq x*y$.

A BCI-algebra X is said to be p-semisimple if $0*(0*x)=x$, for all $x\in X$. A nonempty subset S of a BCI-algebra X is called a subalgebra of X, if $x*y\in S$ whenever $x,y\in S$. A nonempty subset I of a BCI-algebra X is called an ideal if: (i) $0\in I$ (ii) $x*y\in I$ and $y\in I$ imply $x\in I$ for all $x,y\in X$. An ideal I of a BCI-algebra X is called closed if $0*x\in X$, for all $x\in X$. An element x in a BCI-algebra X is called a positive element if it satisfies $0*x=0$. A BCI-algebra X is called commutative if $x\le y$ implies $x\wedge y=x$, where $x\wedge y=y*(y*x)$, for all $x,y\in X$. A BCI-algebra X is called associative if $x*(y*z)=(x*y)*z$, for all $x,y,z\in X$. Let I be an ideal of a BCI-algebra X, then the relation θ defined by $(x,y)\in\theta$ if and only if $x*y\in I$ and $y*x\in I$ is a congruence relation on X. Let x/I denote the class of $x\in X$, then $0/I=I$. Assume that $X/I=\{x/I : x\in X\}$. Then $(X/I,*,0/I)$ is a BCI-algebra, where $x/I*y/I=(x*y)/I$, for all $x,y\in X$ (see [11, 14]).

In what follows, $(X,*,0)$ or simply X would mean a BCI-algebra, unless otherwise specified.

Definition 1. i) ([20]). The set $\{x\in X : 0*(0*x)=x\}=\{x\in X : \phi^2(x)=x\}$ is called the center of X and is denoted by $C(X)$.

ii) ([1]). An element $x_0\in X$ is said to be an initial element of X, if $x\le x_0$ implies $x=x_0$. Let I_x denote the set of all initial elements of X. We call it the center of X.

iii) ([15–18]). Let x,y be two elements of X. Then the element $((y\wedge x)*(x\wedge y))*(0*(x*y))$ of X is called a pseudo-commutator of x and y and is denoted by $[x,y]$.

iv) ([17]). For nonempty subsets A and B of X the commutator of A and B is the set of all finite $*$-products of commutators of kind $[a,b]$ with $a\in A$ and $b\in B$.

$$[A,B]=\{[a_{i1},b_{j_1}]*[a_{i2},b_{j_2}]*\ldots*[a_{in},b_{j_n}] : a_{ik}\in A, b_{j_l}\in B, n\in N\}.$$

When $A=B=X$, $[X,X]$ is called the commutator subalgebra or the derived subalgebra of X and denoted by $X^{'}$. Therefore

$$X^{'}=\{x_1*x_2*\ldots*x_n : n\ge 1,\ \text{each } x_i \text{ is a pseudo-commutator in } X\}.$$

v) ([11]). An element x of X is a nilpotent element if $0*x^n=0$ for some positive integer n, where $x*y^n=\underbrace{(\ldots((x*y)*y)*\ldots)*y}_{n-times}$. If every x in X is nilpotent, then X is called a nilpotent BCI-algebra. For every positive integer k, we define $N_k(X)=\{x\in X : 0*x^k=0\}$. The intersection of all maximal ideals of a BCI-algebra X is called the radical of X and is denoted by $Rad(X)$.

Theorem 1 ([11])**.** *A closed ideal I of X is a commutative ideal if and only if the quotient algebra X/I is a commutative BCI-algebra.*

Theorem 2 ([15, 17])**.** *X is commutative if and only if $X^{'}=\{0\}$.*

Theorem 3 ([11])**.** *Let S be a nonempty subset of X and let*

$$A = \{x \in X : (...((x * a_1) * a_2) * ...) * a_n = 0, \text{ for some } a_1, a_2, ..., a_n \in S\}.$$

Then $< S >= A \cup \{0\}$. Especially, if S contains a positive element of X, or if S contains a nilpotent element of X, then $< S >= A$.

Lemma 1. i) ([17]). Let f be a homomorphism from X to a BCI-algebra Y. Then $f([x,y]) = [f(x), f(y)]$, for all $x, y \in X$.
ii) ([18]). $[x,y]$ is a positive element of X, for all $x, y \in X$.
iii) ([11]). $0 * (x * y) = (0 * x) * (0 * y)$, for all $x, y \in X$, i.e., ϕ is a homomorphism.

Theorem 4 ([11])**.** *The following two types of abstract systems are equivalent: p-semisimple BCI-algebra and Abelian group.*

In the following lemma we show that the center of derived subalgebra $C(X^{'})$ is always $\{0\}$.

Lemma 2. i) *$x \in C(X)$ if and only if $a * (a * x) = x$, for all $a \in X$.*
ii) *If $x \in C(X)$, then $\phi(y * x) = x * y$, for all $y \in X$.*
iii) *$C(X^{'}) = \{0\}$.*

Proof. i) Let $x \in C(X)$. Then $0 * (0 * x) = x$. Since $(a * (a * x)) * x = 0$, $a * (a * x) \leq x$. Conversely, $(a * (a * x)) * x = 0$, then $0 * ((a * (a * x)) * x) = 0 * 0 = 0$. Hence $(0 * (a * (a * x))) * (0 * x) = 0$. Therefore $(0 * (0 * x)) * (a * (a * x)) = 0$. Whence $x * (a * (a * x)) = 0$. Hence $x \leq (a * (a * x))$. Then $a * (a * x) = x$, for all $a \in X$. If $a * (a * x) = x$, for any $a \in X$, then $0 * (0 * x) = x$. Therefore $x \in C(X)$.
ii) Suppose that $x \in C(X)$. Then

$$\phi(y * x) = 0 * (y * x) = (0 * y) * (0 * x) = (0 * (0 * x)) * y = x * y.$$

iii) Let $x \in C(X^{'})$. Then $x \in X^{'}$ and $0 * (0 * x) = x$. Thus there exist $a_i, b_i \in X$ such that $x = \prod[a_i, b_i]$. Hence

$$\begin{aligned} x &= 0 * (0 * x) = 0 * (0 * \prod [a_i, b_i]) \\ &= (0 * (0 * [a_1, b_1])) * (0 * (0 * [a_2, b_2])) * ... * (0 * (0 * [a_n, b_n])) = 0 * ... * 0 = 0. \end{aligned}$$

Therefore, $C(X^{'}) = \{0\}$. □

Theorem 5. *Let $x, y \in X$. Then*
i) $\phi([x,y]) = 0$,
ii) $[\phi(x), \phi(y)] = 0$.

Proof. i) We first show that $[x,y] * y \leq \phi(y)$.

$$\begin{aligned} [x,y] * y &= (((x * (x * y)) * (y * (y * x))) * (0 * (x * y))) * y \\ &\leq ((y * (y * (y * x))) * (0 * (x * y))) * y \\ &= ((y * x) * (0 * (x * y))) * y = ((y * x) * y) * (0 * (x * y)) \end{aligned}$$

$$= ((y*y)*x)*(0*(x*y)) = (0*x)*((0*x)*(0*y))$$
$$\leq 0*y = \phi(y).$$

Therefore $0 = ([x,y]*y)*(0*y) \leq [x,y]*0 = [x,y]$. So $\phi([x,y]) = 0*[x,y] = 0$.
ii) By substitute ϕ for f in Lemma 1 we obtain $0 = \phi([x,y]) = [\phi(x), \phi(y)]$. □

W. A. Dudek presents a new method for studying the ideals and centralizer of 0 element based on the map ϕ in BCI/BCH/BCC-algebra $(X,*,0)$ and some useful facts on these notion are proved in [2–9]. He defines the centralizer of 0 element in X by $Z_0 = \{x \in X : x*0 = 0*x\} = \{x \in X : \phi(x) = x\}$. It is shown that if $(X,*,0)$ is a p-semisimple BCI-algebra, then $(X,.,0)$ is an Abelian group, where $x.y = x*(0*y)$ for $x,y \in X$ and conversely, if $(X,.,0)$ is an Abelian group, then $(X,*,0)$ with $x*y = x.y^{-1}$ is a p-semisimple BCI-algebra [3]. So we expect that the centralizer of 0 to be a fixed element such as neutral element in Abelian groups. But in the following example we see that this is not true in general.

Example 1. We consider Abelian group $(Z_3,+,0)$ and adjoint p-semisimple BCI-algebra $(Z_3^{ad},*,0)$ of it with the following Cayley table:

$+$	0	a	b
0	0	a	b
a	a	b	0
b	b	0	a

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

By definition of centralizer of an element in Abelian group $(Z_3,+,0)$ as neutral element 0 we have $Z_0 = \{x \in Z_3 : x+0 = 0+x\} = Z_3$ and in adjoint p-semisimple BCI-algebra $(Z_3^{ad},*,0)$ we obtain $Z_0 = \{x \in Z_3 : x*0 = 0*x\} = \{0\}$.

We are trying to resolve this disagreement. By attention to $\phi(x)$, we would like to consider centralizer from another perspective.

3. Centralizer of a subset in BCI-algebras

In this section, we introduce the notion of centralizer of a subset of BCI-algebras by using commutators and study it in detail.

Definition 2. Suppose that S is a nonempty subset of X. The centralizer of S in X is defined to be $\{x \in X : [x,s] = [s,x] = 0, \forall s \in S\}$ and denoted by $C_X(S)$.

When $S = \{x\}$ is a singleton set, then $C_X(\{x\})$ can be abbreviated to $C_X(x)$. Symbolically,

$$C_X(x) = \{y \in X : [x,y] = [y,x] = 0\}.$$

$C_X(x)$ is a nonempty set, because $[x,0] = [0,x] = [x,x] = 0$, for any $x \in X$. Specifically, $0, x \in C_X(x)$.

Example 2. Let $X = \{0,a,b,c,d\}$ be a *BCI*-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	c	c
a	a	0	0	c	c
b	b	b	0	d	c
c	c	c	c	0	0
d	d	d	c	b	0

By simple calculations we obtain $C_X(0) = C_X(c) = C_X(d) = X$, $C_X(a) = \{0,a,c,d\}$ and $C_X(b) = \{0,b,c,d\}$. For $S = \{0,a,b\}$ we obtain $C_X(S) = \{0,c,d\}$.

Now we describe the relation between centralizer of a set with centralizer of constituent elements.

Theorem 6. *For subset S of X, $C_X(S) = \bigcap_{a\in S} C_X(a)$.*

Proof.

$$\begin{aligned} x \in C_X(S) &\Leftrightarrow \forall a \in S, [x,a] = [a,x] = 0 \\ &\Leftrightarrow \forall a \in S, x \in C_X(a) \\ &\Leftrightarrow x \in \bigcap_{a\in S} C_X(a). \end{aligned}$$

□

Lemma 3. *Suppose that I is an ideal of X and $a,b \in X$. Then*

i) $C_X(0) = X$,
ii) $a \in C_X(b)$ *iff* $b \in C_X(a)$,
iii) $C_X(a)/I \subseteq C_{X/I}(a/I)$.

Proof. i) $C_X(0) = \{x \in X : [x,0] = [0,x] = 0\} = X$.
ii) $a \in C_X(b)$ if and only if $[a,b] = [b,a] = 0$ if and only if $b \in C_X(a)$.
iii) Let $b/I \in C_X(a)/I$. Then $b \in C_X(a)$ and hence $[a,b] = [b,a] = 0$. Therefore $[a,b]/I = [b,a]/I = 0/I$. But $[a,b]/I = [a/I,b/I]$ and $[b,a]/I = [b/I,a/I]$. Then $[a/I,b/I] = [b/I,a/I] = 0/I$. So $b/I \in C_{X/I}(a/I)$. Hence $C_X(a)/I \subseteq C_{X/I}(a/I)$. □

Lemma 4. *Let $f \in Aut(X)$. Then $f(C_X(a)) = C_X(f(a))$, for every $a \in X$.*

Proof. Let $y \in f(C_X(a))$. Then there exists $x \in C_X(a)$ such that $y = f(x)$. Since $x \in C_X(a)$, $[x,a] = [a,x] = 0$. Therefore, $f([x,a]) = f([a,x]) = f(0) = 0$ and hence $[f(x), f(a)] = [f(a), f(x)] = 0$. Thus $y = f(x) \in C_X(f(a))$. i.e., $f(C_X(a)) \subseteq C_X(f(a))$. If $y = f(x) \in C_X(f(a))$, then $[f(x), f(a)] = [f(a), f(x)] = 0$. Therefore $f([x,a]) = f([a,x]) = f(0) = 0$. But f is one to one, then $[x,a] = [a,x] = 0$. Hence $x \in C_X(a)$

and so $y = f(x) \in f(C_X(a))$, that is, $C_X(f(a)) \subseteq f(C_X(a))$. Thus, $f(C_X(a)) = C_X(f(a))$. □

Corollary 1. *For subset S of X and $f \in Aut(X)$, $f(C_X(S)) = C_X(f(S))$.*

Proof.

$$\begin{aligned} f(C_X(S)) &= f(\bigcap_{a\in S} C_X(a)) \\ &= \bigcap_{a\in S} f(C_X(a)) \\ &= \bigcap_{a\in S} C_X(f(a)) \\ &= C_X(f(S)). \end{aligned}$$

□

In the following two theorems, some of the properties of operator C_X, such as symmetry and decreasing are examined.

Theorem 7. *Suppose that S,T are two subsets of X. Then*

i) *$S \subseteq C_X(T)$ iff $T \subseteq C_X(S)$,*

ii) *If $S \subseteq T$, then $C_X(T) \subseteq C_X(S)$.*

Proof. i) Let $S \subseteq C_X(T)$ and let $t \in T$. To show that $t \in C_X(S)$ we must show that for all $s \in S$, $[t,s] = [s,t] = 0$. Suppose that s is an arbitrary element of S. Therefore $s \in C_X(T)$. By definition $[t,s] = [s,t] = 0$, for every $t \in T$. Hence $t \in C_X(S)$. By symmetry if $T \subseteq C_X(S)$, then we see $S \subseteq C_X(T)$.

ii) Let $S \subseteq T$. If $x \in C_X(T)$, then $[x,t] = [t,x] = 0$, for all $t \in T$. Since $S \subseteq T$, for all $s \in S$, $[x,s] = [s,x] = 0$. Hence $x \in C_X(S)$. □

Theorem 8. *Let S be a subset of X. Then*

i) *$S \subseteq C_X(C_X(S))$,*

ii) *$C_X(C_X(C_X(S))) = C_X(S)$.*

Proof. i) Let $x \in S$. Then $x \in C_X(C_X(S))$ iff for all $s \in C_X(S)$ we have $[x,s] = [s,x] = 0$. Let s be an arbitrary element of $C_X(S)$. Then $[s,s'] = [s',s] = 0$, for all $s' \in S$. Since $x \in S$, $[x,s] = [s,x] = 0$. Therefore $S \subseteq C_X(C_X(S))$.

ii) Since $S \subseteq C_X(C_X(S))$, it follows that $C_X(C_X(C_X(S))) \subseteq C_X(S)$. Also we obtain $C_X(S) \subseteq C_X(C_X(C_X(S)))$ by putting $C_X(S)$ instead of S in (i). Hence $C_X(S) = C_X(C_X(C_X(S)))$. □

Remark 1. $C_X(C_X(S))$ contains S but is not necessarily equal, also $C_X(S)$ need not contain S. For the set $S = \{0,a,b\}$ from Example 2, we obtain $C_X(S) = \{0,c,d\}$ that is not contain S. Also $C_X(C_X(S)) = X \neq S$.

Definition 3. A subalgebra S of X is said to be C-closed if $S = C_X(C_X(S))$.

We denote by $\mathbb{C}(X)$ the set of all C-closed subalgebras of X. Since $C_X(C_X(X))$ contains X, $C_X(C_X(X)) = X$. Then $\mathbb{C}(X)$ is nonempty.

Obvious that if $C_X(S)$ is a subalgebra of X, then $C_X(S)$ is C-closed, because $C_X(S) = C_X(C_X(C_X(S)))$.

Example 3. Let $X = \{0,a,b,c,d\}$ be a *BCI*-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	0
b	b	a	0	b	a
c	c	c	c	0	c
d	d	d	d	d	0

X has 14 subalgebra, but only 4 until of their are C-closed. The sets $S_1 = \{0,c\}$, $S_2 = \{0,c,d\}$, $S_3 = \{0,a,b,c\}, S_4 = X$ are subalgebras of X such that

$$C_X(C_X(S_1)) = S_1,\ C_X(C_X(S_2)) = S_2,\ C_X(C_X(S_3)) = S_3 \text{ and } C_X(C_X(S_4)) = S_4.$$

Hence $\mathbb{C}(X) = \{\{0,c\},\ \{0,c,d\}, \{0,a,b,c\}, X\}$.

Now, we move to the study of C-closed subalgebras of BCI-algebras and it consequences.

Theorem 9. *If S,T are C-closed subalgebras of X, then $S\cap T$ is a C-closed subalgebra of X.*

Proof. Let S,T be C-closed subalgebras of X. Clearly, $S\cap T$ is a subalgebra of X. Since $C_X(C_X(S\cap T))$ contains $S\cap T$, it is sufficient to show that $C_X(C_X(S\cap T)) \subseteq (S\cap T)$. But $S\cap T \subseteq S$ and $S\cap T \subseteq T$, then $C_X(S) \subseteq C_X(S\cap T)$ and $C_X(T) \subseteq C_X(S\cap T)$. Hence $C_X(C_X(S\cap T)) \subseteq C_X(C_X(S)) = S$ and $C_X(C_X(S\cap T)) \subseteq C_X(C_X(T)) = T$. That means $C_X(C_X(S\cap T)) \subseteq (S\cap T)$. □

This proves that the intersection of any two C-closed subalgebra of X is again an C-closed subalgebra of X. The above theorem can be generalized to intersection of any family of C-closed subalgebra of X.

Remark 2. A C-closed subalgebra of a C-closed subalgebra is again C-closed.

Corollary 2. *If X is commutative, then X is only subalgebra that is C-closed.*

Proof. Let X be commutative. Then for every proper subalgebra S of X, $C_X(S) = X$. Therefore $C_X(C_X(S)) = C_X(X) = X \neq S$, for every proper subalgebra S of X. If $S = X$, then $C_X(C_X(X)) = C_X(X) = X$. □

The following example shows that the converse of Corollary 2 is not correct in general.

Example 4. Let $X = \{0,a,b,c,d\}$ be a *BCI*-algebra in which $*$ operation is defined by the following table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	0
c	c	a	a	0	0
d	d	c	b	a	0

By routine calculations we obtain $\mathbb{C}(X) = \{X\}$. But X is not commutative, because $b \wedge c = a \neq c \wedge b = b$.

$(\mathbb{C}(X), \cap, X)$ is a commutative monoid. $\mathbb{C}(X)$ is closed under $\cap$ and for any $S \in \mathbb{C}(X)$, $S \cap X = S$. Moreover the operation $\cap$ is commutative and associative. Also $(\mathbb{C}(X), \cap)$ forms a lower semi-lattice with respect to $\subseteq$. Indeed, $(\mathbb{C}(X), \subseteq)$ is a partially ordered set and for any $S, T \in \mathbb{C}(X)$ we have $inf\{S,T\} = S \cap T$.

In the following lemma we examine the conditions under which the converse of the Theorem 7 (ii) is also true.

Lemma 5. *If $S \in \mathbb{C}(X)$ and $C_X(S) \subseteq C_X(T)$, then $T \subseteq S$.*

Proof. Suppose that $C_X(S) \subseteq C_X(T)$. Then $C_X(C_X(T)) \subseteq C_X(C_X(S))$. Since $S \in \mathbb{C}(X)$, $C_X(C_X(S)) = S$, hence $T \subseteq C_X(C_X(T)) \subseteq S$. □

4. The pseudo center of BCI-algebras

In this section, at first, we recall that the center of a BCI-algebra is defined in several different ways such as Definition 1, but with common results. However, the logical reasons for this definitions is not clear, it is famed that in algebraic structures including groups, rings and Lie algebras the notion of center is defined based on commutators [10, 19]. These motivate us to introduce a new notion of center in BCI-algebras without using the commutators. This concept is different from the center of BCI-algebras previously defined but it is consistent with the center of other mentioned algebras. The new proposed center definition is more general and reliable and is called pseudo center in this manuscript.

Definition 4. The set $\{x \in X : [x,y] = [y,x] = 0, \ \forall y \in X\}$ is called the pseudo center of X and is denoted by $Z(X)$.

Obviously, $0 \in Z(X)$.

Example 5. i) For Example 2, $Z(X) = \{0,c,d\}$, $C(X) = \{0,c\}$ and $X' = \{0,a\}$.
ii) Let $X = \{0,a,b,c,d\}$ be a *BCI*-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	b	0
c	c	a	c	0	c
d	d	d	d	d	0

By simple calculations we obtain $Z(X) = \{0,a,c\}$, $C(X) = \{0\}$ and $X^{'} = \{0,b\}$. Indeed, $Z(X)$ is commutative part of X, and $X^{'}$ evaluates non commutative part and commutative part of X from each other.

Theorem 10. *X is commutative if and only if $Z(X) = X$.*

Proof. Let X be commutative and $x \in X$. Then $[x,y] = [y,x] = 0$, for every $y \in X$. So $x \in Z(X)$ and hence $X \subseteq Z(X)$. Obviously $Z(X) \subseteq X$. Thus $Z(X) = X$.

Conversely, let $Z(X) = X$. Then $[x,y] = [y,x] = 0$, for all $x,y \in X$. Therefore X is a commutative BCI-algebra. □

Corollary 3. *The following conditions are equivalent:*

i) *X is commutative,*
ii) *$Z(X) = X$,*
iii) *$X^{'} = \{0\}$.*

Proof. i) $\leftrightarrow$ ii) follows directly from Theorem 10.
ii) $\leftrightarrow$ iii) Let $Z(X) = X$. Then X is commutative. By Theorem 6, $X^{'} = \{0\}$.
Conversely, let $X^{'} = \{0\}$. Therefore, X is commutative. Hence $Z(X) = X$. □

Remark 3. $Z(X)$ is neither an ideal nor a subalgebra of X, in general. From Example 4 by routine calculations we obtain that $Z(X) = \{0,a,d\}$ is not a subalgebra of X because $a,d \in Z(X)$ but $d*a = c \notin Z(X)$ also $Z(X)$ is not an ideal of X because $c*a \in Z(X)$ and $a \in Z(X)$, but $c \notin Z(X)$. Also, $C_X(I)$ for ideal I is not an ideal (a subalgebra) of X, in general. For instance, in Example 4, X is an ideal of X, but $C_X(X) = Z(X)$ is not an ideal (a subalgebra) of X.

In the following proposition we describe the relationship between center and pseudo center in BCI-algebras.

Proposition 1. $C(X) \subseteq Z(X)$.

Proof. Suppose that $x \in C(X)$. Then $0*(0*x) = x$. We must show that for all $y \in X$, $[x,y] = [y,x] = 0$. But

$$\begin{aligned}[x,y]c &= ((x*(x*y))*(y*(y*x)))*(0*(x*y)) = ((x*(x*y))*(x))*(0*(x*y)) \\ &= ((x*x)*(x*y))*(0*(x*y)) = (0*(x*y))*(0*(x*y)) = 0.\end{aligned}$$

Also

$$\begin{aligned}[y,x] &= ((y*(y*x))*(x*(x*y)))*(0*(y*x)) = (x*(x*(x*y)))*(0*(y*x))\\ &= (x*y)*(x*y) = 0.\end{aligned}$$

Therefore $x \in Z(X)$. Hence $C(X) \subseteq Z(X)$. □

Remark 4. In Example 4 we see $Z(X) = \{0,a,d\}$ and $C(X) = \{0\}$. Then the equality of Proposition 1 does not hold, in general.

As immediate consequences of Definition 2 and Definition 4 we obtain:

Theorem 11. *Suppose that $a,b \in X$ and X_1, X_2 are subsets of X. Then*

i) *$a \in Z(X)$ if and only if $C_X(a) = X$,*
ii) *$Z(X) = \bigcap_{a\in X} C_X(a)$,*
iii) *If $X_1 \subseteq X_2$, then $Z(X_2) \subseteq Z(X_1)$,*
iv) *$C_X(a) \subseteq C_X(b)$ if and only if $b \in Z(C_X(a))$,*
v) *$Z(X) \subseteq C_X(a)$, for every $a \in X$,*
vi) *X is commutative if and only if $C_X(a) = X$, for every $a \in X$,*
vii) *$C_X(X) = Z(X)$,*
viii) *$C_X(a) = C_X(b)$ if and only if $Z(C_X(a)) = Z(C_X(b))$.*

Proof. i) Let $a \in Z(X)$. Then for all $x \in X$, $[x,a] = [a,x] = 0$. Obviously, $C_X(a) \subseteq X$. Now, let $x \in X$. Thus $[x,a] = [a,x] = 0$. Therefore $x \in C_X(a)$, that is, $X \subseteq C_X(a)$. Hence $C_X(a) = X$. Conversely, let $C_X(a) = X$. Since $C_X(a) = \{b \in X : [a,b] = [b,a] = 0\} = X$, it follows that $a \in Z(X)$.

ii) Let $x \in Z(X)$. Then $[x,a] = [a,x] = 0$ for every $a \in X$. So $x \in C_X(a)$, for every $a \in X$. That means $x \in \bigcap_{a\in X} C_X(a)$. Therefore $Z(X) \subseteq \bigcap_{a\in X} C(a)$.
Conversely, let $x \in \bigcap_{a\in X} C_X(a)$. Then $x \in C_X(a)$ for every $a \in X$. Hence $[x,a] = [a,x] = 0$ for every $a \in X$. Then $x \in Z(X)$. So $\bigcap_{a\in X} C_X(a) \subseteq Z(X)$.

iii) Let $x \in Z(X_2)$. Then for every $y \in X_2$, $[x,y] = [y,x] = 0$. Since $X_1 \subseteq X_2$, for every $y \in X_1$ we have $[x,y] = [y,x] = 0$. Therefore $x \in Z(X_1)$.

iv) Let $C_X(a) \subseteq C_X(b)$. Then for any $x \in C_X(a)$, $x \in C_X(b)$. Hence $[x,b] = [b,x] = 0$, for any $x \in C_X(a)$. Therefore $b \in Z(C_X(a))$. Conversely, let $b \in Z(C_X(a))$ and $x \in C_X(a)$. Therefore $[x,b] = [b,x] = 0$. Hence $x \in C_X(b)$. Then $C_X(a) \subseteq C_X(b)$.

v) Since $Z(X) = \bigcap_{a\in X} C_X(a)$, it follows that $Z(X) \subseteq C_X(a)$, for any $a \in X$.

vi) X is commutative iff $Z(X) = X$ iff $X = \bigcap_{a\in X} C_X(a)$ iff $X = C_X(a)$, for every $a \in X$.

vii) $C_X(X) = \{x \in X : [x,y] = [y,x] = 0, \text{ for all } y \in X\} = Z(X)$.

viii) Obviously, if $C_X(a) = C_X(b)$, then $Z(C_X(a)) = Z(C_X(b))$. Conversely, let $Z(C_X(a)) = Z(C_X(b))$. Since $a \in Z(C_X(a))$, $a \in Z(C_X(b))$. Then $C_X(b) \subseteq C_X(a)$. Similarly, since $b \in Z(C_X(b))$, $b \in Z(C_X(a))$, then $C_X(a) \subseteq C_X(b)$. Therefore $C_X(a) = C_X(b)$. □

Theorem 12. *Let I be an ideal of X. Then $Z(X)/I \subseteq Z(X/I)$.*

Proof. Let $x/I \in Z(X)/I$. Then $x \in Z(X)$ and hence for every $y \in X$, $[x,y] = [y,x] = 0$. Therefore $[x,y]/I = [y,x]/I = 0/I$. Hence $[x/I, y/I] = [y/I, x/I] = 0/I$, for every $y/I \in X/I$. So $x/I \in Z(X/I)$. Thus $Z(X)/I \subseteq Z(X/I)$. □

The following example shows that the equality of Theorem 12 may not hold.

Example 6. Let $X = \{0,1,2,3,4,5\}$ be a *BCI*-algebra with the Cayley table as follows:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	1	0	1	5
2	2	2	0	2	0	5
3	3	3	3	0	0	5
4	4	3	4	1	0	5
5	5	5	5	5	5	0

By simple calculations we obtain $Z(X) = \{0,5\} = C(X)$. For ideal $I = \{0,2\}$ of X we have $X/I = \{0/I, 1/I, 3/I, 4/I, 5/I\}$. By routine calculus we obtain $Z(X/I) = \{0/I, 4/I, 5/I\}$ and $Z(X)/I = \{0/I, 5/I\}$. Then $Z(X)/I \subsetneq Z(X/I)$.

Lemma 6. *$f(Z(X)) \subseteq Z(X)$ and $f(C(X)) \subseteq C(X)$, for every $f \in Aut(X)$.*

Proof. Let $y \in f(Z(X))$. Then there exists $x \in Z(X)$ such that $y = f(x)$. Since $x \in Z(X)$, $[x,a] = [a,x] = 0$, for all $a \in X$. Therefore, $f[x,a] = f[a,x] = f(0) = 0$ and hence $[f(x), f(a)] = [f(a), f(x)] = 0$. Since $f \in Aut(X)$, $y = f(x) \in Z(X)$. Hence $f(Z(X)) \subseteq Z(X)$.

If $y \in f(C(X))$, then exists $x \in C(X)$ such that $y = f(x)$. Since $x \in C(X)$, $0*(0*x) = x$. Therefore $f(x) = f(0*(0*x)) = f(0)*(f(0)*f(x)) = 0*(0*f(x))$. Thus $y \in C(X)$. Hence $f(C(X)) \subseteq C(X)$. □

Proposition 2. i) *If X is a p-semisimple, then $Z(X) = X$.*
ii) *If X is an associative, then $Z(X) = X$.*

Proof. i) Let X be a p-semisimple. Since $C(X) \subseteq Z(X)$ and $C(X) = X$ we obtain $Z(X) = X$.

ii) Suppose that X is associative, then for any $x, y \in X$

$$\begin{aligned}
[x,y] &= ((x*(x*y))*((y*(y*x))))*(0*(x*y)) \\
&= (((x*x)*y)*((y*y)*x))*(x*y) \\
&= ((0*y)*(0*x))*(x*y) \\
&= (y*x)*(x*y) = (y*x)*(y*x) = 0.
\end{aligned}$$

Similarly, $[y,x] = 0$. Then $Z(X) = X$. □

In the following example we show that the converse of Proposition 2 is generally not correct.

Example 7. Let $X = \{0, a, b\}$ with $(*)$ be defined by the following table:

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	a	0

X is a commutative BCI-algebra. Therefore $Z(X) = X$. But X is not associative because $a = a*(a*b) \neq (a*a)*b = 0$. Also X is not p-semisimple because $0*(0*a) \neq a$.

Lemma 7. *Let X, Y be two BCI-algebras. Then $Z(X \times Y) = Z(X) \times Z(Y)$.*

Proof. Let $(x,y) \in Z(X \times Y)$. Then for every $(a,b) \in X \times Y$, we obtain

$$[(x,y),(a,b)] = [(a,b),(x,y)] = (0,0).$$

But

$$\begin{aligned}
&[(x,y),(a,b)] \\
&= (((x,y)*((x,y)*(a,b)))*((a,b)*((a,b)*(x,y))))*((0,0)*((x,y)*(a,b))) \\
&= ((x*(x*a), y*(y*b))*(a*(a*x), b*(b*y)))*(0*(x*a), 0*(y*b)) \\
&= ((x*(x*a)*(a*(a*x))), ((y*(y*b))*(b*(b*y))))*(0*(x*a), 0*(y*b)) \\
&= ((x*(x*a)*(a*(a*x))*(0*(x*a))), ((y*(y*b))*(b*(b*y)))*(0*(y*b))) \\
&= ([x,a],[y,b]).
\end{aligned}$$

Since $[(x,y),(a,b)] = [(a,b),(x,y)] = (0,0)$, $([x,a],[y,b]) = ([a,x],[b,y]) = (0,0)$. Therefore, $[x,a] = [a,x] = 0$, for any $a \in X$ and $[y,b] = [b,y] = 0$, for any $b \in Y$. Hence $x \in Z(X)$ and $y \in Z(Y)$. Then $(x,y) \in Z(X) \times Z(Y)$. That means $Z(X \times Y) \subseteq Z(X) \times Z(Y)$.

Conversely, let $(x,y) \in Z(X) \times Z(Y)$. Then $x \in Z(X)$ and $y \in Z(Y)$. Therefore, for every $a \in X$, $[x,a] = [a,x] = 0$ and for every $b \in Y$, $[y,b] = [b,y] = 0$. Thus $[(x,y),(a,b)] = ([x,a],[y,b]) = (0,0)$ and $[(a,b),(x,y)] = ([a,x],[b,y]) = (0,0)$. So $(x,y) \in Z(X \times Y)$. Hence $Z(X) \times Z(Y) \subseteq Z(X \times Y)$. Whence $Z(X \times Y) = Z(X) \times Z(Y)$. □

Lemma 8. *Let S be a subalgebra of X. Then*

i) $S \subseteq Z(Z(S))$,
ii) $Z(Z(Z(S))) = Z(S)$.

Proof. i) Let $s \in S$. Then for any $x \in Z(S)$ we get $[x,s] = [s,x] = 0$. Hence $s \in Z(Z(S))$. Therefore $S \subseteq Z(Z(S))$.

ii) Since $S \subseteq Z(Z(S))$, it follows that $Z(Z(Z(S))) \subseteq Z(S)$. Also by putting $Z(S)$ instead of S in (i) we obtain $S \subseteq Z(Z(Z(S)))$. Hence $Z(Z(Z(S))) = Z(S)$. □

Theorem 13. $Z(X) \subseteq \bigcap_{S\in\mathbb{C}(X)} S$.

Proof. Since $C_X(S) \subseteq X$, for any $S \in \mathbb{C}(X)$, we obtain $C_X(X) \subseteq C_X(C_X(S))$. Hence for any $S \in \mathbb{C}(X)$, $C_X(X) \subseteq S$. Then $Z(X) = C_X(X) \subseteq \bigcap_{S\in\mathbb{C}(X)} S$. □

Theorem 14. *Suppose that S is a nonempty subset of X. Then the following conditions are equivalent:*

i) $C_X(S) = X$,
ii) $S \subseteq Z(X)$,
iii) $[S,X] = [X,S] = \{0\}$.

Proof. i)↔ii) Let $C_X(S) = X$ and $s \in S$. Since $s \in X = C_X(S)$ we obtain $[x,s] = [s,x] = 0$, for every $x \in X$. Therefore $s \in Z(X)$. Conversely, let $S \subseteq Z(X)$. Then $C_X(Z(X)) \subseteq C_X(S)$. But $C_X(Z(X)) = X$ and hence $X \subseteq C_X(S)$. Obviously, $C_X(S) \subseteq X$. Then $C_X(S) = X$.

ii)$leftrightarrow$iii) Let $S \subseteq Z(X)$ and $t \in [S,X]$. Then $t = \prod[s_i,x_i]$ such that $s_i \in S$ and $x_i \in X$. Since $S \subseteq Z(X)$, $s_i \in Z(X)$ and so $t = [s_i,x_i] = 0$. Therefore $[S,X] = \{0\}$. Similarity, $[X,S] = \{0\}$. Conversely, let $[S,X] = [X,S] = \{0\}$ and let $s \in S$. Therefore $[s,x] = [x,s] = 0$, for all $x \in X$. Then $s \in Z(X)$. Hence $S \subseteq Z(X)$. □

5. CENTRAL IDEAL AND DERIVED IDEAL

In this section, we introduce the notions of central ideal and derived ideal and investigate the relation between commutative ideals and the derived subalgebra, central ideal and derived ideal and others ideals of BCI-algebras.

Definition 5. The generated ideal by $Z(X)$ is called the central ideal of X and is denoted by $CI(X)$. i.e.,

$$CI(X) =< Z(X) >= \bigcap_{Z(X)\subseteq I} I.$$

Also the generated ideal by $X^{'}$ is called the derived ideal of X and is denoted by $DI(X)$. i.e.,

$$DI(X) =< X^{'} >= \bigcap_{X^{'}\subseteq I} I,$$

where I is any ideal of X.

Example 8. i) In Example 7 we have $CI(X) = X = DI(X)$ and for Example 5 (ii), we have $CI(X) = \{0,a,c\}, DI(X) = \{0,b\}$.

ii) Let $X = \{0,a,b,c,d\}$ be a *BCI*-algebra in which $*$ operation is defined by the following table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	0
b	b	a	0	b	0
c	c	c	c	0	c
d	d	d	d	d	0

By routine calculations, we obtain $Z(X) = \{0,c\}$. The central ideal of X is $CI(X) = \{0,c\}$. Also $X^{'} = \{0,a,b\}$. Therefore $DI(X) =< X^{'} >= \{0,a,b\}$.

Theorem 15. *Let I be an ideal of X. Then the following hold:*

i) *X/I is commutative if and only if $X^{'} \subseteq I$,*
ii) *$X/DI(X)$ is commutative,*
iii) *If $DI(X) = X^{'}$, then $X^{'}$ is smallest ideal of X such that corresponding quotient algebra is commutative.*

Proof. i) see Theorem 5.1 in [17].
ii) Since $X^{'} \subseteq DI(X)$ by (i) $X/DI(X)$ is a commutative BCI-algebra.
iii) Let $DI(X) = X^{'}$, then $X/DI(X) = X/X^{'}$ is commutative. Now, let I be an ideal of X such that X/I is commutative and $I \subseteq X^{'}$. Since X/I is commutative by (i) $X^{'} \subseteq I$. Therefore, $X^{'} = I$. Hence $X^{'}$ is smallest ideal of X such that corresponding quotient algebra is commutative. □

Remark 5. Since $X^{'}$ is a subalgebra of X, $DI(X) =< X^{'} >$ is a closed ideal [3]. Then $X/DI(X)$ is a commutative BCI-algebra. Hence $DI(X)$ is a commutative ideal. The converse of this statement is not correct, for example X is a commutative ideal of X but is not a derived ideal of X, generally.

The following example shows that the central ideals and derived ideals are different from the other ideals, in general.

Example 9. i) Generally, a central ideal is neither commutative nor positive implicative ideal of X. The central ideal from Example 3 is $CI(X) = \{0,c\}$. Since $a*d = 0 \in CI(X)$ but $d*(a*(a*d)) = d \notin CI(X)$, $CI(X)$ is not a commutative ideal. Also $(b*a)*a = 0 \in CI(X)$ and $a*a = 0 \in CI(X)$ but $b*a = a \notin CI(X)$. Therefore $CI(X)$ is not a positive implicative ideal. Also $(a*(d*a))*c = 0 \in CI(X)$ and $c \in CI(X)$ but $a \notin CI(X)$. Therefore, $CI(X)$ is not an implicative ideal. Since, $b \wedge d = d*(d*b) = d*d = 0 \in CI(X)$ but $d \wedge b = b*(b*d) = a \notin CI(X)$, $CI(X)$ is not a normal ideal. $CI(X)$ is not prime ideal of X, because $b \wedge d = d*(d*b) = d*d = 0 \in CI(X)$ but neither $b \in CI(X)$ and nor $d \in CI(X)$.

ii) Let $X = \{0, a, b, c, d\}$ be a *BCI*-algebra in which $*$ operation is defined as follows

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0

By routine calculations, the central ideal of X is $CI(X) = \{0\}$. Also $X^{'} = \{0, a, b\}$. Therefore, the derived ideal of X is $DI(X) = \{0, a, b\}$. In this example $CI(X) = \{0\}$ is not a maximal ideal of X, because $CI(X) \subsetneq I = \{0, a\}$. Also $CI(X) = \{0, a\} \cap \{0, b\}$ but $CI(X) \neq \{0, a\}$ and $CI(X) \neq \{0, b\}$. Then $CI(X)$ is not an irreducible ideal of X. Since $a, b \notin CI(X)$ and neither $a * b \in CI(X)$ and nor $b * a \in CI(X)$, $CI(X)$ is not an obstinate ideal of X.

iii) In Example 3, X is an implicative, commutative, positive implicative, prime, obstinate, maximal, Varlet, irreducible and normal ideal but is not central ideal of X (for more details see [3, 12]).

Theorem 16. *Let X be a BCI-algebra. Then*

i) $CI(X) = \{x \in X : (...((x * a_1) * a_2) * ...) * a_n = 0,$ *for some* $a_1, \ldots, a_n \in Z(X)\}$.
ii) $DI(X) = \{x \in X : (...((x * a_1) * a_2) * ...) * a_n = 0,$ *for some* $a_1, \ldots, a_n \in X^{'}\}$.

Proof. Since 0 is a positive element of X and $0 \in Z(X)$ and $0 \in X^{'}$ by Theorem 3 (i) and (ii) holds. □

Theorem 17. *Suppose that I is a closed ideal of X. Then I is a commutative ideal if and only if* $[x, y] \in I$, *for all* $x, y \in X$.

Proof. Let I be a closed ideal of X. I is a commutative ideal if and only if X/I is a commutative BCI-algebra if and only if $X^{'} \subseteq I$ if and only if $[x, y] \in I$, for all $x, y \in X$. □

In the following theorem we consider a condition under which the equality of Theorem 12 is correct.

Theorem 18. *Let I be a commutative closed ideal of X and* $I \cap X^{'} = \{0\}$. *Then* $I \subseteq Z(X)$ *and so* $Z(X/I) = Z(X)/I$.

Proof. Let I be a commutative closed ideal of X and let $x \in I$. Then $[x, y] \in I$, for every $y \in X$. Since $[x, y] \in X^{'}$ for every $x, y \in X$, $[x, y] \in I \cap X^{'} = \{0\}$. Therefore, $[x, y] = 0$. Similarity, $[y, x] = 0$. Thus $[x, y] = [y, x] = 0$. Hence $x \in Z(X)$. That is $I \subseteq Z(X)$. But $Z(X/I) = \{x/I : [x/I, y/I] = [y/I, x/I] = 0/I$, for all $y/I \in X/I\} =$

$\{x/I : [x,y]/I = [y,x]/I = 0/I$, for all $y \in X\} = \{x/I : [x,y],[y,x] \in I$, for all $y \in X\}$. Since $[x,y],[y,x] \in X^{'}$, for all $x,y \in X$, then the recent set is equal $\{x/I : [x,y],[y,x] \in I \cap X^{'}$, for all $y \in X\} = \{x/I : [x,y] = [y,x] = 0$, for all $y \in X\} = \{x/I : x \in Z(X)\} = Z(X)/I$. □

Theorem 19. *If I is a commutative closed ideal of X, then I is a normal ideal of X.*

Proof. Let I be a commutative closed ideal of X and $x \wedge y \in I$. Then $[y,x] = (y \wedge x) * (x \wedge y) \in I$ for all $x,y \in X$. But I is an ideal, then $(y \wedge x) \in I$.
Conversely, let $y \wedge x \in I$. Since $[x,y] = (x \wedge y) * (y \wedge x) \in I$ and I is an ideal of X, then $x \wedge y \in I$. Therefore, I is a normal ideal of X. □

In the following example we show that the converse of Theorems 18, 19 are generally not correct.

Example 10. Let $X = \{0,a,b\}$. Define a binary operation $(*)$ on X by

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	b	0

X is a BCI-algebra. The set $I = \{0\}$ is a closed ideal of X. With simple calculations we obtain $Z(X) = \{0\}$ and $X^{'} = \{0,a\}$. Also $I \cap X^{'} = \{0\}$ and $I \subseteq Z(X)$ but I is not a commutative ideal, because $[a,b] = a \notin I$.

In the Wronski algebra [3] $I = \{0\}$ is a normal ideal of X but is not a commutative ideal.

Remark 6. The set $N_k(X) = \{x \in X : 0 * x^k = 0\}$, where k is a fixed natural number is a commutative closed ideal of X [3]. Then $X/N_k(X)$ is a commutative BCI-algebra. Hence by Theorem 15, $X^{'} \subseteq N_k(X)$ and so $DI(X) =< X^{'} >\subseteq< N_k(X) >= N_k(X)$. Therefore, any pseudo commutator element of X is a nilpotent element of X. Also by Corollary 3, we have $Z(X/N_k(X)) = X/N_k(X)$.

In the last theorem a relationship between $Rad(X)$ and $CI(X)$, $DI(X)$ is expressed and proved.

Theorem 20. *$CI(X) \cap DI(X) \subseteq Rad(X)$.*

Proof. It is sufficient to show that every maximal ideal M of X is contains $CI(X) \cap DI(X)$. Since M is a maximal ideal, $DI(X) \subseteq M$ or $X = M \times DI(X)$, because if $DI(X) \nsubseteq M$, then $M \subsetneq M \times DI(X)$, by the maximality of M in this case $X = M \times DI(X)$. If $DI(X) \subseteq M$, then $CI(X) \cap DI(X) \subseteq M$. In the case $X = M \times DI(X)$, $X/M \cong DI(X)$. Then X/M is a commutative BCI-algebra and hence $X^{'} \subseteq M$. Therefore $DI(X) =< X^{'} >\subseteq< M >= M$. Hence $CI(X) \cap DI(X) \subseteq M$. □

Open problem: Under what conditions dose the equality in Theorem 20 hold?

6. Conclusion

In groups, rings, lie algebras, monoids and semigroups the centralizer of a subset is the set of all elements such that commute with all elements of them set and the normalizer are elements that satisfy a weaker condition. This article presented the centralizer of a subset of BCI-algebras as well as the concept of pseudo center of BCI-algebras. The results of this paper show that:

i) X is commutative iff $Z(X) = X$ iff $X^{'} = \{0\}$.
ii) The pseudo center of X is exactly $C_X(X)$ and X is commutative if and only if $C_X(X) = Z(X) = X$.
iii) For subsets S,T of X, $T \subseteq C_X(S)$ if and only if $S \subseteq C_X(T)$.
iv) $C_X(C_X(S))$ contains S, but $C_X(S)$ need not contain S.
v) $CI(X) \cap DI(X) \subseteq Rad(X)$.

Some important topics for future work are:

i) The concept of C-dimension in BCI-algebras using centralization could be an interesting subject for studies.
ii) Making normalizer of a subset S of X such that $N_X(X) = X$ and for singleton sets, $N_X(a) = C_X(a)$, for $a \in X$.
iii) Find subalgebra of X to which $X/CI(X)$ (also $X/DI(X)$) is isomorphic.
iv) Using commutators to construct idealizer of an ideal I of X.

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References

[1] S. A. Bhatti and M. A. Chaudhry, "Ideals in BCI-algebras," *Internat. J. Math. Ed. Sci. Tech.*, vol. 21, no. 4, pp. 637–643, 1990, doi: 10.1080/0020739900210419.

[2] W. A. Dudek, X. Zhang, and Y. Wang, "Ideals and atoms of *BZ*-algebras," *Math. Slovaca*, vol. 59, no. 4, pp. 387–404, 2009, doi: 10.2478/s12175-009-0135-2.

[3] W. A. Dudek, "On some BCI-algebras with the condition (S)," *Math. Japon.*, vol. 31, no. 1, pp. 25–29, 1986.

[4] W. A. Dudek, "On group-like BCI-algebras," *Demonstratio Math.*, vol. 21, no. 2, pp. 369–376, 1988.

[5] W. A. Dudek, "Solid weak BCC-algebras," *Int. J. Comput. Math.*, vol. 88, no. 14, pp. 2915–2925, 2011, doi: 10.1080/00207160.2011.582103.

[6] W. A. Dudek and Y. B. Jun, "Quasi *p*-ideals of quasi BCI-algebras," *Quasigroups Related Systems*, vol. 11, pp. 25–38, 2004.

[7] W. A. Dudek, B. Karamdin, and S. A. Bhatti, "Branches and ideals of weak BCC-algebras," *Algebra Colloq.*, vol. 18, no. Special Issue 1, pp. 899–914, 2011, doi: 10.1142/S1005386711000782.

[8] W. A. Dudek and J. Thomys, "On decompositions of BCH-algebras," *Math. Japon.*, vol. 35, no. 6, pp. 1131–1138, 1990.

[9] W. A. Dudek and J. Thomys, "On some generalizations of BCC-algebras," *Int. J. Comput. Math.*, vol. 89, no. 12, pp. 1596–1616, 2012, doi: 10.1080/00207160.2012.692782.
[10] K. H. Hofmann and S. A. Morris, *The Lie theory of connected pro-Lie groups*, ser. EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich, 2007, vol. 2, doi: 10.4171/032.
[11] Y. Huang, *BCI-algebra*. Elsevier, 2006.
[12] Y. Imai and K. Iséki, "On axiom systems of propositional calculi. XIV," *Proc. Japan Acad.*, vol. 42, pp. 19–22, 1966.
[13] K. Iséki, "An algebra related with a propositional calculus," *Proc. Japan Acad.*, vol. 42, pp. 26–29, 1966.
[14] K. Iséki, "On BCI-algebras," *Math. Sem. Notes Kobe Univ.*, vol. 8, no. 1, pp. 125–130, 1980.
[15] A. Najafi and A. Borumand Saeid, "Solvable BCK-algebras," *Cankaya University Journal of Science and Engineering*, vol. 11, pp. –, 2014.
[16] A. Najafi, "Pseudo-commutators in BCK-algebras," *Pure Mathematical Science*, vol. 2, no. 1, pp. 29–32, 2013.
[17] A. Najafi, A. Borumand Saeid, and E. Eslami, "Commutators in BCI-algebras," *Journal of Intelligent & Fuzzy Systems*, vol. 31, no. 1, pp. 357–366, 2016.
[18] A. Najafi, E. Eslami, and A. Borumand Saeid, "A new type of nilpotent BCI-algebras," *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)*, vol. 64, no. 2, pp. 309–326, 2018.
[19] K. C. O'Meara, J. Clark, and C. I. Vinsonhaler, *Advanced topics in linear algebra: Weaving matrix problems through the Weyr form*. Oxford University Press, Oxford, 2011.
[20] L. H. Shi, "An axiom system of BCI-algebras," *Math. Japon.*, vol. 30, no. 3, pp. 351–352, 1985.

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ESSENTIAL RADICAL SUPPLEMENTED MODULES

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Abstract. In this work, (amply) essential radical supplemented modules are defined and some properties of these modules are investigated. Let M be an R-module and $M = M_1 + M_2 + \cdots + M_n$. If M_i is essential radical supplemented for every $i = 1, 2, \ldots, n$, then M is also essential radical supplemented. It is proved that every factor module and every homomorphic image of an essential radical supplemented module are essential radical supplemented. Let M be an essential radical supplemented R-module. Then every finitely M-generated R-module is essential radical supplemented.

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1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let M be an R-module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*) *submodule* of M and denoted by $N \ll M$. A submodule N of an R-module M is called an *essential submodule* of M and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equivalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. Let M be an R-module and K be a submodule of M. K is called a *generalized small* (briefly, *g-small*) *submodule* of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, then we write $K \ll_g M$. It is clear that every small submodule is a generalized small submodule but the converse is not true generally. Let M be an R-module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is called a *supplemented module* if every submodule of M has a supplement in M. Let M be an R-module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a supplement V' with $V' \leq V$, we say U has *ample supplements* in M. If every submodule of M has *ample supplements* in M, then M is called an *amply supplemented module*. If every essential submodule of M has a supplement in M, then M is called an *essential supplemented* (or briefly, *e-supplemented*) *module*. If every essential submodules of M has ample supplements in M, then M is called an *amply essential supplemented* (or briefly, *amply e-supplemented*) *module*. Let M be an $R-$module and $U,V \leq M$. If $M = U+V$ and $M = U+T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U+V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. The intersection of all maximal submodules of an R-module M is called the *radical* of M and denoted by $\mathrm{Rad}M$. If M have no maximal submodules, then we denote $\mathrm{Rad}M = M$. The intersection of essential maximal submodules of an R-module M is called the *generalized radical* of M and denoted by $\mathrm{Rad}_g M$. If M have no essential maximal submodules, then we denote $\mathrm{Rad}_g M = M$. Let M be an R-module and $U,V \leq M$. If $M = U+V$ and $U \cap V \leq \mathrm{Rad}V$, then V is called a *generalized* (*radical*) *supplement* (or briefly, *Rad-supplement*) of U in M. M is called a *generalized* (*radical*) *supplemented* (or briefly, *Rad-supplemented*) *module* if every submodule of M has a Rad-supplement in M. Let M be an R-module and $U \leq M$. If for every $V \leq M$ such that $M = U+V$, U has a Rad-supplement V' with $V' \leq V$, we say U has *ample Rad-supplements* in M. If every submodule of M has ample Rad-supplements in M, then M is called an *amply generalized* (*radical*) *supplemented* (or briefly, *amply Rad-supplemented*) *module*. Let M be an R-module. We say submodules X and Y of M are β^* *equivalent*, $X\beta^*Y$, if and only if $Y+K = M$ for every $K \leq M$ such that $X+K = M$ and $X+T = M$ for every $T \leq M$ such that $Y+T = M$. Let M be an R-module $X \leq Y \leq M$. If $Y/X \ll M/X$, then we say Y *lies above* X in M.

More information about (amply) supplemented modules are in [3, 9, 10] and [11]. More information about (amply) essential supplemented modules are in [5, 6]. More results about g-small submodules and g-supplemented modules are in [4, 7]. The definitions of (amply) generalized supplemented modules and some properties of them are in [8, 10]. Some properties of (amply) generalized supplemented modules are also in [2]. The definition of β^* equivalence relation and some properties of this relation are in [1].

In this paper, we define (amply) essential radical supplemented modules and investigate some properties about these modules. We constitute relationships between essential radical supplemented modules and amply essential radical supplemented modules by Proposition 3 and Proposition 4. We also constitute relationships between essential radical supplemented modules and π-projective modules by Lemma 12. We give two examples for essential radical supplemented modules separating with essential supplemented modules at the end of this paper.

Lemma 1. *Let M be an R-module and $K \leq N \leq M$. If K is a generalized small submodule of N, then K is a generalized small submodule in submodules of M which contain N.*

Proof. See [4, Lemma 1 (2)]. □

Lemma 2. *Let M be an R-module. Then* $\mathrm{Rad}_g M = \sum_{L \ll_g M} L$.

Proof. See [4, Lemma 5 and Corollary 5]. □

Lemma 3. *Let V be a Rad-supplement of U in M. Then* $\mathrm{Rad}V = V \cap \mathrm{Rad}M$.

Proof. Let T be any maximal submodule of V. Since

$$M/(U+T) = (U+T+V)/(U+T) \cong V/(U \cap V + T) = V/T,$$

then $U+T$ is a maximal submodule of M. Hence $\mathrm{Rad}M \leq U+T$ and $V \cap \mathrm{Rad}M \leq U \cap V + T = T$. Thus $V \cap \mathrm{Rad}M \leq \mathrm{Rad}V$ and since $\mathrm{Rad}V \leq V \cap \mathrm{Rad}M$, $\mathrm{Rad}V = V \cap \mathrm{Rad}M$. □

2. Essential radical supplemented modules

Definition 1. Let M be an R-module. If every essential submodule of M has a Rad-supplement in M, then M is called an *essential radical supplemented* (or briefly, *e-Rad-supplemented*) *module*.

Clearly we see that every essential supplemented module is essential radical supplemented. But the converse is not true in general. (See Examples 1 and 2).

Definition 2. Let M be an R-module and $X \leq M$. If X is a Rad-supplement of an essential submodule in M, then X is called an *essential radical supplement* (or briefly, e-*Rad-supplement*) *submodule* in M.

Lemma 4. *Let M be an R-module, V be an e-Rad-supplement in M and $x \in V$. Then $Rx \ll_g M$ if and only if $Rx \ll_g V$.*

Proof. ($\Longrightarrow$) Let $Rx \ll_g M$. Since V is an e-Rad-supplement in M, there exists $U \trianglelefteq M$ such that V is a Rad-supplement of U in M. Let $Rx + T = V$ with $T \trianglelefteq V$. Then $M = U + V = U + T + Rx$, and since $Rx \ll_g M$ and $(U+T) \trianglelefteq M$, $U + T = M$. Let $x = u + t$ with $u \in U$ and $t \in T$. Since $x, t \in V$, then $u = x - t \in V$. Then $V = Rx + T \leq Ru + Rt + T = Ru + T \leq V$ and $Ru + T = V$. Since $u \in U \cap V \leq \mathrm{Rad}V$, then $Ru \ll V$ and $T = V$. Hence $Rx \ll_g V$.
($\Longleftarrow$) Clear from Lemma 1. □

Corollary 1. *Let M be an R-module and V be an e-Rad-supplement in M. Then* $\mathrm{Rad}_g V = V \cap \mathrm{Rad}_g M$.

Proof. Let $x \in \mathrm{Rad}_g V$. Here $Rx \ll_g V$ and by Lemma 1, $Rx \ll_g M$. Then by Lemma 2, $Rx \leq \mathrm{Rad}_g M$ and $x \in V \cap \mathrm{Rad}_g M$.
Let $y \in V \cap \mathrm{Rad}_g M$. Then $y \in V$ and $Ry \ll_g M$. By Lemma 4, $Ry \ll_g V$. By Lemma 2, $Ry \leq \mathrm{Rad}_g V$ and $y \in \mathrm{Rad}_g V$.
Hence $\mathrm{Rad}_g V = V \cap \mathrm{Rad}_g M$. □

Proposition 1. *Let M be an essential radical supplemented module. Then $M/\mathrm{Rad}M$ have no proper essential submodules.*

Proof. Let $\frac{K}{\text{Rad}M}$ be any essential submodule of $\frac{M}{\text{Rad}M}$. Since $\frac{K}{\text{Rad}M} \trianglelefteq \frac{M}{\text{Rad}M}$, $K \trianglelefteq M$ and since M is essential radical supplemented, K has a Rad-supplement V in M. Then $M = K + V$ and $K \cap V \leq \text{Rad}V$. Since $M = K + V$, $\frac{M}{\text{Rad}M} = \frac{K}{\text{Rad}M} + \frac{V+\text{Rad}M}{\text{Rad}M}$. Since $K \cap V \leq \text{Rad}M$, then $\frac{K}{\text{Rad}M} \cap \frac{V+\text{Rad}M}{\text{Rad}M} = \frac{K \cap V+\text{Rad}M}{\text{Rad}M} = 0$ and $\frac{M}{\text{Rad}M} = \frac{K}{\text{Rad}M} \oplus \frac{V+\text{Rad}M}{\text{Rad}M}$. Since $\frac{M}{\text{Rad}M} = \frac{K}{\text{Rad}M} \oplus \frac{V+\text{Rad}M}{\text{Rad}M}$ and $\frac{K}{\text{Rad}M} \trianglelefteq \frac{M}{\text{Rad}M}$, $\frac{K}{\text{Rad}M} = \frac{M}{\text{Rad}M}$. Hence $\frac{M}{\text{Rad}M}$ have no proper essential submodules. □

Lemma 5. *Let M be an R-module, U be an essential submodule of M and $M_1 \leq M$. If M_1 is e-Rad-supplemented and $U + M_1$ has a Rad-supplement in M, then U has a Rad-supplement in M.*

Proof. Let X be a Rad-supplement of $U + M_1$ in M. Then $M = U + M_1 + X$ and $X \cap (U + M_1) \leq \text{Rad}X$. Since $U \trianglelefteq M$, $(U + X) \trianglelefteq M$ and $(U + X) \cap M_1 \trianglelefteq M_1$. Since M_1 is e-Rad-supplemented, $(U + X) \cap M_1$ has a Rad-supplement Y in M_1. This case $M_1 = (U + X) \cap M_1 + Y$ and $(U + X) \cap Y = (U + X) \cap M_1 \cap Y \leq \text{Rad}Y$. Then $M = U + M_1 + X = U + X + (U + X) \cap M_1 + Y = U + X + Y$ and $U \cap (X + Y) \leq (U + X) \cap Y + (U + Y) \cap X \leq (U + M_1) \cap X + (U + X) \cap Y \leq \text{Rad}X + \text{Rad}Y \leq \text{Rad}(X + Y)$. Hence $X + Y$ is a Rad-supplement of U in M. □

Corollary 2. *Let M be an R-module, U be an essential submodule of M and $M_i \leq M$ for every $i = 1, 2, \ldots, n$. If M_i is e-Rad-supplemented for every $i = 1, 2, \ldots, n$ and $U + M_1 + M_2 + \cdots + M_n$ has a Rad-supplement in M, then U has a Rad-supplement in M.*

Proof. Clear from Lemma 5. □

Lemma 6. *Let $M = M_1 + M_2$. If M_1 and M_2 are e-Rad-supplemented, then M is also e-Rad-supplemented.*

Proof. Let $U \trianglelefteq M$. Then 0 is a Rad-supplement of $U + M_1 + M_2$ in M. Since M_2 is e-Rad-supplemented and $(U + M_1) \trianglelefteq M$, by Lemma 5, $U + M_1$ has a Rad-supplement in M. Since M_1 is e-Rad-supplemented and $U \trianglelefteq M$, by Lemma 5, U has a Rad-supplement in M. Hence M is e-Rad-supplemented. □

Corollary 3. *Let $M = M_1 + M_2 + \cdots + M_n$. If M_i is e-Rad-supplemented for each $i = 1, 2, \ldots, n$, then M is also e-Rad-supplemented.*

Proof. Clear from Lemma 6. □

Lemma 7. *Every factor module of an e-Rad-supplemented module is e-Rad-supplemented.*

Proof. Let M be an e-Rad-supplemented $R-$module and $\frac{M}{K}$ be any factor module of M. Let $\frac{U}{K} \trianglelefteq \frac{M}{K}$. Then $U \trianglelefteq M$ and since M is e-Rad-supplemented, U has a Rad-supplement V in M. Since $K \leq U$, by the proof of [8, Proposition 2.6(1)], $\frac{V+K}{K}$ is a Rad-supplement of $\frac{U}{K}$ in $\frac{M}{K}$. Hence $\frac{M}{K}$ is e-Rad-supplemented. □

Corollary 4. *Every homomorphic image of an e-Rad-supplemented module is e-Rad-supplemented.*

Proof. Clear from Lemma 7. □

Lemma 8. *Let M be an e-Rad-supplemented R-module. Then every finitely M-generated R-module is e-Rad-supplemented.*

Proof. Let N be a finitely M-generated R-module. Then there exist a finite index set Λ and an R-module epimorphism $f: M^{(\Lambda)} \longrightarrow N$. Since M is e-Rad-supplemented, by Corollary 3, $M^{(\Lambda)}$ is e-Rad-supplemented. Then by Corollary 4, N is e-Rad-supplemented. □

Proposition 2. *Let R be a ring. Then ${}_RR$ is essential radical supplemented if and only if every finitely generated R-module is essential radical supplemented.*

Proof. Clear from Lemma 8. □

Lemma 9. *Let M be an R-module. If every essential submodule of M is β^* equivalent to an e-Rad-supplement submodule in M, then M is essential radical supplemented.*

Proof. Let U be an essential submodule of M. By hypothesis there exists an e-Rad-supplement submodule X in M such that $U\beta^*X$. Since X is an e-Rad-supplement submodule in M, there exists an essential submodule Y of M such that X is a Rad-supplement of Y in M. This case $M = X+Y$ and $X \cap Y \leq \text{Rad}X$. Since $Y \trianglelefteq M$, by hypothesis, there exists an e-Rad-supplement submodule V in M such that $Y\beta^*V$. Since $U\beta^*X$ and $M = X+Y$, then $M = U+Y$ and since $Y\beta^*V$, $M = U+V$. Let $x \in U \cap V$ and $Rx+T = M$ with $T \leq M$. Then $U \cap V + T = M$ and since $M = U+V$, $M = U+V\cap T = X+V\cap T$. Since $M = V+T = X+V\cap T$, $M = V+X\cap T$. Then by $Y\beta^*V$, $M = Y+X\cap T$. Since $M = X+T = Y+X\cap T$, $M = X\cap Y+T$. Let $x = y+t$, with $y \in X\cap Y$ and $t \in T$. Since $Rx+T = M$, $Ry+T = M$ also holds. By $y \in X\cap Y \leq \text{Rad}X \leq \text{Rad}M$, $Ry \ll M$ and since $Ry+T = M$, $T = M$. Hence $Rx \ll M$ and $x \in \text{Rad}M$. Since V is a Rad-supplement in M, then by Lemma 3, $V\cap \text{Rad}M = \text{Rad}V$. Since $x \in V$ and $x \in \text{Rad}M$, $x \in V\cap\text{Rad}M = \text{Rad}V$ and $U\cap V \leq \text{Rad}V$. Hence V is a Rad-supplement of U in M and M is essential radical supplemented. □

Corollary 5. *Let M be an R-module. If every essential submodule of M lies above an e-Rad-supplement submodule in M, then M is essential radical supplemented.*

Proof. Clear from Lemma 9. □

3. Amply essential radical supplemented modules

Definition 3. Let M be an R-module. If every essential submodule has ample Rad-supplements in M, then M is called an *amply essential radical supplemented* (or briefly, *amply e-Rad-supplemented*) *module*.

Lemma 10. *Let M be an amply e-Rad-supplemented module. Then every factor module of M is amply e-Rad-supplemented.*

Proof. Let M/K be any factor module of M, $U/K \trianglelefteq M/K$ and $U/K+V/K = M/K$ with $V/K \leq M/K$. Since $U/K \trianglelefteq M/K$, $U \trianglelefteq M$. Since $U/K+V/K = M/K$, $U+V = M$. Because M is amply e-Rad-supplemented, U has a Rad-supplement $V^{'}$ in M with $V^{'} \leq V$. By the proof of [8, Proposition 2.6(1)], $\frac{V^{'}+K}{K}$ is a Rad-supplement of $\frac{U}{K}$ in $\frac{M}{K}$. In addition to this, $\frac{V^{'}+K}{K} \leq \frac{V}{K}$. Hence M/K is amply e-Rad-supplemented. □

Corollary 6. *Let M be an amply e-Rad-supplemented module. Then every homomorphic image of M is amply e-Rad-supplemented.*

Proof. Clear from Lemma 10. □

Lemma 11. *Let M be an R-module. If every submodule of M is e-Rad-supplemented, then M is amply e-Rad-supplemented.*

Proof. Let $M = U+V$ with $U \trianglelefteq M$ and $V \leq M$. By hypothesis, V is e-Rad-supplemented. Since $U \trianglelefteq M$, $U \cap V \trianglelefteq V$. Since V is e-Rad-supplemented, $U \cap V$ has a Rad-supplement K in V. Here $U \cap V + K = V$ and $U \cap K = U \cap V \cap K \leq \mathrm{Rad}K$. Then $M = U+V = U+U \cap V+K = U+K$ and $U \cap K \leq \mathrm{Rad}K$. Hence M is amply e-Rad-supplemented. □

Proposition 3. *Let R be any ring. Then every R-module is e-Rad-supplemented if and only if every R-module is amply e-Rad-supplemented.*

Proof. $(\Longrightarrow)$ Let M be any R-module. Since every R-module is e-Rad-supplemented, every submodule of M is e-Rad-supplemented. Then by Lemma 11, M is amply e-Rad-supplemented.
$(\Longleftarrow)$ Clear. □

Lemma 12. *Let M be a π-projective and e-Rad-supplemented R-module. Then M is amply e-Rad-supplemented.*

Proof. Let $U \trianglelefteq M$, $M = U+V$ and X be a Rad-supplement of U in M. Since M is π-projective and $M = U+V$, there exists an R-module homomorphism $f: M \to M$ such that $\mathrm{Im}f \subset V$ and $\mathrm{Im}(1-f) \subset U$. So, we have $M = f(M)+(1-f)(M) = f(U)+f(X)+U = U+f(X)$. Suppose that $a \in U \cap f(X)$. Since $a \in f(X)$, then there exists $x \in X$ such that $a = f(x)$. Since $a = f(x) = f(x)-x+x = x-(1-f)(x)$ and $(1-f)(x) \in U$, we have $x = a+(1-f)(x) \in U$. Thus $x \in U \cap X$ and so, $a = f(x) \in f(U \cap X)$. Therefore we have $U \cap f(X) \leq f(U \cap X) \leq f(\mathrm{Rad}X) \leq \mathrm{Rad}f(X)$. This means that $f(X)$ is a Rad-supplement of U in M. Moreover, $f(X) \subset V$. Therefore M is amply e-Rad-supplemented. □

Corollary 7. *If M is a projective and e-Rad-supplemented module, then M is an amply e-Rad-supplemented module.*

Proof. Clear from Lemma 12. □

Proposition 4. *Let R be a ring. The following assertions are equivalent.*

(i) ${}_RR$ *is e-Rad-supplemented*
(ii) ${}_RR$ *is amply e-Rad-supplemented.*
(iii) Every finitely generated R-module is e-Rad-supplemented.
(iv) Every finitely generated R-module is amply e-Rad-supplemented.

Proof. $(i) \Longleftrightarrow (ii)$ Clear from Corollary 7, since ${}_RR$ is projective.

$(i) \Longrightarrow (iii)$ Clear from Lemma 8.

$(iii) \Longrightarrow (iv)$ Let M be a finitely generated R-module. Then there exist a finite index set Λ and an R-module epimorphism $f\colon R^{(\Lambda)} \longrightarrow M$. Since every finitely generated R-module is e-Rad-supplemented, $R^{(\Lambda)}$ is e-Rad-supplemented. Since ${}_RR$ is projective, $R^{(\Lambda)}$ is also projective. Then by Corollary 7, $R^{(\Lambda)}$ is amply e-Rad-supplemented. Since $f\colon R^{(\Lambda)} \longrightarrow M$ is an R-module epimorphism, by Corollary 6, M is also amply e-Rad-supplemented.

$(iv) \Longrightarrow (i)$ Clear. □

Example 1. Consider the $\mathbb{Z}$-module $\mathbb{Q}$. Since $Rad\mathbb{Q} = \mathbb{Q}$, ${}_{\mathbb{Z}}\mathbb{Q}$ is essential radical supplemented. But, since ${}_{\mathbb{Z}}\mathbb{Q}$ is not supplemented and every nonzero submodule of ${}_{\mathbb{Z}}\mathbb{Q}$ is essential in ${}_{\mathbb{Z}}\mathbb{Q}$, ${}_{\mathbb{Z}}\mathbb{Q}$ is not essential supplemented.

Example 2. Consider the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}_p$ for a prime p. It is easy to check that $Rad\,(\mathbb{Q} \oplus \mathbb{Z}_p) = \mathbb{Q} \neq \mathbb{Q} \oplus \mathbb{Z}_p$. Since $\mathbb{Q}$ and $\mathbb{Z}_p$ are essential radical supplemented, by Lemma 6, $\mathbb{Q} \oplus \mathbb{Z}_p$ is essential radical supplemented. But $\mathbb{Q} \oplus \mathbb{Z}_p$ is not essential supplemented.

References

[1] G. F. Birkenmeier, F. Takil Mutlu, C. Nebiyev, N. Sokmez, and A. Tercan, "Goldie*-supplemented modules," *Glasg. Math. J.*, vol. 52, no. A, pp. 41–52, 2010, doi: 10.1017/S0017089510000212.

[2] E. Büyükaşik and C. Lomp, "On a recent generalization of semiperfect rings," *Bull. Aust. Math. Soc.*, vol. 78, no. 2, pp. 317–325, 2008, doi: 10.1017/S0004972708000774.

[3] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting modules: Supplements and projectivity in module theory*, ser. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[4] B. Koşar, C. Nebiyev, and N. Sökmez, "g-supplemented modules," *Ukrainian Math. J.*, vol. 67, no. 6, pp. 975–980, 2015, translation of Ukraïn. Mat. Zh. **67** (2015), no. 6, 861–864, doi: 10.1007/s11253-015-1127-8.

[5] C. Nebiyev, H. H. Ökten, and A. Pekin, "Essential supplemented modules," *International Journal of Pure and Applied Mathematics*, vol. 120, no. 2, pp. 253–257, 2018, doi: 10.12732/ijpam.v120i2.9.

[6] C. Nebiyev, H. H. Ökten, and A. Pekin, "Amply essential supplemented modules," *Journal of Scientific Research and Reports*, vol. 21, no. 4, pp. 1–4, 2018, doi: 10.9734/JSRR/2018/45651.

[7] N. Sökmez, B. Koşar, and C. Nebiyev, "Genelleştirilmiş küçük alt modüller," in *XIII. Ulusal Matematik Sempozyumu*. Kayseri: Erciyes Üniversitesi, 2010.

[8] Y. Wang and N. Ding, "Generalized supplemented modules," *Taiwanese J. Math.*, vol. 10, no. 6, pp. 1589–1601, 2006, doi: 10.11650/twjm/1500404577.

[9] R. Wisbauer, *Foundations of module and ring theory*, german ed., ser. Algebra, Logic and Applications. Gordon and Breach Science Publishers, Philadelphia, PA, 1991, vol. 3, a handbook for study and research.

[10] W. Xue, "Characterizations of semiperfect and perfect rings," *Publ. Mat.*, vol. 40, no. 1, pp. 115–125, 1996, doi: 10.5565/PUBLMAT_40196_08.

[11] H. Zöschinger, "Komplementierte Moduln über Dedekindringen," *J. Algebra*, vol. 29, pp. 42–56, 1974, doi: 10.1016/0021-8693(74)90109-4.

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$G-$SUPPLEMENTED LATTICES

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Abstract. In this work, g-supplemented lattices are defined and some properties of these lattices are investigated. g-small submodules and g-supplemented modules are generalized to lattices. Let L be a lattice and $1 = a_1 \vee a_2 \vee ... \vee a_n$ with $a_i \in L$ $(1 \leq i \leq n)$. If $a_i/0$ is g-supplemented for every $i = 1, 2, ..., n$, then L is also g-supplemented. If L is g-supplemented, then $1/a$ is also g-supplemented for every $a \in L$. It is also defined the g-radical of a lattice L and it is shown that if L is g-supplemented, then $1/r_g(L)$ is complemented.

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1. Introduction

In this paper, every lattice is complete modular lattice with the smallest element 0 and the greatest element 1. Let L be a lattice, $x, y \in L$ and $x \leq y$. A sublattice $\{a \in L | x \leq a \leq y\}$ is called a *quotient sublattice* and denoted by y/x. An element y of a lattice L is called a *complement* of x in L if $x \wedge y = 0$ and $x \vee y = 1$, this case we denote $1 = x \oplus y$ (in this case we call x and y are *direct summands* of L). L is said to be *complemented* if each element has at least one complement in L. An element x of L is said to be *small* or *superfluous* and denoted by $x \ll L$ if $y = 1$ for every $y \in L$ such that $x \vee y = 1$. The meet of all maximal $(\neq 1)$ elements of a lattice L is called the *radical* of L and denoted by $r(L)$. An element a of L is called a *supplement* of b in L if it is minimal for $a \vee b = 1$. a is a supplement of b in a lattice L if and only if $a \vee b = 1$ and $a \wedge b \ll a/0$. A lattice L is called a *supplemented lattice* if every element of L has a supplement in L. We say that an element y of L *lies above* an element x of L if $x \leq y$ and $y \ll 1/x$. L is said to be *hollow* if every element distinct from 1 is superfluous in L, and L is said to be *local* if L has the greatest element $(\neq 1)$. We say an element $x \in L$ has *ample supplements* in L if for every $y \in L$ with $x \vee y = 1$, x has a supplement z in L with $z \leq y$. L is said to be *amply supplemented* if every element of L has ample supplements in L. It is clear that every amply supplemented lattice is supplemented. Let L be a lattice and $k \in L$. If $t = 0$ for very $t \in L$ with $k \wedge t = 0$, then k is called an *essential element* of L and denoted by $k \trianglelefteq L$.

More informations about (amply) supplemented lattices are in [1, 2, 5, 9]. More results about (amply) supplemented modules are in [8, 12].

Definition 1. Let L be a lattice and $a \in L$. If $b = 1$ for every $b \trianglelefteq L$ with $a \vee b = 1$, then a is called a generalized small (briefly, g-small) element of L and denoted by $a \ll_g L$.

It is clear that every small element is g-small, but the converse is not true in general (See Example 1 and Example 2).

G-small elements generalize g-small submodules. G-small submodules are studied in [6, 7, 11].

Lemma 1. *Let L be a lattice and $a, b, c, d \in L$. Then the followings are hold.*

(i) *If $a \leq b$ and $b \ll_g L$, then $a \ll_g L$.*
(ii) *If $a \ll_g b/0$, then $a \ll_g t/0$ for every $t \in L$ with $b \leq t$.*
(iii) *If $a \ll_g L$, then $a \vee b \ll_g 1/b$.*
(iv) *If $a \ll_g b/0$ and $c \ll_g d/0$, then $a \vee c \ll_g (b \vee d)/0$.*

Proof. (i) Let $a \vee k = 1$ with $k \trianglelefteq L$. Since $a \leq b$, $b \vee k = 1$ and since $b \ll_g L$, $k = 1$. Hence $a \ll_g L$ as desired.

(ii) Let $t \in L$ with $b \leq t$ and let $a \vee k = t$ with $k \trianglelefteq t/0$. Here $k \wedge b \trianglelefteq b/0$. Since $a \leq b$, by modularity, $b = b \wedge t = b \wedge (a \vee k) = a \vee (k \wedge b)$ and since $a \ll_g b/0$, $k \wedge b = b$ and $b \leq k$. Hence $a \leq k$ and $t = a \vee k = k$. Therefore, $a \ll_g t/0$.

(iii) Let $a \vee b \vee k = 1$ with $k \trianglelefteq 1/b$. Since $k \trianglelefteq 1/b$, we can easily see that $k \trianglelefteq L$. Since $1 = a \vee b \vee k = a \vee k$ and $a \ll_g L$, $k = 1$. Hence $a \vee b \ll_g 1/b$ as desired.

(iv) Let $a \vee c \vee k = b \vee d$ with $k \trianglelefteq (b \vee d)/0$. By (ii) $a \ll_g (b \vee d)/0$ and $c \ll_g (b \vee d)/0$. Since $a \ll_g (b \vee d)/0$ and $c \vee k \trianglelefteq (b \vee d)/0, c \vee k = b \vee d$ and since $c \ll_g (b \vee d)/0$, $k = b \vee d$. Hence $a \vee c \ll_g (b \vee d)/0$ as desired. □

Corollary 1. *If $a_i \ll_g b_i/0$ for $a_i, b_i \in L$ $(i = 1, 2, ..., n)$, then $a_1 \vee a_2 \vee ... \vee a_n \ll_g (b_1 \vee b_2 \vee ... \vee b_n)/0$.*

Proof. Clear from Lemma 1(iv). □

Corollary 2. *Let $a, b \in L$ and $a \leq b$. If $b \ll_g L$, then $b \ll_g 1/a$.*

Proof. Clear from Lemma 1(iii). □

2. G-SUPPLEMENTED LATTICES

Definition 2. Let L be a lattice and $a, b \in L$. If $1 = a \vee b$ and $1 = a \vee t$ with $t \trianglelefteq b/0$ implies that $t = b$, then b is called a g-supplement of a in L. If every element of L has a g-supplement in L, then L is called a g-supplemented lattice.

G-supplemented lattices generalize g-supplemented modules. G-supplemented modules are studied in [7]. Every supplemented lattice is g-supplemented. Hollow and local lattices are g-supplemented.

Lemma 2. *Let L be a lattice and $a,b \in L$. Then b is a g-supplement of a in L if and only if $1 = a \vee b$ and $a \wedge b \ll_g b/0$.*

Proof. $(\Longrightarrow)$ Let $(a \wedge b) \vee k = b$ with $k \trianglelefteq b/0$. Then $1 = a \vee b = a \vee (a \wedge b) \vee k = a \vee k$ hold. Since b is a g-supplement of a in L and $k \trianglelefteq b/0$, by definition, $k = b$. Hence $a \wedge b \ll_g b/0$ as desired.

$(\Longleftarrow)$ Let $1 = a \vee t$ with $t \trianglelefteq b/0$. Since $t \leq b$, by modularity, $b = b \wedge 1 = b \wedge (a \vee t) = (a \wedge b) \vee t$. Since $a \wedge b \ll_g b/0$, $t = b$. Hence b is a g-supplement of a in L as desired. □

Lemma 3. *Let L be a lattice and $a,b \in L$. If $a \vee b$ has a g-supplement x in L and $(a \vee x) \wedge b$ has a g-supplement y in $b/0$, then $x \vee y$ is a g-supplement of a in L.*

Proof. Since x is a g-supplement of $a \vee b$ in L, by Lemma 2,

$$1 = a \vee b \vee x \text{ and } (a \vee b) \wedge x \ll_g x/0.$$

Since y is a g-supplement of $(a \vee x) \wedge b$ in $b/0$, by Lemma 2,

$$b = ((a \vee x) \wedge b) \vee y$$

and

$$(a \vee x) \wedge y = (a \vee x) \wedge b \wedge y \ll_g y/0.$$

Then

$$1 = a \vee b \vee x = a \vee x \vee ((a \vee x) \wedge b) \vee y = a \vee x \vee y$$

and by Lemma 1,

$$\begin{aligned} a \wedge (x \vee y) &\leq ((a \vee x) \wedge y) \vee ((a \vee y) \wedge x) \\ &\leq ((a \vee x) \wedge y) \vee ((a \vee b) \wedge x) \ll_g (x \vee y)/0. \end{aligned}$$

Hence $x \vee y$ is a g-supplement of a in L. □

Corollary 3. *Let L be a lattice and $a,b \in L$. If $a \vee b$ has a g-supplement in L and $b/0$ is g-supplemented, then a has a g-supplement in L.*

Proof. Clear from Lemma 3. □

Lemma 4. *Let $1 = a \vee b$ with $a,b \in L$. If $a/0$ and $b/0$ are g-supplemented, then L is also g-supplemented.*

Proof. Let x be any element of L. Then 0 is a g-supplement of $x \vee a \vee b$ in L and since $b/0$ is g-supplemented, by Corollary 3, $x \vee a$ has a g-supplement in L. Since $a/0$ is g-supplemented, again by Corollary 3, x has a g-supplement in L. Hence L is g-supplemented. □

Corollary 4. *Let $1 = a_1 \vee a_2 \vee \ldots \vee a_n$ with $a_i \in L$ $(1 \leq i \leq n)$. If $a_i/0$ is g-supplemented for every $i = 1,2,\ldots,n$, then L is also g-supplemented.*

Proof. Clear from Lemma 4. □

Lemma 5. *Let L be a lattice and $a,b,c \in L$ with $c \leq a$. If b is a g-supplement of a in L, then $b \vee c$ is a g-supplement of a in $1/c$.*

Proof. Since b is a g-supplement of a in L, $1 = a \vee b$ and $a \wedge b \ll_g b/0$. Since $a \wedge b \ll_g b/0$, by Lemma 1 (ii),

$$a \wedge b \ll_g (b \vee c)/0$$

and by Lemma 1 (iii),

$$(a \wedge b) \vee c \ll_g (b \vee c)/c.$$

Hence $1 = a \vee b = a \vee b \vee c$ and $a \wedge (b \vee c) = (a \wedge b) \vee c \ll_g (b \vee c)/c$ and $b \vee c$ is a g-supplement of a in $1/c$. □

Corollary 5. *Let L be a g-supplemented lattice. Then $1/a$ is g-supplemented for every $a \in L$.*

Proof. Clear from Lemma 5. □

Definition 3. Let L be a lattice and t be a maximal $(\neq 1)$ element of L. If $t \trianglelefteq L$, then t is called a g-maximal element of L. The meet of all g-maximal elements of L is called the g-radical of L and denoted by $r_g(L)$. If L have not any g-maximal elements, then we call $r_g(L) = 1$.

Corollary 6. *Let L be a lattice. Then $r(L) \leq r_g(L)$.*

Proof. Clear from definitions. □

Lemma 6. *Let L be a lattice and $a \in L$. If $a \ll_g L$, then $a \leq r_g(L)$.*

Proof. Assume $a \nleq r_g(L)$. Then there exists a g-maximal element t of L with $a \nleq t$. Since t is maximal $(\neq 1)$ and $a \nleq t$, $a \vee t = 1$ and since $a \ll_g L$ and $t \trianglelefteq L$, $t = 1$. This is contradiction. Hence $a \leq r_g(L)$ as desired. □

Lemma 7. *Let L be a lattice and $a \in L$. Then $r_g(a/0) \leq r_g(L)$.*

Proof. Let t be any g-maximal element of L. If $a \leq t$, then $r_g(a/0) \leq t$. If $a \nleq t$, we can easily see that $a \wedge t$ is a g-maximal element of $a/0$ and hence $r_g(a/0) \leq t$. Therefore, $r_g(a/0) \leq r_g(L)$. □

Lemma 8. *Let L be a g-supplemented lattice. Then $1/r_g(L)$ is complemented.*

Proof. Let x be any element of $1/r_g(L)$. Since L is g-supplemented, x has a g-supplement y in L. Here $1 = x \vee y$ and $x \wedge y \ll_g y/0$. Since $x \wedge y \ll_g y/0$, by Lemma 6 and Lemma 7, $x \wedge y \leq r_g(y/0) \leq r_g(L)$. Hence $1 = x \vee y \vee r_g(L)$ and

$$x \wedge (y \vee r_g(L)) = (x \wedge y) \vee r_g(L) = r_g(L).$$

Therefore, $y \vee r_g(L)$ is a complement of x in $1/r_g(L)$ and $1/r_g(L)$ is complemented. □

Let $x, y \in L$. It is defined a relation β_* on the elements of L by $x\beta_* y$ if and only if for every $t \in L$ with $1 = x \vee t$ then $1 = y \vee t$ and for every $k \in L$ with $1 = y \vee k$ then $1 = x \vee k$. (See [10, Definition 1]. More informations about β_* relation are in [10]. More informations about β^* relation on modules are in [4].

Corollary 7. *Let L be a g-supplemented lattice. Then $1/r_g(L)$ is $\oplus-$supplemented.*

Proof. Clear from [3, Definition 1] and Lemma 8. □

Lemma 9. *Let L be a lattice and $a\beta_* b$ in L. If a and b have g-supplements in L, then they have the same g-supplements in L.*

Proof. Let x be a g-supplement of a in L. Then $1 = a \vee x$ and since $a\beta_* b$, we have $1 = b \vee x$. Let $1 = b \vee t$ with $t \trianglelefteq x/0$. Since $a\beta_* b$, we have $1 = a \vee t$ and since x is a g-supplement of a in L, we have $t = x$. Hence x is a g-supplement of b in L. Similarly, interchanging the roles of a and b we can prove that each g-supplement of b in L is also a g-supplement of a in L. □

Corollary 8. *Let L be a lattice and a lies above b in L. If a and b have g-supplements in L, then they have the same g-supplements in L.*

Proof. By [10, Theorem 3], $a\beta_* b$ and by Lemma 9, the desired is obtained. □

Lemma 10. *Let L be a lattice and $t \ll_g x/0$ for every g-supplement element x in L and for every $t \ll_g L$ with $t \leq x$. If every element of L is β_* equivalent to a g-supplement element in L, then L is g-supplemented.*

Proof. Let $a \in L$. By hypothesis, there exists a g-supplement element x in L such that $a\beta_* x$. Let x be a g-supplement of b in L. By hypothesis, there exists a g-supplement element y in L with $b\beta_* y$. By Lemma 9, x is a g-supplement of y in L. Here $1 = x \vee y$ and $x \wedge y \ll_g x/0$. Since $x \wedge y \ll_g L$ and y is a g-supplement element in L, by hypothesis, $x \wedge y \ll_g y/0$. Then by Lemma 2, y is a g-supplement of x in L. Since $a\beta_* x$, by Lemma 9, y is a g-supplement of a in L. Hence L is g-supplemented. □

Corollary 9. *Let L be a lattice and $t \ll_g x/0$ for every g-supplement element x in L and for every $t \ll_g L$ with $t \leq x$. If every element of L lies above a g-supplement element in L, then L is g-supplemented.*

Proof. Clear from [10, Theorem 3] and Lemma 10. □

Definition 4. Let L be a lattice. If every element of L with distinct from 1 is g-small in L, then L is called a g-hollow lattice.

Clearly we can see that every hollow lattice is g-hollow. But the converse is not true in general (See Example 2).

Proposition 1. *Every g-hollow lattice is g-supplemented.*

Proof. Let L be a g-hollow lattice. Then 1 is a g-supplement of every element of L with distinct from 1 and 0 is a g-supplement of 1 in L. Hence L is g-supplemented. □

Proposition 2. *Let L be a lattice with $r_g(L) \neq 1$. The following conditions are equivalent.*

(i) *L is g-hollow.*
(ii) *L is local.*
(iii) *L is hollow.*

Proof. $(i) \Longrightarrow (ii)$ Let $x \in L$ and $x \neq 1$. Since L is g-hollow, $x \ll_g L$ and by Lemma 6, $x \leq r_g(L)$. By hypothesis, $r_g(L) \neq 1$. Hence $r_g(L)$ is the greatest element $(\neq 1)$ of L and L is local.

$(ii) \Longrightarrow (iii)$ and $(iii) \Longrightarrow (i)$ are clear. □

Example 1. Let L be a nonzero complemented lattice. Here $1 \ll_g L$, but not $1 \ll L$. 1 is a g-supplement of 1 in L, but 1 is not a supplement of 1 in L.

Example 2. Consider the lattice $L = \{0, a, b, 1\}$ given by the following diagram.

$$\begin{tikzcd}
 & 1 & \\
a \arrow[ur] & & b \arrow[ul] \\
 & 0 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Then L is g-hollow but not hollow. Here $1 \ll_g L$, but not $1 \ll L$. 1 is a g-supplement of 1 in L, but 1 is not a supplement of 1 in L. Here also $r(L) = 0 \neq 1 = r_g(L)$ hold.

Example 3. Consider the lattice $L = \{0, a, b, c, 1\}$ given by the following diagram.

$$\begin{tikzcd}
 & 1 & \\
b \arrow[ur] & & c \arrow[ul] \\
 & a \arrow[ul] \arrow[ur] & \\
 & 0 \arrow[u] &
\end{tikzcd}$$

Then L is g-supplemented but not g-hollow.

References

[1] R. Alizade and S. E. Toksoy, "Cofinitely weak supplemented lattices," *Indian J. Pure Appl. Math.*, vol. 40, no. 5, pp. 337–346, 2009.
[2] R. Alizade and S. E. Toksoy, "Cofinitely supplemented modular lattices," *Arab. J. Sci. Eng.*, vol. 36, no. 6, pp. 919–923, 2011, doi: 10.1007/s13369-011-0095-z.

[3] Ç. Biçer and C. Nebiyev, "⊕-supplemented lattices," *Miskolc Mathematical Notes*, vol. 20, no. 2, pp. 773–780, 2019.
[4] G. F. Birkenmeier, F. Takil Mutlu, C. Nebiyev, N. Sokmez, and A. Tercan, "Goldie*-supplemented modules," *Glasg. Math. J.*, vol. 52, no. A, pp. 41–52, 2010, doi: 10.1017/S0017089510000212.
[5] G. Călugăreanu, *Lattice concepts of module theory*, ser. Kluwer Texts in the Mathematical Sciences. Kluwer Academic Publishers, Dordrecht, 2000, vol. 22, doi: 10.1007/978-94-015-9588-9.
[6] B. Koşar, C. Nebiyev, and A. Pekin, "A generalization of g-supplemented modules," *Miskolc Math. Notes*, vol. 20, no. 1, pp. 345–352, 2019, doi: 10.18514/MMN.2019.2586.
[7] B. Koşar, C. Nebiyev, and N. Sökmez, "*g*-supplemented modules," *Ukrainian Mathematical Journal*, vol. 67, no. 6, pp. 861–864, 2015.
[8] C. Nebiyev and A. Pancar, "On supplement submodules," *Ukrainian Math. J.*, vol. 65, no. 7, pp. 1071–1078, 2013, doi: 10.1007/s11253-013-0842-2.
[9] C. Nebiyev, "On supplement elements in lattices," *Miskolc Math. Notes*, vol. 20, no. 1, pp. 441–449, 2019, doi: 10.18514/MMN.2019.2844.
[10] C. Nebiyev and H. H. Ökten, "β_* relation on lattices," *Miskolc Math. Notes*, vol. 18, no. 2, pp. 993–999, 2017, doi: 10.18514/mmn.2017.1782.
[11] C. Nebiyev and H. H. Ökten, "Weakly *G*-supplemented modules," *Eur. J. Pure Appl. Math.*, vol. 10, no. 3, pp. 521–528, 2017.
[12] R. Wisbauer, *Foundations of module and ring theory*, ser. Algebra, Logic and Applications. Gordon and Breach Science Publishers, Philadelphia, PA, 1991, vol. 3, a handbook for study and research.

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A STUDY OF A SPECIAL KIND OF N-FIXED POINT EQUATION SYSTEM AND APPLICATIONS

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Abstract. The purpose of this paper is to consider a system of N-fixed point equations in metric spaces. The existence and uniqueness of solution and an iterative algorithm for approximating the solution are studied. This system of N-fixed point equations is an extension of the classical of fixed point equation $x = Tx$. The results of this paper improve several important works recently published in the literature.

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1. Introduction

Banach's contraction principle is one of the most powerful tools in applied nonlinear analysis. Weak contractions (also called ϕ-contractions), as generalizations of Banach contraction mappings, have been studied by several authors. Let T be a self-map of a metric space (X,d) and $\phi : [0,+\infty) \to [0,+\infty)$ be a function. We say that T is a ϕ-contraction if

$$d(Tx,Ty) \le \phi(d(x,y)), \ \forall\, x,y \in X.$$

In 1968, Browder [2] proved that if ϕ is non-decreasing and right continuous and (X,d) is complete, then T has a unique fixed point $x^* \in X$ and $\lim_{n\to\infty} T^n x_0 = x^*$ for any given $x_0 \in X$. Subsequently, this result was extended in 1969 by Boyd and Wong [1] by weakening the hypothesis on ϕ, in the sense that it is sufficient to assume that ϕ is right upper semi-continuous and not necessarily monotone. For other results of this type see also [3]. For a comprehensive study of relations between several such contraction type conditions, see [7] and [8].

On the other hand, in 2015, Su and Yao [14] proved the following generalized contraction mapping principle.

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Theorem 1. *Let (X,d) be a complete metric space. Let $T : X \to X$ be a mapping such that*

$$\psi(d(Tx,Ty)) \le \phi(d(x,y)),\ \forall\, x,y \in X, \tag{1.1}$$

where $\psi,\phi : [0,+\infty) \to [0,+\infty)$ are two functions satisfying the conditions:

(1) $\psi(a) \le \phi(b) \Rightarrow a \le b$;

(2) $\begin{cases} \psi(a_n) \le \phi(b_n) \\ a_n \to \varepsilon,\ b_n \to \varepsilon \end{cases} \Rightarrow \varepsilon = 0.$

Then, T has a unique fixed point and, for any given $x_0 \in X$, the iterative sequence $T^n x_0$ converges to this fixed point.

In particular, the study of the fixed points and coupled fixed points for weak contractions and generalized contractions was extended to partially ordered metric spaces in [4–6, 10–12, 15]. Among them, some results involve altering distance functions. Such functions were introduced by Khan et al. in [9], where some fixed point theorems are presented.

Recently, Y. Su, A. Petruşel and J. C. Yao [13] proved a multivariate contraction mapping principle in complete metric spaces.

The purpose of this paper is to consider a of system of N-fixed point equations in metric spaces. The existence and uniqueness of solution and the iterative algorithm of solution are studied. We notice that the system of N-fixed point equations is a generalized form of the fixed point equation $x = Tx$. The results of this paper improve several important works published recently in the literature.

2. A SYSTEM OF NONLINEAR EQUATIONS WITH CONTRACTION TYPE OPERATORS

We will start the section with some concepts and results which are useful in our approach.

Definition 1. A multiply metric function $\triangle(a_1,a_2,\cdots,a_N)$ is a continuous N variables non-negative real function with the domain

$$\{(a_1,a_2,\cdots,a_N) \in \mathbb{R}^N : a_i \ge 0,\ i \in \{1,2,3,\cdots,N\}\}$$

which satisfies the following conditions:

(1) $\triangle(a_1,a_2,\cdots,a_N)$ is non-decreasing for each variable $a_i,\ i \in \{1,2,\cdot\cdot,N\}$,

(2) $\triangle(a_1+b_1,a_2+b_2,\cdots,a_N+b_N) \le \triangle(a_1,a_2,\cdots,a_N) + \triangle(b_1,b_2,\cdots,b_N)$,

(3) $\triangle(a,a,\cdots,a) = a$,

(4) $\triangle(a_1,a_2,\cdots,a_N) \to 0 \Leftrightarrow a_i \to 0,\ i \in \{1,2,3,\cdots,N\}$,

(5) $\triangle(a_1,a_2,\cdots,a_N) = \triangle(a_{i_1},a_{i_2},\cdots,a_{i_N})$,

for all $a_i,b_i,a \in \mathbb{R},\ i \in \{1,2,3,\cdots,N\}$, where $a_{i_1},a_{i_2},\cdots,a_{i_N}$ is an arbitrary permutation of elements $a_1,a_2,\cdots,a_N$.

The following are some basic examples of multiply metric functions.

Example 1. $\triangle_1(a_1,a_2,\cdots,a_N) = \frac{1}{N}\sum_{i=1}^{N} a_i$.

Example 2. $\triangle_2(a_1,a_2,\cdots,a_N) = \frac{1}{h}\sum_{i=1}^{N} q_i a_i$, where $q_i \in [0,1),\ i \in \{1,\cdots,N\}$ and $0 < h := \sum_{i=1}^{N} q_i < 1$.

Example 3. $\triangle_3(a_1,a_2,\cdots,a_N) = \sqrt{\frac{1}{N}\sum_{i=1}^{N} a_i^2}$.

Example 4. $\triangle_4(a_1,a_2,\cdots,a_N) = \max\{a_1,a_2,\cdots,a_N\}$.

An important concept is now presented.

Definition 2. Let (X,d) be a metric space, $T : X^N \to X$ be a N-variables mapping, an element $p \in X$ is called a multivariate fixed point (or a fixed point of order N, see [13]) of T if

$$p = T(p,p,\cdots,p).$$

In what follows, we recall the following theorem which is a generalization of Banach's contraction principle. This theorem was proved by Y. Su, A. Petruşel and J. C. Yao in 2016, see [11].

Theorem 2. *Let (X,d) be a complete metric space and $T : X^N \to X$ be a N-variables mapping for which there exists $h \in (0,1)$ such that the following condition holds*

$$d(Tx,Ty) \le h\triangle(d(x_1,y_1),d(x_2,y_2),\cdots,d(x_N,y_N)),\ \forall\, x,y \in X^N,$$

where $\triangle$ is a multiply metric function.

Then, T has a unique multivariate fixed point $p \in X$ and, for any $p_0 \in X^N$, the iterative sequence $\{p_n\} \subset X^N$ defined by

$$\begin{aligned} p_1 &= (Tp_0,Tp_0,\cdots,Tp_0)\\ p_2 &= (Tp_1,Tp_1,\cdots,Tp_1)\\ &\cdots\\ p_{n+1} &= (Tp_n,Tp_n,\cdots,Tp_n) \end{aligned}$$

converges, in the multiply metric $\triangle$, to $(p,p,\cdots,p) \in X^N$ and the iterative sequence $\{Tp_n\} \subset X$ converges, with respect to d, to $p \in X$.

In this article, we will extend Banach's contraction principle in another direction.

Definition 3. Let (X,d) be a metric space, $T : X^N \to X$ be a N-variables mapping, we consider the following N-variables system of equations:

$$\begin{cases} T(x_{1,1},x_{1,2},\cdots,x_{1,N}) = x_1, \\ \quad \cdots \\ T(x_{i,1},x_{i,2},\cdots,x_{i,N}) = x_i, \\ \quad \cdots \\ T(x_{N,1},x_{N,2},\cdots,x_{N,N}) = x_N, \end{cases} \tag{2.1}$$

The system of equations (2.1) is said to be system of N-fixed point equations, where $x_{i,1},x_{i,2},\cdots,x_{i,N},\ i=1,2,\cdots,N$ and $x_{1,j},x_{2,j},\cdots,x_{N,j},\ j=1,2,\cdots,N$ are the permutations of elements $x_1,x_2,x_3,\cdots,x_N$.

Remark 1. It is easy to see that system (2.1) includes $\prod_{n=1}^{N} n!$ systems of equations. For example, if $N=2$, then (2.1) includes $\prod_{n=1}^{2} n! = 2!\cdot 1! = 2$ systems of coupled fixed point operator equations, i.e., the following systems

$$\begin{cases} T(x_1,x_2) = x_1 \\ T(x_2,x_1) = x_2 \end{cases} \quad \begin{cases} T(x_1,x_2) = x_2 \\ T(x_2,x_1) = x_1. \end{cases}$$

Example 5. Let (X,d) be a metric space and $T : X^N \to X$. We consider the following systems of equations:

$$\begin{cases} T(x_1,x_2,\cdots,x_N) = x_1, \\ T(x_2,x_3,\cdots,x_1) = x_2, \\ \quad \cdots \\ \quad \cdots \\ T(x_N,x_1,\cdots,x_{N-1}) = x_N, \end{cases} \tag{2.2}$$

and

$$\begin{cases} T(x_1,x_2,\cdots,x_N) = x_N, \\ T(x_2,x_3,\cdots,x_1) = x_1, \\ \quad \cdots \\ \quad \cdots \\ T(x_N,x_1,\cdots,x_{N-1}) = x_{N-1}. \end{cases} \tag{2.3}$$

The above systems of equations are also special forms of a system of N-fixed point equations. Moreover, the system (2.3) can be re-written as

$$\begin{cases} T(x_2,x_3,\cdots,x_1)=x_1, \\ T(x_3,x_4,\cdots,x_2)=x_2, \\ \quad\cdots \\ \quad\cdots \\ T(x_N,x_1,\cdots,x_{N-1})=x_{N-1}, \\ T(x_1,x_2,\cdots,x_N)=x_N, \end{cases} \tag{2.4}$$

In what follows, we prove our first main theorem, which generalizes Banach's contraction principle. We need first the following auxiliary notions and results.

Definition 4. Let (X,d) be a complete metric space and define the following multiply metric D given by

$$D((x_1,\cdots,x_N),(y_1,\cdots,y_N))=\triangle(d(x_1,y_1),\cdots,d(x_N,y_N))$$

for all $(x_1,\cdots,x_N),\ (y_1,\cdots,y_N)\in X^N$.

Notice that the functional D is a metric on X^N. Indeed, the following two conditions are obvious:

(i) $D((x_1,\cdots,x_N),(y_1,\cdots,y_N))=0 \Leftrightarrow (x_1,\cdots,x_N)=(y_1,\cdots,y_N)$;
(ii) $D((y_1,\cdots,y_N)),(x_1,\cdots,x_N)=D((x_1,\cdots,x_N),(y_1,\cdots,y_N))$,

for all $(x_1,\cdots,x_N),(y_1,\cdots,y_N)\in X^N$.

Next we prove the triangular inequality. For all

$$(x_1,\cdots,x_N),(y_1,\cdots,y_N),(z_1,\cdots,z_N)\in X^N,$$

from the definition of $\triangle$, we have that

$$\begin{aligned} D((x_1,\cdots,x_N),(y_1,\cdots,y_N)) &= \triangle(d(x_1,y_1),\cdots,d(x_N,y_N)) \\ &\le \triangle(d(x_1,z_1)+d(z_1,y_1),\cdots,d(x_N,z_N)+d(z_N,y_N)) \\ &\le \triangle(d(x_1,z_1),\cdots,d(x_N,z_N))+\triangle(d(z_1,y_1),\cdots,d(z_N,y_N)) \\ &= D((x_1,\cdots,x_N),(z_1,\cdots,z_N))+D((z_1,\cdots,z_N),(y_1,\cdots,y_N)). \end{aligned}$$

Moreover, if the metric space (X,d) is complete, then we can prove that (X^N,D) is a complete metric space. Indeed, let $\{p_n\}\subset X^N$ be a Cauchy sequence, then we have

$$\lim_{n,m\to\infty} D(p_n,p_m)=\lim_{n,m\to\infty}\triangle(d(x_{1,n},x_{1,m}),\cdots,d(x_{N,n},x_{N,m}))=0,$$

where $p_n=(x_{1,n},x_{2,n},\cdots,x_{N,n}),\ p_m=(x_{1,m},x_{2,m},\cdots,x_{N,m})$. From the definition of $\triangle$, we have that

$$\lim_{n,m\to\infty} d(x_{i,n},x_{i,m})=0,\ \forall i\in\{1,2,3,\cdots,N\}.$$

Hence each $\{x_{i,n}\}$ $(i \in \{1,2,\cdots,N\})$ is a Cauchy sequence. Since (X,d) is a complete metric space, there exist $x_1,x_2,\cdots,x_N \in X$ such that, for all $i \in \{1,2,\cdots,N\}$ we have $\lim_{n\to\infty} d(x_{i,n},x_i) = 0$. Therefore $\lim_{n\to\infty} D(p_n,x) = 0$, where $x = (x_1,x_2,\cdots,x_N) \in X^N$. Thus, the pair (X^N,D) is a complete metric space.

Theorem 3. *Let (X,d) be a complete metric space and $T : X^N \to X$ be a N-variables mapping for which there exists $h \in (0,1)$ such that, for all $x = (x_1,\cdots,x_N)$, $y = (y_1,\cdots,y_N) \in X^N$, the following condition is satisfied:*

$$d(Tx,Ty) \le h\triangle(d(x_1,y_1),\cdots,d(x_N,y_N)),$$

where $\triangle$ is a multiply metric function.

Then, the system of N-fixed point equations (2.1) has a unique solution $p = (p_1,\cdots,p_N)$ and for any $u_0 = (x_1^0,\cdots,x_N^0) \in X^N$, the Picard iterative sequence $\{u_n\} \subset X^N$ defined by $u_n := T_*^n(u_0)$ converges, with respect to the multiply metric D, to $p \in X^N$, where the operator $T_* : X^N \to X^N$ is defined by

$$T_* : (x_1,x_2,\cdots,x_N) \mapsto (X_1,X_2,\cdots,X_N),$$

where $X_1 := T(x_{1,1},x_{1,2},\cdots,x_{1,N})$, $X_2 := T(x_{2,1},x_{2,2},\cdots,x_{2,N}),\cdots$, and $X_N := T(x_{N,1},x_{N,2},\cdots,x_{N,N})$.

Proof. We consider on X^N the define multiply metric D given in Definition 4. We prove that T_* is a contraction from (X^N,D) into itself. Observe that, for any $x = (x_1,x_2,\cdots,x_N), y = (y_1,y_2,\cdots,y_N) \in X^N$, we have that

$$\begin{aligned}
D(T_*x,T_*y) &= D((T(x_{1,1},x_{1,2},\cdots,x_{1,N}),\cdots,T(x_{N,1},x_{N,2},\cdots,x_{N,N})),\\
&\qquad (T(y_{1,1},y_{1,2},\cdots,y_{1,N}),\cdots,T(x_{N,1},x_{N,2},\cdots,x_{N,N})))\\
&= \triangle(d(T(x_{1,1},x_{1,2},\cdots,x_{1,N}),T(y_{1,1},y_{1,2},\cdots,y_{1,N})),\\
&\qquad d(T(x_{2,1},x_{2,2},\cdots,x_{2,N}),T(y_{2,1},y_{2,2},\cdots,y_{2,N})),\\
&\qquad d(T(x_{N,1},x_{N,2},\cdots,x_{N,N}),T(y_{N,1},y_{N,2},\cdots,y_{N,N})))\\
&\le \triangle(h\triangle(d(x_{1,1},y_{1,1}),d(x_{1,2},y_{1,2}),\cdots,d(x_{1,N},y_{1,N})),\\
&\qquad h\triangle(d(x_{2,1},y_{2,1}),d(x_{2,2},y_{2,2}),\cdots,d(x_{2,N},y_{2,N})),\\
&\qquad h\triangle(d(x_{3,1},y_{3,1}),d(x_{3,2},y_{3,2}),\cdots,d(x_{3,N},y_{3,N}))),\\
&\qquad \cdots\cdots,\\
&\qquad h\triangle(d(x_{N,1},y_{N,1}),d(x_{N,2},y_{N,2}),\cdots,d(x_{N,N},y_{N,N})))
\end{aligned}$$

From the conditions (1), (5) of $\triangle$, we have

$$\begin{aligned}
&\triangle(h\triangle(d(x_{1,1},y_{1,1}),d(x_{1,2},y_{1,2}),\cdots,d(x_{1,N},y_{1,N})),\\
&\quad h\triangle(d(x_{2,1},y_{2,1}),d(x_{2,2},y_{2,2}),\cdots,d(x_{2,N},y_{2,N})),\\
&\quad h\triangle(d(x_{3,1},y_{3,1}),d(x_{3,2},y_{3,2}),\cdots,d(x_{3,N},y_{3,N}))),\\
&\quad \cdots\cdots,
\end{aligned}$$

$$
\begin{aligned}
& h\triangle(d(x_{N,1},y_{N,1}),\quad d(x_{N,2},y_{N,2}),\cdots,d(x_{N,N},y_{N,N}))) \\
&= h\triangle(d(x_{1,1},y_{1,1}),d(x_{1,2},y_{1,2}),\cdots,d(x_{1,N},y_{1,N})) \\
&= hD((x_1,x_2,\cdots,x_N),(y_1,y_2,\cdots,y_N)) \\
&= hD(x,y).
\end{aligned}
$$

Hence

$$D(T_*x,T_*y) \leq hD(x,y), \quad \text{for all } x,y \in X^N.$$

By Banach's contraction principle, we obtain that there exists a unique element $p=(p_1,\cdots,p_N) \in X^N$ such that $p=T_*p$ and, for any element $u_0 \in X^N$, the iterative sequence $u_n = T_*^n(u_0)$ converges, in the multiply metric D, to p. That is

$$T_*(p_1,p_2,p_3,\cdots,p_N) = (p_1,p_2,p_3,\cdots,p_N).$$

From the definition of T_*, we have

$$T_*(p_1,p_2,p_3,\cdots,p_N) = (P_1,P_2,P_3,\cdots,P_N),$$

where

$$
\begin{aligned}
P_1 &= T(p_{1,1},p_{1,2},\cdots,p_{1,N}), \\
P_2 &= T(p_{2,1},p_{2,2},\cdots,p_{2,N}), \\
&\cdots, \\
P_N &= T(p_{N,1},p_{N,2},\cdots,p_{N,N}).
\end{aligned}
$$

Therefore, the vector $p=(p_1,p_2,\cdots,p_N) \in X^N$ is the unique solution of the system of N-fixed point equations (2.1). This completes the proof. $\square$

Notice that taking $N=1, \triangle(a)=a$ in Theorem 3, we obtain Banach's contraction mapping principle. Some other consequences of the above general result are the following corollaries.

Corollary 1. *Let (X,d) be a complete metric space and $T: X\times X \to X$ be a mapping for which there exists $h\in(0,1)$ such that*

$$d(Tx,Ty) \leq h\triangle(d(x_1,y_1),d(x_2,y_2)),\ \forall\, x=(x_1,x_2), y=(y_1,y_2) \in X\times X,$$

where $\triangle$ is a multiply metric function.

Then, the system of coupled fixed point equations

$$
\begin{cases}
T(x_1,x_2) = x_1, \\
T(x_2,x_1) = x_2.
\end{cases}
$$

has a unique solution $p=(p_1,p_2)$ and for any $u_0=(x_1^0,x_2^0)\in X^2$, the Picard iterative sequence $\{u_n\}\subset X^N$ defined by $u_n=T_*^n(u_0)$ converges, in the multiply metric D, to $p\in X^2$, where the operator $T_*: X^2\to X^2$ is defined by

$$T_*: (x_1,x_2) \mapsto (T(x_1,x_2),T(x_2,x_1)).$$

Notice that, in the above theorem, the detailed Picard iterative process is the following:

$$\begin{cases} u_0 = & (x_1^0, x_2^0), \\ u_1 = & (T(x_1^0, x_2^0), T(x_2^0, x_1^0)), \\ u_2 = & (T(T(x_1^0, x_2^0), T(x_2^0, x_1^0)), T(T(x_2^0, x_1^0), T(x_1^0, x_2^0))), \\ \cdots \end{cases}$$

Another consequence follows if we consider the multiply metric given in Example 3.

Corollary 2. *Let (X,d) be a complete metric space, $T : X^N \to X$ be a N-variables mapping for which there exists $h \in (0,1)$ such that*

$$d(Tx,Ty) \leq \frac{h}{N}\sum_{i=1}^{N} d(x_i,y_i),\ \forall x = (x_1,x_2,\cdots,x_N), y = (y_1,y_2,\cdots,y_N) \in X^N.$$

Then, the system of N-fixed point equations (2.1) has a unique solution $p = (p_1,p_2,\cdots,p_N)$ and, for any $u_0 = (x_1^0,x_2^0,\cdots,x_N^0) \in X^N$, the Picard iterative sequence $\{u_n\} \subset X^N$ defined by $u_n = T_*^n(u_0)$ converges, in the multiply metric D, to $p \in X^N$, where the operator $T_* : X^N \to X^N$ is defined by

$$T_* : (x_1,x_2,\cdots,x_N) \mapsto (X_1,X_2,\cdots,X_N),$$

where

$$\begin{aligned} X_1 &= T(x_{1,1},x_{1,2},x_{1,3},\cdots,x_{1,N}), \\ X_2 &= T(x_{2,1},x_{2,2},x_{2,3},\cdots,x_{2,N}), \\ &\cdots, \\ X_N &= T(x_{N,1},x_{N,2},x_{N,3},\cdots,x_{N,N}). \end{aligned}$$

Some other corollaries, given for different other multiply metrics, can be given. For example, we have the following two results.

Corollary 3. *Let (X,d) be a complete metric space and $T : X^N \to X$ be a N-variables mapping for which there exists $h \in (0,1)$ such that the following condition holds*

$$d(Tx,Ty) \leq h\sqrt{\frac{1}{N}\sum_{i=1}^{N} d(x_i,y_i)^2},\ \forall x = (x_1,x_2,\cdots,x_N), y = (y_1,y_2,\cdots y_N) \in X^N.$$

Then, the system of N-fixed point equations (2.1) has a unique solution $p = (p_1,p_2,p_3,\cdots,p_N)$ and for any $u_0 = (x_1^0,x_2^0,\cdots,x_N^0) \in X^N$, the Picard iterative sequence $\{u_n\} \subset X^N$ defined by $u_n = T_*^n(u_0)$ converges, in the multiply metric D, to $p \in X^N$. The operator $T_* : X^N \to X^N$ is defined by

$$T_* : (x_1,x_2,\cdots,x_N) \mapsto (X_1,X_2,\cdots,X_N),$$

where

$$\begin{aligned}
X_1 &= T(x_{1,1}, x_{1,2}, x_{1,3}, \cdots, x_{1,N}),\\
X_2 &= T(x_{2,1}, x_{2,2}, x_{2,3}, \cdots, x_{2,N}),\\
&\cdots,\\
X_N &= T(x_{N,1}, x_{N,2}, x_{N,3}, \cdots, x_{N,N}).
\end{aligned}$$

Corollary 4. *Let (X,d) be a complete metric space and $T : X^N \to X$ be a N-variables mapping for which there exists $h \in (0,1)$ such that, for all $x = (x_1, x_2, \cdots, x_N), y = (y_1, y_2, \cdots y_N) \in X^N$, the following condition holds*

$$d(Tx, Ty) \le h \max\{d(x_1, y_1), \cdots, d(x_N, y_N)\}.$$

Then, the system of N-fixed point equations (2.1) has a unique solution $p = (p_1, p_2, \cdots, p_N)$ and, for any $u_0 = (x_1^0, x_2^0, \cdots, x_N^0) \in X^N$, the Picard iterative sequence $\{u_n\} \subset X^N$ defined by $u_n = T_*^n(u_0)$ converges, in the multiply metric D, to $p \in X^N$, where the operator $T_* : X^N \to X^N$ is defined by

$$T_* : (x_1, x_2, \cdots, x_N) \mapsto (X_1, X_2, \cdots, X_N),$$

where

$$\begin{aligned}
X_1 &= T(x_{1,1}, x_{1,2}, x_{1,3}, \cdots, x_{1,N}),\\
X_2 &= T(x_{2,1}, x_{2,2}, x_{2,3}, \cdots, x_{2,N}),\\
&\cdots,\\
X_N &= T(x_{N,1}, x_{N,2}, x_{N,3}, \cdots, x_{N,N}).
\end{aligned}$$

In particular, for the systems (2.2) and (2.3) we get the following results.

Corollary 5. *Let (X,d) be a complete metric space, $T : X^N \to X$ be a N-variables mapping for which there exists $h \in (0,1)$ such that, for all $x = (x_1, \cdots, x_N)$, $y = (y_1, \cdots, y_N) \in X^N$, the following condition holds*

$$d(Tx, Ty) \le h \triangle(d(x_1, y_1), \cdots, d(x_N, y_N)),$$

where $\triangle$ is a multiply metric function.
Then, the system of N-fixed point equations (2.2) has a unique solution $p = (p_1, p_2, \cdots, p_N)$ and for any $u_0 = (x_1^0, x_2^0, \cdots, x_N^0) \in X^N$, the Picard iterative sequence $\{u_n\} \subset X^N$ defined by $u_n = T_*^n(u_0)$ converges, in the multiply metric D, to $p \in X^N$, where the operator $T_* : X^N \to X^N$ is defined by

$$T_* : (x_1, x_2, \cdots, x_N) \mapsto (X_1, X_2, \cdots, X_N),$$

where

$$\begin{aligned}
X_1 &= T(x_1, x_2, x_3, \cdots, x_N),\\
X_2 &= T(x_2, x_3, x_4, \cdots, x_1),\\
&\cdots,
\end{aligned}$$

$$X_N = T(x_N, x_1, x_2, \cdots, x_{N-1}).$$

Corollary 6. *Let (X,d) be a complete metric space, $T : X^N \to X$ be a N-variables mapping for which there exists $h \in (0,1)$ such that, for all $x = (x_1, x_2, \cdots, x_N), y = (y_1, y_2, \cdots, y_N) \in X^N$, the following condition holds*

$$d(Tx, Ty) \leq h \triangle (d(x_1, y_1), \cdots, d(x_N, y_N)),$$

where $\triangle$ is a multiply metric function.
Then, the system of N-fixed point equations (2.3) has a unique solution $p = (p_1, p_2, \cdots, p_N)$ and, for any $u_0 = (x_1^0, x_2^0, \cdots, x_N^0) \in X^N$, the iterative sequence $\{u_n\} \subset X^N$ defined by $u_n = T_*^n(u_0)$ converges, in the multiply metric D, to $p \in X^N$, where the operator $T_* : X^N \to X^N$ is defined by

$$T_* : (x_1, x_2, \cdots, x_N) \mapsto (X_1, X_2, \cdots, X_N),$$

where

$$\begin{aligned} &X_1 = T(x_2, x_3, x_4, \cdots, x_1), \\ &X_2 = T(x_3, x_4, x_5, \cdots, x_2), \\ &\cdots, \\ &X_N = T(x_1, x_2, x_3, \cdots, x_N). \end{aligned}$$

3. An application to a system of first order differential equations

We will give now an application of the above results to an initial value problem related to a system of first order differential equations of the following form:

$$\begin{cases} \frac{dx_1}{dt} = f(x_{1,1}(t), x_{1,2}(t), \cdots, x_{1,N}(t), t), \\ \quad \cdots \\ \frac{dx_i}{dt} = f(x_{i,1}(t), x_{i,2}(t), \cdots, x_{i,N}(t), t), \\ \quad \cdots \\ \frac{dx_N}{dt} = f(x_{N,1}(t), x_{N,2}(t), \cdots, x_{N,N}(t), t), \\ x_i(t_0) = x^0, \;\; i = 1, 2, 3, \cdots, N. \end{cases} \tag{3.1}$$

where $t_0 \in \mathbb{R}$. We denote $I := [t_0 - \delta, t_0 + \delta]$ (where $\delta > 0$ is a given real number) and consider $f : \mathbb{R}^N \times I \to \mathbb{R}$ be a continuous $(N+1)$-variables function satisfying the following Lipschitz type condition

$$|f(x_1, x_2, \cdots, x_N, t) - f(y_1, y_2, \cdots, y_N, t)| \leq k(t) \sum_{i=1}^{N} |x_i - y_i|,$$

with $k \in L^1(I, \mathbb{R}_+)$. We will consider first the following equivalent system of integral equations

$$\begin{cases} x_1(t) = \int_{t_0}^{t} f(x_{1,1}(\tau), x_{1,2}(\tau), \cdots, x_{1,N}(\tau), \tau) d\tau + x^0, \\ \quad \cdots \\ x_i(t) = \int_{t_0}^{t} f(x_{i,1}(\tau), x_{i,2}(\tau), \cdots, x_{i,N}(\tau), \tau) d\tau + x^0, \\ \quad \cdots \\ x_N(t) = \int_{t_0}^{t} f(x_{N,1}(\tau), x_{N,2}(\tau), \cdots, x_{N,N}(\tau), \tau) d\tau + x^0, \end{cases} \tag{3.2}$$

Let $X := C[t_0 - \delta, t_0 + \delta]$ be the linear space of continuous real functions defined on closed interval $I := [t_0 - \delta, t_0 + \delta]$. We introduce on X a Bielecki type metric, by the relation

$$d_B(x, y) := \max_{t_0 - \delta \leq t \leq t_0 + \delta} |x(t) - y(t)| e^{-LK(t)},$$

where $K(t) := \int_{t_0}^{t} k(s) ds$ and L is a constant greater than N.

Let $T : X \times X \times \cdots \times X \to X$ be a mapping defined by

$$Tx(t) := \int_{t_0}^{t} f(x_1(\tau), x_2(\tau), \cdots, x_N(\tau), \tau) d\tau + x^0.$$

For any $x = (x_1, x_2, \cdots, x_N), y = (y_1, y_2, \cdots, y_N) \in X^N$ and $t \in I$ we have that

$$\begin{aligned} |Tx(t) - Ty(t)| &\leq |\int_{t_0}^{t} |f(x(\tau), \tau) - f(y(\tau), \tau)| d\tau| \\ &\leq |\int_{t_0}^{t} \sum_{i=1}^{N} k(\tau) |x_i(\tau) - y_i(\tau) d\tau| \\ &\leq |\int_{t_0}^{t} |\sum_{i=1}^{N} \max_{\tau \in I} [|x_i(\tau) - y_i(\tau)| e^{-LK(\tau)}] k(\tau) e^{LK(\tau)} d\tau| \\ &= N |\int_{t_0}^{t} (\frac{1}{N} \sum_{i=1}^{N} d_B(x_i, y_i)) k(\tau) e^{LK(\tau)} d\tau| \\ &= N \triangle_1 (d_B(x_1, y_1), \cdots, d_B(x_N, y_N)) |\int_{t_0}^{t} k(\tau) e^{LK(\tau)} d\tau| \\ &\leq \frac{N}{L} \cdot \triangle_1 (d_B(x_1, y_1), \cdots, d_B(x_N, y_N)) e^{LK(t)}. \end{aligned}$$

Thus,

$$|Tx(t) - Ty(t)| e^{-LK(t)} \leq \frac{N}{L} \cdot \triangle_1 (d_B(x_1, y_1), \cdots, d_B(x_N, y_N)), \text{ for all } t \in I.$$

Hence we get that

$$d_B(Tx, Ty) \leq \frac{N}{L} \cdot \triangle_1 (d_B(x_1, y_1), \cdots, d_B(x_N, y_N)), \text{ for all } x, y \in X.$$

Since $h := \frac{N}{L} < 1$, we conclude, by using Theorem 3, that the system of integral equations (3.2) has a unique solution

$$x^* = (x_1^*(t), x_2^*(t), x_3^*(t), \cdots, x_N^*(t)) \in (C[t_0 - \delta, t_0 + \delta])^N.$$

Since the system (3.2) is equivalent to the system (3.1), by our approach, the existence and uniqueness result for (3.1) follows.

On the other hand, by Theorem 3, we also know that, for any initial value

$$u_0 = (x_{0,1}, x_{0,2}, x_{0,3}, \cdots, x_{0,N}) \in (C[t_0 - \delta, t_0 + \delta])^N,$$

the iterative sequence $\{u_n\} \subset (C[t_0 - \delta, t_0 + \delta])^N$ defined by $u_n = T_*^n(u_0)$ converges, in the multiply metric D, to the unique solution of the initial value problem (3.1) (i.e., $x^* \in (C[t_0 - \delta, t_0 + \delta])^N$), where

$$T_* : (C[t_0 - \delta, t_0 + \delta])^N \to (C[t_0 - \delta, t_0 + \delta])^N$$

is defined by

$$T_* : (x_1(t), x_2(t), x_3(t), \cdots, x_N(t)) \mapsto (X_1(t), X_2(t), X_3(t), \cdots, X_N(t)),$$

with

$$\begin{cases} X_1(t) = \int_{t_0}^t f(x_{1,1}(\tau), x_{1,2}(\tau), \cdots, x_{1,N}(\tau), \tau) d\tau + x^0, \\ \quad \cdots \\ X_i(t) = \int_{t_0}^t f(x_{i,1}(\tau), x_{i,2}(\tau), \cdots, x_{i,N}(\tau), \tau) d\tau + x^0, \\ \quad \cdots \\ X_N(t) = \int_{t_0}^t f(x_{N,1}(\tau), x_{N,2}(\tau), \cdots, x_{N,N}(\tau), \tau) d\tau + x^0, \end{cases}$$

and the multiply metric D used here (see Example 1) is defined by

$$D((x_1, x_2, \cdots, x_N), (y_1, y_2, \cdots, y_N)) = \triangle_1(d_B(x_1, y_1), d_B(x_2, y_2), \cdots, d_B(x_N, y_N)),$$

for all $(x_1, x_2, \cdots, x_N)$, $(y_1, y_2, \cdots, y_N) \in (C[t_0 - \delta, t_0 + \delta])^N$.

References

[1] D. Boyd and J. Wong, "On nonlinear contractions," *Proc. Amer. Math. Soc.*, vol. 20, pp. 458–464, 1969, doi: 10.1090/S0002-9939-1969-0239559-9.

[2] F. Browder, "On the convergence of successive approximations for nonlinear functional equations," *Indag. Math.*, vol. 30, no. 1, pp. 27–35, 1968, doi: 10.1016/S1385-7258(68)50004-0.

[3] M. Geraghty, "On contractive mappings," *Proc. Amer. Math. Soc.*, vol. 40, pp. 604–608, 1973, doi: 10.1090/S0002-9939-1973-0334176-5.

[4] T. Gnana Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Anal.*, vol. 65, no. 7, pp. 1379–1393, 2006, doi: 10.1016/j.na.2005.10.017.

[5] J. Harjani and K. Sadarangni, "Fixed point theorems for weakly contraction mappings in partially ordered sets," *Nonlinear Anal.*, vol. 71, pp. 3403–3410, 2009, doi: 10.1016/j.na.2009.01.240.

[6] J. Harjani and K. Sadarangni, "Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations," *Nonlinear Anal.*, vol. 72, pp. 1188–1197, 2010, doi: 10.1016/j.na.2009.08.003.

[7] J. Jachymski, "Equivalence of some contractivity properties over metrical structures," *Proc. Amer. Math. Soc.*, vol. 125, pp. 2327–2335, 1997, doi: 10.1090/S0002-9939-97-03853-7.

[8] J. Jachymski and I. Jozwik, "Nonlinear contractive conditions: a comparison and related problems," in *Fixed Point Theory and its Applications*, ser. Polish Acad. Sci., Banach Center Publ., J. Jachymski and S. Reich, Eds., vol. 77, doi: 10.4064/bc77-0-10. Warsaw: PWN Wrasaw, 2007, pp. 123–146.

[9] M. Khan, M. Swaleh, and S. Sessa, "Fixed point theorems by altering distances between the points," *Bull. Aust. Math. Soc.*, vol. 30, no. 1, pp. 1–9, 1984, doi: 10.1017/S0004972700001659.

[10] V. Lakshmikantham and L. Ciric, "Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces," *Nonlinear Anal.*, vol. 70, no. 12, pp. 4341–4349, 2009, doi: 10.1016/j.na.2008.09.020.

[11] J. Nieto and R. Rodriguez-López, "Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations," *Acta Math. Sinica*, vol. 23, no. 11, pp. 2205–2212, 2007, doi: 10.1007/s10114-005-0769-0.

[12] A. Petrusel, G. Petrusel, B. Samet, and J. Yao, "Coupled fixed point theorems for symmetric contractions in b-metric spaces with applications to operator equation systems," *Fixed Point Theory*, vol. 17, pp. 457–476, 2016.

[13] Y. Su, A. Petrusel, and J.-C. Yao, "Multivariate fixed point theorems for contractions and nonexpansive mappings with applications," *Fixed Point Theory Appl.*, vol. 2016:19, pp. 1–19, 2016, doi: 10.1186/s13663-015-0493-0.

[14] Y. Su and J.-C. Yao, "Further generalized contraction mapping principle and best proximity theorem in metric spaces," *Fixed Point Theory Appl.*, vol. 2015:120, pp. 1–13, 2015, doi: 10.1186/s13663-015-0373-7.

[15] F. Yan, Y. Su, and Q. Feng, "A new contraction mapping principle in partially ordered metric spaces and applications to ordinary differential equations," *Fixed Point Theory Appl.*, vol. 2012:152, pp. 1–13, 2012, doi: 10.1186/1687-1812-2012-152.

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GLOBAL ATTRACTOR FOR THE TIME DISCRETIZED MODIFIED THREE-DIMENSIONAL BÉNARD SYSTEMS

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Abstract. In this paper, we aim to study the existence of global attractors for the time discretized modified three-dimensional (3D) Bénard systems. Using the backward implicit Euler scheme, we obtain the time discretization systems of 3D Bénard systems. Then, by the Galerkin method and the Brouwer fixed point theorem, we prove the existence of the solution to this time-discretized systems. On this basis, we proved the existence of the attractor by the compact embedding theorem of Sobolev. Finally, we discuss the limiting behavior of the solution as N tends to infinity.

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1. INTRODUCTION

In this work, we study the following 3D Bénard system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + F_N(\|\nabla u\|)(u\cdot\nabla)u + \xi\omega = f(x) - \nabla p, \\ \operatorname{div} u = 0, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

$$\begin{cases} \dfrac{\partial \omega}{\partial t} - \Delta\omega + (u\cdot\nabla)\omega = g(x), \\ \omega|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain; $u = u(t,x)$, $\omega = \omega(t,x)$ and $p = p(t,x)$ denote velocity, temperature and pressure of the fluid, respectively; $\nu > 0$, $\xi \in \mathbb{R}^3$ are constants; $f\colon\ \Omega \to \mathbb{R}^3$, $g\colon\ \Omega \to \mathbb{R}$ are given functions, and for $N \geq 1$, $F_N(r) = \min\left\{1, \dfrac{N}{r}\right\}$.

It is well-known that the Bénard system is a dynamic model describing the rate, pressure and temperature of incompressible fluids that are coupled by Navier-Stokes

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equations and convection diffusion equations. This problem is fundamentally important and of both theoretical and practical interest. In recent years, many important achievements have been made in the study of the Bénard system, of which the study of the solution and attractor of the Bénard system is a very important part. For example, in [10], the authors proved the existence of global solution of the equation (1.1)-(1.2) on channel-like domains by the Galerkin method, then they constructed the global φ attractor of this systems. To date, many studies have investigated the case of $F_N = 1$ in the equation (1.1)-(1.2), for example, see [8, 9, 11, 12]. In [8], authors introduced a class of functions which are strongly continuous with respect to the second component of the vector. Then they prove the existence of solutions for the 3D Bénard system, and construct a multi-valued semi-flow generated by such solutions. Moreover, they obtain the existence of a global φ attractor for the weak-strong topology. In [12], authors investigate the regularized 3D Bénard problem. Using the averaging technique which will give us the properties of the mean characteristics of the flow, they prove that the global existence and uniqueness of the solutions, and then obtain the existence of the global attractor. In [9], authors study the asymptotic behaviour of weak solutions for the 3D Bénard problem. They first show some regularity properties of the weak solutions of this systems. Then they construct a one parameter family of multi-valued semi-flow and obtain the existence of a global attractor with respect to the weak topology of the phase space. In [11], authors first establish an energy inequality in the space L^4 for a broader class of weak solutions. Using this inequality, they prove the existence and connectedness of a global attractor in the space $H_w \times L^2$ for the corresponding m-semi-flow.

It is well-known that the discretization method is the basic method to solve the problems of continuum mechanics, which is a method to approximate the physical quantities in continuum mechanics with finite parameters. The laws of continuum mechanics are generally described by differential equations and integral equations. The discretization method approximates the original problem by transforming it into an algebraic equation with finite parameters. The discretization of differential equation mainly refers to the discretization of time and space. The usual discretization methods include finite difference method, finite element method, weighted residual method and so on (see [1, 2, 4, 6, 7, 14, 16]). In [5], the modified 3D Navier-Stokes equations were discretized on the time by finite difference method, then the existence of the global attractor was proved. In the literature [15], the Benjamin-Bona-Mahony equation was discretized on the time by the Crank-Nicolson scheme. Then, using the Galerkin method and the Brouwer fixed point theorem, authors proved that the existence of the solution to this time discretized system. Furthermore, authors showed that the existence of attractor by Sobolev's compact embedding theorem.

The main purpose of this paper is to investigate the long time dynamical behavior of the solution of the discretized, modified 3D Bénard system (1.1)-(1.2) by the idea in [5, 15].

Let us firstly introduce some notations. Set

$$\mathbb{H} = \Big\{ u \in (L^2(\Omega))^3, \text{div } u = 0,\ u \cdot n = 0 \text{ on } \partial\Omega \Big\},$$
$$\mathbb{V} = \Big\{ u \in (H_0^1(\Omega))^3, \text{div } u = 0 \Big\},$$

with norms $\|\cdot\|$, $|||\cdot|||$ and scalar products $(\cdot,\cdot)$, $((\cdot,\cdot))$ (the same notations for norms and scalar products also apply to $L^2(\Omega)$, $H_0^1(\Omega)$), where, n is the unit outward normal on $\partial\Omega$. Let

$$b(u,v,z) = \int_\Omega \sum_{i,j=1}^{3} u_i \frac{\partial v_j}{\partial x_i} z_j dx, \qquad c(u,\omega,\eta) = \int_\Omega \sum_{i=1}^{3} u_i \frac{\partial \omega}{\partial x_i} \eta dx,$$

and $b_N(u,v,z) = F_N(\|\nabla v\|) b(u,v,z)$. Thanks to Poincaré inequality, we can put

$$((u,v)) = (\nabla u, \nabla v), \qquad |||u||| = \|\nabla u\|.$$

We denote by $\mathcal{P}$ the Leray projection of $L^2(\Omega))^d$ onto $\mathbb{H}$ and by $\mathcal{T}$ the Leray projection of $L^2(\Omega))^d$ onto $L^2(\Omega)$. And we denote by $D(A_1)$ the domain of the Stokes operator $A_1 = -\mathcal{P}\Delta$ in $\mathbb{H}$, and by $D(A_2)$ the domain of $A_2 = -\mathcal{T}\Delta$ in $L^2(\Omega)$. Obviously, $A_1 : \mathbb{V} \to \mathbb{V}^*$, $A_2 : H_0^1(\Omega) \to H^{-1}(\Omega)$ are linear continuous operators and such that

$$\langle A_1 u, v \rangle_{\mathbb{V},\mathbb{V}^*} = (\nabla u, \nabla v), \qquad \langle A_2 \omega, \eta \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} = (\nabla \omega, \nabla \eta),$$

where, $u, v \in \mathbb{V}, \omega, \eta \in H_0^1(\Omega)$. From the regularity theory for the Stokes equation, it is proved in [13] that $D(A_1) = H^2(\Omega)^3 \cap \mathbb{V}$, $D(A_2) = H^2(\Omega) \cap H_0^1(\Omega)$, and the following holds true

$$D(A_1) \subset \mathbb{V} \subset \mathbb{H}, \qquad D(A_2) \subset H_0^1(\Omega) \subset L^2(\Omega).$$

Therefore,

$$|||u||| \le \frac{1}{\sqrt{\lambda_1}} \|A_1 u\|, \ \forall u \in D(A_1), \qquad |||\,\omega||| \le \frac{1}{\sqrt{\lambda_2}} \|A_2 \omega\|, \ \forall \omega \in D(A_2),$$
$$\|u\| \le \frac{1}{\sqrt{\lambda_1}} |||u|||, \ \forall u \in \mathbb{V}, \qquad \|\omega\| \le \frac{1}{\sqrt{\lambda_2}} |||\omega|||, \ \forall \omega \in H_0^1(\Omega),$$

where, $\lambda_1 > 0$, $\lambda_2 > 0$ are the first eigenvalues of the Stokes operator A_1, A_2, respectively.

We introduce two bilinear operators $B : \mathbb{V} \times \mathbb{V} \to \mathbb{V}^*$ and $C : \mathbb{V} \times H_0^1(\Omega) \to H^{-1}(\Omega)$, defined as:

$$\langle B(u,v), z \rangle_{\mathbb{V},\mathbb{V}^*} = b(u,v,z), \qquad \langle C(u,\omega), \eta \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} = c(u,\omega,\eta),$$

where, $u,v,z \in \mathbb{V}$, $\omega,\eta \in H_0^1(\Omega)$. From [5],

$$\begin{cases} |b(u,v,z)| \le C_b\|u\|^{\frac{1}{4}}|||u|||^{\frac{3}{4}}|||v|||\|z\|^{\frac{1}{4}}|||z|||^{\frac{3}{4}}, & \forall u,v,z \in \mathbb{V}, \\ |b(u,v,z)| \le C_b|||u|||^{\frac{1}{2}}\|A_1u\|^{\frac{1}{2}}|||v|||\|z\|, & \forall u \in D(A_1), v \in \mathbb{V}, z \in \mathbb{H}, \\ |b(u,v,z)| \le C_b\|u\|^{\frac{1}{4}}\|A_1u\|^{\frac{3}{4}}|||v|||\|z\|, & \forall u \in D(A_1), v \in \mathbb{V}, z \in \mathbb{H}, \\ b(u,v,v) = 0, & \forall u,v \in \mathbb{V}. \end{cases} \tag{1.3}$$

Therefore,

$$b_N(u,v,v) = 0, \;\; \forall u,v \in \mathbb{V} \quad \text{and} \quad \langle B_N(u,v),z\rangle_{\mathbb{V},\mathbb{V}^*} = b_N(u,v,z), \;\; \forall u,v,z \in \mathbb{V}.$$

Since $\Omega \subset \mathbb{R}^3$ is bounded, there exists a constant $c > 0$, which is only related to Ω, such that for all $v \in H^1(\Omega)$ [5],

$$\|v\|_{L^3(\Omega)} \le c\|v\|^{1/2}|||v|||^{1/2}, \qquad \|v\|_{L^6(\Omega)} \le c|||v|||. \tag{1.4}$$

For M, N, p, $q \in \mathbb{R}_+$, there holds [5]

$$|F_N(p) - F_N(q)| \le \frac{|p-q|}{q}, \qquad |F_M(p) - F_N(q)| \le \frac{|M-N|}{q} + \frac{|p-q|}{q}. \tag{1.5}$$

By the notations above, the equations (1.1)-(1.2) can be rewritten in the weak form as

$$\begin{cases} u_t + \nu A_1 u + B_N(u,u) + \xi\omega = f(x), \\ \omega_t + A_2\omega + C(u,\omega) = g(x). \end{cases} \tag{1.6}$$

In this paper, we aim to study the existence of global attractors for the time discretized modified three-dimensional (3D) Bénard systems (1.1)-(1.2). To this end, using the *backward implicit Euler scheme*, we obtain the time discretization systems of (1.6):

$$\begin{cases} \dfrac{u^m - u^{m-1}}{k} + \nu A_1 u^m + B_N(u^m,u^m) + \xi\omega^m = f, & (1.7) \\ \dfrac{\omega^m - \omega^{m-1}}{k} + A_2\omega^m + C(u^m,\omega^m) = g, & (1.8) \end{cases}$$

where k is the time step, and $u^m \sim u(t^m)$, $\omega^m \sim \omega(t^m)$.

The main results of this paper are as follows. Firstly, by the Galerkin method and the Brouwer fixed point theorem, we prove the existence of the solution to this time-discretized systems (1.7)-(1.8).

Theorem 1. *Supposing that $u_0 \in D(A_1)$, $\omega_0 \in D(A_2)$. Let $f \in L^2(\Omega)^3$, $g \in L^2(\Omega)$ be given functions, and let $k > 0$. Then there is at least one set of solutions $\{u^m,\, \omega^m\} \in D(A_1) \times D(A_2)$ to (1.7)-(1.8) for $m \ge 1$ be integers.*

On this basis, by the compact embedding theorem of Sobolev, we proved the existence of the attractor.

Theorem 2. *Supposing that $u_0 \in \mathbb{V}$, $\omega_0 \in H_0^1(\Omega)$. Let $f \in L^2(\Omega)^3$, $g \in L^2(\Omega)$ be given functions and let $k > 0$ small enough. Then the C^0 semigroup S^m defined by the systems (1.7)-(1.8) has global attractors $\mathcal{A}$ in $\mathbb{V} \times H_0^1(\Omega)$.*

Finally, we discuss the limiting behavior of the solution to (1.7)-(1.8) as N tends to infinity.

Theorem 3. *Supposing that $u_0 \in D(A_1)$, $\omega_0 \in D(A_2)$. Let $f \in L^2(\Omega)^3$, $g \in L^2(\Omega)$ be given functions, and let $k > 0$. Then, for $m > 1$ be integers, the solution sequence $\{u_N^m, \omega_N^m\}_N$ of (1.7)-(1.8) converges to the weak solution of the following equations when $N \to \infty$,*

$$\begin{cases} \dfrac{u^m - u^{m-1}}{k} + \nu A_1 u^m + B(u^m, u^m) + \xi \omega^m = f, & (1.9) \\ \dfrac{\omega^m - \omega^{m-1}}{k} + A_2 \omega^m + C(u^m, \omega^m) = g. & (1.10) \end{cases}$$

This paper is organized as follows. Section 2 proves the existence of solutions and completes the proof of Theorem 1. Section 3 proves the boundedness of solution in phase space. Section 4 proves the continuous dependence of solution on initial value and parameter N , and establishes a discrete semigroup S^m to complete the proof of Theorem 2. Section 5 discusses the limit behavior of $\{u_N^m, \omega_N^m\}$ as N tends to infinity and completes the proof of Theorem 3.

2. Existence of solutions

In this section, we construct a weak solution of (1.7)-(1.8) by the Faedo-Galerkin method and the following Brouwer fixed point principle (see [3], 24-29).

Lemma 1 ([3]). *Let X be a finite-dimensional space endowed with a scalar product $[\cdot,\cdot]$ and consider a continuous mapping $F\colon X \to X$. Suppose that there exists $R_0 > 0$ such that $[F(U_0), U_0] > 0$ for all $U_0 \in X$ with $[U_0, U_0] = R_0^2$. Then there exists U with $[U, U] \leq R_0^2$ such that $F(U) = 0$.*

To prove the existence of the solution for (1.7)-(1.8), the following three steps are required:

Step 1: Construct an approximate solution. Let $p \geq 1$ be an integer. For $u^1, \cdots, u^{m-1}, \omega^1, \cdots, \omega^{m-1}$, we can define the approximate solutions of (1.7)-(1.8) by $u_p^m = \sum_{i=1}^{p} g_{ip}{}^m e_i$ and $\omega_p^m = \sum_{i=1}^{p} h_{ip}{}^m \bar{e}_i$:

$$\begin{cases} \dfrac{u_p^m - u^{m-1}}{k} + \nu A_1 u_p^m + B_N\left(u_p^m, u_p^m\right) + \xi \omega_p^m = f, & (2.1) \\ \dfrac{\omega_p^m - \omega^{m-1}}{k} + A_2 \omega_p^m + C\left(u_p^m, \omega_p^m\right) = g, & (2.2) \end{cases}$$

where ${g_{ip}}^m \in \mathbb{R}$, $\{e_i\}_{i=1}^{\infty} \subset D(A_1)$, corresponding to the eigenvectors of the operator A_1, which are ortho-normal base in $\mathbb{H}$ and orthogonal in $\mathbb{V}$; and ${h_{ip}}^m \in \mathbb{R}$, $\{\bar{e}_i\}_{i=1}^{\infty} \subset D(A_2)$, corresponding to the eigenvectors of the operator A_2, which are ortho-normal base in $L^2(\Omega)$ and orthogonal in $H_0^1(\Omega)$. Let $K_p = \langle e_1, e_2, \cdots, e_p\rangle$ is the space generated by $e_1, e_2, \cdots, e_p$ and $M_p = \langle \bar{e}_1, \bar{e}_2, \cdots, \bar{e}_p\rangle$ is the space generated by $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_p$, we define operator Q_1: $K_p \to K_p$ and Q_2: $M_p \to M_p$ satisfy

$$((Q_1(u), v_1)) = (u, v_1) + \nu k(\nabla u, \nabla v_1) + kb_N(u, u, v_1) + k(\xi\omega, v_1) - (u^{m-1}, v_1) - k(f, v_1);$$
$$((Q_2(\omega), v_2)) = (\omega, v_2) + k(\nabla\omega, \nabla v_2) + kc(u, \omega, v_2) - (\omega^{m-1}, v_2) - k(g, v_2).$$

To apply Lemma 1, we introduce the operator $F(u, \omega) = (Q_1(u), Q_2(\omega))^\top$:

$$\begin{aligned}\left(F(u,\omega), (v_1, v_2)^\top\right) &= \left((Q_1(u), Q_2(\omega))^\top, (v_1, v_2)^\top\right)\\ &= ((Q_1(u), v_1)) + ((Q_2(\omega), v_2)).\end{aligned}$$

Now we need to prove that $F(u, \omega)$ is continuous in $\mathbb{V}$. To this end, let $u_1, u_2, v_1 \in K_p$ and $\omega_1, \omega_2, v_2 \in M_p$, we have

$$\begin{aligned}&\left(F(u_1,\omega_1) - F(u_2,\omega_2), (v_1, v_2)^\top\right)\\ &= \left((Q_1(u_1), Q_2(\omega_1))^\top, (v_1, v_2)^\top\right) + \left((Q_1(u_2), Q_2(\omega_2))^\top, (v_1, v_2)^\top\right)\\ &= (u_1, v_1) + \nu k(\nabla u_1, \nabla v_1) + kb_N(u_1, u_1, v_1) + k(\xi\omega_1, v_1) - (u^{m-1}, v_1)\\ &\quad - k(f, v_1) + (\omega_1, v_2) + k(\nabla\omega_1, \nabla v_2) + kc(u_1, \omega_1, v_2) - (\omega^{m-1}, v_2) - k(g, v_2)\\ &\quad - \Big[(u_2, v_1) + \nu k(\nabla u_2, \nabla v_1) + kb_N(u_2, u_2, v_1) + k(\xi\omega_2, v_1) - (u^{m-1}, v_1)\\ &\quad - k(f, v_1) + (\omega_2, v_2) + k(\nabla\omega_2, \nabla v_2) + kc(u_2, \omega_2, v_2) - (\omega^{m-1}, v_2) - k(g, v_2)\Big]\\ &= (u_1 - u_2, v_1) + \nu k(\nabla(u_1 - u_2), \nabla v_1) + k(\xi(\omega_1 - \omega_2), v_1)\\ &\quad + k(\nabla(\omega_1 - \omega_2), \nabla v_2) + (\omega_1 - \omega_2, v_2)\\ &\quad + k\Big[b_N(u_1, u_1, v_1) - b_N(u_2, u_2, v_1)\Big] + k\Big[c(u_1, \omega_1, v_2) - c(u_2, \omega_2, v_2)\Big].\end{aligned} \tag{2.3}$$

Here, by using Poincaré inequality, we can get

$$\begin{aligned}&(u_1 - u_2, v_1) + \nu k(\nabla(u_1 - u_2), \nabla v_1) + k(\xi(\omega_1 - \omega_2), v_1)\\ &+ k(\nabla(\omega_1 - \omega_2), \nabla v_2) + (\omega_1 - \omega_2, v_2)\\ &\le \Big[C\|v_1\| + \nu k|||v_1|||\Big]|||u_1 - u_2||| + \Big[C\|v_2\| + kC|\xi|\,\|v_1\| + k|||v_2|||\Big]|||\omega_1 - \omega_2|||.\end{aligned} \tag{2.4}$$

And by the definition of F_N and (1.3), we obtain

$$\begin{aligned}&b_N(u_1, u_1, v_1) - b_N(u_2, u_2, v_1) = F_N(|||u_1|||)b(u_1, u_1, v_1) - F_N(|||u_2|||)b(u_2, u_2, v_1)\\ &= F_N(|||u_1|||)b(u_1 - u_2, u_1, v_1) + F_N(|||u_2|||)b(u_2, u_1 - u_2, v_1)\end{aligned}$$

$$
\begin{aligned}
&+\Big[F_N(|||u_1|||)-F_N(|||u_2|||)\Big]b(u_2,u_1,v_1)\\
&\le CN|||u_1-u_2|||\,|||v_1|||+C|||u_1-u_2|||\,|||u_1|||\,|||v_1|||\\
&\le \Big[CN|||v_1|||+C|||u_1|||\,|||v_1|||\Big]|||u_1-u_2|||. \qquad (2.5)
\end{aligned}
$$

On the other hand, for $c(u,\omega,\eta)$, we have

$$|c(u,\omega,\eta)|\le \|u\|_{L^6(\Omega)}\|\nabla\omega\|_{L^2(\Omega)}\|\eta\|_{L^3(\Omega)}\le C|||u|||\,|||\omega|||\,|||\eta|||,$$

and $c(u,\omega,\omega)=0$, so we can get

$$
\begin{aligned}
c(u_1,\omega_1,v_2)-c(u_2,\omega_2,v_2)&=c(u_1-u_2,\omega_1,v_2)+c(u_2,\omega_1-\omega_2,v_2)\\
&\le C|||\omega_1|||\,|||v_2|||\,|||u_1-u_2|||+C|||u_2|||\,|||v_2|||\,|||\omega_1-\omega_2|||. \qquad (2.6)
\end{aligned}
$$

Thus, by (2.3)-(2.6), we obtain

$$
\begin{aligned}
&\Big(F(u_1,\omega_1)-F(u_2,\omega_2),(v_1,v_2)^\top\Big)\\
&\le \Big[C\|v_1\|+\nu k|||v_1|||+CNk|||v_1|||+Ck|||u_1|||\,|||v_1|||+C|||\omega_1|||\,|||v_2|||\Big]|||u_1-u_2|||\\
&\quad+\Big[C\|v_2\|+kC\,|\xi|\,\|v_1\|+k|||v_2|||+Ck|||u_2|||\,|||v_2|||\Big]|||\omega_1-\omega_2|||.
\end{aligned}
$$

It is easy to know that $F(u,\omega)$ is continuous in $\mathbb{V}$. Next, let k be small enough such that $1-\frac{k}{2}>0$ and $1-\frac{k}{2}|\xi|^2>0$. For $\{u,\omega\}\in K_p\times M_p$, by Cauchy-Schwarz and Poincaré inequality, we find

$$
\begin{aligned}
&\Big(F(u,\omega),(u,\omega)^\top\Big)=\Big((Q_1(u),Q_2(\omega))^\top,(u,\omega)^\top\Big)\\
&=(u,u)+\nu k(\nabla u,\nabla u)+kb_N(u,u,u)+k(\xi\omega,u)-(u^{m-1},u)-k(f,u)\\
&\quad+(\omega,\omega)+k(\nabla\omega,\nabla\omega)+kc(u,\omega,\omega)-(\omega^{m-1},\omega)-k(g,\omega)\\
&=\|u\|^2+\nu k|||u|||^2+k(\xi\omega,u)-(u^{m-1},u)-k(f,u)\\
&\quad+\|\omega\|^2+k|||\omega|||^2-(\omega^{m-1},\omega)-k(g,\omega)\\
&\ge \|u\|^2+\nu k|||u|||^2+\|\omega\|^2+k|||\omega|||^2-k\|\xi\omega\|\,\|u\|\\
&\quad-\|u^{m-1}\|\,\|u\|-k\|f\|\,\|u\|-\|\omega^{m-1}\|\,\|\omega\|-k\|g\|\,\|\omega\|\\
&\ge \|u\|^2+\nu k|||u|||^2+\|\omega\|^2+k|||\omega|||^2-\frac{k}{2}\Big[\|\xi\omega\|^2+\|u\|^2\Big]\\
&\quad-\frac{\|u^{m-1}\|}{\sqrt{\lambda_1}}|||u|||-k\frac{\|f\|}{\sqrt{\lambda_1}}|||u|||-\frac{\|\omega^{m-1}\|}{\sqrt{\lambda_2}}|||\omega|||-k\frac{\|g\|}{\sqrt{\lambda_2}}|||\omega|||\\
&=\Big[1-\frac{k}{2}\Big]\|u\|^2+\nu k|||u|||^2+\Big[1-\frac{k}{2}|\xi|^2\Big]\|\omega\|^2+k|||\omega|||^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\|u^{m-1}\|}{\sqrt{\lambda_1}}|||u|||-k\frac{\|f\|}{\sqrt{\lambda_1}}|||u|||-\frac{\|\omega^{m-1}\|}{\sqrt{\lambda_2}}|||\omega|||-k\frac{\|g\|}{\sqrt{\lambda_2}}|||\omega|||\\
\geq{}& \nu k|||u|||^2+k|||\omega|||^2-\frac{\|u^{m-1}\|}{\sqrt{\lambda_1}}|||u|||-k\frac{\|f\|}{\sqrt{\lambda_1}}|||u|||-\frac{\|\omega^{m-1}\|}{\sqrt{\lambda_2}}|||\omega|||-k\frac{\|g\|}{\sqrt{\lambda_2}}|||\omega|||\\
={}& |||u|||\left[\nu k|||u|||-\frac{\|u^{m-1}\|}{\sqrt{\lambda_1}}-k\frac{\|f\|}{\sqrt{\lambda_1}}\right]+|||\omega|||\left[k|||\omega|||-\frac{\|\omega^{m-1}\|}{\sqrt{\lambda_2}}-k\frac{\|g\|}{\sqrt{\lambda_2}}\right].
\end{aligned}
$$

Let $r_1>\frac{\|u^{m-1}\|+k\|f\|}{\nu k\sqrt{\lambda_1}}$ and $r_2>\frac{\|\omega^{m-1}\|+k\|g\|}{k\sqrt{\lambda_2}}$, for any $\{u,\omega\}\in K_p\times M_p$ with $|||u|||=r_1$ and $|||\omega|||=r_2$, one has $\left(F(u,\omega),(u,\omega)^\top\right)>0$. Thus, from Lemma 1, we can find (u^*,ω^*) satisfy $F(u^*,\omega^*)=0$, which is $(Q_1(u^*),Q_2(\omega^*))^\top=\mathbf{0}$, and so $\begin{cases}Q_1(u^*)=0,\\ Q_2(\omega^*)=0.\end{cases}$ Therefore, the approximate solution $\{u_p^m,\omega_p^m\}$ exists.

Step 2: Some priori estimates. For k and m are fixed, we want to get a priori estimates independent of p. Multiplying the equation (2.1) by u_p^m and the equation (2.2) by ω_p^m, we obtain

$$
\begin{aligned}
&\|u_p^m\|^2+\|\omega_p^m\|^2+\|u_p^m-u^{m-1}\|^2+\|\omega_p^m-\omega^{m-1}\|^2+2\nu k|||u_p^m|||^2+2k|||\omega_p^m|||^2\\
&=2k\left(f,u_p^m\right)+2k\left(g,\omega_p^m\right)+\|u^{m-1}\|^2+\|\omega^{m-1}\|^2-2k\left(\xi\omega_p^m,u_p^m\right)\\
&\leq 2k\|f\|\|u_p^m\|+2k\|g\|\|\omega_p^m\|+2k|\xi|\,\|\omega_p^m\|\|u_p^m\|+\|u^{m-1}\|^2+\|\omega^{m-1}\|^2\\
&\leq \frac{k}{\nu\lambda_1}\|f\|^2+\nu k|||u_p^m|||^2+\frac{k}{\lambda_2}\|g\|^2+k|||\omega_p^m|||^2+k|\xi|\,\|\omega_p^m\|^2\\
&\quad+k|\xi|\,\|u_p^m\|^2+\|u^{m-1}\|^2+\|\omega^{m-1}\|^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(1-k|\xi|)\,\|u_p^m\|^2+(1-k|\xi|)\,\|\omega_p^m\|^2\\
&\quad+\|u_p^m-u^{m-1}\|^2+\|\omega_p^m-\omega^{m-1}\|^2+\nu k|||u_p^m|||^2+k|||\omega_p^m|||^2\\
&\leq\frac{k}{\nu\lambda_1}\|f\|^2+\frac{k}{\lambda_2}\|g\|^2+\|u^{m-1}\|^2+\|\omega^{m-1}\|^2.
\end{aligned}
$$

Let k be small enough such that $1-k|\xi|>0$, one has

$$
\begin{aligned}
&\|u_p^m\|^2+\|\omega_p^m\|^2+\frac{1}{1-k|\xi|}\|u_p^m-u^{m-1}\|^2\\
&\quad+\frac{1}{1-k|\xi|}\|\omega_p^m-\omega^{m-1}\|^2+\frac{\nu k}{1-k|\xi|}|||u_p^m|||^2+\frac{k}{1-k|\xi|}|||\omega_p^m|||^2\\
&\leq\frac{1}{1-k|\xi|}\left[\frac{k}{\nu\lambda_1}\|f\|^2+\frac{k}{\lambda_2}\|g\|^2+\|u^{m-1}\|^2+\|\omega^{m-1}\|^2\right].
\end{aligned}
\tag{2.7}
$$

Now taking the L^2 inner product of the equation (2.1) with $A_1u_p^m$ and of the equation (2.2) with $A_2\omega_p^m$, we obtain

$$
\begin{aligned}
&\nu k\|A_1u_p^m\|^2+k\|A_2\omega_p^m\|^2\\
&=k\left(f,A_1u_p^m\right)+k\left(g,A_2\omega_p^m\right)-\left(u_p^m-u^{m-1},A_1u_p^m\right)-\left(\omega_p^m-\omega^{m-1},A_2\omega_p^m\right)\\
&\quad -kb_N\left(u_p^m,u_p^m,A_1u_p^m\right)-kc\left(u_p^m,\omega_p^m,A_2\omega_p^m\right)-k\left(\xi\omega_p^m,A_1u_p^m\right)\\
&\leq k\|f\|\|A_1u_p^m\|+k\|g\|\|A_2\omega_p^m\|+\|u_p^m-u^{m-1}\|\|A_1u_p^m\|+\|\omega_p^m-\omega^{m-1}\|\|A_2\omega_p^m\|\\
&\quad +k\frac{N}{|||u_p^m|||}\|\left(u_p^m\cdot\nabla\right)u_p^m\|\|A_1u_p^m\|+k\left|c\left(u_p^m,\omega_p^m,A_2\omega_p^m\right)\right|+k\|\xi\omega_p^m\|\|A_1u_p^m\|\\
&\leq k\|f\|\|A_1u_p^m\|+k\|g\|\|A_2\omega_p^m\|+\|u_p^m-u^{m-1}\|\|A_1u_p^m\|+\|\omega_p^m-\omega^{m-1}\|\|A_2\omega_p^m\|\\
&\quad +k\frac{N}{|||u_p^m|||}C|||u_p^m|||\,|||u_p^m|||^{\frac{1}{2}}\|A_1u_p^m\|^{\frac{3}{2}}+k\left|c\left(u_p^m,\omega_p^m,A_2\omega_p^m\right)\right|+k\left|\xi\right|\|\omega_p^m\|\|A_1u_p^m\|\\
&\leq k\|f\|\|A_1u_p^m\|+k\|g\|\|A_2\omega_p^m\|+\|u_p^m-u^{m-1}\|\|A_1u_p^m\|+\|\omega_p^m-\omega^{m-1}\|\|A_2\omega_p^m\|\\
&\quad +CNk|||u_p^m|||^{\frac{1}{2}}\|A_1u_p^m\|^{\frac{3}{2}}+k\left|c\left(u_p^m,\omega_p^m,A_2\omega_p^m\right)\right|+k\left|\xi\right|\|\omega_p^m\|\|A_1u_p^m\|.
\end{aligned}
$$

Since

$$
\left|c\left(u,\omega,A_2\omega\right)\right|\leq\|u\|_{L^\infty(\Omega)}\|\nabla\omega\|_{L^2(\Omega)}\|A_2\omega\|_{L^2(\Omega)}\leq C\left[\frac{\varepsilon}{2}|||u|||^2|||\omega|||^2+\frac{1}{2\varepsilon}\|A_2\omega\|^2\right],
$$

we can get

$$
\begin{aligned}
\|A_1u_p^m\|^2+\|A_2\omega_p^m\|^2&\leq C\|f\|^2+C\|g\|^2+C\|u_p^m-u^{m-1}\|^2+C\|\omega_p^m-\omega^{m-1}\|^2\\
&\quad +CNk|||u_p^m|||^2+ck\left|\xi\right|\|\omega_p^m\|^2+c|||u_p^m|||^2|||\omega_p^m|||^2.
\end{aligned}\tag{2.8}
$$

By (2.7)-(2.8), we can get

$$
\|A_1u_p^m\|^2+\|A_2\omega_p^m\|^2\leq C\left(\|f\|,\|g\|,\nu,\lambda_1,\lambda_2,k,\|u^{m-1}\|,\|\omega^{m-1}\|,N,|\xi|\right).
$$

Step 3: Passage to the limit. For k and m fixed, from the above inequality we can see $\left\{u_p^m\right\}_p$, $\left\{\omega_p^m\right\}_p$ are bounded in $D(A_1)$ and $D(A_2)$, respectively. Thus one can extract from $\left\{u_p^m\right\}_p$ and $\left\{\omega_p^m\right\}_p$ subsequences respectively, denoted also by $\left\{u_p^m\right\}_p$, $\left\{\omega_p^m\right\}_p$, such that $u_p^m\rightharpoonup u^m$, $p\to\infty$ in $D(A_1)$, and $\omega_p^m\rightharpoonup\omega^m$, $p\to\infty$ in $D(A_2)$. But, $D(A_1)\hookrightarrow\mathbb{V}$ and $D(A_2)\hookrightarrow H_0^1(\Omega)$ are compact, so $u_p^m\to u^m$, $p\to\infty$ in $\mathbb{V}$, and $\omega_p^m\to\omega^m$, $p\to\infty$ in $H_0^1(\Omega)$. Next, we prove that $\left\{u^m,\omega^m\right\}$ is the solution of (2.1)-(2.2). For the purpose, it is enough to show that

$$
\lim_{p\to\infty}b_N\left(u_p^m,u_p^m,v_1\right)=b_N\left(u^m,u^m,v_1\right)\quad\text{and}\quad\lim_{p\to\infty}c\left(u_p^m,\omega_p^m,v_2\right)=c\left(u^m,\omega^m,v_2\right).
$$

To this end, we calculate as follows

$$
\begin{aligned}
&b_N\left(u_p^m,u_p^m,v_1\right)-b_N\left(u^m,u^m,v_1\right)=F_N\left(|||u_p^m|||\right)b\left(u_p^m,u_p^m,v_1\right)-F_N\left(|||u^m|||\right)b\left(u^m,u^m,v_1\right)\\
&=F_N\left(|||u_p^m|||\right)\left[b\left(u_p^m,u_p^m,v_1\right)-b\left(u^m,u^m,v_1\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+F_N\left(|||u_p^m|||\right)b\left(u^m,u^m,v_1\right)-F_N\left(|||u^m|||\right)b\left(u^m,u^m,v_1\right)\\
&=F_N\left(|||u_p^m|||\right)\left[b\left(u_p^m,u_p^m,v_1\right)-b\left(u^m,u^m,v_1\right)\right]+\left[F_N\left(|||u_p^m|||\right)-F_N\left(|||u^m|||\right)\right]b\left(u^m,u^m,v_1\right)\\
&\le\left|b\left(u_p^m,u_p^m,v_1\right)-b\left(u^m,u^m,v_1\right)\right|+\frac{\left||||u_p^m|||-|||u^m|||\right|}{|||u^m|||}\left|b\left(u^m,u^m,v_1\right)\right|.
\end{aligned}
$$

Following [13], one has $\left|b\left(u_p^m,u_p^m,v_1\right)-b\left(u^m,u^m,v_1\right)\right|\to 0,\ p\to\infty$. And since $\left|b\left(u^m,u^m,v_1\right)\right|$ is bounded uniformly with respect to p, one sees that

$$
\frac{\left||||u_p^m|||-|||u^m|||\right|}{|||u^m|||}\left|b\left(u^m,u^m,v_1\right)\right|\to 0, p\to\infty.
$$

Therefore, $\lim\limits_{p\to\infty} b_N\left(u_p^m,u_p^m,v_1\right)=b_N\left(u^m,u^m,v_1\right)$. Similarly, we can get

$$
\lim_{p\to\infty} c\left(u_p^m,\omega_p^m,v_2\right)=c\left(u^m,\omega^m,v_2\right).
$$

So $\left\{u^m,\omega^m\right\}$ is the solution of (1.7)-(1.8). The proof of Theorem 1 is completed. □

3. BOUNDEDNESS

Let $\left\{u^m,\omega^m\right\}$ be the solution sequence of (1.7)-(1.8), we are going to show that the boundedness of $\left\{u^m,\omega^m\right\}$ in $\mathbb{H}\times L^2(\Omega)$, $\mathbb{V}\times H_0^1(\Omega)$ and $D(A_1)\times D(A_2)$ respectively.

3.1. *Boundedness in $\mathbb{H}\times L^2(\Omega)$*

Lemma 2. *Let $\left\{u^m,\omega^m\right\}_m$ be the solution sequence of (1.7)-(1.8), constructed in Theorem 1. Then for all integers $m\ge 1$, $\left\{u^m,\omega^m\right\}$ remain bounded in $\mathbb{H}\times L^2(\Omega)$, in the following sense,*

$$
\|\omega^m\|^2\le K_1, \qquad \forall m\ge 1; \tag{3.1}
$$

$$
\|u^m\|^2\le K_1^*, \qquad \forall m\ge 1; \tag{3.2}
$$

$$
\sum_{m=i}^{L-1}\|\omega^m-\omega^{m-1}\|^2+\sum_{m=i}^{L-1}k|||\omega^m|||^2\le\frac{k}{\lambda_2}\|g\|^2(L-i)+K_1, \qquad L\ge i; \tag{3.3}
$$

$$
\begin{aligned}
&\sum_{m=i}^{L-1}\|u^m-u^{m-1}\|^2+\sum_{m=i}^{L-1}\nu k|||u^m|||^2\\
&\le K_1^*+(L-i)\left[\frac{2k}{\nu\lambda_1}\|f\|^2+\frac{2k|\xi|^2}{\nu\lambda_1}K_1\right], \qquad L\ge i,
\end{aligned} \tag{3.4}
$$

where $K_1\triangleq\|\omega^0\|^2+\dfrac{\|g\|^2}{\lambda_2^2}$, $\quad K_1^*\triangleq\|u^0\|^2+\dfrac{2\|f\|^2}{\nu^2\lambda_1^2}+\dfrac{2|\xi|^2}{\nu^2\lambda_1^2}K_1$.

Proof. Taking the L^2 inner product of the equation (1.8) with $2k\omega^m$, we obtain

$$
\|\omega^m\|^2+\|\omega^m-\omega^{m-1}\|^2+2k|||\omega^m|||^2=2k(g,\omega^m)+\|\omega^{m-1}\|^2
$$

$$\le 2k\frac{\|g\|}{\sqrt{\lambda_2}}|||\omega^m||| + \|\omega^{m-1}\|^2 \le \frac{k}{\lambda_2}\|g\|^2 + k|||\omega^m|||^2 + \|\omega^{m-1}\|^2.$$

Thus, we can get

$$\|\omega^m\|^2 + \|\omega^m - \omega^{m-1}\|^2 + k|||\omega^m|||^2 \le \frac{k}{\lambda_2}\|g\|^2 + \|\omega^{m-1}\|^2. \tag{3.5}$$

By Poincaré inequality, we have

$$\|\omega^m\|^2 \le \frac{1}{1+k\lambda_2}\|\omega^{m-1}\|^2 + \frac{1}{1+k\lambda_2}\frac{k}{\lambda_2}\|g\|^2.$$

Using the above inequality recursively, we find

$$\|\omega^m\|^2 \le \frac{1}{(1+k\lambda_2)^m}\|\omega^0\|^2 + \frac{\|g\|^2}{\lambda_2^2}\left[1 - \frac{1}{(1+k\lambda_2)^m}\right] \le \|\omega^0\|^2 + \frac{\|g\|^2}{\lambda_2^2} \triangleq K_1, \tag{3.6}$$

that is, equation (3.1) holds. On the other hand, taking the L^2 inner product of the equation (1.7) with $2ku^m$, we obtain

$$\begin{aligned}\|u^m\|^2 + \|u^m - u^{m-1}\|^2 + 2\nu k|||u^m|||^2 &= 2k(f,u^m) + \|u^{m-1}\|^2 - 2k(\xi\omega^m, u^m)\\ &\le 2k\|f\|\|u^m\| + \|u^{m-1}\|^2 + 2k\|\xi\omega^m\|\|u^m\|\\ &\le \frac{2k}{\sqrt{\lambda_1}}\|f\||||u^m||| + \|u^{m-1}\|^2 + \frac{2k|\xi|}{\sqrt{\lambda_1}}\|\omega^m\||||u^m|||\\ &\le \frac{2k}{\nu\lambda_1}\|f\|^2 + \nu k|||u^m|||^2 + \|u^{m-1}\|^2 + \frac{2k|\xi|^2}{\nu\lambda_1}\|\omega^m\|^2.\end{aligned}$$

Combination (3.6) imply that

$$\|u^m\|^2 + \|u^m - u^{m-1}\|^2 + \nu k|||u^m|||^2 \le \|u^{m-1}\|^2 + \frac{2k}{\nu\lambda_1}\|f\|^2 + \frac{2k|\xi|^2}{\nu\lambda_1}K_1. \tag{3.7}$$

Using Poincaré inequality again, we can get

$$(1+\nu k\lambda_1)\|u^m\|^2 \le \|u^{m-1}\|^2 + \frac{2k}{\nu\lambda_1}\|f\|^2 + \frac{2k|\xi|^2}{\nu\lambda_1}K_1.$$

Using the above inequality recursively, we find

$$\begin{aligned}\|u^m\|^2 &\le \frac{1}{(1+\nu k\lambda_1)^m}\|u^0\|^2 + \left[1 - \frac{1}{(1+\nu k\lambda_1)^m}\right]\left[\frac{2\|f\|^2}{\nu^2\lambda_1^2} + \frac{2|\xi|^2}{\nu^2\lambda_1^2}K_1\right]\\ &\le \|u^0\|^2 + \frac{2\|f\|^2}{\nu^2\lambda_1^2} + \frac{2|\xi|^2}{\nu^2\lambda_1^2}K_1 \triangleq K_1^*,\end{aligned} \tag{3.8}$$

that is, the equation (3.2) holds. Adding up (3.5) with m from i to $L-1$, we find

$$\|\omega^{L-1}\|^2 + \sum_{m=i}^{L-1}\|\omega^m - \omega^{m-1}\|^2 + \sum_{m=i}^{L-1}k|||\omega^m|||^2 \le \frac{k}{\lambda_2}\|g\|^2(L-i) + \|\omega^{i-1}\|^2.$$

Combination (3.6) imply that (3.3) holds. Adding up (3.7) with m from i to $L-1$, we find

$$\|u^{L-1}\|^2+\sum_{m=i}^{L-1}\|u^m-u^{m-1}\|^2+\sum_{m=i}^{L-1}\nu k|||u^m|||^2 \le \|u^{i-1}\|^2+(L-i)\left[\frac{2k}{\nu\lambda_1}\|f\|^2+\frac{2k|\xi|^2}{\nu\lambda_1}K_1\right].$$

Combination (3.8) imply that (3.4) holds. The proof of Lemma 2 is completed. □

3.2. *Boundedness in* $\mathbb{V}\times H_0^1(\Omega)$

Lemma 3 ([5]). *Let* $\{x_m\}$, $\{y_m\}$, $\{z_m\}$ *be non-negative sequences. Assume that there are integers* m_0, m_1 *such that, for* $k>0$,

$$\begin{aligned} ky_m &< \frac{1}{2}, && \forall m\ge m_0,\\ (1-ky_m)x_m &\le x_{m-1}+kz_m, && \forall m> m_0+m_1. \end{aligned}$$

and that for all integers $m*\ge m_0$, $k\sum_{m=m*}^{m*+m_1} y_m\le a_1$, $k\sum_{m=m*}^{m*+m_1} z_m\le a_2$, $k\sum_{m=m*}^{m*+m_1} x_m\le a_3$. *Then*

$$x_m\le\left[\frac{a_3}{km_1}+a_2\right]e^{4a_1},\qquad \forall m>m_0+m_1.$$

Lemma 4. *Let* $\{u^m,\omega^m\}_m$ *be the solutions sequence of* (1.7)-(1.8), *constructed in Theorem 1. Then for all integers* $m\ge 1$, $\{u^m,\omega^m\}$ *remain bounded in* $\mathbb{V}\times H_0^1(\Omega)$, *in the following sense, there exists positive constants* C, a_1, a_2, a_3, *such that*

$$|||\omega^m|||^2\le K_3,\qquad \forall m\ge 1; \tag{3.9}$$

$$|||u^m|||^2\le K_2,\qquad \forall m\ge 1; \tag{3.10}$$

$$\begin{aligned}&\sum_{m=i}^{L-1}|||u^m-u^{m-1}|||^2+\sum_{m=i}^{L-1}\frac{\nu k}{16}\|A_1u^m\|^2\\ &\quad\le K_2+C\left[\frac{\|f\|^2}{\nu}+\frac{N^8}{\nu^7}K_1^*+\frac{|\xi|^2}{\nu}K_1\right](L-i)k,\qquad L\ge i\ge 1;\end{aligned} \tag{3.11}$$

$$\begin{aligned}&\sum_{m=i}^{L-1}|||\omega^m-\omega^{m-1}|||^2+\sum_{m=i}^{L-1}\frac{1}{2}k\|A_2\omega^m\|^2\\ &\quad\le K_3+\left[k\|g\|^2+CK_2\left(\frac{k}{\lambda_2}\|g\|^2+K_1\right)\right](L-i),\qquad L\ge i\ge 1,\end{aligned} \tag{3.12}$$

where

$$K_2\triangleq |||u^0|||^2+\frac{C}{\nu^2\lambda_1}\|f\|^2+\frac{N^8C}{\nu^8\lambda_1}K_1^*+\frac{C|\xi|^2}{\nu^2\lambda_1}K_1,$$

$$K_3\triangleq\max\left\{|||\omega^0|||^2+\frac{\|g\|^2}{\frac{1}{2}\lambda_2-CK_2},\ \left[\frac{a_3}{km_1}+a_2\right]e^{4a_1},\ C|||\omega^0|||^2+\frac{C}{K_2-\lambda_2}\|g\|^2\right\},$$

$L \geq i \geq 1$.

Proof. Taking the L^2 inner product of the equation (1.7) with $2kA_1u^m$, we obtain

$$|||u^m|||^2 + |||u^m - u^{m-1}|||^2 + 2\nu k\|A_1u^m\|^2$$
$$= |||u^{m-1}|||^2 + 2k(f, A_1u^m) - 2kb_N(u^m, u^m, A_1u^m) - 2k(\xi\omega^m, A_1u^m).$$

Each term of the right hand side of the above equation can be majorize by (1.3) as follows

$$2k(f, A_1u^m) \leq 2k\|f\|\|A_1u^m\| \leq \frac{k}{\nu}\|f\|^2 + \nu k\|A_1u^m\|^2;$$
$$2k\,|(\xi\omega^m, A_1u^m)| \leq \frac{16k}{\nu}\,|\xi|^2\,\|\omega^m\|^2 + \frac{\nu k}{16}\|A_1u^m\|^2;$$
$$\begin{aligned} 2k\,|b_N(u^m, u^m, A_1u^m)| &= 2kF_N(|||u^m|||)\,|b(u^m, u^m, A_1u^m)| \\ &\leq 2k\frac{N}{|||u^m|||}C\|u^m\|^{\frac14}\|A_1u^m\|^{\frac34}|||u^m|||\|A_1u^m\| \\ &= CNk\|u^m\|^{\frac14}\|A_1u^m\|^{\frac74} \leq \frac78\nu k\|A_1u^m\|^2 + \frac{N^8C^8k}{8\nu^7}\|u^m\|^2. \end{aligned}$$

Then

$$\begin{aligned} &|||u^m|||^2 + |||u^m - u^{m-1}|||^2 + 2\nu k\|A_1u^m\|^2 \\ &\leq |||u^{m-1}|||^2 + \frac{k}{\nu}\|f\|^2 + \nu k\|A_1u^m\|^2 + \frac78\nu k\|A_1u^m\|^2 \\ &\quad + \frac{N^8C^8k}{8\nu^7}\|u^m\|^2 + \frac{16k}{\nu}\,|\xi|^2\,\|\omega^m\|^2 + \frac{\nu k}{16}\|A_1u^m\|^2. \end{aligned}$$

By (3.1) and (3.2) gives

$$\begin{aligned} &|||u^m|||^2 + |||u^m - u^{m-1}|||^2 + \frac{\nu k}{16}\|A_1u^m\|^2 \\ &\leq |||u^{m-1}|||^2 + \frac{k}{\nu}\|f\|^2 + \frac{N^8C^8k}{8\nu^7}\|u^m\|^2 + \frac{16k}{\nu}\,|\xi|^2\,\|\omega^m\|^2 \\ &\leq |||u^{m-1}|||^2 + \frac{k}{\nu}\|f\|^2 + \frac{N^8C^8k}{8\nu^7}K_1^* + \frac{16k}{\nu}\,|\xi|^2\,K_1, \end{aligned} \tag{3.13}$$

which together with $|||u^m||| \leq \frac{1}{\sqrt{\lambda_1}}\|A_1u^m\|$ gives

$$|||u^m|||^2 + |||u^m - u^{m-1}|||^2 + \frac{\nu k\lambda_1}{16}|||u^m|||^2 \leq |||u^{m-1}|||^2 + \frac{k}{\nu}\|f\|^2 + \frac{N^8C^8k}{8\nu^7}K_1^* + \frac{16k}{\nu}\,|\xi|^2\,K_1.$$

Therefore

$$\left[1 + \frac{\nu k\lambda_1}{16}\right]|||u^m|||^2 \leq |||u^{m-1}|||^2 + \frac{k}{\nu}\|f\|^2 + \frac{N^8C^8k}{8\nu^7}K_1^* + \frac{16k}{\nu}\,|\xi|^2\,K_1,$$

which is

$$|||u^m|||^2 \le \frac{1}{1+\frac{\nu k\lambda_1}{16}}|||u^{m-1}|||^2 + \frac{1}{1+\frac{\nu k\lambda_1}{16}}\left[\frac{k}{\nu}\|f\|^2 + \frac{N^8C^8k}{8\nu^7}K_1^* + \frac{16k}{\nu}|\xi|^2K_1\right].$$

Using the above inequality recursively, we find

$$\begin{aligned}
|||u^m|||^2 &\le \frac{1}{(1+\frac{\nu k\lambda_1}{16})^m}|||u^0|||^2 + \frac{16}{\nu k\lambda_1}\left[1-\frac{1}{(1+\frac{\nu k\lambda_1}{16})^m}\right] \\
&\quad \times \left[\frac{k}{\nu}\|f\|^2 + \frac{N^8C^8k}{8\nu^7}K_1^* + \frac{16k}{\nu}|\xi|^2K_1\right] \\
&\le |||u^0|||^2 + \frac{16}{\nu k\lambda_1}\left[\frac{k}{\nu}\|f\|^2 + \frac{N^8C^8k}{8\nu^7}K_1^* + \frac{16k}{\nu}|\xi|^2K_1\right] \\
&\le |||u^0|||^2 + \frac{C}{\nu^2\lambda_1}\|f\|^2 + \frac{N^8C}{\nu^8\lambda_1}K_1^* + \frac{C|\xi|^2}{\nu^2\lambda_1}K_1 \triangleq K_2. \qquad (3.14)
\end{aligned}$$

Therefore, we get (3.10), and u^m is bounded in $\mathbb{V}$.

Taking the L^2 inner product of the equation (1.8) with $2kA_2\omega^m$, we find

$$\begin{aligned}
&|||\omega^m|||^2 + |||\omega^m-\omega^{m-1}|||^2 + 2k\|A_2\omega^m\|^2 \\
&= |||\omega^{m-1}|||^2 + 2k(g, A_2\omega^m) - 2kc(u^m,\omega^m,A_2\omega^m) \\
&\le |||\omega^{m-1}|||^2 + k\|g\|^2 + k\|A_2\omega^m\|^2 + CK_2k|||\omega^m|||^2 + \frac{1}{2}k\|A_2\omega^m\|^2.
\end{aligned}$$

Taking k small enough such that $1-CK_2k>0$, we obtain

$$(1-CK_2k)|||\omega^m|||^2 + \frac{1}{2}k\|A_2\omega^m\|^2 + |||\omega^m-\omega^{m-1}|||^2 \le |||\omega^{m-1}|||^2 + k\|g\|^2, \quad (3.15)$$

which, together with $|||\omega^m||| \le \frac{1}{\sqrt{\lambda_2}}\|A_2\omega^m\|$, leads to

$$\left[1+\frac{1}{2}\lambda_2k - CK_2k\right]|||\omega^m|||^2 + |||\omega^m-\omega^{m-1}|||^2 \le |||\omega^{m-1}|||^2 + k\|g\|^2. \quad (3.16)$$

There are two cases to discuss the above inequality (3.16):

Case 1: If $\frac{1}{2}\lambda_2 > CK_2$, which is $1+\frac{1}{2}\lambda_2k - CK_2k > 1$, then from (3.16), one has

$$|||\omega^m|||^2 \le \frac{1}{1+\frac{1}{2}\lambda_2k-CK_2k}|||\omega^{m-1}|||^2 + \frac{k}{1+\frac{1}{2}\lambda_2k-CK_2k}\|g\|^2.$$

Using the above inequality recursively, we find

$$\begin{aligned}
|||\omega^m|||^2 &\le \frac{1}{(1+\frac{1}{2}\lambda_2k-CK_2k)^m}|||\omega^0|||^2 \\
&\quad + \frac{1}{\frac{1}{2}\lambda_2-CK_2}\|g\|^2\left[1-\frac{1}{(1+\frac{1}{2}\lambda_2k-CK_2k)^m}\right]
\end{aligned}$$

$$\leq |||\omega^0|||^2 + \frac{1}{\frac{1}{2}\lambda_2 - CK_2}\|g\|^2.$$

Case 2: If $\frac{1}{2}\lambda_2 \leq CK_2$, which is $1+\frac{1}{2}\lambda_2 k - CK_2 k \leq 1$, then from (3.16), one gets

$$\left[1 - k\left(CK_2 - \tfrac{1}{2}\lambda_2\right)\right] |||\omega^m|||^2 \leq |||\omega^{m-1}|||^2 + k\|g\|^2.$$

In the following, we will use Lemma 3 to discuss the above inequality. Let $x_m = |||\omega^m|||^2$, $y_m = CK_2 - \frac{1}{2}\lambda_2$, $z_m = \|g\|^2$. Obviously, $\{x_m\}$, $\{y_m\}$, $\{z_m\}$ are non-negative sequences, and there exists m_0, m_1 such that

$$\begin{aligned} ky_m = CK_2 k - \frac{1}{2}k\lambda_2 &< \frac{1}{2}, && \forall m \geq m_0,\\ (1-ky_m)x_m \leq x_{m-1} + kz_m&, && \forall m > m_0 + m_1, \end{aligned}$$

and that for all integers $m* \geq m_0$, from (3.3), we get

$$\begin{aligned} k\sum_{m=m*}^{m*+m_1} y_m &= k\sum_{m=m*}^{m*+m_1}\left[CK_2 - \tfrac{1}{2}\lambda_2\right] = k(m_1+1)\left[CK_2 - \tfrac{1}{2}\lambda_2\right] \leq a_1,\\ k\sum_{m=m*}^{m*+m_1} z_m &= k\sum_{m=m*}^{m*+m_1}\|g\|^2 = k(m_1+1)\|g\|^2 \leq a_2,\\ k\sum_{m=m*}^{m*+m_1} x_m &= k\sum_{m=m*}^{m*+m_1}|||\omega^m|||^2 \leq \frac{k}{\lambda_2}\|g\|^2(m_1+1) + K_1 \leq a_3. \end{aligned}$$

By Lemma 3, we get

$$|||\omega^m|||^2 \leq \left[\frac{a_3}{km_1} + a_2\right] e^{4a_1}, \qquad \forall m > m_0 + m_1.$$

When $m \leq m_0 + m_1$, there is $\left[1+\frac{1}{2}\lambda_2 k - CK_2 k\right]^m \geq \left[1+\frac{1}{2}\lambda_2 k - CK_2 k\right]^{m_0+m_1}$, thus

$$\begin{aligned} |||\omega^m|||^2 \leq{}& \frac{1}{(1+\frac{1}{2}\lambda_2 k - CK_2 k)^{m_0+m_1}}|||\omega^0|||^2\\ &+ \frac{1}{CK_2 - \frac{1}{2}\lambda_2}\|g\|^2\left[\frac{1}{(1+\frac{1}{2}\lambda_2 k - CK_2 k)^{m_0+m_1}} - 1\right]\\ \leq{}& C|||\omega^0|||^2 + \frac{C}{K_2 - \lambda_2}\|g\|^2, \qquad \forall m \leq m_0 + m_1. \end{aligned}$$

Above all, we get (3.9), that is, ω^m is bounded in $H_0^1(\Omega)$.

Next, adding up (3.13) with $m = i, i+1, \cdots, L-1$, we find

$$|||u^{L-1}|||^2 + \sum_{m=i}^{L-1}|||u^m - u^{m-1}|||^2 + \sum_{m=i}^{L-1}\frac{\nu k}{16}\|A_1 u^m\|^2$$

$$\leq |||u^{i-1}|||^2 + (L-i)\left[\frac{k}{\nu}\|f\|^2 + \frac{N^8C^8k}{8\nu^7}K_1^* + \frac{16k}{\nu}|\xi|^2 K_1\right].$$

Together with (3.14), leads to (3.11). Using (3.5)-(3.6) and (3.15), we obtain

$$\begin{aligned}
&|||\omega^m|||^2 + \frac{1}{2}k\|A_2\omega^m\|^2 + |||\omega^m - \omega^{m-1}|||^2 \\
&\leq |||\omega^{m-1}|||^2 + k\|g\|^2 + CK_2k|||\omega^m|||^2 \\
&\leq |||\omega^{m-1}|||^2 + k\|g\|^2 + CK_2\left[\frac{k}{\lambda_2}\|g\|^2 + K_1\right].
\end{aligned}$$

Adding up the above inequality with $m = i, i+1, \cdots, L-1$, we get

$$\begin{aligned}
&|||\omega^{L-1}|||^2 + \sum_{m=i}^{L-1}|||\omega^m - \omega^{m-1}|||^2 + \sum_{m=i}^{L-1}\frac{1}{2}k\|A_2\omega^m\|^2 \\
&\qquad \leq |||\omega^{i-1}|||^2 + (L-i)\left[k\|g\|^2 + CK_2\left(\frac{k}{\lambda_2}\|g\|^2 + K_1\right)\right].
\end{aligned}$$

Together with (3.9), one obtains (3.12). Above all, we have $\{u^m, \omega^m\}$ is bounded in $\mathbb{V} \times H_0^1(\Omega)$. The proof of Lemma 4 is completed. □

3.3. *Boundedness in $D(A_1) \times D(A_2)$*

Lemma 5. *Assuming that $f \in L^2(\Omega)^3$, $g \in L^2(\Omega)$, $u_0 \in D(A_1)$, $\omega_0 \in D(A_2)$. Let $\{u^m, \omega^m\}_{m\geq 1}$ be the solutions sequence of (1.7)-(1.8). Then there exist positive constants $C_0 \equiv C_0(\nu, N, |\xi|, K_1, K_2, K_3, \|A_1u^0\|, \|A_2\omega^0\|, \|f\|, \|g\|)$ and $K_4 \equiv K_4(k, \nu, \lambda_1, \lambda_2, N, |\xi|, \|u^0\|, \|\omega^0\|, \|f\|, \|g\|)$ such that*

$$\left\|\frac{u^1 - u^0}{k}\right\| + \left\|\frac{\omega^1 - \omega^0}{k}\right\| \leq C_0; \tag{3.17}$$

$$\left\|\frac{u^m - u^{m-1}}{k}\right\|^2 + \left\|\frac{\omega^m - \omega^{m-1}}{k}\right\|^2 \leq K_4, \quad m > 1. \tag{3.18}$$

Proof. Let $u = u^1 - u^0$, $\omega = \omega^1 - \omega^0$, then from (1.7)-(1.8), one obtains

$$\begin{cases}
\dfrac{u}{k} + \nu A_1u + \nu A_1u^0 + B_N(u+u^0, u+u^0) + \xi(\omega+\omega^0) = f, & (3.19)\\[2ex]
\dfrac{\omega}{k} + A_2\omega + A_2\omega^0 + C(u+u^0, \omega+\omega^0) = g. & (3.20)
\end{cases}$$

Taking the scalar product of the equation (3.19) with A_1u, we obtain

$$\frac{|||u|||^2}{k} + \nu\|A_1u\|^2 = (f - \nu A_1u^0, A_1u) - b_N(u+u^0, u+u^0, A_1u) - (\xi\omega + \xi\omega^0, A_1u). \tag{3.21}$$

Taking the scalar product of the equation (3.20) with $A_2\omega$, we find

$$\frac{|||\omega|||^2}{k} + \|A_2\omega\|^2 = (g - A_2\omega^0, A_2\omega) - c(u+u^0, \omega+\omega^0, A_2\omega). \tag{3.22}$$

Putting together (3.21) and (3.22), one obtains

$$\begin{aligned}
&\frac{|||u|||^2}{k} + \frac{|||\omega|||^2}{k} + \nu\|A_1u\|^2 + \|A_2\omega\|^2 \\
&= (f - \nu A_1u^0, A_1u) + (g - A_2\omega^0, A_2\omega) - (\xi\omega + \xi\omega^0, A_1u) \\
&\quad - b_N(u+u^0, u+u^0, A_1u) - c(u+u^0, \omega+\omega^0, A_2\omega).
\end{aligned}$$

By (3.1) and (3.9)-(3.10) gives

$$\begin{aligned}
|b_N(u+u^0, u+u^0, A_1u)| &= F_N(|||u+u^0|||)|b(u+u^0, u+u^0, A_1u)| \\
&\le F_N(|||u+u^0|||)\|(u+u^0)\cdot\nabla(u+u^0)\|\|A_1u\| \\
&\le F_N(|||u+u^0|||)\|u+u^0\|_{L_6}\|\nabla(u+u^0)\|_{L_3}\|A_1u\| \\
&\le \frac{CN}{|||u+u^0|||}|||u+u^0||||||u+u^0|||^{\frac{1}{2}}\|A_1(u+u^0)\|^{\frac{1}{2}}\|A_1u\| \\
&\le CN|||u+u^0|||^{\frac{1}{2}}\|A_1(u+u^0)\|^{\frac{1}{2}}\|A_1u\| \\
&\le CNK_2^{\frac{1}{4}}\|A_1u\|^{\frac{3}{2}} + CNK_2^{\frac{1}{4}}\|A_1u^0\|^{\frac{1}{2}}\|A_1u\|; \\
(f - \nu A_1u^0, A_1u) &\le \|f\|\|A_1u\| + \nu\|A_1u^0\|\|A_1u\|; \\
(g - A_2\omega^0, A_2\omega) &\le \|g\|\|A_2\omega\| + \|A_2\omega^0\|\|A_2\omega\|; \\
|c(u+u^0, \omega+\omega^0, A_2\omega)| &\le C|||u^1||||||\omega^1|||\|A_2\omega\| \le CK_2^{\frac{1}{2}}K_3^{\frac{1}{2}}\|A_2\omega\|; \\
|(\xi\omega+\xi\omega^0, A_1u)| &\le |\xi|\|\omega^1\|\|A_1u\| \le |\xi|K_1^{\frac{1}{2}}\|A_1u\|.
\end{aligned}$$

Therefore, one has

$$\begin{aligned}
&\frac{|||u|||^2}{k} + \frac{|||\omega|||^2}{k} + \nu\|A_1u\|^2 + \|A_2\omega\|^2 \\
&\le \|f\|\|A_1u\| + \nu\|A_1u^0\|\|A_1u\| + \|g\|\|A_2\omega\| + \|A_2\omega^0\|\|A_2\omega\| \\
&\quad + CNK_2^{\frac{1}{4}}\|A_1u\|^{\frac{3}{2}} + CNK_2^{\frac{1}{4}}\|A_1u^0\|^{\frac{1}{2}}\|A_1u\| \\
&\quad + CK_2^{\frac{1}{2}}K_3^{\frac{1}{2}}\|A_2\omega\| + |\xi|K_1^{\frac{1}{2}}\|A_1u\|,
\end{aligned}$$

which, together with Young's inequality, leads to

$$\begin{aligned}
&\frac{|||u|||^2}{k} + \frac{|||\omega|||^2}{k} + C\|A_1u\|^2 + C\|A_2\omega\|^2 \\
&\le C\|f\|^2 + C\nu\|A_1u^0\|^2 + C\|g\|^2 + C\|A_2\omega^0\|^2
\end{aligned}$$

$$+CNK_2+CNK_2^{\frac{1}{2}}\|A_1u^0\|+CK_2K_3+C|\xi|^2K_1. \tag{3.23}$$

From (1.7)-(1.8), one has

$$\begin{cases} \dfrac{u^1-u^0}{k}=-\nu A_1u^1-B_N(u^1,u^1)-\xi\omega^1+f,\\ \dfrac{\omega^1-\omega^0}{k}=-A_2\omega^1-C(u^1,\omega^1)+g. \end{cases}$$

Thus

$$\begin{aligned} &\left\|\frac{u^1-u^0}{k}\right\|+\left\|\frac{\omega^1-\omega^0}{k}\right\|\\ &\le \nu\|A_1u^1\|+\|A_2\omega^1\|+\|B_N(u^1,u^1)\|+\|C(u^1,\omega^1)\|+\|\xi\omega^1\|+\|f\|+\|g\|\\ &\le \nu\|A_1u\|+\nu\|A_1u^0\|+\|A_2\omega\|+\|A_2\omega^0\|\\ &\quad+\|B_N(u^1,u^1)\|+\|C(u^1,\omega^1)\|+|\xi|\|\omega^1\|+\|f\|+\|g\|\\ &\le \nu\|A_1u\|+\nu\|A_1u^0\|+\|A_2\omega\|+\|A_2\omega^0\|\\ &\quad+CN\|\|u^1\|\|^{\frac{1}{2}}\|A_1u^1\|^{\frac{1}{2}}+\|\|u^1\|\|\|\|\omega^1\|\|+|\xi|\|\omega^1\|+\|f\|+\|g\|\\ &\le \nu\|A_1u\|+\nu\|A_1u^0\|+\|A_2\omega\|+\|A_2\omega^0\|+CNK_2^{\frac{1}{4}}\|A_1u\|^{\frac{1}{2}}\\ &\quad+CNK_2^{\frac{1}{4}}\|A_1u^0\|^{\frac{1}{2}}+K_2^{\frac{1}{2}}K_3^{\frac{1}{2}}+|\xi|K_1^{\frac{1}{2}}+\|f\|+\|g\|. \end{aligned}$$

Which together with (3.23), gives the desired result, that is

$$\left\|\frac{u^1-u^0}{k}\right\|+\left\|\frac{\omega^1-\omega^0}{k}\right\|\le C_0\left(\nu,N,|\xi|,K_1,K_2,K_3,\|A_1u^0\|,\|A_2\omega^0\|,\|f\|,\|g\|\right).$$

Then (3.17) is holds.

For $m>1$, let $u_*^m=\dfrac{u^m-u^{m-1}}{k}$, $\omega_*^m=\dfrac{\omega^m-\omega^{m-1}}{k}$ in (1.7)-(1.8), we obtain

$$\begin{cases} \dfrac{u_*^m-u_*^{m-1}}{k}+\nu A_1u_*^m+\dfrac{1}{k}\left[B_N(u^m,u^m)-B_N(u^{m-1},u^{m-1})\right]+\xi\omega_*^m=0, & (3.24)\\ \dfrac{\omega_*^m-\omega_*^{m-1}}{k}+A_2\omega_*^m+\dfrac{1}{k}\left[C(u^m,\omega^m)-C(u^{m-1},\omega^{m-1})\right]=0. & (3.25) \end{cases}$$

Taking the scalar product of the equation (3.24) with $2ku_*^m$, and taking the scalar product of the equation (3.25) with $2k\omega_*^m$, we obtain

$$\begin{aligned} &\|u_*^m\|^2+\|u_*^m-u_*^{m-1}\|^2+2k\nu\|\|u_*^m\|\|^2+\|\omega_*^m\|^2+\|\omega_*^m-\omega_*^{m-1}\|^2+2k\|\|\omega_*^m\|\|^2\\ &=\|u_*^{m-1}\|^2-2b_N(u^m,u^m,u_*^m)+2b_N(u^{m-1},u^{m-1},u_*^m)-2k(\xi\omega_*^m,u_*^m)\\ &\quad+\|\omega_*^{m-1}\|^2-2c(u^m,\omega^m,\omega_*^m)+2c(u^{m-1},\omega^{m-1},\omega_*^m). \end{aligned} \tag{3.26}$$

We now majorize the right-hand side of (3.26). By (1.3) and (3.10), one gets

$$
\begin{aligned}
&2b_N(u^{m-1},u^{m-1},u_*^m)-2b_N(u^m,u^m,u_*^m)\\
&=2F_N(|||u^{m-1}|||)b(u^{m-1},u^{m-1},u_*^m)-2F_N(|||u^m|||)b(u^m,u^m,u_*^m)\\
&=2\left[F_N(|||u^{m-1}|||)-F_N(|||u^m|||)\right]b(u^{m-1},u^{m-1},u_*^m)\\
&\quad+2F_N(|||u^m|||)\left[b(u^{m-1},u^{m-1},u_*^m)-b(u^m,u^m,u_*^m)\right]\\
&\le 2\left[F_N(|||u^{m-1}|||)-F_N(|||u^m|||)\right]b(u^{m-1},u^{m-1},u_*^m)\\
&\quad+2F_N(|||u^m|||)|kb(u_*^m,u^m,u_*^m)|\\
&\le C\frac{|||u^m-u^{m-1}|||}{|||u^{m-1}|||}|||u^{m-1}||||||u^{m-1}|||\|u_*^m\|^{\frac14}|||u_*^m|||^{\frac34}\\
&\quad+\frac{CNk}{|||u^m|||}\|u_*^m\|^{\frac14}|||u_*^m|||^{\frac34}|||u^m|||\|u_*^m\|^{\frac14}|||u_*^m|||^{\frac34}\\
&\le CK_2^{\frac12}k\|u_*^m\|^{\frac14}|||u_*^m|||^{\frac74}+CNk\|u_*^m\|^{\frac12}|||u_*^m|||^{\frac32},\\
&\quad-2k(\xi\omega_*^m,u_*^m)\le 2k|(\xi\omega_*^m,u_*^m)|\le 2k|\xi|\|\omega_*^m\|\|u_*^m\|.
\end{aligned}
$$

Together with (3.9), one has

$$
\begin{aligned}
2c(u^{m-1},\omega^{m-1},\omega_*^m)-2c(u^m,\omega^m,\omega_*^m)&\le 2|c(ku_*^m,\omega^m,\omega_*^m)|\\
&\le Ck|||u_*^m||||||\omega^m|||\|\omega_*^m\|\le CK_3^{\frac12}k|||u_*^m|||\|\omega_*^m\|.
\end{aligned}
$$

Thus, from (3.26) and above inequality, by Young's inequality, we obtain

$$
\begin{aligned}
&\|u_*^m\|^2+\|u_*^m-u_*^{m-1}\|^2+2k\nu|||u_*^m|||^2+\|\omega_*^m\|^2+\|\omega_*^m-\omega_*^{m-1}\|^2+2k|||\omega_*^m|||^2\\
&\le\|u_*^{m-1}\|^2+CK_2^{\frac12}k\|u_*^m\|^{\frac14}|||u_*^m|||^{\frac74}+CNk\|u_*^m\|^{\frac12}|||u_*^m|||^{\frac32}\\
&\quad+2k|\xi|\|\omega_*^m\|\|u_*^m\|+\|\omega_*^{m-1}\|^2+CK_3^{\frac12}k|||u_*^m|||\|\omega_*^m\|\\
&\le\|u_*^{m-1}\|^2+\|\omega_*^{m-1}\|^2+Ck\|u_*^m\|^2+2k\nu|||u_*^m|||^2+Ck\|\omega_*^m\|^2.
\end{aligned}
$$

Therefore

$$
(1-Ck)\left(\|u_*^m\|^2+\|\omega_*^m\|^2\right)\le\|u_*^{m-1}\|^2+\|\omega_*^{m-1}\|^2. \tag{3.27}
$$

Using the above inequality (3.27) recursively, we find

$$
\|u_*^m\|^2+\|\omega_*^m\|^2\le\frac{1}{(1-Ck)^{m-1}}\left(\|u_*^1\|^2+\|\omega_*^1\|^2\right).
$$

Obviously, when $m\le M_0=ent\left\{\frac{T}{k}\right\}$, then $\|u_*^m\|^2+\|\omega_*^m\|^2$ is bounded, where $ent\left\{\frac{T}{k}\right\}$ is the entire part of $\frac{T}{k}$ with T an arbitrarily fixed constant. Let $x_m=\|u_*^m\|^2+\|\omega_*^m\|^2$, $y_m=C$, $z_m=0$, then by Lemma 2-3 and equation (3.27), we see that $\|u_*^m\|^2+\|\omega_*^m\|^2$

is bounded as $m > M_0$. Thus we have

$$\left\| \frac{u^m - u^{m-1}}{k} \right\|^2 + \left\| \frac{\omega^m - \omega^{m-1}}{k} \right\|^2 \leq K_4 \left(\nu, \lambda_1, \lambda_2, N, |\xi|, \|u^0\|, \|\omega^0\|, \|f\|, \|g\| \right). \tag{3.28}$$

Then (3.18) is holds. The proof of Lemma 5 is completed. □

Theorem 4. *Assuming that $f \in L^2(\Omega)^3$, $g \in L^2(\Omega)$, $u_0 \in D(A_1)$, $\omega_0 \in D(A_2)$. Let $\{u^m, \omega^m\}_{m \geq 1}$ be the solution sequence of (1.7)-(1.8). Then there exists a positive constant C such that, $\forall m \geq 1$,*

$$\|A_1 u^m\|^2 + \|A_2 \omega^m\|^2 \leq K_5. \tag{3.29}$$

Proof. From (1.7)-(1.8), we obtain

$$\begin{cases} \nu A_1 u^m = -\dfrac{u^m - u^{m-1}}{k} - B_N(u^m, u^m) - \xi \omega^m + f, & (3.30) \\ A_2 \omega^m = -\dfrac{\omega^m - \omega^{m-1}}{k} - C(u^m, \omega^m) + g. & (3.31) \end{cases}$$

Two sides of equation (3.30) and (3.31) multiply by $A_1 u^m$ and $A_2 \omega^m$ respectively, and then integrate gives

$$\begin{aligned} &\nu \|A_1 u^m\|^2 + \|A_2 \omega^m\|^2 \\ &= -\left[\frac{u^m - u^{m-1}}{k}, A_1 u^m \right] - \left[\frac{\omega^m - \omega^{m-1}}{k}, A_2 \omega^m \right] - b_N(u^m, u^m, A_1 u^m) \\ &\quad - c(u^m, \omega^m, A_2 \omega^m) - (\xi \omega^m, A_1 u^m) + (f, A_1 u^m) + (g, A_2 \omega^m) \\ &\leq \left\| \frac{u^m - u^{m-1}}{k} \right\| \|A_1 u^m\| + \left\| \frac{\omega^m - \omega^{m-1}}{k} \right\| \|A_2 \omega^m\| \\ &\quad + \frac{CN}{\|\|u^m\|\|} \|(u^m \cdot \nabla) u^m\| \|A_1 u^m\| + C \|\|u^m\|\| \|\|\omega^m\|\| \|A_2 \omega^m\| \\ &\quad + \|\xi \omega^m\| \|A_1 u^m\| + \|f\| \|A_1 u^m\| + \|g\| \|A_2 \omega^m\| \\ &\leq \left\| \frac{u^m - u^{m-1}}{k} \right\| \|A_1 u^m\| + \left\| \frac{\omega^m - \omega^{m-1}}{k} \right\| \|A_2 \omega^m\| + CN \|\|u^m\|\|^{\frac{1}{2}} \|A_1 u^m\|^{\frac{3}{2}} \\ &\quad + C \|\|u^m\|\| \|\|\omega^m\|\| \|A_2 \omega^m\| + \|\xi \omega^m\| \|A_1 u^m\| + \|f\| \|A_1 u^m\| + \|g\| \|A_2 \omega^m\|. \end{aligned}$$

By Young's inequality, one gets

$$\begin{aligned} \|A_1 u^m\|^2 + \|A_2 \omega^m\|^2 &\leq C \left\| \frac{u^m - u^{m-1}}{k} \right\|^2 + C \left\| \frac{\omega^m - \omega^{m-1}}{k} \right\|^2 + C \|\|u^m\|\|^2 \\ &\quad + C \|\|u^m\|\|^2 \|\|\omega^m\|\|^2 + C |\xi|^2 \|\omega^m\|^2 + C \|f\|^2 + C \|g\|^2 \\ &\leq CK_4 + CK_2 + CK_2 K_3 + C |\xi|^2 K_1 + C \|f\|^2 + C \|g\|^2 \triangleq K_5. \end{aligned}$$

The proof of Theorem 4 is completed. □

4. Global Attractor

In this section, we first prove that continuous dependence of solutions on initial data and N.

Lemma 6. *Assuming that M, $N > 0$, $f \in L^2(\Omega)^3$, $g \in L^2(\Omega)$, u_1^0, $u_2^0 \in D(A_1)$, and ω_1^0, $\omega_2^0 \in D(A_2)$.*

Let $\{u_1^m, \omega_1^m\}_m$ be the solution of (1.7)-(1.8), with initial condition $\{u_1^0, \omega_1^0\}$ and parameter N.

Let $\{u_2^m, \omega_2^m\}_m$ be the solution of (1.7)-(1.8), with initial condition $\{u_2^0, \omega_2^0\}$ and parameter M.

Then there exists C, C_*, a_1^*, a_2^*, a_3^*, such that

$$|||u_1^m - u_2^m|||^2 + |||\omega_1^m - \omega_2^m|||^2 \leq \frac{1}{{C_*}^{M_0}}\left[|||u_1^0 - u_2^0|||^2 + |||\omega_1^0 - \omega_2^0|||^2\right] + \frac{kCK_5}{1-C_*}\left[\frac{1}{C_*^{M_0}} - 1\right]|M-N|^2, \quad \forall m \leq M_0; \tag{4.1}$$

$$|||u_1^m - u_2^m|||^2 + |||\omega_1^m - \omega_2^m|||^2 \leq \left[\frac{a_3^*}{k(M_0-3)} + a_2^*\right]e^{4a_1^*}, \qquad \forall m > M_0, \tag{4.2}$$

where $M_0 = ent\left(\frac{T}{k}\right)$, T is an arbitrarily fixed constant.

Proof. Let $u_*^m = u_1^m - u_2^m$, $\omega_*^m = \omega_1^m - \omega_2^m$ in (1.7)-(1.8), we obtain

$$\begin{cases} \dfrac{u_*^m - u_*^{m-1}}{k} + \nu A_1 u_*^m + B_N(u_1^m, u_1^m) - B_M(u_2^m, u_2^m) + \xi\omega_*^m = 0, & (4.3)\\[2ex] \dfrac{\omega_*^m - \omega_*^{m-1}}{k} + A_2\omega_*^m + C(u_1^m, \omega_1^m) - C(u_2^m, \omega_2^m) = 0. & (4.4)\end{cases}$$

Taking the scalar product of (4.3) with $2kA_1u_*^m$, we find

$$\begin{aligned}
&|||u_*^m|||^2 - |||u_*^{m-1}|||^2 + |||u_*^m - u_*^{m-1}|||^2 + 2\nu k\|A_1u_*^m\|^2\\
&= 2k\left[b_M(u_2^m, u_2^m, A_1u_*^m) - b_N(u_1^m, u_1^m, A_1u_*^m)\right] - 2k(\xi\omega_*^m, A_1u_*^m)\\
&= 2kF_M(|||u_2^m|||)b(u_2^m, u_2^m, A_1u_*^m)\\
&\quad - 2kF_N(|||u_1^m|||)b(u_1^m, u_1^m, A_1u_*^m) - 2k(\xi\omega_*^m, A_1u_*^m)\\
&= 2k\left[F_M(|||u_2^m|||) - F_N(|||u_1^m|||)\right]b(u_1^m, u_2^m, A_1u_*^m)\\
&\quad - 2kF_M(|||u_2^m|||)b(u_*^m, u_2^m, A_1u_*^m)\\
&\quad - 2kF_N(|||u_1^m|||)b(u_1^m, u_*^m, A_1u_*^m) - 2k(\xi\omega_*^m, A_1u_*^m).
\end{aligned} \tag{4.5}$$

From (1.3) and Lemma 1, one has

$$F_M(|||u_2^m|||)\,|b(u_*^m, u_2^m, A_1u_*^m)| \leq \frac{M}{|||u_2^m|||}C|||u_*^m|||^{\frac{1}{2}}\,|||u_2^m|||\,\|A_1u_*^m\|^{\frac{1}{2}}\,\|A_1u_*^m\|$$

$$
\begin{aligned}
&= CM|||u_*^m|||^{\frac{1}{2}}\|A_1u_*^m\|^{\frac{3}{2}};\\
F_N(|||u_1^m|||)\,|b(u_1^m,u_*^m,A_1u_*^m)| &\le |b(u_1^m,u_*^m,A_1u_*^m)|\\
&\le C\|A_1u_1^m\|\,|||u_*^m|||\,\|A_1u_*^m\|;
\end{aligned}
$$

and

$$
\begin{aligned}
&[F_M(|||u_2^m|||)-F_N(|||u_1^m|||)]b(u_1^m,u_2^m,A_1u_*^m)\\
&\le \Big[\frac{|M-N|+|||u_1^m-u_2^m|||}{|||u_2^m|||}\Big]C\|A_1u_1^m\|\,|||u_2^m|||\,\|A_1u_*^m\|\\
&\le C\Big[|M-N|+|||u_*^m|||\Big]\|A_1u_1^m\|\|A_1u_*^m\|.
\end{aligned}
$$

From (4.5) and above inequality, we obtain

$$
\begin{aligned}
&|||u_*^m|||^2-|||u_*^{m-1}|||^2+|||u_*^m-u_*^{m-1}|||^2+2\nu k\|A_1u_*^m\|^2\\
&\le 2kC\,[|M-N|+|||u_*^m|||]\,\|A_1u_1^m\|\|A_1u_*^m\|+2kCM|||u_*^m|||^{\frac{1}{2}}\|A_1u_*^m\|^{\frac{3}{2}}\\
&\quad+2kC\|A_1u_1^m\|\,|||u_*^m|||\,\|A_1u_*^m\|+2k\|\xi\omega_*^m\|\|A_1u_*^m\|\\
&\le \Big[\frac{kC\,|M-N|^2}{\varepsilon_1}+\frac{kC|||u_*^m|||^2}{\varepsilon_2}\Big]\|A_1u_1^m\|^2+(kC\varepsilon_1+kC\varepsilon_2)\|A_1u_*^m\|^2\\
&\quad+\frac{kM^4C^4}{2\varepsilon_3}|||u_*^m|||^2+\frac{3k\varepsilon_3^3}{2}\|A_1u_*^m\|^2+\frac{kC^2}{\varepsilon_4}\|A_1u_1^m\|^2|||u_*^m|||^2\\
&\quad+k\varepsilon_4\|A_1u_*^m\|^2+\frac{k\,|\xi|^2}{\varepsilon_5\lambda_2}|||\omega_*^m|||^2+k\varepsilon_5\|A_1u_*^m\|^2.
\end{aligned}
\tag{4.6}
$$

Taking the scalar product of (4.4) with $2kA_2\omega_*^m$, we find

$$
\begin{aligned}
&|||\omega_*^m|||^2-|||\omega_*^{m-1}|||^2+|||\omega_*^m-\omega_*^{m-1}|||^2+2k\|A_2\omega_*^m\|^2\\
&\qquad=2k[c(u_2^m,\omega_2^m,A_2\omega_*^m)-c(u_1^m,\omega_1^m,A_2\omega_*^m)].
\end{aligned}
\tag{4.7}
$$

By Young's inequality and (3.9)-(3.10) gives

$$
\begin{aligned}
&|c(u_2^m,\omega_2^m,A_2\omega_*^m)-c(u_1^m,\omega_1^m,A_2\omega_*^m)|\\
&=|c(u_2^m,\omega_2^m,A_2\omega_*^m)-c(u_*^m,\omega_1^m,A_2\omega_*^m)-c(u_2^m,\omega_1^m,A_2\omega_*^m)|\\
&=|c(u_2^m,\omega_*^m,A_2\omega_*^m)+c(u_*^m,\omega_1^m,A_2\omega_*^m)|\\
&\le C|||u_2^m|||\,|||\omega_*^m|||\,\|A_2\omega_*^m\|+C|||u_*^m|||\,|||\omega_1^m|||\,\|A_2\omega_*^m\|\\
&\le C\Big[\frac{1}{2\varepsilon_6}|||u_2^m|||^2|||\omega_*^m|||^2+\frac{\varepsilon_6}{2}\|A_2\omega_*^m\|^2\Big]\\
&\quad+C\Big[\frac{1}{2\varepsilon_7}|||u_*^m|||^2|||\omega_1^m|||^2+\frac{\varepsilon_7}{2}\|A_2\omega_*^m\|^2\Big]\\
&\le C\Big[\frac{1}{2\varepsilon_6}K_2|||\omega_*^m|||^2+\frac{\varepsilon_6}{2}\|A_2\omega_*^m\|^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+C\Big[\frac{1}{2\varepsilon_7}|||u_*^m|||^2K_3+\frac{\varepsilon_7}{2}\|A_2\omega_*^m\|^2\Big]\\
&=\frac{CK_2}{\varepsilon_6}|||\omega_*^m|||^2+\frac{CK_3}{\varepsilon_7}|||u_*^m|||^2+C(\varepsilon_6+\varepsilon_7)\|A_2\omega_*^m\|^2.
\end{aligned}
$$

Thus, from (4.7), we have

$$
\begin{aligned}
&|||\omega_*^m|||^2-|||\omega_*^{m-1}|||^2+|||\omega_*^m-\omega_*^{m-1}|||^2+2k\|A_2\omega_*^m\|^2\\
&\qquad\le 2k\Big[\frac{CK_2}{\varepsilon_6}|||\omega_*^m|||^2+\frac{CK_3}{\varepsilon_7}|||u_*^m|||^2+C(\varepsilon_6+\varepsilon_7)\|A_2\omega_*^m\|^2\Big].
\end{aligned}
\tag{4.8}
$$

From (4.6) and (4.8), we get

$$
\begin{aligned}
&|||u_*^m|||^2+|||\omega_*^m|||^2-|||u_*^{m-1}|||^2-|||\omega_*^{m-1}|||^2+|||u_*^m-u_*^{m-1}|||^2\\
&\quad+|||\omega_*^m-\omega_*^{m-1}|||^2+2\nu k\|A_1u_*^m\|^2+2k\|A_2\omega_*^m\|^2\\
&\le\Big[\frac{kC\,|M-N|^2}{\varepsilon_1}+\frac{kC|||u_*^m|||^2}{\varepsilon_2}\Big]\|A_1u_1^m\|^2+(kC\varepsilon_1+kC\varepsilon_2)\|A_1u_*^m\|^2\\
&\quad+\frac{kM^4C^4}{2\varepsilon_3}|||u_*^m|||^2+\frac{3k\varepsilon_3^3}{2}\|A_1u_*^m\|^2+\frac{kC^2}{\varepsilon_4}\|A_1u_1^m\|^2|||u_*^m|||^2\\
&\quad+k\varepsilon_4\|A_1u_*^m\|^2+\frac{k\,|\xi|^2}{\varepsilon_5\lambda_2}|||\omega_*^m|||^2+k\varepsilon_5\|A_1u_*^m\|^2\\
&\quad+2k\Big[\frac{CK_2}{\varepsilon_6}|||\omega_*^m|||^2+\frac{CK_3}{\varepsilon_7}|||u_*^m|||^2+C(\varepsilon_6+\varepsilon_7)\|A_2\omega_*^m\|^2\Big]\\
&\le\Big[\frac{kC\,|M-N|^2}{\varepsilon_1}+\frac{kC|||u_*^m|||^2}{\varepsilon_2}\Big]K_5+(kC\varepsilon_1+kC\varepsilon_2)\|A_1u_*^m\|^2\\
&\quad+\frac{kM^4C^4}{2\varepsilon_3}|||u_*^m|||^2+\frac{3k\varepsilon_3^3}{2}\|A_1u_*^m\|^2+\frac{kC^2}{\varepsilon_4}K_5|||u_*^m|||^2\\
&\quad+k\varepsilon_4\|A_1u_*^m\|^2+\frac{k\,|\xi|^2}{\varepsilon_5\lambda_2}|||\omega_*^m|||^2+k\varepsilon_5\|A_1u_*^m\|^2\\
&\quad+2k\Big[\frac{CK_2}{\varepsilon_6}|||\omega_*^m|||^2+\frac{CK_3}{\varepsilon_7}|||u_*^m|||^2+C(\varepsilon_6+\varepsilon_7)\|A_2\omega_*^m\|^2\Big]\\
&=|||u_*^m|||^2kC\Big[\frac{K_5}{\varepsilon_2}+\frac{M^4}{\varepsilon_3}+\frac{K_5}{\varepsilon_4}+\frac{K_3}{\varepsilon_7}\Big]+\|A_1u_*^m\|^2kC(\varepsilon_1+\varepsilon_2+\varepsilon_3^3+\varepsilon_4+\varepsilon_5)\\
&\quad+|||\omega_*^m|||^2kC\Big[\frac{K_2}{\varepsilon_6}+\frac{|\xi|^2}{\varepsilon_5\lambda_2}\Big]+\|A_2\omega_*^m\|^2kC(\varepsilon_6+\varepsilon_7)+\frac{kCK_5}{\varepsilon_1}\,|M-N|^2.
\end{aligned}
$$

That is

$$
\big[1-kC(K_5+M^4+K_3)\big]\,|||u_*^m|||^2+\Big[1-kC\Big(K_2+\frac{|\xi|^2}{\lambda_2}\Big)\Big]|||\omega_*^m|||^2
$$

$$
\begin{aligned}
&+|||u_*^m-u_*^{m-1}|||^2+|||\omega_*^m-\omega_*^{m-1}|||^2+kC\|A_1u_*^m\|^2+kC\|A_2\omega_*^m\|^2\\
&\le |||u_*^{m-1}|||^2+|||\omega_*^{m-1}|||^2+kCK_5\,|M-N|^2. \qquad (4.9)
\end{aligned}
$$

Let $C_*=\min\left\{1-kC(K_5+M^4+K_3),\ \ 1-kC\left(K_2+\frac{|\xi|^2}{\lambda_2}\right)\right\}$, then

$$
|||u_*^m|||^2+|||\omega_*^m|||^2\le\frac{1}{C_*}\left[|||u_*^{m-1}|||^2+|||\omega_*^{m-1}|||^2\right]+\frac{kCK_5}{C_*}\,|M-N|^2.
$$

Using the above inequality recursively, we find

$$
|||u_*^m|||^2+|||\omega_*^m|||^2\le\frac{1}{C_*^m}\left[|||u_*^0|||^2+|||\omega_*^0|||^2\right]+\frac{kCK_5}{1-C_*}\,|M-N|^2\left[\frac{1}{C_*^m}-1\right].
$$

Since $0<C_*<1$, for $m\le M_0=ent\left\{\frac{T}{k}\right\}$, one has

$$
|||u_*^m|||^2+|||\omega_*^m|||^2\le\frac{1}{C_*^{M_0}}\left[|||u_*^0|||^2+|||\omega_*^0|||^2\right]+\frac{kCK_5}{1-C_*}\,|M-N|^2\left[\frac{1}{C_*^{M_0}}-1\right].
$$

For $m>M_0$, let $C_{**}=\max\left\{C(K_5+M^4+K_3),\ \ C\left(K_2+\frac{|\xi|^2}{\lambda_2}\right)\right\}$. From (4.9), one has

$$
(1-kC_{**})|||u_*^m|||^2+(1-kC_{**})|||\omega_*^m|||^2\le|||u_*^{m-1}|||^2+|||\omega_*^{m-1}|||^2+kCK_5\,|M-N|^2.
$$

Let $x_m=|||u_*^m|||^2+|||\omega_*^m|||^2$, $y_m=C_{**}$, $z_m=CK_5\,|M-N|^2$. Obviously, $\{x_m\}$, $\{y_m\}$, $\{z_m\}$ are non-negative sequences, and for $k>0$,

$$
\begin{aligned}
ky_m=kC_{**}&<\frac{1}{2}, && \forall m\ge 2,\\
(1-ky_m)x_m&\le x_{m-1}+kz_m, && \forall m\ge M-1.
\end{aligned}
$$

For all integers $m_*\ge 2$, by Lemma 4, we get

$$
\begin{aligned}
&k\sum_{m=m_*}^{m_*+m_1}y_m=k(M_0-2)C_{**}\le a_1^*;\\
&k\sum_{m=m_*}^{m_*+m_1}z_m=k(M_0-2)CK_5\,|M-N|^2\le a_2^*;\\
&k\sum_{m=m_*}^{m_*+m_1}x_m=k\sum_{m=m_*}^{m_*+m_1}(|||u_1^m-u_2^m|||^2+|||\omega_1^m-\omega_2^m|||^2)\\
&\qquad\le k\sum_{m=m_*}^{m_*+m_1}\left(C|||u_1^m|||^2+C|||u_2^m|||^2+C|||\omega_1^m|||^2+C|||\omega_2^m|||^2\right)\\
&\qquad\le kC\sum_{m=m_*}^{m_*+m_1}(K_2+K_2+K_3+K_3)\le kC(M_0-2)(K_2+K_3)\le a_3^*.
\end{aligned}
$$

Thus, $x_m \le \left[\frac{a_3^*}{k(M_0-3)} + a_2^*\right] e^{4a_1^*}$, which is

$$|||u_*^m|||^2 + |||\omega_*^m|||^2 \le \left[\frac{a_3^*}{k(M_0-3)} + a_2^*\right] e^{4a_1^*}.$$

The proof of Lemma 6 is completed. □

Proof of Theorem 2. Above, we show that the continuous dependence of solutions on initial value and parameter N. It can be seen under the above conditions, when determining the initial value and the parameter N, the system (1.7)-(1.8) has a unique solution. Therefore, we can define a C^0 semigroup S^m, acting on the phase space $\mathbb{V} \times H_0^1(\Omega)$, and defined as follows:

$$S^m\left(u^0, \omega^0\right) = \left(u^m, \omega^m\right), \ \ \forall m \ge 0.$$

From Lemma 4, the semigroup S^m has a bounded absorbing set in $\mathbb{V} \times H_0^1(\Omega)$:

$$B_{\mathbb{V}\times H_0^1(\Omega)} = \left\{\left(u^m, \omega^m\right) \in \mathbb{V} \times H_0^1(\Omega), |||u^m|||^2 + |||\omega^m|||^2 \le K_2 + K_3\right\}.$$

And from Theorem 4 we can know that S^m is bounded in $D(A_1) \times D(A_2)$, and using Sobolev embedding theorem to know that S^m is compact in $\mathbb{V} \times H_0^1(\Omega)$. Hence, S^m has a global attractor $\mathcal{A}$ in $\mathbb{V} \times H_0^1(\Omega)$. The proof of Theorem 2 is completed. □

5. LIMITING BEHAVIOR FOR $N \to \infty$

One sees from Lemma 2 that

$$\nu k|||u^m|||^2 + k|||\omega^m|||^2 \le \frac{2k}{\nu\lambda_1}\|f\|^2 + \frac{k}{\lambda_2}\|g\|^2 + \left[1 + \frac{2k|\xi|^2}{\nu\lambda_1}\right] K_1 + K_1^*.$$

For m and k fixed, let $\left\{u_N^m, \omega_N^m\right\}_N$ be the solution of (1.7)-(1.8), then

$$|||u_N^m|||^2 + |||\omega_N^m|||^2 \le \frac{C}{\nu^2\lambda_1}\|f\|^2 + \frac{C}{\nu\lambda_2}\|g\|^2 + \frac{1}{\nu k}\left[\left(1 + \frac{2k|\xi|^2}{\nu\lambda_1}\right) K_1 + K_1^*\right].$$

Thus the sequence $\left\{u_N^m, \omega_N^m\right\}_N$ is bounded in $\mathbb{V} \times H_0^1(\Omega)$ uniformly in N. Therefore, we can extract from $\left\{u_N^m, \omega_N^m\right\}_N$ a subsequence still denoted by $\left\{u_N^m, \omega_N^m\right\}_N$ such that $u_N^m \rightharpoonup u^m$, as $N \to \infty$ in $\mathbb{V}$, and $\omega_N^m \rightharpoonup \omega^m$, as $N \to \infty$ in $H_0^1(\Omega)$. As the injection $\mathbb{V} \hookrightarrow \mathbb{H}$ and $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ both are compact, we have $u_N^m \to u^m$, as $N \to \infty$ in $\mathbb{H}$, and $\omega_N^m \to \omega^m$ as $N \to \infty$ in $L^2(\Omega)$.

We shall show that

$$\begin{cases} \lim\limits_{N\to\infty} F_N(|||u_N^m|||)b(u_N^m, u_N^m, v) = b(u^m, u^m, v), & \forall v \in D(A_1), \\ \lim\limits_{N\to\infty} C(u_N^m, \omega_N^m, v_*) = c(u^m, \omega^m, v_*), & \forall v_* \in H^2(\Omega). \end{cases}$$

Indeed, a simple computation gives

$$
\begin{aligned}
&F_N(|||u_N^m|||)b(u_N^m,u_N^m,v)-b(u^m,u^m,v)\\
&\qquad\qquad\qquad=[F_N(|||u_N^m|||)-1]\,b(u_N^m,u_N^m,v)+b(u_N^m,u_N^m,v)-b(u^m,u^m,v).
\end{aligned}
$$

First, by the definition of F_N, we have $F_N(|||u_N^m|||)=\min\left\{1,\dfrac{N}{|||u_N^m|||}\right\}\le 1$. And from the above inequality we can see

$$
\frac{N}{|||u_N^m|||}\ge N\left\{\frac{C}{\nu^2\lambda_1}\|f\|^2+\frac{C}{\nu\lambda_2}\|g\|^2+\frac{1}{\nu k}\left[\left(1+\frac{2k|\xi|^2}{\nu\lambda_1}\right)K_1+K_1^*\right]\right\}^{-\frac{1}{2}}.
$$

Hence, if $N>\left\{\dfrac{C}{\nu^2\lambda_1}\|f\|^2+\dfrac{C}{\nu\lambda_2}\|g\|^2+\dfrac{1}{\nu k}\left[\left(1+\dfrac{2k|\xi|^2}{\nu\lambda_1}\right)K_1+K_1^*\right]\right\}^{\frac{1}{2}}$, we find that $F_N(|||u_N^m|||)=1$. Therefore, $\lim\limits_{N\to\infty}F_N(|||u_N^m|||)=1$. Next, from (1.3), we obtain

$$
\begin{aligned}
&b(u_N^m,u_N^m,v)\le C|||u_N^m|||\,|||u_N^m|||\,\|A_1v\|\\
&\qquad\le C\left(\frac{C}{\nu^2\lambda_1}\|f\|^2+\frac{C}{\nu\lambda_2}\|g\|^2+\frac{1}{\nu k}\left[\left(1+\frac{2k|\xi|^2}{\nu\lambda_1}\right)K_1+K_1^*\right]\right)\|A_1v\|,
\end{aligned}
$$

showing that $b(u_N^m,u_N^m,v)$ is bounded uniformly with respect to N, so

$$
\lim_{N\to\infty}[F_N(|||u_N^m|||)-1]b(u_N^m,u_N^m,v)=0.
$$

Using the strong convergence of u_N^m in $\mathbb{H}$, we can prove as in [13], that $b(u_N^m,u_N^m,v)\to b(u^m,u^m,v)$, as $N\to\infty$. Thus $\lim\limits_{N\to\infty}F_N(|||u_N^m|||)b(u_N^m,u_N^m,v)=b(u^m,u^m,v)$, $\forall v\in D(A_1)$. Similarly, we have $\lim\limits_{N\to\infty}c(u_N^m,\omega_N^m,v_*)=c(u^m,\omega^m,v_*)$, $\forall v_*\in H^2(\Omega)$. Therefore, $\{u_N^m,\omega_N^m\}_N$ converges to the weak solution of the following equations when $N\to\infty$,

$$
\begin{cases}
\dfrac{u^m-u^{m-1}}{k}+\nu A_1u^m+B(u^m,u^m)+\xi\omega^m=f,\\
\dfrac{\omega^m-\omega^{m-1}}{k}+A_2\omega^m+C(u^m,\omega^m)=g.
\end{cases}\tag{5.1}
$$

Thus, we have completed the proof of Theorem 3. $\square$

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REFERENCES

[1] A. M. Alghamdi, I. B. Omrane, S. Gala, and M. A. Ragusa, "A regularity criterion to the 3D Boussinesq equations," *Sib. Electron. Math. Rep.*, vol. 16, no. 12, pp. 1795–1804, 2019, doi: 10.33048/semi.2019.16.127.

[2] B. Birnir and N. Svanstedt, "Existence theory and strong attractors for the Rayleigh-Bénard problem with a large aspect ratio," *Discr. Contin. Dynam. Syst. A*, vol. 10, no. 1/2, pp. 53–74, 2012, doi: 10.3934/dcds.2004.10.53.

[3] F. E. Browder, *Existence and uniqueness theorems for solutions of nonlinear boundary value problems*. New York: American Mathematical Society, Providence, 1965. doi: 10.1090/pspm/017/0197933.

[4] F. Cheng and C. J. Xu, "Analytical smoothing effect of solution for the Boussinesq equations," *Acta Math. Sci.*, vol. 39B, no. 1, pp. 165–179, 2019, doi: 10.3969/j.issn.0252-9602.2019.01.014.

[5] G. Deugoue and J. K. Djoko, "On the time discretization for the globally modified three dimensional Navier-Stokes equations," *J. Comput. Appl. Math.*, vol. 235, no. 8, pp. 2015–2029, 2010, doi: 10.1016/j.cam.2010.10.003.

[6] S. Gala and M. A. Ragusa, "A regularity criterion of weak solutions to the 3D Boussinesq equations," *Bullet. Brazilian Math. Society*, vol. 50, no. 3, 2019, doi: 10.1007/s00574-019-00162-z.

[7] A. Huang and W. Huo, "The global attractor of the 2D Boussinesq equations with fractional Laplacian in Subcritical case," *Discr. Contin. Dynam. Syst. B*, vol. 21, no. 8, pp. 2531–2550, 2017, doi: 10.3934/dcdsb.2016059.

[8] A. V. Kapustyan, A. V. Pankov, and J. Valero, "On Global Attractors of Multivalued Semiflows Generated by the 3D Bénard System," *Set-Valued Variat. Analy.*, vol. 20, no. 4, pp. 445–465, 2012, doi: 10.1007/s11228-011-0197-5.

[9] O. V. Kapustyan, V. S. Melnik, and J. Valero, "A weak attractor and properties of solutions for the three-dimensional Bénard problem," *Discr. Contin. Dynam. Syst.*, vol. 18, no. 2/3, pp. 449–481, 2007, doi: 10.3934/dcds.2007.18.449.

[10] O. V. Kapustyan and A. V. Pankov, "Global $\varphi-$attractor for a modified 3D Bénard system on channel-like domains," *Nonauto. Stochastic Dyna. Syst.*, vol. 1, no. 1, pp. 1–9, 2014, doi: 10.2478/msds-2013-0001.

[11] O. V. Kapustyan, A. V. Pankov, and J. Valero, "On the existence and connectedness of a global attractor for solutions of the three-dimensional Bénard system that satisfy a system of energy inequalities," *J. Math. Sci.*, vol. 191, no. 3, pp. 388–402, 2013, doi: 10.1007/s10958-013-1325-6.

[12] M. Kaya and A. O. Çelebi, "Global attractor for the regularized Bénard problem," *Appl. Analy.*, vol. 93, no. 9, pp. 1989–2001, 2014, doi: 10.1080/00036811.2013.867018.

[13] R. Temam, *Theory and Numerical Analysis of the Navier-Stokes Equations*. New York: North Holland Publishing company, 1977.

[14] X. M. Wang, "Asymptotic Behavior of the Global Attractors to the Boussinesq System for Rayleigh-Bénard Convection at Large Prandtl Number," *Commun. Pure Appli. Math.*, vol. 60, no. 9, pp. 1293–1318, 2007, doi: 10.1002/cpa.20170.

[15] C. S. Zhu, "Attractor of a semi-discrete Benjamin-Bona-Mahony equation on $\mathbb{R}^1$," *Annales Polonici Math.*, vol. 115, no. 3, pp. 219–234, 2015, doi: 10.4064/ap115-3-2.

[16] Z. Q. Zhu, "Stable analysis for a delayed population model with survival rate," *Miskolc Math. Notes*, vol. 20, no. 1, pp. 611–624, 2019, doi: 10.18514/MMN.2019.2675.

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