# The asymptotic behavior of solutions to a class of inhomogeneous problems: an Orlicz-Sobolev space approach 

Andrei Grecu ${ }^{1,2}$ and Denisa Stancu-Dumitru ${ }^{\boxtimes 2,3}$<br>${ }^{1}$ Department of Mathematics, University of Craiova, 200585 Craiova, Romania<br>${ }^{2}$ Research group of the project PN-III-P1-1.1-TE-2019-0456, University Politehnica of Bucharest, 060042 Bucharest, Romania<br>${ }^{3}$ Department of Mathematics and Computer Sciences, Politehnica University of Bucharest, 060042 Bucharest, Romania

Received 10 January 2021, appeared 19 April 2021
Communicated by Roberto Livrea


#### Abstract

The asymptotic behavior of the sequence $\left\{v_{n}\right\}$ of nonnegative solutions for a class of inhomogeneous problems settled in Orlicz-Sobolev spaces with prescribed Dirichlet data on the boundary of domain $\Omega$ is analysed. We show that $\left\{v_{n}\right\}$ converges uniformly in $\Omega$ as $n \rightarrow \infty$, to the distance function to the boundary of the domain.


Keywords: weak solution, viscosity solution, nonlinear elliptic equations, asymptotic behavior, Orlicz-Sobolev spaces.
2020 Mathematics Subject Classification: 35D30, 35D40, 35J60, 35J70, 46E30, 46 E 35.

## 1 Introduction

Let $\Omega \subset \mathbb{R}^{N}(N \geq 2)$ be a bounded domain with smooth boundary $\partial \Omega$. We consider the family of problems

$$
\begin{cases}-\operatorname{div}\left(\frac{\varphi_{n}(|\nabla v|)}{|\nabla v|} \nabla v\right)=\lambda e^{v} & \text { in } \Omega  \tag{1.1}\\ v=0 & \text { on } \partial \Omega\end{cases}
$$

where for each positive integer $n$, the mappings $\varphi_{n}: \mathbb{R} \rightarrow \mathbb{R}$ are odd, increasing homeomorphisms of class $C^{1}$ satisfying Lieberman-type condition

$$
\begin{equation*}
N-1<\varphi_{n}^{-}-1 \leq \frac{t \varphi_{n}^{\prime}(t)}{\varphi_{n}(t)} \leq \varphi_{n}^{+}-1<\infty, \quad \forall t \geq 0 \tag{1.2}
\end{equation*}
$$

for some constants $\varphi_{n}^{-}$and $\varphi_{n}^{+}$with $1<\varphi_{n}^{-} \leq \varphi_{n}^{+}<\infty$,

$$
\begin{equation*}
\varphi_{n}^{-} \rightarrow \infty \quad \text { as } n \rightarrow \infty, \tag{1.3}
\end{equation*}
$$

[^0]and such that
\[

$$
\begin{equation*}
\text { there exists a real constant } \beta>1 \text { with the property that } \varphi_{n}^{+} \leq \beta \varphi_{n}^{-}, \forall n \geq 1 \tag{1.4}
\end{equation*}
$$

\]

and

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \varphi_{n}(1)^{1 / \varphi_{n}^{-}}=1 . \tag{1.5}
\end{equation*}
$$

For some examples of functions satisfying conditions (1.2)-(1.5) the reader is referred to [5, p. 4398]. Here we just point out the fact that in the particular case when $\varphi_{n}(t)=|t|^{n-2} t, n \geq 2$, the differential operator involved in problem (1.1) is the $n$-Laplacian, which for sufficiently smooth functions $v$ is defined as $\Delta_{n} v:=\operatorname{div}\left(|\nabla v|^{n-2} \nabla v\right)$. In this particular case problem (1.1) becomes

$$
\begin{cases}-\Delta_{n} v=\lambda e^{v} & \text { in } \Omega  \tag{1.6}\\ v=0 & \text { on } \partial \Omega\end{cases}
$$

which has been extensively studied in the literature (see, e.g. [3,7,12, 14, 15, 18, 19, 32]). An existence result concerning problem (1.6) for each given $n>N$ and $\lambda>0$ sufficiently small was proved by Aguilar Crespo \& Peral Alonso in [3] by using a fixed-point argument while Mihăilescu et al. [32] showed a similar result by using variational techniques. Moreover, in [32] was studied the asymptotic behavoir of solutions as $n \rightarrow \infty$. More precisely, it was proved that there exists $\lambda^{\star}>0$ (which does not depend on $n$ ) such that for each $n>N$ and each $\lambda \in\left(0, \lambda^{\star}\right)$ problem (1.6) possesses a nonnegative solution $u_{n} \in W_{0}^{1, n}(\Omega)$ and the sequence of solutions $\left\{u_{n}\right\}$ converges uniformly in $\bar{\Omega}$, as $n \rightarrow \infty$, to the unique viscosity solution of the problem

$$
\begin{cases}\min \left\{|\nabla u|-1,-\Delta_{\infty} u\right\}=0 & \text { in } \Omega,  \tag{1.7}\\ u=0 & \text { on } \partial \Omega,\end{cases}
$$

which is precisely the distance function to the boundary of the domain $\operatorname{dist}(\cdot, \partial \Omega)$ (see [26, Lemma 6.10]). The result from [32] was extended to the case of equations involving variable exponent growth conditions by Mihăilescu \& Fărcăseanu in [14]. Motivated by these results the goal of this paper is to investigate the asymptotic behaviour of the solutions of the family of problems (1.1), as $n \rightarrow \infty$, for $\lambda>0$ sufficiently small. We will show that the results from [32] and [14] continue to hold true in the case of the family of problems (1.1). In particular, our results generalise the results from [32] and complement the results from [14].

The paper is organized as follows. In Section 2 we give the definitions of the Orlicz and Orlicz-Sobolev spaces which represent the natural functional framework where the problems of type (1.1) should be investigated. Section 3 is devoted to the proof of the existence of weak solutions for problem (1.1) when $\lambda$ is sufficiently small. Finally, in Section 4 we analyse the asymptotic behavior of the sequence of solutions found in the previous section, as $n \rightarrow \infty$, and we prove its uniform convergence to the distance function to the boundary of the domain.

## 2 Orlicz and Orlicz-Sobolev spaces

In this section we provide a brief overview on the Orlicz and Orlicz-Sobolev spaces and we recall the definitions and some of their main properties. For more details about these spaces the reader can consult the books $[2,22,33,34]$ and papers $[4,9,10,20,21]$.

First, we will introduce the Orlicz spaces. We assume that the function $\varphi$ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ of class $C^{1}$. We define $\Phi:[0, \infty) \rightarrow \mathbb{R}$ by

$$
\Phi(t)=\int_{0}^{t} \varphi(s) d s
$$

Note that $\Phi$ is a Young function, that is $\Phi$ vanishes when $t=0, \Phi$ is continuous, $\Phi$ is convex and $\lim _{t \rightarrow \infty} \Phi(t)=\infty$. Moreover, since $\Phi(0)=0$ if and only if $t=0, \lim _{t \rightarrow 0} \frac{\Phi(t)}{t}=0$ and $\lim _{t \rightarrow \infty} \frac{\Phi(t)}{t}=\infty$, then $\Phi$ is called a $N$-function (see $[1,2]$ ). Next, we define the function $\Phi^{\star}:[0, \infty) \rightarrow \mathbb{R}$ given by

$$
\Phi^{\star}(t)=\int_{0}^{t} \varphi^{-1}(s) d s
$$

$\Phi^{\star}$ is called the complementary function of $\Phi$. The functions $\Phi$ and $\Phi^{\star}$ satisfy

$$
\Phi^{\star}(t)=\sup _{s \geq 0}(s t-\Phi(s)) \quad \text { for any } t \geq 0
$$

We note that $\Phi^{\star}$ is also a $N$-function, too.
Throughout this paper, we will assume that

$$
\begin{equation*}
0<\varphi^{-}-1 \leq \frac{t \varphi^{\prime}(t)}{\varphi(t)} \leq \varphi^{+}-1<\infty, \quad \text { for all } t>0 \tag{2.1}
\end{equation*}
$$

for some positive constants $\varphi^{-}$and $\varphi^{+}$. By [28, Lemma 1.1] (see also [31, Lemma 2.1]) we deduce that

$$
\begin{equation*}
1<\varphi^{-} \leq \frac{t \varphi(t)}{\Phi(t)} \leq \varphi^{+}<\infty, \quad \text { for all } t>0 \tag{2.2}
\end{equation*}
$$

By relation (2.2) it follows that for each $t>0$ and $s \in(0,1]$ we have

$$
-\ln s^{\varphi^{-}}=\int_{s t}^{t} \frac{\varphi^{-}}{\tau} d \tau \leq \int_{s t}^{t} \frac{\varphi(\tau)}{\Phi(\tau)} d \tau=\ln \Phi(t)-\ln \Phi(s t) \leq \int_{s t}^{t} \frac{\varphi^{+}}{\tau} d \tau=-\ln s^{\varphi^{+}}
$$

or

$$
\begin{equation*}
s^{\varphi^{+}} \Phi(t) \leq \Phi(s t) \leq s^{\varphi^{-}} \Phi(t), \quad \forall t>0, s \in(0,1] . \tag{2.3}
\end{equation*}
$$

Similarly, for each $t>0$ and $s>1$ we have

$$
\ln s^{\varphi^{-}}=\int_{t}^{s t} \frac{\varphi^{-}}{\tau} d \tau \leq \int_{t}^{s t} \frac{\varphi(\tau)}{\Phi(\tau)} d \tau=\ln \Phi(s t)-\ln \Phi(t) \leq \int_{t}^{s t} \frac{\varphi^{+}}{\tau} d \tau=\ln s^{\varphi^{+}}
$$

or

$$
\begin{equation*}
s^{\varphi^{-}} \Phi(t) \leq \Phi(s t) \leq s^{\varphi^{+}} \Phi(t), \quad \forall t>0, s>1 . \tag{2.4}
\end{equation*}
$$

Inequalities (2.3) and (2.4) can be reformulated as follows

$$
\begin{equation*}
\min \left\{s^{\varphi^{-}}, s^{\varphi^{+}}\right\} \Phi(t) \leq \Phi(s t) \leq \max \left\{s^{\varphi^{-}}, s^{\varphi^{+}}\right\} \Phi(t) \text { for any } s, t>0 . \tag{2.5}
\end{equation*}
$$

Similarly, by [31, Lemma 2.1] we deduce that

$$
\begin{equation*}
\min \left\{s^{\varphi^{-}-1}, s^{\varphi^{+}-1}\right\} \varphi(t) \leq \varphi(s t) \leq \max \left\{s^{\varphi^{-}-1}, s^{\varphi^{+}-1}\right\} \varphi(t), \quad \forall s, t>0 . \tag{2.6}
\end{equation*}
$$

Next, if we let $s=\varphi^{-1}(t)$ then we have

$$
\frac{t\left(\varphi^{-1}\right)^{\prime}(t)}{\varphi^{-1}(t)}=\frac{\varphi(s)}{\varphi^{\prime}(s) s}
$$

By (2.1) we deduce that

$$
\frac{1}{\varphi^{+}-1} \leq \frac{t\left(\varphi^{-1}\right)^{\prime}(t)}{\varphi^{-1}(t)} \leq \frac{1}{\varphi^{-}-1}, \quad \forall t>0
$$

The above relation implies that

$$
\begin{equation*}
1<\frac{\varphi^{+}}{\varphi^{+}-1} \leq \frac{t \varphi^{-1}(t)}{\Phi^{\star}(t)} \leq \frac{\varphi^{-}}{\varphi^{-}-1}<\infty \quad \text { for all } t>0 \tag{2.7}
\end{equation*}
$$

Examples. We point out some example of functions $\varphi$ which are odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, and $\varphi$ and the corresponding primitive $\Phi$ satisfy condition (2.2) (see [10, Examples 1-3, p. 243]):

1. $\varphi(t)=|t|^{p-2} t, \Phi(t)=\frac{|t|^{p}}{p}$ with $p>1$ and $\varphi^{-}=\varphi^{+}=p$.
2. $\varphi(t)=\log \left(1+|t|^{r}\right)|t|^{p-2} t, \Phi(t)=\log \left(1+|t|^{r}\right) \frac{\mid t^{p}}{p}-\frac{r}{p} \int_{0}^{|t|} \frac{s^{p+r-1}}{1+s^{r}} d s$ with $p, r>1$ and $\varphi^{-}=p, \varphi^{+}=p+r$.
3. $\varphi(t)=\frac{|t|^{p-2} t}{\log (1+|t|)}$ for $t \neq 0, \varphi(0)=0, \Phi(t)=\frac{\mid t t^{p}}{p \log (1+|t|)}+\frac{1}{p} \int_{0}^{|t|} \frac{s^{p}}{(1+s)(\log (1+s))^{2}} d s$ with $p>2$ and $\varphi^{-}=p-1, \varphi^{+}=p=\lim \inf _{t \rightarrow \infty} \frac{\log \Phi(t)}{\log t}$.

For each bounded domain $\Omega \subset \mathbb{R}^{N}$, the Orlicz space $L^{\Phi}(\Omega)$ defined by the $N$-function $\Phi$ (see $[1,2,9]$ ) is the set of real-valued measurable functions $u: \Omega \rightarrow \mathbb{R}$ such that

$$
\|u\|_{L^{\Phi}(\Omega)}:=\sup \left\{\int_{\Omega} u(x) v(x) d x ; \quad \int_{\Omega} \Phi^{\star}(|v(x)|) d x \leq 1\right\}<\infty .
$$

Then, the Orlicz space $L^{\Phi}(\Omega)$ endowed with the Orlicz norm $\|\cdot\|_{L^{\Phi}(\Omega)}$ is a Banach space and its Orlicz norm $\|\cdot\|_{L^{\Phi}(\Omega)}$ is equivalent to the so-called Luxemburg norm defined by

$$
\begin{equation*}
\|u\|_{\Phi}:=\inf \left\{\mu>0 ; \int_{\Omega} \Phi\left(\frac{u(x)}{\mu}\right) d x \leq 1\right\} . \tag{2.8}
\end{equation*}
$$

In the case of Orlicz spaces, the following relations hold true (see, e.g. [17, Lemma 2.1]):

$$
\begin{array}{ll}
\|u\|_{\Phi}^{\varphi^{+}} \leq \int_{\Omega} \Phi(|u(x)|) d x \leq\|u\|_{\Phi}^{\varphi^{-}} & \forall u \in L^{\Phi}(\Omega) \text { with }\|u\|_{\Phi}<1, \\
\|u\|_{\Phi}^{\varphi^{-}} \leq \int_{\Omega} \Phi(|u(x)|) d x \leq\|u\|_{\Phi}^{\varphi^{+}} & \forall u \in L^{\Phi}(\Omega) \text { with }\|u\|_{\Phi}>1 \tag{2.10}
\end{array}
$$

and

$$
\begin{equation*}
\int_{\Omega} \Phi(|u(x)|) d x=1 \Longleftrightarrow\|u\|_{\Phi}=1, \quad \forall u \in L^{\Phi}(\Omega) \tag{2.11}
\end{equation*}
$$

Next, we recall that for each bounded domain $\Omega \subset \mathbb{R}^{N}$, the Orlicz-Sobolev space $W^{1, \Phi}(\Omega)$ defined by the $N$-function $\Phi$ is the set of all functions $u$ such that $u$ and its distributional derivatives of order 1 lie in Orlicz space $L^{\Phi}(\Omega)$. More exactly, $W^{1, \Phi}(\Omega)$ is the space given by

$$
W^{1, \Phi}(\Omega)=\left\{u \in L^{\Phi}(\Omega) ; \frac{\partial u}{\partial x_{j}} \in L^{\Phi}(\Omega), j \in\{1, \ldots, N\}\right\} .
$$

It is a Banach space with respect to the following norm

$$
\|u\|_{1, \Phi}:=\|u\|_{\Phi}+\||\nabla u|\|_{\Phi} .
$$

By $W_{0}^{1, \Phi}(\Omega)$ we denoted the closure of all functions of class $C^{\infty}$ with compact support over $\Omega$ with respect to norm of $W^{1, \Phi}(\Omega)$, i.e.

$$
W_{0}^{1, \Phi}(\Omega):=\overline{C_{0}^{\infty}(\Omega)} \|^{\|\cdot\|_{1, \Phi}} .
$$

Note that the norms $\|\cdot\|_{1, \Phi}$ and $\|\cdot\|_{W_{0}^{1, \Phi}}:=\||\nabla \cdot|\|_{\Phi}$ are equivalent on the Orlicz-Sobolev space $W_{0}^{1, \Phi}(\Omega)$ (see [21, Lemma 5.7]).

Under conditions (2.2) and (2.7), $\Phi$ and $\Phi^{\star}$ satisfy the $\Delta_{2}$-condition, i.e.

$$
\begin{equation*}
\Phi(2 t) \leq C \Phi(t), \quad \forall t \geq 0, \tag{2.12}
\end{equation*}
$$

for some constant $C>0$ (see [2, p. 232]). Therefore, $L^{\Phi}(\Omega), W^{1, \Phi}(\Omega)$ and $W_{0}^{1, \Phi}(\Omega)$ are reflexive Banach spaces (see [2, Theorem 8.19] and [2, p. 232]).

Remark 2.1. For each real number $p>1$ let $\varphi(t)=|t|^{p-2} t, t \in \mathbb{R}$. It can be shown that $\varphi^{-}=\varphi^{+}=p$ as mentioned above in Example 1 and the corresponding Orlicz space $L^{\Phi}(\Omega)$ reduces to the classical Lebesgue space $L^{p}(\Omega)$ while the Orlicz-Sobolev spaces $W^{1, \Phi}(\Omega)$ and $W_{0}^{1, \Phi}(\Omega)$ become the classical Sobolev spaces $W^{1, p}(\Omega)$ and $W_{0}^{1, p}(\Omega)$, respectively. Note also that by [2, Theorem 8.12] the Orlicz space $L^{\Phi}(\Omega)$ is continuously embedded in the Lebesgue spaces $L^{q}(\Omega)$ for each $q \in\left(1, \varphi^{-}\right]$.

## 3 Variational solutions for problem (1.1)

In this section we will show that there exists a certain constant $\lambda^{\star}>0$ (independent of $n$ ) such that for each $\lambda \in\left(0, \lambda^{\star}\right)$ problem (1.1) possesses a nonnegative weak solution for each integer $n \geq 1$.

We start by introducing the following notations: for each positive integer $n$ we denote by $\Phi_{n}$ a primitive of the function $\varphi_{n}$. More precisely, we define $\Phi_{n}:[0, \infty) \rightarrow \mathbb{R}$ by

$$
\Phi_{n}(t):=\int_{0}^{t} \varphi_{n}(s) d s
$$

Definition 3.1. We say that $v_{n}$ is a weak solution of problem (1.1) if $v_{n} \in W_{0}^{1, \Phi_{n}}(\Omega)$ and the following relation holds true

$$
\begin{equation*}
\int_{\Omega} \frac{\varphi_{n}\left(\left|\nabla v_{n}\right|\right)}{\left|\nabla v_{n}\right|} \nabla v_{n} \nabla w d x=\lambda \int_{\Omega} e^{v_{n}} w d x, \quad \forall w \in W_{0}^{1, \Phi_{n}}(\Omega) . \tag{3.1}
\end{equation*}
$$

Note that the integral from the right-hand side of relation (3.1) is well-defined since the Orlicz-Sobolev space $W_{0}^{1, \Phi_{n}}(\Omega)$ is continuously embedded in the classical Sobolev space $W_{0}^{1, \varphi_{n}^{-}}(\Omega)$ (see, e.g. [2, Theorem 8.12]) and for $\varphi_{n}^{-}>N$ we have $W_{0}^{1, \varphi_{n}^{-}}(\Omega) \subset L^{\infty}(\Omega)$. Moreover, we recall that Morrey's inequality holds true, i.e. there exists a positive constant $C_{n}$ such that

$$
\begin{equation*}
\|v\|_{L^{\infty}(\Omega)} \leq C_{n}\||\nabla v|\|_{L^{\varphi_{\bar{n}}}(\Omega)^{\prime}} \quad \forall v \in W_{0}^{1, \varphi_{n}^{-}}(\Omega) . \tag{3.2}
\end{equation*}
$$

By [8, Proposition 3.1] we know that we can choose $C_{n}$ as follows

$$
\begin{equation*}
C_{n}:=\varphi_{n}^{-}|B(0,1)|^{-\frac{1}{\varphi_{\bar{n}}}} N^{-\frac{N\left(\varphi_{n}^{-}+1\right)}{\left(\varphi_{\bar{n}}\right)^{2}}}\left(\varphi_{n}^{-}-1\right)^{\frac{N\left(\varphi_{\bar{n}}^{-}-1\right)}{\left(\varphi_{\bar{n}}\right)^{2}}}\left(\varphi_{n}^{-}-N\right)^{\frac{N-\left(\varphi_{\overline{-}}^{-}\right)^{2}}{\left(\varphi_{n}\right)^{2}}}\left[\lambda_{1}\left(\varphi_{n}^{-}\right)\right]^{\frac{N-\varphi_{\bar{n}}^{-}}{\left(\varphi_{\bar{n}}\right)^{2}}}, \tag{3.3}
\end{equation*}
$$

where $|B(0,1)|$ is the volume of the unit ball in $\mathbb{R}^{N}$ and for each real number $p \in(1, \infty), \lambda_{1}(p)$ denotes the first eigenvalue for the $p$-Laplace operator with homogeneous Dirichlet boundary conditions, i.e.

$$
\lambda_{1}(p):=\inf _{u \in C_{0}^{o}(\Omega) \backslash\{0\}} \frac{\int_{\Omega}|\nabla u|^{p} d x}{\int_{\Omega}|u|^{p} d x}, \quad \forall p \in(1, \infty) .
$$

By [8, Proposition 3.1] (see also [13, Theorem 3.2] for a similar result) it is well known that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} C_{n}=\|\operatorname{dist}(\cdot, \partial \Omega)\|_{L^{\infty}(\Omega)}, \tag{3.4}
\end{equation*}
$$

where $\operatorname{dist}(x, \partial \Omega):=\inf _{y \in \partial \Omega}|x-y|, \forall x \in \Omega$, stands for the distance function to the boundary of $\Omega$.

For each positive integer $n$ and each positive real number $\lambda$ we introduce the EulerLagrange functional associated to problem (1.1) as $J_{n, \lambda}: W_{0}^{1, \Phi_{n}}(\Omega) \rightarrow \mathbb{R}$ defined by

$$
J_{n, \lambda}(v):=\int_{\Omega} \Phi_{n}(|\nabla v|) d x-\lambda \int_{\Omega} e^{v} d x, \quad \forall v \in W_{0}^{1, \Phi_{n}}(\Omega) .
$$

Standard arguments can be used in order to show that $J_{n, \lambda} \in C^{1}\left(W_{0}^{1, \Phi_{n}}(\Omega), \mathbb{R}\right)$ and

$$
\left\langle J_{n, \lambda}^{\prime}(v), w\right\rangle=\int_{\Omega} \frac{\varphi_{n}(|\nabla v|)}{|\nabla v|} \nabla v \nabla w d x-\lambda \int_{\Omega} e^{v} w d x, \quad \forall v, w \in W_{0}^{1, \Phi_{n}}(\Omega) .
$$

Thus, it is clear that $v_{n}$ is a weak solution of (1.1) if and only if $v_{n}$ is a critical point of functional $J_{n, \lambda}$.

We point out that we cannot find critical points of $J_{n, \lambda}$ by using the Direct Method in the Calculus of Variations since in the case of our problem $J_{n, \lambda}$ is not coercive. For that reason we propose an analysis of problem (1.1) based on Ekeland's Variational Principle in order to find critical points of $J_{n, \lambda}$.

For each positive integer $n$ we denote

$$
\begin{equation*}
\lambda_{n}^{\star}:=\frac{1}{2|\Omega|} e^{-C_{n}\left[\Omega \left\lvert\,+\frac{1}{\Phi_{n}(1)}\right.\right]^{1 / \varphi_{n}^{\bar{n}}}}, \tag{3.5}
\end{equation*}
$$

where $C_{n}$ is the constant given by relation (3.3) and $|\Omega|$ stands for the $N$-dimensional Lebesgue measure of $\Omega$. The starting point of our approach is the following lemma.

Lemma 3.2. For each positive integer $n$ let $\lambda_{n}^{\star}$ be given by relation (3.5). Then for each $\lambda \in\left(0, \lambda_{n}^{\star}\right)$ we have

$$
J_{n, \lambda}(v) \geq \frac{1}{2}, \quad \forall v \in W_{0}^{1, \Phi_{n}}(\Omega) \quad \text { with } \quad\|v\|_{W_{0}^{1, \Phi_{n}}}=1
$$

Proof. Let $n$ be a positive integer arbitrary fixed. By relation (2.5) we get that $\Phi_{n}(s) \geq$ $\Phi_{n}(1) s^{\varphi_{n}^{-}}$, for all $s>1$ and thus,

$$
s^{\varphi_{n}^{-}} \leq 1+\frac{\Phi_{n}(s)}{\Phi_{n}(1)}, \quad \forall s \geq 0 .
$$

Using this fact we deduce that

$$
\begin{equation*}
\int_{\Omega}|\nabla v|^{\varphi_{n}^{-}} d x \leq|\Omega|+\frac{1}{\Phi_{n}(1)} \int_{\Omega} \Phi_{n}(|\nabla v|) d x, \quad \forall v \in W_{0}^{1, \Phi_{n}}(\Omega) \tag{3.6}
\end{equation*}
$$

By the above inequality, and since for each $v \in W_{0}^{1, \Phi_{n}}(\Omega)$ with $\|v\|_{W_{0}^{1, \Phi_{n}}}:=\||\nabla v|\|_{\Phi_{n}}=1$ we have $\int_{\Omega} \Phi_{n}(|\nabla v|) d x=1$ (via relation (2.11)), it results

$$
\begin{equation*}
\||\nabla v|\|_{L^{\varphi_{n}^{-}}(\Omega)} \leq\left[|\Omega|+\frac{1}{\Phi_{n}(1)}\right]^{1 / \varphi_{n}^{-}}, \quad \forall v \in W_{0}^{1, \Phi_{n}}(\Omega) \quad \text { with }\|v\|_{W_{0}^{1, \Phi_{n}}}=1 \tag{3.7}
\end{equation*}
$$

Next, taking into account that $W_{0}^{1, \Phi_{n}}(\Omega)$ is continuously embedded in $W_{0}^{1, \varphi_{n}^{-}}(\Omega)$ and using the fact that $\varphi_{n}^{-}>N$ and Morrey's inequality (3.2) we obtain

$$
\begin{aligned}
J_{n, \lambda}(v) & =\int_{\Omega} \Phi_{n}(|\nabla v|) d x-\lambda \int_{\Omega} e^{v} d x \\
& \geq 1-\lambda|\Omega| e^{\|v\|_{L^{\infty}(\Omega)}} \\
& \geq 1-\lambda|\Omega| e^{C_{n}\||\nabla v|\|_{L^{\varphi_{n}^{-}}(\Omega)}}, \quad \forall v \in W_{0}^{1, \Phi_{n}}(\Omega) \text { with }\|v\|_{W_{0}^{1, \Phi_{n}}}=1
\end{aligned}
$$

Then for each $\lambda \in\left(0, \lambda_{n}^{\star}\right)$, combining the above estimates with relation (3.7) we get

$$
J_{n, \lambda}(v) \geq 1-\lambda|\Omega| e^{C_{n}\left[|\Omega|+\frac{1}{\Phi_{n}(1)}\right]^{1 / \varphi_{n}^{-}}} \geq 1-\lambda_{n}^{\star}|\Omega| e^{C_{n}\left[|\Omega|+\frac{1}{\Phi_{n}(1)}\right]^{1 / \varphi_{n}^{-}}}=\frac{1}{2}
$$

for all $v \in W_{0}^{1, \Phi_{n}}(\Omega)$ with $\|v\|_{W_{0}^{1, \Phi_{n}}}=1$. The proof of the lemma is complete.
Lemma 3.3. For each positive integer $n$ let $\lambda_{n}^{\star}$ be given by relation (3.5). Define

$$
\begin{equation*}
\lambda^{\star}:=\inf _{n \in \mathbb{N}^{*}} \lambda_{n}^{\star} \tag{3.8}
\end{equation*}
$$

Then $\lambda^{\star}>0$.
Proof. First, we show that there exists a positive constant $K>0$ such that

$$
\begin{equation*}
\left[|\Omega|+\frac{1}{\Phi_{n}(1)}\right]^{1 / \varphi_{n}^{-}}<K, \quad \forall n \geq 1 \tag{3.9}
\end{equation*}
$$

Indeed, since by (1.5) we have

$$
\lim _{n \rightarrow \infty} \varphi_{n}(1)^{1 / \varphi_{n}^{-}}=1
$$

it yields that for each positive integer $n$ large enough we get

$$
\frac{1}{2} \leq \varphi_{n}(1)^{1 / \varphi_{n}^{-}}
$$

which implies that

$$
\frac{1}{\varphi_{n}(1)} \leq 2^{\varphi_{n}^{-}}
$$

By (1.2) (via (2.1) and (2.2)) we find that for each positive integer $n$ large enough the following inequalities hold true

$$
\frac{1}{\Phi_{n}(1)} \leq \frac{\varphi_{n}^{+}}{\varphi_{n}(1)} \leq \varphi_{n}^{+} 2^{\varphi_{n}^{-}} \leq \beta \varphi_{n}^{-} 2^{\varphi_{n}^{-}}
$$

Using the above relations we deduce that for each positive integer $n$ large enough we obtain

$$
\left[|\Omega|+\frac{1}{\Phi_{n}(1)}\right]^{1 / \varphi_{n}^{-}} \leq\left[|\Omega|+\beta \varphi_{n}^{-} 2^{\varphi_{n}^{-}}\right]^{1 / \varphi_{n}^{-}} \leq\left(\beta \varphi_{n}^{-} 2^{\varphi_{n}^{-}+1}\right)^{1 / \varphi_{n}^{-}}
$$

Now, taking into account the fact that $\lim _{n \rightarrow \infty}\left(\beta \varphi_{n}^{-} 2^{\varphi_{n}^{-}+1}\right)^{1 / \varphi_{n}^{-}}=2$, the above approximations imply that relation (3.9) holds true.

Next, using (3.9) and the expression of $\lambda_{n}^{\star}$ we infer that

$$
\lambda_{n}^{\star}=\frac{1}{2|\Omega|} e^{-C_{n}\left[|\Omega|+\frac{1}{\Phi_{n}(1)}\right]^{1 / \varphi_{n}^{-}}}>\frac{1}{2|\Omega|} e^{-K C_{n}}, \quad \forall n \geq 1
$$

Recalling that $\lim _{n \rightarrow \infty} C_{n}=\|\operatorname{dist}(\cdot, \partial \Omega)\|_{L^{\infty}(\Omega)}$ (by (3.4)) and taking into account that function $(1, \infty) \ni p \longrightarrow \lambda_{1}(p)$ is continuous (see, Lindqvist [29] or Huang [23]) we conclude from the above estimates that $\lambda^{\star}=\inf _{n \in \mathbb{N}^{*}} \lambda_{n}^{\star}>0$. The proof of Lemma 3.3 is complete.

The main goal of this section is to prove the existence of weak solutions of problem (1.1) for each positive integer $n$. This result is the core of the following theorem.

Theorem 3.4. Let $\lambda^{\star}>0$ be given by (3.8). Then for each $\lambda \in\left(0, \lambda^{\star}\right)$ and each $n \in \mathbb{N}^{\star}$, problem (1.1) has a nonnegative solution $v_{n} \in \overline{B_{1}(0)} \subset W_{0}^{1, \Phi_{n}}(\Omega)$ identified by $J_{n, \lambda}\left(v_{n}\right)=\inf _{\overline{B_{1}(0)}} J_{n, \lambda}$, where $B_{1}(0)$ is the unit ball centered at the origin in the Orlicz-Sobolev space $W_{0}^{1, \Phi_{n}}(\Omega)$.

Proof. We consider $\lambda \in\left(0, \lambda^{\star}\right)$ and $n \in \mathbb{N}^{\star}$ arbitary fixed. For each $v \in W_{0}^{1, \Phi_{n}}(\Omega)$ with $\|v\|_{W_{0}^{1, \Phi_{n}}} \leq 1$, in view of relations (2.9) and (2.11), we have

$$
\begin{equation*}
\|v\|_{W_{0}^{1, \Phi_{n}}}^{\varphi_{n}^{-}} \geq \int_{\Omega} \Phi_{n}(|\nabla v|) d x \geq\|v\|_{W_{0}^{1, \Phi_{n}}}^{\varphi_{n}^{+}} . \tag{3.10}
\end{equation*}
$$

$\underline{T h u s,}$ taking into account (3.10), Morrey's inequality (3.2) and relation (3.6), for each $v \in$ $\overline{B_{1}(0)} \subset W_{0}^{1, \Phi_{n}}(\Omega)$ we obtain

$$
\begin{aligned}
J_{n, \lambda}(v) & =\int_{\Omega} \Phi_{n}(|\nabla v|) d x-\lambda \int_{\Omega} e^{v} d x \\
& \geq\|v\|_{W_{0}^{1, \Phi_{n}}}^{\varphi_{n}^{+}}-\lambda|\Omega| e^{\|v\|_{L^{\infty}(\Omega)}} \\
& \geq-\lambda|\Omega| e^{C_{n}\||\nabla v|\|_{L^{\varphi_{n}^{-}}(\Omega)}} \\
& \geq-\lambda|\Omega| e^{C_{n}\left[|\Omega|+\frac{1}{\Phi_{n}(1)}\right]^{1 / \varphi_{n}^{-}}}
\end{aligned}
$$

Computing $J_{n, \lambda}(0)=-\lambda|\Omega|$ we deduce that

$$
J_{n, \lambda}(0)<0
$$

while by Lemma 3.2 we get

$$
\inf _{\partial B_{1}(0)} J_{n, \lambda} \geq \frac{1}{2}>0
$$

which imply that

$$
\gamma_{n}:=\frac{\inf }{B_{1}(0)} J_{n, \lambda} \in(-\infty, 0)
$$

We consider $\epsilon>0$ such that

$$
\begin{equation*}
\epsilon<\inf _{\partial B_{1}(0)} J_{n, \lambda}-\inf _{B_{1}(0)} J_{n, \lambda} . \tag{3.11}
\end{equation*}
$$

Ekeland's variational principle applied to $J_{n, \lambda}$ restricted to $\overline{B_{1}(0)}$ provides the existence of $v_{\epsilon} \in \overline{B_{1}(0)}$ having the properties

$$
\begin{aligned}
& \text { i) } J_{n, \lambda}\left(v_{\epsilon}\right)<\frac{\inf }{B_{1}(0)} J_{n, \lambda}+\epsilon, \\
& \text { ii) } J_{n, \lambda}\left(v_{\epsilon}\right)<J_{n, \lambda}(v)+\epsilon\left\|v-v_{\epsilon}\right\|_{W_{0}^{1, \phi_{n}}} \text { for all } v \neq v_{\epsilon} .
\end{aligned}
$$

Since $\inf _{\overline{B_{1}(0)}} J_{n, \lambda} \leq \inf _{B_{1}(0)} J_{n, \lambda}$ and $\epsilon$ is chosen small such that (3.11) holds true, using relation i) above we arrive at

$$
J_{n, \lambda}\left(v_{\epsilon}\right)<\inf _{B_{1}(0)} J_{n, \lambda}+\epsilon \leq \inf _{B_{1}(0)} J_{n, \lambda}+\epsilon<\inf _{\partial B_{1}(0)} J_{n, \lambda},
$$

from which we deduce that $v_{\epsilon}$ is not an element on the boundary of the unit ball of space $W_{0}^{1, \Phi_{n}}(\Omega), v_{\epsilon} \notin \partial B_{1}(0)$, and consequently, $v_{\epsilon}$ is an element in the interior of this ball, that means $v_{\epsilon} \in B_{1}(0)$.

Next, we focus on the functional $F_{n, \lambda}: \overline{B_{1}(0)} \rightarrow \mathbb{R}$ defined by $F_{n, \lambda}(v)=J_{n, \lambda}(v)+$ $\epsilon\left\|v-v_{\epsilon}\right\|_{W_{0}^{1, \Phi_{n}}}$. Obviously, $v_{\epsilon}$ is a minimum point of $F_{n, \lambda}$ (via $\left.i i\right)$ ) that infers

$$
\frac{F_{n, \lambda}\left(v_{\epsilon}+t w\right)-F_{n, \lambda}\left(v_{\epsilon}\right)}{t} \geq 0
$$

for small $t>0$ and any $w \in B_{1}(0)$. Computing the above relation we find

$$
\frac{J_{n, \lambda}\left(v_{\epsilon}+t w\right)-J_{n, \lambda}\left(v_{\epsilon}\right)}{t}+\epsilon\|w\|_{W_{0}^{1, \Phi_{n}}} \geq 0
$$

and then passing to the limit as $t \rightarrow 0^{+}$it yields that $\left\langle J_{n, \lambda}^{\prime}\left(v_{\epsilon}\right), w\right\rangle+\epsilon\|w\|_{W_{0}^{1, \Phi_{n}}} \geq 0$ that implies $\left\|J_{n, \lambda}^{\prime}\left(v_{\epsilon}\right)\right\|_{\left(W_{0}^{1, \Phi_{n}}(\Omega)\right)^{\star}} \leq \epsilon$, where $\left(W_{0}^{1, \Phi_{n}}(\Omega)\right)^{\star}$ is the dual space of $W_{0}^{1, \Phi_{n}}(\Omega)$.

In consideration of that, we draw to the conclusion that there exists a sequence $\left\{v_{m}\right\}_{m} \subset$ $B_{1}(0)$ such that

$$
\begin{equation*}
\lim _{m \rightarrow \infty} J_{n, \lambda}\left(v_{m}\right)=\gamma_{n} \text { and } \lim _{m \rightarrow \infty} J_{n, \lambda}^{\prime}\left(v_{m}\right)=0 . \tag{3.12}
\end{equation*}
$$

The sequence $\left\{v_{m}\right\}_{m}$ is certainly bounded in $W_{0}^{1, \Phi_{n}}(\Omega)$ since $v_{m} \in B_{1}(0)$ for all $m \in \mathbb{N}^{\star}$ and this fact induces the existence of $v_{n} \in W_{0}^{1, \Phi_{n}}(\Omega)$ such that, up to a subsequence, $\left\{v_{m}\right\}_{m}$ converges weakly to $v_{n}$ in $W_{0}^{1, \Phi_{n}}(\Omega)$ and uniformly in $\Omega$, since $\varphi_{n}^{-}>N$, as $m \rightarrow \infty$. Furthermore, we infer that

$$
\lim _{m \rightarrow \infty} \int_{\Omega} e^{v_{m}}\left(v_{m}-v_{n}\right) d x=0
$$

and

$$
\lim _{m \rightarrow \infty}\left\langle J_{n, \lambda}^{\prime}\left(v_{m}\right), v_{m}-v_{n}\right\rangle=0,
$$

which imply that

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \int_{\Omega} \frac{\varphi_{n}\left(\left|\nabla v_{m}\right|\right)}{\left|\nabla v_{m}\right|} \nabla v_{m} \nabla\left(v_{m}-v_{n}\right) d x=0 \tag{3.13}
\end{equation*}
$$

Owing to the weak convergence of sequence $\left\{v_{m}\right\}_{m}$ to $v_{n}$ in $W_{0}^{1, \Phi_{n}}(\Omega)$, as $m \rightarrow \infty$, we have that

$$
\lim _{m \rightarrow \infty}\left\langle J_{n, \lambda}^{\prime}\left(v_{n}\right), v_{m}-v_{n}\right\rangle=0
$$

and it follows that

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \int_{\Omega} \frac{\varphi_{n}\left(\left|\nabla v_{n}\right|\right)}{\left|\nabla v_{n}\right|} \nabla v_{n} \nabla\left(v_{m}-v_{n}\right) d x=0 . \tag{3.14}
\end{equation*}
$$

Assembling relations (3.13) and (3.14), we conclude that

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \int_{\Omega}\left[\frac{\varphi_{n}\left(\left|\nabla v_{m}\right|\right)}{\left|\nabla v_{m}\right|} \nabla v_{m}-\frac{\varphi_{n}\left(\left|\nabla v_{n}\right|\right)}{\left|\nabla v_{n}\right|} \nabla v_{n}\right] \nabla\left(v_{m}-v_{n}\right) d x=0 . \tag{3.15}
\end{equation*}
$$

By [16, Lemma 3.2] we know that there exists a positive constant $k_{n}$ such that

$$
\left[\frac{\varphi_{n}(|\xi|)}{|\xi|} \xi-\frac{\varphi_{n}(|\eta|)}{|\eta|} \eta\right] \cdot(\xi-\eta) \geq k_{n} \frac{\left[\Phi_{n}(|\xi-\eta|) \frac{\varphi_{n}+2}{9^{\bar{n}}+1}\right.}{\left[\Phi_{n}(|\xi|)+\Phi_{n}(|\eta|)\right]^{1 /\left(\varphi_{n}^{-}+1\right)}}, \quad \forall \xi, \eta \in \mathbb{R}^{N}, \xi \neq \eta .
$$

In our case, we established that there exist constant $k_{n}>0$ so that

$$
\begin{aligned}
\int_{\Omega}\left[\frac{\varphi_{n}\left(\left|\nabla v_{m}\right|\right)}{\left|\nabla v_{m}\right|} \nabla v_{m}-\frac{\varphi_{n}\left(\left|\nabla v_{n}\right|\right)}{\left|\nabla v_{n}\right|} \nabla v_{n}\right]\left(\nabla v_{m}-\nabla v_{n}\right) d x & \\
& \geq k_{n} \int_{\Omega} \frac{\left[\Phi_{n}\left(\left|\nabla v_{m}-\nabla v_{n}\right|\right)\right]^{\frac{\varphi_{\bar{n}}+2}{\varphi_{\bar{n}}^{1+1}}}}{\left[\Phi_{n}\left(\left|\nabla v_{m}\right|\right)+\Phi_{n}\left(\left|\nabla v_{n}\right|\right)\right]^{1 /\left(\varphi_{n}^{-}+1\right)}} d x .
\end{aligned}
$$

Due to relation (3.15) we deduce that

$$
\lim _{m \rightarrow \infty} \int_{\Omega} \Phi_{n}\left(\left|\nabla\left(v_{m}-v_{n}\right)\right|\right)\left[\frac{\Phi_{n}\left(\left|\nabla\left(v_{m}-v_{n}\right)\right|\right)}{\Phi_{n}\left(\left|\nabla v_{m}\right|\right)+\Phi_{n}\left(\left|\nabla v_{n}\right|\right)}\right]^{1 /\left(\varphi_{n}^{-}+1\right)} d x=0 .
$$

Since $\Phi_{n}$ is a convex function we obtain by relation (2.5) that

$$
\Phi_{n}\left(\left|\nabla\left(v_{m}-v_{n}\right)\right|\right) \leq \frac{\Phi_{n}\left(2\left|\nabla v_{m}\right|\right)+\Phi_{n}\left(2\left|\nabla v_{n}\right|\right)}{2} \leq 2^{\varphi_{n}^{+}-1}\left[\Phi_{n}\left(\left|\nabla v_{m}\right|\right)+\Phi_{n}\left(\left|\nabla v_{n}\right|\right)\right] .
$$

Using assumption (1.4), the last two relations require

$$
\lim _{m \rightarrow \infty} \int_{\Omega} \Phi_{n}\left(\left|\nabla\left(v_{m}-v_{n}\right)\right|\right) d x=0
$$

and (2.9) generates

$$
\lim _{m \rightarrow \infty}\left\|v_{m}-v_{n}\right\|_{W_{0}^{1, \Phi_{n}}}=0 .
$$

That being the case, $\left\{v_{m}\right\}_{m}$ converges strongly to $v_{n}$ in $W_{0}^{1, \Phi_{n}}(\Omega)$ as $m \rightarrow \infty$. Hence, relation (3.12) contribute to

$$
\begin{equation*}
J_{n, \lambda}\left(v_{n}\right)=\gamma_{n}<0 \quad \text { and } \quad J_{n, \lambda}^{\prime}\left(v_{n}\right)=0 . \tag{3.16}
\end{equation*}
$$

As a result, $v_{n}$ is the minimizer of $J_{n, \lambda}$ on $B_{1}(0)$, and also $v_{n}$ is a critical point of the functional $J_{n, \lambda}$. Of course, $v_{n}$ is really a weak solution of (1.1). Finally, note that $J_{n, \lambda}(|v|) \leq J_{n, \lambda}(v)$ for any $v \in W_{0}^{1, \Phi_{n}}(\Omega)$ and for this reason $v_{n}$ is a nonnegative function on $\Omega$.

The proof of Theorem 3.4 is complete.

## 4 The asymptotic behavior of the sequence of solutions $\left\{v_{n}\right\}_{n}$ of problem (1.1) given by Theorem 3.4 as $n \rightarrow \infty$

The goal of this section is to prove the following result.
Theorem 4.1. Let $\lambda^{\star}>0$ be given by (3.8). For each $\lambda \in\left(0, \lambda^{\star}\right)$ and each $n \in \mathbb{N}^{\star}$ we denote by $v_{n}$ the nonnegative weak solution of problem (1.1) given by Theorem 3.4. The sequence $\left\{v_{n}\right\}$ converges uniformly in $\Omega$ to $\operatorname{dist}(\cdot, \partial \Omega)$, the distance function to the boundary of $\Omega$.

In order to prove Theorem 4.1 we first establish the uniform Hölder estimates for the weak solutions of (1.1).
Lemma 4.2. Let $\lambda^{\star}>0$ be given by (3.8). Fix $\lambda \in\left(0, \lambda^{\star}\right)$ and let $v_{n}$ be the nonnegative solution of problem (1.1) given by Theorem 3.4. Then there is a subsequence $\left\{v_{n}\right\}$ which converges uniformly in $\Omega$, as $n \rightarrow \infty$, to a continuous function $v_{\infty} \in C(\bar{\Omega})$ with $v_{\infty} \geq 0$ in $\Omega$ and $v_{\infty}=0$ on $\partial \Omega$.

Proof. Let $q \geq N$ be an arbitrary real number. By (1.3) we can choose $q<\varphi_{n}^{-}$for sufficiently large positive integer $n$. Using Hölder's inequality, relation (3.6), recalling that $v_{n} \in B_{1}(0) \subset$ $W_{0}^{1, \Phi_{n}}(\Omega)$ and taking into account (2.9) we have

$$
\begin{aligned}
\left(\int_{\Omega}\left|\nabla v_{n}\right|^{q} d x\right)^{1 / q} & \leq\left(\int_{\Omega}\left|\nabla v_{n}\right|^{\varphi_{n}^{-}} d x\right)^{1 / \varphi_{n}^{-}}|\Omega|^{1 / q-1 / \varphi_{n}^{-}} \\
& \leq\left[|\Omega|+\frac{1}{\Phi_{n}(1)} \int_{\Omega} \Phi_{n}\left(\left|\nabla v_{n}\right|\right) d x\right]^{1 / \varphi_{n}^{-}}|\Omega|^{1 / q-1 / \varphi_{n}^{-}} \\
& \leq\left[|\Omega|+\frac{1}{\Phi_{n}(1)}\left\|v_{n}\right\|_{W_{0}^{-1, \phi_{n}}}^{\varphi_{\overline{-}}}\right]^{1 / \varphi_{n}^{-}}|\Omega|^{1 / q-1 / \varphi_{n}^{-}} \\
& \leq\left[|\Omega|+\frac{1}{\Phi_{n}(1)}\right]^{1 / \varphi_{n}^{-}}|\Omega|^{1 / q-1 / \varphi_{n}^{-}}
\end{aligned}
$$

Thereupon, using (3.9) we find that sequence $\left\{\left|\nabla v_{n}\right|\right\}$ is uniformly bounded in $L^{q}(\Omega)$. It is clear that $q>N$ ensures that the embedding of $W_{0}^{1, q}(\Omega)$ into $C(\bar{\Omega})$ is compact. Keeping in mind the reflexivity of the Sobolev space $W_{0}^{1, q}(\Omega)$ we deduce that there exists a subsequence (not relabelled) of $\left\{v_{n}\right\}$ and a function $v_{\infty} \in C(\bar{\Omega})$ such that $v_{n} \rightharpoonup v_{\infty}$ weakly in $W_{0}^{1, q}(\Omega)$ and $v_{n} \rightarrow v_{\infty}$ uniformly in $\Omega$ as $n \rightarrow \infty$. In addition, the facts that $v_{n} \geq 0$ in $\Omega$ and $v_{n}=0$ on $\partial \Omega$ for each $\varphi_{n}^{-}>N$ hint that $v_{\infty} \geq 0$ in $\Omega$ and $v_{\infty}=0$ on $\partial \Omega$. The proof of Lemma 4.2 is complete.

In Theorem 4.5 below we show that function $v_{\infty}$ given by Lemma 4.2 is the solution in the viscosity sense (see, Crandall, Ishii \& Lions [11]) of a certain limiting problem. Accordingly, we adopt the usual strategy of first proving that continuous weak solutions of problem (1.1) at level $n$ are indeed solutions in the viscosity sense. Before recalling the definition of viscosity solutions for this type of problems, let us note that if we assume for a moment that the solutions $v_{n}$ of problem (1.1) are sufficiently smooth so that we can perform the differentiation in the PDE

$$
-\operatorname{div}\left(\frac{\varphi_{n}\left(\left|\nabla v_{n}\right|\right)}{\left|\nabla v_{n}\right|} \nabla v_{n}\right)=\lambda e^{v_{n}}, \quad \text { in } \Omega,
$$

we get

$$
\begin{equation*}
-\frac{\varphi_{n}\left(\left|\nabla v_{n}\right|\right)}{\left|\nabla v_{n}\right|} \Delta v_{n}-\frac{\left|\nabla v_{n}\right| \varphi_{n}^{\prime}\left(\left|\nabla v_{n}\right|\right)-\varphi_{n}\left(\left|\nabla v_{n}\right|\right)}{\left|\nabla v_{n}\right|^{3}} \Delta_{\infty} v_{n}=\lambda e^{v_{n}}, \quad \text { in } \Omega, \tag{4.1}
\end{equation*}
$$

where $\Delta$ stands for the Laplace operator, $\Delta v:=\operatorname{Trace}\left(D^{2} v\right)=\sum_{i=1}^{N} \frac{\partial^{2} v}{\partial x_{i}^{2}}$ and $\Delta_{\infty}$ stands for the $\infty$-Laplace operator,

$$
\Delta_{\infty} v:=\left\langle D^{2} v \nabla v, \nabla v\right\rangle=\sum_{i, j=1}^{N} \frac{\partial v}{\partial x_{i}} \frac{\partial v}{\partial x_{j}} \frac{\partial^{2} v}{\partial x_{i} \partial x_{j}},
$$

while $D^{2} v$ denotes the Hessian matrix of $v$.
Remark that (4.1) can be reformulated as

$$
H_{n}\left(v_{n}, \nabla v_{n}, D^{2} v_{n}\right)=0, \quad \text { in } \Omega
$$

with function $H_{n}$ defined as follows

$$
H_{n}(y, z, S):=-\frac{\varphi_{n}(|z|)}{|z|} \operatorname{Trace} S-\frac{|z| \varphi_{n}^{\prime}(|z|)-\varphi_{n}(|z|)}{|z|^{3}}\langle S z, z\rangle-\lambda e^{y},
$$

where $y \in \mathbb{R}, z$ is a vector in $\mathbb{R}^{N}$ and $S$ stands for a real symmetric matrix in $\mathbb{M}^{N \times N}$.
Since our main objective in this section is the asymptotic analysis of solutions $\left\{v_{n}\right\}$ as $n \rightarrow \infty$, we are now ready to give the definition of viscosity solutions for the homogeneous Dirichlet boundary value problem associated to degenerate elliptic PDE of the type

$$
\begin{cases}H_{n}\left(v, \nabla v, D^{2} v\right)=0 & \text { in } \Omega  \tag{4.2}\\ v=0 & \text { on } \partial \Omega .\end{cases}
$$

## Definition 4.3.

i) An upper semicontinuous function $v$ is a viscosity subsolution of problem (4.2) if $v \leq 0$ on $\partial \Omega$ and, whenever $x_{0} \in \Omega$ and $\Psi \in C^{2}(\Omega)$ are such that $v\left(x_{0}\right)=\Psi\left(x_{0}\right)$ and $v(x)<\Psi(x)$ if $x \in B\left(x_{0}, r\right) \backslash\left\{x_{0}\right\}$ for some $r>0$, we have $H_{n}\left(\Psi\left(x_{0}\right), \nabla \Psi\left(x_{0}\right), D^{2} \Psi\left(x_{0}\right)\right) \leq 0$.
ii) A lower semicontinuous function $v$ is a viscosity supersolution of problem (4.2) if $v \geq 0$ on $\partial \Omega$ and, whenever $x_{0} \in \Omega$ and $\mathrm{Y} \in \mathrm{C}^{2}(\Omega)$ are such that $v\left(x_{0}\right)=\mathrm{Y}\left(x_{0}\right)$ and $v(x)>\mathrm{Y}(x)$ if $x \in B\left(x_{0}, r\right) \backslash\left\{x_{0}\right\}$ for some $r>0$, we have $H_{n}\left(\mathrm{Y}\left(x_{0}\right), \nabla \mathrm{Y}\left(x_{0}\right), D^{2} \mathrm{Y}\left(x_{0}\right)\right) \geq 0$.
iii) A continuous function $v$ is a viscosity solution of problem (4.2) if it is both viscosity supersolution and viscosity subsolution of problem (4.2).

In the sequel, functions $\Psi$ and Y stand for test functions touching the graph of $v$ from above and below, respectively.

Our goal now is to prove that any continuous weak solution of (1.1) is also viscosity solution of (1.1) and in order to establish this result we follow the approach by Juutinen, Lindqvist \& Manfredi in [27, Lemma 1.8] (see also [35, Lemma 1] for a similar approach but in the framework of inhomogeneous differential operators).
Lemma 4.4. A continuous weak solution of problem (1.1) is also a viscosity solution of (1.1).
Proof. Firstly, we prove that if $v_{n}$ is a continuous weak solution of problem (1.1) for a fixed positive integer $n$, then it is a viscosity subsolution of problem (1.1). We begin by considering $x_{n}^{0} \in \bar{\Omega}$ and a test function $\Psi_{n} \in C^{2}(\bar{\Omega})$ such that $v_{n}\left(x_{n}^{0}\right)=\Psi_{n}\left(x_{n}^{0}\right)$ and $v_{n}-\Psi_{n}$ has a strict local maximum at $x_{n}^{0}$, that is $v_{n}(y)<\Psi_{n}(y)$ if $y \in B\left(x_{n}^{0}, \rho\right) \backslash\left\{x_{n}^{0}\right\}$ for some $\rho>0$.

Next, we have to show that

$$
-\operatorname{div}\left(\frac{\varphi_{n}\left(\left|\nabla \Psi_{n}\left(x_{n}^{0}\right)\right|\right)}{\left|\nabla \Psi_{n}\left(x_{n}^{0}\right)\right|} \nabla \Psi_{n}\left(x_{n}^{0}\right)\right) \leq \lambda e^{\Psi_{n}\left(x_{n}^{0}\right)}
$$

or
$-\frac{\varphi_{n}\left(\left|\nabla \Psi_{n}\left(x_{n}^{0}\right)\right|\right)}{\left|\nabla \Psi_{n}\left(x_{n}^{0}\right)\right|} \Delta \Psi_{n}\left(x_{n}^{0}\right)-\frac{\left|\nabla \Psi_{n}\left(x_{n}^{0}\right)\right| \varphi_{n}^{\prime}\left(\left|\nabla \Psi_{n}\left(x_{n}^{0}\right)\right|\right)-\varphi_{n}\left(\left|\nabla \Psi_{n}\left(x_{n}^{0}\right)\right|\right)}{\left|\nabla \Psi_{n}\left(x_{n}^{0}\right)\right|^{3}} \Delta_{\infty} \Psi_{n}\left(x_{n}^{0}\right) \leq \lambda e^{\Psi_{n}\left(x_{n}^{0}\right)}$.
Arguing ad contrarium, suppose that this is not the case of the above assertion. In other words, we admit that there exists a radius $\rho_{n}>0$ such that $B\left(x_{n}^{0}, \rho_{n}\right) \subset \Omega$ from the Euclidean space $\mathbb{R}^{N}$ such that

$$
-\frac{\varphi_{n}\left(\left|\nabla \Psi_{n}(y)\right|\right)}{\left|\nabla \Psi_{n}(y)\right|} \Delta \Psi_{n}(y)-\frac{\left|\nabla \Psi_{n}(y)\right| \varphi_{n}^{\prime}\left(\left|\nabla \Psi_{n}(y)\right|\right)-\varphi_{n}\left(\left|\nabla \Psi_{n}(y)\right|\right)}{\left|\nabla \Psi_{n}(y)\right|^{3}} \Delta_{\infty} \Psi_{n}(y)>\lambda e^{\Psi_{n}(y)}
$$

for all $y \in B\left(x_{n}^{0}, \rho_{n}\right)$. For $\rho_{n}$ small enough, we may presume that $v_{n}-\Psi_{n}$ has a strict local maximum at $x_{n}^{0}$, that is $v_{n}(y)<\Psi_{n}(y)$ if $y \in B\left(x_{n}^{0}, \rho_{n}\right) \backslash\left\{x_{n}^{0}\right\}$. This fact implies that actually

$$
\sup _{\partial B\left(x_{n}^{0}, \rho_{n}\right)}\left(v_{n}-\Psi_{n}\right)<0 .
$$

Thus, we may consider a perturbation of the test function $\Psi_{n}$ defined as

$$
\bar{w}_{n}(y):=\Psi_{n}(y)+\frac{1}{2} \sup _{y \in \partial B\left(x_{n}^{0}, p_{n}\right)}\left[v_{n}-\Psi_{n}\right](y)
$$

that has the properties

- $\bar{w}_{n}\left(x_{n}^{0}\right)<v_{n}\left(x_{n}^{0}\right)$;
- $\bar{w}_{n}>v_{n}$ on $\partial B\left(x_{n}^{0}, \rho_{n}\right)$;
- $-\operatorname{div}\left(\frac{\varphi_{n}\left(\left|\nabla \bar{w}_{n}\right|\right)}{\left|\nabla \bar{w}_{n}\right|} \nabla \bar{w}_{n}\right)>\lambda e^{\Psi_{n}}$ in $B\left(x_{n}^{0}, \rho_{n}\right)$.

Multiplying the above inequality by the positive part of the function $v_{n}-\bar{w}_{n}$, i.e. $\left(v_{n}-\bar{w}_{n}\right)^{+}$, that vanishes on the boundary of the ball $B\left(x_{n}^{0}, \rho_{n}\right)$, and integrating on $B\left(x_{n}^{0}, \rho_{n}\right)$, we get

$$
\begin{equation*}
\int_{\mathcal{M}_{n}} \frac{\varphi_{n}\left(\left|\nabla \bar{w}_{n}(x)\right|\right)}{\left|\nabla \bar{w}_{n}(x)\right|} \nabla \bar{w}_{n}(x)\left[\nabla v_{n}(x)-\nabla \bar{w}_{n}(x)\right] d x>\lambda \int_{\mathcal{M}_{n}} e^{\Psi_{n}(x)}\left[v_{n}(x)-\bar{w}_{n}(x)\right] d x \tag{4.3}
\end{equation*}
$$

where the set $\mathcal{M}_{n}:=\left\{x \in B\left(x_{n}^{0}, \rho_{n}\right) ; \bar{w}_{n}(x)<v_{n}(x)\right\}$.
On the other hand, taking the test function in relation (3.1) to be

$$
w: \Omega \rightarrow \mathbb{R}, w(x)= \begin{cases}\left(v_{n}-\bar{w}_{n}\right)^{+}(x), & \text { if } x \in B\left(x_{n}^{0}, \rho_{n}\right), \\ 0, & \text { if } x \in \Omega \backslash B\left(x_{n}^{0}, \rho_{n}\right),\end{cases}
$$

we obtain

$$
\int_{B\left(x_{n}^{0}, \rho_{n}\right)} \frac{\varphi_{n}\left(\left|\nabla v_{n}(x)\right|\right)}{\left|\nabla v_{n}(x)\right|} \nabla v_{n}(x) \nabla\left(v_{n}-\bar{w}_{n}\right)^{+}(x) d x=\lambda \int_{B\left(x_{n}^{0}, \rho_{n}\right)} e^{v_{n}(x)}\left(v_{n}-\bar{w}_{n}\right)^{+}(x) d x
$$

or

$$
\int_{\mathcal{M}_{n}} \frac{\varphi_{n}\left(\left|\nabla v_{n}(x)\right|\right)}{\left|\nabla v_{n}(x)\right|} \nabla v_{n}(x) \nabla\left(v_{n}-\bar{w}_{n}\right)(x) d x=\lambda \int_{\mathcal{M}_{n}} e^{v_{n}(x)}\left(v_{n}-\bar{w}_{n}\right)(x) d x
$$

since $v_{n} \leq \bar{w}_{n}$ in the ball $B\left(x_{n}^{0}, \rho_{n}\right)$ outside $\mathcal{M}_{n}$.
Applying the subtraction of the above equality from inequality (4.3) it produces

$$
\begin{align*}
\int_{\mathcal{M}_{n}}\left[\frac{\varphi_{n}\left(\left|\nabla \bar{w}_{n}\right|\right)}{\left|\nabla \bar{w}_{n}\right|} \nabla \bar{w}_{n}-\frac{\varphi_{n}\left(\left|\nabla v_{n}\right|\right)}{\left|\nabla v_{n}\right|} \nabla v_{n}\right]\left(\nabla v_{n}\right. & \left.-\nabla \bar{w}_{n}\right) d x \\
& >\lambda \int_{\mathcal{M}_{n}}\left(e^{\Psi_{n}}-e^{v_{n}}\right)\left(v_{n}-\bar{w}_{n}\right) d x \geq 0 \tag{4.4}
\end{align*}
$$

with the aid of the facts that $v_{n}<\Psi_{n}$ on $B\left(x_{n}^{0}, \rho_{n}\right) \backslash\left\{x_{n}^{0}\right\}$ and $\bar{w}_{n}<v_{n}$ on $\mathcal{M}_{n} \subset B\left(x_{n}^{0}, \rho_{n}\right)$.
Cauchy-Schwarz inequality implies

$$
\begin{aligned}
\int_{\mathcal{M}_{n}} & {\left[\varphi_{n}\left(\left|\nabla v_{n}\right|\right)-\varphi_{n}\left(\left|\nabla \bar{w}_{n}\right|\right)\right]\left(\left|\nabla v_{n}\right|-\left|\nabla \bar{w}_{n}\right|\right) d x } \\
& \leq \int_{\mathcal{M}_{n}}\left[\frac{\varphi_{n}\left(\left|\nabla v_{n}\right|\right)}{\left|\nabla v_{n}\right|} \nabla v_{n}-\frac{\varphi_{n}\left(\left|\nabla \bar{w}_{n}\right|\right)}{\left|\nabla \bar{w}_{n}\right|} \nabla \bar{w}_{n}\right] \nabla\left(v_{n}-\bar{w}_{n}\right) d x
\end{aligned}
$$

and combined with relation (4.4) leads to

$$
\int_{\mathcal{M}_{n}}\left[\varphi_{n}\left(\left|\nabla \bar{w}_{n}\right|\right)-\varphi_{n}\left(\left|\nabla v_{n}\right|\right)\right]\left(\left|\nabla \bar{w}_{n}\right|-\left|\nabla v_{n}\right|\right) d x<0
$$

which is a contradiction with the statement that $\varphi_{n}$ is an increasing function on $\mathbb{R}$. Actually, it follows that $v_{n}$ is a viscosity subsolution of problem (1.1).

On the other hand, $v_{n}$ is a viscosity supersolution of problem (1.1) with similar arguments as above adapted for this case and therefore, these details will be omitted. The proof of Lemma 4.4 is complete.

By Lemma 4.2 we may select a subsequence $\left\{v_{n}\right\}$ that converges uniformly to $v_{\infty}$ in $\Omega$ as $n \rightarrow \infty$. Next, we will focus to identify the limit equation verified by $v_{\infty}$. The following theorem encloses the main result regarding the asymptotic behavior of the solutions $\left\{v_{n}\right\}$ of problem (1.1).

Theorem 4.5. Let $v_{\infty}$ be the function achieved as the uniform limit of a subsequence of $\left\{v_{n}\right\}$ in Lemma 4.2. Then $v_{\infty}$ is a solution in the viscosity sense of problem

$$
\begin{cases}\min \left\{-\Delta_{\infty} v,|\nabla v|-1\right\}=0 & \text { in } \Omega  \tag{4.5}\\ v=0 & \text { on } \partial \Omega\end{cases}
$$

Proof. First, we investigate if $v_{\infty}$ is a viscosity supersolution of (4.5). We consider $y_{0} \in \Omega$ and a test function $\mathrm{Y} \in C^{2}(\Omega)$ such that $v_{\infty}-\mathrm{Y}$ has a strict local minimum point at $y_{0}$. We claim that the uniform convergence of $\left\{v_{n}\right\}$ shown in Lemma 4.2 allows us to extract, up to a subsequence, $\left\{y_{n}\right\} \subset \Omega$ such that $y_{n}$ converges to $y_{0}$ and moreover $v_{n}-\mathrm{Y}$ achieves a strict local minimum point at $y_{n}$. Indeed, since $y_{0}$ is a strict minimum point of $v_{\infty}-\mathrm{Y}$ it follows that $v_{\infty}\left(y_{0}\right)=\mathrm{Y}\left(y_{0}\right)$ and $v_{\infty}(y)>\mathrm{Y}(y)$ for every $y$ in a punctured neighborhood of $y_{0}$, let's say $B\left(y_{0}, r\right) \backslash\left\{y_{0}\right\}$ with $r>0$ fixed in such a manner that $B\left(y_{0}, 2 r\right) \subset \Omega$. For any positive $\rho$ with $\rho<r$ we get

$$
\inf _{B\left(y_{0}, r\right) \backslash B\left(y_{0}, \rho\right)}\left(v_{\infty}-Y\right)>0 .
$$

By the uniform convergence of $\left\{v_{n}\right\}$ to $v_{\infty}$ in $\Omega$ and in particular in $\overline{B\left(y_{0}, r\right)}$, for any positive integer $n$ sufficiently large, the function $v_{n}-\mathrm{Y}$ attains its zero minimum value in $B\left(y_{0}, \rho\right)$ and
thus, the minimum point of $v_{n}-\mathrm{Y}$ will be represented by $y_{n} \in B\left(y_{0}, \rho\right)$. Considering a sequence $\rho_{k} \rightarrow 0^{+}$as $k \rightarrow \infty$, we can construct a subsequence $\left\{n_{k}\right\}$ such that $y_{n_{k}}$ converges to $y_{0}$ as $k \rightarrow \infty$. The claim now holds true after an appropriate relabelling of the indices. In other words, taking into account that $v_{n}, v_{\infty} \in C(\bar{\Omega})$ for any positive integer $n$ sufficiently large, the uniform convergence of sequence $\left\{v_{n}\right\}$ to $v_{\infty}$ in $\Omega$ implies that since Y touches $v_{\infty}$ from below at $y_{0}$, then there are points $y_{n} \rightarrow y_{0}$ such that

$$
v_{n}(y)-\mathrm{Y}(y)>0=v_{n}\left(y_{n}\right)-\mathrm{Y}\left(y_{n}\right) \text { for all } y \in B\left(y_{0}, \rho\right) \backslash\left\{y_{0}\right\}
$$

for some subsequence (see [6, Theorem 3.1] or [30, Lemma 11]).
Keeping in mind that in view of Lemma 4.4, $v_{n}$ is a continuous viscosity solution of (1.1) we have

$$
\begin{equation*}
-\frac{\varphi_{n}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)}{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|} \Delta \mathrm{Y}\left(y_{n}\right)-\frac{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right| \varphi_{n}^{\prime}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)-\varphi_{n}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)}{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|^{3}} \Delta_{\infty} \mathrm{Y}\left(y_{n}\right) \geq \lambda e^{\mathrm{Y}\left(y_{n}\right)} . \tag{4.6}
\end{equation*}
$$

Since $\lambda e^{\mathrm{Y}\left(y_{n}\right)}>0$ for any $\lambda \in\left(0, \lambda^{*}\right)$, it follows that $\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|>0$ for each positive integer $n$. Recalling inequality (2.6) states

$$
\begin{equation*}
\min \left\{s^{\varphi_{n}^{-}-1}, s^{\varphi_{n}^{+}-1}\right\} \varphi_{n}(t) \leq \varphi_{n}(s t) \leq \max \left\{s^{\varphi_{n}^{-}-1}, s^{\varphi_{n}^{+}-1}\right\} \varphi_{n}(t), \quad \forall s, t \geq 0 \tag{4.7}
\end{equation*}
$$

and keeping in mind (1.3), for each positive integer $n$ sufficiently large, the functions $A_{n}, B_{n}$ : $[0, \infty) \rightarrow \mathbb{R}$,

$$
A_{n}(t):=\left\{\begin{array}{ll}
\frac{t \varphi_{n}^{\prime}(t)-\varphi_{n}(t)}{t^{3}}, & \text { if } t>0, \\
0, & \text { if } t=0,
\end{array} \quad B_{n}(t):= \begin{cases}\frac{\varphi_{n}(t)}{t}, & \text { if } t>0 \\
0, & \text { if } t=0\end{cases}\right.
$$

are continuous. Moreover, function $B_{n}$ is of class $C^{1}$ since $A_{n}(t)=t^{-1} B_{n}^{\prime}(t)$ for $t>0$. According to (1.2) and (1.3), we deduce that

$$
\frac{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|^{3}}{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right| \varphi_{n}^{\prime}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)-\varphi_{n}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)}>0 .
$$

Inequality (4.6) multiplied with the above positive quantity in both sides becomes

$$
\begin{align*}
-\frac{\varphi_{n}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|^{2}}{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right| \varphi_{n}^{\prime}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)-\varphi_{n}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)} & \Delta \mathrm{Y}\left(y_{n}\right)-\Delta_{\infty} \mathrm{Y}\left(y_{n}\right) \\
& \geq \frac{\lambda e^{\mathrm{Y}\left(y_{n}\right)\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|^{3}}}{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right| \varphi_{n}^{\prime}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)-\varphi_{n}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)} . \tag{4.8}
\end{align*}
$$

On the other hand, we obtain

$$
\begin{equation*}
\frac{\varphi_{n}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|^{2}}{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right| \varphi_{n}^{\prime}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)-\varphi_{n}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)}=\frac{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|^{2}}{\frac{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right| \varphi_{n}^{\prime}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)}{\varphi_{n}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)}-1} \leq \frac{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|^{2}}{\varphi_{n}^{-}-2}, \tag{4.9}
\end{equation*}
$$

where in the latter inequality we use Lieberman-type condition (1.2).
In relation (4.8) we pass to the limit as $n \rightarrow \infty$ and then using (1.3) we infer by relation (4.9) that

$$
\begin{equation*}
-\Delta_{\infty} \mathrm{Y}\left(y_{0}\right) \geq \limsup _{n \rightarrow \infty} \frac{\lambda e^{\mathrm{Y}\left(y_{n}\right)}\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|^{3}}{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right| \varphi_{n}^{\prime}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)-\varphi_{n}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)} \tag{4.10}
\end{equation*}
$$

which hints that

$$
\begin{equation*}
-\Delta_{\infty} \mathrm{Y}\left(y_{0}\right) \geq 0 \tag{4.11}
\end{equation*}
$$

In the following we will show that

$$
\begin{equation*}
\left|\nabla \mathrm{Y}\left(y_{0}\right)\right|-1 \geq 0 . \tag{4.12}
\end{equation*}
$$

If we assume by contradiction that is not the case of the above claim, we get $\left|\nabla \mathrm{Y}\left(y_{0}\right)\right|-$ $1<0$, that implies $\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|<1$ for any positive integer $n$ sufficiently large. Taking into consideration (1.2) and then inequality (4.7) we arrive at

$$
\begin{aligned}
\frac{\lambda e^{\mathrm{Y}\left(y_{n}\right)\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|^{3}}}{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right| \varphi_{n}^{\prime}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)-\varphi_{n}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)} & =\frac{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|^{3}}{\frac{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right| \varphi_{n}^{\prime}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)}{\varphi_{n}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)}-1} \cdot \frac{\lambda e^{\mathrm{Y}\left(y_{n}\right)}}{\varphi_{n}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)} \\
& \geq \frac{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|^{3}}{\varphi_{n}^{+}-2} \cdot \frac{\lambda e^{\mathrm{Y}\left(y_{n}\right)}}{\varphi_{n}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)} \\
& \geq \frac{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|^{3}}{\varphi_{n}^{+}-2} \cdot \frac{\lambda e^{\mathrm{Y}\left(y_{n}\right)}}{\varphi_{n}(1)\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|^{\varphi_{n}^{-}-1}} \\
& =\left[\left(\frac{\lambda e^{\mathrm{Y}\left(y_{n}\right)}}{\left(\varphi_{n}^{+}-2\right) \varphi_{n}(1)}\right)^{1 /\left(\varphi_{n}^{-}-4\right)} \frac{1}{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|}\right]^{\varphi_{n}^{-}-4} .
\end{aligned}
$$

Since by (1.5) we have $\lim _{n \rightarrow \infty} \varphi_{n}(1)^{1 / \varphi_{n}^{-}}=1$ we get using (1.4) that

$$
\lim _{n \rightarrow \infty}\left(\frac{\lambda e^{\mathrm{Y}\left(y_{n}\right)}}{\left(\varphi_{n}^{+}-2\right) \varphi_{n}(1)}\right)^{1 /\left(\varphi_{n}^{-}-4\right)}=1
$$

Next, taking into account that $\lim _{n \rightarrow \infty} \frac{1}{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|}=\frac{1}{\left|\nabla \mathrm{Y}\left(y_{0}\right)\right|}>1$ we obtain

$$
\lim _{n \rightarrow \infty}\left(\frac{\lambda e^{\mathrm{Y}\left(y_{n}\right)}}{\left(\varphi_{n}^{+}-2\right) \varphi_{n}(1)}\right)^{1 /\left(\varphi_{n}^{-}-4\right)} \frac{1}{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|}=\frac{1}{\mid \nabla \mathrm{Y}\left(\left(y_{0}\right) \mid\right.}>1
$$

and then, we deduce that there exists $\epsilon_{0}>0$ such that

$$
\left(\frac{\lambda e^{\mathrm{Y}\left(y_{n}\right)}}{\left(\varphi_{n}^{+}-2\right) \varphi_{n}(1)}\right)^{1 /\left(\varphi_{n}^{-}-4\right)} \frac{1}{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|} \geq 1+\epsilon_{0} \quad \text { for all positive integer } n \text { sufficiently large, }
$$

which yields to

$$
\limsup _{n \rightarrow \infty} \frac{\lambda e^{\mathrm{Y}\left(y_{n}\right)}\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|^{3}}{\left|\nabla \mathrm{Y}\left(y_{n}\right)\right| \varphi_{n}^{\prime}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)-\varphi_{n}\left(\left|\nabla \mathrm{Y}\left(y_{n}\right)\right|\right)} \geq \lim _{n \rightarrow \infty}\left(1+\epsilon_{0}\right)^{\varphi_{n}^{--4}}=+\infty,
$$

a contradiction with (4.10). Thus, inequality (4.12) holds true.
Assembling relations (4.11) and (4.12) we have $\min \left\{-\Delta_{\infty} \mathrm{Y}\left(y_{0}\right),\left|\nabla \mathrm{Y}\left(y_{0}\right)\right|-1\right\} \geq 0$ which leads to the fact that $v_{\infty}$ is a viscosity supersolution of (4.5).

Now, it remains to see that in fact $v_{\infty}$ is a viscosity subsolution of (4.5). We take a test function $\Psi \in C^{2}(\Omega)$ that touches the graph of $v_{\infty}$ from above in a point $x_{0} \in \Omega$, that means $v_{\infty}\left(x_{0}\right)=\Psi\left(x_{0}\right)$ and $v_{\infty}(x)<\Psi(x)$ for every $x$ in a punctured neighborhood of $x_{0}$ and we have
to establish that $\min \left\{-\Delta_{\infty} \Psi\left(x_{0}\right),\left|\nabla \Psi\left(x_{0}\right)\right|-1\right\} \leq 0$. We notice that if $\left|\nabla \Psi\left(x_{0}\right)\right|=0$ then we have $\Delta_{\infty} \Psi\left(x_{0}\right)=0$ and everything is clear. Then, it is sufficient to check that if $\left|\nabla \Psi\left(x_{0}\right)\right|>0$ and also

$$
\begin{equation*}
\left|\nabla \Psi\left(x_{0}\right)\right|-1>0 \tag{4.13}
\end{equation*}
$$

we get $-\Delta_{\infty} \Psi\left(x_{0}\right) \leq 0$. Actually, the uniform convergence of subsequence of $\left\{v_{n}\right\}$ ensures again, as in the first part of this proof, the existence of a sequence $x_{n} \rightarrow x_{0}$ as $n \rightarrow \infty$ such that $v_{n}-\Psi$ has a strict local maximum point at $x_{n}$ and

$$
\begin{align*}
-\frac{\varphi_{n}\left(\left|\nabla \Psi\left(x_{n}\right)\right|\right)\left|\nabla \Psi\left(x_{n}\right)\right|^{2}}{\left|\nabla \Psi\left(x_{n}\right)\right| \varphi_{n}^{\prime}\left(\left|\nabla \Psi\left(x_{n}\right)\right|\right)-\varphi_{n}\left(\left|\nabla \Psi\left(x_{n}\right)\right|\right)} & \Delta \Psi\left(x_{n}\right)-\Delta_{\infty} \Psi\left(x_{n}\right) \\
& \leq \frac{\lambda e^{\Psi\left(x_{n}\right)}\left|\nabla \Psi\left(x_{n}\right)\right|^{3}}{\left|\nabla \Psi\left(x_{n}\right)\right| \varphi_{n}^{\prime}\left(\left|\nabla \Psi\left(x_{n}\right)\right|\right)-\varphi_{n}\left(\left|\nabla \Psi\left(x_{n}\right)\right|\right)} . \tag{4.14}
\end{align*}
$$

Passing to the limit as $n \rightarrow \infty$ in the above relation and using (4.13), inequality (4.7), and assumptions (1.3) and (1.5), we deduce that

$$
-\Delta_{\infty} \Psi\left(x_{0}\right) \leq \liminf _{n \rightarrow \infty}\left[\left(\frac{\lambda e^{\Psi\left(x_{n}\right)}}{\left(\varphi_{n}^{-}-2\right) \varphi_{n}(1)}\right)^{1 /\left(\varphi_{n}^{\bar{n}}-4\right)} \frac{1}{\left|\nabla \Psi\left(x_{n}\right)\right|}\right]^{\varphi_{n}^{-}-4}=0
$$

which implies that $-\Delta_{\infty} \Psi\left(x_{0}\right) \leq 0$. Thus, we conclude that $v_{\infty}$ is a viscosity solution of problem (4.5). The proof of Theorem 4.5 is complete.

Next, we identify the limit of the entire sequence of weak solutions $\left\{v_{n}\right\}$ of problem (1.1).
Proof of Theorem 4.1 (concluded). It is well-known that problem (4.5) has as unique viscosity solution $\operatorname{dist}(\cdot, \partial \Omega)$, namely the distance function to the boundary of $\Omega$ (see Jensen [25], or Juutinen [26, Lemma 6.10], or Ishibashi \& Koike [24, p. 546]). As a consequence, Lemma 4.2 and Theorem 4.5 allow us to reach to the conclusion that the entire sequence $\left\{v_{n}\right\}$ converges uniformly to $\operatorname{dist}(\cdot, \partial \Omega)$ in $\Omega$, as $n \rightarrow \infty$.

## Acknowledgements

The authors have been partially supported by CNCS-UEFISCDI Grant No. PN-III-P1-1.1-TE-2019-0456.

## References

[1] D. R. Adams, L. I. Hedberg, Function spaces and potential theory, Grundlehren der Mathematischen Wissenschaften, Vol. 314, Springer-Verlag, Berlin, 1996. https://doi.org/10. 1007/978-3-662-03282-4; MR1411441; Zbl 0834.46021
[2] R. Adams, Sobolev spaces, Academic Press, New York, 1975. MR0450957; Zbl 0314.46030
[3] J. A. Aguilar Crespo, I. Peral Alonso, On an elliptic equation with exponential growth, Rend. Sem. Mat. Univ. Padova 96(1996), 143-175. MR1438296; Zbl 0887.35055
[4] G. Barletta, E. Tornatore, Elliptic problems with convection terms in Orlicz spaces, J. Math. Anal. Appl. 495(2021), No. 2, 124779, 28 pp. https://doi.org/10.1016/j.jmaa. 2020.124779; MR4182981; Zbl 07315660
[5] M. Bocea, M. Mihăilescu, On a family of inhomogeneous torsional creep problems, Proc. Amer. Math. Soc. 145(2017), 4397-4409. https://doi.org/10.1090/proc/13583; MR3690623; Zbl 1387.35311
[6] M. Bocea, M. Mihăilescu, D. Stancu-Dumitru, The limiting behavior of solutions to inhomogeneous eigenvalue problems in Orlicz-Sobolev spaces, Adv. Nonlinear Stud. 14(2014), 977-990. https://doi.org/10.1515/ans-2014-0409; MR3269381; Zbl 1335.35173
[7] H. Brezis, F. Merle, Uniform esitmates and blow-up behavior for solutions of $\Delta u=$ $v(x) e^{u}$ in two dimensions, Comm. Partial Differential Equations 16(1991), 1223-1253. https : //doi.org/10.1080/03605309108820797; MR1132783; Zbl 0746.35006
[8] F. Charro, E. Parini, Limits as $p \rightarrow \infty$ of $p$-Laplacian problems with a superdiffusive power-type nonlinearity: positive and sign-changing solutions, J. Math. Anal. Appl. 372(2010), 629-644. https://doi.org/10.1016/j.jmaa.2010.07.005; MR2678889; Zbl 1198.35115
[9] Рh. Clément, M. García-Huidobro, R. Manásevich, K. Schmitt, Mountain pass type solutions for quasilinear elliptic equations, Calc. Var. 11(2000), 33-62. https://doi.org/ 10.1007/s005260050002; MR1777463; Zbl 0959.35057
[10] Ph. Clément, B. de Pagter, G. Sweers, F. de Thélin, Existence of solutions to a semilinear elliptic system through Orlicz-Sobolev spaces, Mediterr. J. Math. 1(2004), 241-267. https://doi.org/10.1007/s00009-004-0014-6; MR2094464; Zbl 1167.35352
[11] M. G. Crandall, H. Ishii, P. L. Lions, User's guide to viscosity solutions of second-order partial differential equations, Bull. Am. Math. Soc. 27(1992), 1-67. https://doi.org/10. 1090/S0273-0979-1992-00266-5; MR1118699; Zbl 0755.35015
[12] W. Chen, C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Mathematical Journal 63(1991), 615-622. https ://doi.org/10.1215/S0012-7094-91-06325-8; MR1121147; Zbl 0768.35025
[13] G. Ercole, G. A. Pereira, Asymptotics for the best Sobolev constants and their extremal functions, Math. Nachr. 289(2016), 1433-1449. https://doi.org/10.1002/mana. 201500263; MR3541818; Zbl 1355.35081
[14] M. Fărcășeanu, M. Mihăilescu, The asymptotic behaviour of the sequence of solutions for a family of equations involving $p(\cdot)$-Laplace operators, Mosc. Math. J. 20(2020), 495509. https://doi.org/10.17323/1609-4514-2020-20-3-495-509; MR4100135
[15] H. Fujita, On the nonlinear equation $\Delta u+\exp u=0$ and $v_{t}=\Delta u+\exp u$, Bull. Amer. Math. Soc. 75(1969), 132-135. MR0239258; Zbl 0216.12101
[16] N. Fukagai, K. Narukawa, On the existence of multiple positive solutions of quasilinear elliptic eigenvalue problems, Ann. Mat. Pura Appl. (4) 186(2007), No. 3, 539-564. https: //doi.org/10.1007/s10231-006-0018-x; MR2317653; Zbl 1223.35132
[17] N. Fukagai, M. Ito, K. Narukawa, Positive solutions of quasilinear elliptic equations with critical Orlicz-Sobolev nonlinearity on $\mathbb{R}^{N}$, Funkcial. Ekvac. 49(2006), 235-267. https://doi.org/10.1619/fesi.49.235; MR2271234; Zbl 1387.35405
[18] J. Garcia Azozero, I. Peral Alonso, On a Emden-Fowler type equation, Nonlinear Anal. 18(1992), 1085-1097. https://doi.org/10.1016/0362-546X(92)90197-M; MR1167423; Zbl 0781.35021
[19] J. Garcia Azozero, I. Peral Alonso, J. P. Puel, Quasilinear problems with exponential growth in the reaction term, Nonlinear Anal. 22(1994), 481-498. https://doi.org/10. 1016/0362-546X (94) 90169-4; MR1266373; Zbl 0804.35037
[20] M. García-Huidobro, V. K. Le, R. Manásevich, K. Schmitt, On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz-Sobolev space setting, NoDEA Nonlinear Differential Equations Appl. 6(1999), 207-225. https://doi.org/10. 1007/s000300050073; MR1694787; Zbl 0936.35067
[21] J. P. Gossez, Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients, Trans. Amer. Math. Soc. 190(1974), 163-205. https://doi. org/10.2307/1996957; MR0342854; Zbl 0239.35045
[22] P. Harjulehto, P. Нästö, Orlicz spaces and generalized Orlicz spaces, Lecture Notes in Mathematics, Vol. 2236, Springer, Cham, 2019. https://doi.org/10.1007/978-3-030-15100-3; MR3931352; Zbl 1436.46002
[23] X. Y. Huang, On the eigenvalues of the $p$-Laplacian with varying $p$, Proc. Amer. Math. Soc. 125(1997), 3347-3354. MR1403133; Zbl 0882.35087
[24] T. Ishibashi, S. Koike, On fully nonlinear PDE's derived from variational problems of $L^{p}$ norms, SIAM J. Math. Anal. 33(2001), 545-569. https://doi.org/10.1137/ S0036141000380000; MR1871409; Zbl 1030.35088
[25] R. Jensen, Uniqueness of Lipschitz extensions minimizing the sup-norm of the gradient, Arch. Rat. Mech. Anal. 123(1993), 51-74. https://doi.org/10.1007/BF00386368; MR1218686; Zbl 0789.35008
[26] P. Juutinen, Minimization problems for Lipschitz functions via viscosity solutions, Dissertation, University of Jyväskulä, Jyväskulä, 1998, Ann. Acad. Sci. Fenn. Math. Diss. 115(1998), 53 pp. MR1632063; Zbl 0902.35037
[27] P. Juutinen, P. Lindqvist, J. J. Manfredi, The $\infty$-eigenvalue problem, Arch. Rational Mech. Anal. 148(1999), 89-105. https://doi.org/10.1007/s002050050157; MR1716563; Zbl 0947.35104
[28] G. M. Lieberman, The natural generalization of the natural conditions of Ladyzhensaya and Uraltseva for elliptic equations, Comm. Partial Differential Equations 16(1991), No. 2-3, 311-361. https://doi.org/10.1080/03605309108820761; MR1104103; Zbl 0742.35028
[29] P. LindQvist, On non-linear Rayleigh quotients, Potential Anal. 2(1993), 199-218. https : //doi.org/10.1007/BF01048505; MR1245239; Zbl 0795.35039
[30] P. Lindqvist, Notes on the infinity Laplace equation, BCAM SpringerBriefs in Mathematics, Springer, 2016. https://doi.org/10.1007/978-3-319-31532-4; MR3467690; Zbl 1352.35045
[31] S. Martínez, N. Wolanski, A minimum problem with free boundary in Orlicz spaces, Adv. Math. 218(2008), 1914-1971. https://doi.org/10.1016/j .aim. 2008.03.028; MR2431665; Zbl 1170.35030
[32] M. Mihăilescu, D. Stancu-Dumitru, Cs. Varga, The convergence of nonnegative solutions for the family of problems $-\Delta_{p} u=\lambda e^{u}$ as $p \rightarrow \infty$, ESAIM Control Optim. Calc. Var. 24(2018), No. 2, 569-578. https://doi.org/10.1051/cocv/2017048; MR3816405; Zbl 1404.35311
[33] J. Musielak, Orlicz spaces and modular spaces, Lecture Notes in Mathematics, Vol. 1034, Springer, Berlin, 1983. https://doi.org/10.1007/BFb0072210; MR0724434; Zbl 0557.46020
[34] M. M. Rao, Z. D. Ren, Theory of Orlicz spaces, Marcel Dekker, Inc., New York, 1991. MR1113700; Zbl 0724.46032
[35] D. Stancu-Dumitru, The asymptotic behavior of a class of $\varphi$-harmonic functions in Orlicz-Sobolev spaces, J. Math. Anal. Appl. 463(2018), No. 1, 365-376. https://doi.org/ 10.1016/j.jmaa.2018.03.025; MR3779668; Zbl 1392.35137

# Optimal version of the Picard-Lindelöf theorem 

Jan-Christoph Schlage-Puchta ${ }^{\boxtimes}$<br>Universität Rostock, Mathematisches Institut, Ulmenstraße 69, 18057 Rostock, Germany

Received 20 February 2020, appeared 7 May 2021
Communicated by Mihály Pituk


#### Abstract

Consider the differential equation $y^{\prime}=F(x, y)$. We determine the weakest possible upper bound on $|F(x, y)-F(x, z)|$ which guarantees that this equation has for all initial values a unique solution, which exists globally.


Keywords: ordinary differential equations, Picard-Lindelöf theorem, global existence, uniqueness.
2020 Mathematics Subject Classification: 34A12.

## 1 Introduction

Let $F: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be a continuous function. The well known global Picard-Lindelöf theorem states that if $F$ is Lipschitz continuous with respect to the second variable, then for every real number $y_{0}$, the initial value problem $y^{\prime}=F(x, y), y(0)=y_{0}$ has a unique solution, which exists globally. On the other hand the initial value problem $y^{\prime}=2 \sqrt{|y|}, y(0)=0$ has infinitely many solutions, which can be parametrized by real numbers $-\infty \leq a \leq 0 \leq b \leq \infty$ as

$$
y= \begin{cases}-(x-a)^{2}, & x<a \\ 0, & a \leq x \leq b \\ (x-b)^{2}, & x>b\end{cases}
$$

We conclude that uniqueness does not hold in general without the Lipschitz condition. Similarly the initial value problem $y^{\prime}=1+y^{2}, y(0)=0$ has the solution $\tan x$, which does not exist globally. Thus, global existence also needs some kind of Lipschitz condition. Here we show that while some condition is necessary, being Lipschitz is unnecessarily strict, and determine the optimal condition. We prove the following.

Theorem 1.1. Let $\varphi:[0, \infty] \rightarrow(0, \infty)$ be a non-decreasing function. Then the following are equivalent.
(i) The series $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)}$ diverges;

[^1](ii) For every continuous functions $F: \mathbb{R}^{2} \rightarrow \mathbb{R}$ for which there exists a continuous function $\psi: \mathbb{R} \rightarrow(0, \infty)$ such that
\[

$$
\begin{equation*}
|F(x, y)-F(x, z)|<(z-y) \psi(x) \varphi(|\ln (z-y)|) \tag{1.1}
\end{equation*}
$$

\]

holds for all real numbers $x, y, z$ such that $y<z \leq y+1$, the initial value problem $y^{\prime}=F(x, y)$, $y(0)=y_{0}$ has a unique local solution.
(iii) For every continuous functions $F: \mathbb{R}^{2} \rightarrow \mathbb{R}$ for which there exists a continuous function $\psi: \mathbb{R} \rightarrow(0, \infty)$ such that

$$
\begin{equation*}
|F(x, y)|<|y| \psi(x) \varphi(\ln (2+|y|)) \tag{1.2}
\end{equation*}
$$

holds for all $x$ and $y$, every local solution of the initial value problem $y^{\prime}=F(x, y), y(0)=y_{0}$, where $y_{0}$ is arbitrary, can be continued to a global solution.

In particular it is not possible to prove a general Picard-Lindelöff type theorem with a bound that is strictly weaker than (1.1) or (1.2) for a function $\varphi$ satisfying (i). In this sense our theorem is indeed optimal. It might still be possible to prove existence or uniqueness under weaker conditions on the growth of $F$, if we impose other additional constrictions. However, the counterexamples we construct to prove (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) involve quite well behaved functions $F$, so it is not clear how such an additional assumption could look like.

Cid and Pouso [1, Theorem 1.2] gave a quite ingenious proof for a uniqueness theorem, which is equivalent to the implication (i) $\Rightarrow$ (ii) of our theorem, provided that $F\left(x, y_{0}\right)=0$ for all $x$ in a neighbourhood of 0 . Rudin [5] showed that if every global solution of $y^{\prime}=F(x, y)$ is unique, then there exists a function $h$ such that for all $y_{0}$ there exists some $x_{0}$, such that the solution of $y^{\prime}=F(x, y), y(0)=y_{0}$, satisfies $|y(x)| \leq h(x)$ for all $x>x_{0}$, whereas if solutions are not unique, then there might exist arbitrary fast growing solutions. Although this result is only loosely connected to our theorem, this work is relevant here, because the construction of the counterexamples in [5] is quite similar to our construction. We would like to thank the referee for making us aware of these publications.

The usual proof of the Picard-Lindelöf theorem uses contraction on a suitably defined Banach space. For extensions of the Picard-Lindelöf theorem using contractions we refer the reader to [2], [3] and [4]. A different generalization was given in [6]. However, our proof is more elementary, once some local existence result is available. We will use Peano's theorem in the following form.

Theorem 1.2 (Peano). Let $F: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be a continuous function, $y_{0}$ be some real number. Then there exists some $\epsilon>0$ and a differentiable function $y:(-\epsilon, \epsilon) \rightarrow \mathbb{R}$, which satisfies $y^{\prime}=F(x, y), y(0)=$ $y_{0}$.

We begin with the implication (i) $\Rightarrow$ (ii) in the special case that the zero function is a solution, that is, $F(x, 0)=0$ holds for all $x$. Let $y$ be a function satisfying $y^{\prime}=F(x, y), y(0)=1$. We claim that $y(x)>0$ for all $x>0$. If a solution tends to $+\infty$ in finite time without attaining negative values we say that this statement is also satisfied. For $n \geq 1$ define $x_{n}$ to be the smallest positive solution of the equation $y\left(x_{n}\right)=e^{-n}$. If there is some $n$ such that this equation is not solvable, then $y(x)>e^{-n}$ for this particular $n$ and all $x>0$, and our claim is trivially true, henceforth we assume that this equation is solvable for all $n$. Define $x_{n}^{+}$to be the largest solution of the equation $y(x)=e^{-n}$ with $x \in\left[x_{n}, x_{n+1}\right]$. Clearly, $x_{n}^{+}$exists. In the
interval $\left[x_{n}^{+}, x_{n+1}\right]$ we have $y(x) \in\left[e^{-(n+1)}, e^{-n}\right]$, thus,

$$
\begin{aligned}
x_{n+1}-x_{n}^{+} & \geq \frac{e^{-n}-e^{-(n+1)}}{\max _{x \in\left[x_{n}^{+}, x_{n+1}\right]}\left|y^{\prime}(x)\right|} \\
& \geq \frac{e^{-n}-e^{-(n+1)}}{\max _{x \in\left[x_{n}^{+}, x_{n+1}\right]} \max _{t \in\left[e^{\left.-(n+1), e^{-n}\right]}\right.}|F(x, t)|} \\
& \geq \frac{e^{-n}-e^{-(n+1)}}{e^{-n} \varphi(n+1) \max _{x \in\left[x_{n}^{+}, x_{n+1}\right]} \psi(x)} \\
& \geq \frac{1}{2 \varphi(n+1) \max _{x \in\left[0, x_{n+1}\right]} \psi(x)} .
\end{aligned}
$$

Assume that the sequence $\left(x_{n}\right)$ is bounded. Then $\max _{x \in\left[0, x_{n+1}\right]} \psi(x)$ is bounded by some constant $C$. We conclude that in this case

$$
x_{n+1}-x_{n} \geq x_{n+1}-x_{n}^{+} \geq \frac{1}{2 C \varphi(n+1)}
$$

By assumption we have that $\sum \frac{1}{\varphi(n)}$ diverges, which contradicts the assumption that $\left(x_{n}\right)$ is bounded. Hence, $\left(x_{n}\right)$ tends to infinity. By the definition of $x_{n}$ we have $y(x)>0$ in $\left[0, x_{n}\right]$, and our claim follows.

Next suppose that $F(x, 0)=0$ holds for all $x$, and $y_{1}$ is a solution of $y^{\prime}=F(x, y)$ with $y(0) \neq 0$. Then $\tilde{y}=\frac{y_{1}}{y_{1}(0)}$ is a solution of $y^{\prime}=\frac{1}{y_{1}(0)} F(x, y)$. As $\tilde{y}(0)=1$, we conclude that $\tilde{y}(x)>0$ holds for all $x>0$, and therefore $y_{1}(x) \neq 0$ for all $x>0$.

Now suppose that $F$ satisfies the assumption of the theorem, and $y_{1}, y_{2}$ are solutions of $y^{\prime}=F(x, y)$ with $y_{1}(0) \neq y_{2}(0)$. Then we consider the differential equation

$$
y^{\prime}=F\left(x, y+y_{1}(x)\right)-y_{1}^{\prime}(x)
$$

The constant function $y=0$ is a solution. The function $y(x)=y_{2}(x)-y_{1}(x)$ is also a solution, as for this function we have

$$
F\left(x, y_{2}(x)-y_{1}(x)+y_{1}(x)\right)-y_{1}^{\prime}(x)=F\left(x, y_{2}(x)\right)-y_{1}^{\prime}(x)=y_{2}^{\prime}(x)-y_{1}^{\prime}(x)
$$

The function $G(x, t)=F\left(x, t+y_{1}(x)\right)-y_{1}^{\prime}(x)$ is continuous, and satisfies

$$
\begin{aligned}
\left|G\left(x, t_{1}\right)-G\left(x, t_{2}\right)\right| & =\left|F\left(x, t_{1}+y_{1}(x)\right)-F\left(x, t_{2}+y_{1}(x)\right)\right| \\
& \leq\left(t_{1}-t_{2}\right) \psi(x) \varphi\left(\left|\ln \left(t_{1}-t_{2}\right)\right|\right)
\end{aligned}
$$

as well as $G(x, 0)=F\left(x, y_{1}(x)\right)-y_{1}^{\prime}(x)=0$. In particular we know that the claimed implication holds for $G$, and we obtain that $y_{1}-y_{2}$ does not vanish. As we may revert time, it follows that solutions are unique.

Now we prove the implication (i) $\Rightarrow$ (iii). By symmetry it suffices to consider the range $[0, \infty)$. Let $I \subseteq[0, \infty)$ be the maximal range of a solution. By Peano's theorem we know that solutions exist locally, that is, $I$ is half open. Suppose $I=\left[0, x_{\max }\right)$ with $x_{\max }<\infty$. A computation similar to the one used for uniqueness shows that $y$ is bounded on $\left[0, x_{\max }\right)$. As $\psi$ is continuous and $\left[0, x_{\max }\right]$ is compact, $\psi$ is also bounded. Put $Y=\sup _{x \leq x_{\max }}|y(x)| \psi(x)$. Then $F$ is bounded on $\left[0, x_{\max }\right] \times[-Y, Y]$, that is, $y$ is Lipschitz continuous on $\left[0, x_{\max }\right)$, and we can
extend $y$ continuously to $\left[0, x_{\max }\right]$. Moreover, as $F$ is continuous, this extension satisfies the differential equation in $x_{\max }$ if we interpret the derivative as a one-sided derivative. By Peano there exists a local solution around $x_{\max }$, which by the uniqueness we already know coincides with $y$ for $x<x_{\text {max }}$, hence, $y$ can be extended beyond $x_{\max }$ as a solution of the differential equation. This contradicts the definition of $x_{\max }$, and we conclude that $x_{\max }=\infty$, that is, $y$ exists globally.

We now turn to the reverse implications. For a given function $\varphi$, such that $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)}$ converges, we construct functions $F$ which satisfy the conditions (1.1) resp. (1.2), but for which the solutions of the corresponding differential equation are not unique resp. tend to infinity. We claim that we may assume without loss of generality that $\varphi(n) \leq n^{2}$. In fact, $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)}$ converges if and only if $\sum_{n=1}^{\infty} \frac{1}{\min \left(\varphi(n), n^{2}\right)}$ converges, hence, replacing $\varphi(n)$ by $\min \left(\varphi(n), n^{2}\right)$ does not change condition (i), whereas conditions (ii) and (iii) become weaker.

Next we show (iii) $\Rightarrow$ (i). Suppose that $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)}$ converges. Let $F$ be a piecewise linear function satisfying $F(0)=1, F\left(e^{n}\right)=e^{n} \varphi(n)$. Let $y$ be a solution of the differential equation $y^{\prime}=F(y), y(0)=0$, and let $x_{n}$ be the positive solution of the equation $y(x)=e^{n}$. Note that $x_{n}$ exists and is unique, as $y^{\prime} \geq 1$ for all $x \geq 0$. Then we have

$$
x_{n+1}-x_{n} \leq \frac{e^{n+1}-e^{n}}{\min _{x_{n} \leq x \leq x_{n+1}} y^{\prime}(x)}=\frac{e^{n+1}-e^{n}}{\min _{e^{n} \leq y \leq e^{n+1}} F(y)} \leq \frac{e^{n+1}-e^{n}}{F\left(e^{n}\right)}=\frac{e^{n+1}-e^{n}}{e^{n} \varphi(n)} \leq \frac{2}{\varphi(n)} .
$$

As $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)}$ converges, we conclude that the sequence $x_{n}$ converges to some finite limit $x_{\infty}$, that is, $y(x)$ tends to infinity as $x \rightarrow x_{\infty}$. We conclude that if $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)}$ converges, then there exists a differential equation as in (iii) which does not have a global solution.

Now consider the implication (ii) $\Rightarrow$ (i). Suppose that $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)}$ converges, and let $F$ be the function satisfying $F(t)=0$ for $t \leq 0, F(t)=\varphi(0)$ for $t \geq \frac{1}{e}, F\left(e^{-n}\right)=e^{-n} \varphi(n-1)$, which is continuous and linear on all intervals $\left(e^{-n-1}, e^{-n}\right)$. We claim that $F$ satisfies (2). As $F$ is constant outside $\left[0, \frac{1}{e}\right]$, it suffices to check the case $0 \leq y<z \leq \frac{1}{e}$.

If $y=0$, let $n$ be the unique integer satisfying $e^{-n-1}<z \leq e^{-n}$. Then we have

$$
\begin{aligned}
|F(y)-F(z)| & =F(z) \\
& =\frac{e^{-n}-z}{e^{-n}-e^{-n-1}} e^{-n-1} \varphi(n)+\frac{z-e^{-n-1}}{e^{-n}-e^{-n-1}} e^{-n} \varphi(n-1) \\
& \leq \frac{e^{-n}-z}{e^{-n}-e^{-n-1}} e^{-n-1} \varphi(n)+\frac{z-e^{-n-1}}{e^{-n}-e^{-n-1}} e^{-n} \varphi(n) \\
& =z \varphi(n) \\
& \leq z \varphi(|\ln z|) .
\end{aligned}
$$

If $y>0$, let $m \leq n$ be the unique integers satisfying $e^{-n-1}<y \leq e^{-n}, e^{-m-1}<z \leq e^{-m}$. If $m<n$, then

$$
\begin{aligned}
|F(y)-F(z)|= & F(z)-F(y) \\
= & \frac{e^{-m}-z}{e^{-m}-e^{-m-1}} e^{-m-1} \varphi(m)+\frac{z-e^{-m-1}}{e^{-m}-e^{-m-1}} e^{-m} \varphi(m-1) \\
& -\frac{e^{-n}-y}{e^{-n}-e^{-n-1}} e^{-n-1} \varphi(n)-\frac{y-e^{-n-1}}{e^{-n}-e^{-n-1}} e^{-n} \varphi(n-1) \\
\leq & z \varphi(m)-y \varphi(n-1)
\end{aligned}
$$

$$
\begin{aligned}
& \leq(z-y) \varphi(m) \\
& \leq(z-y) \varphi(|\ln z|) \\
& \leq(z-y) \varphi(|\ln (z-y)|),
\end{aligned}
$$

and our claim follows. If $m=n$, then

$$
\begin{aligned}
|F(y)-F(z)|= & F(z)-F(y) \\
= & \frac{e^{-m}-z}{e^{-m}-e^{-m-1} e^{-m-1} \varphi(m)+\frac{z-e^{-m-1}}{e^{-m}-e^{-m-1}} e^{-m} \varphi(m-1)} \\
& -\frac{e^{-n}-y}{e^{-n}-e^{-n-1}} e^{-n-1} \varphi(n)-\frac{y-e^{-n-1}}{e^{-n}-e^{-n-1}} e^{-n} \varphi(n-1) \\
= & (z-y) \frac{e^{-m} \varphi(m-1)-e^{-m-1} \varphi(m)}{e^{-m}-e^{-m-1}} \\
\leq & (z-y) \varphi(m) \\
\leq & (z-y) \varphi(|\ln (z-y)|)
\end{aligned}
$$

We find that (2) holds in all cases.
Now consider the differential equation $y^{\prime}=-F(y)$. This equation has the obvious solution $y=0$. Now consider the solution with starting value $y(0)=\frac{1}{e}$. As $y^{\prime}(x)<0$ for all $x$ with $y(x)>0$, there is for every $n$ a unique $x_{n}$ solving the equation $y(x)=e^{-n}$. We have

$$
x_{n+1}-x_{n} \leq \frac{e^{-n}-e^{-n-1}}{\min _{x_{n} \leq x \leq x_{n+1}} y^{\prime}(x)}=\frac{e^{-n}-e^{-n-1}}{F\left(e^{-n-1}\right)}=\frac{e-1}{\varphi(n)}
$$

As $\sum \frac{1}{\varphi(n)}$ converges, the sequence $\left(x_{n}\right)$ converges to some limit $x_{\infty}$, and we obtain that $y(x)=$ 0 for $x>x_{\infty}$. Reversing time we find that the equation $y^{\prime}=F(y), y(0)=0$ does not have a unique solution. Hence, if (i) fails, so does (ii), and the proof of the theorem is complete.

We remark that the proof not only yields global existence and uniqueness of solutions, but also gives explicit bounds. Here an explicit measure for uniqueness is a bound how quickly different solutions can diverge. Equivalently we can revert time and ask how quickly solutions with different starting conditions converge. By computing the sequence $\left(x_{n}\right)$ occurring in the proof of the implication (i) $\Rightarrow$ (ii) for specific functions $\varphi$ we obtain the following.

## Proposition 1.3.

(i) Let $F: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be a continuous function satisfying

$$
|F(x, y)-F(x, z)|<L|y-z|
$$

and $F(x, 0)=0$ for all real numbers $x, y$ and $z$. Then every solution of the equation $y^{\prime}=F(x, y)$ satisfies $|y(x)| \leq e^{L x}|y(0)|$ for all $x \geq 0$, and if $y_{1}, y_{2}$ are solutions, and $x \geq 0$, then we have

$$
\left|y_{1}(x)-y_{2}(x)\right| \geq\left|y_{1}(0)-y_{2}(0)\right| e^{-L x} .
$$

(ii) Let $F: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be a continuous function satisfying

$$
\begin{equation*}
|F(x, y)-F(x, z)|<|y-z|(1+\sqrt{|\ln (|y-z|)|}) \tag{1.3}
\end{equation*}
$$

for all real numbers $x, y$ and $z$ such that $y \leq z \leq y+1$. If $y_{1}, y_{2}$ are solutions of the equation $y^{\prime}=F(x, y)$ satisfying $y_{1}(0)-y_{2}(0)=1$, then we have

$$
\left|y_{1}(x)-y_{2}(x)\right| \geq e^{-x^{2}}
$$

for all $x>35$.
(iii) Let $F: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be a continuous function satisfying

$$
\begin{equation*}
|F(x, y)-F(x, z)|<|y-z|(1+|\ln (|y-z|)|) \tag{1.4}
\end{equation*}
$$

for all real numbers $x, y, z$ such that $y \leq z \leq y+1$. If $y_{1}, y_{2}$ are solutions of the equation $y^{\prime}=F(x, y)$ satisfying $y_{1}(0)-y_{2}(0)=1$, then we have

$$
\left|y_{1}(x)-y_{2}(x)\right| \geq e^{-e^{2 x}-4}
$$

for all $x \geq 0$.
Proof. For the upper bound in (i) note that as $F(x, 0)=0$, the Lipschitz condition with $z=0$ reads $\left|y^{\prime}(x)\right| \leq L|y(x)|$, and the upper bound follows. For the lower bound note that $f(x)=$ $y_{1}(x)-y_{2}(x)$ satisfies $\left|f^{\prime}(x)\right| \leq L|f(x)|$. We may assume without loss that $y_{1}(0)>y_{2}(0)$. Then we consider $g(x)=e^{L x} f(x)$. Put $x_{0}=\inf \{x: f(x) \leq 0\}$, we want to show that $x_{0}=\infty$. For $x \in\left[0, x_{0}\right]$ we have $g^{\prime}(x)=e^{L x}\left(L f(x)+f^{\prime}(x)\right) \geq 0$, in particular $g(x) \geq g(0)$ and therefore $f(x) \geq e^{-L x} f(0)$. As $f$ is continuous, we see that $x_{0}$ cannot be finite, and $f(x) \geq e^{-L x} f(0)$ holds for all $x \geq 0$.

For (ii) and (iii) define $x_{n}$ as the least positive $x$, such that $\left|y_{1}(x)-y_{2}(x)\right|=e^{-n}$, and let $x_{n}^{+}$ be the largest real number $x$, such that $x_{n} \leq x \leq x_{n+1}$, and $\left|y_{1}(x)-y_{2}(x)\right|=e^{-n}$. We will give a lower bound for $x_{n+1}-x_{n}$, telescope these bounds to get a lower bound for $x_{n}$, and solve for $n$ to get a lower bound for $y_{1}-y_{2}$.

Suppose first that $F$ satisfies (1.3) for all real numbers $y \leq z \leq y+1$. Then in $\left[x_{n}^{+}, x_{n+1}\right]$ we have

$$
\begin{aligned}
\left|y_{1}^{\prime}(x)-y_{2}^{\prime}(x)\right|=\mid F\left(x, y_{1}(x)\right)-F\left(x, y_{2}(x) \mid\right. & \leq \sup _{x, y_{1}, y_{2}}|F(x, y)| \\
& \leq \max _{e^{-n-1} \leq \delta \leq e^{-n}} \delta(1+\sqrt{|\ln \delta|}) \leq e^{-n}(1+\sqrt{n}),
\end{aligned}
$$

thus, by the mean value theorem,

$$
\frac{e^{-n}-e^{-n-1}}{x_{n+1}-x_{n}} \leq e^{-n}(1+\sqrt{n}),
$$

that is, $x_{n+1}-x_{n} \geq \frac{1-e^{-1}}{1+\sqrt{n}}$. As $x_{0}=0$ we obtain

$$
\begin{aligned}
x_{n} & =\sum_{v=0}^{n-1}\left(x_{v+1}-x_{v}\right) \geq \sum_{v=0}^{n} \frac{1-e^{-1}}{1+\sqrt{v}} \\
& \geq\left(1-e^{-1}\right) \int_{0}^{n} \frac{d t}{1+\sqrt{t}}=\left(1-e^{-1}\right)(2 \sqrt{n}-2 \ln (\sqrt{n}+1)) .
\end{aligned}
$$

By the definition of $x_{n}$ we have $\left|y_{1}(x)-y_{2}(x)\right|>e^{-n}$ for $0 \leq x<x_{n}$, and we obtain $\mid y_{1}(x)-$ $y_{2}(x) \mid>e^{-n}$ for $n \geq 3$ and

$$
x<2\left(1-e^{-1}\right)(\sqrt{n}-\ln n)
$$

The right hand side is larger than $1.264 \sqrt{n}-1.264 \ln n$, and for $n>1200$ we conclude that $\left|y_{1}(x)-y_{2}(x)\right|>e^{-n}$ for $x \leq \sqrt{n+1}$. Choosing $n=\left\lfloor x^{2}\right\rfloor$ we obtain $\left|y_{1}(x)-y_{2}(x)\right|>e^{-n} \geq$ $e^{-x^{2}}$, provided that $x>35$.

Now suppose that $F$ satisfies (1.4) for all real numbers $y \leq z \leq y+1$. Then in $\left[x_{n}^{+}, x_{n+1}\right]$ we have

$$
\left|y_{1}^{\prime}(x)-y_{2}^{\prime}(x)\right|=\left|F\left(x, y_{1}(x)\right)-F\left(x, y_{2}(x)\right)\right| \leq e^{-n}(n+1) .
$$

Using the mean value theorem we obtain

$$
\frac{e^{-n}-e^{-n-1}}{x_{n+1}-x_{n}} \leq e^{-n}(n+1)
$$

that is, $x_{n+1}-x_{n} \geq \frac{1-e^{-1}}{n+1}$. As $x_{0}=0$ we obtain

$$
x_{n}=\sum_{v=0}^{n-1}\left(x_{v+1}-x_{v}\right) \geq \sum_{v=0}^{n-1} \frac{1-e^{-1}}{v+1} \geq\left(1-e^{-1}\right) \int_{1}^{n+1} \frac{d t}{t} \geq\left(1-e^{-1}\right) \ln n
$$

If $n \geq 4$ we obtain $x_{n} \geq \frac{1}{2} \ln (n+1)$. Putting $n=\left\lfloor e^{2 x}\right\rfloor$ we obtain $\left|y_{1}(x)-y_{2}(x)\right| \geq e^{-n} \geq e^{-e^{2 x}}$ provided that $n \geq 4$, which is satisfied for $x>1$.

Since $x_{4} \geq\left(1-e^{-1}\right)\left(1+\frac{1}{2}+\frac{1}{3}\right) \approx 1.159$, we have $\left|y_{1}(x)-y_{2}(x)\right| \geq e^{-4}$ for $x \in[0,1]$, and therefore $\left|y_{1}(x)-y_{2}(x)\right| \geq e^{-4} \cdot e^{-e^{2 x}}$ for all $x \geq 0$.

The constants 35 and $e^{-4}$ have no particular meaning, we just have to capture lower order terms. We can either do so by prescribing a lower bound for $x$, as we did in (ii), or by introducing a factor as we did in (iii).

In the same way we could give upper bounds corresponding to (iii) of Theorem 1.1, however, it turns out that a simple ad hoc argument is much easier.

## Proposition 1.4.

(i) Let $F: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be a continuous function satisfying $|F(x, y)| \leq|y| \sqrt{1+\ln |y|}$ for all $x$ and $y$ such that $|y| \geq 1$. Then every solution of the initial value problem $y^{\prime}=F(x, y), y(0)=0$ satisfies $|y(x)| \leq e^{\frac{x^{2}}{4}+x}$ for all $x \geq 0$.
(ii) Let $F: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be a continuous function satisfying $|F(x, y)| \leq|y| \ln |y|$ for all $x$ and $y$ such that $|y| \geq e$. Then every solution of the initial value problem $y^{\prime}=F(x, y), y(0)=0$ satisfies $|y(x)| \leq e^{e^{x}}$ for all $x \geq 0$.
Proof. Let $F$ and $y$ be as in (i). The function $\tilde{y}(x)=e^{\frac{x^{2}}{4}+x}$ satisfies the equation $y^{\prime}=y \sqrt{1+\ln y}$, $y(0)=1$. We claim that for all $x \geq 0$ we have $|y(x)|<\tilde{y}(x)$. Define $x_{0}=\sup \{x>0:|y(x)|<$ $\tilde{y}(x)\}$. Clearly $x_{0}>0$. For $x \in\left[0, x_{0}\right)$ we have $|y(x)| \leq 1$ or

$$
\begin{aligned}
\tilde{y}^{\prime}(x)-y^{\prime}(x) & =\tilde{y}(x) \sqrt{1+\ln \tilde{y}(x)}-F(x, y(x)) \\
& \geq \tilde{y}(x) \sqrt{1+\ln \tilde{y}(x)}-|y(x)| \sqrt{1+\ln |y(x)|}>0 .
\end{aligned}
$$

If $x_{0} \neq \infty$, it follows that $\tilde{y}\left(x_{0}\right)>y\left(x_{0}\right)$. In the same way we obtain $\tilde{y}\left(x_{0}\right)>-y\left(x_{0}\right)$, and conclude that $x_{0}=\infty$.

Now let $F$ and $y$ be as in (ii). The function $\tilde{y}(x)=e^{e^{x}}$ satisfies the equation $y^{\prime}=y \ln y$, $y(0)=e$, and we obtain

$$
\tilde{y}^{\prime}(x)-y^{\prime}(x)=\tilde{y}(x) \ln \tilde{y}(x)-F(x, y(x)) \geq \tilde{y}(x) \ln \tilde{y}(x)-y(x) \ln y(x)
$$

for all $x$ such that $e \leq y(x)<\tilde{y}(x)$, and our claim follows as in the first case.
In general whenever one can give a lower bound for the growth of the partial sums $\sum_{n \leq N} \frac{1}{\varphi(n)}$, one obtains upper bounds for the growth of solutions and for the convergence of different solutions with different starting values.

## References

[1] J. Á. Cid, R. L. Pouso, Integration by parts and by substitution unified with applications to Green's theorem and uniqueness for ODEs, Amer. Math. Monthly 123(2016), 40-52. https : //doi.org/abs/10.4169/amer.math.monthly.123.1.40; MR3453534; Zbl 1342.26033
[2] Z. Feng, F. Li, Y. Lv, S. Zhang, A note on Cauchy-Lipschitz-Picard theorem, J. Inequal. Appl. 2016, Paper No. 271, 6 pp. https://doi.org/10.1186/s13660-016-1214-x; MR3568263; Zbl 1354.34028
[3] N. Hayek, J. Trujillo, M. Rivero, B. Bonilla, J. C. Moreno, An extension of PicardLindelöff theorem to fractional differential equations, Appl. Anal. 70 (1999), 347-361. https://doi.org/10.1080/00036819808840696; MR1688864; Zbl 1030.34003
[4] C. Mortici, A contractive method for the proof of Picard's theorem, Bul. Stiint. Univ. Baia Mare Ser. B 14(1998), 179-184. MR1686780; Zbl 1006.34005
[5] W. Rudin, Nonuniqueness and growth in first-order differential equations, Amer. Math. Monthly 89(1982), 241-244. https://doi.org/10.2307/2320223; MR0650671; Zbl 0494.34025
[6] S. Siegmund, C. Nowak, J. Diblík, A generalized Picard-Lindelöf theorem, Electron. J. Qual. Theory Differ. Equ. 2016, No. 28. https://doi.org/10.14232/ejqtde.2016.1.28; MR3506819; Zbl 1363.34034

# Infinitely many weak solutions for a fourth-order equation on the whole space 

Mohammad Reza Heidari Tavani ${ }^{\boxtimes 1}$ and Mehdi Khodabakhshi ${ }^{2}$

${ }^{1}$ Department of Mathematics, Ramhormoz branch, Islamic Azad University, Ramhormoz, Iran
${ }^{2}$ Department of Mathematics and Computer Sciences, Amir Kabir University of Technology, Tehran, Iran

Received 30 June 2020, appeared 20 May 2021
Communicated by Gabriele Bonanno


#### Abstract

The existence of infinitely many weak solutions for a fourth-order equation on the whole space with a perturbed nonlinear term is investigated. Our approach is based on variational methods and critical point theory.


Keywords: weak solution, fourth-order equation, critical point theory, variational methods.

2020 Mathematics Subject Classification: 34B40, 34B15, 47H14.

## 1 Introduction

In this paper we consider the following problem

$$
\begin{equation*}
u^{i v}(x)-\left(q(x) u^{\prime}(x)\right)^{\prime}+s(x) u(x)=\lambda f(x, u(x)), \quad \text { a.e. } x \in \mathbb{R}, \tag{1.1}
\end{equation*}
$$

where $\lambda$ is a positive parameter and $q, s \in L^{\infty}(\mathbb{R})$ with $q_{0}=\operatorname{ess}_{\inf }^{\mathbb{R}} q>0$ and $s_{0}=$ ess $\inf _{\mathbb{R}} s>0$. Here the function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is an $L^{1}$-Carathéodory function.

As we know, differential equations have many applications in engineering and mechanical science. Many important engineering topics eventually lead to a differential equation. One of the most important and widely used types of such equations is the fourth-order differential equation. These equations play an essential role in describing the large number of elastic deflections in beams. Due to the importance of these equations in applied sciences, many authors have studied different types of these equations and obtained important results. Research on the existence and multiplicity of solutions for different types of these equations can be seen in the work of many authors. For example, to study fourth-order two-point boundary value problems we refer the reader to references [3-5,8,10-12].

For instance in [3], the authors researched the following problem:

$$
\left\{\begin{array}{l}
u^{i v}+A u^{\prime \prime}+B u=\lambda f(t, u),  \tag{1.2}\\
u(0)=u(1)=u^{\prime \prime}(0)=u^{\prime \prime}(1)=0
\end{array} \quad t \in[0,1],\right.
$$

[^2]where $A$ and $B$ are real constants and they achieved multiplicity results using variational methods and critical point theory. It should be noted that in the study of many important problems such as mathematical models of beam deflection, the differential equation is considered at infinite interval. Also, because the operators used to solve equations such as (1.1) on $\mathbb{R}$ are not compact, so the study of such problems is very important. That is why some authors have turned their attention to the whole space. For example in [9], applying the critical point theory the author has studied the existence and multiplicity of solutions for the following problem:
\[

$$
\begin{equation*}
u^{i v}(x)+A u^{\prime \prime}(x)+B u(x)=\lambda \alpha(x) \cdot f(u(x)), \quad \text { a.e. } x \in \mathbb{R}, \tag{1.3}
\end{equation*}
$$

\]

where $A$ is a real negative constant and $B$ is a real positive constant, $\lambda$ is a positive parameter and $\alpha, f: \mathbb{R} \rightarrow \mathbb{R}$ are two functions such that $\alpha \in L^{1}(\mathbb{R}), \alpha(x) \geq 0$, for a.e. $x \in \mathbb{R}, \alpha \not \equiv 0$ and also $f$ is continuous and non-negative.

In this work, using a critical point theorem obtained in [2] which we recall in the next section (Theorem 2.7), we establish the existence of infinitely many weak solutions for the problem (1.1).

## 2 Preliminaries

Let us recall some basic concepts.
Definition 2.1. A function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is said to be an $L^{1}$-Carathéodory function, if
$\left(C_{1}\right)$ the function $x \mapsto f(x, t)$ is measurable for every $t \in \mathbb{R}$,
$\left(C_{2}\right)$ the function $t \mapsto f(x, t)$ is continuous for almost every $x \in \mathbb{R}$,
$\left(C_{3}\right)$ for every $\rho>0$ there exists a function $l_{\rho}(x) \in L^{1}(\mathbb{R})$ such that

$$
\sup _{|t| \leq \rho}|f(x, t)| \leq l_{\rho}(x),
$$

for a.e. $x \in \mathbb{R}$.
Denote $W_{0}^{2,2}(\mathbb{R})$ is the closure of $C_{0}^{\infty}(\mathbb{R})$ in $W^{2,2}(\mathbb{R})$ and according to the properties of the Sobolev spaces, we know that $W_{0}^{2,2}(\mathbb{R})=W^{2,2}(\mathbb{R}),[1$, Corollary 3.19].

We denote by $|\cdot|_{t}$ the usual norm on $L^{t}(\mathbb{R})$, for all $t \in[1,+\infty]$ and it is well known that $W^{2,2}(\mathbb{R})$ is continuously embedded in $L^{\infty}(\mathbb{R}),[6$, Corollary 9.13].

The Sobolev space $W^{2,2}(\mathbb{R})$ is equipped with the following norm

$$
\|u\|_{W^{2,2}(\mathbb{R})}=\left(\int_{\mathbb{R}}\left(\left|u^{\prime \prime}(x)\right|^{2}+\left|u^{\prime}(x)\right|^{2}+|u(x)|^{2}\right) d x\right)^{1 / 2}
$$

for all $u \in W^{2,2}(\mathbb{R})$. Also, we consider $W^{2,2}(\mathbb{R})$ with the norm

$$
\|u\|=\left(\int_{\mathbb{R}}\left(\left|u^{\prime \prime}(x)\right|^{2}+q(x)\left|u^{\prime}(x)\right|^{2}+s(x)|u(x)|^{2}\right) d x\right)^{1 / 2}
$$

for all $u \in W^{2,2}(\mathbb{R})$. According to

$$
\begin{equation*}
\left(\min \left\{1, q_{0}, s_{0}\right\}\right)^{\frac{1}{2}}\|u\|_{W^{2,2}(\mathbb{R})} \leq\|u\| \leq\left(\max \left\{1,|q|_{\infty},|s|_{\infty}\right\}\right)^{\frac{1}{2}}\|u\|_{W^{2,2}(\mathbb{R})}, \tag{2.1}
\end{equation*}
$$

the norm $\|\cdot\|$ is equivalent to the $\|\cdot\|_{W^{2,2}(\mathbb{R})}$ norm. Since the embedding $W^{2,2}(\mathbb{R}) \rightarrow L^{\infty}(\mathbb{R})$ is continuous hence there exists a constant $C_{q, s}$ (depending on the functions $q$ and $s$ ) such that

$$
|u|_{\infty} \leq C_{q, s}\|u\|, \quad \forall u \in W^{2,2}(\mathbb{R}) .
$$

In the following proposition, we provide an approximation for this constant.
Proposition 2.2. We have

$$
\begin{equation*}
|u|_{\infty} \leq C_{q, s}\|u\| \tag{2.2}
\end{equation*}
$$

where $C_{q, s}=\left(\frac{1}{\left.4 q\right|_{\infty} \mid s s_{\infty}}\right)^{\frac{1}{4}}\left(\frac{\max \left\{1,|q|_{\infty},|s|_{\infty}\right\}}{\min \left\{1, q_{0}, s_{0}\right\}}\right)^{\frac{1}{2}}$.
Proof. Let $v \in W^{1,1}(\mathbb{R})$, then from [7, p. 138, formula 4.64], one has

$$
\begin{equation*}
|v(x)| \leq \frac{1}{2} \int_{\mathbb{R}}\left|v^{\prime}(t)\right| d t . \tag{2.3}
\end{equation*}
$$

Now if $u \in W^{2,2}(\mathbb{R})$ then $v(x)=\left(|q|_{\infty}|s|_{\infty}\right)^{\frac{1}{2}}|u(x)|^{2} \in W^{1,1}(\mathbb{R})$ and thus from (2.3) and Hölder's inequality one has,

$$
\left(|q|_{\infty}|s|_{\infty}\right)^{\frac{1}{2}}|u(x)|^{2} \leq \int_{\mathbb{R}}\left(|q|_{\infty}|s|_{\infty}\right)^{\frac{1}{2}}\left|u^{\prime}(t)\right||u(t)| d t \leq\left(\left(|q|_{\infty}\right)^{\frac{1}{2}}\left|u^{\prime}\right|_{2}\right)\left(|s|_{\infty}^{\frac{1}{2}}|u|_{2}\right)
$$

that is,

$$
\begin{equation*}
|u(x)| \leq\left(\frac{1}{|q|_{\infty}|s|_{\infty}}\right)^{\frac{1}{4}}\left(\left(|q|_{\infty}\right)^{\frac{1}{2}}\left|u^{\prime}\right|_{2}\right)^{\frac{1}{2}}\left(|s|_{\infty}^{\frac{1}{2}}|u|_{2}\right)^{\frac{1}{2}} . \tag{2.4}
\end{equation*}
$$

Now according to $x^{a} y^{1-a} \leq a^{a}(1-a)^{1-a}(x+y), x, y \geq 0,0<a<1$ [7, p. 130, formula 4.47], and classical inequality $a^{\frac{1}{p}}+b^{\frac{1}{p}} \leq 2^{\frac{(p-1)}{p}}(a+b)^{\frac{1}{p}}$, from (2.1) and (2.4) one has

$$
\begin{aligned}
|u(x)| & \leq\left(\frac{1}{|q|_{\infty}|s|_{\infty}}\right)^{\frac{1}{4}}\left(\frac{1}{2}\right)^{\frac{1}{2}}\left(\frac{1}{2}\right)^{\frac{1}{2}}\left[\left(\int_{\mathbb{R}}|q|_{\infty}\left|u^{\prime}(t)\right|^{2} d t\right)^{\frac{1}{2}}+\left(\int_{\mathbb{R}}|s|_{\infty}|u(t)|^{2} d t\right)^{\frac{1}{2}}\right] \\
& \leq\left(\frac{1}{|q|_{\infty}|s|_{\infty}}\right)^{\frac{1}{4}}\left(\frac{1}{2}\right)^{\frac{1}{2}}\left(\frac{1}{2}\right)^{\frac{1}{2}}(2)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\left(|q|_{\infty}\left|u^{\prime}(t)\right|^{2}+|s|_{\infty}|u(t)|^{2}\right) d t\right)^{\frac{1}{2}} \\
& \leq\left(\frac{1}{4|q|_{\infty}|s|_{\infty}}\right)^{\frac{1}{4}}\left(\int_{\mathbb{R}}\left(\left|u^{\prime \prime}(t)\right|^{2}+|q|_{\infty}\left|u^{\prime}(t)\right|^{2}+|s|_{\infty}|u(t)|^{2}\right) d t\right)^{\frac{1}{2}} \\
& \leq\left(\frac{1}{4|q|_{\infty}|s|_{\infty}}\right)^{\frac{1}{4}}\left(\frac{\max \left\{1,|q|_{\infty},|s|_{\infty}\right\}}{\min \left\{1, q_{0}, s_{0}\right\}}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\left(\left|u^{\prime \prime}(t)\right|^{2}+\left|u^{\prime}(t)\right|^{2}+|u(t)|^{2}\right) d t\right)^{\frac{1}{2}}
\end{aligned}
$$

which means that $|u|_{\infty} \leq C_{q, s}\|u\|$.
Let $\Phi, \Psi: W^{2,2}(\mathbb{R}) \rightarrow \mathbb{R}$ be defined by

$$
\begin{equation*}
\Phi(u)=\frac{1}{2}\|u\|^{2}=\frac{1}{2} \int_{\mathbb{R}}\left(\left|u^{\prime \prime}(x)\right|^{2}+q(x)\left|u^{\prime}(x)\right|^{2}+s(x)|u(x)|^{2}\right) d x \tag{2.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\Psi(u)=\int_{\mathbb{R}} F(x, u(x)) d x \tag{2.6}
\end{equation*}
$$

for every $u \in W^{2,2}(\mathbb{R})$ where $F(x, \xi)=\int_{0}^{\xi} f(x, t) d t$ for all $(x, \xi) \in \mathbb{R}^{2}$. It is well known that $\Psi$ is a sequentially weakly upper semicontinuous whose differential at the point $u \in W^{2,2}(\mathbb{R})$ is

$$
\Psi^{\prime}(u)(v)=\int_{\mathbb{R}} f(x, u(x)) v(x) d x
$$

It is clear that $\Phi$ is a strongly continuous and coercive functional. Also since the norm $\|\cdot\|$ on Hilbert space $W^{2,2}(\mathbb{R})$ is a weakly sequentially lower semi-continuous functional in $W^{2,2}(\mathbb{R})$ therefore $\Phi$ is a sequentially weakly lower semicontinuous functional on $W^{2,2}(\mathbb{R})$. Moreover, $\Phi$ is continuously Gâteaux differentiable functional whose differential at the point $u \in W^{2,2}(\mathbb{R})$ is

$$
\Phi^{\prime}(u)(v)=\int_{\mathbb{R}}\left(u^{\prime \prime}(x) v^{\prime \prime}(x)+q(x) u^{\prime}(x) v^{\prime}(x)+s(x) u(x) v(x)\right) d x
$$

for every $v \in W^{2,2}(\mathbb{R})$.
Definition 2.3. Let $\Phi$ and $\Psi$ be defined as above. Put $I_{\lambda}=\Phi-\lambda \Psi, \lambda>0$. We say that $u \in W^{2,2}(\mathbb{R})$ is a critical point of $I_{\lambda}$ when $I_{\lambda}^{\prime}(u)=0_{\left\{W^{2,2}(\mathbb{R})^{*}\right\}}$, that is, $I_{\lambda}^{\prime}(u)(v)=0$ for all $v \in W^{2,2}(\mathbb{R})$.

Definition 2.4. A function $u: \mathbb{R} \rightarrow \mathbb{R}$ is a weak solution to the problem (1.1) if $u \in W^{2,2}(\mathbb{R})$ and

$$
\int_{\mathbb{R}}\left(u^{\prime \prime}(x) v^{\prime \prime}(x)+q(x) u^{\prime}(x) v^{\prime}(x)+s(x) u(x) v(x)-\lambda f(x, u(x)) v(x)\right) d x=0
$$

for all $v \in W^{2,2}(\mathbb{R})$.
Remark 2.5. We clearly observe that the weak solutions of the problem (1.1) are exactly the solutions of the equation $I_{\lambda}^{\prime}(u)(v)=\Phi^{\prime}(u)(v)-\lambda \Psi^{\prime}(u)(v)=0$.
Lemma 2.6. Suppose that $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is a non-negative $L^{1}$-Carathéodory function. If $u_{0} \not \equiv 0$ is a weak solution for problem (1.1) then $u_{0}$ is non-negative.

Proof. From Remark 2.5, one has $\Phi^{\prime}\left(u_{0}\right)(v)-\lambda \Psi^{\prime}\left(u_{0}\right)(v)=0$ for all $v \in W^{2,2}(\mathbb{R})$. Let $v(x)=$ $\bar{u}_{0}=\max \left\{-u_{0}(x), 0\right\}$ and we assume that $E=\left\{x \in \mathbb{R}: u_{0}(x)<0\right\}$. Then we have

$$
\int_{E \cup E^{c}}\left(u_{0}^{\prime \prime}(x) \bar{u}_{0}^{\prime \prime}(x)+q(x) u_{0}^{\prime}(x) \bar{u}_{0}^{\prime}(x)+s(x) u_{0}(x) \bar{u}_{0}(x)\right) d x=\int_{\mathbb{R}} \lambda f\left(x, u_{0}(x)\right) \bar{u}_{0}(x) d x
$$

that is

$$
\int_{E}\left(-\left|\bar{u}_{0}^{\prime \prime}(x)\right|^{2}-q(x)\left|\bar{u}_{0}^{\prime}(x)\right|^{2}-s(x)\left|\bar{u}_{0}(x)\right|^{2}\right) d x \geq 0
$$

which means that $\left\|\bar{u}_{0}\right\|=0$ and hence $u_{0} \geq 0$ and the proof is complete .
Our main tool is the following critical point theorem.
Theorem 2.7 ([2, Theorem 2.1]). Let $X$ be a reflexive real Banach space, let $\Phi, \Psi: X \rightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that $\Phi$ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and $\Psi$ is sequentially weakly upper semicontinuous. For every $r>\inf _{X} \Phi$, let us put

$$
\varphi(r):=\inf _{u \in \Phi^{-1}(]-\infty, r[)} \frac{\sup _{\left.v \in \Phi^{-1}(]-\infty, r\right]} \Psi(v)-\Psi(u)}{r-\Phi(u)}
$$

and

$$
\gamma:=\liminf _{r \rightarrow+\infty} \varphi(r), \quad \delta:=\liminf _{r \rightarrow\left(\inf _{X} \Phi\right)^{+}} \varphi(r) .
$$

Then, one has
(a) for every $r>\inf _{X} \Phi$ and every $\left.\lambda \in\right] 0, \frac{1}{\varphi(r)}$, the restriction of the functional $I_{\lambda}=\Phi-\lambda \Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of $I_{\lambda}$ in $X$.
(b) If $\gamma<+\infty$ then, for each $\lambda \in] 0, \frac{1}{\gamma}[$, the following alternative holds: either
$\left(b_{1}\right) I_{\lambda}$ possesses a global minimum,
or
$\left(b_{2}\right)$ there is a sequence $\left\{u_{n}\right\}$ of critical points (local minima) of $I_{\lambda}$ such that

$$
\lim _{n \rightarrow+\infty} \Phi\left(u_{n}\right)=+\infty
$$

(c) If $\delta<+\infty$ then, for each $\lambda \in] 0, \frac{1}{\delta}[$, the following alternative holds: either
$\left(c_{1}\right)$ there is a global minimum of $\Phi$ which is a local minimum of $I_{\lambda}$,
or
$\left(c_{2}\right)$ there is a sequence of pairwise distinct critical points (local minima) of $I_{\lambda}$, with $\lim _{n \rightarrow+\infty} \Phi\left(u_{n}\right)=\inf _{X} \Phi$, which weakly converges to a global minimum of $\Phi$.

## 3 Main results

Let

$$
\begin{gather*}
\tau:=\frac{540}{86111\left(\max \left\{1,|q|_{\infty},|s|_{\infty}\right\}\right) C_{q, s}^{2}}  \tag{3.1}\\
A:=\liminf _{\rho \rightarrow+\infty} \frac{\int_{\mathbb{R}} \sup _{|t| \leq \rho} F(x, t) d x}{\rho^{2}} \tag{3.2}
\end{gather*}
$$

and

$$
\begin{equation*}
B:=\limsup _{\rho \rightarrow+\infty} \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x, \rho) d x}{\rho^{2}} . \tag{3.3}
\end{equation*}
$$

Now we formulate our main result as follows.
Theorem 3.1. Let $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be an $L^{1}$-Carathéodory function, and assume that
(i) $F(x, t) \geq 0$ for every $(x, t) \in \mathbb{R} \times] 0, \frac{3}{8}[\cup] \frac{5}{8}, 1[$,
(ii) $A<\tau B$, where $\tau$, $A$ and $B$ are given by (3.1), (3.2) and (3.3) respectively.

Then for every

$$
\lambda \in] \frac{\left(\max \left\{1,|q|_{\infty},|s|_{\infty}\right\}\right)}{B} \frac{86111}{1080}, \frac{1}{2 A C_{q, s}{ }^{2}}[
$$

the problem (1.1) admits a sequence of many weak solutions which is unbounded in $W^{2,2}(\mathbb{R})$.

Proof. Fix $\lambda$ as in our conclusion. Our aim is to apply Theorem 2.7, part (b) with $X=W^{2,2}(\mathbb{R})$, and $\Phi, \Psi$ are the functionals introduced in section 2 . As shown in the previous section, the functionals $\Phi$ and $\Psi$ satisfy all regularity assumptions requested in Theorem 2.7. Now, we look on the existence of critical points of the functional $I_{\lambda}$ in $W^{2,2}(\mathbb{R})$. To this end, we take $\left\{\rho_{n}\right\}$ be a sequence of positive numbers such that $\lim _{n \rightarrow \infty} \rho_{n}=+\infty$ and

$$
\lim _{n \rightarrow \infty} \frac{\int_{\mathbb{R}} \sup _{|t| \leq \rho_{n}} F(x, t) d x}{\rho_{n}^{2}}=A
$$

Set $r_{n}:=\frac{1}{2}\left(\frac{\rho_{n}}{C_{q, s}}\right)^{2}$, for every $n \in \mathbb{N}$.
For each $u \in W^{2,2}(\mathbb{R})$ and bearing (2.2) in mind, we see that

$$
\begin{aligned}
\Phi^{-1}(]-\infty, r_{n}[) & =\left\{u \in W^{2,2}(\mathbb{R}) ; \Phi(u)<r_{n}\right\} \\
& =\left\{u \in W^{2,2}(\mathbb{R}) ; \frac{1}{2}\|u\|^{2}<\frac{1}{2}\left(\frac{\rho_{n}}{C_{q, s}}\right)^{2}\right\} \\
& =\left\{u \in X ; C_{q, s}\|u\|<\rho_{n}\right\} \subseteq\left\{u \in W^{2,2}(\mathbb{R}) ;|u|_{\infty} \leq \rho_{n}\right\} .
\end{aligned}
$$

Now, since $0 \in \Phi^{-1}(]-\infty, r_{n}[)$ then we have the following inequalities:

$$
\begin{aligned}
\varphi\left(r_{n}\right) & =\inf _{u \in \Phi^{-1}(]-\infty, r_{n}[)} \frac{\sup _{v \in \Phi^{-1}(]-\infty, r_{n}[)} \int_{\mathbb{R}} F(x, v(x)) d x-\int_{\mathbb{R}} F(x, u(x)) d x}{r_{n}-\frac{\|u\|^{2}}{2}} \\
& \leq \frac{\int_{\mathbb{R}} \sup _{|t| \leq \rho_{n}} F(x, t) d x}{r_{n}}=\frac{\int_{\mathbb{R}} \sup _{|t| \leq \rho_{n}} F(x, t) d x}{\frac{1}{2}\left(\frac{\rho_{n}}{C_{q, s}}\right)^{2}} \\
& =2 C_{q, s} 2 \frac{\int_{\mathbb{R}} \sup _{|t| \leq \rho_{n}} F(x, t) d x}{\rho_{n}^{2}}
\end{aligned}
$$

for every $n \in \mathbb{N}$. Hence, it follows that

$$
\gamma \leq \liminf _{n \rightarrow \infty} \Phi\left(r_{n}\right) \leq 2 C_{q, s}^{2} A<+\infty,
$$

because condition (ii) shows $A<+\infty$. Now, we prove that the functional $I_{\lambda}$ is unbounded from below. For our goal, let $\left\{\eta_{n}\right\}$ be a sequence of positive numbers such that $\lim _{n \rightarrow \infty} \eta_{n}=$ $+\infty$ and

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F\left(x, \eta_{n}\right) d x}{\eta_{n}^{2}}=B \tag{3.4}
\end{equation*}
$$

Let $\left\{v_{n}\right\}$ be a sequence in $W^{2,2}(\mathbb{R})$ which is defined by

$$
v_{n}(x):= \begin{cases}-\frac{64 \eta_{n}}{9}\left(x^{2}-\frac{3}{4} x\right), & \text { if } x \in\left[0, \frac{3}{8}\right]  \tag{3.5}\\ \eta_{n}, & \text { if } \left.x \in] \frac{3}{8}, \frac{5}{8}\right] \\ -\frac{64 \eta_{n}}{9}\left(x^{2}-\frac{5}{4} x+\frac{1}{4}\right), & \text { if } \left.x \in] \frac{5}{8}, 1\right] \\ 0, & \text { otherwise }\end{cases}
$$

One can compute that

$$
\left\|v_{n}\right\|_{W^{2,2}(\mathbb{R})}{ }^{2}=\frac{86111}{540} \eta_{n}^{2}
$$

and so from (2.1) we have

$$
\begin{equation*}
\left(\min \left\{1, q_{0}, s_{0}\right\}\right) \frac{86111}{1080} \eta_{n}^{2} \leq \Phi\left(v_{n}\right) \leq\left(\max \left\{1,|q|_{\infty},|s|_{\infty}\right\}\right) \frac{86111}{1080} \eta_{n}^{2} \tag{3.6}
\end{equation*}
$$

Also, by using condition (i), we infer

$$
\int_{\mathbb{R}} F\left(x, v_{n}(x)\right) d x \geq \int_{\frac{3}{8}}^{\frac{5}{8}} F\left(x, \eta_{n}\right) d x
$$

for every $n \in \mathbb{N}$. Therefore, we have

$$
I_{\lambda}\left(v_{n}\right)=\Phi\left(v_{n}\right)-\lambda \Psi\left(v_{n}\right) \leq\left(\max \left\{1,|q|_{\infty},|s|_{\infty}\right\}\right) \frac{86111}{1080} \eta_{n}^{2}-\lambda \int_{\frac{3}{8}}^{\frac{5}{8}} F\left(x, \eta_{n}\right) d x
$$

for every $n \in \mathbb{N}$. If $B<+\infty$, let

$$
\epsilon \in] \frac{\left(\max \left\{1,|q|_{\infty},|s|_{\infty}\right\}\right)}{\lambda B} \frac{86111}{1080}, 1[
$$

By (3.4) there is $N_{\epsilon}$ such that

$$
\int_{\frac{3}{8}}^{\frac{5}{8}} F\left(x, \eta_{n}\right) d x>\epsilon B \eta_{n}{ }^{2}, \quad\left(\forall n>N_{\epsilon}\right)
$$

Consequently, one has

$$
\begin{aligned}
I_{\lambda}\left(v_{n}\right) & \leq\left(\max \left\{1,|q|_{\infty},|s|_{\infty}\right\}\right) \frac{86111}{1080} \eta_{n}^{2}-\lambda \epsilon B \eta_{n}^{2} \\
& =\eta_{n}^{2}\left(\left(\max \left\{1,|q|_{\infty},|s|_{\infty}\right\}\right) \frac{86111}{1080}-\lambda \epsilon B\right)
\end{aligned}
$$

for every $n>N_{\epsilon}$. Thus, it follows that

$$
\lim _{n \rightarrow \infty} I_{\lambda}\left(v_{n}\right)=-\infty
$$

If $B=+\infty$, then consider

$$
M>\frac{\left(\max \left\{1,|q|_{\infty},|s|_{\infty}\right\}\right)}{\lambda} \frac{86111}{1080}
$$

By (3.4) there is $N(M)$ such that

$$
\int_{\frac{3}{8}}^{\frac{5}{8}} F\left(x, \eta_{n}\right) d x>M \eta_{n}{ }^{2}, \quad(\forall n>N(M))
$$

So, we have

$$
\begin{aligned}
I_{\lambda}\left(v_{n}\right) & \leq\left(\max \left\{1,|q|_{\infty},|s|_{\infty}\right\}\right) \frac{86111}{1080} \eta_{n}^{2}-\lambda M \eta_{n}^{2} \\
& =\eta_{n}^{2}\left(\left(\max \left\{1,|q|_{\infty},|s|_{\infty}\right\}\right) \frac{86111}{1080}-\lambda M\right)
\end{aligned}
$$

for every $n>N(M)$. Taking into account the choice of $M$, also in this case, one has

$$
\lim _{n \rightarrow \infty} I_{\lambda}\left(v_{n}\right)=-\infty
$$

Therefore according to Theorem 2.7, the functional $I_{\lambda}$ admits an unbounded sequence $\left\{u_{n}\right\} \subset$ $W^{2,2}(\mathbb{R})$ of critical points. It means that, problem (1.1) admits a sequence of many weak solutions which is unbounded in $W^{2,2}(\mathbb{R})$.

Now we present the following example to illustrate Theorem 3.1.
Example 3.2. Let $F: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be the function defined as

$$
F(x, t):= \begin{cases}\frac{t^{5}(1-\cos (\ln |t|))}{1+x^{2}}, & \text { if }(x, t) \in \mathbb{R} \times \mathbb{R}-\{0\} \\ 0, & \text { if }(x, t) \in \mathbb{R} \times\{0\}\end{cases}
$$

and therefore we have

$$
f(x, t):= \begin{cases}\frac{5 t^{4}(1-\cos (\ln |t|))+t^{4} \sin (\ln |t|)}{1+x^{2}}, & \text { if }(x, t) \in \mathbb{R} \times \mathbb{R}-\{0\} \\ 0, & \text { if }(x, t) \in \mathbb{R} \times\{0\}\end{cases}
$$

We observe that

$$
\begin{equation*}
A:=\liminf _{\rho \rightarrow+\infty} \frac{\int_{\mathbb{R}} \sup _{|t| \leq \rho} F(x, t) d x}{\rho^{2}}=0 \tag{3.7}
\end{equation*}
$$

and

$$
\begin{equation*}
B:=\limsup _{\rho \rightarrow+\infty} \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x, \rho) d x}{\rho^{2}}=+\infty \tag{3.8}
\end{equation*}
$$

So, by Theorem 3.1, for every $\lambda \in(0,+\infty)$ the problem

$$
\left\{\begin{array}{l}
u^{i v}(x)-\left(\left(1+e^{-x^{2}}\right) u^{\prime}(x)\right)^{\prime}+\left(\pi+\tan ^{-1} x\right) u(x)  \tag{3.9}\\
=\lambda \frac{5 u(x)^{4}(1-\cos (\ln |u(x)|))+u(x)^{4} \sin (\ln |u(x)|)}{1+x^{2}},
\end{array} \quad \text { a.e. } x \in \mathbb{R}\right.
$$

has a sequence of weak solutions which is unbounded in $W^{2,2}(\mathbb{R})$.
Note that, as in the previous example, under appropriate conditions, the existence of infinitely many weak solutions for problem (1.1) will be guaranteed for any $\lambda \in \mathbb{R}^{+}$. For this case, the following result is a consequence of Theorem 3.1.

Corollary 3.3. Suppose that $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is an $L^{1}$-Carathéodory function. Also, assume that the assumption ( $i$ ) in Theorem 3.1 holds and $A=\infty$ and $B=0$ where $A$ and $B$ are given by (3.2) and (3.3) respectively. Then, for every $\lambda>0$, the problem (1.1) possesses a sequence of many weak solutions which is unbounded in $W^{2,2}(\mathbb{R})$.

A special case of Theorem 3.1 is given in the following corollary.
Corollary 3.4. Suppose that $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is an $L^{1}$-Carathéodory function. Also, assume that the assumption (i) in Theorem 3.1 holds and
$\left(i_{1}\right)\left(\max \left\{1,|q|_{\infty},|s|_{\infty}\right\}\right) \frac{86111}{1080}<B$,
$\left(i_{2}\right) A<\frac{1}{2 C_{q, s}{ }^{2}}$.
Then, the problem

$$
\begin{equation*}
u^{i v}(x)-\left(q(x) u^{\prime}(x)\right)^{\prime}+s(x) u(x)=f(x, u(x)), \quad \text { a.e. } x \in \mathbb{R}, \tag{3.10}
\end{equation*}
$$

possesses a sequence of many weak solutions which is unbounded in $W^{2,2}(\mathbb{R})$.
Proof. The corollary is an immediate consequence of Theorem 3.1 when $\lambda=1$.

Remark 3.5. In Theorem 3.1, we can consider $f(x, t)=\beta(x) g(t)$ where $\beta, g: \mathbb{R} \rightarrow \mathbb{R}$ are two functions such that $\beta \in L^{1}(\mathbb{R}), \beta \geq 0$, for a.e. $x \in \mathbb{R}, \beta \not \equiv 0$ and also $g$ is continuous and non-negative. We set $G(t)=\int_{0}^{t} g(\xi) d \xi$ for all $t \in \mathbb{R}$. Since $G^{\prime}(t)=g(t) \geq 0$ then $G$ is non-decreasing function. Therefore (3.2) and (3.3) become the following simpler forms:

$$
\begin{equation*}
A:=\liminf _{\rho \rightarrow+\infty} \frac{\int_{\mathbb{R}} \sup _{|t| \leq \rho} F(x, t) d x}{\rho^{2}}=\liminf _{\rho \rightarrow+\infty} \frac{G(\rho)|\beta|_{1}}{\rho^{2}} \tag{3.11}
\end{equation*}
$$

and

$$
\begin{equation*}
B:=\limsup _{\rho \rightarrow+\infty} \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x, \rho) d x}{\rho^{2}}=\limsup _{\rho \rightarrow+\infty} \frac{G(\rho) \int_{\frac{3}{8}}^{\frac{5}{8}} \beta(x) d x}{\rho^{2}} . \tag{3.12}
\end{equation*}
$$

Now if we assume that $A<\tau B$ where $\tau, A$ and $B$ are given by (3.1), (3.11) and (3.12) respectively, then according to Theorem 3.1 and Lemma 2.6 for every

$$
\lambda \in] \frac{\left(\max \left\{1,|q|_{\infty},|s|_{\infty}\right\}\right)}{B} \frac{86111}{1080}, \frac{1}{2 A C_{q, s}{ }^{2}}[
$$

the problem

$$
\begin{equation*}
u^{i v}(x)-\left(q(x) u^{\prime}(x)\right)^{\prime}+s(x) u(x)=\lambda \beta(x) g(u(x)), \quad \text { a.e. } x \in \mathbb{R}, \tag{3.13}
\end{equation*}
$$

admits a sequence of many non-negative weak solutions which is unbounded in $W^{2,2}(\mathbb{R})$.
Using the conclusion (c) instead of (b) in Theorem 3.1, can be obtained a sequence of pairwise distinct weak solutions to the problem (1.1) which converges uniformly to zero. In this case, by replacing $\rho \rightarrow+\infty$ with $\rho \rightarrow 0^{+}, A$ and $B$ will be converted to the following forms:

$$
\begin{equation*}
A^{\prime}:=\liminf _{\rho \rightarrow 0^{+}} \frac{\int_{\mathbb{R}} \sup _{|t| \leq \rho} F(x, t) d x}{\rho^{2}} \tag{3.14}
\end{equation*}
$$

and

$$
\begin{equation*}
B^{\prime}:=\underset{\rho \rightarrow 0^{+}}{\lim \sup } \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x, \rho) d x}{\rho^{2}} . \tag{3.15}
\end{equation*}
$$

Therefore, we can present the other main result of this section as follows.
Corollary 3.6. Let $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be an $L^{1}$-Carathéodory function, and assume that
(i) $F(x, t) \geq 0$ for every $(x, t) \in \mathbb{R} \times] 0, \frac{3}{8}[\cup] \frac{5}{8}, 1[$,
(ii) $A^{\prime}<\tau B^{\prime}$, where $\tau, A^{\prime}$ and $B^{\prime}$ are given by (3.1), (3.14) and (3.15) respectively.

Then for every

$$
\lambda \in] \frac{\left(\max \left\{1,|q|_{\infty},|s|_{\infty}\right\}\right)}{B^{\prime}} \frac{86111}{1080}, \frac{1}{2 A^{\prime} C_{q, s^{2}}{ }^{2}}[
$$

the problem (1.1) admits a sequence of many weak solutions which strongly converges to zero in $W^{2,2}(\mathbb{R})$.

We present the following example to illustrate Corollary 3.6.

Example 3.7. Let $\alpha>\frac{86111 \sqrt{\frac{\pi}{6}}}{120 \int_{\frac{3}{8}}^{\frac{5}{8}} e^{-x^{2}} d x}-1 \approx 2673$ be a real number and $F: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be a function defined by

$$
F(x, t):= \begin{cases}e^{-x^{2} t^{2}\left(1+\alpha \cos ^{2}\left(\frac{1}{t}\right)\right),} & \text { if }(x, t) \in \mathbb{R} \times] 0,+\infty[ \\ 0, & \text { if }(x, t) \in \mathbb{R} \times]-\infty, 0]\end{cases}
$$

From $F(x, t)=\int_{0}^{t} f(x, \xi) d \xi$ we have

$$
f(x, t):= \begin{cases}e^{-x^{2}}\left(2 t+2 \alpha t \cos ^{2}\left(\frac{1}{t}\right)+\alpha \sin \left(\frac{2}{t}\right)\right), & \text { if }(x, t) \in \mathbb{R} \times] 0,+\infty[ \\ 0, & \text { if }(x, t) \in \mathbb{R} \times]-\infty, 0]\end{cases}
$$

It is clear that $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is an $L^{1}$-Carathéodory function.
Let $q(x)=1+\frac{1}{1+x^{2}}$ and $s(x)=2+\tanh x$ and therefore $|q|_{\infty}=2, q_{0}=1,|s|_{\infty}=3, s_{0}=1$ and $\tau=\frac{120 \sqrt{6}}{86111}$.

Put $a_{n}=\frac{1}{\frac{2 n+1}{2} \pi}$ and $b_{n}=\frac{1}{n \pi}$ for every $n \in \mathbb{N}$, one has

$$
\begin{align*}
A^{\prime} & :=\liminf _{\rho \rightarrow 0^{+}} \frac{\sup _{|t| \leq \rho} t^{2}\left(1+\alpha \cos ^{2}\left(\frac{1}{t}\right)\right) \int_{\mathbb{R}} e^{-x^{2}} d x}{\rho^{2}} \\
& \leq \lim _{n \rightarrow \infty} \frac{a_{n}^{2}\left(1+\alpha \cos ^{2}\left(\frac{1}{a_{n}}\right)\right) \int_{\mathbb{R}} e^{-x^{2}} d x}{a_{n}^{2}}=\sqrt{\pi} \tag{3.16}
\end{align*}
$$

and

$$
\begin{align*}
B^{\prime} & :=\underset{\rho \rightarrow 0^{+}}{\limsup } \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x, \rho) d x}{\rho^{2}} \geq \lim _{n \rightarrow \infty} \frac{b_{n}{ }^{2}\left(1+\alpha \cos ^{2}\left(\frac{1}{b_{n}}\right)\right) \int_{\frac{3}{8}}^{\frac{5}{8}} e^{-x^{2}} d x}{b_{n}{ }^{2}} \\
& =(1+a) \int_{\frac{3}{8}}^{\frac{5}{8}} e^{-x^{2}} d x . \tag{3.17}
\end{align*}
$$

Now, since $\alpha>\frac{86111 \sqrt{\frac{\pi}{6}}}{120 \int_{\frac{3}{8}}^{\frac{5}{8}} e^{-x^{2}} d x}-1$, we have

$$
A^{\prime} \leq \sqrt{\pi}<\frac{120 \sqrt{6}}{86111}(1+a) \int_{\frac{3}{8}}^{\frac{5}{8}} e^{-x^{2}} d x \leq \tau B^{\prime}
$$

and so condition (ii) of the Theorem 3.1 is satisfied. Now, according to the Theorem 3.1 for every

$$
\lambda \in] \frac{86111}{360(1+a)\left(\int_{\frac{3}{8}}^{\frac{5}{8}} e^{-x^{2}} d x\right)}, \frac{1}{3} \sqrt{\frac{6}{\pi}}[
$$

the problem

$$
\left\{\begin{array}{l}
u^{i v}(x)-\left(\left(1+\frac{1}{1+x^{2}}\right) u^{\prime}(x)\right)^{\prime}+(2+\tanh x) u(x)  \tag{3.18}\\
=\lambda e^{-x^{2}}\left(2 u(x)+2 \alpha u(x) \cos ^{2}\left(\frac{1}{u(x)}\right)+\alpha \sin \left(\frac{2}{u(x)}\right)\right), \quad \text { a.e. } x \in \mathbb{R},
\end{array}\right.
$$

admits a sequence of many weak solutions which is converges uniformly to zero.

## 4 Acknowledgments

The second author is sponsored by the National Science Foundation of Iran (INFS). Also the authors are very thankful for the many helpful suggestions and corrections given by the referees who reviewed this paper.

## References

[1] R. A. Adams, Sobolev spaces, Academic Press, 1975. MR0450957; Zbl 0314.46030
[2] G. Bonanno, G. Molica Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, Bound. Value Probl. 2009, Art. ID 670675, 20 pp. https: //doi.org/10.1155/2009/670675; MR2487254; Zbl 1177.34038
[3] G. Bonanno, B. Di Bella, A boundary value problem for fourth-order elastic beam equations, J. Math. Anal. Appl. 343(2008), 1166-1176. https://doi.org/10.1016/j.jmaa. 2008.01.049; MR2417133; Zbl 1145.34005
[4] G. Bonanno, B. Di Bella, Infinitely many solutions for a fourth-order elastic beam equations, NoDEA Nonlinear Differential Equations Appl. 18(2011) 357-368. https://doi. org/ 10.1007/s00030-011-0099-0; MR2811057; Zbl 1222.34023
[5] G. Bonanno, B. Di Bella, D. O’Regan, Non-trivial solutions for nonlinear fourth-order elastic beam equations, Comput. Math. Appl. 62(2011) 1862-1869. https://doi.org/10. 1016/j.camwa.2011.06.029; MR2834811; Zbl 1231.74259
[6] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Springer, New York, 2011. https://doi.org/10.1007/978-0-387-70914-7; MR2759829; Zbl 1220.46002
[7] V. I. Burenkov, Sobolev spaces on domains, Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], Vol. 137, B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1998. https : //doi.org/10.1007/978-3-663-11374-4; MR1622690; Zbl 0893.46024
[8] G. Han, Z. $\mathrm{X}_{\mathrm{u}}$, Multiple solutions of some nonlinear fourth-order beam equations, Nonlinear Anal. 68(2008), No. 12, 3646-3656. https://doi.org/10.1016/j.na.2007.04.007; MR2416072; MR2416072; Zbl 1145.34008
[9] M. R. Heidari Tavani, Existence results for fourth-order elastic beam equations on the real line, Dynam. Systems Appl. 27(2018), 149-163. https://doi.org/10.12732/dsa. v27i1. 8
[10] X. L. Liv, W. T. Li, Existence and multiplicity of solutions for fourth-order boundary values problems with parameters, J. Math. Anal. Appl. 327(2007) 362-375. https://doi. org/10.1016/j.jmaa.2006.04.021; MR2277419; Zbl 1109.34015
[11] X. WANG, Infinitely many solutions for a fourth-order differential equation on a nonlinear elastic foundation, Bound. Value Probl. 2013, 2013:258, 10 pp. https: //doi. org/10.1186/ 1687-2770-2013-258; MR3341376; Zbl 1301.34024
[12] F. WANG, Y. An, Existence and multiplicity of solutions for a fourth-order elliptic equation, Bound. Value Probl. 2012, 2012:6, 9 pp. https ://doi.org/10.1186/1687-2770-20126; MR2891968; Zbl 1278.35066

# Bifurcation curves of positive solutions for the Minkowski-curvature problem with cubic nonlinearity 

Shao-Yuan Huang ${ }^{\boxtimes 1}$ and Min-Shu Hwang ${ }^{2}$<br>${ }^{1}$ Department of Mathematics and Information Education National Taipei University of Education, Taipei 106, Taiwan<br>${ }^{2}$ Department of Mathematics, National Tsing Hua University, Hsinchu 300, Taiwan

Received 25 February 2021, appeared 22 May 2021
Communicated by Paul Eloe


#### Abstract

In this paper, we study the shape of bifurcation curve $S_{L}$ of positive solutions for the Minkowski-curvature problem $$
\left\{\begin{array}{l} -\left(\frac{u^{\prime}(x)}{\sqrt{1-\left(u^{\prime}(x)\right)^{2}}}\right)^{\prime}=\lambda\left(-\varepsilon u^{3}+u^{2}+u+1\right), \quad-L<x<L \\ u(-L)=u(L)=0 \end{array}\right.
$$ where $\lambda, \varepsilon>0$ are bifurcation parameters and $L>0$ is an evolution parameter. We prove that there exists $\varepsilon_{0}>0$ such that the bifurcation curve $S_{L}$ is monotone increasing for all $L>0$ if $\varepsilon \geq \varepsilon_{0}$, and the bifurcation curve $S_{L}$ is from monotone increasing to S-shaped for varying $L>0$ if $0<\varepsilon<\varepsilon_{0}$.


Keywords: bifurcation curve, positive solution, Minkowski-curvature problem.
2020 Mathematics Subject Classification: 34B15, 34B18, 34C23, 74G35.

## 1 Introduction and main result

In this paper, we study the shapes of bifurcation curves of positive solutions $u \in C^{2}(-L, L) \cap$ $C[-L, L]$ for the one-dimensional Minkowski-curvature problem

$$
\left\{\begin{array}{l}
-\left(\frac{u^{\prime}(x)}{\sqrt{1-\left(u^{\prime}(x)\right)^{2}}}\right)^{\prime}=\lambda f(u), \quad-L<x<L  \tag{1.1}\\
u(-L)=u(L)=0
\end{array}\right.
$$

where $\lambda>0$ is a bifurcation parameter, $L>0$ is an evolution parameter and the nonlinearity

$$
\begin{equation*}
f(u) \equiv-\varepsilon u^{3}+u^{2}+u+1, \quad \varepsilon>0 . \tag{1.2}
\end{equation*}
$$

[^3]It is well-known that studying the multiplicity of positive solutions of problem (1.1) is equivalent to studying the shape of bifurcation curve $S_{L}$ of (1.1) where

$$
\begin{equation*}
S_{L} \equiv\left\{\left(\lambda,\left\|u_{\lambda}\right\|_{\infty}\right): \lambda>0 \text { and } u_{\lambda} \text { is a positive solution of (1.1) }\right\} \quad \text { for } L>0 . \tag{1.3}
\end{equation*}
$$

Thus this investigation is essential.
Before going into further discussions on problems (1.1), we give some terminologies in this paper for the shape of bifurcation curve $S_{L}$ on the $\left(\lambda,\|u\|_{\infty}\right)$-plane.

Definition 1.1. Let $S_{L}$ be the bifurcation curve of (1.1) on the $\left(\lambda,\|u\|_{\infty}\right)$-plane.
(i) S-like shaped: The curve $S_{L}$ is said to be $S$-like shaped if $S_{L}$ has at least two turning points at some points $\left(\lambda_{1},\left\|u_{\lambda_{1}}\right\|_{\infty}\right)$ and $\left(\lambda_{2},\left\|u_{\lambda_{2}}\right\|_{\infty}\right)$ where $\lambda_{1}<\lambda_{2}$ are two positive numbers such that:
(a) at $\left(\lambda_{1},\left\|u_{\lambda_{1}}\right\|_{\infty}\right)$ the bifurcation curve $S_{L}$ turns to the right,
(b) $\left\|u_{\lambda_{2}}\right\|_{\infty}<\left\|u_{\lambda_{1}}\right\|_{\infty}$,
(c) at $\left(\lambda_{2},\left\|u_{\lambda_{2}}\right\|_{\infty}\right)$ the bifurcation curve $S_{L}$ turns to the left.
(ii) S-shaped: The curve $S_{L}$ is said to be $S$-shaped if $S_{L}$ is $S$-like shaped, has exactly two turning points, and has at most three intersection points with any vertical line on the ( $\lambda,\|u\|_{\infty}$ )-plane.
(iii) Monotone increasing: The curve $S_{L}$ is said to be monotone increasing if $\lambda_{1}<\lambda_{2}$ for any two points $\left(\lambda_{i},\left\|u_{\lambda_{i}}\right\|_{\infty}\right), i=1,2$, lying in $S_{L}$ with $\left\|u_{\lambda_{1}}\right\|_{\infty} \leq\left\|u_{\lambda_{2}}\right\|_{\infty}$.

Crandall and Rabinowitz [2, p. 177] first considered shape of bifurcation curve of positive solutions for the $n$-dimensional semilinear problem

$$
\begin{cases}-\Delta u(x)=\lambda\left(-\varepsilon u^{3}+u^{2}+u+1\right) & \text { in } \Omega,  \tag{1.4}\\ u(x)=0 & \text { on } \partial \Omega\end{cases}
$$

where $\Omega$ is a general bounded domain in $\mathbb{R}^{n}(n \geq 1)$ with smooth boundary $\partial \Omega$. They applied the implicit function theorem and perturbation arguments to prove that the bifurcation curve of positive solutions of (1.4) is S-like shaped on the $\left(\lambda,\left\|u_{\lambda}\right\|_{\infty}\right)$-plane when $\varepsilon>0$ is sufficiently small. Shi [17, Theorem 4.1] proved that the bifurcation curve of positive solutions of (1.4) is S-shaped when $\varepsilon>0$ is small and $\Omega$ is a ball in $\mathbb{R}^{n}$ with $1 \leq n \leq 6$. Hung and Wang [6] consider the one-dimensional case

$$
\left\{\begin{array}{l}
-u^{\prime \prime}(x)=\lambda\left(-\varepsilon u^{3}+u^{2}+u+1\right), \quad-1<x<1  \tag{1.5}\\
u(-1)=u(1)=0
\end{array}\right.
$$

Then they provided the complete variational process of shape of bifurcation curve $\bar{S}$ of (1.5) with varying $\varepsilon>0$ where

$$
\begin{equation*}
\bar{S} \equiv\left\{\left(\lambda,\left\|u_{\lambda}\right\|_{\infty}\right): \lambda>0 \text { and } u_{\lambda} \text { is a positive solution of }(1.5)\right\} \tag{1.6}
\end{equation*}
$$

see Theorem 1.2.


Figure 1.1: Graphs of bifurcation curves $\bar{S}$ of (1.4). (i) $\varepsilon \geq \varepsilon_{0}$ and (ii) $0<\varepsilon<\varepsilon_{0}$.

Theorem 1.2 ([6, Theorem 3.1]). Consider (1.5). Then the bifurcation curve $\bar{S}$ is continuous on the $\left(\lambda,\left\|u_{\lambda}\right\|_{\infty}\right)$-plane, starts from $(0,0)$ and goes to infinity. Furthermore, there exists a critical bifurcation value $\varepsilon_{0} \in(0,1 / \sqrt{27})$ such that the bifurcation curve $\bar{S}$ is monotone increasing if $\varepsilon \geq \varepsilon_{0}$, and $\bar{S}$ is $S$-shaped if $0<\varepsilon<\varepsilon_{0}$, see Figure 1.1.

To the best of my knowledge, there are no manuscripts to describe the variational process for $S_{L}$ of (1.5) with varying $\varepsilon, L>0$. Hence we start to concern this issue. In addition, references $[7,8,16$ ] provided some sufficient conditions to determine the shape of bifurcation curve or multiplicity of positive solutions of problem (1.1) with general $f(u) \in C[0, \infty)$. However, these results can not be applied in our problem (1.1) because the cubic nonlinearity $f(u)$ defined by (1.2) is not always positive in $[0, \infty)$. So studying the problem (1.1) is worth and interesting.

By elementary analysis, we find that $f(u)$ has unique zero $\beta_{\varepsilon}$ in $[0, \infty)$. Then the main result is as follows:

Theorem 1.3 (See Figure 1.2). Consider (1.1). Let $\varepsilon_{0}$ be defined in Theorem 1.2. Then the following statements (i)-(iii) hold:
(i) For $L>0$, the bifurcation curve $S_{L}$ is continuous on the $\left(\lambda,\left\|u_{\lambda}\right\|_{\infty}\right)$-plane, starts from $(0,0)$ and goes to infinity along the horizontal line $\|u\|_{\infty}=\rho_{L, \varepsilon}$ where $\rho_{L, \varepsilon} \equiv \min \left\{L, \beta_{\varepsilon}\right\}$.
(ii) If $\varepsilon \geq \varepsilon_{0}$, then the bifurcation curve $S_{L}$ is monotone increasing for all $L>0$.
(iii) If $0<\varepsilon<\varepsilon_{0}$, then there exist two positive numbers $L_{\varepsilon}<\tilde{L}_{\varepsilon}$ such that
(a) the bifurcation curve $S_{L}$ is monotone increasing for $0<L \leq L_{\varepsilon}$.
(b) the bifurcation curve $S_{L}$ is $S$-like shaped for $L_{\varepsilon}<L \leq \tilde{L}_{\varepsilon}$.
(c) the bifurcation curve $S_{L}$ is $S$-shaped for $L>\tilde{L}_{\varepsilon}$.

Furthermore, $L_{\varepsilon}$ is a continuous function of $\varepsilon \in\left(0, \varepsilon_{0}\right), \lim _{\varepsilon \rightarrow 0^{+}} L_{\varepsilon} \in(0, \infty)$ and $\lim _{\varepsilon \rightarrow \varepsilon_{0}^{-}} L_{\varepsilon}=$ $\infty$.

Remark 1.4. By numerical simulations to bifurcation curves $S_{L}$ of (1.1), we conjecture that the bifurcation curve $S_{L}$ is also S-shaped on the $\left(\lambda,\left\|u_{\lambda}\right\|_{\infty}\right)$-plane for $L_{\varepsilon}<L \leq \tilde{L}_{\varepsilon}$ and $0<\varepsilon<\varepsilon_{0}$. Further investigations are needed. In addition, by Theorems 1.2 and 1.3, we make a list which shows the different properties for Minkowski-curvature problem (1.1) and semilinear problem (1.4), see Table 1.


Figure 1.2: Graphs of bifurcation curve $S_{L}$ of (1.1) for $\varepsilon>0$.

| Bifurcation curve | $S_{L}$ of (1.1) | $\bar{S}$ of (1.4) |
| :---: | :---: | :---: |
| 1. Shapes $\left(0<\varepsilon<\varepsilon_{0}\right)$ | from monotone increasing to S-shaped with varying $\varepsilon$ | S-shaped |
| 2. Shapes $\left(\varepsilon \geq \varepsilon_{0}\right)$ | monotone increasing | monotone increasing |
| Numbers of <br> 3. turning points | (1). from 0 to 2 varying $L>0$ if $0<\varepsilon<\varepsilon_{0}$ <br> (2). 0 <br> if $\varepsilon \geq \varepsilon_{0}$ | (1). 2 if $0<\varepsilon<\varepsilon_{0}$ <br> (2). $0 \quad$ if $\varepsilon \geq \varepsilon_{0}$ |
| 4. Continuity | continuous | continuous |
| 5. Evolution parameter(s) | $\varepsilon$ and $L$ | $\varepsilon$ |
| 6. Starting point | $(0,0)$ | $(0,0)$ |
| 7. "End point" | $\left(\infty, \rho_{L, \varepsilon}\right)$ | $(\infty, \infty)$ |

Table 1.1: Comparison of properties of $S_{L}$ and $\bar{S}$.

The paper is organized as follows: Section 2 contains the lemmas used for proving the main result. Section 3 contains the proof of main result (Theorem 1.3). Section 4 contains the proof of assertion (2.31).

## 2 Lemmas

To prove Theorem 1.3, we first introduce the time-map method used in Corsato [4, p. 127]. We define the time-map formula for (1.1) by

$$
\begin{equation*}
T_{\lambda}(\alpha) \equiv \int_{0}^{\alpha} \frac{\lambda[F(\alpha)-F(u)]+1}{\sqrt{\{\lambda[F(\alpha)-F(u)]+1\}^{2}-1}} d u \quad \text { for } 0<\alpha<\beta_{\varepsilon} \text { and } \lambda>0 \tag{2.1}
\end{equation*}
$$

where $F(u) \equiv \int_{0}^{u} f(t) d t$. Observe that positive solutions $u_{\lambda} \in C^{2}(-L, L) \cap C[-L, L]$ for (1.1) correspond to

$$
\left\|u_{\lambda}\right\|_{\infty}=\alpha \quad \text { and } \quad T_{\lambda}(\alpha)=L
$$

So by definition of $S_{L}$ in (1.3), we have that

$$
\begin{equation*}
S_{L}=\left\{(\lambda, \alpha): T_{\lambda}(\alpha)=L \text { for some } 0<\alpha<\beta_{\varepsilon} \text { and } \lambda>0\right\} \tag{2.2}
\end{equation*}
$$

Thus, it is important to understand fundamental properties of the time-map $T_{\lambda}(\alpha)$ on $\left(0, \beta_{\varepsilon}\right)$ in order to study the shape of the bifurcation curve $S_{L}$ of (1.1) for any fixed $L>0$. Note that it can be proved that $T_{\lambda}(\alpha)$ is a triple differentiable function of $\varepsilon \in\left(0, \beta_{\varepsilon}\right)$ for $\varepsilon, \lambda>0$, and $T_{\lambda}(\alpha), T_{\lambda}^{\prime}(\alpha)$ are differentiable function of $\lambda>0$ for $0<\alpha<\beta_{\varepsilon}$ and $a>0$. The proofs are easy but tedious and hence we omit them. Similarly, we define the time-map formula for (1.5) by

$$
\begin{equation*}
\bar{T}(\alpha) \equiv \frac{1}{\sqrt{2}} \int_{0}^{\alpha} \frac{1}{\sqrt{F(\alpha)-F(u)}} d u \quad \text { for } \alpha>0 \tag{2.3}
\end{equation*}
$$

see $[12, \mathrm{p} .779]$. Then we have that $\left\|u_{\lambda}\right\|_{\infty}=\alpha$ and $\bar{T}(\alpha)=\sqrt{\lambda}$. So by the definition of $\bar{S}$ in (1.6), we see that

$$
\begin{equation*}
\bar{S}=\{(\lambda, \alpha): \sqrt{\lambda}=\bar{T}(\alpha) \text { for some } \alpha>0\} \tag{2.4}
\end{equation*}
$$

For the sake of convenience, we let

$$
\begin{aligned}
& A=A(\alpha, u) \equiv \alpha f(\alpha)-u f(u), B=B(\alpha, u) \equiv F(\alpha)-F(u) \\
& C=C(\alpha, u) \equiv \alpha^{2} f^{\prime}(\alpha)-u^{2} f^{\prime}(u) \quad \text { and } \quad D=D(\alpha, u) \equiv \alpha^{3} f^{\prime \prime}(\alpha)-u^{3} f^{\prime \prime}(u)
\end{aligned}
$$

Obviously, we have

$$
\begin{equation*}
B(\alpha, u)=\int_{u}^{\alpha} f(t) d t>0 \quad \text { for } 0<u<\alpha<\beta_{\varepsilon} \tag{2.5}
\end{equation*}
$$

because $f(u)>0$ for $0<u<\beta_{\varepsilon}$.
Lemma 2.1. Consider (1.1) with $\varepsilon>0$. Then the following statements (i)-(iii) hold:
(i) $\lim _{\alpha \rightarrow 0^{+}} T_{\lambda}(\alpha)=0$ and $\lim _{\alpha \rightarrow \beta_{\varepsilon}^{-}} T_{\lambda}(\alpha)=\infty$ for $\lambda>0$.
(ii) $\lim _{\lambda \rightarrow 0^{+}} \sqrt{\lambda} T_{\lambda}^{(i)}(\alpha)=\bar{T}^{(i)}(\alpha)$ and $\lim _{\lambda \rightarrow \infty} T_{\lambda}^{\prime}(\alpha)=1$ for $0<\alpha<\beta_{\varepsilon}$ and $i=1,2,3$.
(iii) $\partial T_{\lambda}(\alpha) / \partial \lambda<0$ for $0<\alpha<\beta_{\varepsilon}$ and $\lambda>0$.

Proof. Since

$$
\lim _{u \rightarrow 0^{+}} \frac{F(u)}{u^{2}}=\infty
$$

and by [7, Lemma 3.1], we obtain that $\lim _{\alpha \rightarrow 0^{+}} T_{\lambda}(\alpha)=0$. Since $f\left(\beta_{\varepsilon}\right)=0$, there exist $b, c \in \mathbb{R}$ such that $f(u)=\left(\beta_{\varepsilon}-u\right)\left(\varepsilon u^{2}+b u+c\right)$. Since $f(u)>0$ on $\left(0, \beta_{\varepsilon}\right)$, there exists $M>0$ such that $0<\varepsilon u^{2}+b u+c<M$ for $0<u<\beta_{\varepsilon}$. For $0<t<1$, by the mean-value theorem, there exists $\eta_{t} \in\left(\beta_{\varepsilon} t, \beta_{\varepsilon}\right)$ such that

$$
\begin{align*}
B\left(\beta_{\varepsilon}, \beta_{\varepsilon} t\right) & =\int_{\beta_{\varepsilon} t}^{\beta_{\varepsilon}} f(t) d t=f\left(\eta_{t}\right) \beta_{\varepsilon}(1-t)=\left(\beta_{\varepsilon}-\eta_{t}\right)\left(\varepsilon \eta_{t}^{2}+b \eta_{t}+c\right) \beta_{\varepsilon}(1-t) \\
& <\left(\beta_{\varepsilon}-\beta_{\varepsilon} t\right) M \beta_{\varepsilon}(1-t)=M \beta_{\varepsilon}^{2}(1-t)^{2} \tag{2.6}
\end{align*}
$$

Then there exists $t^{*} \in(0,1)$ such that $B\left(\beta_{\varepsilon}, \beta_{\varepsilon} t\right)<1$ for $t^{*}<t<1$. So by (2.5) and (2.6), we see that

$$
\begin{aligned}
\lim _{\alpha \rightarrow \beta_{\varepsilon}^{-}} T_{\lambda}(\alpha) & =\lim _{\alpha \rightarrow \beta_{\varepsilon}^{-}} \alpha \int_{0}^{1} \frac{\lambda B(\alpha, \alpha t)+1}{\sqrt{\lambda^{2} B^{2}(\alpha, \alpha t)+2 \lambda B(\alpha, \alpha t)}} d t \\
& \geq \lim _{\alpha \rightarrow \beta_{\varepsilon}^{-}} \alpha \int_{t^{*}}^{1} \frac{1}{\sqrt{\lambda^{2} B^{2}(\alpha, \alpha t)+2 \lambda B(\alpha, \alpha t)}} d t \\
& \geq \beta_{\varepsilon} \int_{t^{*}}^{1} \frac{1}{\sqrt{\left(\lambda^{2}+2 \lambda\right) B\left(\beta_{\varepsilon}, \beta_{\varepsilon} t\right)}} d t \geq \frac{1}{\sqrt{\left(\lambda^{2}+2 \lambda\right) M}} \int_{t^{*}}^{1} \frac{1}{1-t} d t=\infty,
\end{aligned}
$$

which implies that statement (i) holds. In addition, we compute that, for $0<\alpha<\beta_{\varepsilon}$ and $\lambda>0$,

$$
\begin{gather*}
T_{\lambda}^{\prime}(\alpha)=\frac{1}{\alpha} \int_{0}^{\alpha} \frac{\lambda^{3} B^{3}+3 \lambda^{2} B^{2}+\lambda(2 B-A)}{\left(\lambda^{2} B^{2}+2 \lambda B\right)^{3 / 2}} d u,  \tag{2.7}\\
T_{\lambda}^{\prime \prime}(\alpha)=\frac{1}{\alpha^{2}} \int_{0}^{\alpha} \frac{\left(3 A^{2} B-B^{2} C-2 A B^{2}\right) \lambda^{3}+\left(3 A^{2}-4 A B-2 B C\right) \lambda^{2}}{\left(\lambda^{2} B^{2}+2 \lambda B\right)^{5 / 2}} d u,  \tag{2.8}\\
T_{\lambda}^{\prime \prime \prime}(\alpha)= \\
\frac{1}{\alpha^{3}} \int_{0}^{\alpha} \frac{\lambda^{3}}{\left[\lambda^{2} B^{2}+2 \lambda B\right]^{7 / 2}}\left[B^{2}\left(9 A^{2} B-3 B^{2} C-B^{2} D-12 A^{3}+9 A B C\right) \lambda^{2}\right. \\
+B\left(27 A^{2} B-12 B^{2} C-4 B^{2} D-24 A^{3}+27 A B C\right) \lambda+18 A^{2} B-12 B^{2} C  \tag{2.9}\\
\left.-4 B^{2} D-15 A^{3}+18 A B C\right] d u .
\end{gather*}
$$

So we observe that, for $0<\alpha<\beta_{\varepsilon}$,

$$
\begin{gathered}
\lim _{\lambda \rightarrow 0^{+}} \sqrt{\lambda} T_{\lambda}^{\prime}(\alpha)=\frac{1}{\alpha} \int_{0}^{\alpha} \frac{2 B-A}{(2 B)^{3 / 2}} d u=\bar{T}^{\prime}(\alpha), \\
\lim _{\lambda \rightarrow 0^{+}} \sqrt{\lambda} T_{\lambda}^{\prime \prime}(\alpha)=\frac{1}{\alpha^{2}} \int_{0}^{\alpha} \frac{3 A^{2}-4 A B-2 B C}{(2 B)^{5 / 2}} d u=\bar{T}^{\prime \prime}(\alpha), \\
\lim _{\lambda \rightarrow 0^{+}} \sqrt{\lambda} T_{\lambda}^{\prime \prime \prime}(\alpha)=\frac{1}{\alpha^{3}} \int_{0}^{\alpha} \frac{18 A^{2} B-12 B^{2} C-4 B^{2} D-15 A^{3}+18 A B C}{(2 B)^{5 / 2}} d u=\bar{T}^{\prime \prime \prime}(\alpha) .
\end{gathered}
$$

Furthermore, $\lim _{\lambda \rightarrow \infty} T_{\lambda}^{\prime}(\alpha)=1$. So statement (ii) holds. The statement (iii) follows immediately by [7, Lemma 4.2(ii)]. The proof is complete.

Lemma 2.2. Consider (1.1) with $\varepsilon>0$. Then the following statements (i) and (ii) hold:
(i) $T_{\lambda}^{\prime}(\alpha)>0$ for $0<\alpha \leq 1$ and $\lambda>0$.
(ii) $T_{\lambda}(\alpha)$ has at most one critical point, a local minimum, on $\left[\frac{5}{12 \varepsilon}, \beta_{\varepsilon}\right)$.

Proof. We can see that $2 B(\alpha, u)-A(\alpha, u)>0$ for $0<u<\alpha \leq 1$ because $2 B(\alpha, \alpha)-A(\alpha, \alpha)=0$ and

$$
\frac{\partial}{\partial u}[2 B(\alpha, u)-A(\alpha, u)]=-2 \varepsilon u^{3}+\left(u^{2}-1\right)<0 \text { for } 0<u<\alpha<1
$$

So by (2.5) and (2.7), we obtain that $T_{\lambda}^{\prime}(\alpha)>0$ for $0<\alpha \leq 1$ and $\lambda>0$. Then statement (i) holds. By (2.5), (2.7) and (2.8), we observe that, for $0<\alpha<\beta_{\varepsilon}$ and $\lambda>0$,

$$
\begin{align*}
& \alpha T_{\lambda}^{\prime \prime}(\alpha)+2 T_{\lambda}^{\prime}(\alpha) \\
&=\frac{1}{\alpha} \int_{0}^{\alpha} \frac{B^{5} \lambda^{3}+5 B^{4} \lambda^{2}+\lambda B\left(3 A^{2}+16 B^{2}-4 A B-B C\right)+3 A^{2}+8 B^{2}-8 A B-2 B C}{\sqrt{\lambda}\left(\lambda B^{2}+2 B\right)^{5 / 2}} d u \\
& \quad=\frac{1}{\alpha} \int_{0}^{\alpha} \frac{\lambda B\left(3 A^{2}+16 B^{2}-4 A B-B C\right)+3 A^{2}+8 B^{2}-8 A B-2 B C}{\sqrt{\lambda}\left(\lambda B^{2}+2 B\right)^{5 / 2}} d u \\
& \quad=\frac{1}{\alpha} \int_{0}^{\alpha} \frac{\lambda B\left[3(A-B)^{2}+5 B^{2}+B(2 A-2 B-C)\right]+3(A-2 B)^{2}+2 B(2 A-2 B-C)}{\sqrt{\lambda}\left(\lambda B^{2}+2 B\right)^{5 / 2}} d u \\
& \quad>\frac{1}{\alpha} \int_{0}^{\alpha} \frac{\lambda B^{2}(2 A-2 B-C)+2 B(2 A-2 B-C)}{\sqrt{\lambda}\left(\lambda B^{2}+2 B\right)^{5 / 2}} d u \\
& \quad=\frac{1}{\alpha} \int_{0}^{\alpha} \frac{\left(\lambda B^{2}+2 B\right)(2 A-2 B-C)}{\sqrt{\lambda}\left(\lambda B^{2}+2 B\right)^{5 / 2}} d u=\frac{1}{\alpha} \int_{0}^{\alpha} \frac{2 A-2 B-C}{\sqrt{\lambda}\left(\lambda B^{2}+2 B\right)^{3 / 2}} d u \\
& \quad=\frac{1}{6 \alpha} \int_{0}^{\alpha} \frac{\phi(\alpha)-\phi(u)}{\sqrt{\lambda}\left(\lambda B^{2}+2 B\right)^{3 / 2}} d u, \tag{2.10}
\end{align*}
$$

where $\phi(u) \equiv u^{3}(9 \varepsilon u-4)$. Clearly, $\phi^{\prime}(u)=12 u^{2}(3 \varepsilon u-1)$. Since

$$
f\left(\frac{4}{9 \varepsilon}\right)=1+\frac{324 \varepsilon+80}{729 \varepsilon^{2}}>0
$$

we see that

$$
\begin{equation*}
\frac{1}{3 \varepsilon}<\frac{4}{9 \varepsilon}<\beta_{\varepsilon} \tag{2.11}
\end{equation*}
$$

So we observe that

$$
\phi(u)\left\{\begin{array} { l l } 
{ < 0 } & { \text { for } 0 < u < \frac { 4 } { 9 \varepsilon } , }  \tag{2.12}\\
{ = 0 } & { \text { for } u = \frac { 4 } { 9 \varepsilon } , } \\
{ > 0 } & { \text { for } \frac { 4 } { 9 \varepsilon } < u < \beta _ { \varepsilon } , }
\end{array} \quad \text { and } \quad \phi ^ { \prime } ( u ) \left\{\begin{array}{ll}
<0 & \text { for } 0<u<\frac{1}{3 \varepsilon} \\
=0 & \text { for } u=\frac{1}{3 \varepsilon} \\
>0 & \text { for } \frac{1}{3 \varepsilon}<u<\beta_{\varepsilon}
\end{array}\right.\right.
$$

Let $\alpha \in\left[\frac{5}{12 \varepsilon}, \beta_{\varepsilon}\right)$ be given. Then we consider two cases.
Case 1. Assume that $\frac{4}{9 \varepsilon} \leq \alpha<\beta_{\varepsilon}$. Since $\phi(0)=0$, and by (2.12), we see that $\phi(\alpha)-\phi(u)>0$ for $0<u<\alpha$. So by (2.10), we obtain $\alpha T_{\lambda}^{\prime \prime}(\alpha)+2 T_{\lambda}^{\prime}(\alpha)>0$ for $\lambda>0$.

Case 2. Assume that $\frac{5}{12 \varepsilon} \leq \alpha<\frac{4}{9 \varepsilon}$. Since $\phi(0)=0$, and by (2.12), there exists $\tilde{\alpha} \in\left(0, \frac{1}{3 \varepsilon}\right)$ such that

$$
\phi(\alpha)-\phi(u) \begin{cases}<0 & \text { for } 0<u<\tilde{\alpha} \\ =0 & \text { for } u=\tilde{\alpha} \\ >0 & \text { for } \tilde{\alpha}<u<\alpha\end{cases}
$$

So by (2.10), we observe that, for $\lambda>0$,

$$
\begin{aligned}
\alpha T_{\lambda}^{\prime \prime}(\alpha) & +2 T_{\lambda}^{\prime}(\alpha) \\
& >\frac{1}{6 \alpha \sqrt{\lambda}}\left[\int_{0}^{\tilde{\alpha}} \frac{\phi(\alpha)-\phi(u)}{\left[\lambda B^{2}+2 B\right]^{3 / 2}} d u+\int_{\tilde{\alpha}}^{\alpha} \frac{\phi(\alpha)-\phi(u)}{\left[\lambda B^{2}+2 B\right]^{3 / 2}} d u\right] \\
& >\frac{1}{6 \alpha \sqrt{\lambda}\left[\lambda B^{2}(\alpha, \tilde{\alpha})+2 B(\alpha, \tilde{\alpha})\right]^{3 / 2}}\left\{\int_{0}^{\tilde{\alpha}}[\phi(\alpha)-\phi(u)] d u+\int_{\tilde{\alpha}}^{\alpha}[\phi(\alpha)-\phi(u)] d u\right\} \\
& =\frac{1}{6 \alpha \sqrt{\lambda}\left[\lambda B^{2}(\alpha, \tilde{\alpha})+2 B(\alpha, \tilde{\alpha})\right]^{3 / 2}} \int_{0}^{\alpha}[\phi(\alpha)-\phi(u)] d u \\
& =\frac{6 \varepsilon \alpha^{3}}{5 \sqrt{\lambda}\left[\lambda B^{2}(\alpha, \tilde{\alpha})+2 B(\alpha, \tilde{\alpha})\right]^{3 / 2}}\left(\alpha-\frac{5}{12 \varepsilon}\right) \geq 0 .
\end{aligned}
$$

Thus by Cases 1-2, we have

$$
\begin{equation*}
\alpha T_{\lambda}^{\prime \prime}(\alpha)+2 T_{\lambda}^{\prime}(\alpha)>0 \text { for } \frac{5}{12 \varepsilon} \leq \alpha<\beta_{\varepsilon} \text { and } \lambda>0 . \tag{2.13}
\end{equation*}
$$

Fixed $\lambda>0$. If $T_{\lambda}(\alpha)$ has a critical point $\breve{\alpha}$ in $\left[\frac{5}{12 \varepsilon}, \beta_{\varepsilon}\right)$, by (2.13), then $\breve{\alpha} T_{\lambda}^{\prime \prime}(\breve{\alpha})=\breve{\alpha} T_{\lambda}^{\prime \prime}(\breve{\alpha})+$ $2 T_{\lambda}^{\prime}(\breve{\alpha})>0$. It implies that $T_{\lambda}(\alpha)$ has at most one critical point, a local minimum, on $\left[\frac{5}{12 \varepsilon}, \beta_{\varepsilon}\right)$ for $\lambda>0$. Then the statement (ii) holds. The proof is complete.

Lemma 2.3. Consider (1.1) with $\varepsilon>0$. Then

$$
\begin{equation*}
\frac{\partial}{\partial \lambda}\left[\sqrt{\lambda} T_{\lambda}^{\prime}(\alpha)\right]>0 \quad \text { for } 0<\alpha \leq \frac{5}{12 \varepsilon} \text { and } \lambda>0 \text {. } \tag{2.14}
\end{equation*}
$$

Proof. By (2.5) and (2.7), we compute and find that

$$
\begin{equation*}
\frac{\partial}{\partial \lambda}\left[\sqrt{\lambda} T_{\lambda}^{\prime}(\alpha)\right]=\frac{1}{2 \alpha} \int_{0}^{\alpha} \frac{B^{2}\left(B^{3} \lambda^{2}+5 B^{2} \lambda+3 A+6 B\right)}{\left(\lambda B^{2}+2 B\right)^{5 / 2}} d u>\frac{1}{2 \alpha} \int_{0}^{\alpha} \frac{3 B^{2}(A+2 B)}{\left(\lambda B^{2}+2 B\right)^{5 / 2}} d u . \tag{2.15}
\end{equation*}
$$

In addition, we compute that

$$
\frac{\partial}{\partial u}[A(\alpha, u)+2 B(\alpha, u)]=R(u),
$$

where $R(u) \equiv 3 \varepsilon u^{3}-3(1-\varepsilon) u^{2}-6 u-4$. Clearly, $R^{\prime}(u)=9 \varepsilon u^{2}-6(1-\varepsilon) u-6$ is a quadratic polynomial of $u$ with positive leading coefficient. Furthermore,

$$
R^{\prime}(0)=-6<0 \quad \text { and } \quad R^{\prime}\left(\frac{5}{12 \varepsilon}\right) \equiv-\frac{56 \varepsilon+15}{16 \varepsilon}<0 .
$$

Thus we observe that $R^{\prime}(u)<0$ for $0 \leq u \leq \frac{5}{12 \varepsilon}$. It follows that

$$
\frac{\partial}{\partial u}[A(\alpha, u)+2 B(\alpha, u)]=R(u) \leq R(0)=-4<0 \quad \text { for } 0 \leq u \leq \frac{5}{12 \varepsilon} .
$$

Then we have

$$
A(\alpha, u)+2 B(\alpha, u)>A(\alpha, \alpha)+2 B(\alpha, \alpha)=0 \quad \text { for } 0<u<\alpha \leq \frac{5}{12 \varepsilon} .
$$

So by (2.15), we obtain (2.14). The proof is complete.

Lemma 2.4. Consider (1.1) with $\varepsilon>0$. Let I be a closed interval in $\left(0, \beta_{\varepsilon}\right)$. Then the following statements (i)-(iii) hold:
(i) If $\bar{T}^{\prime}(\alpha)<0$ for $\alpha \in I$, then there exists $\check{\lambda}>0$ such that $T_{\lambda}^{\prime}(\alpha)<0$ for $\alpha \in I$ and $0<\lambda<\check{\lambda}$.
(ii) If $\alpha \bar{T}^{\prime \prime}(\alpha)+k \bar{T}^{\prime}(\alpha)<0$ for $\alpha \in I$ and some $k>0$, then there exists $\hat{\lambda}>0$ such that $\alpha T_{\lambda}^{\prime \prime}(\alpha)+$ $k T_{\lambda}^{\prime}(\alpha)<0$ for $\alpha \in I$ and $0<\lambda<\hat{\lambda}$.
(iii) If $\left[2 \alpha \bar{T}^{\prime \prime}(\alpha)+3 \bar{T}^{\prime}(\alpha)\right]^{\prime}>0$ for $\alpha \in I$, then there exists $\bar{\lambda}>0$ such that $\left[2 \alpha T_{\lambda}^{\prime \prime}(\alpha)+3 T_{\lambda}^{\prime}(\alpha)\right]^{\prime}>$ 0 for $\alpha \in I$ and $0<\lambda<\bar{\lambda}$.

Proof. (I) Assume that $\bar{T}^{\prime}(\alpha)<0$ for $\alpha \in I$. By Lemma 2.1(ii), we have

$$
\begin{equation*}
\lim _{\lambda \rightarrow 0^{+}} \sqrt{\lambda} T_{\lambda}^{\prime}(\alpha)=\bar{T}^{\prime}(\alpha)<0 \quad \text { for } \alpha \in I . \tag{2.16}
\end{equation*}
$$

For $\alpha \in I$, by (2.16), we define $\lambda_{\alpha}$ by

$$
\lambda_{\alpha} \equiv \begin{cases}1 & \text { if } T_{\lambda}^{\prime}(\alpha)<0 \text { for all } \lambda>0  \tag{2.17}\\ \sup \left\{\lambda_{1}: T_{\lambda}^{\prime}(\alpha)<0 \text { for } 0<\lambda<\lambda_{1}\right\} & \text { if } T_{\lambda}^{\prime}(\alpha) \geq 0 \text { for some } \lambda>0\end{cases}
$$

Clearly, $T_{\lambda}^{\prime}(\alpha)<0$ for $\alpha \in I$ and $0<\lambda<\lambda_{\alpha}$. Let $\check{\lambda} \equiv \inf \left\{\lambda_{\alpha}: \alpha \in I\right\}$. Assume that $\check{\lambda}=0$. By (2.17), there exists a sequence $\left\{\alpha_{k}\right\}_{k \in \mathbb{N}} \subset I$ such that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \lambda_{\alpha_{k}}=0 \quad \text { and } \quad T_{\lambda_{\alpha_{k}}}^{\prime}\left(\alpha_{k}\right) \geq 0 \quad \text { for } k \in \mathbb{N} . \tag{2.18}
\end{equation*}
$$

Without loss of generality, we assume that $\lim _{k \rightarrow \infty} \alpha_{k}=\check{\alpha} \in I$. So by (2.16) and (2.18), we observe that

$$
0 \leq \lim _{k \rightarrow \infty} \sqrt{\lambda_{\alpha_{k}}} T_{\lambda_{\alpha_{k}}}^{\prime}\left(\alpha_{k}\right)=\lim _{k \rightarrow \infty} \sqrt{\lambda_{\alpha_{k}}} T_{\lambda_{\alpha_{k}}}^{\prime}(\breve{\alpha})=\bar{T}^{\prime}(\check{\alpha})<0,
$$

which is a contradiction. It implies that $\check{\lambda}>0$. So statement (i) holds.
(II) Assume that $\alpha \bar{T}^{\prime \prime}(\alpha)+k \bar{T}^{\prime}(\alpha)<0$ for $\alpha \in I$ and some $k>0$. Let $G_{1}(\alpha, \lambda) \equiv \alpha T_{\lambda}^{\prime \prime}(\alpha)+$ $k T_{\lambda}^{\prime}(\alpha)$. By Lemma 2.1(ii), we see that

$$
\begin{equation*}
\lim _{\lambda \rightarrow 0^{+}} \sqrt{\lambda} G_{1}(\alpha, \lambda)=\alpha \bar{T}^{\prime \prime}(\alpha)+k \bar{T}^{\prime}(\alpha)<0 \quad \text { for } \alpha \in I . \tag{2.19}
\end{equation*}
$$

For $\alpha \in I$, by (2.19), we define $\lambda_{\alpha}$ by

$$
\lambda_{\alpha} \equiv \begin{cases}1 & \text { if } G_{1}(\alpha, \lambda)<0 \text { for all } \lambda>0, \\ \sup \left\{\lambda_{2}: G_{1}(\alpha, \lambda)<0 \text { for } 0<\lambda<\lambda_{2}\right\} & \text { if } G_{1}(\alpha, \lambda) \geq 0 \text { for some } \lambda>0\end{cases}
$$

Clearly, $G_{1}(\alpha, \lambda)<0$ for $\alpha \in I$ and $0<\lambda<\lambda_{\alpha}$. Let $\hat{\lambda} \equiv \inf \left\{\lambda_{\alpha}: \alpha \in I\right\}$. We use the similar argument in (I) to obtain that $\hat{\lambda}>0$. So statement (ii) holds.
(III) Assume that $\left[2 \alpha \bar{T}^{\prime \prime}(\alpha)+3 \bar{T}^{\prime}(\alpha)\right]^{\prime}>0$ for $\alpha \in I$. Let $G_{2}(\alpha, \lambda) \equiv\left[2 \alpha T^{\prime \prime}(\alpha)+3 T^{\prime}(\alpha)\right]^{\prime}$. By Lemma 2.1(ii), we see that

$$
\begin{align*}
\lim _{\lambda \rightarrow 0^{+}} \sqrt{\lambda} G_{2}(\alpha, \lambda) & =\lim _{\lambda \rightarrow 0^{+}}\left[2 \alpha \sqrt{\lambda} T_{\lambda}^{\prime \prime \prime}(\alpha)+5 \sqrt{\lambda} T_{\lambda}^{\prime \prime}(\alpha)\right]=2 \alpha \bar{T}^{\prime \prime \prime}(\alpha)+5 \bar{T}^{\prime \prime}(\alpha) \\
& =\left[2 \alpha \bar{T}^{\prime \prime}(\alpha)+3 \bar{T}^{\prime}(\alpha)\right]^{\prime}>0 \quad \text { for } \alpha \in I . \tag{2.20}
\end{align*}
$$

For $\alpha \in I$, by (2.20), we define $\lambda_{\alpha}$ by

$$
\lambda_{\alpha} \equiv \begin{cases}1 & \text { if } G_{2}(\alpha, \lambda)<0 \text { for all } \lambda>0 \\ \sup \left\{\lambda_{3}: G_{2}(\alpha, \lambda)<0 \text { for } 0<\lambda<\lambda_{3}\right\} & \text { if } G_{2}(\alpha, \lambda) \geq 0 \text { for some } \lambda>0\end{cases}
$$

Clearly, $G_{2}(\alpha, \lambda)<0$ for $\alpha \in I$ and $0<\lambda<\lambda_{\alpha}$. Let $\bar{\lambda} \equiv \inf \left\{\lambda_{\alpha}: \alpha \in I\right\}$. We use the similar argument in (I) to obtain that $\bar{\lambda}>0$. So statement (iii) holds. The proof is complete.

Lemma 2.5. Consider (1.5) with $\varepsilon>0$. Let $\varepsilon_{0}$ be defined in Theorem 1.2. Then the following statements (i)-(iii) hold:
(i) $\bar{T}^{\prime}(\alpha) \geq 0$ for $0<\alpha<\beta_{\varepsilon}$ and $\varepsilon \geq \varepsilon_{0}$.
(ii) $\left[2 \alpha \bar{T}^{\prime \prime}(\alpha)+3 \bar{T}^{\prime}(\alpha)\right]^{\prime}>0$ for $\frac{1}{3 \varepsilon} \leq \alpha \leq \frac{5}{12 \varepsilon}$ and $\varepsilon \leq \varepsilon_{0}$.
(iii) There exists $\hat{\varepsilon} \in\left(0, \varepsilon_{0}\right)$ such that $\bar{T}^{\prime}(\alpha) \geq 0$ for $0<\alpha \leq \frac{1}{3 \varepsilon}$ and $\hat{\varepsilon} \leq \varepsilon<\varepsilon_{0}$. Furthermore, $\hat{\varepsilon}<\sqrt{31 / 1000}$.

Proof. The statement (i) follows immediately by Theorem 1.2 and (2.4). The statement (ii) follows immediately by [6, Lemma 3.5]. By [11, Theorem 2.1], there exists $\hat{\varepsilon}>0$ satisfying

$$
\hat{\varepsilon}<\sqrt{\frac{31}{1000}}<\varepsilon_{0}
$$

such that

$$
\bar{T}^{\prime}\left(\frac{1}{3 \varepsilon}\right) \begin{cases}<0 & \text { for } 0<\varepsilon<\hat{\varepsilon}  \tag{2.21}\\ =0 & \text { for } \varepsilon=\hat{\varepsilon} \\ >0 & \text { for } \hat{\varepsilon}<\varepsilon<\varepsilon_{0}\end{cases}
$$

By Theorem 1.2, (2.4) and [6, Lemma 3.3], we see that, for $0<\varepsilon<\varepsilon_{0}$, there exist two positive numbers $\alpha_{*}<\alpha^{*}<\beta_{\varepsilon}$ such that

$$
\bar{T}^{\prime}(\alpha) \begin{cases}>0 & \text { on }\left(0, \alpha_{*}\right) \cup\left(\alpha^{*}, \beta_{\varepsilon}\right),  \tag{2.22}\\ =0 & \text { when } \alpha=\alpha_{*} \text { or } \alpha=\alpha^{*}, \\ <0 & \text { for }\left(\alpha_{*}, \alpha^{*}\right) .\end{cases}
$$

Since $f$ is a convex function on $\left[0, \frac{1}{3 \varepsilon}\right]$, and by [15, Lemma 3.2], we see that $\bar{T}(\alpha)$ is either strictly increasing on $\left(0, \frac{1}{3 \varepsilon}\right)$, or strictly increasing and then strictly decreasing on $\left(0, \frac{1}{3 \varepsilon}\right)$. So by (2.21) and (2.22), we observe that $\frac{1}{3 \varepsilon} \leq \alpha_{*}$ for $\hat{\varepsilon} \leq \varepsilon<\varepsilon_{0}$. It follows that $\bar{T}^{\prime}(\alpha) \geq 0$ for $0<\alpha \leq \frac{1}{3 \varepsilon}$ and $\hat{\varepsilon} \leq \varepsilon<\varepsilon_{0}$. So the statement (iii) holds. The proof is complete.

Lemma 2.6. Consider (1.5) with $0<\varepsilon \leq \hat{\varepsilon}$ where $\hat{\varepsilon}$ is defined in Lemma 2.5. Then $\alpha \bar{T}^{\prime \prime}(\alpha)+\bar{T}^{\prime}(\alpha)<$ 0 for $1 \leq \alpha \leq 1.7$.
Proof. Let $\bar{A} \equiv \varepsilon\left(\alpha^{4}-u^{4}\right), \bar{B} \equiv \alpha^{3}-u^{3}, \bar{C} \equiv \alpha^{2}-u^{2}$ and $\bar{D} \equiv \alpha-u$. We compute that

$$
\begin{equation*}
\alpha \bar{T}^{\prime \prime}(\alpha)+\bar{T}^{\prime}(\alpha)=\frac{1}{4 \sqrt{2} \alpha} \int_{0}^{\alpha} \frac{N_{1}(\alpha, u)}{[F(\alpha)-F(u)]^{5 / 2}} d u, \tag{2.23}
\end{equation*}
$$

where

$$
N_{1}(\alpha, u) \equiv \frac{1}{72}\left(9 \bar{A}^{2}+4 \bar{B}^{2}+36 \bar{D}^{2}-6 \bar{A} \bar{B}+198 \bar{A} \bar{D}-120 \bar{B} \bar{D}+36 \bar{A} \bar{C}-12 \bar{B} \bar{C}-36 \bar{C} \bar{D}\right) .
$$

Let $\alpha \in[1,1.7], u \in(0, \alpha)$ and $\varepsilon \in(0, \tilde{\varepsilon}]$ be given. By Lemma [11, Lemma 3.6], we have

$$
\bar{A}<\frac{4 \varepsilon \alpha}{3} \bar{B} \quad \text { and } \quad \bar{D}>\frac{1}{3 \alpha^{2}} \bar{B}>\frac{1}{3 \alpha^{2}}\left(\frac{3}{4 \varepsilon \alpha} \bar{A}\right)=\frac{\bar{A}}{4 \alpha^{3} \varepsilon} .
$$

Then

$$
\begin{gather*}
1<\alpha^{2}<\frac{\left(\alpha^{2}+\alpha u+u^{2}\right) \bar{D}}{\bar{D}}=\frac{\bar{B}}{\bar{D}}<3 \alpha^{2} \leq 3(1.7)^{2}=8.67,  \tag{2.24}\\
\bar{A}<\frac{4 \varepsilon \alpha}{3} \bar{B}<\frac{4 \hat{\varepsilon}}{3}(1.7) \bar{B}=\frac{34 \hat{\varepsilon}}{15} \bar{B} \quad \text { and } \quad \bar{D}>\frac{\bar{A}}{4 \alpha^{3} \varepsilon}>\frac{\bar{A}}{4(1.7)^{3} \hat{\varepsilon}}=\frac{250}{4913 \hat{\varepsilon}} \bar{A} . \tag{2.25}
\end{gather*}
$$

In addition, by Lemma 2.5(iii), we compute and find that

$$
\begin{gather*}
\frac{34}{15} \hat{\varepsilon}-\frac{2}{3}<\frac{34}{15} \sqrt{\frac{31}{1000}}-\frac{2}{3}(\approx-0.26)<0  \tag{2.26}\\
198\left(\frac{34}{15} \hat{\varepsilon}-\frac{20}{33}\right)<198\left(\frac{34}{15} \sqrt{\frac{31}{1000}}-\frac{20}{33}\right)(\approx-40.98)<-0.40  \tag{2.27}\\
1-\frac{5}{34 \hat{\varepsilon}}-\frac{250}{4913 \hat{\varepsilon}}<1-\frac{5}{34 \sqrt{\frac{31}{1000}}}-\frac{250}{4913 \sqrt{\frac{31}{1000}}}(\approx-0.88)<0 \tag{2.28}
\end{gather*}
$$

By (2.24)-(2.28), we observe that

$$
\begin{aligned}
N_{1}(\alpha, u)= & \frac{1}{72}\left(9 \bar{A}^{2}+4 \bar{B}^{2}+36 \bar{D}^{2}-6 \bar{A} \bar{B}+198 \bar{A} \bar{D}-120 \bar{B} \bar{D}+36 \bar{A} \bar{C}-12 \bar{B} \bar{C}-36 \bar{C} \bar{D}\right) \\
= & \frac{1}{72}\left[9 \bar{A}\left(\bar{A}-\frac{2}{3} \bar{B}\right)+198 \bar{D}\left(\bar{A}-\frac{20}{33} \bar{B}\right)+36 \bar{C}\left(\bar{A}-\frac{1}{3} \bar{B}-\bar{D}\right)+4 \bar{B}^{2}+36 \bar{D}^{2}\right] \\
< & \frac{1}{72}\left[9 \bar{A} \bar{B}\left(\frac{34}{15} \hat{\varepsilon}-\frac{2}{3}\right)+198 \bar{B} \bar{D}\left(\frac{34}{15} \hat{\varepsilon}-\frac{20}{33}\right)\right. \\
& \left.+36 \bar{A} \bar{C}\left(1-\frac{5}{34 \hat{\varepsilon}}-\frac{250}{4913 \hat{\varepsilon}}\right)+4 \bar{B}^{2}+36 \bar{D}^{2}\right] \\
< & \frac{1}{72}\left(-40 \bar{B} \bar{D}+4 \bar{B}^{2}+36 \bar{D}^{2}\right)=\frac{\bar{D}^{2}}{18}\left[\left(\frac{\bar{B}}{\bar{D}}-5\right)^{2}-16\right] \\
< & \frac{\bar{D}^{2}}{18}\left[(1-5)^{2}-16\right]=0 .
\end{aligned}
$$

So by (2.23), we obtain that $\alpha \bar{T}^{\prime \prime}(\alpha)+\bar{T}^{\prime}(\alpha)<0$ for $1 \leq \alpha \leq 1.7$ and $0<\varepsilon \leq \hat{\varepsilon}$. The proof is complete.
Lemma 2.7. Consider (1.5) with $0.07 \leq \varepsilon \leq \hat{\varepsilon}$. Then $\alpha \bar{T}^{\prime \prime}(\alpha)+\frac{5}{2} \bar{T}^{\prime}(\alpha)<0$ for $1.7 \leq \alpha \leq \frac{1}{3 \varepsilon}$.
Proof. We compute that

$$
\begin{equation*}
\alpha \bar{T}^{\prime \prime}(\alpha)+\frac{5}{2} \bar{T}^{\prime}(\alpha)=\frac{1}{4 \sqrt{2} \alpha} \int_{0}^{\alpha} \frac{N_{2}(\alpha, u)}{[F(\alpha)-F(u)]^{5 / 2}} d u \tag{2.29}
\end{equation*}
$$

where

$$
\begin{align*}
N_{2}(\alpha, u) \equiv & \frac{1}{144}\left(-9 \bar{A}^{2}+42 \bar{A} \bar{B}+450 \bar{A} \bar{D}+126 \bar{A} \bar{C}-16 \bar{B}^{2}-240 \bar{B} \bar{D}\right. \\
& \left.-60 \bar{B} \bar{C}+288 \bar{D}^{2}+36 \bar{C} \bar{D}\right) . \tag{2.30}
\end{align*}
$$

Then we assert that

$$
\begin{equation*}
N_{2}(\alpha, u)<0 \quad \text { for } 0<u<\alpha, 1.7 \leq \alpha \leq \frac{1}{3 \varepsilon} \text { and } 0.07 \leq \varepsilon \leq \hat{\varepsilon} . \tag{2.31}
\end{equation*}
$$

The proof of assertion (2.31) is easy but tedious. Thus, we put it in Appendix. So by (2.29)(2.31), we see that $\alpha \bar{T}^{\prime \prime}(\alpha)+\frac{5}{2} \bar{T}^{\prime}(\alpha)<0$ for $1.7 \leq \alpha \leq \frac{1}{3 \varepsilon}$ and $0.07 \leq \varepsilon \leq \hat{\varepsilon}$.

Lemma 2.8. Consider (1.5) with $0<\varepsilon<0.07$. Then $\bar{T}^{\prime}(\alpha)<0$ for $1.7 \leq \alpha \leq \frac{1}{3 \varepsilon}$.
Proof. We compute that

$$
\begin{equation*}
\bar{T}^{\prime}(\alpha)=\frac{1}{2 \sqrt{2} \alpha} \int_{0}^{\alpha} \frac{2 B(\alpha, u)-A(\alpha, u)}{B^{3 / 2}(\alpha, u)} d u=\frac{1}{2 \sqrt{2} \alpha} \int_{0}^{\alpha} \frac{\theta(\alpha)-\theta(u)}{B^{3 / 2}(\alpha, u)} d u, \tag{2.32}
\end{equation*}
$$

where $\theta(u) \equiv 2 F(u)-u f(u)$ for $0 \leq u<\beta_{\varepsilon}$. Since $0<\varepsilon<0.07$, and by [11, Lemma 3.1], there exists $p \in\left(0, \frac{1}{3 \varepsilon}\right)$ such that $\theta^{\prime}(u)>0$ for $(0, p)$ and $\theta^{\prime}(u)<0$ for $\left(p, \frac{1}{3 \varepsilon}\right)$. Let $\alpha \in\left[1.7, \frac{1}{3 \varepsilon}\right]$ be given. Assume that $\theta(\alpha) \leq 0$, see Figure 2.1(i). Since $\theta(0)=0$, we see that $\theta(\alpha)-\theta(u)<0$ for $0<u<\alpha$. So by (2.32), we obtain that $\bar{T}^{\prime}(\alpha)<0$. Assume that $\theta(\alpha)>0$, see Figure 2.1(ii). We compute and find that

$$
\theta^{\prime}(1.7)=2 \varepsilon u^{3}-u^{2}+\left.1\right|_{u=1.7}=\frac{4913}{500} \varepsilon-\frac{189}{100}<0 \quad \text { for } 0<\varepsilon<0.07 .
$$

Since $1.7 \leq \alpha \leq \frac{1}{3 \varepsilon}$, there exists $\bar{\alpha} \in(0, p)$ such that

$$
\theta(\alpha)-\theta(u) \begin{cases}>0 & \text { for } 0<u<\bar{\alpha} \\ =0 & \text { for } u=\bar{\alpha} \\ <0 & \text { for } \bar{\alpha}<u<\alpha\end{cases}
$$


(i)

(ii)

Figure 2.1: Graphs of $\theta(u)$ on $[0, \alpha]$ where $1.7 \leq \alpha \leq \frac{1}{3 \varepsilon}$ and $0<\varepsilon<0.07$.
So by (2.32) and similar argument of [14, (3.11)], we observe that

$$
\begin{equation*}
\bar{T}^{\prime}(\alpha)<\frac{1}{2 \sqrt{2} \alpha B^{3 / 2}(\alpha, \bar{\alpha})} \int_{0}^{\alpha} u \theta^{\prime}(u) d u=\frac{\alpha\left(8 \varepsilon \alpha^{3}-5 \alpha^{2}+10\right)}{40 \sqrt{2} B^{3 / 2}(\alpha, \bar{\alpha})} . \tag{2.33}
\end{equation*}
$$

Since

$$
\frac{\partial}{\partial u}\left(8 \varepsilon u^{3}-5 u^{2}+10\right)=2 u(12 \varepsilon u-5)<0 \quad \text { for } 1.7 \leq u \leq \frac{1}{3 \varepsilon}
$$

we see that, for $1.7 \leq u \leq \frac{1}{3 \varepsilon}$ and $0<\varepsilon<0.07$,

$$
8 \varepsilon u^{3}-5 u^{2}+10<8 \varepsilon u^{3}-5 u^{2}+\left.10\right|_{u=1.7}=\frac{4913}{125} \varepsilon-\frac{89}{20}<0 .
$$

So by (2.33), we obtain that $\bar{T}^{\prime}(\alpha)<0$. The proof is complete.
Lemma 2.9. Consider (1.1) with $0<\varepsilon<\varepsilon_{0}$. Then there exists $\xi_{\varepsilon}>0$ such that

$$
\Gamma_{\varepsilon} \equiv\left\{\lambda>0: T_{\lambda}^{\prime}(\alpha)<0 \text { for some } \alpha \in\left(0, \beta_{\varepsilon}\right)\right\}=\left(0, \xi_{\varepsilon}\right) .
$$

Proof. Let $\varepsilon \in\left(0, \varepsilon_{0}\right)$ be given. By (2.22), there exist two positive numbers $\alpha_{*}<\alpha^{*}<\beta_{\varepsilon}$ such that

$$
\lim _{\lambda \rightarrow 0^{+}} \sqrt{\lambda} T_{\lambda}^{\prime}(\alpha)=\bar{T}^{\prime}(\alpha) \begin{cases}>0 & \text { on }\left(0, \alpha_{*}\right) \cup\left(\alpha^{*}, \beta_{\varepsilon}\right)  \tag{2.34}\\ =0 & \text { when } \alpha=\alpha_{*} \text { or } \alpha^{*} \\ <0 & \text { on }\left(\alpha_{*}, \alpha^{*}\right)\end{cases}
$$

Then we divide this proof into the next four steps.
Step 1. We prove that $\alpha_{*}<\frac{5}{12 \varepsilon}$. Assume that $\alpha_{*} \geq \frac{5}{12 \varepsilon}$. By (2.34) and Lemma 2.3, we see that

$$
\begin{equation*}
0 \leq \bar{T}^{\prime}(\alpha)=\lim _{\lambda \rightarrow 0^{+}} \sqrt{\lambda} T_{\lambda}^{\prime}(\alpha)<\sqrt{\lambda} T_{\lambda}^{\prime}(\alpha) \quad \text { for } 0<\alpha \leq \frac{5}{12 \varepsilon} \text { and } \lambda>0 . \tag{2.35}
\end{equation*}
$$

By Lemma 2.2(ii) and (2.35), we further see that $T_{\lambda}^{\prime}(\alpha)>0$ for $0<\alpha<\beta_{\varepsilon}$ for $\lambda>0$. So by (2.34), we obtain that

$$
0 \leq \lim _{\lambda \rightarrow 0^{+}} \sqrt{\lambda} T_{\lambda}^{\prime}\left(\frac{\alpha_{*}+\alpha^{*}}{2}\right)=\bar{T}^{\prime}\left(\frac{\alpha_{*}+\alpha^{*}}{2}\right)<0
$$

which is a contradiction. It implies that $\alpha_{*}<\frac{5}{12 \varepsilon}$.
Step 2. We prove that, for $\alpha \in\left(\alpha_{*}, \alpha^{*}\right) \cap\left(0, \frac{5}{12 \varepsilon}\right]$, there exists a continuously differential function $\tilde{\lambda}_{\alpha}>0$ of $\alpha$ such that

$$
\sqrt{\lambda} T_{\lambda}^{\prime}(\alpha) \begin{cases}<0 & \text { if } 0<\lambda<\tilde{\lambda}_{\alpha}  \tag{2.36}\\ =0 & \text { if } \lambda=\tilde{\lambda}_{\alpha} \\ >0 & \text { if } \lambda>\tilde{\lambda}_{\alpha}\end{cases}
$$

By Lemma 2.1(ii), we see that

$$
\begin{equation*}
\lim _{\lambda \rightarrow \infty} \sqrt{\lambda} T_{\lambda}^{\prime}(\alpha)=\infty \cdot 1=\infty \quad \text { for } \alpha \in\left(0, \beta_{\varepsilon}\right) \tag{2.37}
\end{equation*}
$$

By (2.34), (2.37), Lemma 2.3 and implicit function theorem, we observe that, for $\alpha \in\left(\alpha_{*}, \alpha^{*}\right) \cap$ $\left(0, \frac{5}{12 \varepsilon}\right]$, there exists a continuously differential function $\tilde{\lambda}_{\alpha}>0$ of $\alpha$ such that (2.36) holds.
Step 3. We prove that

$$
\xi_{\varepsilon} \equiv \sup \left\{\tilde{\lambda}_{\alpha}: \alpha \in\left(\alpha_{*}, \alpha^{*}\right) \cap\left(0, \frac{5}{12 \varepsilon}\right]\right\} \in(0, \infty)
$$

Clearly, $\mathcal{\xi}_{\varepsilon}>0$. By (2.34) and Lemma 2.3, we see that

$$
0=\lim _{\lambda \rightarrow 0^{+}} \sqrt{\lambda} T_{\lambda}^{\prime}\left(\alpha_{*}\right)<T_{\lambda=1}^{\prime}\left(\alpha_{*}\right)
$$

So by Lemma 2.3 and continuity of $T_{\lambda=1}^{\prime}(\alpha)$ with respect to $\alpha$, there exists $\delta>0$ such that

$$
0<T_{\lambda=1}^{\prime}(\alpha) \leq \sqrt{\lambda} T_{\lambda}^{\prime}(\alpha) \quad \text { for } \alpha_{*}<\alpha<\alpha_{*}+\delta<\frac{5}{12 \varepsilon} \text { and } \lambda \geq 1
$$

from which it follows that $\tilde{\lambda}_{\alpha}<1$ for $\alpha_{*}<\alpha<\alpha_{*}+\delta$. Thus $\lim _{\alpha \rightarrow \alpha_{*}^{+}} \tilde{\lambda}_{\alpha} \leq 1<\infty$. By similar argument, we obtain that

$$
\lim _{\alpha \rightarrow\left(\alpha^{*}\right)^{-}} \tilde{\lambda}_{\alpha}<\infty \quad \text { if } \alpha^{*}<\frac{5}{12 \varepsilon} .
$$

So by Step 2, we observe that $\xi_{\varepsilon} \in(0, \infty)$.
Step 4. We prove that $\Gamma_{\varepsilon}=\left(0, \xi_{\varepsilon}\right)$. Let $\lambda_{1} \in\left(0, \xi_{\varepsilon}\right)$. There exists $\alpha_{1} \in\left(\alpha_{*}, \alpha^{*}\right) \cap\left(0, \frac{5}{12 \varepsilon}\right]$ such that $\lambda_{1}<\tilde{\lambda}_{\alpha_{1}}$. Then by (2.36), we see that $T_{\lambda_{1}}^{\prime}\left(\alpha_{1}\right)<0$, which implies that $\lambda_{1} \in \Gamma_{\varepsilon}$. Thus $\left(0, \xi_{\varepsilon}\right) \subseteq \Gamma_{\varepsilon}$. Let $\lambda_{2} \in \Gamma_{\varepsilon}$. There exists $\alpha_{2} \in\left(0, \beta_{\varepsilon}\right)$ such that $T_{\lambda_{2}}^{\prime}\left(\alpha_{2}\right)<0$. Next, we consider two cases.
Case 1. Assume that $\frac{5}{12 \varepsilon}<\alpha^{*}$. By (2.34) and Lemma 2.3, we see that

$$
\begin{equation*}
0 \leq \lim _{\lambda \rightarrow 0^{+}} \sqrt{\lambda} T_{\lambda}^{\prime}(\alpha)<\sqrt{\lambda} T_{\lambda}^{\prime}(\alpha) \quad \text { for } \alpha \in\left(0, \alpha_{*}\right] \text { and } \lambda>0 . \tag{2.38}
\end{equation*}
$$

By Steps 2 and 3, we see that

$$
\begin{equation*}
\sqrt{\lambda} T_{\lambda}^{\prime}(\alpha) \geq 0 \quad \text { for } \alpha \in\left(\alpha_{*}, \frac{5}{12 \varepsilon}\right] \quad \text { if } \lambda \geq \xi_{\varepsilon} . \tag{2.39}
\end{equation*}
$$

By (2.39) and Lemma 2.2, we see that

$$
\begin{equation*}
T_{\lambda}^{\prime}(\alpha)>0 \quad \text { for } \frac{5}{12 \varepsilon} \leq \alpha<\beta_{\varepsilon} \text { and } \lambda \geq \xi_{\varepsilon} \tag{2.40}
\end{equation*}
$$

So by (2.38)-(2.40), we obtain that $T_{\lambda}^{\prime}(\alpha) \geq 0$ for $\alpha \in\left(0, \beta_{\varepsilon}\right)$ if $\lambda \geq \xi_{\varepsilon}$. It implies that $\lambda_{2}<\xi_{\varepsilon}$. Thus $\Gamma_{\varepsilon} \subseteq\left(0, \xi_{\varepsilon}\right)$.
Case 2. Assume that $\alpha^{*}<\frac{5}{12 \varepsilon}$. By (2.34) and Lemma 2.3, we see that

$$
\begin{equation*}
0 \leq \lim _{\lambda \rightarrow 0^{+}} \sqrt{\lambda} T_{\lambda}^{\prime}(\alpha)<\sqrt{\lambda} T_{\lambda}^{\prime}(\alpha) \quad \text { for } \alpha \in\left(0, \alpha_{*}\right] \cup\left[\alpha^{*}, \frac{5}{12 \varepsilon}\right] \text { and } \lambda>0 \tag{2.41}
\end{equation*}
$$

By Steps 2 and 3, we see that

$$
\begin{equation*}
\sqrt{\lambda} T_{\lambda}^{\prime}(\alpha) \geq 0 \quad \text { for } \alpha \in\left(\alpha_{*}, \alpha^{*}\right) \text { if } \lambda \geq \xi_{\varepsilon} . \tag{2.42}
\end{equation*}
$$

By (2.41) and Lemma 2.2(ii), we see that

$$
\begin{equation*}
T_{\lambda}^{\prime}(\alpha)>0 \quad \text { for } \frac{5}{12 \varepsilon} \leq \alpha<\beta_{\varepsilon} \text { and } \lambda>0 \tag{2.43}
\end{equation*}
$$

So by (2.41)-(2.43), we obtain that $T_{\lambda}^{\prime}(\alpha) \geq 0$ for $\alpha \in\left(0, \beta_{\varepsilon}\right)$ if $\lambda \geq \xi_{\varepsilon}$. It implies that $\lambda_{2}<\xi_{\varepsilon}$. Thus $\Gamma_{\varepsilon} \subseteq\left(0, \xi_{\varepsilon}\right)$.

By the above discussions, we obtain that $\Gamma_{\varepsilon}=\left(0, \xi_{\varepsilon}\right)$. The proof is complete.
Lemma 2.10. Consider (1.1) with $0<\varepsilon<\varepsilon_{0}$. Then there exists $\kappa_{\varepsilon} \in\left(0, \xi_{\varepsilon}\right)$ such that $T_{\lambda}(\alpha)$ has exactly two critical points, a local maximum at $\alpha_{M}(\lambda)$ and a local minimum at $\alpha_{m}(\lambda)\left(>\alpha_{M}(\lambda)\right)$, on $\left(0, \beta_{\varepsilon}\right)$ if $0<\lambda<\kappa_{\varepsilon}$.

Proof. Let $\varepsilon \in\left(0, \varepsilon_{0}\right)$ be given. By (2.34) and Lemma 2.1(ii), there exists $\lambda_{1}>0$ such that

$$
\begin{equation*}
T_{\lambda}^{\prime}\left(\frac{\alpha_{*}+\alpha^{*}}{2}\right)<0 \quad \text { for } 0<\lambda<\lambda_{1} \tag{2.44}
\end{equation*}
$$

We divide this proof into the next four steps.
Step 1. We prove that there exists $\lambda_{2} \in\left(0, \lambda_{1}\right)$ such that, for $0<\lambda<\lambda_{2}$, either $T_{\lambda}^{\prime}(\alpha)>0$ on $\left(0, \frac{1}{3 \varepsilon}\right]$, or $T_{\lambda}(\alpha)$ has exactly one critical point, a local maximum, on $\left(0, \frac{1}{3 \varepsilon}\right]$, see Figure 2.2. By Lemma 2.2(i), we have

$$
\begin{equation*}
T_{\lambda}^{\prime}(\alpha)>0 \quad \text { for } 0<\alpha \leq 1 \text { and } \lambda>0 \tag{2.45}
\end{equation*}
$$



Figure 2.2: Graphs of $T_{\lambda}(\alpha)$ on $\left(0, \frac{1}{3 \varepsilon}\right]$ for $0<\lambda<\lambda_{2}$.
Then we consider the following three cases.
Case 1. Assume that $\hat{\varepsilon} \leq \varepsilon<\varepsilon_{0}$. By Lemmas 2.1(ii), 2.3 and 2.5(iii), we see that

$$
0 \leq \bar{T}^{\prime}(\alpha)=\lim _{\lambda \rightarrow 0^{+}} \sqrt{\lambda} T_{\lambda}^{\prime}(\alpha)<\sqrt{\lambda} T_{\lambda}^{\prime}(\alpha) \quad \text { for } 1<\alpha \leq \frac{1}{3 \varepsilon} \text { and } \lambda>0
$$

So by (2.45), $T_{\lambda}^{\prime}(\alpha)>0$ on $\left(0, \frac{1}{3 \varepsilon}\right]$ for $\lambda>0$, see Figure 2.2(i).
Case 2. Assume that $0.07 \leq \varepsilon<\hat{\varepsilon}$. By (2.21), Lemmas 2.1(ii), 2.4(ii), 2.6 and 2.7, there exists $\lambda_{2} \in\left(0, \lambda_{1}\right)$ such that

$$
\begin{equation*}
T_{\lambda}^{\prime}\left(\frac{1}{3 \varepsilon}\right)<0 \quad \text { and } \quad \alpha T_{\lambda}^{\prime \prime}(\alpha)+K(\alpha) T_{\lambda}^{\prime}(\alpha)<0 \quad \text { for } 1 \leq \alpha \leq \frac{1}{3 \varepsilon} \text { and } 0<\lambda<\lambda_{2} \tag{2.46}
\end{equation*}
$$

where $K(\alpha) \equiv 1$ if $1 \leq \alpha \leq 1.7$, and $K(\alpha) \equiv 5 / 2$ if $1.7<\alpha \leq \frac{1}{3 \varepsilon}$. By (2.45) and (2.46), there exists $\alpha_{\lambda} \in\left(1, \frac{1}{3 \varepsilon}\right)$ such that $T_{\lambda}^{\prime}\left(\alpha_{\lambda}\right)=0$ for $0<\lambda<\lambda_{2}$. Furthermore,

$$
\alpha_{\lambda} T_{\lambda}^{\prime \prime}\left(\alpha_{\lambda}\right)=\alpha_{\lambda} T_{\lambda}^{\prime \prime}\left(\alpha_{\lambda}\right)+K\left(\alpha_{\lambda}\right) T_{\lambda}^{\prime}\left(\alpha_{\lambda}\right)<0 \quad \text { for } 0<\lambda<\lambda_{2}
$$

Thus $T_{\lambda}(\alpha)$ has exactly one local maximum at $\alpha_{\lambda}$ on $\left(0, \frac{1}{3 \varepsilon}\right]$ for $0<\lambda<\lambda_{2}$, see Figure 2.2(ii).
Case 3. Assume that $0<\varepsilon<0.07$. By Lemmas 2.4, 2.6 and 2.8 , there exists $\lambda_{2} \in\left(0, \lambda_{1}\right)$ such that

$$
\begin{gather*}
\alpha T_{\lambda}^{\prime \prime}(\alpha)+T_{\lambda}^{\prime}(\alpha)<0 \quad \text { for } 1 \leq \alpha \leq 1.7 \text { and } 0<\lambda<\lambda_{2}  \tag{2.47}\\
T_{\lambda}^{\prime}(\alpha)<0 \quad \text { for } 1.7 \leq \alpha \leq \frac{1}{3 \varepsilon} \text { and } 0<\lambda<\lambda_{2} . \tag{2.48}
\end{gather*}
$$

So by (2.45), (2.47) and (2.48), there exists $\alpha_{\lambda} \in(1,1.7)$ such that $T_{\lambda}^{\prime}\left(\alpha_{\lambda}\right)=0$ for $0<\lambda<\lambda_{2}$. Furthermore,

$$
\alpha_{\lambda} T_{\lambda}^{\prime \prime}\left(\alpha_{\lambda}\right)=\alpha_{\lambda} T_{\lambda}^{\prime \prime}\left(\alpha_{\lambda}\right)+T_{\lambda}^{\prime}\left(\alpha_{\lambda}\right)<0 \quad \text { for } 0<\lambda<\lambda_{2}
$$

Thus $T_{\lambda}(\alpha)$ has exactly one local maximum at $\alpha_{\lambda}$ on $\left(0, \frac{1}{3 \varepsilon}\right]$ for $0<\lambda<\lambda_{2}$, see Figure 2.2(ii).
Step 2. We prove that there exists $\lambda_{3} \in\left(0, \lambda_{2}\right)$ such that, for $\lambda \in\left(0, \lambda_{3}\right)$, one of the following cases holds:
(ci) $T_{\lambda}^{\prime}(\alpha)>0$ on $\left(\frac{1}{3 \varepsilon}, \frac{5}{12 \varepsilon}\right)$.
(cii) $T_{\lambda}^{\prime}(\alpha)<0$ on $\left(\frac{1}{3 \varepsilon}, \frac{5}{12 \varepsilon}\right)$.
(ciii) $T_{\lambda}^{\prime}(\alpha)<0$ on $\left(\frac{1}{3 \varepsilon}, \check{\alpha}\right)$ and $T_{\lambda}^{\prime}(\alpha)>0$ on $\left(\check{\alpha}, \frac{5}{12 \varepsilon}\right)$ for some $\check{\alpha} \in\left(\frac{1}{3 \varepsilon}, \frac{5}{12 \varepsilon}\right)$.
(civ) $T_{\lambda}^{\prime}(\alpha)>0$ on $\left(\frac{1}{3 \varepsilon}, \check{\alpha}\right)$ and $T_{\lambda}^{\prime}(\alpha)<0$ on $\left(\check{\alpha}, \frac{5}{12 \varepsilon}\right)$ for some $\check{\alpha} \in\left(\frac{1}{3 \varepsilon}, \frac{5}{12 \varepsilon}\right)$.
(cv) $T_{\lambda}^{\prime}(\alpha)>0$ on $\left(\frac{1}{3 \varepsilon}, \check{\alpha}\right) \cup\left(\hat{\alpha}, \frac{5}{12 \varepsilon}\right)$ and $T_{\lambda}^{\prime}(\alpha)<0$ on $(\check{\alpha}, \hat{\alpha})$ for some $\check{\alpha}, \hat{\alpha} \in\left(\frac{1}{3 \varepsilon}, \frac{5}{12 \varepsilon}\right)$.

See Figure 2.3.


Figure 2.3: Graphs of $T_{\lambda}(\alpha)$ on $\left(\frac{1}{3 \varepsilon}, \frac{5}{12 \varepsilon}\right)$ for $0<\lambda<\lambda_{3}$.
Let $H(\alpha, \lambda) \equiv 2 \alpha T_{\lambda}^{\prime \prime}(\alpha)+3 T_{\lambda}^{\prime}(\alpha)$. By Lemmas 2.4(iii) and 2.5(ii), there exists $\lambda_{3} \in\left(0, \lambda_{2}\right)$ such that

$$
\begin{equation*}
\frac{\partial}{\partial \alpha} H(\alpha, \lambda)>0 \quad \text { for } \frac{1}{3 \varepsilon} \leq \alpha \leq \frac{5}{12 \varepsilon} \text { and } 0<\lambda \leq \lambda_{3} . \tag{2.49}
\end{equation*}
$$

Fixed $\lambda \in\left(0, \lambda_{3}\right)$. Then we consider three cases.
Case 1. Assume that $H(\alpha, \lambda)<0$ for $\frac{1}{3 \varepsilon} \leq \alpha<\frac{5}{12 \varepsilon}$. If $T_{\lambda}(\alpha)$ has a critical point $\alpha_{1}$ in $\left(\frac{1}{3 \varepsilon}, \frac{5}{12 \varepsilon}\right)$, then

$$
2 \alpha_{1} T_{\lambda}^{\prime \prime}\left(\alpha_{1}\right)=H\left(\alpha_{1}, \lambda\right)<0 .
$$

It implies that $T_{\lambda}(\alpha)$ has at most one critical point, a local maximum, on $\left(\frac{1}{3 \varepsilon}, \frac{5}{12 \varepsilon}\right)$. Thus one of (ci), (cii) and (civ) holds.

Case 2. Assume that $H(\alpha, \lambda)>0$ for $\frac{1}{3 \varepsilon}<\alpha \leq \frac{5}{12 \varepsilon}$. If $T_{\lambda}(\alpha)$ has a critical point $\alpha_{2}$ in $\left(\frac{1}{3 \varepsilon}, \frac{5}{12 \varepsilon}\right)$, then

$$
2 \alpha_{2} T_{\lambda}^{\prime \prime}\left(\alpha_{2}\right)=H\left(\alpha_{2}, \lambda\right)>0
$$

It implies that $T_{\lambda}(\alpha)$ has at most one critical point, a local minimum, on $\left(\frac{1}{3 \varepsilon}, \frac{5}{12 \varepsilon}\right)$. Thus one of (ci), (cii) and (ciii) holds.

Case 3. Assume that there exists $\alpha_{*} \in\left(\frac{1}{3 \varepsilon}, \frac{5}{12 \varepsilon}\right)$ such that $H(\alpha, \lambda)<0$ for $\frac{1}{3 \varepsilon}<\alpha<\alpha_{*}$ and $H(\alpha, \lambda)>0$ for $\alpha_{*}<\alpha<\frac{5}{12 \varepsilon}$. If $T_{\lambda}(\alpha)$ has a critical point in $\left(\frac{1}{3 \varepsilon}, \alpha_{*}\right)$, by above similar argument, $T_{\lambda}(\alpha)$ has at most one critical point, a local maximum, on $\left(\frac{1}{3 \varepsilon}, \alpha_{*}\right)$. If $T_{\lambda}(\alpha)$ has a critical point in $\left(\alpha_{*}, \frac{5}{12 \varepsilon}\right)$, by above similar argument, $T_{\lambda}(\alpha)$ has at most one critical point, a local minimum, on $\left(\alpha_{*}, \frac{5}{12 \varepsilon}\right)$. Thus one of (ci)-(cv) holds.
Step 3. We prove Lemma 2.10. By Lemmas 2.1(i) and 2.2(ii), we see that, for $\lambda>0$, either $T_{\lambda}^{\prime}(\alpha)>0$ on $\left[\frac{5}{12 \varepsilon}, \beta_{\varepsilon}\right)$, or there exists $\dot{\alpha} \in\left(\frac{5}{12 \varepsilon}, \beta_{\varepsilon}\right)$ such that $T_{\lambda}^{\prime}(\alpha)<0$ on $\left[\frac{5}{12 \varepsilon}, \stackrel{\circ}{\alpha}\right)$ and $T_{\lambda}^{\prime}(\alpha)>0$ on $\left(\stackrel{\alpha}{\alpha}, \beta_{\varepsilon}\right)$, see Figure 2.4.

(i)

(ii)

Figure 2.4: Graphs of $T_{\lambda}(\alpha)$ on [5/(12e), $\left.\beta_{\varepsilon}\right)$ for $\lambda>0$.
Then by (2.44) and Steps 1-2, we observe that $T_{\lambda}(\alpha)$ has exactly two critical points, a local maximum at $\alpha_{M}(\lambda)$ and a local minimum at $\alpha_{m}(\lambda)\left(>\alpha_{M}(\lambda)\right)$, on $\left(0, \beta_{\varepsilon}\right)$ if $0<\lambda<\kappa_{\varepsilon}=\lambda_{3}$.

The proof is complete.
Lemma 2.11. Consider (1.1) with $0<\varepsilon<\varepsilon_{0}$. Let $\alpha_{M}(\lambda)$ and $\alpha_{m}(\lambda)$ be defined in Lemma 2.10. Then $\alpha_{M}(\lambda)$ is a strictly increasing function of $\lambda \in\left(0, \kappa_{\varepsilon}\right)$ and

$$
\begin{equation*}
\lim _{\lambda \rightarrow 0^{+}} \alpha_{M}(\lambda)<\alpha_{M}(\lambda)<\lim _{\lambda \rightarrow \kappa_{\varepsilon}^{-}} \alpha_{M}(\lambda) \leq \alpha_{m}(\lambda) \text { for } \lambda \in\left(0, \kappa_{\varepsilon}\right) \text {. } \tag{2.50}
\end{equation*}
$$

Proof. By Lemma 2.10, we have that

$$
T_{\lambda}^{\prime}(\alpha)\left\{\begin{array}{ll}
>0 & \text { for } \alpha \in\left(0, \alpha_{M}(\lambda)\right) \cup\left(\alpha_{m}(\lambda), \infty\right),  \tag{2.51}\\
=0 & \text { for } \alpha=\alpha_{M}(\lambda) \text { or } \alpha=\alpha_{m}(\lambda), \\
<0 & \text { for } \alpha \in\left(\alpha_{M}(\lambda), \alpha_{m}(\lambda)\right),
\end{array} \quad \text { if } 0<\lambda<\mathcal{K}_{\varepsilon} .\right.
$$

By Lemma 2.2, we see that $0<\alpha_{M}(\lambda)<\frac{5}{12 \varepsilon}$ for $0<\lambda<\kappa_{\varepsilon}$. Let $0<\lambda_{1}<\lambda_{2}<\kappa_{\varepsilon}$. By Lemma 2.3, we obtain that

$$
\sqrt{\lambda_{1}} T_{\lambda_{1}}^{\prime}\left(\alpha_{M}\left(\lambda_{2}\right)\right)<\sqrt{\lambda_{2}} T_{\lambda_{2}}^{\prime}\left(\alpha_{M}\left(\lambda_{2}\right)\right)=0
$$

which implies that $\alpha_{M}\left(\lambda_{1}\right)<\alpha_{M}\left(\lambda_{2}\right)$ by (2.51). So $\alpha_{M}(\lambda)$ is a strictly increasing function of $\lambda \in\left(0, \kappa_{\varepsilon}\right)$. It follows that

$$
\lim _{\lambda \rightarrow 0^{+}} \alpha_{M}(\lambda)<\alpha_{M}(\lambda)<\lim _{\lambda \rightarrow \kappa_{\varepsilon}^{-}} \alpha_{M}(\lambda) \quad \text { for } \lambda \in\left(0, \kappa_{\varepsilon}\right) .
$$

Assume that there exists $\lambda_{3} \in\left(0, \kappa_{\varepsilon}\right)$ such that $\lim _{\lambda \rightarrow 0^{+}} \alpha_{M}(\lambda)<\alpha_{m}\left(\lambda_{3}\right)<\lim _{\lambda \rightarrow \kappa_{\varepsilon}^{-}} \alpha_{M}(\lambda)$. Then there exists $\lambda_{4} \in\left(\lambda_{3}, \kappa_{\varepsilon}\right)$ such that

$$
\begin{equation*}
\alpha_{M}\left(\lambda_{3}\right)<\alpha_{m}\left(\lambda_{3}\right)<\alpha_{M}\left(\lambda_{4}\right)<\frac{5}{12 \varepsilon} . \tag{2.52}
\end{equation*}
$$

By (2.51), there exists $\alpha_{1} \in\left(\alpha_{M}\left(\lambda_{4}\right), \frac{5}{12 \varepsilon}\right)$ such that $T_{\lambda_{4}}^{\prime}\left(\alpha_{1}\right)<0$. Then by (2.51), (2.52) and Lemma 2.3, we observe that

$$
0<\sqrt{\lambda_{3}} T_{\lambda_{3}}^{\prime}\left(\alpha_{1}\right)<\sqrt{\lambda_{4}} T_{\lambda_{4}}^{\prime}\left(\alpha_{1}\right)<0
$$

which is a contradiction. So (2.50) holds. The proof is complete.
Lemma 2.12 ([9, Lemma 4.6]). Consider (1.1) with fixed $L>0$. Let $\rho_{L, \varepsilon} \equiv \min \left\{L, \beta_{\varepsilon}\right\}$ and $\operatorname{sgn}(u)$ be the signum function. Then the following statements (i)-(iii) hold:
(i) There exists a positive function $\lambda_{L}(\alpha) \in C^{1}\left(0, \rho_{L, \varepsilon}\right)$ such that $T_{\lambda_{L}(\alpha)}(\alpha)=L$. Moreover, the bifurcation curve $S_{L}=\left\{\left(\lambda_{L}(\alpha), \alpha\right): \alpha \in\left(0, \rho_{L, \varepsilon}\right)\right\}$ is continuous on the $\left(\lambda,\|u\|_{\infty}\right)$-plane.
(ii) $\lim _{\alpha \rightarrow 0^{+}} \lambda_{L}(\alpha)=0$ and $\lim _{\alpha \rightarrow p_{L, e}^{-}}^{-} \lambda_{L}(\alpha)=\infty$.
(iii) $\operatorname{sgn}\left(\lambda_{L}^{\prime}(\alpha)\right)=\operatorname{sgn}\left(T_{\lambda_{L}(\alpha)}^{\prime}(\alpha)\right)$ for $\alpha \in\left(0, \rho_{L, \varepsilon}\right)$.

Lemma 2.13 ([10, Lemma 3.5]). Consider (1.1). Let $L>0$. Then the following statements (i) and (ii) hold:
(i) If $\lambda_{L}(\alpha)$ has a local maximum at $\alpha_{M}$, then $T_{\lambda_{L}\left(\alpha_{M}\right)}(\alpha)$ has a local maximum at $\alpha_{M}$. Conversely, if $T_{\lambda}(\alpha)$ has a local maximum at $\alpha_{M}$ and $T_{\lambda}\left(\alpha_{M}\right)=L$, then $\lambda_{L}(\alpha)$ has a local maximum at $\alpha_{M}$.
(ii) If $\lambda_{L}(\alpha)$ has a local minimum at $\alpha_{m}$, then $T_{\lambda_{L}\left(\alpha_{m}\right)}(\alpha)$ has a local minimum at $\alpha_{m}$. Conversely, if $T_{\lambda}(\alpha)$ has a local minimum at $\alpha_{m}$ and $T_{\lambda}\left(\alpha_{m}\right)=L$, then $\lambda_{L}(\alpha)$ has a local minimum at $\alpha_{m}$.

Lemma 2.14. Consider (1.1) with $0<\varepsilon<\varepsilon_{0}$. Then there exists a continuous function $L_{\varepsilon} \in(0, \infty)$ of $\varepsilon$ such that

$$
\Lambda_{\varepsilon} \equiv\left\{L>0: \lambda_{L}^{\prime}(\alpha)<0 \text { for some } \alpha \in\left(0, \rho_{L, \varepsilon}\right)\right\}=\left(L_{\varepsilon}, \infty\right) .
$$

Furthermore, $\lambda_{L}^{\prime}(\alpha)>0$ for $\alpha \in\left(0, \rho_{L, \varepsilon}\right)$ where $0<L<L_{\varepsilon}$.
Proof. Let $\varepsilon \in\left(0, \varepsilon_{0}\right)$ be given. By Lemma 2.9 and similar argument in the proof of [7, Lemma 4.7], there exists $L_{\varepsilon} \in[0, \infty)$ such that $\Lambda_{\varepsilon}=\left(L_{\varepsilon}, \infty\right)$. We divide the rest of the proof into the next three steps.
Step 1. We prove that $L_{\varepsilon}>0$. Assume that $L_{\varepsilon}=0$. By Lemma 2.9, we have

$$
\begin{equation*}
T_{\lambda}^{\prime}(\alpha) \geq 0 \quad \text { for } 0<\alpha<\beta_{\varepsilon} \text { and } \lambda \geq \xi_{\varepsilon} . \tag{2.53}
\end{equation*}
$$

Let $L=T_{\xi_{\varepsilon}}(1)$. It implies that $L \in \Lambda_{\varepsilon}=(0, \infty)$. Then there exists $\alpha_{1} \in\left(0, \rho_{L, \varepsilon}\right)$ such that $\lambda_{L}^{\prime}\left(\alpha_{1}\right)<0$. It follows that $T_{\lambda_{L}\left(\alpha_{1}\right)}^{\prime}\left(\alpha_{1}\right)<0$ by Lemma 2.12(iii). By (2.45) and (2.53), we observe that $\alpha_{1}>1$ and $0<\lambda_{L}\left(\alpha_{1}\right)<\xi_{\varepsilon}$. By Lemmas 2.1(iii), 2.12(i) and (2.53), we further observe that

$$
L=T_{\lambda_{L}\left(\alpha_{1}\right)}\left(\alpha_{1}\right)>T_{\xi_{\varepsilon}}\left(\alpha_{1}\right) \geq T_{\xi_{\varepsilon}}(1)=L,
$$

which is a contradiction. Thus $L_{\varepsilon}>0$.
Step 2. We prove that $\lambda_{L}^{\prime}(\alpha)>0$ for $\alpha \in\left(0, \rho_{L, \varepsilon}\right)$ where $0<L<L_{\varepsilon}$. Let $L \in\left(0, L_{\varepsilon}\right)$ be given. Assume that there exists $\alpha_{2} \in\left(0, \rho_{L, \varepsilon}\right)$ such that $\lambda_{L}^{\prime}\left(\alpha_{2}\right)=0$. So by Lemma 2.12(iii), we obtain that $T_{\lambda_{L}\left(\alpha_{2}\right)}^{\prime}\left(\alpha_{2}\right)=0$. Since

$$
0<\alpha_{2}<\rho_{L, \varepsilon}=\min \left\{L, \beta_{\varepsilon}\right\}<\min \left\{L_{\varepsilon}, \beta_{\varepsilon}\right\}=\rho_{L_{\varepsilon}, \varepsilon}
$$

we see that $T_{\lambda_{L}\left(\alpha_{2}\right)}\left(\alpha_{2}\right)=L<L_{\varepsilon}=T_{\lambda_{L_{\varepsilon}}\left(\alpha_{2}\right)}\left(\alpha_{2}\right)$. So by Lemma 2.1(iii), we obtain that $\lambda_{L}\left(\alpha_{2}\right)>$ $\lambda_{L_{\varepsilon}}\left(\alpha_{2}\right)$. Assume that $\alpha_{2} \geq \frac{5 \varepsilon}{12}$. Since $T_{\lambda_{L}\left(\alpha_{2}\right)}^{\prime}\left(\alpha_{2}\right)=0$, and by Lemma 2.2(ii), $T_{\lambda_{L}\left(\alpha_{2}\right)}(\alpha)$ has a local minimum at $\alpha_{2}$. By Lemma 2.13, we find that $\lambda_{L}(\alpha)$ has a local minimum at $\alpha_{2}$, which is a contradiction since $L<L_{\varepsilon}$. So $0<\alpha_{2}<\frac{5 \varepsilon}{12}$. By Lemma 2.3, we see that

$$
\sqrt{\lambda_{L_{\varepsilon}}\left(\alpha_{2}\right)} T_{\lambda_{L_{\varepsilon}}\left(\alpha_{2}\right)}^{\prime}\left(\alpha_{2}\right)<\sqrt{\lambda_{L}\left(\alpha_{2}\right)} T_{\lambda_{L}\left(\alpha_{2}\right)}^{\prime}\left(\alpha_{2}\right)=0,
$$

from which it follows that by Lemma 2.12(iii), $\lambda_{L_{\varepsilon}}^{\prime}\left(\alpha_{2}\right)<0$. It is a contradiction since $\lambda_{L_{\varepsilon}}^{\prime}(\alpha) \geq$ 0 for $\alpha \in\left(0, \rho_{L, \varepsilon}\right)$. Thus $\lambda_{L}^{\prime}(\alpha)>0$ for $\alpha \in\left(0, \rho_{L, \varepsilon}\right)$ where $0<L<L_{\varepsilon}$.
Step 3. We prove the continuity of $L_{\varepsilon}$. Let $\bar{\varepsilon} \in\left(0, \varepsilon_{0}\right)$ be given. For the sake of convenience, we let $T_{\lambda}(\alpha, \varepsilon)=T_{\lambda}(\alpha)$ and $\lambda_{L}(\alpha, \varepsilon)=\lambda_{L}(\alpha)$. We consider the following two cases and prove they would not occur.

Case 1. Assume that $\liminf _{\varepsilon \rightarrow \bar{\varepsilon}} L_{\varepsilon}<L_{\bar{\varepsilon}}$. Let $L \in\left(\liminf _{\varepsilon \rightarrow \bar{\varepsilon}} L_{\varepsilon}, L_{\bar{\varepsilon}}\right)$ be given. Then there exists $\left\{\varepsilon_{n}\right\}_{n \in \mathbb{N}} \subset\left(0, \varepsilon_{0}\right)$ such that

$$
\lim _{n \rightarrow \infty} \varepsilon_{n}=\bar{\varepsilon} \text { and } L_{\varepsilon_{n}}<L<L_{\bar{\varepsilon}} \text { for } n \in \mathbb{N} .
$$

So there exists $\left\{\alpha_{n}\right\}_{n \in \mathbb{N}} \subset\left(0, \rho_{L, \varepsilon_{n}}\right)$ such that

$$
\begin{equation*}
\frac{\partial}{\partial \alpha} \lambda_{L}(\alpha, \bar{\varepsilon})>0 \quad \text { for } 0<\alpha<\rho_{L, \varepsilon} \quad \text { and } \quad \frac{\partial}{\partial \alpha} \lambda_{L}\left(\alpha_{n}, \varepsilon_{n}\right)<0 \quad \text { for } n \in \mathbb{N} . \tag{2.54}
\end{equation*}
$$

By Lemmas 2.2(i) and 2.12(iii), we have

$$
\begin{equation*}
\frac{\partial}{\partial \alpha} \lambda_{L}(\alpha, \varepsilon)>0 \quad \text { for } 0<\alpha \leq 1 \text { and } 0<\varepsilon<\varepsilon_{0} . \tag{2.55}
\end{equation*}
$$

By (2.54) and (2.55), we see that $\alpha_{n} \in\left(1, \rho_{L, \varepsilon_{n}}\right)$. We assume without loss of generality that $\lim _{n \rightarrow \infty} \alpha_{n}=\bar{\alpha} \in\left[1, \rho_{L, \varepsilon_{n}}\right]$. If $\bar{\alpha}<\rho_{L, \varepsilon_{n}}$, by (2.54), we observe that

$$
0<\frac{\partial}{\partial \alpha} \lambda_{L}(\bar{\alpha}, \bar{\varepsilon})=\lim _{n \rightarrow \infty} \frac{\partial}{\partial \alpha} \lambda_{L}\left(\alpha_{n}, \varepsilon_{n}\right) \leq 0,
$$

which is a contradiction. If $\bar{\alpha}=\rho_{L, \varepsilon_{n}}$, by (2.54) and Lemma 2.12(ii), we observe that

$$
\lim _{\alpha \rightarrow \rho_{L, \varepsilon}^{\bar{\varepsilon}}} \lambda_{L}(\alpha, \bar{\varepsilon})=\infty \quad \text { and } \quad \lim _{\alpha \rightarrow \rho_{L, \varepsilon}^{\bar{\varepsilon}}} \frac{\partial}{\partial \alpha} \lambda_{L}(\alpha, \bar{\varepsilon}) \leq 0,
$$

which is a contradiction.
Case 2. Assume that $\lim \sup _{\varepsilon \rightarrow \bar{\varepsilon}} L_{\varepsilon}>L_{\bar{\varepsilon}}$. Let $L \in\left(L_{\bar{\varepsilon}} \lim _{\sup }^{\varepsilon \rightarrow \bar{\varepsilon}}, L_{\varepsilon}\right)$ be given. Then there exists $\left\{\varepsilon_{n}\right\}_{n \in \mathbb{N}} \subset\left(0, \varepsilon_{0}\right)$ such that

$$
\lim _{n \rightarrow \infty} \varepsilon_{n}=\bar{\varepsilon} \text { and } L_{\bar{\varepsilon}}<L<L_{\varepsilon_{n}} \text { for } n \in \mathbb{N} .
$$

So there exists $\bar{\alpha} \in\left(0, \rho_{L, \bar{\varepsilon}}\right)$ such that

$$
\begin{equation*}
\frac{\partial}{\partial \alpha} \lambda_{L}(\bar{\alpha}, \bar{\varepsilon})<0 \quad \text { and } \quad \frac{\partial}{\partial \alpha} \lambda_{L}\left(\alpha, \varepsilon_{n}\right)>0 \quad \text { for } 0<\alpha<\rho_{L, \varepsilon_{n}} \text { and } n \in \mathbb{N} . \tag{2.56}
\end{equation*}
$$

Since $f\left(\beta_{\varepsilon}\right)=0$, and by implicit function theorem, $\beta_{\varepsilon}$ is a strictly decreasing and continuous function of $\varepsilon>0$. So we see that $\bar{\alpha}<\rho_{L, \bar{\varepsilon}} \leq \beta_{\bar{\varepsilon}}<\beta_{\varepsilon_{n}}$ for $n \in \mathbb{N}$. It implies that $0<\bar{\alpha}<\rho_{L, \varepsilon_{n}}$ for $n \in \mathbb{N}$. By (2.56), we observe that

$$
0>\frac{\partial}{\partial \alpha} \lambda_{L}(\bar{\alpha}, \bar{\varepsilon})=\lim _{n \rightarrow \infty} \frac{\partial}{\partial \alpha} \lambda_{L}\left(\bar{\alpha}, \varepsilon_{n}\right) \geq 0,
$$

which is a contradiction.
So by Cases 1 and 2, we see that $\limsup _{\varepsilon \rightarrow \bar{\varepsilon}} L_{\varepsilon} \leq L_{\bar{\varepsilon}} \leq \liminf _{\varepsilon \rightarrow \bar{\varepsilon}} L_{\varepsilon}$. It follows that $L_{\bar{\varepsilon}}=$ $\lim _{a \rightarrow \bar{a}} L_{\varepsilon}$. Thus $L_{\varepsilon}$ is a continuous function on $\left(0, \varepsilon_{0}\right)$.

The proof is complete.
Lemma 2.15. Consider (1.1) with $0<\varepsilon<\varepsilon_{0}$. Then there exists $\tilde{L}_{\varepsilon}>L_{\varepsilon}$ such that $\lambda_{L}(\alpha)$ has exactly one local maximum and exactly one local minimum on $\left(0, \rho_{L, \varepsilon}\right)$ for $L>\tilde{L}_{\varepsilon}$.

Proof. Let $\lambda_{*} \in\left(0, \kappa_{\varepsilon}\right)$ be given. By Lemma 2.10, then

$$
T_{\lambda}^{\prime}(\alpha)\left\{\begin{array}{ll}
>0 & \text { for } \alpha \in\left(0, \alpha_{M}(\lambda)\right) \cup\left(\alpha_{m}(\lambda), \beta_{\varepsilon}\right),  \tag{2.57}\\
=0 & \text { for } \alpha=\alpha_{M}(\lambda) \text { or } \alpha=\alpha_{m}(\lambda), \\
<0 & \text { for } \alpha \in\left(\alpha_{M}(\lambda), \alpha_{m}(\lambda)\right),
\end{array} \quad \text { if } 0<\lambda \leq \lambda^{*} .\right.
$$

Let $\tilde{L}_{\varepsilon} \equiv T_{\lambda^{*}}\left(\alpha_{M}\left(\lambda^{*}\right)\right)$. We divide this proof into the next three steps.
Step 1. We prove that $\tilde{L}_{\varepsilon}>L_{\varepsilon}$. Let $L \geq \tilde{L}_{\varepsilon}$ and

$$
\begin{equation*}
\alpha_{1} \in\left(\alpha_{M}\left(\lambda^{*}\right), \min \left\{\alpha_{m}\left(\lambda^{*}\right), \frac{5}{12 \varepsilon}\right\}\right) \tag{2.58}
\end{equation*}
$$

By (2.57) and (2.58), we see that

$$
\lim _{\lambda \rightarrow 0^{+}} T_{\lambda}(\alpha)=\infty>L \geq T_{\lambda^{*}}\left(\alpha_{M}\left(\lambda^{*}\right)\right)>T_{\lambda^{*}}\left(\alpha_{1}\right) .
$$

So by Lemma 2.1(iii) and continuity of $T_{\lambda}(\alpha)$ with respect to $\lambda$, there exists $\lambda_{*} \in\left(0, \lambda^{*}\right)$ such that $L=T_{\lambda_{*}}\left(\alpha_{1}\right)$. Clearly, $\lambda_{*}=\lambda_{L}\left(\alpha_{1}\right)$ by Lemma 2.12(i). Then by (2.57), (2.58) and Lemma 2.3, we observe that

$$
\sqrt{\lambda_{*}} T_{\lambda_{L}\left(\alpha_{1}\right)}^{\prime}\left(\alpha_{1}\right)=\sqrt{\lambda_{*}} T_{\lambda_{*}}^{\prime}\left(\alpha_{1}\right)<\sqrt{\lambda^{*}} T_{\lambda^{*}}^{\prime}\left(\alpha_{1}\right)<0
$$

So by Lemma 2.12(iii), we obtain that $\lambda_{L}^{\prime}\left(\alpha_{1}\right)<0$. It implies that $L>L_{\varepsilon}$ by Lemma 2.14. Thus $\tilde{L}_{\varepsilon}>L_{\varepsilon}$.
Step 2. We prove that $\lambda_{L}(\alpha)$ has exactly one local maximum in $\left(0, \rho_{L, \varepsilon}\right)$ for $L>\tilde{L}_{\varepsilon}$. Let $L>\tilde{L}_{\varepsilon}$ be given. By Lemmas 2.2(i) and 2.12(iii), we see that $\lambda_{L}^{\prime}(\alpha)>0$ for $0<\alpha \leq 1$. Since
$L>\tilde{L}_{\varepsilon}$, and by Lemma 2.14, $\lambda_{L}(\alpha)$ has at least one local maximum in $\left(0, \rho_{L, \varepsilon}\right)$. Assume that $\lambda_{L}(\alpha)$ has two local maximums at $\alpha_{M}^{1}$ and $\alpha_{M}^{2}\left(>\alpha_{M}^{1}\right)$. Then $\lambda_{L}(\alpha)$ has a local minimum at $\alpha_{m} \in\left(\alpha_{M}^{1}, \alpha_{M}^{2}\right)$. Without loss of generality, we assume that $\lambda_{L}\left(\alpha_{M}^{1}\right)>\lambda_{L}\left(\alpha_{m}\right)$. For the sake of convenience, we let

$$
\lambda_{1}=\lambda_{L}\left(\alpha_{M}^{1}\right), \quad \lambda_{2}=\lambda_{L}\left(\alpha_{M}^{2}\right) \quad \text { and } \quad \lambda_{3}=\lambda_{L}\left(\alpha_{m}\right)
$$

So by Lemma 2.13, we see that $T_{\lambda_{1}}\left(\alpha_{M}^{1}\right)$ and $T_{\lambda_{2}}\left(\alpha_{M}^{2}\right)$ are local maximum values and $T_{\lambda_{3}}\left(\alpha_{m}\right)$ is a local minimum value. In addition, we note that

$$
\begin{equation*}
T_{\lambda_{1}}\left(\alpha_{M}^{1}\right)=T_{\lambda_{L}\left(\alpha_{M}^{1}\right)}\left(\alpha_{M}^{1}\right)=L>\tilde{L}_{\varepsilon}=T_{\lambda^{*}}\left(\alpha_{M}\left(\lambda^{*}\right)\right) \tag{2.59}
\end{equation*}
$$

Assume that $\lambda_{1} \geq \lambda^{*}$. By Lemma 2.1(iii) and (2.59), we observe that $T_{\lambda^{*}}\left(\alpha_{M}^{1}\right) \geq T_{\lambda_{1}}\left(\alpha_{M}^{1}\right)>$ $T_{\lambda^{*}}\left(\alpha_{M}\left(\lambda^{*}\right)\right)$. It implies that

$$
\begin{equation*}
\alpha_{m}\left(\lambda^{*}\right)<\alpha_{M}^{1} \quad \text { and } \quad T_{\lambda^{*}}^{\prime}\left(\alpha_{M}^{1}\right)>0 \tag{2.60}
\end{equation*}
$$

By Lemma 2.2(ii), we have $\alpha_{M}^{1}<\alpha_{M}^{2}<\frac{5}{12 \varepsilon}$. So by Lemma 2.3 and (2.60), we observe hat

$$
0<\sqrt{\lambda^{*}} T_{\lambda^{*}}^{\prime}\left(\alpha_{M}^{1}\right) \leq \sqrt{\lambda_{1}} T_{\lambda_{1}}^{\prime}\left(\alpha_{M}^{1}\right)=0
$$

which is a contradiction. So $\lambda_{1}<\lambda^{*}$. Similarly, we obtain that $\lambda_{2}<\lambda^{*}$. So by (2.57) and Lemma 2.10, we see that

$$
\alpha_{M}\left(\lambda_{1}\right)=\alpha_{M}^{1}<\alpha_{m}=\alpha_{m}\left(\lambda_{3}\right)<\alpha_{M}^{2}=\alpha_{M}\left(\lambda_{2}\right)
$$

which is a contradiction by Lemma 2.11. Thus $\lambda_{L}(\alpha)$ has exactly one local maximum in $\left(0, \rho_{L, \varepsilon}\right)$.

Step 3. We prove Lemma 2.15. Since $\lambda_{L}^{\prime}(\alpha)>0$ for $0<\alpha \leq 1$, and by Lemma 2.12(ii) and Step 2 , we see that $\lambda_{L}(\alpha)$ has exactly one local maximum and one local minimum on $\left(0, \rho_{L, \varepsilon}\right)$ for $L>\tilde{L}_{\varepsilon}$.

The proof is complete.

## 3 Proof of the main result

Proof of Theorem 1.3. (I) The statement (i) follows immediately by Lemma 2.12(i)(ii).
(II) Assume that $\varepsilon \geq \varepsilon_{0}$. By Theorem 1.2 and (2.4), we obtain that $\bar{T}^{\prime}(\alpha) \geq 0$ for $0<\alpha<\beta_{\varepsilon}$. So by Lemmas 2.1(ii) and 2.3, we see that

$$
\begin{equation*}
0 \leq \bar{T}^{\prime}(\alpha)=\lim _{\lambda \rightarrow 0^{+}} \sqrt{\lambda} T_{\lambda}^{\prime}(\alpha)<\sqrt{\lambda} T_{\lambda}^{\prime}(\alpha) \quad \text { for } 0<\alpha \leq \frac{5}{12 \varepsilon} \text { and } \lambda>0 \tag{3.1}
\end{equation*}
$$

Since $T_{\lambda}^{\prime}\left(\frac{5}{12 \varepsilon}\right)>0$ for $\lambda>0$, and by Lemma 2.2(ii), we further see that

$$
\begin{equation*}
T_{\lambda}^{\prime}(\alpha)>0 \quad \text { for } \frac{5}{12 \varepsilon}<\alpha<\beta_{\varepsilon} \text { and } \lambda>0 \tag{3.2}
\end{equation*}
$$

So by (3.1), (3.2) and Lemma 2.12(iii), we obtain that

$$
\lambda_{L}^{\prime}(\alpha)>0 \quad \text { for } 0<\alpha<\rho_{L, \varepsilon} \text { and } \lambda>0
$$

Then the statement (ii) holds.
(III) Assume that $0<\varepsilon<\varepsilon_{0}$. By Lemma 2.14, there exists a continuous function $L_{\varepsilon} \in(0, \infty)$ of $\varepsilon$ such that

$$
\Lambda_{\varepsilon}=\left\{L>0: \lambda_{L}^{\prime}(\alpha)<0 \text { for some } \alpha \in\left(0, \rho_{L, \varepsilon}\right)\right\}=\left(L_{\varepsilon}, \infty\right) .
$$

So by Lemma 2.12(i), the bifurcation curve $S_{L}$ is monotone increasing if $0<L \leq L_{\varepsilon}$, and is S-like shaped if $L>L_{\varepsilon}$. In addition, by Lemma 2.15, there exists $\tilde{L}_{\varepsilon}>L_{\varepsilon}$ such that $\lambda_{L}(\alpha)$ has one local maximum and one local minimum on $\left(0, \rho_{L, \varepsilon}\right)$ for $L>\tilde{L}_{\varepsilon}$. So by Lemma 2.12(i), the bifurcation curve $S_{L}$ is S-shaped if $L>\tilde{L}_{\varepsilon}$. Next, we divide into the next two steps to prove that $\lim _{\varepsilon \rightarrow 0^{+}} L_{\varepsilon} \in(0, \infty)$ and $\lim _{\varepsilon \rightarrow \varepsilon_{0}^{-}} L_{\varepsilon}=\infty$.
Step 1. We prove that $\lim _{\varepsilon \rightarrow \varepsilon_{0}^{-}} L_{\varepsilon}=\infty$. Assume that $\lim _{\varepsilon \rightarrow \varepsilon_{0}^{-}} L_{\varepsilon}<\infty$. Let $L>\lim _{\varepsilon \rightarrow \varepsilon_{0}^{-}} L_{\varepsilon}$. For the sake of convenience, we let

$$
\lambda_{L}(\alpha, \varepsilon)=\lambda_{L}(\alpha), \quad T_{\lambda}(\alpha, \varepsilon)=T_{\lambda}(\alpha) \quad \text { and } \quad \bar{T}(\alpha, \varepsilon)=\bar{T}(\alpha) .
$$

Since $L>\lim _{\varepsilon \rightarrow \varepsilon_{0}^{-}} L_{\varepsilon}$, there exists $\delta>0$ such that $L>L_{\varepsilon}$ for $\varepsilon \in\left(\varepsilon_{0}-\delta, \varepsilon_{0}\right)$. So for $\varepsilon \in$ $\left(\varepsilon_{0}-\delta, \varepsilon_{0}\right)$, by Lemmas 2.2(ii) and 2.14, there exists $\alpha_{\varepsilon} \in\left[1, \frac{5}{12 \varepsilon}\right]$ such that $\frac{\partial}{\partial \alpha} \lambda_{L}\left(\alpha_{\varepsilon}, \varepsilon\right)<0$. Without loss of generality, we assume that $\lim _{\varepsilon \rightarrow \varepsilon_{0}^{+}} \alpha_{\varepsilon}=\alpha_{0} \in\left[1, \frac{5}{12 \varepsilon}\right]$. By Theorem 1.2 and (2.4), we see that $\bar{T}^{\prime}\left(\alpha_{0}, \varepsilon_{0}\right) \geq 0$. So by Lemma 2.3, we further see that

$$
0 \leq \bar{T}^{\prime}\left(\alpha_{0}, \varepsilon_{0}\right)=\lim _{\lambda \rightarrow 0^{+}} \sqrt{\lambda} T_{\lambda}^{\prime}\left(\alpha_{0}, \varepsilon_{0}\right)<\sqrt{\lambda} T_{\lambda}^{\prime}\left(\alpha_{0}, \varepsilon_{0}\right) \quad \text { for } \lambda>0
$$

Then by Lemma 2.12(iii), we obtain that $\frac{\partial}{\partial \alpha} \lambda_{L}\left(\alpha_{0}, \varepsilon_{0}\right)>0$. It follows that

$$
0 \geq \lim _{\varepsilon \rightarrow \varepsilon_{0}^{+}} \frac{\partial}{\partial \alpha} \lambda_{L}\left(\alpha_{\varepsilon}, \varepsilon\right)=\frac{\partial}{\partial \alpha} \lambda_{L}\left(\alpha_{0}, \varepsilon_{0}\right)>0,
$$

which is a contradiction. So $\lim _{\varepsilon \rightarrow \varepsilon_{0}^{-}} L_{\varepsilon}=\infty$.
Step 2. We prove that $\lim _{\varepsilon \rightarrow 0^{+}} L_{\varepsilon} \in(0, \infty)$. Notice that as $\varepsilon \rightarrow 0^{+}$, the cubic polynomial $f(u)$ reduces to the quadratic polynomial $u^{2}+u+1$. So we consider the equation

$$
\left\{\begin{array}{l}
-\left(\frac{u^{\prime}(x)}{\sqrt{1-\left(u^{\prime}(x)\right)^{2}}}\right)^{\prime}=\lambda\left(u^{2}+u+1\right), \quad-L<x<L,  \tag{3.3}\\
u(-L)=u(L)=0 .
\end{array}\right.
$$

Since $u^{2}+u+1$ satisfies all hypotheses of [7, Theorem 3.2], there exists $L_{0}>0$ such that the bifurcation curve $S_{L}$ of (3.3) is S-like shaped for $L>L_{0}$, monotone increasing for $0<L \leq L_{0}$, and has no vertical tangent lines for $0<L<L_{0}$. Thus we have the following assertions (i)-(iii):
(i) if $L>L_{0}$, then $\lambda_{L}^{\prime}(\alpha, 0)<0$ for some $\alpha>0$.
(ii) if $L=L_{0}$, then $\lambda_{L}^{\prime}(\alpha, 0) \geq 0$ for $\alpha>0$.
(iii) if $0<L<L_{0}$, then $\lambda_{L}^{\prime}(\alpha, 0)>0$ for $\alpha>0$.

By a similar argument as in the proof of Lemma 2.14, we can prove that $L_{\varepsilon}$ is a continuous function of $\varepsilon \in\left[0, \varepsilon_{0}\right)$. Thus $\lim _{\varepsilon \rightarrow 0^{+}} L_{\varepsilon}=L_{0} \in(0, \infty)$.

The proof is complete.

## 4 Appendix

In this section, we prove assertion (2.31). Let $\bar{\varepsilon}=\sqrt{\frac{31}{1000}}(\approx 0.176)$. By Lemma 2.5(iii), we have $\hat{\varepsilon}<\bar{\varepsilon}$. To prove (2.31), it is sufficient to prove that

$$
\begin{equation*}
N_{2}(\alpha, u)<0 \quad \text { for } \quad 0<u<\alpha, 1.7 \leq \alpha \leq \frac{1}{3 \varepsilon} \quad \text { and } \quad 0.07 \leq \varepsilon \leq \bar{\varepsilon}(\approx 0.176) \tag{4.1}
\end{equation*}
$$

Let $\alpha \in\left[1.7, \frac{1}{3 \varepsilon}\right]$ be given and $N_{2}(u)=N_{2}(\alpha, u)$. It is easy to compute that

$$
\begin{aligned}
N_{2}^{\prime}(u)= & -\frac{1}{2} \varepsilon^{2} u^{7}+\frac{49}{24} \varepsilon u^{6}+\left(\frac{21}{4} \varepsilon-\frac{2}{3}\right) u^{5}+\left(\frac{125}{8} \varepsilon-\frac{25}{12}\right) u^{4}+\left(\frac{1}{2} \varepsilon^{2} \alpha^{4}-\frac{7}{6} \varepsilon \alpha^{3}\right. \\
& \left.-\frac{7}{2} \varepsilon \alpha^{2}-\frac{25}{2} \varepsilon \alpha-\frac{20}{3}\right) u^{3}+\left(-\frac{7}{8} \varepsilon \alpha^{4}+\frac{2}{3} \alpha^{3}+\frac{5}{4} \alpha^{2}+5 \alpha+\frac{3}{4}\right) u^{2} \\
& +\left(-\frac{7}{4} \varepsilon \alpha^{4}+\frac{5}{6} \alpha^{3}-\frac{1}{2} \alpha+4\right) u-\frac{25}{8} \varepsilon \alpha^{4}+\frac{5}{3} \alpha^{3}-\frac{1}{4} \alpha^{2}-4 \alpha, \\
N_{2}^{\prime \prime}(u)=- & \frac{7}{2} \varepsilon^{2} u^{6}+\frac{49}{4} \varepsilon u^{5}+\left(\frac{105}{4} \varepsilon-\frac{10}{3}\right) u^{4}+\left(\frac{125}{2} \varepsilon-\frac{25}{3}\right) u^{3}+\left(\frac{3}{2} \varepsilon^{2} \alpha^{4}-\frac{7}{2} \varepsilon \alpha^{3}\right. \\
& \left.-\frac{21}{2} \varepsilon \alpha^{2}-\frac{75}{2} \varepsilon \alpha-20\right) u^{2}+\left(-\frac{7}{4} \varepsilon \alpha^{4}+\frac{4}{3} \alpha^{3}+\frac{5}{2} \alpha^{2}+10 \alpha+\frac{3}{2}\right) u \\
& -\frac{7}{4} \varepsilon \alpha^{4}+\frac{5}{6} \alpha^{3}-\frac{1}{2} \alpha+4, \\
N_{2}^{\prime \prime \prime}(u)= & -21 \varepsilon^{2} u^{5}+\frac{245}{4} \varepsilon u^{4}+\left(105 \varepsilon-\frac{40}{3}\right) u^{3}+\left(\frac{375}{2} \varepsilon-25\right) u^{2}+\left(3 \varepsilon^{2} \alpha^{4}-7 \varepsilon \alpha^{3}\right. \\
& \left.-21 \varepsilon \alpha^{2}-75 \varepsilon \alpha-40\right) u-\frac{7}{4} \varepsilon \alpha^{4}+\frac{4}{3} \alpha^{3}+\frac{5}{2} \alpha^{2}+10 \alpha+\frac{3}{2}, \\
& \quad-21 \varepsilon \alpha^{2}-75 \varepsilon \alpha-40, \\
N_{2}^{(4)}(u)= & -105 \varepsilon^{2} u^{4}+245 \varepsilon u^{3}+(315 \varepsilon-40) u^{2}+(375 \varepsilon-50) u+3 \varepsilon^{2} \alpha^{4}-7 \varepsilon \alpha^{3} \\
& N_{2}^{(5)}(u)=-420 \varepsilon^{2} u^{3}+735 \varepsilon u^{2}+(630 \varepsilon-80) u+375 \varepsilon-50, \\
& N_{2}^{(6)}(u)=-1260 \varepsilon^{2} u^{2}+1470 \varepsilon u+630 \varepsilon-80 .
\end{aligned}
$$

Then we divide the proof into the next four steps.
Step 1. We prove that, for $0.07 \leq \varepsilon \leq \bar{\varepsilon}$,

$$
\begin{equation*}
N_{2}^{\prime \prime}(0)=-\frac{7}{4} \varepsilon \alpha^{4}+\frac{5}{6} \alpha^{3}-\frac{1}{2} \alpha+4>0 \tag{4.2}
\end{equation*}
$$

It is easy to see that

$$
\begin{equation*}
1.7 \leq \alpha \leq \frac{1}{3 \varepsilon} \leq \frac{1}{3(0.07)}=\frac{100}{21} \text { for } 0.07 \leq \varepsilon \leq \bar{\varepsilon} \tag{4.3}
\end{equation*}
$$

Since $\varepsilon \leq \frac{1}{3 \alpha}$, and by (4.3), we observe that

$$
\begin{aligned}
N_{2}^{\prime \prime}(0) & \geq-\frac{7}{4}\left(\frac{1}{3 \alpha}\right) \alpha^{4}+\frac{5}{6} \alpha^{3}-\frac{1}{2} \alpha+4=\frac{1}{4}\left(\alpha^{3}-2 \alpha+16\right) \\
& >\frac{1}{4}\left[(1.7)^{3}-2\left(\frac{100}{21}\right)+16\right]=\frac{239173}{84000}>0 .
\end{aligned}
$$

Step 2. We prove that, for $0.07 \leq \varepsilon \leq \bar{\varepsilon}$,

$$
\begin{equation*}
N_{2}^{\prime \prime}(\alpha)=-2 \alpha^{6} \varepsilon^{2}+\alpha^{3}\left(7 \alpha^{2}+14 \alpha+25\right) \varepsilon-2 \alpha^{4}-5 \alpha^{3}-10 \alpha^{2}+\alpha+4<0 . \tag{4.4}
\end{equation*}
$$

Clearly,

$$
\left\{(\alpha, \varepsilon): 1.7 \leq \alpha \leq \frac{1}{3 \varepsilon} \text { and } 0.07 \leq \varepsilon \leq \bar{\varepsilon}\right\}=\Omega_{1} \cup \Omega_{2}
$$

where

$$
\begin{align*}
& \Omega_{1} \equiv\left\{(\alpha, \varepsilon): 1.7 \leq \alpha \leq \frac{1}{3 \bar{\varepsilon}} \text { and } 0.07 \leq \varepsilon \leq \bar{\varepsilon}\right\}  \tag{4.5}\\
& \Omega_{2} \equiv\left\{(\alpha, \varepsilon): \frac{1}{3 \bar{\varepsilon}} \leq \alpha \leq \frac{1}{3 \varepsilon} \text { and } 0.07 \leq \varepsilon \leq \bar{\varepsilon}\right\}, \tag{4.6}
\end{align*}
$$

see Figure 4.1. So we consider the following two cases.


Figure 4.1: The sets $\Omega_{1}$ and $\Omega_{2} .0$
Case 1. Assume that $(\alpha, \varepsilon) \in \Omega_{1}$. It implies that

$$
\begin{equation*}
1.7 \leq \alpha \leq \frac{1}{3 \bar{\varepsilon}}(\approx 1.893)<1.9 . \tag{4.7}
\end{equation*}
$$

So we observe that

$$
\begin{aligned}
\frac{\partial}{\partial \varepsilon} N_{2}^{\prime \prime}(\alpha) & =-4 \varepsilon \alpha^{6}+7 \alpha^{5}+14 \alpha^{4}+25 \alpha^{3}>-4 \bar{\varepsilon} \alpha^{6}+7 \alpha^{5}+14 \alpha^{4}+25 \alpha^{3} \\
& >-4 \bar{\varepsilon}(1.9)^{6}+7(1.7)^{5}+14(1.7)^{4}+25(1.7)^{3} \\
& =\frac{33914439}{10^{5}}-\frac{47045881}{25 \times 10^{6}} \sqrt{310}(\approx 306.01)>0 .
\end{aligned}
$$

Then by (4.7),

$$
\begin{aligned}
N_{2}^{\prime \prime}(\alpha)< & \left.N_{2}^{\prime \prime}(\alpha)\right|_{\varepsilon=\bar{\varepsilon}} \\
= & -\frac{31}{500} \alpha^{6}+\frac{7}{10} \sqrt{\frac{31}{10}} \alpha^{5}+\left(\frac{7}{5} \sqrt{\frac{31}{10}}-2\right) \alpha^{4}+\left(\frac{5}{2} \sqrt{\frac{31}{10}}-5\right) \alpha^{3} \\
& -10 \alpha^{2}+\alpha+4 \\
< & 0
\end{aligned}
$$

see Figure 4.2(i).
Case 2. Assume that $(\alpha, \varepsilon) \in \Omega_{2}$. It implies that

$$
(\alpha, \varepsilon) \in \Omega_{2}=\left\{(\alpha, \varepsilon): \frac{1}{3 \bar{\varepsilon}} \leq \alpha \leq \frac{1}{0.21} \text { and } 0<\varepsilon<\frac{1}{3 \alpha}\right\}
$$

Then we observe that

$$
\begin{align*}
\frac{\partial}{\partial \varepsilon} N_{2}^{\prime \prime}(\alpha) & =-4 \alpha^{6} \varepsilon+\alpha^{3}\left(7 \alpha^{2}+14 \alpha+25\right)>-4 \alpha^{6}\left(\frac{1}{3 \alpha}\right)+\alpha^{3}\left(7 \alpha^{2}+14 \alpha+25\right) \\
& =\frac{1}{3}\left(17 \alpha^{2}+75+42 \alpha\right) \alpha^{3}>0 \tag{4.8}
\end{align*}
$$

Since

$$
\begin{equation*}
1.8<(1.89 \approx) \frac{1}{3 \bar{\varepsilon}} \leq \alpha \leq \frac{1}{0.21}<5 \tag{4.9}
\end{equation*}
$$

and by (4.8), we observe that

$$
N_{2}^{\prime \prime}(\alpha)<\left.N_{2}^{\prime \prime}(\alpha)\right|_{\varepsilon=\frac{1}{3 \alpha}}=\frac{1}{9}\left(\alpha^{2}-3\right)\left(\alpha^{2}-3 \alpha-12\right)<0
$$

see Figure 4.2(ii).
Thus (4.4) holds by Cases 1-2.

(i)

(ii)

Figure 4.2: (i) The graph of $-\frac{31}{500} \alpha^{6}+\frac{7}{10} \sqrt{\frac{31}{10}} \alpha^{5}+\left(\frac{7}{5} \sqrt{\frac{31}{10}}-2\right) \alpha^{4}+\left(\frac{5}{2} \sqrt{\frac{31}{10}}-5\right) \alpha^{3}$ $-10 \alpha^{2}+\alpha+4$ on $[1.7,9]$. (ii) The graph of $\left(\alpha^{2}-3\right)\left(\alpha^{2}-3 \alpha-12\right)$ on $[1.8,5]$.

Step 3. We prove that, for $0.07 \leq \varepsilon \leq \bar{\varepsilon}$,
$N_{2}^{\prime \prime}(u)$ is strictly increasing, or strictly increasing-decreasing, or strictly increasing-decreasing-increasing on $(0, \alpha)$.

Clearly, $N_{2}^{(6)}(u)$ is a quadratic polynomial of $u$ with negative leading coefficient. Since, for $\varepsilon>0$,

$$
N_{2}^{(6)}(0)=630 \varepsilon-80\left\{\begin{array}{ll}
<0 & \text { if } 0.07 \leq \varepsilon<\frac{8}{63}, \\
\geq 0 & \text { if } \frac{8}{63} \leq \varepsilon \leq \bar{\varepsilon},
\end{array} \quad \text { and } \quad N_{2}^{(6)}\left(\frac{1}{3 \varepsilon}\right)=90(7 \varepsilon+3)>0,\right.
$$

we see that

$$
\left\{\begin{array}{l}
N_{2}^{(5)}(u) \text { is strictly decreasing-increasing on }(0, \alpha) \text { if } 0.07 \leq \varepsilon<\frac{8}{63},  \tag{4.11}\\
N_{2}^{(5)}(u) \text { is strictly increasing on }(0, \alpha) \text { if }(0.126 \approx) \frac{8}{63} \leq \varepsilon \leq \bar{\varepsilon} .
\end{array}\right.
$$

In addition, we compute and find that

$$
\begin{gather*}
N_{2}^{(5)}(0)=375 \varepsilon-50 \begin{cases}<0 & \text { for } 0.07 \leq \varepsilon<\frac{2}{15}(\approx 0.133), \\
\geq 0 & \text { for } \frac{2}{15} \leq \varepsilon \leq \bar{\varepsilon},\end{cases}  \tag{4.12}\\
N_{2}^{(5)}(1.7)=-\frac{103173}{50} \varepsilon^{2}+\frac{71403}{20} \varepsilon-186>0 \quad \text { for } 0.07 \leq \varepsilon \leq \bar{\varepsilon} . \tag{4.13}
\end{gather*}
$$

Since $0<u<\alpha$ and $1.7 \leq \alpha \leq \frac{1}{3 \varepsilon}$, and by (4.11)-(4.13), we obtain that

$$
\begin{equation*}
N_{2}^{(4)}(u) \text { is either strictly decreasing-increasing, or strictly increasing on }(0, \alpha) \text {. } \tag{4.14}
\end{equation*}
$$

Since $1.7 \leq \alpha \leq \frac{1}{3 \varepsilon}$ and $0.07 \leq \varepsilon \leq \bar{\varepsilon}$, we compute and find that

$$
\begin{align*}
N_{2}^{(4)}(0) & =3 \varepsilon^{2} \alpha^{4}-7 \varepsilon \alpha^{3}-21 \varepsilon \alpha^{2}-75 \varepsilon \alpha-40 \\
& <3 \varepsilon^{2}\left(\frac{1}{3 \varepsilon}\right)^{4}-7 \varepsilon(1.7)^{3}-21 \varepsilon(1.7)^{2}-75 \varepsilon(1.7)-40 \\
& =\frac{1}{27000 \varepsilon^{2}}\left(-6009687 \varepsilon^{3}-1080000 \varepsilon^{2}+1000\right) \\
& <\frac{1}{27000 \varepsilon^{2}}\left[-6009687(0.07)^{3}-1080000(0.07)^{2}+1000\right] \\
& =-\frac{6353322641}{27 \times 10^{9} \varepsilon^{2}}<0 . \tag{4.15}
\end{align*}
$$

So by (4.14) and (4.15), we obtain that

$$
\begin{equation*}
N_{2}^{\prime \prime \prime}(u) \text { is either strictly decreasing, or strictly decreasing-increasing on }(0, \alpha) \text {. } \tag{4.16}
\end{equation*}
$$

Since $0.07 \leq \varepsilon \leq \bar{\varepsilon}$, and by (4.3), we see that

$$
\begin{align*}
N_{2}^{\prime \prime \prime}(0) & =-\frac{7}{4} \varepsilon \alpha^{4}+\frac{4}{3} \alpha^{3}+\frac{5}{2} \alpha^{2}+10 \alpha+\frac{3}{2} \geq-\frac{7}{4} \hat{\varepsilon} \alpha^{4}+\frac{4}{3} \alpha^{3}+\frac{5}{2} \alpha^{2}+10 \alpha+\frac{3}{2} \\
& =\frac{1}{12}\left(-\frac{21}{10} \sqrt{\frac{31}{10}} \alpha^{4}+16 \alpha^{3}+30 \alpha^{2}+120 \alpha+18\right)>0, \tag{4.17}
\end{align*}
$$

see Figure 4.3. Then by (4.16) and (4.17), we obtain (4.10).
Step 4. We prove (4.1). By Steps 1-2 and (4.10), we obtain that

$$
\begin{equation*}
N_{2}^{\prime}(u) \text { is strictly increasing-decreasing on }(0, \alpha) \text {. } \tag{4.18}
\end{equation*}
$$



Figure 4.3: The graph of $-\frac{21}{10} \sqrt{\frac{31}{10}} \alpha^{4}+16 \alpha^{3}+30 \alpha^{2}+120 \alpha+18$ on [1.7,5].

Since $N_{2}^{\prime}(\alpha)=0$ for $1.7 \leq \alpha \leq \frac{1}{3 \varepsilon}$, and by (4.18), we obtain that
$N_{2}(u)$ is either strictly increasing, or strictly decreasing-increasing on $(0, \alpha)$.
We assert that

$$
\begin{equation*}
N_{2}(0)=-\frac{1}{16} \varepsilon^{2} \alpha^{8}+\frac{7}{24} \varepsilon \alpha^{7}+\frac{7}{8} \varepsilon \alpha^{6}+\frac{25}{8} \varepsilon \alpha^{5}-\frac{1}{9} \alpha^{6}-\frac{5}{12} \alpha^{5}-\frac{5}{3} \alpha^{4}+\frac{1}{4} \alpha^{3}+2 \alpha^{2} \leq 0 . \tag{4.20}
\end{equation*}
$$

Since $N_{2}(\alpha)=0$, and by (4.19) and (4.20), we see that (4.1) holds. Next, we prove assertion (4.20). Since $1.7 \leq \alpha \leq \frac{1}{3 \varepsilon}$ and $0.07 \leq \varepsilon \leq \bar{\varepsilon}$, we compute and find that

$$
\begin{align*}
\frac{\partial}{\partial \varepsilon} N_{2}(0) & =\left(-\frac{1}{8} \varepsilon \alpha^{3}+\frac{7}{24} \alpha^{2}+\frac{7}{8} \alpha+\frac{25}{8}\right) \alpha^{5} \\
& \geq\left[-\frac{1}{8} \varepsilon\left(\frac{1}{3 \varepsilon}\right)^{3}+\frac{7}{24}(1.7)^{2}+\frac{7}{8}(1.7)+\frac{25}{8}\right] \alpha^{5} \\
& =\frac{117837 \varepsilon^{2}-100}{21600 \varepsilon^{2}} \alpha^{5} \geq \frac{117837(0.07)^{2}-100}{21600 \varepsilon^{2}} \alpha^{5} \\
& =\frac{4774013}{216 \times 10^{6} \varepsilon^{2}} \alpha^{5}>0 . \tag{4.21}
\end{align*}
$$

Recall the sets $\Omega_{1}$ and $\Omega_{2}$ defined by (4.5) and (4.6) respectively, see Figure 4.1. Then we consider the following two cases.
Case 1. Assume that $(\alpha, \varepsilon) \in \Omega_{1}$. By (4.7) and (4.21), we see that

$$
N_{2}(0) \leq\left. N_{2}(0)\right|_{\varepsilon=\bar{\varepsilon}}=Q_{1}(\alpha)<0 \quad \text { for } 0.07 \leq \varepsilon \leq \bar{\varepsilon},
$$

where

$$
\begin{aligned}
Q_{1}(\alpha) \equiv & -\frac{31}{16000} \alpha^{8}+\frac{7}{240} \sqrt{\frac{31}{10}} \alpha^{7}+\left(\frac{7}{80} \sqrt{\frac{31}{10}}-\frac{1}{9}\right) \alpha^{6} \\
& +\left(\frac{5}{16} \sqrt{\frac{31}{10}}-\frac{5}{12}\right) \alpha^{5}-\frac{5}{3} \alpha^{4}+\frac{1}{4} \alpha^{3}+2 \alpha^{2},
\end{aligned}
$$

see Figure 4.4(i).

Case 2. Assume that $(\alpha, \varepsilon) \in \Omega_{2}$. By (4.9) and (4.21), we see that

$$
N_{2}(0) \leq\left. N_{2}(0)\right|_{\varepsilon=\frac{1}{3 \alpha}}=Q_{2}(\alpha)<0 \quad \text { for } \frac{1}{3 \bar{\varepsilon}} \leq \alpha \leq \frac{1}{0.21},
$$

where

$$
Q_{2}(\alpha) \equiv-\frac{1}{48} \alpha^{6}-\frac{1}{8} \alpha^{5}-\frac{5}{8} \alpha^{4}+\frac{1}{4} \alpha^{3}+2 \alpha^{2},
$$

see Figure 4.4(ii).

(i)

(ii)

Figure 4.4: (i) The graph of $Q_{1}(\alpha)$ on [1.7,1.9]. (ii) The graph of $Q_{2}(\alpha)$ on [1.8,5].
Thus, by Cases 1 and 2, assertion (4.20) holds. The proof is complete.

## References

[1] R. Bartnik, L. Simon, Spacelike hypersurfaces with prescribed boundary values and mean curvature, Comm. Math. Phys. 87(1982), 131-152. https://doi.org/10.1007/ BF01211061; MR0680653; Zbl 0512.53055
[2] M. G. Crandall, P. H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability Arch. Rational Mech. Anal. 52(1973), 161-180. https://doi.org/10. 1007/BF00282325; MR0341212; Zbl 0275.47044
[3] I. Coelho, C. Corsato, F. Obersnel, P. Omari, Positive solutions of the Dirichlet problem for the one-dimensional Minkowski-curvature equation Adv. Nonlinear Stud. 12(2012), 621-638. https://doi.org/10.1007/BF00282325; MR2976056; Zbl 1263.34028
[4] C. Corsato, Mathematical analysis of some differential models involving the Euclidean or the Minkowski mean curvature operator, PhD thesis, University of Trieste, 2015.
[5] R. P. Feynman, R. B. Leighton, M. Sands, The Feynman lectures on physics. Vol. 2: Mainly electromagnetism and matter, Addison-Wesley Publishing Co., Inc., Reading, Mass.London, 1964. MR0213078; Zbl 0131.38703
[6] K.-C. Hung, S.-H. Wang, Global bifurcation and exact multiplicity of positive solutions for a positone problem with cubic nonlinearity and their applications Trans. Amer. Math. Soc. 365(2013), 1933-1956. https://doi.org/10.1090/S0002-9947-2012-05670-4; MR3009649; Zbl 1282.34031
[7] S.-Y. Huang, Classification and evolution of bifurcation curves for the one-dimensional Minkowski-curvature problem and its applications J. Differential Equations 264(2018), 5977-6011. https://doi.org/10.1016/j.jde.2018.01.021; MR3765772; Zbl 1390.34051
[8] S.-Y. Huang, Exact multiplicity and bifurcation curves of positive solutions of a one-dimensional Minkowski-curvature problem and its application Commun. Pure Appl. Anal. 17(2018), 1271-1294. https://doi.org/10.3934/cpaa.2018061; MR3809123; Zbl 1398.34034
[9] S.-Y. Huang, Bifurcation diagrams of positive solutions for one-dimensional Minkowskicurvature problem and its applications Discrete Contin. Dyn. Syst. 39(2019), 3443-3462. https://doi.org/10.3934/dcds.2019142; MR3959436; Zbl 1419.34086
[10] S.-Y. Huang, Global bifurcation diagrams for Liouville-Bratu-Gelfand problem with Minkowski-curvature operator, J. Dynam. Differential Equations, accepted. https://doi. org/10.1007/s10884-021-09982-4
[11] S.-Y. Huang, S.-H. Wang, An evolutionary property of the bifurcation curves for a positone problem with cubic nonlinearity Taiwanese J. Math. 20(2016), 639-661. https: //doi.org/10.11650/tjm.20.2016.6563; MR3512001; Zbl 1383.34029
[12] S.-Y. Huang, S.-H. Wang, Proof of a conjecture for the one-dimensional perturbed Gelfand problem from combustion theory Arch. Ration. Mech. Anal. 222(2016), 769-825. https://doi.org/10.1007/s00205-016-1011-1; MR3544317; Zbl 1354.34041
[13] K.-C. Hung, S.-Y. Huang, S.-H. Wang, A global bifurcation theorem for a positone multiparameter problem and its application Discrete Contin. Dyn. Syst. 37(2017), 5127-5149. https://doi.org/10.3934/dcds.2017222; MR3668355; Zbl 1378.34041
[14] K.-C. Hung, S.-H. Wang, A theorem on S-shaped bifurcation curve for a positone problem with convex-concave nonlinearity and its applications to the perturbed Gelfand problem J. Differential Equations 251(2011), 223-237. https://doi.org/10.1016/j.jde.2011. 03.017; MR2800152; Zbl 1229.34037
[15] T. Laetsch, The number of solutions of a nonlinear two point boundary value problem Indiana Univ. Math. J. 20(1970), 1-13. https://doi.org/10.1512/iumj.1971.20.20001; MR0269922; Zbl 0215.14602
[16] R. MA, Y. Lu, Multiplicity of positive solutions for second order nonlinear Dirichlet problem with one-dimension Minkowski-curvature operator Adv. Nonlinear Stud. 15(2015), 789-803. https://doi.org/10.1515/ans-2015-0403; MR3405816; Zbl 1344.34041
[17] J. Shi, Multi-parameter bifurcation and applications, in: Topological methods, variational methods and their applications (Taiyuan, 2002), World Sci. Publ., River Edge, NJ, 2003, pp. 211-221. MR2011699; Zbl 1210.35020

# Notes on the linear equation with Stieltjes derivatives 

Ignacio Márquez Albés ${ }^{\boxtimes}$<br>Departamento de Estatística, Análise Matemática e Optimización, Facultade de Matemáticas, Universidade de Santiago de Compostela, Santiago de Compostela, Campus Vida, 15782<br>Instituto de Matemáticas, Universidade de Santiago de Compostela, Santiago de Compostela, Campus Vida, 15782

Received 17 November 2020, appeared 2 June 2021
Communicated by Josef Diblík


#### Abstract

In this paper we continue the study of the linear equation with Stieltjes derivatives in [M. Frigon, R. López Pouso, Adv. Nonlinear Anal. 6(2017), 13-36]. Specifically, we revisit some of the results there presented, removing some of the required conditions as well as amending some mistakes. Furthermore, following the classical setting, we use the connection between the linear equation and the Gronwall inequality to obtain a new version of this type of inequalities in the context of Lebesgue-Stieltjes integrals. From there, we obtain a uniqueness criterion for initial value problems.


Keywords: Stieltjes integration, Stieltjes differentiation, linear equation, uniqueness, Gronwall inequality.

2020 Mathematics Subject Classification: 26A24, 26D10, 34A12, 34A30, 34A34.

## 1 Introduction

In this paper we explore the linear equation with Stieltjes derivatives in its homogeneous and nonhomogeneous formulation. Specifically, we will be looking at the initial value problem

$$
\begin{equation*}
x_{g}^{\prime}(t)+d(t) x(t)=h(t), \quad t \in\left[t_{0}, t_{0}+T\right), \quad x\left(t_{0}\right)=x_{0} \tag{1.1}
\end{equation*}
$$

where $t_{0}, T, x_{0} \in \mathbb{R}, T>0$, are fixed, $d, h:\left[t_{0}, t_{0}+T\right) \rightarrow \mathbb{R}$ are given functions and $x_{g}^{\prime}$ stands for the Stieltjes derivative of $x$ with respect to a nondecreasing and left-continuous function $g: \mathbb{R} \rightarrow \mathbb{R}$, usually called derivator, see $[3,10]$. Note that [3] provides some information regarding (1.1) in its homogeneous form (i.e. $h=0$ ) as well as for the nonhomogeneous case. Nevertheless, the results obtained there present some limitations, as the authors make use of the product rule for Stieltjes derivatives in [10] which, unfortunately, is wrongly stated. Here, we amend the mistakes in [10] as well as we simplify the required hypotheses for the existence and uniqueness of solution of (1.1).

Furthermore, given the close relation existing between the Gronwall inequality and the linear equation in the setting of ordinary differential equations, we will prove a new version

[^4]of the inequality in the context of Stieltjes integrals, generalizing the classical result in [5], as well as other existing formulations in the context of Stieltjes integrals, see $[6,8,11,12,17]$.

The paper is structure as follows. In Section 2 we gather and revisit some of the information available in $[3,10]$ regarding the definition of Stieltjes derivatives and its properties, as well as some other basic definitions necessary for this paper. In particular, it is at this point that we correct the formula for the product and the quotient rule in [10]. Next, in Section 3 we study the linear equation (1.1), providing explicit expressions for its solutions as well as some of their properties. Finally, in Section 4 we establish a Gronwall-type inequality for the Lebesgue-Stieltjes integral using the solution of the homogeneous linear equation. Then, we discuss the relations with other existing inequalities available in the literature and we complete the revision of the results in [3] for (1.1) through a uniqueness result based on our version of Gronwall's inequality.

## 2 Preliminaries

Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing and left-continuous function. Let us introduce some notation before including the definition of Stieltjes derivative in [10]. In what follows, we will consider $\mu_{g}$ to be the Lebesgue-Stieltjes measure associated to $g$, given by

$$
\mu_{g}([a, b))=g(b)-g(a), \quad a, b \in \mathbb{R}, a<b
$$

see $[1,13,15]$; we will use the term " $g$-measurable" for a set or function to refer to $\mu_{g}$ measurability in the corresponding sense; and we will denote the integration with respect to $\mu_{g}$ as

$$
\int_{X} f(s) \mathrm{d} g(s)
$$

Similarly, we will talk about properties holding $g$-almost everywhere in a set $X$, shortened to $g$-a.e. in $X$, as a simplified way to express that they hold $\mu_{g}$-almost everywhere in $X$. In an analogous way, we will write that a property holds for $g$-almost all (or simply, $g$-a.a.) $x \in X$ meaning that it holds for $\mu_{g}$-almost all $x \in X$. Along those lines, we find the following interesting set:

$$
C_{g}:=\{t \in \mathbb{R}: g \text { is constant on }(t-\varepsilon, t+\varepsilon) \text { for some } \varepsilon>0\}
$$

The set $C_{g}$ is the set of points around which $g$ is constant and, as pointed out in [10, Proposition 2.5], we have that $\mu_{g}\left(C_{g}\right)=0$. Hence, this set can be disregarded when it comes to properties holding $g$-almost everywhere in a set. Observe that, as pointed out in [10], the set $C_{g}$ is open in the usual topology, so it can be uniquely expressed as the countable union of open disjoint intervals, say

$$
\begin{equation*}
C_{g}=\bigcup_{n \in \mathbb{N}}\left(a_{n}, b_{n}\right) \tag{2.1}
\end{equation*}
$$

Another fundamental set for the work that lies ahead is the set $D_{g}$ of all discontinuity points of $g$. Observe that, given that $g$ is nondecreasing, we can write

$$
D_{g}=\left\{t \in \mathbb{R}: \Delta^{+} g(t)>0\right\}
$$

where $\Delta^{+} g(t)=g\left(t^{+}\right)-g(t), t \in \mathbb{R}$, and $g\left(t^{+}\right)$denotes the right-hand side limit of $g$ at $t$. Recall that Froda's Theorem, [4], ensures that the set $D_{g}$ is at most countable. Finally, given the previous definitions, we can define the sets $N_{g}^{-}$and $N_{g}^{+}$introduced in [9] as

$$
N_{g}^{-}=\left\{a_{n}: n \in \mathbb{N}\right\} \backslash D_{g}, \quad N_{g}^{+}=\left\{b_{n}: n \in \mathbb{R}\right\} \backslash D_{g}
$$

where $a_{n}, b_{n} \in \mathbb{R}$ are as in (2.1). We denote $N_{g}=N_{g}^{-} \cup N_{g}^{+}$.
We have now all the information required to properly introduce the definition of Stieltjes derivative in [10]. In order to clarify its definition, we have included a brief remark explaining the limits involved.

Definition 2.1. Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing and left-continuous function and consider a map $f: \mathbb{R} \rightarrow \mathbb{R}$. We define the Stieltjes derivative, or $g$-derivative, of $f$ at a point $t \in \mathbb{R} \backslash C_{g}$ as

$$
f_{g}^{\prime}(t)= \begin{cases}\lim _{s \rightarrow t} \frac{f(s)-f(t)}{g(s)-g(t)}, & t \notin D_{g} \\ \lim _{s \rightarrow t^{+}} \frac{f(s)-f(t)}{g(s)-g(t)}, & t \in D_{g}\end{cases}
$$

provided the corresponding limits exist. In that case, we say that $f$ is $g$-differentiable at $t$.
Remark 2.2. Given a function $f: \mathbb{R} \rightarrow \mathbb{R}$ and a point $t \in \mathbb{R}$, we define the function

$$
F_{t}(\cdot)=\frac{f(\cdot)-f(t)}{g(\cdot)-g(t)},
$$

which we will assume to be defined in a neighbourhood of $t$ in which the expression makes sense, namely, at the points $s$ such that $g(s)-g(t) \neq 0$. The limits in Definition 2.1 are welldefined when $t$ is an accumulation point of the domain of the function $F_{t}$. This explains why the points of $C_{g}$ are excluded in the definition as if $t \in C_{g}$, then there exists $\varepsilon_{t}>0$ such that the expression of $F_{t}$ does not make sense for any neighbourhood $(t-\varepsilon, t+\varepsilon), \varepsilon \in\left(0, \varepsilon_{t}\right)$. Moreover, the limits in Definition 2.1 should be properly understood at some other conflicting points. For example, imagine there exists $\delta>0$ such that $g(s)=g(t)$ for $s \in(t-\delta, t)$, and $g(s)>g(t)$ for $s>t$. Then

$$
\lim _{s \rightarrow t} F_{t}(s)=\lim _{s \rightarrow t^{+}} F_{t}(s),
$$

since $F_{t}$ is not defined at the left of $t$. Similarly, if there exists $\delta>0$ such that $g(s)=g(t)$ for $s \in(t, t+\delta)$, the function $F_{t}$ is not defined at the right of $t$, so if $g(s)<g(t)$ for $s<t$, then

$$
\lim _{s \rightarrow t} F_{t}(s)=\lim _{s \rightarrow t^{-}} F_{t}(s) .
$$

Therefore, the $g$-derivative of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ at a point $t \in \mathbb{R} \backslash C_{g}$ is computed as

$$
f_{g}^{\prime}(t)= \begin{cases}\lim _{s \rightarrow t} \frac{f(s)-f(t)}{g(s)-g(t)}, & t \notin D_{g} \cup N_{g} \\ \lim _{s \rightarrow t^{-}} \frac{f(s)-f(t)}{g(s)-g(t)}, & t \in N_{g}^{-} \\ \lim _{s \rightarrow t^{+}} \frac{f(s)-f(t)}{g(s)-g(t)^{\prime}}, & t \in D_{g} \cup N_{g}^{+}\end{cases}
$$

provided the corresponding limits exist.
Remark 2.3. Since $g$ is a regulated function, it follows that the $g$-derivative of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ at a point $t \in D_{g}$ exists if and only if the limit of $f$ from the right of $t, f\left(t^{+}\right)$, exists. In that case, we have that

$$
f_{g}^{\prime}(t)=\frac{f\left(t^{+}\right)-f(t)}{\Delta^{+} g(t)}
$$

First, we include some information available in [10] regarding the Stieltjes derivatives of functions. Specifically, we include a result about the continuity of differentiable functions, [10, Proposition 2.1], that we will use to revisit the product and quotient rule in [10], as the formulas there included are not correct.

Proposition 2.4. Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing and left-continuous function and $f$ be a realvalued function defined on a neighborhood of $t$ such that $f_{g}^{\prime}(t)$ exists. Then $t \notin C_{g}$ and, if $g$ is continuous at $t$ :

- $f$ is continuous from the left at $t$ provided that

$$
\begin{equation*}
g(s)<g(t) \quad \text { for all } s<t \tag{2.2}
\end{equation*}
$$

- $f$ is continuous from the right at t provided that

$$
\begin{equation*}
g(s)>g(t) \quad \text { for all } s>t . \tag{2.3}
\end{equation*}
$$

Proposition 2.4 is a fundamental tool for the proof of [10, Proposition 2.2], where the authors included some basic properties of the Stieltjes derivatives, such as the linearity of the derivative or the product and the quotient rule. However, the authors did not include the proof of the result, which led to an incorrect formulation of the product and the quotient rule. Here, we amend these mistakes and, later, we show the limitations of the formulas in [10, Proposition 2.2].

Proposition 2.5. Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing and left-continuous function, $t \in \mathbb{R}$, and $f_{1}, f_{2}$ be two real-valued functions defined on a neighborhood of $t, U_{t}$. If $f_{1}$ and $f_{2}$ are $g$-differentiable at $t$, then:
(i) The product $f_{1} f_{2}$ is $g$-differentiable at $t$ and

$$
\begin{equation*}
\left(f_{1} f_{2}\right)_{g}^{\prime}(t)=\left(f_{1}\right)_{g}^{\prime}(t) f_{2}(t)+\left(f_{2}\right)_{g}^{\prime}(t) f_{1}(t)+\left(f_{1}\right)_{g}^{\prime}(t)\left(f_{2}\right)_{g}^{\prime}(t) \Delta^{+} g(t) . \tag{2.4}
\end{equation*}
$$

(ii) If $\left(f_{2}(t)\right)^{2}+\left(f_{2}\right)_{g}^{\prime}(t) f_{2}(t) \Delta^{+} g(t) \neq 0$, the quotient $f_{1} / f_{2}$ is $g$-differentiable at $t$ and

$$
\begin{equation*}
\left(\frac{f_{1}}{f_{2}}\right)_{g}^{\prime}(t)=\frac{\left(f_{1}\right)_{g}^{\prime}(t) f_{2}(t)-f_{1}(t)\left(f_{2}\right)_{g}^{\prime}(t)}{\left.\left(f_{2}(t)\right)^{2}+\left(f_{2}\right)_{g}^{\prime}(t) f_{2}(t) \Delta^{+} g(t)\right)} . \tag{2.5}
\end{equation*}
$$

Proof. First, observe that $t \notin C_{g}$ since $\left(f_{1}\right)_{g}^{\prime}(t)$ and $\left(f_{2}\right)_{g}^{\prime}(t)$ exist. Hence, we have that (2.2) and/or (2.3) hold.

Let us show that (2.4) holds. First, observe that we can rewrite $f_{1} f_{2}(s)-f_{1} f_{2}(t), s \in U_{t}$, as

$$
\begin{equation*}
\frac{\left(f_{1}(s)-f_{1}(t)\right)\left(f_{2}(t)+f_{2}(s)\right)+\left(f_{2}(s)-f_{2}(t)\right)\left(f_{1}(t)+f_{1}(s)\right)}{2}, \quad s \in U_{t} . \tag{2.6}
\end{equation*}
$$

Assume that (2.3) holds. Then, it follows from (2.6) that the following limit exists and

$$
\begin{equation*}
\lim _{s \rightarrow t^{+}} \frac{f_{1} f_{2}(s)-f_{1} f_{2}(t)}{g(s)-g(t)}=\frac{\left(f_{1}\right)_{g}^{\prime}(t)\left(f_{2}(t)+f_{2}\left(t^{+}\right)\right)+\left(f_{2}\right)_{g}^{\prime}(t)\left(f_{1}(t)+f_{1}\left(t^{+}\right)\right)}{2} . \tag{2.7}
\end{equation*}
$$

Now, if $t \in D_{g}$, it follows from Remark 2.3 that

$$
\begin{equation*}
f_{i}\left(t^{+}\right)=\left(f_{i}\right)_{g}^{\prime}(t) \Delta^{+} g(t)+f_{i}(t), \quad i=1,2 . \tag{2.8}
\end{equation*}
$$

Thus, (2.7) yields that

$$
\begin{equation*}
\lim _{s \rightarrow t^{+}} \frac{f_{1} f_{2}(s)-f_{1} f_{2}(t)}{g(s)-g(t)}=\left(f_{1}\right)_{g}^{\prime}(t) f_{2}(t)+\left(f_{2}\right)_{g}^{\prime}(t) f_{1}(t)+\left(f_{1}\right)_{g}^{\prime}(t)\left(f_{2}\right)_{g}^{\prime}(t) \Delta^{+} g(t) \tag{2.9}
\end{equation*}
$$

On the other hand, if $t \notin D_{g}$, it follows from Proposition 2.4 and (2.7) that

$$
\lim _{s \rightarrow t^{+}} \frac{f_{1} f_{2}(s)-f_{1} f_{2}(t)}{g(s)-g(t)}=\left(f_{1}\right)_{g}^{\prime}(t) f_{2}(t)+\left(f_{2}\right)_{g}^{\prime}(t) f_{1}(t)
$$

which matches (2.9) since $\Delta^{+} g(t)=0$. In other words, (2.9) holds in both cases. Hence, if $t \in D_{g}$ or $g(s)>g(t), s \in[t-\delta, t]$ for some $\delta>0$, then the limit in (2.9) coincides with $\left(f_{1} f_{2}\right)_{g}^{\prime}(t)$ and the proof is complete. Otherwise, $t \notin D_{g}$ and (2.2) holds. In that case, we obtain from (2.6) that the following limit exists and

$$
\lim _{s \rightarrow t^{-}} \frac{f_{1} f_{2}(s)-f_{1} f_{2}(t)}{g(s)-g(t)}=\frac{\left(f_{1}\right)_{g}^{\prime}(t)\left(f_{2}(t)+f_{2}\left(t^{-}\right)\right)+\left(f_{2}\right)_{g}^{\prime}(t)\left(f_{1}(t)+f_{1}\left(t^{-}\right)\right)}{2}
$$

However, in that case, Proposition 2.4 ensures that the previous limit equals $\left(f_{1}\right)_{g}^{\prime}(t) f_{2}(t)+$ $\left(f_{2}\right)_{g}^{\prime}(t) f_{1}(t)$, and so $f_{1} f_{2}$ is $g$-differentiable at $t$ and (2.4) holds.

Now, we show that (2.5) holds. First, observe that the extra hypothesis in (ii) guarantees that $f_{2}(t) \neq 0$. Furthermore, we also have that $f_{2}(t)+\left(f_{2}\right)_{g}^{\prime}(t) \Delta^{+} g(t) \neq 0$ which, provided that $t \in D_{g}$, ensures that $f_{2}\left(t^{+}\right) \neq 0$, see (2.8).

Assume that (2.3) holds. Since $f_{2}(t) \neq 0$, it follows from Proposition 2.4 (if $t \notin D_{g}$ ) and the definition of limit from the right (if $t \in D_{g}$ ) that there exists $\varepsilon>0$ such that $f_{2}$ does not vanish in $[t, t+\varepsilon) \cap U_{t}$. Hence, the following expression is well-defined for any $s \in[t, t+\varepsilon) \cap U_{t}$,

$$
\begin{equation*}
\frac{f_{1}(s)}{f_{2}(s)}-\frac{f_{1}(t)}{f_{2}(t)}=\frac{f_{1}(s) f_{2}(t)-f_{1}(t) f_{2}(s)}{f_{2}(t) f_{2}(s)}=\frac{\left(f_{1}(s)-f_{1}(t)\right) f_{2}(t)+f_{1}(t)\left(f_{2}(t)-f_{2}(s)\right)}{f_{2}(t) f_{2}(s)} . \tag{2.10}
\end{equation*}
$$

Taking the corresponding limit from the right, we have that

$$
\begin{equation*}
\lim _{s \rightarrow t^{+}} \frac{\left(f_{1} / f_{2}\right)(s)-\left(f_{1} / f_{2}\right)(t)}{g(s)-g(t)}=\frac{\left(f_{1}\right)_{g}^{\prime}(t) f_{2}(t)-f_{1}(t)\left(f_{2}\right)_{g}^{\prime}(t)}{f_{2}(t) f_{2}\left(t^{+}\right)} . \tag{2.11}
\end{equation*}
$$

Now, if $t \in D_{g}$, it follows from (2.8) that

$$
\begin{equation*}
\lim _{s \rightarrow t^{+}} \frac{\left(f_{1} / f_{2}\right)(s)-\left(f_{1} / f_{2}\right)(t)}{g(s)-g(t)}=\frac{\left(f_{1}\right)_{g}^{\prime}(t) f_{2}(t)-f_{1}(t)\left(f_{2}\right)_{g}^{\prime}(t)}{\left.\left(f_{2}(t)\right)^{2}+\left(f_{2}\right)_{g}^{\prime}(t) f_{2}(t) \Delta^{+} g(t)\right)} \tag{2.12}
\end{equation*}
$$

On the other hand, if $t \notin D_{g}$, it follows from Proposition 2.4 and (2.11) that

$$
\lim _{s \rightarrow t^{+}} \frac{\left(f_{1} / f_{2}\right)(s)-\left(f_{1} / f_{2}\right)(t)}{g(s)-g(t)}=\frac{\left(f_{1}\right)_{g}^{\prime}(t) f_{2}(t)-f_{1}(t)\left(f_{2}\right)_{g}^{\prime}(t)}{\left(f_{2}(t)\right)^{2}}
$$

which matches (2.12). That is, (2.12) holds in both cases. Hence, if $t \in D_{g}$ or $g(s)>g(t)$, $s \in[t-\delta, t]$ for some $\delta>0$, then the limit in (2.12) coincides with $\left(f_{1} / f_{2}\right)_{g}^{\prime}(t)$ and the proof is complete. Otherwise, $t \notin D_{g}$ and (2.2) holds. In that case, given that $f_{2}(t) \neq 0$, it follows from Proposition 2.4 that there exists $\varepsilon^{\prime}>0$ such that $f_{2}$ does not vanish in $\left(t-\varepsilon^{\prime}, t\right] \cap U_{t}$. Hence, (2.10) is valid for all $s \in\left(t-\varepsilon^{\prime}, t\right] \cap U_{t}$. As a consequence, we obtain that the following limit exists and

$$
\lim _{s \rightarrow t^{-}} \frac{\left(f_{1} / f_{2}\right)(s)-\left(f_{1} / f_{2}\right)(t)}{g(s)-g(t)}=\frac{\left(f_{1}\right)_{g}^{\prime}(t) f_{2}(t)-f_{1}(t)\left(f_{2}\right)_{g}^{\prime}(t)}{f_{2}(t) f_{2}\left(t^{-}\right)}=\frac{\left(f_{1}\right)_{g}^{\prime}(t) f_{2}(t)-f_{1}(t)\left(f_{2}\right)_{g}^{\prime}(t)}{\left(f_{2}(t)\right)^{2}}
$$

where the last equality follows, once again, from Proposition 2.4. This guarantees that $f_{1} / f_{2}$ is $g$-differentiable at $t$ and (2.5) holds.

Remark 2.6. Observe that the formulas here presented reduce to the usual formulation when $g=$ Id. Furthermore, note that the expressions in Proposition 2.5 do not match those in [10]. Let us illustrate that the formulas there presented are not correct with some examples.

Consider $g^{\prime}, f_{1}, f_{2}: \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$
g(t)=\left\{\begin{array}{ll}
t & \text { if } t \leq 0, \\
0 & \text { if } 0<t \leq 1, \\
t & \text { if } t>1,
\end{array} \quad f_{1}(t)=t+2, \quad f_{2}(t)= \begin{cases}1 & \text { if } t \leq 0 \\
t+2 & \text { if } t>0\end{cases}\right.
$$

For this choice of functions, we have that $f_{1} \cdot f_{2}, f_{1} / f_{2}: \mathbb{R} \rightarrow \mathbb{R}$ are defined as

$$
f_{1} \cdot f_{2}(t)=\left\{\begin{array}{ll}
t+2 & \text { if } t \leq 0, \\
(t+2)^{2} & \text { if } t>0,
\end{array} \quad \quad \frac{f_{1}}{f_{2}}(t)= \begin{cases}t+2 & \text { if } t \leq 0 \\
1 & \text { if } t>0\end{cases}\right.
$$

Observe that, given that $g(t)=g(0)$ for $t \in(0,1)$, the derivatives at 0 are computed as the limit from the left, as pointed out by Remark 2.2. In particular, we have that

$$
\begin{aligned}
& \left(f_{1}\right)_{g}^{\prime}(0)=\lim _{s \rightarrow 0^{-}} \frac{f_{1}(s)-f_{1}(0)}{g(s)-g(0)}=\lim _{s \rightarrow 0^{-}} \frac{s+2-2}{s-0}=1, \\
& \left(f_{2}\right)_{g}^{\prime}(0)=\lim _{s \rightarrow 0^{-}} \frac{f_{2}(s)-f_{2}(0)}{g(s)-g(0)}=\lim _{s \rightarrow 0^{-}} \frac{1-1}{s-0}=0,
\end{aligned}
$$

and, since $f_{1} \cdot f_{2}=f_{1} / f_{2}=f_{1}$ on $(-\infty, 0]$, we have that $\left(f_{1} \cdot f_{2}\right)_{g}^{\prime}(0)=\left(f_{1} / f_{2}\right)_{g}^{\prime}(0)=1$. Observe that (2.4) and (2.5) hold at $t=0$.

First, let us show that the formula for the product of two functions in [10],

$$
\left(f_{1} f_{2}\right)_{g}^{\prime}(t)=\left(f_{1}\right)_{g}^{\prime}(t) f_{2}\left(t^{+}\right)+\left(f_{2}\right)_{g}^{\prime}(t) f_{1}\left(t^{+}\right),
$$

is not correct. Indeed, at $t=0$ we have that

$$
\left(f_{1}\right)_{g}^{\prime}(0) f_{2}\left(0^{+}\right)+\left(f_{2}\right)_{g}^{\prime}(0) f_{1}\left(0^{+}\right)=1 \cdot 2+0 \cdot 2=2 \neq 1=\left(f_{1} \cdot f_{2}\right)_{g}^{\prime}(0) .
$$

Furthemore, this example also shows that the formula in [14, Lemma 13],

$$
\begin{equation*}
\left(f_{1} \cdot f_{2}\right)_{g}^{\prime}(t)=\left(f_{1}\right)_{g}^{\prime}(t) f_{2}\left(t^{+}\right)+\left(f_{2}\right)_{g}^{\prime}(t) f_{1}(t), \quad t \in D_{g} \tag{2.13}
\end{equation*}
$$

cannot be valid for a generic point in $\mathbb{R} \backslash C_{g}$, as the only difference with respect to the previous formula is that $f_{1}\left(0^{+}\right)$is replaced by $f_{1}(0)$, which has no effect as both terms are multiplied by zero. Nevertheless, observe that (2.4) yields (2.13) for $t \in D_{g}$ as a consequence of (2.8).

Now, for the quotient formula in [10],

$$
\left(\frac{f_{1}}{f_{2}}\right)_{g}^{\prime}(t)=\frac{\left(f_{1}\right)_{g}^{\prime}(t) f_{2}(t)-\left(f_{2}\right)_{g}^{\prime}(t) f_{1}(t)}{f_{2}(t) f_{2}\left(t^{+}\right)}
$$

Once again, this formula fails to be true as

$$
\frac{\left(f_{1}\right)_{g}^{\prime}(0) f_{2}(0)-\left(f_{2}\right)_{g}^{\prime}(0) f_{1}(0)}{f_{2}(0) f_{2}\left(0^{+}\right)}=\frac{1 \cdot 1-0 \cdot 2}{1 \cdot 2}=\frac{1}{2} \neq 1=\left(\frac{f_{1}}{f_{2}}\right)_{g}^{\prime}(0) .
$$

Finally, we include the last pieces of information required for this paper, the two formulations of the Fundamental Theorem of Calculus for the Lebesgue-Stieltjes integral. The next result is a reformulation of [10, Theorem 5.4], where we have added the definition of $g$-absolute continuity, [10, Definition 5.1], to its statement.

Theorem 2.7. Let $a, b \in \mathbb{R}, a<b$, and $F:[a, b] \rightarrow \mathbb{R}$. The following conditions are equivalent:

1. The function $F$ is $g$-absolutely continuous on $[a, b]$ according to the following definition: for every $\varepsilon>0$, there exists $\delta>0$ such that for every open pairwise disjoint family of subintervals $\left\{\left(a_{n}, b_{n}\right)\right\}_{n=1}^{m}$ verifying

$$
\sum_{n=1}^{m}\left(g\left(b_{n}\right)-g\left(a_{n}\right)\right)<\delta,
$$

we have that

$$
\sum_{n=1}^{m}\left|F\left(b_{n}\right)-F\left(a_{n}\right)\right|<\varepsilon .
$$

2. The function $F$ satisfies the following conditions:
(i) there exists $F_{g}^{\prime}(t)$ for $g$-a.a. $t \in[a, b)$;
(ii) $F_{g}^{\prime} \in \mathcal{L}_{g}^{1}([a, b), \mathbb{R})$, the set of Lebesgue-Stieltjes integrable functions with respect to $\mu_{g}$;
(iii) for each $t \in[a, b]$,

$$
F(t)=F(a)+\int_{[a, t)} F_{g}^{\prime}(s) \mathrm{d} g(s) .
$$

Remark 2.8. Observe that in statement 2 (iii) of Theorem 2.7, for $t=a$, we are considering the integral over $[a, a)=\{x \in \mathbb{R}: a \leq x<a\}=\varnothing$, which makes the integral null, thus giving the equality.

The other formulation of the Fundamental Theorem of Calculus that we include here is a combination of Theorem 2.4 and Proposition 5.2 in [10] and it reads as follows.

Theorem 2.9. Let $f \in \mathcal{L}_{g}^{1}([a, b), \mathbb{R})$. Then, the function $F:[a, b] \rightarrow \mathbb{R}$, defined as

$$
F(t)=\int_{[a, t)} f(s) \mathrm{d} g(s),
$$

is well-defined, $g$-absolutely continuous on $[a, b]$ and

$$
F_{g}^{\prime}(t)=f(t), \quad \text { for } g-a . a . t \in[a, b) .
$$

In the work that follows, we shall use some known properties for $g$-absolutely continuous functions, most of which are analogous to those of absolutely continuous functions in the usual sense. For convenience, we refer the reader to $[3,10]$ for more information on the topic.

## 3 Linear equation

In this section we focus on the study of the linear equation with Stieltjes derivatives on the real line in its homogeneous and nonhomogeneous formulation. Specifically, given a nondecreasing and left-continuous map, $g: \mathbb{R} \rightarrow \mathbb{R}$, we consider the initial value problem

$$
\begin{equation*}
x_{g}^{\prime}(t)+d(t) x(t)=h(t), \quad t \in\left[t_{0}, t_{0}+T\right), \quad x\left(t_{0}\right)=x_{0} \tag{3.1}
\end{equation*}
$$

with $x_{0} \in \mathbb{R}$ and $d, h:\left[t_{0}, t_{0}+T\right) \rightarrow \mathbb{R}$. Naturally, (3.1) yields the homogeneous formulation of the problem when $h=0$. In that case, for simplicity and in order to simplify the connections with [3], we shall write $c(t)=-d(t), t \in\left[t_{0}, t_{0}+T\right)$, so that (3.1) reads as

$$
\begin{equation*}
x_{g}^{\prime}(t)=c(t) x(t), \quad t \in\left[t_{0}, t_{0}+T\right), \quad x\left(t_{0}\right)=x_{0} . \tag{3.2}
\end{equation*}
$$

It is important to note, nevertheless, that [3] is not the only paper available in the study of linear equations in a Stieltjes sense. For example, in $[7,16,17]$ we find linear integral equations in more general settings, for which the different authors were able to obtain the existence and uniqueness of solution. In some cases, an explicit solution is given provided one can find a fundamental matrix for the corresponding problem, which might be hard to obtain. Here, we limit ourselves to a scalar version of the linear differential equation for which we obtain an explicit solution in terms of elemental functions. Interestingly enough, the relations between the different linear problems in the Stieltjes sense arises naturally. For example, condition (6.13) in [17] is a necessary condition for the existence of solution, which yields the condition required in Theorem 3.5 for our solution when both contexts are compatible.

Following [3], we start our study of the linear equation studying the homogeneous formulation. A first reasonable guess for a solution for (3.2) would be to consider, under the assumption that $c \in \mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+T\right), \mathbb{R}\right)$, the map

$$
\begin{equation*}
x(t)=x_{0} \exp \left(\int_{\left[t_{0}, t\right)} c(s) \mathrm{d} g(s)\right), \quad t \in\left[t_{0}, t_{0}+T\right], \tag{3.3}
\end{equation*}
$$

as this is the solution for $g=$ Id. Nevertheless, note that this cannot be a solution of (3.2) as for any $t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}$,

$$
\begin{aligned}
x_{g}^{\prime}(t) & =\lim _{s \rightarrow t^{+}} \frac{x(s)-x(t)}{g(s)-g(t)} \\
& =\lim _{s \rightarrow t^{+}} \frac{x(t)\left(\exp \left(\int_{[t, s)} c(r) \mathrm{d} g(r)\right)-1\right)}{g(s)-g(t)}=x(t) \frac{\exp \left(\int_{\{t\}} c(r) \mathrm{d} g(r)\right)-1}{\Delta^{+} g(t)} .
\end{aligned}
$$

Therefore, we have that

$$
x_{g}^{\prime}(t)=x(t) \frac{\exp \left(c(t) \Delta^{+} g(t)\right)-1}{\Delta^{+} g(t)}, \quad t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}
$$

which is not, in general, equal to $x(t) c(t)$. Therefore, the map $x$ in (3.3) cannot be a solution of (3.2). Nevertheless, it is easy to see using the chain rule for the Stieltjes derivative, [10, Theorem 2.3] that $x$ solves the problem in $\left[t_{0}, t_{0}+T\right) \backslash D_{g}$. All this ideas resulted in the modification of the map in (3.3) presented in [3, Definition 6.1]. It is at this point that we encounter the first improvement on the results of [3]. The mentioned modification is subject to a condition regarding the convergence of a series, namely, condition (3.5) in this paper. In the following result we show that such condition is redundant in the considered context.

Lemma 3.1. Let $c \in \mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+T\right), \mathbb{R}\right)$ be such that $1+c(t) \Delta^{+} g(t) \neq 0$ for all $t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}$. Then

$$
\begin{equation*}
\sum_{t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}}|\log | 1+c(t) \Delta^{+} g(t)| |<+\infty . \tag{3.4}
\end{equation*}
$$

In particular, if $1+c(t) \Delta^{+} g(t)>0$ for all $t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}$, then

$$
\begin{equation*}
\sum_{t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}}\left|\log \left(1+c(t) \Delta^{+} g(t)\right)\right|<+\infty . \tag{3.5}
\end{equation*}
$$

Proof. First, observe that the hypotheses ensure that the logarithms in the corresponding expressions are well-defined and finite for each $t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}$.

Now, elementary calculations show that $\lim _{s \rightarrow 0}|\log | 1+s|/ s|=1$. Hence, the definition of limit guarantees the existence of some $r>0$ such that

$$
\left|\left|\frac{\log |1+s|}{s}\right|-1\right|<1, \quad s \in(-r, r)
$$

In particular, this implies that $|\log | 1+s| |<2|s|$ for all $s \in(-r, r)$.
On the other hand, since $c$ is $g$-integrable on $\left[t_{0}, t_{0}+T\right)$, we have that

$$
\sum_{t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}}\left|c(t) \Delta^{+} g(t)\right| \leq \int_{\left[t_{0}, t_{0}+T\right)}|c(s)| \mathrm{d} g(s)<+\infty .
$$

Therefore, the set $A_{r}=\left\{t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}:\left|c(t) \Delta^{+} g(t)\right| \geq r\right\}$ must be finite. Hence, denoting $B_{r}=\left(\left[t_{0}, t_{0}+T\right) \cap D_{g}\right) \backslash A_{r}$, we have that

$$
\begin{aligned}
\sum_{t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}}|\log | 1+c(t) \Delta^{+} g(t)| | & =\sum_{t \in A_{r}}|\log | 1+c(t) \Delta^{+} g(t)| |+\sum_{t \in B_{r}}|\log | 1+c(t) \Delta^{+} g(t)| | \\
& \leq \sum_{t \in A_{r}}|\log | 1+c(t) \Delta^{+} g(t)| |+2 \sum_{t \in B_{r}}\left|c(t) \Delta^{+} g(t)\right|<+\infty .
\end{aligned}
$$

This shows that (3.4) holds. Now (3.5) follows from the extra hypothesis.
As a consequence of Lemma 3.1 and the product differentiation rule, we can reformulate Lemmas 6.2 and 6.3 in [3] into the following results.

Theorem 3.2. Let $c \in \mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+T\right), \mathbb{R}\right)$ be such that $1+c(t) \Delta^{+} g(t)>0$ for all $t \in\left[t_{0}, t_{0}+T\right) \cap$ $D_{g}$. Then, the map $\tilde{c}:\left[t_{0}, t_{0}+T\right) \rightarrow \mathbb{R}$, defined as

$$
\widetilde{c}(t)= \begin{cases}c(t) & \text { if } t \in\left[t_{0}, t_{0}+T\right) \backslash D_{g}  \tag{3.6}\\ \frac{\log \left(1+c(t) \Delta^{+} g(t)\right)}{\Delta^{+} g(t)} & \text { if } t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}\end{cases}
$$

belongs to $\mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+T\right), \mathbb{R}\right)$; the map $e_{c}\left(\cdot, t_{0}\right):\left[t_{0}, t_{0}+T\right] \rightarrow(0,+\infty)$,

$$
\begin{equation*}
e_{c}\left(t, t_{0}\right):=\exp \left(\int_{\left[t_{0}, t\right)} \widetilde{c}(s) \mathrm{d} g(s)\right), \quad t \in\left[t_{0}, t_{0}+T\right], \tag{3.7}
\end{equation*}
$$

is well-defined and $g$-absolutely continuous on $\left[t_{0}, t_{0}+T\right]$; and the map $x:\left[t_{0}, t_{0}+T\right] \rightarrow \mathbb{R}$, given by $x(t)=x_{0} e_{c}\left(t, t_{0}\right), t \in\left[t_{0}, t_{0}+T\right]$, solves the initial value problem (3.2) $g$-a.e. in $\left[t_{0}, t_{0}+T\right)$.

Remark 3.3. Observe that, for any $t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}$,

$$
\begin{aligned}
e_{c}\left(t^{+}, t_{0}\right) & =\lim _{s \rightarrow t^{+}} \exp \left(\int_{\left[t_{0}, s\right]} \widetilde{c}(s) \mathrm{d} g(s)\right)=\lim _{s \rightarrow t^{+}}\left(\exp \left(\int_{\left[t_{0}, t\right)} \widetilde{c}(s) \mathrm{d} g(s)\right) \exp \left(\int_{[t, s]} \widetilde{c}(s) \mathrm{d} g(s)\right)\right) \\
& =e_{c}\left(t, t_{0}\right) \exp \left(\int_{\{t\}} \widetilde{c}(s) \mathrm{d} g(s)\right)=e_{c}\left(t, t_{0}\right)\left(1+c(t) \Delta^{+} g(t)\right) .
\end{aligned}
$$

Essentially, this shows that the limitations that the map in (3.3) had at the discontinuity points are avoided for $e_{c}\left(\cdot, t_{0}\right)$.

An analogous improvement to the more general result [3, Lemma 6.5] can be obtained making use of the information in Lemma 3.1 regarding (3.4) instead of (3.5). In that case, we obtain the following result.

Theorem 3.4. Let $c \in \mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+T\right), \mathbb{R}\right)$ be such that $1+c(t) \Delta^{+} g(t) \neq 0$ for all $t \in\left[t_{0}, t_{0}+T\right) \cap$ $D_{g}$. Then, the set

$$
T_{c}^{-}=\left\{t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}: 1+c(t) \Delta^{+} g(t)<0\right\}
$$

has finite cardinality. Furthermore, if $T_{c}^{-}=\left\{t_{1}, \ldots, t_{k}\right\}, t_{0} \leq t_{1}<t_{2}<\cdots<t_{k}<t_{k+1}=t_{0}+T$, then the map $\widehat{c}:\left[t_{0}, t_{0}+T\right) \rightarrow \mathbb{R}$, defined as

$$
\widehat{c}(t)= \begin{cases}c(t) & \text { if } t \in\left[t_{0}, t_{0}+T\right) \backslash D_{g} \\ \frac{\log \left|1+c(t) \Delta^{+} g(t)\right|}{\Delta^{+} g(t)} & \text { if } t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}\end{cases}
$$

belongs to $\mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+T\right), \mathbb{R}\right)$; the map $\widehat{e}_{c}\left(\cdot, t_{0}\right):\left[t_{0}, t_{0}+T\right] \rightarrow \mathbb{R} \backslash\{0\}$, given by

$$
\widehat{e}_{c}\left(t, t_{0}\right)= \begin{cases}\exp \left(\int_{\left[t_{0}, t\right)} \widehat{c}(s) \mathrm{d} g(s)\right) & \text { if } t_{0} \leq t \leq t_{1} \\ (-1)^{j} \exp \left(\int_{\left[t_{0}, t\right)} \widehat{c}(s) \mathrm{d} g(s)\right) & \text { if } t_{j}<t \leq t_{j+1}, j=1, \ldots, k\end{cases}
$$

is well-defined and $g$-absolutely continuous on $\left[t_{0}, t_{0}+T\right]$; and the map $x:\left[t_{0}, t_{0}+T\right] \rightarrow \mathbb{R}$, given by $x(t)=x_{0} \widehat{e}_{c}\left(t, t_{0}\right), t \in\left[t_{0}, t_{0}+T\right]$, solves the initial value problem (3.2) g-a.e. in $\left[t_{0}, t_{0}+T\right)$.

Now, we move on to the study of the nonhomogeneous case. The study of this problem was also carried out in [3]. In particular, [3, Proposition 6.8] guarantees the existence of a unique solution of (3.2) under certain hypothesis. Furthermore, although it is not explicitly stated in the result, its proof provides a way to obtain it through the connection with the problem in [3, Proposition 6.7], and they have been made explicit in [2]. However, the proof of [3, Proposition 6.7] relays on the product rule for Stieltjes derivatives which, as it has been pointed out before, was not correct in that paper. Specifically, it is equation (6.16) in [3] that makes use of this property. It is possible to show that such expression remains true with the product formula in Proposition 2.5. Nevertheless, here we will use a different approach to the study of (3.2). Namely, we will recreate the method of variation of constants in this context.

Roughly speaking, the method of variation of constants revolves around the idea that the solution of a nonhomogeneous linear equation can be expressed as the sum of a solution of the homogeneous linear equation plus a particular solution of the nonhomogeneous one. In order to obtain the particular solution, we consider the following family of functions

$$
x_{C}(t)=C x_{h}(t), \quad t \in\left[t_{0}, t_{0}+T\right], \quad C \in \mathbb{R}
$$

where $x_{h}$ is a given solution of $x_{g}^{\prime}(t)=c(t) x(t)$. Observe that each element of the family $x_{C}$, $C \in \mathbb{R}$, also solves the same problem. From there, we make a guess that a particular solution is similar to that one, where we allow the constants to vary, i.e. we consider them as a function. Explicitly, we guess that the solution is of the form

$$
x(t)=C(t) x_{h}(t), \quad t \in\left[t_{0}, t_{0}+T\right]
$$

for some function $C:\left[t_{0}, t_{0}+T\right] \rightarrow \mathbb{R}$. Then, we try our guess on the corresponding nonhomogeneous linear equation. In order to do so, we need to make use of the product rule for Stieltjes derivatives, statement (ii) in Proposition 2.5. Let $t \in\left[t_{0}, t_{0}+T\right)$ be such that $x_{g}^{\prime}(t)$ exists. In that case,

$$
\begin{aligned}
x_{g}^{\prime}(t) & =C_{g}^{\prime}(t) x_{h}(t)+C(t) x_{h}(t)(-d(t))+C_{g}^{\prime}(t) x_{h}(t)(-d(t)) \Delta^{+} g(t) \\
& =x_{h}(t)\left(C_{g}^{\prime}(t)\left(1-d(t) \Delta^{+} g(t)\right)-C(t) d(t)\right)
\end{aligned}
$$

Hence, for such $t \in\left[t_{0}, t_{0}+T\right)$, it follows that $x_{g}^{\prime}(t)+d(t) x(t)=x_{h}(t) C_{g}^{\prime}(t)\left(1-d(t) \Delta^{+} g(t)\right)$. Therefore, if $x$ solves the nonhomogeneous linear equation, we must have that for such $t \in$ $\left[t_{0}, t_{0}+T\right)$,

$$
h(t)=x_{h}(t) C_{g}^{\prime}(t)\left(1-d(t) \Delta^{+} g(t)\right) .
$$

Therefore, if we can find a function $C$ satisfying the equation above, we obtain a particular solution of the nonhomogeneous linear equation and, as a consequence, the general solution of the same problem. Then, imposing the initial condition, we obtain the following result.

Theorem 3.5. Let $d, h \in \mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+T\right), \mathbb{R}\right)$ be such that $1-d(t) \Delta^{+} g(t) \neq 0$ for all $t \in\left[t_{0}, t_{0}+\right.$ $T) \cap D_{g}$. Then the map $x:\left[t_{0}, t_{0}+T\right] \rightarrow \mathbb{R}$, defined as

$$
\begin{equation*}
x(t)=\widehat{e}_{-d}\left(t, t_{0}\right)\left(x_{0}+\int_{\left[t_{0}, t\right)} \frac{h(s)}{\widehat{e}_{-d}\left(s, t_{0}\right)\left(1-d(s) \Delta^{+} g(s)\right)} \mathrm{d} g(s)\right), \quad t \in\left[t_{0}, t_{0}+T\right] \tag{3.8}
\end{equation*}
$$

is well-defined, $g$-absolutely continuous on $\left[t_{0}, t_{0}+T\right]$ and it solves (3.1) $g$-a.e. in $\left[t_{0}, t_{0}+T\right)$.
If, in particular, $1-d(t) \Delta^{+} g(t)>0$ for all $t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}$, then

$$
\begin{equation*}
x(t)=e_{-d}\left(t, t_{0}\right)\left(x_{0}+\int_{\left[t_{0}, t\right)} \frac{h(s)}{e_{-d}\left(s, t_{0}\right)\left(1-d(s) \Delta^{+} g(s)\right)} \mathrm{d} g(s)\right), \quad t \in\left[t_{0}, t_{0}+T\right] . \tag{3.9}
\end{equation*}
$$

Proof. First of all, note that, under the corresponding hypotheses, the maps $\widehat{e}_{-d}\left(\cdot, t_{0}\right)$ and $e_{-d}\left(\cdot, t_{0}\right)$ are well-defined. Let us show that the map $x$ in (3.8) has the stated properties.

Consider the maps $E, H:\left[t_{0}, t_{0}+T\right) \rightarrow \mathbb{R}$ defined as

$$
E(t)=\widehat{e}_{-d}\left(t, t_{0}\right)\left(1-d(t) \Delta^{+} g(t)\right), \quad H(t)=\frac{h(t)}{E(t)}, \quad t \in\left[t_{0}, t_{0}+T\right) .
$$

Since $E(t)=\widehat{e}_{-d}\left(t, t_{0}\right)$ for all $t \in I \backslash D_{g}$ and $D_{g}$ is countable, $E$ is $g$-measurable. Moreover, since $E \neq 0$ by definition, and $h$ and $E$ are $g$-measurable, it follows that $H$ is $g$-measurable. Furthermore, $H$ belongs to $\mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+T\right), \mathbb{R}\right)$. Indeed, first of all note that for each $t \in\left[t_{0}, t_{0}+T\right)$,

$$
\begin{aligned}
\left|\widehat{e}_{-d}\left(t, t_{0}\right)\right| & =\exp \left(\int_{\left[t_{0}, t\right)} \widehat{c}(s) \mathrm{d} g(s)\right) \\
& \geq \exp \left(-\int_{\left[t_{0}, t\right)}|\widehat{c}(s)| \mathrm{d} g(s)\right) \geq \exp \left(-\int_{\left[t_{0}, t_{0}+T\right)}|\widehat{c}(s)| \mathrm{d} g(s)\right) .
\end{aligned}
$$

Observe that $m:=\exp \left(-\int_{\left[t_{0}, t_{0}+T\right)}|\widehat{c}(s)| \mathrm{d} g(s)\right)>0$. Hence,

$$
|H(t)| \leq \frac{1}{m} \frac{|h(t)|}{\left|1-d(t) \Delta^{+} g(t)\right|}, \quad t \in\left[t_{0}, t_{0}+T\right) .
$$

Therefore, it is enough to show that the map $\bar{h}:\left[t_{0}, t_{0}+T\right) \rightarrow \mathbb{R}$, defined as

$$
\bar{h}(t)=\frac{h(t)}{1-d(t) \Delta^{+} g(t)}, \quad t \in\left[t_{0}, t_{0}+T\right)
$$

is $g$-integrable to prove that $H \in \mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+T\right), \mathbb{R}\right)$. In order to see that $\bar{h}$ is $g$-integrable, observe that the set $A=\left\{t \in\left[t_{0}, t_{0}+T\right): d(t) \Delta^{+} g(t)>1 / 2\right\}$ has finite cardinality as

$$
\sum_{t \in A} \frac{1}{2}<\sum_{t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}}\left|d(t) \Delta^{+} g(t)\right| \leq \int_{\left[t_{0}, t_{0}+T\right)}|d(s)| \mathrm{d} g(s)<+\infty .
$$

As a consequence, we have that $|\bar{h}(t)| \leq 2|h(t)|$ for all $t \in\left[t_{0}, t_{0}+T\right) \backslash A$, from which the $g$-integrability of $\bar{h}$ follows. Hence, $H \in \mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+T\right), \mathbb{R}\right)$. Now, Theorem 2.9 yields that $x$ is $g$-absolutely continuous on $\left[t_{0}, t_{0}+T\right]$. Hence, all that is left to do is to check that $x$ solves (3.1).

By definition, we have that $x\left(t_{0}\right)=x_{0}$. Furthermore, (i) in Proposition 2.5 and Theorems 2.9 and 3.4, ensure that for $g$-a.a. $t \in\left[t_{0}, t_{0}+T\right)$,

$$
\begin{aligned}
x_{g}^{\prime}(t) & =-d(t) \widehat{e}_{-d}\left(t, t_{0}\right)\left(x_{0}+\int_{\left[t_{0}, t\right)} H(s) \mathrm{d} g(s)\right)+\widehat{e}_{-d}\left(t, t_{0}\right) H(t)-\widehat{e}_{-d}\left(t, t_{0}\right) d(t) H(t) \Delta^{+} g(t) \\
& =-d(t) x(t)+\widehat{e}_{-d}\left(t, t_{0}\right) H(t)\left(1-d(t) \Delta^{+} g(t)\right)=-d(t) x(t)+h(t),
\end{aligned}
$$

i.e. $x$ solves (3.1).

Now, the expression of $x$ in (3.9) follows from the extra hypothesis and the definition of $\widehat{e}_{-d}\left(\cdot, t_{0}\right)$ and $e_{-d}\left(\cdot, t_{0}\right)$

Observe that, unlike [3, Proposition 6.8], Theorem 3.5 does not guarantee the uniqueness of solution of (3.1) but it offers an explicit expression for a solution of the problem under simpler conditions as condition (3.4) is not required. Nevertheless, using the results in the next section, we will be able to show that (3.1) has a unique solution under the assumption that $d \in \mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+T\right), \mathbb{R}\right)$.

## 4 Gronwall's inequality for Lebesgue-Stieltjes integrals

In this section we turn our attention to the Gronwall inequality in the setting of LebesgueStieltjes integrals. Here, following the ideas [5], we obtain an integral inequality involving the solution of the linear problem with Stieltjes derivatives. This argument improves, as we show later, the corresponding results existing in the literature, such as those in [6,8,11,12,17].

In order to simplify the proof of the main result of this section, Proposition 4.3, we include the following result. By doing this, we can also reflect on the meaning of Proposition 4.1 for the study of the corresponding linear equation in (3.2).

Proposition 4.1. Let $c \in \mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+T\right), \mathbb{R}\right)$ be such that $1+c(t) \Delta^{+} g(t)>0$ for all $t \in\left[t_{0}, t_{0}+\right.$ $T) \cap D_{g}$. Then the map $h:\left[t_{0}, t_{0}+T\right] \rightarrow \mathbb{R}$, defined as

$$
\begin{equation*}
h(t)=\left(e_{c}\left(t, t_{0}\right)\right)^{-1}, \quad t \in\left[t_{0}, t_{0}+T\right], \tag{4.1}
\end{equation*}
$$

is well-defined, $g$-absolutely continuous on $\left[t_{0}, t_{0}+T\right]$ and

$$
\begin{equation*}
h_{g}^{\prime}(t)=\frac{-c(t)}{e_{c}\left(t, t_{0}\right)\left(1+c(t) \Delta^{+} g(t)\right)}, \quad g \text {-a.a. } t \in\left[t_{0}, t_{0}+T\right) . \tag{4.2}
\end{equation*}
$$

Proof. Define $h_{1}(t)=e_{c}\left(t, t_{0}\right), t \in\left[t_{0}, t_{0}+T\right]$. Since $h_{1}$ is $g$-absolutely continuous on $\left[t_{0}, t_{0}+T\right]$, it has bounded variation on that interval (see [10, Proposition 5.3]) and thus, it is bounded on $\left[t_{0}, t_{0}+T\right]$. In particular, if we take

$$
m:=\exp \left(-\int_{\left[t_{0}, t_{0}+T\right)}|\widetilde{c}(s)| \mathrm{d} g(s)\right), \quad M:=\exp \left(\int_{\left[t_{0}, t_{0}+T\right)}|\widetilde{c}(s)| \mathrm{d} g(s)\right),
$$

where $\widetilde{c}$ is the modified function in (3.6), we have that $0<m \leq h(t) \leq M<+\infty, t \in$ $\left[t_{0}, t_{0}+T\right]$. Hence, taking $h_{2}(t)=1 / t, t \in[m, M]$, we can rewrite $h$ as $h(t)=h_{2} \circ h_{1}$, which
shows that it is well-defined. Now, as in the classical setting, this is enough to ensure that $h$ is $g$-absolutely continuous on $\left[t_{0}, t_{0}+T\right]$, see [3, Proposition 5.3].

Let $t \in\left[t_{0}, t_{0}+T\right)$ be such that $h_{g}^{\prime}(t)$ exists. If $t \notin D_{g}$, then by the chain rule, [10, Theorem 2.3],

$$
h_{g}^{\prime}(t)=h_{2}^{\prime}\left(h_{1}(t)\right)\left(h_{2}\right)_{g}^{\prime}(t)=\frac{-1}{\left(e_{c}\left(t, t_{0}\right)\right)^{2}} e_{c}\left(t, t_{0}\right) c(t)=\frac{-c(t)}{e_{c}\left(t, t_{0}\right)^{\prime}}
$$

which coincides with (4.2) since $\Delta^{+} g(t)=0$. On the other hand, if $t \in D_{g}$, using Remarks 2.3 and 3.3 we have that

$$
h_{g}^{\prime}(t)=\frac{\left(e_{c}\left(t^{+}, t_{0}\right)\right)^{-1}-\left(e_{c}\left(t, t_{0}\right)\right)^{-1}}{\Delta^{+} g(t)}=\frac{\left(1+c(t) \Delta^{+} g(t)\right)^{-1}-1}{e_{c}\left(t, t_{0}\right) \Delta^{+} g(t)}=\frac{-c(t)}{e_{c}\left(t, t_{0}\right)\left(1+c(t) \Delta^{+} g(t)\right)}
$$

which concludes the proof.
Remark 4.2. Observe that Proposition 4.1 shows that, under the corresponding hypotheses, $\left(e_{c}\left(t, t_{0}\right)\right)^{-1}$ solves the Stieltjes differential equation $x_{g}^{\prime}(t)=-c(t) x(t)$ except at the discontinuity points of the derivator, presenting the limitations that the map in (3.3) had. In order to obtain an equality at those points, one would have to modify the map $c$ in an analogous way to (3.6), which would lead to Theorem 3.2 under the corresponding hypotheses for $-c$.

As we mentioned before, Proposition 4.1 allows us to derive a version of Gronwall's inequality in the context of Lebesgue-Stieltjes integrals. Naturally, in this context, the exponential map involved in the inequality is the one in (3.7). However, as we will see later, we can obtain a different version of Gronwall's inequality involving the usual exponential map. Let us state and prove our first version of Gronwall's inequality for the Lebesgue-Stieltjes integral.

Proposition 4.3. Let $u, K, L:\left[t_{0}, t_{0}+T\right) \rightarrow[0,+\infty)$ be such that $L, K \cdot L, u \cdot L \in \mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+T\right)\right.$, $[0,+\infty)$ ). If

$$
\begin{equation*}
u(t) \leq K(t)+\int_{\left[t_{0}, t\right)} L(s) u(s) \mathrm{d} g(s), \quad t \in\left[t_{0}, t_{0}+T\right) \tag{4.3}
\end{equation*}
$$

then

$$
\begin{equation*}
u(t) \leq K(t)+\int_{\left[t_{0}, t\right)} K(s) L(s) \exp \left(\int_{[s, t)} \widetilde{L}(r) \mathrm{d} g(r)\right) \mathrm{d} g(s), \quad t \in\left[t_{0}, t_{0}+T\right) \tag{4.4}
\end{equation*}
$$

where $\widetilde{L}$ is the modified function in (3.6). Moreover, if the map $\varphi:\left[t_{0}, t_{0}+T\right) \rightarrow \mathbb{R}$, defined as $\varphi(t)=K(t)\left(1+L(t) \Delta^{+} g(t)\right)$, is nondecreasing, then

$$
\begin{equation*}
u(t) \leq \varphi(t) e_{L}\left(t, t_{0}\right), \quad t \in\left[t_{0}, t_{0}+T\right) . \tag{4.5}
\end{equation*}
$$

 $\widetilde{L}$ and $e_{L}\left(\cdot, t_{0}\right)$ are well-defined.

Define $U(t)=\int_{\left[t_{0}, t\right)} L(s) u(s) \mathrm{d} g(s), t \in\left[t_{0}, t_{0}+T\right]$. It follows from the hypotheses and Theorem 2.9 that $U$ is well-defined, $g$-absolutely continuous on $\left[t_{0}, t_{0}+T\right]$ and

$$
U_{g}^{\prime}(t)=L(t) u(t), \quad g \text {-a.a. } t \in\left[t_{0}, t_{0}+T\right) .
$$

Let $h:\left[t_{0}, t_{0}+T\right] \rightarrow \mathbb{R}$ be as in (4.1) for $c=L$ and define $v(t)=U(t) h(t), t \in\left[t_{0}, t_{0}+T\right]$. This is enough to ensure that $v$ is $g$-absolutely continuous on $\left[t_{0}, t_{0}+T\right.$ ], which guarantees that $v_{g}^{\prime}(t)$ exists $g$-almost everywhere in $\left[t_{0}, t_{0}+T\right)$.

Given $t \in\left[t_{0}, t_{0}+T\right)$ such that $v_{g}^{\prime}(t)$ exists, Propositions 2.5 and 4.1 yield

$$
\begin{aligned}
v_{g}^{\prime}(t) & =U_{g}^{\prime}(t)\left(h(t)+h_{g}^{\prime}(t) \Delta^{+} g(t)\right)+h_{g}^{\prime}(t) U(t) \\
& =u(t) L(t)\left(h(t)-\frac{L(t) h(t)}{1+L(t) \Delta^{+} g(t)} \Delta^{+} g(t)\right)-\frac{L(t) h(t)}{1+L(t) \Delta^{+} g(t)} U(t) \\
& =u(t) L(t) h(t) \frac{1}{1+L(t) \Delta^{+} g(t)}-\frac{L(t) h(t) U(t)}{1+L(t) \Delta^{+} g(t)} \\
& =\frac{L(t) h(t)}{1+L(t) \Delta^{+} g(t)}\left(u(t)-\int_{\left[t_{0}, t\right)} L(s) u(s) \mathrm{d} g(s)\right)
\end{aligned}
$$

Thus, inequality (4.3) and the fact that $1+L(t) \Delta^{+} g(t) \geq 1$ for all $t \in\left[t_{0}, t_{0}+T\right)$, ensure that

$$
v_{g}^{\prime}(t) \leq \frac{K(t) L(t) h(t)}{1+L(t) \Delta^{+} g(t)} \leq K(t) L(t) h(t), \quad g \text {-a.a. } t \in\left[t_{0}, t_{0}+T\right)
$$

Therefore, it follows from Fundamental Theorem of Calculus for the Lebesgue-Stieltjes integral, Theorem 2.7, that

$$
v(t)=v\left(t_{0}\right)+\int_{\left[t_{0}, t\right)} v_{g}^{\prime}(s) \mathrm{d} g(s) \leq \int_{\left[t_{0}, t\right)} K(s) L(s) h(s) \mathrm{d} g(s), \quad t \in\left[t_{0}, t_{0}+T\right]
$$

and, as a consequence, for all $t \in\left[t_{0}, t_{0}+T\right]$ we have

$$
\begin{aligned}
\int_{\left[t_{0}, t\right)} L(s) u(s) \mathrm{d} g(s) & =e_{L}\left(t, t_{0}\right) v(t) \\
& \leq e_{L}\left(t, t_{0}\right) \int_{\left[t_{0}, t\right)} K(s) L(s) h(s) \mathrm{d} g(s) \\
& =e_{L}\left(t, t_{0}\right) \int_{\left[t_{0}, t\right)} K(s) L(s)\left(e_{L}\left(s, t_{0}\right)\right)^{-1} \mathrm{~d} g(s) \\
& =\int_{\left[t_{0}, t\right)} K(s) L(s) \exp \left(\int_{[s, t)} \widetilde{L}(r) \mathrm{d} g(r)\right) \mathrm{d} g(s)
\end{aligned}
$$

Thus, it follows from 4.3 that

$$
u(t) \leq K(t)+\int_{\left[t_{0}, t\right)} K(s) L(s) \exp \left(\int_{[s, t)} \widetilde{L}(r) \mathrm{d} g(r)\right) \mathrm{d} g(s), \quad t \in\left[t_{0}, t_{0}+T\right)
$$

that is, (4.4) holds.
To prove (4.5), for each $t \in\left[t_{0}, t_{0}+T\right)$, define

$$
\psi_{t}(s)=\exp \left(\int_{[s, t)} \widetilde{L}(r) \mathrm{d} g(r)\right)=\frac{e_{L}\left(t, t_{0}\right)}{e_{L}\left(s, t_{0}\right)}, \quad s \in\left[t_{0}, t\right] .
$$

Then, it follows from (4.4) that for all $t \in\left[t_{0}, t_{0}+T\right)$,

$$
\begin{aligned}
u(t) & \leq K(t)+\int_{\left[t_{0}, t\right)} K(s) L(s) \psi_{t}(s) \mathrm{d} g(s) \\
& \leq K(t)\left(1+L(t) \Delta^{+} g(t)\right)+\int_{\left[t_{0}, t\right)} K(s)\left(1+L(s) \Delta^{+} g(s)\right) \frac{L(s) \psi_{t}(s)}{1+L(s) \Delta^{+} g(s)} \mathrm{d} g(s)
\end{aligned}
$$

Now, since $\varphi(t)=K(t)\left(1+L(t) \Delta^{+} g(t)\right)$ is nondecreasing, we have that

$$
u(t) \leq \varphi(t)\left(1+\int_{\left[t_{0}, t\right)} \frac{L(s) \psi_{t}(s)}{1+L(s) \Delta g(s)} \mathrm{d} g(s)\right), \quad t \in\left[t_{0}, t_{0}+T\right]
$$

On the other hand, Proposition 4.1 ensures that for all $t \in\left[t_{0}, t_{0}+T\right)$, the map $\psi_{t}$ is $g$ absolutely continuous on $\left[t_{0}, t\right]$ and

$$
\left(\psi_{t}\right)_{g}^{\prime}(s)=\frac{-L(s)}{1+L(s) \Delta g(s)} \psi_{t}(s) \quad g \text {-a.a. } s \in\left[t_{0}, t\right)
$$

This fact, together with the Fundamental Theorem of Calculus, Theorem 2.7, yields that

$$
u(t) \leq \varphi(t)\left(1-\int_{\left[t_{0}, t\right)}\left(\psi_{t}\right)_{g}^{\prime}(s) \mathrm{d} g(s)\right)=\varphi(t)\left(1-\left(\psi_{t}(t)-\psi_{t}\left(t_{0}\right)\right)\right)=\varphi(t) \psi_{t}\left(t_{0}\right)
$$

for all $t \in\left[t_{0}, t_{0}+T\right)$, from which the result follows.
Remark 4.4. The bound (4.4), under the corresponding hypotheses, is sharp. Indeed, let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing and left-continuous function, $K:\left[t_{0}, t_{0}+T\right) \rightarrow[0,+\infty)$ be constant and $L$ be $g$-integrable on $\left[t_{0}, t_{0}+T\right)$. The map $x(t)=K e_{L}\left(t, t_{0}\right), t \in\left[t_{0}, t_{0}+T\right]$, is $g$-absolutely continuous on $\left[t_{0}, t_{0}+T\right]$. As a consequence, and with the aid of Theorems 2.7 and 3.2 , we have that

$$
x(t)=K+\int_{\left[t_{0}, t\right)} L(s) K e_{L}\left(s, t_{0}\right) \mathrm{d} g(s), \quad t \in\left[t_{0}, t_{0}+T\right]
$$

that is, (4.3) holds. Furthermore, that same expression shows that (4.4) also holds with the equality.

This type of inequalities for Stieltjes integrals already exist in the literature, see for example $[6,8,11,12,17]$. Let us briefly discuss the relations between the mentioned references and Proposition 4.3. First, in [12, Theorem 7.5.3], the authors worked in the more general context of the Kurzweil-Stieltjes integral. Nevertheless, the results can be compared in the context of the Lebesgue-Stieltjes integral as the integrability in this sense implies the integrability in the Kurzweil-Stieltjes sense. In that case, we can see that the hypotheses required there are stronger than the ones in Proposition 4.3. Furthermore, it is possible to deduce through our next result, Corollary 4.5, that (4.5) gives a sharper bound than the one [12, Theorem 7.5.3]. A similar argument can be done for [8, Chapter 22], where the authors imposes some condition regarding the length of the jumps that the map $g$ presents to arrive to a similar inequality that is not as sharp as the one provided in Proposition 4.3. The same thing happens when we consider the generalized version of the Gronwall inequality in [17, Theorem 1.40]. For the particular setting in which we recover the usual Gronwall inequality (namely, when $\omega(r)=r)$ then we obtain the same inequality as in [12, Theorem 7.5.3], which we have already discussed. Now, for $[6,11]$, the authors obtained a Gronwall type inequality in the context of a certain family of linear operators. The operators there considered can be the LebesgueStieltjes integrals in this paper. In that case, the authors impose some conditions on the discontinuities of the map $g$, and moreover, the inequality is expressed using an unknown function introduced in [6], called Gronwall majorant. Hence, in the context of our work, the inequality in Proposition 4.3 provides more information.

Note that (4.5) in Proposition 4.3 becomes the usual Gronwall's inequality when the derivator $g$ is the identity map. Furthermore, as we mentioned before, we can obtain a different Gronwall type inequality involving the usual exponential map, i.e. not involving the modified map in (3.6). However, the bound in Proposition 4.3 is sharper than the one in the following result.

Corollary 4.5. Let $u, K, L:\left[t_{0}, t_{0}+T\right) \rightarrow[0,+\infty)$ be such that $L, K \cdot L, u \cdot L \in \mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+T\right)\right.$, $[0,+\infty)$ ). If (4.3) holds, then

$$
u(t) \leq K(t)+\int_{\left[t_{0}, t\right)} K(s) L(s) \exp \left(\int_{[s, t)} L(r) \mathrm{d} g(r)\right) \mathrm{d} g(s), \quad t \in\left[t_{0}, t_{0}+T\right)
$$

Moreover, if the map $\varphi(t)=K(t)(1+L(t) \Delta g(t))$ is nondecreasing, then

$$
u(t) \leq \varphi(t) \exp \left(\int_{\left[t_{0}, t\right)} L(r) \mathrm{d} g(r)\right), \quad t \in\left[t_{0}, t_{0}+T\right)
$$

Proof. Given the inequalities in Proposition 4.3, it is enough to show that $\widetilde{L} \leq L$ on $\left[t_{0}, t_{0}+T\right)$. Observe that $\widetilde{L}=L$ on $\left[t_{0}, t_{0}+T\right) \backslash D_{g}$. Thus, we only need to show the inequality for $\left[t_{0}, t_{0}+T\right) \cap D_{g}$.

For $t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}$, we have that $1+L(t) \Delta g(t)>0$. Now, since $\log (1+s) \leq s$ for $s \in(-1,+\infty)$, it follows that

$$
\widetilde{L}(t)=\frac{\log (1+L(t) \Delta g(t))}{\Delta g(t)} \leq \frac{L(t) \Delta g(t)}{\Delta g(t)}=L(t)
$$

which concludes the proof.
As in the classical setting, Gronwall's inequality allows us to obtain a uniqueness result for a general initial value problem under the assumption that the map defining the problem satisfies a Lipschitz condition. We present this information in the following result.

Theorem 4.6. Let $X \subset \mathbb{R}^{n}, x_{0} \in X$ and $f:\left[t_{0}, t_{0}+T\right) \times X \rightarrow \mathbb{R}^{n}$. If there exists $\tau \in(0, T]$ and $L \in \mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+\tau\right),[0,+\infty)\right)$ such that

$$
\|f(t, x)-f(t, y)\| \leq L(t)\|x-y\|, \quad \text { g-a.a. } t \in\left[t_{0}, t_{0}+\tau\right), \quad x, y \in X
$$

then the initial value problem

$$
\begin{equation*}
x_{g}^{\prime}(t)=f(t, x(t)), \quad \text { g-a.a. } t \in\left[t_{0}, t_{0}+T\right), \quad x\left(t_{0}\right)=x_{0} \tag{4.6}
\end{equation*}
$$

has at most one $g$-absolutely continuous solution on $\left[t_{0}, t_{0}+\tau\right)$.
Proof. Suppose that $x_{1}, x_{2} \in \mathcal{A C}_{g}\left(\left[t_{0}, t_{0}+\tau\right], \mathbb{R}^{n}\right)$ are two solutions of (4.6) on $\left[t_{0}, t_{0}+\tau\right)$. It follows from Theorem 2.7 that $f\left(\cdot, x_{i}(\cdot)\right) \in \mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+\tau\right), \mathbb{R}^{n}\right), i=1,2$. As a consequence, we have that the map $\left\|f\left(\cdot, x_{1}(\cdot)\right)-f\left(\cdot, x_{2}(\cdot)\right)\right\|$ is $g$-integrable over $\left[t_{0}, t_{0}+\tau\right)$.

Define $u(t)=\left\|x_{1}(t)-x_{2}(t)\right\|, t \in\left[t_{0}, t_{0}+\tau\right]$. Clearly, $u$ is nonnegative and bounded on $\left[t_{0}, t_{0}+\tau\right]$ as $x_{1}$ and $x_{2}$ are bounded, see [10, Proposition 5.3]. Hence, it follows that $u, u \cdot L \in \mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+\tau\right),[0,+\infty)\right)$. Furthermore, the Fundamental Theorem of Calculus yields that for $t \in\left[t_{0}, t_{0}+\tau\right]$,

$$
\begin{aligned}
u(t) & =\left\|\int_{\left[t_{0}, t\right)} f\left(s, x_{1}(s)\right) \mathrm{d} g(s)-\int_{\left[t_{0}, t\right)} f\left(s, x_{2}(s)\right) \mathrm{d} g(s)\right\| \\
& \leq \int_{\left[t_{0}, t\right)}\left\|f\left(s, x_{1}(s)\right)-f\left(s, x_{2}(s)\right)\right\| \mathrm{d} g(s) \leq \int_{\left[t_{0}, t\right)} L(s) u(s) \mathrm{d} g(s)
\end{aligned}
$$

Hence, (4.3) holds with $K=0$. As a consequence, (4.4) holds for $K=0$, which implies that $u=0$ on $\left[t_{0}, t_{0}+\tau\right)$, or equivalently, $x_{1}=x_{2}$ on that interval.

We can now combine Theorems 3.5 and 4.6 to obtain the following result which is, to some extend, a revision of [3, Proposition 6.8].

Theorem 4.7. Let $d, h \in \mathcal{L}_{g}^{1}\left(\left[t_{0}, t_{0}+T\right), \mathbb{R}\right)$ be such that $1-d(t) \Delta g(t) \neq 0$ for all $t \in$ $\left[t_{0}, t_{0}+T\right) \cap D_{g}$. Then the unique g-absolutely continuous solution of (3.1) is given by the map in (3.8). If, in particular, $1-d(t) \Delta g(t)>0$ for all $t \in\left[t_{0}, t_{0}+T\right) \cap D_{g}$, then the unique $g$-absolutely continuous solution of (3.1) matches (3.9).

## Acknowledgements

Ignacio Márquez Albés was partially supported by Xunta de Galicia under grants ED481A2017/095 and ED431C 2019/02.

## References

[1] F. E. Burk, A garden of integrals, Mathematical Association of America, Washington D.C., 2007. MR2311537
[2] F. J. Fernández, F. A. F. Tojo, Numerical solution of Stieltjes differential equations, Mathematics 8(2020), 1571. https://doi.org/10.3390/math8091571
[3] M. Frigon, R. López Pouso, Theory and applications of first-order systems of Stieltjes differential equations, Adv. Nonlinear Anal. 6(2017), No. 1, 13-36. https://doi.org/10. 1515/anona-2015-0158; MR3604936
[4] A. Froda, Sur la distribution des propriétés de voisinage des fonctions de variables réelles (in French), PhD thesis, Hermann, Paris, 1929.
[5] T. H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, Ann. Math. 20(1919), No. 4, 292-296. https://doi.org/ 10.2307/1967124; MR1502565
[6] J. V. Herod, A Gronwall inequality for linear Stieltjes integrals, Proc. Amer. Math. Soc. 23(1969), 34-36. https://doi.org/10. 2307/2037481; MR249557
[7] T. H. Hildebrandt, On systems of linear differentio-Stieltjes-integral equations, Illinois J. Math. 3(1959), No. 3, 352-373. MR105600
[8] J. Kurzweil, Generalized ordinary differential equations. Not absolutely continuous solutions, World Scientific Publishing, Singapore, 2012. https://doi .org/10.1142/9789814324038; MR2906899
[9] R. López Pouso, I. Márquez Albés, Resolution methods for mathematical models based on differential equations with Stieltjes derivatives, Electron. J. Qual. Theory Differ. Equ. 2019, No. 72, 1-15. https://doi.org/10.14232/ejqtde.2019.1.72; MR4019523
[10] R. López Pouso, A. Rodríguez, A new unification of continuous, discrete, and impulsive calculus through Stieltjes derivatives, Real Anal. Exchange 40(2014/15), No. 2, 319-353. https://doi.org/10.14321/realanalexch.40.2.0319; MR3499768
[11] A. B. Mingarelli, On a Stieltjes version of Gronwall's inequality, Proc. Amer. Math. Soc. 82(1981), 249-252. https://doi. org/10.2307/2043318; MR609660
[12] G. A. Monteiro, A. Slavík, M. Tvrdý, Kurzweil-Stieltjes integral. Theory and applications, World Scientific Publishing, Singapore, 2019. MR3839599
[13] W. Rudin, Real and complex analysis, McGraw-Hill, Singapore, 1987. MR924157
[14] B. Satco, G. Smyrlis, Periodic boundary value problems involving Stieltjes derivatives, J. Fixed Point Theory Appl. 22(2020), No. 94, 1-23. https://doi.org/10.1007/ s11784-020-00825-1; MR4161917
[15] E. Schechter, Handbook of analysis and its foundations, Academic Press, San Diego, California, 1997. MR1417259
[16] Š. Schwabik, Linear Stieltjes integral equations in Banach spaces, Math. Bohem. 124(1999), No. 4, 433-457. MR1722877
[17] Š. Schwabik, Generalized ordinary differential equations, World Scientific Publishing Co., River Edge, New Jersey, 1992. https://doi.org/10.1142/1875; MR1200241

# Weak damping for the Korteweg-de Vries equation* 

Roberto de A. Capistrano-Filho ${ }^{\boxtimes}$<br>Departamento de Matemática, Universidade Federal de Pernambuco (UFPE), 50740-545, Recife (PE), Brazil

Received 2 February 2021, appeared 2 June 2021
Communicated by Vilmos Komornik


#### Abstract

For more than 20 years, the Korteweg-de Vries equation has been intensively explored from the mathematical point of view. Regarding control theory, when adding an internal force term in this equation, it is well known that the Korteweg-de Vries equation is exponentially stable in a bounded domain. In this work, we propose a weak forcing mechanism, with a lower cost than that already existing in the literature, to achieve the result of the global exponential stability to the Korteweg-de Vries equation.


Keywords: KdV equation, stabilization, observability inequality, unique continuation property.
2020 Mathematics Subject Classification: 35Q53, 93B07, 93D15.

## 1 Introduction

### 1.1 Historical review

In 1834 John Scott Russell, a Scottish naval engineer, was observing the Union Canal in Scotland when he unexpectedly witnessed a very special physical phenomenon which he called a wave of translation [35]. He saw a particular wave traveling through this channel without losing its shape or velocity, and was so captivated by this event that he focused his attention on these waves for several years, not only built water wave tanks at his home conducting practical and theoretical research into these types of waves, but also challenged the mathematical community to prove theoretically the existence of his solitary waves and to give an a priori demonstration a posteriori.

A number of researchers took up Russell's challenge. Boussinesq was the first to explain the existence of Scott Russell's solitary wave mathematically. He employed a variety of asymptotically equivalent equations to describe water waves in the small-amplitude, longwave regime. In fact, several works presented to the Paris Academy of Sciences in 1871 and 1872, Boussinesq addressed the problem of the persistence of solitary waves of permanent form on a fluid interface [4-7]. It is important to mention that in 1876, the English physicist Lord Rayleigh obtained a different result [31].

[^5]After Boussinesq theory, the Dutch mathematicians D. J. Korteweg and his student G. de Vries [22] derived a nonlinear partial differential equation in 1895 that possesses a solution describing the phenomenon discovered by Russell,

$$
\begin{equation*}
\frac{\partial \eta}{\partial t}=\frac{3}{2} \sqrt{\frac{g}{l}} \frac{\partial}{\partial x}\left(\frac{1}{2} \eta^{2}+\frac{3}{2} \alpha \eta+\frac{1}{3} \beta \frac{\partial^{2} \eta}{\partial x^{2}}\right), \tag{1.1}
\end{equation*}
$$

in which $\eta$ is the surface elevation above the equilibrium level, $l$ is an arbitrary constant related to the motion of the liquid, $g$ is the gravitational constant, and $\beta=\frac{l^{3}}{3}-\frac{T l}{\rho g}$ with surface capillary tension $T$ and density $\rho$. The equation (1.1) is called the Korteweg-de Vries equation in the literature, often abbreviated as the KdV equation, although it had appeared explicitly in [7], as equation (283bis) in a footnote on page $360^{*}$.

Eliminating the physical constants by using the following change of variables

$$
t \rightarrow \frac{1}{2} \sqrt{\frac{g}{l \beta}} t, \quad x \rightarrow-\frac{x}{\beta}, \quad u \rightarrow-\left(\frac{1}{2} \eta+\frac{1}{3} \alpha\right)
$$

one obtains the standard Korteweg-de Vries (KdV) equation

$$
\begin{equation*}
u_{t}+6 u u_{x}+u_{x x x}=0 \tag{1.2}
\end{equation*}
$$

which is now commonly accepted as a mathematical model for the unidirectional propagation of small-amplitude long waves in nonlinear dispersive systems. It turns out that the equation is not only a good model for some water waves but also a very useful approximation model in nonlinear studies whenever one wishes to include and balance a weak nonlinearity and weak dispersive effects [27].

### 1.2 Motivation and setting of the problem

Consider the KdV equation (1.2). Let us introduce a source term in this equation as follows:

$$
\begin{equation*}
u_{t}+6 u u_{x}+u_{x x x}+f=0, \tag{1.3}
\end{equation*}
$$

where $f$ will be defined as

$$
\begin{equation*}
f:=G u(x, t)=1_{\omega}\left(u(x, t)-\frac{1}{|\omega|} \int_{\omega} u(x, t) d x\right) . \tag{1.4}
\end{equation*}
$$

Here, $1_{\omega}$ denotes the characteristic function of the set $\omega$. Notice that this term can be seen as a damping mechanism, which helps the energy of the system to dissipate. In fact, let us consider $\omega$ subset of a domain $\mathcal{M}:=\mathbb{T}$ or $\mathcal{M}:=\mathbb{R}$ and the total energy of the linear equation associated to (1.3), in this case, is given by

$$
\begin{equation*}
E_{s}(t)=\frac{1}{2} \int_{\mathcal{M}}|u|^{2}(x, t) d x \tag{1.5}
\end{equation*}
$$

Then, we can (formally) verify that

$$
\frac{d}{d t} \int_{\mathcal{M}}|u|^{2}(x, t) d x=-\|G u\|_{L^{2}(\mathcal{M})}^{2}, \text { for any } t \in \mathbb{R}
$$

[^6]The inequality above shows that the term $G$ plays the role of feedback mechanism and, consequently, we can investigate whether the solutions of (1.3) tend to zero as $t \rightarrow \infty$ and under what rate they decay.

Inspired by this, in our work we will study the full KdV equation from a control point of view posed in a bounded domain $(0, L) \subset \mathbb{R}$ with a weak forcing term $G h$ added as a control input, namely:

$$
\begin{cases}u_{t}+u_{x}+u u_{x}+u_{x x x}+G h=0 & \text { in }(0, L) \times(0, T),  \tag{1.6}\\ u(0, t)=u(L, t)=u_{x}(L, t)=0, & \text { in }(0, T), \\ u(x, 0)=u_{0}(x), & \text { in }(0, L) .\end{cases}
$$

Here, $G$ is the operator defined by

$$
\begin{equation*}
G h(x, t)=1_{\omega}\left(h(x, t)-\frac{1}{|\omega|} \int_{\omega} h(x, t) d x\right), \tag{1.7}
\end{equation*}
$$

where $h$ is considered as a new control input with $\omega \subset(0, L)$ and $1_{\omega}$ denotes the characteristic function of the set $\omega$.

Thus, we are interested in proving the stability for solutions of (1.6), which can be expressed in the following natural issue.
Stabilization problem: Can one find a feedback control law $h$ so that the resulting closed-loop system (1.6) is asymptotically stable when $t \rightarrow \infty$ ?

### 1.3 Previous results

The study of the controllability and stabilization to the KdV equation started with the works of Russell and Zhang [37] for a system with periodic boundary conditions and an internal control. Since then, both the controllability and the stabilization have been intensively studied. In particular, the exact boundary controllability of $K d V$ on a finite domain was investigated in e.g. [10, 11, 14-16, 32, 33, 39].

Most of these works deal with the following system

$$
\begin{cases}u_{t}+u_{x}+u_{x x x}+u u_{x}=0 & \text { in }(0, T) \times(0, L),  \tag{1.8}\\ u(t, 0)=h_{1}(t), u(t, L)=h_{2}(t), u_{x}(t, L)=h_{3}(t) & \text { in }(0, T),\end{cases}
$$

in which the boundary data $h_{1}, h_{2}, h_{3}$ can be chosen as control inputs.
The boundary control problem of the KdV equation was first studied by Rosier [32] who considered system (1.8) with only one boundary control input $h_{3}$ (i.e., $h_{1}=h_{2}=0$ ) in action. He showed that the system (1.8) is locally exactly controllable in the space $L^{2}(0, L)$. Precisely, the result can be read as follows:

Theorem $\mathcal{A}$ ([32]). Let $T>0$ be given and assume

$$
\begin{equation*}
L \notin \mathcal{N}:=\left\{2 \pi \sqrt{\frac{j^{2}+l^{2}+j l}{3}}: j, l \in \mathbb{N}^{*}\right\} . \tag{1.9}
\end{equation*}
$$

There exists a $\delta>0$ such that if $\phi, \psi \in L^{2}(0, L)$ satisfies

$$
\|\phi\|_{L^{2}(0, L)}+\|\psi\|_{L^{2}(0, L)} \leq \delta
$$

then one can find a control input $h_{3} \in L^{2}(0, T)$ such that the system (1.8), with $h_{1}=h_{2}=0$, admits a solution

$$
u \in C\left([0, T] ; L^{2}(0, L)\right) \cap L^{2}\left(0, T ; H^{1}(0, L)\right)
$$

satisfying

$$
u(x, 0)=\phi(x), u(x, T)=\psi(x) .
$$

Theorem $\mathcal{A}$ was first proved for the associated linear system using the Hilbert Uniqueness Method due J.-L. Lions [24] without the smallness assumption on the initial state $\phi$ and the terminal state $\psi$. The linear result was then extended to the nonlinear system to obtain Theorem $\mathcal{A}$ by using the contraction mapping principle.

Still regarding the KdV equation in a bounded domain, Chapouly [12] studied the exact controllability to the trajectories and the global exact controllability of a nonlinear KdV in a bounded interval. Precisely, first, she introduced two more controls as follows

$$
\begin{cases}u_{t}+u_{x}+u u_{x}+u_{x x x}=g(t), & x \in(0, L), t>0,  \tag{1.10}\\ u(0, t)=h_{1}(t), u(L, t)=h_{2}(t), u_{x}(L, t)=0, & t>0,\end{cases}
$$

where $g=g(t)$ is independent of the spatial variable $x$ and is considered as a new control input. Then, Chapouly proved that, thanks to these three controls, the global controllability to the trajectories, for any positive time T, holds. Finally, she introduced a fourth control on the first derivative at the right endpoint, namely,

$$
\begin{cases}u_{t}+u_{x}+u u_{x}+u_{x x x}=g(t), & x \in(0, L), t>0, \\ u(0, t)=h_{1}(t), u(L, t)=h_{2}(t), u_{x}(L, t)=h_{3}(t), & t>0,\end{cases}
$$

where $g=g(t)$ has the same structure as in (1.10). With this equation in hand, she showed the global exact controllability, for any positive time T .

Considering now a periodic domain $\mathbb{T}$, Laurent et al. in [23] worked with the following equation:

$$
\begin{equation*}
u_{t}+u u_{x}+u_{x x x}=0, \quad x \in \mathbb{T}, t \in \mathbb{R} . \tag{1.11}
\end{equation*}
$$

Equation (1.11) is known to possess an infinite set of conserved integral quantities, of which the first three are

$$
I_{1}(t)=\int_{\mathbb{T}} u(x, t) d x, \quad I_{2}(t)=\int_{\mathbb{T}} u^{2}(x, t) d x
$$

and

$$
I_{3}(t)=\int_{\mathbb{T}}\left(u_{x}^{2}(x, t)-\frac{1}{3} u^{3}(x, t)\right) d x .
$$

From the historical origins [4,22,27] of the KdV equation, involving the behavior of water waves in a shallow channel, it is natural to think of $I_{1}$ and $I_{2}$ as expressing conservation of volume (or mass) and energy, respectively. The Cauchy problem for equation (1.11) has been intensively studied for many years (see $[3,19,21,38]$ and the references therein).

With respect to control theory, Laurent et al. [23] studied the equation (1.11) from a control point of view with a forcing term $f=f(x, t)$ added to the equation as a control input:

$$
\begin{equation*}
u_{t}+u u_{x}+u_{x x x}=f, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}, \tag{1.12}
\end{equation*}
$$

where $f$ is assumed to be supported in a given open set $\omega \subset \mathbb{T}$. However, in the periodic domain, control problems were first studied by Russell and Zhang in [36,37]. In their works, in order to keep the mass $I_{1}(t)$ conserved, the control input $f(x, t)$ is chosen to be of the form

$$
\begin{equation*}
f(x, t)=[G h](x, t):=g(x)\left(h(x, t)-\int_{\mathbb{T}} g(y) h(y, t) d y\right), \tag{1.13}
\end{equation*}
$$

where $h$ is considered as a new control input, and $g(x)$ is a given non-negative smooth function such that $\{g>0\}=\omega$ and

$$
2 \pi[g]=\int_{\mathbb{T}} g(x) d x=1
$$

For the chosen $g$, it is easy to see that

$$
\frac{d}{d t} \int_{\mathbb{T}} u(x, t) d x=\int_{\mathbb{T}} f(x, t) d x=0, \quad \text { for any } t \in \mathbb{R}
$$

for any solution $u=u(x, t)$ of the system

$$
\begin{equation*}
u_{t}+u u_{x}+u_{x x x}=G h . \tag{1.14}
\end{equation*}
$$

Thus, the mass of the system is indeed conserved. Therefore, the following results are due to Russell and Zhang.

Theorem $\mathcal{B}$ ([37]). Let $s \geq 0$ and $T>0$ be given. There exists a $\delta>0$ such that for any $u_{0}, u_{1} \in$ $H^{s}(\mathbb{T})$ with $\left[u_{0}\right]=\left[u_{1}\right]$ satisfying

$$
\left\|u_{0}\right\|_{H^{s}} \leq \delta, \quad\left\|u_{1}\right\|_{H^{s}} \leq \delta,
$$

one can find a control input $h \in L^{2}\left(0, T ; H^{s}(\mathbb{T})\right)$ such that the system (1.14) admits a solution $u \in$ $C\left([0, T] ; H^{s}(\mathbb{T})\right)$ satisfying $u(x, 0)=u_{0}(x), u(x, T)=u_{1}(x)$.

Note that one can always find an appropriate control input $h$ to guide system (1.12) from a given initial state $u_{0}$ to a terminal state $u_{1}$ so long as their amplitudes are small and $\left[u_{0}\right]=\left[u_{1}\right]$. With this result the two following questions arise naturally, which have already been cited in this work.

Question 1: Can one still guide the system by choosing appropriate control input h from a given initial state $u_{0}$ to a given terminal state $u_{1}$ when $u_{0}$ or $u_{1}$ have large amplitude?
Question 2: Do the large amplitude solutions of the closed-loop system (1.12) decay exponentially as $t \rightarrow \infty$ ?

Laurent et al. gave the positive answers to these questions:
Theorem $\mathcal{C}$ ([23]). Let $s \geq 0, R>0$ and $\mu \in \mathbb{R}$ be given. There exists a $T>0$ such that for any $u_{0}, u_{1} \in H^{s}(\mathbb{T})$ with $\left[u_{0}\right]=\left[u_{1}\right]=\mu$ are such that

$$
\left\|u_{0}\right\|_{H^{s}} \leq R, \quad\left\|u_{1}\right\|_{H^{s}} \leq R,
$$

then one can find a control input $h \in L^{2}\left(0, T ; H^{s}(\mathbb{T})\right)$ such that the system (1.12) admits a solution $u \in C\left([0, T] ; H^{s}(\mathbb{T})\right)$ satisfying

$$
u(x, 0)=u_{0}(x) \quad \text { and } \quad u(x, T)=u_{1}(x) .
$$

Theorem $\mathcal{D}$ ([23]). Let $s \geq 0, R>0$ and $\mu \in \mathbb{R}$ be given. There exists a $k>0$ such that for any $u_{0} \in H^{s}(\mathbb{T})$ with $\left[u_{0}\right]=\mu$ the corresponding solution of the system (1.12) satisfies

$$
\left\|u(\cdot, t)-\left[u_{0}\right]\right\|_{H^{s}} \leq \alpha_{s, \mu}\left(\left\|u_{0}-\left[u_{0}\right]\right\|_{H^{0}}\right) e^{-k t}\left\|u_{0}-\left[u_{0}\right]\right\|_{H^{s}} \quad \text { for all } t>0,
$$

where $\alpha_{s, \mu}: \mathbb{R}^{+} \longrightarrow \mathbb{R}^{+}$is a nondecreasing continuous function depending on $s$ and $\mu$.
These results are established with the aid of certain properties of propagation of compactness and regularity in Bourgain spaces for the solutions of the associated linear system. Finally, with Slemrod's feedback law, the resulting closed-loop system is shown to be locally exponentially stable with an arbitrarily large decay rate.

Still with respect to problems of stabilization, Pazoto [28] proved the exponential decay for the energy of solutions of the Korteweg-de Vries equation in a bounded interval with a localized damping term, precisely, with a term $a=a(x)$ satisfying

$$
\left\{\begin{array}{l}
a \in L^{\infty}(0, L) \text { and } a(x) \geq a_{0}>0 \text { a.e. in } \omega  \tag{1.15}\\
\text { where } \omega \text { is a nonempty open subset of }(0, L)
\end{array}\right.
$$

With this mechanism the author showed that

$$
\frac{\mathrm{d} E}{\mathrm{~d} t}=-\int_{0}^{L} a(x)|u(x, t)|^{2} \mathrm{~d} x-\frac{1}{2}\left|u_{x}(0, t)\right|^{2}
$$

with

$$
E(t)=\frac{1}{2} \int_{0}^{L}|u(x, t)|^{2} \mathrm{~d} x
$$

This indicates that the term $a(x) u$ in the equation plays the role of a feedback damping mechanism. Finally, following the method in Menzala et al. [26] which combines energy estimates, multipliers and compactness arguments, the problem is reduced to prove the unique continuation of weak solutions. The result proved by the author can be read as follows.

Theorem $\mathcal{E}$ ([28]). For any $L>0$, any damping potential a satisfying (1.15) and $R>0$, there exist $c=c(R)>0$ and $\mu=\mu(R)>0$ such that

$$
E(t) \leq c\left\|u_{0}\right\|_{L^{2}(0, L)}^{2} \mathrm{e}^{-\mu t}
$$

holds for all $t \geq 0$ and any solution of

$$
\begin{cases}u_{t}+u_{x}+u u_{x}+u_{x x x}+a(x) u=0 & \text { in }(0, T) \times(0, L)  \tag{1.16}\\ u(t, 0)=u(t, L)=u_{x}(t, L)=0 & \text { in }(0, T) \\ u(0, x)=u_{0}(x) & \text { in }(0, L)\end{cases}
$$

with $u_{0} \in L^{2}(0, L)$ such that $\left\|u_{0}\right\|_{L^{2}(0, L)} \leq R$.
Massarolo et al. showed in [25] that a very weak amount of additional damping stabilizes the KdV equation. In particular, a damping mechanism dissipating the $L^{2}-$ norm as $a()$ does is not needed. Dissipating the $H^{-1}$ - norm proves to be. For instance, one can take the damping term $B u$ instead of $a(x) u$, where $B u$ is defined by

$$
B=1_{\omega}\left(-\frac{\mathrm{d}^{2}}{\mathrm{~d} x^{2}}\right)^{-1}=1_{\omega}(-\Delta)^{-1}
$$

where $1_{\omega}$ denotes the characteristic function of the set $\omega,\left(-d^{2} / d x^{2}\right)^{-1}$ is the inverse of the Laplace operator with Dirichlet boundary conditions (on the boundary of $\omega \subset(0, L)$ ). Under the above considerations, they observed that (formally) the operator $B$ satisfies

$$
\begin{aligned}
\int_{0}^{L} u B u \mathrm{~d} x & =\int_{0}^{L} u\left[-1_{\omega} \Delta^{-1} u\right] \mathrm{d} x=-\int_{\omega}\left(\Delta^{-1} u\right) \Delta\left(\Delta^{-1} u\right) \mathrm{d} x \\
& =-\left.\Delta^{-1} u\left[\Delta^{-1} u\right]_{x}\right|_{\partial \omega}+\int_{\omega}\left|\left[\Delta^{-1} u\right]_{x}\right|^{2} \mathrm{~d} x \\
& =\left\|\left[\Delta^{-1} u\right]_{x}\right\|_{L^{2}(\omega)}^{2}=\left\|\Delta^{-1} u\right\|_{H_{0}^{1}(\omega)}^{2}=\|u\|_{H^{-1}(\omega)}^{2} .
\end{aligned}
$$

Consequently, the total energy $E(t)$ associated with (1.16) with $B u$ instead of $a(x) u$, satisfies

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \int_{0}^{L}|u(x, t)|^{2} \mathrm{~d} x=-u_{x}^{2}(0, t)-\|u\|_{H^{-1}(\omega)}^{2}
$$

where

$$
E(t)=\int_{0}^{L}|u(x, t)|^{2} \mathrm{~d} x .
$$

This indicates that the term $B u$ plays the role of a feedback damping mechanism. Consequently, they investigated whether $E(t)$ tends to zero as $t \rightarrow \infty$ and the uniform rate at which it may decay, showing the similar result as in Theorem $\mathcal{E}$.

To finish that small sample of the previous works, let us present another result of controllability for the KdV equation posed on a bounded domain. Recently, the author in collaboration with Pazoto and Rosier, showed in [9] results for the following system,

$$
\begin{cases}u_{t}+u_{x}+u u_{x}+u_{x x x}=1_{\omega} f(t, x) & \text { in }(0, T) \times(0, L),  \tag{1.17}\\ u(t, 0)=u(t, L)=u_{x}(t, L)=0 & \text { in }(0, T), \\ u(0, x)=u_{0}(x) & \text { in }(0, L),\end{cases}
$$

considering $f$ as a control input and $1_{\omega}$ is a characteristic function supported on $\omega \subset(0, L)$. Precisely, when the control acts in a neighborhood of $x=L$, they obtained the exact controllability in the weighted Sobolev space $L_{\frac{1}{L-x} d x}^{2}$ defined as

$$
L_{\frac{1}{L-x} d x}^{2}:=\left\{u \in L_{l o c}^{1}(0, L) ; \int_{0}^{L} \frac{|u(x)|^{2}}{L-x} d x<\infty\right\} .
$$

More precisely, they proved the following result:
Theorem $\mathcal{F}$ [9]: Let $T>0, \omega=\left(l_{1}, l_{2}\right)=(L-v, L)$ where $0<v<L$. Then, there exists $\delta>0$ such that for any $u_{0}, u_{1} \in L_{\frac{1}{L-x} d x}^{2}$ with

$$
\left\|u_{0}\right\|_{L^{2} \frac{1}{L-x} d x} \leq \delta \text { and }\left\|u_{1}\right\|_{L^{2} \frac{1}{L-x} d x} \leq \delta
$$

one can find a control input $f \in L^{2}\left(0, T ; H^{-1}(0, L)\right)$ with $\operatorname{supp}(f) \subset(0, T) \times \omega$ such that the solution $u \in C^{0}\left([0, L], L^{2}(0, L)\right) \cap L^{2}\left(0, T, H^{1}(0, L)\right)$ of (1.17) satisfies $u(T,)=.u_{1}$ in $(0, L)$ and $u \in C^{0}\left([0, T], L_{L-x}^{2} d x\right)$. Furthermore, $f \in L_{(T-t) d t}^{2}\left(0, T, L^{2}(0, L)\right)$.

We caution that this is only a small sample of the extant works in this field. Now, we are able to present our result in this manuscript.

### 1.4 Main result and heuristic of the paper

The aim of this manuscript is to address the stabilization issue for the KdV equation on a bounded domain with a weak source (or forcing) term, as a distributed control, namely

$$
\begin{cases}u_{t}+u_{x}+u u_{x}+u_{x x x}+G h=0, & \text { in }(0, L) \times(0, T),  \tag{1.18}\\ u(0, t)=u(L, t)=u_{x}(L, t)=0, & \text { in }(0, T), \\ u(x, 0)=u_{0}(x), & \text { in }(0, L),\end{cases}
$$

where $G$ is the operator defined by (1.7).
Notice that with a good choose of $G h$, that is,

$$
\begin{equation*}
G h:=G u(x, t)=1_{\omega}\left(u(x, t)-\frac{1}{|\omega|} \int_{\omega} u(x, t) d x\right), \tag{1.19}
\end{equation*}
$$

the energy associate

$$
I_{2}(t)=\int_{0}^{L} u^{2}(x, t) d x
$$

verify that

$$
\frac{d}{d t} \int_{0}^{L} u^{2}(x, t) d x \leq-\|G u\|_{L^{2}(0, L)}^{2}, \quad \text { for any } t>0
$$

at least for the linear system

$$
u_{t}+u_{x}+u_{x x x}+G h=0, \quad \text { in }(0, L) \times\{t>0\} .
$$

Consequently, we can investigate whether the solutions of this equation tend to zero as $t \rightarrow \infty$ and under what rate they decay. To be precise, the main result of the work, give us an answer to the stabilization problem for the system (1.6)-(1.7), proposed on the beginning of this paper, and will be state in the following form.

Theorem 1.1. Let $T>0$. Then, for every $R_{0}>0$ there exist constants $C>0$ and $k>0$, such that, for any $u_{0} \in L^{2}(0, L)$ with

$$
\left\|u_{0}\right\|_{L^{2}(0, L)} \leq R_{0},
$$

the corresponding solution $u$ of (1.6) satisfies

$$
\|u(\cdot, t)\|_{L^{2}(0, L)} \leq C e^{-k t}\left\|u_{0}\right\|_{L^{2}(0, L)}, \quad \forall t>0 .
$$

Note that our goal in this work is to give an answer for the stabilization problem that was mentioned at the beginning of this introduction. Is important to point out that a similar feedback law was used in [37] and, more recently, in [23] for the Korteweg-de Vries equation, to prove a globally uniform exponential result in a periodic domain. In [23,37] the damping with a null mean was introduced to conserve the integral of the solution, which for KdV represents the mass (or volume) of the fluid.

In the context presented in this manuscript, our result improves earlier works on the subject, for example, [28]. Roughly speaking, differently from what was proposed by [23,37], in this work, the weak damping (1.7) is to have a lower cost than the one presented in [28] in the sense of that we can remove a medium term in the mechanisms proposed in these works and still have positive result of stabilization of the KdV equation.

Observe that the control used in [28], is formally the first part of the following forcing term:

$$
G h(x, t)=1_{\omega}\left(h(x, t)-\frac{1}{|\omega|} \int_{\omega} h(x, t) d x\right),
$$

where $\omega \subset(0, L)$. In fact, to see this, in [28], define $a(x):=-1_{\omega}$ in the above equality and just forget the remaining term. Thus, due to these considerations, we do not need a strong mechanism acting as control input. Surely, of what was shown in this article, to achieve the stability result for the KdV equation, is that the forcing operator Gh can be taken as a function supported in $\omega$ removing the medium term associated to the first term of the control mechanism.

Here, it is important point out that, the week damping mechanism is related with respect to the cost of the stabilization, as mentioned previously, which is different in the context of [25], where the authors proves that the energy of the system dissipates in the $H^{-1}$-norm instead of $L^{2}$-norm.

Concerning to the stabilization problem, the main ingredient to prove Theorem 1.1 is the Carleman estimate for the linear problem proved by Capistrano-Filho et al. in [9]. This estimate together with the energy estimate and compactness arguments reduces the problem to prove the Unique Continuation Property (UCP) for the solutions of the nonlinear problem, precisely, the following result is showed.
UCP: Let $L>0$ and $T>0$ be two real numbers, and let $\omega \subset(0, L)$ be a nonempty open set. If $v \in L^{\infty}\left(0, T ; H^{1}(0, L)\right)$ solves

$$
\begin{cases}v_{t}+v_{x}+v_{x x x}+v v_{x}=0, & \text { in }(0, L) \times(0, T), \\ v(0, t)=v(L, t)=0, & \text { in }(0, T), \\ v=c, & \text { in } \omega \times(0, T),\end{cases}
$$

for some $c \in \mathbb{R}$. Thus, $v \equiv c$ in $(0, L) \times(0, T)$, where $c \in \mathbb{R}$.
It is important to point out here that the previous UCP was first proved by Rosier and Zhang in [34]. In this way, to sake of completeness, we revisited this result now using the Carleman estimate proved by the author in [9].

### 1.5 Structure of the work

To end our introduction, we present the outline of the manuscript: In Section 2, we present some estimates for the KdV equation which will be used in the course of the work. Section 3 is devoted to present the proof of Theorem 1.1, that is, give the answer to the stabilization problem. Comments of our result as well as some extensions for other models are presented in Section 4. Finally, on the Appendix A, we will give a sketch how to prove the unique continuation property (UCP) presented above.

## 2 Well-posedness for KdV equation

In this section, we will review a series of estimates for the KdV equation, namely,

$$
\begin{cases}u_{t}+u_{x}+u u_{x}+u_{x x x}=f, & \text { in }(0, L) \times(0, T),  \tag{2.1}\\ u(0, t)=u(L, t)=u_{x}(L, t)=0, & \text { in }(0, T), \\ u(x, 0)=u_{0}(x), & \text { in }(0, L),\end{cases}
$$

which will be borrowed of [32]. Here $f=f(t, x)$ is a function which stands for the control of the system.

### 2.1 The linearized $K d V$ equation

The well-posedness of the problem (2.1), with $f \equiv 0$, was proved by Rosier [32]. He notice that operator $A=-\frac{\partial^{3}}{\partial x^{3}}-\frac{\partial}{\partial x}$ with domain

$$
D(A)=\left\{w \in H^{3}(0, L) ; w(0)=w(L)=w_{x}(L)=0\right\} \subseteq L^{2}(0, L)
$$

is the infinitesimal generator of a strongly continuous semigroup of contractions in $L^{2}(0, L)$.
Theorem 2.1. Let $u_{0} \in L^{2}(0, L)$ and $f \equiv 0$. There exists a unique weak solution $u=S(\cdot) u_{0}$ of (2.1) such that

$$
\begin{equation*}
u \in C\left([0, T] ; L^{2}(0, L)\right) \cap H^{1}\left(0, T ; H^{-2}(0, L)\right) . \tag{2.2}
\end{equation*}
$$

Moreover, if $u_{0} \in D(A)$, then (2.1) has a unique (classical) solution $u$ such that

$$
\begin{equation*}
u \in C([0, T] ; D(A)) \cap C^{1}\left(0, T ; L^{2}(0, L)\right) . \tag{2.3}
\end{equation*}
$$

An additional regularity result for the weak solutions of the linear system associated to system (2.1) was also established in [32]. The result can be read as follows.

Theorem 2.2. Let $u_{0} \in L^{2}(0, L), G w \equiv 0$ and $u=S(\cdot) u_{0}$ the weak solution of (2.1). Then, $u \in$ $L^{2}\left(0, T ; H^{1}(0, L)\right)$ and there exists a positive constant $c_{0}$ such that

$$
\begin{equation*}
\|u\|_{L^{2}\left(0, T ; H^{1}(0, L)\right)} \leq c_{0}\left\|u_{0}\right\|_{L^{2}(0, L)} . \tag{2.4}
\end{equation*}
$$

Moreover, there exist two positive constants $c_{1}$ and $c_{2}$ such that

$$
\begin{equation*}
\left\|u_{x}(\cdot, 0)\right\|_{L^{2}(0, T)}^{2} \leq c_{1}\left\|u_{0}\right\|_{L^{2}(0, L)} \tag{2.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|u_{0}\right\|_{L^{2}(0, L)} \leq \frac{1}{T}\|u\|_{L^{2}\left(0, T ; L^{2}(0, L)\right)}^{2}+c_{2}\left\|u_{x}(\cdot, 0)\right\|_{L^{2}(0, T)}^{2} \tag{2.6}
\end{equation*}
$$

### 2.2 The nonlinear $K d V$ equation

In this section we prove the well-posedness of the following system

$$
\begin{cases}u_{t}+u_{x}+u u_{x}+u_{x x x}=G w, & \text { in }(0, L) \times(0, T),  \tag{2.7}\\ u(0, t)=u(L, t)=u_{x}(L, t)=0, & \text { in }(0, T), \\ u(x, 0)=u^{0}(x), & \text { in }(0, L) .\end{cases}
$$

To solve the problem we write the solution of (2.7) as follows

$$
u=S(t) u_{0}+u_{1}+u_{2},
$$

where $(S(t))_{t \geq 0}$ denotes the semigroup associated with the operator $A u=-u^{\prime \prime \prime}-u^{\prime}$ with domain $\mathcal{D}(A)$ dense in $L^{2}(0, L)$ defined by

$$
\mathcal{D}(A)=\left\{v \in H^{3}(0, L) ; v(0)=v(L)=v^{\prime}(L)=0\right\},
$$

and $u_{1}$ and $u_{2}$ are (respectively) solutions of two non-homogeneous problems

$$
\begin{cases}u_{1 t}+u_{1 x}+u_{1 x x x}=G w, & \text { in } \omega \times(0, T),  \tag{2.8}\\ u_{1}(0, t)=u_{1}(L, t)=u_{1 x}(L, t)=0, & \text { in }(0, T), \\ u_{1}(x, 0)=0, & \text { in }(0, L)\end{cases}
$$

and

$$
\begin{cases}u_{2 t}+u_{2 x}+u_{2 x x x}=f, & \text { in }(0, L) \times(0, T),  \tag{2.9}\\ u_{2}(0, t)=u_{2}(L, t)=u_{2 x}(L, t)=0, & \text { in }(0, T), \\ u_{2}(x, 0)=0, & \text { in }(0, L),\end{cases}
$$

where $f=-u_{2} u_{2 x}$ and $w$ is solution of the following adjoint system

$$
\begin{cases}-w_{t}-w_{x}-w_{x x x}=0, & \text { in }(0, L) \times(0, T),  \tag{2.10}\\ w(0, t)=w(L, t)=w_{x}(0, t)=0, & \text { in }(0, T) \\ w(x, T)=0(x), & \text { in }(0, L)\end{cases}
$$

Let us define the following map

$$
\Psi: w \in L^{2}\left(0, T ; L^{2}(0, L)\right) \longmapsto u_{1} \in C\left([0, T] ; L^{2}(0, L)\right) \cap L^{2}\left(0, T ; H^{1}(0, L)\right)=: B
$$

endowed with norm

$$
\left\|u_{1}\right\|_{B}:=\sup _{t \in[0, T]}\left\|u_{1}(\cdot, t)\right\|_{L^{2}(0, L)}+\left(\int_{0}^{T}\left\|u_{1}(\cdot, t)\right\|_{H^{1}(0, L)}^{2} d t\right)^{\frac{1}{2}}
$$

be the map which associates with $w$ the weak solution of (2.8). Observe that, by using Theorem 2.2 the map $u_{0} \in L^{2}(0, L) \mapsto S(\cdot) u^{0} \in B$ is continuous. Furthermore, the following proposition holds true.

Proposition 2.3. The function $\Psi$ is a (linear) continuous map.
Proof. Indeed, let us divide the proof in two parts.

## First part.

Notice that in (2.8) $w$ is the solution of (2.10), thus,

$$
g(x, t)=G w(x, t) \in C^{1}\left([0, T] ; L^{2}(0, L)\right)
$$

and from classical results concerning such non-homogeneous problems (see [29]) we obtain a unique solution

$$
\begin{equation*}
u_{1} \in C([0, T] ; \mathcal{D}(A)) \cap C^{1}\left([0, T] ; L^{2}(0, L)\right) \tag{2.11}
\end{equation*}
$$

of (2.8). Additionally, the following estimate can be proved:

$$
\begin{equation*}
\int_{0}^{T}\|G u\|_{L^{2}(0, L)} d t \leq C T\|u\|_{Y_{0, T}}, \tag{2.12}
\end{equation*}
$$

where,

$$
Y_{0, T}=C\left([0, T] ; L^{2}(0, T)\right) \cap L^{2}\left([0, T] ; H^{1}(0, L)\right) .
$$

In fact, by a direct computation, we have

$$
\begin{aligned}
\int_{0}^{T}\|G u\|_{L^{2}(0, L)}^{2} d t & =\int_{0}^{T}\left(\int_{\omega} u^{2} d x-|\omega|^{-1}\left(\int_{\omega} u d x\right)^{2}\right)^{1 / 2} d t \\
& \leq \int_{0}^{T}\left(\int_{0}^{L} u^{2} d x\right)^{1 / 2} d t \leq T| | u \|_{\gamma_{0, T}}
\end{aligned}
$$

Thus, (2.12) follows.

## Second part.

Now, we will prove some estimates by multipliers method. Consider $u_{0}(x) \in \mathcal{D}(A)$. Let $w \in L^{2}\left(0, T ; L^{2}(0, L)\right)$ and $q \in C^{\infty}([0, T] \times[0, L])$. Multiplying (2.8) by $q u_{1}$, we obtain

$$
\begin{equation*}
\int_{0}^{S} \int_{0}^{L} q u_{1}\left(u_{1 t}+u_{1 x}+u_{1 x x x}\right) d x d t=\int_{0}^{S} \int_{0}^{L} q u_{1}(G w) d x d t \tag{2.13}
\end{equation*}
$$

where $S \in[0, T]$. Using (2.8) (and Fubini's theorem) we get:

$$
\begin{align*}
& -\int_{0}^{S} \int_{0}^{L}\left(q_{t}+q_{x}+q_{x x x}\right) \frac{u_{1}^{2}}{2} d x d t+\int_{0}^{L}\left(\frac{q u_{1}^{2}}{2}\right)(x, S) d x  \tag{2.1}\\
& \quad+\frac{3}{2} \int_{0}^{S} \int_{0}^{L} q_{x} u_{1 x}^{2} d x d t+\frac{1}{2} \int_{0}^{S}\left(q u_{1 x}^{2}\right)(0, t) d t=\int_{0}^{S} \int_{0}^{L}\left(q u_{1}\right)(G w) d x d t .
\end{align*}
$$

Choosing $q=1$ it follows that

$$
\begin{aligned}
\int_{0}^{L} u_{1}(x, S)^{2} d x+\int_{0}^{S} u_{1 x}(0, t)^{2} d t & =\int_{0}^{S} \int_{0}^{L} u_{1}(G w) d x d t \\
& \leq \frac{1}{2}\|u\|_{L^{2}\left(0, S ; L^{2}(0, L)\right)}+\frac{1}{2}\|G w\|_{L^{2}\left(0, S ; L^{2}(0, L)\right)}^{2}
\end{aligned}
$$

Then, we get

$$
\begin{equation*}
\left\|u_{1}\right\|_{C\left([0, T] ; L^{2}(0, L)\right)} \leq C\|G w\|_{L^{2}\left(0, T ; L^{2}(0, L)\right)}, \tag{2.15}
\end{equation*}
$$

which yields

$$
\begin{equation*}
\left\|u_{1}\right\|_{L^{2}((0, T) \times(0, L))} \leq C\|G w\|_{L^{2}\left(0, T ; L^{2}(0, L)\right)} \tag{2.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|u_{1 x}(0, \cdot)\right\|_{L^{2}(0, T)} \leq C\|G w\|_{L^{2}\left(0, T ; L^{2}(0, L)\right)} . \tag{2.17}
\end{equation*}
$$

Now take $q(x, t)=x$ and $S=T$, (2.14) gives,

$$
\begin{equation*}
-\int_{0}^{T} \int_{0}^{L} \frac{u_{1}^{2}}{2} d x d t+\int_{0}^{L} \frac{x}{2} u_{1}^{2}(x, T) d x+\frac{3}{2} \int_{0}^{T} \int_{0}^{L} u_{1 x}^{2} d x d t=\int_{0}^{T} \int_{0}^{L} x u_{1}(G w) d x d t \tag{2.18}
\end{equation*}
$$

Hence

$$
\int_{0}^{T} \int_{0}^{L} u_{1 x}^{2} d x d t \leq \frac{1}{3}\left(\int_{0}^{T} \int_{0}^{L} u_{1}^{2} d x d t+L\left\{\int_{0}^{T} \int_{0}^{L} u^{2} d x d t+\int_{0}^{T} \int_{0}^{L}(G w)^{2} d x d t\right\}\right)
$$

and then, using (2.16),

$$
\begin{equation*}
\left\|u_{1}\right\|_{L^{2}\left(0, T ; H^{1}(0, L)\right)} \leq C(T, L)\|G w\|_{L^{2}\left(0, T ; L^{2}(0, L)\right)} . \tag{2.19}
\end{equation*}
$$

Using (2.15), (2.19), (2.12) and the density of $\mathcal{D}(A)$ in $L^{2}(0, L)$, we deduce that $\Psi$ is a linear continuous map, proving thus the proposition.

The next result, proved in [32, Proposition 4.1], give us that nonlinear system (2.9) is wellposed.

Proposition 2.4. The following items can be proved.

1. If $u \in L^{2}\left(0, T ; H^{1}(0, L)\right), u u_{x} \in L^{1}\left(0, T ; L^{2}(0, L)\right)$ and $u \mapsto u u_{x}$ is continuous.
2. For $f \in L^{1}\left(0, T ; L^{2}(0, L)\right)$ the mild solution $u_{2}$ of (2.9) belongs to $B$. Moreover, the linear map

$$
\Theta: f \longmapsto u_{2}
$$

is continuous.
Remark 2.5. Recall that for $f \in L^{1}\left(0, T ; L^{2}(0, L)\right)$ the mild solution $u_{2}$ of (2.9) is given by

$$
\begin{equation*}
u_{2}(\cdot, t)=\int_{0}^{t} S(t-s) f(\cdot, s) d s \tag{2.20}
\end{equation*}
$$

## 3 Stabilization of KdV equation

In this section we study the stabilization of the system

$$
\begin{cases}u_{t}+u_{x}+u u_{x}+u_{x x x}+G u=0, & \text { in }(0, L) \times\{t>0\}  \tag{3.1}\\ u(0, t)=u(L, t)=u_{x}(L, t)=0, & t>0 \\ u(x, 0)=u_{0}(x), & \text { in }(0, L)\end{cases}
$$

Here, $G u$ is defined by (1.19). Precisely, the issue in this section is the following one:
Stabilization problem: Can one find a feedback control law h so that the resulting closed-loop system (3.1) is asymptotically stable when $t \rightarrow \infty$ ?

The answer to the stability problem is given by the theorem below.
Theorem 3.1. Let $T>0$. Then, there exist constants $k>0, R_{0}>0$ and $C>0$, such that for any $u_{0} \in L^{2}(0, L)$ with

$$
\left\|u_{0}\right\|_{L^{2}(0, L)} \leq R_{0}
$$

the corresponding solution $u$ of (3.1) satisfies

$$
\begin{equation*}
\|u(\cdot, t)\|_{L^{2}(0, L)} \leq C e^{-k t}\left\|u_{0}\right\|_{L^{2}(0, L)}, \quad \forall t \geq 0 \tag{3.2}
\end{equation*}
$$

As usual in the stabilization problem, Theorem 3.1 is a direct consequence of the following observability inequality.

Proposition 3.2. Let $T>0$ and $R_{0}>0$ be given. There exists a constant $C>1$, such that, for any $u_{0} \in L^{2}(0, L)$ satisfying

$$
\left\|u_{0}\right\|_{L^{2}(0, L)} \leq R_{0}
$$

the corresponding solution $u$ of (3.1) satisfies

$$
\begin{equation*}
\left\|u_{0}\right\|_{L^{2}(0, L)}^{2} \leq C \int_{0}^{T}\|G u\|_{L^{2}(0, L)}^{2} d t \tag{3.3}
\end{equation*}
$$

Indeed, if (3.3) holds, then it follows from the energy estimate that

$$
\begin{equation*}
\|u(\cdot, T)\|_{L^{2}(0, L)}^{2} \leq\left\|u_{0}\right\|_{L^{2}(0, L)}^{2}-\int_{0}^{T}\|G u\|_{L^{2}(0, L)}^{2} d t \tag{3.4}
\end{equation*}
$$

or, more precisely,

$$
\|u(\cdot, T)\|_{L^{2}(0, L)}^{2} \leq\left(1-C^{-1}\right)\left\|u_{0}\right\|_{L^{2}(0, L)}^{2} .
$$

Thus,

$$
\|u(\cdot, m T)\|_{L^{2}(0, L)}^{2} \leq\left(1-C^{-1}\right)^{m}\left\|u_{0}\right\|_{L^{2}(0, L)}^{2}
$$

which gives (3.2) by the semigroup property. In (3.2), we obtain a constant $k$ independent of $R_{0}$ by noticing that for $t>c\left(\left\|u_{0}\right\|_{L^{2}(0, L)}\right)$, the $L^{2}$-norm of $u(\cdot, t)$ is smaller than 1 , so that we can take the $k$ corresponding to $R_{0}=1$.

Proof of Proposition 3.2. We prove (3.3) by contradiction. Suppose that (3.3) does not occurs. Thus, for any $n \geq 1$, (3.1) admits a solution $u_{n} \in C\left([0, T] ; L^{2}(0, L)\right) \cap L^{2}\left(0, T ; H^{1}(0, L)\right)$ satisfying

$$
\left\|u_{n}(0)\right\|_{L^{2}(0, L)} \leq R_{0}
$$

and

$$
\begin{equation*}
\int_{0}^{T}\left\|G u_{n}\right\|_{L^{2}(0, L)}^{2} d t \leq \frac{1}{n}\left\|u_{0, n}\right\|_{L^{2}(0, L)}^{2} \tag{3.5}
\end{equation*}
$$

where $u_{0, n}=u_{n}(0)$. Since $\alpha_{n}:=\left\|u_{0, n}\right\|_{L^{2}(0, L)} \leq R_{0}$, one can choose a subsequence of $\left\{\alpha_{n}\right\}$, still denoted by $\left\{\alpha_{n}\right\}$, such that

$$
\lim _{n \rightarrow \infty} \alpha_{n}=\alpha
$$

There are two possible cases: $i . \alpha>0$ and $i i . \alpha=0$.
i. $\alpha>0$.

Note that the sequence $\left\{u_{n}\right\}$ is bounded in $L^{\infty}\left(0, T ; L^{2}(0, L)\right) \cap L^{2}\left(0, T ; H^{1}(0, L)\right)$. On the other hand,

$$
u_{n, t}=-\left(u_{n, x}+\frac{1}{2} \partial_{x}\left(u_{n}^{2}\right)+u_{n, x x x}-G u_{n}\right),
$$

is bounded in $L^{2}\left(0, T ; H^{-2}(0, L)\right)$. As the first immersion of

$$
H^{1}(0, L) \hookrightarrow L^{2}(0, L) \hookrightarrow H^{-2}(0, L),
$$

is compact, exists a subsequence, still denoted by $\left\{u_{n}\right\}$, such that

$$
\begin{align*}
u_{n} & \longrightarrow u \quad \text { in } L^{2}\left(0, T ; L^{2}(0, L)\right), \\
-\frac{1}{2} \partial_{x}\left(u_{n}^{2}\right) & \rightharpoonup-\frac{1}{2} \partial_{x}\left(u^{2}\right) \quad \text { in } L^{2}\left(0, T ; H^{-1}(0, L)\right) . \tag{3.6}
\end{align*}
$$

It follows from (3.5) and (3.6) that

$$
\begin{equation*}
\int_{0}^{T}\left\|G u_{n}\right\|_{L^{2}(0, L)}^{2} d t \xrightarrow{n \rightarrow \infty} \int_{0}^{T}\|G u\|_{L^{2}(0, L)}^{2}=0, \tag{3.7}
\end{equation*}
$$

which implies that

$$
G u=0,
$$

i.e.,

$$
u(x, t)-\frac{1}{|\omega|} \int_{\omega} u(x, t) d x=0 \Rightarrow u(x, t)=\frac{1}{|\omega|} \int_{\omega} u(x, t) d x .
$$

Consequently,

$$
u(x, t)=c(t) \quad \text { in } \omega \times(0, T),
$$

for some function $c(t)$. Thus, letting $n \rightarrow \infty$, we obtain from (3.1) that

$$
\begin{cases}u_{t}+u_{x}+u_{x x x}=f, & \text { in }(0, L) \times(0, T),  \tag{3.8}\\ u=c(t), & \text { in } \omega \times(0, T)\end{cases}
$$

Let $w_{n}=u_{n}-u$ and $f_{n}=-\frac{1}{2} \partial_{x}\left(u_{n}^{2}\right)-f-G u_{n}$. Note first that,

$$
\begin{align*}
& \int_{0}^{T}\left\|G w_{n}\right\|_{L^{2}(0, L)}^{2} d t \\
& \quad \quad=\int_{0}^{T}\left\|G u_{n}\right\|_{L^{2}(0, L)}^{2} d t+\int_{0}^{T}\|G u\|_{L^{2}(0, L)}^{2} d t-2 \int_{0}^{T}\left(G u_{n}, G u\right)_{L^{2}(0, L)} d t \rightarrow 0 . \tag{3.9}
\end{align*}
$$

Since $w_{n} \rightharpoonup 0$ in $L^{2}\left(0, T ; H^{1}(0, L)\right)$, we infer from Rellich's Theorem that $\int_{0}^{L} w_{n}(y, t) d y \rightarrow 0$ strongly in $L^{2}(0, T)$. Combining (3.6) and (3.9), we have that

$$
\int_{0}^{T} \int_{0}^{L}\left|w_{n}\right|^{2} \longrightarrow 0
$$

Thus,

$$
\begin{gathered}
w_{n, t}+w_{n, x}+w_{n, x x x}=f_{n} \\
f_{n} \rightharpoonup 0 \quad \text { in } L^{2}\left(0, T ; H^{-1}(0, L)\right),
\end{gathered}
$$

and,

$$
w_{n} \longrightarrow 0 \quad \text { in } L^{2}\left(0, T ; L^{2}(0, L)\right),
$$

so,

$$
\partial_{x}\left(w_{n}^{2}\right) \longrightarrow w_{x}^{2}
$$

in the sense of distributions. Therefore, $f=-\frac{1}{2} \partial_{x}\left(u^{2}\right)$ e $u \in L^{2}\left(0, T ; L^{2}(0, L)\right)$ satisfies

$$
\begin{cases}u_{t}+u_{x}+u_{x x x}+\frac{1}{2}\left(u^{2}\right)_{x}=0, & \text { in }(0, L) \times(0, T), \\ u=c(t), & \text { in } \omega \times(0, T) .\end{cases}
$$

The first equation gives $c^{\prime}(t)=0$ which, combined with unique continuation property (see Appendix A), yields that $u(x, t)=c$ for some constant $c \in \mathbb{R}$. Since $u(L, t)=0$, we deduce that

$$
0=u(L, t)=c,
$$

and $u_{n}$ converges strongly to 0 in $L^{2}\left(0, T ; L^{2}(0, L)\right)$. We can pick some time $t_{0} \in[0, T]$ such that $u_{n}\left(t_{0}\right)$ tends to 0 strongly in $L^{2}(0, L)$. Since

$$
\left\|u_{n}(0)\right\|_{L^{2}(0, L)}^{2} \leq\left\|u_{n}\left(t_{0}\right)\right\|_{L^{2}(0, L)}^{2}+\int_{0}^{t_{0}}\left\|G u_{n}\right\|_{L^{2}(0, L)}^{2} d t
$$

it is inferred that $\alpha_{n}=\left\|u_{n}(0)\right\|_{L^{2}(0, L)} \longrightarrow 0$, as $n \rightarrow \infty$, which is in contradiction with the assumption $\alpha>0$.
ii. $\alpha=0$.

First, note that $\alpha_{n}>0$, for all $n$. Set $v_{n}=u_{n} / \alpha_{n}$, for all $n \geq 1$. Then,

$$
v_{n, t}+v_{n, x}+v_{n, x x x}-G v_{n}+\frac{\alpha_{n}}{2}\left(v_{n}^{2}\right)_{x}=0
$$

and

$$
\begin{equation*}
\int_{0}^{T}\left\|G v_{n}\right\|_{L^{2}(0, L)}^{2} d t<\frac{1}{n} . \tag{3.10}
\end{equation*}
$$

Since

$$
\begin{equation*}
\left\|v_{n}(0)\right\|_{L^{2}(0, L)}=1, \tag{3.11}
\end{equation*}
$$

the sequence $\left\{v_{n}\right\}$ is bounded in $L^{2} 0, T ; L^{2}(0, L) \cap L^{2}\left(0, T ; H^{1}(0, L)\right)$, and, therefore, $\left\{\partial_{x}\left(v_{n}^{2}\right)\right\}$ is bounded in $L^{2}\left(0, T ; L^{2}(0, L)\right)$. Then, $\alpha_{n} \partial_{x}\left(v_{n}^{2}\right)$ tends to 0 in this space. Finally,

$$
\int_{0}^{T}\|G v\|_{L^{2}(0, L)}^{2} d t=0
$$

Thus, $v$ is solution of

$$
\begin{cases}v_{t}+v_{x}+v_{x x x}=0, & \text { in }(0, L) \times(0, T), \\ v=c(t), & \text { in } \omega \times(0, T)\end{cases}
$$

We infer that $v(x, t)=c(t)=c$, thanks to Holmgren's Theorem, and that $c=0$ due the fact that $v(L, t)=0$.

According to the previous fact, pick a time $t_{0} \in[0, T]$ such that $v_{n}\left(t_{0}\right)$ converges to 0 strongly in $L^{2}(0, L)$. Since

$$
\left\|v_{n}(0)\right\|_{L^{2}(0, L)}^{2} \leq\left\|v_{n}\left(t_{0}\right)\right\|_{L^{2}(0, L)}^{2}+\int_{0}^{t_{0}}\left\|G v_{n}\right\|_{L^{2}(0, L)}^{2} d t
$$

we infer from (3.10) that $\left\|v_{n}(0)\right\|_{L^{2}(0, L)} \rightarrow 0$, which contradicts to (3.11). The proof is complete.

## 4 Comments and extensions for other models

In this section we intend to analyze the results obtained in this manuscript as well as to present some extensions of these results for other models.

### 4.1 Comments of the results

In this work we deal with the KdV equation from a control point of view posed in a bounded domain $(0, L) \subset \mathbb{R}$ with a forcing term $G h$ added as a control input, namely:

$$
\begin{cases}u_{t}+u_{x}+u u_{x}+u_{x x x}+G h=0, & \text { in }(0, L) \times(0, T),  \tag{4.1}\\ u(0, t)=u(L, t)=u_{x}(L, t)=0, & \text { in }(0, T), \\ u(x, 0)=u_{0}(x), & \text { in }(0, L) .\end{cases}
$$

Here $G$ is the operator defined by (1.4).
The result presented in this manuscript gives us a new "weak" forcing mechanism that ensures global stability to the system (4.1). In fact, Theorem 1.1 guarantees a lower cost to control
the system proposed in this work and, consequently, to derive a good result related with the stabilization problem as compared with existing results in the literature.

The interested readers can look at the following article [28], related to what we call "strong" forcing mechanism. Indeed, in this article, the author proposed the source term as $1_{\omega} h(x, t)$, that is, the mechanism proposed does not remove a medium term as seen in Gh defined by (1.4).

Finally, observe that the approach used to prove our main result as well as the weak mechanism can be extended for KdV-type equation and for a model of strong interaction between internal solitary waves. Let us breviary describe these systems and the results that can be derived by using the same approach applied in this work.

### 4.2 KdV-type equation

Fifth-order KdV type equation can be written as

$$
\begin{equation*}
u_{t}+u_{x}+\beta u_{x x x}+\alpha u_{x x x x x}+u u_{x}=0, \tag{4.2}
\end{equation*}
$$

where $u=u(t, x)$ is a real-valued function of two real variables $t$ and $x, \alpha$ and $\beta$ are real constants. When we consider, in (4.2), $\beta=1$ and $\alpha=-1$, T. Kawahara [20] introduced a dispersive partial differential equation which describes one-dimensional propagation of smallamplitude long waves in various problems of fluid dynamics and plasma physics, the so-called Kawahara equation.

With the damping mechanism proposed in this manuscript, we can investigate the stabilization problem, already mentioned in this article, for the following system

$$
\begin{cases}u_{t}+u_{x}+u u_{x}+u_{x x x}-u_{x x x x x}+G h=0, & \text { in }(0, T) \times(0, L)  \tag{4.3}\\ u(t, 0)=u(t, L)=u_{x}(t, 0)=u_{x}(t, L)=u_{x x}(t, L)=0, & \text { in }(0, T) \\ u(0, x)=u_{0}(x) & \text { in }(0, L)\end{cases}
$$

and $G$ as in (1.7).
In fact, a similar result can be obtained with respect to global stabilization. Obviously, we need to pay attention to the unique continuation property for this case (for our case see Appendix A). However, due the Carleman estimate provided by Chen in [13], it is possible to show the unique continuation property for the Kawahara operator.

### 4.3 Model of strong interaction between internal solitary waves

We can consider a model of two KdV equations types. Precisely, in [17], a complex system of equations was derived by Gear and Grimshaw to model the strong interaction of two-dimensional, long, internal gravity waves propagating on neighboring pycnoclines in a stratified fluid. It has the structure of a pair of Korteweg-de Vries equations coupled through both dispersive and nonlinear effects and has been the object of intensive research in recent years. In particular, we also refer to [2] for an extensive discussion on the physical relevance of the system.

An interesting possibility now presents itself is the study of the stability properties when the model is posed on a bounded domain $(0, L)$, that is, to study the Gear-Grimshaw system
with only a weak damping mechanism, namely,

$$
\begin{cases}u_{t}+u u_{x}+u_{x x x}+a_{3} v_{x x x}+a_{1} v v_{x}+a_{2}(u v)_{x}=0, & \text { in }(0, L) \times(0, \infty),  \tag{4.4}\\ c v_{t}+r v_{x}+v v_{x}+a_{3} b_{2} u_{x x x}+v_{x x x}+a_{2} b_{2} u u_{x}+a_{1} b_{2}(u v)_{x}+G v=0, & \text { in }(0, L) \times(0, \infty), \\ u(x, 0)=u^{0}(x), \quad v(x, 0)=v^{0}(x), & \text { in }(0, L),\end{cases}
$$

satisfying the following boundary conditions

$$
\begin{cases}u(0, t)=0, u(L, t)=0, u_{x}(L, t)=0, & \text { in }(0, \infty),  \tag{4.5}\\ v(0, t)=0, v(L, t)=0, v_{x}(L, t)=0, & \text { in }(0, \infty),\end{cases}
$$

where $a_{1}, a_{2}, a_{3}, b_{2}, c, r$ are constants in $\mathbb{R}$ assuming physical relations. Here, as in all work, $G v$ is the weak forcing term defined in (1.7).

The stabilization problem for the system (4.4)-(4.5) was addressed in [8]. The author showed that the total energy associated with the model decay exponentially when $t$ tends to $\infty$, considering two damping mechanisms $G u$ and $G v$ acting in both equations of (4.4). However, even though the system (4.4) has the structure of a pair of KdV equations, it cannot be decoupled into two single KdV equations** and, in this case, the result shown in this work is not a consequence of the results proved in [8].

Lastly, Bárcena-Petisco et al. in a recent work [1], addressed the controllability problem for the system (4.5), by means of a control $1_{\omega} f(x, t)$, supported in an interior open subset of the domain and acting on one equation only. The proof consists mainly on proving the controllability of the linearized system, which is done by getting a Carleman estimate for the adjoint system. With this result in hand, by using $G v$ as a control mechanism, instead of $1_{\omega} f(x, t)$, it is possible to prove the global stabilization for the model (4.5). As in the KdV (see Appendix A) and Kawahara cases, we need to prove a unique continuation property to achieve the stabilization problem, however with the Carleman estimate [1, Proposition 3.2], we are able to derive this property for the Gear-Grimshaw operator.

### 4.4 About exact controllability results

Now, we will discuss the exact controllability property of the KdV system

$$
\begin{cases}u_{t}+u_{x}+u u_{x}+u_{x x x}=G w, & \text { in }(0, L) \times(0, T),  \tag{4.6}\\ u(0, t)=u(L, t)=u_{x}(L, t)=0, & \text { in }(0, T), \\ u(x, 0)=u_{0}(x), & \text { in }(0, L) .\end{cases}
$$

with weak source term $G$ defined by

$$
G w(x, t)=1_{\omega}\left(w(x, t)-\frac{1}{|\omega|} \int_{\omega} w(x, t) d x\right)
$$

where $\omega \subset(0, L)$ and $1_{\omega}$ denotes the characteristic function of the set $\omega$. We raise the following open question:
Control problem: Given an initial state $u_{0}$ and a terminal state $u_{1}$ in $L^{2}(0, L)$, can one find an appropriate control input $w \in L^{2}(\omega \times(0, T))$ so that the equation (4.6) admits a solution $u$ which satisfies $u(\cdot, 0)=u_{0}$ and $u(\cdot, T)=u_{1}$ ?

[^7]It is important to point out that we do not expect that system (4.6) has the exact control property as above mentioned when we consider the control $w$ in $L^{2}(\omega \times(0, T))$. Roughly speaking, (large) negative waves propagate from the right to the left. Therefore, a negative wave cannot be generated by a left control, that means, when the control is acting far from the endpoint $x=L$, i.e. in some interval $\omega=\left(l_{1}, l_{2}\right)$ with $0<l_{1}<l_{2}<L$, then there is no chance to control exactly the state function on $\left(l_{2}, L\right)$, (see e.g. [33]). However, we believe that using the techniques proposed in [9] (or in [15]), i.e., considering the weight Sobolev spaces (or control more regular), there is a chance to get positive answer for exact control problem in the right hand side of the domain, precisely, considering $\omega=(L-\epsilon, L)$, with the weak control as defined in (1.19).

### 4.5 A natural damping mechanism

When we consider the boundary condition of (4.1) with $G=0$, a natural feedback law is revealed as we can see below

$$
\begin{equation*}
\frac{\mathrm{d} E}{\mathrm{~d} t}=-\frac{1}{2}\left|u_{x}(0, t)\right|^{2} \tag{4.7}
\end{equation*}
$$

with

$$
E(t)=\frac{1}{2} \int_{0}^{L}|u(x, t)|^{2} \mathrm{~d} x .
$$

The energy dissipation law (4.7) shows that the boundary value problem under consideration is dissipated through the extreme $x=0$ and leads one to guess that any solution of (4.1), with $G=0$, may decay to zero as $t \rightarrow \infty$. In order to answer this question, a really nonlinear method is needed, and the method applied here can not be addressed to solve it.

## A Unique continuation property

This appendix aims to provide a sketch of how to obtain the unique continuation property through a Carleman estimate.

## A. 1 Carleman inequality

Pick any function $\psi \in C^{3}([0, L])$ with

$$
\begin{align*}
\psi>0 & \text { in }[0, L], \quad\left|\psi^{\prime}\right|>0, \quad \psi^{\prime \prime}<0, \quad \text { and } \quad \psi^{\prime} \psi^{\prime \prime \prime}<0 \quad \text { in }[0, L],  \tag{A.1}\\
\psi^{\prime}(0)<0, & \psi^{\prime}(L)>0, \quad \text { and } \quad \max _{x \in[0, L]} \psi(x)=\psi(0)=\psi(L) . \tag{A.2}
\end{align*}
$$

Set

$$
\begin{equation*}
\varphi(t, x)=\frac{\psi(x)}{t(T-t)} \tag{A.3}
\end{equation*}
$$

For $f \in L^{2}\left(0, T ; L^{2}(0, L)\right)$ and $q_{0} \in L^{2}(0, L)$, let $q$ denote the solution of the system

$$
\begin{cases}q_{t}+q_{x}+q_{x x x}=f, & t \in(0, T), x \in(0, L)  \tag{A.4}\\ q(t, 0)=q(t, L)=q_{x}(t, L)=0 & t \in(0, T) \\ q(0, x)=q_{0}(x), & \text { in }(0, L)\end{cases}
$$

Thus, the following result is a direct consequence of the Carleman estimate proved by [9].

Proposition A.1. Pick any $T>0$. There exist two constants $C>0$ and $s_{0}>0$ such that any $f \in L^{2}\left(0, T ; L^{2}(0, L)\right)$, any $q_{0} \in L^{2}(0, L)$ and any $s \geq s_{0}$, the solution $q$ of (A.4) fulfills

$$
\begin{equation*}
\int_{0}^{T} \int_{0}^{L}\left[s \varphi\left|q_{x x}\right|^{2}+(s \varphi)^{3}\left|q_{x}\right|^{2}+(s \varphi)^{5}|q|^{2}\right] e^{-2 s \varphi} d x d t \leq C\left(\int_{0}^{T} \int_{0}^{L}|f|^{2} e^{-2 s \varphi} d x d t\right) \tag{A.5}
\end{equation*}
$$

where $\varphi$ is defined by (A.4) and $\psi$ satisfies (A.1)-(A.2).
Actually, Proposition A. 1 will play a great role in establishing the unique continuation property describes below.

Corollary A.2. Let $L>0$ and $T>0$ be two real numbers, and let $\omega \subset(0, L)$ be a nonempty open set. If $v \in L^{\infty}\left(0, T ; H^{1}(0, L)\right)$ solves

$$
\begin{cases}v_{t}+v_{x}+v_{x x x}+v v_{x}=0, & \text { in }(0, L) \times(0, T), \\ v(0, t)=0, & \text { in }(0, T), \\ v=c, & \text { in }\left(l^{\prime}, L\right) \times(0, T),\end{cases}
$$

with $0<l^{\prime}<L$ and $c \in \mathbb{R}$, then $v \equiv c$ in $(0, L) \times(0, T)$.
Proof. We do not expect that $v$ belongs to

$$
L^{2}\left(0, T ; H^{3}(0, l)\right) \cap H^{1}\left(0, T ; L^{2}(0, l)\right) .
$$

In this way, we have to smooth it. For any function $v=v(x, t)$ and any $h>0$, let us consider $v^{[h]}(x, t)$ defined by

$$
v^{[h]}(x, t):=\frac{1}{h} \int_{t}^{t+h} v(x, s) d s
$$

Remember that if $v \in L^{p}(0, T ; V)$, where $1 \leq p \leq+\infty$ and $V$ denotes any Banach space, we have that

$$
\begin{gathered}
v^{[h]} \in W^{1, p}(0, T-h ; V) \\
\left\|v^{[h]}\right\|_{L^{p}(0, T-h ; V)} \leq\|v\|_{L^{p}(0, T ; V)}
\end{gathered}
$$

and

$$
v^{[h]} \rightarrow v \quad \text { in } L^{p}\left(0, T^{\prime} ; V\right) \text { as } h \rightarrow 0,
$$

for $p<\infty$ and $T^{\prime}<T$.
Choose any $T^{\prime}<T$. Thus, for a small enough number $h$,

$$
v^{[h]} \in W^{1, \infty}\left(0, T^{\prime} ; H_{0}^{1}(0, l)\right)
$$

and $v^{[h]}$ is solution of

$$
\begin{gather*}
v_{t}^{[h]}+v_{x}^{[h]}+v_{x x x}^{[h]}+\left(v v_{x}\right)^{[h]}=0 \quad \text { in }(0, l) \times\left(0, T^{\prime}\right),  \tag{A.6}\\
v^{[h]}(0, t)=0 \quad \text { in }\left(0, T^{\prime}\right) \tag{A.7}
\end{gather*}
$$

and

$$
\begin{equation*}
v^{[h]} \equiv c \quad \text { in }\left(l^{\prime}, l\right) \times\left(0, T^{\prime}\right) \tag{A.8}
\end{equation*}
$$

for some $c \in \mathbb{R}$. Since $v \in L^{\infty}\left(0, T ; H^{1}(0, l)\right)$ and $v v_{x} \in L^{\infty}\left(0, T ; L^{2}(0, l)\right)$, therefore, it follows from (A.6), that

$$
v_{x x x}^{[h]} \in L^{\infty}\left(0, T^{\prime} ; L^{2}(0, l)\right)
$$

and thus

$$
v^{[h]} \in L^{\infty}\left(0, T^{\prime} ; H^{3}(0, l)\right)
$$

Thanks to the Carleman estimate (A.5), we get that

$$
\begin{align*}
& \int_{0}^{T^{\prime}} \quad \int_{0}^{L}\left[s \varphi\left|v_{x x}^{[h]}\right|^{2}+(s \varphi)^{3}\left|v_{x}^{[h]}\right|^{2}+(s \varphi)^{5}\left|v^{[h]}\right|^{2}\right] e^{-2 s \varphi} d x d t \\
& \quad \leq C\left(\int_{0}^{T^{\prime}} \int_{0}^{L}|f|^{2} e^{-2 s \varphi} d x d t\right)  \tag{A.9}\\
& \quad \leq 2 C_{0} \int_{0}^{T^{\prime}} \int_{0}^{l}\left|v v_{x}^{[h]}\right|^{2} e^{-2 s \varphi} d x d t+2 C_{0} \int_{0}^{T^{\prime}} \int_{0}^{l}\left|\left(v v_{x}\right)^{[h]}-v v_{x}^{[h]}\right|^{2} e^{-2 s \varphi} d x d t \\
& \quad=: I_{1}+I_{2}
\end{align*}
$$

for any $s \geq s_{0}$ and $\varphi(t, x)$ defined by (A.3).
Claim 1: $I_{1}$ is bounded and can be absorbed by the left-hand side of (A.9).
In fact, since $v \in L^{\infty}\left(0, T ; L^{\infty}(0, l)\right)$, we have

$$
\begin{equation*}
I_{1} \leq C \int_{0}^{T^{\prime}} \int_{0}^{l}\left|v_{x}^{[h]}\right|^{2} e^{-2 s \varphi} d x d t \tag{A.10}
\end{equation*}
$$

for some constant $C>0$ which does not depend on $h$. Comparing the powers of $s$ in the right-hand side of (A.10) with those in the left-hand side of (A.9) we deduce that the term $I_{1}$ in (A.9) may be dropped by increasing the constants $C_{0}$ and $s_{0}$ in a convenient way, getting Claim 1.

Claim 2: $I_{2} \rightarrow 0$, as $h \rightarrow 0$.
From now on, fix $s$, which means, to the value $s_{0}$. Thanks to the fact that $e^{-2 s_{0} \varphi} \leq 1$, it is sufficient to prove that

$$
\begin{equation*}
\left(v v_{x}\right)^{[h]} \rightarrow v v_{x} \quad \text { in } L^{2}\left(0, T^{\prime} ; L^{2}(0, l)\right) \tag{A.11}
\end{equation*}
$$

and

$$
\begin{equation*}
v v_{x}^{[h]} \rightarrow v v_{x} \quad \text { in } L^{2}\left(0, T^{\prime} ; L^{2}(0, l)\right) \tag{A.12}
\end{equation*}
$$

In fact, since

$$
v v_{x} \in L^{2}\left(0, T^{\prime} ; L^{2}(0, l)\right)
$$

(A.11) holds and, from the fact that $v \in L^{\infty}\left(0, T^{\prime} ; L^{\infty}(0, l)\right) \cap L^{2}\left(0, T^{\prime} ; H^{1}(0, l)\right)$, (A.12) follows, showing the Claim 2.

By Claims 1 and 2, as $h \rightarrow 0$, the integral term

$$
\int_{0}^{T^{\prime}} \int_{0}^{L}\left[s \varphi\left|v_{x x}^{[h]}\right|^{2}+(s \varphi)^{3}\left|v_{x}^{[h]}\right|^{2}+(s \varphi)^{5}\left|v^{[h]}\right|^{2}\right] e^{-2 s \varphi} d x d t \rightarrow 0
$$

On the other hand, $v^{[h]} \rightarrow v$ in $L^{2}\left(0, T^{\prime} ; L^{2}(0, l)\right)$. It follows that $v \equiv c$ in $(0, l) \times\left(0, T^{\prime}\right)$, for $c \in \mathbb{R}$. As $T^{\prime}$ may be taken arbitrarily close to $T$, we infer that $v \equiv c$ in $(0, l) \times(0, T)$, for some $c \in \mathbb{R}$. This completes the proof of Corollary A.2.

As a consequence of Corollary A.2, we give below the unique continuation property.

Corollary A.3. Let $L>0, T>0$ be real numbers, and $\omega \subset(0, L)$ be a nonempty open set. If $v \in L^{\infty}\left(0, T ; H^{1}(0, L)\right)$ is solution of

$$
\begin{cases}v_{t}+v_{x}+v_{x x x}+v v_{x}=0, & \text { in }(0, L) \times(0, T), \\ v(0, t)=v(L, t)=0, & \text { in }(0, T), \\ v=c, & \text { in } \omega \times(0, T),\end{cases}
$$

where $c \in \mathbb{R}$, then $v \equiv c$ in $(0, L) \times(0, T)$.
Proof. Without loss of generality we may assume that $\omega=\left(l_{1}, l_{2}\right)$ with $0 \leq l_{1}<l_{2} \leq L$. Pick $l=\left(l_{1}+l_{2}\right) / 2$. First, apply Corollary A. 2 to the function $v(x, t)$ on $(0, l) \times(0, T)$. After that, we use the following change of variable $v(L-x, T-t)$ on $(0, L-l) \times(0, T)$, to conclude that $v \equiv c$ on $(0, L) \times(0, T)$, achieving the result.

## Acknowledgements

The author wish to thank the referee for his/her valuable comments which improved this paper.

Roberto de A. Capistrano-Filho was partially supported by CNPq 408181/2018-4, CAPESPRINT 88881.311964/2018-01, CAPES-MATHAMSUD 88881.520205/2020-01, MATHAMSUD 21-MATH-03 and Propesqi (UFPE).

## References

[1] J. Bárcena-Petisco, S. Guerreiro, A. F. Pazoto, Local null controllability of a model system for strong interaction between internal solitary waves, Commun. Contemp. Math., 2150003 (2021). https://doi.org/10.1142/S0219199721500036;
[2] J. Bona, G. Ponce, J.-C. Saut, M.M. Tom, A model system for strong interaction between internal solitary waves, Comm. Math. Phys. 143(1992), 287-313. https://doi.org/ 10.1007/BF02099010; Zbl 0752.35056
[3] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to non-linear evolution equations, part II: the KdV equation, Geom. Funct. Anal. 3(1993), 209-262. https://doi.org/10.1007/BF01895688; Zbl 0787.35098
[4] J. Boussinesq, Théorie de l'intumescence liquide, applelée onde solitaire ou de, translation, se propageant dans un canal rectangulaire (in French), C. R. Acad. Sci. Paris 72(1871), 755-759. Zbl 03.0486.02
[5] J. Boussinesq, Théorie générale des mouvements qui sont propagés dans un canal rectangulaire horizontal (in French), C. R. Acad. Sci. Paris 73(1871), 256-260. Zbl 03.0486.02
[6] J. Boussinesq, Théorie des ondes et des remous qui se propagent le long d'un canal rectangularie horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond (in French), J. Math. Pures Appl. 17(1872), 55-108.
[7] J. Boussinesq, Essai sur la théorie des eaux courantes (in French), Mémoires présentés par divers savants á l'Académie des Sciences Inst 23(1877), 666-680. Zbl 09.0680.04
[8] R. A. Capistrano-Filho, Stabilization of the Gear-Grimshaw system with weak damping, Journal of Dynamics and Control Systems 24(2018), 145-166. https://doi.org/10. 1007/s10883-017-9363-x; Zbl 1387.35524
[9] R. A. Capistrano-Filho, A. F. Pazoto, L. Rosier, Internal controllability for the Korteweg-de Vries equation on a bounded domain, ESAIM Control Optim. Calc. Var. 21(2015), 1076-1107. https://doi.org/10.1051/cocv/2014059; Zbl 1331.35302
[10] E. Cerpa, Exact controllability of a nonlinear Korteweg-de Vries equation on a critical spatial domain, SIAM J. Control Optim. 43(2007), 877-899. https://doi.org/10.1137/ 06065369x; Zbl 1147.93005
[11] E. Cerpa, E. Crépeau, Boundary controllability for the nonlinear Korteweg-de Vries equation on any critical domain, Ann. Inst. Henri Poincaré Anal. Non Linéaire 26(2009), 457-475. https://doi.org/10.1016/j.anihpc.2007.11.003; Zbl 1158.93006
[12] M. Chapouly, Global controllability of a nonlinear Korteweg-de Vries equation, Commun. Contemp. Math. 11(2009), 495-521. https://doi.org/10.1142/S0219199709003454; Zbl 1170.93006
[13] M. Chen, Internal controllability of the Kawahara equation on a bounded domain, Nonlinear Anal. 185(2019), 356-373. https://doi.org/10.1016/j.na.2019.03.016; Zbl 1418.35322
[14] E. Crépeau, J.-M. Coron, Exact boundary controllability of a nonlinear KdV equation with a critical length, J. Eur. Math. Soc. 6(2005), 367-398. https://doi. org/10.4171/JEMS/ 13; Zbl 1061.93054
[15] O. Glass, S. Guerrero, Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit, Asymptot. Anal. 60(2008), 61-100. https://doi.org/10.3233/ASY-2008-0900; Zbl 1160.35063
[16] O. Glass, S. Guerrero, Controllability of the Korteweg-de Vries equation from the right Dirichlet boundary condition, Syst. Control Lett. 59(2010), 390-395. https://doi.org/10. 1016/j.sysconle.2010.05.001; Zbl 1200.93019
[17] J. A. Gear, R. Grimshaw, Weak and strong interaction between internal solitary waves, Stud. Appl. Math. 70(1984), 235-258. https://doi.org/10.1002/sapm1984703235; Zbl 0548.76020
[18] E. M. Jager, On the origin of the Korteweg-de Vries equation, arXiv:math/0602661 [math.HO] (2006).
[19] T. Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equations, in: Studies in applied mathematics, Advances in Mathematics Supplementary Studies, Vol. 8, 1983, pp. 93-128. MR759907
[20] T. Kawahara, Oscillatory solitary waves in dispersive media, J. Phys. Soc. Japan. 33(1972), 260-264. https://doi.org/10.1143/JPSJ. 33.260
[21] C. E. Kenig, G. Ponce, L. Vega, A bilinear estimate with applications to the KdV equation, J. Amer. Math. Soc. 9(1996), 573-603. https://doi.org/10.1090/S0894-0347-96-00200-7; MR1329387; Zbl 0848.35114
[22] D. J. Korteweg, G. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, Philos. Mag. 39(1895), 422-443. Zbl 26.0881.02
[23] C. Laurent, L. Rosier, B.-Y. Zhang, Control and stabilization of the Korteweg-de Vries Equation on a periodic domain, Commun. Partial Differential Equations 35(2010), 707-744. 39(1895), 422-443. https ://doi.org/10.1080/03605300903585336; Zbl 1213.93015
[24] J.-L Lions, Exact controllability, stabilization and perturbations for distributed systems, SIAM Rev. 30(1988), 1-68. https://doi.org/10.1137/1030001; Zbl 0644.49028
[25] C. P. Massarolo, G. P. Menzala, A. F. Pazoto, On the uniform decay for the Korteweg-de Vries equation with weak damping, Math. Methods Appl. Sci. 30(2007), 14191435. https://doi.org/10.1002/mma.847; Zbl 1114.93080
[26] G. P. Menzala, C. F. Vasconcellos, E. Zuazua, Stabilization of the Korteweg-de Vries equation with localized damping, Quarterly Appl. Math. 60(2002), 111-129. https://doi. org/10.1090/qam/1878262; MR1878262; Zbl 1039.35107
[27] R. M. Miura, The Korteweg-de Vries equation: A survey of results, SIAM Rev. 18(1976), 412-459. https://doi.org/10.1137/1018076; Zbl 0333.35021
[28] A. F. Рazoto, Unique continuation and decay for the Korteweg-de Vries equation with localized damping, ESAIM Control Optim. Calc. Var. 11(2005), 473-486. https : //doi .org/ 10.1051/cocv: 2005015; Zbl 1148.35348
[29] A. Pazy, Semigoups of linear operators and applications to partial differential equations, Springer, New York, 1983. https://doi.org/10.1007/978-1-4612-5561-1
[30] R. Pego, Origin of the KdV equation, Notices Amer. Math. Soc. 45(1998), p. 358.
[31] Rayleigh (J. W. Strutt), On waves, Phil. Mag. 1(1876), 257-271.
[32] L. Rosier, Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain, ESAIM Control Optim. Calc. Var. 2(1997), 33-55. https://doi. org/10. 1051/cocv: 1997102; Zbl 0873.93008
[33] L. Rosier, Control of the surface of a fluid by a wavemaker, ESAIM Control Optim. Calc. Var. 10(2004), 346-380. https://doi .org/10.1051/cocv:2004012; Zbl 1094.93014
[34] L. Rosier, B.-Y. Zhang, Global stabilization of the generalized Korteweg-de Vries equation posed on a finite domain, SIAM J. Control Optim. 45(2006), 927-956. https: //doi.org/10.1137/050631409; Zbl 1116.35108
[35] J. S. Russell, Report on waves, Fourteenth meeting of the British Association for the Advancement of Science, 1844.
[36] D. L. Russell, B.-Y. Zhang, Controllability and stabilizability of the third order linear dispersion equation on a periodic domain, SIAM J. Cont. Optim. 31(1993), 659-676. https : //doi.org/10.1137/0331030; Zbl 0771.93073
[37] D. L. Russell, B.-Y. Zhang, Exact controllability and stabilizability of the Korteweg-de Vries equation, Trans. Amer. Math. Soc. 348(1996), 3643-3672. https://doi. org/10.1090/ S0002-9947-96-01672-8; MR1360229; Zbl 0862.93035
[38] J.-C. Saut, R. Temam, Remarks on the Korteweg-de Vries equation, Israel J. Math. 24(1976), 78-87. https://doi.org/10.1007/BF02761431; Zbl 0334.35062
[39] B.-Y. Zhang, Exact boundary controllability of the Korteweg-de Vries equation, SIAM J. Cont. Optim. 37(1999), 543-565. https://doi.org/10.1137/S0363012997327501; Zbl 0930.35160

# Errata article for "Three point boundary value problems for ordinary differential equations, uniqueness implies existence" 

Paul W. Eloe ${ }^{\boxtimes 1,2}$, Johnny Henderson ${ }^{2}$ and Jeffrey T. Neugebauer ${ }^{3}$<br>${ }^{1}$ University of Dayton, Department of Mathematics, Dayton, OH 45469, USA<br>${ }^{2}$ Baylor University, Department of Mathematics, Waco, TX 76798, USA<br>${ }^{3}$ Eastern Kentucky University, Department of Mathematics and Statistics, Richmond, KY 40475, USA

Received 3 May 2021, appeared 8 July 2021
Communicated by Gennaro Infante


#### Abstract

This paper serves as an errata for the article "P. W. Eloe, J. Henderson, J. Neugebauer, Electron. J. Qual. Theory Differ. Equ. 2020, No. 74, 1-15." In particular, the proof the authors give in that paper of Theorem 3.6 is incorrect, and so, that alleged theorem remains a conjecture. In this erratum, the authors state and prove a correct theorem.


Keywords: uniqueness implies existence, nonlinear interpolation, ordinary differential equations, three point boundary value problems.
2020 Mathematics Subject Classification: 34B15, 34B10.

## 1 Introduction

Let $n \geq 2$ denote an integer and let $a<T_{1}<T_{2}<T_{3}<b$. Let $a_{i} \in \mathbb{R}, i=1, \ldots, n$. We shall consider the ordinary differential equation

$$
\begin{equation*}
y^{(n)}(t)=f\left(t, y(t), \ldots, y^{(n-1)}(t)\right), \quad t \in\left[T_{1}, T_{3}\right], \tag{1.1}
\end{equation*}
$$

where $f:(a, b) \times \mathbb{R}^{n} \rightarrow \mathbb{R}$, or the ordinary differential equation

$$
\begin{equation*}
y^{(n)}(t)=f(t, y(t)), \quad t \in\left[T_{1}, T_{3}\right], \tag{1.2}
\end{equation*}
$$

where $f:(a, b) \times \mathbb{R} \rightarrow \mathbb{R}$. We shall consider three point boundary value problems for either (1.1) or (1.2) with the boundary conditions, for $j \in\{1,2\}$,

$$
\begin{equation*}
y^{(i-1)}\left(T_{1}\right)=a_{i}, \quad i=1, \ldots, n-2, \quad y\left(T_{2}\right)=a_{n-1}, \quad y^{(j-1)}\left(T_{3}\right)=a_{n}, \tag{1.3}
\end{equation*}
$$

and we shall have need to consider two point boundary value problems for either (1.1) or (1.2) with the boundary conditions, for $j \in\{1,2\}$,

$$
\begin{equation*}
y^{(i-1)}\left(T_{1}\right)=a_{i}, \quad i=1, \ldots, n-1, \quad y^{(j-1)}\left(T_{2}\right)=a_{n} . \tag{1.4}
\end{equation*}
$$

With respect to (1.1), common assumptions for the types of results that we consider are:

[^8](A) $f\left(t, y_{1}, \ldots, y_{n}\right):(a, b) \times \mathbb{R}^{n} \rightarrow \mathbb{R}$ is continuous;
(B) Solutions of initial value problems for (1.1) are unique and extend to $(a, b)$;
(C) For $j \in\{1,2\}$, solutions of the two-point boundary value problems (1.1), (1.3) are unique if they exist;
(D) For $j \in\{1,2\}$, solutions of the two-point boundary value problems (1.1), (1.4) are unique if they exist.

With respect to (1.2), the assumptions (A, (B), (C) and (D) are replaced, respectively, by
( $\left.A^{\prime}\right) f(t, y):(a, b) \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous;
$\left(B^{\prime}\right)$ Solutions of initial value problems for (1.2) are unique and extend to $(a, b)$.
( $C^{\prime}$ ) For $j \in\{1,2\}$, solutions of the two-point boundary value problems (1.2), (1.3) are unique if they exist.
( $D^{\prime}$ ) For $j \in\{1,2\}$, solutions of the two-point boundary value problems (1.2), (1.4) are unique if they exist.

In [3, Theorem 3.6], the authors claimed to have proved the following theorem.
Theorem 1.1. Assume that with respect to (1.1), Conditions (A), (B), (C) and (D) are satisfied. Then for each $a<T_{1}<T_{2}<T_{3}<b, a_{i} \in \mathbb{R}, i=1, \ldots, n$, and for $j=1$, the three point boundary value problem (1.1), (1.3) has a solution.

The proof that is offered in [3] is incorrect and so, the alleged theorem remains a conjecture. In this erratum, we state and prove a correct theorem. With the statement and proof of this correct theorem, the remainder of the results produced in [3] are correct.

Theorem 1.2. Assume that with respect to (1.2), Conditions $\left(A^{\prime}\right),\left(B^{\prime}\right),\left(C^{\prime}\right)$ and $\left(D^{\prime}\right)$ are satisfied. Then for each $a<T_{1}<T_{2}<T_{3}<b, a_{i} \in \mathbb{R}, i=1, \ldots, n$, and for $j=1$, the three point boundary value problem (1.2), (1.3) has a solution.

Before proving Theorem 1.2, we state several results to which we refer in the proof. The first two are results about the continuous dependence of solutions of (1.1), (1.4) or (1.2), (1.4) on boundary conditions. The third is a known generalized mean value theorem.

Theorem 1.3. Assume that with respect to (1.1), Conditions (A), (B), and (D) are satisfied. Let $j \in\{1,2\}$.
(i) Given any $a<T_{1}<T_{2}<T_{3}<b$, and any solution $y$ of (1.1), there exists $\epsilon>0$ such that if $\left|T_{11}-T_{1}\right|<\epsilon,\left|y^{(i-1)}\left(T_{1}\right)-y_{i 1}\right|<\epsilon, i=1, \ldots, n-2,\left|T_{21}-T_{2}\right|<\epsilon$, and $\left|T_{31}-T_{3}\right|<\epsilon$, $\left|y\left(T_{2}\right)-y_{(n-1) 1}\right|<\epsilon,\left|y\left(T_{3}\right)-y_{n 1}\right|<\epsilon$, then there exists a solution $z$ of (1.1) such that $z^{(i-1)}\left(T_{11}\right)=y_{l 1}, i=1, \ldots, n-2, z\left(T_{21}\right)=y_{(n-1) 1}$, and $z^{(j-1)}\left(T_{31}\right)=y_{n 1}$.
(ii) If $T_{1 k} \rightarrow T_{1}, T_{2 k} \rightarrow T_{2}, T_{3 k} \rightarrow T_{3}, y_{i k} \rightarrow y_{i}, i=1, \ldots, n$ and $z_{k}$ is a sequence of solutions of (1.1) satisfying $z_{k}^{(i-1)}\left(T_{1 k}\right)=y_{i k}, i=1, \ldots, n-2, z_{k}\left(T_{2 k}\right)=y_{(n-1) k}, z_{k}^{(j-1)}\left(T_{3 k}\right)=y_{n k}$, then for each $i \in\{1, \ldots, n\}, z_{k}^{(i-1)}$ converges uniformly to $y^{(i-1)}$ on compact subintervals of $(a, b)$.

Theorem 1.3 was proved in [3] with a standard application of the Brouwer invariance of domain theorem; technically we shall apply the following theorem for which the proof is completely analogous to the proof of Theorem 1.3.

Theorem 1.4. Assume that with respect to (1.2), Conditions $\left(A^{\prime}\right),\left(B^{\prime}\right)$, and $\left(D^{\prime}\right)$ are satisfied. Let $j \in\{1,2\}$.
(i) Given any $a<T_{1}<T_{2}<T_{3}<b$, and any solution $y$ of (1.1), there exists $\epsilon>0$ such that if $\left|T_{11}-T_{1}\right|<\epsilon,\left|y^{(i-1)}\left(T_{1}\right)-y_{i 1}\right|<\epsilon, i=1, \ldots, n-2,\left|T_{21}-T_{2}\right|<\epsilon$, and $\left|T_{31}-T_{3}\right|<\epsilon$, $\left|y\left(T_{2}\right)-y_{(n-1) 1}\right|<\epsilon,\left|y\left(T_{3}\right)-y_{n 1}\right|<\epsilon$, then there exists a solution $z$ of (1.1) such that $z^{(i-1)}\left(T_{11}\right)=y_{l 1}, i=1, \ldots, n-2, z\left(T_{21}\right)=y_{(n-1) 1}$, and $z^{(j-1)}\left(T_{31}\right)=y_{n 1}$.
(ii) If $T_{1 k} \rightarrow T_{1}, T_{2 k} \rightarrow T_{2}, T_{3 k} \rightarrow T_{3}, y_{i k} \rightarrow y_{i}, i=1, \ldots, n$ and $z_{k}$ is a sequence of solutions of (1.1) satisfying $z_{k}^{(i-1)}\left(T_{1 k}\right)=y_{i k}, i=1, \ldots, n-2, z_{k}\left(T_{2 k}\right)=y_{(n-1) k}, z_{k}^{(j-1)}\left(T_{3 k}\right)=y_{n k}$, then for each $i \in\{1, \ldots, n\}, z_{k}^{(i-1)}$ converges uniformly to $y^{(i-1)}$ on compact subintervals of $(a, b)$.

For a proof of a generalized mean value theorem, we refer the reader to the text by Conte and de Boor [1, Theorem 2.2]. Let $t_{0}, \ldots, t_{i}$ denote $i+1$ distinct real numbers and let $z: \mathbb{R} \rightarrow \mathbb{R}$. Define $z\left[t_{l}\right]=z\left(t_{l}\right), l=0, \ldots, i$ and if $t_{l}, \ldots, t_{k+1}$ denote $k-l+2$ distinct points, define

$$
z\left[t_{l}, \ldots, t_{k+1}\right]=\frac{z\left[t_{l+1}, \ldots, t_{k+1}\right]-z\left[t_{l}, \ldots, t_{k}\right]}{t_{k+1}-t_{l}} .
$$

Theorem 1.5. Assume $z(t)$ is a real-valued function, defined on $[a, b]$ and $i$ times differentiable in $(a, b)$. If $t_{0}, \ldots, t_{i}$ are $i+1$ distinct points in $[a, b]$, then there exists

$$
c \in\left(\min \left\{t_{0}, \ldots, t_{i}\right\}, \max \left\{t_{0}, \ldots, t_{i}\right\}\right)
$$

such that

$$
z\left[t_{0}, \ldots, t_{i}\right]=\frac{z^{(i)}(c)}{i!}
$$

For our purposes, we shall set $h>0$ and choose $t_{0}=T_{1}, t_{1}=T_{1}+h, \ldots, t_{i}=T_{1}+i h$ to be equally spaced. In this setting

$$
z\left[T_{1}, T_{1}+h, \ldots, T_{1}+i h\right]=\frac{\sum_{l=0}^{i}(-1)^{i-l}\binom{i}{l} z\left(T_{1}+l h\right)}{i!h^{i}}
$$

and, in general there exists $c \in\left(T_{1}, T_{1}+i h\right)$ such that

$$
\begin{equation*}
\frac{\sum_{l=0}^{i}(-1)^{i-l}\binom{i}{l_{2}} z(T+i h)}{h^{i}}=z^{(i)}(c) . \tag{1.5}
\end{equation*}
$$

We now proceed to the proof of Theorem 1.2.
Proof. Let $a<T_{1}<T_{2}<T_{3}<b$, and $a_{i} \in \mathbb{R}, i=1, \ldots, n$. Let $m \in \mathbb{R}$ and denote by $y(t ; m)$ the solution of the initial value problem (1.2), with initial conditions

$$
y^{(i-1)}\left(T_{1} ; m\right)=a_{i}, \quad i=1, \ldots, n-1, \quad y^{(n-2)}\left(T_{1} ; m\right)=m, \quad y\left(T_{2}\right)=a_{n-1} .
$$

Let

$$
\Omega=\left\{p \in \mathbb{R}: \text { there exists } m \in \mathbb{R} \text { with } y\left(T_{3} ; m\right)=p\right\} .
$$

The theorem is proved by showing $\Omega=\mathbb{R}$. It follows by Conditions $\left(A^{\prime}\right)$, ( $B^{\prime}$ ) and ( $D^{\prime}$ ) (see [2]), $\Omega \neq \varnothing$; thus, the theorem is proved by showing $\Omega$ is open and closed. That $\Omega$ is open follows from Theorem 1.4.

To show $\Omega$ is closed, let $p_{0}$ denote a limit point of $\Omega$ and without loss of generality let $p_{k}$ denote a strictly increasing sequence of reals in $\Omega$ converging to $p_{0}$. Assume $y\left(T_{3} ; m_{k}\right)=p_{k}$ for each $k \in \mathbb{N}_{1}$. It follows by the uniqueness of solutions, Condition $\left(C^{\prime}\right)$, that

$$
\begin{equation*}
y^{(j-1)}\left(t ; m_{k_{1}}\right) \neq y^{(j-1)}\left(t ; m_{k_{2}}\right), \quad t \in\left(T_{2}, b\right) \tag{1.6}
\end{equation*}
$$

for each $j \in\{1,2\}$, if $k_{1}<k_{2}$, and in particular,

$$
\begin{equation*}
y\left(t ; m_{1}\right)<y\left(t ; m_{k}\right) \quad t \in\left(T_{2}, b\right) \tag{1.7}
\end{equation*}
$$

for each $k$.
Either $y^{\prime}\left(T_{3} ; m_{k}\right) \leq 0$ infinitely often or $y^{\prime}\left(T_{3} ; m_{k}\right) \geq 0$ infinitely often. Relabel if necessary and assume $y^{\prime}\left(T_{3} ; m_{k}\right) \leq 0$ or $y^{\prime}\left(T_{3} ; m_{k}\right) \geq 0$ for each $k$.

We first assume the case $y^{\prime}\left(T_{3} ; m_{k}\right) \leq 0$ for each $k$. We now consider two subcases. For the first subcase, assume $y^{\prime}\left(T_{3} ; m_{k}\right)<y^{\prime}\left(T_{3} ; m_{1}\right) \leq 0$ infinitely often. Relabeling if necessary, assume $y^{\prime}\left(T_{3} ; m_{k}\right)<y^{\prime}\left(T_{3} ; m_{1}\right)<0$ for each $k$. Find $T_{3}<T_{4}<b$ such that $y^{\prime}\left(t ; m_{1}\right) \leq 0$, for $t \in\left[T_{3}, T_{4}\right]$. Then $y\left(t ; m_{1}\right)$ is decreasing on $\left[T_{3}, T_{4}\right]$. Set $L=y\left(T_{4} ; m_{1}\right)$; then, for $t \in\left[T_{3}, T_{4}\right]$,

$$
L=y\left(T_{4} ; m_{1}\right) \leq y\left(t ; m_{1}\right) \leq y\left(T_{3} ; m_{1}\right) \leq p_{0}
$$

Since $y^{\prime}\left(T_{2} ; m_{k}\right)<y^{\prime}\left(T_{2} ; m_{1}\right)$, then analogous to (1.7), it follows that

$$
y^{\prime}\left(t ; m_{k}\right)<y^{\prime}\left(t ; m_{1}\right), \quad t \in\left(T_{2}, b\right)
$$

and $y\left(t ; m_{k}\right)$ is decreasing on $\left[T_{3}, T_{4}\right]$. Then for $t \in\left[T_{3}, T_{4}\right]$,

$$
\begin{equation*}
L=y\left(T_{4} ; m_{1}\right) \leq y\left(t ; m_{1}\right) \leq y\left(t ; m_{k}\right) \leq y\left(T_{3} ; m_{k}\right) \leq p_{0} \tag{1.8}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
\left\{\left(t, y\left(t ; m_{k}\right): t \in\left[T_{3}, T_{4}\right], k \in \mathbb{N}_{1}\right\} \subset\left[T_{3}, T_{4}\right] \times\left[L, p_{0}\right] .\right. \tag{1.9}
\end{equation*}
$$

Since $f:(a, b) \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, there exists $M>0$ such that

$$
\begin{equation*}
\max _{t \in\left[T_{3}, T_{4}\right], k \in \mathbb{N}_{1}}\left|y^{(n)}\left(t ; m_{k}\right)\right| \leq M \tag{1.10}
\end{equation*}
$$

We now proceed to adapt an observation made by Lasota and Opial [4] and apply the adapted observation to higher order derivatives. Lasota and Opial essentially observed that

$$
\begin{equation*}
0>\frac{y\left(T_{4} ; m_{k}\right)-y\left(T_{3} ; m_{k}\right)}{T_{4}-T_{3}} \geq \frac{L-p_{0}}{T_{4}-T_{3}}=-K_{1} \tag{1.11}
\end{equation*}
$$

which implies

$$
\left\{t \in\left[T_{3}, T_{4}\right]:-K_{1} \leq y^{\prime}\left(t ; m_{k}\right)<0\right\} \neq \varnothing
$$

For our purposes, define

$$
S_{k 1}=\left\{t \in\left[T_{3}, T_{4}\right]:\left|y^{\prime}\left(t ; m_{k}\right)\right| \leq K_{1}\right\}
$$

and $S_{k 1} \neq \varnothing$.
To proceed to higher order derivatives, employ Theorem 1.5. For example, set

$$
h=\frac{T_{4}-T_{3}}{2}
$$

and consider

$$
\frac{y\left(T_{3} ; m_{k}\right)-2 y\left(T_{3}+h ; m_{k}\right)+y\left(T_{3}+2 h ; m_{k}\right)}{h^{2}}
$$

Employing (1.8), it follows that

$$
\begin{aligned}
\left|\frac{y\left(T_{3} ; m_{k}\right)-2 y\left(T_{3}+h ; m_{k}\right)+y\left(T_{3}+2 h ; m_{k}\right)}{h^{2}}\right| & \leq \frac{2\left(p_{0}-L\right)}{h^{2}} \\
& =\frac{2^{3}\left(p_{0}-L\right)}{\left(T_{4}-T_{3}\right)^{2}}=K_{2}
\end{aligned}
$$

Thus,

$$
S_{k 2}=\left\{t \in\left[T_{3}, T_{4}\right]:\left|y^{\prime \prime}\left(t ; m_{k}\right)\right| \leq K_{2}\right\} \neq \varnothing
$$

So, in general, let $i \in\{1, \ldots n-1\}$. Set $h=\frac{T_{4}-T_{3}}{i}$. Then,

$$
\left|\frac{\sum_{l=0}^{i}(-1)^{i-l}\binom{i}{l} y\left(T_{3}+l h ; m_{k}\right)}{h^{i}}\right| \leq \frac{(i)^{i} 2^{i-1}\left(p_{0}-L\right)}{\left(T_{4}-T_{3}\right)^{i}}=K_{i}
$$

Apply (1.5) and the set,

$$
S_{k i}=\left\{t \in\left[T_{3}, T_{4}\right]:\left|y^{(i)}\left(t ; m_{k}\right)\right| \leq K_{i}\right\} \neq \varnothing
$$

Let $c_{n-1} \in S_{k(n-1)}$. Then for $t \in\left[T_{3}, T_{4}\right]$,

$$
y^{(n-1)}\left(t ; m_{k}\right)=y^{(n-1)}\left(c_{n-1} ; m_{k}\right)+\int_{c_{n-1}}^{t} y^{(n)}\left(s ; m_{k}\right) d s
$$

which implies

$$
\max _{t \in\left[T_{3}, T_{4}\right]}\left|y^{(n-1)}\left(t ; m_{k}\right)\right| \leq K_{n-1}+M\left(T_{4}-T_{3}\right)=M_{n-1}
$$

Since $S_{k(n-2)} \neq \varnothing$, the same argument implies that

$$
\max _{t \in\left[T_{3}, T_{4}\right]}\left|y^{(n-2)}\left(t ; m_{k}\right)\right| \leq K_{n-2}+M_{n-1}\left(T_{4}-T_{3}\right)=M_{n-2}
$$

Continuing with the same argument, define for $i \in\{n-2, \ldots, 1\}$,

$$
M_{i}=K_{i}+M_{i+1}\left(T_{4}-T_{3}\right)
$$

Then

$$
\max _{t \in\left[T_{3}, T_{4}\right]}\left|y^{(i)}\left(t ; m_{k}\right)\right| \leq M_{i}, \quad i=1, \ldots, n-1
$$

For each $k$, choose $t_{k} \in\left[T_{3}, T_{4}\right]$. Then

$$
\begin{equation*}
\left(t_{k}, y\left(t_{k} ; m_{k}\right), y^{\prime}\left(t_{k} ; m_{k}\right), \ldots, y^{(n-1)}\left(t_{k} ; m_{k}\right)\right) \in\left[T_{3}, T_{4}\right] \times\left[L, p_{0}\right] \times \Pi_{i=1}^{n-1}\left[-M_{i}, M_{i}\right] \tag{1.12}
\end{equation*}
$$

The set on the righthand side of (1.12) is a compact subset of $\mathbb{R}^{n+1}$ and independent of $k$. Thus, there exists a convergent subsequence (relabeling if necessary)

$$
\left\{\left(t_{k}, y\left(t_{k} ; m_{k}\right), y^{\prime}\left(t_{k} ; m_{k}\right), \ldots, y^{(n-1)}\left(t_{k} ; m_{k}\right)\right)\right\} \rightarrow\left(t_{0}, c_{1}, \ldots, c_{n}\right)
$$

where $t_{0} \in\left[T_{3}, T_{4}\right]$. Since $t_{0} \in(a, b)$, by the continuous dependence of solutions of initial value problems, $y\left(t ; m_{k}\right)$ converges in $C^{n-1}\left[T_{1}, T_{3}\right]$ to a solution, say $z(t)$, of the initial value problem
(1.2), with initial conditions, $y^{(i-1)}\left(t_{0}\right)=c_{i}, i=1, \ldots, n$. Thus, $p_{0}=z\left(T_{3}\right)$ which implies $p_{0} \in \Omega$ and $\Omega$ is closed. This completes the proof if, for each $k$,

$$
y^{\prime}\left(T_{3} ; m_{k}\right)<y^{\prime}\left(T_{3} ; m_{1}\right) \leq 0 .
$$

Moving to the second subcase, assume $y^{\prime}\left(T_{3} ; m_{1}\right)<y^{\prime}\left(T_{3} ; m_{k}\right) \leq 0$ infinitely often. Relabeling if necessary, assume $y^{\prime}\left(T_{3} ; m_{1}\right)<y^{\prime}\left(T_{3} ; m_{k}\right) \leq 0$ for each $k$. For this case, we work on an interval to the left of $T_{3}$. Find $T_{2}<T_{4}<T_{3}$ such that $y^{\prime}\left(t ; m_{1}\right) \leq 0$ and $y\left(T_{3} ; m_{1}\right) \leq y\left(t ; m_{1}\right) \leq p_{0}$ for $t \in\left[T_{4}, T_{3}\right]$. The inequality (1.7) remains valid and

$$
y^{\prime}\left(t ; m_{1}\right)<y^{\prime}\left(t ; m_{k}\right), \quad t \in\left(T_{2}, b\right) .
$$

So, for $t \in\left[T_{4}, T_{3}\right]$,

$$
y\left(T_{3} ; m_{1}\right) \leq y\left(t ; m_{1}\right)<y\left(t ; m_{k}\right)
$$

and there exists $c_{k} \in\left(t, T_{3}\right)$ such that

$$
\begin{aligned}
y\left(t ; m_{k}\right) & =y\left(T_{3} ; m_{k}\right)+y^{\prime}\left(c_{k} ; m_{k}\right)\left(t-T_{3}\right) \leq y\left(T_{3} ; m_{k}\right)+y^{\prime}\left(c_{k} ; m_{1}\right)\left(t-T_{3}\right) \\
& \leq p_{0}+\max _{t \in\left[T_{4}, T_{3}\right]}\left|y^{\prime}\left(t ; m_{1}\right)\right|\left(T_{3}-T_{4}\right) .
\end{aligned}
$$

Set $L=y\left(T_{3} ; m_{1}\right)$ and $P_{0}=p_{0}+\max _{t \in\left[T_{4}, T_{3}\right]}\left|y^{\prime}\left(t ; m_{1}\right)\right|\left(T_{3}-T_{4}\right)$ and analogous to (1.8) we have for $t \in\left[T_{4}, T_{3}\right], k \in \mathbb{N}_{1}$,

$$
L \leq y\left(t ; m_{k}\right) \leq P_{0} .
$$

The proof of the second subcase now proceeds precisely as the proof of the first case.
For these two subcases we have assumed $y^{\prime}\left(T_{3} ; m_{k}\right) \leq 0$ for each $k$. If $y^{\prime}\left(T_{3} ; m_{k}\right)>0$ for each $k$, one again considers two subcases, $y^{\prime}\left(T_{3} ; m_{k}\right)>y^{\prime}\left(T_{3} ; m_{1}\right)>0$ for each $k$, or $y^{\prime}\left(T_{3} ; m_{1}\right)>y^{\prime}\left(T_{3} ; m_{k}\right) \geq 0$ for each $k$. If $y^{\prime}\left(T_{3} ; m_{k}\right)>y^{\prime}\left(T_{3} ; m_{1}\right)>0$ for each $k$, produce an analogue to the preceding first subcase on an interval $\left[T_{4}, T_{3}\right]$ where $T_{2}<T_{4}<T_{3}$ and define $L=y\left(T_{4} ; m_{1}\right)$. If $y^{\prime}\left(T_{3} ; m_{1}\right)>y^{\prime}\left(T_{4} ; m_{k}\right) \geq 0$ for each $k$, produce an analogue to the preceding second subcase on an interval $\left[T_{3}, T_{4}\right]$ where $T_{3}<T_{4}<b$. The proof is complete.

Remark 1.6. In [3], the authors claim to have constructed a sequence of solutions of (1.1), (1.3) for $j=1$ and a compact set analogous to (1.12). The calculations to obtain an interval analogous to $\left[T_{3}, T_{4}\right]$ of positive length are incorrect which in turn implies the calculations to obtain a priori bounds on higher order derivatives are incorrect. Thus, the conjecture, stated as Theorem 3.6 in [3] is unproven.

## References

[1] S. D. Conte, Carl de Boor, Elementary numerical analysis: An algorithmic approach, Third edition, McGraw-Hill Book Co., New York, 1981. MR0202267; Zbl 0494.65001
[2] P. W. Eloe, J. Henderson, Two point boundary value problems for ordinary differential equations, uniqueness implies existence, Proc. Amer. Math. Soc., 148(2020), No. 10, 43774387. https://doi.org/10.1090/proc/15115; MR4135304; Zbl 1452.34035
[3] P. W. Eloe, J. Henderson, J. Neugebauer, Three point boundary value problems for ordinary differential equations, uniqueness implies existence, Electron. J. Qual. Theory Differ. Equ. 2020, No. 74, 1-15. https://doi.org/10.14232/ejqtde.2020.1.74; MR4208481; Zbl 07334628
[4] A. Lasota, Z. Opial, On the existence and uniqueness of solutions of a boundary value problem for an ordinary second order differential equation, Colloq. Math. 18(1967), 1-5. https://doi.org/10.4064/cm-18-1-1-5; MR0219792; Zbl 0155.41401

# Geometry, integrability and bifurcation diagrams of a family of quadratic differential systems as application of the Darboux theory of integrability 

Regilene Oliveira ${ }^{\boxtimes 1}$, Dana Schlomiuk ${ }^{2}$, Ana Maria Travaglini ${ }^{1}$ and Claudia Valls ${ }^{3}$<br>${ }^{1}$ Departamento de Matemática, ICMC-Universidade de São Paulo, Avenida Trabalhador São-carlense, 400-13566-590, São Carlos, SP, Brazil<br>${ }^{2}$ Département de Mathématiques et de Statistique, Université de Montréal, CP 6128 succ. Centre-Ville, Montréal QC H3C 3J7, Canada<br>${ }^{3}$ Departamento de Matemática, Instituto Superior Técnico, Universidade Técnica de Lisboa, Avenida Rovisco Pais 1049-001, Lisboa, Portugal

Received 23 December 2020, appeared 9 July 2021 Communicated by Gabriele Villari


#### Abstract

During the last forty years the theory of integrability of Darboux, in terms of algebraic invariant curves of polynomial systems has been very much extended and it is now an active area of research. These developments are covered in numerous papers and several books, not always following the conceptual historical evolution of the subject and its significant connections to Poincaré's problem of the center. Our first goal is to give in a concise way, following the history of the subject, its conceptual development. Our second goal is to display the many aspects of the theory of Darboux we have today, by using it for studying the special family of planar quadratic differential systems possessing an invariant hyperbola, and having either two singular points at infinity or the infinity filled up with singularities. We prove the integrability for systems in 11 of the 13 normal forms of the family and the generic non-integrability for the other 2 normal forms. We construct phase portraits and bifurcation diagrams for 5 of the normal forms of the family, show how they impact the changes in the geometry of the systems expressed in their configurations of their invariant algebraic curves and point out some intriguing questions on the interplay between this geometry and the integrability of the systems. We also solve the problem of Poincaré of algebraic integrability for 4 of the normal forms we study.


Keywords: quadratic differential system, invariant algebraic curve, invariant hyperbola, Darboux integrability, Liouvillian integrability, configuration of invariant algebraic curves, bifurcation of configuration, singularity and bifurcation.
2020 Mathematics Subject Classification: 58K45, 34A26, 34C23

[^9]
## 1 Introduction

Let $\mathbb{R}[x, y]$ be the set of all real polynomials in the variables $x$ and $y$. Consider the planar system

$$
\begin{align*}
& \dot{x}=P(x, y),  \tag{1.1}\\
& \dot{y}=Q(x, y),
\end{align*}
$$

where $\dot{x}=d x / d t, \dot{y}=d y / d t$ and $P, Q \in \mathbb{R}[x, y]$. We define the degree of a system (1.1) as $\max \{\operatorname{deg} P, \operatorname{deg} Q\}$. In the case where the polynomials $P$ and $Q$ are relatively prime i.e. they do not have a non-constant common factor, we say that (1.1) is non-degenerate.

Consider

$$
\begin{equation*}
\chi=P(x, y) \frac{\partial}{\partial x}+Q(x, y) \frac{\partial}{\partial y} \tag{1.2}
\end{equation*}
$$

the polynomial vector field associated to (1.1).
A real quadratic differential system is a polynomial differential system of degree 2, i.e.

$$
\begin{align*}
& \dot{x}=p_{0}+p_{1}(\tilde{a}, x, y)+p_{2}(\tilde{a}, x, y) \equiv p(\tilde{a}, x, y)  \tag{1.3}\\
& \dot{y}=q_{0}+q_{1}(\tilde{a}, x, y)+q_{2}(\tilde{a}, x, y) \equiv q(\tilde{a}, x, y)
\end{align*}
$$

with $\max \{\operatorname{deg} p, \operatorname{deg} q\}=2$ and

$$
\begin{array}{lll}
p_{0}=a, & p_{1}(\tilde{a}, x, y)=c x+d y, & p_{2}(\tilde{a}, x, y)=g x^{2}+2 h x y+k y^{2}, \\
q_{0}=b, & q_{1}(\tilde{a}, x, y)=e x+f y, & q_{2}(\tilde{a}, x, y)=l x^{2}+2 m x y+n y^{2} .
\end{array}
$$

Here we denote by $\tilde{a}=(a, c, d, g, h, k, b, e, f, l, m, n)$ the 12 -tuple of the coefficients of system (1.3). Thus a quadratic system can be identified with a point $\tilde{a}$ in $\mathbb{R}^{12}$.

We denote the class of all quadratic differential systems with QS.
Planar polynomial differential systems occur very often in various branches of applied mathematics, in modeling natural phenomena, for example, modeling the time evolution of conflicting species, in biology, in chemical reactions, in economics, in astrophysics, in the equations of continuity describing the interactions of ions, electrons and neutral species in plasma physics (see, for example: [45], [73], [8] and [55]). Such differential systems have also theoretical importance. Several problems on polynomial differential systems, which were stated more than one hundred years ago, are still open: the second part of Hilbert's 16th problem stated by Hilbert in 1900 [32], the problem of the center stated by Poincaré in 1885 [50], the problem of algebraic integrability stated by Poincaré in 1891 [51], [52] (both problems later discussed in this work), and problems on integrability resulting from the work of Darboux [20] published in 1878. With the exception of the problem of the center for quadratic differential systems that was solved, all the other problems mentioned above, are still unsolved even in the quadratic case.

The theory of Darboux [20] (1878) was built for complex polynomial differential equations over the complex projective plane. Here we are interested in polynomial differential systems over the real affine plane. But every system (1.1) with real coefficient can be extended over the complex affine plane and it leads to a polynomial differential equation with homogeneous coefficients over the complex projective plane (see for example [40], pp. 316-317). As a consequence, the theory of Darboux can be applied to real polynomial differential systems. This is a theory of integrability of polynomial differential systems (1.1) which is based on the existence of particular solutions that are algebraic. The cases of integrable systems are rare but as Arnold said in [2, p. 405] ". . . these integrable cases allow us to collect a large amount of information
about the motion in more important systems...." Poincare was enthusiastic about the theory of Darboux and called it "admirable" in [51] and "oeuvre magistrale" in [52]. In [52] Poincaré stated his problem of algebraic integrability on systems (1.1), which is still open today. The French Academy of Sciences proposed this problem for a prize which was won by Painlevé and Autonne received an honorable mention but although the new results were interesting they have not provided a complete solution to the problem posed by Poincaré. After the research done by Poincaré, Painlevé and Autonne at the end of the 19th century we have the work of Dulac and of Lagutinskiĭ at the beginning of the 20th century. The work of Dulac [24] will later be briefly discussed in this work. Lagutinskii's work is not well known because except for one paper written in French, all of his other 16 papers, published between 1903 and 1914, were written in Russian. He died in 1915 at the age of 44 . The interested reader could find information about his life and work in [21], [22]. Almost a century passed before Darboux theory began again to significantly attract researchers. It started to flower towards the end of the last century and the beginning of the 21th century when in numerous works Darboux's theory has been enriched with new notions and results. Now this is a very active field with new results scattered in many articles and several books. The various aspects of this extended theory appear in the literature in surveys, some incomplete as they were published earlier, some containing the latest additions to the theory such as [44]. These surveys are mainly concerned with results and not with the historical conceptual development of the subject, which is fascinating. For example we mentioned above the 1908 work of Dulac on Poincaré's problem of the center where connections with Darboux integrability are present. These connections go deep. They allowed Dulac to solve the problem of the center for complex quadratic systems with a center, the only case where the problem was solved. The method of Darboux is also powerful in unifying proofs of integrability for whole families of systems with centers or for other families of systems like the ones we consider in this paper. For other applications of the theory of Darboux see the survey article of Llibre and Zhang [44].

One of the goals of this article is to make this task easier by providing here a brief conceptual survey of this beautiful theory, which closely follows the historical evolution of the subject. We also prove here that even when trying to understand the integrability of real systems, their complex invariant curves are essential (see in Section 2, Example 40).

Definition 1.1 ([20]). An algebraic curve $f(x, y)=0$ with $f(x, y) \in \mathbb{C}[x, y]$ is called an invariant algebraic curve of system (1.1) if it satisfies the following identity:

$$
\begin{equation*}
f_{x} P+f_{y} Q=K f \tag{1.4}
\end{equation*}
$$

for some $K \in \mathbb{C}[x, y]$ where $f_{x}$ and $f_{y}$ are the derivatives of $f$ with respect to $x$ and $y . K$ is called the cofactor of the curve $f=0$.

For simplicity we write the curve $f$ instead of the curve $f=0$ in $\mathbb{C}^{2}$. Note that if system (1.1) has degree $m$ then the cofactor of an invariant algebraic curve $f$ of the system has degree $m-1$.

Definition 1.2 ([20]). Consider a planar polynomial system (1.1). An algebraic solution of (1.1) is an algebraic invariant curve $f$ which is irreducible over $\mathbb{C}$.

Definition 1.3. Let $U$ be an open subset of $\mathbb{R}^{2}$. A real function $H: U \rightarrow \mathbb{R}$ is a first integral of system (1.1) if it is constant on all solution curves $(x(t), y(t))$ of system (1.1), i.e., $H(x(t), y(t))=k$, where $k$ is a real constant, for all values of $t$ for which the solution $(x(t), y(t))$ is defined on $U$.

If $H$ is differentiable in $U$ then $H$ is a first integral on $U$ if and only if

$$
\begin{equation*}
H_{x} P+H_{y} Q=0 . \tag{1.5}
\end{equation*}
$$

The problem of integrating a polynomial system by using its algebraic invariant curves over $\mathbb{C}$ was considered for the first time by Darboux in [20].

Theorem 1.4 (Darboux [20]). Suppose that a polynomial system (1.1) has $m$ invariant algebraic curves $f_{i}(x, y)=0, i \leq m$, with $f_{i} \in \mathbb{C}[x, y]$ and with $m>n(n+1) / 2$ where $n$ is the degree of the system. Then we can compute complex numbers $\lambda_{1}, \ldots, \lambda_{m}$ such that $f_{1}^{\lambda_{1}} \ldots f_{m}^{\lambda_{m}}$ is a first integral of the system.

Definition 1.5. If a system (1.1) has a first integral of the form

$$
\begin{equation*}
H(x, y)=f_{1}^{\lambda_{1}} \ldots f_{p}^{\lambda_{p}} \tag{1.6}
\end{equation*}
$$

where $f_{i}$ are the invariant algebraic curves of system (1.1) and $\lambda_{i} \in \mathbb{C}$ then we say that system (1.1) is Darboux integrable and we call the function $H$ a Darboux function.

Remark 1.6. We stress that the theorem of Darboux gives only a sufficient condition for Darboux integrability of a system (1.1) (see example below), expressed in a relation between the number of invariant algebraic curves the system possesses and the degree of the system.

Example 1.7. Consider the system

$$
\left\{\begin{array}{l}
\dot{x}=3+2 x^{2}+x y \\
\dot{y}=3+x y+2 y^{2}
\end{array}\right.
$$

This system admits the invariant line $x-y=0$ and the invariant hyperbola $2+x y=0$. Then, $m=2<3=n(n+1) / 2$. However we still have here a Darboux first integral $H(x, y)=(x-$ $y)^{-3 / 2}(2+x y)$. Thus the lower bound on the number of invariant curves sufficient for Darboux integrability in his theorem is in general greater than necessary. The following question arises then naturally: Could we find a necessary and sufficient condition for Darboux integrability?

Definition 1.8. Let $U$ be an open subset of $\mathbb{R}^{2}$ and let $R: U \rightarrow \mathbb{R}$ be an analytic function which is not identically zero on $U$. The function R is an integrating factor of a polynomial system (1.1) on $U$ if one of the following two equivalent conditions on $U$ holds:

$$
\operatorname{div}(R P, R Q)=0, \quad R_{x} P+R_{y} Q=-R \operatorname{div}(P, Q)
$$

where $\operatorname{div}(P, Q)=P_{x}+Q_{y}$.
A first integral $H$ of

$$
\dot{x}=R P, \quad \dot{y}=R Q
$$

associated to the integrating factor $R$ is then given by

$$
H(x, y)=\int R(x, y) P(x, y) d y+h(x)
$$

where $H(x, y)$ is a function satisfying $H_{x}=-R Q$. Then,

$$
\dot{x}=H_{y}, \quad \dot{y}=-H_{x} .
$$

In order that this function $H$ be well defined the open set $U$ must be simply connected.
The simplest integrable systems (1.1) are the Hamiltonian ones having a polynomial first integral. Next we have the systems (1.1) which admit a rational first integral. These were called by Poincaré algebraically integrable systems. Such a first integral yields a foliation with singularities of the plane in algebraic phase curves. The question asked by Poincaré in [52] is the following:

Can we recognize when a system (1.1) admits a rational first integral?
This is Poincarés problem of algebraic integrability and it is not even solved for quadratic differential systems. We say more on this question in the next section.

To advance knowledge on algebraic, Darboux or more general types of integrability it is useful to have a large number of examples to analyze. In the literature, scattered isolated examples were analyzed, among them is the family of quadratic differential systems possessing a center, i.e. a singular point surrounded by closed phase curves. There is a rather strong relationship between the problem of the center and the theory of Darboux. In particular, every quadratic system with a center possesses invariant algebraic curves and in the generic case it possesses a Darboux first integral. For non-generic cases such a system is still integrable but with a more general type of a first integral.

A more systematic approach for studying families of integrable systems was initiated in the papers of Schlomiuk and Vulpe [65], [66], [67], [68] and [64] where they classified topologically the phase portraits of quadratic systems with invariant lines of at least four total multiplicity (including the line at infinity) as well as the quadratic systems with the line at infinity filled up with singularities and proved their integrability. These results were applied by Schlomiuk and Vulpe $[69,70]$ to the family of Lotka-Volterra differential systems (the L-V family), important for so many applications. Not all the systems in this family are integrable but since the L-V systems always have at least three invariant lines (including the line at infinity), numerous systems in this family also belong to the family of systems possessing at least four invariant lines and using this fact and the results in the papers above indicated, simplified the classification. There are thus many L-V systems that are integrable according to the method of Darboux. For the Liouvillian integrability of L-V systems see [6]. The case of quadratic systems possessing two complex invariant lines intersecting at a real finite point was completed in [71]. Systems in this family are not always integrable but as the authors show, for a large subfamily we can apply the Darboux theory of integrability. Work is in progress for completing the study of the family of systems possessing three invariant lines, including the line at infinity. In the above studies, the properties of the "configuration" of invariant lines (term we will later define) were important to distinguish the types of integrability of the systems. A natural question which arises is the following one:

What is the relation between the geometry of a "configuration" of invariant algebraic curves of a system (1.1) and its integrability?

In order to be able to provide responses to such a question, data involving only invariant lines is insufficient. Data involving more general curves and in particular conics and cubics, is needed. In [47] the authors classified the family QSH of non-degenerate quadratic differential systems possessing an invariant hyperbola according to "configurations of invariant hyperbolas and lines". They proved that the family QSH is geometrically rich as it has 205 distinct such configurations. The problem of integrability of systems in QSH according to the theory of Darboux was not considered in [47]. This is the problem we study in the second part of this paper. Considered from the viewpoint of integrability, the family QSH is also very rich displaying a vast array of systems of various kinds of integrability as we see in the examples we
provide in this paper. This data will be precious in deeper exploring the Darboux theory of integrability. Here by "deeper" we mean understanding the relationship between the integrability of the systems and the geometry of the "configurations" of invariant algebraic curves they possess.

Since its creation by Darboux in 1878 [20], this theory has evolved and it has been significantly extended. Much of this development occurred during the past forty years. The literature on this extended theory is scattered in many articles and also some books, not necessarily following the history of the conceptual evolution of the subject with its connections with the problem of the center. These connections were important for drawing attention to the role of the theory of Darboux and its unifying capacity for proving integrability of families of polynomial differential systems as we explain in the next Section and for classifying families of vector fields not necessarily integrable such as the family of L-V systems previousy discussed.

The second goal is to study the systems of the family QSH from the viewpoint of what we call today the Darboux theory of integrability. This adds a lot of integrability data next to the data we have from the work of Schlomiuk and Vulpe, mentioned above, on quadratic systems with invariant straight lines by allowing us to also include conics. Apart from richly illustrating the theory and pointing out some rather subtle issues, this testing ground provides us with the possibility of asking new questions relating the geometry of the configuration of invariant algebraic curves and the Darboux theory of integrability. It is this relationship that is our main motivation.

Our paper is organized as follows:
In Section 2 we give a short conceptual and historical overview of Darboux theory as we have it today, including all essential new notions not used in Darboux's work, as well as new results, extensions of his theory. We also recall the unifying character of the method of Darboux in proving integrability for some families of vector fields and we prove that the theory of Darboux is essentially a theory over the complex field even when we search to calculate real first integrals of real systems (see Example 2.34 in this Section).

In Section 3 we discuss the class QSH from the viewpoint of the relationship between integrability and the geometry of the "configuration of invariant algebraic curves" which the systems possess. In particular we are concerned here with the family $\mathbf{Q S H}_{\eta=0}$ of systems in QSH which possess either exactly two distinct real singularities at infinity or the line at infinity filled up with singularities. In [47] the authors calculated the invariant lines and hyperbolas of each normal form in $\mathbf{Q S H}_{\eta=0}$.

In Section 4 we introduce a number of geometrical concepts which are very helpful in understanding the relation between the geometry of the configuration of invariant algebraic curves and the integrability of the systems.

In Section 5 we prove that for the 11 of the 13 normal forms for the systems in $\mathbf{Q S H}_{\eta=0}$ all systems have a Liouvillian first integral. We present the invariant algebraic curves, exponential factors, integrating factors and first integrals for each one of these 11 normal forms for $\mathbf{Q S H}_{\eta=0}$.

In Section 6 we prove the generic non-integrability for the remaining two normal forms for $\mathbf{Q S H}_{\eta=0}$, cases where the number of invariant curves and exponential factors are not sufficient for finding a first integral or integrating factor.

In Section 7 we apply the Darboux theory of integrability to the geometric analysis of five families of systems in $\mathbf{Q S H}_{\eta=0}$. We exhibit the bifurcation diagrams of the configurations of invariant algebraic curves as well as the bifurcation diagrams of the systems and raise the problem of interaction between these two kinds of bifurcations. Phase portraits for quadratic
system with an invariant hyperbola and an invariant straight line were also constructed in [41]. However, we point out that the authors of [41] did not get all of the phase portraits, in particular, in Section 7 we point out some of their missing phase portraits. This is due to the fact that their normal form for this family misses some of the systems in the family. We also solve the Poincaré problem of algebraic integrability for four of the families we studied.

In Section 8 we highlight some significant points raised in this paper, explain the relation between the bifurcations of configurations of invariant curves and topological bifurcations, raise a number of questions and state some problems. Finally we mention that we also obtained, as limiting cases of the family (D), three other normal forms, i.e. (F), (G) and (I).

## 2 Brief conceptual and historical overview of the theory of Darboux [20] as it is understood today

After the publication of the works of Poincaré, Painlevé and Autonne in the 1890's originating in the work of Darboux [20], the first article using the method of integration of Darboux was Dulac's paper [24] (1908) in which he solved Poincaré's problem of the center [50] for quadratic differential systems (see more on this problem on page 9). After the publication of Dulac's paper, the next important result concerning the Darboux theory of integrability is Jouanolou's who in [34] (1979) gave a sufficient condition for algebraic integrability.

Theorem 2.1 (Jouanolou [34]). Consider a polynomial system (1.1) of degree $n$ and suppose that it admits $m$ invariant algebraic curves $f_{i}(x, y)=0$ where $1 \leq i \leq m$, then if $m \geq 2+\frac{n(n+1)}{2}$, there exists integers $N_{1}, N_{2}, \ldots, N_{m}$ such that $I(x, y)=\prod_{i=1}^{m} f_{i}^{N_{i}}$ is a first integral of (1.1).

If a differential system (1.1) has a rational first integral $H(x, y)=f(x, y) / g(x, y)$ with $f, g \in \mathbb{C}[x, y]$, then the solution curves are located on its level curves $H(x, y)=C$ where $C$ is a constant, i.e. on the algebraic curves $f(x, y)-C g(x, y)=0$. We call degree of the first integral $H$ the number $\max (\operatorname{deg}(f), \operatorname{deg}(g))$. Then all the algebraic invariant curves of the system have a degree bounded by the degree of $H$.

We can argue that in case we can show that a system has invariant algebraic curves of bounded degree, in order to decide whether the system is algebraically integrable it remains to compute, by solving algebraic equations, a sufficient amount of invariant algebraic curves. This is true because we know that a finite number of steps will be sufficient. For this reason the problem of Poincaré is sometimes understood as the problem of bounding the degrees of the invariant algebraic curves the system possesses. Thus, in [7] the problem of Poincaré is stated as follows:

Let $\mathcal{F}$ be a holomorphic foliation by curves of the complex projective plane $\mathbb{P}_{\mathbb{C}}^{2}$. Let $C$ be an algebraic curve in $\mathbb{P}_{\mathrm{C}}^{2}$. Is it possible to bound the degree of C in terms of the degree of $\mathcal{F}$ ?

The problem of Poincare is understood in this way elsewhere in the literature, see for example [33], page 242. But solving this problem is far from solving the problem as initially formulated by Poincaré. Indeed, the algebraic equations we would need to solve in order to find a sufficient amount of algebraic invariant curves of the systems, to obtain algebraic integrability, can easily surpass the present day capacity of computers. Besides, the problem of bounding the degree of an algebraic invariant curve is not even solved in the general case. For a solution of this problem under restrictive conditions see [7].

So far we mentioned only three steps in the hierarchy of first integrals: polynomial, rational and Darboux first integrals which could be rational or transcendental. What other kinds
of first integrals can we have next in this hierarchy? We can have elementary first integrals. Roughly speaking these are functions which are constructed by using addition, multiplication, composition of finitely many rational functions, trigonometric and exponential functions and their inverses.

The next important result, obtained in 1983, involves elementary first integrals and is due to Prelle and Singer. It was stated for more general vector fields in $\mathbb{C}^{n}$ in differential algebra language. Here we consider only the case of planar differential systems (1.1).

Theorem 2.2 (Prelle-Singer [53]). If a polynomial differential system (1.1) has an elementary first integral, then the system has a first integral of the following form:

$$
f(x, y)+c_{1} \log \left(f_{1}(x, y)\right)+c_{2} \log \left(f_{2}(x, y)\right)+\cdots+c_{k} \log \left(f_{k}(x, y)\right)
$$

where $f$ and $f_{i}$, are algebraic functions over $\mathbb{C}(x, y)$ and $c_{i} \in \mathbb{C}, i=1,2, \ldots k$.
Taking the exponential of the above expression we obtain the following corollary.
Corollary 2.3. If a polynomial differential system (1.1) possesses an elementary first integral then it also admits a first integral of the form:

$$
e^{f(x, y)} f_{1}(x, y)^{c_{1}} f_{2}(x, y)^{c_{2}} \ldots f_{k}^{c_{k}}
$$

where $f$ and $f_{i}$, are algebraic functions over $\mathbb{C}(x, y)$ and $c_{i} \in \mathbb{C}, i=1,2, \ldots k$.
In particular we can take for $f(x, y)$ a rational function and for all $f_{i}^{\prime} s$ polynomial functions over $\mathbb{C}$. This kind of expression differs from a Darboux first integral by the exponential factor $e^{f(x, y)}$ which appears though not explicitly, in Prelle-Singer's paper [53] and also $f_{i}$ 's are here algebraic and not just polynomials over $\mathbb{C}$.

The above expression is a more general first integral that includes the case of a Darboux first integral when $f$ is the zero-function and $f_{i}^{\prime}$ s are polynomials. Although this kind of expression does not appear in [20], nowadays a first integral of this more general kind, with $f$ rational and all $f_{i}^{\prime}$ s polynomial functions, is still called a Darboux first integral in the literature.

In Section 3 of their paper [53] Prelle and Singer talk about "Algorithmic considerations" and they say:

The preceding work was motivated by our desire to develop a decision procedure for finding elementary first integrals. These results show that we need only look for elementary integrals of a prescribed form. In this section we shall discuss the problem of finding an elementary first integral for a twodimensional autonomous system of differential equations and reduce this problem to that of bounding the degrees of algebraic solutions of this system.

They base their algorithm on the following two propositions.
Proposition 2.4 ([53]). If the planar system (1.1) has an elementary first integral, then there exists an integer $n$ and an invariant algebraic curve $f$ such that

$$
P f_{x}+Q f_{y}=-n\left(P_{x}+Q_{y}\right) f
$$

Proposition 2.5. If the equations of (1.1) have an elementary first integral, then there exists an element $R$ algebraic over $\mathbb{C}(x, y)$ such that $R_{x} P+R_{y} Q=-\left(P_{x}+Q_{y}\right) R$.

We use here a version of the Prelle-Singer algorithm provided in [31].

Theorem 2.6 (The Prelle-Singer algorithm [53] (1983), as presented in [31] (2001)).
(1) Let $N=1$.
(2) Find all the invariant algebraic curves $C: f(x, y)=0$ with

$$
P f_{x}+Q f_{y}=K f
$$

such that $K(x, y) \in \mathbb{C}[x, y]$ and $\operatorname{deg}(f) \leq N$.
(3) Decide if there exist constants $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{m} \in \mathbb{C}$, not all zero, such as

$$
\sum_{i=0}^{m} \lambda_{i} K_{i}=0
$$

where $K_{i}$ is cofactor of a curve $f_{i}$ found in (2). If such $\lambda_{i}$ 's exist, then $I=\prod_{i=0}^{m} f_{i}^{\lambda_{i}}$ is a first integral. Otherwise, go to (4).
(4) Decide if there exist constants $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{m} \in \mathbb{C}$, not all zero, such as

$$
\sum_{i=0}^{m} \lambda_{i} K_{i}=-\left(P_{x}+Q_{y}\right),
$$

where $K_{i}$ is cofactor of a curve $f_{i}$ found in (2).
If such $\lambda_{i}$ 's exist, then $R=\prod_{i=0}^{m} f_{i}^{\lambda_{i}}$ is an integrating factor and a first integral can be obtained by integrating the equations:

$$
\begin{gathered}
I_{x}=R Q \\
I_{y}=-R P .
\end{gathered}
$$

If such $\lambda_{i}$ 's do not exist, return to (1) increasing $N$ by 1 and continue the process.
In further exploring the evolution of ideas and development of the theory of Darboux it is important to mention the connections between this theory and the problem of the center stated by Poincaré in [50] in 1885. These connections have done much to draw attention to the theory of Darboux and its unifying power in proving integrability of polynomial systems. We indicate here some of these connections as well as the story of the solution of the problem of the center for quadratic systems and in proving their integrability in a unified way by the method of Darboux.

For quadratic systems the problem of the center as already mentioned at the beginning of this section was solved by Dulac. Unlike Poincaré, Dulac considered differential systems defined over $\mathbf{C}$. In [23] he defined the following notion of center: A singular point of a planar holomorphic differential system with non-zero eigenvalues is a center if and only if the quotient of its eigenvalues is negative and rational and the system has a local analytic first integral. In his paper [24], Dulac mentions that the general case is more difficult to treat, he supposes that the quotient of the eigenvalues is -1 . Placing the singular point at the origin, he used the following normal form for quadratic systems:

$$
\begin{array}{r}
\dot{x}=x+a_{20} x^{2}+a_{11} x y+a_{02} y^{2}, \\
\dot{y}=-y+b_{20} x^{2}+b_{11} x y+b_{02} y^{2} .
\end{array}
$$

To solve the problem of the center for quadratic systems means to find necessary and sufficient conditions in terms the coefficients $a_{i j}$ and $b_{i j}$ so that the origin be a center. He solved this problem in 1908 [24] and used the method of integration of Darboux in one case.

This work of Dulac could not be readily applied for real system. Indeed, in the normal form considered by Dulac, if we assume that the coefficients of the equations are real than this real system has a saddle at the origin and we cannot pass from this normal form to the normal form used by Poincaré (where the linear terms of the two equations are respectively $-y, x)$ by a real linear transformation. Thus the conditions for the center obtained by Dulac cannot be readily used in the case of real systems for centers as defined by Poincaré.

In 1985, working on perturbations of quadratic Hamiltonian systems with a center, Guckenheimer, Rand and Schlomiuk needed the conditions on a real quadratic differential system to have a center. Exploring the literature they found that it is very messy, containing many errors. In [58] (1990), after making a historical survey pointing out the errors, they proved by diverse ad hoc methods that each real quadratic system with a center is integrable. The correct conditions for a center were obtained by Kapteyn and Bautin (see $[35,36]$ ) thus solving the problem of the center for a real quadratic differential systems. At the suggestion of Guckenheimer, Schlomiuk then tried to give a geometric interpretation of the Kapteyn-Bautin conditions for a center. This geometric interpretation was revealed by studying the bifurcation diagram of the family QSC of quadratic systems with a center (see [61]). The conditions for the center can be interpreted in terms of the types of invariant algebraic curves the systems possess.

These results were presented for the first time by Schlomiuk at the Luminy conference in France on differential equations in 1989 and later in 1992 at the NATO Advanced Study Institute in Montreal where she also presented the work of Prelle-Singer (see [60]). Meetings are always very useful for disseminating information. Thus, it was in 1989 at the Luminy conference that Moussu, present at that meeting, told Schlomiuk about the work of Darboux. Specialists in integrability in the audience at the Montreal meeting in 1992, not previously aware of this work of Prelle and Singer, found out about this work from Schlomiuk's lectures. A unified proof of integrability based on the theory of Darboux, for all systems in QSC was obtained (see [59,60]) (1993). While the proof in [58] was done by using diverse ad hoc methods, in the new proof all the cases were treated in the same way, by the method of Darboux in terms of invariant algebraic curves. These and other articles mentioned further below played a role in drawing attention to the unifying role the method of integration of Darboux played in proving integrability for entire families of certain planar polynomial differential systems. The articles [59,61] were read by a number of people, in particular they were cited in [10] (1997), which contains an extension of the theory of Darboux to be later discussed, and also in [5].

In his PhD Thesis (1990) entitled Invariant algebraic curves in polynomial differential systems as well as later in his paper [11] (1994) Christopher had independently explored the relationship between the presence of invariant algebraic curves and conditions for the center in quadratic and also some cubic differential systems such as the cubic Kukles systems without a term in $y^{3}$ in the equation for $d y / d t$, or the cubic system of Dolov. He showed that the conditions for the center given by Kukles were incomplete and proved that the system of Dolov was integrable by using four invariant lines and a circle.

Work on these connections between the problem of the center and the Darboux theory of integrability continued to be published. We only mention here a few of the earliest papers such as [38] (1992) of Cozma and Șubă on cubic differential systems and of Żoła̧dek [76] (1994) on quadratic systems and their perturbations. More work on cubic systems done by Cozma
and Șubă and also by Żołądek alone or together with some of his students, can be accessed through MathSciNet. The cubic symmetric systems were proven to be integrable using the method of Darboux by Rousseau and Schlomiuk in [57] (1995) and they also had integrability results on the reduced cubic Kukles systems [56] (1995).

To get to a higher echelon in the hierarchy of first integrals, we need to consider Liouvillian first integrals. In [72] Singer describes Liouvillian functions as follows:

Liouvillian functions are functions that are built up from rational functions using exponentiation, integration, and algebraic functions.

Thus, the logarithm as a function of one variable is a Liouvillian function being defined as the integral from 0 to $x$ of $1 / x$. In general, Liouvillian functions are defined in the context of differential algebra.

The following result was proved by Singer in 1992.
Theorem 2.7 ([72]). If the system (1.1) has a Liouvillian first integral, then it has an integrating factor of the form

$$
e^{\int U d x+V d y}, \quad U_{y}=V_{x}
$$

where $U$ and $V$ are rational functions over $\mathbb{C}[x, y]$.
A consequence of Singer's theorem is the following.
Corollary 2.8 ([72]). A system of differential equations (1.1) has a Liouvillian first integral if and only if it has an integrating factor of the form

$$
R(x, y)=e^{\int U d x+V d y}, \quad U_{y}=V_{x}(U, V \text { are rational function over } \mathbb{C}[x, y])
$$

in which case

$$
F(x, y)=\int R(x, y) Q(x, y) d x-R(x, y) P(x, y) d y
$$

## is a Liouvillian first integral.

It is important to mention that a Liouvillian integrable system does not necessarily have an affine invariant algebraic curve. An example of such a polynomial differential system is presented in [30].

The following notion was defined by Christopher in 1994 (see [11]) where he called it "degenerate invariant algebraic curve".
Definition 2.9. Let $F(x, y)=\exp \left(\frac{G(x, y)}{H(x, y)}\right)$ with $G, H \in \mathbb{C}[x, y]$ coprime. We say that $F$ is an exponential factor of system (1.1) if it satisfies the equality

$$
\begin{equation*}
F_{x} P+F_{y} Q=L F \tag{2.1}
\end{equation*}
$$

for some $L \in \mathbb{C}[x, y]$. The polynomial $L$ is called the cofactor of the exponential factor $F$.
Proposition 2.10 ([11]). If $F=\exp (G / H)$ is an exponential factor of system (1.1) with cofactor $L$ then $H=0$ is an invariant algebraic curve of the system (1.1) with cofactor $K_{H}$ and $G$ satisfies the equation

$$
\begin{equation*}
P G_{x}+Q G_{y}=K_{H} G+L H, \quad \text { where } G, H, L, K_{H} \in \mathbb{C}[x, y] . \tag{2.2}
\end{equation*}
$$

See [15] for a detailed proof.
A theorem of Darboux was rephrased by Chavarriga, Llibre and Sotomayor [10] (1997) by introducing in [10] the notion of independent points.

If $S(x, y)=\sum_{i+j=0}^{m-1} a_{i j} x^{i} y^{j}$ is a polynomial of degree at most $m-1$ with $m(m+1) / 2$ coefficients in $\mathbb{C}$, then we write $S \in \mathbb{C}_{m-1}[x, y]$. We identify the linear space $\mathbb{C}_{m-1}[x, y]$ with $\mathbb{C}^{m(m+1) / 2}$ through the isomorphism

$$
S \rightarrow\left(a_{00}, a_{10}, a_{01}, \ldots, a_{m-1,0}, a_{m-2,1}, \ldots, a_{0, m-1}\right)
$$

Definition 2.11 ([10]). We say that $r$ singular points $\left(x_{k}, y_{k}\right) \in \mathbb{C}^{2}, k=1, \ldots, r$ of a differential system (1.1) of degree $m$ are independent with respect to $\mathbb{C}_{m-1}[x, y]$ if the intersection of the $r$ hyperplanes

$$
\sum_{i+j=0}^{m-1} x_{k}^{i} y_{k}^{j} a_{i j}=0, \quad k=1, \ldots, r
$$

in $\mathbb{C}^{m(m+1) / 2}$ is a linear subspace of dimension $[m(m+1) / 2]-r$.
We remark that the maximum number of isolated singular points of the polynomial system (1.1) of degree $m$ is $m^{2}$ (by Bézout's Theorem), that the maximum number of independent isolated singular points of the system is $m(m+1) / 2$, and that $m(m+1) / 2<m^{2}$ for $m \geq 2$.

The following is a theorem of Darboux as stated by Chavarriga, Llibre and Sotomayor proved in [10].

Theorem 2.12 ([20]). Assume that a real (complex) polynomial system of degree $m$ admits $q=$ $m(m+1) / 2+1-p$ algebraic solutions $f_{i}=0, i=1,2, \ldots, q$, not passing through $p$ real (com$p l e x)$ independent singular points $\left(x_{k}, y_{k}\right), k=1,2, \ldots, p$, then the system has a first integral of the form $f_{1}^{\lambda_{1}} f_{2}^{\lambda_{2}} \ldots f_{q}^{\lambda_{q}}$ with $\lambda_{i} \in \mathbb{R}$ (C).

Remark 2.13. The above theorem is interesting because it reduces the number of invariant algebraic curves we need to have, that according to Darboux's theorem is $m(m+1) / 2+1$, to just $m(m+1) / 2+1-p$.

Let us consider again Example 1.7:

$$
\left\{\begin{array}{l}
\dot{x}=3+2 x^{2}+x y, \\
\dot{y}=3+x y+2 y^{2} .
\end{array}\right.
$$

The line $f_{1}(x, y)=x-y=0$ and the hyperbola $f_{2}(x, y)=2+x y=0$ are invariant for this system with co-factors $K_{1}(x, y)=2 x+2 y$ and $K_{2}(x, y)=3 x+3 y$. Here $m=2=n$ and hence $m<n(n+1) / 2$. Still, the number of curves suffices to compute the first integral $H(x, y)=$ $(x-y)^{-3 / 2}(2+x y)$ although the condition in the theorem of Darboux is not satisfied by this number. But here we have that the singular points $P_{1,2}= \pm(-i \sqrt{3}, i \sqrt{3})$ of the system are independent. Indeed, solving the system $H_{1}=a_{00}-i \sqrt{3} a_{10}+i \sqrt{3} a_{01}=0, H_{2}=a_{00}+$ $i \sqrt{3} a_{10}-i \sqrt{3} a_{01}=0$, we get $a_{00}=0$ and $a_{10}=a_{01}$ and hence $\operatorname{dim}\left(H_{1} \cap H_{2}\right)=1$. Also $f_{1}\left(P_{i}\right) \neq 0$ and $f_{2}\left(P_{i}\right) \neq 0$. So the points $P_{i}^{\prime}$ 's are independent. Applying the above theorem we have $q=2, p=2, n=2$ and we have $q=n(n+1) / 2+1-p$.

Definition 2.14. A singular point $\left(x_{0}, y_{0}\right)$ of system (1.1) is called weak if the divergence of system (1.1) at $\left(x_{0}, y_{0}\right)$ is zero.

In what follows we state a generalization of Darboux's theorem taking into account exponential factors, independent points and invariants. The result was stated and proved by Christopher and Llibre in 2000 in [15]. An earlier version appeared in [5] (1999).

Theorem 2.15 ([15]). Suppose that a $\mathbb{C}$-polynomial system (1.1) of degree $m$ admits $p$ algebraic solutions $f_{i}=0$ with cofactors $K_{i}$ for $i=1, \ldots, p, q$ exponential factors $F_{j}=\exp \left(g_{j} / h_{j}\right)$ with cofactors $L_{j}$ for $j=1, \ldots, q$, and $r$ independent singular points $\left(x_{k}, y_{k}\right) \in \mathbb{C}^{2}$ such that $f_{i}\left(x_{k}, y_{k}\right) \neq 0$ for $i=1, \ldots, p$ and for $k=1, \ldots, r$.
(i) There exist $\lambda_{i}, \mu_{j} \in \mathbb{C}$ not all zero such that

$$
\sum_{i=1}^{p} \lambda_{i} K_{i}+\sum_{j=1}^{q} \mu_{j} L_{j}=0
$$

if and only if the (multi-valued) function

$$
\begin{equation*}
f_{1}^{\lambda_{1}} \ldots f_{p}^{\lambda_{p}} F_{1}^{\mu_{1}} \ldots F_{q}^{\mu_{q}} \tag{2.3}
\end{equation*}
$$

is a first integral of system (1.1).
(ii) If $p+q+r \geq[m(m+1) / 2]+1$, then there exist $\lambda_{i}, \mu_{j} \in \mathbb{C}$ not all zero such that

$$
\sum_{i=1}^{p} \lambda_{i} K_{i}+\sum_{i=1}^{q} \mu_{j} L_{j}=0
$$

(iii) If $p+q+r \geq[m(m+1) / 2]+2$, then system (1.1) has a rational first integral, and consequently all trajectories of the system are contained in invariant algebraic curves.
(iv) There exist $\lambda_{i}, \mu_{j} \in \mathbb{C}$ not all zero such that

$$
\sum_{i=1}^{p} \lambda_{i} K_{i}+\sum_{j=1}^{q} \mu_{j} L_{j}=-\operatorname{div}(P, Q)
$$

if and only if function (2.3) is an integrating factor of system (1.1).
(v) If $p+q+r=m(m+1) / 2$ and the $r$ independent singular points are weak, then function (2.3) is a first integral if

$$
\sum_{i=1}^{p} \lambda_{i} K_{i}+\sum_{i=1}^{q} \mu_{j} L_{j}=0
$$

or an integrating factor if

$$
\sum_{i=1}^{p} \lambda_{i} K_{i}+\sum_{j=1}^{q} \mu_{j} L_{j}=-\operatorname{div}(P, Q)
$$

under the condition that not all $\lambda_{i}, \mu_{j} \in \mathbb{C}$ are zero.
(vi) If there exist $\lambda_{i}, \mu_{j} \in \mathbb{C}$ not all zero such that

$$
\sum_{i=1}^{p} \lambda_{i} K_{i}+\sum_{j=1}^{q} \mu_{j} L_{j}=-s
$$

for some $s \in \mathbb{C} \backslash\{0\}$, then the (multi-valued) function

$$
\begin{equation*}
f_{1}^{\lambda_{1}} \ldots f_{p}^{\lambda_{p}} F_{1}^{\mu_{1}} \ldots F_{q}^{\mu_{q}} \exp (s t) \tag{2.4}
\end{equation*}
$$

is an invariant of system (1.1).

Of course, each irreducible factors of each $h_{j}$ is one of the $f_{i}$ 's.
Definition 2.16. If system (1.1) has a first integral of the form

$$
\begin{equation*}
H(x, y)=f_{1}{ }^{\lambda_{1}} \ldots f_{p}{ }^{\lambda_{p}} F_{1}{ }^{\mu_{1}} \ldots F_{q}{ }^{\mu_{q}} \tag{2.5}
\end{equation*}
$$

where $f_{i}$ and $F_{j}$ are respectively the invariant algebraic curves and exponential factors of a system (1.1) and $\lambda_{i}, \mu_{j} \in \mathbb{C}$, then we say that the system is generalized Darboux integrable. We call the function $H$ a generalized Darboux function.

Remark 2.17. In [20] Darboux considered functions of the type (1.6), not of type (2.5). In recent works functions of type (2.5) were called Darboux functions. Since in this work we need to pay attention to the distinctions among the various kinds of first integral we call (1.6) a Darboux and (2.5) a generalized Darboux first integral.

Proposition 2.18 ([25]). For a real polynomial system (1.1) the function $\exp (G / H)$ is an exponential factor with cofactor $K$ if and only if the function $\exp (\bar{G} / \bar{H})$ is an exponential factor with cofactor $\bar{K}$.

Remark 2.19 ([25]). If among exponential factors of the real system (1.1) a complex pair $F=$ $\exp (G / H)$ and $\bar{F}=\exp (\bar{G} / \bar{H})$ occurs, then the first integral (2.5) has a real factor of the form

$$
(\exp (G / H))^{\mu}(\exp (\bar{G} / \bar{H}))^{\bar{\mu}}=\exp (2 \operatorname{Re}(\mu(G / H))),
$$

where $\mu \in \mathbb{C}$ and $\operatorname{Im}(\mu) \operatorname{Im}(F) \neq 0$. This means that function (2.5) is real when system (1.1) is real.

Considering the definition of generalized Darboux function we can rewrite Corollary 2.8 as follows.

Theorem 2.20 ([11,72]). A planar polynomial differential system (1.1) has a Liouvillian first integral if and only if it has a generalized Darboux integrating factor.

For a proof see [75], page 134.
We can also state easily the following result of Preller-Singer.
Theorem 2.21 ([9,53]). If a planar polynomial vector field (1.2) has a generalized Darboux first integral, then it has a rational integrating factor.

In 2019, a converse of the previous result was proved in [16] as a consequence of [54].
Theorem 2.22 ([16]). If a planar polynomial vector field (1.2) has a rational integrating factor, then it has a generalized Darboux first integral.

We have the following table summing up these results.

| First integral |  | Integrating factor |
| :---: | :---: | :---: |
| Generalized Darboux | $\Leftrightarrow$ | Rational |
| Liouvillian | $\Leftrightarrow$ | Generalized Darboux |

To study the way integrable systems vary within families of polynomial differential systems (1.1) using the theory of Darboux, one needs to consider perturbations of a system within such a family. An algebraic invariant curve $f(x, y)=0$ of such a system could split in several
algebraic invariant curves occurring in nearby systems. In [11] (1994) C. Christopher considered in an example the coalescence of two such curves and its relationship with exponential factors but in [11] he did not yet talk about multiplicity of an invariant algebraic curve.

In [62] (1997) Schlomiuk introduced a general notion of multiplicity of an invariant algebraic curve $f=0$ of a polynomial differential system (1.1). This definition was given in terms of the multiplicities of singularities of the system located on the projective completion of the curve (Definition 4.1 in [62]).

A notion of multiplicity was defined by Schlomiuk and Vulpe in 2004 for invariant lines of quadratic differential systems and in [64] they classified the family of quadratic systems with invariant lines of total multiplicity at least five, including the line at infinity, according to configurations of straight lines of such systems. Around the same time this study was in progress, Christopher, Llibre and Pereira were working on their important paper [18] (2007) and produced a preprint, earlier version of their work, containing several notions of multiplicity of an invariant algebraic curve. In [18] they gave a condition for these notions to coincide. In this work, as we see later, we use three of the notions introduced in [18].

Suppose that a polynomial differential system has an algebraic solution $f(x, y)=0$ where $f(x, y) \in \mathbb{C}[x, y]$ is of degree $n$ given by

$$
f(x, y)=c_{0}+c_{10} x+c_{01} y+c_{20} x^{2}+c_{11} x y+c_{02} y^{2}+\cdots+c_{n 0} x^{n}+c_{n-1,1} x^{n-1} y+\cdots+c_{0 n} y^{n}
$$

with $\hat{c}=\left(c_{0}, c_{10}, \ldots, c_{0 n}\right) \in \mathbb{C}^{N}$ where $N=(n+1)(n+2) / 2$. We note that the equation

$$
\lambda f(x, y)=0, \quad \lambda \in \mathbb{C}^{*}=\mathbb{C}-\{0\}
$$

yields the same locus of complex points in the plane as the locus induced by $f(x, y)=0$. Therefore, a curve of degree $n$ is defined by $\hat{c}$ where

$$
[\hat{c}]=\left[c_{0}: c_{10}: \cdots: c_{0 n}\right] \in P_{N-1}(\mathbb{C}) .
$$

We say that a sequence of curves $f_{i}(x, y)=0$, each one of degree $n$, converges to a curve $f(x, y)=0$ if and only if the sequence of points $\left[c_{i}\right]=\left[c_{i 0}: c_{i 10}: \cdots: c_{i 0 n}\right]$ converges to $[\hat{c}]=\left[c_{0}: c_{10}: \cdots: c_{0 n}\right]$ in the topology of $P_{N-1}(\mathbb{C})$.

We observe that if we rescale the time $t^{\prime}=\lambda t$ by a positive constant $\lambda$ the geometry of the systems (1.1) (phase curves) does not change. So for our purposes we can identify a system (1.1) of degree $n$ with a point

$$
\left[a_{0}: a_{10}: \cdots: a_{0 n}: b_{0}: b_{10}: \cdots: b_{0 n}\right] \in \mathbb{S}^{N-1}(\mathbb{R})
$$

where $N=(n+1)(n+2)$.

## Definition 2.23 ([64]).

(1) We say that an invariant curve

$$
\mathcal{L}: f(x, y)=0, \quad f \in \mathbb{C}[x, y]
$$

for a polynomial system $(S)$ of degree $n$ has geometric multiplicity $m$ if there exists a sequence of real polynomial systems $\left(S_{k}\right)$ of degree $n$ converging to $(S)$ in the topology of $\mathbb{S}^{N-1}(\mathbb{R})$ where $N=(n+1)(n+2)$ such that each $\left(S_{k}\right)$ has $m$ distinct invariant curves

$$
\mathcal{L}_{1, k}: f_{1, k}(x, y)=0, \ldots, \mathcal{L}_{m, k}: f_{m, k}(x, y)=0
$$

over $\mathbb{C}, \operatorname{deg}(f)=\operatorname{deg}\left(f_{i, k}\right)=r$, converging to $\mathcal{L}$ as $k \rightarrow \infty$, in the topology of $P_{R-1}(\mathbb{C})$, with $R=(r+1)(r+2) / 2$ and this does not occur for $m+1$.
(2) We say that the line at infinity

$$
\mathcal{L}_{\infty}: Z=0
$$

of a polynomial system $(S)$ of degree $n$ has geometric multiplicity $m$ if there exists a sequence of real polynomial systems $\left(S_{k}\right)$ of degree $n$ converging to $(S)$ in the topology of $\mathbb{S}^{N-1}(\mathbb{R})$ where $N=(n+1)(n+2)$ such that each $\left(S_{k}\right)$ has $m-1$ distinct invariant lines

$$
\mathcal{L}_{1, k}: f_{1, k}(x, y)=0, \ldots, \mathcal{L}_{m-1, k}: f_{m-1, k}(x, y)=0
$$

over $\mathbb{C}$, converging to the line at infinity $\mathcal{L}_{\infty}$ as $k \rightarrow \infty$, in the topology of $P_{2}(\mathbb{C})$ and this does not occur for $m$.

In 2007 the authors of [18] introduced the following notion of geometric multiplicity:
Definition 2.24 ([18]). Consider $\chi$ a polynomial vector field of degree $d$. An invariant algebraic curve $f=0$ of degree $n$ of the vector field $\chi$ has geometric multiplicity $m$ if $m$ is the largest integer for which there exists a sequence of vector fields $\left(\chi_{i}\right)_{i>0}$ of bounded degree, converging to $h \chi$, for some polynomial $h$, not divisible by $f$, such that each $\chi_{r}$ has $m$ distinct invariant algebraic curves, $f_{r, 1}=0, f_{r, 2}=0, \ldots, f_{r, m}=0$, of degree at most $n$, which converge to $f=0$ as $r$ goes to infinity. If $h=1$, then we say that the curve has strong geometric multiplicity $m$.

Definition 2.25 ([18,49]). Let $\mathbb{C}_{m}[x, y]$ be the $\mathbb{C}$-vector space of polynomials in $\mathbb{C}[x, y]$ of degree at most $m$ and of dimension $R=(m+1)(m+2) / 2$. Let $\left\{v_{1}, v_{2}, \ldots, v_{R}\right\}$ be a base of $\mathbb{C}_{m}[x, y]$. We denote by $M_{R}(m)$ the $R \times R$ matrix

$$
M_{R}(m)=\left(\begin{array}{cccc}
v_{1} & v_{2} & \cdots & v_{R}  \tag{2.6}\\
\chi\left(v_{1}\right) & \chi\left(v_{2}\right) & \cdots & \chi\left(v_{R}\right) \\
\vdots & \vdots & \ddots & \vdots \\
\chi^{R-1}\left(v_{1}\right) & \chi^{R-1}\left(v_{2}\right) & \cdots & \chi^{R-1}\left(v_{R}\right)
\end{array}\right)
$$

where $\chi^{k+1}\left(v_{i}\right)=\chi\left(\chi^{k}\left(v_{i}\right)\right)$. The $m$ th extactic curve of $\chi, \mathcal{E}_{m}(\chi)$, is given by the equation $\operatorname{det} M_{R}(m)=0$. We also call $\mathcal{E}_{m}(\chi)$ the $m$ th extactic polynomial.

From the properties of the determinant we note that the extactic curve is independent of the choice of the base of $\mathbb{C}_{m}[x, y]$.

Theorem 2.26 ([49]). Consider a planar vector field (1.2). We have $\mathcal{E}_{m}(\chi)=0$ and $\mathcal{E}_{m-1}(\chi) \neq 0$ if and only if $\chi$ admits a rational first integral of exact degree $m$.

Observe that if $f=0$ is an invariant algebraic curve of degree $m$ of $\chi$, then $f$ divides $\mathcal{E}_{m}(\chi)$. This is due to the fact that if $f$ is a member of a base of $\mathbb{C}_{m}[x, y]$, then $f$ divides the whole column in which $f$ is located.

Definition 2.27 ([18]). We say that an invariant algebraic curve $f=0$ of degree $m \geq 1$ has algebraic multiplicity $k$ if $\operatorname{det} M_{R}(m) \neq 0$ and $k$ is the maximum positive integer such that $f^{k}$ divides $\operatorname{det} M_{R}(m)$; and it has no defined algebraic multiplicity if $\operatorname{det} M_{R}(m) \equiv 0$.

Definition 2.28 ([18]). We say that an invariant algebraic curve $f=0$ of degree $m \geq 1$ has integrable multiplicity $k$ with respect to $\chi$ if $k$ is the largest integer for which the following is true: there are $k-1$ exponential factors $\exp \left(g_{j} / f^{j}\right), j=1, \ldots, k-1$, with $\operatorname{deg}\left(g_{j}\right) \leq j_{m}$, such that each $g_{j}$ is not a multiple of $f$.

In the next result we see that the algebraic and integrable multiplicity coincide if $f=0$ is an irreducible invariant algebraic curve.

Theorem 2.29 ([18]). Consider an algebraic solution $f=0$ of degree $m \geq 1$ of $\chi$. Then $f$ has algebraic multiplicity $k$ if and only if the vector field (1.2) has $k-1$ exponential factors $\exp \left(g_{j} / f^{j}\right)$, where $\left(g_{j}, f\right)=1$ and $g_{j}$ is a polynomial of degree at most $j m$, for $j=1, \ldots, k-1$.

In 2007 Christopher, Llibre and Pereira showed in [18] that the definitions of geometric (see Definition 2.24), algebraic and integrable multiplicity are equivalent when $f=0$ is an algebraic solution of the vector field (1.2). The algebraic multiplicity has the advantage that we have the possibility of calculating it via the extactic curve and if the curve is irreducible then this coincides with either the integrable (reflected in the exponential factors) or the geometric one. Christopher, Llibre and Pereira also stated and proved the following theorem about Darboux theory of integrability that takes into account the multiplicity of the invariant algebraic curves.

Theorem 2.30 ([18], see Theorem 8.3.). Consider a planar vector field (1.2). Assume that (1.2) has $p$ distinct irreducible invariant algebraic curves $f_{i}=0, i=1, \ldots, p$ of multiplicity $m_{i}$, and let $N=\sum_{i=1}^{p} m_{i}$. Suppose, furthermore, that there are $q$ critical points $p_{1}, \ldots, p_{q}$ which are independent with respect to $\mathbb{C}_{m-1}[x, y]$, and $f_{j}\left(p_{k}\right) \neq 0$ for $j=1, \ldots, p$ and $k=1, \ldots, q$. We have:
(a) If $N+q \geq[m(m+1) / 2]+2$, then $\chi$ has a rational first integral.
(b) If $N+q \geq[m(m+1) / 2]+1$, then $\chi$ has a Darboux first integral.
(c) If $N+q \geq[m(m+1) / 2]$ and $p_{i}$ 's are weak, then $\chi$ has either a Darboux first integral or a Darboux integrating factor.

This theorem was generalized by Llibre and Zhang in [42] for invariant hypersurfaces in $\mathbb{C}^{n}$. In the same paper they also generalized the theorem of Jouanolou and gave a simplified, elementary proof.

The term of total multiplicity of invariant curves, finite and infinite, of a polynomial differential system was used for the first time in the theory of Darboux by Schlomiuk and Vulpe in [64], in the specific context of invariant straight lines of quadratic differential systems. In [18] the total multiplicity $N$ of the finite number of affine (finite) invariant algebraic curves appeared for the first time in the general context of the theory of Darboux in the above quoted theorem. This number is clearly not the total multiplicity of invariant algebraic curves of the system as the line at infinity is invariant and could have multiplicity (for examples see [64,68]).

The total multiplicity of the invariant algebraic curves finite and infinite occurs for the first time in the general setting in the work of Llibre and Zhang (2009) but only for invariant hyper-surfaces of polynomial vector fields in $\mathbb{R}^{n}$ and to this day we do not have the analog of this theorem for multiple invariant hypersurfaces, both finite and the hyper plane at infinity.

We consider now the result of Llibre and Zhang in [43]. To state it the authors generalized the Poincaré compactification on the sphere for planar differential systems to the Poincaré compactification of polynomial differential systems in $\mathbb{R}^{n}$ which they constructed in the Appendix of [43].

To talk about multiplicity of the hyperplane at infinity they only needed to pass by central projection from the systems in $\mathbb{R}^{n}$, considered as the hyperplane $Z=1$ in $\mathbb{R}^{n+1}$ tangent to the $n$-sphere with radius 1 centered at the origin of $\mathbb{R}^{n+1}$, and then further into the chart $x_{1}=1$ and obtain $\left(x_{1}, \ldots, x_{n}, 1\right)=\lambda\left(1, y_{2}, \ldots, y_{n}, Z\right)$ for some non-zero real $\lambda$. Hence we must have $\lambda=x_{1}$ and therefore $y_{2}=x_{2} / x_{1}, \ldots, y_{n}=x_{n} / x_{1}, Z=1 / x_{1}$ and $x_{1}=1 / Z, x_{2}=y_{2} / Z, \ldots$, $x_{n}=y_{n} / Z$. Transferring the vector field in this chart we obtain that it has a pole on $Z=0$. In complete analogy with the compactification of the plane we can obtain an analytic vector field on the $n$-sphere which is conjugate to the vector field thus obtained. In this way our initial hyper-surface at infinity, becomes just an affine hypersurface in the chart $x_{1}=1$ and hence we can apply to it our notions of multiplicity. Let $\bar{\chi}=\left(P_{1}(x), P_{2}(x), \ldots, P_{n}(x)\right)$ be the expression of the compactified vector field $\chi$. We say that the infinity of $\chi$ has algebraic multiplicity $k$ if $Z=0$ has algebraic multiplicity $k$ for the vector field $\bar{\chi}$; and that it has no defined algebraic multiplicity if $Z=0$ has no defined algebraic multiplicity for $\chi$. One thing the authors did not say is that this definition of the multiplicity of the infinite hypersurface does not depend on the chart $x_{1}$ we chose, and that it leads to the same value if we replace this chart by any other chart $x_{i}=1$ with $i \neq 1$.

Theorem 2.31 ([43]). Let $\bar{\chi}$ be the expression of the compactified vector field $\chi$. Assume that $\chi$ restricted to $\mathrm{Z}=0$ has no rational first integral. Then $\mathrm{Z}=0$ has algebraic multiplicity $k$ for $\bar{\chi}$ if and only if $\bar{\chi}$ has $k-1$ exponential factors $\exp \left(\overline{g_{j}} / Z^{j}\right)$ where $j=1, \ldots, k-1$ with $\overline{g_{j}} \in \mathbb{C}_{j}\left[Z, y_{2}, \ldots, y_{n}\right]$ having no factor $Z$.

The next result provides a relation between the exponential factors of $\chi$ and those of $\bar{\chi}$ associated with $Z=0$.

Proposition 2.32 ([43]). For the exponential factors associated with the hyperplane at infinity the following statements hold.
(a) If $E=\exp (g(x))$ with $g$ a polynomial of degree $k$ is an exponential factor of $\chi$ with cofactor $L_{E}(x)$, then $\bar{E}=\exp \left(\frac{\bar{g}}{Z^{k}}\right)$ with $\bar{g}=Z^{k} g\left(\frac{1}{Z}, \frac{y_{2}}{Z}, \ldots, \frac{y_{n}}{Z}\right)$ is an exponential factor of $\bar{\chi}$ with cofactor $L_{\bar{E}}=Z^{d-1} L_{E}\left(\frac{1}{Z}, \frac{y_{2}}{Z}, \ldots, \frac{y_{n}}{Z}\right)$.
(b) Conversely if $\bar{F}=\exp \left(\frac{\bar{h}}{Z^{k}}\right)$ with $\bar{h} \in \mathbb{R}_{k}\left[Z, y_{2}, \ldots, y_{n}\right]$ is an exponential factor of $\bar{\chi}$ with cofactor $L_{\bar{F}}$, then $F=\exp (h(x))$ with $h(x)=x^{k} \bar{h}\left(\frac{1}{x_{1}}, \frac{x_{2}}{x_{1}}, \ldots, \frac{x_{n}}{x_{1}}\right)$ is an exponential factor of $\chi$ with cofactor $L_{F}=x^{d-1} L_{\bar{F}}\left(\frac{1}{x_{1}}, \frac{x_{2}}{x_{1}}, \ldots, \frac{x_{n}}{x_{1}}\right)$.
The following result was proved in 2009 by Llibre and Zhang.
Theorem 2.33 ([43]). Assume that the polynomial vector field $\chi$ in $\mathbb{R}^{n}$ of degree $d>0$ has irreducible invariant algebraic hypersurfaces $f_{i}=0$ for $i=1, \ldots, p$ and the invariant hyperplane at infinity.
(i) If one of these irreducible invariant algebraic hypersurfaces or the invariant hyperplane at infinity has no defined algebraic multiplicity, then the vector field $\chi$ has a rational first integral.
(ii) Suppose that each irreducible invariant algebraic hypersurfaces $f_{i}=0$ has algebraic multiplicity $m_{i}$ for $i=1, \ldots, p$ and that the invariant hyperplane at infinity has algebraic multiplicity $k$. If the vector field restricted to the hyperplane at infinity or to any invariant hypersurface with multiplicity larger than 1 has no rational first integral, then the following hold
(a) If $\sum_{i=1}^{p} m_{i}+k=N+2$, then the vector field $\chi$ has a real Darboux first integral, where $N=\left({ }_{n}^{n+d-1}\right)$.
(b) If $\sum_{i=1}^{p} m_{i}+k=N+n+1$, then (1.2) has a real rational first integral.

We remark that by "real Darboux first integral" in Theorem 2.33 the authors mean generalized Darboux first integral. For two-dimensional polynomial vector fields, the additional condition in Theorem 2.33 on the nonexistence of rational first integrals of the vector field restricted to the invariant algebraic curves including the line at infinity is not necessary.

We end our conceptual and historical survey with some comments about this result over the reals. Darboux constructed his theory over the complex projective space which we think is the natural field and natural space for this theory. Firstly the complex numbers form an algebraically closed field. So an essential ingredient in his theory, the theory of algebraic curves, can be properly done. Indeed, Bézout's theorem cannot be proved over the reals. Secondly the complex projective plane is a compact space and in particular "the line at infinity" of the affine plane completely looses its special status in the projective plane. It is like any other line.

On the other hand it is important to observe that when we consider the theory of Darboux for real systems, we can go to their complexification and these systems could have complex invariant algebraic curves $f(x, y)=0$ with $f \in \mathbb{C}[x, y]$. We can therefore end up with more invariant curves than those with real coefficients. Let us consider an example.

## Example 2.34.

$$
\left\{\begin{array}{l}
\dot{x}=x^{2}+1 \\
\dot{y}=x+y
\end{array}\right.
$$

This system clearly has two invariant lines which are complex $x \pm i=0$ with respective co-factors $x \mp i$. It can easily be checked that the line at infinity has the multiplicity two. So the total multiplicity of invariant lines over $\mathbb{C}$ is four. This system was proved to be integrable in [66] having the inverse Darboux integrating factor $(x+i)^{1+i / 2}(x-i)^{1-i / 2}$.

Let us now consider this real system without taking into consideration its complexification. Suppose now that we want to prove just by using real curves that the system is integrable. The lines $x \pm i y=0$ are defined over $\mathbb{C}$ and it is only their union, the conic $x^{2}+1=0$ which is defined over $\mathbb{R}$. This is an invariant curve of the real system with the cofactor $2 x$. We also have an exponential factor $e^{1+2 y}$ with co-factor $2(x+y)$. However this is insufficient for proving integrability as we can check by trying to apply the usual algorithm for computing an integrating factor. Indeed, given $\lambda_{1}, \lambda_{2} \in \mathbb{C}$ we have $2 \lambda_{1} x+2 \lambda_{2}(x+y)=-\operatorname{div}(P, Q)=$ $-2 x-1$, which has no solution. So although the real system is integrable and has a real first integral, we cannot compute this real first integral without considering the two invariant lines. So this supports the idea that the full real extension of the Darboux theory that also covers the line at infinity with its own multiplicity cannot produce all the real integrable systems.

In conclusion we really need an extension of the Darboux theory over $\mathbb{C}$ that includes the multiplicity of the line at infinity and work in this direction is in progress.

## 3 Applications of Darboux's theory to the family QSH of quadratic differential systems with an invariant hyperbola, case $\eta=0$

The notion of configuration of invariant curves of a polynomial differential system appears in several works, see for instance [64].

Definition 3.1 ([64]). Consider a real planar polynomial system (1.1) with a finite number of singular points. By configuration of algebraic solutions of the system we mean a set of algebraic solutions over $\mathbb{C}$ of the system, each one of these curves endowed with its own multiplicity and together with all the real singular points of this system located on these curves, each one of these singularities endowed with its own multiplicity.

The notion of equivalence of configurations was used in many papers (see for instance, [47,64-68]) to classify systems in QS possessing invariant algebraic curves according to the kind of configurations these systems.

In particular, in [47], QSH was classified according to the configuration of invariant hyperbolas and lines the systems possess. The equivalence of configurations in the class QSH depends on whether the systems admits a finite or an infinite number of invariant hyperbolas. See Definition 1.9 of [47] in case the system has a finite number of invariant hyperbolas and Definition 1.10 of [47] for the case the system has an infinite family of invariant hyperbolas. The classification of QSH led to 205 distinct configurations.

Here we introduce some invariant polynomials that play an important role in the study of polynomial vector fields. Considering $C_{2}(\tilde{a}, x, y)=y p_{2}(\tilde{a}, x, y)-x q_{2}(\tilde{a}, x, y)$ as a cubic binary form of $x$ and $y$ we calculate

$$
\eta(\tilde{a})=\operatorname{Discrim}\left[C_{2}, \tilde{\zeta}\right], \quad M(\tilde{a}, x, y)=\operatorname{Hessian}\left[C_{2}\right],
$$

where $\xi=y / x$ or $\xi=x / y$. It is known that the singular points at infinity of quadratic systems are given by the solutions in $x$ and $y$ of $C_{2}(\tilde{a}, x, y)=0$. If $\eta<0$ then this means we have one real singular point at infinity and two complex ones.

Remark 3.2. We note that since a system in QSH always has an invariant hyperbola then clearly we always have at least 2 real singular points at infinity. So we must have $\eta \geq 0$.

The family QSH can be split as follows: QSH $_{\eta=0}$ of systems which possess either exactly two distinct real singularities at infinity or the line at infinity filled up with singularities and QSH $_{\eta>0}$ of systems which possess three distinct real singularities at infinity in $P_{2}(\mathbb{C})$. In this paper we present a study of $\mathbf{Q S H}_{\eta=0}$.

In [47] the authors gave necessary and sufficient conditions for a quadratic system to have an invariant hyperbola. These conditions were given in terms of 40 affine invariant polynomials and hence these conditions are independent of the normal forms in which the systems may be presented. For the sake of completeness, we give below in the following tables these conditions for $\mathbf{Q S H}_{\eta=0}$.

In the next table we present in the first column the number associated to the equations in [47], which are the normal forms for the systems in QSH. In the second column are the necessary and sufficient conditions. For a proof see [47].

| Equations in [47] | Invariants |
| :---: | :---: |
| (4.4) | $\eta=0, M \neq 0, \theta \neq 0, \beta_{2} \neq 0, \beta_{1} \neq 0, \mathcal{R}_{1} \neq 0, B_{1} \neq 0$ |
| (4.10) | $\eta=0, M \neq 0, \theta \neq 0, \beta_{2} \neq 0, \beta_{1} \neq 0, \mathcal{R}_{1} \neq 0, B_{1}=0$ |
| (4.11) | $\eta=0, M \neq 0, \theta \neq 0, \beta_{2} \neq 0, \beta_{1}=0, \gamma_{1}=0, \mathcal{R}_{3} \neq 0$ |
| (4.13) | $\eta=0, M \neq 0, \theta \neq 0, \beta_{2}=0, \beta_{1}=\gamma_{14}=0, \mathcal{R}_{10} \neq 0$ |
| (4.13) $g=1 / 4$ | $\begin{aligned} & \eta=0, M \neq 0, \theta \neq 0, \beta_{2}=0, \beta_{1}=\gamma_{14}=0, \mathcal{R}_{10} \neq 0, \beta_{7} \beta_{8}=0, \mathcal{R}_{10}<0, \beta_{8}=0 \\ & \eta=0, M \neq 0, \theta \neq 0, \beta_{2}=0, \beta_{1}=\gamma_{14}=0, \mathcal{R}_{10} \neq 0, \beta_{7} \beta_{8}=0, \mathcal{R}_{10}>0, \beta_{8}=0 \end{aligned}$ |
| (4.13) $g=1 / 2$ | $\begin{aligned} & \eta=0, M \neq 0, \theta \neq 0, \beta_{2}=0, \beta_{1}=\gamma_{14}=0, \mathcal{R}_{10} \neq 0, \beta_{7} \beta_{8}=0, \mathcal{R}_{10}<0, \beta_{7}=0 \\ & \eta=0, M \neq 0, \theta \neq 0, \beta_{2}=0, \beta_{1}=\gamma_{14}=0, \mathcal{R}_{10} \neq 0, \beta_{7} \beta_{8}=0, \mathcal{R}_{10}>0, \beta_{7}=0 \end{aligned}$ |
| (4.16) | $\begin{aligned} & \eta=0, \mathcal{A}_{1}[M \neq 0, \theta=0], \mu_{0} \neq 0, \beta_{3}=\gamma_{8}=0, \mathcal{R}_{7} \neq 0, \chi_{A}^{(7)}<0 \\ & \eta=0, \mathcal{A}_{1}[M \neq 0, \theta=0], \mu_{0} \neq 0, \beta_{3}=\gamma_{8}=0, \mathcal{R}_{7} \neq 0, \chi_{A}^{(7)}>0 \end{aligned}$ |
| (4.16) $c^{2}=a$ | $\eta=0, \mathcal{A}_{1}[M \neq 0, \theta=0], \mu_{0} \neq 0, \beta_{3}=\gamma_{8}=0, \mathcal{R}_{7} \neq 0, \chi_{A}^{(7)}=0$ |
| (4.18) | $\eta=0, \mathcal{A}_{1}[M \neq 0, \theta=0], \mu_{0}=0, N \neq 0,\left(\mathfrak{C}_{1}\right), \beta_{12} \neq 0, \mu_{2} \neq 0$ |
| (4.18) $g=0$ | $\eta=0, \mathcal{A}_{1}[M \neq 0, \theta=0], \mu_{0}=0, N \neq 0,\left(\mathfrak{C}_{1}\right), \beta_{12} \neq 0, \mu_{2}=0, \gamma_{16} \neq 0$ |
| $(4.18) c=0$ | $\eta=0, \mathcal{A}_{1}[M \neq 0, \theta=0], \mu_{0}=0, N \neq 0,\left(\mathfrak{C}_{1}\right), \beta_{12} \neq 0, \mu_{2}=0, \gamma_{16}=0$ |
| (4.22) | $\begin{aligned} & \eta=0, \mathcal{A}_{1}[M \neq 0, \theta=0], \mu_{0}=0, N \neq 0,\left(\mathfrak{C}_{1}\right), \beta_{12}=0, \gamma_{17}<0 \\ & \eta=0, \mathcal{A}_{1}[M \neq 0, \theta=0], \mu_{0}=0, N \neq 0,\left(\mathfrak{C}_{1}\right), \beta_{12}=0, \gamma_{17}>0 \end{aligned}$ |
| (4.22) $\epsilon=0$ | $\eta=0, \mathcal{A}_{1}[M \neq 0, \theta=0], \mu_{0}=0, N \neq 0,\left(\mathfrak{C}_{1}\right), \beta_{12}=0, \gamma_{17}=0$ |
| (4.25) $c \neq 0$ | $\eta=0, \mathcal{A}_{1}[M \neq 0, \theta=0], \mu_{0}=0, N=0, \beta_{13} \neq 0, \gamma_{10}=\gamma_{17}=0, \mathcal{R}_{11} \neq 0, \gamma_{16} \neq 0$ |
| (4.25) $c=0$ | $\eta=0, \mathcal{A}_{1}[M \neq 0, \theta=0], \mu_{0}=0, N=0, \beta_{13} \neq 0, \gamma_{10}=\gamma_{17}=0, \mathcal{R}_{11} \neq 0, \gamma_{16}=0$ |
| (4.27) | $\eta=0, \mathcal{A}_{1}[M \neq 0, \theta=0], \mu_{0}=0, N=0, \beta_{13}=0, \tilde{\gamma}_{18}=\tilde{\gamma}_{19}=0, \mu_{2} \neq 0$ |
| (4.28) | $\eta=0, \mathcal{A}_{1}[M \neq 0, \theta=0], \mu_{0}=0, N=0, \beta_{13}=0, \tilde{\gamma}_{18}=\tilde{\gamma}_{19}=0, \mu_{2}=0$ |
| (4.30) | $\begin{aligned} & \eta=0, \mathcal{A}_{2}\left[C_{2}=0, M=0\right], N_{7}=0, H_{10} \neq 0, H_{9}<0 \\ & \eta=0, \mathcal{A}_{2}\left[C_{2}=0, M=0\right], N_{7}=0, H_{10} \neq 0, H_{9}>0 \end{aligned}$ |
| (4.31) | $\eta=0, \mathcal{A}_{2}\left[C_{2}=0, M=0\right], N_{7}=0, H_{10} \neq 0, H_{9}=0$ |
| (4.34) | $\eta=0, \mathcal{A}_{2}\left[C_{2}=0, M=0\right], N_{7}=0, H_{10}=0, H_{12} \neq 0, H_{2} \neq 0$ |
| (4.36) | $\eta=0, \mathcal{A}_{2}\left[C_{2}=0, M=0\right], N_{7}=0, H_{10}=0, H_{12} \neq 0, H_{2}=0$ |
| (4.38) | $\eta=0, \mathcal{A}_{2}\left[C_{2}=0, M=0\right], N_{7}=0, H_{10}=0, H_{12}=0$ |

In this table we denote by (4.25) the following system that appears in [47] without number

$$
\dot{x}=-\frac{3 c^{2}}{16}+c x-x^{2}, \quad \dot{y}=1-2 x y
$$

If $c \neq 0$ we may assume $c=4$ by the rescaling $(x, y, t) \mapsto(c x / 4,4 y / c, 4 t / c)$. So we obtain the system denoted by (4.25) in [47] which we denote here by (4.25) $c \neq 0$.

The normal forms numbered in the table from (4.4) and up to (4.38), that were obtained in [47], appear below in a condensed table in the following proposition.

Proposition 3.3 ([47]). Every system in QSH $_{\eta=0}$ can be brought by an affine transformation and time rescaling to one of the following 13 normal forms, where $a, g, c, \epsilon$ are real parameters. Next to each normal forms we present the respective invariant hyperbola.

$$
\left\{\begin{array}{l}
\dot{x}=2 a+x+g x^{2}+x y,  \tag{A}\\
\dot{y}=a(2 g-1)-y+(g-1) x y+y^{2},
\end{array} \quad \Phi(x, y)=a+x y\right.
$$

where $a(g-1) \neq 0$
$\left\{\begin{array}{l}\dot{x}=2 a+g x^{2}+x y, \\ \dot{y}=a(2 g-1)+(g-1) x y+y^{2},\end{array} \quad \Phi(x, y)=a+x y\right.$
where $a(g-1) \neq 0$
$\left\{\begin{array}{l}\dot{x}=2 a+3 c x+x^{2}+x y \\ \dot{y}=a-c^{2}+y^{2},\end{array}\right.$

$$
\begin{equation*}
\Phi(x, y)=a+c x+x y \tag{C}
\end{equation*}
$$

where $a \neq 0$
$\left\{\begin{array}{l}\dot{x}=(c+x)(c(2 g-1)+g x) \\ \dot{y}=1+(g-1) x y,\end{array} \quad \Phi(x, y)=\frac{1}{(-1+2 g)}+c y+x y\right.$
where $(g \pm 1)(3 g-1)(2 g-1) \neq 0$
$\left\{\begin{array}{ll}\dot{x}=x^{2}+\epsilon \\ \dot{y}=1-2 x y\end{array} \quad \Phi_{1,2}(x, y)=-1 \pm i \sqrt{\epsilon} y+x y\right.$
$\left\{\begin{array}{l}\dot{x}=(x-1)(3-x) \\ \dot{y}=1-2 x y\end{array} \quad \Phi(x, y)=\frac{1}{3}+y-x y\right.$
$\left\{\begin{array}{l}\dot{x}=-x^{2} \\ \dot{y}=1-2 x y\end{array} \quad \Phi(x, y)=-1+3 x y\right.$
$\left\{\begin{array}{l}\dot{x}=(2 x-1)(2 x+1) / 4 \\ \dot{y}=y\end{array} \quad \Phi(x, y)=-\frac{q}{2}+q x+\frac{y}{2}+2 x y, q \neq 0\right.$
$\left\{\begin{array}{l}\dot{x}=x^{2} \\ \dot{y}=1\end{array}\right.$

$$
\begin{equation*}
\Phi(x, y)=1+r x+x y \tag{I}
\end{equation*}
$$

$\begin{cases}\dot{x}=a+y+x^{2} \\ \dot{y}=x y & \Phi(x, y)=a+2 y+x^{2}-m^{2} y^{2}\end{cases}$
$\begin{cases}\dot{x}=(1+3 x)(2+3 x) / 9 \\ \dot{y}=x y & \\ \end{cases}$
$\left\{\begin{array}{l}\dot{x}=a+x^{2} \\ \dot{y}=x y,\end{array} \quad \Phi(x, y)=a+x^{2}-m^{2} x y\right.$
where $a \neq 0$

$$
\left\{\begin{array}{l}
\dot{x}=x^{2}  \tag{M}\\
\dot{y}=1+x y
\end{array}\right.
$$

Using the invariants described previously which are powerful so as to give the necessary and sufficient conditions for systems (1.3) to have an invariant hyperbola, in [47] the authors considered the two possibilities: $M(\tilde{a}, x, y) \neq 0$ (i.e. at infinity we have two distinct real singularities) and $M=0=C_{2}$ (when we have an infinite number of singularities at infinity).
(i) $M(\tilde{a}, x, y) \neq 0$ : This brings systems (1.3) to the systems

$$
\left\{\begin{array}{l}
\dot{x}=a+c x+d y+g x^{2}+h x y,  \tag{3.1}\\
\dot{y}=b+e x+f y+(g-1) x y+h y^{2} .
\end{array}\right.
$$

with invariants $C_{2}(x, y)=x^{2} y$ and $\theta=-h^{2}(g-1) / 2$.
(i.1) The case $\theta \neq 0$ gives the condition $h(g-1) \neq 0$ for (3.1) and via the bifurcation diagram in [47] we arrive at the normal forms

- (A) where $a(g-1) \neq 0$,
- (B) where $a(g-1) \neq 0$.
(i.2) The case $\theta=0$ gives the condition $h(g-1)=0$ for (3.1) and via the bifurcation diagram in [47] we arrive at another invariant $\mu_{0}=g h^{2}$.
(i.2.1) For $\mu_{0} \neq 0$ we have the normal form
- (C) where $a \neq 0$.
(i.2.2) For $\mu_{0}=0$ they calculated the invariant $N=9(g-1)(g+1) x^{2}$ and we need to consider two possibilities.
(i.2.2.1) For the case $N \neq 0$ we have the normal forms
- (D) where $(g-1)(g+1)(2 g-1)(3 g-1) \neq 0$,
- (E) where $\epsilon \neq 0$.
(i.2.2.2) The case $N=0$ gives the condition $(g-1)(g+1)=0$ and we have the normal forms
- (F),
- (G),
- (H),
- (I).
(ii) $M(\tilde{a}, x, y)=0=C_{2}$ : We arrive at the normal forms
- (J),
- (K),
- (L) where $a \neq 0$,
- (M).

Remark 3.4. The invariant hyperbolas involve:
(i) sometimes all the parameters of the system (such as (C));
(ii) sometimes only some parameters (such as (A)) and
(iii) sometimes additional parameters (such as (J)).

The next theorem is the main result of this paper.
Consider the following sets:
$\mathrm{L}_{1}=\cup_{k \in \mathbb{N}} \mathrm{~L}_{1, k}$, where $\mathrm{L}_{1, k}=\left\{(a, g) \in \mathbb{R}^{2}: g=k / 2\right.$ and $\left.a \neq 0\right\}, k \in \mathbb{N}$,
$\mathrm{L}_{2}=\cup_{k \in \mathbb{N}} \mathrm{~L}_{2, k}$, where $\mathrm{L}_{2, k}=\left\{(a, g) \in \mathbb{R}^{2}: g=k / 3\right.$ and $\left.a \neq 0\right\}, k \in \mathbb{N}$,
$\mathrm{L}_{3}=\left\{(a, g) \in \mathbb{R}^{2}: g=1 / 4\right.$ and $\left.a \neq 0\right\}$,
$\mathrm{C}^{\prime}=\cup_{k \in \mathbb{N}} \mathrm{C}_{k}$, where $\mathrm{C}_{k}=\left\{(a, g) \in \mathbb{R}^{2}: g=(2+a-2 a k) / 4 a\right.$ and $\left.a \neq 0\right\}, k \in \mathbb{N}$.

Main Theorem. Consider the polynomial systems in QSH $_{\eta=0}$.
(a) The 11 normal forms (C)-(M) are all Liouvillian integrable. The following table sums up the results regarding the types of integrability:

| Systems | Parameters | Type of first integral |
| :--- | :--- | :--- |
| (C) | $a=8 c^{2} / 9$ and $c \neq 0$ | Generalized Darboux |
| (C) | $a\left(a-8 c^{2} / 9\right) \neq 0$ | Liouvillian |
| (D) | $g(g \pm 1)(2 g-1)(3 g-1) \neq 0$ | Darboux |
| (D) | $g=0$ and $c \neq 0$ | Generalized Darboux |
| (E) | $\epsilon \in \mathbb{R}$ | Polynomial (hamiltonian) |
| (F) | - | Rational |
| (G) | - | Rational |
| (H) | - | Rational |
| (I) | - | Rational |
| (J) | $a \in \mathbb{R}$ | Rational |
| (K) | - | Rational |
| (L) | $a \neq 0$ | Rational |
| (M) | - | Rational |
|  |  |  |

(b) For the normal forms (A) and (B) we have the following:
(i) If $(a, g) \in \mathbb{R}^{2}-\left(\mathrm{L}_{1} \cup \mathrm{~L}_{2} \cup \mathrm{C}^{\prime}\right)$, then systems (A) are not Liouvillian integrable.
(i.1) If $(a, g) \in \mathrm{L}_{1,1}$ then systems $(\mathrm{A})$ are not Liouvillian integrable.
(ii) If $(a, g) \in \mathbb{R}^{2}-\left(\mathrm{L}_{1} \cup \mathrm{~L}_{3}\right)$, then systems (B) are not Liouvillian integrable.
(ii.1) If $(a, g) \in \mathrm{L}_{1,1}$ then systems (B) are generalized Darboux integrable.
(ii.2) If $(a, g) \in \mathrm{L}_{3}$ then systems (B) are Liouvillian integrable.

The following table sums up the results regarding the types of integrability:

| Systems | Parameters | Type of first integral |
| :--- | :--- | :--- |
| $(\mathrm{A})$ | $(a, g) \in \mathbb{R}-\left(\mathrm{L}_{1} \cup \mathrm{~L}_{2} \cup \mathrm{C}^{\prime}\right)$ | Not Liouvillian integrable |
| $(\mathrm{B})$ | $g=1 / 2$ and $a \neq 0$ | Generalized Darboux |
| (B) | $g=1 / 4$ and $a \neq 0$ | Liouvillian |
| (B) | $(a, g) \in \mathbb{R}-\left(\mathrm{L}_{1} \cup \mathrm{~L}_{3}\right)$ | Not Liouvillian integrable |

Observation 3.5. The Liouvillian integrability of any system in class (A) with $(a, g) \in\left(\mathrm{L}_{1}-\mathrm{L}_{1,1}\right) \cup$ $\mathrm{L}_{2} \cup \mathrm{C}^{\prime}$ or in class $(\mathrm{B})$ with $(a, g) \in\left(\mathrm{L}_{1}-\mathrm{L}_{1,1}\right)$ is still open. The reason is that the methods applied in this paper for proving the existence or non-existence of a Liouvillian first integral do not work in these cases and so new ideas are needed for proving or disproving their Liouvillian integrability.

For a proof, see Section 5 (integrable cases) and Section 6 (non integrable cases).

## 4 Geometrical concepts and results useful for studying the geometry of the configurations of invariant curves and their bifurcations

Remark 4.1. In the theory of Darboux presented in the preceding section what counts is mainly the number of invariant curves, their multiplicities, the number of independent points. When certain inequalities involving these numbers are satisfied then we have integrability of the system.

However in 1993 Christopher and Kooij stated a theorem in [13] where, if we reformulate the theorem in geometric terms, we see a beautiful relation between the geometry of the "configuration of invariant curves" and their Darboux integrability. This theorem was proved in [17].

Theorem 4.2 ([13]). Consider a polynomial system (1.1) that has $k$ algebraic solutions $C_{i}=0$ such that
(a) all curves $C_{i}=0$ are non-singular and have no repeated factor in their highest order terms,
(b) no more than two curves meet at any point in the finite plane and are not tangent at these points,
(c) no two curves have a common factor in their highest order terms,
(d) the sum of the degrees of the curves is $n+1$, where $n$ is the degree of system (1.1).

Then system (1.1) has an integrating factor

$$
\mu(x, y)=1 /\left(C_{1} C_{2} \ldots C_{k}\right) .
$$

This theorem has a geometric content which is not completely explicit in the algebraic way they stated the result. We rewrite the theorem above in geometric terms as follows:

Theorem 4.3. Consider a polynomial system (1.1) that has kalgebraic solutions $C_{i}=0$ such that
(a) all curves $C_{i}=0$ are non-singular and they intersect transversally the line at infinity $Z=0$,
(b) no more than two curves meet at any point in the finite plane and are not tangent at these points,
(c) no two curves intersect at a point on the line at infinity $Z=0$,
(d) the sum of the degrees of the curves is $n+1$, where $n$ is the degree of system (1.1).

Then system (1.1) has an integrating factor

$$
\mu(x, y)=1 /\left(C_{1} C_{2} \ldots C_{k}\right) .
$$

In the hypotheses of this theorem the way the curves are placed with respect to one another in the totality of the curves, in other words the "geometry of the configuration of invariant algebraic curves" has an impact of the kind of integrating factor we could have.

We are interested in relating the geometry of the invariant algebraic curves curves taken in their totality with the various kinds of integrability. To begin doing this we need to recall some concepts and in particular those introduced by Poincaré in [52]. Among them we have the following.

Let $H=f / g$ be a rational first integral of the polynomial vector field (1.2). We say that $H$ has degree $n$ if $n$ is the maximum of the degrees of $f$ and $g$. We say that the degree of $H$
is minimal among all the degrees of the rational first integrals of $\chi$ if any other rational first integral of $\chi$ has a degree greater than or equal to $n$. Let $H=f / g$ be a rational first integral of $\chi$. According to Poincaré [52] we say that $c \in \mathbb{C} \cup\{\infty\}$ is a remarkable value of $H$ if $f+c g$ is a reducible polynomial in $\mathbb{C}[x, y]$. Here, if $c=\infty$, then $f+c g$ denotes $g$. Note that for all $c \in \mathbb{C}$ the algebraic curve $f+c g=0$ is invariant. The curves in the factorization of $f+c g$, when c is a remarkable value, are called remarkable curves.

Now suppose that $c$ is a remarkable value of a rational first integral $H$ and that $u_{1}^{\alpha_{1}} \ldots u_{r}^{\alpha}$ is the factorization of the polynomial $f+c g$ into reducible factors in $\mathbb{C}[x, y]$. If at least one of the $\alpha_{i}$ is larger than 1 then we say, following again Poincare (see for instance [28]), that $c$ is a critical remarkable value of $H$, and that $u_{i}=0$ having $\alpha_{i}>1$ is a critical remarkable curve of the vector field (1.2) with exponent $\alpha_{i}$.

Since we can think of $\mathbb{C} \cup\{\infty\}$ as the projective line $P_{1}(\mathbb{R})$ we can also use the following definition.

Definition 4.4. Consider $\mathcal{F}_{\left(c_{1}, c_{2}\right)}: c_{1} f-c_{2} g=0$ where $f / g$ is a rational first integral of (1.2). We say that $\left[c_{1}: c_{2}\right]$ is a remarkable value of the curve $\mathcal{F}_{\left(c_{1}, c_{2}\right)}$ if $\mathcal{F}_{\left(c_{1}, c_{2}\right)}$ is reducible over $\mathbb{C}$.

It was proved in [9] that there are finitely many remarkable values for a given rational first integral $H$ and if (1.2) has a rational first integral and has no polynomial first integrals, then it has a polynomial inverse integrating factor if and only if the first integral has at most two critical remarkable values.

Given $H=f / g$ a rational first integral, consider $F_{\left(c_{1}, c_{2}\right)}=c_{1} f-c_{2} g$ where $\operatorname{deg} F_{\left(c_{1}, c_{2}\right)}=n$. If $F_{\left(c_{1}, c_{2}\right)}=f_{1} f_{2}$ where $f_{1}, f_{2} \in \mathbb{C}[x, y]$ and $\operatorname{deg} f_{i}=n_{i}<n$ then necessarily the points on the intersection of $f_{1}=0$ and $f_{2}=0$ must be singular points of the curve $F_{\left(c_{1}, c_{2}\right)}$.

Lemma 4.5 ([11]). Assume that system (1.1) with degree $m$ has an invariant algebraic curve $f$ of degree $n$. Let $f_{n}, P_{m}$ and $Q_{m}$ be the homogeneous parts of $f$ with degree $n, P$ and $Q$ with degree $m$. Then each one of the irreducible factors of $f_{n}$ divides $y P_{m}-x Q_{m}$.

In geometric terms, this lemma means that the points at infinity of any invariant algebraic curve $f=0$ of a system (1.1) are singularities of this system.

Let us recall the algebraic-geometric definition of an r-cycle on an irreducible algebraic variety of dimension $n$.

Definition 4.6. Let $V$ be an irreducible algebraic variety of dimension $n$ over a field $\mathbb{K}$. A cycle of dimension $r$ or r-cycle on $V$ is a formal sum

$$
\sum_{W} n_{W} W
$$

where $W$ is a subvariety of $V$ of dimension $r$ which is not contained in the singular locus of $V, n_{W} \in \mathbb{Z}$, and only a finite number of $n_{W}$ 's are non-zero. We call degree of an r-cycle the sum

$$
\sum_{W} n_{W}
$$

An $(n-1)$-cycle is called a divisor.
Definition 4.7. For a non-degenerate polynomial differential system $(S)$ possessing a finite number of algebraic solutions

$$
\mathcal{F}=\left\{f_{i}\right\}_{i=1}^{m}, \quad f_{i}(x, y)=0, \quad f_{i}(x, y) \in \mathbb{C}[x, y],
$$

each with multiplicity $n_{i}$ and a finite number of singularities at infinity, we define the algebraic solutions divisor (also called the invariant curves divisor) on the projective plane attached to the family $\mathcal{F}$,

$$
\text { ICD }_{\mathcal{F}}=\sum_{n_{i}} n_{i} \mathcal{C}_{i}+n_{\infty} \mathcal{L}_{\infty}
$$

where $C_{i}: F_{i}(X, Y, Z)=0$ are the projective completions of $f_{i}(x, y)=0, n_{i}$ is the multiplicity of the curve $C_{i}=0$ and $n_{\infty}$ is the multiplicity of the line at infinity $\mathcal{L}_{\infty}: Z=0$.
Proposition 4.8 ([2]). Every polynomial differential system of degree $n$ and with a finite number of invariant lines has at most $3 n$ invariant straight lines, including the line at infinity.

In particular the maximum number of invariant lines for a quadratic system with a finite number of invariant lines is six. In the case we consider here, we have a particular instance of the divisor ICD because the invariant curves we consider are invariant hyperbolas and invariant lines of a quadratic differential system, in case these are in finite number. In case we have an infinite number of hyperbolas we can construct the divisor of the invariant straight lines which are always in finite number.

Another ingredient of the configuration of algebraic solutions are the real singularities situated on these curves. We also need to use here the notion of multiplicity divisor of real singularities of a system, located on the algebraic solutions of the system.

## Definition 4.9.

1. Suppose a real quadratic system (1.3) has a non-empty finite set of invariant hyperbolas $\mathcal{H}_{i}$ and a finite number of affine invariant lines $\mathcal{L}_{j}$, where $\mathcal{H}_{i}: h_{i}(x, y)=0, i=1,2, \ldots, k$, $\mathcal{L}_{j}: f_{j}(x, y)=0, j=1,2, \ldots, l$ and $h_{i}, f_{j} \in \mathbb{C}[x, y]$.
We denote the line at infinity $\mathcal{L}_{\infty}: Z=0$ and suppose that on this line we have a finite number of singularities. The divisor of invariant hyperbolas and invariant lines on the complex projective plane of the system is the following

$$
\text { ICD }=n_{1} \mathcal{H}_{1}+\cdots+n_{k} \mathcal{H}_{k}+m_{1} \mathcal{L}_{1}+\cdots+m_{l} \mathcal{L}_{l}+m_{\infty} \mathcal{L}_{\infty}
$$

where $n_{i}$ (respectively $m_{j}$ ) is the multiplicity of the hyperbola $\mathcal{H}_{i}$ (respectively $m_{j}$ of the line $\mathcal{L}_{j}$ ), and $m_{\infty}$ is the multiplicity of $\mathcal{L}_{\infty}$. We mark the complex (non-real) invariant hyperbolas (respectively lines) denoting them by $\mathcal{H}_{i}^{C}$ (respectively $\mathcal{L}_{i}^{C}$ ). We define the total multiplicity $T M$ of the divisor as the sum $\sum_{i} n_{i}+\sum_{j} m_{j}+m_{\infty}$.
2. The zero-cycle on the real projective plane, of singularities of a quadratic system (1.3) located on a configuration of invariant lines and invariant hyperbolas, is given by

$$
M_{0 C S}=r_{1} P_{1}+\cdots+r_{l} P_{l}+v_{1} P_{1}^{\infty}+\cdots+v_{n} P_{n}^{\infty}
$$

where $P_{i}$ (respectively $P_{j}^{\infty}$ ) are all the finite (respectively infinite) real singularities of the system and $r_{i}$ (respectively $v_{j}$ ) are their corresponding multiplicities. We mark the complex singular points denoting them by $P_{i}^{C}$. We define the total multiplicity $T M$ of zero-cycles as the sum $\sum_{i} r_{i}+\sum_{j} v_{j}$.

## Definition 4.10.

(1) In case we have an infinite number of hyperbolas and just two or three singular points at infinity but we have a finite number of invariant straight lines we define the invariant lines divisor as

$$
\text { ILD }=m_{1} \mathcal{L}_{1}+\cdots+m_{l} \mathcal{L}_{l}+m_{\infty} \mathcal{L}_{\infty}
$$

where $m_{i}$ denotes the multiplicity of the line $\mathcal{L}_{i}$ and $m_{\infty}$ the multiplicity of $\mathcal{L}_{\infty}$.
(2) In case we have an infinite number of hyperbolas, the line at infinity is filled up with singularities and we have a finite number of affine lines, we define the invariant lines divisor

$$
\text { ILD }=m_{1} \mathcal{L}_{1}+\cdots+m_{l} \mathcal{L}_{l} .
$$

## Definition 4.11.

(1) Suppose we have a finite number of invariant hyperbolas and invariant straight lines of a system $(S)$ and that they are given by equations

$$
f_{i}(x, y)=0, \quad i \in\{1,2, \ldots, k\}, \quad f_{i} \in \mathbb{C}[x, y] .
$$

Set $F_{i}(X, Y, Z)=0$ the projection completion of the invariant curves $f_{i}=0$ in $P_{2}(\mathbb{C})$. The total invariant algebraic curve of the system $(S)$ in $Q S H$, on $P_{2}(\mathbb{R})$, is the curve

$$
T(S)=\prod_{i} F_{i}(X, Y, Z)^{m_{i}} Z^{m_{\infty}}=0
$$

where $m_{i}$ is the multiplicity of $f_{i}=0, i=1, \ldots, k$ and $m_{\infty}$ is the multiplicity of the line at infinity.
(2) Suppose that a system $(S)$ has an infinite number of invariant hyperbola. Then the system $(S)$ has a finite number of invariant affine straight lines (see [47]). Set $L_{i}(X, Y, Z)=$ 0 the projective completions of the invariant lines $l_{i}(x, y)=0, i \in\{1,2, \ldots, k\}$ in $P_{2}(\mathbb{C})$.
(i) If there are a finite number of singular points at infinity, the total invariant curve of system (S) is

$$
T(S)=\prod_{i} L_{i}(X, Y, Z)^{m_{i}} Z^{m_{\infty}}=0,
$$

where $m_{i}$ is the multiplicity of the line $l_{i}=0, i=1, \ldots, k$ and $m_{\infty}$ is the multiplicity of the line at infinity.
(ii) If the line at infinity is filled up with singularities, the total invariant curve of system $(S)$ is

$$
T(S)=\prod_{i} L_{i}(X, Y, Z)^{m_{i}}=0
$$

where $m_{i}$ is the multiplicity of the line $l_{i}=0, i=1, \ldots, k$.

The singular points of the system $(S)$ situated on $T(S)$ are of two kinds: those which are simple (or smooth) points of $T(S)$ and those which are multiple points of $T(S)$.

Remark 4.12. To each singular point of the system we have its associated multiplicity as a singular point of the system. In addition, when these singular points are situated on the total curve, we also have the multiplicity of these points as points on the total curve $T(S)$. Through a singular point of the systems there may pass several of the curves $F_{i}=0$ and $Z=0$. Also we may have the case when this point is a singular point of one or even of several of the curves in case we work with invariant curves with singularities. This leads to the multiplicity of the point as point of the curve $T(S)$. The simple points of the curve $T(S)$ are those of multiplicity one. They are also the smooth points of this curve.

## Definition 4.13.

(i) Suppose a quadratic system $(S)$ has a finite number of singularities finite or infinite. The zero-cycle of singularities of the total curve $T(S)$ of system $(S)$ is given by

$$
M_{0 C T}=r_{1} P_{1}+\cdots+r_{l} P_{l}+v_{1} P_{1}^{\infty}+\cdots+v_{n} P_{n}^{\infty}
$$

where $P_{i}$ (respectively $P_{j}^{\infty}$ ) are all the finite (respectively infinite) singularities situated on $T(S)$ and $r_{i}$ (respectively $v_{j}$ ) are their corresponding multiplicities as points on the total curve $T(S)$. We mark the complex singular points denoting them by $P_{i}^{C}$. We define the total multiplicity $T M$ of the zero-cycle $M_{0 C T}$ as the sum $\sum_{i} r_{i}+\sum_{j} v_{j}$.
(ii) Suppose a system $(S)$ possessess the line at infinity filled up with singularities. The zero-cycle of the total curve $T(S)$ of system $(S)$ is given by

$$
M_{0 C T}=r_{1} P_{1}+\cdots+r_{l} P_{l}
$$

where $P_{i}$ are all the finite singularities situated on $T(S)$ and $r_{i}$ are their corresponding multiplicities as points on the total curve $T(S)$. We mark the complex singular points denoting them by $P_{i}^{C}$. The total multiplicity $T M$ of the zero-cycle $M_{0 C T}$ as the sum $\sum_{i} r_{i}$.

Definition 4.14. If the intersection multiplicity [29] of two curves is one then we say that the curves intersect transversally or that this point is a simple point of intersection.

If at a point two curves are tangent, we have an intersection multiplicity higher than or equal to two.

Definition 4.15 ([63]). Two polynomial differential systems $S_{1}$ and $S_{2}$ are topologically equivalent if and only if there exists a homeomorphism of the plane carrying the oriented phase curves of $S_{1}$ to the oriented phase curves of $S_{2}$ and preserving the orientation.

To cut the number of non equivalent phase portraits in half we use here another equivalence relation.

Definition 4.16. Two polynomial differential systems $S_{1}$ and $S_{2}$ are topologically equivalent if and only if there exists a homeomorphism of the plane carrying the oriented phase curves of $S_{1}$ to the oriented phase curves of $S_{2}$, preserving or reversing the orientation.

Notation: $\cong_{\text {top }}$.
In [4] the authors provide a complete classification of $\mathbf{Q S}$ according to the geometric equivalence relation of topological configurations of singularities, finite or infinite. Here we use the same terminology and notation for singularities introduced in [4].

We say that a singular point is elemental if it possesses two non-zero eigenvalues; semielemental if it possess exactly one eigenvalue equal to zero and nilpotent if it possesses two zero eigenvalues and the linear part is not zero. We call intricate a singular point with its Jacobian matrix identically zero.

We place first the finite singular points denoted with lower case letters and secondly the infinite singular points denoted by capital letters, separating them by a semicolon ';'.

In our study we have real and complex finite singular points for real systems and from the topological viewpoint only the real ones are interesting. When we have a complex finite singular point we use the notation $\odot$. For the elemental singular points we use the notation
' $s$ ', ' $S$ ' for saddles, ' $n$ ', ' $N$ ' for nodes, ' $f$ ' for foci and ' $c$ ' for centers. We also denote by ' $a$ ' (antisaddle) for either a focus or any type of node when the local phase portraits are topologically equivalent.

Non-elemental singular points are multiple points. We denote by $\left({ }_{b}^{a}\right)$ the maximum number $a$ (respectively $b$ ) of finite (respectively infinite) singularities which can be obtained by perturbation of the multiple point at infinity. For example, $\left({ }_{1}^{1}\right) S N$ and $\left.{ }_{2}^{0}\right) S N$ correspond to two saddle-nodes at infinity which are locally topologically distinct since the first arises from the coalescence of a finite with an infinite singularity and the second from the coalescence of two infinite singularities.

The semi-elemental singular points can either be nodes, saddles or saddle-nodes (finite or infinite). If they are finite singular points we denote them by ' $n_{(3)}{ }^{\prime}$, ' $s_{(3)}$ ' and ' $s n_{(2)}$ ', respec-
 their multiplicity. We note that semi-elemental nodes and saddles are respectively topologically equivalent with elemental nodes and saddles.

The nilpotent singular points can either be saddles, nodes, saddle-nodes, elliptic-saddles, cusps, foci or centers. The only finite nilpotent points for which we need to introduce notation are the elliptic-saddles and cusps which we denote respectively by ' $e s^{\prime}$ and ' $c p$ '.

In the case of nilpotent infinite points, the relative positions of the sectors with respect to the line at infinity, can produce topologically different phase portraits. Then we use a notation for these points similar to the notation which we will use for the intricate points.

The intricate singular points are degenerate singular points. It is known that the neighbourhood of any singular point of a polynomial vector field (except for foci and centers) is formed by a finite number of sectors which could only be of three types: parabolic ( $p$ ), hyperbolic ( $h$ ) and elliptic (e) (see [25]). Then, a reasonable way to describe intricate and nilpotent points at infinity is to use a sequence formed by the types of their sectors. From the topological view point, any two adjacent parabolic geometrical sectors merge into one and any elliptic sector, in a small vicinity of the singularity, always has two parabolic sectors one of each side. We make the convention to eliminate the parabolic sectors adjacent to the elliptic sectors, according to the notation in [4].

In quadratic systems, we have just four topological possibilities for finite intricate singular points of multiplicity four:

- phph;
- hh;
- hhhhhh;
- ee.

For intricate and nilpotent singular points at infinity, we insert a dash (hyphen) between the sectors to split those which appear on one side or the other of the equator of the sphere. When describing a single finite nilpotent or intricate singular point, one can always apply an affine change of coordinates to the system, so it does not really matter which sector starts the sequence, or the direction (clockwise or counter-clockwise) we choose. If it is an infinite nilpotent or intricate singular point, then we always start with a sector bordering the infinity (to avoid using two dashes).

If the line at infinity is filled up with singularities, then it is known that any such system has in a sufficiently small neighbourhood of infinity one of 7 topological distinct phase
portraits (see [67]). The way to determine these portraits is by studying the reduced systems on the infinite local charts after removing the degeneracy of the systems within these charts. Following [3] we use the notation $[\infty ; \varnothing],[\infty ; N],\left[\infty ; N^{d}\right]$ (one-direction node, that is a node with two identical eigenvalues whose Jacobian matrix cannot be diagonal), $[\infty ; S]$, $[\infty ; C],\left[\infty ;\left({ }_{2}^{2}\right) S N\right],\left[\infty ;\left({ }_{3}^{0}\right) E S\right]$ indicating the kinds of singularities obtained after removing the line filled with singularities.

The degenerate systems are systems with a common factor in the polynomials defining the system. We denote this case with the symbol $\ominus$. The degeneracy can be produced by a nonconstant common factor of degree one which defines a straight line or a common quadratic factor which defines a conic. In this paper we have just the second case happening.

Moreover, we also want to determine whether after removing the common factor of the polynomials, singular points remain on the curve defined by this common factor. If the reduced system has no finite singularity on this curve, we use the symbol $\varnothing$ to describe this situation. If some singular points remain on this curve we use the corresponding notation of their various kinds. In this situation, the geometrical properties of the singularity that remains after the removal of the degeneracy, may produce topologically different phenomena, even if they are topologically equivalent singularities. So, we need to keep the geometrical information associated to that singularity. In this paper we use the notation $(\ominus[)(] ; \varnothing)$ which denotes the presence of a hyperbola filled up with singular points in the system such that the reduced system has no finite singularity on this curve.

The existence of a common factor of the polynomials defining the differential system also affects the infinite singular points. We point out that the projective completion of a real affine line filled up with singular points has a point on the line at infinity which will then be also a non-isolated singularity. There is a detailed description of this notation in [3]. In case that after the removal of the finite degeneracy, a singular point at infinity remains at the same place, we must denote it with all its geometrical properties since they may influence the local topological phase portrait. In this paper we use the notation $(\ominus[)(] ; N, \varnothing)$ that means that the system has at infinity a node, and one non-isolated singular point which is part of a real hyperbola filled up with singularities and that the reduced linear system has no infinite singular point in that position.

See [4] for more details on the notation for singularities.
In order to distinguish topologically the phase portraits of the systems we obtained, we also use some invariants introduced in [66]. Let SC be the total number of separatrix connections, i.e. of phase curves connecting two singularities which are local separatrices of the two singular points. We denote by

- $S C_{f}^{f}$ the total number of $S C$ connecting two finite singularities,
- $S C_{f}^{\infty}$ the total number of $S C$ connecting a finite with an infinite singularity,
- $S C_{\infty}^{\infty}$ the total number of $S C$ connecting two infinite.

A graphic as defined in [26] is formed by a finite sequence of singular points $p_{1}, p_{2}, \ldots, p_{n}$, $p_{n+1}=p_{1}$ and oriented regular orbits $s_{1}, \ldots, s_{n}$ connecting them such that $s_{j}$ has $p_{j}$ as $\alpha$-limit set and $p_{j+1}$ as $\omega$-limit set for $j<n$ and $s_{n}$ has $p_{n}$ as $\alpha$-limit set and $p_{1}$ as $\omega$-limit set. Graphics may or may not have a return map. Particular graphics are given special names. A loop is a graphic through a unique singular point and with a return map. A polycycle is a graphic through several singular points and with a return map. A degenerate graphic as defined in [26] is formed by singular points $p_{1}, p_{2}, \ldots, p_{n}, p_{n+1}=p_{1}$, oriented regular orbits and segments
$s_{1}, \ldots, s_{n}$ of curves of singular points (which are also oriented) such that either $s_{j}$ is a orbit that has $p_{j}$ as $\alpha$-limit and $p_{j+1}$ as $\omega$-limit for $j<n$ and $s_{n}$ has $p_{n}$ as $\alpha$-limit set and $p_{1}$ as $\omega$-limit set or an open segment of a curve of singular points with end points $p_{j}$ and $p_{j+1}$, for each $j<n$. Moreover, the regular orbits and the curves of singular points have coherent orientations in the sense that if $s_{j-1}$ has left hand orientation then so does $s_{j}$. For more details, see [26].

In what follows we present an example of the notation used in paper to describe the global configuration of singularities of QSH.

- Stable node
- Unstable node
- Saddle
$\Delta$ Semi-elemental saddle-node
- Non-elemental
.... Curve of singularities
_ Separatrices
..... Orbits
- Graphics

Figure 4.1: Notations used on the phase portraits.


Figure 4.2: Some examples of phase portraits.
The notation used to describe the topological type of the singularities in Figure 4.2 is

$$
\begin{array}{r}
\left.\left(a, s, a, s ;{ }_{2}^{0}\right) S N, N\right) \\
\left.\left(s, s n, a ;{ }_{2}^{( }\right) S N, N\right) \\
\left.\left(s ;{ }_{(2}^{2}\right) E-E,\left({ }_{1}^{2}\right) S N\right)
\end{array}
$$

for each phase portrait appearing in the respective order. The first letters appearing with lower case represents the topological type of the finite singularities. Here 'sn' denotes a saddle-node which arises from the coalescence of a finite saddle with a finite node so this is a singularity of multiplicity two, ' $a$ ' denotes an elemental anti-saddle and ' $s$ ' denotes an elemental saddle. The capital letters give the topological type of the singularities at infinity: ${ }^{( }\left({ }_{2}\right) S N$ ' denotes a saddle-node which arises from the coalescence of two infinite singularities (saddle and node) so this is a double singularity, ( ${ }_{1}^{1}$ (1) $S N^{\prime}$ also denotes a saddle-node but here this multiplicity arises from the coalescence of a finite with an infinite singularity, ' $\left.{ }_{2}{ }_{2}\right) E-E^{\prime}$ denotes an intricate singularity arising from the coalesce of two finite singularities with two infinite singularities and the neighbourhood of this singularity is formed by an elliptic sector which has, in a small
vicinity of the singularity, two parabolic sectors one of each side. The cases where we do no indicate the multiplicity means the singularity is simple, which is the case of ' $S$ ' (elemental saddle) and ' $N$ ' (elemental node).

## 5 Proof of the Main theorem for the integrable cases

The data described in Table 5.2 led us to the proof of Main theorem for the integrable case.
We begin by using the Prelle-Singer algorithm (including the exponential factors, when they exist) in order to prove integrability.

The result of our calculations are given in Table 5.2 where we have the invariant algebraic curves, exponential factors and their cofactors, first integrals or integrating factors for each normal form of Proposition 3.3 obtained using the software Mathematica.

In the first column are the normal forms for $\mathbf{Q S H}_{\eta=0}$.
In the second column are the invariant algebraic curves, the exponential factors and the respective cofactors.

In the third column are the expressions of the first integrals or the expressions of the integrating factors. If we give the expression for the first integral then it is not necessary to give the integrating factor to guarantee the integrability. When we give the expression for the integrating factor instead of the first integral this means that we could not compute the expression for the first integral using Mathematica and we use the notation "-". When "-" appears in both the first integral and integrating factor this means that we could find neither of them applying the Prelle-Singer algorithm.

In the fourth and fifth columns are the normal forms and their possible configurations as in [47]. The notation "-" appears when we do not have them appearing in [47].

In the sixth column are indicated the types of integrability of each normal form using the notations in Table 5.1.

The precise integrating factor, first integral that did not fit in the table will be given in the text following the table.

Thereby, the proof of the Main theorem follows except for the non integrable cases, that will be done in Section 6.

Table 5.1: Notations used in Table 5.2.

| Notation |
| :--- |
| N-I : Systems admit neither a Darboux nor a Liouvillian first integral; |
| D: Systems are Darboux integrable; |
| GD: Systems are generalized Darboux integrable; |
| L: Systems are Liouvillian integrable; |
| P: Systems admit a polynomial first integral; |
| R: Systems admit a rational first integral; |
| HAM: Systems are Hamiltonian. |
| open case : We could prove neither the integrability nor the non-integrability; |
| $\mathcal{R}:$ Represents an integrating factor; |
| $\mathcal{F}:$ Represents a first integral; |

Table 5.2: Proof of the Main theorem for the integrable cases.

| Orbit representative$a, g, c, \epsilon \in \mathbb{R}: a \neq 0$ | Invariant curves/ ExpFac | Integrating Factor $\mathcal{R}_{i}$ | Eq. [47] | Config. $\tilde{H}$ | Integ. |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | Respective cofactors | First integral $\mathcal{F}_{i}$ |  |  |  |
| (A) $\left\{\begin{array}{l}\dot{x}=2 a+x+g x^{2}+x y, \\ \dot{y}=a(2 g-1)-y+(g-1) x y+y^{2},\end{array}\right.$ <br> where $a(g-1)(2 g-1)(3 g-1)(g-2)(25 a-3) \neq 0$ | $a+x y$ | - | (4.4) | $\begin{aligned} & 1,3,4,5 \\ & 6,7,8,9 \\ & 10,11 \end{aligned}$ | N-I |
|  | $(-1+2 g) x+2 y$ | - |  |  |  |
| (A) where $g=1 / 2$ | $y, a+x y$ | - | (4.10) | $\begin{aligned} & \hline 12,13,14 \\ & 15,16,17 \end{aligned}$ | N-I |
|  | $-1-\frac{x}{2}+y, 2 y$ | - |  |  |  |
| (A) where $g=2$ and $a=3 / 25$ | $x y+\frac{3}{25}, \quad x+\frac{5 y^{2}}{9}-y+\frac{3}{5}$ | - | - | - | open case |
|  | $3 x+2 y, 2 x+2 y-\frac{1}{5}$ | - |  |  |  |
| (A) where $g=1 / 3$ and $a=\bar{a} / 2$ | $\bar{a}+2 x y$ | - | (4.11) | $\begin{aligned} & 1,4,5,6, \\ & 10 \end{aligned}$ | open case |
|  | $2 y-\frac{x}{3}$ | - |  |  |  |
| (B) $\left\{\begin{array}{l}\dot{x}=2 a+g x^{2}+x y, \\ \dot{y}=a(2 g-1)+(g-1) x y+y^{2},\end{array}\right.$ <br> where $a(g-1)(2 g-1)(4 g-1) \neq 0$ | $a+x y$ | - | (4.13) | 1,2,6 | N-I |
|  | $(-1+2 g) x+2 y$ | - |  |  |  |
| (B) where $g=1 / 2$ | $y, a+x y, \quad e^{-\frac{a+22^{2}}{2(a+x y)}}$ |  | - | 31,32 | GD |
|  | $-\frac{x}{2}+y, 2 y, y$ | $\mathcal{F}_{B, 1}=e^{\frac{a+2 y^{2}}{a+x y}}(a+x y)$ |  |  |  |
| (B) where $g=1 / 4$ | $a+x y, \quad e^{\frac{y^{2}}{a+x y}}$ | $\mathcal{R}_{B, 3}=\frac{e^{\frac{2 y^{2}}{a+x y}}}{\sqrt{a+x y}}$ | - | 29,30 | L |
|  | $-\frac{x}{2}+2 y,-y$ | - |  |  |  |


| (C) $\left\{\begin{array}{l}\dot{x}=2 a+3 c x+x^{2}+x y, \\ \dot{y}=a-c^{2}+y^{2},\end{array}\right.$ <br> where $a\left(c^{2}-a\right)\left(9 a-8 c^{2}\right) \neq 0$ | $\begin{aligned} & y+\sqrt{c^{2}-a}, \\ & \sqrt{c^{2}-a}, a+c x+x y \\ & y-\sqrt{c^{2}-a}, \\ & \sqrt{c^{2}-a}, 2 c+x+2 y \end{aligned}$ | $\mathcal{R}_{C}$ - | (4.16) | $\begin{aligned} & 18,20,21, \\ & 22,33 \end{aligned}$ | L |
| :---: | :---: | :---: | :---: | :---: | :---: |
| (C) where $a=c^{2}$ | $y, c^{2}+c x+x y, \quad e^{\frac{1}{y}}$ | $\mathcal{R}_{C, 1}$ | - | 23 | L |
|  | $y, 2 c+x+2 y,-1$ | - |  |  |  |
| (C) where $a=8 c^{2} / 9$ | $\begin{aligned} & \hline 3 y-c, \quad 3 y+c, \quad 8 c^{2}+ \\ & 9 c x+9 x y, \quad e^{\frac{-c x+48 c y+63 x y-24 y^{2}}{48 c\left(8 c^{2}+9 c x+9 x y\right)}} \\ & \hline \end{aligned}$ |  | - | 33 | GD |
|  | $\begin{aligned} & \frac{c}{3}+y, \quad y-\frac{c}{3}, \quad 2 c+x+ \\ & 2 y, \frac{y}{18 c}-\frac{1}{54} \end{aligned}$ | $\mathcal{F}_{C, 2}$ |  |  |  |
| (D) $\left\{\begin{array}{l}\dot{x}=(c+x)(c(2 g-1)+g x), \\ \dot{y}=1+(g-1) x y,\end{array}\right.$ where $\operatorname{cg}(g \pm 1)(2 g-1)(3 g-1) \neq 0$ | $\begin{aligned} & x+c, \quad c(2 g-1)+g x, \\ & \frac{1}{(-1+2 g)}+c y+x y \end{aligned}$ |  | (4.18) | 19 | D |
|  | $\begin{aligned} & c(-1+2 g)+g x, \quad c g+ \\ & g x, \quad c(-1+2 g)+(-1+ \\ & 2 g) x \end{aligned}$ | $\mathcal{F}_{D}$ |  |  |  |
| (D) where $g=0$ and $c \neq 0$ | $c+x,-1+c y+x y, e^{x+1}$ |  | - | 24 | GD |
|  | $-c,-c-x,-c^{2}-c x$ | $\mathcal{F}_{D, 1}=e^{x+1}(y(c+x)-1)^{-c}$ |  |  |  |
| (D) where $c=0$ and $g \neq 0,-1 / 2$ | $\begin{aligned} & x, \frac{1}{-1+2 g}+x y, \quad e^{\frac{1}{x}}, \\ & e^{\frac{2 g x y+1}{x^{2}}} \end{aligned}$ |  |  | 25,34 | D |
|  | $\begin{aligned} & g x, \quad(-1+2 g) x,-g, \\ & -2 g y \end{aligned}$ | $\mathcal{F}_{D, 2}=\frac{x^{\frac{1}{8}-2}(2 g x y-x y+1)}{2 g-1}$ |  |  |  |
| (E) $\left\{\begin{array}{l}\dot{x}=x^{2}+\epsilon, \\ \dot{y}=1-2 x y,\end{array}\right.$ <br> where $\epsilon \neq 0$ | $\begin{aligned} & x+i \sqrt{\epsilon}, \quad x-i \sqrt{\epsilon}, \\ & -1+i \sqrt{\epsilon} y+x y \\ & -1-i \sqrt{\epsilon} y+x y \end{aligned}$ |  | (4.22) | 27, 28 | P/HAM |
|  | $\begin{aligned} & x-i \sqrt{\epsilon}, \quad x+i \sqrt{\epsilon} \\ & -x-i \sqrt{\epsilon}, \quad-x+i \sqrt{\epsilon} \end{aligned}$ | $\mathcal{F}_{E}=\left(x^{2}+\epsilon\right)\left((x y-1)^{2}+y^{2} \epsilon\right)$ |  |  |  |


| (E) where $\epsilon=0$ | $\begin{aligned} & x,-1+x y, e^{\frac{1}{x}}, \\ & e^{\frac{2 x y+1}{x^{2}}}, e^{\frac{y}{x y-1}}, e^{\frac{y^{2}(2 x y-3)}{(x y-1)^{2}}} \\ & x,-x,-1, \\ & -6 y,-1,-6 y \end{aligned}$ | $\mathcal{F}_{E, 1}=x(-1+x y)$ | - | 34 | P/HAM |
| :---: | :---: | :---: | :---: | :---: | :---: |
| (F) $\left\{\begin{array}{l}\dot{x}=(x-1)(3-x), \\ \dot{y}=1-2 x y\end{array}\right.$ | $\begin{aligned} & 1-x, \quad 3-x, \\ & -\frac{1}{3}-y+x y, \\ & -\frac{19}{8}+x+3 y-\frac{x^{2}}{8} \\ & 3-x, \quad 1-x, \quad 3-3 x \\ & -2 x \end{aligned}$ | $\mathcal{F}_{F}=-\frac{(x-3)^{2}}{3(x-1) y-1}$ | (4.25) | 19 | R |
| (G) $\left\{\begin{array}{l}\dot{x}=-x^{2}, \\ \dot{y}=1-2 x y\end{array}\right.$ | $\begin{aligned} & x,-1+3 x y, e^{\frac{1}{x}}, \\ & e^{\frac{1-2 x y}{x^{2}}}, e^{\frac{1-3 x y-2 x^{2} y+x}{x^{3}}} \\ & -x,-3 x, 1,2 y, \\ & 2 y \end{aligned}$ | $\mathcal{F}_{G}=\frac{x^{3}}{3 x y-1}$ | - | 26 | R |
| (H) $\left\{\begin{array}{l}\dot{x}=(2 x-1)(2 x+1) / 4, \\ \dot{y}=y\end{array}\right.$ | $\begin{aligned} & 1+2 x, \quad 1-2 x, \quad y, \\ & -\frac{q}{2}+q x+y+2 x y, \\ & e^{y}, e^{\frac{-2 x+y+1}{1-2 x}} \\ & -\frac{1}{2}+x, \quad \frac{1}{2}+x, \quad 1, \\ & \frac{1}{2}+x, \quad y, \quad \frac{y}{2} \end{aligned}$ | $\mathcal{F}_{H}=\frac{(2 x+1) y}{q\left(x-\frac{1}{2}\right)+2 x y+y}$ | (4.27) | 35 | R |
| (I) $\left\{\begin{array}{l}\dot{x}=x^{2}, \\ \dot{y}=1\end{array}\right.$ | $\begin{aligned} & x, 1+r x+x y, \quad e^{\frac{x+1}{x}} \\ & e^{\frac{x^{2}+2 x y+x+1}{x^{2}}}, \quad e^{y^{2}+y+1} \\ & x, x,-1 \\ & -1-2 y, \quad 2 y+1 \end{aligned}$ | $\mathcal{F}_{I}=\frac{x}{1+r x+x y}$ | (4.28) | 36 | R |
| (J) $\left\{\begin{array}{l}\dot{x}=a+y+x^{2}, \\ \dot{y}=x y,\end{array}\right.$ <br> where $a \neq 0$ | $\begin{aligned} & \hline y, \quad-i \sqrt{a}+x-\frac{i y}{\sqrt{a}} \\ & i \sqrt{a}+x+\frac{i y}{\sqrt{a}}, \\ & a+2 y+x^{2}-m^{2} y^{2} \\ & x, \quad i \sqrt{a}+x, \\ & -i \sqrt{a}+x, \quad 2 x \\ & \hline \end{aligned}$ | $\mathcal{F}_{J}=\frac{y^{2}}{a+2 y+x^{2}-m^{2} y^{2}}$ | (4.30) | 39, 41 | R |


| (J) where $a=0$ | $\begin{aligned} & y, 2 y+x^{2}-m^{2} y^{2} \\ & e^{\frac{x}{y}}, e^{\frac{x^{2}+2 x y+2 y^{2}}{2 y^{2}}} \end{aligned}$ |  | - | 43 | R |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $x, 2 x, 1,1$ | $\mathcal{F}_{J, 1}=\frac{y^{2}}{2 y+x^{2}-m^{2} y^{2}}$ |  |  |  |
| (K) $\left\{\begin{array}{l}\dot{x}=(1+3 x)(2+3 x) / 9, \\ \dot{y}=x y\end{array}\right.$ | $\begin{aligned} & y, \quad 2+3 x, \quad 1+3 x, \quad 4+ \\ & 12 x+9 x^{2}+m y+3 m x y \\ & \hline \end{aligned}$ |  | (4.34) | 37 | R |
|  | $x, \frac{1}{3}+x, \frac{2}{3}+x, \frac{2}{3}+2 x$ | $\mathcal{F}_{K}=\frac{(3 x+1) y}{(3 x+2)^{2}}$ |  |  |  |
| (L) $\left\{\begin{array}{l}\dot{x}=a+x^{2}, \\ \dot{y}=x y,\end{array}\right.$ <br> where $a \neq 0$ | $\begin{aligned} & y, \quad 1-\frac{i x}{\sqrt{a}}, 1+\frac{i x}{\sqrt{a}} \\ & a+x^{2}-m^{2} y^{2} \end{aligned}$ |  | (4.36) | 38, 40 | R |
|  | $x, \quad-i \sqrt{a}+x, \quad i \sqrt{a}+x, 2 x$ | $\mathcal{F}_{L}=\frac{x^{2}+a}{a y^{2}}$ |  |  |  |
| (M) $\left\{\begin{array}{l}\dot{x}=x^{2}, \\ \dot{y}=1+x y\end{array}\right.$ | $\begin{aligned} & x, 1+m x^{2}+2 x y, \\ & e^{1 / x}, e^{\frac{x^{2}+2 x y+x+1}{x^{2}}} \end{aligned}$ |  | (4.38) | 42 | R |
|  | $x, 2 x,-1,-1$ | $\mathcal{F}_{M}=\frac{x^{2}}{1+m x^{2}+2 x y}$ |  |  |  |

$$
\begin{aligned}
\mathcal{R}_{C} & =\left(y+\sqrt{c^{2}-a}\right)^{\frac{1}{2}}\left(1+\frac{c}{\sqrt{c^{2}-a}}\right)\left(y-\sqrt{c^{2}-a}\right)^{\frac{1}{2}\left(1-\frac{c}{\sqrt{c^{2}-a}}\right)}(a+c x+x y)^{-2} ; \\
\mathcal{R}_{C, 1} & =y\left(c+x+\frac{x y}{c}\right)^{-2} e^{\frac{-c}{y}} ; \\
\mathcal{F}_{C, 2} & =(c+3 y)\left(e^{\frac{-c x+48 c y+63 y-24)^{2}}{48 c\left(8 c^{c}+9 c x+9 x y\right)}}\right)^{-18 c} ; \\
\mathcal{F}_{D} & =(c(2 g-1)+g x)\left(y(c+x)+\frac{1}{2 g-1}\right)^{-\frac{g}{2 g-1}} .
\end{aligned}
$$

## 6 Proof of the Main theorem for the non integrable cases

Consider the sets:

$$
\begin{aligned}
& \mathrm{L}_{1}=\cup_{k \in \mathbb{N}} \mathrm{~L}_{1, k}, \text { where } \mathrm{L}_{1, k}=\left\{(a, g) \in \mathbb{R}^{2}: g=k / 2 \text { and } a \neq 0\right\}, k \in \mathbb{N}, \\
& \mathrm{~L}_{2}=\cup_{k \in \mathbb{N}} \mathrm{~L}_{2, k}, \text { where } \mathrm{L}_{2, k}=\left\{(a, g) \in \mathbb{R}^{2}: g=k / 3 \text { and } a \neq 0\right\}, k \in \mathbb{N}, \\
& \mathrm{~L}_{3}=\left\{(a, g) \in \mathbb{R}^{2}: g=1 / 4 \text { and } a \neq 0\right\}, \\
& \mathrm{C}^{\prime}=\cup_{k \in \mathbb{N}} \mathrm{C}_{k}, \text { where } \mathrm{C}_{k}=\left\{(a, g) \in \mathbb{R}^{2}: g=(2+a-2 a k) / 4 a, a \neq 0\right\}, k \in \mathbb{N} .
\end{aligned}
$$

### 6.1 The systems (A)

$$
\left\{\begin{array}{l}
\dot{x}=2 a+x+g x^{2}+x y \\
\dot{y}=a(2 g-1)-y+(g-1) x y+y^{2}
\end{array}\right.
$$

where $a(g-1) \neq 0$.

## Theorem 6.1.

(a) If $(a, g) \notin \mathrm{L}_{1}$ then the only invariant algebraic curves of a system in the family $(\mathrm{A})$ are of the form $J_{1}^{m}=0$ where $J_{1}(x, y)=a+x y$ and $m$ is a positive integer.
(b) If $(a, g) \notin\left(\mathrm{L}_{1} \cup \mathrm{~L}_{2} \cup \mathrm{C}^{\prime}\right)$ then any system in the family (A) has no exponential factors.
(c) If $(a, g) \notin\left(\mathrm{L}_{1} \cup \mathrm{~L}_{2} \cup \mathrm{C}^{\prime}\right)$ then any system in the family $(\mathrm{A})$ is not Liouvillian integrable.

Remark: When $g=1 / 2$ the systems posses the invariant line $y=0$ but this invariant curve is still not enough to prove the integrability. The non integrability in this case can be done just by adapting $g=1 / 2$ in the proof below. In (a) we find $C=y^{m-l}(a+x y)^{l}$.

Proof. (a) By a straightforward computation, we can verify that $J_{1}(x, y)=a+x y$ is an invariant hyperbola with cofactor $\alpha_{1}(x, y)=(-1+2 g) x+2 y$. Assume that

$$
C=\sum_{i=0}^{n} C_{i}(x, y)=0
$$

is an invariant algebraic curve of the system (A) with cofactor $K=K_{0}+K_{1} x+K_{2} y$, where $C_{i}$ are homogeneous polynomial of degree $i$ where $0 \leq i \leq n$. From the definition of invariant
algebraic curve (1.4), we have:

$$
\begin{align*}
\left(2 a+x+g x^{2}+x y\right) \sum_{i=0}^{n} C_{i, x}+\left(a(2 g-1)-y+(g-1) x y+y^{2}\right) & \sum_{i=0}^{n} C_{i, y} \\
& =\left(K_{0}+K_{1} x+K_{2} y\right) \sum_{i=0}^{n} C_{i} \tag{6.1}
\end{align*}
$$

Taking from (6.1) the terms of degree $n+1$ we have:

$$
\begin{equation*}
\left(g x^{2}+x y\right) C_{n, x}+\left((g-1) x y+y^{2}\right) C_{n, y}=\left(K_{1} x+K_{2} y\right) C_{n} \tag{6.2}
\end{equation*}
$$

For this system we have

$$
y P_{2}-x Q_{2}=x^{2} y
$$

Then, from Lemma 4.5 we can assume that

$$
C_{n}=x^{m} y^{l}, \quad \text { where } n=m+l
$$

Substituting $C_{n}$ in (6.2) and doing some computations we terminate that

$$
K_{1}=g m+(g-1) l ; \quad K_{2}=m+l
$$

Now, taking from (6.1) the terms of degree $n$ we have:

$$
\begin{align*}
x C_{n, x}+\left(g x^{2}+x y\right) C_{n-1, x}-y C_{n, y}+ & \left((g-1) x y+y^{2}\right) C_{n-1, y} \\
& =K_{0} C_{n}+[(g m+(g-1) l) x+(m+l) y] C_{n-1} . \tag{6.3}
\end{align*}
$$

Set $C_{n-1}=\sum_{i=0}^{n-1} c_{n-1-i} x^{n-1-i} y^{i}$. Replacing $C_{n}, C_{n-1}$ in (6.3) and doing some calculations, we obtain

$$
\sum_{i=0}^{m+l-1}(l-i-g) c_{m+l-1-i} x^{m+l-i} y^{i}-\sum_{i=0}^{m+l-1} c_{m+l-1-i} x^{m+l-1-i} y^{i+1}=\left(K_{0}-m+l\right) x^{m} y^{l}
$$

Note that this equation can be written as

$$
\sum_{i=0}^{m+l}\left[(l-i-g) c_{m+l-1-i}-c_{m+l-i}\right] x^{m+l-i} y^{i}=\left(K_{0}-m+l\right) x^{m} y^{l}
$$

where $c_{i}=0$ for $i<0$ and $i>m+l-1$. Equating the coefficients of $x^{i} y^{j}$ in the above equation, we get:

$$
\left\{\begin{array}{l}
(l-i-g) c_{m+l-1-i}-c_{m+l-i}=0, \text { where } i=0,1, \ldots, l-1, l+1, \ldots, m+l  \tag{6.4}\\
(-g) c_{m-1}-c_{m}=K_{0}-m+l
\end{array}\right.
$$

For $i=m+l, m+l-1, \ldots, l+1$ we have

$$
c_{0}=c_{1}=\cdots=c_{m-1}=0
$$

Then $c_{m}=-K_{0}+m-l$. Working recursively we have

$$
\begin{aligned}
c_{m+1} & =(-g+1)\left(-K_{0}+m-l\right) \\
c_{m+2} & =(-g+2)(-g+1)\left(-K_{0}+m-l\right), \ldots \\
c_{m+l-1} & =(-g+l-1) \ldots(-g+2)(-g+1)\left(-K_{0}+m-l\right)
\end{aligned}
$$

Replacing $c_{m+l-1}$ in (6.4) where $i=0$, we get

$$
(-g+l)(-g+l-1) \ldots(-g+2)(-g+1)\left(-K_{0}+m-l\right)=0 .
$$

Note that

$$
(-g+l)(-g+l-1) \ldots(-g+2)(-g+1)
$$

is a polynomial of degree $l$ in the variable $g$, which has at most $l$ real roots. Denote by $S_{1}^{l}$ the set of roots. If $g \notin S_{1}^{l}$ then $K_{0}=m-l$. Therefore, we can conclude that

$$
K=(m-l)+(g m+(g-1) l) x+(m+l) y,
$$

since $g \notin S_{1}^{l}$. This is

$$
C_{n-1} \equiv 0 .
$$

Now, taking from (6.1) the terms of degree $n-1$ we have:

$$
\begin{align*}
2 a C_{n, x}+\left(g x^{2}+x y\right) C_{n-2, x}+a(2 g-1) C_{n, y}+ & \left((g-1) x y+y^{2}\right) C_{n-2, y} \\
& =[(g m+(g-1) l) x+(m+l) y] C_{n-2} . \tag{6.5}
\end{align*}
$$

Setting $C_{n-2}=\sum_{i=0}^{n-2} c_{n-2-i} x^{n-2-i} y^{i}$ and replacing $C_{n}, C_{n-2}$ in (6.5) we obtain

$$
\begin{aligned}
\sum_{i=0}^{m+l-2}(l-i-2 g) c_{m+l-2-i} x^{m+l-1-i} y^{i}+\sum_{i=0}^{m+l-2}(-2) & c_{m+l-2-i} x^{m+l-2-i} y^{i+1} \\
& =-a(2 g-1) l x^{m} y^{l-1}-2 a m x^{m-1} y^{l}
\end{aligned}
$$

This equation can be written as

$$
\begin{aligned}
\sum_{i=0}^{m+l-2}\left[(l-i-2 g) c_{m+l-2-i}+(-2) c_{m+l-1-i}\right] x^{m+l-1-i} & y^{i} \\
& =-a(2 g-1) l x^{m} y^{l-1}-2 a m x^{m-1} y^{l}
\end{aligned}
$$

where $c_{i}=0$ for $i<0$ and $i>m+l-2$. Equating the coefficients of $x^{i} y^{j}$ in the above equation, we get

$$
\left\{\begin{array}{l}
(l-i-2 g) c_{m+l-2-i}-2 c_{m+l-1-i}=0, \text { where } i=0, \ldots, l-2, l+1, \ldots, m+l-1,  \tag{6.6}\\
(-2 g+1) c_{m-1}-2 c_{m}=-a(2 g-1) l, \\
(-2 g) c_{m-2}-2 c_{m-1}=-2 a m .
\end{array}\right.
$$

For $i=m+l-1, m+l-2, \ldots, l+1$ we have

$$
c_{0}=c_{1}=\cdots=c_{m-2}=0 .
$$

Then $c_{m-1}=a m$. Working recursively we have

$$
\begin{aligned}
& c_{m}=\frac{a(-2 g+1)(m-l)}{2}, \quad c_{m+1}=\frac{a(-2 g+1)(-2 g+2)(m-l)}{4}, \ldots \\
& c_{m+l-2}=\frac{a(-2 g+1)(-2 g+2) \ldots(-2 g+l-1)(m-l)}{2^{l-1}} .
\end{aligned}
$$

Replacing $c_{m+l-2}$ in (6.6) where $i=0$, we get

$$
\frac{a(-2 g+1)(-2 g+2) \ldots(-2 g+l-1)(-2 g+l)(m-l)}{2^{l-1}}=0 .
$$

Note that

$$
(-2 g+1)(-2 g+2) \ldots(-2 g+l-1)(-2 g+l)
$$

is a polynomial of degree $l$ in the variable $g$, which has at most $l$ real roots. Denote by $S_{2}^{l}$ the set of roots. If $g \notin S_{1}^{l} \cup S_{2}^{l}$ then $a=0$ or $m=l$.

The hyperbola is not an invariant algebraic curve when $a=0$ and this cases does not matter for us. We assume $m=l$. Then, we have the following:

$$
\begin{align*}
K & =(2 g-1) m x+2 m y, \\
C_{n} & =x^{m} y^{m}, \quad C_{n-1} \equiv 0, \quad C_{n-2}=a m x^{m-1} y^{m-1} \tag{6.7}
\end{align*}
$$

for $g \notin S_{1}^{m} \cup S_{2}^{m}$ which is a numerable set.
Following similar arguments for terms of degree $n-2, n-3, \ldots$ in (6.1) we conjecture that $C=(a+x y)^{m}$. Now we prove this statement by induction:

Suppose that for $k=1,2 \ldots, L$ we have

$$
\begin{equation*}
C_{n-(2 k-1)} \equiv 0, \quad C_{n-2 k}=\frac{a^{k}(m-(k-1))!}{k!} x^{m-k} y^{m-k} \tag{6.8}
\end{equation*}
$$

We shall prove that:

$$
C_{n-2 L-1} \equiv 0, \quad C_{n-2 L-2}=\frac{a^{L+1}(m-L)!}{(L+1)!} x^{m-L-1} y^{m-L-1} .
$$

Considering in (6.1) the terms of degree $n-2 L$ we have:

$$
\begin{aligned}
2 a C_{n-2 L+1, x}+ & x C_{n-2 L, x}+\left(g x^{2}+x y\right) C_{n-2 L-1, x}+a(2 g-1) C_{n-2 L+1, y} \\
& -y C_{n-2 L, y}+\left((g-1) x y+y^{2}\right) C_{n-2 L-1, y}=((2 g-1) m x+2 m y) C_{n-2 L-1} .
\end{aligned}
$$

By the induction hypothesis $C_{n-2 L+1} \equiv 0$ then:

$$
\begin{align*}
& x C_{n-2 L, x}+\left(g x^{2}+x y\right) C_{n-2 L-1, x}-y C_{n-2 L, y}+\left((g-1) x y+y^{2}\right) C_{n-2 L-1, y} \\
&=[(2 g-1) m x+2 m y] C_{n-2 L-1} . \tag{6.9}
\end{align*}
$$

Setting $C_{n-2 L-1}=\sum_{i=0}^{2 m-2 L-1} c_{2 m-2 L-1-i} x^{2 m-2 L-1-i} y^{i}$ and replacing $C_{n-2 L}, C_{n-2 L-1}$ in (6.9) we obtain

$$
\begin{aligned}
\sum_{i=0}^{2 m-2 L-1}(m-i-g(2 L+1)) c_{2 m-2 L-1-i} & x^{2 m-2 L-i} y^{i} \\
& +\sum_{i=0}^{2 m-2 L-1}(-2 L-1) c_{2 m-2 L-1-i} x^{2 m-2 L-1-i} y^{i+1}=0 .
\end{aligned}
$$

This equation can be written as

$$
\sum_{i=0}^{2 m-2 L}\left[(m-i-g(2 L+1)) c_{2 m-2 L-1-i}+(-2 L-1) c_{2 m-2 L-i}\right] x^{2 m-2 L-i} y^{i}=0
$$

where $c_{i}=0$ for $i<0$ and $i>2 m-2 L-1$. Equating the coefficients of $x^{i} y^{j}$ in the above equation, we have:

$$
(m-i-g(2 L+1)) c_{2 m-2 L-1-i}+(-2 L-1) c_{2 m-2 L-i}=0,
$$

for $i=0,1, \ldots, 2 m-2 L$. As $L \in \mathbb{N}$ then $L \neq-1 / 2$ and:

$$
c_{2 m-2 L-1}=c_{2 m-2 L-2}=\cdots=c_{1}=c_{0}=0 .
$$

Therefore,

$$
C_{n-2 L-1} \equiv 0 .
$$

Now, considering in (6.1) the terms of degree $n-2 L-1$ we have:

$$
\text { 2a } \begin{aligned}
C_{n-2 L, x}+ & x C_{n-2 L-1, x}+\left(g x^{2}+x y\right) C_{n-2 L-2, x}+a(2 g-1) C_{n-2 L, y} \\
& -y C_{n-2 L-1, y}+\left((g-1) x y+y^{2}\right) C_{n-2 L-2, y}=[(2 g-1) m x+2 m y] C_{n-2 L-2} .
\end{aligned}
$$

We just proved that $C_{n-2 L-1} \equiv 0$, then we have:

$$
\begin{align*}
2 a C_{n-2 L, x}+\left(g x^{2}+x y\right) & C_{n-2 L-2, x}+a(2 g-1) C_{n-2 L, y} \\
& +\left((g-1) x y+y^{2}\right) C_{n-2 L-2, y}=[(2 g-1) m x+2 m y] C_{n-2 L-2} \tag{6.10}
\end{align*}
$$

By the induction hypothesis it follows that $C_{n-2 L}=\frac{a^{L}(m-(L-1))!}{L!} x^{m-L} y^{m-L}$. Setting $C_{n-2 L-2}=$ $\sum_{i=0}^{2 m-2 L-2} c_{2 m-2 L-2-i} x^{2 m-2 L-2-i} y^{i}$ and replacing $C_{n-2 L}, C_{n-2 L-2}$ in (6.10) we have:

$$
\begin{aligned}
& \sum_{i=0}^{2 m-2 L-2}\left(m-i-g(2(L+1)) c_{2 m-2 L-2-i} x^{2 m-2 L-1-i} y^{i}\right. \\
& \quad+\quad \sum_{i=0}^{2 m-2 L-2}(-2 L-2) c_{2 m-2 L-2-i} x^{2 m-2 L-2-i} y^{i+1} \\
& \quad=-(2 g-1) \frac{a^{L+1}(m-L)!}{L!} x^{m-L} y^{m-L-1}-\frac{2 a^{L+1}(m-L)!}{L!} x^{m-L-1} y^{m-L} .
\end{aligned}
$$

This equation can be rewritten as

$$
\begin{array}{r}
\sum_{i=0}^{2 m-2 L-1}\left[\left(m-i-g(2(L+1)) c_{2 m-2 L-2-i}+(-2 L-2) c_{2 m-2 L-1-i}\right] x^{2 m-2 L-1-i} y^{i}\right. \\
=-(2 g-1) \frac{a^{L+1}(m-L)!}{L!} x^{m-L} y^{m-L-1}-\frac{2 a^{L+1}(m-L)!}{L!} x^{m-L-1} y^{m-L}
\end{array}
$$

where $c_{i}=0$ for $i<0$ and $i>2 m-2 L-2$. Equating the coefficients of $x^{i} y^{j}$ in the above equation, we get the following equations

$$
\left\{\begin{array}{l}
(m-i-g(2(L+1))) c_{2 m-2 L-2-i}+(-2 L-2) c_{2 m-2 L-1-i}=0, \\
(L+1-g(2(L+1))) c_{m-L-1}+(-2 L-2) c_{m-L}=-(2 g-1) \frac{a^{L+1}(m-L)!}{L!}, \\
(L-g(2(L+1))) c_{m-L-2}+(-2 L-2) c_{m-L-1}=-\frac{2 a^{L+1}(m-L)!}{L!},
\end{array}\right.
$$

for $i=0,1, \ldots, m-L-2, m-L+1, \ldots, 2 m-2 L-1$. As $L \in \mathbb{N}$ then $L \neq-1$ and

$$
c_{m-L-2}=\cdots=c_{1}=c_{0}=0
$$

Then,

$$
c_{m-L-1}=\frac{a^{L+1}(m-L)!}{(L+1)!}, \quad c_{m-L}=0
$$

When $i=m-L-2, \ldots, 0$, we obtain

$$
c_{m-L+1}=c_{m-L+2}=\cdots=c_{2 m-2 L-2}=0 .
$$

Therefore,

$$
C_{n-2 L-2}=\frac{a^{L+1}(m-L)!}{(L+1)!} x^{m-L-1} y^{m-L-1}
$$

This finishes the induction proof. It follows that

$$
C=J_{1}^{m}, \quad m \in \mathbb{N}
$$

for all $(a, g) \notin \mathrm{L}_{1}$, where $\mathrm{L}_{1}=\cup_{k \in \mathbb{N}} \mathrm{~L}_{1, k}=\cup_{k \in \mathbb{N}}\left\{(a, g) \in \mathbb{R}^{2}: g=\frac{k}{2}\right.$ and $\left.a \neq 0\right\}$.
(b) From (a) systems (A) have only the algebraic solution $J_{1}(x, y)=a+x y$ for $(a, g) \in$ $\mathbb{R}^{2}-\mathrm{L}_{1}$. Then by Proposition 2.10, if systems (A) have an exponential factor, it must have the form:

$$
F=\exp \left(G / J_{1}^{l}\right)
$$

with cofactor $L=\bar{L}_{0}+\bar{L}_{1} x+\bar{L}_{2} y$ and where $l$ is non-negative integer. Since the invariant algebraic curve $J_{1}^{l}=0$ has the cofactor

$$
K=l \alpha_{1}=l(-1+2 g) x+2 l y
$$

it follows by (2.2) that $G$ satisfies the following equation:

$$
\begin{align*}
{\left[2 a+x+g x^{2}+x y\right] G_{x} } & +\left[a(2 g-1)-y+(g-1) x y+y^{2}\right] G_{y} \\
& +[l(1-2 g) x-2 l y] G=\left[\bar{L}_{0}+\bar{L}_{1} x+\bar{L}_{2} y\right] \sum_{k=0}^{l}\binom{l}{k} a^{k} x^{l-k} y^{l-k} \tag{6.11}
\end{align*}
$$

From now on we assume $G(x, y)=\sum_{i=0}^{n} G_{i}(x, y)$, where $G_{i}$ is a homogeneous polynomial of degree $i$ and split the study in cases.

Case 1: $n+1<2 l$.
By equating the homogeneous terms of highest degree in (6.11) we obtain that

$$
\bar{L}_{1}=\bar{L}_{2}=0 \quad \text { and } \quad \bar{L}_{0}=0
$$

Thus, $G$ is an invariant algebraic curve. Then, $G=c J_{1}^{l}$ where $c$ is a constant. Therefore, $F$ is constant and it cannot be an exponential factor of system (A).

Case 2: $n+1=2 l$.
By equating the homogeneous terms of highest degree in (6.11) we obtain that

$$
\bar{L}_{1}=\bar{L}_{2}=0 .
$$

Set $G_{n}=\sum_{i=0}^{n} c_{n-i} x^{n-i} y^{i}$ where $c_{n-i}$ are constants. Equating the terms of degree $n+1$ in (6.11) and using that $n+1=2 l$, we have:

$$
\sum_{i=0}^{2 l}\left[(-g-i+l) c_{2 l-i-1}+(-1) c_{2 l-i}\right] x^{2 l-i} y^{i}=\bar{L}_{0} x^{l} y^{l}
$$

where $c_{i}=0$ for $i<0$ and $i>n$. Equating the coefficients of $x^{i} y^{j}$ in the above equation, we have:

$$
\left\{\begin{array}{l}
(-g-i+l) c_{2 l-i-1}-c_{2 l-i}=0, \text { where } i=0,1, \ldots, l-1, l+1, \ldots, 2 l \\
(-g) c_{l-1}-c_{l}=\bar{L}_{0} .
\end{array}\right.
$$

For $i=2 l, 2 l-1, \ldots, l+1$ we obtain: $c_{0}=c_{1}=\cdots=c_{l-1}=0$. Then $c_{l}=-L_{0}$. Working recursively,

$$
\begin{aligned}
c_{l+1} & =(-g+1)\left(-\bar{L}_{0}\right), \quad c_{l+2}=(-g+2)(-g+1)\left(-\bar{L}_{0}\right), \ldots \\
c_{2 l-1} & =(-g+l)(-g+l-1) \ldots(-g+1)\left(-\bar{L}_{0}\right) .
\end{aligned}
$$

Therefore, if $g \notin\{l, l-1, \ldots, 1\}$, this is $(a, g) \notin \mathrm{L}_{1}$ we have $\bar{L}_{0}=0$ then

$$
L=0
$$

Consequently, system (A) has no exponential factors for $(a, g) \in \mathbb{R}^{2}-\mathrm{L}_{1}$.
Case 3: $n=2 l$.
Consider the notation for $G_{n}$ introduced in the study of Case 2. Equating the terms of degree $n+1$ in (6.11) we have

$$
\sum_{i=0}^{2 l+1}(l-i) c_{2 l-i} x^{2 l-i+1} y^{i}=\bar{L}_{1} x^{l+1} y^{l}+\bar{L}_{2} x^{l} y^{l+1}
$$

where $c_{i}=0$ for $i<0$ and $i>n$. These equations are equivalent to

$$
\left\{\begin{array}{l}
(l-i) c_{2 l-i}=0, \text { where } i=0,1, \ldots, l-1, l+2, l+3, \ldots, 2 l+1  \tag{6.12}\\
0 c_{l}=\bar{L}_{1}, \\
(-1) c_{l-1}=\bar{L}_{2}
\end{array}\right.
$$

For $i=2 l, 2 l-1, \ldots, l+2$ we obtain: $c_{0}=c_{1}=\cdots=c_{l-2}=0$. Then,

$$
c_{l-1}=-\bar{L}_{2}, \quad c_{l} \text { is free, } \bar{L}_{1}=0 \text { and } c_{l+1}=c_{l+2}=\cdots=c_{2 l}=0
$$

Therefore,

$$
G_{n}=c_{l} x^{l} y^{l}-\bar{L}_{2} x^{l-1} y^{l+1}
$$

Since $c_{l} \neq 0$, without loss of generality, we assume that $c_{l}=1$.
Equating the terms of degree $n$ in (6.11) we have

$$
\begin{aligned}
x G_{n, x}+\left[g x^{2}+x y\right] G_{n-1, x}-y G_{n, y}+\left[(g-1) x y+y^{2}\right] & G_{n-1, y} \\
& +[l(1-2 g) x-2 l y] G_{n-1}=\bar{L}_{0} x^{l} y^{l} .
\end{aligned}
$$

Set $G_{n-1}=\sum_{i=0}^{n-1} c_{n-i-1} x^{n-i-1} y^{i}$. Replacing $G_{n-1}$ in the above equation and using that $n=2 l$ we obtain:

$$
\sum_{i=0}^{2 l}\left[(-g-i+l) c_{2 l-i-1}-c_{2 l-i}\right] x^{2 l-i} y^{i}=\bar{L}_{0} x^{l} y^{l}+2 \bar{L}_{2} x^{l-1} y^{l+1}
$$

where $c_{i}=0$ for $i<0$ and $i>2 l-1$. Equating the coefficients of $x^{i} y^{j}$ in the above equation, we have:

$$
\left\{\begin{array}{l}
(-g-i+l) c_{2 l-i-1}-c_{2 l-i}=0, \text { where } i=0,1, \ldots, l-1, l+2, \ldots, 2 l  \tag{6.13}\\
(-g) c_{l-1}-c_{l}=\bar{L}_{0}, \\
(-g-1) c_{l-2}-c_{l-1}=2 \bar{L}_{2} .
\end{array}\right.
$$

For $i=2 l, \ldots, l+2$ we have: $c_{0}=c_{1}=\cdots=c_{l-2}=0$. Then, $c_{l-1}=-2 \bar{L}_{2}, c_{l}=2 g \bar{L}_{2}-\bar{L}_{0}$. Working recursively,

$$
\begin{aligned}
c_{l+1} & =(-g+1)\left(2 g \bar{L}_{2}-\bar{L}_{0}\right), \quad c_{l+2}=(-g+1)(-g+2)\left(2 g \bar{L}_{2}-\bar{L}_{0}\right), \ldots \\
c_{2 l-1} & =(-g+1)(-g+2) \ldots(-g+l-1)\left(2 g \bar{L}_{2}-\bar{L}_{0}\right) .
\end{aligned}
$$

Replacing $c_{2 l-1}$ in (6.13) where $i=0$, we have:

$$
(-g+l)(-g+l-1) \ldots(-g+1)\left(2 g \bar{L}_{2}-\bar{L}_{0}\right)=0 .
$$

Then, if $g \notin\{l, l-1, \ldots, 1\}$ we must have $\bar{L}_{0}=2 g \bar{L}_{2}$ and

$$
G_{n-1}=-2 \bar{L}_{2} x^{l-1} y^{l} .
$$

Therefore, $L=2 g \bar{L}_{2}+\bar{L}_{2} y$.
Equating the terms of degree $n-1$ in (6.11) we have:

$$
\begin{aligned}
2 a G_{n, x}+x G_{n-1, x}+\left[g x^{2}\right. & +x y] G_{n-2, x}+a(2 g-1) G_{n, y}-y G_{n-1, y} \\
& +\left[(g-1) x y+y^{2}\right] G_{n-2, y}+[l(1-2 g) x-2 l y] G_{n-2}=a l \bar{L}_{2} x^{l-1} y^{l}
\end{aligned}
$$

Set $G_{n-2}=\sum_{i=0}^{n-2} c_{n-i-2} x^{n-i-2} y^{i}$. Replacing $G_{n-2}$ in the above equation and using that $n=2 l$ we obtain:

$$
\begin{aligned}
& \sum_{i=0}^{2 l}\left[(-2 g-i+l) c_{2 l-i-2}-2 c_{2 l-i-1}\right] x^{2 l-i-1} y^{i}=a l(-2 g+1) \bar{L}_{2} x^{l} y^{l-1} \\
& \quad+\left(\bar{L}_{2}(a l-2)-2 a l+a(2 g-1)(l+1) \bar{L}_{2}\right) x^{l-1} y^{l}+2 a(l-1) \bar{L}_{2} x^{l-2} y^{l+1}
\end{aligned}
$$

where $c_{i}=0$ for $i<0$ and $i>2 l-2$. Equating the coefficients of $x^{i} y^{j}$ in the above equation, we have:

$$
\left\{\begin{array}{l}
(-2 g-i+l) c_{2 l-i-2}-2 c_{2 l-i-1}=0, \text { where } i=0,1, \ldots, l-2, l+2, \ldots, 2 l-1  \tag{6.14}\\
(-2 g+1) c_{l-1}-2 c_{l}=a l(-2 g+1) \bar{L}_{2} \\
(-2 g) c_{l-2}-2 c_{l-1}=\bar{L}_{2}(a l-2)-2 a l+a(2 g-1)(l+1) \bar{L}_{2}, \\
(-2 g-1) c_{l-3}-2 c_{l-2}=2 a(l-1) \bar{L}_{2} .
\end{array}\right.
$$

For $i=2 l-1, \ldots, l+2$, we obtain: $c_{0}=c_{1}=\cdots=c_{l-3}=0$. Then,

$$
\begin{aligned}
c_{l-2} & =-a(l-1) \bar{L}_{2}, \quad c_{l-1}=a l+\bar{L}_{2}\left(1+\frac{a}{2}-2 a g\right) \\
c_{l} & =\frac{a l}{2}(2 g-1) \bar{L}_{2}+\frac{(-2 g+1)}{2}\left(\bar{L}_{2}+a l+\frac{a \bar{L}_{2}}{2}-2 a g \bar{L}_{2}\right) \doteq A
\end{aligned}
$$

Working recursively, we have:

$$
\begin{aligned}
& c_{l+1}=\frac{(-2 g+2) A}{2}, \quad c_{l+2}=\frac{(-2 g+2)(-2 g+3) A}{2^{2}}, \ldots \\
& c_{2 l-2}=\frac{(-2 g+l-1)(-2 g+l-2) \ldots(-2 g+2) A}{2^{l-2}}
\end{aligned}
$$

Replacing $c_{2 l-2}$ in (6.14) where $i=0$ we obtain:

$$
\begin{gathered}
\frac{(-2 g+l)(-2 g+l-1) \ldots(-2 g+2) A}{2^{l-2}}=0, \text { or } \\
\frac{(-2 g+l)(-2 g+l-1) \ldots(-2 g+2)(-2 g+1)}{2^{l-1}}\left(-a l \bar{L}_{2}+\bar{L}_{2}+a l+\frac{a \bar{L}_{2}}{2}-2 a g \bar{L}_{2}\right)=0
\end{gathered}
$$

If $g \notin\left\{\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \ldots\right\}$, this is $(a, g) \notin \mathrm{L}_{1}$ then we must have:

$$
-a l \bar{L}_{2}+\bar{L}_{2}+a l+\frac{a \bar{L}_{2}}{2}-2 a g \bar{L}_{2}=0,
$$

that happens if, and only if,

$$
\bar{L}_{2}=\frac{-2 a l}{-2 a l+2+a-4 a g}, \quad \text { for }-2 a l+2+a-4 a g \neq 0, \quad \text { or } \quad g=\frac{2+a-2 a l}{4 a} .
$$

Suppose that $(a, g) \notin \mathrm{L}_{1} \cup \mathrm{C}^{\prime}$ where

$$
\mathrm{C}^{\prime}=\cup_{k \in \mathbb{N}} \mathrm{C}_{k}=\cup_{k \in \mathbb{N}}\left\{(a, g): g=\frac{2+a-2 a k}{4 a} \text { and } a \neq 0\right\} .
$$

Therefore, we have:

$$
\begin{aligned}
G_{n} & =x^{l} y^{l}+\frac{2 a l}{(-2 a l+2+a-4 a g)} x^{l-1} y^{l+1}, \quad G_{n-1}=\frac{4 a l}{(-2 a l+2+a-4 a g)} x^{l-1} y^{l}, \\
G_{n-2} & =\frac{2 a^{2} l(l-1)}{(-2 a l+2+a-4 a g)} x^{l-2} y^{l}-\frac{2 a^{2} l^{2}}{(-2 a l+2+a-4 a g)} x^{l-1} y^{l-1}, \\
L & =\frac{-4 a g l}{(-2 a l+2+a-4 a g)}-\frac{2 a l}{(-2 a l+2+a-4 a g)} y .
\end{aligned}
$$

Equating the terms of degree $n-2$ in (6.11) we have:

$$
\begin{aligned}
& 2 a G_{n-1, x}+x G_{n-2, x}+\left[g x^{2}+x y\right] G_{n-3, x}+a(2 g-1) G_{n-1, y}-y G_{n-2, y} \\
& \quad+\left[(g-1) x y+y^{2}\right] G_{n-3, y}+[l(1-2 g) x-2 l y] G_{n-3}=\frac{-4 a^{2} g l^{2}}{(-2 a l+2+a-4 a g)} x^{l-1} y^{l-1} .
\end{aligned}
$$

Set $G_{n-3}=\sum_{i=0}^{n-3} c_{n-i-3} x^{n-i-3} y^{i}$. Replacing $G_{n-3}$ in the above equation and using that $n=2 l$ we obtain:

$$
\begin{aligned}
\sum_{i=0}^{2 l-2}\left[(-3 g-i+l) c_{2 l-i-3}-\right. & \left.3 c_{2 l-i-2}\right] x^{2 l-i-2} y^{i} \\
& =\frac{4 a^{2} l^{2}(1-3 g)}{(-2 a l+2+a-4 a g)} x^{l-1} y^{l-1}-\frac{4 a^{2} l(l-1)}{(-2 a l+2+a-4 a g)} x^{l-2} y^{l}
\end{aligned}
$$

where $c_{i}=0$ for $i<0$ and $i>2 l-3$. Equating the coefficients of $x^{i} y^{j}$ in the above equation, we have:

$$
\left\{\begin{array}{l}
(-3 g-i+l) c_{2 l-i-3}-3 c_{2 l-i-2}=0, \text { where } i=0,1, \ldots, l-2, l+1, \ldots, 2 l-2  \tag{6.15}\\
(-3 g+1) c_{l-2}-3 c_{l-1}=\frac{4 a^{2} l^{2}(1-3 g)}{(-2 a l+2+a-4 a g)}, \\
(-3 g) c_{l-3}-3 c_{l-2}=-\frac{4 a^{2} l(l-1)}{(-2 a l+2+a-4 a g)} .
\end{array}\right.
$$

For $i=2 l-2, \ldots, l+1: c_{0}=c_{1}=\cdots=c_{l-3}=0$. Then,

$$
c_{l-2}=\frac{4 a^{2} l(l-1)}{3(-2 a l+2+a-4 a g)}, \quad c_{l-1}=\frac{\left(4 a^{2} l(2 l-1)\right)(1-3 g)}{(-9)(-2 a l+2+a-4 a g)} \doteq B .
$$

Working recursively, we obtain:

$$
\begin{aligned}
c_{l} & =\frac{(-3 g+2) B}{3}, \quad c_{l+1}=\frac{(-3 g+2)(-3 g+3) B}{3^{2}}, \ldots \\
c_{2 l-3} & =\frac{(-3 g+2)(-3 g+3) \ldots(-3 g+l-1) B}{3^{l-3}} .
\end{aligned}
$$

Replacing $c_{2 l-3}$ in (6.15) where $i=0$ :

$$
\begin{gathered}
\frac{(-3 g+l)(-3 g+l-1) \ldots(-3 g+2) B}{3^{l-2}}=0, \quad \text { or } \\
\frac{(-3 g+l)(-3 g+l-1) \ldots(-3 g+2)(-3 g+1)}{3^{l}}\left(\frac{4 a^{2} l(2 l-1)}{2 a l-2-a+4 a g}\right)=0
\end{gathered}
$$

Consider $\mathrm{L}_{2}=\cup_{k \in \mathbb{N}} \mathrm{~L}_{2, k}=\cup_{k \in \mathbb{N}}\left\{(a, g): g=\frac{k}{3}\right\}$. Then, if $(a, g) \notin\left(\mathrm{L}_{1} \cup \mathrm{~L}_{2} \cup \mathrm{C}^{\prime}\right)$ we must have

$$
4 a^{2} l(2 l-1)=0
$$

What happens if, and only if $a=0$ or $l=0$ or $l=1 / 2$. Therefore, systems (A) have no exponential factors for $(a, g) \in \mathbb{R}^{2}-\left(\mathrm{L}_{1} \cup \mathrm{~L}_{2} \cup \mathrm{C}^{\prime}\right)$

Case 4: $n>2 l$.
Consider the notation for $G_{n}$ introduced in the study of Case 2. Equating the terms of degree $n+1$ in (6.11) we have:

$$
\begin{aligned}
{\left[g x^{2}+x y\right] \sum_{i=0}^{n}(n-i) c_{n-i} x^{n-i-1} y^{i}+\left[(g-1) x y+y^{2}\right] } & \sum_{i=0}^{n} i c_{n-i} x^{n-i} y^{i-1} \\
& +[l(1-2 g) x-2 l y] \sum_{i=0}^{n} c_{n-i} x^{n-i} y^{i}=0 .
\end{aligned}
$$

Working in a similar way to the previous cases, we obtain:

$$
\sum_{i=0}^{n+1}\left[(g n-i+l(1-2 g)) c_{n-i}+(n-2 l) c_{n-i+1}\right] x^{n-i+1} y^{i}=0
$$

when $c_{i}=0$ for $i<0$ and $i>n$. Therefore,

$$
(g n-i+l(1-2 g)) c_{n-i}+(n-2 l) c_{n-i+1}=0,
$$

for $i=0,1, \ldots, n+1$. As $n \neq 2 l$ we have

$$
c_{0}=c_{1}=\cdots=c_{n}=0 .
$$

Then, $G_{n}=0$.
Summing up these four cases the proof follows.
(c) Suppose $(a, g) \in \mathbb{R}^{2}-\left(\mathrm{L}_{1} \cup \mathrm{~L}_{2} \cup \mathrm{C}^{\prime}\right)$. Then, by (a) and (b) we get that the systems (A) have only the algebraic solution

$$
J_{1}(x, y)=a+x y
$$

with cofactor $\alpha_{1}=(-1+2 g) x+2 y$ and they have no exponential factor. Under these assumptions

$$
\left\{\begin{array}{l}
\lambda_{1} \alpha_{1}=0 \Leftrightarrow \lambda_{1}=0 \quad \text { and } \\
\lambda_{1} \alpha_{1}=-\operatorname{div}(P, Q)=-(1+2 g x+y)-(-1+(g-1) x+2 y) \text { has no solution. }
\end{array}\right.
$$

Hence, from the Darboux theory of integrability it follows that systems (A) are not Liouvillian integrable.

### 6.2 The systems (B)

$$
\left\{\begin{array}{l}
\dot{x}=2 a+g x^{2}+x y, \\
\dot{y}=a(2 g-1)+(g-1) x y+y^{2},
\end{array}\right.
$$

where $a(g-1) \neq 0$.

## Theorem 6.2.

(a) If $(a, g) \notin \mathrm{L}_{1}$ then the only invariant algebraic curves of a system in the family (B) are of the form $J_{1}^{m}=0$ where $J_{1}(x, y)=a+x y$ and $m$ is a positive integer.
(b) If $(a, g) \notin\left(\mathrm{L}_{1} \cup \mathrm{~L}_{3}\right)$ then any system in the family (B) has no exponential factors.
(c) If $(a, g) \notin\left(\mathrm{L}_{1} \cup \mathrm{~L}_{3}\right)$ then any system in the family (B) is not Liouvillian integrable.

Proof. (a) By a straightforward computation, we can verify that $J_{1}(x, y)=a+x y$ is an invariant hyperbola with cofactor $\alpha_{1}(x, y)=(-1+2 g) x+2 y$. Assume that

$$
C=\sum_{i=0}^{n} C_{i}(x, y)=0
$$

is an invariant algebraic curve of the systems (B) with cofactor $K=K_{0}+K_{1} x+K_{2} y$, where $C_{i}$ are homogeneous polynomial of degree $i$ where $0 \leq i \leq n$. From the definition of the invariant algebraic curve (1.4), we have:

$$
\begin{align*}
\left(2 a+g x^{2}+x y\right) \sum_{i=0}^{n} C_{i, x}+\left(a(2 g-1)+(g-1) x y+y^{2}\right) \sum_{i=0}^{n} & C_{i, y} \\
& =\left(K_{0}+K_{1} x+K_{2} y\right) \sum_{i=0}^{n} C_{i} \tag{6.16}
\end{align*}
$$

The step where we take the terms of degree $n+1$ in (6.16) is exactly the same made for system (A). Then we have:

$$
\begin{aligned}
& C_{n}=x^{m} y^{l}, \quad \text { where } n=m+l, \\
& K_{1}=g m+(g-1) l ; \quad K_{2}=m+l .
\end{aligned}
$$

Now, taking from (6.16) the terms of degree $n$ we have:

$$
\begin{align*}
\left(g x^{2}+x y\right) C_{n-1, x}+\left((g-1) x y+y^{2}\right) & C_{n-1, y} \\
& =K_{0} C_{n}+[(g m+(g-1) l) x+(m+l) y] C_{n-1} \tag{6.17}
\end{align*}
$$

Set $C_{n-1}=\sum_{i=0}^{n-1} c_{n-1-i} x^{n-1-i} y^{i}$. Replacing $C_{n}, C_{n-1}$ in (6.17) and doing some calculations, we obtain:

$$
\sum_{i=0}^{m+l-1}(l-i-g) c_{m+l-1-i} x^{m+l-i} y^{i}-\sum_{i=0}^{m+l-1} c_{m+l-1-i} x^{m+l-1-i} y^{i+1}=K_{0} x^{m} y^{l}
$$

This equation can be written as

$$
\sum_{i=0}^{m+l}\left[(l-i-g) c_{m+l-1-i}-c_{m+l-i}\right] x^{m+l-i} y^{i}=K_{0} x^{m} y^{l}
$$

where $c_{i}=0$ for $i<0$ and $i>m+l-1$. Equating the coefficients of $x^{i} y^{j}$ in the above equation, we get:

$$
\left\{\begin{array}{l}
(l-i-g) c_{m+l-1-i}-c_{m+l-i}=0, \text { where } i=0,1, \ldots, l-1, l+1, \ldots, m+l  \tag{6.18}\\
(-g) c_{m-1}-c_{m}=K_{0}
\end{array}\right.
$$

For $i=m+l, m+l-1, \ldots, l+1$ we have:

$$
c_{0}=c_{1}=\cdots=c_{m-1}=0
$$

Then $c_{m}=-K_{0}$. Working recursively we have

$$
\begin{aligned}
c_{m+1} & =(-g+1)\left(-K_{0}\right) \\
c_{m+2} & =(-g+1)(-g+2)\left(-K_{0}\right), \ldots \\
c_{m+l-1} & =(-g+1)(-g+2) \ldots(-g+l-1)\left(-K_{0}\right) .
\end{aligned}
$$

Replacing $c_{m+l-1}$ in (6.18) where $i=0$, we get

$$
(-g+l)(-g+l-1) \ldots(-g+2)(-g+1)\left(-K_{0}\right)=0
$$

Note that

$$
(-g+l)(-g+l-1) \ldots(-g+2)(-g+1)
$$

is a polynomial of degree $l$ in the variable $g$, which has at most $l$ real roots. Denote by $S_{1}^{l}$ the set of roots. If $g \notin S_{1}^{l}$ we have that $K_{0}=0$. Therefore, we can conclude that

$$
K=(g m+(g-1) l) x+(m+l) y
$$

since $g \notin S_{1}^{l}$. This is $C_{n-1} \equiv 0$.
The step where we take the terms of degree $n-1$ in (6.16) is exactly the same made for system (A). Then we have $m=l$ which leads us to

$$
\begin{align*}
K & =(2 g-1) m x+2 m y \\
C_{n} & =x^{m} y^{m}, \quad C_{n-1} \equiv 0, \quad C_{n-2}=a m x^{m-1} y^{m-1} \tag{6.19}
\end{align*}
$$

for $g \notin S_{1}^{m} \cup S_{2}^{m}$ numerable set, where $S_{2}^{m}$ is the set of roots of the polynomial

$$
(-2 g+1)(-2 g+2) \ldots(-2 g+m-1)(-2 g+m)
$$

Following similar arguments for terms of degree $n-2, n-3, \ldots$ in (6.16) we can prove that $C=(a+x y)^{m}$. It follows that $C=J_{1}^{m}, \quad m \in \mathbb{N}$ for all $(a, g) \notin \mathrm{L}_{1}$, where $\mathrm{L}_{1}=\cup_{k \in \mathbb{N}} \mathrm{~L}_{1, k}=$ $\cup_{k \in \mathbb{N}}\left\{(a, g) \in \mathbb{R}^{2}: g=\frac{k}{2}\right.$ and $\left.a \neq 0\right\}$.
(b) From (a) systems (B) have only the algebraic solution $J_{1}(x, y)=a+x y$ for $(a, g) \in \mathbb{R}^{2}-$ $\mathrm{L}_{1}$. Then, by Proposition 2.10, if systems (B) have an exponential factor, it must have the form

$$
F=\exp \left(G / J_{1}^{l}\right)
$$

with a cofactor $L=\bar{L}_{0}+\bar{L}_{1} x+\bar{L}_{2} y$ and where $l$ is non-negative integers. Since the invariant algebraic curve $J_{1}^{l}=0$ has the cofactor

$$
K=l \alpha_{1}=l(-1+2 g) x+2 l y
$$

it follows by (2.2) that $G$ satisfies the following equation:

$$
\begin{align*}
{\left[2 a+g x^{2}+x y\right] G_{x}+[a(2 g-1)+(g-1) x y} & \left.+y^{2}\right] G_{y}+[l(1-2 g) x-2 l y] G \\
& =\left[\bar{L}_{0}+\bar{L}_{1} x+\bar{L}_{2} y\right] \sum_{k=0}^{l}\binom{l}{k} a^{k} x^{l-k} y^{l-k} \tag{6.20}
\end{align*}
$$

From now on we assume $G(x, y)=\sum_{i=0}^{n} G_{i}(x, y)$, where $G_{i}$ is a homogeneous polynomial of degree $i$ and split the study in cases.

Case 1: $n+1<2 l$.
This case is exactly the same proved for systems (A). We have that $F$ is constant, what cannot happen.

Case 2: $n+1=2 l$.
This case is also the same proved for systems (A). We have that if $g \notin\{l, l-1, \ldots, 1\}$, this is $(a, g) \notin \mathrm{L}_{1}$ then $L=0$. Consequently, system (B) has no exponential factors for $(a, g) \in$ $\mathbb{R}^{2}-\mathrm{L}_{1}$.

Case 3: $n=2 l$.
Set $G_{n}=\sum_{i=0}^{n} c_{n-i} x^{n-i} y^{i}$, where $c_{n-i}$ are constants. Equating the terms of (6.20) with degree $n+1$ we have

$$
\sum_{i=0}^{2 l+1}(l-i) c_{2 l-i} x^{2 l-i+1} y^{i}=\bar{L}_{1} x^{l+1} y^{l}+\bar{L}_{2} x^{l} y^{l+1}
$$

where $c_{i}=0$ for $i<0$ and $i>n$. This is the same equation solved for systems (A) in case 3 . Then we have:

$$
G_{n}=x^{l} y^{l}-\bar{L}_{2} x^{l-1} y^{l+1} \quad \text { and } \quad \bar{L}_{1}=0
$$

Equating the terms of degree $n$ in (6.20) we have

$$
\left[g x^{2}+x y\right] G_{n-1, x}+\left[(g-1) x y+y^{2}\right] G_{n-1, y}+[l(1-2 g) x-2 l y] G_{n-1}=\bar{L}_{0} x^{l} y^{l}
$$

Set $G_{n-1}=\sum_{i=0}^{n-1} c_{n-i-1} x^{n-i-1} y^{i}$. Replacing $G_{n-1}$ in the above equation and using that $n=2 l$ we obtain:

$$
\sum_{i=0}^{2 l}\left[(-g-i+l) c_{2 l-i-1}-c_{2 l-i}\right] x^{2 l-i} y^{i}=\bar{L}_{0} x^{l} y^{l}
$$

where $c_{i}=0$ for $i<0$ and $i>2 l-1$. Equating the coefficients of $x^{i} y^{j}$ in the above equation, we have:

$$
\left\{\begin{array}{l}
(-g-i+l) c_{2 l-i-1}-c_{2 l-i}=0, \text { where } i=0,1, \ldots, l-1, l+1, \ldots, 2 l  \tag{6.21}\\
(-g) c_{l-1}-c_{l}=\bar{L}_{0} .
\end{array}\right.
$$

Then

$$
\begin{aligned}
c_{0} & =c_{1}=\cdots=c_{l-1}=0, \quad c_{l}=-\bar{L}_{0} \\
c_{l+1} & =(-g+1)\left(-\bar{L}_{0}\right), \quad c_{l+2}=(-g+1)(-g+2)\left(-\bar{L}_{0}\right), \ldots \\
c_{2 l-1} & =(-g+1)(-g+2) \ldots(-g+l-1)\left(-\bar{L}_{0}\right)
\end{aligned}
$$

Replacing $c_{2 l-1}$ in (6.21) where $i=0$, we have:

$$
(g+l)(-g+l-1) \ldots(-g+1)\left(-\bar{L}_{0}\right)=0
$$

Then, if $g \notin\{l, l-1, \ldots, 1\}$ we must have $\bar{L}_{0}=0$. Therefore,

$$
G_{n-1} \equiv 0 \quad \text { and } \quad L=\bar{L}_{2} y
$$

Equating the terms of degree $n-1$ in (6.20) we have:

$$
\begin{aligned}
& 2 a G_{n, x}+\left[g x^{2}+x y\right] G_{n-2, x}+a(2 g-1) G_{n, y}+\left[(g-1) x y+y^{2}\right] G_{n-2, y} \\
&+[l(1-2 g) x-2 l y] G_{n-2}=a l \bar{L}_{2} x^{l-1} y^{l}
\end{aligned}
$$

Set $G_{n-2}=\sum_{i=0}^{n-2} c_{n-i-2} x^{n-i-2} y^{i}$. Replacing $G_{n-2}$ in the above equation and using that $n=2 l$ we obtain:

$$
\begin{aligned}
\sum_{i=0}^{2 l}\left[(-2 g-i+l) c_{2 l-i-2}-\right. & \left.2 c_{2 l-i-1}\right] x^{2 l-i-1} y^{i}=(a l(1-2 g)) x^{l} y^{l-1} \\
& +\left(\bar{L}_{2} a l-2 a l+a(2 g-1)(l+1) \bar{L}_{2}\right) x^{l-1} y^{l}+2 a(l-1) \bar{L}_{2} x^{l-2} y^{l+1}
\end{aligned}
$$

where $c_{i}=0$ for $i<0$ and $i>2 l-2$. Equating the coefficients of $x^{i} y^{j}$ in the above equation, we have:

$$
\left\{\begin{array}{l}
(-2 g-i+l) c_{2 l-i-1}+(-2) c_{2 l-i}=0, \text { where } i=0,1, \ldots, l-2, l+2, \ldots, 2 l-1  \tag{6.22}\\
(-2 g+1) c_{l-1}+(-2) c_{l}=a l(1-2 g), \\
(-2 g) c_{l-2}+(-2) c_{l-1}=\bar{L}_{2} a l-2 a l+a(2 g-1)(l+1) \bar{L}_{2}, \\
(-2 g-1) c_{l-3}+(-2) c_{l-2}=2 a(l-1) \bar{L}_{2} .
\end{array}\right.
$$

Therefore,

$$
\begin{aligned}
c_{0} & =c_{1}=\cdots=c_{l-3}=0, \quad c_{l-2}=-a(l-1) \bar{L}_{2}, \quad c_{l-1}=\frac{a}{2}\left(2 l+\bar{L}_{2}\right)-2 a g \bar{L}_{2} \\
c_{l} & =\frac{\bar{L}_{2} a}{2}\left(-2 g+\frac{1}{2}\right)(-2 g+1) \doteq A, \quad c_{l+1}=\frac{(-2 g+2) A}{2} \\
c_{l+2} & =\frac{(-2 g+2)(-2 g+3) A}{4}, \ldots, c_{2 l-2}=\frac{(-2 g+l-1)(-2 g+l-2) \ldots(-2 g+2)}{2^{l-2}} A .
\end{aligned}
$$

Replacing $c_{2 l-2}$ in (6.22) where $i=0$ we have:

$$
\begin{gathered}
\frac{(-2 g+l)(-2 g+l-1) \ldots(-2 g+2)}{2^{l-2}} A=0, \quad \text { or } \\
\frac{(-2 g+l)(-2 g+l-1) \ldots(-2 g+2)(-2 g+1)}{2^{l-2}}\left[\frac{\bar{L}_{2} a}{2}\left(-2 g+\frac{1}{2}\right)\right]=0 .
\end{gathered}
$$

Then, if $g \notin\{1,2, \ldots, 1\} \cup\left\{\frac{1}{2}, \frac{2}{2}, \ldots, \frac{l}{2}\right\} \cup\{1 / 4\}$, this is $(a, g) \notin\left(\mathrm{L}_{1} \cup \mathrm{~L}_{3}\right)$ we must have

$$
\bar{L}_{2}=0 .
$$

Therefore, $L=0$ and systems (B) have no exponential factors for $(a, g) \in \mathbb{R}^{2}-\left(\mathrm{L}_{1} \cup \mathrm{~L}_{3}\right)$.
Case 4: $n>2 l$.
This case is the same proved for systems (A). Then, $G_{n}=0$ that cannot happen.
Summing up these four cases the proof follows.
(c) Suppose $(a, g) \in \mathbb{R}^{2}-\left(\mathrm{L}_{1} \cup \mathrm{~L}_{3}\right)$. Then, by (a) and (b) we get that the systems (B) have only the algebraic solution

$$
J_{1}(x, y)=a+x y
$$

with cofactor $\alpha_{1}=(-1+2 g) x+2 y$ and they have no exponential factor. Under these assumptions

$$
\left\{\begin{array}{l}
\lambda_{1} \alpha_{1}=0 \Leftrightarrow \lambda_{1}=0 \text { and } \\
\lambda_{1} \alpha_{1}=-\operatorname{div}(P, Q)=-(2 g x+y)-((g-1) x+2 y) \text { has no solution. }
\end{array}\right.
$$

Hence, from the Darboux theory of integrability it follows that systems (B) are not Liouvillian integrable.

## 7 Geometric study of the families (C) and (D)

In this section we present a detailed study of two of the normal forms (C) and (D) for the family $\mathbf{Q S H}_{\eta=0}$. We note that we obtained, as limiting cases of the family (D), three other normal forms, i.e. (F), (G) and (I). This is part of the more ample project of gathering data on the geometry of polynomial systems as expressed in the configurations of invariant algebraic curves and their impact on integrability. This data is useful in gaining more insight into the Darboux theory of integrability in order to enable us to answer some questions and in particular to give an answer to the problem of Poincaré for specific families of polynomial differential systems. We are also interested in the topological phase portraits of systems in QSH $_{\eta=0}$ and their bifurcation diagrams. Is there any relationship between the two kinds of bifurcation diagrams? Can we determine when we have algebraically integrable systems? All these motivate our study in this section and we answer some of these questions here or in the last section.

We first present the results of our calculations of the geometric features of the configurations as well as the information on the singularities. Afterwards we sum up these in a proposition and in pictures of the two bifurcation diagrams: one of the changes in the configurations, the other on the topological bifurcation of the systems.

### 7.1 Geometric analysis of family (C)

Consider the family

$$
\text { (C) }\left\{\begin{array}{l}
\dot{x}=2 a+3 c x+x^{2}+x y \\
\dot{y}=a-c^{2}+y^{2}
\end{array}\right.
$$

where $a \neq 0$.
For a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations (C) we study here also the limit case $a=0$ where the equations are still defined.

In the generic case

$$
a\left(a-c^{2}\right)\left(a-8 c^{2} / 9\right) \neq 0
$$

the systems have two invariant lines $J_{1}$ and $J_{2}$ and only one invariant hyperbolas $J_{3}$ with respective cofactors $\alpha_{i}, 1 \leq i \leq 3$ where

$$
\begin{array}{ll}
J_{1}=y-\sqrt{c^{2}-a}, & \alpha_{1}=y+\sqrt{c^{2}-a}, \\
J_{2}=y+\sqrt{c^{2}-a}, & \alpha_{2}=y-\sqrt{c^{2}-a}, \\
J_{3}=a+c x+x y, & \alpha_{3}=2 c+x+2 y .
\end{array}
$$

We note that if $a=c^{2}$ the two lines coincide and we get a double line. Also if $a=8 v^{2} / 9$ we get a double hyperbola as we later prove.

The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using the 1st extactic polynomial for the lines and the 2nd extactic polynomial for the hyperbola.
(i) The generic case: $a\left(a-c^{2}\right)\left(a-8 c^{2} / 9\right) \neq 0$.

Table 7.1: Invariant curves, cofactors, singularities and intersection points of family (C) for the generic case.

| Inv. curves and cofactors | Singularities | Intersection points |
| :---: | :---: | :---: |
| $\begin{aligned} & J_{1}=y-\sqrt{c^{2}-a} \\ & J_{2}=y+\sqrt{c^{2}-a} \\ & J_{3}=a+c x+x y \\ & \alpha_{1}=y+\sqrt{c^{2}-a} \\ & \alpha_{2}=y-\sqrt{c^{2}-a} \\ & \alpha_{3}=2 c+x+2 y \end{aligned}$ | $\begin{aligned} & P_{1}=\left(-\sqrt{c^{2}-a}-c,-\sqrt{c^{2}-a}\right) \\ & P_{2}=\left(-2\left(\sqrt{c^{2}-a}+c\right), \sqrt{c^{2}-a}\right) \\ & P_{3}=\left(\sqrt{c^{2}-a}-c, \sqrt{c^{2}-a}\right) \\ & P_{4}=\left(2\left(\sqrt{c^{2}-a}-c\right),-\sqrt{c^{2}-a}\right) \\ & P_{1}^{\infty}=[0: 1: 0] \\ & P_{2}^{\infty}=[1: 0: 0] \end{aligned}$ <br> For $a<\frac{8 c^{2}}{9}$ we have $\left.a, s, a, s ;{ }_{2}^{0}\right) S N, N$ <br> For $\frac{8 c^{2}}{9}<a<c^{2}$ we have <br> $s, s, a, a ;\left(_{2}^{0}\right) S N, N$ if $c<0$ <br> $a, a, s, s ;\left(_{2}^{0}\right) S N, N$ if $c>0$ <br> For $c^{2}<a$ we have ©, ©, ©, ©; ${ }_{2}^{0}$ ) SN,$N$ | $\begin{aligned} & \bar{J}_{1} \cap \bar{J}_{2}=P_{2}^{\infty} \text { simple } \\ & \bar{J}_{1} \cap \bar{J}_{3}=\left\{\begin{array}{l} P_{2}^{\infty} \text { simple } \\ P_{3} \text { simple } \end{array}\right. \\ & \bar{J}_{1} \cap \mathcal{L}_{\infty}=P_{2}^{\infty} \text { simple } \\ & \bar{J}_{2} \cap \bar{J}_{3}=\left\{\begin{array}{l} P_{2}^{\infty} \text { simple } \\ P_{1} \text { simple } \end{array}\right. \\ & \bar{J}_{2} \cap \mathcal{L}_{\infty}=P_{2}^{\infty} \text { simple } \\ & \bar{J}_{3} \cap \mathcal{L}_{\infty}=\left\{\begin{array}{l} P_{1}^{\infty} \text { simple } \\ P_{2}^{\infty} \text { simple } \end{array}\right. \end{aligned}$ |

Table 7.2: Divisor and zero-cycles of family (C) for the generic.

| Divisor and zero-cycles |
| :---: |
| $I C D=\left\{\begin{array}{l\|c\|}\hline J_{1}+J_{2}+J_{3}+\mathcal{L}_{\infty} \text { if } a<c^{2} \\ J_{1}^{C}+J_{2}^{C}+J_{3}+\mathcal{L}_{\infty} \text { if } a>c^{2} & 4 \\ M_{0 C S}=\left\{\begin{array}{l}P_{1}+P_{2}+P_{3}+P_{4}+2 P_{1}^{\infty}+P_{2}^{\infty} \text { if } a<c^{2} \\ P_{1}^{C}+P_{2}^{C}+P_{3}^{C}+P_{4}^{C}+2 P_{1}^{\infty}+P_{2}^{\infty} \text { if } a>c^{2}\end{array}\right. & 7 \\ T=Z \bar{J}_{1} \bar{J}_{2} \bar{J}_{3}=0 & 5 \\ M_{0 C T}=\left\{\begin{array}{l}2 P_{1}+P_{2}+2 P_{3}+P_{4}+2 P_{1}^{\infty}+4 P_{2}^{\infty} \text { if } a<c^{2} \\ 2 P_{1}^{C}+P_{2}^{C}+2 P_{3}^{C}+P_{4}^{C}+2 P_{1}^{\infty}+4 P_{2}^{\infty} \text { if } a<c^{2}\end{array}\right. & 12 \\ \hline\end{array}\right.$ |

where the total curve $T$ has four distinct tangents at $P_{2}^{\infty}$.
Remark 7.1. Mathematica could not give a response for the computation of the first integral of family (C) in this generic case.

Table 7.3: Integrating factor of family (C) for the generic case.

|  | Integrating Factor |
| :---: | :---: |
| General | $R=J_{1}^{\frac{c+\sqrt{c^{2}-a}}{2 \sqrt{c^{2}-a}}} J_{2}^{\frac{-c+\sqrt{c^{2}-a}}{2 \sqrt{c^{2}-a}}} J_{3}^{-2}$ |
| Simple <br> example | $\mathcal{R}=J_{1}^{\frac{c+\sqrt{c^{2}-a}}{2 \sqrt{c^{2}-a}}} J_{2}^{\frac{-c+\sqrt{c^{2}-a}}{2 \sqrt{c^{2}-a}}} J_{3}^{-2}$ |

(ii) The non-generic cases: $a\left(a-c^{2}\right)\left(a-8 c^{2} / 9\right)=0$.
(ii.1) $a=c^{2}$ and $c \neq 0$.

Table 7.4: Invariant curves, exponential factor, cofactors, singularities and intersection points of family $(C)$ for $a=c^{2}$ and $c \neq 0$.

| Inv. curves and cofactors | Singularities | Intersection points |
| :--- | :--- | :---: |
| $J_{1}=y$ | $P_{1}=(-2 c, 0)$ | $\bar{J}_{1} \cap \bar{J}_{2}=\left\{\begin{array}{l}P_{2}^{\infty} \text { simple } \\ P_{2} \text { simple }\end{array}\right.$ |
| $J_{2}=\frac{x y}{c}+c+x$ | $P_{2}=(-c, 0)$ |  |
| $E_{3}=e^{\frac{g_{0}+g_{1} y}{y}}$ | $P_{1}^{\infty}=[0: 1: 0]$ | $\bar{J}_{1} \cap \mathcal{L}_{\infty}=P_{2}^{\infty}$ simple |
| $\alpha_{1}=y$ | $P_{2}^{\infty}=[1: 0: 0]$ |  |
| $\alpha_{2}=2 c+x+2 y$ | $s n, s n ;\left({ }_{2}^{0}\right) S N, N$ | $\bar{J}_{2} \cap \mathcal{L}_{\infty}=\left\{\begin{array}{l}P_{1}^{\infty} \text { simple } \\ P_{2}^{\infty} \text { simple }\end{array}\right.$ |
| $\alpha_{3}=-g_{0}$ |  |  |

Table 7.5: Divisor and zero-cycles of family (C) for $a=c^{2}$ and $c \neq 0$.

| Divisor and zero-cycles | Degree |
| :--- | :---: |
| $I C D=2 J_{1}+J_{2}+\mathcal{L}_{\infty}$ | 4 |
| $M_{0 C S}=2 P_{1}+2 P_{2}+2 P_{1}^{\infty}+P_{2}^{\infty}$ | 7 |
| $T=Z \bar{J}_{1}^{2} \bar{J}_{2}=0$ | 5 |
| $M_{0 C T}=2 P_{1}+3 P_{2}+2 P_{1}^{\infty}+4 P_{2}^{\infty}$ | 11 |

Table 7.6: First integral and integrating factor of family (C) for $a=c^{2}$ and $c \neq 0$.

|  | First integral | Integrating Factor |
| :---: | :---: | :---: |
| General | $I$ | $R=J_{1} J_{2}^{-2} E_{3}^{-\frac{c}{\delta 口_{0}}}$ |
| Simple <br> example | $\mathcal{I}$ | $\mathcal{R}=J_{1} J_{2}^{-2} E_{3}^{-c}$ |

$$
I=\mathcal{I}=\frac{c^{2}\left(e^{c / y} E_{i}\left(-\frac{c}{y}\right)\left(c^{2}+c x+x y\right)+y(c+x-y)\right)\left(e^{\frac{g_{0}}{y}+g_{1}}\right)^{-\frac{c}{g_{0}}}}{c^{2}+c x+x y}
$$

where $E_{i}(z)=-\int_{-z}^{\infty} \frac{e^{-t}}{t} d t$ is the exponential integral function that has a branch cut discontinuity in the complex z plane running from $-\infty$ to 0 .
(ii.2) $a=8 c^{2} / 9$ and $c \neq 0$.

Table 7.7: Invariant curves, exponential factor, cofactors, singularities and intersection points of family (C) for $a=8 c^{2} / 9$ and $c \neq 0$.

| Invariant curves and cofactors | Singularities | Intersection points |
| :---: | :---: | :---: |
| $J_{1}=-c+3 y$ |  | $\bar{J}_{1} \cap \bar{J}_{2}=P_{2}^{\infty}$ simple |
| $J_{2}=c+3 y$ | $P_{1}=\left(-\frac{8 c}{3}, \frac{c}{3}\right)$ | $\bar{J}^{\sim} \bar{J}_{3}=\left\{\begin{array}{l}P_{2}^{\infty} \text { simple }\end{array}\right.$ |
| $J_{3}=8 c^{2}+9 c x+9 x y$ | $P_{2}=\left(-\frac{4 c}{3},-\frac{c}{3}\right)$ | $J_{1} \cap J_{3}=\left\{\begin{array}{l}\text { P } \\ P_{3} \text { simple }\end{array}\right.$ |
| $E_{4}=e^{\frac{c^{2}\left(488_{0}-88_{1} x+488_{1 y} y+5488 x+3 c_{9} y(21 x x-8 y)+58_{80} x y\right.}{4 c^{2}\left(8 c^{2}+9 c x+9 x y\right)}}$ | $P_{3}=\left(-\frac{2 c}{3}, \frac{c}{3}\right)$ | $\bar{J}_{1} \cap \mathcal{L}_{\infty}=P_{2}^{\infty}$ simple |
|  | $P_{1}^{\infty}=[0: 1: 0]$ | $\bar{J}_{2} \cap \bar{J}_{3}=\left\{\begin{array}{l} P_{2}^{\infty} \text { simple } \end{array}\right.$ |
| $\alpha_{1}=y+\frac{c}{3}$ | $P_{2}^{\infty}=[1: 0: 0]$ | $P_{2}$ simple |
| $\alpha_{2}=y-\frac{c}{3}$ |  | $J_{2} \cap \mathcal{L}_{\infty}=P_{2}^{\infty}$ simple |
| $\begin{aligned} & \alpha_{3}=2 c+x+2 y \\ & \alpha_{4}=-\frac{g_{1}(c-3 y)}{54 c} \end{aligned}$ | $\left.s, s n, a ;{ }_{2}^{0}\right) S N, N$ | $\bar{J}_{3} \cap \mathcal{L}_{\infty}=\left\{\begin{array}{l} P_{1}^{\infty} \text { simple } \\ P_{2}^{\infty} \text { simple } \end{array}\right.$ |

Table 7.8: Divisor and zero-cycles of family (C) for $a=8 c^{2} / 9$ and $c \neq 0$.

| Divisor and zero-cycles | Degree |
| :--- | :---: |
| $I C D=J_{1}+J_{2}+2 J_{3}+\mathcal{L}_{\infty}$ | 5 |
| $M_{0 C S}=P_{1}+2 P_{2}+P_{3}+2 P_{1}^{\infty}+P_{2}^{\infty}$ | 7 |
| $T=Z \bar{J}_{1} \bar{J}_{2} \bar{J}_{3}^{2}=0$ | 7 |
| $M_{0 C T}=P_{1}+3 P_{2}+3 P_{3}+3 P_{1}^{\infty}+5 P_{2}^{\infty}$ | 15 |

Table 7.9: First integral and integrating factor of family (C) for $a=8 c^{2} / 9$ and $c \neq 0$.

|  | First integral | Integrating Factor |
| :---: | :---: | :---: |
| General | $I=J_{1}^{0} J_{2}^{\lambda_{2}} J_{3}^{0} E_{4}^{-\frac{18 c \lambda_{2}}{81}}$ | $R=J_{1}^{2} J_{2}^{\lambda_{2}^{\prime}} J_{3}^{-2} E_{4}^{-\frac{18\left(\lambda_{2}^{\prime}+c\right)}{81}}$ |
| Simple <br> example | $\mathcal{I}=J_{2} E_{4}^{-18 c}$ | $\mathcal{R}=\frac{J_{1}^{2}}{J_{2} /_{3}^{2}}$ |

(ii.3) $a=0$ and $c \neq 0$.

Under this condition, systems defined by (C) do not belong to QSH. All the invariant lines are $x=0$ and $\pm c+y=0$ that are simple. By perturbing the reducible conic $x(c+y)=0$ we can produce the hyperbola $a+c x+x y=0$. Furthermore, the conic $x(c+y)=0$ has integrable multiplicity two.

Table 7.10: Invariant curves, exponential factor, cofactors, singularities and intersection points of family (C) for $a=0$ and $c \neq 0$.

| Invariant curves and cofactors | Singularities | Intersection points |
| :--- | :--- | :--- |
| $J_{1}=-c+y$ | $P_{1}=(-4 c, c)$ |  |
| $J_{2}=c+y$ | $P_{2}=(-2 c,-c)$ |  |
| $J_{3}=x$ | $\bar{J}_{1} \cap \bar{J}_{2}=P_{2}^{\infty}$ simple |  |
| $E_{4}=e^{\left(-\frac{c^{2}\left(80-3 g_{1} x\right)+2 c c_{0}\left(x-y y-3 s c_{1} x y+80 y^{2}\right.}{3(x(c+y)}\right)}$ | $P_{3}=(0,-c)$ | $\bar{J}_{1} \cap \bar{J}_{3}=P_{4}$ simple |
|  | $P_{4}=(0, c)$ | $\bar{J}_{1} \cap \mathcal{L}_{\infty}=P_{2}^{\infty}$ simple |
| $\alpha_{1}=c+y$ | $P_{1}^{\infty}=[0: 1: 0]$ | $\bar{J}_{2} \cap \bar{J}_{3}=P_{3}$ simple |
| $\alpha_{2}=-c+y$ | $P_{2}^{\infty}=[1: 0: 0]$ | $\bar{J}_{2} \cap \mathcal{L}_{\infty}=P_{2}^{\infty}$ simple |
| $\alpha_{3}=3 c+x+y$ | $s, a, s, a ;\left(_{2}^{0}\right) S N, N$ |  |
| $\alpha_{4}=\frac{g_{0}(y-c)}{3 c}$ |  |  |

Table 7.11: Divisor and zero-cycles of family (C) for $a=0$ and $c \neq 0$.

| Divisor and zero-cycles | Degree |
| :--- | :---: |
| $I C D=J_{1}+J_{2}+J_{3}+\mathcal{L}_{\infty}$ | 4 |
| $M_{0 C S}=P_{1}+P_{2}+P_{3}+P_{4}+2 P_{1}^{\infty}+P_{2}^{\infty}$ | 7 |
| $T=Z \bar{J}_{1} \bar{J}_{2} \bar{J}_{3}=0$ | 4 |
| $M_{0 C T}=P_{1}+P_{2}+2 P_{3}+2 P_{4}+2 P_{1}^{\infty}+3 P_{2}^{\infty}$ | 11 |

Table 7.12: First integral and integrating factor of family (C) for $a=0$ and $c \neq 0$.

|  | First integral | Integrating Factor |
| :---: | :---: | :---: |
| General | $I=J_{1}^{0} J_{2}^{\lambda_{2}} J_{3}^{0} E_{4}^{-\frac{3 c c_{2}}{80}}$ | $R=J_{1} J_{2}^{\lambda_{2}^{\prime}} J_{3}^{-2} E_{4}^{-\frac{3\left(2 c+c \lambda_{2}^{\prime}\right)}{80}}$ |
| Simple <br> example | $\mathcal{I}=J_{2} E_{4}^{-3 c}$ | $\mathcal{R}=\frac{I_{1}}{J_{2}^{2} 2_{3}^{2}}$ |

(ii.4) $a=c=0$.

Under this condition, systems defined by (C) do not belong to QSH. Here we have a single system which has a generalized Darboux first integral. The affine invariant lines $x=0$ and $y=0$ are both double. By perturbing the reducible conic $x y=0$ we can produce the hyperbola $a+c x+x y=0$.

Table 7.13: Invariant curves, exponential factor, cofactors, singularities and intersection points of family (C) for $a=c=0$.

| Invariant curves and cofactors | Singularities | Intersection points |
| :---: | :--- | :--- |
| $J_{1}=x$ |  |  |
| $J_{2}=y$ |  |  |
| $E_{3}=e^{g_{0}+\frac{g_{1} y}{x}}$ | $P_{1}=(0,0)$ |  |
| $E_{4}=e^{h_{0}+\frac{h_{1}}{y}}$ |  | $\bar{J}_{1}^{\infty}=[0: 1: 0]$ |
|  | $\bar{J}_{2} \cap P_{1}$ simple |  |
| $\alpha_{1}=c+y$ | $\mathcal{L}_{\infty}=P_{1}^{\infty}$ simple |  |
| $\alpha_{2}=-c+y$ | $p h p h ; 0]$ | $\left.\bar{J}_{2}\right) S N, N$ |
| $\alpha_{3} \cap-\mathcal{L}_{\infty}=P_{2}^{\infty}$ simple |  |  |
| $\alpha_{4}=-h_{1} y$ |  |  |

Table 7.14: Divisor and zero-cycles of family (C) for $a=c=0$.

| Divisor and zero-cycles | Degree |
| :--- | :---: |
| $I C D=2 J_{1}+2 J_{2}+\mathcal{L}_{\infty}$ | 5 |
| $M_{0 C S}=4 P_{1}+2 P_{1}^{\infty}+P_{2}^{\infty}$ | 7 |
| $T=Z \bar{J}_{1}^{2} \bar{J}_{2}^{2}=0$ | 5 |
| $M_{0 C T}=4 P_{1}+3 P_{1}^{\infty}+3 P_{2}^{\infty}$ | 10 |

Table 7.15: First integral and integrating factor of family (C) for $a=c=0$.

|  | First integral | Integrating Factor |
| :---: | :---: | :---: |
| General | $I=J_{1}^{0} J_{2}^{g_{1} \lambda_{3}} E_{3}^{\lambda_{3}} E_{4}^{0}$ | $R=J_{1}^{-2} J_{2}^{-1+g_{1} \lambda_{3}^{\prime}} E_{3}^{\lambda_{3}} E_{4}^{0}$ |
| Simple <br> example | $\mathcal{I}=J_{2} E_{3}$ | $\mathcal{R}=\frac{1}{J_{1}^{2} J_{2}}$ |

We sum up the topological, dynamical and algebraic geometric features of family (C) and also confront our results with previous results in literature in the following proposition. We show that all the phase portraits for family (C) are missing in [41].

## Proposition 7.2.

(a) For the family (C) we obtained six distinct configurations $C_{1}^{(C)}-C_{6}^{(C)}$ of invariant hyperbolas and lines (see Figure 7.1 for the complete bifurcation diagram of configurations of this family). The bifurcation set of configurations in the full parameter space is is $a\left(a-c^{2}\right)\left(a-8 c^{2} / 9\right)=0$ and it is made of points of bifurcation due to change in the multiplicities of the invariant algebraic invariant curves: On $a=c^{2}$ and $c \neq 0$ the invariant lines coalesce into a double line. On $a=8 c^{2} / 9$ and $c \neq 0$ the hyperbola becomes double. Outside the parameter space, i.e. on $a=0$ the invariant hyperbola becomes reducible producing the lines $x=0$ and $c+y=0$ and when also $c=0$ then $x=0$ and $y=0$ become double lines.
(b) The family (C) is Liouvillian integrable for $a\left(a-8 c^{2} / 9\right) \neq 0$ and generalized Darboux integrable for $a=8 c^{2} / 9$. All systems in family (C) do not have a polynomial inverse integrating factor. Outside the parameter space, i.e. on $a=0$ we have a polynomial inverse integrating factor only when $c=0$.
(c) For the family (C) we have five topologically distinct phase portraits $P_{1}^{(C)}-P_{5}^{(C)}$. The topological bifurcation set is the same as the one for configurations, i.e. it is $a\left(a-c^{2}\right)\left(a-8 c^{2} / 9\right)=0$ (see Figure 7.2 for the complete topological bifurcation diagram). The parabolas $a=c^{2}$ and $a=8 c^{2} / 9$ are bifurcation sets of singularities and the line $a=0$ is a bifurcation of separatrices connection. The phase portraits $P_{1}^{(\mathrm{C})}-P_{5}^{(\mathrm{C})}$ are not topologically equivalent with anyone of the phase portraits in [41].

Proof of Proposition 7.2. (a) We have the following type of divisors and zero-cycles of the total invariant curve $T$ for the configurations of family (C):

Table 7.16: Configurations for family (C).

| Configurations | Divisors and zero-cycles of the total inv. curve $T$ |
| :---: | :---: |
| $C_{1}^{(C)}$ | $I C D=J_{1}+J_{2}+J_{3}+\mathcal{L}_{\infty}$ |
| $C_{2}^{(C)}$ | $M_{0 C T}=2 P_{1}+P_{2}+2 P_{3}+P_{4}+2 P_{1}^{\infty}+4 P_{2}^{\infty}$ |
| $C_{3}^{(C)}$ | $I C D=J_{1}+J_{2}+J_{3}+\mathcal{L}_{\infty}$ |
| $M_{0 C T}=2 P_{1}+P_{2}+2 P_{3}+P_{4}+2 P_{1}^{\infty}+4 P_{2}^{\infty}$ |  |
| $C_{4}^{(C)}$ | $I C D=J_{1}+J_{2}+J_{3}+\mathcal{L}_{\infty}$ |
| $M_{0 C T}=2 P_{1}+P_{2}+2 P_{3}+P_{4}+2 P_{1}^{\infty}+4 P_{2}^{\infty}$ |  |
| $C_{5}^{(C)}$ | $I C D=J_{1}^{C}+J_{2}^{C}+J_{3}+\mathcal{L}_{\infty}$ |
| $M_{0 C T}=2 P_{1}^{C}+P_{2}^{C}+2 P_{3}^{C}+P_{4}^{C}+2 P_{1}^{\infty}+4 P_{2}^{\infty}$ |  |
| $C_{6}^{(C)}$ | $I C D=2 J_{1}+J_{2}+\mathcal{L}_{\infty}$ |
|  | $M_{0 C T}=2 P_{1}+3 P_{2}+2 P_{1}^{\infty}+4 P_{2}^{\infty}$ |

Note that $C_{1}^{(C)}, C_{2}^{(C)}$ and $C_{3}^{(C)}$ admit the same type of divisor and zero-cycles but the configurations are non equivalent. In fact, consider the convex quadrilateral in Figure 7.1 formed by the four finite singularities in these configurations. In $C_{1}^{(C)}$ any two consecutive or opposite points of this quadrilateral are not joined by anyone of the two branches of the hyperbola, in $C_{2}^{(C)}$, two opposite points are joined by a branch of the hyperbola and in $C_{3}^{(C)}$ two consecutive points of this quadrilateral is joined by a branch of the hyperbola.

Therefore, the configurations $C_{1}^{(C)}$ up to $C_{6}^{(C)}$ are all distinct. For the limit cases of family (C) we have the following configurations:

Table 7.17: Configurations for the limit cases of family (C).

| Configurations | Divisors and zero-cycles of the total inv. curve $T$ |
| :---: | :---: |
| $c_{1}$ | $I C D=J_{1}+J_{2}+J_{3}+\mathcal{L}_{\infty}$ |
|  | $M_{0 C T}=P_{1}+P_{2}+2 P_{3}+2 P_{4}+2 P_{1}^{\infty}+3 P_{2}^{\infty}$ |
| $c_{2}$ | $I C D=2 J_{1}+2 J_{2}+\mathcal{L}_{\infty}$ |
|  | $M_{0 C T}=4 P_{1}+3 P_{1}^{\infty}+3 P_{2}^{\infty}$ |

Therefore, we have two distinct configurations for the limit cases.
(b) In the generic case $a\left(a-c^{2}\right)\left(a-8 c^{2} / 9\right) \neq 0$ the three cofactors $\alpha_{1}, \alpha_{2}, \alpha_{3}$ of $J_{1}, J_{2}, J_{3}$ are linearly independent. Hence we cannot get a Darboux first integral by using these curves. Furthermore the curves are each of multiplicity 1 and hence we cannot have exponential factors attached to them. However we obtained an integrating factor for (C) in the generic case. Using Mathematica we could not obtain an expression for the first integral of these systems but we know that it exists and it is Liouvillian. For the non-generic cases we obtained first integrals and they were given in previously exhibited tables.

Let us show that the family does not admit a polynomial inverse integrating factor.
(i) The generic case: $a\left(a-c^{2}\right)\left(a-8 c^{2} / 9\right) \neq 0$.

We have the following integrating factor

$$
R=J_{1}^{\frac{c+\sqrt{c^{2}-a}}{2 \sqrt{c^{2}-a}}} J_{2}^{\frac{-c+\sqrt{c^{2}-a}}{2 \sqrt{c^{2}-a}}} J_{3}^{-2} .
$$

In order to $R^{-1}$ to be polynomial we must have that

$$
\left\{\begin{array}{l}
\frac{c+\sqrt{c^{2}-a}}{2 \sqrt{c^{2}-a}}=\frac{c}{2 \sqrt{c^{2}-a}}+\frac{1}{2}=-m_{1}, m_{1} \in \mathbb{N} \\
\frac{-c+\sqrt{c^{2}-a}}{2 \sqrt{c^{2}-a}}=\frac{-c}{2 \sqrt{c^{2}-a}}+\frac{1}{2}=-m_{2}, m_{2} \in \mathbb{N}
\end{array}\right.
$$

Adding up these two expressions we have

$$
1=-\left(m_{1}+m_{2}\right), \quad m_{1}, m_{2} \in \mathbb{N}
$$

and this equation does not have a solution. Therefore, $R^{-1}$ cannot be polynomial.
(ii) The non-generic case: $a\left(a-c^{2}\right)\left(a-8 c^{2} / 9\right)=0$.
(ii.1) $a=c^{2}$ : We have the integrating factor

$$
R=J_{1} J_{2}^{-2} E_{3}^{-c / g_{0}}
$$

and it is clear that $R^{-1}$ cannot be polynomial.
(ii.2) $a=8 c^{2} / 9$ : We have the integrating factor

$$
R=J_{1}^{2} J_{2}^{\lambda_{2}^{\prime}} J_{3}^{-2} E_{4}^{-18\left(c \lambda_{2}^{\prime}+c\right) / g_{1}}
$$

again it is clear that $R^{-1}$ cannot be polynomial.
(ii.3) $a=0$ and $c \neq 0$. We have the integrating factor

$$
R=J_{1} J_{2}^{\lambda_{2}^{\prime}} J_{3}^{-2} E_{4}^{-3\left(2 c+c \lambda_{2}^{\prime}\right) / g_{0}}
$$

again it is clear that $R^{-1}$ cannot be polynomial.
(ii.4) $a=0$ and $c=0:$ We have the integrating factor

$$
R=J_{1}^{-2} J_{2}^{-1+g_{1} \lambda_{3}^{\prime}} E_{3}^{\lambda_{3}^{\prime}} E_{4}^{0}
$$

Taking $\lambda_{3}^{\prime}=0$ we have that $R^{-1}=J_{1}^{2} J_{2}$ which is polynomial.
(c) We have:

Table 7.18: Phase portraits for family (C).

| Phase Portraits | Sing. at $\infty$ | Finite sing. | Separatrix connections |
| :---: | :---: | :---: | :---: |
| $P_{1}^{(C)}$ | $\left(\left(_{2}^{0}\right) S N, N\right)$ | $(a, s, a, s)$ | $4 S C_{f}^{f} \quad 6 S C_{f}^{\infty} \quad 0 S C_{\infty}^{\infty}$ |
| $P_{2}^{(C)}$ | $\left(\left(_{2}^{0}\right) S N, N\right)$ | $(a, s, a, s)$ | $4 S C_{f}^{f} 5 S C_{f}^{\infty} \quad 1 S C_{\infty}^{\infty}$ |
| $P_{3}^{(C)}$ | $\left(\left(_{2}^{0}\right) S N, N\right)$ | $(\odot, \odot, \odot, \odot)$ | $0 S C_{f}^{f} 0 S C_{f}^{\infty} \quad 2 S C_{\infty}^{\infty}$ |
| $P_{4}^{(C)}$ | $\left(\left(_{2}^{0}\right) S N, N\right)$ | $(s n, s n)$ | $1 S C_{f}^{f} 5 S C_{f}^{\infty} \quad 1 S C_{\infty}^{\infty}$ |
| $P_{5}^{(C)}$ | $\left(\left(_{2}^{0}\right) S N, N\right)$ | $(s, s n, a)$ | $3 S C_{f}^{f} 5 S C_{f}^{\infty} \quad 1 S C_{\infty}^{\infty}$ |

Therefore, we have five distinct phase portraits for systems (C). For the limit case of family (C) we have the following phase portraits:

Table 7.19: Phase portraits for the limit case of family (C).

| Phase portraits | Sing. at $\infty$ | Finite sing. | Separatrix connections |
| :---: | :---: | :---: | :---: |
| $p_{1}$ | $\left(\left(_{2}^{0}\right) S N, N\right)$ | $(s, a, s, a)$ | $4 S C_{f}^{f} 5 S C_{f}^{\infty} 0 S C_{\infty}^{\infty}$ |
| $p_{2}$ | $\left(\left(_{2}^{0}\right) S N, N\right)$ | $p h p h$ | $0 S C_{f}^{f} 4 S C_{f}^{\infty} 0 S C_{\infty}^{\infty}$ |

Therefore, we have two topologically distinct phase portraits for the limit cases.
Table 7.20: Phase portraits in [41] that admit 2 singular points at infinity with the type $(S N, N)$.

| Phase Portrait | Sing. at $\infty$ | Real finite sing. | Separatrix connections |
| :---: | :---: | :---: | :---: |
| $L_{01}$ | $\left(\left(_{2}^{0}\right) S N, N\right)$ | $\varnothing$ | $0 S C_{f}^{f} 0 S C_{f}^{\infty} \quad 3 S C_{\infty}^{\infty}$ |
| $L_{03}$ | $\left(\left(_{2}^{0}\right) S N, N\right)$ | $\varnothing$ | $0 S C_{f}^{f} \quad 0 S C_{f}^{\infty} \quad 3 S C_{\infty}^{\infty}$ |
| $\omega_{1}$ | $\left(\left(_{2}^{0}\right) S N, N\right)$ | $(s, a)$ | $1 S C_{f}^{f} \quad 6 S C_{f}^{\infty} \quad 0 S C_{\infty}^{\infty}$ |

Therefore, the phase portraits $P_{1}^{(\mathrm{C})}-P_{5}^{(\mathrm{C})}$ are not topologically equivalent with anyone of the phase portraits in [41].

Remark 7.3. Note that $c_{1}$ (for example, for $c>0$ ) has three distinct lines, each line is an irreducible curve and for these lines the algebraic, integrable and geometric multiplicities coincide and this multiplicity is one. Hence in perturbations the line $y+c=0$ can produce at most one line and in this case, it produces the line $y+\sqrt{c^{2}-a}=0$.

Remark 7.4. Note that the necessary and sufficient condition for systems defined by the equations (C) to have a double hyperbola or a double line is that it has two singularities of the system of multiplicity two or just one singularity of multiplicity four.


Figure 7.1: Bifurcation diagram of configurations for family (C). The dashed line $a=0$ is a limit case of this family. The multiple invariant curves are emphasized and the complex curves are drawn as dashed in the drawings of the configurations.


Figure 7.2: Topological bifurcation diagram for family (C). The continuous curves in the phase portraits are separatrices. The dashed curves are the orbits given in each region of the phase portraits. The green bullet represents an elemental saddle, the red bullet an elemental unstable node and the blue an elemental stable node. The yellow triangle represents a saddle-node (semielemental) and the black bullet is an intricate singularity.

### 7.2 Geometric analysis of family (D)

Consider the family:

$$
\text { (D) }\left\{\begin{array}{l}
\dot{x}=(c+x)(c(2 g-1)+g x) \\
\dot{y}=1+(g-1) x y,
\end{array}\right.
$$

where $(g \pm 1)(3 g-1)(2 g-1) \neq 0$ and $c^{2}+g^{2} \neq 0$.
For a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations (D) we study here also the limit cases $(g \pm 1)(3 g-1)(2 g-1)=0$ where the equations are still defined.

In the generic case

$$
c g(g \pm 1)(3 g-1)(2 g-1) \neq 0
$$

the systems have two invariant lines $J_{1}$ and $J_{2}$ and only one invariant hyperbola $J_{3}$ with respective cofactors $\alpha_{i}, 1 \leq i \leq 3$ where

$$
\begin{array}{ll}
J_{1}=c+x, & \alpha_{1}=c(-1+2 g)+g x, \\
J_{2}=c(-1+2 g)+g x, & \alpha_{2}=-c g+g x, \\
J_{3}=\frac{1}{2 g-1}+y(c+x), & \alpha_{3}=c(-1+2 g)+(-1+2 g) x .
\end{array}
$$

We note that if $g=1$ or $c=0$ then the two lines coincide and we get a multiple line.
The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using the 1st extactic polynomial for the line and the 2nd extactic polynomial for the hyperbola.
(i) The generic case: $\operatorname{cg}(g \pm 1)(3 g-1)(2 g-1) \neq 0$.

Table 7.21: Invariant curves, cofactors, singularities and intersection points of family (D) for the generic case.

| Invariant curves and cofactors | Singularities | Intersection points |
| :---: | :---: | :---: |
| $\begin{aligned} & J_{1}=c+x \\ & J_{2}=c(-1+2 g)+g x \\ & J_{3}=\frac{1}{2 g-1}+y(c+x) \\ & \alpha_{1}=c(-1+2 g)+g x \\ & \alpha_{2}=-c g+g x \\ & \alpha_{3}=c(-1+2 g)+(-1+2 g) x \end{aligned}$ | $\begin{aligned} & P_{1}=\left(-c, \frac{1}{c(g-1)}\right) \\ & P_{2}=\left(c\left(\frac{1}{g}-2\right), \frac{g}{2 c g^{2}-3 c g+c}\right) \end{aligned}$ |  |
|  | $\begin{aligned} & P_{1}^{\infty}=[0: 1: 0] \\ & P_{2}^{\infty}=[1: 0: 0] \end{aligned}$ | $\begin{aligned} & \bar{J}_{1} \cap \bar{J}_{2}=P_{1}^{\infty} \text { simple } \\ & \bar{J}_{1} \cap \bar{J}_{3}=P_{1}^{\infty} \text { double } \end{aligned}$ |
|  | For $c<0$ we have | $\begin{aligned} & \bar{J}_{1} \cap \mathcal{L}_{\infty}=P_{1}^{\infty} \text { simple } \\ & \bar{I}_{2} \cap \bar{I}_{2}=\left\{P_{1}^{\infty}\right. \text { simple } \end{aligned}$ |
|  | $s, a ;\left({ }_{2}^{2}\right) E-E, S$ if $g<0$ <br> $s, s ;\left({ }_{2}^{2}\right) E-E, N$ if $0<g<1 / 2$ <br> $\left.s, a ;{ }_{2}^{2}\right) P H-P H, N$ if $g>1 / 2$ <br> For $c>0$ we have | $\begin{aligned} & J_{2} \cap J_{3}=\left\{\begin{array}{l} P_{2} \text { simple } \end{array}\right. \\ & \bar{J}_{2} \cap \mathcal{L}_{\infty}=P_{1}^{\infty} \text { simple } \\ & \bar{J}_{3} \cap \mathcal{L}_{\infty}=\left\{\begin{array}{l} P_{1}^{\infty} \text { simple } \\ P_{2}^{\infty} \text { simple } \end{array}\right. \end{aligned}$ |
|  | $\begin{aligned} & s, a ;\binom{2}{2} E-E, S \text { if } g<0 \\ & s, s ;(2) E-E, N \text { if } 0<g<1 / 2 \\ & s, a ;\left(2_{2}^{2}\right) P H-P H, N \text { if } g>1 / 2 \end{aligned}$ |  |

Table 7.22: Divisor and zero-cycles of family (D) for the generic.

| Divisor and zero-cycles | Degree |
| :--- | :---: |
| ICD $=J_{1}+J_{2}+J_{3}+\mathcal{L}_{\infty}$ | 4 |
| $M_{0 C S}=P_{1}+P_{2}+4 P_{1}^{\infty}+P_{2}^{\infty}$ | 7 |
| $T=Z \bar{J}_{1} \bar{J}_{2} \bar{J}_{3}=0$ | 5 |
| $M_{0 C T}=P_{1}+2 P_{2}+4 P_{1}^{\infty}+2 P_{2}^{\infty}$ | 9 |

where the total curve $T$ has only three distinct tangents at $P_{1}^{\infty}$, but one of them is double.
Table 7.23: First integral and integrating factor of family (D) for the generic case.

|  | First integral | Integrating Factor |
| :---: | :---: | :---: |
| General | $I=J_{1}^{0} J_{2}^{\lambda_{2}} J_{3}^{-\frac{g \lambda_{2}}{28-1}}$ | $R=J_{1}^{0} J_{2}^{\lambda_{2}^{\prime}} J_{3}^{\frac{1-g\left(\lambda_{2} \lambda_{2}+3\right)}{2 g-1}}$ |
| Simple <br> example | $\mathcal{I}=J_{2} J_{3}^{-\frac{g}{28-1}}$ | $\mathcal{R}=\frac{1}{J_{1} J_{2}}$ |

(ii) The non-generic case: $\operatorname{cg}(g \pm 1)(3 g-1)(2 g-1)=0$
(ii.1) $c=0$ and $g(g \pm 1)(3 g-1)(2 g-1) \neq 0$.

Here the invariant line $x=0$ has multiplicity 3 so we compute the exponential factors $E_{3}$ and $E_{4}$. The line at infinity and the hyperbolas are both simple. Thus the total multiplicity of the hyperbolas and lines is 5 and by the Theorem 2.33 we must have a Darboux first integral. We do the calculations to effectively compute the first integrals and the geometric features of this family.

Table 7.24: Invariant curves, exponential factors, cofactors, singularities and intersection points of family (D) for $c=0$ and $g \neq 0, \pm 1,1 / 3,1 / 2$.

| Inv.curves/exp.fac. <br> and cofactors | Singularities | Intersection points |
| :--- | :--- | :--- |
| $J_{1}=x$ | $P_{1}^{\infty}=[0: 1: 0]$ |  |
| $J_{2}=\frac{1}{2 g-1}+x y$ | $P_{2}^{\infty}=[1: 0: 0]$ |  |
| $E_{3}=e^{80+g_{1} x}$ | For $g<0$ we have |  |
| $E_{4}=e^{\frac{2 g h_{0} x y+h_{0}+x\left(h_{1}+h_{2} x\right)}{x^{2}}}$ |  | $\bar{J}_{1} \cap \bar{J}_{2}=P_{1}^{\infty}$ double |
| $\alpha_{1}=g x$ | $\bar{J}_{1} \cap \mathcal{L}_{\infty}=P_{1}^{\infty}$ simple |  |
| $\alpha_{2}=(2 g-1) x$ | For $g>0$ we have | $\bar{J}_{2} \cap \mathcal{L}_{\infty}=\left\{\begin{array}{l}P_{1}^{\infty} \text { simple } \\ P_{2}^{\infty} \text { simple } \\ \alpha_{3}=-g g_{0}\end{array}\right.$ |
| $\alpha_{4}=-g\left(h_{1}+2 h_{0} y\right)$ | $\varnothing ;\left(\frac{4}{2}\right) P H P-P H P, N$ if $g<1 / 2$ |  |
|  | $\varnothing ;\left(\frac{4}{2}\right) P H-H P, N$ if $g>1 / 2$ |  |

Table 7.25: Divisor and zero-cycles of family (D) for $c=0$ and $g \neq$ $0, \pm 1,1 / 3,1 / 2$.

| Divisor and zero-cycles | Degree |
| :---: | :---: |
| $I C D=3 J_{1}+J_{2}+\mathcal{L}_{\infty}$ | 5 |
| $M_{0 C S}=6 P_{1}^{\infty}+P_{2}^{\infty}$ | 7 |
| $T=Z \bar{J}_{1}^{3} \bar{J}_{2}=0$. | 6 |
| $M_{0 C T}=5 P_{1}^{\infty}+2 P_{2}^{\infty}$ | 7 |

where the total curve $T$ has only two distinct tangents at $P_{1}^{\infty}$, but one of them is quadruple.

Table 7.26: First integral and integrating factor of family (D) when $c=0$ and $g \neq 0,1 / 2$.

|  | First integral | Integrating Factor |
| :---: | :---: | :---: |
| General | $I=J_{1}^{\lambda_{1}} J_{2}^{-\frac{g \lambda_{1}}{2 g-1}} E_{3}^{0} E_{4}^{0}$ | $R=J_{1}^{\lambda_{1}^{\prime}} J_{2}^{\frac{1-g\left(\lambda_{1}+3\right)}{2 g-1}} E_{3}^{0} E_{4}^{0}$ |
| Simple <br> example | $\mathcal{I}=J_{1} J_{2}^{-\frac{g}{2 g-1}}$ | $\mathcal{R}=\frac{1}{J_{1} J_{2}}$ |

(ii.2) $c=0$ and $g(g \pm 1)(3 g-1)(2 g-1)=0$.
(ii.2.1) $c=0$ and $g=-1$.

Under this condition, systems defined by the equations (D) do not belong to family (D). We note that system defined by the equation (D) when $c=0$ and $g=-1$ is exactly the family (G). Here we have a single system that possess an invariant line which has multiplicity four. The line at infinity and the hyperbolas are both simple. Thus the total multiplicity of hyperbolas and lines is 6 and by the Theorem 2.33 we must have a rational first integral. We do the calculations to effectively compute the first integrals and the geometric features of this family.

Table 7.27: Invariant curves, cofactors, singularities and intersection points of system (D) when $c=0$ and $g=-1$.

| Inv.curves and cofactors | Singularities | Intersection points |
| :---: | :---: | :---: |
| $\begin{aligned} & \hline \hline J_{1}=x \\ & J_{2}=-1+3 x y \\ & E_{3}=e^{\frac{g_{0}+\xi_{1} x}{x}} \\ & E_{4}=e^{\frac{-2 h_{0} x y+h_{0}+h_{1} x+h_{2} x^{2}}{x^{2}}} \\ & E_{5}=e^{\left(\frac{-3 l_{0} x y+l_{0}+x\left(-2 h_{1} x y+l_{1}+x\left(l_{2}+l_{3} x\right)\right)}{x^{3}}\right)} \\ & \alpha_{1}=-x \\ & \alpha_{2}=-3 x \\ & \alpha_{3}=g_{0} \\ & \alpha_{4}=h_{1}+2 h_{0} y \\ & \alpha_{5}=l_{2}+2 l_{1} y \end{aligned}$ | $\begin{aligned} & P_{1}^{\infty}=[0: 1: 0] \\ & P_{2}^{\infty}=[1: 0: 0] \\ & \varnothing ;\left(\frac{4}{2}\right) E-E, S \end{aligned}$ | $\begin{aligned} & \bar{J}_{1} \cap \bar{J}_{2}=P_{1}^{\infty} \text { double } \\ & \bar{J}_{1} \cap \mathcal{L}_{\infty}=P_{1}^{\infty} \text { simple } \\ & \bar{J}_{2} \cap \mathcal{L}_{\infty}=\left\{\begin{array}{l} P_{1}^{\infty} \text { simple } \\ P_{2}^{\infty} \text { simple } \end{array}\right. \end{aligned}$ |

Table 7.28: Divisor and zero-cycles of system (D) when $c=0$ and $g=-1$.

| Divisor and zero-cycles | Degree |
| :---: | :---: |
| $I C D=4 J_{1}+J_{2}+\mathcal{L}_{\infty}$ | 6 |
| $M_{0 C S}=6 P_{1}^{\infty}+P_{2}^{\infty}$ | 7 |
| $T=Z \bar{J}_{1}^{4} \bar{J}_{2}=0$. | 7 |
| $M_{0 C T}=6 P_{1}^{\infty}+2 P_{2}^{\infty}$ | 8 |

where the total curve $T$ has only two distinct tangents at $P_{1}^{\infty}$, but one of them is quintuple.

Table 7.29: First integral and integrating factor of system(D) when $c=0$ and $g=-1$.

|  | First integral | Integrating Factor |
| :---: | :---: | :---: |
| General | $I=J_{1}^{\lambda_{1}} J_{2}^{-\frac{\lambda_{1}}{3}} E_{3}^{-\frac{\left(h_{1} l_{1}-h_{0} l_{2}\right) \lambda_{4}}{g_{0} l_{1}}} E_{4}^{\lambda_{4}} E_{5}^{-\frac{h_{0} \lambda_{4}}{l_{1}}}$ | $R=J_{1}^{\lambda_{1}^{\prime}} J_{2}^{-\frac{4}{3}-\frac{\lambda_{1}^{\prime}}{3}} E_{3}^{-\frac{\left(h_{1} l_{1}-h_{0} I_{2}\right) \lambda_{4}^{\prime}}{g_{0} l_{1}}} E_{4}^{\lambda_{4}^{\prime}} E_{5}^{-\frac{h_{0} \lambda_{4}^{\prime}}{l_{1}}}$ |
| Simple <br> example | $\mathcal{I}=\frac{J_{1}^{3}}{J_{2}}$ | $\mathcal{R}=\frac{1}{J_{1} J_{2}}$ |

Remark 7.5. Consider $\mathcal{F}_{\left(c_{1}, c_{2}\right)}^{1}=c_{1} J_{1}^{3}-c_{2} J_{2}=0, \operatorname{deg} \mathcal{F}_{\left(c_{1}, c_{2}\right)}^{1}=3$. The remarkable value of $\mathcal{F}_{\left(c_{1}, c_{2}\right)}^{1}$ is $[1: 0]$ for which we have

$$
\mathcal{F}_{(1,0)}^{1}=J_{1}^{3}
$$

Therefore, $J_{1}$ is a critical remarkable curves and $[1: 0]$ is a critical remarkable value of $\mathcal{I}_{1}$.
(ii.2.2) $c=0$ and $g=0$.

Under this condition, systems defined by (D) do not belong to QSH. The hyperbola $-1+$ $x y=0$ filled up with singularities and the following study is done with the reduced system.

Table 7.30: Singularities of the reduced system (D) when $c=g=0 .(\ominus[)(] ; \varnothing)$ denotes the presence of a hyperbola filled up with singular points in the system such that the reduced system has no finite singularity on this curve and $(\ominus[)(] ; N, \varnothing)$ denotes that the system has at infinity a node, and one non-isolated singular point which is part of a real hyperbola filled up with singularities and that the reduced linear system has no infinite singular point in that position.

| Singularities |
| :---: |
| $P_{1}^{\infty}=[0: 1: 0]$ |
| $(\ominus[)(] ; \varnothing) ;(\ominus[)(] ; N, \varnothing)$ |

Table 7.31: First integral and integrating factor of the reduced system (D) when $c=g=0$.

|  | First integral | Integrating Factor |
| :---: | :---: | :---: |
| General | $I=-x$ | $R=-1$ |
| Simple <br> example | $\mathcal{I}=-x$ | $\mathcal{R}=-1$ |

(ii.2.3) $c=0$ and $g=1 / 3$.

Under this condition, systems defined by the equations (D) do not belong to family (D). Here we have one affine invariant line and one invariant hyperbola, both of them are triple so we compute the exponential factors $E_{3}, E_{4}, E_{5}$ and $E_{6}$. This system is Hamiltonian so it admits a polynomial first integral.

Table 7.32: Invariant curves, exponential factors, cofactors, singularities and intersection points of family (D) when $c=0$ and $g=1 / 3$.

| Inv.curves/exp.fac. and cofactors | Singularities | Intersection points |
| :---: | :---: | :---: |
| $\begin{aligned} & J_{1}=x \\ & J_{2}=-3+x y \\ & E_{3}=e^{\frac{8_{0}+8 g_{1} x}{x}} \\ & E_{4}=e^{\frac{h_{0}+h_{1} x+h_{2} x^{2}+\frac{2 h_{0} x y}{3}}{x^{2}}} \\ & E_{5}=e^{\frac{-0_{0} y}{-3+x y}} \\ & E_{6}=e^{\frac{m_{0}}{9}+\frac{y\left(m_{1}(6-2 x y)+3 m-2 y(x x y-9)\right)}{6(x y-3)^{2}}} \\ & \alpha_{1}=\frac{x}{3} \\ & \alpha_{2}=-\frac{x}{3} \\ & \alpha_{3}=-\frac{80}{3} \\ & \alpha_{4}=-\frac{h_{1}}{3}-\frac{2 h_{0} y}{3} \\ & \alpha_{5}=-t t_{0} \\ & \alpha_{6}=\frac{m_{1}}{9}-m_{2} y \end{aligned}$ | $\begin{aligned} & P_{1}^{\infty}=[0: 1: 0] \\ & P_{2}^{\infty}=[1: 0: 0] \\ & \varnothing ;\left(\frac{4}{2}\right) P H P-P H P, N \end{aligned}$ | $\begin{aligned} & \bar{J}_{1} \cap \bar{J}_{2}=P_{2}^{\infty} \text { double } \\ & \bar{J}_{1} \cap \mathcal{L}_{\infty}=P_{1}^{\infty} \text { simple } \\ & \bar{J}_{2} \cap \mathcal{L}_{\infty}=\left\{\begin{array}{l} P_{1}^{\infty} \text { simple } \\ P_{2}^{\infty} \text { simple } \end{array}\right. \end{aligned}$ |

Table 7.33: Divisor and zero-cycles of family (D) when $c=0$ and $g=1 / 3$.

| Divisor and zero-cycles | Degree |
| :--- | :---: |
| $I C D=3 J_{1}+3 J_{2}+\mathcal{L}_{\infty}$ | 7 |
| $M_{0 C S}=6 P_{1}^{\infty}+P_{2}^{\infty}$ | 7 |
| $T=Z \bar{J}_{1}^{3} J_{2}^{3}=0$ | 10 |
| $M_{0 C T}=7 P_{1}^{\infty}+4 P_{2}^{\infty}$ | 11 |

where the total curve $T$ has

1) only three distinct tangents at $P_{1}^{\infty}$, but two of them are triple;

2 ) only two distinct tangents at $P_{2}^{\infty}$, but one of them is triple.
Table 7.34: First integral and integrating factor of family (D) when $c=0$ and $g=1 / 3$.

|  | First integral | Integrating Factor |
| :---: | :---: | :---: |
| General | $I=J_{1}^{\lambda_{1}} J_{2}^{\lambda_{1}} E_{3}^{\lambda_{3}} E_{4}^{\lambda_{4}} E_{5}^{-\frac{g_{1} \Lambda_{3}}{10}-\frac{\lambda_{4}\left(2 h_{0} m_{1}+9 h_{1} m_{1}\right)}{9_{0} m_{2}}} E_{6}^{-\frac{2 h_{0} \lambda_{1}}{3 m_{2}}}$ | $R$ |
| Simple <br> example | $\mathcal{I}_{1}=J_{1} J_{2}$ | $\mathcal{R}_{1}=\frac{1}{\left.J_{1}\right]_{2}}$ |

where $R=J_{1}^{\lambda_{1}^{\prime}} J_{2}^{\lambda_{1}^{\prime}} E_{3}^{\lambda_{3}^{\prime}} E_{4}^{\lambda_{4}^{\lambda_{4}^{\prime}}} E_{5}^{-\frac{g_{0} \nu_{3}^{\prime}}{l_{0}}-\frac{\lambda_{4}^{\prime}\left(2 h_{0} m_{1}+h_{1} m_{1}\right)}{9_{0} m_{2}}} E_{6}^{-\frac{2 h_{0} \lambda_{4}^{\prime}}{3 m_{2}}}$.
(ii.2.4) $c=0$ and $g=1 / 2$.

Under this condition, systems defined by (D) do not belong to QSH. Here we have only one triple invariant line so we compute the exponential factors $E_{2}$ and $E_{3}$. We have that the line at infinity $Z=0$ also is triple.

Table 7.35: Invariant curves, exponential factors, cofactors, singularities and intersection points of family (D) when $c=0$ and $g=1 / 2$.

| Inv.curves/exp.fac. and cofactors | Singularities | Intersection points |
| :--- | :--- | :--- |
| $J_{1}=x$ |  |  |
| $E_{2}=e^{\frac{g_{0}+g_{1} x}{x}}$ |  |  |
| $E_{3}=e^{\frac{h_{0} x y+h_{0}+x\left(h_{1}+h_{2} x\right)}{}}$ |  |  |
| $E_{4}=e^{l_{0}+l_{1} x y}$ | $P_{1}^{\infty}=[0: 1: 0]$ |  |
| $\alpha_{1}=\frac{x}{2}$ | $P_{2}^{\infty}=[1: 0: 0]$ | $\bar{J}_{1} \cap \mathcal{L}_{\infty}=P_{1}^{\infty}$ simple |
| $\alpha_{2}=-\frac{g_{0}}{2}$ | $\varnothing ;\left(_{2}^{4}\right) P H-H P, N$ |  |
| $\alpha_{3}=-h_{0} y-\frac{h_{1}}{2}$ |  |  |
| $\alpha_{4}=l_{1} x$ |  |  |

Table 7.36: Divisor and zero-cycles of family (D) when $c=0$ and $g=1 / 2$.

| Divisor and zero-cycles | Degree |
| :---: | :---: |
| $I C D=3 J_{1}+\mathcal{L}_{\infty}$ | 4 |
| $M_{0 C S}=6 P_{1}^{\infty}+P_{2}^{\infty}$ | 7 |
| $T=Z \bar{J}_{1}^{3}=0$. | 4 |
| $M_{0 C T}=4 P_{1}^{\infty}+P_{2}^{\infty}$ | 5 |

where the total curve $T$ has only two distinct tangents at $P_{1}^{\infty}$, one of them triple.

Table 7.37: First integral and integrating factor of family (D) when $c=0$ and $g=1 / 2$.

|  | First integral | Integrating Factor |
| :---: | :---: | :---: |
| General | $I=J_{1}^{\lambda_{1}} E_{2}^{0} E_{3}^{0} E_{4}^{-\frac{\lambda_{1}}{L_{1}}}$ | $R=J_{1}^{\lambda_{1}^{\prime}} E_{2}^{0} E_{3}^{0} E_{4}^{-\frac{\left(1+\lambda_{1}^{\prime}\right)}{2_{1}}}$ |
| Simple <br> example | $\mathcal{I}=J_{1}^{2} E_{4}^{-1}$ | $\mathcal{R}=\frac{1}{J_{1}}$ |

(ii.2.5) $c=0$ and $g=1$.

Under this condition, systems defined by the equations (D) do not belong to family (D). The system defined by the equations (D) when $c=0$ and $g=1$ is exactly the family (I). Here the systems possess an invariant line with multiplicity three and a family of invariant hyperbolas

$$
1+r x+x y
$$

where $r \in \mathbb{R}$. The line at infinity $\mathcal{L}_{\infty}: Z=0$ has multiplicity 3 .
Table 7.38: Invariant curves, exponential factors, cofactors, singularities and intersection points of system (D) when $c=0$ and $g=1$.

| Inv.cur./exp.fac and cofactors | Singularities | Intersection points |
| :--- | :--- | :--- |
| $J_{1}=x$ |  |  |
| $J_{2, r}=1+r x+x y$ |  |  |
| $E_{3}=e^{g_{0}+g_{1} y+g_{2} y^{2}}$ |  |  |
| $E_{4}=e^{\frac{n_{0}+h_{1} x}{c+x}}$ |  |  |
| $E_{5}=e^{\frac{l_{0}+l_{1} x+l_{2} x^{2}+2 h_{0} x y}{x^{2}}}$ | $P_{1}^{\infty}=[0: 1: 0]$ | $\bar{J}_{1} \cap \bar{J}_{2, r}=P_{1}^{\infty}$ double |
|  | $P_{2}^{\infty}=[1: 0: 0]$ | $\bar{J}_{1} \cap \mathcal{L}_{\infty}=P_{1}^{\infty}$ simple |
| $\alpha_{1}=x$ |  |  |
| $\alpha_{2}=x$ | $\left.\bar{J}_{2}^{4}\right) P H-H P, N$ | $\bar{J}_{2, r} \cap \mathcal{L}_{\infty}=\left\{\begin{array}{l}P_{1}^{\infty} \text { simple } \\ P_{2}^{\infty} \text { simple } \\ \alpha_{3}=g_{1}+2 g_{2} x\end{array}\right.$ |
| $\alpha_{4}=-h_{0}$ |  |  |
| $\alpha_{5}=-l_{1}-2 l_{0} y$ |  |  |

Table 7.39: Divisor and zero-cycles of system (D) when $c=0$ and $g=1$.

| Divisor and zero-cycles | Degree |
| :---: | :---: |
| $I L D=3 J_{1}+3 \mathcal{L}_{\infty}$ | 6 |
| $M_{O C S}=6 P_{1}^{\infty}+P_{2}^{\infty}$ | 7 |
| $T=Z^{3} \bar{J}_{1}^{3}=0$ | 6 |
| $M_{O C T}=6 P_{1}^{\infty}+3 P_{2}^{\infty}$ | 9 |

where the total curve $T$ has

1) only two distinct tangents at $P_{1}^{\infty}$, both of them triple and
2) only one triple tangent at $P_{2}^{\infty}$.

Table 7.40: First integral and integrating factor of system (D) when $c=0$ and $g=1$.

|  | First integral | Integrating Factor |
| :---: | :---: | :---: |
| General | $I=J_{1}^{\lambda_{1}} J_{2, r}^{-\lambda_{1}} E_{3}^{\lambda_{3}} E_{4}^{-\frac{\lambda_{3}\left(g_{2} l_{1}-g_{1} l_{0}\right)}{h_{0} I_{0}}} E_{5}^{\frac{g_{2} \lambda_{3}}{L_{0}}}$ | $R=J_{1}^{\lambda_{1}^{\prime}} J_{2, r}^{-2-\lambda_{1}^{\prime}} E_{3}^{\lambda_{3}^{\prime}} E_{4}^{-\frac{\lambda_{3}\left(g_{2} l_{1}-\text { g1 } 1_{0}\right)}{h_{0} L_{0}}} E_{5}^{\frac{g_{2} \lambda_{3}}{I_{0}}}$ |
| Simple <br> example | $\mathcal{I}_{1}=\frac{J_{1}}{J_{2,1}}$ | $\mathcal{R}_{1}=\frac{1}{J_{1}^{2}}$ |

Remark 7.6. Consider $\mathcal{F}_{\left(c_{1}, c_{2}\right)}^{1}=c_{1} J_{1}-c_{2} J_{2,1}=0, \operatorname{deg} \mathcal{F}_{\left(c_{1}, c_{2}\right)}^{1}=2$. We do not have any remarkable values and remarkable curves for $\mathcal{I}_{1}$.
(ii.3) $c \neq 0$ and $g(g \pm 1)(3 g-1)(2 g-1)=0$.
(ii.3.1) $g=-1$ and $c \neq 0$.

Under this condition, systems defined by the equations (D) do not belong to family (D). Here we have two invariant lines, one invariant hyperbola and one invariant parabola. We note that in the case $c=g=-1$ this system is exactly family ( F ).

Table 7.41: Invariant curves, cofactors, singularities and intersection points of system (D) when $g=-1$ and $c \neq 0$.

| Inv.curves and cofactors | Singularities | Intersection points |
| :---: | :---: | :---: |
| $\begin{aligned} & J_{1}=c+x \\ & J_{2}=3 c+x \\ & J_{3}=-1+3 c y+3 x y \\ & J_{4}=3 c^{2} y+\frac{x^{2}}{8 c}+\frac{19 c}{8}+x \\ & \alpha_{1}=-3 c-x \\ & \alpha_{2}=-c-x \\ & \alpha_{3}=-3 c-3 x \\ & \alpha_{4}=-2 x \end{aligned}$ | $\begin{aligned} & P_{1}=\left(-c,-\frac{1}{2 c}\right) \\ & P_{2}=\left(-3 c,-\frac{1}{6 c}\right) \\ & P_{1}^{\infty}=[0: 1: 0] \\ & P_{2}^{\infty}=[1: 0: 0] \\ & s, a ;\left({ }_{2}^{2}\right) E-E, S \end{aligned}$ | $\left.\left.\begin{array}{l} \hline \hline \bar{J}_{1} \cap \bar{J}_{2}=P_{1}^{\infty} \text { simple } \\ \bar{J}_{1} \cap \bar{J}_{3}=P_{1}^{\infty} \text { double } \end{array}\right] \begin{array}{l} \bar{J}_{1} \cap \bar{J}_{4}=\left\{\begin{array}{l} P_{1}^{\infty} \text { simple } \\ P_{1} \text { simple } \end{array}\right. \\ \bar{J}_{1} \cap \mathcal{L}_{\infty}=P_{1}^{\infty} \text { simple } \end{array}\right\} \begin{aligned} & \bar{J}_{2} \cap \bar{J}_{3}=\left\{\begin{array}{l} P_{1}^{\infty} \text { simple } \\ P_{2} \text { simple } \end{array}\right. \\ & \bar{J}_{2} \cap \bar{J}_{4}=\left\{\begin{array}{l} P_{1}^{\infty} \text { simple } \\ P_{2} \text { simple } \end{array}\right. \\ & \bar{J}_{2} \cap \mathcal{L}_{\infty}=P_{1}^{\infty} \text { simple } \end{aligned}$ |

Table 7.42: Divisor and zero-cycles of system (D) when $g=-1$ and $c \neq 0$.

| Divisor and zero-cycles | Degree |
| :--- | :---: |
| $I C D=J_{1}+J_{2}+J_{3}+J_{4}+\mathcal{L}_{\infty}$ | 5 |
| $M_{0 C S}=P_{1}+P_{2}+4 P_{1}^{\infty}+P_{2}^{\infty}$ | 7 |
| $T=Z \bar{J}_{1} \bar{J}_{2} \bar{J}_{3} \bar{J}_{4}=0$. | 7 |
| $M_{0 C T}=2 P_{1}+3 P_{2}+5 P_{1}^{\infty}+2 P_{2}^{\infty}$ | 12 |

where the total curve $T$ has

1) only three distinct tangents at $P_{1}^{\infty}$, two of them double and one simple,
2) two distinct tangents at $P_{2}$, but one of them is double.

Table 7.43: First integral and integrating factor of system (D) when $g=-1$ and $c \neq 0$.

|  | First integral | Integrating Factor |
| :---: | :---: | :---: |
| General | $I=J_{1}^{\lambda_{1}} J_{2}^{\lambda_{2}} J_{3}^{-\lambda_{1}-\frac{\lambda_{2}}{3}} J_{4}^{\lambda_{1}}$ | $R=J_{1}^{\lambda_{1}^{\prime}} J_{2}^{\lambda_{2}^{\prime}} J_{3}^{-\frac{4}{3}-\lambda_{1}^{\prime}-\frac{\lambda_{2}^{\prime}}{3}} J_{4}^{\lambda_{1}^{\prime}}$ |
| Simple <br> example | $\mathcal{I}_{1}=\frac{J_{2}^{3}}{J_{3}}$ | $\mathcal{R}=\frac{1}{J_{1} J_{2} J_{4}}$ |

Remark 7.7. Consider $\mathcal{F}_{\left(c_{1}, c_{2}\right)}^{1}=c_{1} J_{2}^{3}-c_{2} J_{3}=0, \operatorname{deg} \mathcal{F}_{\left(c_{1}, c_{2}\right)}^{1}=3$. The remarkable value of $\mathcal{F}_{\left(c_{1}, c_{2}\right)}^{1}$ are $\left[1:-8 c^{3}\right]$ and $[1: 0]$ for which we have

$$
\mathcal{F}_{\left(1,-8 c^{3}\right)}^{1}=8 c J_{1} J_{4}, \quad \mathcal{F}_{(1,0)}^{1}=J_{2}^{3} .
$$

Therefore, $J_{1}, J_{2}, J_{4}$ are remarkable curves and $\left[1:-8 c^{3}\right],[1: 0]$ are remarkable values of $\mathcal{I}_{1}$. Moreover, $[1: 0]$ is a critical remarkable values and $J_{2}$ is critical remarkable curve of $\mathcal{I}_{1}$. The singular points are $P_{1}$ for $\mathcal{F}_{\left(1,-8 c^{3}\right)}^{1}$ and $P_{2}$ for $\mathcal{F}_{(1,0)}^{1}$.
(ii.3.2) $g=0$ and $c \neq 0$.

Here we have only one affine invariant line and one invariant hyperbola both of them are simple. The line at infinity $Z=0$ is double so we compute the exponential factor $E_{3}$.

Table 7.44: Invariant curves, exponential factors, cofactors, singularities and intersection points of family (D) for $g=0$ and $c \neq 0$.

| Inv.curves/exp.fac. and cofactors | Singularities | Intersection points |
| :--- | :--- | :--- |
|  | $P_{1}=\left(-c,-\frac{1}{c}\right)$ |  |
|  | $P_{1}^{\infty}=[0: 1: 0]$ |  |
| $J_{1}=c+x$ | $P_{2}^{\infty}=[1: 0: 0]$ |  |
| $J_{2}=-1+c y+x y$ |  | $\bar{J}_{1} \cap \bar{J}_{2}=P_{1}^{\infty}$ double |
| $E_{3}=e^{g_{0}+g_{1} x}$ | For $c<0$ we have | $\bar{J}_{1} \cap \mathcal{L}_{\infty}=P_{1}^{\infty}$ simple |
| $\alpha_{1}=-c$ | $\left.s ;{ }_{(2)}^{2}\right) E-E,\left({ }_{1}^{1}\right) S N$ | $\bar{J}_{2} \cap \mathcal{L}_{\infty}=\left\{\begin{array}{l}P_{1}^{\infty} \text { simple } \\ P_{2}^{\infty} \text { simple } \\ \alpha_{2}=-c-x\end{array}\right.$ |
| $\alpha_{3}=-c^{2} g_{1}-c g_{1} x$ | For $c>0$ we have |  |
|  | $s ;\left({ }_{2}^{2}\right) E-E,\left({ }_{1}^{1}\right) S N$ |  |

Table 7.45: Divisor and zero-cycles of family (D) for $g=0$ and $c \neq 0$.

| Divisor and zero-cycles | Degree |
| :--- | :---: |
| ICD $=J_{1}+J_{2}+2 \mathcal{L}_{\infty}$ | 4 |
| $M_{0 C S}=P_{1}+4 P_{1}^{\infty}+2 P_{2}^{\infty}$ | 7 |
| $T=Z^{2} \bar{J}_{1} \bar{J}_{2}=0$. | 4 |
| $M_{0 C T}=P_{1}+4 P_{1}^{\infty}+3 P_{2}^{\infty}$ | 8 |

where the total curve $T$ has

1) only three distinct tangents at $P_{1}^{\infty}$, but one of them is double and

1 ) only two distinct tangents at $P_{2}^{\infty}$, but one of them is double.
Table 7.46: First integral and integrating factor of family (D) when $g=0$ and $c \neq 0$.

|  | First integral | Integrating Factor |
| :---: | :---: | :---: |
| General | $I=J_{1}^{0} J_{2}^{\lambda_{2}} E_{3}^{-\frac{\lambda_{2}}{g_{1}}}$ | $R=J_{1}^{0} J_{2}^{\lambda_{2}^{\prime}} E_{3}^{-\frac{1+\lambda_{2}^{\prime}}{c g_{1}}}$ |
| Simple <br> example | $\mathcal{I}=J_{2}^{c} E_{3}^{-1}$ | $\mathcal{R}=\frac{1}{J_{2}}$ |

(ii.3.3) $g=1 / 3$ and $c \neq 0$.

Under this condition, systems defined by the equations (D) do not belong to family (D).
Here we have two invariant lines and two hyperbolas. These systems are Hamiltonian so they admit a polynomial first integral.

Table 7.47: Invariant curves, exponential factors, cofactors, singularities and intersection points of family (D) when $g=1 / 3$ and $c \neq 0$.

| Inv.curves and cofactors | Singularities | Intersection points |
| :---: | :---: | :---: |
| $\begin{aligned} & J_{1}=-c+x \\ & J_{2}=c+x \\ & J_{3}=-3-c y+x y \\ & J_{4}=-3+c y+x y \\ & \alpha_{1}=\frac{c}{3}+\frac{x}{3} \\ & \alpha_{2}=\frac{x}{3}-\frac{c}{3} \\ & \alpha_{3}=\frac{c}{3}-\frac{x}{3} \\ & \alpha_{4}=-\frac{c}{3}-\frac{x}{3} \end{aligned}$ | $\begin{aligned} & P_{1}=\left(-c,-\frac{3}{2 c}\right) \\ & P_{2}=\left(c, \frac{3}{2 c}\right) \\ & P_{1}^{\infty}=[0: 1: 0] \\ & P_{2}^{\infty}=[1: 0: 0] \\ & \left.s, s ;{ }_{2}^{2}\right) E-E, N \end{aligned}$ | $\left.\begin{array}{l} \hline \bar{J}_{1} \cap \bar{J}_{2}=P_{1}^{\infty} \text { simple } \\ \bar{J}_{1} \cap \bar{J}_{3}=P_{1}^{\infty} \text { double } \\ \bar{J}_{1} \cap \bar{J}_{4}=\left\{\begin{array}{l} P_{1}^{\infty} \text { simple } \\ P_{1} \text { simple } \end{array}\right. \\ \bar{J}_{1} \cap \mathcal{L}_{\infty}=P_{1}^{\infty} \text { simple } \end{array}\right\} \begin{aligned} & \bar{J}_{2} \cap \bar{J}_{3}=\left\{\begin{array}{l} P_{1}^{\infty} \text { simple } \\ P_{2} \text { simple } \end{array}\right. \\ & \bar{J}_{2} \cap \bar{J}_{4}=P_{1}^{\infty} \text { double } \\ & \bar{J}_{2} \cap \mathcal{L}_{\infty}=P_{1}^{\infty} \text { simple } \\ & \bar{J}_{3} \cap \bar{J}_{4}=\left\{\begin{array}{l} P_{1}^{\infty} \text { simple } \\ P_{2}^{\infty} \text { triple } \end{array}\right. \\ & \bar{J}_{3} \cap \mathcal{L}_{\infty}=\left\{\begin{array}{l} P_{1}^{\infty} \text { simple } \\ P_{2}^{\infty} \text { simple } \end{array}\right. \\ & \bar{J}_{4} \cap \mathcal{L}_{\infty}=\left\{\begin{array}{l} P_{1}^{\infty} \text { simple } \\ P_{2}^{\infty} \text { simple } \end{array}\right. \end{aligned}$ |

Table 7.48: Divisor and zero-cycles of family (D) when $g=1 / 3$ and $c \neq 0$.

| Divisor and zero-cycles | Degree |
| :--- | :---: |
| ICD $=J_{1}+J_{2}+J_{3}+J_{4}+\mathcal{L}_{\infty}$ | 5 |
| $M_{0 C S}=P_{1}+P_{2}+4 P_{1}^{\infty}+P_{2}^{\infty}$ | 7 |
| $T=Z \bar{J}_{1} \bar{J}_{2} \bar{J}_{3} \bar{J}_{4}=0$. | 7 |
| $M_{0 C T}=2 P_{1}+2 P_{2}+5 P_{1}^{\infty}+3 P_{2}^{\infty}$ | 12 |

where the total curve $T$ has

1) only three distinct tangents at $P_{1}^{\infty}$, but two of them are double;
2) three distinct tangents at $P_{2}^{\infty}$.

Table 7.49: First integral and integrating factor of family (D) when $g=1 / 3$ and $c \neq 0$.

|  | First integral | Integrating Factor |
| :---: | :---: | :---: |
| General | $I=J_{1}^{\lambda_{1}} J_{2}^{\lambda_{2}} J_{3}^{\lambda_{2}} J_{4}^{\lambda_{1}}$ | $R=J_{1}^{\lambda_{1}} J_{2}^{\lambda_{2}^{\prime}} J_{3}^{\lambda_{2}^{\prime}} J_{4}^{\lambda_{1}^{\prime}}$ |
| Simple <br> example | $\mathcal{I}_{1}=J_{1} J_{4}$ | $\mathcal{R}_{1}=\frac{1}{J_{1} J_{4}}$ |

(ii.3.4) $g=1 / 2$ and $c \neq 0$.

Under this condition, systems defined by (D) do not belong to QSH. Here we have two invariant lines. We also could find an exponential factor but it did not arise from multiple curves since by calculating the 1st extactic polynomial we checked that the multiplicity of the affine invariant lines is one. We also checked the multiplicity of the line at infinity and it is also simple.

Table 7.50: Invariant curves, exponential factors, cofactors, singularities and intersection points of family (D) for $g=1 / 2$ and $c \neq 0$.

| Inv.curves/exp.fac. and cofactors | Singularities | Intersection points |
| :---: | :---: | :---: |
| $J_{1}=x$ | $P_{1}=\left(-c,-\frac{2}{c}\right)$ |  |
| $J_{2}=\frac{x}{c}+1$ |  |  |
| $E_{3}=e^{c g_{1} y+g_{0}+g_{1} x y}$ | $P_{1}^{\infty}=[0: 1: 0]$ | $\bar{J}_{1} \cap \bar{J}_{2}=P_{1}^{\infty}$ simple |
|  | $P_{2}^{\infty}=[1: 0: 0]$ | $\bar{J}_{1} \cap \mathcal{L}_{\infty}=P_{1}^{\infty}$ simple |
| $\alpha_{1}=\frac{c}{2}+\frac{x}{2}$ |  | $\bar{J}_{2} \cap \mathcal{L}_{\infty}=P_{1}^{\infty}$ simple |
| $\alpha_{2}=\frac{x}{2}$ | $\left.s ;{ }_{2}^{3}\right) E-P H, N$ |  |
| $\alpha_{3}=c g_{1}+g_{1} x$ |  |  |

Table 7.51: Divisor and zero-cycles of family (D) for $g=1 / 2$ and $c \neq 0$.

| Divisor and zero-cycles | Degree |
| :--- | :---: |
| $I C D=J_{1}+J_{2}+\mathcal{L}_{\infty}$ | 3 |
| $M_{0 C S}=P_{1}+5 P_{1}^{\infty}+P_{2}^{\infty}$ | 7 |
| $T=Z \bar{J}_{1} \bar{J}_{2}=0$. | 3 |
| $M_{0 C T}=P_{1}+3 P_{1}^{\infty}+P_{2}^{\infty}$ | 5 |

where the total curve $T$ has three distinct tangents at $P_{1}^{\infty}$.
Table 7.52: First integral and integrating factor of family (D) when $g=1 / 2$ and $c \neq 0$.

|  | First integral | Integrating Factor |
| :---: | :---: | :---: |
| General | $I=J_{1}^{\lambda_{1}} J_{2}^{0} E_{3}^{-\frac{\lambda_{1}}{28_{1}}}$ | $R=J_{1}^{\lambda_{1}^{\prime}} J_{2}^{0} E_{3}^{-\frac{\lambda_{1}^{\prime}}{2 g_{1}}-\frac{1}{28_{1}}}$ |
| Simple <br> example | $\mathcal{I}=J_{1}^{2} E_{3}^{-1}$ | $\mathcal{R}=\frac{1}{J_{1}}$ |

(ii.3.5) $g=1$ and $c \neq 0$.

Note that this case can be reduced to the case $c=0$ and $g=1$ due to the translation $(x, y) \rightarrow(x-c, y)$. Therefore, on the line $g=1$ and $c \neq 0$ we have the same configuration and the phase portrait as at the point $(g, c)=(1,0)$.

We sum up the topological, dynamical and algebraic geometric features of family (D) and also confront our results with previous results in literature in the following proposition.

## Proposition 7.8.

(a) For the family (D) we obtained three distinct configurations $C_{1}^{(\mathrm{D})}-C_{3}^{(\mathrm{D})}$ of invariant hyperbolas and lines (see Figure 7.3 for the complete bifurcation diagram of configurations of this family). The parameter space for this family is $(g \pm 1)(2 g-1)(3 g-1) \neq 0, c^{2}+g^{2} \neq 0$ and the bifurcation set for the full family is $c g=0$. On $c=0$ and $g \neq 0$ the invariant lines coalesce and two finite singularities coalesced with an infinite singularity. On $g=0$ and $c \neq 0$ the line at infinity has multiplicity two and we have just one invariant line. The bifurcation set of configurations in the full parameter space is $\operatorname{cg}(g \pm 1)(2 g-1)(3 g-1)=0$. Outside the parameter space occurs the following: On $g=1 / 3$ and $c \neq 0$ we have an additional invariant hyperbola. On $g=1 / 3$ and $c=0$ we have one triple line and one triple hyperbola. On $g=-1$ and $c \neq 0$ we have an invariant parabola. On $g=-1$ and $c=0$ we have one quadruple line. On $g=1$ we have a family of invariant hyperbolas and the invariant lines coalesce. On $g=1 / 2$ the invariant hyperbola becomes reducible and we have two invariant lines $c+x=0$ and $x=0$ when $c \neq 0$ and only one triple line $x=0$ when $c=0$. On $c=g=0$ the hyperbola is filled up with singularities.
(b) The family (D) is Darboux integrable in the generic case $\operatorname{cg}(g \pm 1)(3 g-1)(2 g-1) \neq 0$ and also when $c=0$ and $g \neq 0, \pm 1,1 / 3,1 / 2$. The family (D) is generalized Darboux integrable when $g=0$ and $c \neq 0$. All systems in family (D) have an inverse integrating factor which is polynomial.
(c) For the family (D) we have seven topologically distinct phase portraits $P_{1}^{(\mathrm{D})}-P_{7}^{(\mathrm{D})}$. The topological bifurcation diagram of family (D) is done in Figure 7.4. The bifurcation set is $\operatorname{cg}(2 g-1)(g-$ $1)=0$ and it is the bifurcation set of singularities. The phase portrait $P_{7}^{(\mathrm{D})}$ is not topologically equivalent with anyone of the phase portraits in [41].

Proof of Proposition 7.8. (a) We have the following type of divisors and zero-cycles of the total invariant curve $T$ for the configurations of family (D):

Table 7.53: Configurations for family (D).

| Configurations | Divisors and zero-cycles of the total inv. curve $T$ |
| :---: | :---: |
| $C_{1}^{(\mathrm{D})}$ | $I C D=J_{1}+J_{2}+J_{3}+\mathcal{L}_{\infty}$ |
| $\mathrm{M}_{0 C T}=P_{1}+2 P_{2}+4 P_{1}^{\infty}+2 P_{2}^{\infty}$ |  |
| $C_{2}^{(\mathrm{D})}$ | $I C D=3 J_{1}+J_{2}+\mathcal{L}_{\infty}$ |
| $C_{3}^{(\mathrm{D})}$ | $I C D=J_{1}+J_{2}+2 \mathcal{L}_{\infty}$ |
|  | $M_{0 C T}=P_{1}+4 P_{1}^{\infty}+3 P_{2}^{\infty}$ |

Therefore, the configurations $C_{1}^{(\mathrm{D})}$ up to $C_{3}^{(\mathrm{D})}$ are all distinct. For the limit cases of family (D) we have the following configurations:

Table 7.54: Configurations for the limit cases of family (D).

| Configurations | Divisors and zero-cycles of the total inv. curve $T$ |
| :---: | :---: |
| $c_{1}$ | $I C D=J_{1}+J_{2}+J_{3}+J_{4}+\mathcal{L}_{\infty}$ <br> $M_{0 C T}=2 P_{1}+2 P_{2}+5 P_{1}^{\infty}+3 P_{2}^{\infty}$ |
| $c_{2}$ | $I C D=3 J_{1}+3 J_{2}+\mathcal{L}_{\infty}$ <br> $M_{0 C T}=7 P_{1}^{\infty}+4 P_{2}^{\infty}$ |
| $c_{3}$ | $I C D=J_{1}+J_{2}+J_{3}+J_{4}+\mathcal{L}_{\infty}$ <br> $M_{0 C T}=2 P_{1}+3 P_{2}+6 P_{1}^{\infty}+2 P_{2}^{\infty}$ |
| $c_{4}$ | $I C D=4 J_{1}+J_{2}+\mathcal{L}_{\infty}$ <br> $M_{0 C T}=6 P_{1}^{\infty}+2 P_{2}^{\infty}$ |
| $c_{5}$ | $I L D=3 J_{1}+3 \mathcal{L}_{\infty}$ <br> $M_{0 C T}=6 P_{1}^{\infty}+3 P_{2}^{\infty}$ |
| $c_{6}$ | $I C D=J_{1}+J_{2}+\mathcal{L}_{\infty}$ <br> $M_{0 C T}=P_{1}+3 P_{1}^{\infty}+P_{2}^{\infty}$ |
| $c_{7}$ | $I C D=3 J_{1}+3 \mathcal{L}_{\infty}$ <br> $M_{0 C T}=6 P_{1}^{\infty}+3 P_{2}^{\infty}$ |
| $c_{8}$ | $I C D=\mathcal{L}_{\infty}$ <br> $M_{0 C T}=P_{1}^{\infty}$ |

The other statements on (a) follows from the study done previously.
(b) It follows directly from the tables.
(c) We have:

Table 7.55: Phase portraits for family (D).

| Phase Portraits | Sing. at $\infty$ | Finite sing. | Separatrix connections |
| :---: | :---: | :---: | :---: |
| $P_{1}^{(\mathrm{D})}$ | $\left.\binom{2}{2}-E, S\right)$ | $(s, a)$ | $1 S C_{f}^{f} 4 S C_{f}^{\infty} 1 S C_{\infty}^{\infty}$ |
| $P_{2}^{(D)}$ | $\left.\left({ }_{2}^{2}\right) E-E, N\right)$ | $(s, s)$ | $0 S C_{f}^{f} 8 S C_{f}^{\infty} 0 S C_{\infty}^{\infty}$ |
| $P_{3}^{(\mathrm{D})}$ | ( ${ }_{2}^{2}$ ) $P H-P H, N$ ) | (s,a) | $1 S C_{f}^{f} 5 S C_{f}^{\infty} 0 S C_{\infty}^{\infty}$ |
| $P_{4}^{(D)}$ | $\left.\left({ }_{(2)}^{4}\right) E-E, S\right)$ | $\varnothing$ | $0 S C_{f}^{f} 0 S C_{f}^{\infty} \quad 2 S C_{\infty}^{\infty}$ |
| $P_{5}^{(D)}$ | $\left.\left({ }_{2}^{4}\right) P H P-P H P, N\right)$ | $\varnothing$ | $0 S C_{f}^{f} 0 S C_{f}^{\infty} 3 S C_{\infty}^{\infty}$ |
| $P_{6}^{(\mathrm{D})}$ | $\left.{ }_{\left({ }_{2}^{4}\right)}^{4} \mathrm{PH}-\mathrm{HP}, \mathrm{N}\right)$ | $\varnothing$ | $0 S C_{f}^{f} 0 S C_{f}^{\infty} 1 S C_{\infty}^{\infty}$ |
| $P_{7}^{(\mathrm{D})}$ | $\left({ }_{2}^{2}\right) E-E,{ }_{1}^{1}$ ) $S N$ ) | $s$ | $0 S C_{f}^{f} 4 S C_{f}^{\infty} 1 S C_{\infty}^{\infty}$ |

Therefore, we have seven distinct phase portraits for systems (D). For the limit case of family (D) we have the following phase portraits:

Table 7.56: Phase portraits for the limit case of family (D).

| Phase Portraits | Sing. at $\infty$ | Finite sing. | Separatrix connections |
| :---: | :---: | :---: | :---: |
| $P_{1}^{(\mathrm{D})}$ | $\left.\binom{2}{2} E-E, S\right)$ | $(s, a)$ | $1 S C_{f}^{f} 4 S C_{f}^{\infty} 1 S C_{\infty}^{\infty}$ |
| $P_{2}^{(D)}$ | $\left.\left({ }_{2}^{2}\right) E-E, N\right)$ | $(s, s)$ | $0 S C_{f}^{f} 8 S_{f}^{\infty} \quad 0 S C_{\infty}^{\infty}$ |
| $P_{4}^{(D)}$ | $\left.\left({ }_{2}^{4}\right) E-E, S\right)$ | $\varnothing$ | $0 S C_{f}^{f} 0 S C_{f}^{\infty} 2 S C_{\infty}^{\infty}$ |
| $P_{5}^{(\mathrm{D})}$ | $\left.\left.{ }_{(2)}^{4}\right) P H P-P H P, N\right)$ | $\varnothing$ | $0 S C_{f}^{f} 0 S C_{f}^{\infty} 3 S C_{\infty}^{\infty}$ |
| $P_{6}^{(\mathrm{D})}$ | $\left.\left({ }_{2}^{4}\right) P H-H P, N\right)$ | $\varnothing$ | $0 S C_{f}^{f} 0 S C_{f}^{\infty} 1 S C_{\infty}^{\infty}$ |
| $p_{1}$ | $\left.\binom{3}{2} E-P H, N\right)$ | $s$ | $0 S C_{f}^{f} 4 S C_{f}^{\infty} 1 S C_{\infty}^{\infty}$ |
| $p_{2}$ | $(\ominus[)(] ; N, \varnothing)$ | $(\ominus[)(] ; \varnothing)$ | $0 S C_{f}^{f} 0 S C_{f}^{\infty} 0 S C_{\infty}^{\infty}$ |

In [41] the authors do not have any phase portrait with 2 singular points at infinity and with 1 singular point in the finite plane. Therefore, the phase portrait $P_{7}^{(\mathrm{D})}$ is missing.

### 7.2.1 The solution of the Poincaré problem for the family (D)

The following theorem solves the problem of Poincare for the family defined by the equations (D) with $(c, g) \in \mathbb{R}^{2}$.

Theorem 7.9. The necessary and sufficient condition for a system (S) defined by the equations (D) with $(c, g) \in \mathbb{R}^{2}$ to have a rational first integral given by invariant algebraic curves of degree at most two, is that $g \in \mathbb{Q}$ and either $g(2 g-1) \neq 0$ or $g=0=c$.

Proof. The proof of this result is based on all the formulas for the first integrals contained in the Tables calculated for the family of systems defined by the equations (D) with $(c, g) \in \mathbb{R}^{2}$.

Case 1. This is the generic case, $\operatorname{cg}(g \pm 1)(2 g-1)(3 g-1) \neq 0$. We first show the necessity of the condition, so suppose that the system has a rational first integral after fixing the corresponding value for the free parameter $\lambda_{2}$. Then according to Table 7.23 the exponents of $J_{2}$ and $J_{3}$ in this Table need to be integers. This implies that $g /(2 g-1)=r \in \mathbb{Q}$ and hence $g=r /(2 r-1) \in \mathbb{Q}$ (here we have $2 r-1=2 g /(2 g-1)-1=1 /(2 g-1) \neq 0)$. By hypothesis we have that $c g \neq 0$. To prove necessity we need to show that $g(2 g-1) \neq 0$ but this we have by the hypothesis of Case 1 .

To prove the sufficiency we now suppose that $g \in \mathbb{Q}$, that is $g=m / n$ where $m, n \in \mathbb{Z}$ with $n \neq 0$ and $m$ and $n$ are relatively prime. We also suppose that $(c, g)$ satisfies the above generic condition. Then the general first integral is given in Table 7.23. The exponents of $J_{2}$, respectively $J_{3}$ are $\lambda_{2}$ and $-g \lambda_{2} /(2 g-1)$. But $g /(2 g-1)=m /(2 m-n)$ where $2 m-n \neq 0$ since otherwise the fraction $m / n$ would be reducible. So by taking $\lambda_{2}=2 m-n$ we get the first integral $J_{2}^{2 m-n} J_{3}^{-m}$ which is a rational first integral. Hence the condition is also sufficient.


Figure 7.3: Bifurcation diagram of configurations for family (D). The dashed lines $g= \pm 1, g=1 / 2$ and $g=1 / 3$ are limit cases of this family. The multiple invariant curves are emphasized in the drawings of the configurations. When $g=1$ we have a family of invariant hyperbolas that are drawn in colors. The dotted hyperbola on $c=g=0$ represents an invariant hyperbola filled up with singularities.


Figure 7.4: Topological bifurcation diagram for family (D). Note that the phase portraits $P_{1}^{(\mathrm{D})}, P_{2}^{(\mathrm{D})}, P_{4}^{(\mathrm{D})}, P_{7}^{(\mathrm{D})}, p_{1}$ and $p_{2}$ possess graphics (drew in green). The continuous curves in the phase portraits are separatrices. The dashed curves are the orbits given in each region of the phase portraits. The green bullet represents an elemental saddle, the red bullet an elemental unstable node and the blue an elemental stable node. The yellow triangle represents a saddle-node (semielemental) and the black bullet is an intricate singularity. In $c=g=0$ we have an hyperbola filled up with singularities represented by a dotted hyperbola.

Case 2. This is the non-generic case, $\operatorname{cg}(g \pm 1)(2 g-1)(3 g-1)=0$.
Case 2.1. Consider $c=0 \neq g(g \pm 1)(2 g-1)(3 g-1)$. We look for the general expression of first integrals for this case which is $J_{1}^{\lambda_{1}} J_{2}^{-g \lambda_{1} /(2 g-1)}$. Suppose that we have a rational first integral. Then in this integral the exponents $\lambda_{1},-g \lambda_{1} /(2 g-1)$ must be integers. But this implies that $g /(2 g-1)=r \in \mathbb{Q}$. Then $g=r /(2 r-1) \in \mathbb{Q}$. By hypothesis we have that $g(2 g-1) \neq 0$. Hence the necessity is proved. For the sufficiency we suppose that $g \in \mathbb{Q}$. Then again $g=m / n$ with $m, n \in \mathbb{Q}, n \neq 0$ and $m, n$ relatively prime. The general form of the first integral is $J_{1}^{\lambda_{1}} J_{2}^{-g \lambda_{1} /(2 g-1)}$. Replacing here $g$ by $m / n$ we obtain $J_{1}^{\lambda_{1}} J_{2}^{-m \lambda_{1} /(2 m-n)}$. Hence by taking $\lambda_{1}=2 m-n$ the first integral becomes $J_{1}^{2 m-n} J_{2}^{-m}$ which is rational.

Case 2.2. Consider $c=0$ and $g(g \pm 1)(2 g-1)(3 g-1)=0$. This is a set of five points all with a rational coordinate $g$. In this case a necessary and sufficient condition to have a rational first integral is that $g \neq 1 / 2$. Indeed, by checking the Tables for first integrals we see that except for the point $g=1 / 2$ at all four other points, we have a rational first integral. So the condition to have a rational first integral is satisfied in this case too.

Case 2.3. Consider $c \neq 0$ and $g(g \pm 1)(2 g-1)(3 g-1)=0$. This is a collection of five lines out of which we exclude their intersection with $c=0$.
Case 2.3.1. Suppose $c \neq 0$ and $g(2 g-1)=0$ then we see that the general first integrals for either $g=0$ or $g=1 / 2$ must contain an exponential factor and hence they are never rational. Therefore if $c \neq 0$ then to have a rational first integral it is necessary and sufficient to have $g \neq 0,1 / 2$.
Case 2.3.2. Suppose $c \neq 0$ and $(g \pm 1)(3 g-1)=0$.
Case 2.3.2.1. We consider first $c \neq 0$ and $g=-1$. In this case two half lines. For each $c$ the system $(c, g)=(c,-1)$ has a rational first integral as the Table for this case indicates and the condition $g \in \mathbb{Q}$ and $g \neq 0,1 / 2$ is clearly satisfied.
Case 2.3.2.2. The case $c \neq 0$ and $(g-1)(3 g-1)$ is treated in analogous manner as the Case 2.3.2.1.

We note that the set of systems defined by the equations (D) for $(c, g) \in \mathbb{R}^{2}$ that are algebraically integrable is dense in $\mathbb{R}^{2}$.

## 8 Concluding comments and problems

There are many papers on the Darboux theory of integrability for planar polynomial vector fields but it is actually impossible to find a single source summing up this theory with all its extended features, as we know it today. The literature also often overlooks some significant moments in its developments as well as its most useful consequences such as its unifying character in proofs of integrability for whole classes of polynomial differential systems as well as its help in topologically classifying some families of systems. In this paper we covered all these aspects and proved that its complex character is essential.

In this paper the study of integrability of the family $\mathbf{Q S H}_{\eta=0}$ displayed all the types from algebraic to Liouvillian integrability as well as non-integrability, proved for the generic case.

We then pursued the geometric analysis of two of the four 2-parameter sub-families of QSH $_{\eta=0}$ by applying the method of Darboux and obtained all their phase portraits as well
as their bifurcation diagrams and the bifurcation diagrams of the configurations of invariant hyperbolas and lines.

In this Section we are interested in the relationship between these two bifurcation diagrams, more precisely we show how the dynamics of the systems expressed in their topological bifurcations impacts the bifurcations of the geometry of the configurations and the resulting bifurcations in integrability.

### 8.1 Family (C)

The parameter space for this family is $\left\{(a, c) \in \mathbb{R}^{2}: a \neq 0\right\}$, its topological bifurcation set is $\left(a-8 c^{2} / 9\right)\left(a-c^{2}\right)=0$ and it is formed of bifurcation points of finite singularities. On $a-8 c^{2} / 9=0$ we see coalescence of two finite singularities, both situated on the same one of the two invariant lines, yielding a double singular point on this line. On $a-c^{2}=0$ we have two coalesces, but each one of them being a coalescence of two points situated on distinct lines yielding a double singular point situated on a double line.

The bifurcation set for configurations of invariant lines and hyperbolas is also ( $a-8 c^{2} / 9$ ). $\left(a-c^{2}\right)=0$. On $a-8 c^{2} / 9=0$ the coalescence of the two singularities on the same line yielding a double singular point generates a distinct configuration than the one in the generic case surrounding points on this parabola where none of the singularities is double. As already mentioned above, on the parabola $a-c^{2}=0$ we get a double line due to the coalescence of the four singularities located on the two lines into two double singularities on the double line. Can we explain in a similar way the appearance of a double hyperbola on the parabola $a-8 c^{2} / 9=0$ ? Within this family this is however not possible. Indeed moving in all directions from a point on this parabola, we always get just one hyperbola, no two hyperbolas coalesce when on $a-8 c^{2} / 9=0$ in the parameter space of this family.

We claim however that the same kind of phenomenon occurs as on $a-c^{2}=0$, namely that a bifurcation of singularities does occur on $a-8 c^{2} / 9=0$ but when we unfold these systems in a larger family that includes systems with three distinct singular points at infinity and hence for these systems we have $\eta>0$. Looking at the families of systems in the set of systems with $\eta>0$ in [47] we find the configuration denoted by Config. H. 139 (see Figure 8.1) with three singular points at infinity in the real projective plane and with 4 singular points in the affine plane. This configuration has two hyperbolas that coalesce when two of the three singular points at infinity collide and we also have collision of two finite singular points located on distinct hyperbolas. To prove this, consider the systems:

$$
\left\{\begin{array}{l}
\dot{x}=\frac{72 c^{2}(1-\epsilon)(2+\epsilon)}{\left(-9+\epsilon^{2}\right)^{2}}+3 c x+x^{2}+(1+\epsilon) x y  \tag{8.1}\\
\dot{y}=-\frac{9 c^{2}\left(1+\epsilon^{2}\right)}{\left(-9+\epsilon^{2}\right)^{2}}+y^{2},
\end{array}\right.
$$

where $\epsilon$ is sufficiently small. These systems possess the configuration Config. H. 139 (see Figure 8.1) of [47] for any value of $\epsilon>0$ as we can show that it satisfies the required conditions on the polynomial invariants. On the other hand, the systems (8.1) form a perturbation of the system obtained by setting $\epsilon=0$ which has the configuration Config. $\tilde{H} .33$ (see Figure 8.1) of [47] (here family (C) when $a=8 v^{2} / 9$ with configuration $C_{6}^{(C)}$ ).

We conclude that on both parabolas $a-c^{2}=0$ and $a-8 c^{2} / 9=0$ bifurcation of multiple singular points produce bifurcation points of configurations corresponding to multiple invariant curves but this time we have apart from coalescence of finite singularities, also coalescence of two infinite singularities.

In the article [47] the classification of QSH according to the configurations of invariant hyperbolas and lines was done separately for the two subfamilies corresponding to $\eta>0$ and $\eta=0$ leading to two (non-integrated) bifurcation diagrams in terms of invariant polynomials.

As the above example clearly illustrates there is the need of obtaining an integrated bifurcation diagram of QSH. We thus propose the following problem:

Problem: Obtain an integrated bifurcation diagram for the family QSH of the configurations of invariant hyperbolas and lines that systems in QSH have, by finding a common set of invariant polynomials to be applied jointly to both subfamilies $\eta>0$ and $\eta=0$.

Finally, in the full (extended) parameter space we observe that on $a=0$ the hyperbola becomes reducible. For $c \neq 0$ the hyperbola splits into the lines $x=0$ and $c+y=0$. On $a=0=c$, the two lines coincide yielding a double line $x=0$ and in addition the hyperbola splits into the lines $x=0$ and $\mathrm{y}=0$.


Figure 8.1: Config. H. 139 and Config. H. 33 (respectively) from [47]. The left configuration becomes the right one when the hyperbola with infinite points point $[1: 1: 0]$ and $[1: 0: 0]$ is identified with the other hyperbola by moving the point here at $[1: 0: 0]$ to coincide with $[0: 1: 0]$ in $P_{2}(\mathbb{C})$.

### 8.2 The family (D)

The parameter space for this family is $\left\{(c, g) \in \mathbb{R}^{2}:(g \pm 1)(3 g-1)(2 g-1) \neq 0\right.$ and $c^{2}+g^{2} \neq$ $0\}$. The topological bifurcation set for this family is the set $c g=0$, with $c^{2}+g^{2} \neq 0$. On $c=0$ and $g \neq 0$ we have that all the singularities of the systems are at infinity and this occurs nowhere else. Moreover, on $c=0$ and $g \neq 0$ we have that $[0: 1: 0]$ is of multiplicity $\left({ }_{4}^{2}\right)$ while $[1: 0: 1]$ is of multiplicity one. On $g=0$ and $c \neq 0$ the singular point $[1: 0: 1]$ is of multiplicity $\binom{1}{1}$ while for neighbouring parameters this point is of multiplicity 1.

The bifurcation set of the configurations is again $c g=0$, with $c^{2}+g^{2} \neq 0$. On $c=0$ the line $x=0$ is a triple line, except for the value $(c, g)=(0,-1)$ where $x=0$ is a quadruple line. This phenomenon is forced by the topological bifurcation of singularities. Indeed, on this line two of the finite singularities, one on a line and one at the intersection of the hyperbola with the line coalesced with $[0: 1: 0]$ producing the a line of multiplicity at least two. In fact calculation indicates that the multiplicity of $x=0$ is actually 3 for $g \neq 0$. Everywhere else in the parameter space of (D) we either have just one simple invariant line (this occurs on $g=0$ ) or two simple invariant lines. This proves that $g=0$ is a bifurcation line of configurations.

Thus for both families of systems (C) and (D) the bifurcation of configurations is produced by coalescence of singularities either finite or infinite or coalescence of a finite with an infinite singularities.

The following problem was stated in the article [48].
Problem: Generalize the Theorem 4.2 so as to cover more cases than the ones imposed by the hypotheses of this theorem.

The study of the families (C) and (D) give more motivation for solving this problem. For example the systems in the family (D) with $c=0 \neq(g \pm 1)(2 g-1)(3 g-1)$ have the invariant line $J_{1}=x$ and the invariant hyperbola $J_{2}=1 /(2 g-1)+x y$ but these curves do not satisfy the ( $\mathrm{C}-\mathrm{K}$ ) conditions because the line intersects the hyperbola at infinity but we still have the inverse integrating factor $J_{1} J_{2}$. We also have other examples.
Remark 8.1. Finally we observe that if we take in the family of systems with equations (D) $c=0$ and $g=-1$ we obtain exactly the system denoted by $(G)$ in the list of normal forms. The normal form ( F ) is also for just one system. This system coincides with the system in the family (D) when $g=c=-1$. If we take $c=0$ and $g=1$ in the systems defined by equations (D) we obtain exactly (I). Hence in this paper we covered five of the normal forms listed in Proposition 3.3: (C), (D), (F) and (G), (I).

## Acknowledgements

We are grateful to Professor Nicolae Vulpe for suggestions and his help in constructing the perturbation (8.1). We are also thankful to the referees for the corrections and comments they made.
R. Oliveira is partially supported by FAPESP grants "Projeto Temático" 2019/21181-0 and CNPq grant Processo 304766/2019-4. D. Schlomiuk is partially supported by NSERC Grant RN000355. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001 (A. M. Travaglini is partially supported by this grant). C. Valls is partially supported by FCT/Portugal through UID/MAT/04459/2019.

## References

[1] V. Arnold, Mathematical methods of classical mechanics, Graduate Texts in Mathematics, Vol. 60, Springer-Verlag, New York-Heidelberg, 1978. https://doi.org/10.1007/978-1-4757-2063-1; MR0690288.
[2] J. C. Artés, B. Grunbaum, J. Llibre, On the number of invariant straight lines for polynomial differential systems, Pacific J. Math. 184(1998), 317-327. https ://doi .org/10.2140/ pjm. 1998.184.207; MR1628583;
[3] J. C. Artés, J. Llibre, D. Schlomiuk, N. Vulpe Geometric configurations of singularities of planar polynomial differential systems. A global classification in the quadratic case, Birkhäuser, 2021. https://doi.org/10.1007/978-3-030-50570-7
[4] J. C. Artés, J. Llibre, D. Schlomiuk, N. Vulpe, Global topological configurations of singularities for the whole family of quadratic differential systems, Qual. Theory Dyn. Syst. 19(2020), No. 51, 1-32. https://doi.org/10.1007/s12346-020-00372-7; MR4067332
[5] L. Cairó, M. R. Feix, J. Llibre, Integrability and algebraic solutions for planar polynomial differential systems with emphasis on the quadratic systems, Resenhas 4(1999), No. 2, 127-161. MR1751553
[6] L. Cairó, H. Giacomini, J. Llibre, Liouvillian first integrals for the planar Lotka-Volterra system, Rend. Circ. Mat. Palermo (2) 52(2003), No. 3, 389-418. https://doi .org/10.1007/ BF02872763; MR2029552
[7] M. M. Carnicer, The Poincaré problem in the nondicritical case, Ann. of Math. (2) 140(1994), No. 2, 289-294. https://doi.org/10.2307/2118601; MR1298714
[8] S. Chandrasekhar, An introduction to the study of stellar structure, University of Chicago press, 1939. MR0092663
[9] J. Chavarriga, H. Giacomini, J. Giné, J. Llibre Darboux integrability and the inverse integrating factor, J. Differential Equations 194(2003), 116-139. https://doi .org/10.1016/ S0022-0396(03)00190-6; MR2001031
[10] J. Chavarriga, J. Llibre, J. Sotomayor, Algebraic solutions for polynomial systems with emphasis in the quadratic case, Exposition. Math. 15(1997), 161-173. MR1458763
[11] C. J. Christopher Invariant algebraic curves and conditions for a centre, Proc. Roy. Soc. Edinburgh Sect. A 124(1994), 1209-1229. https://doi.org/10.1017/S0308210500030213; MR1313199
[12] C. J. Christopher, Liouvillian first integrals of second order polynomials differential equations, Electron. J. Differential Equations 1999, No. 49, 1-7. MR1729833
[13] C. Christopher, R. E. Kooij, Algebraic invariant curves and the integrability of polynomial systems, Appl. Math. Lett. 6(1993), No. 4, 51-53. https://doi.org/10.1016/0893-9659(93)90123-5; MR1348232
[14] C. Christopher, J. Llibre, Algebraic aspects of integrability for polynomial systems. Qual. Theory Dyn. Syst. 1(1999), 71-95. https://doi.org/10.1007/BF02969405; MR1747198
[15] C. Christopher, J. Llibre, Integrability via invariant algebraic curves for planar polynomial differential systems, Ann. Differential Equations 14(2000), 5-19. MR1768817
[16] C. Christopher, J. Llibre, C. Pantazi, S. Walcher, On planar polynomial vector fields with elementary first integrals, J. Differential Equations 267(2019), 4572-4588. https:// doi.org/10.1016/j.jde.2019.05.007; MR3983046
[17] C. Christopher, J. Llibre, C. Pantazi, X. Zhang, Darboux integrability and invariant algebraic curves for planar polynomial systems, J. Phys. A 35(2002), 2457-2476. https: //doi.org/10.1088/0305-4470/35/10/310; MR1909404
[18] C. Christopher, J. Llibre, J. V. Pereira, Multiplicity of invariant algebraic curves in polynomial vector fields, Pacific J. Math. 229(2007), No. 1, 63-117. https://doi.org/10. 2140/pjm. 2007.229.63; MR2276503.
[19] B. Coll, A. Ferragut, J. Llibre, Phase portraits of the quadratic systems with a polynomial inverse integrating factor, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 19(2008), No. 3, 765-783. https://doi.org/10.1142/S0218127409023299; MR2533481
[20] G. Darboux, Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré, Bull. Sci. Math. 2(1878), 60-96; 123-144; 151-200. http://www . numdam. org/item?id=BSMA_1878_2_2_1_60_1
[21] V. A. Dobrovol'skiĬ, Zh. Strel'tsyn, N. V. Lokot', Mikhail Nikolaevich Lagutinskiĭ (1871-1915), Istor.-Mat. Issled. (2) 6(41)(2001), 111-127. MR1931090
[22] V. A. Dobrovol'skĭ̆, N. V. Lokot', J. Strelcyn, Mikhail Nikolaevich Lagutinskii (1871-1915): un mathématicien méconnu, Historia Math. 25(1998), No. 3, 245-264. https: //doi.org/10.1006/hmat.1998.2194; MR1649949
[23] M. H. Dulac, Recherches sur les points singuliers des équations différentielles, J. de l'Éc. Pol. (2), 9(1904), 5-125. Zbl 35.0331.02
[24] M. H. Dulac, Détermination et integration d'une certaine classe d'équations différentielles ayant pour un point singulier un centre, Bull. Sci. Math. 32(1908), No. 1, 230-252.
[25] F. Dumortier, J. Llibre, J. C. Artés, Qualitative theory of planar differential systems, Universitext, Springer-Verlag, Berlin, 2006. https://doi.org/10.1007/978-3-540-32902-2; MR2256001
[26] F. Dumortier, R. Roussarie, C. Rousseau Hilbert's 16th problem for quadratic vector fields, J. Differential Equations 110(1994), 86-133. https://doi.org/10.1006/jdeq. 1994. 1061; MR1275749
[27] A. Ferragut, A. Gasull, Seeking Darboux polynomials, Acta Appl. Math. 139(2015), 167-186. https://doi.org/10.1007/s10440-014-9974-0; MR3400587
[28] A. Ferragut, J. Llibre, On the remarkable values of the rational first integrals of polynomial vector fields, J. Differential Equations 241(2007), 399-417. https://doi. org/10.1016/ j.jde.2007.05.002; MR2358899.
[29] W. Fulton, Algebraic curves. An introduction to algebraic geometry, Author's version, 2008. http://xahlee.info/math/algebraic_geometry_99109.html; MR1042981
[30] J. Giné, J. Llibre, A note on Liouvillian integrability, J. Math. Anal. Appl. 387(2012), 1044-1049. https://doi.org/10.1016/j.jmaa.2011.10.009; MR2853194
[31] A. Goriely, Integrability and nonintegrability of dynamical systems, Advanced Series in Nonlinear Dynamics, Vol. 19, World Scientific Publishing Co., Inc., River Edge, NJ, 2001. https://doi.org/10.1142/9789812811943; MR1857742
[32] D. Hilbert Mathematical problems, reprinted from Bull. Amer. Mat. Soc. 8(1902), 437479, in Bull. Amer. Math. 37(2000), 407-436. https://doi.org/10.1090/S0273-0979-00-00881-8; MR1779412
[33] Y. Ilyashenko, S. Yakovenko, Lectures on analytic differential equations, Graduate Studies in Mathematics, Vol. 86, American Mathematical Society, Providence, RI, 2008. https: //doi.org/10.1090/gsm/086; MR2363178
[34] J. P. Jouanolou, Équations de Pfaff algébriques, Lectures Notes in Mathematics, Vol. 708, Springer, Berlin, 1979. MR537038
[35] W. Kapteyn, On the midpoints of integral curves of differential equations of the first order and the first degree, Nederl. Acad. Wetensch. Verslangen, Afd. Natuurkunde Koninkl. Nederland 19(1911), No. 2, 1446-1457.
[36] W. Kapteyn, New investigations on the midpoints of integrals of differential equations of the first order and the first degree, Nederl. Acad. Wetensch. Verslangen, Afd. Natuurkunde 20(1912), No. 2, 1354-1365.
[37] K. Knopp, Theory of functions. Parts I and II, Two volumes bound as one, New York: Dover, Part I pp. 103 and Part II 93-146, 1996. MR0012656, MR0019722
[38] D. V. Koz'ma, A. S. Shubé, Center conditions of a cubic system with four integral lines, Izv. Akad. Nauk Respub. Moldova Mat. 95(1992), No. 3, 62-67. MR1210689.
[39] J. Llibre, Integrability of polynomial differential systems, in: Handbook of differential equations: ordinary differential equations, Elsevier/North-Holland, Amsterdam, 2004, pp. 437-532. https://doi.org/10.1016/S1874-5725(00)80007-3; MR2166493
[40] J. Llibre, D. Schlomiuk, The geometry of quadratic differential systems with a weak focus of third order, Canad J. Math. 56(2004), No. 2, 310-343. https://doi. org/10.4153/ CJM-2004-015-2; MR2040918;
[41] J. Llibre, J. Yu, Global phase portraits for quadratic systems with a hyperbola and a straight line as invariant algebraic curves, Electron. J. Differential Equations 2018, No. 141 , 1-19. MR3831887
[42] J. Llibre, X. Zhang, Darboux theory of integrability in $\mathbb{C}^{n}$ taking into account the multiplicity, J. Differential Equations 246(2009), No. 2, 541-551. https://doi.org/10.1016/j . jde.2008.07.020; MR2468727
[43] J. Llibre, X. Zhang, Darboux theory of integrability in $\mathbb{R}^{n}$ taking into account the multiplicity at infinity. J. Differential Equations 133(2009), 765-778. https://doi.org/10.1016/j. bulsci.2009.06.002; MR2557407;
[44] J. Llibre, X. Zhang, On the Darboux integrability of polynomial differential systems, Qual. Theory Dyn. Syst. 11(2012), 129-144. https://doi.org/10.1007/s12346-011-0053x; MR2902728;
[45] A. J. Lotka, Analytical note on certain rhythmic relations in organic systems, Proc. Natl. Acad. Sci. U.S.A. 6(1920), 410-415. https://doi.org/10.1073/pnas.6.7.410
[46] Y. K. Man, M. A. H. MacCallum, A rational approach to the Prelle-Singer algorithm, J. Symbolic Comput. 24(1997), 31-43. https://doi.org/10.1006/jsco.1997.0111; MR1459668
[47] R. D. S. Oliveira, A. C. Rezende, D. Schlomiuk, N. Vulpe, Geometric and algebraic classification of quadratic differential systems with invariant hyperbolas, Electron. J. Differential Equations 2017, No. 295, 1-112. MR3748013.
[48] R. D. S. Oliveira, D. Schlomiuk, A. M. Travaglini, Geometry and integrability of quadratic systems with invariant hyperbolas, Electron. J. Qual. Theory Differ. Equ. 2021, No. 6, 1-56. https://doi.org/10.14232/ejqtde.2021.1.6; MR4204947
[49] J. V. Pereira, Vector fields, invariant varieties and linear systems, Ann. Inst. Fourier (Grenoble) 51(2001), No. 5, 1385-1405. MR1860669
[50] H. Poincaré, Mémoire sur les courbes définies par les équations différentialles, J. Math. Pures Appl. (4) 1(1885), 165-244; Ouvres de Henri Poincaré 1, Gauthier-Villard, Paris, 1951, 95-114. Zbl 17.0680.01
[51] H. Poincaré, Sur l'intégration algébrique des équations différentielles, C. R. Acad. Sci. Paris 112(1891), 761-764. Zbl 23.0319.01
[52] H. Poincaré, Sur l'intégration algébrique des équations différentielles du premier ordre et du premier degré, Rend. Circ. Mat. Palermo 5(1891), 169-191. Zbl 23.0319.01
[53] M. J. Prelle, M. F. Singer, Elementary first integrals of differential equations, Trans. Amer. Math. Soc. 279(1983), 215-229. https://doi.org/10.2307/1999380; MR704611
[54] M. Rosenlicht, On Liouville's theory of elementary functions, Pacific J. Math. 65(1976), 485-492. https://doi.org/10.2140/pjm.1976.65.485; MR0447199
[55] J. R. Roth, Periodic small-amplitude solutions to Volterra's problem of two conflicting population and their application to the plasma continuity equations, J. Math. Phys. 10(1969), 1-43. Zbl 0176.38902;
[56] C. Rousseau, D. Schlomiuk, P. Thibaudeau, The centres in the reduced Kukles system, Nonlinearity 8(1995), No. 4, 541-569. https://doi.org/10.1088/0951-7715/8/4/005; MR1342503
[57] C. Rousseau, D. Schlomiuk, Cubic vector fields symmetric with respect to a center, J. Differential Equations 123(1995), No. 2, 388-436. https://doi.org/10.1006/jdeq. 1995. 1168; MR1362881
[58] D. Schlomiuk, J. Guckenheimer, R. Rand, Integrability of plane quadratic vector fields, Exposition. Math. 8(1990), 3-25. MR1042200
[59] D. Schlomiuk, Elementary first integrals and algebraic invariant curves of differential equations, Exposition. Math. 11(1993), 433-454. MR1249165; Zbl 0791.34004
[60] D. Schlomiuk, Algebraic and geometric aspects of the theory of polynomial vector fields, in: Bifurcations and periodic orbits of vector fields (Montreal, PQ, 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Vol. 408, Kluwer, Dordrecht, 1993, pp. 429-467. MR1258526; Zbl 0790.34031
[61] D. Schlomiuk, Algebraic particular integrals, integrability and the problem of the center, Trans. Amer. Math. Soc. 338(1993), No. 2, 799-841. https://doi.org/10.2307/2154430; MR1106193
[62] D. Schlomiuk, Basic algebro-geometric concepts in the study of planar polynomial vector fields, Publ. Mat. 41(1997), 269-295. https://doi.org/10.5565/PUBLMAT_41197_16; MR1461655
[63] D. Schlomiuk, Topological and polynomial invariants, moduli spaces, in classification problems of polynomial vector fields, Publ. Mat. 58(2014), suppl., 461-496. https: //doi. org/10.5565/PUBLMAT_EXTRA14_23; MR3211846
[64] D. Schlomiuk, N. Vulpe, Planar quadratic vector fields with invariant lines of total multiplicity at least five, Qual. Theory Dyn. Syst. 5(2004), 135-194. https ://doi. org/10.1007/ BF02968134; MR2197428
[65] D. Schlomiuk, N. Vulpe, Integrals and phase portraits of planar quadratic systems with invariant lines of at least five total multiplicity, Rocky Mountain J. Math. 38(2008), No. 6, 2015-2075. https://doi.org/10.1216/RMJ-2008-38-6-2015; MR2467367
[66] D. Schlomiuk, N. Vulpe, Integrals and phase portraits of planar quadratic systems with invariant lines of total multiplicity four, Bul. Acad. Sțiințe Repub. Mold. Mat. 1(2008), No. 56, 27-83. MR2392678; Zbl 1159.34329
[67] D. Schlomiuk, N. Vulpe, The full study of planar quadratic differential systems possessing a line of singularities at infinity, J. Dynam. Differential Equations 20(2008), 737-775. https://doi.org/10.1007/s10884-008-9117-2; MR2448210; Zbl 1168.34024
[68] D. Schlomiuk, N. Vulpe, Planar quadratic differential systems with invariant straight lines of the total multiplicity four, Nonlinear Anal. 68(2008), 681-715. https://doi.org/ 10.1016/j.na.2006.11.028; MR10.1016/j.na.2006.11.028; Zbl 1136.34037
[69] D. Schlomiuk, N. Vulpe, Global classification of the planar Lotka-Volterra differential systems according to their configurations of invariant straight lines, J. Fixed Point Theory Appl. 8(2010), No. 1, 177-245. https://doi.org/10.1007/s11784-010-0031-y; MR2735491; Zbl 1205.34073
[70] D. Schlomiuk, N. Vulpe, Global topological classification of Lotka-Volterra quadratic differential systems, Electron. J. Differential Equations 2012, No. 64, 1-69. MR2927799; Zbl 1256.34036
[71] D. Schlomiuk, X. Zhang, Quadratic differential systems with complex conjugate invariant lines meeting at a finite point, J. Differential Equations 265(2018), No. 8, 3650-3684. https://doi.org/10.1016/j.jde.2018.05.014; MR3823981; Zbl 1393.37023
[72] M. F. Singer, Liouvillian first integrals of differential equations, Trans. Amer. Math. Soc. 333(1992), 673-688. https://doi.org/10.1090/S0002-9947-1992-1062869-X; MR1062869
[73] V. Volterra, Leçons sur la théorie mathématique de la lutte pour la vie, Gauthier Villars, Paris, 1931. MR1189803; Zbl 0002.04202
[74] E. W. Weisstein, Euler's homogeneous function theorem. From: MathWorld - A Wolfram Web Resource, 2011, http://mathworld.wolfram.com/EulersHomogeneousFunctionTheorem. html
[75] X. Zhang, Integrability of dynamical systems: algebra and analysis, Developments in Mathematics, Vol. 47, Springer, Singapore, 2017. https://doi .org/10.1007/978-981-10-42263; MR3642375
[76] H. ŻoŁadek, Quadratic systems with center and their perturbations, J. Differential Equations 109(1994), No. 2, 223-273. https://doi.org/10.1006/jdeq.1994.1049; MR1273302; Zbl 0797.34044

# New oscillation criteria for third-order differential equations with bounded and unbounded neutral coefficients 

Ercan Tunç ${ }^{1}$, Serpil Şahin ${ }^{2}$, John R. Graef ${ }^{\boxtimes 3}$ and Sandra Pinelas ${ }^{4}$<br>${ }^{1}$ Department of Mathematics, Faculty of Arts and Sciences, Tokat Gaziosmanpasa University, 60240, Tokat, Turkey<br>${ }^{2}$ Department of Mathematics, Faculty of Arts and Sciences, Amasya University, 05000, Amasya, Turkey<br>${ }^{3}$ Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA<br>${ }^{4}$ RUDN 6 Miklukho-Maklaya St, Moscow, 117198, Russian Federation

Received 23 April 2021, appeared 10 July 2021
Communicated by Zuzana Došlá


#### Abstract

This paper examines the oscillatory behavior of solutions to a class of thirdorder differential equations with bounded and unbounded neutral coefficients. Sufficient conditions for all solutions to be oscillatory are given. Some examples are considered to illustrate the main results and suggestions for future research are also included.


Keywords: oscillation, third-order, neutral differential equation.
2020 Mathematics Subject Classification: 34C10, 34K11, 34K40.

## 1 Introduction

In this paper, we wish to obtain some new criteria for the oscillation of all solutions of the third-order differential equations with bounded and unbounded neutral coefficients of the form

$$
\begin{equation*}
(x(t)+p(t) x(\tau(t)))^{\prime \prime \prime}+q(t) x^{\beta}(\sigma(t))=0, \tag{1.1}
\end{equation*}
$$

where $t \geq t_{0}>0$, and $\beta$ is the ratio of odd positive integers with $0<\beta \leq 1$. Throughout the paper, we will always assume that:
(C1) $p, q:\left[t_{0}, \infty\right) \rightarrow \mathbb{R}$ are continuous functions with $p(t) \geq 1, p(t) \not \equiv 1$ for large $t, q(t) \geq 0$, and $q(t)$ not identically zero for large $t$;
(C2) $\tau, \sigma:\left[t_{0}, \infty\right) \rightarrow \mathbb{R}$ are continuous functions such that $\tau(t) \leq t, \tau$ is strictly increasing, $\sigma$ is nondecreasing, and $\lim _{t \rightarrow \infty} \tau(t)=\lim _{t \rightarrow \infty} \sigma(t)=\infty ;$

[^10](C3) there exist a constant $\theta \in(0,1)$ and $t_{\theta} \geq t_{0}$ such that
\[

$$
\begin{equation*}
\left(\frac{t}{\tau(t)}\right)^{2 / \theta} \frac{1}{p(t)} \leq 1, \quad t \geq t_{\theta} . \tag{1.2}
\end{equation*}
$$

\]

By a solution of equation (1.1), we mean a function $x \in C\left(\left[t_{x}, \infty\right), \mathbb{R}\right)$ for some $t_{x} \geq t_{0}$ such that $x(t)+p(t) x(\tau(t)) \in C^{3}\left(\left[t_{x}, \infty\right), \mathbb{R}\right)$ and $x$ satisfies (1.1) on $\left[t_{x}, \infty\right)$. We only consider those solutions of (1.1) that exist on some half-line $\left[t_{x}, \infty\right)$ and satisfy the condition

$$
\sup \left\{|x(t)|: T_{1} \leq t<\infty\right\}>0 \text { for any } T_{1} \geq t_{x}
$$

we tacitly assume that (1.1) possesses such solutions. Such a solution $x(t)$ of equation (1.1) is said to be oscillatory if it has arbitrarily large zeros, and it is called nonoscillatory otherwise. Equation (1.1) is termed oscillatory if all its solutions are oscillatory.

Neutral differential equations are differential equations in which the highest order derivative of the unknown function appears both with and without deviating arguments. As stated in many sources, besides their theoretical interest, equations of this type have numerous applications in the natural sciences and technology. For example, they appear in networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar, and as the Euler equation in some variational problems; we refer the reader to the monograph by Hale [14] for these and other applications.

Oscillatory and asymptotic behavior of solutions to various classes of third and higher odd-order neutral differential equations have been attracting attention of researchers during the last few decades, and we mention the papers $[1,3-13,15,18-26]$ and the references cited therein for examples of some recent contributions in this area. However, except for the papers [3,4,12,23,26], all the above cited papers were concerned with the case where $p(t)$ is bounded, i.e., the cases where $0 \leq p(t) \leq p_{0}<1,-1<p_{0} \leq p(t) \leq 0$, and $0 \leq p(t) \leq p_{0}<\infty$ were considered, and so the results established in these papers cannot be applied to the case $p(t) \rightarrow \infty$ as $t \rightarrow \infty$. Based on this observation, the aim of this paper is to establish some new oscillation criteria that can be applied not only to the case where $p(t) \rightarrow \infty$ as $t \rightarrow \infty$ but also to the case where $p(t)$ is a bounded function. We would like to point out that the results established here are motivated by oscillation results of Koplatadze et all. [17], where a $n$th order linear differential equation with a deviating argument was considered. Since our equation considered here is fairly simple, it would be possible to extend our results to the more general equations studied in the papers cited above and to the others types that include equation (1.1) as a special case. For these reasons, it is our hope that the present paper will stimulate additional interest in research on third and higher odd-order neutral differential equations in general, and those with unbounded neutral coefficients in particular.

In the sequel, all functional inequalities are supposed to hold for all $t$ large enough. Without loss of generality, we deal only with positive solutions of (1.1); since if $x(t)$ is a solution of (1.1), then $-x(t)$ is also a solution.

## 2 Main results

For the reader's convenience, we define:

$$
z(t):=x(t)+p(t) x(\tau(t)),
$$

$$
\begin{aligned}
h(t) & :=\tau^{-1}(\sigma(t)), \quad g(t):=\tau^{-1}(\eta(t)), \quad \eta \in C^{1}\left(\left[t_{0}, \infty\right)\right), \\
\pi_{1}(t) & :=\frac{1}{p\left(\tau^{-1}(t)\right)}\left[1-\left(\frac{\tau^{-1}\left(\tau^{-1}(t)\right)}{\tau^{-1}(t)}\right)^{2 / \theta} \frac{1}{p\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}\right]
\end{aligned}
$$

and

$$
\pi_{2}(t):=\frac{1}{p\left(\tau^{-1}(t)\right)}\left[1-\frac{1}{p\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}\right],
$$

where $\tau^{-1}$ is the inverse function of $\tau$ (if $\tau$ is invertible) and $\theta \in(0,1)$. It is also important to notice that condition (1.2) in (C3) ensures the nonnegativity of the functions $\pi_{1}(t)$.
Lemma 2.1 (See [2, Lemma 1]). Suppose that the function $h$ satisfies $h^{(i)}(t)>0, i=0,1,2, \ldots, m$, and $h^{(m+1)}(t) \leq 0$ on $[T, \infty)$ and $h^{(m+1)}(t)$ is not identically zero on any interval of the form $\left[T^{\prime}, \infty\right)$, $T^{\prime} \geq T$. Then for every $\theta \in(0,1)$,

$$
\frac{h(t)}{h^{\prime}(t)} \geq \theta \frac{t}{m^{\prime}}
$$

eventually.
Lemma 2.2. Assume that $x$ is an eventually positive solution of (1.1), say for $t_{1} \geq t_{0}$. Then there exists a $t_{2} \geq t_{1}$ such that the corresponding function $z$ satisfies one of the following two cases:
(I) $z(t)>0, z^{\prime}(t)>0, z^{\prime \prime}(t)>0, z^{\prime \prime \prime}(t) \leq 0$,
(II) $z(t)>0, z^{\prime}(t)<0, z^{\prime \prime}(t)>0, z^{\prime \prime \prime}(t) \leq 0$
for $t \geq t_{2}$.
Proof. This result follows immediately from Kiguradze's lemma [16], so we omit its proof.
Lemma 2.3. Let $x(t)$ be an eventually positive solution of (1.1) with $z(t)$ satisfying case (I) of Lemma 2.2 for $t \geq t_{2}$ for some $t_{2} \geq t_{1}$. Then for every $\theta \in(0,1)$ there exists a $t_{\theta} \geq t_{2}$ such that

$$
\begin{equation*}
\left(\frac{z(t)}{t^{2 / \theta}}\right)^{\prime} \leq 0 \text { for } t \geq t_{\theta} . \tag{2.1}
\end{equation*}
$$

Proof. Since $z$ satisfies case (I) of Lemma 2.2 for $t \geq t_{2}$ for some $t_{2} \geq t_{1}$, by Lemma 2.1, there exists a $t_{\theta} \geq t_{2}$ for every $\theta \in(0,1)$ such that

$$
\begin{equation*}
z(t) \geq \frac{\theta}{2} t z^{\prime}(t) \text { for } t \geq t_{\theta} . \tag{2.2}
\end{equation*}
$$

It follows from (2.2) that

$$
\left(\frac{z(t)}{t^{2 / \theta}}\right)^{\prime}=\frac{\theta t z^{\prime}(t)-2 z(t)}{\theta t^{2 / \theta+1}} \leq 0 \text { for } t \geq t_{\theta} .
$$

This completes the proof of the lemma.
Lemma 2.4. Let $x(t)$ be an eventually positive solution of (1.1) with $z(t)$ satisfying case (I) of Lemma 2.2. Assume that

$$
\begin{equation*}
\int_{t_{0}}^{\infty} \int_{u}^{\infty} q(s) \pi_{1}^{\beta}(\sigma(s)) h^{\beta}(s) d s d u=\infty . \tag{2.3}
\end{equation*}
$$

Then:
(i) $z$ satisfies the inequality

$$
\begin{equation*}
z^{\prime \prime \prime}(t)+q(t) \pi_{1}^{\beta}(\sigma(t)) z^{\beta}(h(t)) \leq 0 \tag{2.4}
\end{equation*}
$$

for large t;
(ii) $z^{\prime}(t) \rightarrow \infty$ as $t \rightarrow \infty$;
(iii) $z(t) / t$ is increasing.

Proof. Let $x(t)$ be an eventually positive solution of (1.1) such that $x(t)>0, x(\tau(t))>0$, and $x(\sigma(t))>0$ for $t \geq t_{1}$ for some $t_{1} \geq t_{0}$. From the definition of $z$, we have

$$
\begin{align*}
x(t) & =\frac{1}{p\left(\tau^{-1}(t)\right)}\left[z\left(\tau^{-1}(t)\right)-x\left(\tau^{-1}(t)\right)\right] \\
& \geq \frac{z\left(\tau^{-1}(t)\right)}{p\left(\tau^{-1}(t)\right)}-\frac{1}{p\left(\tau^{-1}(t)\right) p\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)} z\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right) . \tag{2.5}
\end{align*}
$$

Now $\tau(t) \leq t$ and $\tau$ is strictly increasing, so $\tau^{-1}$ is increasing and $t \leq \tau^{-1}(t)$. Thus,

$$
\begin{equation*}
\tau^{-1}(t) \leq \tau^{-1}\left(\tau^{-1}(t)\right) \tag{2.6}
\end{equation*}
$$

Since $z(t)$ satisfies case (I) for $t \geq t_{2}$, by Lemma 2.3, there exists a $t_{\theta} \geq t_{2}$ such that (2.1) holds for $t \geq t_{\theta}$. From (2.1) and (2.6), we observe that

$$
\begin{equation*}
z\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right) \leq \frac{\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)^{2 / \theta} z\left(\tau^{-1}(t)\right)}{\left(\tau^{-1}(t)\right)^{2 / \theta}} \tag{2.7}
\end{equation*}
$$

Using (2.7) in (2.5) yields

$$
\begin{equation*}
x(t) \geq \pi_{1}(t) z\left(\tau^{-1}(t)\right) \quad \text { for } t \geq t_{\theta} . \tag{2.8}
\end{equation*}
$$

Since $\lim _{t \rightarrow \infty} \sigma(t)=\infty$, we can choose $t_{3} \geq t_{\theta}$ such that $\sigma(t) \geq t_{\theta}$ for all $t \geq t_{3}$. Thus, it follows from (2.8) that

$$
\begin{equation*}
x(\sigma(t)) \geq \pi_{1}(\sigma(t)) z\left(\tau^{-1}(\sigma(t))\right) \quad \text { for } t \geq t_{3} . \tag{2.9}
\end{equation*}
$$

Using (2.9) in (1.1) gives

$$
\begin{equation*}
z^{\prime \prime \prime}(t)+q(t) \pi_{1}^{\beta}(\sigma(t)) z^{\beta}(h(t)) \leq 0 \text { for } t \geq t_{3} \tag{2.10}
\end{equation*}
$$

i.e., (2.4) holds.

Next, we claim that condition (2.3) implies $z^{\prime}(t) \rightarrow \infty$ as $t \rightarrow \infty$. If this is not the case, then there exists a constant $k>0$ such that $\lim _{t \rightarrow \infty} z^{\prime}(t)=k$, and so $z^{\prime}(t) \leq k$. Since $z^{\prime}(t)$ is positive and increasing on $\left[t_{2}, \infty\right)$, there exist a $t_{3} \geq t_{2}$ and a constant $c>0$ such that

$$
z^{\prime}(t) \geq c \quad \text { for } t \geq t_{3}
$$

which implies

$$
z(t) \geq d t
$$

for $t \geq t_{4}$, for some $t_{4} \geq t_{3}$ and some $d>0$. Since $\lim _{t \rightarrow \infty} h(t)=\infty$, we can choose $t_{5} \geq t_{4}$ such that $h(t) \geq t_{4}$ for all $t \geq t_{5}$, so

$$
z(h(t)) \geq d h(t) .
$$

Using this in (2.10) gives

$$
z^{\prime \prime \prime}(t)+d^{\beta} q(t) \pi_{1}^{\beta}(\sigma(t)) h^{\beta}(t) \leq 0 \text { for } t \geq t_{5} .
$$

Integrating this inequality from $t$ to $\infty$, we obtain

$$
z^{\prime \prime}(t) \geq d^{\beta} \int_{t}^{\infty} q(s) \pi_{1}^{\beta}(\sigma(s)) h^{\beta}(s) d s .
$$

Now integrating from $t_{5}$ to $t$ yields

$$
k \geq z^{\prime}(t) \geq d^{\beta} \int_{t_{5}}^{t} \int_{u}^{\infty} q(s) \pi_{1}^{\beta}(\sigma(s)) h^{\beta}(s) d s d u
$$

which contradicts (2.3) and proves the claim.
Finally, from the fact that $z^{\prime}(t) \rightarrow \infty$ as $t \rightarrow \infty$, we see that

$$
z(t)=z\left(t_{2}\right)+\int_{t_{2}}^{t} z^{\prime}(s) d s \leq z\left(t_{2}\right)+\left(t-t_{2}\right) z^{\prime}(t) \leq t z^{\prime}(t)
$$

which implies

$$
\left(\frac{z(t)}{t}\right)^{\prime}=\frac{t z^{\prime}(t)-z(t)}{t^{2}} \geq 0
$$

i.e., (iii) holds. The proof of the lemma is now complete.

Lemma 2.5. Let $x(t)$ be an eventually positive solution of (1.1) with $z(t)$ satisfying case (I) of Lemma 2.2. If

$$
\begin{equation*}
\int_{t_{0}}^{\infty} q(s) \pi_{1}^{\beta}(\sigma(s)) h^{2 \beta / \theta}(s) d s=\infty, \tag{2.11}
\end{equation*}
$$

then

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{z(t)}{t^{2 / \theta}}=0 \tag{2.12}
\end{equation*}
$$

Proof. Since $z(t)$ satisfies case (I) for $t \geq t_{2}$ for some $t_{2} \geq t_{1}$, by Lemma 2.3, there exists a $t_{\theta} \geq t_{2}$ such that (2.1) holds for $t \geq t_{\theta}$, i.e., $z(t) / t^{2 / \theta}$ is decreasing for $t \geq t_{\theta}$. We now claim that (2.11) implies

$$
\lim _{t \rightarrow \infty} \frac{z(t)}{t^{2 / \theta}}=0 .
$$

If this is not the case, then there exist a constant $b>0$ and a $t_{3} \geq t_{\theta}$ such that

$$
\begin{equation*}
z(t) \geq b t^{2 / \theta} \quad \text { for } t \geq t_{3} . \tag{2.13}
\end{equation*}
$$

Since case (I) holds, we again arrive at (2.10) for $t \geq t_{3}$. Using (2.13) in (2.10) gives

$$
\begin{equation*}
z^{\prime \prime \prime}(t)+b^{\beta} q(t) \pi_{1}^{\beta}(\sigma(t)) h^{2 \beta / \theta}(t) \leq 0 \tag{2.14}
\end{equation*}
$$

for $t \geq t_{4}$ for some $t_{4} \geq t_{3}$. Integrating (2.14) from $t_{4}$ to $t$ yields

$$
\int_{t_{4}}^{t} q(s) \pi_{1}^{\beta}(\sigma(s)) h^{2 \beta / \theta}(s) d s \leq \frac{z^{\prime \prime}\left(t_{4}\right)}{b^{\beta}}
$$

which contradicts (2.11) and completes the proof.

Lemma 2.6. Let $x(t)$ be an eventually positive solution of (1.1) with $z(t)$ satisfying case (II) of Lemma 2.2. Suppose also that there exists a nondecreasing function $\eta \in C^{1}\left(\left[t_{0}, \infty\right), \mathbb{R}\right)$ such that $\sigma(t) \leq \eta(t)<\tau(t)$ for $t \geq t_{0}$. If

$$
\begin{equation*}
\int_{t_{0}}^{\infty} q(s) \pi_{2}(\sigma(s))(g(s)-h(s))^{2 \beta} d s=\infty \tag{2.15}
\end{equation*}
$$

then

$$
\begin{equation*}
\lim _{t \rightarrow \infty} z^{\prime \prime}(t)=0 \tag{2.16}
\end{equation*}
$$

Proof. Let $x(t)$ be an eventually positive solution of (1.1) such that $x(t)>0, x(\tau(t))>0$, and $x(\sigma(t))>0$ for $t \geq t_{1}$ for some $t_{1} \geq t_{0}$. As in Lemma 2.4, we again see that (2.5) and (2.6) hold. Since $z^{\prime}(t)<0$, it follows from (2.6) that

$$
z\left(\tau^{-1}(t)\right) \geq z\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)
$$

so inequality (2.5) takes the form

$$
\begin{equation*}
x(t) \geq \pi_{2}(t) z\left(\tau^{-1}(t)\right) \tag{2.17}
\end{equation*}
$$

Using (2.17) in (1.1) gives

$$
\begin{equation*}
z^{\prime \prime \prime}(t)+q(t) \pi_{2}^{\beta}(\sigma(t)) z^{\beta}(h(t)) \leq 0 \tag{2.18}
\end{equation*}
$$

for $t \geq t_{3}$ for some $t_{3} \geq t_{2}$. Since $(-1)^{k} z^{(k)}(t)>0$ for $k=0,1,2$ and $z^{\prime \prime \prime}(t) \leq 0$, for $t_{3} \leq u \leq v$, it is easy to see that

$$
\begin{equation*}
z(u) \geq \frac{(v-u)^{2}}{2} z^{\prime \prime}(v) \tag{2.19}
\end{equation*}
$$

Since $\sigma(t) \leq \eta(t)$ and $\tau$ is increasing, we conclude that $\tau^{-1}(\sigma(t)) \leq \tau^{-1}(\eta(t))$, i.e, $h(t) \leq g(t)$. Letting $u=h(t)$ and $v=g(t)$ in (2.19), we obtain

$$
z(h(t)) \geq \frac{(g(t)-h(t))^{2}}{2} z^{\prime \prime}(g(t))
$$

Using the latter inequality in (2.18) gives

$$
\begin{equation*}
z^{\prime \prime \prime}(t)+\frac{1}{2^{\beta}} q(t) \pi_{2}^{\beta}(\sigma(t))(g(t)-h(t))^{2 \beta}\left(z^{\prime \prime}(g(t))\right)^{\beta} \leq 0 . \tag{2.20}
\end{equation*}
$$

Since $\pi_{2}(t)<1$, we have $\pi_{2}^{\beta}(t) \geq \pi_{2}(t)$. So, inequality (2.20) takes the form

$$
\begin{equation*}
z^{\prime \prime \prime}(t)+\frac{1}{2^{\beta}} q(t) \pi_{2}(\sigma(t))(g(t)-h(t))^{2 \beta}\left(z^{\prime \prime}(g(t))\right)^{\beta} \leq 0 \tag{2.21}
\end{equation*}
$$

Now, we claim that (2.15) implies $z^{\prime \prime}(t) \rightarrow 0$ as $t \rightarrow \infty$. Suppose to the contrary that

$$
\lim _{t \rightarrow \infty} z^{\prime \prime}(t)=\ell>0 .
$$

Then, $z^{\prime \prime}(t) \geq \ell$ for $t \geq t_{3}$ for some $t_{3} \geq t_{2}$. Since $\lim _{t \rightarrow \infty} g(t)=\infty$, we can choose $t_{4} \geq t_{3}$ such that $g(t) \geq t_{3}$ for all $t \geq t_{4}$. Hence, $z^{\prime \prime}(g(t)) \geq \ell$ for $t \geq t_{4}$. Using this in (2.21) gives

$$
\begin{equation*}
z^{\prime \prime \prime}(t)+\frac{\ell^{\beta}}{2^{\beta}} q(t) \pi_{2}(\sigma(t))(g(t)-h(t))^{2 \beta} \leq 0 \quad \text { for } t \geq t_{4} . \tag{2.22}
\end{equation*}
$$

Integrating (2.22) from $t_{4}$ to $t$ yields

$$
\int_{t_{4}}^{t} q(s) \pi_{2}(\sigma(s))(g(s)-h(s))^{2 \beta} d s \leq\left(\frac{2}{\ell}\right)^{\beta} z^{\prime \prime}\left(t_{4}\right)
$$

which contradicts (2.15) and completes the proof.

Now, we are ready to present our main results. Our first result is concerned with equation (1.1) in the case where $\beta=1$, i.e., equation (1.1) is linear.

Theorem 2.7. Let (2.3) hold and assume that there exists a nondecreasing function $\eta \in C^{1}\left(\left[t_{0}, \infty\right), \mathbb{R}\right)$ such that $\sigma(t) \leq \eta(t)<\tau(t)$ for $t \geq t_{0}$. If there exist constants $\alpha, \theta \in(0,1)$ such that

$$
\begin{align*}
& \limsup _{t \rightarrow \infty}\left(\frac{\alpha \theta h^{1-\frac{2}{\theta}}(t)}{2} \int_{t_{0}}^{h(t)} s q(s) \pi_{1}(\sigma(s))(h(s))^{2 / \theta} d s\right. \\
& +\frac{\alpha \theta h^{2-\frac{2}{\theta}}(t)}{2} \int_{h(t)}^{t} q(s) \pi_{1}(\sigma(s))(h(s))^{2 / \theta} d s \\
& \left.+\frac{\alpha \theta h(t)}{2} \int_{t}^{\infty} q(s) \pi_{1}(\sigma(s)) h(s) d s\right)>1, \tag{2.23}
\end{align*}
$$

and

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \int_{g(t)}^{t} \frac{1}{2} q(s) \pi_{2}(\sigma(s))(g(s)-h(s))^{2} d s>1 \tag{2.24}
\end{equation*}
$$

then equation (1.1) is oscillatory.
Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t)>0, x(\tau(t))>0$, and $x(\sigma(t))>0$ for $t \geq t_{1}$ for some $t_{1} \geq t_{0}$. Then, from Lemma 2.2, the corresponding function $z$ satisfies either case (I) or case (II) for $t \geq t_{2}$ for some $t_{2} \geq t_{1}$.

First, we consider case (I). By Lemma 2.4, we again arrive at (2.10) for $t \geq t_{3}$, which, for $\beta=1$, takes the form

$$
\begin{equation*}
z^{\prime \prime \prime}(t)+q(t) \pi_{1}(\sigma(t)) z(h(t)) \leq 0 \text { for } t \geq t_{3} \tag{2.25}
\end{equation*}
$$

Integrating (2.25) from $t$ to $\infty$ yields

$$
\begin{equation*}
z^{\prime \prime}(t) \geq \int_{t}^{\infty} q(s) \pi_{1}(\sigma(s)) z(h(s)) d s \tag{2.26}
\end{equation*}
$$

and integrating again from $t_{3}$ to $t$ yields

$$
\begin{aligned}
z^{\prime}(t) & \geq \int_{t_{3}}^{t} \int_{u}^{\infty} q(s) \pi_{1}(\sigma(s)) z(h(s)) d s d u \\
& =\int_{t_{3}}^{t} \int_{u}^{t} q(s) \pi_{1}(\sigma(s)) z(h(s)) d s d u+\int_{t_{3}}^{t} \int_{t}^{\infty} q(s) \pi_{1}(\sigma(s)) z(h(s)) d s d u \\
& =\int_{t_{3}}^{t}\left(s-t_{3}\right) q(s) \pi_{1}(\sigma(s)) z(h(s)) d s+\left(t-t_{3}\right) \int_{t}^{\infty} q(s) \pi_{1}(\sigma(s)) z(h(s)) d s
\end{aligned}
$$

For any $\alpha \in(0,1)$ there exists $t_{4} \geq t_{3}$ such that $s-t_{3} \geq \alpha$ s and $t-t_{3} \geq \alpha t$ for $t \geq s \geq t_{4}$. Thus, from the last inequality we see that

$$
\begin{equation*}
z^{\prime}(t) \geq \alpha \int_{t_{4}}^{t} s q(s) \pi_{1}(\sigma(s)) z(h(s)) d s+\alpha t \int_{t}^{\infty} q(s) \pi_{1}(\sigma(s)) z(h(s)) d s \tag{2.27}
\end{equation*}
$$

In view of (2.2), it follows that

$$
\begin{equation*}
\frac{2 z(t)}{\theta t} \geq \alpha \int_{t_{4}}^{t} s q(s) \pi_{1}(\sigma(s)) z(h(s)) d s+\alpha t \int_{t}^{\infty} q(s) \pi_{1}(\sigma(s)) z(h(s)) d s \tag{2.28}
\end{equation*}
$$

From (2.28), we see that

$$
\begin{align*}
& \frac{2 z(h(t))}{\theta h(t)} \geq \alpha \int_{t_{4}}^{h(t)} s q(s) \pi_{1}(\sigma(s)) z(h(s)) d s \\
& \\
& \quad+\alpha h(t) \int_{h(t)}^{t} q(s) \pi_{1}(\sigma(s)) z(h(s)) d s  \tag{2.29}\\
& \quad+\alpha h(t) \int_{t}^{\infty} q(s) \pi_{1}(\sigma(s)) z(h(s)) d s .
\end{align*}
$$

Also, for $t \leq s$, we have $h(t) \leq h(s)$. Since $z(t) / t$ is increasing (see Lemma 2.4 (iii)),

$$
\begin{equation*}
z(h(s)) \geq \frac{h(s) z(h(t))}{h(t)} . \tag{2.30}
\end{equation*}
$$

For $h(t) \leq s \leq t$, we have $h(h(t)) \leq h(s) \leq h(t)$. Since $z(t) / t^{2 / \theta}$ is decreasing (see (2.1)),

$$
\begin{equation*}
z(h(s)) \geq h^{2 / \theta}(s) \frac{z(h(t))}{h^{2 / \theta}(t)} . \tag{2.31}
\end{equation*}
$$

For $t_{4} \leq s \leq h(t)$ and $h(t) \leq t$, we have $h(s) \leq h(h(t)) \leq h(t)$. Since $z(t) / t^{2 / \theta}$ is decreasing, we again obtain (2.31). Using (2.30) and (2.31) in (2.29) gives

$$
\begin{align*}
& \frac{2 z(h(t))}{\theta h(t)} \geq\left(\alpha \int_{t_{4}}^{h(t)} s q(s) \pi_{1}(\sigma(s))(h(s))^{2 / \theta} d s\right) \frac{z(h(t))}{(h(t))^{\frac{2}{\theta}}} \\
& +\left(\alpha h(t) \int_{h(t)}^{t} q(s) \pi_{1}(\sigma(s))(h(s))^{2 / \theta} d s\right) \frac{z(h(t))}{(h(t))^{\frac{2}{\theta}}} \\
& \quad+\left(\alpha h(t) \int_{t}^{\infty} q(s) \pi_{1}(\sigma(s)) h(s) d s\right) \frac{z(h(t))}{h(t)} . \tag{2.32}
\end{align*}
$$

From (2.32), we see that

$$
\begin{aligned}
& \frac{\alpha \theta h^{1-\frac{2}{\theta}}(t)}{2} \int_{t_{4}}^{h(t)} s q(s) \pi_{1}(\sigma(s))(h(s))^{2 / \theta} d s \\
& \quad+\frac{\alpha \theta h^{2-\frac{2}{\theta}}(t)}{2} \int_{h(t)}^{t} q(s) \pi_{1}(\sigma(s))(h(s))^{2 / \theta} d s+\frac{\alpha \theta h(t)}{2} \int_{t}^{\infty} q(s) \pi_{1}(\sigma(s)) h(s) d s \leq 1 .
\end{aligned}
$$

Taking the limsup the on both sides of the above inequality, we obtain a contradiction to condition (2.23),

Next, we consider case (II). As in Lemma 2.6, we again arrive at (2.20), which, for $\beta=1$, takes the form

$$
\begin{equation*}
z^{\prime \prime \prime}(t)+\frac{1}{2} q(t) \pi_{2}(\sigma(t))(g(t)-h(t))^{2} z^{\prime \prime}(g(t)) \leq 0 . \tag{2.33}
\end{equation*}
$$

Integrating (2.33) from $g(t)$ to $t$ yields

$$
z^{\prime \prime}(t)+\left[\int_{g(t)}^{t} \frac{1}{2} q(s) \pi_{2}(\sigma(s))(g(s)-h(s))^{2} d s-1\right] z^{\prime \prime}(g(t)) \leq 0,
$$

which, by (2.24), leads to a contradiction. This completes the proof of the theorem.
Our next results is for equation (1.1) in the case where $\beta<1$, i.e., equation (1.1) is sublinear.

Theorem 2.8. Let (2.3) and (2.11) hold. Assume that there exists a nondecreasing function $\eta \in$ $C^{1}\left(\left[t_{0}, \infty\right), \mathbb{R}\right)$ such that $\sigma(t) \leq \eta(t)<\tau(t)$ for $t \geq t_{0}$. If there exists $\theta \in(0,1)$ such that

$$
\begin{align*}
& \quad \limsup _{t \rightarrow \infty}\left(h^{1-\frac{2}{\theta}}(t) \int_{t_{0}}^{h(t)} s q(s) \pi_{1}^{\beta}(\sigma(s))(h(s))^{2 \beta / \theta} d s\right. \\
& \quad+h^{2-\frac{2}{\theta}}(t) \int_{h(t)}^{t} q(s) \pi_{1}^{\beta}(\sigma(s))(h(s))^{2 \beta / \theta} d s \\
&  \tag{2.34}\\
& \left.\quad+\frac{h^{2-\beta}(t)}{h^{2(1-\beta) / \theta}(t)} \int_{t}^{\infty} q(s) \pi_{1}^{\beta}(\sigma(s)) h^{\beta}(s) d s\right)>0
\end{align*}
$$

and

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \int_{g(t)}^{t} q(s) \pi_{2}(\sigma(s))(g(s)-h(s))^{2 \beta} d s>0 \tag{2.35}
\end{equation*}
$$

then equation (1.1) is oscillatory.
Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t)>0, x(\tau(t))>0$, and $x(\sigma(t))>0$ for $t \geq t_{1}$ for some $t_{1} \geq t_{0}$. Then, by Lemma 2.2, the corresponding function $z$ satisfies either case (I) or case (II) for $t \geq t_{2}$ for some $t_{2} \geq t_{1}$.

First, we consider case (I). By Lemma 2.4, we again arrive at (2.10) for $t \geq t_{3}$. Integrating (2.10) from $t$ to $\infty$ gives

$$
\begin{equation*}
z^{\prime \prime}(t) \geq \int_{t}^{\infty} q(s) \pi_{1}^{\beta}(\sigma(s)) z^{\beta}(h(s)) d s \tag{2.36}
\end{equation*}
$$

Integrating (2.36) from $t_{3}$ to $t$ yields

$$
\begin{aligned}
z^{\prime}(t) & \geq \int_{t_{3}}^{t} \int_{u}^{\infty} q(s) \pi_{1}^{\beta}(\sigma(s)) z^{\beta}(h(s)) d s d u \\
& =\int_{t_{3}}^{t} \int_{u}^{t} q(s) \pi_{1}^{\beta}(\sigma(s)) z^{\beta}(h(s)) d s d u+\int_{t_{3}}^{t} \int_{t}^{\infty} q(s) \pi_{1}^{\beta}(\sigma(s)) z^{\beta}(h(s)) d s d u \\
& =\int_{t_{3}}^{t}\left(s-t_{3}\right) q(s) \pi_{1}^{\beta}(\sigma(s)) z^{\beta}(h(s)) d s+\left(t-t_{3}\right) \int_{t}^{\infty} q(s) \pi_{1}^{\beta}(\sigma(s)) z^{\beta}(h(s)) d s .
\end{aligned}
$$

For any $\alpha \in(0,1)$ there exists $t_{4} \geq t_{3}$ such that $s-t_{3} \geq \alpha s$ and $t-t_{3} \geq \alpha t$ for $t \geq s \geq t_{4}$. Thus,

$$
\begin{equation*}
z^{\prime}(t) \geq \alpha \int_{t_{4}}^{t} s q(s) \pi_{1}^{\beta}(\sigma(s)) z^{\beta}(h(s)) d s+\alpha t \int_{t}^{\infty} q(s) \pi_{1}^{\beta}(\sigma(s)) z^{\beta}(h(s)) d s . \tag{2.37}
\end{equation*}
$$

By (2.2) and (2.37), we observe that

$$
\begin{equation*}
\frac{2 z(t)}{\theta t} \geq \alpha \int_{t_{4}}^{t} s q(s) \pi_{1}^{\beta}(\sigma(s)) z^{\beta}(h(s)) d s+\alpha t \int_{t}^{\infty} q(s) \pi_{1}^{\beta}(\sigma(s)) z^{\beta}(h(s)) d s . \tag{2.38}
\end{equation*}
$$

It follows from (2.38) that

$$
\begin{align*}
& \frac{2 z(h(t))}{\theta h(t)} \geq \alpha \int_{t_{4}}^{h(t)} s q(s) \pi_{1}^{\beta}(\sigma(s)) z^{\beta}(h(s)) d s \\
& \qquad \begin{aligned}
& \\
& \quad \alpha h(t) \int_{h(t)}^{t} q(s) \pi_{1}^{\beta}(\sigma(s)) z^{\beta}(h(s)) d s \\
& \quad+\alpha h(t) \int_{t}^{\infty} q(s) \pi_{1}^{\beta}(\sigma(s)) z^{\beta}(h(s)) d s .
\end{aligned}
\end{align*}
$$

Using (2.30) and (2.31) in (2.39) gives

$$
\begin{align*}
& \frac{2 z(h(t))}{\theta h(t)} \geq\left(\alpha \int_{t_{4}}^{h(t)} s q(s) \pi_{1}^{\beta}(\sigma(s))(h(s))^{2 \beta / \theta} d s\right) \frac{z^{\beta}(h(t))}{h^{2 \beta / \theta}(t)} \\
& \quad+\left(\alpha h(t) \int_{h(t)}^{t} q(s) \pi_{1}^{\beta}(\sigma(s))(h(s))^{2 \beta / \theta} d s\right) \frac{z^{\beta}(h(t))}{h^{2 \beta / \theta}(t)} \\
& \quad+\left(\alpha h(t) \int_{t}^{\infty} q(s) \pi_{1}^{\beta}(\sigma(s)) h^{\beta}(s) d s\right) \frac{z^{\beta}(h(t))}{h^{\beta}(t)} . \tag{2.40}
\end{align*}
$$

Letting

$$
w(t)=\frac{z(h(t))}{(h(t))^{2 / \theta}},
$$

it follows from (2.40) that

$$
\begin{align*}
& \frac{2}{\alpha \theta} w^{1-\beta}(t) \geq h^{1-\frac{2}{\theta}}(t)\left(\int_{t_{4}}^{h(t)} s q(s) \pi_{1}^{\beta}(\sigma(s))(h(s))^{2 \beta / \theta} d s\right) \\
& \quad+h^{2-\frac{2}{\theta}}(t)\left(\int_{h(t)}^{t} q(s) \pi_{1}^{\beta}(\sigma(s))(h(s))^{2 \beta / \theta} d s\right) \\
& \quad+\frac{h^{2-\beta}(t)}{h^{2(1-\beta) / \theta}}\left(\int_{t}^{\infty} q(s) \pi_{1}^{\beta}(\sigma(s)) h^{\beta}(s) d s\right) . \tag{2.41}
\end{align*}
$$

Taking the lim sup ${ }_{t \rightarrow \infty}$ on both sides of the above inequality and using (2.12), we obtain a contradiction to condition (2.34).

Next, we consider case (II). As in the proof of Lemma 2.6, we again arrive at (2.21). Integrating (2.21) from $g(t)$ to $t$ yields

$$
\int_{g(t)}^{t} q(s) \pi_{2}(\sigma(s))(g(s)-h(s))^{2 \beta} d s \leq 2^{\beta}\left(z^{\prime \prime}(g(t))\right)^{1-\beta}
$$

Noting that (2.35) implies (2.15), we see that (2.16) holds. Taking the $\lim _{\sup }^{t \rightarrow \infty}$ on both sides of the above inequality and using (2.16), we obtain a contradiction to condition (2.35), and this proves the theorem.

We conclude this paper with the following examples and remarks to illustrate the above results. Our first example is concerned with an equation with bounded neutral coefficients in the case where $p$ is a constant function; the second example is for an equation with unbounded neutral coefficients in the case where $p(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Example 2.9. Consider the third-order differential equation of Euler type

$$
\begin{equation*}
\left(x(t)+16 x\left(\frac{t}{2}\right)\right)^{\prime \prime \prime}+\frac{q_{0}}{t^{3}} x\left(\frac{t}{4}\right)=0, \quad t \geq 1 . \tag{2.42}
\end{equation*}
$$

Here $p(t)=16, q(t)=q_{0} / t^{3}, \beta=1, \tau(t)=t / 2$, and $\sigma(t)=t / 4$. Then, it is easy to see that conditions (C1)-(C2) hold, and

$$
\tau^{-1}(t)=2 t, \tau^{-1}\left(\tau^{-1}(t)\right)=4 t, h(t)=t / 2, \text { and } g(t)=2 t / 3 \text { with } \eta(t)=t / 3 \text {. }
$$

Choosing $\theta=2 / 3$, we see that

$$
\left(\frac{t}{\tau(t)}\right)^{2 / \theta} \frac{1}{p(t)}=\frac{1}{2}
$$

i.e., condition (C3) holds, $\pi_{1}(t)=1 / 32$ and $\pi_{2}(t)=15 / 256$. Letting $\alpha=\theta=2 / 3$, by Theorem 2.7 , Eq. (2.42) is oscillatory for

$$
q_{0}>\frac{3 \times 2^{11}}{5 \ln \frac{3}{2}}
$$

Example 2.10. Consider the sublinear equation

$$
\begin{equation*}
\left(x(t)+t x\left(\frac{t}{2}\right)\right)^{\prime \prime \prime}+\frac{q_{0}}{t^{6 / 5}} x^{3 / 5}\left(\frac{t}{10}\right)=0, \quad t \geq 16 \tag{2.43}
\end{equation*}
$$

Here $p(t)=t, q(t)=q_{0} / t^{6 / 5}, \beta=3 / 5, \tau(t)=t / 2$, and $\sigma(t)=t / 10$. Then, it is easy to see that conditions (C1)-(C2) hold, and

$$
\tau^{-1}(t)=2 t, \tau^{-1}\left(\tau^{-1}(t)\right)=4 t, h(t)=t / 5, \text { and } g(t)=t / 4 \text { with } \eta(t)=t / 8
$$

Choosing $\theta=2 / 3$, we see that

$$
\left(\frac{t}{\tau(t)}\right)^{2 / \theta} \frac{1}{p(t)}=\frac{8}{t} \leq \frac{1}{2}
$$

i.e., condition (C3) holds. Since $\pi_{1}(t) \geq 7 / 16 t$ and $\pi_{2}(t) \geq 63 / 128 t$, by Theorem 2.8, Eq. (2.43) is oscillatory for all $q_{0}>0$.

Remark 2.11. The results of this paper can be extended to the odd-order equation

$$
\left(r(t)\left(z^{(n-1)}(t)\right)^{\gamma}\right)^{\prime}+q(t) x^{\beta}(\sigma(t))=0, \quad t \geq t_{0}>0
$$

under either of the conditions

$$
\int_{t_{0}}^{\infty} r^{-1 / \gamma}(t) d t=\infty
$$

or

$$
\int_{t_{0}}^{\infty} r^{-1 / \gamma}(t) d t<\infty
$$

where $n \geq 3$ is an odd natural number, $r \in C\left(\left[t_{0}, \infty\right),(0, \infty)\right), \gamma$ is the ratio of odd positive integers, and the other functions in the equation are defined as in this paper.

Remark 2.12. It would be of interest to study the oscillatory behavior of all solutions of (1.1) for $p(t) \leq-1$ with $p(t) \not \equiv-1$ for large $t$.

## Acknowledgments

This paper has been supported by the RUDN University Strategic Academic Leadership Program.

## References

[1] B. Baculíková, J. Džurina, Oscillation of third-order neutral differential equations, Math. Comput. Model. 52(2010), 215-226. https://doi.org/10.1016/j.mcm.2010.02.011
[2] B. Baculíková, J. Džurina, On certain inequalities and their applications in the oscillation theory, Adv. Differ. Equ. 2013, Article ID 165, 1-8. https://doi.org/10.1186/ 1687-1847-2013-165
[3] G. E. Chatzarakis, J. Džurina, I. Jadlovská, Oscillatory properties of third-order neutral delay differential equations with noncanonical operators, Mathematics 7(2019), No. 12, 1-12.
[4] G. E. Chatzarakis, S. R. Grace, I. Jadlovská, T. Li, E. Tunç, Oscillation criteria for third-order Emden-Fowler differential equations with unbounded neutral coefficients, Complexity 2019, Article ID 5691758, 1-7.
[5] D.-X. Chen, J.-C. Liu, Asymptotic behavior and oscillation of solutions of third-order nonlinear neutral delay dynamic equations on time scales, Can. Appl. Math. Q. 16(2008), 19-43. MR2508451
[6] P. Das, Oscillation criteria for odd order neutral equations, J. Math. Anal. Appl. 188(1994), 245-257. https://doi.org/10.1006/jmaa.1994.1425
[7] J. Džurina, S. R. Grace, I. Jadlovská, On nonexistence of Kneser solutions of thirdorder neutral delay differential equations, Appl. Math. Lett. 88(2019), 193-200. https: //doi.org/10.1016/j.aml.2018.08.016
[8] Z. Došlá P. Liška, Comparison theorems for third-order neutral differential equations, Electron. J. Differential Equations 2016, No. 38, 1-13. MR3466509
[9] S. R. Grace, J. Alzabut, A. Özbekler, New criteria on oscillatory and asymptotic behavior of third-order nonlinear dynamic equations with nonlinear neutral terms, Entropy 23(2021), 1-11. https://doi. org/10.3390/e23020227
[10] S. R. Grace, J. R. Graef, E. Tunç, Oscillatory behavior of third order nonlinear differential equations with a nonlinear nonpositive neutral term, J. Taibah Univ. Sci. 13(2019), 704-710.
[11] J. R. Graef, R. Savithri, E. Thandapani, Oscillatory properties of third order neutral delay differential equations, Proceedings of the Fourth International Conference on Dynamical Systems and Differential Equations, May 24-27, 2002, Wilmington, NC, USA, 342-350. MR2018134
[12] J. R. Graef, E. Tunç, S. R. Grace, Oscillatory and asymptotic behavior of a third-order nonlinear neutral differential equation, Opuscula Math. 37(2017), 839-852. https://doi. org/10.7494/OpMath.2017.37.6.839
[13] J. R. Graef, P. W. Spikes, M. K. Grammatikopoulos, Asymptotic behavior of nonoscillatory solutions of neutral delay differential equations of arbitrary order, Nonlinear Anal. 21(1993), 23-42. https://doi. org/10.1016/0362-546X (93) 90175-R
[14] J. K. Hale, Theory of functional differential equations, Springer-Verlag, New York, 1977. MR0508721
[15] Y. Jiang, C. Jiang, T. Li, Oscillatory behavior of third-order nonlinear neutral delay differential equations, Adv. Differ. Equ. 2016, Article ID 171, 1-12.
[16] I. T. Kiguradze, On the oscillatory character of solutions of the equation $d^{m} u / d t^{m}+$ $a(t)|u|^{n} \operatorname{sign} u=0$, Mat. Sb. (N.S.) 65(1964), 172-187. MR0173060,
[17] R. Koplatadze, G. Kvinikadze, I. P. Stavroulakis, Properties A and B of $n$th order linear differential equations with deviating argument, Georgian Math. J. 6(1999), 553-566. https://doi.org/10.1023/A:1022962129926
[18] T. Li, Yu. V. Rogovchenko, Asymptotic behavior of higher-order quasilinear neutral differential equations, Abstr. Appl. Anal. 2014, Article ID 395368, 1-11. https://doi.org/10. 1155/2014/395368
[19] B. Mihalíková, E. Kostiкová, Boundedness and oscillation of third order neutral differential equations, Tatra Mt. Math. Publ. 43(2009), 137-144. https://doi .org/10.2478/ v10127-009-0033-6; MR2588884
[20] O. Moaaz, J. Awrejcewicz, A. Muhib, Establishing new criteria for oscillation of oddorder nonlinear differential equations, Mathematics 8(2020), No. 6, Article No. 937, 15 pp. https://doi.org/10.3390/math8060937
[21] S. H. Saker, J. R. Graef, Oscillation of third-order nonlinear neutral functional dynamic equations on time scales, Dynam. Syst. Appl. 21(2012), 583-606. MR3026094
[22] Y. Sun, T. S. Hassan, Comparison criteria for odd order forced nonlinear functional neutral dynamic equations, Appl. Math. Comput. 251(2015), 387-395. https://doi.org/ 10.1016/j.amc.2014.11.095
[23] Y. Sun, Y. Zhao, Oscillatory behavior of third-order neutral delay differential equations with distributed deviating arguments, J. Inequal. Appl. 2019, Article ID 207, 1-16. https: //doi.org/10.1186/s13661-019-01301-7
[24] E. Thandapani, T. Li, On the oscillation of third-order quasi-linear neutral functional differential equations, Arch. Math. (Brno) 47(2011), 181-199. MR2852380
[25] E. Thandapani, S. Padmavathy, S. Pinelas, Oscillation criteria for odd-order nonlinear differential equations with advanced and delayed arguments, Electron. J. Differential Equations 2014, No. 174, 1-13. MR3262045
[26] E. Tunç, Oscillatory and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments, Electron. J. Differential Equations 2017, No. 16, 1-12. MR3609144

Electronic Journal of Qualitative Theory of Differential Equations

# Multiple positive solutions for singular anisotropic Dirichlet problems 

Zhenhai Liu ${ }^{\boxtimes 1,2}$ and Nikolaos S. Papageorgiou ${ }^{3}$<br>${ }^{1}$ Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, P.R. China.<br>${ }^{2}$ Guangxi Key Laboratory of Hybrid Computation and IC Design Analysis, Guangxi University for Nationalities, Nanning, Guangxi, 530006, P.R. China<br>${ }^{3}$ Department of Mathematics, National Technical University, Zografou Campus, 15780 Athens, Greece

Received 22 April 2021, appeared 10 July 2021
Communicated by Gabriele Bonanno


#### Abstract

We consider a nonlinear Dirichlet problem driven by the variable exponent (anisotropic) $p$-Laplacian and a reaction that has the competing effects of a singular term and of a superlinear perturbation. There is no parameter in the equation (nonparametric problem). Using variational tools together with truncation and comparison techniques, we show that the problem has at least two positive smooth solutions.


Keywords: variable exponent, anisotropic regularity, anisotropic maximum principle, positive solutions, critical point theory.
2020 Mathematics Subject Classification: 35J75, 35J60.

## 1 Introduction

Let $\Omega \subseteq \mathbb{R}^{N}$ be a bounded domain with a $C^{2}$-boundary $\partial \Omega$. In this paper we study the following anisotropic singular Dirichlet problem

$$
\begin{equation*}
-\Delta_{p(z)} u(z)=u(z)^{-\eta(z)}+f(z, u(z)) \quad \text { in } \Omega,\left.\quad u\right|_{\partial \Omega}=0, \quad u>0 \tag{1.1}
\end{equation*}
$$

In this problem the exponent $p: \bar{\Omega} \rightarrow \mathbb{R}$ in the differential operator, is Lipschitz continuous (that is $p \in C^{0,1}(\bar{\Omega})$ ) and $1<p_{-}=\min _{\bar{\Omega}} p$. By $\Delta_{p(z)}$ we denote the anisotropic $p$-Laplace operator defined by

$$
\Delta_{p(z)} u=\operatorname{div}\left(|D u|^{p(z)-2} D u\right) \quad \forall u \in W_{0}^{1, p(z)}(\Omega)
$$

In problem (1.1) we have the competing effects of a singular term $x^{-\eta(z)}$ with $\eta \in C(\bar{\Omega}), 0<$ $\eta(z)<1$ for all $z \in \bar{\Omega}$ and a Carathéodory perturbation $f(z, x)$ (that is, for all $x \in \mathbb{R}, z \rightarrow$ $f(z, x)$ is measurable and for a.a. $z \in \Omega, x \rightarrow f(z, x)$ is continuous), which is $\left(p_{+}-1\right)$ superlinear as $x \rightarrow+\infty$ (here $p_{+}=\max _{\bar{\Omega}} p$ ), but need not satisfy the usual for superlinear

[^11]problems Ambrosetti-Rabinowitz condition (the AR-condition for short). We are looking for positive solutions. Using a combination of variational tools based on the critical point theory, together with truncation and comparison techniques, we show that the problem has at least two positive smooth solutions.

While anisotropic boundary value problems have been studied extensively in the last few years (see the books of Diening-Harjulehto-Hästö-Růžička [2] and of Rădulescu-Repovš [12] and the references therein), the study of singular anisotropic problems is lagging behind. Only a very limited number of works exist on this subject and they all concern parametric problems (see the works of Byun-Ko [1] and Saoudi-Ghanmi [13]). The presence of parameter in the equation is very helpful, since by varying the parameter, we achieve certain desirable geometric configurations which in turn permit the use of the minimax theorems of critical point theory. In problem (1.1) there is no parameter to facilitate the analysis.

## 2 Mathematical background - hypotheses

The study of problem (1.1) requires the use of Lebesgue and Sobolev spaces with variable exponents. A comprehensive presentation of these spaces can be found in the book of Diening-Harjulehto-Hästö-Růžička [2].

For every $r \in C(\bar{\Omega})$ we set

$$
r_{-}=\min _{\bar{\Omega}} r \text { and } r_{+}=\max _{\bar{\Omega}} r .
$$

Let $E_{1}=\left\{r \in C(\bar{\Omega}): 1<r_{-}\right\}$and $M(\Omega)=\{u: \Omega \rightarrow \mathbb{R}$ measurable $\}$. As usual, we identify two such functions which differ only on a Lebesgue-null set. For $r \in E_{1}$, the variable exponent Lebesgue space $L^{r(z)}(\Omega)$ is defined by

$$
L^{r(z)}(\Omega)=\left\{u \in M(\Omega): \int_{\Omega}|u|^{r(z)} d z<\infty\right\} .
$$

We equip this space with the so-called "Luxemburg norm" defined by

$$
\|u\|_{r(z)}=\inf \left[\lambda>0: \int_{\Omega}\left(\frac{|u(z)|}{\lambda}\right)^{r(z)} d z \leq 1\right], \quad u \in L^{r(z)}(\Omega) .
$$

With this norm the space $L^{r(z)}(\Omega)$ is a Banach space which is separable and reflexive (in fact uniformly convex). Let $r^{\prime} \in E_{1}$ be defined by $r^{\prime}(z)=\frac{r(z)}{r(z)-1}$ for all $z \in \bar{\Omega}$ (that is, $\frac{1}{r(z)}+\frac{1}{r^{\prime}(z)}=1$ for all $\left.z \in \bar{\Omega}\right)$. Then we have

$$
L^{r(z)}(\Omega)^{*}=L^{r^{\prime}(z)}(\Omega)
$$

and the following version of Hölder's inequality is true

$$
\int_{\Omega}|u v| d z \leq\left[\frac{1}{r_{-}}+\frac{1}{r_{-}^{\prime}}\right]\|u\|_{r(z)}\|v\|_{r^{\prime}(z)}, \quad \forall u \in L^{r(z)}(\Omega), \forall v \in L^{L^{\prime}(z)}(\Omega) .
$$

Note that if $r_{1}, r_{2} \in E_{1}$ and $r_{1}(z) \leq r_{2}(z)$ for all $z \in \bar{\Omega}$, then we have

$$
L^{r_{2}(z)}(\Omega) \hookrightarrow L^{r_{1}(z)}(\Omega) \quad \text { continuously. }
$$

Using the variable exponent Lebesgue spaces, we can introduce variable exponent Sobolev spaces. Given $r \in E_{1}$, the anisotropic Sobolev space $W^{1, r(z)}(\Omega)$ is defined by

$$
W^{1, r(z)}(\Omega)=\left\{u \in L^{r(z)}(\Omega):|D u| \in L^{r(z)}(\Omega)\right\},
$$

where $D u$ denotes the gradient of $u$ in the weak sense. This space is equipped with the norm

$$
\|u\|_{1, r(z)}=\|u\|_{r(z)}+\|D u\|_{r(z)}, \quad u \in W^{1, r(z)}(\Omega) \quad\left(\text { here }\|D u\|_{r(z)}=\|\mid D u\|_{r(z)}\right) .
$$

If $r \in E_{1} \cap C^{0,1}(\bar{\Omega})$, then we define

$$
W_{0}^{1, r(z)}(\Omega)={\overline{C_{c}^{\infty}(\Omega)}}^{\|\cdot\|_{1, r(z)}}
$$

The spaces $W^{1, r(z)}(\Omega)$ and $W_{0}^{1, r(z)}(\Omega)$ are separable, reflexive (in fact uniformly convex). For the space $W_{0}^{1, r(z)}(\Omega)$ the Poincaré inequality holds, that is, there exists $\hat{c}>0$ such that

$$
\|u\|_{r(z)} \leq \widehat{c}\|D u\|_{r(z)} \quad \text { for all } u \in W_{0}^{1, r(z)}(\Omega) .
$$

This implies that on $W_{0}^{1, r(z)}(\Omega)$ we can use the equivalent norm

$$
|u|_{1, r(z)}=\|D u\|_{r(z)}, \quad u \in W_{0}^{1, r(z)}(\Omega) .
$$

For $r \in E_{1}$, we set

$$
r^{*}(z)=\left\{\begin{array}{ll}
\frac{N r(z)}{N-r(z)}, & \text { if } r(z)<N \\
+\infty, & \text { if } N \leq r(z)
\end{array} \quad \forall z \in \bar{\Omega} .\right.
$$

Let $r, q \in E_{1} \cap C^{0,1}(\bar{\Omega})$ and suppose that $q(z) \leq r^{*}(z)\left(\right.$ resp. $\left.q(z)<r^{*}(z)\right)$ for all $z \in \bar{\Omega}$. Then we have the anisotropic Sobolev embedding theorem

$$
\begin{gathered}
W_{0}^{1, r(z)}(\Omega) \hookrightarrow L^{q(z)}(\Omega) \quad \text { continuously } \\
\text { (resp. } W_{0}^{1, r(z)}(\Omega) \hookrightarrow L^{q(z)}(\Omega) \quad \text { compactly). }
\end{gathered}
$$

In the study of these spaces, central role plays the following modular function

$$
\rho_{r}(u)=\int_{\Omega}|u|^{r(z)} d z \quad \text { for all } u \in L^{r(z)}(\Omega) \text {. }
$$

If $u \in W_{0}^{1, r(z)}(\Omega)$ or $u \in W^{1, r(z)}(\Omega)$, then $\rho_{r}(D u)=\rho_{r}(|D u|)$.
This modular function is closely related to the Luxemburg norm.
Proposition 2.1. If $r \in E_{1}$ and $\left\{u_{n}, u\right\}_{n \in \mathbb{N}} \subseteq L^{r(z)}(\Omega)$, then we have
(a) For all $\lambda>0$,
$\|u\|_{r(z)}=\lambda \Leftrightarrow \rho_{r}\left(\frac{u}{\lambda}\right)=1 ;$
(b) $\|u\|_{r(z)}<1 \Leftrightarrow\|u\|_{r(z)}^{r_{+}} \leq \rho_{r}(u) \leq\|u\|_{r(z)}^{r_{-}}$,
$\|u\|_{r(z)}>1 \Leftrightarrow\|u\|_{r(z)}^{r_{-}} \leq \rho_{r}(u) \leq\|u\|_{r(z)}^{r_{+}} ;$
(c) $\left\|u_{n}\right\|_{r(z)} \rightarrow 0 \Leftrightarrow \rho_{r}\left(u_{n}\right) \rightarrow 0$;
(d) $\left\|u_{n}\right\|_{r(z)} \rightarrow \infty \Leftrightarrow \rho_{r}\left(u_{n}\right) \rightarrow+\infty$.

Also for $r \in E_{1} \cap C^{0,1}(\bar{\Omega})$, we have

$$
W_{0}^{1, r(z)}(\Omega)^{*}=W^{-1, r^{\prime}(z)}(\Omega) .
$$

Consider the operator $A_{r}: W_{0}^{1, r(z)}(\Omega) \rightarrow W^{-1, r^{\prime}(z)}(\Omega)$ defined by

$$
\left\langle A_{r(z)}(u), h\right\rangle=\int_{\Omega}|D u|^{r(z)-2}(D u, D h)_{\mathbb{R}^{v}} d z, \quad \text { for all } u, h \in W_{0}^{1, r(z)}(\Omega) .
$$

This operator has the following properties (see Gasiński-Papageorgiou [5], Proposition 2.5 and Rădulescu-Repovš [12], p.40).

Proposition 2.2. If $r \in E_{1} \cap C^{0,1}(\bar{\Omega})$ and $A_{r}: W_{0}^{1, r(z)}(\Omega) \rightarrow W^{-1, r^{\prime}(z)}(\Omega)$ is defined as above, then $A_{r}(\cdot)$ is bounded (maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and is of type $(S)_{+}$, that is, it has the following property:

$$
\text { "if } u_{n} \xrightarrow{w} u \text { in } W_{0}^{1, r(z)}(\Omega) \text { and } \limsup _{n \rightarrow \infty}\left\langle A_{r(z)}\left(u_{n}\right), u_{n}-u\right\rangle \leq 0 \text {, then } u_{n} \rightarrow u \text { in } W_{0}^{1, r(z)}(\Omega) . "
$$

For every $u \in W_{0}^{1, r(z)}(\Omega)$, we define $u^{ \pm}=\max \{ \pm u, 0\}$. Then

$$
u^{ \pm} \in W_{0}^{1, r(z)}(\Omega), \quad u=u^{+}-u^{-}, \quad|u|=u^{+}+u^{-} .
$$

Suppose $u, v: \Omega \rightarrow \mathbb{R}$ are measurable functions such that $u(z) \leq v(z)$ for a.a $z \in \Omega$. We define

$$
\begin{aligned}
{[u, v] } & =\left\{h \in W_{0}^{1, r(z)}(\Omega): u(z) \leq h(z) \leq v(z) \quad \text { for a.a. } z \in \Omega\right\}, \\
{[u) } & =\left\{h \in W_{0}^{1, r(z)}(\Omega): u(z) \leq h(z) \quad \text { for a.a. } z \in \Omega\right\} .
\end{aligned}
$$

Another space that we will need is $C_{0}^{1}(\bar{\Omega})=\left\{u \in C^{1}(\bar{\Omega}):\left.u\right|_{\partial \Omega}=0\right\}$. This is an ordered Banach space with positive (order) cone $C_{+}=\left\{u \in C_{0}^{1}(\bar{\Omega}): u(z) \geq 0\right.$ for all $\left.z \in \bar{\Omega}\right\}$. This cone has a nonempty interior given by

$$
\operatorname{int} C_{+}=\left\{u \in C_{+}: u(z)>0 \quad \text { for all } z \in \Omega,\left.\frac{\partial u}{\partial n}\right|_{\partial \Omega}<0\right\} .
$$

with $n(\cdot)$ being the outward unit normal on $\partial \Omega$.
Let $X$ be a Banach space and $\varphi \in C^{1}(X)$. We introduce the set

$$
K_{\varphi}=\left\{u \in X: \varphi^{\prime}(u)=0\right\} \quad(\text { the critical set of } \varphi) .
$$

We say that $\varphi(\cdot)$ satisfies the "C-condition", if it has the following property:
"Every sequence $\left\{u_{n}\right\}_{n \in \mathbb{N}} \subseteq X$ such that $\left\{\varphi\left(u_{n}\right)\right\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is bounded and $\left(1+\left\|u_{n}\right\|_{X}\right) \varphi^{\prime}\left(u_{n}\right) \rightarrow 0$ in $X^{*}$ as $n \rightarrow \infty$, admits a strongly convergent subsequence."

Now we are ready to introduce our hypotheses on the data of problem (1.1).
$H_{0}: p \in C^{0,1}(\bar{\Omega}), 1<p_{-}=\min _{\bar{\Omega}} p, \eta \in C(\bar{\Omega}), 0<\eta(z)<1$ for all $z \in \bar{\Omega}$.
$H_{1}: f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0)=0$ for a.a. $z \in \Omega$ and
(i) $|f(z, x)| \leq a(z)\left[1+x^{r(z)-1}\right]$ for a.a. $z \in \Omega$, all $x \geq 0$, with $r \in C(\bar{\Omega})$ and $p(z)<$ $r(z)<p^{*}(z)$ for all $z \in \bar{\Omega}$;
(ii) if $F(z, x)=\int_{0}^{x} f(z, s) d s$, then $\lim _{x \rightarrow+\infty} \frac{F(z, x)}{x^{p+}}=+\infty \quad$ uniformly for a.a. $z \in \Omega$ and there exists $\tau \in C(\bar{\Omega})$ such that

$$
\begin{aligned}
& \tau(z) \in\left(\left(r_{+}-p_{-}\right) \max \left\{\frac{N}{p_{-}}, 1\right\}, p^{*}(z)\right) \quad \text { for all } z \in \bar{\Omega} \\
& 0<\widehat{\eta}_{0} \leq \liminf _{x \rightarrow+\infty} \frac{f(z, x) x-p_{+} F(z, x)}{x^{\tau(z)}} \quad \text { uniformly for a.a. } z \in \Omega
\end{aligned}
$$

(iii) there exists $\theta>0$ such that

$$
\theta^{-\eta(z)}+f(z, \theta) \leq-\widehat{c}<0 \text { for a.a. } z \in \Omega
$$

(iv) there exist $\delta>0$ and $q \in E_{1}$ such that $q_{+}<p_{-}$such that

$$
c_{1} x^{q(x)-1} \leq f(z, x) \quad \text { for a.a. } z \in \Omega, \text { all } 0 \leq x \leq \delta, \text { with } c_{1}>0
$$

(v) there exists $\widehat{\xi}_{\theta}>0$ such that for a.a. $z \in \Omega$, the function

$$
x \rightarrow f(z, x)+\widehat{\xi}_{\theta} x^{p(z)-1} \quad \text { is nondecreasing on }[0, \theta]
$$

Remark 2.3. Since we look for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_{+}=[0, \infty)$, we can always assume without any loss of generality that $f(z, x)=0$ for a.a. $z \in \Omega$, all $x \leq 0$. Hypotheses $H_{1}(i i)$ implies that for a.a. $z \in \Omega f(z, \cdot)$ is $\left(p_{+}-1\right)$-superlinear. However, it need not satisfy the AR-condition which is common in the literature when dealing with superlinear problems (see, for example, Saoudi-Ghanmi [13], hypothesis (H4) and Byun-Ko [1, p. 76]). Condition $H_{1}(i i)$ is less restrictive and incorporates in our framework also superlinear nonlinearities with "slower" growth as $x \rightarrow+\infty$, which fail to satisfy the AR-condition. For example, the following function $f(z, x)$ satisfies hypotheses $H_{1}$ but fails to satisfy the AR-condition:

$$
f(z, x)= \begin{cases}\left(x^{+}\right)^{q(z)-1}-2\left(x^{+}\right)^{k(z)-1} & \text { if } x \leq 1 \\ x^{p_{+}-1} \ln x-x^{p(z)-1} & \text { if } 1<x\end{cases}
$$

with $q \in E_{1}$ as in hypothesis $H_{1}(i v), k \in C(\bar{\Omega}), \tau(z)<k(z)$ for a $z \in \bar{\Omega}$. Evidently for this $f(z, x)$ we can choose $\theta=1$. Hypotheses $H_{1}(i i i)$, (iv) dictate an oscillatory behavior for $f(z, \cdot)$ near $0^{+}$since it starts positive near zero (see hypothesis $H_{1}(v)$ ) and drops to negative values as we approach $\theta>0$ (see hypothesis $H_{1}(i i i)$ ). Also, hypothesis $H_{1}(v)$ implies the presence of a concave term near zero.

## 3 An auxiliary problem

When dealing with singular problems, a major difficulty that we encounter, is that the presence of the singularity leads to an energy functional which is not $C^{1}$. This fact prevents us
from using the results of critical point theory. So, we need to find a way to bypass the singularity and deal with $C^{1}$-functions in order to use the minimax theorems of critical point theory. This is done by using the solution of an auxiliary problem which we introduce and solve in this section. The auxiliary problem is suggested by a unilateral growth condition satisfied by $f(z, \cdot)$. More precisely note that on account of hypotheses $H_{1}(i),(i v)$, we can find $c_{2}>0$ such that

$$
\begin{equation*}
f(z, x) \geq c_{1} x^{q(z)-1}-c_{2} x^{r(z)-1} \quad \text { for a.a } z \in \Omega, \text { all } x \geq 0 . \tag{3.1}
\end{equation*}
$$

Motivated by this unilateral growth condition on $f(z, \cdot)$ and using hypothesis $H_{1}(i i i)$, we introduce the Carathéodory function $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$
g(z, x)= \begin{cases}c_{1}\left(x^{+}\right)^{q(z)-1}-c_{2}\left(x^{+}\right)^{r(z)-1} & \text { if } x \leq \theta  \tag{3.2}\\ c_{1} \theta^{q(z)-1}-c_{2} \theta^{r(z)-1} & \text { if } \theta<x\end{cases}
$$

Then we consider the following Dirichlet problem

$$
\begin{equation*}
-\Delta_{p(z)} u(z)=g(z, u(z)) \quad \text { in } \Omega,\left.u\right|_{\partial \Omega}=0, u>0 . \tag{3.3}
\end{equation*}
$$

Proposition 3.1. If hypotheses $H_{0}$ hold, then problem (3.3) has a unique positive solution $\bar{u} \in \operatorname{int} C_{+}$ and $0 \leq \bar{u}(z) \leq \theta$ for all $z \in \bar{\Omega}$.

Proof. First we show the existence of a positive solution for problem (3.3). To this end, let $\psi_{0}: W_{0}^{1, p(z)}(\Omega) \rightarrow \mathbb{R}$ be the $C^{1}$-functional defined by $\psi_{0}(u)=\int_{\Omega} \frac{1}{p(z)}|D u|^{p(z)} d z-\int_{\Omega} G(z, u) d z$ for all $u \in W_{0}^{1, p(z)}(\Omega)$, where $G(z, x)=\int_{0}^{x} g(z, s) d s$. From (3.2), we see that

$$
\begin{aligned}
& \psi_{0}(u) \geq \frac{1}{p} \rho_{p}(D u)-c_{3} \quad \text { for some } c_{3}>0, \\
\Rightarrow & \psi_{0}(\cdot) \quad \text { is coercive (see Proposition 2.1). }
\end{aligned}
$$

Also, from the anisotropic Sobolev embedding theorem, we see that $\psi_{0}(\cdot)$ is sequentially weakly lower semicontinuous.

So, by the Weierstrass-Tonelli theorem, we can find $\bar{u} \in W_{0}^{1, p(z)}(\Omega)$ such that

$$
\begin{equation*}
\psi_{0}(\bar{u})=\min \left[\psi_{0}(u): u \in W_{0}^{1, p(z)}(\Omega)\right] . \tag{3.4}
\end{equation*}
$$

Let $u \in \operatorname{int} C_{+}$and choose $t \in(0,1)$ small so that $0 \leq t u(z) \leq \theta$ for all $z \in \bar{\Omega}$. Then using (3.2), we have

$$
\begin{align*}
\psi_{0}(t u) \leq & \frac{t^{p_{-}}}{p_{-}} \rho_{p}(D u)+\frac{t^{r_{-}}}{r_{-}} \rho_{\tau}(u)-\frac{t^{q_{+}}}{q_{+}} \rho_{q}(u) \\
\leq & c_{4} t^{p_{-}}-c_{5} t^{q_{+}} \quad \text { for some } c_{4}, c_{5}>0 .  \tag{3.5}\\
& \left(\text { since } 1<q_{+}<p_{-}<r_{-} \text {and } t \in(0,1)\right) .
\end{align*}
$$

From (3.5) we see that by taking $t \in(0,1)$ even smaller if necessary, we have

$$
\begin{aligned}
& \psi_{0}(t u)<0, \\
\Rightarrow & \psi_{0}(\bar{u})<0=\psi_{0}(0) \quad(\text { see }(3.4)), \\
\Rightarrow & \bar{u} \neq 0 .
\end{aligned}
$$

From (3.4) we have that

$$
\begin{align*}
& \psi_{0}^{\prime}(\bar{u})=0 \\
\Rightarrow & \left\langle A_{p(z)}(\bar{u}), h\right\rangle=\int_{\Omega} g(z, \bar{u}) h d z, \quad \text { for all } h \in W_{0}^{1, p(z)}(\Omega) . \tag{3.6}
\end{align*}
$$

In (3.6) first we choose $h=-\bar{u}^{-} \in W_{0}^{1, p(z)}(\Omega)$ and obtain

$$
\begin{aligned}
& \rho_{p}\left(D \bar{u}^{-}\right)=0 \quad(\text { see }(3.2)), \\
\Rightarrow & \bar{u} \geq 0, \bar{u} \neq 0 .
\end{aligned}
$$

Next in (3.6) first we choose $h=[\bar{u}-\theta]^{+} \in W_{0}^{1, p(z)}(\Omega)$. We obtain

$$
\begin{aligned}
\left\langle A_{p(z)}(\bar{u}),(\bar{u}-\theta)^{+}\right\rangle & =\int_{\Omega}\left[c_{1} \theta^{q(z)-1}-c_{2} \theta^{r(z)-1}\right](\bar{u}-\theta)^{+} d z \quad \text { (see (3.2)) } \\
& \leq \int_{\Omega} f(z, \theta)(\bar{u}-\theta)^{+} d z \quad(\text { see }(3.1)) \\
& \leq 0=\left\langle A_{p(z)}(\theta),(\bar{u}-\theta)^{+}\right\rangle \quad\left(\text { see } H_{1}(i i i)\right) \\
& \Rightarrow \bar{u} \leq \theta .
\end{aligned}
$$

So, we have proved that

$$
\begin{equation*}
\bar{u} \in[0, \theta], \quad \bar{u} \neq 0 . \tag{3.7}
\end{equation*}
$$

From (3.7),(3.2) and (3.6), we infer that $\bar{u} \neq 0$ is a positive solution of problem (3.3). From Fan [3] (Theorem 1.3), we have that $\bar{u} \in C_{+} \backslash\{0\}$. Moreover, we have

$$
\Delta_{p(z)}(\bar{u}) \leq c_{2} \theta^{r(z)-p(z)} \bar{u}(z)^{p(z)-1} \leq c_{6} \bar{u}(z)^{p(z)-1} \quad \text { in } \Omega \text { for some } c_{6}>0
$$

Then the anisotropic maximum principle of Zhang [15, Theorem 1.2] implies that

$$
\begin{equation*}
\bar{u} \in \operatorname{int} C_{+} . \tag{3.8}
\end{equation*}
$$

Next we show that this positive solution of (3.3) is in fact unique. Let $\bar{v} \in W_{0}^{1, p(z)}(\Omega)$ be another positive solution of (3.3). Again we have

$$
\begin{equation*}
\bar{v} \in \operatorname{int} C_{+} . \tag{3.9}
\end{equation*}
$$

From (3.8) and (3.9) and using Proposition 4.1.22, p. 274, of Papageorgiou-RădulescuRepovš [9], we have that

$$
\begin{equation*}
\frac{\bar{u}}{\bar{v}} \in L^{\infty}(\Omega) \quad \text { and } \quad \frac{\bar{v}}{\bar{u}} \in L^{\infty}(\Omega) . \tag{3.10}
\end{equation*}
$$

Let $j: L^{1}(\Omega) \rightarrow \overline{\mathbb{R}}=\mathbb{R} \cup\{+\infty\}$ be the integral functional defined by

$$
j(u)= \begin{cases}\int_{\Omega} \frac{1}{p(z)}\left|D u^{1 / p_{-}}\right| p(z) & d z \quad \text { if } u \geq 0, u^{1 / p_{-}} \in W_{0}^{1, p(z)}(\Omega) \\ +\infty \text { otherwise }\end{cases}
$$

Let $\operatorname{dom} j=\left\{u \in L^{1}(\Omega): j(u)<\infty\right\}$ (the effective domain of $j(\cdot)$ ). From Theorem 2.2 of Takač-Giacomoni [14], we know that $j(\cdot)$ is convex. Let $h=\bar{u}^{p_{-}-} \bar{v}^{p_{-}} \in W_{0}^{1, p(z)}(\Omega)$. On account of (3.10), for $|t|<1$ small, we have

$$
\bar{u}^{p_{-}}+t h \in \operatorname{dom} j \text { and } \bar{v}^{p_{-}}+t h \in \operatorname{dom} j .
$$

Then the convexity of $j(\cdot)$ implies the Gateaux differentiability of $j(\cdot)$ at $\bar{u}^{p_{-}}$and at $\bar{v}^{p_{-}}$in the direction $h$. Moreover, using Green's theorem, we obtain

$$
\begin{aligned}
j^{\prime}\left(\bar{u}^{p_{-}}\right)(h) & =\frac{1}{p_{-}} \int_{\Omega} \frac{-\Delta_{p(z)} \bar{u}}{\bar{u}^{p_{-}-1}} h d z=\frac{1}{p_{-}} \int_{\Omega}\left[\frac{c_{1}}{\bar{u}^{p_{-} q(z)}}-c_{2} \bar{u}^{r(z)-p_{-}}\right] h d z \\
j^{\prime}\left(\bar{v}^{p_{-}}\right)(h) & =\frac{1}{p_{-}} \int_{\Omega} \frac{-\Delta_{p(z)} \bar{v}}{\bar{v}^{p_{-}-1}} h d z=\frac{1}{p_{-}} \int_{\Omega}\left[\frac{c_{1}}{\bar{v}^{p_{-}-q(z)}}-c_{2} \bar{v}^{r(z)-p_{-}}\right] h d z .
\end{aligned}
$$

The convexity of $j(\cdot)$ implies the monotonicity of $j^{\prime}(\cdot)$. So, we have

$$
\begin{aligned}
0 \leq & \int_{\Omega}\left[c_{1}\left(\frac{1}{\bar{u}^{p_{-}-q(z)}}-\frac{1}{\bar{v}^{p_{-}-q(z)}}\right)-c_{2}\left(\bar{u}^{r(z)-p_{-}}-\bar{v}^{r(z)-p_{-}}\right)\right]\left(\bar{u}^{p_{-}}-\bar{v}^{p_{-}}\right) d z \leq 0 \\
& \left(\text { since } q_{+}<p_{-}<r_{-}\right) \\
\Rightarrow & \bar{u}=\bar{v}
\end{aligned}
$$

This proves the uniqueness of the positive solution $\bar{u} \in \operatorname{int} C_{+}$.
In what follows, let $\widehat{d}(\cdot)=d(\cdot, \partial \Omega)$ and $\widehat{u}_{1}$ is the positive, $L^{p_{+}}$_normalized (that is, $\left\|\widehat{u}_{1}\right\|_{p_{+}}=$ 1) eigenfunction corresponding to the principal eigenvalue of $\left(-\Delta_{p_{+}}, W^{1, p_{+}}(\Omega)\right)$. We know that $\widehat{u}_{1} \in \operatorname{int} C_{+}$(see, for example, Gasiński-Papageorgiou [4, p. 739]).

Proposition 3.2. If Hypotheses $H_{0}$ hold and $\bar{u} \in \operatorname{int} C_{+}$is the unique solution of problem (3.3), then $\bar{u}(\cdot)^{-\eta(\cdot)} \in L^{1}(\Omega)$ and for every $h \in W_{0}^{1, p(z)}(\Omega), \bar{u}(\cdot)^{-\eta(\cdot)} h(\cdot) \in L^{1}(\Omega)$.

Proof. From Lemma 14.16, p. 355 of Gilbarg-Trudinger [6], we can find $\delta_{0}>0$ such that, if $\Omega_{\delta_{0}}=\left\{z \in \bar{\Omega}: \widehat{d}(z)<\delta_{0}\right\}$, then $\widehat{d} \in C^{2}\left(\Omega_{\delta_{0}}\right)$. If follows that $\widehat{d} \in \operatorname{int} C_{+}$and so by Proposition 4.1.22, p. 274, of Papageorgiou-Rădulescu-Repovš [9], we can find $c_{7}>0$ such that

$$
\begin{equation*}
c_{7} \widehat{u}_{1} \leq \widehat{d} \quad \text { and } \quad c_{7} \widehat{d} \leq \bar{u} \quad\left(\text { recall } \bar{u} \in \operatorname{int} C_{+}\right) \tag{3.11}
\end{equation*}
$$

From (3.11) we infer that

$$
\bar{u}^{-\eta(\cdot)} \leq c_{8} \widehat{u}_{1}^{-\eta(\cdot)} \quad \text { for some } c_{8}>0
$$

Then the Lemma (in fact its proof to be precise) of Lazer-McKenna [8], implies that $\widehat{u}_{1}^{-\eta(\cdot)} \in$ $L^{1}(\Omega)$. Therefore we have

$$
\bar{u}^{-\eta(\cdot)} \in L^{1}(\Omega)
$$

On the other hand, for every $h \in W_{0}^{1, p(z)}(\Omega)$, we have

$$
\begin{aligned}
\int_{\Omega}\left|\bar{u}^{-\eta(z)} h\right| d z= & \int_{\Omega} \bar{u}^{1-\eta(z)} \frac{|h|}{\bar{u}} d z \\
\leq & c_{9} \int_{\Omega} \frac{|h|}{\bar{u}} d z \quad \text { for some } c_{9}>0 \\
& \left(\text { recall that } \bar{u} \in \operatorname{int} C_{+} \text {and see hypotheses } H_{0}\right) \\
\leq & c_{10} \int_{\Omega} \frac{|h|}{\widehat{d}} d z \quad \text { for some } c_{10}>0 \quad(\text { see (3.11)) } \\
\leq & c_{11}\left\|\frac{h}{\widehat{d}}\right\|_{p(z)} \quad \text { for some } c_{11}>0 \\
\leq & c_{12}\|D h\|_{p(z)} \quad \text { for some } c_{12}>0
\end{aligned}
$$

This last inequality is a consequence of the anisotropic Hardy inequality due to Harjulehto-Hästö-Koskenoja [7]. So, finally we have

$$
\bar{u}(\cdot)^{-\eta(\cdot)} h(\cdot) \in L^{1}(\Omega) \quad \text { for all } h \in W_{0}^{1, p(z)}(\Omega)
$$

## 4 Multiple positive solutions

In this section using $\bar{u} \in \operatorname{int} C_{+}$, the unique positive solution of (3.3), we are able to bypass the singularity and have $C^{1}$-functionals. Working with them, we show that problem (1.1) has at least two positive smooth solutions.

Theorem 4.1. If hypotheses $H_{0}, H_{1}$ hold, then problem (1.1) has at least two positive solutions $u_{0}, \widehat{u} \in$ $\operatorname{int} C_{+}, u_{0} \neq \widehat{u}, u_{0}(z)<\theta$ for all $z \in \bar{\Omega}$.

Proof. Let $\bar{u} \in \operatorname{int} C^{+}$be the unique positive solution of problem (3.3) produced in Proposition 3.1. We introduce the Carathéodory function $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$
g(z, x)= \begin{cases}\bar{u}^{-\eta(z)}+f(z, \bar{u}(z)) & \text { if } x \leq \bar{u}(z)  \tag{4.1}\\ x^{-\eta(z)}+f(z, x) & \text { if } \bar{u}(z)<x .\end{cases}
$$

From Proposition 3.1 we know that $0 \leq \bar{u}(z) \leq \theta$ for all $z \in \bar{\Omega}$. Hence we can consider the truncation of $g(z, \cdot)$ at $\theta$, that is, the Carathéodory function $\widehat{g}: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$
\widehat{g}(z, x)= \begin{cases}g(z, x) & \text { if } x \leq \theta  \tag{4.2}\\ g(z, \theta) & \text { if } \theta<x\end{cases}
$$

We set $G(z, x)=\int_{0}^{x} g(z, s) d s$ and $\widehat{G}(z, x)=\int_{0}^{x} \widehat{g}(z, s) d s$ and consider the functions $\psi, \widehat{\psi}$ : $W_{0}^{1, p(z)}(\Omega) \rightarrow \mathbb{R}$ defined by

$$
\begin{aligned}
& \psi(u)=\int_{\Omega} \frac{1}{p(z)}|D u|^{p(z)} d z-\int_{\Omega} G(z, u) d z, \\
& \widehat{\psi}(u)=\int_{\Omega} \frac{1}{p(z)}|D u|^{p(z)} d z-\int_{\Omega} \widehat{G}(z, u) d z, \quad \text { for all } u \in W_{0}^{1, p(z)}(\Omega) .
\end{aligned}
$$

On account of Proposition 3.2, these functionals are well-defined and in fact Proposition 3.1 of Papageorgiou-Smyrlis [11] implies that $\psi, \widehat{\psi} \in C^{1}\left(W_{0}^{1, p(z)}(\Omega)\right)$.

For every $u \in W_{0}^{1, p(z)}(\Omega)$, we have

$$
\begin{aligned}
\widehat{\psi}(u) \geq & \frac{1}{p_{+}} \rho_{p}(D u)-c_{13} \quad \text { for some } c_{13}>0 \\
& (\text { see }(4.1),(4.2) \text { and Proposition 3.2) } \\
\Rightarrow & \widehat{\psi}(\cdot) \text { is coercive. } \\
& \quad(\text { see Proposition } 2.1 \text { and use Poincaré's inequality }) .
\end{aligned}
$$

The anisotropic Sobolev embedding theorem implies that $\widehat{\psi}(\cdot)$ is sequentially weakly lower semicontinuous.

So, by the Weierstrass-Tonelli theorem, we can find $u_{0} \in W_{0}^{1, p(z)}(\Omega)$ such that

$$
\begin{equation*}
\widehat{\psi}\left(u_{0}\right)=\min \left[\widehat{\psi}(u): u \in W_{0}^{1, p(z)}(\Omega)\right], \tag{4.3}
\end{equation*}
$$

From (4.3) we have

$$
\begin{align*}
& \left\langle\widehat{\psi}^{\prime}\left(u_{0}\right), h\right\rangle=0 \quad \text { for all } h \in W_{0}^{1, p(z)}(\Omega), \\
\Rightarrow & \left\langle A_{p(z)}\left(u_{0}\right), h\right\rangle=\int_{\Omega} \widehat{g}\left(z, u_{0}\right) h d z \quad \text { for all } h \in W_{0}^{1, p(z)}(\Omega) . \tag{4.4}
\end{align*}
$$

In (4.4) first we choose $h=\left[\bar{u}-u_{0}\right]^{+} \in W_{0}^{1, p(z)}(\Omega)$. We have

$$
\begin{aligned}
\left\langle A_{p(z)}\left(u_{0}\right),\left(\bar{u}-u_{0}\right)^{+}\right\rangle & =\int_{\Omega}\left[\bar{u}^{-\eta(z)}+f(z, \bar{u})\right]\left(\bar{u}-u_{0}\right)^{+} d z \quad(\text { see (4.1),(4.2)) } \\
& \geq \int_{\Omega} f(z, \bar{u})\left(\bar{u}-u_{0}\right)^{+} d z \quad\left(\text { since } \bar{u} \in \operatorname{int} C_{+}\right) \\
& =\left\langle A_{p(z)}(\bar{u}),\left(\bar{u}-u_{0}\right)^{+}\right\rangle \quad \text { (see Proposition 3.1) }, \\
& \Rightarrow \bar{u} \leq u_{0} \quad(\text { see Proposition 2.2). }
\end{aligned}
$$

Next in (4.4) we choose $h=\left[u_{0}-\theta\right]^{+} \in W_{0}^{1, p(z)}(\Omega)$. We have

$$
\begin{aligned}
\left\langle A_{p(z)}\left(u_{0}\right),\left(u_{0}-\theta\right)^{+}\right\rangle & =\int_{\Omega}\left[\theta^{-\eta}+f(z, \theta)\right]\left(u_{0}-\theta\right)^{+} d z \quad(\text { see }(4.1),(4.2)) \\
& \left.\leq 0=\left\langle A_{p(z)}(\theta),\left(u_{0}-\theta\right)^{+}\right\rangle \quad \text { (see hypothesis } H_{1}(i i i)\right), \\
& \Rightarrow u_{0} \leq \theta .
\end{aligned}
$$

So, we have proved that

$$
\begin{equation*}
u_{0} \in[\bar{u}, \theta] . \tag{4.5}
\end{equation*}
$$

From (4.5), (4.1), (4.2) and (4.4), we have that $u_{0}$ is a positive solution of (1.1). Invoking Theorem 13.1 of Saaudi-Ghanmi [13] (see also Theorem 3.2 of Byun-Ko [1]), we have that $u_{0} \in \operatorname{int} C_{+}\left(\right.$recall $\left.\bar{u} \in \operatorname{int} C_{+}\right)$.

Now let $\widehat{\xi}_{\theta}>0$ be as postulated by hypothesis $H_{1}(v)$. We have

$$
\begin{align*}
- & \Delta_{p(z)} u_{0}+\widehat{\xi}_{\theta} u_{0}^{p(z)-1}-u_{0}^{-\eta(z)} \\
& =f\left(z, u_{0}\right)+\widehat{\xi}_{\theta} u_{0}^{p(z)-1} \\
& \leq f(z, \theta)+\widehat{\xi}_{\theta} \theta^{p(z)-1} \quad\left(\text { see }(4.5) \text { and hypothesis } H_{1}(v)\right) \\
& \left.\leq-\Delta_{p(z)} \theta+\widehat{\xi}_{\theta} \theta^{p(z)-1}-\theta^{-\eta(z)} \quad \text { (see hypothesis } H_{1}(i i i)\right), \\
\Rightarrow & u_{0}(z)<\theta \quad \text { for all } z \in \bar{\Omega}  \tag{4.6}\\
& \quad \text { (from Proposition A4 of Papageorgiou-Rădulescu-Zhang [10]). }
\end{align*}
$$

It is clear from (4.1) and (4.2) that

$$
\left.\psi\right|_{[0, \theta]}=\left.\widehat{\psi}\right|_{[0, \theta]} .
$$

Since $u_{0} \in \operatorname{int} C_{+}$, we infer that

$$
\begin{align*}
& u_{0} \text { is a local } C_{0}^{1}(\bar{\Omega}) \text { minimizer of } \psi(\cdot) \quad(\text { see }(4.6)), \\
\Rightarrow & u_{0} \text { is a local } W_{0}^{1, p(z)}(\Omega) \text { minimizer of } \psi(\cdot) \quad(\text { see }[10,13]) . \tag{4.7}
\end{align*}
$$

Using (4.1) and the anisotropic regularity theory, we can see that $K_{\psi} \subseteq[\bar{u}) \cap \operatorname{int} C_{+}$. So, we may assume that $K_{\psi}$ is finite or otherwise on account of (4.1) we see that we already have
a whole sequence of distinct positive smooth solutions and so we are done. Then from (4.7) and Theorem 5.7.6, p. 449, of Papageorgiou-Rădulescu-Repovš [9], we know that we can find $\rho \in(0,1)$ small such that

$$
\begin{equation*}
\psi\left(u_{0}\right)<\inf \left[\psi(u):\left\|u-u_{0}\right\|=\rho\right]=m_{\rho} . \tag{4.8}
\end{equation*}
$$

Moreover, hypothesis $H_{1}(i i)$ implies that if $u \in \operatorname{int} C_{+}$, then

$$
\begin{equation*}
\psi(t u) \rightarrow-\infty \quad \text { as } t \rightarrow+\infty . \tag{4.9}
\end{equation*}
$$

Finally from Proposition 4.1 of Gasiński-Papageorgiou [5] (see hypothesis $H_{1}(i i)$ ), we have that

$$
\begin{equation*}
\psi(\cdot) \quad \text { satisfies the C-condition. } \tag{4.10}
\end{equation*}
$$

Then (4.8), (4.9) and (4.10) permit the use of the mountain pass theorem. Therefore we can find $\widehat{u} \in W_{0}^{1, p(z)}(\Omega)$ such that

$$
\begin{align*}
& \widehat{u} \in K_{\psi} \subseteq[\bar{u}) \cap \operatorname{int} C_{+}, m_{\rho} \leq \psi(\widehat{u}),  \tag{4.11}\\
\Rightarrow & \widehat{u} \in \operatorname{int} C_{+} \text {is a positive solution of (1.1) (see (4.1)), } \\
& \widehat{u} \neq u_{0} \quad(\text { see }(4.8) \text { and }(4.11)), u_{0}(z)<\theta \text { for all } z \in \bar{\Omega} .
\end{align*}
$$

## Acknowledgements

The work was supported by NNSF of China Grant No. 12071413, NSF of Guangxi Grant No. 2018GXNSFDA138002.

## References

[1] S. S. Byun, E. Ko, Global $C^{1, \alpha}$ regularity and existence of multiple solutions for singular $p(x)$-Laplacian equations, Calc. Var. 56(2017), Paper No. 76, 29 pp. https://doi.org/10. 1007/s00526-017-1152-6; MR3641923; Zbl 1375.35222
[2] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, Lebesgue and sobolev Spaces with variable exponent, Lecture Notes in Mathematics, Vol. 2017, Springer, Heidelberg 2011. https://doi.org/10.1007/978-3-642-18363-8; MR2790542
[3] X. Fan, Global $C^{1, \alpha}$ regularity for variable exponent elliptic equations in divergence form, J. Differential Equations 235(2007), 397-417. https://doi.org/10.1016/j.jde.2007.01. 008; MR2317489; Zbl 1143.35040
[4] L. Gasiński, N. S. Papageorgiou, Nonlinear analysis, Series in Mathematical Analysis and Applications, Vol. 9, Chapman \& Hall/CRC, Boca Raton, FL, 2006. MR2168068; Zbl 1086.47001
[5] L. Gasiński, N. S. Papageorgiou, Anisotropic nonlinear Neumann problems, Calc. Var. 42(2011), 323-354. https://doi.org/10.1007/s00526-011-0390-2; MR2846259; Zbl 1271.35011
[6] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, 2nd edition, Springer, Berlin, 2001. https://doi.org/10.1007/978-3-642-61798-0; MR1814364
[7] P. Harjulehto, P. Hästö, M. Koskenoja, Hardy's inequality in a variable exponent Sobolev space, Georgian Math. J. 12(2005) 431-442. MR2174945; Zbl 1096.46017
[8] A. C. Lazer, P. McKenna, On a singular nonlinear elliptic boundary-value problem, Proc. Amer. Math. Soc. 111(1991), 721-730. https://doi.org/10.1090/S0002-9939-1991-1037213-9; MR1037213; Zbl 0727.35057
[9] N. S. Papageorgiou, V. D. Rădulescu, D. Repovš, Nonlinear analysis - theory and methods, Springer Nature, Swizerland AG, 2019. https://doi.org/10.1007/978-3-030-03430-6; MR3890060; Zbl 1414.46003
[10] N. S. Papageorgiou, V. D. Rădulescu, Y. Zhang, Anisotropic singular double phase Dirichlet problems, submitted.
[11] N. S. Papageorgiou, G. Smyrlis, Bifurcation-type theorem for singular nonlinear elliptic equations, Methods Appl. Anal. 22(2015), 147-170. https://doi.org/10.4310/MAA. 2015. v22.n2.a2; MR3352702; Zbl 1323.35042
[12] V. D. Rădulescu, D. Repovš, Partial differential equations with variable exponents. Variational methods and qualitative analysis, CRC Press, Boca Raton, FL, 2015. https://doi.org/10. 1201/b18601; MR3379920; Zbl 1343.35003
[13] K. Saoudi, A. Ghanmi, A multiplicity result for a singular equation involving the $p(x)$ Laplace operator, Complex Var. Elliptic Equ. 62(2017), 695-725. https ://doi .org/10.1080/ 17476933.2016.1238466; MR3613680; Zbl 1365.35032
[14] P. Taкač, J. Giacomoni, A $p(x)$-Laplacian extension of the Díaz-Saa inequality and some applications, Proc. Roy. Soc. Edinburgh Sect. A 150(2020), No. 1, 205-232. https://doi. org/10.1017/prm. 2516; MR4065080; Zbl 1436.35210.
[15] Q. Zhang, A strong maximum principle for differential equations with nonstandard $p(x)$-growth conditions. J. Math. Anal. Appl. 312(2005), 24-32. https://doi.org/10.1016/ j.jmaa.2005.03.013; MR2175201; Zbl 1162.35374.

# Necessary and sufficient conditions for the existence of invariant algebraic curves 

Maria V. Demina ${ }^{\boxtimes}$<br>HSE University, 34 Tallinskaya Street, 123458, Moscow, Russian Federation

Received 28 August 2020, appeared 13 July 2021
Communicated by Gabriele Villari


#### Abstract

We present a set of conditions enabling a polynomial system of ordinary differential equations in the plane to have invariant algebraic curves. These conditions are necessary and sufficient. Our main tools include factorizations over the field of Puiseux series near infinity of bivariate polynomials generating invariant algebraic curves. The set of conditions can be algorithmically verified. This fact gives rise to a method, which is able not only to find some irreducible invariant algebraic curves, but also to perform their classification. We study in details the problem of classifying invariant algebraic curves in the most difficult case: we consider differential systems with infinite number of trajectories passing through infinity. As an example, we find necessary and sufficient conditions such that a general polynomial Liénard differential system has invariant algebraic curves. We present a set of all irreducible invariant algebraic curves for quintic Liénard differential systems with a linear damping function. It is supposed in scientific literature that the degrees of their irreducible invariant algebraic curves are bounded by 6 . While we derive irreducible invariant algebraic curves of degree 9.


Keywords: invariant algebraic curves, Darboux polynomials, Liénard differential systems, Puiseux series.
2020 Mathematics Subject Classification: 34C05, 37C80.

## 1 Introduction

Performing the complete classification of trajectories contained in algebraic curves or surfaces for a given polynomial system of ordinary differential equations is a very difficult problem. Such algebraic curves and surfaces producing trajectories of a differential system are called invariants. The knowledge of the set of all irreducible invariants is very important in describing dynamical properties and establishing integrability of a system under consideration. It was noted by Jean Gaston Darboux and Henri Poincaré that the main difficulty in finding irreducible invariants lies in the fact that their degrees are unknown in advance. Nowadays the problem of defining an upper bound on the degrees of irreducible invariant algebraic curves is known as the Poincaré problem. This problem is very difficult in general settings. Solutions are only available in restricted cases, for more details see [20] and references therein.

[^12]Let us consider the following polynomial system of ordinary differential equations in the plane

$$
\begin{equation*}
x_{t}=P(x, y), \quad y_{t}=Q(x, y) \tag{1.1}
\end{equation*}
$$

with coprime polynomials $P(x, y)$ and $Q(x, y) \in \mathbb{C}[x, y]$. By $\mathbb{C}[x, y]$ we denote the ring of bivariate polynomials with coefficients from the field of complex numbers $\mathbb{C}$. The curve $F(x, y)=0$ with $F(x, y) \in \mathbb{C}[x, y] \backslash \mathbb{C}$ is an invariant algebraic curves of this system whenever the following condition is valid $\left.F_{t}\right|_{F=0}=\left.\left(P F_{x}+Q F_{y}\right)\right|_{F=0}=0$. If $F(x, y)$ is irreducible in $\mathbb{C}[x, y]$, then the ideal generated by $F(x, y)$ is radical. Consequently, there exists an element $\lambda(x, y)$ of the ring $\mathbb{C}[x, y]$ such that the following linear partial differential equation $P(x, y) F_{x}+Q(x, y) F_{y}=\lambda(x, y) F$ is satisfied. The polynomial $\lambda(x, y)$ is called the cofactor of the invariant algebraic curve $F(x, y)=0$. The degree of $\lambda(x, y)$ is at most $d-1$, where $d$ is the maximum between the degrees of the polynomials $P(x, y)$ and $Q(x, y)$. Let the variable $y$ be privileged with respect to the variable $x$, then the function $y(x)$ satisfies the following algebraic first-order ordinary differential equation

$$
\begin{equation*}
P(x, y) y_{x}-Q(x, y)=0 \tag{1.2}
\end{equation*}
$$

The aim of the present article is to present new necessary and sufficient conditions for the existence of invariant algebraic curves. Our main tools include asymptotic analysis of solutions to equation (1.2) and some results of algebraic geometry. The problem of finding a set of conditions satisfied by a polynomial system of ordinary differential equations in the plane with invariant algebraic curves was previously considered by J. Chavarriga et al. [3]. The method of article [3] also uses the local properties of solutions of differential system (1.2). The conditions obtained by J. Chavarriga et al. are necessary conditions, but not sufficient. Let us name some other works [15-17], which deal with algebraic functions, asymptotic series and their role in finding first integrals and invariant algebraic curves of system (1.1).

Puiseux (or fractional power) series generalize Laurent series and can be used if one needs to find local representations of solutions for algebraic equations of the form $F(x, y(x))=0$ with $F(x, y) \in \mathbb{C}[x, y] \backslash \mathbb{C}[x]$. A Puiseux series in a neighborhood of the point $x=\infty$ reads as

$$
\begin{equation*}
y(x)=\sum_{l=0}^{+\infty} c_{l} x^{\frac{l_{0}-l}{n_{0}}} \tag{1.3}
\end{equation*}
$$

where $l_{0} \in \mathbb{Z}, n_{0} \in \mathbb{N}$. The set of formal Puiseux series given by (1.3) produces an algebraically closed field, which we denote by $\mathbb{C}_{\infty}\{x\}$. In addition, we shall consider the ring $\mathbb{C}_{\infty}\{x\}[y]$ of polynomials in one variable with coefficients from the field $\mathbb{C}_{\infty}\{x\}$. It follows from the algebraic closeness of the field $\mathbb{C}_{\infty}\{x\}$ that every element from the ring $\mathbb{C}_{\infty}\{x\}[y]$ is a product of polynomials in $y$ of degree at most one. The differentiation in the field $\mathbb{C}_{\infty}\{x\}$ is defined as a formal operation with most of the properties similar to those valid for convergent Puiseux series. Any bivariate polynomial $F(x, y) \in \mathbb{C}[x, y]$ can be viewed as an element of the ring $\mathbb{C}_{\infty}\{x\}[y]$. Consequently, for the algebraic curve $F(x, y)=0$ given by the polynomial $F(x, y)$, we can construct a factorization into a zero-degree and first-degree factors in the ring $\mathbb{C}_{\infty}\{x\}[y]$, see $[5,6,10,24]$.

All the Puiseux series solving equation (1.2) can be found using algorithms of the power geometry [1,2] and Painlevé methods [19]. After the classification of Puiseux series satisfying equation (1.2) is completed, the computation of invariant algebraic curves $F(x, y)=0$ can be made purely algebraic. Indeed, one should require that the non-polynomial part of the factorization for the polynomial $F(x, y)$ in the ring $\mathbb{C}_{\infty}\{x\}[y]$ vanishes. Generally speaking,
this approach gives an infinite algebraic system. Due to the Hilbert's basis theorem only finite number of equations can be considered in practice. Note that the roles of $x$ and $y$ can be changed.

Let us name other methods of finding invariant algebraic curves. The most commonly used methods include the method of undetermined coefficients, the method of the extactic polynomial $[4,21]$, and an algorithm based on decomposing the vector field related to the original differential system into weight-homogenous components [19]. The method of undetermined coefficients is able to find invariant algebraic curves of fixed degrees only. In addition, the computations may be sufficiently involved. The method of the extactic polynomial was introduced by M. N. Lagutinski [21] and further developed by C. Christopher et al. [4] This method requires calculating certain determinants that are as a rule sufficiently huge. In addition, the method needs a priori information about an upper bound on the degrees of irreducible invariant algebraic curves. The algorithm of decomposing the vector field related to the original system into weight-homogenous components gives an infinite sequence of partial differential equations. On the contrary, the second part of the method of Puiseux series is purely algebraic. Moreover, the latter method is capable to solve the Poincaré for a given polynomial differential system. This comparison shows that the method of Puiseux series presented in works $[5,6,10]$ and developed in this article is a natural and visual method of finding and classifying invariant algebraic curves of polynomial differential systems in the plane (1.1). Let us mention that the problem of finding all irreducible invariant algebraic curves of differential systems (1.1) with infinite number of trajectories passing through infinity was not considered in articles $[5,6,10]$. Meanwhile this case turns out to be the most difficult. In this work our goal is to fill this gap. In other words we shall examine the situation with infinite number of Puiseux series near the point $x=\infty$ that satisfy equation (1.2).

As an application of our method we shall consider the famous Liénard differential systems. The systems of first-order ordinary differential equations given by

$$
\begin{equation*}
x_{t}=y, \quad y_{t}=-f(x) y-g(x) \tag{1.4}
\end{equation*}
$$

are commonly referred to as Liénard differential systems. These systems are used to model different phenomena in physics, chemistry, biology, economics, etc. In this article we consider polynomial Liénard differential systems, i.e. $f(x)$ and $g(x)$ are polynomials

$$
\begin{equation*}
f(x)=f_{0} x^{m}+\cdots+f_{m}, \quad g(x)=g_{0} x^{n}+\cdots+g_{n}, \quad f_{0} g_{0} \neq 0 \tag{1.5}
\end{equation*}
$$

with coefficients in the field C. K. Odani proved that Liénard systems with $n \leq m$ have no invariant algebraic curves with the exception for some trivial cases [23]. Integrability properties of these families of systems under the condition $n \leq m$ were studied by J. Llibre and C. Valls [22]. H. Żoladek considered the problem of finding limit cycles contained in the ovals of hyperelliptic invariant algebraic curves $(y-p(x))^{2}-q(x)=0$ with $p(x), q(x) \in \mathbb{C}[x]$, see [25]. The general structure of irreducible invariant algebraic curves and some other properties in the case $m<n<2 m+1$ were investigated in articles [6,10]. Explicit expressions of invariant algebraic curves for Liénard differential systems with $m=1$ and $n=2$ where presented in work [14]. This article is devoted to the leftover cases: $n \geq 2 m+1$. Let us note that the case $n=2 m+1$ is in certain sense degenerate and the problem of classifying invariant algebraic curves for $n=2 m+1$ is very complicated. This degeneracy can be explained analyzing properties of Puiseux series satisfying an algebraic first-order ordinary differential equation of the form (1.2) related to associated Liénard differential systems.

This article is organized as follows. In Section 2 we present and prove our main results and consider some computational aspects of solving an algebraic system resulting form our theorems. In Section 3 we study Liénard differential systems with $n \geq 2 m+1$ in details. In particular, we present the general structure of their invariant algebraic curves and cofactors. Finally, in Section 4 we derive the complete classification of irreducible invariant algebraic curves of systems (1.4) with $m=1(\operatorname{deg} f(x)=1)$ and $n=5(\operatorname{deg} g(x)=5)$. In the Appendix, an algorithm of finding Puiseux series solving an algebraic first-order ordinary differential equation is described.

## 2 Computational aspects of the Puiseux series method

Let us begin this section with some preliminary observations resulting from a factorization of an invariant algebraic curve $F(x, y)=0$ of differential system (1.1) in the ring $\mathbb{C}_{\infty}\{x\}[y]$. It is straightforward to show that invariant algebraic curves of differential system (1.1) capture Puiseux series satisfying equation (1.2).

Lemma 2.1 ([5]). Let $y(x)$ be a Puiseux series near the point $x=\infty$ that satisfies the equation $F(x, y)=0$ with $F(x, y)=0$ being an invariant algebraic curve of differential system (1.1) such that $F(x, y) \in \mathbb{C}[x, y] \backslash \mathbb{C}[x]$. Then the series $y(x)$ solves equation (1.2).

Suppose $S(x, y)$ is an element of the ring $\mathbb{C}_{\infty}\{x\}[y]$. Let us introduce two operators of projection acting in this ring. The first operator $\{S(x, y)\}_{+}$gives the sum of the monomials of $S(x, y)$ with non-negative integer powers. In other words, $\{S(x, y)\}_{+}$yields the polynomial part of $S(x, y)$. Analogously, the projection $\{S(x, y)\}_{-}=S(x, y)-\{S(x, y)\}_{+}$produces the non-polynomial part of $S(x, y)$. It is straightforward to show that these projections are linear operators. The action of the projection operators can be extended to the ring of Puiseux series in $y$ near the point $y=\infty$ with coefficients from the field $\mathbb{C}_{\infty}\{x\}$.

By $\mu(x)$ we shall denote the highest-order coefficient (with respect to $y$ ) of the bivariate polynomial $F(x, y)$ producing the invariant algebraic curve $F(x, y)=0$ of differential system (1.1). The following theorem was proved in articles [5,10].

Theorem 2.2 ([5, 10]). Let $F(x, y)=0$ with $F(x, y) \in \mathbb{C}[x, y] \backslash \mathbb{C}[x]$ be an irreducible invariant algebraic curve of differential system (1.1). Then $F(x, y)$ and its cofactor $\lambda(x, y)$ take the form

$$
\begin{align*}
& F(x, y)=\left\{\mu(x) \prod_{j=1}^{N}\left\{y-y_{j}(x)\right\}\right\}_{+} \\
& \lambda(x, y)=\left\{P(x, y) \sum_{m=0}^{\infty} \sum_{l=1}^{L} \frac{v_{l} x_{l}^{m}}{x^{m+1}}+\sum_{m=0}^{\infty} \sum_{j=1}^{N} \frac{\left\{Q(x, y)-P(x, y) y_{j, x}\right\} y_{j}^{m}}{y^{m+1}}\right\}_{+} \tag{2.1}
\end{align*}
$$

where $y_{1}(x), \ldots, y_{N}(x)$ are pairwise distinct Puiseux series in a neighborhood of the point $x=\infty$ that satisfy equation (1.2), $x_{1}, \ldots, x_{L}$ are pairwise distinct zeros of the polynomial $\mu(x) \in \mathbb{C}[x]$ with multiplicities $v_{1}, \ldots, v_{L} \in \mathbb{N}$ and $L \in \mathbb{N} \cup\{0\}$. The degree of $F(x, y)$ with respect to $y$ does not exceed the number of distinct Puiseux series of the from (1.3) satisfying equation (1.2) whenever the latter is finite. If $\mu(x)=\mu_{0}$, where $\mu_{0} \in \mathbb{C}$, then we suppose that $L=0$ and the first series is absent in the expression for the cofactor $\lambda(x, y)$.

Theorem 2.2 gives rise to the following algorithm of finding invariant algebraic curves $F(x, y)=0$ with $F(x, y) \in \mathbb{C}[x, y] \backslash \mathbb{C}[x]$.

At the first step one should construct all the Puiseux series (near finite points and infinity) that satisfy equation (1.2). Algorithms of classifying Puiseux series solving an algebraic ordinary differential equation are available in the framework of the power geometry $[1,2]$ and the Painlevé methods [19], see Appendix.

At the second step one uses Theorem 2.2 in order to derive the structure of an irreducible invariant algebraic curve and its cofactor, see relations (2.1). Possible zeros of the polynomial $\mu(x)$ can be obtained using Puiseux series near finite points possessing certain properties. We shall not discuss this problem here, for more details see [10]. Note that at this step all possible combinations of Puiseux series near infinity found at the first step should be considered if one wishes to classify irreducible invariant algebraic curves. Requiring that the following condition

$$
\begin{equation*}
\left\{\mu(x) \prod_{j=1}^{N}\left\{y-y_{j}(x)\right\}\right\}_{-}=0 \tag{2.2}
\end{equation*}
$$

is satisfied yields a system of algebraic equations.
At the third step one solves the algebraic system and makes the verification substituting the resulting polynomial $F(x, y)$ related to the invariant algebraic curve and its cofactor $\lambda(x, y)$ into equation

$$
\begin{equation*}
P(x, y) F_{x}+Q(x, y) F_{y}=\lambda(x, y) F . \tag{2.3}
\end{equation*}
$$

Interestingly, we do not need to consider the convergence of formal Puiseux series solving equation (1.2). Indeed, we perform all the steps of the method working with formal series, and finally, if some formal Puiseux series enters the factorization in the ring $\mathbb{C}_{\infty}\{x\}[y]$ of the resulting polynomial $F(x, y)$ giving the invariant algebraic curve $F(x, y)=0$, then this series is convergent in some domain by a Newton-Puiseux theorem.

The aim of the present article is to consider the problem of constructing and solving the system arising at the third step of the method.

Let us leave for a while the $x$-dependence of the elements $y_{j}(x)$ from the field $\mathbb{C}_{\infty}\{x\}$ and consider the ring Sym $\subset \mathbb{C}\left[y_{1}, \ldots, y_{N}\right]$ of symmetric polynomials in $N$ variables. It is a classical result that Sym is isomorphic to a polynomial ring with $N$ generators. The most commonly used generators include elementary symmetric polynomials given by

$$
\begin{equation*}
s_{k}=\sum_{1 \leq j_{1}<j_{2}<\cdots<j_{k} \leq N} y_{j_{1}} y_{j_{2}} \cdots y_{j_{k}} \quad 1 \leq k \leq N \tag{2.4}
\end{equation*}
$$

and power-sum symmetric polynomials

$$
\begin{equation*}
S_{k}=\sum_{j=1}^{N} y_{j}^{k}, \quad 1 \leq k \leq N . \tag{2.5}
\end{equation*}
$$

These generators are related via the Newton's identities of the form

$$
\begin{align*}
k s_{k} & =\sum_{j=1}^{k}(-1)^{j-1} s_{k-j} S_{j}, \quad 1 \leq k \leq N  \tag{2.6}\\
s_{k} & =(-1)^{k-1} k s_{k}+\sum_{j=1}^{k-1}(-1)^{k+j-1} s_{k-j} S_{j}, \quad 1 \leq k \leq N
\end{align*}
$$

where additionally should be set $s_{0}=1$. It is not an easy problem to find the coefficients of the Puiseux series given by the elementary symmetric polynomials $s_{k}\left(y_{1}(x), \ldots, y_{N}(x)\right)$ with
$k>1$ if $N$ is not known in advance. This is due to the fact that the coefficients of Puiseux series satisfying an algebraic ordinary differential equation are defined via recurrence relations. At the same time computing coefficients of symmetric polynomials $S_{k}\left(y_{1}(x), \ldots, y_{N}(x)\right)$ is straightforward. The following theorem contains necessary and sufficient conditions enabling the existence of invariant algebraic curves.

Theorem 2.3. The polynomial $F(x, y) \in \mathbb{C}[x, y] \backslash \mathbb{C}[x]$ of degree $N>0$ with respect to $y$ gives an invariant algebraic curve $F(x, y)=0$ of differential system (1.1) if and only if there exist $N$ Puiseux series $y_{1}(x), \ldots, y_{N}(x)$ from the field $\mathbb{C}_{\infty}\{x\}$ that solve equation (1.2) and satisfy the conditions

$$
\begin{equation*}
\left\{\sum_{j=1}^{k}(-1)^{j-1} w_{k-j}(x) S_{j}\left(y_{1}(x), \ldots, y_{N}(x)\right)\right\}_{-}=0, \quad 1 \leq k \leq N \tag{2.7}
\end{equation*}
$$

where $w_{m}(x) \in \mathbb{C}[x]$ are defined as

$$
\begin{equation*}
w_{m}(x)=\left\{\frac{1}{m} \sum_{j=1}^{m}(-1)^{j-1} w_{m-j}(x) S_{j}\left(y_{1}(x), \ldots, y_{N}(x)\right)\right\}_{+}, 1 \leq m \leq N \tag{2.8}
\end{equation*}
$$

and $w_{0}(x)=\mu(x)$ with $\mu(x) \in \mathbb{C}[x]$ being the highest-order coefficient with respect to $y$ of the polynomial $F(x, y)$.

Proof. Let us prove necessity of conditions (2.7). Factorizing the polynomial $F(x, y)$ giving an invariant algebraic curve $F(x, y)=0$ of differential system (1.1) in the ring $\mathbb{C}_{\infty}\{x\}[y]$ yields

$$
\begin{equation*}
F(x, y)=\mu(x) \prod_{j=1}^{N}\left\{y-y_{j}(x)\right\} \tag{2.9}
\end{equation*}
$$

where it follows from Lemma 2.1 that the Puiseux series $y_{1}(x), \ldots, y_{N}(x)$ satisfy equation (1.2). It is straightforward to rewrite relation (2.9) in the form

$$
\begin{equation*}
F(x, y)=\mu(x) \sum_{j=0}^{N}(-1)^{j} s_{j}\left(y_{1}(x), \ldots, y_{N}(x)\right) y^{N-j} \tag{2.10}
\end{equation*}
$$

The non-polynomial part of this expression vanishes and the elements $\mu(x) s_{m}\left(y_{1}(x), \ldots, y_{N}(x)\right)$ should be polynomials coinciding with $w_{m}(x)$ given in (2.8). Considering the non-polynomial coefficients at $y^{N-k}$, we obtain the conditions

$$
\begin{equation*}
\left\{\mu(x) s_{k}\left(y_{1}(x), \ldots, y_{N}(x)\right)\right\}_{-}=0, \quad 1 \leq k \leq N \tag{2.11}
\end{equation*}
$$

Using relations (2.6), we see that conditions (2.11) are equivalent to (2.7).
In order to verify sufficiency of conditions (2.7), let us consider a formal expression (2.9) and at first prove that it is a polynomial in $\mathbb{C}[x, y]$. We need to establish that for each $k$ from 1 to $N$ the coefficient at $y^{N-k}$ in expression (2.9) is a polynomial. We shall use induction on $k$. If $k=1$, then condition (2.7) reads as $\left\{\mu(x) S_{1}\left(y_{1}(x), \ldots, y_{N}(x)\right)\right\}_{-}=0$ and we see that the coefficient at $y^{N-1}$ in relations (2.9) and (2.10) is a polynomial in $x$ taking the form $-w_{1}(x)$, where $w_{1}(x)=\left\{\mu(x) S_{1}\left(y_{1}(x), \ldots, y_{N}(x)\right)\right\}_{+}$. Let us suppose that the coefficients at $y^{N-k}$ with $1<k \leq l$ are polynomials in $x$. These polynomials we denote as $(-1)^{k} w_{k}(x)$. It is straightforward to prove that they are given by relations (2.8) with $1<k \leq l$.

The coefficient at $y^{N-(l+1)}$ in relation (2.9) is equal to $(-1)^{l+1} \mu(x) s_{l+1}\left(y_{1}(x), \ldots, y_{N}(x)\right)$. Using expression (2.6), we find

$$
\begin{equation*}
\mu(x) s_{l+1}\left(y_{1}(x), \ldots, y_{N}(x)\right)=\frac{1}{l+1} \sum_{j=1}^{l+1}(-1)^{j-1} \mu(x) s_{l+1-j} S_{j} \tag{2.12}
\end{equation*}
$$

According to the induction hypothesis, we see that the elements $\mu(x) s_{l+1-j}, 1 \leq j \leq l+1$ are polynomials in $x$ coinciding with $w_{l+1-j}(x), 1 \leq j \leq l+1$ and consequently it follows from condition (2.7) at $k=l+1$ that the coefficient at $y^{N-(l+1)}$ is a polynomial in $x$. Thus we conclude that expression (2.9) gives a bivariate polynomial $F(x, y)$ from the ring $\mathbb{C}[x, y]$.

Finally, let us establish that the polynomial $F(x, y)$ indeed gives an invariant algebraic curve $F(x, y)=0$ of differential system (1.1). Let $f(x, y) \in \mathbb{C}[x, y] \backslash \mathbb{C}[x]$ be an irreducible factor of the polynomial $F(x, y)$. The element $P f_{x}+Q f_{y}$ is also a polynomial, which we denote as $h(x, y)$, i.e. $h(x, y)=P f_{x}+Q f_{y}$. Let us take one of the Puiseux series near infinity $y_{j}(x)$ that satisfies the equation $f\left(x, y_{j}(x)\right)=0$. Differentiating this equation, we obtain $f_{x}\left(x, y_{j}(x)\right)+$ $f_{y}\left(x, y_{j}(x)\right) y_{j, x}=0$. Since $f(x, y)$ divides $F(x, y)$, we see that the series $y_{j}(x)$ solves equation (1.2) and we get $P\left(x, y_{j}(x)\right) y_{j, x}-Q\left(x, y_{j}(x)\right)=0$. Combining the equations $f_{x}\left(x, y_{j}(x)\right)+$ $f_{y}\left(x, y_{j}(x)\right) y_{j, x}=0$ and $P\left(x, y_{j}(x)\right) y_{j, x}-Q\left(x, y_{j}(x)\right)=0$ yields the relation $h\left(x, y_{j}(x)\right)=O$, where $O$ is the zero element of the field $\mathbb{C}_{\infty}\{x\}$. Note that $P\left(x, y_{j}(x)\right) \neq O$. Indeed, assuming the converse, we find from equation (1.2) that $Q\left(x, y_{j}(x)\right)=O$. This fact contradicts the assumption that the polynomials $P(x, y)$ and $Q(x, y)$ are coprime in the ring $\mathbb{C}[x, y]$. It follows from the relations $f\left(x, y_{j}(x)\right)=0$ and $h\left(x, y_{j}(x)\right)=0$, that two algebraic curves $f(x, y)=0$ and $h(x, y)=0$ intersect in an infinite number of points inside the domain of convergence of the series $y_{j}(x)$. Using the Bézout's theorem, we see that there exists a polynomial both dividing $f(x, y)$ and $h(x, y)$. Since $f(x, y)$ is irreducible, we find that $h(x, y)=\lambda_{0}(x, y) f(x, y)$ with $\lambda_{0}(x, y) \in \mathbb{C}[x, y]$. Recalling the definition of $h(x, y)$, we conclude that the polynomial $f(x, y)$ gives an invariant algebraic curve of differential system (1.1) and the same is true for all other irreducible divisors of $F(x, y)$. Thus, so does $F(x, y)$. This completes the proof.

If the highest-order coefficient (with respect to $y$ ) of the polynomial $F(x, y)$ is a constant, then there is no loss of generality in setting $\mu(x)=1$. Repeating the reasoning of Theorem 2.3 for this particular case we obtain the following lemma.

Lemma 2.4. The polynomial $F(x, y) \in \mathbb{C}[x, y] \backslash \mathbb{C}[x]$ of degree $N>0$ with respect to $y$ and with $\mu(x)=1$ gives an invariant algebraic curve $F(x, y)=0$ of differential system (1.1) if and only if there exist $N$ Puiseux series $y_{1}(x), \ldots, y_{N}(x)$ defined in a neighborhood of the point $x=\infty$ that solve equation (1.2) and satisfy the conditions

$$
\begin{equation*}
\left\{\sum_{j=1}^{N} y_{j}^{k}(x)\right\}_{-}=0, \quad 1 \leq k \leq N \tag{2.13}
\end{equation*}
$$

Again we remark that an algorithm of finding Puiseux series solving a first-order algebraic ordinary differential equation is presented in the Appendix. It follows from Theorem 2.2 that the Puiseux series in Theorem 2.3 and in Lemma 2.4 should be pairwise distinct whenever one wishes to find irreducible invariant algebraic curves.

If all the Puiseux series near the point $x=\infty$ satisfying equation (1.2) have uniquely determined coefficients, then the degrees with respect to $y$ of bivariate polynomials giving irreducible invariant algebraic curves of differential system (1.1) are bounded by the number of distinct Puiseux series. This fact was established in Theorem 2.2. Consequently, the
algebraic system produced by Theorem 2.3 involves only the parameters of the original system and possibly the zeros of the polynomial $\mu(x)$, which are connected with the existence of Puiseux series near finite points that solve equation (1.2) and have certain properties [10]. While if there exists a family of Puiseux series near the point $x=\infty$ solving equation (1.2) such that these series possess arbitrary coefficients resulting from the presence of a rational non-negative Fuchs index, then it is unknown in advance how many times this family should be taken in representation (2.1) of the polynomial $F(x, y)$ producing irreducible invariant algebraic curve $F(x, y)=0$. Let us consider one of such families with an arbitrary coefficient $c_{m}$ where $m \in \mathbb{N}_{0}$. The coefficient $c_{m}$ is arbitrary in the sense that it is not provided by equation (1.2). Suppose that representation (2.1) involves this family of series $M$ times with $M \in \mathbb{N}$. The coefficients $c_{m}^{(1)}, \ldots, c_{m}^{(M)}$ will entre the algebraic system. The problem is to find not only $c_{m}^{(1)}, \ldots, c_{m}^{(M)}$, but also the number $M$. Note that the coefficients $c_{m}^{(1)}, \ldots, c_{m}^{(M)}$ should be pairwise distinct whenever the resulting invariant algebraic curve is irreducible. Due to the invariance of the polynomial $F(x, y)$ with respect to permutations of the Puiseux series $y_{1}(x), \ldots, y_{N}(x)$ and the structure of recurrence relations satisfied by coefficients of a Puiseux series solving an algebraic first-order ordinary differential equation, we conclude that the polynomial $F(x, y)$ inherits the invariance with respect to the permutations of $c_{m}^{(1)}, \ldots, c_{m}^{(M)}$. Consequently, the algebraic system with the exception for some degenerate cases can be rewritten in terms of invariants

$$
\begin{equation*}
C_{k}=\sum_{j=1}^{M}\left(c_{m}^{(j)}\right)^{k} . \tag{2.14}
\end{equation*}
$$

The same result follows from Theorem 2.3. In relation (2.14) we should set $k \in \mathbb{N}$ whenever the family of Puiseux series under consideration corresponds to an edge of the Newton polygon related to equation (1.2). While $k \in \mathbb{Z}$ provided that the family of Puiseux series in question corresponds to a vertex of the Newton polygon. Thus, we conclude that the variables $M$ and $\left\{C_{k}\right\}$ should be added to the list of variables. Further, one needs to study the structure of the polynomial ideal generated by the algebraic system in the ring of polynomials in the variables including the parameters of the original system, possible zeroes of the polynomial $\mu(x),\left\{C_{k}\right\}$, and $M$. Solutions with $M \in \mathbb{N}$ should be selected. If several families of Puiseux series near the point $x=\infty$ that have arbitrary coefficients take part in representation (2.1), then the variables $\left\{C_{k}\right\}$ and $M$ should be introduced for each family of series.

It was proved in article [10] that there exists at most one irreducible invariant algebraic curve $F(x, y)=0$ of differential system (1.1) such that a Puiseux series near the point $x=\infty$ that solves equation (1.2) and possesses uniquely determined coefficients enters the representation of the polynomial $F(x, y)$ in the field $\mathbb{C}_{\infty}\{x\}$. Consequently, the most difficult problem is finding irreducible invariant algebraic curves given by representation (2.1) with all the Puiseux series possessing coefficients not provided by equation (1.2).

The following theorem is very important for practical solving the algebraic system in the latter case.

Theorem 2.5. Let us consider the algebraic system of equations

$$
\begin{equation*}
\sum_{j=1}^{M}\left(a_{j}\right)^{k}=M g_{k}, \quad k \in \mathbb{N}, \tag{2.15}
\end{equation*}
$$

where $a_{1}, \ldots, a_{M} \in \mathbb{C}$ and $M \in \mathbb{N}$ are unknown variables, $\left\{g_{k}\right\}$ are given complex numbers. If for some $M_{0} \in \mathbb{N}$ this system has a solution $\left(a_{1}, \ldots, a_{M_{0}}\right.$ ) with $a_{j_{1}} \neq a_{j_{2}}$ whenever $j_{1} \neq j_{2}$, then there are
no other solutions of this system except for $M=l M_{0}$, where $l \in \mathbb{N} \backslash\{1\}$. The latter solutions involve $l$ multiple roots for each element of the tuple $\left(a_{1}, \ldots, a_{M_{0}}\right)$. Note that tuples obtained from each other by permutations of their elements are supposed to be equivalent. We consider only one representative from each equivalence class.

Proof. It is straightforward to verify that there exist "multiple" solutions for any solution with pairwise distinct elements of the tuple $\left(a_{1}, \ldots, a_{M_{0}}\right)$. Let us establish that there are no other solutions. The proof is by contradiction. Suppose that system (2.15) possesses a solution ( $\tilde{a}_{1}, \ldots, \tilde{a}_{M_{1}}$ ) with $M=M_{1}$, where either $M_{1} \neq l M_{0}$ or $M_{1}=l M_{0}$ and the tuple ( $\tilde{a}_{1}$, $\left.\ldots, \tilde{a}_{M_{1}}\right)$ does not coincide with that described in the statement of the theorem. We recall that the left-hand side of relations (2.15) represent power-sum symmetric polynomials in the ring $\mathbb{C}\left[a_{1}, \ldots, a_{M}\right]$ :

$$
\begin{equation*}
p_{k}=\sum_{j=1}^{M}\left(a_{j}\right)^{k} . \tag{2.16}
\end{equation*}
$$

Let us introduce the elementary symmetric polynomials

$$
\begin{equation*}
e_{k}=\sum_{1 \leq j_{1}<j_{2}<\cdots<j_{k} \leq M} a_{j_{1}} a_{j_{2}} \cdots a_{j_{k^{\prime}}} \tag{2.17}
\end{equation*}
$$

which are uniquely expressible via power-sum symmetric polynomials. Further, we consider the following algebraic equation of degree $M_{2}=M_{0} M_{1}$

$$
\begin{align*}
& a^{M_{2}}-e_{1}\left(a_{1}, \ldots, a_{M_{2}}\right) a^{M_{2}-1}+e_{2}\left(a_{1}, \ldots, a_{M_{2}}\right) a^{M_{2}-2} \\
& \quad+\cdots+(-1)^{M_{2}} e_{M_{2}}\left(a_{1}, \ldots, a_{M_{2}}\right)=0 \tag{2.18}
\end{align*}
$$

It is straightforward to show that this equation possesses two distinct sets of solutions: $M_{1}$ multiple roots for each element of the tuple ( $a_{1}, \ldots, a_{M_{0}}$ ) and $M_{0}$ multiple roots for each element of the tuple ( $\tilde{a}_{1}, \ldots, \tilde{a}_{M_{1}}$ ). The set of solutions of a polynomial equation in one variable over the field $\mathbb{C}$ is unique up to the permutation of the roots. This contradiction completes the proof.

If all the Puiseux series in representation (2.1) possess arbitrary coefficients, then the elements $C_{k}$ given in (2.14) are of the form $C_{k}=M g_{k}$. It follows from the fact that $F^{l}(x, y)=0$ with $l \in \mathbb{N}$ is an invariant algebraic curve whenever so does $F(x, y)=0$. Consequently, Theorem 2.5 can be used for establishing uniqueness of irreducible invariant algebraic curves. Indeed, as soon as a solution $\left(a_{1}, \ldots, a_{M_{0}}\right)$ with $a_{j_{1}} \neq a_{j_{2}}$ and $M_{0} \in \mathbb{N}$ is found one should stop calculations because other solutions will give reducible invariant algebraic curves. Examples will be given in Section 4.

## 3 Invariant algebraic curves for Liénard differential systems

Now our aim is to apply the general results of the previous section to polynomial Liénard differential systems (1.4). Supposing that the variable $y$ is dependent and the variable $x$ is independent, we see that the function $y(x)$ satisfies the following first-order ordinary differential equation

$$
\begin{equation*}
y y_{x}+f(x) y+g(x)=0 \tag{3.1}
\end{equation*}
$$

Let us begin with simple properties of invariant algebraic curves and their cofactors.

Lemma 3.1. Suppose $F(x, y)=0$ with $F(x, y) \in \mathbb{C}[x, y] \backslash \mathbb{C}$ is an invariant algebraic curve of a Liénard differential system. The following statements are valid.

1. There are no invariant algebraic curves such that $F(x, y) \in \mathbb{C}[x]$.
2. The highest-order coefficient with respect to $y$ of the polynomial $F(x, y)$ is a constant.
3. The cofactors of invariant algebraic curves are independent of $y$.

Proof. Substituting $\lambda(x, y)=\lambda_{0}(x) y^{l}, F(x, y)=\mu(x) y^{N}$ with $l, N \in \mathbb{N} \cup\{0\}$ into the partial differential equation

$$
\begin{equation*}
y F_{x}-\{f(x) y+g(x)\} F_{y}=\lambda(x, y) F \tag{3.2}
\end{equation*}
$$

and balancing the highest-order terms with respect to $y$, we conclude that $\mu(x) \in \mathbb{C}, l=0$, and $N \in \mathbb{N}$. This means that cofactors of invariant algebraic curves do not depend on $y$ and there are no invariant algebraic curves independent of $y$. In addition, we observe that the highest-order coefficient (with respect to $y$ ) of $F(x, y)$ is a constant. Without loss of generality we set $\mu(x)=1$. This result can be also obtained using the structure of Puiseux series near finite points that satisfy equation (3.1), for more details see [10].

Our next step is to establish that the necessary and sufficient conditions of Theorem 2.3 and Lemma 2.4 become very easy in the case of Liénard differential systems.

Theorem 3.2. The polynomial $F(x, y) \in \mathbb{C}[x, y] \backslash \mathbb{C}$ of degree $N \in \mathbb{N}$ with respect to $y$ gives an invariant algebraic curve of a Liénard differential system if and only if there exist $N$ Puiseux series $y_{1}(x), \ldots, y_{N}(x)$ defined in a neighborhood of the point $x=\infty$ that solve equation (3.1) and satisfy the conditions

$$
\begin{equation*}
\left\{\sum_{j=1}^{N} y_{j}(x)\right\}_{-}=0 \tag{3.3}
\end{equation*}
$$

Proof. It follows from Lemma 3.1 that Liénard differential systems do not have invariant algebraic curves with generating polynomials independent of $y$. Let us suppose that $F(x, y)=0$ is an invariant algebraic curve of a system (1.4) such that $F(x, y) \in \mathbb{C}[x, y] \backslash \mathbb{C}[x]$.

We shall use the results of Lemma 2.4. Let us show that if conditions (2.13) are satisfied at $k=1$, then they are also satisfied for all $k \in \mathbb{N}$. Our proof is by induction on $k$. Suppose that conditions (2.13) with $k \leq m$ hold. The Puiseux series appearing in these conditions solve equation (3.1). Substituting $y(x)=y_{j}(x)$ into equation (3.1) and multiplying the result by $y_{j}^{m-1}$, we get

$$
\begin{equation*}
\frac{1}{m+1} \frac{d}{d x}\left(y_{j}^{m+1}\right)=-f(x) y_{j}^{m}-g(x) y_{j}^{m-1} \tag{3.4}
\end{equation*}
$$

Performing the summation, we obtain

$$
\begin{equation*}
\frac{1}{m+1} \frac{d}{d x}\left(\sum_{j=1}^{N} y_{j}^{m+1}\right)=-f(x) \sum_{j=1}^{N} y_{j}^{m}-g(x) \sum_{j=1}^{N} y_{j}^{m-1} \tag{3.5}
\end{equation*}
$$

It follows from the induction hypothesis that the right-hand side in (3.5) is a polynomial. This yields

$$
\begin{equation*}
\frac{1}{m+1}\left\{\frac{d}{d x}\left(\sum_{j=1}^{N} y_{j}^{m+1}\right)\right\}_{-}=0 \tag{3.6}
\end{equation*}
$$

It is straightforward to see that for any element $y(x)$ of the field $\mathbb{C}_{\infty}\{x\}$ the following relation $\{y(x)\}_{-}=0$ is valid whenever $\left\{y_{x}(x)\right\}_{-}=0$. Consequently, we get

$$
\begin{equation*}
\left\{\sum_{j=1}^{N} y_{j}^{m+1}\right\}_{-}=0 \tag{3.7}
\end{equation*}
$$

Finally, the necessity and sufficiency of condition (3.3) follows from the results of Lemma 2.4 and the calculations carried out above.

In Section 4 we shall use this lemma to perform the classification of irreducible invariant cases for Liénard differential systems with $n=1(\operatorname{deg} f(x)=1)$ and $m=5(\operatorname{deg} g(x)=5)$.

Now let us present the general structure of invariant algebraic curves and their cofactors for Liénard systems satisfying the condition $n \geq 2 m+1$. Recall that systems (1.4) with $n<$ $2 m+1$ were considered in articles [5,10]. We begin with the case $n>2 m+1$.

Theorem 3.3. Let $F(x, y)=0$ with $F(x, y) \in \mathbb{C}[x, y] \backslash \mathbb{C}$ be an irreducible invariant algebraic curve of a Liénard differential system from the family (1.4) with $n>2 m+1$. Then $F(x, y)$ and its cofactor take the form

$$
\begin{gather*}
F(x, y)=\left\{\prod_{j=1}^{N_{1}}\left\{y-y_{j}^{(1)}(x)\right\} \prod_{j=1}^{N_{2}}\left\{y-y_{j}^{(2)}(x)\right\}\right\}_{+}  \tag{3.8}\\
\lambda(x, y)=-\left(N_{1}+N_{2}\right) f-\left\{N_{1} h_{x}^{(1)}+N_{2} h_{x}^{(2)}\right\}_{+} \tag{3.9}
\end{gather*}
$$

where the Puiseux series $y_{j}^{(1,2)}(x)$ are given by the relations

$$
\begin{equation*}
y_{j}^{(1,2)}(x)=h^{(1,2)}(x)+\sum_{k=2(n+1)}^{\infty} c_{k, j}^{(1,2)} x^{\frac{n+1}{2}-\frac{k}{2}}, \quad h^{(1,2)}(x)=\sum_{k=0}^{2 n+1} c_{k}^{(1,2)} x^{\frac{n+1}{2}-\frac{k}{2}} \tag{3.10}
\end{equation*}
$$

and $N_{1}, N_{2} \in \mathbb{N} \cup\{0\}, N_{1}+N_{2} \geq 1$. The coefficients $c_{2(n+1), j}^{(1,2)}$ with the same upper index are pairwise distinct and all the coefficients $c_{m, j}^{(1,2)}$ with $m>2(n+1)$ are expressible via $c_{2(n+1), j}^{(1,2)}$. If $n$ is an odd number, then the corresponding Puiseux series are Laurent series and $c_{2 l-1}^{(1,2)}=0, c_{2 l-1, j}^{(1,2)}=0$ with $l \in \mathbb{N}$. In addition, $N_{k}=1$ whenever $n$ is odd and $N_{l}=0$, where $k, l=1,2$ and $k \neq l$. If $n$ is an even number, then $N_{1}=N_{2}$.

Proof. It follows from Lemma 3.1 that we can set $\mu(x)=1$. By Theorem 2.2 Puiseux series from the field $\mathbb{C}_{\infty}\{x\}$ that arise in representation (2.1) are those satisfying equation (3.1). Let us perform the classification of Puiseux series near the point $x=\infty$ solving equation (3.1) with the restriction $n>2 m+1$. For this aim we shall use the algorithm presented in the Appendix. There exists only one dominant balance that produce Puiseux series in a neighborhood of the point $x=\infty$. The ordinary differential equation related to this balance and its solutions are the following

$$
\begin{equation*}
y y_{x}+g_{0} x^{n}=0, \quad y^{(1,2)}(x)=c_{0}^{(1,2)} x^{\frac{n+1}{2}}, \quad c_{0}^{(1,2)}= \pm \frac{\sqrt{-2(n+1) g_{0}}}{(n+1)} . \tag{3.11}
\end{equation*}
$$

Calculating the Gâteaux derivative of the balance at its power solutions yields the Fuchs index: $p=n+1$. Definitions of dominant balances and Fuchs indices can be found in $[1,2,5,19]$, see also Appendix. Thus, we conclude that the Puiseux series corresponding to asymptotics (3.11)
exist and have arbitrary coefficients at $x^{-(n+1) / 2}$ provided that the compatibility conditions related to the unique Fuchs are satisfied. If $n$ is an odd number, then Puiseux series (3.10) are Laurent series.

Finding the factorization of $F(x, y)$ in the ring $\mathrm{C}_{\infty}\{x\}[y]$ and taking the polynomial part of this representation, we obtain (3.8). Since the polynomial $F(x, y)$ in (3.8) is irreducible, we get the condition of the theorem on the coefficients $c_{2(n+1), j}^{(1,2)}$ with the same upper index.

Now let us suppose that $n$ is an odd number and $N_{2}=0$. Our aim is to show that $N_{1}=1$. All the Puiseux series near the point $x=\infty$ arising in expression (3.8) are Laurent series with the same initial part of the series. Further, we introduce the new variable $z$ by the rule

$$
\begin{equation*}
z=y-\sum_{l=0}^{\frac{n+1}{2}} c_{2 l}^{(1)} x^{\frac{n+1}{2}-l} \tag{3.12}
\end{equation*}
$$

Calculating the projection in expression (3.8) yields

$$
\begin{equation*}
\left\{\prod_{j=1}^{N_{1}}\left(z-c_{n+3}^{(1)} x^{-1}-\ldots-c_{2(n+1), j}^{(1)} x^{-\frac{n+1}{2}}-\ldots\right)\right\}_{+}=z^{N_{1}} \tag{3.13}
\end{equation*}
$$

Requiring that the resulting invariant algebraic curve be given by an irreducible polynomial, we get $N_{1}=1$. The same can be done if $N_{1}=0$ and $n$ is odd.

Substituting $L=0$ and series (3.10) into expression (2.1), we find the cofactor as given in (3.9). Finally, if $n$ is even, we calculate the coefficient at $y^{N_{1}+N_{2}-1} x^{(n+1) / 2}$. The result is $\left(N_{1}-N_{2}\right) c_{0}^{(1)}$. Since $y^{N_{1}+N_{2}-1} x^{(n+1) / 2}$ is not an element of the ring $\mathbb{C}[x, y]$ and $c_{0}^{(1)} \neq 0$, we get $N_{1}=N_{2}$. The proof is completed.

Let us turn to Liénard differential systems satisfying the condition $n=2 m+1$. We shall see that the Fuchs indices of the dominant balances near the point $x=\infty$ for equation (3.1) depend on the parameters $f_{0}$ and $g_{0}$. It was proved in article [10] and in Theorem 3.3 that such a situation cannot take place for other Liénard differential systems. This fact makes classification of irreducible invariant algebraic curves sufficiently difficult in the case $n=2 m+1$. The method of Puiseux series can deal with each case of a fixed positive rational Fuchs index separately.

We shall demonstrate that the structure of polynomials producing invariant algebraic curves is in strong correlation with the properties of the following quadratic equation

$$
\begin{equation*}
p^{2}-\varrho p+(m+1) \varrho=0, \tag{3.14}
\end{equation*}
$$

where we have introduced notation

$$
\begin{equation*}
\varrho=4(m+1)-\frac{f_{0}^{2}}{g_{0}} . \tag{3.15}
\end{equation*}
$$

The set of all positive rational numbers will be denoted as $\mathbb{Q}^{+}$. Let $p_{1}$ and $p_{2}$ be the roots of equation (3.14).

Theorem 3.4. Suppose $F(x, y)=0$ with $F(x, y) \in \mathbb{C}[x, y] \backslash \mathbb{C}$ is an irreducible invariant algebraic curve of a Liénard differential system from family (1.4) with $n=2 m+1$. One of the following statements holds.

1. If $p_{1}, p_{2} \notin \mathbb{Q}^{+} \cup\{0\}$, then the polynomial $F(x, y)$ is of degree at most two with respect to $y$ and

$$
\begin{align*}
& F(x, y)=\left\{\left\{y-y^{(1)}(x)\right\}^{s_{1}}\left\{y-y^{(2)}(x)\right\}^{s_{2}}\right\}_{+}, \\
& \lambda(x, y)=-\left(s_{1}+s_{2}\right) f(x)-\left\{s_{1} y_{x}^{(1)}+s_{2} y_{x}^{(2)}\right\}_{+}^{\prime}  \tag{3.16}\\
& y^{(k)}(x)=\sum_{l=0}^{\infty} c_{l}^{(k)} x^{m+1-l}, \quad c_{0}^{(k)}=\frac{f_{0}}{p_{k}-2(m+1)}, \quad k=1,2
\end{align*}
$$

where $s_{1}$ and $s_{2}$ are either 0 or 1 independently, $s_{1}+s_{2}>0$. The Puiseux series $y^{(k)}(x), k=1$, 2 are Laurent series and possess uniquely determined coefficients.
2. If $p_{k} \in \mathbb{Q}^{+}, p_{q} \notin \mathbb{Q}^{+}$, where either $k=1, q=2$ or $k=2, q=1$, then the polynomial $F(x, y)$ and the cofactor $\lambda(x, y)$ take the form

$$
\begin{align*}
F(x, y) & =\left\{\prod_{j=1}^{N_{k}}\left\{y-y_{j}^{(k)}(x)\right\}\left\{y-y^{(q)}(x)\right\}^{s_{q}}\right\}_{+}, \\
\lambda(x, y) & =-\left(N_{k}+s_{q}\right) f(x)-\left\{\sum_{j=1}^{N_{k}} y_{j, x}^{(k)}+s_{q} y_{x}^{(q)}\right\}_{+},  \tag{3.17}\\
y_{j}^{(k)}(x) & =\sum_{l=0}^{\infty} c_{l, j}^{(k)} x^{m+1-\frac{l}{n_{k}}}, \quad y^{(q)}(x)=\sum_{l=0}^{\infty} c_{l}^{(q)} x^{m+1-l}, \\
c_{0, j}^{(k)} & =\frac{f_{0}}{p_{k}-2(m+1)}, \quad c_{0}^{(q)}=\frac{f_{0}}{p_{q}-2(m+1)},
\end{align*}
$$

where $N_{k} \in \mathbb{N} \cup\{0\}, s_{q}$ is either 0 or $1, N_{k}+s_{q}>0$. The Puiseux series $y^{(q)}(x)$ is a Laurent series and possesses uniquely determined coefficients. The Puiseux series $y_{j}^{(k)}(x)$ have pairwise distinct coefficients $c_{n_{k} p_{k}, j}^{(k)}$. The number $n_{k}$ is defined as $p_{k}=l_{k} / n_{k}$, where $l_{k}$ and $n_{k}$ are coprime natural numbers.
3. If $p_{1}, p_{2} \in \mathbb{Q}^{+}$, then the polynomial $F(x, y)$ and the cofactor $\lambda(x, y)$ take the form

$$
\begin{align*}
& F(x, y)=\left\{\prod_{j=1}^{N_{1}}\left\{y-y_{j}^{(1)}(x)\right\} \prod_{j=1}^{N_{2}}\left\{y-y_{j}^{(2)}(x)\right\}\right\}_{+} \\
& \lambda(x, y)=-\left(N_{1}+N_{2}\right) f(x)-\left\{\sum_{j=1}^{N_{1}} y_{j, x}^{(1)}+\sum_{j=1}^{N_{2}} y_{j, x}^{(2)}\right\}_{+},  \tag{3.18}\\
& y_{j}^{(k)}(x)=\sum_{l=0}^{\infty} c_{l, j}^{(k)} x^{m+1-\frac{l}{n_{k}}}, \quad c_{0, j}^{(k)}=\frac{f_{0}}{p_{k}-2(m+1)}, \quad k=1,2,
\end{align*}
$$

where $N_{1}, N_{2} \in \mathbb{N} \cup\{0\}, N_{1}+N_{2}>0$. The Puiseux series $y_{j}^{(k)}(x)$ possess pairwise distinct coefficients $c_{n_{k} p_{k}, j}^{(k)}$. The number $n_{k}$ is defined as $p_{k}=l_{k} / n_{k}$, where $l_{k}$ and $n_{k}$ are coprime natural numbers, $k=1,2$.
4. If $p_{1}=p_{2}=0$, then the polynomial $F(x, y)$ and the cofactor $\lambda(x, y)$ take the form

$$
\begin{align*}
& F(x, y)=y+\frac{f_{0}}{2(m+1)} x^{m+1}-\sum_{l=1}^{m+1} c_{l} x^{m+1-l}  \tag{3.19}\\
& \lambda(x, y)=-f(x)+\frac{f_{0}}{2} x^{m}-\sum_{l=1}^{m}(m+1-l) c_{l} x^{m-l}
\end{align*}
$$

where the coefficients $c_{1}, \ldots, c_{m+1}$ are uniquely determined. In addition, the following relation $4(m+1) g_{0}-f_{0}^{2}=0$ is valid.

There are no other irreducible invariant algebraic curves than those described above.
Proof. Again we use Theorem 2.2 and Lemma 3.1. Let us find Puiseux series near the point $x=\infty$ that satisfy equation (3.1) with the restriction $n=2 m+1$. There exists only one dominant balance producing power asymptotics near the point $x=\infty$. The ordinary differential equation related to this balance and its power solutions are of the form

$$
\begin{equation*}
y y_{x}+f_{0} x^{m} y+g_{0} x^{2 m+1}=0: \quad y^{(k)}(x)=c_{0}^{(k)} x^{m+1}, \quad k=1,2 \tag{3.20}
\end{equation*}
$$

where the coefficients $c_{0}^{(1,2)}$ satisfy the following equation $(m+1) c_{0}^{2}+f_{0} c_{0}+g_{0}=0$. Calculating the Gâteaux derivative of the balance at its power solutions yields the following equation for the Fuchs indices $p:(2(m+1)-p) c_{0}+f_{0}=0$. Expressing $c_{0}$ from this equation and substituting the result into the equation $(m+1) c_{0}^{2}+f_{0} c_{0}+g_{0}=0$, we get relation (3.14). Starting from power asymptotics we can derive asymptotic series possessing these asymptotics as leading-order terms. We are interested in Puiseux asymptotic series.

If equation (3.14) does not have positive rational solutions, then both Puiseux series related to asymptotics (3.20) possess uniquely determined coefficients. Since the number of distinct Puiseux series near the point $x=\infty$ satisfying equation (3.1) is finite and equals 2 , it follows from Theorem 2.2 that the degree with respect to $y$ of the polynomial $F(x, y)$ is bounded by 2. Constructing the factorization of the polynomial $F(x, y)$ in the ring $\mathbb{C}_{\infty}\{x\}[y]$ yields representation (3.16).

Further, if one of the solutions of equation (3.14) defining the Fuchs indices is a positive rational number and another one is not, then the Puiseux series related to the former case possesses an arbitrary coefficient provided that the compatibility condition for this Fuchs index is satisfied. Another Puiseux series possesses uniquely determined coefficients. As a result we obtain relation (3.17). Since the polynomial giving the invariant algebraic curve under consideration is irreducible, the coefficients $c_{n_{k} p_{k}, j}^{(k)}$ corresponding to the positive rational Fuchs index should be pairwise distinct. The number $n_{k}$ can be obtained from the relation $p_{k}=l_{k} / n_{k}$, where $l_{k}$ and $n_{k}$ are coprime natural numbers. For more details see the Appendix.

If both solutions of equation (3.14) are positive rational numbers, then the Puiseux series have arbitrary coefficients and exist whenever the corresponding compatibility conditions for the Fuchs indices hold. We get expression (3.18). Since polynomials generating the invariant algebraic curves in question are irreducible, we conclude that the coefficients with the same upper index $c_{n_{k} p_{k}, j}^{(k)}, k=1,2$ should be pairwise distinct. The numbers $n_{k}, k=1,2$ are found similarly to the previous case.

Finally, we need to examine the situation, when two roots of the equation $(m+1) c_{0}^{2}+$ $f_{0} c_{0}+g_{0}=0$ merge. This gives $4(m+1) g_{0}-f_{0}^{2}=0$ and $c_{0}=-f_{0} /(2\{m+1\})$. Substituting this relation into the equation $(2(m+1)-p) c_{0}+f_{0}=0$ for the Fuchs index yields $p=0$. Consequently, we obtain the Puiseux series with integer exponents and uniquely determined coefficients. This gives the unique irreducible invariant algebraic curve as given in (3.19).

The cofactors $\lambda(x, y)$ we find from expression (2.1). Since we have considered all possible combinations of the Puiseux series from the field $\mathrm{C}_{\infty}\{x\}$ that solve equation (3.1), we conclude that other irreducible invariant algebraic curves cannot exist.

Proving the above theorem, we have also established that if the compatibility condition for the Puiseux series $y_{j}^{(k)}(x)$ to exist is not satisfied and $p_{k} \in \mathbb{N}$ in the case of representa-
tion (3.17), then the irreducible invariant algebraic curve, if exists, is given by the polynomial $F(x, y)=y-c_{0}^{(q)} x^{m+1}-c_{1}^{(q)} x^{m}-\ldots-c_{m+1}^{(q)}$. If a similar situation occurs for representation (3.18), then either $N_{1}=0$ or $N_{2}=0$ and the product in expression (3.18) involving the corresponding series is absent. Moreover, if $p_{1}, p_{2} \in \mathbb{N}$ and the compatibility conditions for both Puiseux series are not satisfied, then there are no invariant algebraic curves.

Let us note that invariant algebraic curves of Theorems 3.3 and 3.4 exist under certain restrictions on the parameters of the original differential systems.

## 4 Examples

The most interesting families of Liénard differential systems (1.4) satisfying the condition $n \geq$ $2 m+1$ are those with the smallest degrees of the polynomial $g(x)$. They include cubic, quartic, and quintic systems with a constant or linear damping function. In addition, a quadratic damping function is allowed if $g(x)$ is a fifth degree polynomial. According to Theorem 3.4 the cases $\operatorname{deg} f(x)=1, \operatorname{deg} g(x)=3$ and $\operatorname{deg} f(x)=2, \operatorname{deg} g(x)=5$ are degenerate. Partial results were obtained in articles [12,13].

We have studied other Liénard differential systems from those listed above. All of them with the exception for the family satisfying the conditions $\operatorname{deg} f(x)=1$ and $\operatorname{deg} g(x)=5$ have irreducible invariant algebraic curves given by bivariate polynomials of degrees at most 2 with respect to $y$. While in the case $\operatorname{deg} f(x)=1$ and $\operatorname{deg} g(x)=5$ there exist irreducible invariant algebraic curves of higher degrees. The aim of the present section is to perform a classification of irreducible invariant algebraic curves of Liénard differential systems (1.4) satisfying the conditions $\operatorname{deg} f(x)=1$ and $\operatorname{deg} g(x)=5$. We shall prove that algebraic curves of degree 3 with respect to $y$ arise. It is sometimes supposed that Liénard systems satisfying under the restriction $\operatorname{deg} g \neq 2 \operatorname{deg} f+1(n \neq 2 m+1)$ do not have such invariant algebraic curves.

Liénard differential systems with $\operatorname{deg} f(x)=1$ and $\operatorname{deg} g(x)=5$ are of the form

$$
\begin{equation*}
x_{t}=y, \quad y_{t}=-(\alpha x+\beta) y-\left(\varepsilon x^{5}+r x^{4}+v x^{3}+e x^{2}+\sigma x+\delta\right), \quad \alpha \varepsilon \neq 0 \tag{4.1}
\end{equation*}
$$

A change of variables $x \mapsto X\left(x+x_{0}\right), y \mapsto Y y, T \mapsto T t, X Y T \neq 0$ relates systems (4.1) with their simplified version at $\alpha=5, \varepsilon=-3, r=0$. Thus, without loss of generality, we obtain the systems

$$
\begin{equation*}
x_{t}=y, \quad y_{t}=-(5 x+\beta) y+\left(3 x^{5}-v x^{3}-e x^{2}-\sigma x-\delta\right) \tag{4.2}
\end{equation*}
$$

where all the parameters are from the field $\mathbb{C}$.
Theorem 4.1. Differential systems (4.2) admit invariant algebraic curves if and only if restrictions on the parameters given below are satisfied. Generating polynomials of irreducible algebraic invariants and their cofactors are of the form:

## invariant algebraic curves of the first degree with respect to $y$

1. $e=\sigma+\frac{1}{8} v-\frac{15}{16}+\frac{15}{8} \beta+\frac{1}{16} v^{2}-\frac{1}{8} \beta v-\frac{3}{16} \beta^{2}$,

$$
\begin{aligned}
& \delta=\frac{1}{192}(3 \beta-v+3)\left(v^{2}-6 v-2 \beta v+6 \beta-3 \beta^{2}+9+16 \sigma\right) \\
& F(x, y)=y-x^{3}+x^{2}+\frac{1}{4}(\beta+v-3) x+\frac{1}{3} \sigma+\frac{1}{48}(\beta+v-3)(-3 \beta+v-3) \\
& \lambda(x, y)=-3 x^{2}-3 x+\frac{1}{4}(v-3 \beta-3)
\end{aligned}
$$

2. $e=\frac{15}{16}+\frac{15}{8} \beta-\sigma-\frac{1}{8} v-\frac{1}{16} v^{2}-\frac{1}{8} \beta v+\frac{3}{16} \beta^{2}$,
$\delta=\frac{1}{192}(3 \beta+v-3)\left(v^{2}-6 v+2 \beta v-6 \beta-3 \beta^{2}+9+16 \sigma\right)$,
$F(x, y)=y+x^{3}+x^{2}+\frac{1}{4}(\beta-v+3) x+\frac{1}{3} \sigma+\frac{1}{48}(\beta-v+3)(3-3 \beta-v)$,
$\lambda(x, y)=3 x^{2}-3 x+\frac{1}{4}(3-v-3 \beta) ;$
invariant algebraic curves of the second degree with respect to $y$
3. $e=0, \quad \delta=0, \quad \sigma=\frac{1}{12}\left(9-v^{2}\right), \quad \beta=0$,
$F(x, y)=y^{2}+\left(2 x^{2}+1-\frac{1}{3} v\right) y-x^{6}+\frac{1}{2}(v-1) x^{4}-\frac{1}{12}(v+1)(v-3) x^{2}$
$+\frac{1}{216}(v+3)(v-3)^{2}, \quad \lambda(x, y)=-6 x ;$
4. $e=\frac{3}{1024} \beta\left(512-5 \beta^{2}\right), \delta=-\frac{3}{262144} \beta^{3}\left(\beta^{2}+1280\right)$,
$\sigma=\frac{3}{65536} \beta^{2}\left(2816+15 \beta^{2}\right), \nu=\frac{15}{128} \beta^{2}+3$,
$F(x, y)=y^{2}+\left(2 x^{2}+\frac{1}{2} \beta x-\frac{5}{128} \beta^{2}\right) y-x^{6}+\left(\frac{15}{256} \beta^{2}+1\right) x^{4}-\frac{1}{512} \beta\left(5 \beta^{2}-256\right) x^{3}$
$+\frac{3}{65536} \beta^{2}\left(512+15 \beta^{2}\right) x^{2}-\frac{1}{131072} \beta^{3}\left(1280+3 \beta^{2}\right) x+\frac{5}{16777216} \beta^{4}\left(\beta^{2}+1280\right)$,
$\lambda(x, y)=-6 x-\frac{3}{2} \beta ;$
invariant algebraic curves of the third degree with respect to $y$
5. $e=\frac{28511847}{62500}, \delta=-\frac{94714508889}{19531250}, \sigma=-\frac{8628822111}{1562500}, v=\frac{133188}{625}, \beta=\frac{91}{5}$,
$F(x, y)=y^{3}+\left(x^{3}+3 x^{2}-\frac{24297}{625} x-\frac{15500849}{62500}\right) y^{2}+\left(2 x^{5}-x^{6}+\frac{73219}{625} x^{4}+\frac{4316949}{31250} x^{3}\right.$
$\left.-\frac{11403548611}{1562500} x^{2}-\frac{7670383903}{19531250} x+\frac{109912617846031}{976562500}\right) y-x^{9}-x^{8}+\frac{96266}{625} x^{7}+\frac{36191047}{62500} x^{6}$
$-\frac{17544478133}{1562500} x^{5}-\frac{812450830009}{19531250} x^{4}+\frac{138358719104879}{390625000} x^{3}+\frac{131625246607012067}{97656250000} x^{2}$
$-\frac{925725907851168424}{152587890625} x-\frac{356383541131462914069}{61035156250000}, \quad \lambda(x, y)=3 x^{2}-9 x-\frac{58422}{625}$;
6. $e=-\frac{28511847}{62500}, \delta=\frac{94714508889}{19531250}, \sigma=-\frac{8628822111}{1562500}, v=\frac{133188}{625}, \beta=-\frac{91}{5}$,
$F(x, y)=y^{3}+\left(3 x^{2}-x^{3}+\frac{24297}{625} x-\frac{15500849}{62500}\right) y^{2}-\left(x^{6}+2 x^{5}-\frac{73219}{625} x^{4}+\frac{4316949}{31250} x^{3}\right.$
$\left.+\frac{11403548611}{1562500} x^{2}-\frac{7670383903}{19531250} x-\frac{109912617846031}{976562500}\right) y+x^{9}-x^{8}-\frac{96266}{625} x^{7}+\frac{36191047}{62500} x^{6}$
$+\frac{17544478133}{1562500} x^{5}-\frac{812450830009}{19531250} x^{4}-\frac{138358719104879}{390625000} x^{3}+\frac{131625246607012067}{97656250000} x^{2}$
$+\frac{925725907851168424}{152587890625} x-\frac{356383541131462914069}{61035156250000}, \quad \lambda(x, y)=-3 x^{2}-9 x+\frac{58422}{625}$.

Proof. The structure of the polynomials $F(x, y)$ producing irreducible invariant algebraic curves has been presented in Theorem 3.3. The Puiseux series of Theorem 3.3 take the following form

$$
\begin{align*}
& y^{(1)}(x)=x^{3}-x^{2}+\frac{1}{4}(3-\beta-v) x+\frac{1}{6}(v+3 \beta-3-2 e)+\sum_{l=4}^{\infty} c_{l}^{(1)} x^{3-l} \\
& y^{(2)}(x)=-x^{3}-x^{2}+\frac{1}{4}(v-\beta-3) x+\frac{1}{6}(v-3 \beta-3+2 e)+\sum_{l=4}^{\infty} c_{l}^{(2)} x^{3-l} \tag{4.3}
\end{align*}
$$

These Puiseux series have arbitrary coefficients $c_{6}^{(1,2)}$ and exist whenever the following conditions are satisfied

$$
\begin{align*}
y^{(1)}(x): \quad \delta= & \frac{3}{160} \beta^{3}+\left(\frac{1}{80} v-\frac{15}{16}\right) \beta^{2}+\left(\frac{123}{32}-\frac{21}{80} v-\frac{1}{160} v^{2}\right) \beta \\
& +\left(2-\frac{1}{10} \beta\right) \sigma+\left(\frac{7}{20} \beta-\frac{7}{4}-\frac{1}{12} v\right) e+\frac{1}{6}(v-3)(v+3) ; \\
y^{(2)}(x): \quad \delta= & \frac{3}{160} \beta^{3}+\left(\frac{15}{16}-\frac{1}{80} v\right) \beta^{2}+\left(\frac{123}{32}-\frac{21}{80} v-\frac{1}{160} v^{2}\right) \beta  \tag{4.4}\\
& -\left(2+\frac{1}{10} \beta\right) \sigma-\left(\frac{7}{4}+\frac{1}{12} v+\frac{7}{20} \beta\right) e-\frac{1}{6}(v-3)(v+3)
\end{align*}
$$

Further, we suppose that the series $y^{(1)}(x)$ enters the factorization of the polynomial $F(x, y)$ $N_{1}$ times with pairwise distinct values of $c_{6, j}^{(1)}$. Analogously, we suppose that the series $y^{(2)}(x)$ enters the factorization of the polynomial $F(x, y) N_{2}$ times with pairwise distinct values of $c_{6, j}^{(2)}$. If $N_{2}=0$, then it follows from Theorem 3.3 that $N_{1}=1$. The resulting irreducible invariant algebraic curve exists whenever the series $y^{(1)}(x)$ terminates at the zero term. This gives the restriction

$$
\begin{equation*}
e=\sigma-\frac{3}{16} \beta^{2}+\frac{1}{8}(15-v) \beta+\frac{1}{16}(v+5)(v-3) . \tag{4.5}
\end{equation*}
$$

Further, we do the same for the case $N_{1}=0$. In such a way we construct irreducible invariant algebraic curves of the first degree with respect to $y$.

Now let us suppose that $N_{1}>0$ and $N_{2}>0$. We introduce the following variables

$$
\begin{equation*}
C_{k}^{(1)}=\sum_{j=1}^{N_{1}}\left(c_{6, j}^{(1)}\right)^{k} ; \quad C_{k}^{(2)}=\sum_{j=1}^{N_{1}}\left(c_{6, j}^{(2)}\right)^{k} . \tag{4.6}
\end{equation*}
$$

According to the results of Theorem 3.2 we need to consider the algebraic system

$$
\begin{equation*}
\sum_{j=1}^{N_{1}} c_{l, j}^{(1)}+\sum_{j=1}^{N_{2}} c_{l, j}^{(2)}=0, \quad l \geq 4 \tag{4.7}
\end{equation*}
$$

We take the first eleven equations from this system. In addition, the compatibility conditions for both series to exist should be considered. Solving the algebraic sub-system, we obtain three possibilities: $N_{1}=N_{2}, N_{1}=2 N_{2}$, and $N_{2}=2 N_{1}$. If the first possibility takes place, then we find

$$
\begin{equation*}
C_{1}^{(1)}=\varrho_{1} N_{1}, \quad C_{2}^{(1)}=\varrho_{1}^{2} N_{1}, \quad C_{1}^{(2)}=\varrho_{2} N_{2}, \quad C_{2}^{(2)}=\varrho_{2}^{2} N_{2} \tag{4.8}
\end{equation*}
$$

and restrictions on the parameters as given in items 3 and 4. There exist two families of irreducible invariant algebraic curves $F(x, y)=0$ with $N_{1}=1$ and $N_{2}=1$. The irreducible
invariant algebraic curves are presented in items 3 and 4 . This fact proves the validity of the following conditions

$$
\begin{equation*}
C_{k}^{(1)}=e_{1}^{k} N_{1}, \quad C_{k}^{(2)}=\varrho_{2}^{k} N_{2}, \quad k \in \mathbb{N} . \tag{4.9}
\end{equation*}
$$

According to Theorem 2.5, we see that the algebraic system in question has no other solutions.
If $N_{1}=2 N_{2}$, then we get $C_{1}^{(2)}=\varrho_{2} N_{2}$ and $C_{2}^{(2)}=\varrho_{2}^{2} N_{2}$. Arguing as above, we find $N_{2}=1$. Analogously the case $N_{2}=2 N_{1}$ can be studied. In expressions (4.8) and (4.9) the values $\varrho_{1}$ and $\varrho_{2}$ either are constants or depend on the parameters of the original differential systems, but not on $N_{1}$ and $N_{2}$.

The cofactors can be obtained with the help of expression (3.9).

Note that it is a difficult computational problem to find invariant algebraic curves of items 5 and 6 using the method of undetermined coefficients, the method of extactic polynomial or an algorithm of decomposing the vector field related to the original system into weighthomogenous components. It seems that the classification of irreducible invariant algebraic curves for quintic Liénard differential systems with a linear damping function is presented here for the first time.

## 5 Conclusion

In this article we have derived necessary and sufficient conditions enabling a planar polynomial differential system (1.1) to have invariant algebraic curves. Our conditions give rise to an algorithm, which is able to perform a classification of irreducible invariant algebraic curves for a given differential system. The algorithm can be easily implemented with the help of computer systems of symbolic computations.

We have presented the general structure in the ring $\mathrm{C}_{\infty}\{x\}[y]$ for the bivariate polynomials generating irreducible invariant algebraic curves of Liénard differential systems (1.4) with $\operatorname{deg} g \geq 2 \operatorname{deg} f+1$. Their cofactors have been calculated in an explicit form. Let us emphasize that the method of Puiseux series is also applicable in the case of systems (1.1) with the parameters affecting degrees of the polynomials $P(x, y)$ and $Q(x, y)$. Some examples are given in articles [9,11]. In addition, the method enables one to find algebraic first-order ordinary differential equations compatible with a higher-order autonomous ordinary differential equation [8]. Moreover, the method of Puiseux series admits a non-autonomous generalization, for more details see [7]. We conclude that the method presented in works $[5,6,10$ ] and developed in this article is a powerful tool of finding invariants for ordinary differential equations and systems of ordinary differential equations.

Another way to derive an algebraic system similar to that presented in expression (2.7) is to require that the non-polynomial part in the expression of the cofactor $\lambda(x, y)$ in (2.1) vanishes. This algebraic system coincides with that arising from Theorems 2.3 and 3.2 in the case of Liénard differential systems. For other polynomial differential systems this approach may lead to finding generalized (non-polynomial in $x$ ) invariant curves possessing polynomial cofactors. It seems that this topic is also worth studying. Some results concerning nonalgebraic invariant curves with polynomial cofactors were obtained in article [18].

## 6 Acknowledgments

The author would like to thank the reviewers for their valuable comments, which contributed to the improvement of the article. This research was supported by Russian Science Foundation grant 19-71-10003.

## 7 Appendix

Let us describe a method, which can be used to perform the classification of Puiseux series satisfying an algebraic first-order ordinary differential equation $E\left(x, y, y_{x}\right)=0$. The left-hand side of this expression can be regarded as a sum of differential monomials given by

$$
\begin{equation*}
M[y(x), x]=C x^{l} y^{j_{0}}\left\{\frac{d y}{d x}\right\}^{j_{1}}, \quad C \in \mathbb{C} \backslash\{0\}, \quad l, j_{0}, j_{1} \in \mathbb{N}_{0} . \tag{7.1}
\end{equation*}
$$

The set of all the differential monomials of the form (7.1) will be denoted as $\mathbb{M}$. In order to simplify notation the expression $W[x, y(x)]$ will stand for a polynomial in $x, y(x)$, and $y_{x}(x)$ with coefficients from the field $\mathbb{C}$.

Let us define the map $q: \mathbb{M} \rightarrow \mathbb{R}^{2}$ by the following rules

$$
C x^{q_{1}} y^{q_{2}} \mapsto q=\left(q_{1}, q_{2}\right), \quad \frac{d^{k} y}{d x^{k}} \mapsto q=(-k, 1), \quad q\left(M_{1} M_{2}\right)=q\left(M_{1}\right)+q\left(M_{2}\right),
$$

where $C \in \mathbb{C} \backslash\{0\}$ is a constant, $M_{1}$ and $M_{2}$ are differential monomials. We denote the set of all points $q \in \mathbb{R}^{2}$ corresponding to the differential monomials of equation $E\left(x, y, y_{x}\right)=0$ as $S(E)$. The convex hull of $S(E)$ is known as the Newton polygon of the equation under consideration.

The boundary of the Newton polygon consists of vertices and edges. Selecting all the differential monomials of the original equation that generate the vertices and the edges of the Newton polygon, we obtain a number of balances. The balance for a vertex is defined as the sum of those differential monomials in $E\left(x, y, y_{x}\right)$ that are mapped into the vertex. The balance for an edge is defined as the sum of differential monomials in $E\left(x, y, y_{x}\right)$ whose images belong to the edge. If solutions of the equation $E\left(x, y, y_{x}\right)=0$ possess an asymptotics of the form $y(x)=c_{0} x^{r}$ with $x \rightarrow 0$ or $x \rightarrow \infty$, then there exists a balance $W[x, y(x)]$ such that the function $y(x)=c_{0} x^{r}$ satisfies the equation $W[x, y(x)]=0$. Conversely, the function $y(x)=c_{0} x^{r}$ solving equation $W[x, y(x)]=0$, where $W[x, y(x)]$ is a balance, is an asymptotics at $x \rightarrow 0$ (or $x \rightarrow \infty$ ) for solutions of equation (1.2) whenever for all the differential monomials $M[x, y(x)]$ of the original equation not involved into $W[x, y(x)]$ we have $\operatorname{Re} \varkappa>\operatorname{Re} \varkappa_{0}$ (or $\operatorname{Re} \varkappa<\operatorname{Re} \varkappa_{0}$ ), where $M\left[x, c_{0} x^{r}\right]=B x^{\varkappa}$ and $M_{0}\left[x, c_{0} x^{r}\right]=B_{0} x^{\varkappa_{0}}$ with $M_{0}[x, y(x)]$ being a differential monomial of the balance $W[x, y(x)]$.

Thus, having found all the power solutions $y(x)=c_{0} x^{r}$ for all the balances, one needs to select those that give asymptotics at $x \rightarrow 0$ or $x \rightarrow \infty$. Using power asymptotics it is possible to derive asymptotic series possessing these asymptotics as leading-order terms [1,2]. In this article we are interested in Puiseux series near $x=\infty$ that satisfy equation (1.2), therefore we shall focus at the case $r \in \mathbb{Q}$ and $x \rightarrow \infty$. Let us suppose that a balance $W[y(x), x]$ of the equation $E\left(x, y, y_{x}\right)=0$ has a solution $y(x)=c_{0} x^{r}$, which is an asymptotics at $x \rightarrow \infty$ and $r \in \mathbb{Q}$.

In order to obtain the structure of the corresponding series one should find the Gâteaux derivative of the balance $W[y(x), x]$ at the solution $y(x)=c_{0} x^{r}$ :

$$
\frac{\delta W}{\delta y}\left[c_{0} x^{r}\right]=\lim _{s \rightarrow 0} \frac{W\left[c_{0} x^{r}+s x^{r-p}, x\right]-W\left[c_{0} x^{r}, x\right]}{s}=V(p) x^{\tilde{r}}, \quad \tilde{r} \in \mathbf{Q} .
$$

In this expression $V(p)$ is a first-degree polynomial with respect to $p$. The coefficients of this polynomial depend on $c_{0}$ and on the parameters (if any) of the original equation involved into the balance $W[y(x), x]$. The zero $p_{0}$ of $V(p)$ is called the Fuchs index (or the resonance) of the balance $W[y(x), x]$ and its power solution $y(x)=c_{0} x^{r}$. Let $\operatorname{lcm}(n, m)$ be the lowest common multiple of two integer numbers $n$ and $m$. If the Fuchs index $p_{0}$ is not a positive rational number, then the number $n_{0}$ in expression (1.3) is given by $n_{0}=r_{2}$ where $r_{2}$ is defined as $r=r_{1} / r_{2}$ with $r_{1}$ and $r_{2}$ being coprime numbers, $r_{1} \in \mathbb{Z}$ and $r_{2} \in \mathbb{N}$. Otherwise we obtain $n_{0}=\operatorname{lcm}\left(g_{2}, r_{2}\right)$, where $r_{2}$ was defined previously and $g_{2}$ is given by $p_{0}=g_{1} / g_{2}$ with coprime natural numbers $g_{1}$ and $g_{2}$.

Finally, it is important to verify the existence of the Puiseux series of the form (1.3) with $l_{0}=r n_{0}$. If the balance $W[y(x), x]$ corresponds to a vertex of the Newton polygon, then the Puiseux series always exists and possesses an arbitrary coefficient $c_{0}$. In this case the Fuchs index is equal to zero. Now let us suppose that the balance $W[y(x), x]$ corresponds to an edge of the Newton polygon. Substituting series (1.3) into the equation $E\left(x, y, y_{x}\right)=0$ one can find the recurrence relation for its coefficients. This relation takes the form

$$
V\left(\frac{k}{n_{0}}\right) c_{k}=U_{k}\left(c_{0}, \ldots, c_{k-1}\right), \quad k \in \mathbb{N}
$$

where $U_{k}$ is a polynomial of its arguments. Note that $U_{k}$ can also depend on the parameters (if any) of the original equation. The equation $U_{n_{0} p_{0}}=0$ is called the compatibility condition. If the compatibility condition is not satisfied, then the Puiseux series under consideration does not exist. Otherwise the corresponding Puiseux series exists and possesses an arbitrary coefficient $c_{n_{0} p_{0}}$. Consequently, we conclude that the Puiseux series in question has uniquely determined coefficients provided that there are no non-negative rational Fuchs indices.

We note that if one wishes to find all the Puiseux series of the form (1.3) that satisfy the original equation, then it is necessary to implement the procedure described above for all the dominant balances and for all their power solutions $y(x)=c_{0} x^{r}$ with $r \in \mathbb{Q}$ and $x \rightarrow \infty$.

Asymptotic Puiseux series near the point $x_{0} \in \mathbb{C}$ can be found introducing the change of variables $w(s)=y\left(s+x_{0}\right), s=x-x_{0}$ and considering the case $s \rightarrow 0$ in the resulting ordinary differential equations.

We also observe that there may exist balances and their power solutions such that the following condition $V(p) \equiv 0$ is valid. If $V(p)$ is identically zero, then one should make the substitution $y(x)=c_{0} x^{r}+w(x)$ in equation $E\left(x, y, y_{x}\right)=0$ and find all the Puiseux series $w(x)=c_{1} x^{r_{1}}+\ldots$ of the latter such that $r_{1}<r, r_{1} \in \mathrm{Q}$ and $x \rightarrow \infty$. More details and some generalizations can be found in the works by A. D. Bruno [1,2].

## References

[1] A. D. Bruno, Asymptotic behaviour and expansions of solutions of an ordinary differential equation, Russ. Math. Surv. 59(2004), No. 3, 429-481. https://doi.org/10.1070/ RM2004v059n03ABEH000736; MR2116535; Zbl 1068.34054
[2] A. D. Bruno, Power geometry in algebraic and differential equations, Elsevier Science (NorthHolland), 2000. MR1773512
[3] J. Chavarriga, H. Giacomini, M. Grau, Necessary conditions for the existence of invariant algebraic curves for planar polynomial systems, Bull. Sci. Math. 129(2005), No. 1, 99-126. https://doi.org/10.1016/j.bulsci.2004.09.002; MR2123262; Zbl 1091.34017
[4] C. Christopher, J. Llibre, J. V. Pereira, Multiplicity of invariant algebraic curves in polynomial vector fields, Pacific J. Math. 229(2007), No. 1, 63-117. https://doi.org/10. 2140/pjm.2007.229.63; MR2276503; Zbl 1160.34003
[5] M. V. Demina, Novel algebraic aspects of Liouvillian integrability for two-dimensional polynomial dynamical systems, Phys. Lett. A 382(2018), No. 20, 1353-1360. https: //doi. org/10.1016/j.physleta.2018.03.037; MR3782570; Zbl 1398.34050
[6] M. V. Demina, Invariant algebraic curves for Liénard dynamical systems revisited, Appl. Math. Lett. 84(2018), 42-48. https://doi.org/10.1016/j.aml.2018.04.013; MR3808495; Zbl 1391.37041
[7] M. V. Demina, Invariant surfaces and Darboux integrability for non-autonomous dynamical systems in the plane, J. Phys. A 51(2018), 505202. https://doi.org/10.1088/ 1751-8121/aaecca; MR3895572; Zbl 1411.70025
[8] M. V. Demina, Classifying algebraic invariants and algebraically invariant solutions, Chaos Solitons Fractals 140(2020), 110219. https://doi.org/10.1016/j.chaos. 2020. 110219; MR4138303
[9] M. V. Demina, Liouvillian integrability of the generalized Duffing oscillators, Anal. Math. Phys. 11(2021), No. 1, Paper No. 25. https://doi.org/10.1007/s13324-020-00459-z; MR4195118; Zbl 07301487
[10] M. V. Demina, The method of Puiseux series and invariant algebraic curves, Commun. Contemp. Math. (2021), 2150007. https://doi.org/10.1142/S0219199721500073
[11] M. V. Demina, N. S. Kuznetsov, Liouvillian integrability and the Poincaré problem for nonlinear oscillators with quadratic damping and polynomial forces, J. Dyn. Control Syst. 27(2021), No. 2, 403-415. https://doi.org/10.1007/s10883-020-09513-2; MR4231678; Zbl 07329777
[12] M. V. Demina, D. I. Sinelshchiкov, Integrability properties of cubic Liénard oscillators with linear damping, Symmetry 11(2019), 1378. https://doi. org/10.3390/sym11111378
[13] M. V. Demina, D. I. Sinelshchikov, Darboux first integrals and linearizability of quadratic-quintic Duffing-van der Pol oscillators, J. Geom. Phys. 165(2021), 104215. https://doi.org/10.1016/j.geomphys.2021.104215; MR4236454
[14] M. V. Demina, C. Valls, On the Poincaré problem and Liouvillian integrability of quadratic Liénard differential equations, Proc. Roy. Soc. Edinburgh Sect. A 150(2020), No. 6, 3231-3251.https://doi.org/10.1017/prm.2019.63; MR4190110; Zbl 07316379
[15] A. Ferragut, H. Giacomini, A new algorithm for finding rational first integrals of polynomial vector fields, Qual. Theory Dyn. Syst. 9(2010), 89-99. https://doi.org/10.1007/ s12346-010-0021-x; MR2737363; Zbl 1216.34001
[16] I. A. García, H. Giacomini, J. Giné, Generalized nonlinear superposition principles for polynomial planar vector fields, J. Lie Theory 15(2005), No. 1, 89-104. MR2115230; Zbl 1073.34003
[17] H. Giacomini, J. Giné, M. Grau, The role of algebraic solutions in planar polynomial differential systems, Math. Proc. Cambridge Philos. Soc. 143(2007), No. 2, 487-508. https: //doi.org/10.1017/S0305004107000497; MR2364665; Zbl 1136.34036
[18] I. A. García, J. Giné, Non-algebraic invariant curves for polynomial planar vector fields, Discrete Contin. Dyn. Syst. 10(2004), No. 3, 755-768. https://doi.org/10.3934/dcds. 2004.10.755; MR2018878; Zbl 1050.34033
[19] A. Goriely, Integrability and nonintegrability of dynamical systems, World Scientific, 2001. https://doi.org/10.1142/9789812811943; MR1857742
[20] Yu. Ilyashenko, S. Yakovenko, Lectures on analytic differential equations, Graduate Studies in Mathematics, Vol. 86, American Mathematical Society, 2008. https://doi.org/10. 1090/gsm/086; MR2363178
[21] M. N. Lagutinski, On some polynomials and their application for algebraic integration of ordinary differential algebraic equations (in Russian), Communications of the Kharkov Mathematical Society. The second series 13(1912), 200-224.
[22] J. Llibre, C. Valls, Liouvillian first integrals for generalized Liénard polynomial differential systems, Adv. Nonlinear Stud. 13(2013), 819-829. https://doi.org/10.1515/ ans-2013-0404; MR3115140; Zbl 1369.34003
[23] K. Odani, The limit cycle of the van der Pol equation is not algebraic, J. Differential Equations 115(1995), No. 1, 146-152. https ://doi.org/10.1006/jdeq.1995.1008; MR1308609; Zbl 0816.34023
[24] R. J. Walker, Algebraic curves, Springer-Verlag, New York, 1978. MR513824
[25] H. Żoॄadek, Algebraic invariant curves for the Liénard equation, Trans. Amer. Math. Soc. 350(1998), No. 4, 1681-1701. https://doi.org/10.1090/S0002-9947-98-02002-9; MR1433130; Zbl 0895.34026

Electronic Journal of Qualitative Theory of Differential Equations

# S-shaped bifurcations in a two-dimensional Hamiltonian system 

André Zegeling ${ }^{\boxtimes 1}$ and Paul Andries Zegeling ${ }^{2}$<br>${ }^{1}$ Guilin University of Aerospace Technology, Jinji Road 2, Guilin, China<br>${ }^{2}$ Utrecht University, Department of Mathematics, Budapestlaan 6, De Uithof, the Netherlands

Received 3 December 2020, appeared 13 July 2021
Communicated by Hans-Otto Walther


#### Abstract

We study the solutions to the following Dirichlet boundary problem:


$$
\frac{d^{2} x(t)}{d t^{2}}+\lambda f(x(t))=0
$$

where $x \in \mathbb{R}, t \in \mathbb{R}, \lambda \in \mathbb{R}^{+}$, with boundary conditions:

$$
x(0)=x(1)=A \in \mathbb{R}
$$

Especially we focus on varying the parameters $\lambda$ and $A$ in the case where the phase plane representation of the equation contains a saddle loop filled with a period annulus surrounding a center.

We introduce the concept of mixed solutions which take on values above and below $x=A$, generalizing the concept of the well-studied positive solutions.

This leads to a generalization of the so-called period function for a period annulus. We derive expansions of these functions and formulas for the derivatives of these generalized period functions.

The main result is that under generic conditions on $f(x)$ so-called S-shaped bifurcations of mixed solutions occur.

As a consequence there exists an open interval for sufficiently small $A$ for which $\lambda$ can be found such that three solutions of the same mixed type exist.

We show how these concepts relate to the simplest possible case $f(x)=x(x+1)$ where despite its simple form difficult open problems remain.
Keywords: ordinary differential equations, boundary value problem, period function.
2020 Mathematics Subject Classification: 34B15, 34C08, 34C23, 37C10.

## 1 Introduction

We study the existence and bifurcation of solutions to a Dirichlet boundary problem:

$$
\begin{equation*}
\frac{d^{2} x(t)}{d t^{2}}+\lambda f(x(t))=0 \tag{1.1}
\end{equation*}
$$

[^13]where $x \in \mathbb{R}, t \in \mathbb{R}, \lambda \in \mathbb{R}^{+}$, with boundary conditions:
\[

$$
\begin{equation*}
x(0)=x(1)=A \in \mathbb{R} . \tag{1.2}
\end{equation*}
$$

\]

The differential equation can be interpreted as the scalar motion of a particle in a conservative potential field depending on its position only. The boundary condition implies that a particle returns to its initial position after one second.

A possible interpretation of this problem is to find the initial speed $\left.\frac{d x(t)}{d t}\right|_{t=0}$ such that the solution with initial conditions $x(0)=A,\left.\frac{d x(t)}{d t}\right|_{t=0}$ returns to $x=A$ after one second.

In [5] Chicone studied a similar problem for Neumann and Dirichlet boundary problems. In this paper we generalize his analysis. The different types of mixed solutions which we study in this paper were not considered there. It turns out that these mixed solutions lead to a richer and more complex solution structure than the cases studied in [5].

## Conditions on $f(x)$

The function $f(x)$ is taken to be real analytic. For some results this condition could be weakened but for the clarity of reading we will assume that $f(x)$ is real analytic in all the cases of this paper.

In particular we will consider the case where the corresponding system in the phase plane has a center at the origin:

$$
\begin{equation*}
f(0)=0, \quad f^{\prime}(0)>0 . \tag{1.3}
\end{equation*}
$$

This will ensure that a continuum of periodic orbits exists, i.e. a period annulus surrounding a singularity of center type.

Furthermore to obtain global results we will typically impose that the corresponding system in the phase plane has a saddle for $x=x_{s}$ :

$$
\begin{equation*}
f\left(x_{s}\right)=0, \quad f^{\prime}\left(x_{s}\right)<0 \tag{1.4}
\end{equation*}
$$

Finally we impose that outside these two singularities the following relation holds:

$$
\begin{equation*}
x\left(x-x_{s}\right) f(x)>0, \quad x \neq 0, x \neq x_{s} . \tag{1.5}
\end{equation*}
$$

which ensures that no other singularities exist.

## Conservative forces and applications

Boundary problems of the type (1.1), (1.2) have been studied extensively in the literature. Typical choices for the conservative force $f(x)$ are $e^{x}$ (see $\left.[2,10]\right),(x-a)(x-b)(x-c)$ (see $[14,27,29]$ ), $e^{\frac{x}{1+\epsilon x}}$ (see [11,30]), $\sum_{k=0}^{k=n} \frac{x^{k}}{k!}$ (see [32]), convex $f(x)$ (see [19, 20]), quadratic $f(x)$ (see [4]). Applications of the BVP typically appear in the steady-state solutions of diffusion equations, see [15] and [16] for an extensive discussion. Other examples of applications can be found in the theory of combustion, see e.g. [3,11,30]. For other interesting flavours of boundary value problems, see [1] and [25], where a constant damping term $c \frac{d x(t)}{d t}$ was added to the equation, and [9], where another type of damping was introduced. These cases with damping are out of scope for this paper and require a different kind of analysis, since in general no first integral of the differential equation is known. The analysis of the systems with constant damping can be related to the study of limit cycles in so-called Liénard systems after a Filippov transformation. The discussion of this relation is outside the scope of this paper.

## Positive and negative solutions

In most of the papers on this subject $A=0$ and only so-called positive solutions are studied, where $x(t)>0$ for $0<t<1$. This generalizes to our formulation as the requirement that $x(t)>A$ for $0<t<1$ : the solution does not return to its initial value before $t=1$. We will refer to this as a positive solution to be consistent with the literature. Similarly a negative solution can be defined as a solution for which $x(t)<A$ for $0<t<1$.

In the study of positive solutions many deep results have been proved in recent years. In particular we refer to the papers $[11,12,14]$, where upper bounds were found for the number of solutions to the boundary value problems for general classes of potential functions.

In the phase plane $(x, y)$, where $y=\frac{d x(t)}{d t}$, positive (negative) solutions are identified by the property that the solution curve stays to the right (left) of the vertical line $x=A$ before returning to $x=A$.

Another important property of these types of solutions is that $\left.\frac{d x(t)}{d t}\right|_{t=0}>0(<0)$ for positive (negative) solutions. Positive (negative) solutions necessarily start at a point $x=A$, $y=y_{0}>0\left(y=y_{0}<0\right)$ in the phase plane.

The study of negative solutions is essentially the same as for positive solutions. Similar techniques can be applied. In this paper we typically prove results for the positive case and state the results for the negative cases if needed without giving the detailed proofs.

## Periodic orbits and mixed solutions

The main novelty of the research presented in this paper is the study of mixed solutions, crossing the line $x=A$ in the phase plane before their final return to $x=A$. Formally a mixed solution is a solution such that $\exists \bar{t} \in(0,1)$ with $x(\bar{t})=A$, i.e. the solution will return at least once to the initial value $x(0)=A$ before $t=1$. It implies that there exist values $t_{1}, t_{2} \in(0,1)$ such that $x\left(t_{1}\right)<A$ and $x\left(t_{2}\right)>A$, hence the terminology mixed solution. A necessary condition for this situation to be possible is that the solution lies on a periodic orbit in the phase plane of (1.1). In systems of the type (1.1), because of its conservative nature, no isolated period orbits (limit cycles) can occur and therefore necessarily we are looking at systems which have a continuum of period orbits, a so-called period annulus.

In the fundamental paper on this subject [27] mixed solutions were studied for the case $f(x)=(x-a)(x-b)(x-c)$. The case of mixed solutions has not received much attention in the literature since and we will show that new complex phenomena may occur even for the simplest cases of $f(x)$. In particular we will argue that the argument in [27] where it was stated that for sufficiently large $\lambda$ no bifurcation values will occur is not necessarily true in general. Even for the simple quadratic case $f(x)=x(x+1)$ there are values of $A$ such that bifurcations exist no matter how large $\lambda$ is chosen.

## Time-to-return functions

Our approach will be to study the problem by a simple rescaling of the time parameter after which we can continue the analysis by studying the time-to-return functions of system (1.1) with $\lambda=1$. These are functions depending on the integration constant (or energy level in terms of the mechanical interpretation of the system) representing the time it takes to return to the vertical line $x=A$ in the phase plane. Returning to the initial $x$-coordinate can be done in many different ways when the orbit in the phase plane is a closed curve representing
a periodic solution. Part of the purpose of this paper is to categorize these different return mechanisms and to analyze the corresponding time-to-return functions.

## S-shaped bifurcations

In the literature one particular bifurcation phenomenon was observed for this type of boundary value problem: the occurrence of S-shaped bifurcations for positive solutions, see [11, 13, 30]. Essentially this corresponds to the existence of two different critical $\lambda$ values where solutions to the equations bifurcate under a change of $\lambda$, while there exist $\lambda$-values for which three solutions occur. We will show in this paper that $S$-shaped bifurcations occur for mixed solutions under generic conditions on the function $f(x)$, if the phase plane contains a period annulus which is bounded on the outside by solution containing a saddle singularity (i.e. a saddle loop) and on the inside by a singularity of center type.

## Quadratic Hamiltonian

As illustration of the results for the general case we consider the simplest example by taking $f(x)=x(x+1)$. For this quadratic Hamiltonian system several results have been obtained in the past. It is well-known that for the case of positive and negative solutions at most two solutions can occur for given $\lambda$, see $[4,19,20]$. The full period function is monotonic (see e.g. [8]). The case of mixed solutions leads to more complicated situations. It will be shown that for the mixed solution types with $f(x)=x(1+x)$, there exist $\lambda$-values for which at least three mixed solutions occur and that S -shaped bifurcations occur.

## Period functions

The problems addressed in this paper can be viewed as a generalization of the work on the so-called period function of a period annulus. There is a rich literature on this subject (see for example the pioneering work of [6] in the field of so-called quadratic systems and more recent work in $[21-24,31]$ ). In a sense, problems related to the period function can be interpreted as a subset of the problems presented in this paper. We will show that in a generic setting at least two local extreme values of the time-to-return functions can occur in the case of a mixed solution, showing the increased complexity compared to the study of the period function.

## Results

The main results of this paper are:

- a full classification of the solution types of system (1.1) with boundary conditions (1.2);
- analytical expressions for the corresponding time-to-return functions for each solution type and their expansions near the center singularity;
- a new recursive formula for the derivatives of the full period function;
- existence of an S-shaped bifurcation phenomenon for systems with a generic form $f(x)$ under the condition that $f^{\prime \prime}(0) \neq 0$ and that a period annulus exists with a center and saddle loop on its boundaries;
- finiteness of the number of solutions for each mixed solution type for a generic class of $f(x)$.


## 2 Time-to-return functions

It is more convenient to study the boundary value problem (1.1), (1.2) in its equivalent form in the phase plane, through the introduction of the auxiliary variable $y(t) \equiv \frac{d x(t)}{d t}$ :

$$
\begin{align*}
\frac{d x(t)}{d t} & =y(t), \\
\frac{d y(t)}{d t} & =-\lambda f(x(t)),  \tag{2.1}\\
x(0) & =x(1)=A . \tag{2.2}
\end{align*}
$$

A simple scaling of the variables changes the boundary value problem (2.1), (2.2) into a more tractable and traditional form, where a straightforward time-traversal can be studied for all solutions. This is a well-known procedure, see e.g. [19].

Introducing new variables $t=\frac{\bar{t}}{\sqrt{\lambda}}, y(t)=\bar{y}(t) \sqrt{\lambda}$, the boundary problem (2.1), (2.2) becomes:

$$
\begin{align*}
& \frac{d x(\bar{t})}{d \bar{t}}=\bar{y}(\bar{t}), \\
& \frac{d \bar{y}(\bar{t})}{d \bar{t}}=-f(x(\bar{t})), \tag{2.3}
\end{align*}
$$

with boundary conditions

$$
\begin{equation*}
x(\bar{t}=0)=x(\bar{t}=\sqrt{\lambda})=A . \tag{2.4}
\end{equation*}
$$

i.e. the dependency on $\lambda$ has been removed from the system of differential equations and has been put into the boundary condition. In the following we will focus on this system and drop the bars for notational convenience. We will refer to trajectories of (2.3) in the $(x, y)$ phase plane as orbits while we will refer to those trajectories satisfying not only (2.3), but the additional boundary condition (2.4) as well, as solutions. So the set of solutions to (2.3), (2.4) is contained in the set of orbits defined by (2.3) but not every orbit in the phase plane will necessarily correspond to a solution.

### 2.1 Reformulating the original boundary value problem

In order to find solutions to the original boundary problem (1.1), (1.2), according to (2.4) we need to find the time it takes an orbit of (2.3) starting at the line $x=A$ in the phase plane to reach the same line $x=A$ again: for given $\lambda$ those orbits of (2.3) returning to the original vertical line $x=A$ in $\sqrt{\lambda}$-time correspond to solutions of the original boundary problem (1.1), (1.2). Depending on the nature of the solution curves in the phase plane, there is not necessarily a unique way (if any) to achieve this. If the solution curve returns to $x=A$, then we refer to the time it takes to traverse back to its original $x$-value as the time-to-return function. Typically for periodic orbits there will not be a unique way to return to the original $x$-value and therefore we will have to consider multiple time-to-return functions, each distinguished by the way the solution returns to $x=A$.

The terminology function is used here to indicate that the time it takes to return to the original $x$-value is a function of the initial starting point in the phase plane, i.e. depending on the initial velocity (the initial $y$-value in the phase plane, i.e. $\left.\frac{d x(t)}{d t}\right|_{t=0}$ in system (2.3)).

### 2.2 Phase plane interpretation of the Hamiltonian system

The orbits of the solutions of (2.3) in the phase plane can be written down explicitly:

$$
\begin{equation*}
h=\frac{1}{2} y^{2}(t)+F(x(t)), \tag{2.5}
\end{equation*}
$$

where

$$
F(u) \equiv \int_{0}^{u} f(x) d x
$$

Each $h$ corresponds to an integral curve in the phase plane. We will assume that conditions


Figure 2.1: Phase portrait for system (2.3) with conditions (1.3), (1.4) and (1.5) on $f(x)$.
(1.3), (1.4) and (1.5) hold. This implies that the phase portrait of the system contains two singularities: a saddle at $\left(x=x_{s}, y=0\right)$ and a center at $(x=0, y=0)$. Through a change of variables $x \rightarrow-x$ (if necessary) the saddle can be positioned to the left of the center, i.e. $x_{s}<0$, which we will assume to hold true in the following for convenience of discussion.

The integration constant $h \equiv h_{\text {sep }}=F\left(x_{s}\right)$ corresponds to a saddle loop, passing through the saddle, see Figure 2.1. The integration constant $h=0$ corresponds to the center point. For the values $0<h<h_{\text {sep }}$ the region between center and saddle loop is filled with closed orbits corresponding to periodic solutions, i.e. each $h$ in this interval corresponds to one closed orbit, which is symmetrical with respect to the $x$-axis as the integral formula (2.5) shows. The time it takes to traverse a solution in the region $y>0$ is the same as it takes to traverse the reflected path for $y<0$. Therefore when we consider traversal times along orbits we can always restrict our attention to the part of the curve lying in $y>0$.

The saddle loop intersects the $x$-axis in two points: through the saddle itself located at $x=$ $x_{s}$ and at the regular point $x=x_{s}^{(2)}>0$ as long as $\exists x_{s}^{(2)}>0$ such that $F\left(x_{s}\right)=F\left(x_{s}^{(2)}\right)$. We will assume that such a point exists, i.e. that the original system has a saddle loop. The arguments of this paper generalize to the situation where $x_{s}^{(2)} \rightarrow \infty$ but for notational convenience we will omit this case here.


Figure 2.2: Properties of a periodic orbit of (2.3).

Since we are interested in the behaviour of the solutions to (1.1), (1.2) related to the periodic orbits, we restrict the value of $A$ to the interval $x_{s}<A<x_{s}^{(2)}$. For values of $A$ outside this interval no periodic orbits can reach the vertical line $x=A$ in the phase plane.

The set of periodic orbits for $h \in\left(0, h_{\text {sep }}\right)$ is referred to as a period annulus in the literature. The closed orbit representing a periodic orbit in the phase plane is denoted in the following by $\gamma_{h}$.

Well-known properties of $\gamma_{h}$ are:

- The orbit $\gamma_{h}$ satisfies (2.5) for some integration constant $h \in\left(0, h_{\text {sep }}\right)$.
- The periodic orbit $\gamma_{h}$ is symmetrical with respect to the $x$-axis. The time it takes to traverse the periodic orbit for $y>0$ is the same as for $y<0$.
- For each $h \in\left(0, h_{\text {sep }}\right) \gamma_{h}$ crosses the $x$-axis in exactly two points, of which the coordinates $x_{-}(h)<0$ and $x_{+}(h)>0$ satisfy $F\left(x_{ \pm}(h)\right)=h$.
- A periodic orbit $\gamma_{h}$ intersecting a line $x=B$ will do so in exactly two points $(x=B, y=$ $\sqrt{2(h-F(B)}),(x=B, y=-\sqrt{2(h-F(B)})$, except when $x=B$ coincides with the crossing of the $x$-axis by $\gamma_{h}$ at $x=x_{-}(h)$ or $x=x_{+}(h)$. In those latter cases there is only one intersection point: the vertical line $x=B$ is tangent to $\gamma_{h}$ at the crossing of the $x$-axis at $\left(x_{ \pm}(h)=B, 0\right)$.

These properties are summarized in Figure 2.2.

## 3 Categorization of solution types

### 3.1 Types of solutions

With the results from the previous section in mind we can categorize the different ways in which a solution to (2.3), (2.4) can start and end on the vertical line $x=A$. In Figure 3.1 the full
list of possible solution types are displayed. Assume that the line $x=A$ intersects the period



- starting point

Figure 3.1: Solution types for the boundary value problem on a periodic orbit of (1.1).
orbit $\gamma_{h}$ in two points $\left(x=A, y=y_{A} \equiv \sqrt{2(h-F(A))},\left(x=A, y=-y_{A} \equiv-\sqrt{2(h-F(A))}\right.\right.$, see Figure 2.2. If there is no such intersection, then the orbit $\gamma_{h}$ cannot generate solutions to (2.3), (2.4).

Positive solutions. First we discuss the case of starting at $\left(x=A, y=y_{A}>0\right)$, i.e. above the $x$-axis. The solution starts at $\left(x=A, y=y_{A}\right)$ on the periodic orbit. It will cross the $x$-axis at $\left(x=x_{+}(h), 0\right)$ and return to $x=A$ for the first time by reaching the reflected point $\left(x=A, y=-y_{A}\right)$. We denote this part of $\gamma_{h}$ by $S_{+}^{A}(h)$. We call it the positive part of the periodic orbit because all $x$-values are larger than $A$ in correspondence with the notation in the literature. The time to reach this first point of return we refer to as $T_{+}^{A}(h)$.

Negative solutions. These solutions have the same properties as the positive solutions except that the solutions have to stay on the left of the line $x=A$. It translates into a starting point $\left(x=A, y=y_{A}<0\right)$, i.e. below the $x$-axis, with the solution returning to its reflected point above the $x$-axis. We denote this part of $\gamma_{h}$ by $S_{-}^{A}(h)$ and the time to reach the other side by $T_{-}^{A}(h)$.

Full solutions. A full solution returns to its original starting point $\left(x=A, y=y_{A}\right)$, i.e. a full period rotation has been made in the phase plane. We denote the time to make a full rotation by $T_{\text {full }}(h)$ (the period of $\gamma_{h}$ ) and the trajectory itself by $S_{n}(h)$, where $n=1,2, .$. indicates the number of full rotations that were made. The corresponding time-to-return function is written as $T_{n}(h) \equiv n T_{\text {full }}(h)$. The function $T_{\text {full }}(h)$ is what in the literature is referred to as the so-called period function. Clearly from the definition $S_{1}(h)=S_{+}^{A}(h) \oplus S_{-}^{A}(h)$ and $T_{\text {full }}(h)=T_{+}^{A}(h)+T_{-}^{A}(h)$.
Mixed solutions. The argument can be continued by considering a positive solution starting at ( $x=A, y=y_{A}>0$ ), returning to ( $x=A, y=-y_{A}<0$ ) and then making one full rotation.

This orbit type is a combination of a partial rotation $S_{+}^{A}(h)$ followed by a full rotation along $S_{\text {full }}(h)$. For notational convenience we label this trajectory by $S_{3 / 2}^{A}(h)$ to indicate that it is a union of the two trajectories $S_{1}(h)$ and $S_{+}^{A}(h)$. It is important to note that this trajectory contains parts where $x<A$ and $x>A$ before returning. Therefore we refer to this type of solution as a mixed solution. The full rotations are mixed as well, but these solutions we will keep referring to as $S_{n}(h)$.

Similarly we can define mixed solution types that start below the $x$-axis. For example starting at $\left(x=A, y=y_{A}<0\right)$, a partial trajectory is followed by one full rotation. This is denoted by $S_{-3 / 2}^{A}(h)$.

In this way we find a countably infinite number of ways of returning to the line $x=A$, starting at $\left(x=A, y=y_{A}\right)$ above and below the $x$-axis. In Figure 3.1 the different solution types are indicated with the corresponding trajectories on the periodic orbit. We summarize the possibilities as follows (the dependency on the parameter $h$ was dropped for convenience of reading):

- $S_{+}^{A}$ : one partial rotation from $y>0$ to $y<0$, ending at the reflection in the $x$-axis of the starting point.
- $S_{\text {full }} \equiv S_{1}$ : one full rotation on the period orbit returning to its original point.
- $S_{3 / 2}^{A}=S_{+}^{A} \oplus S_{\text {full }}$.
- $S_{2}=S_{\text {full }}^{A} \oplus S_{\text {full }}$.
- $S_{5 / 2}^{A}=S_{+}^{A} \oplus S_{\text {full }} \oplus S_{\text {full }}$.
- ...
- $S_{-}^{A}$, similar to $S_{+}^{A}$ but starting at $y<0$ and ending at $y>0$.
- $S_{-3 / 2}^{A}=S_{-}^{A} \oplus S_{\text {full }}$.
- $S_{-5 / 2}^{A}=S_{-}^{A} \oplus S_{\text {full }} \oplus S_{\text {full }}$.

The full set of solutions can be categorized by the following types:
Proposition 3.1. Solutions to the boundary value problem (2.3) and (2.4) corresponding to a given period orbit $\gamma_{h}$, where $h$ is the integration constant in (2.5), can be categorized by:

- Positive solution: $S_{+}^{A}$
- Full solutions: $S_{n}$, where $n=1,2,3, \ldots$
- Mixed solutions: $S_{n+1 / 2}^{A} S_{-n-1 / 2}^{A}$, where $n=1,2,3, \ldots$

Remark 3.2. In the proposition we grouped all full period solutions under the same label as a full solution. For all these cases the time-to-return function to the starting point in the phase plane does not depend on $A$. The behaviour of the solutions solely depends on the structure of the period function of the period annulus.

Each of the solution types in Proposition 3.1 is characterized by the number of times it crosses the $x$-axis in the phase plane and where it crosses the $x$-axis, i.e. for $x<A$ or $x>A$. The way to choose the solution types was chosen to have an easy reference to these crossings. In terms of the original boundary problem (1.1) and (1.2) a crossing of the $x$-axis corresponds with a local minimum $(x<A)$ or local maximum $(x>A)$ of the solution as a function of $t$. This is due to the interpretation of the variable $y$ in the phase plane as $\frac{d x}{d t}$. Therefore the number of $x$-axis crossings equals the number of local extrema of the original solution. Obviously an increase in rotations along the period orbit $\gamma_{h}$ in the phase plane increases the number of local extrema (i.e. each full rotation adds a local maximum and local minimum). The conclusion is:

Proposition 3.3. According to the categorization of solutions in Proposition 3.1 to the original boundary value problem (1.1) and (1.2) each type of solution is characterized by the number of crossings of the $x$-axis by the periodic orbit $\gamma_{h}$ in the phase plane of system (2.3):

- Positive solution: $S_{+}^{A}$ : one local maximum
- Negative solution: $S_{-}^{A}$ : one local minimum
- Full solutions: $S_{n}$, where $n=1,2,3, \ldots$ : $n$ local minima and $n$ local maxima.
- Mixed solutions: $S_{n+1 / 2}^{A}$, where $n=1,2,3, \ldots: 2 n+1$ local extreme points, $n+1$ local maxima and $n$ local minima, the first local extreme point being a local maximum.
- Mixed solutions: $S_{-n-1 / 2}^{A}$, where $n=1,2,3, \ldots: 2 n+1$ local extreme points, $n$ local maxima and $n+1$ local minima, the first local extreme point being a local minimum.


### 3.2 Time-to-return functions for the different types of solutions

To each of the solution types as described in Proposition 3.1 we can associate the time it takes to follow the trajectory from start to end point. As noted before, the corresponding trajectories reflected in the $x$-axis are traversed in the same time span. The time-to-return functions can be written as a linear combination of three fundamental functions:

Lemma 3.4. The time-to-return function for a positive solution of the type $S_{+}^{A}$ is given by:

$$
\begin{equation*}
T_{+}^{A}(h)=2 \int_{x=A}^{x=x_{+}(h)} \frac{d x}{y_{h}(x)} \tag{3.1}
\end{equation*}
$$

where $y_{h}(x) \equiv \sqrt{2(h-F(x))}$ and $F\left(x_{+}(h)\right)=h, x_{+}(h)>A$.
The time-to-return function for a negative solution of the type $S_{-}^{A}$ is given by:

$$
\begin{equation*}
T_{-}^{A}(h)=2 \int_{x=x_{-}(h)}^{x=A} \frac{d x}{y_{h}(x)} \tag{3.2}
\end{equation*}
$$

where $F\left(x_{-}(h)\right)=h, x_{-}(h)<A$.
The time-to-return function for a full solution of the type $S_{1}$ is given by:

$$
\begin{equation*}
T_{\text {full }}(h)=2 \int_{x=x_{-}(h)}^{x_{+}(h)} \frac{d x}{y_{h}(x)} \tag{3.3}
\end{equation*}
$$

Proof. Consider the case of a positive solution $S_{+}^{A}$, i.e. $x(t)>A$. The solution starts at $x=A$, crosses the $x$-axis at $\left(x_{+}(h), 0\right)$ and then returns to $x=A$ along a trajectory which is the reflection of the trajectory above the $x$-axis. The time it takes to traverse the trajectory above the $x$-axis is the same as the time it takes to traverse the trajectory below the $x$-axis. Therefore the total return time is twice the time it takes to reach $\left(x_{+}(h), 0\right)$. The formula in the lemma follows by using the relation $\frac{d x(t)}{d t}=y_{h}(x(t))$ which implies $t_{1}-t_{0}=\int_{x\left(t_{0}\right)}^{x\left(t_{1}\right)} \frac{d x}{y_{h}(x)}$, where we defined $y_{h}(x)=\sqrt{2(h-F(x))}$ for trajectories above the $x$-axis. Here $x\left(t_{0}\right)=A$ and $x\left(t_{1}\right)=x_{+}(h)$.

The proof for the negative solution follows the same arguments.
The formula for the full period is well-known in the literature (see e.g. [4]) .
Remark 3.5. The three functions are related by the obvious relation $T_{\text {full }}(h)=T_{+}^{A}(h)+T_{-}^{A}(h)$.
As a direct consequence of the previous lemma and the solution structure as given in Proposition 3.1, we can write down the time-to-return functions for all solution types:

Proposition 3.6. The time it takes to traverse the trajectories as defined by Proposition 3.1 of mixed type can be expressed in terms of the three fundamental time-to-return functions of Lemma 3.4 in the following way:

$$
\begin{align*}
T_{n+1 / 2}^{A}(h) & =T_{+}^{A}(h)+n T_{\text {full }}(h),  \tag{3.4}\\
T_{-n-1 / 2}^{A}(h) & =T_{-}^{A}(h)+n T_{\text {full }}(h), \tag{3.5}
\end{align*}
$$

where $n=1,2, \ldots$
Remark 3.7. Obviously $T_{1 / 2}^{A}(h)<T_{\text {full }}(h)$, so there is a natural ordering of the values in the proposition:

$$
\begin{aligned}
T_{1 / 2}^{A}(h) & <T_{1}(h)<T_{3 / 2}^{A}(h)<T_{2}(h)<\cdots \\
T_{-1 / 2}^{A}(h) & <T_{1}(h)<T_{-3 / 2}^{A}(h)<T_{2}(h)<\cdots
\end{aligned}
$$

Due to the symmetry in the formulas for the negative and positive time-to-return functions, we will focus on the functions $T_{+}^{A}(h), T_{n}^{A}(h)$ and $T_{n+1 / 2}^{A}(h)$ in this paper. The results for the other two types $T_{-}^{A}(h), T_{-n-1 / 2}^{A}(h)$ can be derived in a similar way and will differ only by introduction of some additional minus signs in the expressions. The simplest way to achieve this is by changing $x \rightarrow-x$ in (2.1), essentially changing $f(x)$ into $f(-x)$. Application of the formulas for $T_{+}^{A}(h)$ and $T_{n+1 / 2}^{A}(h)$ to the new system leads to the formulas for $T_{-}^{A}(h)$ and $T_{-n-1 / 2}^{A}(h)$ in the original system.

## 4 Positive solutions

### 4.1 Expansion of the positive time-to-return function for small $h$ and $A=0$

Proposition 4.1. If $f(x)$ is real analytic and condition (1.3) holds (i.e. a center exists at the origin of the phase plane), then the positive time-to-return function (3.1) for $A=0$ can be expanded for small $h$ as:

$$
\begin{equation*}
T_{+}^{0}(h)=d_{0}+d_{1} h^{\frac{1}{2}}+d_{2} h+d_{3} h^{\frac{3}{2}}+d_{4} h^{2}+\ldots \tag{4.1}
\end{equation*}
$$

The first two terms explicitly take the following form:

$$
\begin{equation*}
T_{+}^{0}(h)=\frac{\pi}{\sqrt{a_{0}}}-\frac{2 a_{1}}{a_{0}^{2}} \sqrt{h}+\ldots \tag{4.2}
\end{equation*}
$$

where $a_{i}$ are the coefficients of the expansion of the potential function: $F(x)=x^{2}\left(a_{0}+a_{1} x+a_{2} x^{2}+\right.$ ...) near $x=0$.

Proof. We write $T_{+}^{0}(h)=2 \int_{0}^{x_{+}(h)} \frac{d x}{y(x, h)}$ in the following convenient way (introduced in [27]):

$$
T_{+}^{0}(h)=\tilde{T}_{+}^{0}\left(x_{+}\right)=\sqrt{2} \int_{x=0}^{x=x_{+}} \frac{d x}{\sqrt{F\left(x_{+}\right)-F(x)}} .
$$

For convenience of reading (and because in the literature such a variable is used) we will write $x_{+} \equiv \alpha$.

With this notation we can rewrite the integral using a scaling of the integration variable $x=\alpha u$. The integral becomes:

$$
\begin{equation*}
\tilde{T}_{+}^{0}(\alpha)=\sqrt{2} \int_{u=0}^{u=1} \frac{\alpha d u}{\sqrt{F(\alpha)-F(\alpha u)}} . \tag{4.3}
\end{equation*}
$$

By assumption we know that $F(x)$ has an expansion of the form $F(x)=x^{2}\left(a_{0}+a_{1} x+a_{2} x^{2}+\right.$ ...) Substitution of this expansion into the integral leads to:

$$
\tilde{T}_{+}^{0}(\alpha)=\sqrt{2} \int_{u=0}^{u=1} \frac{d u}{R(u, \alpha) \sqrt{1-u}},
$$

where $R(u, \alpha)=\sqrt{Z_{0}(u)+Z_{1}(u) \alpha+Z_{2}(u) \alpha^{2}+\ldots}, Z_{0}(u)=a_{0}(1+u), Z_{1}(u)=a_{1}\left(1+u+u^{2}\right)$, $\ldots, Z_{i}(u)=a_{i}\left(1+u+u^{2}+\cdots+u^{i}\right)$.

The function $\frac{1}{R(u, \alpha)}$ is analytical on the interval of integration, because the function $F(x)$ does not have any other zeroes on the interval of integration (we consider only $x$-values close to the isolated zero at $x=0$ ), i.e. $R(u, \alpha) \neq 0$ for $0 \leq u \leq 1$.

It leads to the following expansion in $\alpha$ :

$$
\begin{equation*}
\tilde{T}_{+}^{0}(\alpha)=C_{0}+C_{1} \alpha+C_{2} \alpha^{2}+\ldots \tag{4.4}
\end{equation*}
$$

where

$$
\begin{gathered}
C_{0}=\sqrt{2} \int_{0}^{1} \frac{d u}{\sqrt{Z_{0}(u)} \sqrt{1-u}}=\frac{\pi}{\sqrt{2 a_{0}}}, \\
C_{1}=\sqrt{2} \int_{0}^{1} \frac{-Z_{1}(u) d u}{2 Z_{0}(u)^{\frac{3}{2}} \sqrt{1-u}}=-\frac{\sqrt{2 a_{1}}}{a_{0}^{\frac{3}{2}}}, \\
C_{2}=\sqrt{2} \int_{0}^{1} \frac{\left(3 Z_{1}(u)^{2}-4 Z_{0}(u) Z_{2}(u)\right) d u}{8 Z_{0}(u)^{\frac{5}{2}} \sqrt{1-u}}=\frac{\sqrt{2}\left(15 \pi a_{1}^{2}-12 \pi a_{0} a_{2}-16 a_{1}^{2}\right)}{8 a_{0}^{\frac{5}{2}}} .
\end{gathered}
$$

In order to find the expansion of the positive time-to-return function in terms of $h$ we need to find the relation between $h$ and $\alpha$ for small $h$. After substitution of the above expansion for $F(x)$ into the relation $\sqrt{F(\alpha)}=\sqrt{h}$ we get:

$$
\alpha \sqrt{a_{0}+a_{1} \alpha+a_{2} \alpha^{2}+\ldots}=\sqrt{h} .
$$

Note that the term $\mu(\alpha) \equiv \sqrt{a_{0}+a_{1} \alpha+a_{2} \alpha^{2}+\ldots}$ is analytical in $\alpha$ and $\mu(0) \neq 0$. It means that we can apply the Lagrange inversion theorem to get:

$$
\begin{equation*}
\alpha(u)=\sum_{n=1}^{\infty} g_{n} \frac{u^{n}}{n!}, \tag{4.5}
\end{equation*}
$$

where $u=\sqrt{h}$ and

$$
g_{n}=\lim _{z \rightarrow 0} \frac{d^{n-1}}{d z^{n-1}}\left[\frac{1}{\mu^{n}(z)}\right] .
$$

It follows that the first coefficients of the expansion are:

$$
\begin{gathered}
g_{1}=\lim _{z \rightarrow 0} \frac{1}{\mu(z)}=\frac{1}{\sqrt{a_{0}}}, \\
g_{2}=\lim _{z \rightarrow 0} \frac{d}{d z}\left[\frac{1}{\mu^{2}(z)}\right]=-\frac{a_{1}}{a_{0}^{2}} \\
g_{3}=\lim _{z \rightarrow 0} \frac{d^{2}}{d z^{2}}\left[\frac{1}{\mu^{3}(z)}\right]=\frac{15 a_{1}^{2}-12 a_{0} a_{2}}{4 a_{0}^{\frac{7}{2}}} .
\end{gathered}
$$

Substitution into (4.4) gives the expansion of the positive time-to-return function in terms of $h$.

$$
\begin{equation*}
T_{+}^{0}(h)=C_{0}+C_{1} \alpha(h)+C_{2} \alpha(h)^{2}+\cdots=\frac{\pi}{\sqrt{a_{0}}}-\frac{2 a_{1}}{a_{0}^{2}} \sqrt{h}+\ldots \tag{4.6}
\end{equation*}
$$

### 4.2 Derivative of the positive time-to-return function for $A \neq 0$

Lemma 4.2. For $A \neq 0$ the derivative of the time-to-return functions $T_{+}^{A}(h)$ (3.1) with respect to $h$ is given by the following equivalent expressions:

$$
\begin{equation*}
h \frac{d T_{+}^{A}(h)}{d h}=\frac{\sqrt{2} h_{A}}{f(A) \sqrt{h-h_{A}}}+\int_{y=0}^{y=\sqrt{h-h_{A}}} \omega^{\prime}(x(y)) d y \tag{4.7}
\end{equation*}
$$

where $h_{A}=F(A), \omega(u) \equiv \frac{F(u)}{f(u)^{2}}, \omega^{\prime}(u)=\frac{d \omega(u)}{d u}, x(y)$ satisfies $h=\frac{1}{2} y^{2}+F(x(y)), y>0$.

$$
\begin{equation*}
\frac{d T_{+}^{A}(h)}{d h}=\int_{\bar{A}=0}^{\bar{A}=A} \frac{1}{\sqrt{2}\left(h-h_{\bar{A}}\right)^{\frac{3}{2}}} d \bar{A}+\frac{d T_{+}^{0}(h)}{d h} . \tag{4.8}
\end{equation*}
$$

Proof. Multiply the expression for $T_{+}^{A}(h)$ as given in (3.1) by $h$ and use that $h=\frac{1}{2} y^{2}+F(x)$ to write it in the form:

$$
h T_{+}^{A}(h)=2 \int_{x=A}^{x=x_{+}(h)}\left(\frac{1}{2} y_{h}(x)+\frac{F(x)}{y_{h}(x)}\right) d x .
$$

Integration by parts using $\frac{d y_{h}(x)}{d x}=\frac{-f(x)}{y_{h}(x)}$ leads to:

$$
h T_{+}^{A}(h)=2\left[\frac{h_{A} y_{h}(A)}{f(A)}+\int_{x=A}^{x=x_{+}(h)}\left(\frac{1}{2}+\left(\frac{F(x)^{\prime}}{f(x)}\right)\right) y_{h}(x) d x\right],
$$

where $y_{h}(A)=\sqrt{2\left(h-h_{A}\right)}$.

Taking the derivative of this expression with respect to $h$ leads to:

$$
h \frac{d T_{+}^{A}(h)}{d h}=\frac{\sqrt{2} h_{A}}{f(A) \sqrt{h-h_{A}}}+2\left[\int_{x=A}^{x=x_{+}(h)}\left(-\frac{1}{2}+\left(\frac{F(x)^{\prime}}{f(x)}\right)\right) \frac{1}{y_{h}(x)} d x\right]
$$

where the relation $\frac{\partial y_{h}(x)}{\partial h}=\frac{1}{y_{h}(x)}$ was used. Next we write $-\frac{1}{2}+\left(\frac{F(x)}{f(x)}\right)^{\prime}=\frac{1}{2} f(x) \omega^{\prime}(x)$ and change the integration variable from $x$ to $y$ to obtain the first equation of the lemma:

$$
\begin{aligned}
h \frac{d T_{+}^{A}(h)}{d h} & =\frac{\sqrt{2} h_{A}}{f(A) \sqrt{h-h_{A}}}+\int_{x=A}^{x=x_{+}(h)} f(x) \omega^{\prime}(x) \frac{1}{y_{h}(x)} d x \\
& =\frac{\sqrt{2} h_{A}}{f(A) \sqrt{h-h_{A}}}-\int_{y=\sqrt{h-h_{A}}}^{y=0} \omega^{\prime}(x) d y,
\end{aligned}
$$

where the last step uses the fact that $y(x)$ satisfies the differential equation $\frac{d y}{d x}=-\frac{f(x)}{y}$.
The first step in proving the second equation (4.8) in the lemma is to differentiate the expression for $T_{+}^{A}(h)$ with respect to $A$ :

$$
\begin{equation*}
\frac{\partial T_{+}^{A}(h)}{\partial A}=-\frac{\sqrt{2}}{\sqrt{h-h_{A}}} . \tag{4.9}
\end{equation*}
$$

Differentiating this expression with respect to $h$ gives:

$$
\frac{\partial^{2} T_{+}^{A}(h)}{\partial h \partial A}=\frac{1}{\sqrt{2}\left(h-h_{A}\right)^{\frac{3}{2}}} .
$$

The second equation (4.8) in the lemma then follows by integration over the variable $A$ with the notation that $\frac{d T_{d}^{0}(h)}{d h}$ represents the derivative of the positive time-to-return function for $A=0$.

### 4.3 Limits of the positive time-to-return function

This section contains the limits of the positive time-to-return function $T_{+}^{A}(h)$ near the boundary of its definition, i.e. $h=0$, the center, and $h=h_{\text {sep }}$, the saddle loop.

Proposition 4.3. The behaviour near $h=h_{A}$ of the positive time-to-return function $T_{+}^{A}(h)$ in (3.1) and its derivative, defined on $h \in\left(h_{A}, h_{\text {sep }}\right)$ for system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f^{\prime \prime}(0) \neq 0$ is as follows:

For $A<0$ :

$$
\begin{equation*}
\lim _{h \backslash h_{A}} T_{+}^{A}(h)=T_{\text {full }}^{A}>0 . \tag{4.10}
\end{equation*}
$$

For $A=0$ :

$$
\begin{equation*}
\lim _{h \downarrow 0} T_{+}^{0}(h)=\frac{1}{2} T_{0}>0 . \tag{4.11}
\end{equation*}
$$

For $A>0$ :

$$
\begin{equation*}
\lim _{h \downarrow h_{A}} T_{+}^{A}(h)=0, \tag{4.12}
\end{equation*}
$$

where the period of the periodic orbit $\gamma_{h_{A}}$ is abbreviated as $T_{\text {full }}^{A}$ and is given by the expression:

$$
T_{\text {full }}^{A} \equiv 2 \int_{x=x_{-}\left(h_{A}\right)}^{x=x_{+}\left(h_{A}\right)} \frac{d x}{y_{h_{A}}(x)^{\prime}},
$$

where $h_{A} \equiv F(A)$. The orbit $\gamma_{h_{A}}$ is the periodic orbit tangent to the vertical line $x=A$, passing through the point ( $x=A, y=0$ ) in the phase plane.

The limiting value $T_{0}$ is given by the expression $\frac{2 \pi}{f^{\prime}(0)}$ and is the limiting period of the period orbits in the period annulus when approaching the center in the phase plane.
The limits of the derivative are:
For $A<0$ :

$$
\begin{equation*}
\lim _{h \backslash h_{A}} \frac{d T_{+}^{A}(h)}{d h}=-\infty . \tag{4.13}
\end{equation*}
$$

For $A=0$ :

$$
\begin{equation*}
\lim _{h \downarrow 0} \frac{d T_{+}^{0}(h)}{d h}=-\operatorname{sign}\left(f^{\prime \prime}(0)\right) \infty . \tag{4.14}
\end{equation*}
$$

For $A>0$ :

$$
\begin{equation*}
\lim _{h \downarrow h_{A}} \frac{d T_{+}^{A}(h)}{d h}=\infty . \tag{4.15}
\end{equation*}
$$

Proof. For $A \neq 0$ the limits for $h \downarrow h_{A}$ for $T_{+}^{A}(h)$ follow from the facts that:
For $A>0$ the curve $S_{+}$shrinks and approaches the point $(x=A, y=0)$, i.e. in (3.1) the upper integral limit $x_{+}(h)$ approaches $A$ and the integral approaches 0 .

For $A<0$, the curve $S_{+}$approaches the periodic orbit tangent to $x=A$ if $h \downarrow h_{A}$ and therefore the value of the positive time-to-return function approaches the full period of this periodic orbit.

For $A \neq 0$ the limits for $h \downarrow h_{A}$ for the derivative $\frac{d T_{f}^{A}(h)}{d h}$ follow from the expression (4.7) in Lemma 4.2. The integral expression is bounded (and actually approaches 0 in the limit) because of the continuity and boundedness of the function $\omega^{\prime}(x)$ in the integrand and therefore the behaviour of the derivative is dominated by the first term $\frac{\sqrt{2} h_{A}}{f(A) \sqrt{h-h_{A}}}$ which approaches $\pm \infty$ with the sign depending on the sign of $f(A)$ which is positive (negative) for $A>0(A<0)$.

For $A=0$ we can use the expansion of Proposition 4.1, i.e. expansion (4.2). The equations of the lemma follow taking into account that the sign of $a_{1}$ is determined by $f^{\prime \prime}(0)$. If $f^{\prime \prime}(0)=0$ higher order contributions of the expansion need to be taken into account, which can be achieved by a straightforward procedure which is outside the scope of the paper.

The limits for the different cases are summarized in Figure 4.1.
Note 4.4. The crucial observation in Proposition 4.3 is that the limits in (4.10), (4.11) and (4.12) are not continuous as a function of $A$. The value in (4.10) approaches $T_{0}$ when $A \uparrow 0$, while the value is equal to $\frac{1}{2} T_{0}$ for $A=0$ and is identically equal to 0 for $A>0$. The change in the sign of the derivatives (4.13), (4.14), (4.15) while crossing $A=0$ is exactly the cause of the occurrence of S-shaped bifurcations in the mixed solution cases of this paper.

At the end point of the interval for $h$, i.e. $h=h_{\text {sep }}$, we can use the position of the saddle loop to arrive at:
Lemma 4.5. The limiting behaviour of the positive time-to-return function in $T_{+}^{A}(h)$ (3.1) and its derivative, defined on $h \in\left(h_{A}, h_{\text {sep }}\right)$ for system (2.3) near $h=h_{\text {sep, }}$, is as follows:

$$
\begin{align*}
& \lim _{h \downarrow h_{s p}} T_{+}^{A}(h)=C(A)>0  \tag{4.16}\\
& \lim _{h \downarrow h_{s p}} \frac{d T_{+}^{A}(h)}{d h}=C_{2}(A) \tag{4.17}
\end{align*}
$$



Figure 4.1: The three different cases for the limits of the function $T_{+}^{0}(h)$ depending on the sign of $A$. The case depicted here is for $f^{\prime \prime}(0)>0$.

Proof. These limits follow from the fact that the part of the saddle loop surrounding the period annulus for $x>A$ is traversed in a finite positive time, because the saddle is positioned at $x=x_{s}<A$. Note that the sign of $C_{2}(A)$ is undetermined, which is of no further importance for the discussion in this paper.

## 5 Full solutions

The full period solutions as defined in (3.3) correspond to the traditional period function of the period annulus. First we derive a new iterative procedure for determining the derivatives of all order for the period function.

### 5.1 Derivatives of the period function

Proposition 5.1. The $n$-th derivative $\frac{d^{n} T_{\text {full }}(h)}{d h^{n}} \equiv T_{\text {full }}^{(n)}(h), n \geq 0$ of (3.3) can be expressed in the form (with $n=0$ referring to the function $T_{\text {full }}(h)$ itself):

$$
\begin{equation*}
h^{n} T_{\text {full }}^{(n)}(h)=c_{n} \int_{x_{-}(h)}^{x_{+}(h)} y_{h}(x)^{2 n-1} \psi_{n}(x) d x, \tag{5.1}
\end{equation*}
$$

where

$$
\begin{gathered}
c_{n}=\frac{1}{2^{n-1}} \frac{1}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot(2 n-1)^{\prime}}, \\
\psi_{n}(x)=\mathcal{L}[\mathcal{I}]_{\omega(x)}^{(n)}(x), \\
\mathcal{L}[g]_{\omega(x)}^{(n)}(x) \equiv \mathcal{L}[\mathcal{L}[\ldots[\mathcal{L}[g]] \ldots]](x),
\end{gathered}
$$

$$
\begin{gathered}
\mathcal{L}[g]_{\omega(x)}(x) \equiv\left[(\omega(x) g(x))^{\prime}+\omega(x) g^{\prime}(x)\right]^{\prime} \\
\omega(x) \equiv \frac{F(x)}{f(x)^{2}}
\end{gathered}
$$

and the identity function $\mathcal{I}$ is defined by:

$$
\mathcal{I}(x) \equiv 1
$$

The initial values for the iterations are:

$$
\begin{aligned}
c_{0} & =2 \\
\psi_{0}(x) & =1
\end{aligned}
$$

Proof. The proof is by induction. The formula is true for $n=0$, because of (3.3). It implies that:

$$
\begin{aligned}
c_{0} & =2 \\
\psi_{0}(x) & =1
\end{aligned}
$$

Next we show that it will hold true for $n+1$ if the formula is true for $n$. For notational simplicity we write $T_{\text {full }}=T$ and suppress the dependency of $x_{-}$and $x_{+}$on $h$.

Multiply (5.1) with respect to $h$ on both sides to obtain:

$$
h^{n+1} T^{(n)}(h)=c_{n} \int_{x_{-}}^{x_{+}}\left[\frac{1}{2} y_{h}(x)^{2 n+1} \psi_{n}(x)+y_{h}(x)^{2 n-1} F(x) \psi_{n}(x)\right] d x
$$

where we used (2.5), the expression relating $h, x, y$ on an integral curve.
To the second term on the right hand side we apply integration by parts using $\frac{d y}{d x}=-\frac{f(x)}{y_{h}(x)}$, which is allowed since $F(x)$ has a double zero at $x=0$ compensating for the zero of $f(x)$ at $x=0$ introduced in the denominator:

$$
c_{n} \int_{x_{-}}^{x_{+}} y_{h}(x)^{2 n-1} F(x) \psi_{n}(x) d x=-c_{n} \int_{x_{-}}^{x_{+}} \frac{F(x) \psi_{n}(x)}{(2 n+1) f(x)} d y_{h}(x)^{2 n+1}
$$

which leads to (since the boundary terms vanish):

$$
h^{n+1} T^{(n)}(h)=c_{n} \int_{x_{-}}^{x_{+}} y_{h}(x)^{2 n+1}\left[\frac{1}{2} \psi_{n}(x)+\frac{1}{(2 n+1)}\left(\frac{F(x) \psi_{n}(x)}{f(x)}\right)^{\prime}\right] d x
$$

Differentiating this expression with respect to $h$ gives:

$$
h^{n+1} T^{(n+1)}(h)=c_{n} \int_{x_{-}}^{x_{+}} y_{h}(x)^{2 n-1}\left[\left(\frac{F(x) \psi_{n}(x)}{f(x)}\right)^{\prime}-\frac{1}{2} \psi_{n}(x)\right] d x
$$

In this expression the integrand can be rewritten in the following convenient form:

$$
\left(\frac{F(x) \psi_{n}(x)}{f(x)}\right)^{\prime}-\frac{1}{2} \psi_{n}(x)=\frac{1}{2}\left[f(x)\left(\frac{F(x) \psi_{n}(x)}{f(x)^{2}}\right)^{\prime}+\frac{F(x) \psi_{n}^{\prime}(x)}{f(x)}\right]
$$

It follows that:

$$
h^{n+1} T^{(n+1)}(h)=\frac{1}{2} c_{n} \int_{x_{-}}^{x_{+}} y_{h}(x)^{2 n-1}\left[f(x)\left(\frac{F(x) \psi_{n}(x)}{f(x)^{2}}\right)^{\prime}+\frac{F(x) \psi_{n}^{\prime}(x)}{f(x)}\right] d x
$$

Note that $F(x)$ has a double zero at $x=0$ and therefore the expression $\frac{F(x)}{f(x)^{2}}$ should be wellbehaved near $x=0$.

Again with the use of the relation $\frac{d y}{d x}=-\frac{f(x)}{y_{h}(x)}$, another integration by parts leads to:

$$
h^{n+1} T^{(n+1)}(h)=\frac{c_{n}}{2(2 n+1)} \int_{x_{-}}^{x_{+}} y_{h}(x)^{2 n+1}\left[\left(\frac{F(x) \psi_{n}(x)}{f(x)^{2}}\right)^{\prime}+\frac{F(x) \psi_{n}^{\prime}(x)}{f(x)^{2}}\right]^{\prime} d x
$$

This confirms the general form for the $n$th derivative of the period function as indicated in equation (5.1):

$$
h^{n+1} T^{(n+1)}(h)=c_{n+1} \int_{x_{-}}^{x_{+}} y_{h}(x)^{2 n+1} \psi_{n+1}(x) d x,
$$

where

$$
\begin{gathered}
c_{n+1}=\frac{1}{2(2 n+1)} c_{n} \\
\psi_{n+1}(x)=\left(\frac{F(x) \psi_{n}(x)}{f(x)^{2}}\right)^{\prime \prime}+\left(\frac{F(x) \psi_{n}^{\prime}(x)}{f(x)^{2}}\right)^{\prime} .
\end{gathered}
$$

According to this iterative procedure the first couple of derivatives take the following form:

$$
\begin{aligned}
h T_{\text {full }}^{\prime}(h) & =c_{1} \int_{x_{-}(h)}^{x_{+}(h)} y_{h}(x) \psi_{1}(x) d x, \\
h^{2} T_{\text {full }}^{\prime \prime}(h) & =c_{2} \int_{x_{-}(h)}^{x_{+}(h)}\left[y_{h}(x)\right]^{3} \psi_{2}(x) d x, \\
h^{3} T_{\text {full }}^{\prime \prime \prime}(h) & =c_{3} \int_{x_{-}(h)}^{x_{+}(h)}\left[y_{h}(x)\right]^{5} \psi_{3}(x) d x,
\end{aligned}
$$

$$
\psi_{1}(x)=\omega^{\prime \prime}(x),
$$

$$
\psi_{2}(x)=\left(\omega^{\prime \prime}(x)\right)^{2}+3 \omega^{\prime}(x) \omega^{\prime \prime \prime}(x)+2 \omega(x) \omega^{i v}(x)
$$

$$
\begin{equation*}
\psi_{3}(x)=\left(\omega^{\prime \prime}(x)\right)^{3}+22 \omega(x) \omega^{\prime \prime}(x) \omega^{i v}(x)+18 \omega^{\prime}(x) \omega^{\prime \prime}(x) \omega^{\prime \prime \prime}(x)+15\left(\omega^{\prime}(x)\right)^{2} \omega^{i v}(x) \tag{5.2}
\end{equation*}
$$

$$
+10 \omega(x)\left(\omega^{\prime \prime \prime}(x)\right)^{2}+20 \omega(x) \omega^{\prime}(x) \omega^{v}(x)+4(\omega(x))^{2} \omega^{v i}(x)
$$

where $\omega(x) \equiv \frac{F(x)}{f(x)^{2}}$.
The first derivative corresponds with the well-known expression used in the literature, e.g. see [8]. The expressions for the higher order derivatives seem to be new.

### 5.2 Properties of the full period function

Proposition 5.2. The behaviour of the full time-to-return function $T_{\text {full }}(h)$ in (3.3) and its derivative, defined on $h \in\left(h_{A}, h_{\text {sep }}\right)$ for system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5), near the boundaries of its domain is as follows:

For $A \neq 0$ :

$$
\begin{gather*}
\lim _{h \downarrow h_{A}} T_{\text {full }}(h)=T_{\text {full }}^{A}>0, \\
\lim _{h \backslash h_{A}} \frac{d T_{\text {full }}(h)}{d h}=C_{3}(A)<\infty . \tag{5.3}
\end{gather*}
$$

For $A=0$ :

$$
\begin{gather*}
\lim _{h \downarrow 0} T_{\text {full }}(h)=T_{0}>0, \\
\lim _{h \downarrow 0} \frac{d T_{\text {full }}(h)}{d h}=C_{3}(0)<\infty, \tag{5.4}
\end{gather*}
$$

where the period of the periodic orbit $\gamma_{h_{\text {eff }}}$ tangent to $x=A$ is abbreviated as $T_{\text {full }}^{A}$ and is given by the expression:

$$
T_{\text {full }}^{A} \equiv 2 \int_{x=x_{-}\left(h_{e f f}\right)}^{x=x_{+}\left(h_{e f f}\right)} \frac{d x}{y_{h_{e f f}}(x)},
$$

where $h_{\text {eff }} \equiv F(A)$. Notice that for $A>0(A<0)$, we have the relation $x_{+}\left(h_{\text {eff }}\right)=A\left(x_{-}\left(h_{\text {eff }}\right)=A\right)$. The limiting value $T_{0}$ is given by the expression $\frac{2 \pi}{f^{\prime}(0)}$.

Near the outer boundary of the period annulus enclosed by a saddle loop we have the straightforward result:

$$
\begin{align*}
& \lim _{h \uparrow h_{\text {sep }}} T_{\text {full }}(h)=\infty, \\
& \lim _{h \uparrow h_{\text {sep }}} \frac{d T_{\text {full }}(h)}{d h}=\infty . \tag{5.5}
\end{align*}
$$

Proof. The results for $\lim _{h \downarrow h_{A}}$ and $A \neq 0$ follow from the definition of the full period function. $T_{\text {full }}(h)$ does not depend on $A$ and will therefore assume the value of the function at $h_{A}$ due to continuity. The limit for $A=0$ is a classical result for the period of a periodic solution near an elementary center, see e.g. [4]. In particular for the results of this paper it is important to notice that the derivative remains bounded when approaching the center point at $h=0$.

The outer boundary $h=h_{\text {sep }}$ is the saddle loop. The integrand inside the integral defining the period function $T_{\text {full }}(h)$ approaches an essential singularity for $\lim _{h \uparrow \uparrow_{s e p}}$ and the value of the integral goes to $\infty$. The intuition behind this is that the periodic orbits near the outer boundary of the period annulus approach the saddle singularity where the solutions of the ODE become slower and the passage time approaches $\infty$. A similar argument can be used for the derivatives.

The limits for the full period functions are shown in Figure 5.1.


Figure 5.1: The limits of the full period functions $n T_{\text {full }}(h)$ with $n=1,2,3$.

## 6 Mixed solutions

### 6.1 Properties of the mixed time-to-return functions

Using the limits for $T_{+}^{A}(h)$ and $T_{\text {full }}^{A}(h)$ we can immediately write down the limits for $T_{n+1 / 2}^{A}(h)$.
Proposition 6.1. The behaviour of the mixed time-to-return function $T_{n+1 / 2}^{A}(h)$ in (3.4) and its derivative, defined on $h \in\left(h_{A}, h_{\text {sep }}\right)$ for system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5), near $h=h_{A}$ and $h=h_{\text {sep }}$, is as follows:

For $A<0$ :

$$
\begin{equation*}
\lim _{h \backslash h_{A}} T_{n+1 / 2}^{A}(h)=(n+1) T_{\text {full }}^{A}>0 . \tag{6.1}
\end{equation*}
$$

For $A=0$ :

$$
\begin{equation*}
\lim _{h \downarrow 0} T_{n+1 / 2}^{A}(h)=\left(n+\frac{1}{2}\right) T_{0} . \tag{6.2}
\end{equation*}
$$

For $A>0$ :

$$
\begin{equation*}
\lim _{h \downarrow h_{A}} T_{n+1 / 2}^{A}(h)=0 . \tag{6.3}
\end{equation*}
$$

The limits of the derivative are:
For $A<0$ :

$$
\begin{equation*}
\lim _{h \downarrow h_{A}} \frac{d T_{n+1 / 2}^{A}(h)}{d h}=-\infty . \tag{6.4}
\end{equation*}
$$

For $A=0$ :

$$
\begin{equation*}
\lim _{h \downarrow 0} \frac{d T_{n+1 / 2}^{0}(h)}{d h}=-\operatorname{sign}\left(f^{\prime \prime}(0)\right) \infty . \tag{6.5}
\end{equation*}
$$

For $A>0$ :

$$
\begin{equation*}
\lim _{h \downarrow h_{A}} \frac{d T_{n+1 / 2}^{A}(h)}{d h}=\infty . \tag{6.6}
\end{equation*}
$$

Near the outer boundary the functions and their derivatives approach infinity:

$$
\begin{gather*}
\lim _{h \uparrow \uparrow_{s e p}} T_{n+1 / 2}^{A}(h)=\infty, \\
\lim _{h \uparrow h_{s p}} \frac{d T_{n+1 / 2}^{A}(h)}{d h}=\infty . \tag{6.7}
\end{gather*}
$$

Proof. Since the mixed time-to-return functions are linear combinations of the positive time-to-return function and the full period function through:

$$
T_{n+1 / 2}^{A}(h)=T_{+}^{A}(h)+n T_{\text {full }}(h),
$$

the limiting behaviour near the boundaries follows in a straightforward way from the previously derived limits for $T_{1 / 2}^{A}(h)$ (Proposition 4.3) and $T_{\text {full }}(h)$ (Proposition 5.2). The critical quantity is the limit in (6.5) which is determined by the sign of $f^{\prime \prime}(0)$.

The limits for the mixed period functions are shown in Figure 6.1 for the case $f^{\prime \prime}(0)>0$.
Note 6.2. As was indicated in Note 4.4, the functions display a discontinuity while changing $A$, i.e. while crossing $A=0$, as indicated in the limits (6.1), (6.2) and (6.3). Moreover, the derivatives (6.4), (6.5) and (6.6) will change sign while crossing $A=0$. This is the cause for the occurrence of an additional local extreme value of the mixed period functions while changing $A$. The discontinuity requires an accurate analysis and is the reason for the technical proofs in the next sections.


Figure 6.1: The limits of the mixed period functions $T_{n+1 / 2}^{A}(h)$ with $n=1,2,3$ for the case $f^{\prime \prime}(0)>0$.

### 6.2 Existence of a local minimum for the mixed time-to-return functions $T_{n+1 / 2}^{A}(h)$ for sufficiently small $A$ and $f^{\prime \prime}(0)>0$

For convenience of the exposition we will discuss the case $f^{\prime \prime}(0)>0$. The case $f^{\prime \prime}(0)<0$ can be analysed in a similar fashion. The function $T_{n+1 / 2}^{A}(h)$ for $A=0$ tends to $+\infty$ for $h \uparrow h_{\text {sep }}$ according to (6.7). Since at $h=0$ the derivative is $-\infty$ according to (6.5), there must exist a local minimum for some $h \in\left(0, h_{\text {sep }}\right)$.
Proposition 6.3. For each $n>0$ and with $A=0$ the mixed time-to-return function $T_{n+1 / 2}^{A}(h)$ in (3.4) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f^{\prime \prime}(0)>0$, defined on $h \in\left(0, h_{\text {sep }}\right)$ for system (2.3) has at least one local minimum for $h=h_{1}^{0}$.

Proof. The proof follows easily from the limiting behaviour of the mixed time-to-return functions as given in Proposition 6.1. For each $n$ we have $\lim _{h \downarrow 0} \frac{d T_{n+1 / 2}^{0}(h)}{d h}=-\infty, \lim _{h \uparrow h_{s p}} \frac{d T_{n+1 / 2}^{0}(h)}{d h}=$ $\infty$.

It follows from the continuity of $T_{n+1 / 2}^{0}(h)$ that there exists at least one value $0<h_{1}(n)<$ $h_{\text {sep }}$ such that $\left.\frac{d T_{n+1 / 2}^{0}(h)}{d h}\right|_{h=h_{1}(n)}=0$ and $\left(h-h_{1}(n)\right) d T_{n+1 / 2}^{0}(h)>0$ in a sufficiently small neighborhood of $h=h_{1}(n)$, i.e. $T_{n+1 / 2}^{0}(h)$ has a local minimum at $h=h_{1}^{0}(n)$.

The visualization of the proof is shown in Figure 6.2. Next we prove that the minimum of the previous proposition persists when $A$ is perturbed. For the following we also need estimates on the location of this minimum.

Proposition 6.4. For each $n>0$ there exists $A=\epsilon(n)$ such that the mixed time-to-return function $T_{n+1 / 2}^{A}(h)$ in (3.4) with $A \in(0, \epsilon(n))$ defined on $h \in\left(h_{A}, h_{\text {sep }}\right)$ for system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f^{\prime \prime}(0)>0$, has at least one local minimum for $h=h_{1}^{\epsilon(n)}(n)$ with $\left|h_{1}^{\epsilon(n)}(n)-h_{1}^{0}(n)\right|<\delta(\epsilon(n))$, where $h_{1}^{0}(n)$ corresponds to the local minimum of $T_{n+1 / 2}^{0}(h)$ in Proposition 6.3.


Figure 6.2: The existence of a minimum $h_{1}^{0}(n)$ for the mixed time-to-return functions $T_{n+1 / 2}^{A}(h)$ with $n=1,2,3$ for $A=0$ in system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f^{\prime \prime}(0)>0$.

Proof. From the representation (4.8) in Lemma 4.2 of the derivative of the positive time-toreturn function it follows immediately that:

$$
\frac{d T_{+}^{\epsilon}(h)}{d h}-\frac{d T_{+}^{0}(h)}{d h}=\int_{\bar{A}=0}^{\bar{A}=\epsilon} \frac{1}{\sqrt{2}\left(h-h_{\bar{A}}\right)^{\frac{3}{2}}} d \bar{A} .
$$

Since $T_{n+1 / 2}^{\epsilon}(h)=T_{+}^{\epsilon}(h)+n T_{\text {full }}(h)$ where $T_{\text {full }}(h)$ does not depend on $\epsilon$, the above relationship implies that:

$$
\begin{equation*}
\frac{d T_{n+1 / 2}^{\epsilon}(h)}{d h}-\frac{d T_{n+1 / 2}^{0}(h)}{d h}=\int_{\bar{A}=0}^{\bar{A}=\epsilon} \frac{1}{\sqrt{2}\left(h-h_{\bar{A}}\right)^{\frac{3}{2}}} d \bar{A} . \tag{6.8}
\end{equation*}
$$

Consider an interval $h_{\delta_{1}}<h_{1}^{0}(n)<h_{\delta_{2}}$ such that $\frac{d T_{n+1 / 2}^{0}(h)}{d h}<0(>0)$ for $h_{\delta_{1}}<h_{1}^{0}(n)\left(h_{1}^{0}(n)<\right.$ $h_{\delta_{2}}$ ). According to Proposition 6.3 this is possible. For given $n$ choose $h^{*} \in\left(h_{\delta_{1}}, h_{1}^{0}(n)\right)$. Since in (6.8) the integrand on the right hand side is bounded for fixed $h^{*}>h_{\bar{A}}$, for all $0<\bar{A}<\epsilon$, we can choose $\epsilon$ small such that:

$$
\left.\frac{d T_{n+1 / 2}^{\epsilon}(h)}{d h}\right|_{h=h^{*}}=\left.\frac{d T_{n+1 / 2}^{0}(h)}{d h}\right|_{h=h^{*}}+\int_{\bar{A}=0}^{\bar{A}=\epsilon} \frac{1}{\sqrt{2}\left(h^{*}-h_{\bar{A}}\right)^{\frac{3}{2}}} d \bar{A}<0 .
$$

Moreover we have by construction that:

$$
\left.\frac{d T_{(n+1 / 2)}^{\epsilon}(h)}{d h}\right|_{h=h^{* *}}=\left.\frac{d T_{(n+1 / 2)}^{0}(h)}{d h}\right|_{h=h^{* *}}+\int_{\bar{A}=0}^{\bar{A}=\epsilon} \frac{1}{\sqrt{2}\left(h^{* *}-h_{\bar{A}}\right)^{\frac{3}{2}}} d \bar{A}>0,
$$

for $h^{* *} \in\left(h_{1}^{0}(n), h_{\delta_{2}}\right)$.

From these two equations we conclude that for each $n$ we can find a value $h=h_{1}^{\epsilon(n)}(n)$ such that $\frac{d T_{n+1 / 2}^{A}(h)}{d h}$ has a zero where the sign changes from minus to plus for increasing $h$, i.e. $T_{n+1 / 2}^{A}(h)$ has a local minimum for a value of $h$ close to the local minimum $h_{1}^{0}(n)$ of $T_{n+1 / 2}^{0}(h)$.

The persistence of the minimum of the mixed period functions for small $\epsilon$ is shown in Figure 6.3.


Figure 6.3: The persistence of a minimum $h_{1}^{\epsilon}(n)$ for the mixed time-to-return functions $T_{n+1 / 2}^{A}(h)$ for $A=\epsilon>0$ in system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f^{\prime \prime}(0)>0$.

Note 6.5. This proposition basically states that for sufficiently small $A>0$ the local minimum of the mixed time-to-return function $T_{n+1 / 2}^{A}(h)$ persists as would be expected from continuity.

### 6.3 Existence of a local maximum for the mixed time-to-return functions $T_{n+1 / 2}^{A}(h)$ for sufficiently small $A$ and $f^{\prime \prime}(0)>0$

The result of the previous section showed that a local minimum exists for $T_{n+1 / 2}^{A}(h)$ when $A$ is sufficiently small. However, the results of the limits for the derivatives in Figure 6.1 show that crossing $A=0$ the derivative changes sign. This can only be explained by the creation of a local maximum on the function $T_{n+1 / 2}^{A}(h)$.
Proposition 6.6. For each $n>0$ there exists $A=\epsilon(n)$ such that the mixed time-to-return function $T_{n+1 / 2}^{A}(h)$ in (3.4) with $A \in(0, \epsilon(n))$ defined on $h \in\left(h_{A}, h_{\text {sep }}\right)$ for system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f^{\prime \prime}(0)>0$, has at least one local maximum for $h=h_{2}^{\epsilon(n)}(n)$ with $h_{A}<h_{2}^{\epsilon(n)}(n)<\delta_{2}(\epsilon(n))$.
Proof. In the proof of Proposition 6.4 it was shown that for sufficiently small $A$, there will be a value $h=h^{*}$ depending on $n$ for which the derivative of the mixed time-to-return function $T_{n+1 / 2}^{A}(h)$ is negative. From Proposition 6.1 we know that for $A>0$ :

$$
\lim _{h \downarrow h_{A}} \frac{d T_{n+1 / 2}^{A}(h)}{d h}=\infty,
$$

implying that there exists $h^{* *}$ close enough to $h_{A}$ such that $\left.\frac{d T_{n+1 / 2}^{A}(h)}{d h}\right|_{h^{* *}}>0$. Therefore there is (at least one) a value $h_{2}^{\epsilon(n)}(n) \in\left(h^{* *}, h^{*}\right)$ such that $\left.\frac{d T_{n+1 / 2}^{A}(h)}{d h}\right|_{h_{2}^{\epsilon(n)}(n)}=0$. Moreover the derivative changes sign from minus to plus around this zero, showing that it represents a local maximum of the function $T_{n+1 / 2}^{A}(h)$ as we set out to prove.

The creation of a local maximum of the mixed period functions for small $\epsilon$ is shown in Figure 6.4.


Figure 6.4: The creation of a maximum $h_{2}^{\epsilon}(n)$ for the mixed time-to-return functions $T_{n+1 / 2}^{A}(h)$ for $A=\epsilon>0$ in system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f^{\prime \prime}(0)>0$.

Note 6.7. This proposition basically states that for sufficiently small $A>0$ a local maximum of the mixed time-to-return function $T_{n+1 / 2}^{A}(h)$ is created from the center point at $(x=0, y=0)$ in the phase plane by changing the parameter $A$. The condition $f^{\prime \prime}(0)>0$ ensures that a local maximum is created. This is a critical ingredient for the $S$-shaped bifurcation of the next section.

### 6.4 Co-existence of a local maximum and a local minimum for the mixed time-toreturn functions $T_{n+1 / 2}^{A}(h)$ for sufficiently small $A$ and $f^{\prime \prime}(0)>0$

The results of the previous two sections showed the existence of a local minimum and maximum for the function $T_{n+1 / 2}^{A}(h)$. Combining these results we immediately get our main result.

Theorem 6.8. For each $n>0$ there exists $A=\epsilon(n)$ such that the mixed time-to-return function $T_{n+1 / 2}^{A}(h)$ in (3.4) with $A \in(0, \epsilon(n))$ defined on $h \in\left(h_{A}, h_{\text {sep }}\right)$ for system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f^{\prime \prime}(0)>0$, has at least one local minimum $h=h_{1}^{\epsilon(n)}(n)$ and one local maximum for $h=h_{2}^{\epsilon(n)}(n)$ for small enough $\epsilon(n)$.

Proof. The theorem is a direct consequence of the statements in Proposition 6.4 and Proposition 6.6.

The co-existence of a local maximum and local minimum of the mixed period functions for small $\epsilon$ is shown in Figure 6.5.


Figure 6.5: The S-shaped mixed time-to-return functions $T_{n+1 / 2}^{A}(h)$ with $n=$ $1,2, \ldots$ for $A=\epsilon>0$ and $f^{\prime \prime}(0)>0$.

Note 6.9. This proposition basically states that for sufficiently small $A>0$ an S-shaped bifurcation occurs for each type of mixed time-to-return function $T_{n+1 / 2}^{A}(h)$ with $n=1,2, \ldots$ This S-shape does not occur if $f^{\prime \prime}(0)<0$. In that case it is not difficult to verify that the other type of mixed time-to-return function $T_{-n-1 / 2}^{A}(h)$ will exhibit an $S$-shaped bifurcation. The case $f^{\prime \prime}(0)=0$ is more difficult because it would require taking higher order contributions into account. We believe that even in these cases S -shaped bifurcations will take place.

## 7 Example of a quadratic Hamiltonian system

In this section we provide an application of the previous sections to the simplest possible nonlinear case $f(x)=x(x+1)$. The conditions (1.3), (1.4) and (1.5) and $f^{\prime \prime}(0)>0$ are satisfied, since $f(0)=0, f^{\prime}(0)=1, f^{\prime \prime}(0)=2>0$ and $x_{s}=-1<0, f(-1)=0, f^{\prime}(-1)=-1$ and $f(x) x\left(x-x_{s}\right)=x^{2}(x+1)^{2}$. The integral of this system is given by:

$$
\begin{equation*}
h=\frac{1}{2} y^{2}(t)+\frac{1}{2} x(t)^{2}+\frac{1}{3} x(t)^{3} . \tag{7.1}
\end{equation*}
$$

with the saddle loop represented by $h=h_{\text {sep }}=\frac{1}{6}$. The saddle loop passes through the saddle singularity at $x=-1$ and the regular point $x=\frac{1}{2}$, i.e. we consider $-1<A<\frac{1}{2}$.

The derivatives of the full period function satisfy the following relations.
Lemma 7.1. On the interval $h \in\left(0, \frac{1}{6}\right)$ the period function $T_{\text {full }}(h)$ for the quadratic Hamiltonian system (2.3) with $f(x)=x(1+x)$ satisfies:

$$
\begin{gathered}
\frac{d T_{\text {full }}(h)}{d h}>0, \\
\frac{d^{2} T_{\text {full }}(h)}{d h^{2}}>0, \\
\frac{d^{3} T_{\text {full }}(h)}{d h^{3}}>0 .
\end{gathered}
$$

Proof. In Proposition (5.1) a recurrence relation was derived for the full period function of orbits in (2.3). The first three derivatives are given by:

$$
\begin{aligned}
h \frac{d T_{\text {full }}(h)}{d h} & =c_{1} \int_{x_{-}}^{x_{+}} y_{h}(x) \psi_{1}(x) d x, \\
h^{2} \frac{d^{2} T_{\text {full }}(h)}{d h^{2}} & =c_{2} \int_{x_{-}}^{x_{+}} y_{h}^{3}(x) \psi_{2}(x) d x, \\
h^{3} \frac{d^{3} T_{\text {full }}(h)}{d h^{3}} & =c_{3} \int_{x_{-}}^{x_{+}} y_{h}^{5}(x) \psi_{3}(x) d x,
\end{aligned}
$$

where $y_{h}(x)=\sqrt{2(h-F(x))}, c_{1}, c_{2}, c_{3}$ positive constants,

$$
\begin{aligned}
& \psi_{1}(x)=\left(\frac{F(x)}{f(x)^{2}}\right)^{\prime \prime}, \\
& \psi_{2}(x)=\left(\frac{F(x) \psi_{1}(x)}{f(x)^{2}}\right)^{\prime \prime}+\left(\frac{F(x) \psi_{1}^{\prime}(x)}{f(x)^{2}}\right)^{\prime}, \\
& \psi_{3}(x)=\left(\frac{F(x) \psi_{2}(x)}{f(x)^{2}}\right)^{\prime \prime}+\left(\frac{F(x) \psi_{2}^{\prime}(x)}{f(x)^{2}}\right)^{\prime} .
\end{aligned}
$$

The result of the lemma follows by proving that the three functions $\psi_{1}(x), \psi_{1}(x), \psi_{1}(x)$ are positive on the interval of interest. In our case $x \in\left(-1, \frac{1}{2}\right)$ and $\frac{f(x)}{f(x)^{2}}=\frac{\frac{1}{2} x^{2}-\frac{1}{3} x^{3}}{x^{2}(x+1)^{2}}=\frac{\frac{1}{2}-\frac{1}{3} x}{(x+1)^{2}}$. With this the three functions can be written out to become:

$$
\begin{gathered}
\psi_{1}(x)=\frac{1}{3} \frac{(5+2 x)}{(1+x)^{4}}, \\
\psi_{2}(x)=\frac{35}{9} \frac{\left(11+10 x+2 x^{2}\right)}{(1+x)^{8}}, \\
\psi_{3}(x)=\frac{35}{27} \frac{\left(2431+3486 x+1560 x^{2}+208 x^{3}\right)}{(1+x)^{12}} .
\end{gathered}
$$

These expressions are easily seen to be positive on the interval $x \in\left(-1, \frac{1}{2}\right)$.
In particular this lemma proves that the full period function is convex and monotonically increasing as a function of $h$. The convexity property seems to be new result. The positive time-to-return function is more difficult to analyze, even for this simple case. The following results were already established in the literature.

Lemma 7.2. The positive time-to-return function $T_{+}^{A}(h)$ in (3.1) for $A=0$ is monotonically decreasing for (2.3) with $f(x)=x(1+x)$.

Proof. As before we first write:

$$
T_{+}^{0}(h)=\tilde{T}_{+}^{0}\left(x_{+}\right)=\sqrt{2} \int_{x=0}^{x=x_{+}} \frac{d x}{\sqrt{F\left(x_{+}\right)-F(x)}} .
$$

Again we will write $x_{+} \equiv \alpha$.
With this notation we can rewrite the integral using a scaling of the integration variable $x=\alpha u$. The integral becomes:

$$
\tilde{T}_{+}^{0}(\alpha)=\sqrt{2} \int_{u=0}^{u=1} \frac{\alpha d u}{\sqrt{F(\alpha)-F(\alpha u)}}
$$

We formally differentiate with respect to $\alpha$ to get:

$$
\frac{d \tilde{T}_{+}^{0}(\alpha)}{d \alpha}=\sqrt{2} \int_{u=0}^{u=1} \frac{\sqrt{F(\alpha)-F(\alpha u)}-\alpha \frac{f(\alpha)-u f(\alpha u)}{2 \sqrt{F(\alpha)-F(\alpha u)}}}{F(\alpha)-F(\alpha u)} d u .
$$

Rewriting the integrand we get

$$
\frac{d \tilde{T}_{+}^{0}(\alpha)}{d \alpha}=\frac{\sqrt{2}}{2} \int_{u=0}^{u=1} \frac{2(F(\alpha)-F(\alpha u))-(\alpha f(\alpha)-\alpha u f(\alpha u))}{(F(\alpha)-F(\alpha u))^{\frac{3}{2}}} d u .
$$

This can be rewritten in the following compact form (after changing back to the integration variable $x=\alpha u$ ):

$$
\begin{equation*}
\frac{d \tilde{T}_{+}^{0}(\alpha)}{d \alpha}=\frac{\sqrt{2}}{2} \int_{x=0}^{x=\alpha} \frac{\Delta[\theta(x)]}{\Delta[F(x)]^{\frac{3}{2}}} \frac{d x}{\alpha} \tag{7.2}
\end{equation*}
$$

where $\theta(x) \equiv 2 F(x)-x f(x)$ and $\Delta[Z(x)] \equiv Z(\alpha)-Z(x)$.
In the case of the quadratic Hamiltonian $f(x)=x(1+x), F(x)=\frac{1}{2} x^{2}+\frac{1}{3} x^{3}$ which leads to: $\theta(x)=x^{2}+\frac{2}{3} x^{3}-x^{2}(1+x)=-\frac{1}{3} x^{3}<0$ and $\theta^{\prime}(x)<0$ for $x>0$.

This implies that the integrand of (7.2) will be negative for $0<x<\alpha$. Therefore $\frac{d \tilde{T}_{9}^{0}(\alpha)}{d \alpha}<0$ implying $\frac{d T_{+}^{0}(h)}{d h}<0$ as we set out to prove.

Note 7.3. It is straightforward by using the same type of argument to prove that the positive time-to-return function $T_{+}^{0}(h)$ is monotonically decreasing for $-1<A<0$ as well.

If $A>0$ the situation is slightly more complicated, because the function $T_{+}^{A}(h)$ will have a local maximum for some $A$. For our discussion of mixed time-to-return functions it is important to establish that there exists a value $0<A^{*}<\frac{1}{2}$ such that $T_{+}^{0}(h)$ is monotonically increasing for $0<A^{*} \leq A<\frac{1}{2}$.

Lemma 7.4. $\exists A^{*} \in\left(0, \frac{1}{2}\right)$ such that the positive time-to-return function $T_{+}^{A}(h)$ in (3.1) is monotonically increasing for a quadratic Hamiltonian system (2.3) with $f(x)=x(1+x)$.

Proof. The proof is similar to the case $A=0$, except that we cannot use the formula introduced above which was only valid for $A=0$. Formally, the same procedure leads to the following expression for the derivative $\frac{d T_{A}^{A}(\alpha)}{d \alpha}$ :

$$
\begin{equation*}
\frac{d \tilde{T}_{+}^{A}(\alpha)}{d \alpha}=\frac{\sqrt{2}}{2} \int_{x=0}^{x=\alpha} \frac{\Delta\left[\theta^{A}(x)\right]}{\Delta[F(x)]^{\frac{3}{2}}} \frac{d x}{\alpha} \tag{7.3}
\end{equation*}
$$

where $\theta^{A}(x) \equiv 2 F(x)-(x-A) f(x)$.
Substitution of $f(x)=x(1+x)$ gives: $\theta^{A}(x)=\frac{1}{3} x\left(-x^{2}+3 A x+3 A\right)$ and $\frac{d \theta^{A}(x)}{d x}=-x^{2}+$ $2 A x+A$. A straightforward calculation shows that $\theta^{A}(x)$ is monotonically increasing on the interval $x \in\left(A, \frac{1}{2}\right)$ if $\frac{1}{8}<A<\frac{1}{2}$. Therefore for this range of $A, \Delta\left[\theta^{A}(x)\right]$ is positive and $\frac{d \tilde{T}_{A}^{A}(\alpha)}{d \alpha}>0$ proving that the derivative of $T_{+}^{A}(h)$ is positive. The lemma follows with
$A^{*}<\frac{1}{8}$. $A^{*}<\frac{1}{8}$.

Note 7.5. It is not so straightforward to prove that the positive time-to-return function $T_{+}^{0}(h)$ has a unique local maximum for $0<A<A^{*}$ with this technique. It follows from other results in the literature, i.e. see $[4,19]$.

The application of Theorem 6.8 to the case $f(x)=x(x+1)$ shows that the case of the mixed solutions is much more complicated. A full proof of the exact number of local maxima and minima seems to be difficult even for this case.

Theorem 7.6. For each $n>0$ there exists $A=\epsilon(n)$ such that the mixed time-to-return function $T_{n+1 / 2}^{A}(h)$ in (3.4) with $A \in(0, \epsilon(n))$ defined on $h \in\left(h_{A}, h_{\text {sep }}\right)$ for system (2.3) with $f(x)=x(x+1)$ has at least one local minimum $h=h_{1}^{\epsilon(n)}(n)$ and one local maximum for $h=h_{2}^{\epsilon(n)}(n)$ for sufficiently small $\epsilon(n)$.

## 8 Number of solutions for fixed $\lambda$

The previous sections showed the existence of an S-shaped bifurcation for mixed time-toreturn functions. This section is aimed at investigating how $\lambda$ affects the number of solutions to the original boundary value problem (1.1), (1.2) with conditions (1.3), (1.4) and (1.5). For this we need to consider the intersection of a horizontal line $T=\lambda^{2}=$ constant with the different time-to-return functions $T_{+}^{A}(h), T_{n}(h), T_{n+1 / 2}^{A}(h)$. Each intersection will correspond to a solution to the original problem for such a value of $\lambda$. Each tangency of the horizontal line with such a function (i.e. tangent to a local minimum or maximum of the graph of the function) corresponds to a bifurcation value of $\lambda$.

### 8.1 Number of solutions for fixed $\lambda$, fixed solution type

First we consider each type separately and find an estimate on the number of possible solutions as a function of $\lambda$.

According to the results of the previous sections we know that at least three solutions of the type $S_{n+1 / 2}^{A}$ (for each $n$ ) exist for a proper choice of the parameter $\lambda$ according to Theorem 6.8. Since for small $A>0$ the function has at least one local maximum and local minimum, there must exist a horizontal line which crosses the graph of the function in at least three points, i.e. $T_{n+1 / 2}^{A}(h)=\lambda^{2}$ has at least three solutions for an appropriate choice of $\lambda$.

Proposition 8.1. Boundary value problem (1.1) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f^{\prime \prime}(0)>0$, for sufficiently small positive $A$, has at least three solutions of mixed type $S_{n+1 / 2}^{A}($ for each $n)$ by choosing $\lambda$ appropriately.

In Figure 8.1 a numerical example is shown for this situation, i.e. an example of system (1.1) with $f(x)=x(1+x)$ and boundary condition (1.2) having three solutions of the same type. Figure 8.2 displays the mixed period function for the case $T_{3 / 2}^{A}$, i.e. an example of system (1.1) with $f(x)=x(1+x)$ and boundary condition (1.2) while varying the parameter $A$. It is clearly visible that for the parameters $A=0.001$ and $A=0.002$, a local maximum and local minimum occur. Both disappear by increasing $A$ further as shown for the value $A=0.0035$.

### 8.2 Number of simultaneous solutions for fixed $\lambda$

The first step in estimating the number of simultaneous solutions for the different solution types is to determine the range of the functions. From the properties of the previous sections, the following results are straightforward.

Lemma 8.2. For system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f^{\prime \prime}(0)>0$, the ranges of the time-to-return functions are:


Figure 8.1: Numerical example of the co-existence of three solutions of type $S_{3 / 2}^{A}$ to the boundary value problem for $\lambda=86.23908, A=0.002$ for the quadratic Hamiltonian case $f(x)=x(1+x)$. The initial conditions for the three solutions are $\left.\frac{d x(t)}{d t}\right|_{t=0}=0.46,\left.\frac{d x(t)}{d t}\right|_{t=0}=1.03,\left.\frac{d x(t)}{d t}\right|_{t=0}=1.71$.


Figure 8.2: Numerical example for the mixed period function $T_{3 / 2}^{A}$ for the quadratic Hamiltonian case $f(x)=x(1+x)$ while varying the parameter $A$. Five cases are shown for $A$. For $A=-0.0005$ and $A=0$ only one local minimum occurs. For small positive $A$, i.e. $A=0.001$ and $A=0.002$ an additional local maximum occurs. For larger $A$, i.e. $A=0.0035$, the function is monotonically increasing and both local extreme points have disappeared.

- $T_{+}^{A}(h) \in\left(C_{1}(A), C_{2}(A)\right)$ with $0<C_{2}(A)<\infty ; 0<C_{1}(A)<\infty$, for $x_{s}<A \leq 0$, and $C_{1}(A)=0$ for $0<A<x_{s}^{(2)}$,
- $T_{n}(h) \in\left(n T_{\text {full }}\left(h_{\text {min }}\right), \infty\right)$, where $h_{\text {min }}$ corresponds to the global minimum $T_{\text {full }}\left(h_{\text {min }}\right)>0$ of $T_{\text {full }}(h)$.
- $T_{n+1 / 2}^{A}(h) \in\left(C_{3}(A, n), \infty\right)$, with $C_{3}(A, n)>0$.

Proof. The function $T_{\text {full }}(h)$ tends to $\infty$ for $h \rightarrow h_{\text {sep }}$ according to Lemma 5.2. Since it is positive on a bounded interval and $T_{\text {full }}(0)>0$, a global minimum of the function must exist, denoted
by $h_{\text {min }}$.
This establishes the results for $T_{n}(h)$ and $T_{n+1 / 2}^{A}(h)$ due to the continuity of the functions on the bounded open interval for $h$. In the latter case the minimum value $C_{3}(A, n)$ is not trivial to find explicitly, except when $A>0$ : we have $T_{n+1 / 2}^{A}(h)>T_{n}(h)=n T_{\text {full }}(h)$ and $\lim _{h \backslash h_{A}} T_{n+1 / 2}^{A}\left(h_{A}\right)=n T_{\text {full }}\left(h_{A}\right)$. See Figures 8.5, 8.6 for the case of $f(x)=x(x+1)$ where the function $T_{\text {full }}(h)$ is monotonically increasing according to Lemma 7.1.


Figure 8.3: Conjectured time-to-return functions for simultaneous solutions to the boundary value problem for (2.3) with $f(x)=x(1+x), x_{s}<A \leq 0$.


Figure 8.4: Conjectured time-to-return functions for simultaneous solutions to the boundary value problem for (2.3) with $f(x)=x(1+x), A=0$.

The result for the remaining case $T_{+}^{A}(h)$ follows from the fact that $T_{+}^{A}(h)$ approaches 0 for $h \downarrow h_{A}$ for $0<A<\frac{1}{2}$, while it approaches a positive constant when $-1<A \leq 0$ according to Proposition 4.3.


Figure 8.5: Conjectured time-to-return functions for simultaneous solutions to the boundary value problem for (2.3) with $f(x)=x(1+x), 0<A=\epsilon \ll 1$.


Figure 8.6: Conjectured time-to-return functions for simultaneous solutions to the boundary value problem for (2.3) with $f(x)=x(1+x), 0<A^{*}<A<\frac{1}{2}$.

Note 8.3. The important feature of the lemma is that the range of each of the countably many functions $T_{n}(h)$ and $T_{n+1 / 2}^{A}(h)$ extends to $+\infty$. This is due to the fact that the period annulus is bounded on the exterior by a saddle loop. The other important feature is that the lower bounds of the functions $T_{n}(h)$ and $T_{n+1 / 2}^{A}(h)$ grow with increasing $n$ as we will show below.

The implication of this lemma is that for each fixed sufficiently large $\lambda$ the original boundary value problem has at least one solution. In the case $x_{s}<A \leq 0$ there is an open interval ( $0, C_{1}(A)$ ) such that for $\lambda$ in this interval no solutions exist for the boundary value problem. For $A=0$ there is a second interval $\left(C_{2}(A), T_{\text {full }}(0)\right)$ such that no solutions exist for $\lambda$ in this range. The different possibilities for the relative positions of the functions are shown in Figures 8.3, 8.4, 8.5, 8.6 for the case of $f(x)=x(x+1)$ where the function $T_{\text {full }}(h)$ is monotonically increasing according to Lemma 7.1. The figures assume that the maximum number
of local extreme values on each time-to-return functions is three. Therefore the figures are labelled as conjectured and have not been verified numerically.

The next proposition follows from the fact that in Lemma 8.2 the range of the countably infinite functions $T_{n}(h)$ and $T_{n+1 / 2}^{A}(h)$ is bounded below by a number which is monotonically increasing as a function of $n$. This is obvious for the functions $T_{n}(h)$ which are bounded below by $n T_{\text {full }}\left(h_{\text {min }}\right)$. For the function $T_{n+1 / 2}^{A}(h)$ we have the trivial estimate $T_{n+1 / 2}^{A}(h)=$ $T_{n}(h)+T_{+}^{A}(h)>T_{n}(h)>n T_{\text {full }}\left(h_{\text {min }}\right)$. The consequence of these lower bounds is that for given $\lambda$, there are only finitely many functions which have a lower bound below $\lambda$. It implies the finiteness of solutions of the boundary problem:
Proposition 8.4. Boundary value problem (1.1) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f^{\prime \prime}(0)>0$, has finitely many solutions for each $\lambda>0$ if $0<A<x_{s}^{(2)}$ and for each $\lambda \in\left(C_{1}(A), \infty\right)$ (with $C_{1}(A)$ defined in Lemma 8.2) if $x_{s}<A \leq 0$.

The exact number is not easy to verify since we do not have an upper bound on the number of local maxima and minima of the mixed time-to-return functions. Figures 8.3, 8.4, 8.5, 8.6 show for the case $f(x)=x(x+1)$ that for increasing $\lambda$ the number of solutions will grow with discrete jumps even though the exact number has not been proved, or verified numerically. The number of solutions will jump when $\lambda$ will cross a value of $T_{n+1 / 2}^{A}(h)$ corresponding to a local minimum or maximum. For $-x_{s}<A \leq 0$ the functions each have (at least) a minimum value $C_{3}(A, n)$ which increases without an upper bound as a function of $n$. It shows that for any chosen $\lambda_{c}$ countably infinite bifurcation values $\lambda_{n}^{*}>\lambda_{c}$ can be found. This contradicts the statement in the paper [27] where it was stated that only for small $\lambda$ bifurcations would occur for mixed solutions.

### 8.3 Systems with an infinite number of solutions

The previous section showed that boundary value problem (1.1) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f^{\prime \prime}(0)>0$, has finitely many solutions for given $\lambda$. It is not difficult to point out the reason why this number is finite. The period annulus is bounded on the outside by a saddle loop. It causes the full time-to-return functions $T_{n}(h)$ to become unbounded when $h \uparrow h_{\text {sep }}$. The functions will have a discrete set of distinct values when $h \downarrow h_{A}$. These two properties combined with the continuity of the functions causes the finiteness of solutions, i.e. a finite number of intersections for any horizontal line with the collection of graphs of $T_{n+1 / 2}^{A}(h)$ and $T_{n}(h)$.

It leaves the problem to determine in which situations this conclusion cannot be drawn. This could happen in the case when the period annulus is not bounded by a finite solution curve. A typical example is the case of an unbounded period annulus with the property that $f(x) \rightarrow \pm \infty$ as $\mathcal{O}\left(x^{1+\alpha}\right)$ when $x \rightarrow \pm \infty$ with $\alpha>0$. In such a case the time-to-return function will approach 0 for large $h$ instead of $\infty$ (as was the case for a saddle loop). If $T_{\text {full }}(h)$ tends to 0 instead of $\infty$, then each of the functions $T_{n+1 / 2}^{A}(h), T_{n}(h)$ will approach 0 . It implies that for each $\lambda$ there will be an infinite number of intersections with the graphs of the functions $T_{n+1 / 2}^{A}(h), T_{n}(h)$. Therefore the original boundary value problem has an infinite number of solutions. It is outside the scope of this paper to give a full classification of all the different structure types for the simultaneous solutions of equation (1.1) with boundary condition (1.2) for arbitrary $f(x)$, but the above argument can be extended to achieve this. Moreover, the existence of an S-shaped bifurcation can be generalized as well to any case of $f(x)$ such that a period annulus occurs with a center singularity on the inside and a finite loop formed by the separatrices of two saddles.

In Figure 8.7 we sketch an example of a case with infinitely many solutions for $f(x)=$ $x+\frac{1}{2} x^{2}+\frac{1}{6} x^{3}, A=0$ as numerically discussed in [32].


Figure 8.7: Simultaneous solutions to a boundary value problem with $f(x)=$ $x+\frac{1}{2} x^{2}+\frac{1}{6} x^{3}, A=0$ where the period annulus is unbounded.

## 9 Representation of bifurcations in phase plane

It is possible to represent the bifurcations of the different mixed solution types in a transparent way as a function of $A$ by making the following observation:

Lemma 9.1. Suppose system (2.3) has a period annulus in the phase plane surrounding a singularity of center type, represented by the integral curve given in (2.5) on some interval $h \in\left(h_{\min }, h_{\max }\right)$. Then for given $h$ on the orbit $\gamma_{h}$ in the phase plane there exists for each of the functions $T_{n+1 / 2}^{A}(h)$ exactly one point $\left(x=A_{(2 n+1) / 2}^{b i f} y=y\left(A_{(2 n+1) / 2}^{b i f}, h\right)\right) \equiv\left(x^{b i f}, y^{b i f}\right)$, such that the boundary value problem (1.1) with boundary condition (1.2) where $A=x^{b i f}$ and $\left.\frac{d x(t)}{d t}\right|_{t=0}=y^{b i f}$ has a bifurcation value $\lambda=\lambda^{b i f}$. The bifurcation points of the boundary value problem can be represented by a curve $\mu^{n+1 / 2}(h)$ in the phase plane intersecting the period annulus transversally. The case $n=0$ is included representing the positive time-to-return function $T_{+}^{A}(h)$.

Proof. For a given periodic orbit, i.e. fixed $h$, the domain of $A$-values is given by $\left(A_{-}, A_{+}\right)$ where $F\left(A_{ \pm}\right)=0$. The periodic orbit in a period annulus needs to intersect the $x$-axis in exactly two points defined by $F(x)=0$ and we indicate those two $x$-values by $A_{-}$and $A_{+}$. See Figure 9.1. Equation (4.7) in Lemma 4.2 shows an expression for $\frac{d T_{T}^{A}(h)}{d h}$. At the end points necessarily $F(A)=h_{A}$ and in the expression $\frac{d T_{A}^{A}(h)}{d h}=\frac{\sqrt{2} h_{A}}{f(A) \sqrt{h-h_{A}}}+\int_{y=0}^{y=\sqrt{h-h_{A}}} \omega^{\prime}(x(y)) d y$ the term $\frac{\sqrt{2} h_{A}}{f(A) \sqrt{h-h_{A}}}$ will blow up when $A$ approaches the boundary values. For a periodic orbit in a period annulus necessarily $f\left(A_{-}\right)<0$ and $f\left(A_{+}\right)>0$ and we conclude that $\lim _{A \rightarrow A_{-}, A+} \frac{d T_{+}^{A}(h)}{d h}=\operatorname{sign}\left(f\left(A_{ \pm}\right)\right) \infty=\mp \infty$. According to (4.9) $T_{+}^{A}(h)$ is monotonically decreasing as a function of $A$, showing that there exists exactly one value $A$ such that $\frac{d T_{4}^{A}(h)}{d h}=0$. This argument holds true if $f(x)$ has a unique zero on the relevant $x$-interval. If $f(x)$ has multiple zeroes, the period annulus is bounded on the inside by a solution curve consisting of
separatrices from one or more saddle(s). In that case the conclusion will be more complicated, which is outside the scope of this paper.

The same argument applies to the function $T_{n+1 / 2}^{A}(h)=T_{+}^{A}(h)+n T_{\text {full }}(h)$ because $T_{\text {full }}(h)$ does not depend on $A$ and does not influence the behaviour near the end points $A_{-}$and $A_{+}$ and the monotonicity of the derivative with respect to $A$. It follows that for each $n=1,2,3 \ldots$, fixed $h$, there is a unique $A_{(2 n+1) / 2}^{b i f}$ such that $\frac{d T_{n+1 / 2}^{A}(h)}{d h}=0$, i.e. a bifurcation value for the original boundary value problem.

There are many other properties of the bifurcation curves $\mu^{n+1 / 2}(h)$ mentioned in Lemma 9.1 that can be derived, but they are out of scope for this paper. We briefly indicate some results which are not difficult to prove using the formulas in Lemma 4.2:

- If $\frac{d T_{\text {full }}(h)}{d h}>0(<0)$ then the bifurcation points $\left(x=A_{(2 n+1) / 2^{\prime}}^{b i f} y=y\left(A_{(2 n+1) / 2^{\prime}}^{b i f} h\right)\right)$ are ordered counter-clockwise (clockwise) on the periodic orbit for increasing $n$. If $\frac{d T_{\text {full }}(h)}{d h}=$ 0 the points $\left(x=A_{(2 n+1) / 2^{\prime}}^{b i f} y=y\left(A_{(2 n+1) / 2^{\prime}}^{b i f} h\right)\right)$ collapse into a single point on the periodic orbit. Figure 9.1 shows the three situations.
- If the period annulus has a singularity of center type as its inner boundary, then the curve $\mu^{n+1 / 2}(h)$ approaches the center in the phase plane along a vertical tangent direction. Figure 9.2 shows a sketch of this for the case $f(x)=x(1+x)$.
- If the curve $\mu^{n+1 / 2}(h)$ moves to the right (left) for increasing $h$, then the bifurcation point corresponds to a local maximum (minimum) of the function $T_{n+1 / 2}^{A}(h)$. If the curve $\mu^{n+1 / 2}(h)$ has a vertical tangent line (i.e. it is changing direction in the phase plane), then a local maximum and minimum coincide to form a inflection point on $T_{n+1 / 2}^{A}(h)$.
- If a vertical line $x=A$ in the phase plane intersects $\mu^{n+1 / 2}(h)$ in two points, then an $S$-shaped bifurcation takes place. See Figure 9.2 where the situation is sketched for the case of $f(x)=x(1+x)$. The results in the figure have been confirmed numerically. It is clearly visible how for $A=\epsilon>0$ the situation occurs as was discussed in the previous sections.


Figure 9.1: Ordered bifurcation points on a periodic orbit in a period annulus corresponding to the different types of time-to-return functions.


Figure 9.2: Schematic display of the different types of bifurcation curves shown in the phase plane for the quadratic Hamiltonian case $f(x)=x(1+x)$.

## 10 Discussion

In this paper we studied mixed solutions of a nonlinear ordinary differential equation with Dirichlet boundary conditions. The purpose was to show that generically complex bifurcation phenomena occur, even for the most simple nonlinear choice i.e. $f(x)=x(1+x)$. The obvious question remains how these results extend to more complex cases. The following topics for further study come to our mind.

## 1) Generalizations

The results of this paper do not only apply to the case of a saddle loop surrounding the period annulus with a center inside. A full categorization for all solution types in the case of the general structure of $f(x)$ is feasible and should lead to similar results as in this paper. In particular we would like to point out the condition $f^{\prime \prime}(0)>0$ which is necessary for the mixed solutions to have an S-shaped bifurcation near the center singularity. If $f^{\prime \prime}(0)<0$, then it is not difficult to show that an S -shaped bifurcation will occur for the negative mixed solutions $S_{-(n+1 / 2)}^{A}$, where $n=1,2,3, \ldots$ It implies that if $f^{\prime \prime}(0) \neq 0$ near a center singularity then always an S -shaped bifurcation can be found among the mixed solutions.

## 2) Relation between the different time-to-return functions

There is a relation between the positive, negative and full time-to-return functions: $T_{\text {full }}(h)=T_{+}^{A}(h)+T_{-}^{A}(h)$. This indicates that even though for all solution types different phenomena occur there is still some intrinsic relation between them. For example, the expansion of the functions near the center singularity, i.e. $h \downarrow 0$ has an interesting structure caused by this relationship. The full period function is analytical in $h$, while the positive time-toreturn function $T_{+}^{A}(h)$ is analytical in the variable $\sqrt{h}$ (see the expansion in Proposition 4.1). For the negative time-to-return function a similar result holds. The structure becomes:

$$
\begin{gathered}
T_{\text {full }}(h)=T_{0}+c_{1} h+c_{2} h^{2}+\ldots \\
T_{+}^{A}(h)=\frac{1}{2}\left(T_{0}+c_{1} h+c_{2} h^{2}+\ldots\right)+\sqrt{h}\left(d_{0}+d_{1} h+d_{2} h^{2}+\ldots\right)
\end{gathered}
$$

$$
T_{-}^{A}(h)=\frac{1}{2}\left(T_{0}+c_{1} h+c_{2} h^{2}+\ldots\right)-\sqrt{h}\left(d_{0}+d_{1} h+d_{2} h^{2}+\ldots\right)
$$

It would be interesting to extend the analysis for the local bifurcation of small-amplitude critical periods for the full period function (for which an extensive literature exists) to the cases of the positive and negative time-to-return functions and the different types of mixed time-to-return functions.

## 3) Proving upper bounds

This paper mainly addressed the existence of solutions without considering the upper bounds on the number of solutions. For example in the case of the quadratic Hamiltonian $x(1+x)$ the conjecture is that at most three solutions can occur for each type of mixed solution. The difficulty in proving this lies in the fact that the function contains the full period function for which the depending parameter is $h$ and the positive time-to-return function for which the natural depending parameter is $x_{+}(h)$ (see the proof of Proposition 4.1). In order to study the mixed functions a way must be found to combine the different techniques for these two functions.

## Acknowledgements

The authors would like to thank the referee for his useful comments which helped improve this paper.

## References

[1] T. Bakri, Y. A. Kuznetsov, F. Verhulst, E. Doedel, Multiple solutions of a generalized singularly perturbed Bratu problem, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 22(2012), No. 4, 1250095, 10 pp. https://doi.org/10.1142/S0218127412500952; MR2926072
[2] G. Bratu, Sur les équations intégrales non linéaires (in French), Bull. Soc. Math. France 42(1914), 113-142. MR1504727
[3] J. Bebernes, Solid fuel combustion-some mathematical problems, Rocky Mount. J. Math. 16(1986), No. 3, 417-433. https://doi.org/10.1216/RMJ-1986-16-3-417; MR862274
[4] C. Chicone, The monotonicity of the period function for planar Hamiltonian vector fields, J. Differential Equations 69(1987), No. 3, 310-321. https://doi.org/10.1016/00220396 (87)90122-7; MR903390
[5] C. Chicone, Geometric methods for two-point nonlinear boundary value problems, J. Differential Equations 72(1988), 360-407. https://doi.org/10.1016/0022-0396(88) 90160-X; MR932371
[6] C. Chicone, M. Jacobs, Bifurcation of critical periods for plane vector fields, Trans. Amer. Math. Soc. 312(1989), No. 2, 433-486. https://doi.org/10. 2307/2000999; MR930075
[7] C. V. Coffman, On the positive solutions of boundary-value problems for a class of nonlinear differential equations, J. Differential Equations 3(1967), 92-111. https://doi. org/10.1016/0022-0396(67)90009-5; MR0204755
[8] W. A. Coppel, L. Gavrilov, The period function of a Hamiltonian quadratic system, Differential Integral Equations 6(1993), No. 6, 1357-1365. MR1235199
[9] F. Dumortier, B. Smits, Transition time analysis in singularly perturbed boundary value problems, Trans. of Amer. Math. Soc. 347(1995), No. 10, 4129-4145. https://doi.org/10. 2307/2155217; MR1308009
[10] I. M. Gel'fand, Some problems in the theory of quasi-linear equations (in Russian), Uspehi Mat. Nauk 14(86)1959, No. 2, 87-158, translated in Amer. Math. Soc. Transl. Ser. 2 29(1963), 295-381. MR0110868
[11] S.-Y. Huang, S.-H. Wang, Proof of a conjecture for the one-dimensional perturbed Gelfand problem from combustion theory, Arch. Ration. Mech. Anal. 222(2016), No. 2, 769-825. https://doi.org/10.1007/s00205-016-1011-1; MR3544317
[12] S.-Y. Huang, K.-C. Hung, S.-H. Wang, A global bifurcation theorem for a multiparameter positone problem and its application to the one-dimensional perturbed Gelfand problem, Electron. J. Qual. Theory Differ. Equ. 2019, No. 99, 1-25. https://doi.org/10.14232/ ejqtde.2019.1.99; MR4049574
[13] K.-C. Hung, S.-H. Wang, A theorem on S-shaped bifurcation curve for a positone problem with convex-concave nonlinearity and its applications to the perturbed Gelfand problem, J. Differential Equations 251(2011), No. 2, 223-237. https://doi.org/10.1016/ j.jde.2011.03.017; MR2800152
[14] K.-C. Hung, S.-H. Wang, Global bifurcation and exact multiplicity of positive solutions for a positone problem with cubic nonlinearity and their applications, Trans. Amer. Math. Soc. 365(2013), No. 4, 1933-1956. https://doi.org/10.1090/S0002-9947-2012-05670-4; MR3009649
[15] D. D. Joseph, Non-linear heat generation and stability of the temperature distribution in conducting solids, Int. J. Heat Mass Transf. 8(1965), 281-288. https://doi.org/10.1016/ 0017-9310(65) 90115-8
[16] H. B. Keller, D. S. Cohen, Some positone problems suggested by nonlinear heat generation, J. Math. Mech. 16(1967), 1361-1376. MR0213694
[17] M. A. Krasnosel'skiı, Positive solutions of operator equations, P. Noordhoff, Ltd, Groningen, the Netherlands, 1964. MR0181881
[18] M. A. Krasnosel'skir, Topological methods in the theory of nonlinear integral equations, Macmillam, New York, 1964. MR0159197
[19] T. Laetsch, On the number of solutions of boundary value problems with convex nonlinearities, J. Math. Anal. Appl. 35(1971), 389-404. https://doi.org/10.1016/0022247X (71) 90226-5; MR0280869
[20] T. Laetsch, The number of solutions of a nonlinear two point boundary value problem, Indiana Univ. Math. J. 20(1970/1971), No. 1, 1-13. https://doi.org/10.1512/iumj. 1970. 20.20001; MR0269922
[21] C. Li, K. Lu, The period function of hyperelliptic Hamiltonian of degree 5 with real critical points, Nonlinearity 21(2008), No. 3, 465-483. https://doi.org/10.1088/09517715/21/3/006; MR2396613
[22] P. De Maesschalck, F. Dumortier, The period function of classical Liénard equations, J. Differential Equations 233(2007), No. 2, 380-403. https://doi.org/10.1016/j.jde. 2006. 09.015; MR2292512
[23] F. Mañosas, J. Villadelprat, Criteria to bound the number of critical periods, J. Differential Equations 246(2009), No. 6, 2415-2433. https://doi.org/10.1016/j.jde.2008.07. 002; MR2498846
[24] P. Mardešić, D. Marin, J. Villadelprat, The period function of reversible quadratic centers, J. Differential Equations 2242006, No. 1, 120-171. https://doi.org/10.1016/j.jde. 2005. 07.024; MR2220066
[25] J. B. Mcleod, S. Sadhu, Existence of solutions and asymptotic analysis of a class of singularly perturbed odes with boundary conditions, Adv. Differential Equations 18(2013), No. 9-10, 825-848. MR3100053
[26] G. H. Pimbley, H. George, A sublinear Sturm-Liouville problem, J. Math. Mech. 11(1962), 121-138. MR0138820
[27] J. Smoller, A. Wasserman, Global bifurcation of steady-state solutions, J. Differential Equations 39(1981), No. 2, 269-290. https://doi.org/10.1016/0022-0396(81)90077-2; MR607786
[28] C.-C. Tzeng, K.-C. Hung, S.-H. Wang, Global bifurcation and exact multiplicity of positive solutions for a positone problem with cubic nonlinearity, J. Differential Equations 252(2012), No. 12, 6250-6274. https://doi.org/10.1016/j.jde.2012.02.020; MR2911833
[29] S.-H. Wang, A correction for a paper by J. Smoller and A. Wasserman, J. Differential Equations 77(1989), No. 1, 199-202. https://doi.org/10.1016/0022-0396(89)90162-9; MR980548
[30] S.-H. Wang, On S-shaped bifurcation curves, Nonlinear Anal. 22(1994), No. 12, 1475-1485. https://doi.org/10.1016/0362-546X (94) 90183-X; MR1285087
[31] L. Yang, X. Zeng, The period function of potential systems of polynomials with real zeros, Bull. Sci. Math. 133(2009), No. 6, 555-577. https://doi.org/10.1016/j.bulsci. 2009.05.002; MR2561363
[32] P. A. Zegeling, S. Iqbal, Nonstandard finite differences for a truncated Bratu-Picard model, Appl. Math. Comput. 324(2018), 266-284. MR3743672; https://doi.org/10.1016/ j.amc.2017.12.005
[33] Ya. B. Zeldovich, G. I. Barenblatt, V. B. Librovich, G. M. Makhviladze, The mathematical theory of combustion and explosions, Consultants Bureau [Plenum], New York, 1985. MR781350

# Analysis of an age-structured dengue model with multiple strains and cross immunity 

Ting-Ting Zheng, Lin-Fei Nie ${ }^{\boxtimes}$, Zhi-Dong Teng, Yan-Tao Luo and Sheng-Fu Wang

College of Mathematics and System Science, Xinjiang University, Urumqi 830046, P.R. China
Received 27 November 2019, appeared 14 July 2021
Communicated by Péter L. Simon


#### Abstract

Dengue fever is a typical mosquito-borne infectious disease, and four strains of it are currently found. Clinical medical research has shown that the infected person can provide life-long immunity against the strain after recovering from infection with one strain, but only provide partial and temporary immunity against other strains. On the basis of the complexity of transmission and the diversity of pathogens, in this paper, a multi-strain dengue transmission model with latency age and cross immunity age is proposed. We discuss the well-posedness of this model and give the terms of the basic reproduction number $\mathcal{R}_{0}=\max \left\{\mathcal{R}_{1}, \mathcal{R}_{2}\right\}$, where $\mathcal{R}_{i}$ is the basic reproduction number of strain $i(i=1,2)$. Particularly, we obtain that the model always has a unique diseasefree equilibrium $P_{0}$ which is locally stable for $\mathcal{R}_{0}<1$. And same time, an explicit condition of the global asymptotic stability of $P_{0}$ is obtained by constructing a suitable Lyapunov functional. Furthermore, we also shown that if $\mathcal{R}_{i}>1$, the strain- $i$ dominant equilibrium $P_{i}$ is locally stable for $\mathcal{R}_{j}<\mathcal{R}_{i}^{*}(i, j=1,2, i \neq j)$. Additionally, the threshold criteria on the uniformly persistence, the existence and global asymptotically stability of coexistence equilibrium are also obtained. Finally, these theoretical results and interesting conclusions are illustrated with some numerical simulations.


Keywords: dengue fever, age-structured model, cross immunity, uniform persistence, stability.
2020 Mathematics Subject Classification: 35E99, 92D30.

## 1 Introduction

Dengue is a vector-borne disease which was first described in 1779, and is common in more than 100 countries around the world [16]. Dengue viruses are spread to humans through the bite of an infected female mosquito (mainly Aedes aegypti and Aedes albopictus, which are known as the principal vector of Zika, chikungunya, and other viruses). In recent decades, the global incidence of dengue fever has increased dramatically and about half the world's population is now at risk. Each year, up to 400 million infections occur particularly in tropical and subtropical regions [1]. Due to its high morbidity and mortality, the World Health Organization has identified dengue as one of ten threats to global health in 2019 [44]. In order to understand

[^14]the mechanism of dengue fever transmission, a lot of mathematical models have been used to analyze its epidemiological characteristics $[10,12,20,24,26,30]$. For example, Esteva et al. [10] proposed an ordinary differential equations for the transmission of dengue fever with variable human population size, found three threshold parameters that control the development of this disease and the growth of the human population. Lee et al. [24] formulated a two-patch model to assess the impact of dengue transmission dynamics in heterogeneous environments, and found that reducing traffic is likely to take a host-vector system into the world of manageable outbreaks.

It is well know that dengue fever is caused by the dengue virus, which contains four different but closely relevant serotypes (DEN1-DEN4), for more details, see [9,11,43]. Medical statistic results show that recovery from infection with one virus provides lifelong immunity to that virus, but just temporal cross immunity to the other viruses. Subsequent infection with other viruses increases the risk of severe dengue (including Dengue Hemorrhagic Fever and Dengue Shock Syndrome) which can be life-threatening [43]. According to the diversity and transmission mechanism of dengue fever virus, some multi-strain dengue fever models have been established to investigate the effect of immunological interactions between heterotypic infections on disease dynamics. One example can be found in Ref. [9], Esteva et al. proposed a multi-strain dengue fever model, where the authors assumed that the primary infection with a specific strain changes the probability of being infected by a heterologous strain. Another example is that Feng et al. [11] established a multi-strain dengue fever model and found that there exists competitive exclusion phenomenon between different strains. More research can be found in $[9,11,17,19,27,29,32,34,41,42]$ and the references therein. Of course, there is still a lot of research that has not been mentioned, and the research continues.

The patterns of transmission, infectivity and latent period of infectious diseases play an important role in the process of transmission. It is well known that the period for individuals in latent compartment is different from one to one, which depends on individuals situation. For dengue fever, the period for individuals in latent compartment varies from 3 to 14 days and its distributions usually peak around their mean [3,7]. And for tuberculosis, the latent period for individuals in latent compartment may take months, years or even decades. Therefore, several epidemic models with latent age (time since entry into latent compartment) have been proposed by many famous experts and scholars [5,21,37,40]. Particularly, Wang et al. [37] proposed an SVEIR epidemic model with age-dependent vaccination and latency, found that the latency age not only impacts on the basic reproduction number but also could affect the values of the endemic steady state. They also showed that the introduction of age structure may change the dynamics of the corresponding model without age structure. Additionally, recent studies $[3,15]$ pointed out cross immunity starts immediately after the primary infectious period and prevents individuals from becoming infected by another strain for a period ranging from 6 months to 9 months, even to lifelong. To the best of our knowledge, there is currently no work on the effect of cross immunity age on the dynamics of dengue fever model.

Based on the discussion above, it is necessary to incorporate latency age and cross immunity age in the modeling of dengue fever. In this paper, we formulate a multi-strain dengue model with latency age and cross immunity age to assess the effects of latency age and cross immunity age on the transmission of dengue fever. The paper is structured as follows. The model is proposed in Section 2, and the nonnegative, boundedness and smoothness of the solution of this model are presented in Section 3. Section 4 analyzes the existence and stability of the boundary equilibria of model, which includes the disease-free equilibrium and stain dominant equilibrium. In Section 5, the uniform persistence of disease is discussed and the existence of coexistence equilibrium is obtained, and the theoretical results are illustrated with numerical simulations in Section 6. The paper ends with a brief conclusion.

## 2 Model formulation

Studies have shown that the number of dengue admissions caused by a third and fourth dengue virus infection have relatively few reported cases, accounting for only $0.08 \%--0.80 \%$ of the number of cases [14]. Therefore, it is reasonable to consider two strains in our model denote by strain 1 and strain 2, where 1 and 2 can be DEN1-DEN4. The infected individuals are divided into primary infected and secondary infected, and ignore further infections. Let $S(t)$ represent the number of susceptible individuals who are susceptible to both strain 1 and strain 2 at time $t$. $\widehat{E}_{i}(t), I_{i}(t)$ and $R_{i}(t)$ represent the number of latent, primary infected and recovered individuals with strain $i(i=1,2)$ at time $t$, respectively. Likewise, $Y_{i}(t)$ be the number of secondary infected individuals with strain $i$ after being recovered from strain $j(i, j=1,2, j \neq i)$ at time $t$. Let $R(t)$ represent the number of recovered individuals from secondary infection at time $t$ (to be permanently immune to both strains and hence there is no need to consider the evolution of $R(t)$ ). At the same time, due to the short length of mosquitoes' life cycle, assuming that a mosquito, once infected, never recovers and no secondary infection occurs. The mosquito population is subdivided into susceptible class $U(t)$, and infectious with strain $i$ class $V_{i}(t)$ ( $i=1,2$ ). Based on the transmission characteristics of dengue fever, we further propose two basic assumptions:
(A1) For latent individuals, the latent age (time since entry into latent class) is denoted by $a$. Let $E_{i}(t, a)$ denote the number of strain $i$ latent individuals with latent age $a$ at time $t$. Then the total number of strain $i$ latent individuals at time $t$ is given by $\widehat{E}_{i}(t)=\int_{0}^{\infty} E_{i}(t, a) \mathrm{d} a$. The conversion rate at which the latent individuals become infectious depends on the latent age, and is denoted by $\varepsilon_{i}(a), i=1,2$.
(A2) For recovered individuals, assume that the cross immunity wanes with time. Denote the cross immunity age, i.e., time since entry into recovered class $\widehat{R}_{i}(i=1,2)$, by $b$. Let $R_{i}(t, b)$ represent the number of the recovered individuals from strain $i(i=1,2)$ at time $t$ and cross immunity age $b$. Then the total number of strain $i$ recovered individuals at time $t$ is given by $\widehat{R}_{i}(t)=\int_{0}^{\infty} R_{i}(t, b) \mathrm{d} b, i=1,2$. The rate at which the cross immunity wanes of $\widehat{R}_{i}$ $(i=1,2)$ depends on cross immunity age, and is denoted by $\theta_{j}(b), j=1,2$.

Based on the above assumptions, the model can be written as the following,

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} S(t)}{\mathrm{d} t}=\Lambda_{h}-\beta_{1} S(t) V_{1}(t)-\beta_{2} S(t) V_{2}(t)-\mu_{h} S(t),  \tag{2.1}\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) E_{i}(t, a)=-\left(\mu_{h}+\varepsilon_{i}(a)\right) E_{i}(t, a), \\
\frac{\mathrm{d} I_{i}(t)}{\mathrm{d} t}=\int_{0}^{\infty} \varepsilon_{i}(a) E_{i}(t, a) \mathrm{d} a-\left(\gamma_{i}+\mu_{h}\right) I_{i}(t), \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial b}\right) R_{i}(t, b)=-\beta_{j} \theta_{j}(b) V_{j}(t) R_{i}(t, b)-\mu_{h} R_{i}(t, b), \\
\frac{\mathrm{d} Y_{i}(t)}{\mathrm{d} t}=\beta_{i} V_{i}(t) \int_{0}^{\infty} \theta_{i}(b) R_{j}(t, b) \mathrm{d} b-\left(\gamma_{i}+d_{i}+\mu_{h}\right) Y_{i}, \\
\frac{\mathrm{~d} U(t)}{\mathrm{d} t}=\Lambda_{m}-\alpha_{1}\left(I_{1}(t)+Y_{1}(t)\right) U(t)-\alpha_{2}\left(I_{2}(t)+Y_{2}(t)\right) U(t)-\mu_{m} U(t), \\
\frac{\mathrm{d} V_{i}(t)}{\mathrm{d} t}=\alpha_{i}\left(I_{i}(t)+Y_{i}(t)\right) U(t)-\mu_{m} V_{i}(t), \\
E_{i}(t, 0)=\beta_{i} S(t) V_{i}(t), R_{i}(t, 0)=\gamma_{i} I_{i}(t), \quad i, j=1,2, i \neq j,
\end{array}\right.
$$

with the initial condition

$$
\begin{align*}
& S(0)=S_{0} \geq 0, \quad E_{i}(0, a)=E_{i 0}(a) \geq 0, \quad I_{i}(0)=I_{i 0} \geq 0, \quad R_{i}(0, b)=R_{i 0}(b) \geq 0, \\
& Y_{i}(0)=Y_{i 0} \geq 0, \quad U(0)=U_{0} \geq 0, \quad V_{i}(0)=V_{i 0} \geq 0, \quad i=1,2, \tag{2.2}
\end{align*}
$$

where $E_{i 0}(a), R_{i 0}(b) \in L_{+}^{1}(0, \infty)$, and $L_{+}^{1}(0, \infty)$ is the space of nonnegative and Lebesgue integrable functions on $(0, \infty)$. In model (2.1), $\Lambda_{h}$ and $\Lambda_{m}$ are the recruitment rates of human and mosquito population, respectively; $1 / \mu_{h}$ and $1 / \mu_{m}$ denote the life expectancy for human and the average lifespan of mosquito, respectively; $\beta_{i}$ is the infectious rate from mosquito to human with strain $i ; \gamma_{i}$ is the recovery rate of human with strain $i ; d_{i}$ is the disease induced death rate in human with strain $i$ and $\alpha_{i}$ is the infectious rate from human to mosquito with strain $i$, $i=1,2$. All these parameters are assumed to be positive.

For model (2.1), the following hypotheses are reasonable.
(H1) $\varepsilon_{i}(\cdot), \theta_{i}(\cdot) \in L_{+}^{1}(0, \infty)$ are bounded with essential upper bound $\bar{\varepsilon}_{i}, \bar{\theta}_{i}$, and Lipschitz continuous on $\mathbb{R}_{+}$with Lipschitz coefficients $M_{\varepsilon i}, M_{\theta i}, i=1,2$, respectively. Besides, assuming that $\theta_{i}(\cdot) \in[0,1)$, if $\theta_{i}(\cdot) \in(0,1)$, then there exists cross-immunity between the two strains; if $\theta_{i}(\cdot)=0$, then individuals recovered from primary infection with one strain confer lifelong immunity to both strains.
(H2) $\bar{a}_{i}$ and $\bar{b}_{i}$ are the maximum ages of latency and cross immunity, the $\int_{\bar{a}_{i}}^{\infty} E_{i 0}(a) \mathrm{d} a=0$ and $\int_{\bar{b}_{i}}^{\infty} R_{i 0}(b) \mathrm{d} b=0, i=1,2$.
The state space of model (2.1) is defined as follows, $\mathbb{X}=\mathbb{R}_{+} \times L_{+}^{1}(0, \infty) \times L_{+}^{1}(0, \infty) \times \mathbb{R}_{+}^{2} \times$ $L_{+}^{1}(0, \infty) \times L_{+}^{1}(0, \infty) \times \mathbb{R}_{+}^{5}$. For any $X=\left(x_{1}, \phi_{1}, \phi_{2}, x_{2}, x_{3}, \psi_{1}, \psi_{2}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}\right) \in \mathbb{X}$ the norm is defined by

$$
\|X\|_{\mathrm{X}}=\sum_{i=1}^{8}\left|x_{i}\right|+\int_{0}^{\infty}\left|\varphi_{1}(a)\right| \mathrm{d} a+\int_{0}^{\infty}\left|\varphi_{2}(a)\right| \mathrm{d} a+\int_{0}^{\infty}\left|\psi_{1}(b)\right| \mathrm{d} b+\int_{0}^{\infty}\left|\psi_{2}(b)\right| \mathrm{d} b .
$$

For the convenience, we denote the solution of model (2.1) by $X(t)=\left(S(t), E_{1}(t, \cdot), E_{2}(t, \cdot)\right.$, $\left.I_{1}(t), I_{2}(t), R_{1}(t, \cdot), R_{2}(t, \cdot), Y_{1}(t), Y_{2}(t), U(t), V_{1}(t), V_{2}(t)\right)$. Let $X_{0}:=\left(S_{0}, E_{10}(\cdot), E_{20}(\cdot), I_{10}\right.$, $I_{20}, R_{10}(\cdot), R_{20}(\cdot), Y_{10}, Y_{20}, U_{0}, V_{10}, V_{20}$ ), then the initial condition (2.2) is rewritten as $X(0)=$ $X_{0}$. Furthermore, we denote by $X\left(t, X_{0}\right)$ the solution of model (2.1) with the initial condition $X(0)=X_{0}$.

## 3 The well-posedness

Solving $E_{i}(t, a)$ and $R_{i}(t, b)$ in the second and fourth equations of model (2.1) along the characteristic line $t-a=$ const and $t-b=$ const, respectively, we have

$$
\begin{align*}
& E_{i}(t, a)= \begin{cases}\beta_{i} S(t-a) V_{i}(t-a) \eta_{i}(a), & 0 \leq a<t, \\
E_{i 0}(a-t) \frac{\eta_{i}(a)}{\eta_{i}(a-t)}, & 0 \leq t \leq a,\end{cases} \\
& R_{i}(t, b)= \begin{cases}\gamma_{i} I_{i}(t-b) \Omega_{j}(t, b), & 0 \leq b<t, \\
R_{i 0}(b-t) \frac{\Omega_{j}(t, b)}{\Omega_{j}(t, b-t)}, & 0 \leq t \leq b,\end{cases} \tag{3.1}
\end{align*}
$$

where $\eta_{i}(a)=e^{-\int_{0}^{a}\left(\mu_{h}+\varepsilon_{i}(s)\right) \mathrm{d} s}, \Omega_{i}(t, b)=e^{-\int_{0}^{b}\left[\beta_{i} \theta_{i}(s) V_{i}(t-b+s)+\mu_{h}\right] \mathrm{d} s}, i, j=1,2, i \neq j$.
On the existence and nonnegativity of solution for model (2.1), we have the following result.

## Theorem 3.1.

(i) For any $X_{0} \in \mathbb{X}$, model (2.1) has a unique solution $X(t)$ with the initial condition $X(0)=X_{0}$ defined in maximal existence interval $\left[0, t_{0}\right)$ with $t_{0}>0$.
(ii) $X(t)$ is non-negative for all $t \in\left[0, t_{0}\right)$.
(iii) If $S_{0}>0, E_{i 0}(a)>0, I_{i 0}>0, R_{i 0}(b)>0, Y_{i 0}>0, U_{0}>0, V_{i 0}>0(i=1,2)$, then $X(t)$ also is positive for all $t \in\left[0, t_{0}\right)$.
Proof. From the Ref. [39], it is clear that conclusion (i) holds. From (3.1), we directly yield that $E_{i}(t, a)>0$ and $R_{i}(t, b)>0(i=1,2)$ for all $t \in\left[0, t_{0}\right)$. We can obtain that the solution $X(t)$ of model (2.1) with positive initial value remains is positive by the method of Ref. [38]. From the continuous dependence of solutions with respect to initial value, we immediately obtain that $X(t)$ is non-negative for all $t \in\left[0, t_{0}\right)$. This completes the proof.

## Denote

$$
\begin{aligned}
& \mathbb{D}=\left\{X=\left(S, E_{1}(a), E_{2}(a), I_{1}, I_{2}, R_{1}(b), R_{2}(b), Y_{1}, Y_{2}, U, V_{1}, V_{2}\right) \in \mathbb{X}:\right. \\
& \left.S+\sum_{i=1}^{2}\left(\left\|E_{i}(a)\right\|_{L^{1}}+I_{i}+\left\|R_{i}(b)\right\|_{L^{1}}+Y_{i}\right) \leq \frac{\Lambda_{h}}{\mu_{h}}, U+V_{1}+V_{2} \leq \frac{\Lambda_{m}}{\mu_{m}}\right\} .
\end{aligned}
$$

The following result is on the boundedness of solutions of model (2.1).
Theorem 3.2. For any initial value $X_{0} \in \mathbb{X}$, solution $X\left(t, X_{0}\right)$ of model (2.1) is defined for all $t \geq 0$ and is ultimately bounded. Further, $\mathbb{D}$ is positively invariant for model (2.1), i.e., $X\left(t, X_{0}\right) \in \mathbb{D}$ for all $t \geq 0$ and $X_{0} \in \mathbb{D}$, and $\mathbb{D}$ attracts all points in $\mathbb{X}$.
Proof. From Theorem 3.1, it is obvious that $X\left(t, X_{0}\right) \geq 0$ for all $t \in\left[0, t_{0}\right)$. Define

$$
N_{h}(t)=S(t)+\sum_{i=1}^{2}\left(\int_{0}^{\infty} E_{i}(t, a) \mathrm{d} a+I_{i}(t)+\int_{0}^{\infty} R_{i}(t, b) \mathrm{d} b+Y_{i}(t)\right)
$$

and $N_{m}(t)=U(t)+V_{1}(t)+V_{2}(t)$, from model (2.1), we have

$$
\begin{equation*}
\frac{\mathrm{d} N_{h}(t)}{\mathrm{d} t}=\Lambda_{h}-\mu_{h} N_{h}(t)-d\left(Y_{1}(t)+\Upsilon_{2}(t)\right) \leq \Lambda_{h}-\mu_{h} N_{h}(t), \quad \frac{\mathrm{d} N_{m}(t)}{\mathrm{d} t}=\Lambda_{m}-\mu_{m} N_{m}(t) \tag{3.2}
\end{equation*}
$$

which implies that

$$
N_{h}(t) \leq \max \left\{N_{h}^{0}, \frac{\Lambda_{h}}{\mu_{h}}\right\}, \quad N_{m}(t) \leq \max \left\{N_{m}^{0}, \frac{\Lambda_{m}}{\mu_{m}}\right\} .
$$

Hence, $N_{h}(t)$ and $N_{m}(t)$ are bounded on $\left[0, t_{0}\right)$, which implies that $X\left(t, X_{0}\right)$ is defined for any $t \geq 0$. Further, from (3.2), we have $\lim \sup _{t \rightarrow \infty} N_{h}(t) \leq \Lambda_{h} / \mu_{h}, \lim \sup _{t \rightarrow \infty} N_{m}(t) \leq \Lambda_{m} / \mu_{m}$. It follows that $X\left(t, X_{0}\right)$ is ultimately bounded. Furthermore, $\mathbb{D}$ is positively invariant for model (2.1), and $D$ attracts each point in $\mathbb{X}$. The proof is complete.

From Theorems 3.1 and 3.2, we obtain that all nonnegative solutions $X\left(t, X_{0}\right)$ of model (2.1) with the initial condition $X(0)=X_{0}$ generate a continuous semi-flow $\Phi: \mathbb{R}_{+} \times \mathbb{X} \rightarrow \mathbb{X}$ as $\Phi_{t}\left(X_{0}\right)=X\left(t, X_{0}\right), t \geq 0, X_{0} \in \mathbb{X}$.

On the asymptotically smoothness of the semi-flow $\left\{\Phi_{t}\right\}_{t \geq 0}$, we have the following result.
Theorem 3.3. The semi-flow $\left\{\Phi_{t}\right\}_{t \geq 0}$ generated by model (2.1) is asymptotically smooth. Furthermore, model (2.1) has a compact global attractor $\mathcal{A}$ contained in $\mathbb{X}$.

This theorem can be proved by using the standard argument, see [40] for detailed proof methods.

## 4 The existence and stability of boundary equilibria

Model (2.1) always has a disease-free equilibrium $P_{0}=\left(S^{*}, 0,0,0,0,0,0,0,0,0, U^{*}, 0,0\right)$, where $S^{*}=\Lambda_{h} / \mu_{h}, U^{*}=\Lambda_{m} / \mu_{m}$. For the convenience, denote $K_{i}=\int_{0}^{\infty} \varepsilon_{i}(a) \eta_{i}(a) \mathrm{d} a, i=1,2$. It is clear that $K_{i} \in(0,1), i=1,2$.

Denote the basic reproduction number $\mathcal{R}_{0}$ by

$$
\begin{equation*}
\mathcal{R}_{0}=\max \left\{\mathcal{R}_{1}, \mathcal{R}_{2}\right\}, \quad \mathcal{R}_{i}=\frac{\Lambda_{h} \Lambda_{m} \alpha_{i} \beta_{i} K_{i}}{\mu_{h} \mu_{m}^{2}\left(\gamma_{i}+\mu_{h}\right)}=\frac{\Lambda_{h}}{\mu_{h}} \times \frac{\Lambda_{m}}{\mu_{m}} \times \frac{\beta_{i}}{\mu_{m}} \times \frac{\alpha_{i}}{\gamma_{i}+\mu_{h}} \times K_{i}, \quad i=1,2 . \tag{4.1}
\end{equation*}
$$

Here, $\beta_{i} / \mu_{m}$ represents the number of secondary infections one infectious mosquito will produce in a completely susceptible human population, $\alpha_{i} /\left(\gamma_{i}+\mu_{h}\right)$ represents the number of effective contact human to mosquito during the infectious period of human and $K_{i}$ represents the probability of an exposed individual becomes infective. Therefore, $\mathcal{R}_{i}$ can be considered as the basic reproduction number of strain $i$, which is defined as the average number of secondary infective of strain $i$, produced by a single infective of strain $i$ in a completely susceptible population.

Let $E_{2}(t, a)=I_{2}(t)=R_{2}(t, b)=Y_{2}(t)=V_{2}(t)=0$ in model (2.1), then we obtain the subsystem that only strain 1 exists as follows

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} S(t)}{\mathrm{d} t}=\Lambda_{h}-\beta_{1} S(t) V_{1}(t)-\mu_{h} S(t)  \tag{4.2}\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) E_{1}(t, a)=-\left(\mu_{h}+\varepsilon_{1}(a)\right) E_{1}(t, a), E_{1}(t, 0)=\beta_{1} S(t) V_{1}(t) \\
\frac{\mathrm{d} I_{1}(t)}{\mathrm{d} t}=\int_{0}^{\infty} \varepsilon_{1}(a) E_{1}(t, a) \mathrm{d} a-\left(\gamma_{1}+\mu_{h}\right) I_{1}(t) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial b}\right) R_{1}(t, b)=-\mu_{h} R_{1}(t, b), R_{1}(t, 0)=\gamma_{1} I_{1}(t) \\
\frac{\mathrm{d} U(t)}{\mathrm{d} t}=\Lambda_{m}(t)-\alpha_{1} I_{1}(t) U(t)-\mu_{m} U(t) \\
\frac{\mathrm{d} V_{1}(t)}{\mathrm{d} t}=\alpha_{1} I_{1}(t) U(t)-\mu_{m} V_{1}(t)
\end{array}\right.
$$

Clearly, model (4.2) always has a disease-free equilibrium $p_{0}=\left(\Lambda_{h} / \mu_{h}, 0,0,0, \Lambda_{m} / \mu_{m}, 0\right)$. Let $p_{1}=\left(S_{1}^{*}, E_{1}^{*}(a), I_{1}^{*}, R_{1}^{*}(b), U_{1}^{*}, V_{1}^{*}\right)$ be the positive equilibrium of model (4.2), then

$$
\begin{array}{ll}
\Lambda_{h}-\beta_{1} S_{1}^{*} V_{1}^{*}-\mu_{h} S_{1}^{*}=0, & \Lambda_{m}-\alpha_{1} I_{1}^{*} U_{1}^{*}-\mu_{m} U_{1}^{*}=0 \\
\frac{\mathrm{~d}}{\mathrm{~d} a} E_{1}^{*}(a)=-\left(\mu_{h}+\varepsilon_{1}(a)\right) E_{1}^{*}(a), & \frac{\mathrm{d}}{\mathrm{~d} b} R_{1}^{*}(b)=-\mu_{h} R_{1}^{*}(b),  \tag{4.3}\\
\int_{0}^{\infty} \varepsilon_{1}(a) E_{1}^{*}(a) \mathrm{d} a-\left(\gamma_{1}+\mu_{h}\right) I_{1}^{*}=0, & E_{1}^{*}(0)=\beta_{1} S_{1}^{*} V_{1}^{*}, \quad R_{1}^{*}(0)=\gamma_{1} I_{1}^{*}
\end{array}
$$

From (4.3),

$$
\begin{array}{ll}
R_{1}^{*}(b)=R_{1}^{*}(0) e^{-\mu_{h} b}=\gamma_{1} I_{1}^{*} e^{-\mu_{h} b}, \quad U_{1}^{*}=\frac{\Lambda_{m}}{\alpha_{1} I_{1}^{*}+\mu_{m}}, & V_{1}^{*}=\frac{\alpha_{1} I_{1}^{*} \Lambda_{m}}{\mu_{m}\left(\alpha_{1} I_{1}^{*}+\mu_{m}\right)}, \\
E_{1}^{*}(a)=E_{1}^{*}(0) \eta_{1}(a)=\beta_{1} S_{1}^{*} V_{1}^{*} \eta_{1}(a), & S_{1}^{*}=\frac{\mu_{m}\left(\alpha_{1} I_{1}^{*}+\mu_{m}\right)\left(\gamma_{1}+\mu_{h}\right)}{\alpha_{1} \beta_{1} \Lambda_{m} K_{1}} .
\end{array}
$$

Substituting the above formulas for $V_{1}^{*}, E_{1}^{*}(a)$ and $S_{1}^{*}$ into the first equation of (4.3) yields

$$
I_{1}^{*}=\frac{\alpha_{1} \beta_{1} \Lambda_{h} \Lambda_{m} K_{1}-\mu_{h} \mu_{m}^{2}\left(\gamma_{1}+\mu_{h}\right)}{\alpha_{1}\left(\gamma_{1}+\mu_{h}\right)\left(\beta_{1} \Lambda_{m}+\mu_{h} \mu_{m}\right)}=\frac{\mu_{h} \mu_{m}^{2}\left(\mathcal{R}_{1}-1\right)}{\alpha_{1}\left(\beta_{1} \Lambda_{m}+\mu_{m} \mu_{h}\right)} .
$$

Thus, from the expressions of $S_{1}^{*}, E_{1}^{*}(a), I_{1}^{*}, R_{1}^{*}(b), U_{1}^{*}$ and $V_{1}^{*}$, it can be easily seen that model (4.2) has a unique positive equilibrium $p_{1}$ if and only if $\mathcal{R}_{1}>1$. Therefore, model (2.1) has a strain 1 dominant boundary equilibrium $P_{1}=\left(S_{1}^{*}, E_{1}^{*}, 0, I_{1}^{*}, 0, R_{1}^{*}(b), 0,0,0, U_{1}^{*}, V_{1}^{*}, 0\right)$ when $\mathcal{R}_{1}>1$, where

$$
\begin{aligned}
S_{1}^{*} & =\frac{\mu_{m}^{2}\left(\gamma_{1}+\mu_{h}\right)\left(\mathcal{R}_{1} \mu_{h} \mu_{m}+\beta_{1} \Lambda_{m}\right)}{\alpha_{1} \beta_{1} \Lambda_{m}\left(\beta_{1} \Lambda_{m}+\mu_{m} \mu_{h}\right) K_{1}}, & E_{1}^{*}(a) & =\frac{\mu_{m}^{2} \mu_{h} \eta_{1}(a)\left(\gamma_{1}+\mu_{h}\right)\left(\mathcal{R}_{1}-1\right)}{\alpha_{1}\left(\beta_{1} \Lambda_{m}+\mu_{m} \mu_{h}\right) K_{1}}, \\
I_{i}^{*} & =\frac{\left(\mathcal{R}_{1}-1\right) \mu_{h} \mu_{m}^{2}}{\alpha_{1}\left(\beta_{1} \Lambda_{m}+\mu_{m} \mu_{h}\right)}, & R_{1}^{*}(b) & =\frac{\left(\mathcal{R}_{1}-1\right) \gamma_{1} \mu_{h} \mu_{m}^{2} e^{-\mu_{h} b}}{\alpha_{1}\left(\beta_{1} \Lambda_{m}+\mu_{m} \mu_{h}\right)}, \\
U_{1}^{*} & =\frac{\Lambda_{m}\left(\beta_{1} \Lambda_{m}+\mu_{m} \mu_{h}\right)}{\mu_{m}\left(\mathcal{R}_{1} \mu_{h} \mu_{m}+\beta_{1} \Lambda_{m}\right)^{\prime}}, & V_{1}^{*} & =\frac{\Lambda_{m} \mu_{h}\left(\mathcal{R}_{1}-1\right)}{\left(\mathcal{R}_{1} \mu_{h} \mu_{m}+\beta_{1} \Lambda_{m}\right)} .
\end{aligned}
$$

Similarly, model (2.1) has a strain-2 dominant boundary equilibrium $P_{2}=\left(S_{2}^{*}, 0, E_{2}^{*}, 0, I_{2}^{*}, 0\right.$, $\left.R_{2}^{*}(b), 0,0, U_{2}^{*}, 0, V_{2}^{*}\right)$ when $\mathcal{R}_{2}>1$, where

$$
\begin{aligned}
S_{2}^{*} & =\frac{\mu_{m}^{2}\left(\gamma_{2}+\mu_{h}\right)\left(\mathcal{R}_{2} \mu_{h} \mu_{m}+\beta_{2} \Lambda_{m}\right)}{\alpha_{2} \beta_{2} \Lambda_{m}\left(\beta_{2} \Lambda_{m}+\mu_{m} \mu_{h}\right) K_{2}}, & E_{2}^{*}(a) & =\frac{\mu_{m}^{2} \mu_{h} \eta_{2}(a)\left(\gamma_{2}+\mu_{h}\right)\left(\mathcal{R}_{2}-1\right)}{\alpha_{2}\left(\beta_{2} \Lambda_{m}+\mu_{m} \mu_{h}\right) K_{2}}, \\
I_{2}^{*} & =\frac{\left(\mathcal{R}_{2}-1\right) \mu_{h} \mu_{m}^{2}}{\alpha_{2}\left(\beta_{2} \Lambda_{m}+\mu_{m} \mu_{h}\right)}, & R_{2}^{*}(b) & =\frac{\left(\mathcal{R}_{2}-1\right) \gamma_{2} \mu_{h} \mu_{m}^{2} e^{-\mu_{h} b}}{\alpha_{2}\left(\beta_{2} \Lambda_{m}+\mu_{m} \mu_{h}\right)}, \\
U_{2}^{*} & =\frac{\Lambda_{m}\left(\beta_{2} \Lambda_{m}+\mu_{m} \mu_{h}\right)}{\mu_{m}\left(\mathcal{R}_{2} \mu_{h} \mu_{m}+\beta_{2} \Lambda_{m}\right)}, & V_{2}^{*} & =\frac{\Lambda_{m} \mu_{h}\left(\mathcal{R}_{2}-1\right)}{\left(\mathcal{R}_{2} \mu_{h} \mu_{m}+\beta_{2} \Lambda_{m}\right)} .
\end{aligned}
$$

Summarizing the discussions above, we have the following theorem.

## Theorem 4.1.

(i) Model (2.1) always has a disease-free equilibrium $P_{0}$.
(ii) If $\mathcal{R}_{1}>1$, then model (2.1) has a strain 1 dominant equilibrium $P_{1}$.
(iii) If $\mathcal{R}_{2}>1$, then model (2.1) has a strain 2 dominant equilibrium $P_{2}$.

On the stability of boundary equilibria of model (2.1), we first obtain the following results.
Theorem 4.2. If $\mathcal{R}_{0}<1$, then the disease-free equilibrium $P_{0}$ of model (2.1) is locally asymptotically stable, and if $\mathcal{R}_{0}>1$, then $P_{0}$ is unstable.
Proof. Let $S(t)=S^{*}+s(t), E_{i}(t, a)=e_{i}(t, a), I_{i}(t)=i_{i}(t), R_{i}(t, a)=r_{i}(t, a), Y_{i}(t)=y_{i}(t)$, $U(t)=U^{*}+u(t)$ and $V_{i}(t)=v_{i}(t), i=1,2$. Linearizing model (2.1) at equilibrium $P_{0}$, one has

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} s(t)}{\mathrm{d} t}=-\beta_{1}(t) S^{*} v_{1}(t)-\beta_{2}(t) S^{*} v_{2}(t)-\mu_{h} s(t)  \tag{4.4}\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) e_{i}(t, a)=-\left(\mu_{h}+\varepsilon_{i}(a)\right) e_{i}(t, a), e_{i}(t, 0)=\beta_{i} S^{*} v_{i}(t) \\
\frac{\mathrm{d} i_{i}(t)}{\mathrm{d} t}=\int_{0}^{\infty} \varepsilon_{i}(a) e_{i}(t, a) \mathrm{d} a-\left(\gamma_{i}+\mu_{h}\right) i_{i}(t), \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial b}\right) r_{i}(t, b)=-\mu_{h} r_{i}(t, b), r_{i}(t, 0)=\gamma_{i} i_{i}(t), i=1,2
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} y_{i}(t)}{\mathrm{d} t}=-\left(\gamma_{i}+d_{i}+\mu_{h}\right) y_{i}(t),  \tag{4.5}\\
\frac{\mathrm{d} u(t)}{\mathrm{d} t}=-\alpha_{1}\left(i_{1}(t)+y_{1}(t)\right) U^{*}-\alpha_{2}\left(i_{2}(t)+y_{2}(t)\right) U^{*}-\mu_{m} u(t), \\
\frac{\mathrm{d} v_{i}(t)}{\mathrm{d} t}=\alpha_{i}\left(i_{i}(t)+y_{i}(t)\right) U^{*}-\mu_{m} v_{i}(t), i=1,2 .
\end{array}\right.
$$

It is easy to obtain that $\lim _{t \rightarrow \infty} y_{i}(t)=0, i=1,2$ from the first equation of model (4.5). Thus, we only need to consider model (4.4) and the following limit system of model (4.5)

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} u(t)}{\mathrm{d} t}=-\alpha_{1} i_{1}(t) U^{*}-\alpha_{2} i_{2}(t) U^{*}-\mu_{m} u(t)  \tag{4.6}\\
\frac{\mathrm{d} v_{i}(t)}{\mathrm{d} t}=\alpha_{i} i_{i}(t) U^{*}-\mu_{m} v_{i}(t), i=1,2
\end{array}\right.
$$

Let $s(t)=\bar{s} e^{\lambda t}, e_{i}(t, a)=\bar{e}_{i}(a) e^{\lambda t}, i_{i}(t)=\bar{i}_{i} e^{\lambda t}, r_{i}(t, b)=\bar{r}_{i}(b) e^{\lambda t}, u(t)=\bar{u} e^{\lambda t}$ and $v_{i}(t)=\bar{v}_{i} e^{\lambda t}$, where $\bar{s}, \bar{i}_{i}, \bar{y}_{i}, \bar{u}$ and $\bar{v}_{i}(i=1,2)$ are positive constants, $\bar{e}_{i}(a)$ and $\bar{r}_{i}(b)$ are nonnegative functions, then we obtain the following eigenvalue problem

$$
\begin{equation*}
\left(\lambda+\mu_{h}\right) \bar{s}=-\beta_{1} S^{*} \bar{v}_{1}-\beta_{2} S^{*} \bar{v}_{2}, \quad\left(\lambda+\mu_{m}\right) \bar{u}=-\alpha_{1} \bar{i}_{1} U^{*}-\alpha_{2} \bar{i}_{2} U^{*} \tag{4.7}
\end{equation*}
$$

and

$$
\begin{cases}\left(\lambda+\gamma_{i}+\mu_{h}\right) \bar{i}_{i}=\int_{0}^{\infty} \varepsilon_{i}(a) \bar{e}_{i}(a) \mathrm{d} a, & \left(\lambda+\mu_{m}\right) \bar{v}_{i}=\alpha_{i} \overline{\bar{i}}_{i} U^{*}  \tag{4.8}\\ \frac{\mathrm{~d} \bar{e}_{i}(a)}{\mathrm{d} a}=-\left(\mu_{h}+\varepsilon_{i}(a)+\lambda\right) \bar{e}_{i}(a), & \frac{\mathrm{d} \bar{r}_{i}(b)}{\mathrm{d} b}=-\left(\mu_{h}+\lambda\right) \bar{r}_{i}(b), \\ \bar{e}_{i}(0)=\beta_{i} S^{*} \bar{v}_{i}, \bar{r}_{i}(0)=\gamma_{i} \bar{i}_{i}, i=1,2 . & \end{cases}
$$

From (4.7), it follows that

$$
\lambda_{1}=-\frac{\beta_{1} S^{*} \bar{v}_{1}+\beta_{2} S^{*} \bar{v}_{2}}{\bar{s}}-\mu_{h}<0, \quad \lambda_{2}=-\frac{\alpha_{1} \bar{i}_{1} U^{*}+\alpha_{2} \bar{i}_{2} U^{*}}{\bar{u}}-\mu_{m}<0 .
$$

Therefore, the stability of $P_{0}$ depends on the eigenvalues of (4.8). Directly calculating from the equations of $\bar{i}_{i}, \bar{e}_{i}(a)$ and $\bar{v}_{i}$ in problem (4.8) yields the following characteristic equation

$$
\begin{equation*}
\lambda+\gamma_{i}+\mu_{h}=\frac{\alpha_{i} \beta_{i} \Lambda_{h} \Lambda_{m}}{\mu_{h} \mu_{m}\left(\lambda+\mu_{m}\right)} \int_{0}^{\infty} \varepsilon_{i}(a) e^{-\int_{0}^{a}\left(\lambda+\mu_{h}+\varepsilon_{i}(s)\right) \mathrm{d} s} \mathrm{~d} a, \quad i=1,2 . \tag{4.9}
\end{equation*}
$$

Denote

$$
L H S=\lambda+\gamma_{i}+\mu_{h}, \quad R H S=\mathcal{G}(\lambda)=\frac{\alpha_{i} \beta_{i} \Lambda_{h} \Lambda_{m}}{\mu_{h} \mu_{m}\left(\lambda+\mu_{m}\right)} \int_{0}^{\infty} \varepsilon_{i}(a) e^{-\int_{0}^{a}\left(\lambda+\mu_{h}+\varepsilon_{i}(s)\right) \mathrm{d} s} \mathrm{~d} a
$$

It is easy to verify that for any eigenvalue $\lambda$, if $\operatorname{Re}(\lambda) \geq 0$, when $\mathcal{R}_{0}<1$, then

$$
|L H S| \geq \gamma_{i}+\mu_{h}, \quad|R H S| \leq \mathcal{G}(\operatorname{Re} \lambda) \leq \mathcal{G}(0)=\mathcal{R}_{i}\left(\gamma_{i}+\mu_{h}\right)<|L H S|, \quad i=1,2 .
$$

This leads to a contradiction. Thus, all eigenvalues $\lambda$ of problem (4.8) have negative real parts, which implies that $\lim _{t \rightarrow \infty} i_{i}(t)=0, \lim _{t \rightarrow \infty} e_{i}(t, a)=0, \lim _{t \rightarrow \infty} v_{i}(t)=0$ and $\lim _{t \rightarrow \infty} r_{i}(t, b)=$ 0 . Therefore, $P_{0}$ is locally asymptotically stable when $\mathcal{R}_{0}<1$.

Now, assume that $\mathcal{R}_{0}>1$ and rewrite the characteristic equation (4.9) in the form

$$
\mathcal{G}_{1 i}(\lambda)=\left(\lambda+\gamma_{i}+\mu_{h}\right)-\frac{\alpha_{i} \beta_{i} \Lambda_{h} \Lambda_{m}}{\mu_{h} \mu_{m}\left(\lambda+\mu_{m}\right)} \int_{0}^{\infty} \varepsilon_{i}(a) e^{-\int_{0}^{a}\left(\lambda+\mu_{h}+\varepsilon_{i}(s)\right) \mathrm{d} s} \mathrm{~d} a=0, \quad i=1,2 .
$$

Obviously,

$$
\mathcal{G}_{1 i}(0)=\left(\gamma_{i}+\mu_{h}\right)-\frac{\alpha_{i} \beta_{i} \Lambda_{h} \Lambda_{m}}{\mu_{h} \mu_{m}^{2}} \int_{0}^{\infty} \varepsilon_{i}(a) e^{-\int_{0}^{a}\left(\mu_{h}+\varepsilon_{i}(s)\right) \mathrm{d} s} \mathrm{~d} a=\left(\gamma_{i}+\mu_{h}\right)\left(1-\mathcal{R}_{i}\right)<0,
$$

and $\lim _{\lambda \rightarrow \infty} \mathcal{G}_{1 i}(\lambda)=+\infty$. Hence, the characteristic equation (4.9) at least has a positive real root. It implies that equilibrium $P_{0}$ is unstable. This completes the proof.

Next, we discuss the global stability of equilibrium $P_{0}$. To do so, define

$$
q_{i}(a)=\int_{a}^{\infty} \varepsilon_{i}(s) e^{-\int_{a}^{s}\left(\mu_{h}+\varepsilon_{i}(\xi)\right) \mathrm{d} \xi} \mathrm{~d} s, \quad i=1,2
$$

It is easy to obtain that

$$
\frac{\mathrm{d} q_{i}(a)}{\mathrm{d} a}=\left(\mu_{h}+\varepsilon_{i}(a)\right) q_{i}(a)-\varepsilon_{i}(a), \quad q_{i}(0)=K_{i}, i=1,2 .
$$

Theorem 4.3. If $\mathcal{R}_{0} \leq \min \left\{K_{1}, K_{2}\right\}$, then disease-free equilibrium $P_{0}$ of model (2.1) is globally asymptotically stable.

Proof. Define a Lyapunov functional as follows

$$
L(t)=\sum_{i=1}^{2}\left(\int_{0}^{\infty} q_{i}(a) E_{i}(t, a) \mathrm{d} a+I_{i}(t)+K_{i} Y_{i}(t)+\frac{\beta_{i} \Lambda_{h}}{\mu_{m} \mu_{h}} K_{i} V_{i}(t)\right)
$$

Calculating the time derivative of $L(t)$ along the solution of model (2.1), it can be easily obtained that

$$
\begin{aligned}
\frac{\mathrm{d} L(t)}{\mathrm{d} t}= & \sum_{i=1}^{2}\left(\beta_{i} K_{i} S(t) V_{i}(t)-\left(\gamma_{i}+\mu_{h}\right) I_{i}(t)\right)+\sum_{i=1}^{2}\left(\frac{\alpha_{i} \beta_{i} \Lambda_{h} K_{i}}{\mu_{m} \mu_{h}} U(t)\left(I_{i}(t)+Y_{i}(t)\right)\right. \\
& \left.-\frac{\beta_{i} \Lambda_{h} K_{i}}{\mu_{h}} V_{i}(t)\right)+\sum_{i=1}^{2}\left(K_{i} \beta_{i} V_{i} \int_{0}^{\infty} \theta_{i}(b) R_{j}(t, b) \mathrm{d} b-\left(\gamma_{i}+d_{i}+\mu_{h}\right) K_{i} Y_{i}(t)\right) \\
\leq & \sum_{i=1}^{2}\left(\beta_{i} K_{i} S(t) V_{i}(t)-\left(\gamma_{i}+\mu_{h}\right) I_{i}(t)\right)+\sum_{i=1}^{2}\left(\frac{\alpha_{i} \beta_{i} \Lambda_{h} \Lambda_{m} K_{i}}{\mu_{m}^{2} \mu_{h}}\left(I_{i}(t)+Y_{i}(t)\right)\right. \\
& \left.-\frac{\beta_{i} \Lambda_{h} K_{i}}{\mu_{h}} V_{i}(t)\right)+\sum_{i=1}^{2}\left(K_{i} \beta_{i} V_{i} \int_{0}^{\infty} R_{j}(t, b) \mathrm{d} b-\left(\gamma_{i}+d_{i}+\mu_{h}\right) K_{i} Y_{i}(t)\right) \\
\leq & \sum_{i=1}^{2}\left(\beta_{i} K_{i} V_{i}(t)\left(S(t)+\int_{0}^{\infty} R_{j}(t, b) \mathrm{d} b-\frac{\Lambda_{h}}{\mu_{h}}\right)\right)+\sum_{i=1}^{2}\left(\left(\gamma_{i}+\mu_{h}\right)\left(\mathcal{R}_{i}-1\right) I_{i}(t)\right) \\
& +\sum_{i=1}^{2}\left(\left(\gamma_{i}+\mu_{h}\right)\left(\mathcal{R}_{i}-K_{i}\right) Y_{i}(t)\right)-d_{1} Y_{1}(t)-d_{2} Y_{2}(t) .
\end{aligned}
$$

Restricting to set $\mathbb{D}$, we have $S(t)+\int_{0}^{\infty} R_{j}(t, b) \mathrm{d} b-\Lambda_{h} / \mu_{h} \leq 0$ for all $t \geq 0$. Hence, when $\mathcal{R}_{i} \leq K_{i}(i=1,2)$, we have $\mathrm{d} L(t) / \mathrm{d} t \leq 0$, and the equality holds only if $I_{i}(t)=Y_{i}(t)=0$ and

$$
V_{i}(t)\left(S(t)+\int_{0}^{\infty} R_{j}(t, b) \mathrm{d} b-\frac{\Lambda_{h}}{\mu_{h}}\right)=0
$$

When $I_{i}(t)=Y_{i}(t)=0$, it follows that $\lim _{t \rightarrow \infty} V_{i}(t)=0$ and $\lim _{t \rightarrow \infty} U(t)=U^{*}$ from the sixth and seventh equations model (2.1). Further, it is clearly that $\lim _{t \rightarrow \infty} S(t)=S^{*}$ from the first equation model (2.1). Then, from the second and fourth equations of model (2.1), we obtain that $\lim _{t \rightarrow \infty} E_{i}(t, a)=0$ and $\lim _{t \rightarrow \infty} R_{i}(t, b)=0$. Thus, $\left\{P_{0}\right\}$ is the largest invariant subset of set $\{X \in \mathbb{D}: \mathrm{d} L(t) / \mathrm{d} t=0\}$. By the LaSalle's invariance principle, $P_{0}$ is globally asymptotically stable. The proof is complete.

Remark 4.4. In the Section 6, by the numerical example, we verify the disease-free equilibrium $P_{0}$ is globally asymptotically stable when $\mathcal{R}_{0}<1$. However, our theoretical analysis can only obtain the global stability of $P_{0}$ when $\mathcal{R}_{0}<\min \left\{K_{1}, K_{2}\right\}$. This is an open question, and we will continue to work on it in future studies.

Now, we show the local stability of equilibrium $P_{1}$ of model (2.1). Let $S(t)=s(t)+S_{1}^{*}$, $E_{1}(t, a)=E_{1}^{*}(a)+e_{1}(t, a), I_{1}(t)=I_{1}^{*}+i_{1}(t), R_{1}(t, b)=R_{1}^{*}(b)+r_{1}(t, b), U(t)=U_{1}^{*}+u(t)$, $V_{1}(t)=V_{1}^{*}+v(t), I_{2}(t)=i_{2}(t), R_{2}(t, b)=r_{2}(t, b), Y_{i}(t)=y_{i}(t)$ and $V_{2}(t)=v_{2}(t), i=1,2$, then the linearized system of model (2.1) at equilibrium $P_{1}$ is as follows

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} s(t)}{\mathrm{d} t}=-\beta_{1} S_{1}^{*} v_{1}(t)-\beta_{1} s(t) V_{1}^{*}-\beta_{2} S_{1}^{*} v_{2}(t)-\mu_{h} s(t),  \tag{4.10}\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) e_{1}(t, a)=-\left(\mu_{h}+\varepsilon_{1}(a)\right) e_{1}(t, a), e_{1}(t, 0)=\beta_{1} S_{1}^{*} v_{1}(t)+\beta_{1} s(t) V_{1}^{*}, \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) e_{2}(t, a)=-\left(\mu_{h}+\varepsilon_{2}(a)\right) e_{2}(t, a), e_{2}(t, 0)=\beta_{2} S_{1}^{*} v_{2}(t), \\
\frac{\mathrm{d} i_{i}(t)}{\mathrm{d} t}=\int_{0}^{\infty} \varepsilon_{i}(a) e_{i}(t, a) \mathrm{d} a-\left(\gamma_{i}+\mu_{h}\right) i_{i}(t), i=1,2, \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial b}\right) r_{1}(t, b)=-\beta_{2} \theta_{2}(b) v_{2}(t) R_{1}^{*}(b)-\mu_{h} r_{1}(t, b), r_{1}(t, 0)=\gamma_{1} i_{1}(t), \\
\frac{\partial}{\left.\frac{\partial}{\partial t}+\frac{\partial}{\partial b}\right) r_{2}(t, b)=-\beta_{1} \theta_{1}(b) V_{1}^{*} r_{2}(t, b)-\mu_{h} r_{2}(t, b), r_{2}(t, 0)=\gamma_{2} i_{2}(t),} \\
\frac{\mathrm{d} y_{1}(t)}{\mathrm{d} t}=\beta_{1} V_{1}^{*} \int_{0}^{\infty} \theta_{1}(b) r_{2}(t, b) \mathrm{d} b-\left(\gamma_{1}+d_{1}+\mu_{h}\right) y_{1}, \\
\frac{\mathrm{~d} y_{2}(t)}{\mathrm{d} t}=\beta_{2} v_{2}(t) \int_{0}^{\infty} \theta_{2}(b) R_{1}^{*}(b) \mathrm{d} b-\left(\gamma_{2}+d_{2}+\mu_{h}\right) y_{2}, \\
\frac{\mathrm{~d} u(t)}{\mathrm{d} t}=-\alpha_{1}\left(i_{1}(t)+y_{1}(t)\right) U_{1}^{*}-\alpha_{1} u(t) I_{1}^{*}-\alpha_{2}\left(i_{2}(t)+y_{2}(t)\right) U_{1}^{*}-\mu_{m} u(t), \\
\frac{\mathrm{d} v_{1}(t)}{\mathrm{d} t}=\alpha_{1}\left(i_{1}(t)+y_{1}(t)\right) U_{1}^{*}+\alpha_{1} u(t) I_{1}^{*}-\mu_{m} v_{1}(t), \\
\frac{\mathrm{d} v_{2}(t)}{\mathrm{d} t}=\alpha_{2}\left(i_{2}(t)+y_{2}(t)\right) U_{1}^{*}-\mu_{m} v_{2}(t) .
\end{array}\right.
$$

Firstly, we discuss the equations with strain 2 in model (4.10). Let $e_{2}(t, a)=\tilde{e}_{2}(a) e^{\lambda t}, i_{2}(t)=$ $\tilde{i}_{2} e^{\lambda t}, r_{2}(t, b)=\tilde{r}_{2}(b) e^{\lambda t}$ and $v_{2}(t)=\tilde{v}_{2} e^{\lambda t}$, where $\tilde{i}_{2}, \tilde{y}_{2}$ and $\tilde{v}_{2}$ are positive constants, $\tilde{e}_{2}(a)$ and $\tilde{r}_{2}(b)$ are nonnegative functions, then we can get the following eigenvalue problem

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} \tilde{e}_{2}(a)}{\mathrm{d} a}=-\left(\lambda+\mu_{h}+\varepsilon_{2}(a)\right) \tilde{e}_{2}(a), \quad \tilde{e}_{2}(0)=\beta_{2} S_{1}^{*} \tilde{v}_{2}  \tag{4.11}\\
\left(\lambda+\gamma_{2}+\mu_{h}\right) \tilde{i}_{2}=\int_{0}^{\infty} \varepsilon_{2}(a) \tilde{e}_{2}(a) \mathrm{d} a \\
\frac{\mathrm{~d} \tilde{\mathscr{r}}_{2}(b)}{\mathrm{d} b}=-\left(\lambda+\beta_{1} \theta_{1}(b) V_{1}^{*}+\mu_{h}\right) \tilde{r}_{2}(b), \quad \tilde{r}_{2}(0)=\gamma_{2} \tilde{i}_{2} \\
\left(\lambda+\gamma_{2}+d_{2}+\mu_{h}\right) \tilde{y}_{2}=\beta_{2} \tilde{v}_{2} \int_{0}^{\infty} \theta_{2}(b) R_{1}^{*}(b) \mathrm{d} b \\
\left(\lambda+\mu_{m}\right) \tilde{v}_{2}=\alpha_{2}\left(\tilde{i}_{2}+\tilde{y}_{2}\right) U_{1}^{*},
\end{array}\right.
$$

and characteristic equation

$$
\begin{align*}
\mathcal{G}_{2}(\lambda)= & \left(\lambda+\mu_{m}\right)\left(\lambda+\gamma_{2}+d_{2}+\mu_{h}\right)-\left\{\alpha_{2} \beta_{2} U_{1}^{*} \int_{0}^{\infty} \theta_{2}(b) R_{1}^{*}(b) \mathrm{d} b\right. \\
& \left.-\frac{\alpha_{2} \beta_{2} \mu_{m}\left(\gamma_{1}+\mu_{h}\right)\left(\lambda+\gamma_{2}+d_{2}+\mu_{h}\right)}{\alpha_{1} \beta_{1} K_{1}\left(\lambda+\gamma_{2}+\mu_{h}\right)} \int_{0}^{\infty} \varepsilon_{2}(a) e^{-\int_{0}^{a}\left(\lambda+\mu_{h}+\varepsilon_{2}(s)\right) \mathrm{d} s} \mathrm{~d} a\right\}  \tag{4.12}\\
= & \mathcal{G}_{3}(\lambda)-\mathcal{G}_{4}(\lambda)=0 .
\end{align*}
$$

Suppose that

$$
\mathcal{R}_{2}>\mathcal{R}_{1}^{*}=\mathcal{R}_{1}\left(1-\frac{\alpha_{2} \beta_{2} U_{1}^{*} \int_{0}^{\infty} \theta_{2}(b) R_{1}^{*}(b) \mathrm{d} b}{\mu_{m}\left(\gamma_{2}+d_{2}+\mu_{h}\right)}\right),
$$

then, $\mathcal{G}_{2}(0)=\mathcal{R}_{1} \mu_{m}\left(\gamma_{2}+d_{2}+\mu_{h}\right)\left(\mathcal{R}_{1}^{*}-\mathcal{R}_{2}\right)<0$. Furthermore, it is easy to verify that $\mathcal{G}_{2}(\lambda)$ is increasing with $\lambda$, and $\lim _{\lambda \rightarrow+\infty} \mathcal{G}_{2}(\lambda)=+\infty$. Hence, the equation (4.12) at least has a positive real root, which implies that $P_{1}$ is unstable.

On the other hand, if $\mathcal{R}_{2}<\mathcal{R}_{1}^{*}$, then

$$
\begin{aligned}
& \left|\mathcal{G}_{3}(\lambda)\right| \geq \mu_{m}\left(\gamma_{2}+d_{2}+\mu_{h}\right), \\
& \left|\mathcal{G}_{4}(\lambda)\right| \leq \mathcal{G}_{4}(\operatorname{Re} \lambda) \leq \mathcal{G}_{4}(0)=\mu_{m}\left(\gamma_{2}+d_{2}+\mu_{h}\right) \frac{\mathcal{R}_{2}}{\mathcal{R}_{1}}+\alpha_{2} \beta_{2} U_{1}^{*} \int_{0}^{\infty} \theta_{2}(b) R_{1}^{*}(b) \mathrm{d} b<\left|\mathcal{G}_{3}(\lambda)\right|,
\end{aligned}
$$

for the eigenvalue $\lambda$ with $\operatorname{Re}(\lambda) \geq 0$. This leads to a contradiction. Hence, all eigenvalues of equation (4.12) have negative real parts when $\mathcal{R}_{2}<\mathcal{R}_{1}^{*}$. In this case, the stability of $P_{1}$ depends on the eigenvalues of the following problem,

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} s(t)}{\mathrm{d} t}=-\beta_{1} S_{1}^{*} v_{1}(t)-\beta_{1} s(t) V_{1}^{*}-\mu_{h} s(t),  \tag{4.13}\\
\frac{\mathrm{d} i_{1}(t)}{\mathrm{d} t}=\int_{0}^{\infty} \varepsilon_{1}(a) e_{1}(t, a) \mathrm{d} a-\left(\gamma_{1}+\mu_{h}\right) i_{1}(t), \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) e_{1}(t, a)=-\left(\mu_{h}+\varepsilon_{1}(a)\right) e_{1}(t, a), \quad e_{1}(t, 0)=\beta_{1} S_{1}^{*} v_{1}(t)+\beta_{1} s(t) V_{1}^{*}, \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial b}\right) r_{1}(t, b)=-\mu_{h} r_{1}(t, b), \quad r_{1}(t, 0)=\gamma_{1} i_{1}(t), \\
\frac{\mathrm{d} y_{1}(t)}{\mathrm{d} t}=\beta_{1} V_{1}^{*} \int_{0}^{\infty} \theta_{1}(b) r_{2}(t, b) \mathrm{d} b-\left(\gamma_{1}+d_{1}+\mu_{h}\right) y_{1}, \\
\frac{\mathrm{~d} u(t)}{\mathrm{d} t}=-\alpha_{1}\left(i_{1}(t)+y_{1}(t)\right) U_{1}^{*}-\alpha_{1} u(t) I_{1}^{*}-\mu_{m} u(t), \\
\frac{\mathrm{d} v_{1}(t)}{\mathrm{d} t}=\alpha_{1}\left(i_{1}(t)+y_{1}(t)\right) U_{1}^{*}+\alpha_{1} u(t) I_{1}^{*}-\mu_{m} v_{1}(t) .
\end{array}\right.
$$

The corresponding characteristic equation of problem (4.13) is as follow

$$
\begin{align*}
\left(\lambda+\gamma_{1}+\mu_{h}\right)\left(\lambda+\beta_{1} V_{1}^{*}+\mu_{h}\right)(\lambda+ & \left.\alpha_{1} I_{1}^{*}+\mu_{m}\right) \\
& =\alpha_{1} \beta_{1} S_{1}^{*} U_{1}^{*}\left(\lambda+\mu_{h}\right) \int_{0}^{\infty} \varepsilon_{1}(a) e^{-\int_{0}^{a}\left(\lambda+\varepsilon_{1}(a)+\mu_{h}\right) \mathrm{ds}} \mathrm{~d} a . \tag{4.14}
\end{align*}
$$

Dividing both sides of (4.14) by $\left(\lambda+\mu_{h}\right)\left(\lambda+\mu_{m}\right)$, we obtain

$$
\frac{\left(\lambda+\gamma_{1}+\mu_{h}\right)\left(\lambda+\beta_{1} V_{1}^{*}+\mu_{h}\right)\left(\lambda+\alpha_{1} I_{1}^{*}+\mu_{m}\right)}{\left(\lambda+\mu_{h}\right)\left(\lambda+\mu_{m}\right)}=\frac{\mu_{m}\left(\gamma_{1}+\mu_{h}\right)}{\left(\lambda+\mu_{m}\right) K_{1}} \int_{0}^{\infty} \varepsilon_{1}(a) e^{-\int_{0}^{a}\left(\lambda+\varepsilon_{1}(a)+\mu_{h}\right) \mathrm{ds}} \mathrm{~d} a,
$$

where, we also use the expressions of $S_{1}^{*}$ and $U_{1}^{*}$. Denote

$$
\begin{aligned}
\mathcal{G}_{5}(\lambda) & =\frac{\left(\lambda+\gamma_{1}+\mu_{h}\right)\left(\lambda+\beta_{1} V_{1}^{*}+\mu_{h}\right)\left(\lambda+\alpha_{1} I_{1}^{*}+\mu_{m}\right)}{\left(\lambda+\mu_{h}\right)\left(\lambda+\mu_{m}\right)}, \\
\mathcal{G}_{6}(\lambda) & =\frac{\mu_{m}\left(\gamma_{1}+\mu_{h}\right)}{\left(\lambda+\mu_{m}\right) K_{1}} \int_{0}^{\infty} \varepsilon_{1}(a) e^{-\int_{0}^{a}\left(\lambda+\varepsilon_{1}(a)+\mu_{h}\right) \mathrm{d} s} \mathrm{~d} a .
\end{aligned}
$$

If $\lambda$ is a root of equation (4.14) with $\operatorname{Re} \lambda \geq 0$, then one further have

$$
\left|\mathcal{G}_{5}(\lambda)\right|>\gamma_{1}+\mu_{h},\left|\mathcal{G}_{6}(\lambda)\right| \leq\left|\mathcal{G}_{6}(\operatorname{Re} \lambda)\right| \leq \mathcal{G}_{6}(0)=\gamma_{1}+\mu_{h}<\left|\mathcal{G}_{5}(\lambda)\right|,
$$

which leads to a contradiction. Hence, equation (4.14) has no any root with nonnegative real part. This shows that characteristic equation corresponding to model (4.10) has only roots with negative real parts. Consequently, the boundary equilibrium $P_{1}$ is locally asymptotically stable if $\mathcal{R}_{1}>1$ and $\mathcal{R}_{2}<\mathcal{R}_{1}^{*}$. To sum up, the following results are true.
Theorem 4.5. Assume $\mathcal{R}_{1}>1$, the boundary equilibrium $P_{1}$ of model (2.1) is locally asymptotically stable when $\mathcal{R}_{2}<\mathcal{R}_{1}^{*}$. Moreover, if the inequality is reversed, then $P_{1}$ is unstable.
Remark 4.6. If $\theta_{2}(b)=0$ for all $b \geq 0$, then there is perfect cross-immunity and primary infection with strain 1 prevents secondary infection with strain 2. In this case, from Theorem 4.5 we have that the boundary equilibrium $P_{1}$ is locally asymptotically stable when $\mathcal{R}_{1}>1$ and $\mathcal{R}_{1}>\mathcal{R}_{2}$, i.e., strain 1 is dominant.

On the boundary equilibrium $P_{2}$, we can also obtain similar result as follows.
Theorem 4.7. Assume $\mathcal{R}_{2}>1$, the boundary equilibrium $P_{2}$ of model (2.1) is locally asymptotically stable when

$$
\mathcal{R}_{1}<\mathcal{R}_{2}^{*}=\mathcal{R}_{2}\left(1-\frac{\alpha_{1} \beta_{1} U_{2}^{*} \int_{0}^{\infty} \theta_{1}(b) R_{2}^{*}(b) \mathrm{d} b}{\mu_{m}\left(\gamma_{1}+d_{1}+\mu_{h}\right)}\right) .
$$

Moreover, if $\mathcal{R}_{1} \geq \mathcal{R}_{2}^{*}$, then $P_{2}$ is unstable.
Remark 4.8. If $\theta_{1}(b)=0$ for all $b \geq 0$, then there is perfect cross-immunity and primary infection with strain 2 prevents secondary infection with strain 1. In this case, from Theorem 4.7 we have that the boundary equilibrium $P_{2}$ is locally asymptotically stable when $\mathcal{R}_{2}>1$ and $\mathcal{R}_{2}>\mathcal{R}_{1}$, i.e., strain 2 is dominant.

Remark 4.9. Based on Remark 4.6 and Remark 4.8, we can conclude the following results. That is, if $\theta_{1}(b)=\theta_{2}(b)=0$ for all $b \geq 0$, then there is no secondary infection in model (2.1) and there is competitive exclusion between strain 1 and strain 2 .

## 5 Uniform persistence

Define $\hat{\mathbb{X}}=L_{+}^{1}(0, \infty) \times L_{+}^{1}(0, \infty) \times \mathbb{R}_{+}^{6}$ and

$$
\begin{aligned}
& \hat{\mathcal{Y}}=\left\{\left(E_{1}(\cdot), E_{2}(\cdot), I_{1}, I_{2}, Y_{1}, Y_{2}, V_{1}, V_{2}\right) \in \hat{\mathbb{X}}: \int_{0}^{\bar{a}_{i}} E_{i}(\cdot, a) \mathrm{d} a+I_{i}(\cdot)+Y_{i}(\cdot)+V_{i}(\cdot)>0, i=1,2\right\}, \\
& \mathcal{Y}=\mathbb{R}_{+} \times \hat{\mathcal{Y}} \times L_{+}^{1}(0, \infty) \times L_{+}^{1}(0, \infty) \times \mathbb{R}_{+} .
\end{aligned}
$$

Obviously, $\partial \mathcal{Y}=\mathbb{X} \backslash \mathcal{Y}$ and

$$
\begin{aligned}
& \partial \hat{\mathcal{Y}}=\hat{\mathbb{X}} \backslash \hat{\mathcal{Y}}=\left\{\left(E_{1}(\cdot), E_{2}(\cdot), I_{1}, I_{2}, Y_{1}, Y_{2}, V_{1}, V_{2}\right) \in \hat{\mathbb{X}}: \int_{0}^{\bar{a}_{1}} E_{1}(\cdot, a) \mathrm{d} a+I_{1}(\cdot)+Y_{1}(\cdot)\right. \\
&\left.+V_{1}(\cdot)=0 \text { or } \int_{0}^{\bar{a}_{2}} E_{2}(\cdot, a) \mathrm{d} a+I_{2}(\cdot)+Y_{2}(\cdot)+V_{2}(\cdot)=0\right\}, \\
& \partial \hat{\mathcal{Y}}_{0}=\left\{\left(E_{1}(\cdot), E_{2}(\cdot), I_{1}, I_{2}, Y_{1}, Y_{2}, V_{1}, V_{2}\right) \in \hat{\mathbb{X}}:\right. \\
&\left.\int_{0}^{\bar{a}_{i}} E_{i}(\cdot, a) \mathrm{d} a+I_{i}(\cdot)+Y_{i}(\cdot)+V_{i}(\cdot)=0, i=1,2\right\}, \\
& \partial \hat{\mathcal{Y}}_{i}=\left\{\left(E_{1}(\cdot), E_{2}(\cdot), I_{1}, I_{2}, Y_{1}, Y_{2}, V_{1}, V_{2}\right) \in \hat{\mathbb{X}}: \int_{0}^{\bar{a}_{i}} E_{i}(\cdot, a) \mathrm{d} a+I_{i}(\cdot)+Y_{i}(\cdot)\right. \\
&\left.+V_{i}(\cdot)>0, \int_{0}^{\bar{a}_{j}} E_{j}(\cdot, a) \mathrm{d} a+I_{j}(\cdot)+Y_{j}(\cdot)+V_{j}(\cdot)=0\right\}, \quad i, j=1,2, i \neq j .
\end{aligned}
$$

It is clear that

$$
\partial \mathcal{Y}=\partial \mathcal{Y}_{0} \cup \partial \mathcal{Y}_{1} \cup \partial \mathcal{Y}_{2}, \quad \partial \mathcal{Y}_{i}=\mathbb{R}_{+} \times \partial \hat{\mathcal{Y}}_{i} \times L_{+}^{1}(0, \infty) \times L_{+}^{1}(0, \infty) \times \mathbb{R}_{+}, i=0,1,2
$$

Theorem 5.1. If $\mathcal{R}_{1}>1, \mathcal{R}_{2}>1, \mathcal{R}_{2}>\mathcal{R}_{1}^{*}$ and $\mathcal{R}_{1}>\mathcal{R}_{2}^{*}$, then the semi-flow $\{\Phi(t)\}_{t \geq 0}$ is uniformly persistent with respect to the pair $(\mathcal{Y}, \partial \mathcal{Y})$, i.e., the disease of model (2.1) is uniformly persistent.

Proof. We prove, firstly, the following conclusions:
(i) The disease-free equilibrium $P_{0}$ is globally asymptotically stable for semi-flow $\{\Phi(t)\}_{t \geq 0}$ restricted to $\partial \mathcal{Y}_{0}$.
(ii) The boundary equilibrium $P_{i}$ is globally asymptotically stable and $P_{0}$ is unstable for model (2.1) restricted to $\partial \mathcal{Y}_{i}$ when $\mathcal{R}_{i}>1, i=1,2$.

For conclusion $(i)$. If model (2.1) is restricted to $\partial \mathcal{Y}_{0}$, then it degenerates into

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} S(t)}{\mathrm{d} t}=\Lambda_{h}-\mu_{h} S(t)  \tag{5.1}\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial b}\right) R_{i}(t, b)=-\mu_{h} R_{i}(t, b), R_{i}(t, 0)=0, i=1,2 \\
\frac{\mathrm{~d} U(t)}{\mathrm{d} t}=\Lambda_{m}(t)-\mu_{m} U(t)
\end{array}\right.
$$

We can obtain that $\lim _{t \rightarrow+\infty} S(t)=\Lambda_{h} / \mu_{h}, \lim _{t \rightarrow+\infty} R_{i}(t, b)=0$ and $\lim _{t \rightarrow+\infty} U(t)=\Lambda_{m} / \mu_{m}$. Therefore, $P_{0}$ is globally asymptotically stable for model (2.1) restricted to $\partial \mathcal{Y}_{0}$. Then the conclusion $(i)$ is true.

For conclusion (ii). If model (2.1) restricted in $\partial \mathcal{Y}_{1}$, then it degenerates into

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} S(t)}{\mathrm{d} t}=\Lambda_{h}-\beta_{1} S(t) V_{1}(t)-\mu_{h} S(t),  \tag{5.2}\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) E_{1}(t, a)=-\left(\mu_{h}+\varepsilon_{1}(a)\right) E_{1}(t, a), E_{1}(t, 0)=\beta_{1} S(t) V_{1}(t), \\
\frac{\mathrm{d} I_{1}(t)}{\mathrm{d} t}=\int_{0}^{\infty} \varepsilon_{1}(a) E_{1}(t, a) \mathrm{d} a-\left(\gamma_{1}+\mu_{h}\right) I_{1}(t), \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial b}\right) R_{1}(t, b)=-\mu_{h} R_{1}(t, b), R_{1}(t, 0)=\gamma_{1} I_{1}(t), \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial b}\right) R_{2}(t, b)=-\left(\beta_{1} \theta_{1}(b) V_{1}(t)+\mu_{h}\right) R_{2}(t, b), R_{2}(t, 0)=0, \\
\frac{\mathrm{~d} Y_{1}(t)}{\mathrm{d} t}=\beta_{1} V_{1}(t) \int_{0}^{\infty} \theta_{1}(b) R_{2}(t, b) \mathrm{d} b-\left(\gamma_{1}+d_{1}+\mu_{h}\right) Y_{1}(t), \\
\frac{\mathrm{d} Y_{2}(t)}{\mathrm{d} t}=-\left(\gamma_{2}+d_{2}+\mu_{h}\right) Y_{2}(t), \\
\frac{\mathrm{d} U(t)}{\mathrm{d} t}=\Lambda_{m}(t)-\alpha_{1}\left(I_{1}(t)+Y_{1}(t)\right) U(t)-\mu_{m} U(t), \\
\frac{\mathrm{d} V_{1}(t)}{\mathrm{d} t}=\alpha_{1}\left(I_{1}(t)+Y_{1}(t)\right) U(t)-\mu_{m} V_{1}(t) .
\end{array}\right.
$$

Obviously, $\lim _{t \rightarrow+\infty} R_{2}(t, b)=\lim _{t \rightarrow+\infty} Y_{1}(t)=\lim _{t \rightarrow+\infty} Y_{2}(t)=0$. Since the equation of $R_{1}(t, b)$ is decoupled from the other equations in model (5.2), we can consider the following
system

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} S(t)}{\mathrm{d} t}=\Lambda_{h}-\beta_{1} S(t) V_{1}(t)-\mu_{h} S(t)  \tag{5.3}\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) E_{1}(t, a)=-\left(\mu_{h}+\varepsilon_{1}(a)\right) E_{1}(t, a), E_{1}(t, 0)=\beta_{1} S(t) V_{1}(t) \\
\frac{\mathrm{d} I_{1}(t)}{\mathrm{d} t}=\int_{0}^{\infty} \varepsilon_{1}(a) E_{1}(t, a) \mathrm{d} a-\left(\gamma_{1}+\mu_{h}\right) I_{1}(t) \\
\frac{\mathrm{d} U(t)}{\mathrm{d} t}=\Lambda_{m}(t)-\alpha_{1} I_{1}(t) U(t)-\mu_{m} U(t) \\
\frac{\mathrm{d} V_{1}(t)}{\mathrm{d} t}=\alpha_{1} I_{1}(t) U(t)-\mu_{m} V_{1}(t)
\end{array}\right.
$$

Model (5.3) has the equilibrium $\left(S_{1}^{*}, E_{1}^{*}(a), I_{1}^{*}, U_{1}^{*}, V_{1}^{*}\right)$. Define Lyapunov functional

$$
\mathcal{W}(t)=\mathcal{W}_{1}(t)+\mathcal{W}_{2}(t)+\mathcal{W}_{3}(t)+\mathcal{W}_{4}(t)+\mathcal{W}_{5}(t),
$$

where

$$
\begin{aligned}
& \mathcal{W}_{1}(t)=K_{1} S_{1}^{*} \phi\left(\frac{S(t)}{S_{1}^{*}}\right), \quad \mathcal{W}_{2}(t)=\int_{0}^{\infty} q_{1}(a) E_{1}^{*}(a) \phi\left(\frac{E_{1}(t, a)}{E_{1}^{*}(a)}\right) \mathrm{d} a, \\
& \mathcal{W}_{3}(t)=I_{1}^{*} \phi\left(\frac{I_{1}(t)}{I_{1}^{*}}\right), \quad \mathcal{W}_{4}(t)=\frac{K_{1} E_{1}^{*}(0)}{\alpha_{1} I_{1}^{*}} \phi\left(\frac{U(t)}{U_{1}^{*}}\right), \quad \mathcal{W}_{5}(t)=\frac{K_{1} E_{1}^{*}(0) V_{1}^{*}}{\alpha_{1} I_{1}^{*} U_{1}^{*}} \phi\left(\frac{V_{1}(t)}{V_{1}^{*}}\right),
\end{aligned}
$$

with $\phi(x)=x-1-\ln x$. Then, it yields that

$$
\begin{aligned}
\frac{\mathrm{d} \mathcal{W}_{1}(t)}{\mathrm{d} t}= & K_{1} S_{1}^{*}\left(\frac{1}{S_{1}^{*}}-\frac{1}{S(t)}\right)\left[\Lambda_{h}-\frac{\Lambda_{h} S(t)}{S_{1}^{*}}+\beta_{1} S(t) V_{1}^{*}-\beta_{1} S(t) V_{1}(t)\right] \\
= & -\frac{\Lambda_{h}\left(S(t)-S_{1}^{*}\right)^{2} K_{1}}{S(t) S_{1}^{*}}+K_{1} \beta_{1} S_{1}^{*} V_{1}^{*}\left(\frac{1}{S_{1}^{*}}-\frac{1}{S(t)}\right)\left(S(t)-\frac{S(t) V_{1}(t)}{V_{1}^{*}}\right) \\
= & -K_{1} S_{1}^{*}\left(\beta_{1} V_{1}^{*}+\mu_{h}\right)\left(\phi\left(\frac{S(t)}{S_{1}^{*}}\right)+\phi\left(\frac{S_{1}^{*}}{S(t)}\right)\right) \\
& +K_{1} \beta_{1} S_{1}^{*} V_{1}^{*}\left(\frac{S(t)}{S_{1}^{*}}-1-S(t) V_{1}(t) S_{1}^{*} V_{1}^{*}+V_{1}(t) V_{1}^{*}\right), \\
\frac{\mathrm{d} \mathcal{W}_{2}(t)}{\mathrm{d} t}= & \int_{0}^{\infty} q_{1}(a) E_{1}^{*}(a) \frac{\partial}{\partial t} \phi\left(\frac{E_{1}(t, a)}{E_{1}^{*}(a)}\right) \mathrm{d} a \\
= & -\int_{0}^{\infty} q_{1}(a) E_{1}^{*}(a)\left(\frac{E_{1}(t, a)}{E_{1}^{*}(a)}-1\right)\left(\frac{E_{1 a}(t, a)}{E_{1}(t, a)}+\mu_{h}+\varepsilon_{1}(a)\right) \mathrm{d} a
\end{aligned}
$$

where $E_{1 a}(t, a)=\mathrm{d} E_{1}(t, a) / \mathrm{d} a$. Since

$$
\frac{\partial}{\partial a} \phi\left(\frac{E_{1}(t, a)}{E_{1}^{*}(a)}\right)=\left(\frac{E_{1}(t, a)}{E_{1}^{*}(a)}-1\right)\left(\frac{E_{1 a}(t, a)}{E_{1}(t, a)}+\mu_{h}+\varepsilon_{1}(a)\right)
$$

then

$$
\begin{aligned}
\frac{\mathrm{d} \mathcal{W}_{2}(t)}{\mathrm{d} t}= & -\int_{0}^{\infty} q_{1}(a) E_{1}^{*}(a) \frac{\partial}{\partial a} \phi\left(\frac{E_{1}(t, a)}{E_{1}^{*}(a)}\right) \mathrm{d} a \\
= & -\left.q_{1}(a) E_{1}^{*}(a) \phi\left(\frac{E_{1}(t, a)}{E_{1}^{*}(a)}\right)\right|_{a=\infty}+q_{1}(0) E_{1}^{*}(0) \phi\left(\frac{E_{1}(t, 0)}{E_{1}^{*}(0)}\right) \\
& -\int_{0}^{\infty} \varepsilon_{1}(a) E_{1}^{*}(a) \phi\left(\frac{E_{1}(t, a)}{E_{1}^{*}(a)}\right) \mathrm{d} a \\
= & K_{1} E_{1}(t, 0)-K_{1} E_{1}^{*}(0)\left(1-\ln \left(\frac{E_{1}(t, 0)}{E_{1}^{*}(0)}\right)\right)-\int_{0}^{\infty} \varepsilon_{1}(a) E_{1}^{*}(a) \phi\left(\frac{E_{1}(t, a)}{E_{1}^{*}(a)}\right) \mathrm{d} a
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\frac{\mathrm{d} \mathcal{W}_{3}(t)}{\mathrm{d} t} & =\int_{0}^{\infty} \varepsilon_{1}(a)\left(1-\frac{I_{1}^{*}}{I_{1}(t)}\right)\left(E_{1}(t, a)-E_{1}^{*}(a) \frac{I_{1}(t)}{I_{1}^{*}}\right) \mathrm{d} a \\
& =\int_{0}^{\infty} \varepsilon_{1}(a) E_{1}^{*}(a)\left(\frac{E_{1}(t, a)}{E_{1}^{*}(a)}-\frac{I_{1}^{*}}{I_{1}(t)}-\frac{I_{1}^{*} E_{1}(t, a)}{I_{1}(t) E_{1}^{*}(a)}+1\right) \mathrm{d} a, \\
\frac{\mathrm{~d} \mathcal{W}_{4}(t)}{\mathrm{d} t} & =\frac{K_{1} E_{1}^{*}(0)}{\alpha_{1} I_{1}^{*}}\left(\frac{1}{U_{1}^{*}}-\frac{1}{U(t)}\right)\left(\mu_{m}\left(U_{1}^{*}-U(t)\right)+\alpha_{1} U(t) I_{1}^{*}-\alpha_{1} U(t) I_{1}(t)\right) \\
& =-\frac{K_{1} E_{1}^{*}(0) \mu_{m}\left(U(t)-U_{1}^{*}\right)^{2}}{U(t) I_{1}^{*} U_{1}^{*}}+K_{1} E_{1}^{*}(0)\left(1-\frac{U(t)}{U_{1}^{*}}+\frac{I_{1}(t)}{I_{1}^{*}}-\frac{U(t) I_{1}(t)}{U_{1}^{*} I_{1}^{*}}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d} \mathcal{W}_{5}(t)}{\mathrm{d} t} & =\frac{K_{1} E_{1}^{*}(0) V_{1}^{*}}{\alpha_{1} I_{1}^{*} U_{1}^{*}}\left(\frac{1}{V_{1}^{*}}-\frac{1}{V_{1}(t)}\right)\left(\alpha_{1} U(t) I_{1}(t)-\alpha_{1} U_{1}^{*} I_{1}^{*} \frac{V_{1}(t)}{V_{1}^{*}}\right) \\
& =K_{1} E_{1}^{*}(0)\left(\frac{I_{1}(t) U(t)}{I_{1}^{*} U_{1}^{*}}-\frac{V_{1}(t)}{V_{1}^{*}}-\frac{U(t) I_{1}(t) V_{1}^{*}}{U_{1}^{*} I_{1}^{*} V_{1}(t)}+1\right) .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\mathrm{d} \mathcal{W}(t)}{\mathrm{d} t}= & -K_{1} S_{1}^{*}\left(\beta_{1} V_{1}^{*}+\mu_{h}\right)\left[\phi\left(\frac{S(t)}{S_{1}^{*}}\right)+\phi\left(\frac{S_{1}^{*}}{S(t)}\right)\right]+K_{1} \beta_{1} S_{1}^{*} V_{1}^{*}\left(\frac{S(t)}{S_{1}^{*}}+\frac{V_{1}(t)}{V_{1}^{*}}\right. \\
& \left.-\ln \frac{S(t) V_{1}(t)}{S_{1}^{*} V_{1}^{*}}-2\right)+\int_{0}^{\infty} \varepsilon_{1}(a) E_{1}^{*}(a)\left(2-\frac{I_{1}(t)}{I_{1}^{*}}-\frac{I_{1}^{*} E_{1}(t, a)}{I_{1}(t) E_{1}^{*}(a)}+\ln \frac{E_{1}(t, a)}{E_{1}^{*}(a)}\right) \mathrm{d} a \\
& -\frac{K_{1} E_{1}^{*}(0) \mu_{m}\left(U(t)-U_{1}^{*}\right)^{2}}{U(t) I_{1}^{*} U_{1}^{*}}+K_{1} E_{1}^{*}(0)\left(2-\frac{U(t)}{U_{1}^{*}}+\frac{I_{1}(t)}{I_{1}^{*}}-\frac{V_{1}(t)}{V_{1}^{*}}-\frac{U(t) I_{1}(t) V_{1}^{*}}{U_{1}^{*} I_{1}^{*} V_{1}(t)}\right) \\
= & -K_{1} S_{1}^{*}\left(\beta_{1} V_{1}^{*}+\mu_{h}\right)\left[\phi\left(\frac{S(t)}{S_{1}^{*}}\right)+\phi\left(\frac{S_{1}^{*}}{S(t)}\right)\right]+K_{1} S_{1}^{*} \beta_{1} V_{1}^{*}\left[\phi\left(\frac{S(t)}{S_{1}^{*}}\right)+\phi\left(\frac{V_{1}^{*}}{V_{1}(t)}\right)\right] \\
& -\int_{0}^{\infty} \varepsilon_{1}(a) E_{1}^{*}(a)\left[\phi\left(\frac{E_{1}(t, a) I_{1}^{*}}{E_{1}^{*}(a) I_{1}(t)}\right)+\phi\left(\frac{I_{1}(t)}{I_{1}^{*}}\right)\right] \mathrm{d} a-\frac{K_{1} E_{1}^{*}(0) \mu_{m}\left(U(t)-U_{1}^{*}\right)^{2}}{U(t) I_{1}^{*} U_{1}^{*}} \\
& -K_{1} E_{1}^{*}(0)\left[\phi\left(\frac{U_{1}^{*}}{U(t)}\right)+\phi\left(\frac{V_{1}(t)}{V_{1}^{*}}\right)+\phi\left(\frac{U(t) I_{1}(t) V_{1}^{*}}{U_{1}^{*} I_{1}^{*} V_{1}(t)}\right)\right]+K_{1} E_{1}^{*}(0) \phi\left(\frac{I_{1}(t)}{I_{1}^{*}}\right) \\
= & -K_{1} S_{1}^{*} \beta_{1} V_{1}^{*} \phi\left(\frac{S_{1}^{*}}{S(t)}\right)-K_{1} S_{1}^{*} \mu_{h}\left[\phi\left(\frac{S(t)}{S_{1}^{*}}\right)+\phi\left(\frac{S_{1}^{*}}{S(t)}\right)\right] \\
& -\int_{0}^{\infty} \varepsilon_{1}(a) E_{1}^{*}(a) \phi\left(\frac{E_{1}(t, a) I_{1}^{*}}{E_{1}^{*}(a) I_{1}(t)}\right) \mathrm{d} a-\frac{K_{1} E_{1}^{*}(0) \mu_{m}\left(U(t)-U_{1}^{*}\right)^{2}}{U(t) I_{1}^{*} U_{1}^{*}} \\
& -K_{1} E_{1}^{*}(0)\left[\phi\left(\frac{U_{1}^{*}}{U(t)}\right)+\phi\left(\frac{V_{1}(t)}{V_{1}^{*}}\right)+\phi\left(\frac{U(t) I_{1}(t) V_{1}^{*}}{U_{1}^{*} I_{1}^{*} V_{1}(t)}\right)\right] \leq 0 .
\end{aligned}
$$

It is clear that $\mathrm{d} \mathcal{W}(t) / \mathrm{d} t \leq 0$ for any $S(t)>0, E_{1}(t, a)>0, I_{1}(t)>0, U(t)>0$ and $V_{1}(t)>0$, and $\mathrm{d} \mathcal{W}(t) / \mathrm{d} t=0$ implies that $\left(S(t), E_{1}(t, a), I_{1}(t), U(t), V_{1}(t)\right) \equiv\left(S_{1}^{*}, E_{1}^{*}(a), I_{1}^{*}, U_{1}^{*}, V_{1}^{*}\right)$ for all $t>0$. Thus, by LaSalle's invariance principle, equilibrium $\left(S_{1}^{*}, E_{1}^{*}(a), I_{1}^{*}, U_{1}^{*}, V_{1}^{*}\right)$ is globally asymptotically stable for model (5.3) when $\mathcal{R}_{1}>1$. From model (5.2), we easily obtain that $\lim _{t \rightarrow \infty} R_{1}(t, b)=R_{1}^{*}(b)$. This shows that equilibrium $P_{1}$ is globally asymptotically stable for model (2.1) restricted to $\partial \mathcal{Y}_{1}$ when $\mathcal{R}_{1}>1$. Moreover, from Theorem 4.2, we can obtain that $P_{0}$ is unstable for model (2.1) restricted to $\partial \mathcal{Y}_{1}$ when $\mathcal{R}_{1}>1$. That is, conclusion (ii) is hold.

Similarly, we can show that $P_{2}$ is globally asymptotically stable and $P_{0}$ is unstable for model (2.1) restricted to $\partial \mathcal{Y}_{2}$ when $\mathcal{R}_{2}>1$.

Next, we claim that $W^{s}\left(P_{0}\right) \cap \mathcal{Y}=\varnothing, W^{s}\left(P_{i}\right) \cap \mathcal{Y}=\varnothing, i=1$, 2, where $W^{s}\left(P_{0}\right)=\left\{X_{0} \in \mathcal{Y}\right.$ : $\left.\lim _{t \rightarrow \infty} X\left(t, X_{0}\right)=P_{0}\right\}$ and $W^{s}\left(P_{i}\right)=\left\{X_{0} \in \mathcal{Y}: \lim _{t \rightarrow \infty} X\left(t, X_{0}\right)=P_{i}\right\}, i=1,2$.

For $W^{s}\left(P_{0}\right) \cap \mathcal{Y}=\varnothing$. Suppose that there exists a $X_{0} \in \mathcal{Y}$ such that $\lim _{t \rightarrow \infty} X\left(t, X_{0}\right)=P_{0}$. Then, for any constant $\epsilon>0$, there exists a $T_{0}>0$ such that

$$
\begin{aligned}
& S^{*}-\epsilon<S(t)<S^{*}+\epsilon, 0<E_{i}(t, a)<\epsilon, 0<I_{i}(t)<\epsilon, 0<R_{i}(t, b)<\epsilon, \\
& 0<Y_{i}(t)<\epsilon, U^{*}-\epsilon<U(t)<U^{*}+\epsilon, 0<V_{i}(t)<\epsilon, i=1,2
\end{aligned}
$$

for all $t>T_{0}$. From the third and eighth equations of model (2.1), it follows that

$$
\begin{aligned}
\frac{\mathrm{d} I_{i}(t)}{\mathrm{d} t} & =\int_{0}^{\infty} \varepsilon_{i}(a) E_{i}(t, a) \mathrm{d} a-\left(\gamma_{i}+\mu_{h}\right) I_{i}(t) \\
& \geq \beta_{i}\left(S^{*}-\epsilon_{1}\right) \int_{0}^{t} \varepsilon_{i}(a) V_{i}(t-a) \eta_{i}(a) \mathrm{d} a-\left(\gamma_{i}+\mu_{h}\right) I_{i}(t) \\
\frac{\mathrm{d} V_{i}(t)}{\mathrm{d} t} & \geq \alpha_{i} I_{i}(t) U(t)-\mu_{m} V_{i}(t) \geq \alpha_{i}\left(U^{*}-\epsilon\right) I_{i}(t)-\mu_{m} V_{i}(t)
\end{aligned}
$$

Let us take the Laplace transform of both sides of above inequalities. Since all functions above are bounded, the Laplace transform of the functions exist for $\lambda>0$. Denote the Laplace transform of the function $f(t)$ by $\mathcal{L}[f(t)]$. Using the convolution property of the Laplace transform, we obtain the following inequalities for $\mathcal{L}\left[I_{i}(t)\right]$ and $\mathcal{L}\left[V_{i}(t)\right]$,

$$
\left\{\begin{array}{l}
\lambda \mathcal{L}\left[I_{i}(t)\right]-I_{i}(0) \geq \beta_{i}\left(S^{*}-\epsilon\right) \int_{0}^{\infty} \varepsilon_{i}(a) \eta_{i}(a) e^{-\lambda a} \mathrm{~d} a \mathcal{L}\left[V_{i}(t)\right]-\left(\gamma_{i}+\mu_{h}\right) \mathcal{L}\left[I_{i}(t)\right] \\
\lambda \mathcal{L}\left[V_{i}(t)\right]-V_{i}(0) \geq \alpha_{i}\left(U^{*}-\epsilon\right) \mathcal{L}\left[I_{i}(t)\right]-\mu_{m} \mathcal{L}\left[V_{i}(t)\right]
\end{array}\right.
$$

Eliminating $\mathcal{L}\left[V_{i}(t)\right]$ yields

$$
\begin{equation*}
\mathcal{L}\left[I_{i}(t)\right] \geq \frac{\alpha_{i} \beta_{i}\left(S^{*}-\epsilon\right)\left(U^{*}-\epsilon\right) \int_{0}^{\infty} \varepsilon_{i}(a) \eta_{i}(a) e^{-\lambda a} \mathrm{~d} a}{\left(\lambda+\mu_{m}\right)\left(\lambda+\gamma_{i}+\mu_{h}\right)} \mathcal{L}\left[I_{i}(t)\right]+\frac{I_{i}(0)}{\lambda+\gamma_{i}+\mu_{h}} . \tag{5.4}
\end{equation*}
$$

Since $\mathcal{R}_{i}>1, i=1,2$, we can choose $\lambda$ and $\epsilon$ small enough such that

$$
\frac{\alpha_{i} \beta_{i}\left(S^{*}-\epsilon\right)\left(U^{*}-\epsilon\right) \int_{0}^{\infty} \varepsilon_{i}(a) \eta_{i}(a) e^{-\lambda a} \mathrm{~d} a}{\left(\lambda+\mu_{m}\right)\left(\lambda+\gamma_{i}+\mu_{h}\right)}>1, \quad i=1,2
$$

Therefore, inequality (5.4) does not hold. This implies that $W^{s}\left(P_{0}\right) \cap \mathcal{Y}=\varnothing$.
For $W^{s}\left(P_{1}\right) \cap \mathcal{Y}=\varnothing$. Suppose that there exists a $X_{1} \in \mathcal{Y}$ such that $\lim _{t \rightarrow \infty} X\left(t, X_{1}\right)=P_{1}$. Then, for any constant $\epsilon>0$ there exists a $T_{1}>0$ such that for all $t>T_{1}$ one have

$$
\begin{aligned}
& S_{1}^{*}-\epsilon_{1}<S(t)<S_{1}^{*}+\epsilon_{1}, \quad E_{1}^{*}-\epsilon_{1}<E_{1}(t, a)<E_{1}^{*}+\epsilon_{1}, \quad 0<E_{2}(t, a)<\epsilon_{1}, \\
& 0<I_{2}(t)<\epsilon_{1}, \quad I_{1}^{*}-\epsilon_{1}<I_{1}(t)<I_{1}^{*}+\epsilon_{1}, \quad R_{1}^{*}(b)-\epsilon_{1}<R_{1}(t, b)<R_{1}^{*}(b)+\epsilon_{1}, \\
& 0<R_{2}(t, b)<\epsilon_{1}, \quad 0<Y_{i}(t)<\epsilon_{1}, \quad U_{1}^{*}-\epsilon_{1}<U(t)<U_{1}^{*}+\epsilon_{1}, \\
& V_{1}^{*}-\epsilon_{1}<V_{1}(t)<V_{1}^{*}+\epsilon_{1}, \quad 0<V_{2}(t)<\epsilon_{1}, \quad i=1,2 .
\end{aligned}
$$

From model (2.1), we can obtain

$$
\left\{\begin{align*}
\frac{\mathrm{d} I_{2}(t)}{\mathrm{d} t} & =\int_{0}^{\infty} \varepsilon_{2}(a) E_{2}(t, a) \mathrm{d} a-\left(\gamma_{2}+\mu_{h}\right) I_{2}(t)  \tag{5.5}\\
& \geq \beta_{2}\left(S_{1}^{*}-\epsilon_{1}\right) \int_{0}^{t} \varepsilon_{2}(a) V_{2}(t-a) \eta_{2}(a) \mathrm{d} a-\left(\gamma_{2}+\mu_{h}\right) I_{2}(t), \\
\frac{\mathrm{d} Y_{2}(t)}{\mathrm{d} t} & =\beta_{2} V_{2}(t) \int_{0}^{\infty} \theta_{2}(b) R_{1}(t, b) \mathrm{d} b-\left(\gamma_{2}+d_{2}+\mu_{h}\right) Y_{2}(t) \\
& \geq \beta_{2} V_{2}(t) \int_{0}^{\infty} \theta_{2}(b)\left(R_{1}^{*}(b)-\epsilon_{1}\right) \mathrm{d} b-\left(\gamma_{2}+d_{2}+\mu_{h}\right) Y_{2}(t), \\
\frac{\mathrm{d} V_{2}(t)}{\mathrm{d} t} & \geq \alpha_{2}\left(U_{1}^{*}-\epsilon_{1}\right)\left(I_{2}(t)+\Upsilon_{2}(t)\right)-\mu_{m} V_{2}(t) .
\end{align*}\right.
$$

Take the Laplace transform of both sides of inequalities (5.5). Since all functions above are bounded, the Laplace transform of the functions exist for $\lambda>0$. Then, we can get the following inequalities for $\mathcal{L}\left[I_{2}(t)\right], \mathcal{L}\left[Y_{2}(t)\right]$ and $\mathcal{L}\left[V_{2}(t)\right]$,

$$
\left\{\begin{array}{l}
\lambda \mathcal{L}\left[I_{2}(t)\right]-I_{2}(0) \geq \beta_{2}\left(S_{1}^{*}-\epsilon_{1}\right) \int_{0}^{\infty} \varepsilon_{2}(a) \eta_{2}(a) e^{-\lambda a} \mathrm{~d} a \mathcal{L}\left[V_{2}(t)\right]-\left(\gamma_{2}+\mu_{h}\right) \mathcal{L}\left[I_{2}(t)\right], \\
\lambda \mathcal{L}\left[Y_{2}(t)\right]-Y_{2}(0) \geq \beta_{2} \int_{0}^{\infty} \theta_{2}(b)\left(R_{1}^{*}(b)-\epsilon_{1}\right) \mathrm{d} b \mathcal{L}\left[V_{2}(t)\right]-\left(\gamma_{2}+d_{2}+\mu_{h}\right) \mathcal{L}\left[Y_{2}(t)\right], \\
\lambda \mathcal{L}\left[V_{2}(t)\right]-V_{2}(0) \geq \alpha_{2}\left(U_{1}^{*}-\epsilon_{1}\right)\left(\mathcal{L}\left[I_{2}(t)\right]+\mathcal{L}\left[Y_{2}(t)\right]\right)-\mu_{m} \mathcal{L}\left[V_{2}(t)\right] .
\end{array}\right.
$$

Eliminating $\mathcal{L}\left[I_{2}(t)\right]$ and $\mathcal{L}\left[Y_{2}(t)\right]$ yields

$$
\begin{aligned}
\mathcal{L}\left[V_{2}(t)\right] \geq & \alpha_{2} \beta_{2}\left(U_{1}^{*}-\epsilon_{1}\right) \mathcal{L}\left[V_{2}(t)\right]\left\{\frac{\left(S_{1}^{*}-\epsilon_{1}\right) \int_{0}^{\infty} \varepsilon_{2}(a) \eta_{2}(a) \eta_{2}(a) e^{-\lambda a} \mathrm{~d} a}{\left(\lambda+\mu_{m}\right)\left(\lambda+\gamma_{2}+\mu_{h}\right)}\right. \\
& \left.+\frac{\int_{0}^{\infty} \theta_{2}(b)\left(R_{1}^{*}(b)-\epsilon_{1}\right) \mathrm{d} b}{\left(\lambda+\mu_{m}\right)\left(\lambda+\gamma_{2}+d_{2}+\mu_{h}\right)}\right\}+\frac{V_{2}(0)}{\lambda+\mu_{m}} \\
& +\frac{\alpha_{2}\left(U_{1}^{*}-\epsilon_{1}\right)}{\lambda+\mu_{m}}\left\{\frac{I_{2}(0)}{\lambda+\gamma_{2}+\mu_{h}}+\frac{Y_{2}(0)}{\lambda+\gamma_{2}+d_{2}+\mu_{h}}\right\} .
\end{aligned}
$$

This is impossible when $\mathcal{R}_{2}>\mathcal{R}_{1}^{*}$. By calculation, we have $S_{1}^{*} U_{1}^{*}=\mu_{m}\left(\gamma_{1}+\mu_{h}\right) / \alpha_{1} \beta_{1} K_{1}$, and

$$
\frac{\alpha_{2} \beta_{2} S_{1}^{*} U_{1}^{*} \int_{0}^{\infty} \varepsilon_{2} \eta_{2}(a) \mathrm{d} a}{\mu_{m}\left(\gamma_{2}+\mu_{h}\right)}+\frac{\alpha_{2} \beta_{2} U_{1}^{*} \int_{0}^{\infty} \theta_{2}(b) R_{1}^{*}(b) \mathrm{d} b}{\mu_{m}\left(\gamma_{2}+d_{2}+\mu_{h}\right)}=\frac{\mathcal{R}_{2}}{\mathcal{R}_{1}}+\frac{\alpha_{2} \beta_{2} U_{1}^{*} \int_{0}^{\infty} \theta_{2}(b) R_{1}^{*}(b) \mathrm{d} b}{\mu_{m}\left(\gamma_{2}+d_{2}+\mu_{h}\right)}>1 .
$$

Therefore, we can choose $\lambda$ and $\epsilon_{1}$ small enough such that

$$
\alpha_{2} \beta_{2}\left(U_{1}^{*}-\epsilon_{1}\right)\left\{\frac{\left(S_{1}^{*}-\epsilon_{1}\right) \int_{0}^{\infty} \varepsilon_{2}(a) \eta_{2}(a) \eta_{2}(a) e^{-\lambda a} \mathrm{~d} a}{\left(\lambda+\mu_{m}\right)\left(\lambda+\gamma_{2}+\mu_{h}\right)}+\frac{\int_{0}^{\infty} \theta_{2}(b)\left(R_{1}^{*}(b)-\epsilon_{1}\right) \mathrm{d} b}{\left(\lambda+\mu_{m}\right)\left(\lambda+\gamma_{2}+d_{2}+\mu_{h}\right)}\right\}>1 .
$$

This contradiction implies that $W^{s}\left(P_{1}\right) \cap \mathcal{Y}=\varnothing$.
Similarly, we can verify $W^{s}\left(P_{2}\right) \cap \mathcal{Y}=\varnothing$, when $\mathcal{R}_{1}>\mathcal{R}_{2}^{*}$. Thus, Theorem 4.2 in Hale and Waltman [18] implies the semi-flow $\{\Phi(t)\}_{t \geq 0}$ is uniformly persistent with respect to the pair $(\mathcal{Y}, \partial \mathcal{Y})$ if $\mathcal{R}_{1}>1, \mathcal{R}_{2}>1, \mathcal{R}_{1}>\mathcal{R}_{2}^{*}$ and $\mathcal{R}_{2}>\mathcal{R}_{1}^{*}$. This completes the proof.

As a consequence of Theorem 5.1, we have the following Corollary 5.2.
Corollary 5.2. If $\mathcal{R}_{1}>1, \mathcal{R}_{2}>1, \mathcal{R}_{1}>\mathcal{R}_{2}^{*}$ and $\mathcal{R}_{2}>\mathcal{R}_{1}^{*}$, then model (2.1) has at least a coexistence equilibrium denoted by $P_{3}=\left(\widetilde{S}^{*}, \widetilde{E}_{1}^{*}(a), \widetilde{E}_{2}^{*}(a), \widetilde{I}_{1}^{*}, \widetilde{I}_{2}^{*}, \widetilde{R}_{1}^{*}(b), \widetilde{R}_{2}^{*}(b), \widetilde{Y}_{1}^{*}, \widetilde{Y}_{2}^{*}, \widetilde{U}^{*}, \widetilde{V}_{1}^{*}, \widetilde{V}_{2}^{*}\right)$.

Based on the discussion in Section 4 and Section 5, we can conclude the existence and stability of the equilibria of model (2.1), as shown in Table 5.1. Here, LAS and GAS denote locally asymptotically stable and globally asymptotically stable, respectively.

Remark 5.3. It should be pointed out that the numerical simulations show that if the coexistence equilibrium of model (2.1) is existence, then it is stable. In fact, we have also obtained the sufficient conditions for the stability of the coexistence equilibrium by constructing the Lyapunov function. Due to additional technical conditions, we put this result in the appendix.

| Case | Existence or stability | Case | Existence or stability |
| :--- | :---: | :--- | :---: |
| $\mathcal{R}_{0}<1$ | $P_{0}$ is LAS | $\mathcal{R}_{0}<\min \left\{K_{1}, K_{2}\right\}$ | $P_{0}$ is GAS |
| $\mathcal{R}_{1}>1$ | $P_{1}$ exists | $\mathcal{R}_{2}>1$ | $P_{2}$ exists |
| $\mathcal{R}_{1}>1, \mathcal{R}_{2}<\mathcal{R}_{1}^{*}$ | $P_{1}$ is LAS | $\mathcal{R}_{2}>1, \mathcal{R}_{1}<\mathcal{R}_{2}^{*}$ | $P_{2}$ is LAS |
| $\mathcal{R}_{1}>1, \mathcal{R}_{2}>1$, | coexistence equilibrium |  |  |
| $\mathcal{R}_{2}>\mathcal{R}_{1}^{*}, \mathcal{R}_{1}>\mathcal{R}_{2}^{*}$ | exists |  |  |

Table 5.1: Summarizing the different scenarios depending on the threshold parameters.

## 6 Numerical simulation and discussions

In this section, some numerical simulations are conducted to illustrate our theoretical analysis results. Since the longer one stay in the latency stage, the more one is likely to exposed to the disease, and the risk of infection will increase, we assume that the age-dependent removal rate $\varepsilon_{i}(a)$ in model (2.1) takes the form $\varepsilon_{i}(a)=x_{i} a^{2} \exp \left(-y_{i} a\right)$, where $x_{i}, y_{i}>0, i=1,2$, see [21]. Similarly, in order to describe the primary recovery period and the level about losing cross vaccine protection, we choose cross immunity waning rate function as $\theta_{i}(b)=$ $u_{i}\left(1+5 \exp \left(-v_{i} b\right)\right)^{-1}$, where $u_{i}, v_{i}>0, i=1,2$. Furthermore, the values of other parameters of the model (2.1) are based on Refs. [6,36,42] and the references cited therein.

Example 6.1. The global asymptotic stability of the disease-free equilibrium of model (2.1).
We choose model parameters as follows: $\Lambda_{h}=25, \beta_{1}=2.38 \times 10^{-6}, \beta_{2}=2.25 \times 10^{-6}, \mu_{h}=$ $0.004, \gamma_{1}=\gamma_{2}=0.14, d_{1}=d_{2}=0.0001, \Lambda_{m}=21000, \alpha_{1}=3.75 \times 10^{-6}, \alpha_{2}=3.95 \times 10^{-6}, \mu_{m}=$ $0.09, \varepsilon_{1}(a)=0.01 a^{2} \exp (-0.2 a), \varepsilon_{2}(a)=0.01 \exp (-0.18 a) a^{2}, \theta_{1}(b)=0.45(1+5 \exp (-0.026 b))^{-1}$ and $\theta_{2}(b)=0.48(1+5 \exp (-0.026 b))^{-1}$ in model (2.1). By numerical calculations, we obtain $K_{1} \approx 0.882, K_{2} \approx 0.931$, and basic reproduction number $\mathcal{R}_{1} \approx 0.8687<K_{1}$ and $\mathcal{R}_{2} \approx 0.913<$ $K_{2}$. Then, by Theorem 4.3, the disease-free equilibrium $P_{0}$ of model (2.1) is globally asymptotically stable. The plots in Figures 6.1(a)-(c) show this theoretical result.

Further, we only adjust the values of transmission rates $\beta_{1}$ and $\beta_{2}$ and let $\beta_{1}=2.48 \times 10^{-6}$ and $\beta_{2}=2.32 \times 10^{-6}$ in model (2.1), then by numerical calculations it is obtained that the basic reproduction numbers $\mathcal{R}_{1} \approx 0.9052$ and $\mathcal{R}_{2} \approx 0.9414$. The values of $K_{1}$ and $K_{2}$ remain the same as above, then $\mathcal{R}_{1}>K_{1}$ and $\mathcal{R}_{2}>K_{2}$. In this case, numerical simulations show that the disease-free equilibrium $P_{0}$ is globally asymptotically stable, as shown in Figure 6.1(d). However, numerical simulations show the disease-free equilibrium is globally asymptotically stable if $\mathcal{R}_{0}<1$ without additional conditions. Therefore, we put forward an interesting open question: If $\mathcal{R}_{0}<1$, then the disease-free equilibrium is globally asymptotically stable.

Example 6.2. The existence and stability of strain $i(i=1,2)$ dominant equilibrium of (2.1).
Let $\Lambda_{h}=25, \beta_{1}=9.85 \times 10^{-6}, \beta_{2}=6.85 \times 10^{-6}, \mu_{h}=0.004, \gamma_{1}=0.07, \gamma_{2}=0.14$, $d_{1}=d_{2}=0.0001, \Lambda_{m}=21000, \alpha_{1}=1.75 \times 10^{-6}, \alpha_{2}=3.75 \times 10^{-6}, \mu_{m}=0.07, \varepsilon_{1}(a)=\varepsilon_{2}(a)=$ $0.01 a^{2} \exp (-0.28 a), \theta_{1}(b)=0.45(1+5 \exp (-0.026 b))^{-1}$ and $\theta_{2}(b)=0.48(1+5 \exp (-0.028 b))^{-1}$ in model (2.1). It is easy to calculate that parameter values satisfy all conditions of Theorem 4.5, that is, $\mathcal{R}_{1} \approx 3.528>1$ and $\mathcal{R}_{2} \approx 2.7<\mathcal{R}_{1}^{*} \approx 3.5055$. By Theorem 4.5, the strain 1 dominant equilibrium $P_{1}$ is locally asymptotically stable which is consistent with the simulation results as shown in Figures 6.2(a)-(d). As we can see, in Figure 6.2(e), solution curves of $I_{1}(t), Y_{1}(t)$ and $S(t)$ from different initial values all tend to a point in the first quadrant various, and the number of $I_{2}(t), Y_{1}(t)+Y_{2}(t)$ and $V_{2}(t)$ all tend to zero, which is shown Figure 6.2(f). Therefore,


Figure 6.1: The global asymptotical stability of disease-free equilibrium of model (2.1) with the basic reproduction number $\mathcal{R}_{0}<1$, which implies that the disease dies out.
numerical simulations imply that the strain 1 dominant equilibrium $P_{1}$ of model (2.1) is globally asymptotically stable. In addition, the numerical simulation for the existence and stability of $P_{2}$ is similar to that of $P_{1}$, hence we omit it here.

Example 6.3. The persistence of disease, the existence and stability of coexistence equilibrium for model (2.1).

We choose $\Lambda_{h}=100, \beta_{1}=2.85 \times 10^{-5}, \beta_{2}=4.25 \times 10^{-5}, \mu_{h}=0.004, \gamma_{1}=\gamma_{2}=0.07$, $d_{1}=d_{2}=0.0001, \Lambda_{m}=5500, \alpha_{1}=8.75 \times 10^{-6}, \alpha_{2}=8.45 \times 10^{-6}, \mu_{m}=0.05, \varepsilon_{1}(a)=$ $0.01 \exp (-0.25 a) a^{2}, \varepsilon_{2}(a)=0.01 \exp (-0.31 a) a^{2}$ and $\theta_{1}(b)=\theta_{2}(b)=0.4(1+50 \exp (-0.05 b))^{-1}$ in model (2.1). Numerical calculation follows that $\mathcal{R}_{1} \approx 48.22, \mathcal{R}_{2} \approx 47.23, \mathcal{R}_{1}^{*} \approx 41.3200$ and $\mathcal{R}_{2}^{*} \approx 40.8999$, which satisfy the conditions of Theorem 5.1 and Corollary 5.2. Therefore, the disease is uniformly persistent and model (2.1) exists coexistence equilibrium which is consistent with the simulation results as shown in Figures 6.3(a)-(d). Particularly, as we can see, in Figures 6.3(c) and (d), solution curves of $I_{1}(t)$ and $V_{2}(t)$ from different initial values all tend to a positive constants rather than zero. This implies that model (2.1) exists a globally asymptotically stable coexistence equilibrium. Of course, we also verify the globally asymptotically stability of coexistence equilibrium by constructing a Lyapunov functional in the Appendix with some strong constraint conditions, but these conditions are difficult to verify. This may be related to our research methods and the selection of Lyapunov functional. This encourages us to propose new research methods or construct more suitable Lyapunov functional to solve this problem in the future research.

In additional, we fixed parameter values of model (2.1) as above, and only adjust the value of cross immunity wane rate $\theta_{2}(b)$ to be $0.2(1+50 \exp (-0.05 b))^{-1}, 0.3(1+50 \exp (-0.05 b))^{-1}$, $0.4(1+50 \exp (-0.05 b))^{-1}$ and $0.6(1+50 \exp (-0.05 b))^{-1}$, respectively, we obtain the Fig.6.3(e).


Figure 6.2: The numerical simulation of the stability of strain 1 dominant equilibrium for model (2.1) with the basic reproduction number are $\mathcal{R}_{1} \approx 3.528$ and $\mathcal{R}_{2} \approx 2.7$.

It is easily to see that $\theta_{i}(b)$ does not appear in the expression of the basic reproduction number (i.e., the value of $\theta_{i}(b)$ does not affect the dynamic behavior of the model) from equation (4.1). However, the plot in Figure 6.3(e) show that the peak of secondary infection individuals number with strain 2 increases remarkably with $\theta_{2}(b)$ increases when the model persistent. This illustrates that the value of cross immunity wane rate still play very important role in the transmission of dengue fever.


Figure 6.3: Numerical simulations of the persistence, the existence and stability of coexistence equilibrium of model (2.1) with $\mathcal{R}_{1}>1, \mathcal{R}_{2}>1, \mathcal{R}_{1}>\mathcal{R}_{2}^{*}$ and $\mathcal{R}_{1}>\mathcal{R}_{1}^{*}$.

## 7 Conclusion

In recent years, many scholars have established lots of multi-strain dengue fever transmission models, studied the existence and stability of the disease-free equilibria, endemic equilibria, stain dominant equilibria, and competitive exclusion, and discussed the effects of the and mutual immune of strains on on the spread and control of dengue fever $[9,11,17,19,27,29,32$, 34, 41, 42]. However, most of which are described by ordinary differential equations (ODEs). In this paper, based on the two-strain dengue fever model proposed in Ref. [42], we propose a two-strain dengue fever transmission model with age structure to investigate the effects of latency age and cross immunity on the transmission dynamics of dengue virus. This extends the existing single-strain age structure models $[4,5,8,36]$, which is a highlight of our paper.

By using these methods proposed in Refs. [18,33,38-40], we first obtain the non-negativity, boundedness and asymptotic smoothness for solutions of our model. Further, the basic repro-
duction number $\mathcal{R}_{0}=\max \left\{\mathcal{R}_{1}, \mathcal{R}_{2}\right\}$ are defined, which plays a sharp threshold role in the process of this disease outbreaks. That is, if $\mathcal{R}_{0}<1$, then the disease-free equilibrium $P_{0}$ is locally stable, and $P_{0}$ is unstable for $\mathcal{R}_{0}>1$. Further, we also obtained sufficient conditions for the global asymptotic stability of $P_{0}$. To be specific, if $\mathcal{R}_{0}<\min \left\{K_{1}, K_{2}\right\}$, then $P_{0}$ is globally asymptotically stable. Of course, our numerical simulations suggest that $P_{0}$ is also globally asymptotically stable when $\mathcal{R}_{0}<1$ (see Figure 6.1(d)). In addition, if $\mathcal{R}_{i}>1$, this model has a strain- $i$ dominant equilibrium $P_{i}$ which is locally stable for $\mathcal{R}_{j}<\mathcal{R}_{i}^{*}(i, j=1,2, i \neq j)$. This condition is similar to the threshold condition for the stability of strain- $i$ dominant equilibria of these multi strain ordinary differential equations [11,42]. And we have given sufficient conditions for the uniform persistence of disease and the coexistence of the two strains. Finally, the numerical simulation implies that the strain- $i$ dominant equilibrium is global asymptotic stability for $\mathcal{R}_{i}>1$ and $\mathcal{R}_{j}<\mathcal{R}_{i}^{*}$ (see Figures 6.3(c)-(d)). However, due to the limitations of these research methods, the global attractivity of coexistence equilibrium obtained by us is subject to certain technical conditions. Therefore, this issue needs further research.

From the expression of $\mathcal{R}_{i}$, it is easy to observe that their value depends on $\varepsilon_{i}(a)$. Numerical simulations also shows that if the period of cross-immunity between the two strains increased (i.e., the rate of cross immunity waning decreased), the number of individuals with secondary infection decreased, and then the number of severe dengue cases decreased (see Figure 6.3(d)). This means that the latent age and cross immunity age play a important role in the transmission of dengue fever. Additionally, other model parameters also have an impact on the value of $\mathcal{R}_{i}$, such as the rates of transmission ( $\alpha_{i}$ and $\beta_{i}$ ), the death rate and the recruitment rate of mosquito ( $\mu_{m}$ and $\Lambda_{m}$ ), and so on. Therefore, control or prevent the transmission of dengue fever is mainly to reduce the number of mosquito and to increase personal protect awareness.

## Acknowledgements

This work was supported by the Natural Science Foundation of Xinjiang Uygur Autonomous Region (Grant Nos. 2021D01E12 and 2021D01C070), the National Natural Science Foundation of China (Grant Nos. 11961066 and 11771373), the Scientific Research Programmes of Colleges in Xinjiang (Grant Nos. XJEDU2018I001, XJEDU2017T001, XJUBSCX-201925), the Natural Science Foundation of Xinjiang (Grant No. 2016D01C046).

## Competing interests

The authors declare that they have no competing interests.

## Appendix

According to the Corollary 5.2 and the Figure 6.3, the coexistence equilibrium $P_{3}$ is globally asymptotically stable. Hence, we attempt to construct a Lyapunov functional to obtain the theoretical analysis.

Theorem A.1. If the condition of Corollary 5.2 and the following inequalities hold

$$
\begin{gathered}
\widetilde{S}^{*}+\int_{0}^{\infty} \theta_{1}(b) \widetilde{R}_{2}^{*}(b) \mathrm{d} b<\frac{\mu_{m}}{\beta_{1}}, \quad \widetilde{S}^{*}+\int_{0}^{\infty} \theta_{2}(b) \widetilde{R}_{1}^{*}(b) \mathrm{d} b<\frac{\mu_{m}}{\beta_{2}}, \\
\widetilde{U}^{*}<\min \left\{\frac{\mu_{h}+\gamma_{1}\left(1-K_{1}\right)}{\alpha_{1} K_{1}}, \frac{1}{\alpha_{1}}\left(\gamma_{1}+d_{1}+\mu_{h}\right), \frac{\mu_{h}+\gamma_{2}\left(1-K_{2}\right)}{\alpha_{2} K_{2}}, \frac{1}{\alpha_{2}}\left(\gamma_{2}+d_{2}+\mu_{h}\right)\right\},
\end{gathered}
$$

then model (2.1) has a unique coexistence equilibrium $P_{3}$ which is globally attractive.
Proof. Consider the Lyapunov functional as follows

$$
\mathcal{L}(t)=\mathcal{L}_{1}(t)+\mathcal{L}_{6}(t)+\sum_{i=1}^{2}\left\{\mathcal{L}_{2 i}(t)+\mathcal{L}_{3 i}(t)+\mathcal{L}_{4 i}(t)+\mathcal{L}_{5 i}(t)+\mathcal{L}_{7 i}(t)\right\},
$$

where

$$
\begin{array}{ll}
\mathcal{L}_{1}(t)=\widetilde{S}^{*} \phi\left(\frac{S(t)}{\widetilde{S}^{*}}\right), & \mathcal{L}_{2 i}(t)=\frac{1}{K_{i}} \int_{0}^{\infty} q_{i}(a) \widetilde{E}_{i}^{*}(a) \phi\left(\frac{E_{i}(t, a)}{\widetilde{E}_{i}^{*}(a)}\right) \mathrm{d} a, \\
\mathcal{L}_{3 i}(t)=\frac{1}{K_{i}} \widetilde{I}_{i}^{*} \phi\left(\frac{I_{i}(t)}{\widetilde{I}_{i}^{*}}\right), & \mathcal{L}_{4 i}(t)=\int_{0}^{\infty} \widetilde{R}_{i}^{*}(b) \phi\left(\frac{R_{i}(t, b)}{\widetilde{R}_{i}^{*}(b)}\right) \mathrm{d} a, \\
\mathcal{L}_{5 i}(t)=\widetilde{Y}_{i}^{*} \phi\left(\frac{Y_{i}(t)}{\widetilde{Y}_{i}^{*}}\right), & \mathcal{L}_{6}(t)=\widetilde{U}^{*} \phi\left(\frac{U(t)}{\widetilde{U}^{*}}\right), \quad \mathcal{L}_{7 i}(t)=\widetilde{V}_{i}^{*} \phi\left(\frac{V_{i}(t)}{\widetilde{V}_{i}^{*}}\right) .
\end{array}
$$

Because of the complexity of the expressions, we make the derive of each component of the Lyapunov functional separately.

$$
\begin{aligned}
\frac{\mathrm{d} \mathcal{L}_{1}(t)}{\mathrm{d} t}= & \widetilde{S}^{*}\left(\frac{1}{\widetilde{S}^{*}}-\frac{1}{S(t)}\right)\left(\Lambda_{h}-\frac{\Lambda_{h} S(t)}{\widetilde{S}^{*}}+\beta_{1} S(t) \widetilde{V}_{1}^{*}+\beta_{1} S(t) \widetilde{V}_{2}^{*}-\beta_{1} S(t) V_{1}(t)-\beta_{2} S(t) V_{2}(t)\right) \\
= & \Lambda_{h}\left(2-\frac{\widetilde{S}^{*}}{S(t)}-\frac{S(t)}{\widetilde{S}^{*}}\right)+\beta_{1} \widetilde{S}^{*} \widetilde{V}_{1}^{*}\left(1-\frac{\widetilde{S}^{*}}{S(t)}\right)\left(\frac{S(t)}{\widetilde{S}^{*}}-\frac{S(t) V_{1}(t)}{\widetilde{S}^{*} \widetilde{V}_{1}^{*}}\right) \\
& +\beta_{2} \widetilde{S}^{*} \widetilde{V}_{2}^{*}\left(1-\frac{\widetilde{S}^{*}}{S(t)}\right)\left(\frac{S(t)}{\widetilde{S}^{*}}-\frac{S(t) V_{2}(t)}{\widetilde{S}^{*} \widetilde{V}_{2}^{*}}\right) \\
= & -\Lambda_{h}\left[\phi\left(\frac{S(t)}{\widetilde{S}^{*}}\right)+\phi\left(\frac{\widetilde{S}^{*}}{S(t)}\right)\right]+\beta_{1} \widetilde{S}^{*} \widetilde{V}_{1}^{*}\left(\frac{S(t)}{\widetilde{S}^{*}}-\frac{S(t) V_{1}(t)}{\widetilde{S}^{*} \widetilde{V}_{1}^{*}}-1+\frac{V_{1}(t)}{\widetilde{V}_{1}^{*}}\right) \\
& +\beta_{2} \widetilde{S}^{*} \widetilde{V}_{2}^{*}\left(\frac{S(t)}{\widetilde{S}^{*}}-\frac{S(t) V_{2}(t)}{\widetilde{S}^{*} \widetilde{V}_{2}^{*}}-1+\frac{V_{2}(t)}{\widetilde{V}_{2}^{*}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d} \mathcal{L}_{2 i}(t)}{\mathrm{d} t} & =\frac{1}{K_{i}} \int_{0}^{\infty} q_{i}(a) \widetilde{E}_{i}^{*}(a) \frac{\partial}{\partial t} \phi\left(\frac{E_{i}(t, a)}{\widetilde{E}_{i}^{*}(a)}\right) \mathrm{d} a \\
& =-\frac{1}{K_{i}} \int_{0}^{\infty} q_{i}(a) \widetilde{E}_{i}^{*}(a)\left(\frac{E_{i}(t, a)}{\widetilde{E}_{i}^{*}(a)}-1\right)\left(\frac{E_{i a}(t, a)}{E_{i}(t, a)}+\mu_{h}+\varepsilon_{i}(a)\right) \mathrm{d} a,
\end{aligned}
$$

where $E_{i a}(t, a)=\mathrm{d} E_{i}(t, a) / \mathrm{d} a$. Since

$$
\frac{\partial}{\partial a} \phi\left(\frac{E_{i}(t, a)}{\widetilde{E}_{i}^{*}(a)}\right)=\left(\frac{E_{i}(t, a)}{\widetilde{E}_{i}^{*}(a)}-1\right)\left(\frac{E_{i a}(t, a)}{E_{i}(t, a)}+\mu_{h}+\varepsilon_{i}(a)\right),
$$

it can be easily shown that

$$
\begin{aligned}
\frac{\mathrm{d} \mathcal{L}_{2 i}(t)}{\mathrm{d} t} & =-\frac{1}{K_{i}} \int_{0}^{\infty} q_{i}(a) \widetilde{E}_{i}^{*}(a) \frac{\partial}{\partial a} \phi\left(\frac{E_{i}(t, a)}{\widetilde{E}_{i}^{*}(a)}\right) \mathrm{d} a \\
& =-\left.\frac{1}{K_{i}} q_{i}(a) \widetilde{E}_{i}^{*}(a) \phi\left(\frac{E_{i}(t, a)}{\widetilde{E}_{i}^{*}(a)}\right)\right|_{0} ^{\infty}-\frac{1}{K_{i}} \int_{0}^{\infty} \varepsilon_{i}(a) \widetilde{E}_{i}^{*}(a) \phi\left(\frac{E_{i}(t, a)}{\widetilde{E}_{i}^{*}(a)}\right) \mathrm{d} a \\
& =\beta_{i} \widetilde{S}^{*} \widetilde{V}_{i}^{*} \phi\left(\frac{S(t) V_{i}(t)}{\widetilde{S}^{*} \widetilde{V}_{i}^{*}}\right)-\frac{1}{K_{i}} \int_{0}^{\infty} \varepsilon_{i}(a) \widetilde{E}_{i}^{*}(a) \phi\left(\frac{E_{i}(t, a)}{\widetilde{E}_{i}^{*}(a)}\right) \mathrm{d} a .
\end{aligned}
$$

By directly calculating, we have

$$
\begin{aligned}
\frac{\mathrm{d} \mathcal{L}_{3 i}(t)}{\mathrm{d} t} & =\frac{1}{K_{i}} \int_{0}^{\infty} \varepsilon_{i}(a)\left(1-\frac{\widetilde{I}_{i}^{*}}{I_{i}(t)}\right)\left(E_{i}(t, a)-\widetilde{E}_{i}^{*}(a) \frac{I_{i}(t)}{\widetilde{I}_{i}^{*}}\right) \mathrm{d} a \\
& =\frac{1}{K_{i}} \int_{0}^{\infty} \varepsilon_{i}(a) \widetilde{E}_{i}^{*}(a)\left(\frac{E_{i}(t, a)}{\widetilde{E}_{i}^{*}(a)}-\frac{\widetilde{I}_{i}^{*}}{I_{i}(t)}-\frac{\widetilde{I}_{i}^{*} E_{i}(t, a)}{I_{i}(t) \widetilde{E}_{i}^{*}(a)}+1\right) \mathrm{d} a
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d} \mathcal{L}_{4 i}(t)}{\mathrm{d} t}= & \int_{0}^{\infty} \widetilde{R}_{i}^{*}(b) \frac{\partial}{\partial b} \phi\left(\frac{R_{i}(t, b)}{\widetilde{R}_{i}^{*}(b)}\right) \mathrm{d} b \\
= & -\int_{0}^{\infty} \widetilde{R}_{i}^{*}(b)\left(\frac{1}{\widetilde{R}_{i}^{*}(b)}-\frac{1}{R_{i}(t, b)}\right)\left(\frac{\partial}{\partial b} R_{i}(t, b)+\beta_{j} \theta_{j}(b) V_{j}(t) R_{i}(t, b)+\mu_{h} R_{i}(t, b)\right) \mathrm{d} b \\
= & -\int_{0}^{\infty} \widetilde{R}_{i}^{*}(b)\left(\frac{R_{i}(t, b)}{\widetilde{R}_{i}^{*}(b)}-1\right) \\
& \times\left[\left(\frac{R_{i b}(t, b)}{R_{i}(t, b)}+\beta_{j} \theta_{j}(b) \widetilde{V}_{j}^{*}+\mu_{h}\right)+\beta_{j} \theta_{j}(b) \widetilde{V}_{j}^{*}\left(\frac{V_{j}(t)}{\widetilde{V}_{j}^{*}}-1\right)\right] \mathrm{d} b,
\end{aligned}
$$

where $R_{i b}(t, b)=\mathrm{d} R_{i}(t, b) / \mathrm{d} b, i, j=1,2, i \neq j$. Since

$$
\frac{\partial}{\partial b} \phi\left(\frac{R_{i}(t, b)}{\widetilde{R}_{i}^{*}(b)}\right)=\left(\frac{R_{i}(t, b)}{\widetilde{R}_{i}^{*}(b)}-1\right)\left(\frac{R_{i b}(t, b)}{R_{i}(t, b)}+\beta_{j} \theta_{j}(b) \widetilde{V}_{j}^{*}+\mu_{h}\right),
$$

this gives

$$
\begin{aligned}
\frac{\mathrm{d} \mathcal{L}_{4 i}(t)}{\mathrm{d} t}= & -\int_{0}^{\infty}\left[\widetilde{R}_{i}^{*}(b) \frac{\partial}{\partial b} \phi\left(\frac{R_{i}(t, b)}{\widetilde{R}_{i}^{*}(b)}\right)+\beta_{j} \theta_{j}(b) \widetilde{V}_{j}^{*} \widetilde{R}_{i}^{*}(b)\left(\frac{R_{i}(t, b)}{\widetilde{R}_{i}^{*}(b)}-1\right)\left(\frac{V_{j}(t)}{\widetilde{V}_{j}^{*}}-1\right)\right] \mathrm{d} b \\
= & -\left.\widetilde{R}_{i}^{*}(b) \phi\left(\frac{R_{i}(t, b)}{\widetilde{R}_{i}^{*}(b)}\right)\right|_{0} ^{\infty}-\int_{0}^{\infty} \phi\left(\frac{R_{i}(t, b)}{\widetilde{R}_{i}^{*}(b)}\right)\left(\beta_{j} \theta_{j}(b) \widetilde{V}_{j}^{*} \widetilde{R}_{i}^{*}(b)+\mu_{h} \widetilde{R}_{i}^{*}(b)\right) \mathrm{d} b \\
= & \gamma_{i} \widetilde{I}_{i}^{*} \phi\left(\frac{I_{i}(t)}{\widetilde{I}_{i}^{*}}\right)-\mu_{h} \int_{0}^{\infty} \widetilde{R}_{i}^{*}(b) \phi\left(\frac{R_{i}(t, b)}{\widetilde{R}_{i}^{*}(b)}\right) \mathrm{d} b \\
& +\int_{0}^{\infty} \beta_{j} \theta_{j}(b) \widetilde{R}_{i}^{*}(b) \widetilde{V}_{j}^{*}\left(\frac{R_{i}(t, b) V_{j}(t)}{\widetilde{R}_{i}^{*}(b) \widetilde{V}_{j}^{*}}-\frac{V_{j}(t)}{\widetilde{V}_{j}^{*}}-\ln \frac{R_{i}(t, b)}{\widetilde{R}_{i}^{*}(b)}\right) \mathrm{d} b .
\end{aligned}
$$

Furthermore, it can be easily calculated that

$$
\begin{aligned}
\frac{\mathrm{d} \mathcal{L}_{5 i}(t)}{\mathrm{d} t} & =\beta_{i} \int_{0}^{\infty} \theta_{i}(b)\left(1-\frac{\widetilde{Y}_{i}^{*}}{Y_{i}(t)}\right)\left(V_{i}(t) R_{j}(t, b)-\frac{Y_{i}(t)}{\widetilde{Y}_{i}^{*}} \widetilde{V}_{i}^{*} \widetilde{R}_{j}^{*}(b)\right) \mathrm{d} b \\
& =\beta_{i} \int_{0}^{\infty} \theta_{i}(b) \widetilde{V}_{i}^{*} \widetilde{R}_{j}^{*}(b)\left(\frac{V_{i}(t) R_{j}(t, b)}{\widetilde{V}_{i}^{*} \widetilde{R}_{j}^{*}(b)}-\frac{Y_{i}(t)}{\widetilde{Y}_{i}^{*}}-\frac{\widetilde{Y}_{i}^{*} V_{i}(t) R_{j}(t, b)}{Y_{i}(t) \widetilde{V}_{i}^{*} \widetilde{R}_{j}^{*}(b)}+1\right) \mathrm{d} b .
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d} \mathcal{L}_{6}(t)}{\mathrm{d} t}= & \Lambda_{m}\left(1-\frac{\widetilde{U}^{*}}{U(t)}\right)\left(1-\frac{U(t)}{\widetilde{U}^{*}}\right)+\left(1-\frac{\widetilde{U}^{*}}{U(t)}\right)\left(\alpha_{1}\left(\widetilde{I}_{1}^{*}+\widetilde{Y}_{1}^{*}\right) U(t)-\alpha_{1}\left(I_{1}(t)\right.\right. \\
& \left.\left.+Y_{1}(t)\right) U(t)\right)+\left(1-\frac{\widetilde{U}^{*}}{U(t)}\right)\left(\alpha_{2}\left(\widetilde{I}_{2}^{*}+\widetilde{Y}_{2}^{*}\right) U(t)-\alpha_{2}\left(I_{2}(t)+Y_{2}(t)\right) U(t)\right) \\
= & -\Lambda_{m}\left[\phi\left(\frac{U(t)}{\widetilde{U}^{*}}\right)+\phi\left(\frac{\widetilde{U}^{*}}{U(t)}\right)\right]+\alpha_{1} \widetilde{I}_{1}^{*} \widetilde{U}^{*}\left(\frac{U(t)}{\widetilde{U}^{*}}-1-\frac{U(t) I_{1}(t)}{\widetilde{U}^{*} \widetilde{I}_{1}^{*}}+\frac{I_{1}(t)}{\widetilde{I}_{1}^{*}}\right) \\
& +\alpha_{1} \widetilde{Y}_{1}^{*} \widetilde{U}^{*}\left(\frac{U(t)}{\widetilde{U}^{*}}-1-\frac{U(t) Y_{1}(t)}{\widetilde{U}^{*} \widetilde{Y}_{1}^{*}}+\frac{Y_{1}(t)}{\widetilde{Y}_{1}^{*}}\right)+\alpha_{2} \widetilde{I}_{2}^{*} \widetilde{U}^{*}\left(\frac{U(t)}{\widetilde{U}^{*}}-1-\frac{U(t) I_{2}(t)}{\widetilde{U}^{*} \widetilde{I}_{2}^{*}}\right. \\
& \left.+\frac{I_{2}(t)}{\widetilde{I}_{2}^{*}}\right)+\alpha_{2} \widetilde{Y}_{2}^{*} \widetilde{U}^{*}\left(\frac{U(t)}{\widetilde{U}^{*}}-1-\frac{U(t) Y_{2}(t)}{\widetilde{U}^{*} \widetilde{Y}_{2}^{*}}+\frac{Y_{2}(t)}{\widetilde{Y}_{2}^{*}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d} \mathcal{L}_{7 i}(t)}{\mathrm{d} t}= & \alpha_{i}\left(1-\frac{V_{i}(t)}{\widetilde{V}_{i}^{*}}\right)\left(I_{i}(t) U(t)-\frac{V_{i}(t)}{\widetilde{V}_{i}^{*}} \widetilde{I}_{i}^{*} \widetilde{U}^{*}+Y_{i}(t) U(t)-\frac{V_{i}(t)}{\widetilde{V}_{i}^{*}} \widetilde{Y}_{i}^{*} \widetilde{U}^{*}\right) \\
= & \alpha_{i} \widetilde{I}_{i}^{*} \widetilde{U}^{*}\left(\frac{I_{i}(t) U(t)}{\widetilde{I}_{i}^{*} \widetilde{U}^{*}}-\frac{V_{i}(t)}{\widetilde{V}_{i}^{*}}-\frac{I_{i}(t) U(t) \widetilde{V}_{i}^{*}}{\widetilde{I}_{i}^{*} \widetilde{U}^{*} V_{i}(t)}+1\right) \\
& +\alpha_{i} \widetilde{Y}_{i}^{*} \widetilde{U}^{*}\left(\frac{Y_{i}(t) U(t)}{\widetilde{Y}_{i}^{*} \widetilde{U}^{*}}-\frac{V_{i}(t)}{\widetilde{V}_{i}^{*}}-\frac{Y_{i}(t) U(t) \widetilde{V}_{i}^{*}}{\widetilde{Y}_{i}^{*} \widetilde{U}^{*} V_{i}(t)}+1\right) .
\end{aligned}
$$

Thus, to sum up, we can get

$$
\begin{aligned}
\frac{\mathrm{d} \mathcal{L}(t)}{\mathrm{d} t}= & \frac{\mathrm{d} \mathcal{L}_{1}(t)}{\mathrm{d} t}+\frac{\mathrm{d} \mathcal{L}_{6}(t)}{\mathrm{d} t}+\sum_{i=1}^{2}\left\{\frac{\mathrm{~d} \mathcal{L}_{2 i}(t)}{\mathrm{d} t}+\frac{\mathrm{d} \mathcal{L}_{3 i}(t)}{\mathrm{d} t}+\frac{\mathrm{d} \mathcal{L}_{4 i}(t)}{\mathrm{d} t}+\frac{\mathrm{d} \mathcal{L}_{5 i}(t)}{\mathrm{d} t}+\frac{\mathrm{d} \mathcal{L}_{7 i}(t)}{\mathrm{d} t}\right\} \\
= & -\Lambda_{h}\left[\phi\left(\frac{S(t)}{\widetilde{S}^{*}}\right)+\phi\left(\frac{\widetilde{S}^{*}}{S(t)}\right)\right]-\Lambda_{m}\left[\phi\left(\frac{U(t)}{\widetilde{U}^{*}}\right)+\phi\left(\frac{\widetilde{U}^{*}}{U(t)}\right)\right]+\gamma_{1} \widetilde{I}_{1}^{*} \phi\left(\frac{I_{1}(t)}{\widetilde{I}_{1}^{*}}\right) \\
& -\mu_{h} \int_{0}^{\infty} \widetilde{R}_{1}^{*}(b) \phi\left(\frac{R_{1}(t, b)}{\widetilde{R}_{1}^{*}(b)}\right) \mathrm{d} b+\gamma_{2} \widetilde{I}_{2}^{*} \phi\left(\frac{I_{2}(t)}{\widetilde{I}_{2}^{*}}\right)-\mu_{h} \int_{0}^{\infty} \widetilde{R}_{2}^{*}(b) \phi\left(\frac{R_{2}(t, b)}{\widetilde{R}_{2}^{*}(b)}\right) \mathrm{d} b \\
& +\sum_{i=1}^{2}\left(\mathcal{H}_{1 i}(t)+\mathcal{H}_{2 i}(t)+\mathcal{H}_{3 i}(t)+\mathcal{H}_{4 i}(t)+\mathcal{H}_{5 i}(t)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}_{1 i}(t) & :=\beta_{i} \widetilde{S}^{*} \widetilde{V}_{i}^{*}\left[\frac{S(t)}{\widetilde{S}^{*}}-\frac{S(t) V_{i}(t)}{\widetilde{S}^{*} \widetilde{V}_{i}^{*}}-1+\frac{V_{i}(t)}{\widetilde{V}_{i}^{*}}+\phi\left(\frac{S(t) V_{i}(t)}{\widetilde{S}^{*} \widetilde{V}_{i}^{*}}\right)\right] \\
& =\beta_{i} \widetilde{S}^{*} \widetilde{V}_{i}^{*}\left[\phi\left(\frac{S(t)}{\widetilde{S}^{*}}\right)+\phi\left(\frac{\widetilde{V}_{i}^{*}}{V_{i}(t)}\right)\right], \\
\mathcal{H}_{2 i}(t) & :=\frac{1}{K_{i}} \int_{0}^{\infty} \varepsilon_{i}(a) \widetilde{E}_{i}^{*}(a)\left[\frac{E_{i}(t, a)}{\widetilde{E}_{i}^{*}(a)}-\frac{\widetilde{I}_{i}^{*}}{I_{i}(t)}-\frac{\widetilde{I}_{i}^{*} E_{i}(t, a)}{I_{i}(t) \widetilde{E}_{i}^{*}(a)}+1-\phi\left(\frac{E_{i}(t, a)}{\widetilde{E}_{i}^{*}(a)}\right)\right] \mathrm{d} a \\
& =\frac{1}{K_{i}} \int_{0}^{\infty} \varepsilon_{i}(a) \widetilde{E}_{i}^{*}(a)\left[\phi\left(\frac{E_{i}(t, a) \widetilde{I}_{i}^{*}}{\widetilde{E}_{i}^{*}(a) I_{i}(t)}\right)+\phi\left(\frac{I_{i}(t)}{\widetilde{I}_{i}^{*}}\right)\right] \mathrm{d} a, \\
\mathcal{H}_{3 i}(t) & :=\int_{0}^{\infty} \beta_{i} \theta_{i}(b) \widetilde{R}_{j}^{*}(b) \widetilde{V}_{i}^{*}\left(1+\frac{V_{i}(t)}{\widetilde{V}_{i}^{*}}+\ln \frac{R_{j}(t, b)}{\widetilde{R}_{j}^{*}(b)}-\frac{Y_{i}(t)}{\widetilde{Y}_{i}^{*}}-\frac{\widetilde{Y}_{i}^{*} V_{i}(t) R_{j}(t, b)}{Y_{i}(t) \widetilde{V}_{i}^{*} \widetilde{R}_{j}^{*}(b)}\right) \mathrm{d} b
\end{aligned}
$$

$$
\begin{aligned}
& =\int_{0}^{\infty} \beta_{i} \theta_{i}(b) \widetilde{R}_{j}^{*}(b) \widetilde{V}_{i}^{*}\left[\phi\left(\frac{V_{i}(t)}{\widetilde{V}_{i}^{*}}\right)-\phi\left(\frac{\widetilde{Y}_{i}^{*} V_{i}(t) R_{j}(t, b)}{Y_{i}(t) \widetilde{V}_{i}^{*} \widetilde{R}_{j}^{*}(b)}\right)-\phi\left(\frac{Y_{i}(t)}{\widetilde{Y}_{i}^{*}}\right)\right] \mathrm{d} b, \\
\mathcal{H}_{4 i}(t) & :=\alpha_{i} \widetilde{I}_{i}^{*} \widetilde{U}^{*}\left(\frac{U(t)}{\widetilde{U}^{*}}+\frac{I_{i}(t)}{\widetilde{I}_{i}^{*}}-\frac{V_{i}(t)}{\widetilde{V}_{i}^{*}}-\frac{\widetilde{V}_{i}^{*} U(t) I_{i}(t)}{V_{i}(t) \widetilde{U}^{*} \widetilde{I}_{i}^{*}}\right) \\
& =\alpha_{i} \widetilde{I}_{i}^{*} \widetilde{U}^{*}\left[\phi\left(\frac{U(t)}{\widetilde{U}^{*}}\right)+\phi\left(\frac{I_{i}(t)}{\widetilde{I}_{i}^{*}}\right)-\phi\left(\frac{V_{i}(t)}{\widetilde{V}_{i}^{*}}\right)-\phi\left(\frac{\widetilde{V}_{i}^{*} U(t) I_{i}(t)}{V_{i}(t) \widetilde{U}^{*} \widetilde{I}_{i}^{*}}\right)\right], \\
\mathcal{H}_{5 i}(t) & :=\alpha_{i} \widetilde{Y}_{i}^{*} \widetilde{U}^{*}\left(\frac{U(t)}{\widetilde{U}^{*}}+\frac{Y_{i}(t)}{\widetilde{Y}_{i}^{*}}-\frac{V_{i}(t)}{\widetilde{V}_{i}^{*}}-\frac{\widetilde{V}_{i}^{*} U(t) Y_{i}(t)}{V_{i}(t) \widetilde{U}^{*} \widetilde{Y}_{i}^{*}}\right) \\
& =\alpha_{i} \widetilde{Y}_{i}^{*} \widetilde{U}^{*}\left[\phi\left(\frac{U(t)}{\widetilde{U}^{*}}\right)+\phi\left(\frac{Y_{i}(t)}{\widetilde{Y}_{i}^{*}}\right)-\phi\left(\frac{V_{i}(t)}{\widetilde{V}_{i}^{*}}\right)-\phi\left(\frac{\widetilde{V}_{i}^{*} U(t) Y_{i}(t)}{V_{i}(t) \widetilde{U}^{*} \widetilde{Y}_{i}^{*}}\right)\right] .
\end{aligned}
$$

Note that equilibrium $P_{3}$ satisfies

$$
\begin{aligned}
& \Lambda_{h}=\beta_{1} \widetilde{S}^{*} \widetilde{V}_{1}^{*}+\beta_{2} \widetilde{S}^{*} \widetilde{V}_{2}^{*}+\mu_{h} \widetilde{S}^{*}, \quad \int_{0}^{\infty} \varepsilon_{i}(a) \widetilde{E}_{i}^{*}(a) \mathrm{d} a=\left(\gamma_{i}+\mu_{h}\right) \widetilde{I}_{i}^{*}, \\
& \mu_{m} \widetilde{V}_{i}^{*}=\alpha_{i}\left(\widetilde{I}_{i}^{*}+\widetilde{Y}_{i}^{*}\right) \widetilde{U}^{*}, \quad \beta_{i} \int_{0}^{\infty} \theta_{i}(b) \widetilde{V}_{i}^{*} \widetilde{R}_{j}^{*}(b) \mathrm{d} b=\left(\gamma_{i}+d_{i}+\mu_{h}\right) Y_{i}, \\
& \Lambda_{m}=\alpha_{1}\left(\widetilde{I}_{1}^{*}+\widetilde{Y}_{1}^{*}\right) \widetilde{U}^{*}+\alpha_{2}\left(\widetilde{I}_{2}^{*}+\widetilde{Y}_{2}^{*}\right) \widetilde{U}^{*}+\mu_{m} \widetilde{U}^{*}, \quad i, j=1,2, i \neq j .
\end{aligned}
$$

Therefore, we finally obtain

$$
\begin{aligned}
\frac{\mathrm{d} \mathcal{L}(t)}{\mathrm{d} t}= & -\mu_{h} \widetilde{S}^{*} \phi\left(\frac{S(t)}{\widetilde{S}^{*}}\right)-\Lambda_{h} \phi\left(\frac{\widetilde{S}^{*}}{S(t)}\right)-\frac{1}{K_{1}} \int_{0}^{\infty} \varepsilon_{1}(a) \widetilde{E}_{1}^{*}(a) \phi\left(\frac{E_{1}(t, a) \widetilde{I}_{1}^{*}}{\widetilde{E}_{1}^{*}(a) I_{1}(t)}\right) \mathrm{d} a \\
& -\mu_{h} \int_{0}^{\infty} \widetilde{R}_{1}^{*}(b) \phi\left(\frac{R_{1}(t, b)}{\widetilde{R}_{1}^{*}(b)}\right) \mathrm{d} b-\mu_{h} \int_{0}^{\infty} \widetilde{R}_{2}^{*}(b) \phi\left(\frac{R_{2}(t, b)}{\widetilde{R}_{2}^{*}(b)}\right) \mathrm{d} b \\
& -\frac{1}{K_{2}} \int_{0}^{\infty} \varepsilon_{2}(a) \widetilde{E}_{2}^{*}(a) \phi\left(\frac{E_{2}(t, a) \widetilde{I}_{2}^{*}}{\widetilde{E}_{2}^{*}(a) I_{2}(t)}\right) \mathrm{d} a-\mu_{m} \widetilde{U}^{*} \phi\left(\frac{U(t)}{\widetilde{U}^{*}}\right)-\Lambda_{m} \phi\left(\frac{\widetilde{U}^{*}}{U(t)}\right) \\
& -\int_{0}^{\infty} \beta_{1} \theta_{1}(b) \widetilde{R}_{2}^{*}(b) \widetilde{V}_{1}^{*} \phi\left(\frac{\widetilde{Y}_{1}^{*} V_{1}(t) R_{2}(t, b)}{Y_{1}(t) \widetilde{V}_{1}^{*} \widetilde{R}_{2}^{*}(b)}\right) \mathrm{d} b-\alpha_{1} \widetilde{I}_{1}^{*} \widetilde{U}^{*} \phi\left(\frac{\widetilde{V}_{1}^{*} U(t) I_{1}(t)}{V_{1}(t) \widetilde{U}^{*} \widetilde{I}_{1}^{*}}\right) \\
& -\int_{0}^{\infty} \beta_{2} \theta_{2}(b) \widetilde{R}_{1}^{*}(b) \widetilde{V}_{2}^{*} \phi\left(\frac{\widetilde{Y}_{2}^{*} V_{2}(t) R_{1}(t, b)}{Y_{2}(t) \widetilde{V}_{2}^{*} \widetilde{R}_{1}^{*}(b)}\right) \mathrm{d} b-\alpha_{2} \widetilde{I}_{2}^{*} \widetilde{U}^{*} \phi\left(\frac{\widetilde{V}_{2}^{*} U(t) I_{2}(t)}{V_{2}(t) \widetilde{U}^{*} \widetilde{I}_{2}^{*}}\right) \\
& +\widetilde{I}_{1}^{*} \phi\left(\frac{I_{1}(t)}{\widetilde{I}_{1}^{*}}\right)\left[\alpha_{1} \widetilde{U}^{*}+\gamma_{1}\left(1-\frac{1}{K_{1}}\right)-\frac{\mu_{h}}{K_{1}}\right]+\left(\alpha_{1} \widetilde{U}^{*}-\left(\gamma_{1}+d_{1}+\mu_{h}\right)\right) \widetilde{Y}_{1}^{*} \\
& \times \phi\left(\frac{Y_{1}(t)}{\widetilde{Y}_{1}^{*}}\right)+\widetilde{I}_{1}^{*} \phi\left(\frac{I_{1}(t)}{\widetilde{I}_{1}^{*}}\right)\left[\alpha_{1} \widetilde{U}^{*}+\gamma_{1}\left(1-\frac{1}{K_{1}}\right)-\frac{\mu_{h}}{K_{1}}\right] \\
& +\left(\alpha_{1} \widetilde{U}^{*}-\left(\gamma_{1}+d_{1}+\mu_{h}\right)\right) \widetilde{Y}_{1}^{*} \phi\left(\frac{Y_{1}(t)}{\widetilde{Y}_{1}^{*}}\right)-\alpha_{1} \widetilde{Y}_{1}^{*} \widetilde{U}^{*} \phi\left(\frac{\widetilde{V}_{1}^{*} U(t) Y_{1}(t)}{V_{1}(t) \widetilde{U}^{*} \widetilde{Y}_{1}^{*}}\right) \\
& -\alpha_{2} \widetilde{Y}_{2}^{*} \widetilde{U}^{*} \phi\left(\frac{\widetilde{V}_{2}^{*} U(t) \Upsilon_{2}(t)}{V_{2}(t) \widetilde{U}^{*} \widetilde{Y}_{2}^{*}}\right)+\left(\beta_{1} \widetilde{S}^{*}+\beta_{1} \int_{0}^{\infty} \theta_{1}(b) \widetilde{R}_{2}^{*}(b) \mathrm{d} b-\mu_{m}\right) \\
& \times \widetilde{V}_{1}^{*} \phi\left(\frac{V_{1}(t)}{\widetilde{V}_{1}^{*}}\right)+\left(\beta_{2} \widetilde{S}^{*}+\beta_{2} \int_{0}^{\infty} \theta_{2}(b) \widetilde{R}_{1}^{*}(b) \mathrm{d} b-\mu_{m}\right) \widetilde{V}_{2}^{*} \phi\left(\frac{V_{2}(t)}{\widetilde{V}_{2}^{*}}\right) .
\end{aligned}
$$

It is easy to see that the sufficient condition for $\mathrm{d} \mathcal{L}(t) / \mathrm{d} t<0$ are

$$
\begin{aligned}
& \alpha_{1} \widetilde{U}^{*}+\gamma_{1}\left(1-\frac{1}{K_{1}}\right)-\frac{\mu_{h}}{K_{1}}<0, \quad \alpha_{1} \widetilde{U}^{*}-\left(\gamma_{1}+d_{1}+\mu_{h}\right)<0, \\
& \alpha_{2} \widetilde{U}^{*}+\gamma_{2}\left(1-\frac{1}{K_{2}}\right)-\frac{\mu_{h}}{K_{2}}<0, \quad \alpha_{2} \widetilde{U}^{*}-\left(\gamma_{2}+d_{2}+\mu_{h}\right)<0, \\
& \beta_{1} \widetilde{S}^{*}+\beta_{1} \int_{0}^{\infty} \theta_{1}(b) \widetilde{R}_{2}^{*}(b) \mathrm{d} b-\mu_{m}<0, \quad \beta_{2} \widetilde{S}^{*}+\beta_{2} \int_{0}^{\infty} \theta_{2}(b) \widetilde{R}_{1}^{*}(b) \mathrm{d} b-\mu_{m}<0 .
\end{aligned}
$$

That is,

$$
\begin{aligned}
& \widetilde{S}^{*}+\int_{0}^{\infty} \theta_{1}(b) \widetilde{R}_{2}^{*}(b) \mathrm{d} b<\frac{\mu_{m}}{\beta_{1}}, \quad \widetilde{S}^{*}+\int_{0}^{\infty} \theta_{2}(b) \widetilde{R}_{1}^{*}(b) \mathrm{d} b<\frac{\mu_{m}}{\beta_{2}}, \\
& \widetilde{U}^{*}<\min \left\{\frac{\mu_{h}+\gamma_{1}\left(1-K_{1}\right)}{\alpha_{1} K_{1}}, \frac{1}{\alpha_{1}}\left(\gamma_{1}+d_{1}+\mu_{h}\right), \frac{\mu_{h}+\gamma_{2}\left(1-K_{2}\right)}{\alpha_{2} K_{2}}, \frac{1}{\alpha_{2}}\left(\gamma_{2}+d_{2}+\mu_{h}\right)\right\} .
\end{aligned}
$$

This shows that equilibrium $P_{3}$ is globally attractive. This completes the proof.

## References

[1] S. Bhatt, P. W. Gething, O. J. Brady et al., The global distribution and burden of dengue. Nature 496(2013), 504-507. https: //doi . org/10.1038/nature12060
[2] C. Y. Cheng, Y. P. Dong, Y. Takeuchi, An age-structured virus model with two routes of infection in heterogeneous environments. Nonlinear Anal. Real World Appl. 39(2018), 464491. https://doi.org/10.1016/j .nonrwa.2017.07.013; MR3698151
[3] L. Coudeville, G. P. Garnett, Transmission dynamics of the four dengue serotypes in southern Vietnam and the potential impact of vaccination, Plos One 7(2012), e51244. https: //doi.org/10.1371/journal.pone. 0051244
[4] V.-D.-L. Cruz, L. Esteva, A. Korobeinikov, Age-dependency in host-vector models: a global analysis, Appl. Math. Comput 243(2014), 969-981. https ://doi. org/10.1016/j . amc . 2014.06.042; MR3244544
[5] Y.-X. Dang, Z.-P. Qiu, X.-Z. Li, M. Martcheva, Global dynamics of a vector-host epidemic model with age of infection. Math. Biosci. Eng. 14(2017), No. 5-6, 1159-1186. https://doi. org/10.3934/mbe.2017060; MR3657122
[6] M. Derouich, A. Boutayeb, Dengue fever: Mathematical modelling and computer simulation. Appl. Math. Comput. 177(2006), No. 2, 528-544. https://doi.org/10.1016/j.amc. 2005.11.031; MR2291978
[7] C. A. Donnelly, A. C. Ghani, G. M. Leung et al., Epidemiological determinants of spread of causal agent of severe acute respiratory syndrome in Hong Kong, Lancet 361(2003), 1761-1766. https://doi.org/10.1016/S0140-6736(03)13410-1
[8] X. Duan, S. Yuan, Z. Qiu, J. Ma, Global stability of an SVEIR epidemic model with ages of vaccination and latency, Comput. Math. Appl. 68(2014), No. 3, 288-308. https://doi.org/ 10.1016/j.camwa.2014.06.002; MR3231960
[9] L. Esteva, C. Vargas, Coexistence of different serotypes of dengue virus, J. Math. Biol. 46(2003), No. 1, 31-47. https://doi.org/10.1007/s00285-002-0168-4; MR1957008
[10] L. Esteva, C. Vargas, A model for dengue disease with variable human population, J. Math. Biol. 38(1999), No. 3, 220-240. https://doi.org/10.1007/s002850050147; MR1684881
[11] Z. Feng, J. X. Velasco-Hernández, Competitive exclusion in a vector-host model for the dengue fever, J. Math. Biol. 35(1997), No. 5, 523-544. https://doi.org/10.1007/ s002850050064; MR1479326
[12] N. Ganegoda, T. Götz, K. P. Wijaya, An age-dependent model for dengue transmission: analysis and comparison to field data, Appl. Math. Comput. 388(2021), 125538. https:// doi.org/10.1016/j.amc.2020.125538; MR4130888
[13] S. M. Garba, A. B. Gumel, M. R. A. Bakar, Backward bifurcations in dengue transmission dynamics, Math Biosci. 215(2008), No. 1, 11-25. https://doi.org/10.1016/j.mbs. 2008. 05.002; MR2459525
[14] R. V. Gibbons, S. Kalanarooj, R. G. Jarman, A. Nisalak, D. W. Vaughn, T. P. Endy, M. P. Mammen Jr., A. Srikiatkhachorn, Analysis of repeat hospital admissions for dengue to estimate the frequency of third or fourth dengue infections resulting in admissions and dengue hemorrhagic fever, and serotype sequences, Am. J. Trop. Med. Hyg. 77(2007), No. 5, 910-913.
[15] N. L. González Morales, M. Núñez-López, J. Ramos-Castañeda, J. X. VelascoHernández, Transmission dynamics of two dengue serotypes with vaccination scenarios, Math. Biosci. 287(2017), 54-71. https://doi.org/10.1016/j.mbs.2016.10.001; MR3634153
[16] D. J. Gubler, G. G. Clark, Dengue/dengue hemorrhagic fever: the emergence of a global health problem, Emerg. Infect. Dis. 1(1995), No. 2, 55-57. https : //doi . org/10.3201/ EID0102.952004
[17] H. Gulbudak, C. J. Browne, Infection severity across scales in multi-strain immunoepidemiological Dengue model structured by host antibody level, J. Math. Biol. 80(2020), No. 6, 1803-1843. https://doi.org/10.1007/s00285-020-01480-3; MR4096764
[18] J. K. Hale, P. Waltman, Persistence in infinite-dimensional systems, SIAM J. Math. Anal. 20(1989), No. 2, 388-395. https://doi.org/10.1137/0520025; MR0982666
[19] B. W. Kooi, M. Aguiar, N. Stollenwerk, Bifurcation analysis of a family of multi-strain epidemiology models, J. Comput. Appl. Math. 252(2013), 148-158. https://doi.org/10. 1016/j.cam. 2012.08.008; MR3054565
[20] L. Zou, J. Chen, X. Feng, S. Ruan, Analysis of a dengue model with vertical transmission and application to the 2014 dengue outbreak in Guangdong province, China, Bull. Math. Biol. 80(2018), No. 10, 2633-2651. https://doi.org/10.1007/s11538-018-0480-9; MR3856992
[21] Y. Li, Z. Teng, C. Hu, Q. Ge, Global stability of an epidemic model with age-dependent vaccination, latent and relapse, Chaos Solitons Fractals 105(2017), 195-207. https://doi. org/10.1016/j.chaos.2017.10.027; MR3725991
[22] Z. Liu, J. Hu, L. Wang, Modelling and analysis of global resurgence of mumps: A multigroup epidemic model with asymptomatic infection, general vaccinated and exposed distributions, Nonlinear Anal. Real World Appl. 37(2017), 137-161. https://doi.org/10.1016/ j.nonrwa.2017.02.009; MR3648375
[23] L. Liu J. Wang, X. Liu, Global stability of an SEIR epidemic model with age-dependent latency and relapse, Nonlinear Anal. Real World Appl. 24(2015), 18-35. https://doi.org/ 10.1016/j.nonrwa.2015.01.001; MR3332879
[24] S. Lee, C. Castillo-Chavez, The role of residence times in two-patch dengue transmission dynamics and optimal strategies, J. Theoret. Biol. 374(2015), 152-164. https://doi.org/10. 1016/j.jtbi.2015.03.005; MR3341840
[25] P. Magal, X.-Q. Zhao, Global attractors and steady states for uniformly persistent dynamical systems. SIAM J. Math. Anal. 37(2005), No. 1, 251-275. https://doi . org/10.1137/ S0036141003439173; MR2172756
[26] T. A. McLennan-Smith, G. N. Mercer, Complex behaviour in a dengue model with a seasonally varying vector population, Math. Biosci. 248(2014), 22-30. https://doi.org/ 10.1016/j.mbs.2013.11.003; MR3162637
[27] A. Mishra S. Gakkhar, The effects of awareness and vector control on two strains dengue dynamics, Appl. Math. Comput. 246(2014), 159-167. https://doi.org/10.1016/ j.amc.2014.07.115; MR3265857
[28] P. Nguipdor-Djomo, E. Heldal, L. Cunha Rodrigues, I. Abubakar, P. Mangtani, Duration of BCG protection against tuberculosis and change in effectiveness with time since vaccination in Norway: a retrospective population-based cohort study, Lancet Infect. Dis. 16(2016), No. 2, 219-226. https://doi.org/10.1016/S1473-3099(15) 00400-4
[29] N. Nuraini, E. Soewono, K. A. Sidarto, Mathematical model of dengue disease transmission with severe DHF compartment. Bull. Malays. Math. Sci. Soc. 30(2007), No. 2, 143-157. MR2368041
[30] J. Páez Chávez, T. Götz, S. Siegmund, K. P. Wijaya, An SIR-dengue transmission model with seasonal effects and impulsive control, Math. Biosci. 289(2017), 29-39. https://doi. org/10.1016/j.mbs.2017.04.005; MR3660246
[31] D.-L. Qian, X.-Z. Li, M. Ghosh, Coexistence of the strains induced by mutation, Int. J. Biomath. 5(2012), No. 3, 1260016, 25 pp. https://doi.org/10.1142/S1793524512600169; MR2922595
[32] P. Rashkov, B. W. Kooi, Complexity of host-vector dynamics in a two-strain dengue model, J. Biol. Dyn. 15(2021), No. 1, 35-72. https://doi.org/10.1080/17513758.2020. 1864038; MR4193651
[33] H. L. Smith, H. R. Thieme, Dynamical systems and population persistence, American Mathematical Society, 2011. MR2731633
[34] M. Sriprom, P. Barbazan, I. M. Tang, Destabilizing effect of the host immune status on the sequential transmission dynamic of the dengue virus infection, Math. Comput. Model. 45(2007), No. 9-10, 1053-1066. https://doi.org/10.1016/j.mcm. 2006.09.011; MR2300577
[35] H. R. Thieme, C. Castillo-Chavez, How may infection-age-dependent infectivity affect the dynamics of HIV/AIDS?, SIAM J. Appl. Math. 53(1993), No. 5, 1447-1479. https:// doi.org/10.1137/0153068; MR1239414
[36] X. Wang, Y. Chen, S. Liu, Dynamics of an age-structured host-vector model for malaria transmission, Math. Meth. Appl. Sci. 41(2018), 1966-1987. https://doi.org/10.1002/mma. 4723; MR3778100
[37] L. Wang, Z. Liu, X. Zhang, Global dynamics for an age-structured epidemic model with media impact and incomplete vaccination, Nonlinear Anal. Real World Appl. 32(2016), 136158. https://doi.org/10.1016/j.nonrwa.2016.04.009; MR3514918
[38] L. Wang, Z. Teng, T. Zhang, Threshold dynamics of a malaria transmission model in periodic environment, Commun. Nonlinear. Sci. 18(2013), No. 5, 1288-1303. https://doi. org/10.1016/j.cnsns.2012.09.007; MR2998588
[39] G. F. Webb, Theory of nonlinear age-dependent population dynamics, Marcel Dekker, New York, 1985. MR0772205
[40] R. Xu, X. Tian, F. Zhang, Global dynamics of a tuberculosis transmission model with age of infection and incomplete treatment, Adv. Difference Equ. 2017, Paper No. 242, 34 pp. https://doi.org/10.1186/s13662-017-1294-z; MR3687414
[41] L. Xue, H. Zhang, W. Sun, C. Scoglio, Transmission dynamics of multi-strain dengue virus with cross-immunity, Appl. Math. Comput. 392(2021), 125742. https://doi.org/10. 1016/j.amc.2020.125742; MR4168923
[42] T.-T. Zheng, L.-F. Nie, Modelling the transmission dynamics of two-strain Dengue in the presence awareness and vector control, J. Theoret. Biol. 443(2018), 82-91. https: //doi.org/ 10.1016/j.jtbi.2018.01.017; MR3765526
[43] Dengue and severe dengue. https://www.who.int/news-room/fact-sheets/detail/ dengue-and-severe-dengue
[44] Ten threats to global health in 2019. https://www.who.int/emergencies/ten-threats-to-global-health-in-2019

# The algebraic curves of planar polynomial differential systems with homogeneous nonlinearities 

Vladimir M. Cheresiz ${ }^{1}$ and Evgenii P. Volokitin ${ }^{\boxtimes 1,2}$<br>${ }^{1}$ Sobolev Institute of Mathematics, Novosibirsk, 630090, Russia<br>${ }^{2}$ Novosibirsk State University, Novosibirsk, 630090, Russia

Received 7 March 2021, appeared 15 July 2021
Communicated by Armengol Gasull


#### Abstract

We consider planar polynomial systems of ordinary differential equations of the form $\dot{x}=x+P_{n}(x, y), \dot{y}=y+Q_{n}(x, y)$, where $P_{n}(x, y), Q_{n}(x, y)$ are homogeneous polynomials of degree $n$. We study the algebraic and non-algebraic invariant curves of these systems with emphasis on limit cycles.


Keywords: polynomial systems, algebraic limit cycles.
2020 Mathematics Subject Classification: 34C05, 34A34.

## 1 Introduction

The problem of studying the limit cycles is one of the central problems in the theory of ordinary differential equations. A significant subarea in this area is the study of the limit cycles of autonomous planar polynomial systems

$$
\begin{equation*}
\dot{x}=P(x, y), \quad \dot{y}=Q(x, y) . \tag{1.1}
\end{equation*}
$$

Here $P(x, y), Q(x, y)$ are real polynomials of the variables $x, y ; t \in \mathbb{R}$ acts as an independent variable. The degree of the system is the maximum of the degrees of the polynomials $P(x, y), Q(x, y)$.

A limit cycle of system (1.1) is a periodic solution whose trajectory is isolated among the trajectories of all periodic solutions. A limit cycle of system (1.1) is called algebraic of degree $m$ if it is the real oval of an irreducible algebraic curve $H(x, y)=0$ of degree $m$.

The problems of finding algebraic solutions to polynomial systems, in particular, of algebraic cycles, goes back to H. Poincaré and J.-G. Darboux, and are actively developing at present (see [5] and the literature cited therein).

In the present article, we study these problems in application to a differential system of the form

$$
\begin{equation*}
\dot{x}=x+P_{n}(x, y), \quad \dot{y}=y+Q_{n}(x, y) \tag{1.2}
\end{equation*}
$$

[^15]which we call a Darboux-type system. Here $P_{n}(x, y), Q_{n}(x, y)$ are homogeneous real polynomials of degree $n$ of the variables $x, y$. Systems of such a kind appeared in Darboux's works on geometry.

We study necessary and sufficient conditions for the existence of a hyperbolic limit cycle for system (1.2), and this cycle turns out to be unique. The remaining trajectories (except for the singular point at the origin) have a limit cycle as the $\alpha$ - or $\omega$-limit set and cannot be algebraic curves.

We prove that the degree of an algebraic limit cycle of system (1.2) is equal to 2 and obtain necessary and sufficient conditions for the existence of an algebraic limit cycle for (1.2).

The obtained results are illustrated by examples.

## 2 The main part

Consider a Darboux-type system of a more general kind than (1.2):

$$
\begin{equation*}
\dot{x}=s x+P_{n}(x, y), \quad \dot{y}=s y+Q_{n}(x, y), \quad s \neq 0, n>1 . \tag{2.1}
\end{equation*}
$$

Consider the functions

$$
\begin{aligned}
& f(\vartheta)=\cos \vartheta P_{n}(\cos \vartheta, \sin \vartheta)+\sin \vartheta Q_{n}(\cos \vartheta, \sin \vartheta), \\
& g(\vartheta)=\cos \vartheta Q_{n}(\cos \vartheta, \sin \vartheta)-\sin \vartheta P_{n}(\cos \vartheta, \sin \vartheta) .
\end{aligned}
$$

## Theorem 2.1.

(1) If $n$ is even then system (2.1) has no periodic solutions.
(2) If $n$ is odd and $g(\vartheta)$ has zeros on $[0,2 \pi]$ then system (2.1) has no periodic solutions.
(3) If there is a closed trajectory of system (2.1), then it contains inside itself the only singular point that coincides with the origin.
(4) System (2.1) has at most one limit cycle.
(5) For system (2.1) to have a unique limit cycle $\Gamma$, it is necessary and sufficient that the following conditions hold:

$$
\begin{equation*}
g(\vartheta) \neq 0, \quad \vartheta \in[0,2 \pi] ; \quad \operatorname{sg}(0) \int_{0}^{2 \pi} \frac{f(\vartheta)}{g(\vartheta)} d \vartheta<0 . \tag{2.2}
\end{equation*}
$$

(6) The cycle $\Gamma$ is hyperbolic.
(7) If the cycle $\Gamma$ is algebraic then its degree is equal to 2 and it is defined by an algebraic curve $H(x, y)=0, H(x, y)=1+a x^{2}+2 b x y+c y^{2}$.

Proof. Items (1)-(5) are proved in [1,7].
We will briefly give fragments of this proof in order to use them for proving items (6)-(7), which supplement the results of [1,7] concerning Darboux-type systems.

Henceforth, unless otherwise specified, we assume that $n$ is odd and conditions (4) are fulfilled.

After passing to the polar coordinates $x=r \cos \vartheta, y=r \sin \vartheta$, system (2.1) turns into the system

$$
\begin{equation*}
\dot{r}=s r+r^{n} f(\vartheta), \quad \dot{\vartheta}=r^{n-1} g(\vartheta), \tag{2.3}
\end{equation*}
$$

which we replace by the linear equation

$$
\begin{equation*}
\frac{d \rho}{d \vartheta}=\frac{(n-1) f(\vartheta)}{g(\vartheta)} \rho+\frac{s(n-1)}{g(\vartheta)}, \rho=r^{n-1}, \tag{2.4}
\end{equation*}
$$

where the functions $f(\vartheta), g(\vartheta)$ are defined above.
The periodic solutions to this equation (with period $T=2 \pi$ ) generate periodic solutions to system (2.1).

Introduce the function

$$
F(\vartheta)=\exp \left((n-1) \int_{0}^{\vartheta} \frac{f(\tau)}{g(\tau)} d \tau\right) .
$$

Denote the solution to equation (2.4) with $\rho(0)=\rho_{0}$ by $\rho\left(\vartheta ; \rho_{0}\right)$ :

$$
\begin{equation*}
\rho\left(\vartheta ; \rho_{0}\right)=\left(\rho_{0}+s(n-1) \int_{0}^{\vartheta} \frac{d \tau}{g(\tau) F(\tau)}\right) F(\vartheta) . \tag{2.5}
\end{equation*}
$$

For a periodic solution, we must take $\rho_{0}=\rho_{0}^{*}$ from the condition $\rho\left(2 \pi ; \rho_{0}\right)=\rho_{0}$. In the case under consideration, there exists a unique such value:

$$
\begin{equation*}
\rho_{0}^{*}=s(n-1) \frac{F(2 \pi)}{1-F(2 \pi)} \int_{0}^{2 \pi} \frac{d \tau}{g(\tau) F(\tau)} . \tag{2.6}
\end{equation*}
$$

For a solution $\rho=\rho\left(\vartheta, \rho_{0}^{*}\right)$ to define a periodic solution to (2.1), the condition $\rho\left(\vartheta, \rho_{0}^{*}\right)>$ $0, \vartheta \in[0,2 \pi]$ must be fulfilled. This condition holds by (2.2).

The orbits of system (2.1) have the parametric definition

$$
\begin{equation*}
x=\sqrt[n-1]{\rho\left(\vartheta ; \rho_{0}\right)} \cos \vartheta, \quad y=\sqrt[n-1]{\rho\left(\vartheta ; \rho_{0}\right)} \sin \vartheta, \quad \vartheta \in[0,2 \pi], \tag{2.7}
\end{equation*}
$$

where $\rho\left(\vartheta ; \rho_{0}\right)$ is from (7).
For a periodic orbit, we must take $\rho_{0}=\rho_{0}^{*}$ in (2.6).
The cycle under consideration is hyperbolic. For showing this, calculate the derivative of the solution (7) with respect to $\rho_{0}$ for $\vartheta=2 \pi$ at the point $\rho_{0}=\rho_{0}^{*}$.

$$
\mu=\left.\frac{\partial \rho\left(\vartheta ; \rho_{0}\right)}{\partial \rho_{0}}\right|_{\left\{\vartheta=2 \pi, \rho_{0}=\rho_{0}^{*}\right\}}=F(2 \pi) .
$$

By (2.2), we have that $\mu \neq 1$, i.e., the cycle $\Gamma$ is hyperbolic.
Item (6) is proved.
For proving item (7), make use of another trick, proposed in [4].
The direction field of system (2.1) is symmetric with respect to the origin. In this case, the trajectories of this system and the formulas defining them must also possess the symmetry property. In particular, closed algebraic curves are defined by polynomials of the form $H(x, y)=h_{0}+h_{2}(x, y)+h_{4}(x, y)+\ldots$, where $h_{0}=$ const $\neq 0, h_{2}(x, y), h_{4}(x, y), \ldots$ are homogeneous polynomials of even degrees. Without loss of generality, we may assume that $h_{0}=1$.

If the limit cycle of system (2.1) is algebraic then consider an irreducible polynomial $H(x, y)=1+h_{2}(x, y)+\cdots+h_{2 k}(x, y)$ such that $H(x, y)=0$ contains the oval defined by (2.7)
with $\rho_{0}=\rho_{0}^{*}$. $H(x, y)=0$ is an invariant algebraic curve of (2.1), [4]. Then $H(r \cos \vartheta, r \sin \vartheta)=$ 0 is an invariant curve of system (2.3). After inserting $R=r^{2}$ in $H(r \cos \vartheta, r \sin \vartheta)$, we obtain a polynomial $\tilde{H}(R, \vartheta)$ of the variable $R$ whose coefficients are $h_{i}(\cos \vartheta, \sin \vartheta)$.

Note that the polynomial $\tilde{H}(R, \vartheta)$ has only positive roots. Moreover, $R=\left(\rho\left(\vartheta ; \rho_{0}^{*}\right)\right)^{\frac{2}{n-1}}$ is its $R$-root. Each $R$-root of $\tilde{H}(R, \vartheta)$ generates a solution $\rho\left(\vartheta ; \rho_{0}\right)=R^{\frac{n-1}{2}}$ to (2.4). If $\rho_{0} \neq$ $\rho_{0}^{*}$ then the solution $\rho\left(\vartheta ; \rho_{0}\right)$ takes infinitely many different values at the points $\vartheta=2 \pi k$ for all integers $k$. On the other hand, $\tilde{H}(R, 2 \pi k)=\tilde{H}(R, 0)$ for all integers $k$; therefore, these polynomials have the same roots. Hence, the polynomial $\tilde{H}(R, 0)$ has infinitely many roots $R=\left(\rho\left(2 \pi k ; \rho_{0}\right)\right)^{\frac{2}{n-1}}$, which is impossible. We see that the $R$-root necessarily corresponds to a unique limit cycle system $R^{(n-1) / 2}=\rho\left(\vartheta ; \rho_{0}^{*}\right)$. Since the polynomial $\tilde{H}$ has only positive roots, we conclude that $\tilde{H}(R, \vartheta)$ has one and only one $R$-root and, thus, it takes the form $\tilde{H}(R, \vartheta)=1+h_{2}(\cos \vartheta, \sin \vartheta) R$. Then $H(r \cos \vartheta, r \sin \vartheta)=1+r^{2} h_{2}(\cos \vartheta, \sin \vartheta)$, which implies that $H(x, y)=1+a x^{2}+2 b x y+c y^{2}$.

Item (7) is proved.
Theorem 2.2. System (2.1) has an algebraic limit cycle $H(x, y) \equiv 1+h_{2}(x, y)=0, h_{2}(x, y)<0$ if and only if the conditions

$$
P_{n} \frac{\partial h_{2}}{\partial x}+Q_{n} \frac{\partial h_{2}}{\partial y}=2 s\left(-h_{2}\right)^{\frac{n+1}{2}} ; \quad x P_{n}-y Q_{n} \neq 0 \quad \text { for }(x, y) \neq(0,0)
$$

are satisfied.
Proof. Recall that if the trajectory of a planar polynomial system of differential equations

$$
\begin{equation*}
\dot{x}=P(x, y), \quad \dot{y}=Q(x, y) \tag{1.1}
\end{equation*}
$$

is a part of an irreducible algebraic curve $H(x, y)=0$ then there exists a polynomial $k(x, y)$ (cofactor) such that

$$
\begin{equation*}
\frac{\partial H(x, y)}{\partial x} P(x, y)+\frac{\partial H(x, y)}{\partial y} Q(x, y)=k(x, y) H(x, y) . \tag{2.8}
\end{equation*}
$$

Obviously, the degree of a cofactor is at most $n-1$ if $n$ is the maximum of the degrees of the polynomials $P(x, y), Q(x, y)$.

As follows from Theorem 2.1, for system (2.1), the closed algebraic curve (an algebraic limit cycle) is defined by the polynomial of the form $H(x, y)=1+h_{2}(x, y)$, where $h_{2}(x, y)$ is a homogeneous polynomial of degree 2 .

Condition (2.8) takes the form

$$
\left(s x+P_{n}\right) \frac{\partial}{\partial x}\left(1+h_{2}\right)+\left(s y+Q_{n}\right) \frac{\partial}{\partial y}\left(1+h_{2}\right)=k\left(1+h_{2}\right) .
$$

Setting $n=2 m+1$, we have $k=k_{2}+k_{4}+\cdots+k_{2 m}$, and

$$
\begin{aligned}
\left(s x+P_{2 m+1}\right) \frac{\partial h_{2}}{\partial x}+(s y+ & \left.Q_{2 m+1}\right) \frac{\partial h_{2}}{\partial y} \\
& =k_{2}+\left(h_{2} k_{2}+k_{4}\right)+\left(h_{2} k_{4}+k_{6}\right)+\cdots+\left(h_{2} k_{2 m-2}+k_{2 m}\right)+h_{2} k_{2 m} .
\end{aligned}
$$

We get

$$
\begin{aligned}
s x \frac{\partial h_{2}}{\partial x}+s y \frac{\partial h_{2}}{\partial y} & =k_{2} \\
h_{2} k_{2}+k_{4} & =0 \\
h_{2} k_{4}+k_{6} & =0 \\
& \vdots \\
h_{2} k_{2 m-2}+k_{2 m} & =0 \\
P_{2 m+1} \frac{\partial h_{2}}{\partial x}+Q_{2 m+1} \frac{\partial h_{2}}{\partial y} & =h_{2} k_{2 m}
\end{aligned}
$$

Using Euler's formula for the homogeneous polynomial $h_{2}(x, y)$, we have

$$
2 s h_{2}=k_{2}
$$

Further, we put

$$
k_{4}=-h_{2} k_{2}=-2 s h_{2}^{2}, \quad k_{6}=-h_{2} k_{4}=2 s h_{2}^{3}, \ldots, k_{2 m}=-h_{2} k_{2 m-2}=(-1)^{m+1} 2 s h_{2}^{m} .
$$

Based on these equalities, we have

$$
P_{2 m+1} \frac{\partial h_{2}}{\partial x}+Q_{2 m+1} \frac{\partial h_{2}}{\partial y}=2 s(-1)^{m+1} h^{2 m+1}
$$

The last equality proves the invariance of the curve $H(x, y)=0$.
There are no singular points on the curve.
The theorem is proved.
In the case of a cubic Darboux system $(n=3)$, we can give an exhaustive solution the problem under consideration.

In [2], Theorem 3.2 was proved, which classifies planar homogeneous cubic vector fields. In our case, this theorem has the following consequence:

Proposition 2.3. The system

$$
\begin{equation*}
\dot{x}=x+P_{3}(x, y), \quad \dot{y}=y+Q_{3}(x, y) \tag{2.9}
\end{equation*}
$$

has a limit cycle only if there exists a linear transformation $\sigma \in G L(2 ; \mathbb{R})$ and a time scaling taking system (2.9) into the system of the form

$$
\begin{align*}
\dot{x} & =s x+p_{1} x^{3}+\left(p_{2}-\alpha\right) x^{2} y+p_{3} x y^{2}-\alpha y^{3} \equiv s x+\tilde{P}_{3}(x, y), \\
\dot{y} & =s y+\alpha x^{3}+p_{1} x^{2} y+\left(p_{2}+\alpha\right) x y^{2}+p_{3} y^{3} \equiv s y+\tilde{Q}_{3}(x, y),  \tag{2.10}\\
p_{1}, \quad p_{2}, \quad p_{3}, \quad s & \in \mathbb{R}, \quad \alpha= \pm 1 .
\end{align*}
$$

For system (2.10),

$$
\begin{aligned}
f(\vartheta) & =\frac{1}{2}\left(p_{1}+p_{3}+\left(p_{1}-p_{3}\right) \cos 2 \vartheta+p_{2} \sin 2 \vartheta\right), \quad g(\vartheta) \equiv \alpha, \\
s g(0) \int_{0}^{2 \pi} \frac{f(\vartheta)}{g(\vartheta)} d \vartheta & =\pi s\left(p_{1}+p_{3}\right) .
\end{aligned}
$$

By Theorem 2.1, we infer that, for the existence of a unique hyperbolic limit cycle for system (2.10), it is necessary and sufficient that

$$
\begin{equation*}
s\left(p_{1}+p_{3}\right)<0 \tag{2.11}
\end{equation*}
$$

Consider the question of the existence of an algebraic limit cycle for system (2.10).
Suppose that system (2.10) has a quadratic limit cycle $H=1+h_{2} \equiv 1+a x^{2}+2 b x y+c y^{2}$. Condition (2.8) takes the form

$$
\frac{\partial H}{\partial x}\left(s x+\tilde{P}_{3}\right)+\frac{\partial H}{\partial y}\left(s y+\tilde{Q}_{3}\right)=k_{2} H .
$$

From Theorem 2.2 we obtain

$$
\begin{equation*}
\tilde{P}_{3} \frac{\partial h_{2}}{\partial x}+\tilde{Q}_{3} \frac{\partial h_{2}}{\partial y}=2 s h_{2}^{2} . \tag{2.12}
\end{equation*}
$$

Equating the coefficients at the same degrees of the variables $x, y$ on the left- and righthand sides of (2.12), after easy transformations, we obtain the system of equalities

$$
\begin{align*}
-a^{2} s+a p_{1}+\alpha b & =0, \\
-4 a b s+a p_{2}-\alpha a+2 b p_{1}+\alpha c & =0, \\
-2 a c s+a p_{3}-4 b^{2} s+2 b p_{2}+c p_{1} & =0,  \tag{2.13}\\
-\alpha a-4 b c s+2 b p_{3}+c p_{2}+\alpha c & =0, \\
-\alpha b-c^{2} s+c p_{3} & =0 .
\end{align*}
$$

Here and below, we have used the Mathematica system for implementing symbols and numerical calculations.

System (2.13) can be regarded as an inhomogeneous system of linear equations $A X=B$ with respect to the variables $p_{1}, p_{2}, p_{3}$ with the parameters $a, b, c, s, \alpha$

$$
A=\left(\begin{array}{ccc}
a & 0 & 0 \\
2 b & a & 0 \\
c & 2 b & a \\
0 & c & 2 b \\
0 & 0 & c
\end{array}\right), \quad B=\left(\begin{array}{c}
a^{2} s-\alpha b \\
\alpha(a-c)+4 a b s \\
2 a c s+4 b^{2} s \\
\alpha(a-c)+4 b c s \\
\alpha b+c^{2} s
\end{array}\right), \quad X=\left(\begin{array}{l}
p_{1} \\
p_{2} \\
p_{3}
\end{array}\right) .
$$

The system is solvable if and only if the rank of the matrix $A$ is equal to the rank of the extended matrix $(A \mid B)$.

Obviously, $a \neq 0$ in (2.13). Therefore, rank $A=3$. Then all minors of order 4 in $(A \mid B)$ must be zero. These minors are

$$
\begin{aligned}
& (A \mid B)_{1}=\alpha\left(a^{2} c^{2}-8 a b^{2} c-2 a c^{3}+8 b^{4}+4 b^{2} c^{2}+c^{4}\right), \\
& (A \mid B)_{2}=\alpha\left(-3 a^{2} b c+4 a b^{3}+2 a b c^{2}+b c^{3}\right), \\
& (A \mid B)_{3}=\alpha\left(-a^{3} c+2 a^{2} b^{2}+2 a^{2} c^{2}-a c^{3}+2 b^{2} c^{2}\right), \\
& (A \mid B)_{4}=\alpha\left(a^{3} b+2 a^{2} b c-3 a b c^{2}+4 b^{3} c\right), \\
& (A \mid B)_{5}=\alpha\left(a^{4}-2 a^{3} c+4 a^{2} b^{2}+a^{2} c^{2}-8 a b^{2} c+8 b^{4}\right),
\end{aligned}
$$

where $(A \mid B)_{i}$ stands for the minor obtained from $(A \mid B)$ by deleting the $i$ th row. Note that the obtained expressions do not depend on $s$ and contain $\alpha$ as a factor.

Since $\alpha= \pm 1$, it suffices to consider the system of homogeneous equations

$$
\begin{align*}
a^{2} c^{2}-8 a b^{2} c-2 a c^{3}+8 b^{4}+4 b^{2} c^{2}+c^{4} & =0 \\
-3 a^{2} b c+4 a b^{3}+2 a b c^{2}+b c^{3} & =0 \\
-a^{3} c+2 a^{2} b^{2}+2 a^{2} c^{2}-a c^{3}+2 b^{2} c^{2} & =0  \tag{2.14}\\
a^{3} b+2 a^{2} b c-3 a b c^{2}+4 b^{3} c & =0 \\
a^{4}-2 a^{3} c+4 a^{2} b^{2}+a^{2} c^{2}-8 a b^{2} c+8 b^{4} & =0
\end{align*}
$$

The second and fourth equations have the form

$$
\begin{equation*}
b\left(-3 a^{2} c+4 a b^{2}+2 a c^{2}+c^{3}\right)=0, \quad b\left(a^{3}+2 a^{2} c-3 a c^{2}+4 b^{2} c\right)=0 \tag{2.15}
\end{equation*}
$$

Case 1. $b=0$.
In this case, system (2.14) is reduced to the system

$$
c^{2}(a-c)^{2}=a c(a-c)^{2}=a^{2}(a-c)^{2}=0
$$

which implies that we have a nonzero solution $a=c \neq 0, b=0$ to system (2.14).
Case 2.
If $b \neq 0$ then $-3 a^{2} c+4 a b^{2}+2 a c^{2}+c^{3}=a^{3}+2 a^{2} c-3 a c^{2}+4 b^{2} c=0$, which implies that $(a-c)(a+c)^{3}=0$. If $a=c$ then the fourth equation of (2.14) gives $c=0$, and hence the fifth equation yields $b=0$, which contradicts the assumption $b \neq 0$. If $a=-c$ then the only real solution to (2.15) is $a=b=c=0$, which again leads to a contradiction.

For the above-found values of the parameters $a, b, c$, system (2.13) is reduced to the form

$$
a p_{1}=a^{2} s, \quad a p_{2}=0, \quad a p_{1}+a p_{3}=2 a^{2} s, \quad a p_{2}=0, \quad a p_{3}=a^{2} s, \quad a \neq 0
$$

and has a nonzero solution $p_{1}=p_{3} \neq 0, p_{2}=0, a=c=p_{1} / s, b=0$.
Then $H(x, y)=1+\frac{p_{1}}{s}\left(x^{2}+y^{2}\right)$, and the algebraic curve $H(x, y)=0$ defines a real oval (circle) under the condition $p_{1} / s<0$ (cf. (2.11)).

Thus, we have proved
Theorem 2.4. System (2.10) admits a hyperbolic algebraic cycle if and only if

$$
\begin{equation*}
p_{1}=p_{3}, \quad p_{2}=0, \quad p_{1} s<0 \tag{2.16}
\end{equation*}
$$

Moreover, the cycle is defined by the algebraic curve

$$
H \equiv 1+\frac{p_{1}}{s}\left(x^{2}+y^{2}\right)=0
$$

System (2.10) for which conditions (2.16) are fulfilled has the form

$$
\begin{equation*}
\dot{x}=s x+p x^{3}-\alpha x^{2} y+p x y^{2}-\alpha y^{3}, \quad \dot{y}=s y+\alpha x^{3}+p x^{2} y+\alpha x y^{2}+p y^{3} . \tag{2.17}
\end{equation*}
$$

Put $\delta=\sqrt{-s / p}$.
A straightforward check shows that

$$
x(t)=\delta \cos \delta^{2} t, \quad y(t)=\alpha \delta \sin \delta^{2} t
$$

is a periodic solution to system (2.17). This solution is a suitable parametrization for the circle $H=0$ mentioned in the theorem. The period of the obtained cycle is equal to $T=2 \pi / \delta^{2}$.

The cycle is stable if $s>0$ and unstable if $s<0$.
Using Therem 2.2 and Theorem 2.4, in [8], we proved

Theorem 2.5. System (2.10) has an algebraic limit cycle if and only if the coefficients $p_{i j}, q_{i j}, i, j=$ $0,1,2,3, i+j=3$ are representable as

$$
\begin{aligned}
& p_{30}=-s \frac{\left(c^{2}+d^{2}\right)(\alpha(a c+b d)+p(a d-b c))}{(b c-a d)^{3}}, \\
& p_{21}=s \frac{a^{2}\left(\alpha\left(3 c^{2}+d^{2}\right)+2 c d p\right)+2 a b\left(2 \alpha c d+p\left(d^{2}-c^{2}\right)\right)+b^{2}\left(\alpha\left(c^{2}+3 d^{2}\right)-2 c d p\right)}{(b c-a d)^{3}}, \\
& p_{12}=-s \frac{\left(a^{2}+b^{2}\right)(3 \alpha a c+a d p+3 \alpha b d-b c p)}{(b c-a d)^{3}}, \\
& p_{03}=s \frac{\alpha\left(a^{2}+b^{2}\right)^{2}}{(b c-a d)^{3}}, \quad q_{30}=-s \frac{\alpha\left(c^{2}+d^{2}\right)^{2}}{(b c-a d)^{3}}, \\
& q_{21}=s \frac{\left(c^{2}+d^{2}\right)(3 \alpha(a c+b d)-p(a d-b c))}{(b c-a d)^{3}}, \\
& q_{12}=-s \frac{a^{2}\left(\alpha\left(3 c^{2}+d^{2}\right)-2 c d p\right)+2 a b\left(2 \alpha c d+p\left(c^{2}-d^{2}\right)\right)+b^{2}\left(\alpha\left(c^{2}+3 d^{2}\right)+2 c d p\right)}{(b c-a d)^{3}}, \\
& q_{03}=s \frac{\left(a^{2}+b^{2}\right)(\alpha(a c+b d)-p(a d-b c))}{(b c-a d)^{3}},
\end{aligned}
$$

where $a, b, c, d, p, s \in \mathbb{R}, a d-b c \neq 0, p s<0, \alpha= \pm 1$.
Moreover, the cycle is defined by the algebraic curve

$$
H \equiv 1+\frac{p s}{(a d-b c)^{2}}\left((a y-c x)^{2}+(b y-d x)^{2}\right) .
$$

Consider several examples illustrating the obtained results.
Example 2.6. ([1]).

$$
\begin{equation*}
\dot{x}=-x+x^{3}-x^{2} y+x y^{2}-y^{3}, \quad \dot{y}=-y+x^{3}+x^{2} y+x y^{2}+y^{3} . \tag{2.18}
\end{equation*}
$$

The system can be written down in the form (2.10) if we take $s=-1, p_{1}=p_{3}=1, p_{2}=$ $0, \alpha=1$. Hence, by Theorem 2.4, system (2.18) has the hyperbolic algebraic limit cycle $1-x^{2}-y^{2}=0$. The cycle is unstable since it contains a stable singular point, the origin. This cycle was presented in [1].

## Example 2.7.

$$
\begin{equation*}
\dot{x}=x-2 x^{2} y-4 x y^{2}-2 y^{3}, \quad \dot{y}=y+2 x^{3}+2 x y^{2}-4 y^{3} . \tag{2.19}
\end{equation*}
$$

We have

$$
f(\vartheta)=-4 \sin ^{2} \vartheta, \quad g(\vartheta) \equiv 2, \quad F(\vartheta)=e^{\sin 2 \vartheta-2 \theta} .
$$

By Theorem 2.1, we conclude that the system has a hyperbolic stable limit cycle.
By (2.5), (2.6), the cycle is written as

$$
\begin{aligned}
r & =\sqrt{\rho}, \quad \rho=\left(\rho_{0}^{*}+\int_{0}^{\vartheta} e^{2 \tau-\sin 2 \tau} d \tau\right) e^{\sin 2 \theta-2 \theta}, \\
\rho_{0}^{*} & =\frac{1}{e^{4 \pi}-1} \int_{0}^{2 \pi} e^{\sin 2 \tau-2 \tau} d \tau .
\end{aligned}
$$

An attempt to find a quadratic limit cycle in the form $1+a x^{2}+2 b x y+c y^{2}=0$ leads to the system of equations

$$
a^{2}-2 b=0, \quad a+2 a b-c=0, \quad 2 a+2 b^{2}+a c=0, \quad a+4 b-c+2 b c=0, \quad 2 b+4 c+c^{2}=0 .
$$

Express $b, c$ in terms of $a$ from the first two equations and insert them in the remaining three equations. After easy calculations, it is not hard to see that the system has only the zero solution $a=b=c=0$. Hence, the limit cycle of system (2.19) is non-algebraic.

Example 2.8.

$$
\dot{x}=x+(x-y)\left(x^{2}+2 y^{2}\right)^{2}, \quad \dot{y}=y-(x+y)\left(x^{2}+2 y^{2}\right)^{2} .
$$

We have

$$
f(\vartheta)=g(\vartheta)=-\frac{1}{4}(\cos 2 \vartheta-3)^{2}, \quad F(\vartheta)=e^{4 \vartheta} .
$$

By Theorem 2.1, we conclude that system has a hyperbolic stable limit cycle.
The cycle is written as

$$
\begin{aligned}
r & =\sqrt{\rho}, \quad \rho=\left(\rho_{0}^{*}-8 \int_{0}^{\vartheta} \frac{d \tau}{e^{4 \tau}(\cos 2 \tau-3)^{2}}\right) e^{4 \vartheta}, \\
\rho_{0}^{*} & =\frac{8 e^{8 \pi}}{e^{8 \pi}-1} \int_{0}^{2 \pi} \frac{d \tau}{e^{4 \tau}(\cos 2 \tau-3)^{2}} .
\end{aligned}
$$

Just as in the previous example, it can be shown that the cycle is non-algebraic due to Theorem 2.2.

Using Example 2.6 and Theorem 2.2, we can construct a Darboux system of any (odd) degree with an algebraic limit cycle.

## Example 2.9.

$$
\dot{x}=-x+\left(x^{3}-x^{2} y+x y^{2}-y^{3}\right)\left(x^{2}+y^{2}\right)^{2 k}, \quad \dot{y}=-y+\left(x^{3}+x^{2} y+x y^{2}+y^{3}\right)\left(x^{2}+y^{2}\right)^{2 k} .
$$

The system has the algebraic limit cycle $H(x, y) \equiv 1-x^{2}-y^{2}=0$.
As we already observed, in the presence of a limit cycle, all the remaining trajectories of system (1.2) are non-algebraic curves. In a neighborhood of the cycle, these trajectories are spirals with infinitely many helices and intersect a straight line transversal to the cycle infinitely many times. In this case, the corresponding algebraic equation has infinitely many roots, which is impossible.

Example 2.6 shows that the coexistence of algebraic and non-algebraic curves is possible for cubic Darboux systems. Example 2.7 demonstrates that there exist cubic Darboux systems having no algebraic curve. With account taken of Theorem 2.2, this property is possessed by all Darboux systems (1.2) with $n \neq 3$ for which $g(\vartheta) \neq 0, \vartheta \in[0,2 \pi]$ (see Example 2.8).

It was proved in [1] that system (1.2) is Darboux integrable and a formula for its first integral was given.

System (1.2) has no polynomial integral (and, generally, no integral defined on the whole phase plane) since its singular point at the origin is a node.

Under certain conditions, system (1.2) has a rational integral. In this case, all its trajectories are algebraic curves.

Recall that if system (1.1) has $N=n(n+1)+2$ algebraic invariants then it admits a rational first integral (see [3]).

## Example 2.10.

$$
\begin{equation*}
\dot{x}=x-x^{3}, \quad \dot{y}=y-y^{3} . \tag{2.20}
\end{equation*}
$$

The singular points of system (2.20) are:
$O(0,0), O_{1}(1,1), O_{2}(-1,1), O_{3}(-1,-1), O_{4}(1,-1)$ are star nodes;
$O_{5}(1,0), O_{6}(0,1), O_{7}(-1,0), O_{8}(0,-1)$ are hyperbolic saddles.
System (2.20) has 8 invariant straight lines $x=0, x= \pm 1, y=0, y= \pm 1, y= \pm x$, using which one can construct the rational Darboux integral

$$
V(x, y)=\frac{y^{2}\left(1-x^{2}\right)}{x^{2}\left(1-y^{2}\right)} .
$$

The phase portrait of system (2.20) is given in Figure 2.1.


Figure 2.1: The phase portrait of system (2.20)
In some cases, a rational integral for system (1.2) can be found with the use of a smaller number of invariants than $N$.

Suppose that the homogeneous polynomials $P_{n}(x, y), Q_{n}(x, y)$ in system (1.2) satisfy the Cauchy-Riemann conditions: $P_{n x}=Q_{n y}, P_{n y}=-Q_{n x}$. Introducing the complex variable $z=x+i y$, we can write down this system in the form

$$
\begin{equation*}
\dot{z}=z+\mathcal{P}_{n}(z), \tag{2.21}
\end{equation*}
$$

where $\mathcal{P}_{n}(z)$ is a complex polynomial of degree $n: \mathcal{P}_{n}(z)=P_{n}((z+\bar{z}) / 2,(z-\bar{z}) / 2 i)+$ $i Q_{n}((z+\bar{z}) / 2,(z-\bar{z}) / 2 i)$.

Theorem 2.11 ([6]). Suppose that all the singular points of system (2.21) are star nodes and all the eigenvalues $\lambda_{1,2}^{k}=\omega_{k}$ are rationally commensurable: $\omega_{k} \in \omega \mathbb{Q}, \omega \in \mathbb{R}, k=1,2, \ldots, n$. Then system (2.21) admits a rational first integral.

In this case, the integral can be constructed with the use of $n$ complex invariants of the form $f_{k}=z-z_{k}$, where $z_{k}$ are the roots of the equation $z+\mathcal{P}_{n}(z)=0$ (see [6]).

Example 2.12. Consider the system

$$
\begin{equation*}
\dot{x}=x-x^{5}+10 x^{3} y^{2}-5 x^{4} y, \quad \dot{y}=y-5 x^{4} y-10 x^{2} y^{3}-y^{5}, \tag{2.22}
\end{equation*}
$$

which corresponds to the complex system $\dot{z}=z-z^{5}$.


Figure 2.2: The phase portrait of the system (2.22)

The singular points of system (2.22) are found from the relation $z-z^{5}=0$ and have the form $O(0,0), O_{1}(1,0), O_{2}(0,1), O_{3}(-1,0), O_{4}(0,-1)$. The origin is an unstable star node; the remaining four singular points are stable star nodes. The system has 5 invariants $f_{1}=z, f_{2,3}=z \pm 1, f_{4,5}=z \pm i$, using which we can construct the rational first integral

$$
V(x, y)=\frac{x y\left(x^{2}-y^{2}\right)}{\left(x^{2}+y^{2}\right)^{4}-\left(x^{2}+y^{2}\right)^{2}+8 x^{2} y^{2}} .
$$

Figure 2.2 contains a phase portrait of system (2.22).

## Acknowledgments

V. M. Cheresiz carried out research within the framework of the state contract of Sobolev Institute of Mathematics (project no. 0314-2019-0010). E. P. Volokitin carried out research within the framework of the state contract of Sobolev Institute of Mathematics (project no. 0314-20190007).

## References

[1] A. Bendjeddou, J. Llibre, T. Salhi, Dynamics of the polynomial differential systems with homogeneous nonlinearities and a star node, J. Differential Equations 254(2013), No. 8, 3530-3537. https://doi.org/10.1016/j.jde.2013.01.032; MR3020886
[2] A. Cima, J. Llibre, Algebraic and topological classification of the homogeneous cubic vector fields in the plane, J. Math. Anal. Appl. 147(1990), No. 2, 420-448. https://doi. org/10.1016/0022-247X (90) 90359-N; MR1047119
[3] I. Garcia, M. Grau, A survey on the inverse integrating factor, Qual. Theory Dyn. Syst. 9(2010), No. 1-2, 115-166. https://doi.org/10.1007/s12346-010-0023-8; MR2737365
[4] J. Giné, M. Grau, Coexistence of algebraic and non-algebraic limit cycles, explicitly given, using Riccati equations, Nonlinearity 19(2006), 1939-1950. https://doi.org/10. 1088/0951-7715-19/8/009; MR2250800
[5] J. Llibre, X. Zhang, A survey on algebraic and explicit non-algebraic limit cycles in planar differential systems, Expo. Math. 39(2020), No. 1, 48-61. https://doi.org/10. 1016/j.exmath. 2020.03.001; MR4229369
[6] E. Volokitin, Algebraic first integrals of the polynomial systems satisfying the CauchyRiemann conditions, Qual. Theory Dyn. Syst., 1(2016), No. 2, 575-596. https://doi.org/ 10.1007/s12346-015-0174-8; MR3563437
[7] E. Volokitin, V. Cheresiz, Qualitative investigation of plane polynomial Darboux-type systems (in Russian), Sib. Electron. Math. Izv. 13(2016), 1170-1186. https://doi.org/ 1017377/semi.2016.13.008; MR3592221
[8] E. Volokitin, V. Cheresiz, The algebraic limit cycles of planar cubic systems, Sib. Electron. Math. Izv. 17(2020), 2045-2054. https://doi.org/10.33048/semi.2020.17.136; MR4239151

# Existence of global solutions to chemotaxis fluid system with logistic source 

Harumi Hattori ${ }^{\boxtimes}$ and Aesha Lagha<br>Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

Received 17 February 2021, appeared 23 July 2021
Communicated by Maria Alessandra Ragusa


#### Abstract

We establish the existence of global solutions and $L^{q}$ time-decay of a three dimensional chemotaxis system with chemoattractant and repellent. We show the existence of global solutions by the energy method. We also study $L^{q}$ time-decay for the linear homogeneous system by using Fourier transform and finding Green's matrix. Then, we find $L^{q}$ time-decay for the nonlinear system using solution representation by Duhamel's principle and time-weighted estimate.


Keywords: chemotaxis system, energy method, a priori estimates, Fourier transform, time-decay rates.
2020 Mathematics Subject Classification: Primary: 35Q31, 35Q35; Secondary: 35Q92, 76N10.

## 1 Introduction

Chemotaxis is the oriented movement of biological cells or microscopic organisms toward or away from the concentration gradient of certain chemicals in their environment. We may use cells to denote the biological objects whose movement we are interested in and chemo attractants or repellents to denote chemicals which attract or repell the cells. This type of movement exists in many biological phenomena, such as the movement of bacteria toward certain chemicals [1], or the movement of endothelial cells toward the higher concentration of chemoattractant that cancer cells produce [4].

Keller and Segel $[11,12]$ derived a mathematical model to describe the aggregation of certain types of bacteria, which consists of the equations for the cell density $n=n(x, t)$ and the concentration of chemical attractant $c=c(x, t)$ and is given by

$$
\left\{\begin{array}{l}
n_{t}=\Delta n-\nabla \cdot(n \chi \nabla c), \\
\alpha c_{t}=\Delta c+f(c, n)
\end{array}\right.
$$

where $\chi$ is the sensitivity of the cell movement to the density gradient of the attractant, $\alpha$ is a positive constant, and the reaction term $f$ is a smooth function of the arguments. Since then,

[^16]many mathematical approaches to describe chemotaxis using systems of partial differential equations have emerged, some of which will be discussed later in this section.

In this paper, we use the equations for continuum mechanics to describe the movement of cells and for the chemoattractant and repellent, we use diffusion equations. The combined effects of chemoattractant and repellent for chemotaxis are studied in diseases such as Alzheimer's disease [2].

We consider the initial value problem for the system in $\mathbb{R}^{3}$ given by

$$
\left\{\begin{array}{l}
\partial_{t} n+\nabla \cdot(n u)=n\left(n_{\infty}-n\right)  \tag{1.1}\\
\partial_{t} u+u \cdot \nabla u+\frac{\nabla p(n)}{n}=\chi_{1} \nabla c_{1}-\chi_{2} \nabla c_{2}+\delta \Delta u \\
\partial_{t} c_{1}=\Delta c_{1}-a_{12} c_{1}+a_{11} c_{1} n \\
\partial_{t} c_{2}=\Delta c_{2}-a_{22} c_{2}+a_{21} c_{2} n,
\end{array}\right.
$$

where $n(x, t), u(x, t), c_{1}(x, t), c_{2}(x, t)$ for $t>0, x \in \mathbb{R}^{3}$, are the cell concentration, velocity of cells, chemoattractant concentration, and chemorepellent concentration, respectively. The initial data is given by

$$
\begin{equation*}
\left.\left(n, u, c_{1}, c_{2}\right)\right|_{t=0}=\left(n_{0}, u_{0}, c_{1,0}, c_{2,0}\right)(x), \quad x \in \mathbb{R}^{3} \tag{1.2}
\end{equation*}
$$

where it is supposed to hold that

$$
\left(n_{0}, u_{0}, c_{1,0}, c_{2,0}\right)(x) \rightarrow\left(n_{\infty}, 0,0,0\right) \quad \text { as }|x| \rightarrow \infty,
$$

for some constant $n_{\infty}>0$.
In this model the cells follow a convective logistic equation, the velocity is given by the compressible Navier-Stokes type equations with the added effects of chemoattractants and -repellents. The pressure for the cells $p(n)$ is a smooth function of $n$ and $p^{\prime}(n)>0$, a positive constant $\delta$ is the coefficient for the viscosity term, and $\chi_{1}$ and $\chi_{2}$ express the sensitivity of the cell movement to the density gradients of the attractants and repellents, respectively. Usually $\chi_{i},(i=1,2)$ are functions of $c_{i}$ and in this paper we consider the case $\chi_{i}=K_{i} c_{i}$, where $K_{i}$ are positive constants, so that the sensitivity is proportional to the concentration of the attractants and repellents. We choose $K_{i}=2$ for simplicity. We may equally use $\chi_{i}=K_{i} c_{i}^{\alpha_{i}}$, where $\alpha_{i}$ are positive constants. For chemical substances, we use the reaction diffusion equations. The reaction terms are based on a Lotka-Volterra type model in which the nonnegative regions of $c_{i}$ are invariant in the sense that if the initial conditions for $c_{i}$ are nonnegative, they are nonnegative for positive $t$. This can be verified by the maximum principle. The couplings between $c_{i}$ and $n$ are given as nonlinear terms.

The main goal of this paper is to establish the local and global existence of smooth solutions in three dimensions around a constant state ( $n_{\infty}, 0,0,0$ ) and the decay rate of global smooth solutions for the above system (1.1). The main result of this paper is stated as follows.

Theorem 1.1. Let $N \geq 4$ be an integer. There exists a positive numbers $\epsilon_{0}, C_{0}$ such that if

$$
\left\|\left[n_{0}-n_{\infty}, u_{0}, c_{1,0}, c_{2,0}\right]\right\|_{H^{N}} \leq \epsilon_{0},
$$

then, the Cauchy problem (1.1)-(1.2) has a unique solution $\left(n, u, c_{1}, c_{2}\right)(t)$ globally in time which satisfies

$$
\begin{aligned}
\left(u, c_{1}, c_{2}\right)(t) & \in C\left([0, \infty) ; H^{N}\left(\mathbb{R}^{3}\right)\right) \cap C^{1}\left([0, \infty) ; H^{N-2}\left(\mathbb{R}^{3}\right)\right), \\
n-n_{\infty} & \in C\left([0, \infty) ; H^{N}\left(\mathbb{R}^{3}\right)\right) \cap C^{1}\left([0, \infty) ; H^{N-1}\left(\mathbb{R}^{3}\right)\right)
\end{aligned}
$$

and there are constants $\lambda_{1}>0$ and $\lambda_{2}>0$ such that

$$
\begin{align*}
& \left\|\left[n-n_{\infty}, u, c_{1}, c_{2}\right]\right\|_{H^{N}}^{2}+\lambda_{1} \int_{0}^{t}\left\|\nabla\left[u, c_{1}, c_{2}\right]\right\|_{H^{N}}^{2}+\lambda_{2} \int_{0}^{t}\left\|\left[n-n_{\infty}, c_{1}, c_{2}\right]\right\|_{H^{N}}^{2} \\
& \quad \leq C_{0}\left\|\left[n_{0}-n_{\infty}, u_{0}, c_{1,0}, c_{2,0}\right]\right\|_{H^{N}}^{2} . \tag{1.3}
\end{align*}
$$

Furthermore, the global solution $\left[n, u, c_{1}, c_{2}\right]$ satisfies the following time-decay rates for $t \geq 0$ :

$$
\begin{align*}
\left\|n-n_{\infty}\right\|_{L^{9}} & \leq C(1+t)^{-2+\frac{3}{29}}  \tag{1.4}\\
\|u\|_{L^{9}} & \leq C(1+t)^{\frac{-3}{2}+\frac{3}{29}},  \tag{1.5}\\
\left\|c_{1}, c_{2}\right\|_{L^{9}} & \leq C(1+t)^{\frac{-3}{2}}, \tag{1.6}
\end{align*}
$$

with $2 \leq q<\infty, C>0$.
The proof of the existence of global solutions in Theorem 1.1 is based on the local existence and an a priori estimates. We show the local solutions by constructing a sequence of approximation functions based on iteration. To obtain the a priori estimates we use the energy method. Moreover, to obtain the time-decay rate in $L^{q}$ norm of solutions in Theorem 1.1, we first find the Green's matrix for the linear system using the Fourier transform and then obtain the refined energy estimates with the help of Duhamel's principle.

To motivate our study, we present previous related work on chemotaxis models. Many of them are based on the Keller-Segel system. Wang [21] explored the interactions between the nonlinear diffusion and logistic source on the solutions of the attraction-repulsion chemotaxis system in three dimensions. E. Lankeit and J. Lankeit [13] proved the global existence of classical solutions to a chemotaxis system with singular sensitivity. Liu and Wang [14] established the existence of global classical solutions and steady states to an attraction-repulsion chemotaxis model in one dimension based on the energy methods.

Concerning the chemotaxis models based on fluid dynamics, there are two approaches, incompressible and compressible. For the incompressible case, Chae, Kang and Lee [3], and Duan, Lorz, and Markowich [8] showed the global-in-time existence for the incompressible chemotaxis equations near the constant states, if the initial data is sufficiently small. Rodriguez, Ferreira, and Villamizar-Roa [19] showed the global existence for an attractionrepulsion chemotaxis fluid model with logistic source. Tan and Zhou [20] proved the global existence and time decay estimate of solutions to the Keller-Segel system in $R^{3}$ with the small initial data. For the compressible case, Ambrosi, Bussolino, and Preziosi [2] discussed the vasculogenesis using the compressible fluid dynamics for the cells and the diffusion equation for the attractant.

Many related approaches use the Fourier transform, and we only mention that Duan [6] and Duan, Liu, and Zhu [7] proved the time-decay rate by the combination of energy estimates and spectral analysis. Also by using Green's function and Schauder fixed point theorem, one can study the existence and regularity of solution for these kinds of equations (see [9,10, 17, 18]).

For later use in this paper, we give some notations. C denotes some positive constant and $\lambda_{i}$, where $i=1,2$, denotes some positive (generally small) constant, where both C and $\lambda_{i}$ may take different values in different places. For any integer $m \geq 0$, we use $H^{m}$ to denote the Sobolev space $H^{m}\left(\mathbb{R}^{3}\right)$. Set $L^{2}=H^{0}$. We set $\partial^{\alpha}=\partial_{x_{1}}^{\alpha_{1}} \alpha_{x_{2}}^{\alpha_{2}} \partial_{x_{3}}^{\alpha_{3}}$ for a multi-index $\alpha=\left[\alpha_{1}, \alpha_{2}, \alpha_{3}\right]$. The length of $\alpha$ is $||=.\alpha_{1}+\alpha_{2}+\alpha_{3}$; we also set $\partial_{j}=\partial_{x_{j}}$ for $j=1,2,3$. For an integrable
function $f: \mathbb{R}^{3} \rightarrow \mathbb{R}$, its Fourier transform is defined by $\hat{f}=\int_{\mathbb{R}^{3}} e^{-i x \cdot \xi} f(x) d x, x \cdot \xi=\sum_{i=0}^{3} x_{j} \xi ;$ and $x \in \mathbb{R}^{3}$, where $i=\sqrt{-1}$ is the imaginary unit. Let us denote the space

$$
\begin{aligned}
X(0, T)=\left\{\left(u, c_{1}, c_{2}\right) \in C\left([0, T] ; H^{N}\left(\mathbb{R}^{3}\right)\right)\right. & \cap C^{1}\left([0, T] ; H^{N-2}\left(\mathbb{R}^{3}\right)\right), \\
n & \left.n-n_{\infty} \in C\left([0, T] ; H^{N}\left(\mathbb{R}^{3}\right)\right) \cap C^{1}\left([0, T] ; H^{N-1}\left(\mathbb{R}^{3}\right)\right)\right\} .
\end{aligned}
$$

This paper is organized as follows. In Section 2, we reformulate the Cauchy problem under consideration. In Section 3, we prove the global existence and uniqueness of solutions. In Section 4, we investigate the linearized homogeneous system to obtain the $L^{2}-L^{9}$ timedecay property and the explicit representation of solutions. In Section 5, we study the $L^{q}$ time-decay rates of solutions to the reformulated nonlinear system and finish the proof of Theorem1.1.

## 2 Reformulation of the system (1.1)

Let $U(t)=\left[n, u, c_{1}, c_{2}\right]$ be a smooth solution to the Cauchy problem of the chemotaxis system (1.1) with initial data $U_{0}=\left[n_{0}, u_{0}, c_{1,0}, c_{2,0}\right]$. We introduce the transformation:

$$
\begin{equation*}
n(x, t)=n_{\infty}+\rho(x, t) . \tag{2.1}
\end{equation*}
$$

Then the Cauchy problem (1.1) is reformulated as

$$
\left\{\begin{array}{l}
\partial_{t} \rho+n_{\infty} \nabla \cdot u+n_{\infty} \rho=-\nabla \cdot(\rho u)-\rho^{2}  \tag{2.2}\\
\partial_{t} u+u \cdot \nabla u-\delta \Delta u+\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}} \nabla \rho=\nabla\left(c_{1}\right)^{2}-\nabla\left(c_{2}\right)^{2}-\left(\frac{p^{\prime}\left(\rho+n_{\infty}\right)}{\rho+n_{\infty}}-\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}}\right) \nabla \rho \\
\partial_{t} c_{1}=\Delta c_{1}-\left(a_{12}-a_{11} n_{\infty}\right) c_{1}+a_{11} \rho c_{1} \\
\partial_{t} c_{2}=\Delta c_{2}-\left(a_{22}-a_{21} n_{\infty}\right) c_{2}+a_{21} \rho c_{2},
\end{array}\right.
$$

with initial data

$$
\begin{equation*}
\left.\left(\rho, u, c_{1}, c_{2}\right)\right|_{t=0}=\left(\rho_{0}, u_{0}, c_{1,0}, c_{2,0}\right) \rightarrow(0,0,0,0) \tag{2.3}
\end{equation*}
$$

as $|x| \rightarrow \infty$, where $\rho_{0}=n_{0}-n_{\infty}$. We assume that $a_{12}-a_{11} n_{\infty}>0$ and $a_{22}-a_{21} n_{\infty}>0$.
In what follows, the integer $N \geq 4$ is always assumed.
Proposition 2.1. There exists a positive number $\epsilon_{0}$ which is small enough such that if

$$
\left\|\left[\rho_{0}, u_{0}, c_{1,0}, c_{2,0}\right]\right\|_{H^{N}} \leq \epsilon_{0},
$$

then the Cauchy problem (2.2)-(2.3) has a unique solution $\left(\rho, u, c_{1}, c_{2}\right)(t)$ globally in time which satisfies $\left(\rho, u, c_{1}, c_{2}\right)(t) \in X(0, \infty)$ and there are constants $C_{0}>0, \lambda_{1}>0$ and $\lambda_{1}>0$ such that

$$
\begin{equation*}
\left\|\left[\rho, u, c_{1}, c_{2}\right]\right\|_{H^{N}}^{2}+\lambda_{1} \int_{0}^{t}\left\|\nabla\left[u, c_{1}, c_{2}\right]\right\|_{H^{N}}^{2}+\lambda_{2} \int_{0}^{t}\left\|\left[\rho, c_{1}, c_{2}\right]\right\|_{H^{N}}^{2} \leq C_{0}\left\|\left[\rho_{0}, u_{0}, c_{1,0}, c_{2,0}\right]\right\|_{H^{N}}^{2} . \tag{2.4}
\end{equation*}
$$

Proposition 2.2. Let $U(t)=\left[\rho, u, c_{1}, c_{2}\right]$ be the solution to the Cauchy problem (2.2)-(2.3) obtained in Proposition 2.1, which satisfies the following $L^{q}$-time decay estimates for any $t \geq 0$ :

$$
\begin{align*}
\|\rho\|_{L^{q}} & \leq C(1+t)^{-2+\frac{3}{2 q}}  \tag{2.5}\\
\|u\|_{L^{q}} & \leq C(1+t)^{\frac{-3}{2}+\frac{3}{2 q}}  \tag{2.6}\\
\left\|c_{1}, c_{2}\right\|_{L^{q}} & \leq C(1+t)^{\frac{-3}{2}}, \tag{2.7}
\end{align*}
$$

with $2 \leq q<\infty$ and $C>0$.
The proof of Theorem 1.1 obtained directly from the global existence proof in Proposition 2.1 and the derivation of rates in Theorem 1.1 is based on Proposition 2.2.

## 3 Global solution of the nonlinear system (2.2)

The goal of this section is to prove the global existence of solutions to the Cauchy problem (2.2) when initial data is a small, smooth perturbation near the steady state ( $n_{\infty}, 0,0,0$ ). The proof is based on some uniform a priori estimates combined with the local existence, which will be shown in Subsections 3.1 and 3.2.

### 3.1 Existence of local solutions

In this subsection, we show the proof of the existence of local solutions $\left[\rho, u, c_{1}, c_{2}\right]$ by constructing a sequence of functions that converges to a function satisfying the Cauchy problem. We construct a solution sequence $\left(\rho^{j}, u^{j}, c_{1}^{j}, c_{2}^{j}\right)_{j \geq 0}$ by iteratively solving the Cauchy problem on the following

$$
\left\{\begin{array}{l}
\partial_{t} \rho^{j+1}+n_{\infty} \nabla \cdot u^{j+1}+n_{\infty} \rho^{j+1}=-\rho^{j} \nabla \cdot u^{j+1}-\nabla \rho^{j+1} u^{j}-\rho^{j^{2}}  \tag{3.1}\\
\partial_{t} u^{j+1}-\delta \Delta u^{j+1}=-u^{j} \cdot \nabla u^{j}+\nabla\left(c_{1}^{j}\right)^{2}-\nabla\left(c_{2}^{j}\right)^{2}-\frac{p^{\prime}\left(\rho^{j}+n_{\infty}\right)}{\rho^{j+n_{\infty}}} \nabla \rho^{j} \\
\partial_{t} c_{1}^{j+1}-\Delta c_{1}^{j+1}+\left(a_{12}-a_{11} n_{\infty}\right) c_{1}^{j+1}=a_{11} \rho^{j} c_{1}^{j+1} \\
\partial_{t} c_{2}^{j+1}-\Delta c_{2}^{j+1}+\left(a_{22}-a_{21} n_{\infty}\right) c_{2}^{j+1}=a_{21} \rho^{j} c_{2}^{j+1},
\end{array}\right.
$$

with initial data

$$
\begin{equation*}
\left.\left(\rho^{j+1}, u^{j+1}, c_{1}^{j+1}, c_{2}^{j+1}\right)\right|_{t=0}=U_{0}=\left(\rho_{0}, u_{0}, c_{1,0}, c_{2,0}\right) \rightarrow(0,0,0,0) \tag{3.2}
\end{equation*}
$$

as $|x| \rightarrow \infty$, for $j \geq 0$. For simplicity, in what follows, we write $U^{j}=\left(\rho^{j}, u^{j}, c_{1}^{j}, c_{2}^{j}\right)$ and $U_{0}=\left(\rho_{0}, u_{0}, c_{1,0}, c_{2,0}\right)$, where $U^{0}=(0,0,0,0)$.
Now, we can start the following Lemma.
Lemma 3.1. There are constants $T_{1}$ and $\epsilon_{0}>0$ such that if the initial data $U_{0} \in H^{N}\left(\mathbb{R}^{3}\right)$ and $\left\|U_{0}\right\|_{H^{N}} \leq \epsilon_{0}$, then there exists a unique solution $U=\left(\rho, u, c_{1}, c_{2}\right)$ of the Cauchy problem (2.2)-(2.3) on $\left[0, T_{1}\right]$ with $U \in X\left(0, T_{1}\right)$.

Proof. We first set $U^{0}=(0,0,0,0)$. Then, we use $U^{0}$ to solve the equations for $U^{1}$. The first equation is the first order partial differential equation and the second, third, and fourth equations are the second order parabolic equations. We obtain $u^{1}(x, t), c_{1}^{1}(x, t), c_{2}^{1}(x, t)$, and $\rho^{1}(x, t)$ in this order. Similarly, we define $\left(u^{j}, c_{1}^{j}, c_{2}^{j}, \rho^{j}\right)$ iteratively. Now, we prove the existence and uniqueness of solutions in space $C\left(\left[0, T_{1}\right] ; H^{N}\left(\mathbb{R}^{3}\right)\right)$, where $T_{1}>0$ is suitably small. The proof is divided into four steps as follows.

In the first step, we show the uniform boundedness of the sequence of functions under our construction via energy estimates. We show that there exists a constant $M>0$ such that $U^{j} \in C\left(\left[0, T_{1}\right] ; H^{N}\left(\mathbb{R}^{3}\right)\right)$ is well defined and

$$
\begin{equation*}
\sup _{0 \leq t \leq T_{1}}\left\|U^{j}(t)\right\|_{H^{N}} \leq M \tag{3.3}
\end{equation*}
$$

for all $j \geq 0$. We use the induction to prove (3.3). It is trivial when $j=0$. Suppose that it is true for $j \geq 0$ where $M$ is small enough. To prove for $j+1$, we need some energy estimate for $U^{j+1}$. Applying $\partial^{\alpha}$ to the first equation of (3.1), multiplying it by $\partial^{\alpha} \rho^{j+1}$ and integrating in $x$,
we obtain

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t} \int_{\mathbb{R}^{3}}\left(\partial^{\alpha} \rho^{j+1}\right)^{2} \mathrm{~d} x+n_{\infty} \int_{\mathbb{R}^{3}}\left|\partial^{\alpha} \rho^{j+1}\right|^{2} \mathrm{~d} x \\
& \quad=-n_{\infty} \int_{\mathbb{R}^{3}} \partial^{\alpha} \rho^{j+1} \partial^{\alpha} \nabla \cdot u^{j+1} \mathrm{~d} x-\int_{\mathbb{R}^{3}} \partial^{\alpha} \rho^{j+1} \partial^{\alpha}\left(\nabla \rho^{j+1} \cdot u^{j}\right) \mathrm{d} x \\
& \quad+\int_{\mathbb{R}^{3}} \partial^{\alpha} \rho^{j+1} \partial^{\alpha}\left(\rho^{j} \nabla \cdot u^{j+1}\right) \mathrm{d} x-\int_{\mathbb{R}^{3}} \partial^{\alpha} \rho^{j+1} \partial^{\alpha} \rho^{j 2} \mathrm{~d} x .
\end{aligned}
$$

The terms on the right hand side are further bounded by

$$
\begin{aligned}
& C\left\|\nabla \cdot u^{j+1}\right\|_{H^{N}}\left\|\rho^{j+1}\right\|_{H^{N}}+C\left\|\nabla \cdot u^{j}\right\|_{L^{\infty}}\left\|\rho^{j+1}\right\|_{H^{N}}^{2} \\
& \quad+\left\|u^{j}\right\|_{H^{N}}\left\|\rho^{j+1}\right\|_{H^{N}}\left\|\nabla \rho^{j+1}\right\|_{H^{N-2}}+\left\|\rho^{j}\right\|_{H^{N}}\left\|\rho^{j+1}\right\|_{H^{N}}\left\|\nabla \cdot u^{j+1}\right\|_{H^{N}} \\
& \\
& \quad+C\left\|\rho^{j}\right\|_{H^{N-2}}\left\|\rho^{j+1}\right\|_{H^{N}}\left\|\rho^{j}\right\|_{H^{N}} .
\end{aligned}
$$

Then, after taking the summation over $|\alpha| \leq N$ and using the Cauchy inequality, one has

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t}\left\|\rho^{j+1}\right\|_{H^{N}}^{2}+\lambda_{2}\left\|\rho^{j+1}\right\|_{H^{N}}^{2} \\
& \quad \leq C\left\|\nabla \cdot u^{j+1}\right\|_{H^{N}}^{2}+C\left\|u^{j}\right\|_{H^{N}}^{2}\left\|\rho^{j+1}\right\|_{H^{N}}^{2}+C\left\|\rho^{j}\right\|_{H^{N}}^{2}\left\|\rho^{j+1}\right\|_{H^{N}}^{2}+C\left\|\rho^{j}\right\|_{H^{N}}^{2} . \tag{3.4}
\end{align*}
$$

Similarly, applying $\partial^{\alpha}$ to the second equation of (3.1), multiplying it by $\partial^{\alpha} u^{j+1}$, taking integrations in $x$, and then using integration by parts, we have

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t} \int_{\mathbb{R}^{3}}\left(\partial^{\alpha} u^{j+1}\right)^{2} \mathrm{~d} x+\delta \int_{\mathbb{R}^{3}}\left|\partial^{\alpha} \nabla \cdot u^{j+1}\right|^{2} \mathrm{~d} x=\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}} \int_{\mathbb{R}^{3}} \nabla \cdot \partial^{\alpha} u^{j+1} \partial^{\alpha} \rho^{j+1} \mathrm{~d} x \\
& \quad-\int_{\mathbb{R}^{3}} \nabla \cdot \partial^{\alpha} u^{j+1} \partial^{\alpha} c_{1}^{j^{2}} \mathrm{~d} x+\int_{\mathbb{R}^{3}} \nabla \cdot \partial^{\alpha} u^{j+1} \partial^{\alpha} c_{2}^{j^{2}} \mathrm{~d} x \\
& \quad-\int_{\mathbb{R}^{3}} \partial^{\alpha} u^{j+1} \cdot \partial^{\alpha}\left(u^{j} \cdot \nabla u^{j}\right) \mathrm{d} x-\int_{\mathbb{R}^{3}} \partial^{\alpha} u^{j+1} \cdot \partial^{\alpha}\left(\frac{\nabla p\left(\rho^{j}+n_{\infty}\right)}{\rho^{j}+n_{\infty}}\right) \mathrm{d} x .
\end{aligned}
$$

Then, after taking the summation over $|\alpha| \leq N$, the terms on the right side of the previous equation are bounded by

$$
\begin{aligned}
& C\left\|\nabla \cdot u^{j+1}\right\|_{H^{N}}\left\|\rho^{j+1}\right\|_{H^{N}}+C\left\|c_{1}^{j}\right\|_{H^{N-3}}\left\|\nabla \cdot u^{j+1}\right\|_{H^{N}}\left\|c_{1}^{j}\right\|_{H^{N}} \\
& \quad+C\left\|c_{2}^{j}\right\|_{H^{N-3}}\left\|\nabla \cdot u^{j+1}\right\|_{H^{N}}\left\|c_{2}^{j}\right\|_{H^{N}}+\left\|u^{j}\right\|_{H^{N}}^{2}\left\|\nabla \cdot u^{j+1}\right\|_{H^{N}}+C\left\|\rho^{j}\right\|_{H^{N}}\left\|\nabla \cdot u^{j+1}\right\|_{H^{N}} .
\end{aligned}
$$

By using the Cauchy inequality, we obtain

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t}\left\|u^{j+1}\right\|_{H^{N}}^{2}+\lambda_{1}\left\|\nabla \cdot u^{j+1}\right\|_{H^{N}}^{2} \leq C\left\|\rho^{j+1}\right\|_{H^{N}}^{2}+C\left\|c_{1}^{j}\right\|_{H^{N}}^{2}+C\left\|c_{1}^{j}\right\|_{H^{N}}^{2}\left\|\nabla \cdot u^{j+1}\right\|_{H^{N}}^{2}+C\left\|c_{2}^{j}\right\|_{H^{N}}^{2} \\
& \quad+C\left\|c_{2}^{j}\right\|_{H^{N}}^{2}\left\|\nabla \cdot u^{j+1}\right\|_{H^{N}}^{2}+C\left\|u^{j}\right\|_{H^{N}}^{2}\left\|\nabla \cdot u^{j+1}\right\|_{H^{N}}^{2}+\left\|\rho^{j}\right\|_{H^{N}}^{2} . \tag{3.5}
\end{align*}
$$

In a similar way as above, we can estimate $c_{1}$ and $c_{2}$ as

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t}\left\|c_{1}^{j+1}\right\|_{H^{N}}^{2}+\left\|\nabla c_{1}^{j+1}\right\|_{H^{N}}^{2}+\lambda_{2}\left\|c_{1}^{j+1}\right\|_{H^{N}}^{2} \leq C\left\|\rho^{j}\right\|_{H^{N}}^{2}\left\|c_{1}^{j+1}\right\|_{H^{N}}^{2}  \tag{3.6}\\
& \frac{1}{2} \frac{d}{d t}\left\|c_{2}^{j+1}\right\|_{H^{N}}^{2}+\left\|\nabla c_{2}^{j+1}\right\|_{H^{N}}^{2}+\lambda_{2}\left\|c_{2}^{j+1}\right\|_{H^{N}}^{2} \leq C\left\|\rho^{j}\right\|_{H^{N}}^{2}\left\|c_{2}^{j+1}\right\|_{H^{N}}^{2} . \tag{3.7}
\end{align*}
$$

Taking the linear combination of inequalities (3.4)-(3.7), we have

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t}\left(\left\|\rho^{j+1}\right\|_{H^{N}}^{2}+\left\|u^{j+1}\right\|_{H^{N}}^{2}+\left\|c_{1}^{j+1}\right\|_{H^{N}}^{2}+\left\|c_{2}^{j+1}\right\|_{H^{N}}^{2}\right)+\lambda_{1}\left\|\nabla\left[u^{j+1}, c_{1}^{j+1}, c_{2}^{j+1}\right]\right\|_{H^{N}}^{2} \\
& \quad+\lambda_{2}\left\|\left[\rho^{j+1}, c_{1}^{j+1}, c_{2}^{j+1}\right]\right\|_{H^{N}}^{2} \leq C\left\|\left[\rho^{j}, u^{j}, c_{1^{j}}^{j}, c_{2}^{j}\right]\right\|_{H^{N}}^{2}+C\left\|\left[\rho^{j}, u^{j}\right]\right\|_{H^{N}}^{2}\left\|\rho^{j+1}\right\|_{H^{N}}^{2} \\
& \quad+C\left\|\left[u^{j}, c_{1}^{j}, c_{2}^{j}\right]\right\|_{H^{N}}^{2}\left\|\nabla \cdot u^{j+1}\right\|_{H^{N}}^{2}+C\left\|\rho^{j}\right\|_{H^{N}}^{2}\left\|\left[c_{1}^{j+1}, c_{2}^{j+1}\right]\right\|_{H^{N}}^{2}
\end{aligned}
$$

Thus, after integrating with respect to $t$, we have

$$
\begin{align*}
& \left\|U^{j+1}(t)\right\|_{H^{N}}^{2}+\lambda_{1} \int_{0}^{t}\left\|\nabla\left[u^{j+1}, c_{1}^{j+1}, c_{2}^{j+1}\right]\right\|_{H^{N}}^{2} \mathrm{~d} s+\lambda_{2} \int_{0}^{t}\left\|\left[\rho^{j+1}, c_{1}^{j+1}, c_{2}^{j+1}\right]\right\|_{H^{N}}^{2} \mathrm{~d} s \\
& \leq C\left\|U^{j+1}(0)\right\|_{H^{N}}^{2}+C \int_{0}^{t}\left\|U^{j}(s)\right\|_{H^{N}}^{2} \mathrm{~d} s+C \int_{0}^{t}\left\|U^{j}(s)\right\|_{H^{N}}^{2}\left\|\left[\rho^{j+1}, \nabla \cdot u^{j+1}, c_{1}^{j+1}, c_{2}^{j+1}\right]\right\|_{H^{N}}^{2} \mathrm{~d} s \tag{3.8}
\end{align*}
$$

In the last inequality, we use the induction hypothesis. We obtain

$$
\begin{aligned}
& \left\|U^{j+1}(t)\right\|_{H^{N}}^{2}+\lambda_{1} \int_{0}^{t}\left\|\nabla\left[u^{j+1}, c_{1}^{j+1}, c_{2}^{j+1}\right]\right\|_{H^{N}}^{2} \mathrm{~d} s+\lambda_{2} \int_{0}^{t}\left\|\left[\rho^{j+1}, c_{1}^{j+1}, c_{2}^{j+1}\right]\right\|_{H^{N}}^{2} \mathrm{~d} s \\
& \quad \leq C \epsilon_{0}^{2}+C M^{2} T_{1}+C M^{2} \int_{0}^{t}\left\|\left[\rho^{j+1}, \nabla \cdot u^{j+1}, c_{1}^{j+1}, c_{2}^{j+1}\right]\right\|_{H^{N}}^{2} \mathrm{~d} s
\end{aligned}
$$

for $0 \leq t \leq T_{1}$. Now, we take the small constants $\epsilon_{0}>0, T_{1}>0$ and $M>0$. Then we have

$$
\begin{equation*}
\left\|U^{j+1}(t)\right\|_{H^{N}}^{2}+\lambda_{1} \int_{0}^{t}\left\|\nabla\left[u^{j+1}, c_{1}^{j+1}, c_{2}^{j+1}\right]\right\|_{H^{N}}^{2} \mathrm{~d} s+\lambda_{2} \int_{0}^{t}\left\|\left[\rho^{j+1}, c_{1}^{j+1}, c_{2}^{j+1}\right]\right\|_{H^{N}}^{2} \mathrm{~d} s \leq M^{2} \tag{3.9}
\end{equation*}
$$

for $0 \leq t \leq T_{1}$. This implies that (3.3) holds true for $j+1$. Hence (3.3) is proved for all $j \geq 0$.
For the second step, we prove that the sequence $\left(U^{j}\right)_{j \geq 0}$ is a Cauchy sequence in the Banach space $C\left(\left[0, T_{1}\right] ; H^{N-1}\left(\mathbb{R}^{3}\right)\right)$, which converges to the solution $U=\left(\rho, u, c_{1}, c_{2}\right)$ of the Cauchy problem (2.2)-(2.3), and satisfies $\sup _{0 \leq t \leq T_{1}}\left\|\left[U^{j}(t)\right]\right\|_{H^{N-1}} \leq M$. See for example [16].

For simplicity, we denote $\delta f^{j+1}:=f^{j+1}-f^{j}$. Subtracting the $j$-th equations from the $(j+1)$-th equations, we have the following equations for $\delta \rho^{j+1}, \delta u^{j+1}, \delta c_{1}^{j+1}$ and $\delta c_{1}^{j+1}$ :

$$
\left\{\begin{array}{l}
\partial_{t} \delta \rho^{j+1}+n_{\infty} \nabla \cdot\left(\delta u^{j+1}\right)+n_{\infty} \delta \rho^{j+1}=-\rho^{j} \nabla \cdot \delta u^{j+1}-\delta \rho^{j} \nabla \cdot u^{j} \\
-u^{j} \nabla \delta \rho^{j+1}-\delta u^{j} \nabla \rho^{j}+\left(\rho^{j}+\rho^{j-1}\right) \delta \rho^{j} \\
\partial_{t} \delta u^{j+1}-\delta \Delta \delta u^{j+1}=-u^{j} \cdot \nabla \delta u^{j}-\delta u^{j} \cdot \nabla u^{j-1}+\nabla\left(\left(c_{1}^{j}+c_{1}^{j-1}\right) \delta c_{1}^{j}\right) \\
-\nabla\left(\left(c_{2}^{j}+c_{2}^{j-1}\right) \delta c_{2}^{j}\right)-\left(\frac{\nabla p\left(\rho^{j}+n_{\infty}\right)}{\rho^{j}+n_{\infty}}-\frac{\nabla p\left(\rho^{j-1}+n_{\infty}\right)}{\rho^{j-1}+n_{\infty}}\right) \\
\partial_{t} \delta c_{1}^{j+1}+\Delta \delta c_{1}^{j+1}+\left(a_{12}-a_{11} n_{\infty}\right) \delta c_{1}^{j+1}=a_{11} \rho^{j} \delta c_{1}^{j+1}+a_{11} \delta \rho^{j} c_{1}^{j} \\
\partial_{t} \delta c_{2}^{j+1}+\Delta \delta c_{2}^{j+1}+\left(a_{22}-a_{21} n_{\infty}\right) \delta c_{2}^{j+1}=a_{21} \rho^{j} \delta c_{2}^{j+1}+a_{21} \delta \rho^{j} c_{2}^{j}
\end{array}\right.
$$

The estimate of $\delta \rho^{j+1}$ is as follows:

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t}\left\|\delta \rho^{j+1}\right\|_{H^{N-1}}^{2}+n_{\infty}\left\|\delta \rho^{j+1}\right\|_{H^{N-1}}^{2} \leq C\left\|\nabla \cdot \delta u^{j+1}\right\|_{H^{N-1}}\left\|\delta \rho^{j+1}\right\|_{H^{N-1}} \\
& \quad+C\left\|\rho^{j}\right\|_{H^{N-1}}\left\|\delta \rho^{j+1}\right\|_{H^{N-1}}\left\|\nabla \cdot \delta u^{j+1}\right\|_{H^{N-1}}+C\left\|\delta \rho^{j}\right\|_{H^{N-1}}\left\|\nabla \cdot u^{j}\right\|_{H^{N-1}}\left\|\delta \rho^{j+1}\right\|_{H^{N-1}} \\
& \quad+C\left\|\nabla \cdot u^{j}\right\|_{L^{\infty}}\left\|\delta \rho^{j+1}\right\|_{H^{N-1}}^{2}+C\left\|\delta \rho^{j+1}\right\|_{H^{N-2}}\left\|u^{j}\right\|_{H^{N-1}}\left\|\delta \rho^{j+1}\right\|_{H^{N-1}} \\
& \quad+C\left\|\delta \rho^{j+1}\right\|_{H^{N-1}}\left\|\delta u^{j}\right\|_{H^{N-1}}\left\|\nabla \rho^{j}\right\|_{H^{N-1}}+C\left\|\delta \rho^{j+1}\right\|_{H^{N-1}}\left\|\delta \rho^{j}\right\|_{H^{N-1}} .
\end{aligned}
$$

Then

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t}\left\|\delta \rho^{j+1}\right\|_{H^{N-1}}^{2}+\lambda_{2}\left\|\delta \rho^{j+1}\right\|_{H^{N-1}}^{2} \leq C\left\|\nabla \cdot \delta u^{j+1}\right\|_{H^{N-1}}^{2}+C\left\|\rho^{j}\right\|_{H^{N-1}}^{2}\left\|\delta \rho^{j+1}\right\|_{H^{N-1}}^{2} \\
& \quad+C\left\|\nabla \cdot u^{j}\right\|_{H^{N-1}}^{2}\left\|\delta \rho^{j}\right\|_{H^{N-1}}^{2}+C\left\|u^{j}\right\|_{H^{N-1}}^{2}\left\|\delta \rho^{j+1}\right\|_{H^{N-1}}^{2} \\
& \quad+C\left\|\nabla \rho^{j}\right\|_{H^{N-1}}^{2}\left\|\delta u^{j}\right\|_{H^{N-1}}^{2}+C\left\|\delta \rho^{j}\right\|_{H^{N-1}}^{2} . \tag{3.10}
\end{align*}
$$

The estimate of $\delta u^{j+1}$ is

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t}\left\|\delta u^{j+1}\right\|_{H^{N-1}}^{2}+\delta\left\|\nabla \cdot \delta u^{j+1}\right\|_{H^{N-1}}^{2} \leq C\left\|\nabla \cdot \delta u^{j+1}\right\|_{H^{N-1}}\left\|u^{j}\right\|_{H^{N-1}}\left\|\delta u^{j}\right\|_{H^{N-1}} \\
&+\left\|\delta u^{j+1}\right\|_{H^{N-1}}\left\|\nabla \cdot u^{j}\right\|_{H^{N-1}}\left\|\delta u^{j}\right\|_{H^{N-1}}+C\left\|\delta u^{j+1}\right\|_{H^{N-1}}\left\|\delta u^{j}\right\|_{H^{N-1}}\left\|\nabla \cdot u^{j-1}\right\|_{H^{N-1}} \\
&+C\left\|\delta c_{1}^{j}\right\|_{H^{N-1}}^{2}\left\|\nabla \cdot \delta u^{j+1}\right\|_{H^{N-1}}^{2}+C\left\|\delta c_{2}^{j+1}\right\|_{H^{N-1}}^{2}\left\|\nabla \cdot \delta u^{j+1}\right\|_{H^{N-1}}^{2}+C\left\|\delta \rho^{j+1}\right\|_{H^{N}}^{2} \\
& \quad+C\left\|\delta \rho^{j}\right\|_{H^{N-1}}\left\|\nabla \cdot \delta u^{j+1}\right\|_{H^{N-1}} .
\end{aligned}
$$

Then

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t}\left\|\delta u^{j+1}\right\|_{H^{N-1}}^{2}+\lambda_{1}\left\|\nabla \cdot \delta u^{j+1}\right\|_{H^{N-1}}^{2} \leq C\left\|u^{j}\right\|_{H^{N-1}}^{2}\left\|\delta u^{j}\right\|_{H^{N-1}}^{2}+\left\|\delta u^{j+1}\right\|_{H^{N-1}}^{2}\left\|\nabla \cdot u^{j}\right\|_{H^{N-1}}^{2} \\
& \quad+C\left\|\delta u^{j+1}\right\|_{H^{N-1}}^{2}\left\|\nabla \cdot u^{j-1}\right\|_{H^{N-1}}^{2}+C\left\|\delta u^{j}\right\|_{H^{N-1}}^{2} \\
& \quad+C\left\|\delta c_{1}^{j}\right\|_{H^{N-1}}^{2}+C\left\|\delta c_{2}^{j}\right\|_{H^{N-1}}^{2}+\left\|\delta \rho^{j}\right\|_{H^{N-1}}^{2} . \tag{3.11}
\end{align*}
$$

We have a similar way to estimate $\delta c_{1}^{j+1}$ and $\delta c_{2}^{j+1}$ as follows:

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t}\left\|\delta c_{1}^{j+1}\right\|_{H^{N-1}}^{2}+\left\|\nabla \delta c_{1}^{j+1}\right\|_{H^{N-1}}^{2}+\lambda_{2}\left\|\delta c_{1}^{j+1}\right\|_{H^{N-1}}^{2} \\
& \quad \leq C\left\|\rho^{j}\right\|_{H^{N-1}}^{2}\left\|\delta c_{1}^{j+1}\right\|_{H^{N-1}}^{2}+C\left\|\delta c_{1}^{j}\right\|_{H^{N-1}}^{2}\left\|\rho^{j}\right\|_{H^{N-1}}^{2} \tag{3.12}
\end{align*}
$$

and

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t}\left\|\delta c_{2}^{j+1}\right\|_{H^{N-1}}^{2}+\left\|\nabla \delta c_{2}^{j+1}\right\|_{H^{N-1}}^{2}+\lambda_{2}\left\|\delta c_{2}^{j+1}\right\|_{H^{N-1}}^{2} \\
& \quad \leq C\left\|\rho^{j}\right\|_{H^{N-1}}^{2}\left\|\delta c_{2}^{j+1}\right\|_{H^{N-1}}^{2}+C\left\|\delta c_{2}^{j}\right\|_{H^{N-1}}^{2}\left\|\rho^{j}\right\|_{H^{N-1}}^{2} . \tag{3.13}
\end{align*}
$$

We combine the equations (3.10)-(3.13) to obtain

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t}\left.\left\|\delta \rho^{j+1}\right\|_{H^{N-1}}^{2}+\left\|\delta u^{j+1}\right\|_{H^{N-1}}^{2}+\left\|\delta c_{1}^{j+1}\right\|_{H^{N-1}}^{2}+\left\|\delta c_{2}^{j+1}\right\|_{H^{N-1}}^{2}\right) \\
&+\lambda_{1}\left(\left\|\nabla \cdot \delta u^{j+1}\right\|_{H^{N-1}}^{2}+\left\|\nabla \delta c_{1}^{j+1}\right\|_{H^{N-1}}^{2}+\left\|\nabla \delta c_{2}^{j+1}\right\|_{H^{N-1}}^{2}\right) \\
& \quad+\lambda_{2}\left(\left\|\delta \rho^{j+1}\right\|_{H^{N-1}}^{2}+\left\|\delta c_{1}^{j+1}\right\|_{H^{N-1}}^{2}+C\left\|\delta c_{2}^{j+1}\right\|_{H^{N-1}}^{2}\right) \\
& \leq C\left(\left\|\delta u^{j+1}\right\|_{H^{N-1}}^{2}+\left\|\delta \rho^{j+1}\right\|_{H^{N-1}}^{2}+\left\|\delta c_{1}^{j+1}\right\|_{H^{N-1}}^{2}+\left\|\delta c_{2}^{j+1}\right\|_{H^{N-1}}^{2}\right) \\
&+C\left(\left\|\delta u^{j}\right\|_{H^{N-1}}^{2}+C\left\|\delta \rho^{j}\right\|_{H^{N-1}}^{2}+C\left\|\delta c_{1}^{j}\right\|_{H^{N-1}}^{2}+C\left\|\delta c_{2}^{j}\right\|_{H^{N-1}}^{2}\right) .
\end{aligned}
$$

By using Gronwall's inequality, we obtain

$$
\begin{aligned}
& \sup _{0 \leq t \leq T_{1}}\left(\left\|\delta \rho^{j+1}\right\|_{H^{N-1}}^{2}+\left\|\delta u^{j+1}\right\|_{H^{N-1}}^{2}+\left\|\delta c_{1}^{j+1}\right\|_{H^{N-1}}^{2}+\left\|\delta c_{2}^{j+1}\right\|_{H^{N-1}}^{2}\right) \\
& \quad \leq e^{\int_{0}^{t} c \mathrm{~d} s} \int_{0}^{t}\left\|\delta U^{j}(s)\right\|_{H^{N-1}}^{2} d s+e^{\int_{0}^{t} c \mathrm{cs} s}\left\|\delta U^{j+1}(0)\right\|_{H^{N-1}}^{2} d s \\
& \quad \leq C T_{1}\left(e^{C T_{1}}\right) \sup _{0 \leq t \leq T_{1}}\left\|\delta U^{j}\right\|_{H^{N-1}}^{2} .
\end{aligned}
$$

By taking $T_{1}>0$ sufficiently small we find that $\left(U^{j}\right)_{j \geq 0}$ is a Cauchy sequence in the Banach space $C\left(\left[0, T_{1}\right] ; H^{N-1}\left(\mathbb{R}^{3}\right)\right)$. Thus, we have the limit function

$$
U=U^{0}+\lim _{m \rightarrow \infty} \sum_{j=0}^{m}\left(U^{j+1}-U^{j}\right)
$$

in the same space $C\left(\left[0, T_{1}\right] ; H^{N-1}\left(\mathbb{R}^{3}\right)\right)$, and satisfies

$$
\begin{equation*}
\sup _{0 \leq t \leq T_{1}}\|U\|_{H^{N-1}} \leq \sup _{0 \leq t \leq T_{1}} \lim _{j \rightarrow \infty} \inf \left\|U^{j}\right\|_{H^{N-1}} \leq M . \tag{3.14}
\end{equation*}
$$

Thus, as $j \rightarrow \infty$ the limit exists such that

$$
(U)_{j \geq 0} \rightarrow U(t)
$$

strongly in $C\left(\left[0, T_{1}\right] ; H^{N-1}\right)$ and as $j^{\prime} \rightarrow \infty$, where $\left\{j^{\prime}\right\}$ is a subsequence of $\{j\}$, we have

$$
D\left(u, c_{1}, c_{2}\right)_{j^{\prime}} \rightarrow D\left(u, c_{1}, c_{2}\right)
$$

weakly in $L_{2}\left(\left[0, T_{1}\right] ; H^{N}\right)$ by step one. Also by step one, we know

$$
(U)_{j^{\prime \prime}}(t) \rightarrow U(t)
$$

weakly in $H^{N}$ for every fixed $t \in\left[0, T_{1}\right]$, where $j^{\prime \prime}=j^{\prime \prime}(t)$ is a subsequence of $\left\{j^{\prime}\right\}$, depending on $t$. Thus, we have a solution $U(t) \in L_{\infty}\left(\left[0, T_{1}\right] ; H^{N}\right)$ for the problem (2.2)-(2.3).

For the third step, we show that $\left\|U^{j+1}(t)\right\|_{H^{N}}^{2}$ is continuous in time for each $j \geq 0$.
For simplicity, let us define the equivalent energy functional

$$
\mathfrak{E}\left(U^{j+1}(t)\right)=\left\|\rho^{j+1}\right\|_{H^{N}}^{2}+\left\|u^{j+1}\right\|_{H^{N}}^{2}+\left\|c_{1}^{j+1}\right\|_{H^{N}}^{2}+\left\|c_{2}^{j+1}\right\|_{H^{N}}^{2}
$$

Similarly to how we proved (3.8), we have

$$
\begin{aligned}
& \left|\mathfrak{E} U^{j+1}(t)-\mathfrak{E} U^{j+1}(s)\right|=\left|\int_{s}^{t} \mathfrak{E} U^{j+1}(\theta) \mathrm{d} \theta\right| \leq \int_{s}^{t}\left\|U^{j}(s)\right\|_{H^{N}}^{2} \mathrm{~d} \theta \\
& \quad+C \int_{0}^{t}\left(1+\left\|U^{j}(s)\right\|_{H^{N}}^{2}\right)\left\|\left[\rho^{j+1}, \nabla \cdot u^{j+1}, c_{1}^{j+1}, c_{2}^{j+1}\right]\right\|_{H^{N}}^{2} \mathrm{~d} s+C \int_{s}^{t}\left\|\nabla\left[c_{1}^{j+1}, c_{2}^{j+1}\right]\right\|_{H^{N}}^{2} \mathrm{~d} s \\
& \leq \\
& \quad C M^{2}(t-s)+C\left(M^{2}+1\right) \int_{s}^{t}\left\|\left[\rho^{j+1}, \nabla \cdot u^{j+1}, c_{1}^{j+1}, c_{2}^{j+1}\right]\right\|_{H^{N}}^{2} \mathrm{~d} s \\
& \quad+C \int_{s}^{t}\left\|\nabla\left[c_{1}^{j+1}, c_{2}^{j+1}\right]\right\|_{H^{N}}^{2} \mathrm{~d} s,
\end{aligned}
$$

for any $0 \leq s \leq t \leq T_{1}$. The time integral on the right-hand side from the above inequality is bounded by (3.9), and hence $\mathfrak{E} U^{j+1}(t)$ is continuous in $t$ for each $j \geq 0$. Therefore, $\left\|U^{j}(t)\right\|_{H^{N}}^{2}$ is continuous in time for each $j \geq 1$. Furthermore, $U=\left(\rho, u, c_{1}, c_{2}\right)$ is a local solution to the Cauchy problem (2.2)-(2.3).

For the fourth step, we show that the Cauchy problem (2.2)-(2.3) admits at most one solution in $C\left(\left[0, T_{1}\right] ; H^{N}\left(\mathbb{R}^{3}\right)\right)$. We assume that there exist two local solutions $U, \tilde{U}$ in $C\left(\left[0, T_{1}\right] ; H^{N}\right)$ which satisfy (3.2). Let $\tilde{\rho}=\rho_{1}(x, t)-\rho_{2}(x, t), \tilde{u}(x, t)=u_{1}(x, t)-u_{2}(x, t), \tilde{c}_{1}(x, t)=c_{1,1}(x, t)-$ $c_{1,2}(x, t)$ and $\tilde{c}_{2}(x, t)=c_{2,1}(x, t)-c_{2,2}(x, t)$ solve

$$
\left\{\begin{array}{l}
\partial_{t} \tilde{\rho}+n_{\infty} \nabla \cdot \tilde{u}+n_{\infty} \tilde{\rho}=-\nabla \cdot\left(\tilde{\rho} u_{1}\right)-\nabla \cdot\left(\rho_{2} \tilde{u}\right)-\left(\rho_{1}+\rho_{2}\right) \tilde{\rho}  \tag{3.15}\\
\partial_{t} \tilde{u}+u_{1} \cdot \nabla \tilde{u}-\delta \Delta \tilde{u}=-\tilde{u} \cdot \nabla u_{2}-\frac{p^{\prime}\left(\rho_{1}+n_{\infty}\right)}{\rho_{1}+n_{\infty}} \nabla \tilde{\rho}+\nabla\left(\left(c_{1,1}+c_{1,2}\right) \tilde{c}_{1}\right) \\
-\nabla\left(\left(c_{2,1}+c_{2,2}\right)\right) \tilde{c}_{2}-\left(\frac{p^{\prime}\left(\rho_{1}+n_{\infty}\right)}{\rho_{1}+n_{\infty}}-\frac{p^{\prime}\left(\rho_{2}+n_{\infty}\right)}{\rho_{2}+n_{\infty}}\right) \nabla \rho_{2} \\
\partial_{t} \tilde{c}_{1}=\Delta \tilde{c}_{1}-a_{12} \tilde{c}_{1}+a_{11} \rho_{1} \tilde{c}_{1}+a_{11} \tilde{\rho}_{1} c_{1,2} \\
\partial_{t} \tilde{c}_{2}=\Delta \tilde{c}_{2}-a_{22} \tilde{c}_{2}+a_{21} \rho_{1} \tilde{c}_{2}+a_{21} \tilde{\rho} c_{2,2} .
\end{array}\right.
$$

Multiplying $\tilde{\rho}$ to both sides of the first equation of (3.15) and integrating over $\mathbb{R}^{3}$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^{3}} \tilde{\rho} \partial_{t} \tilde{\rho} d x+n_{\infty} \int_{\mathbb{R}^{3}} \tilde{\rho} \nabla \cdot \tilde{u} d x+ & n_{\infty} \int_{\mathbb{R}^{3}}|\tilde{\rho}|^{2} d x \\
& =-\int_{\mathbb{R}^{3}} \tilde{\rho} \nabla \cdot\left(\tilde{\rho} u_{1}\right) d x+\int_{\mathbb{R}^{3}} \tilde{\rho} \nabla \cdot\left(\rho_{2} \tilde{u}\right) d x+\int_{\mathbb{R}^{3}}\left(\rho_{1}+\rho_{2}\right) \tilde{\rho}^{2} .
\end{aligned}
$$

Using integration by parts and the Cauchy-Schwarz inequality, we have

$$
\begin{align*}
\frac{1}{2} \frac{d}{d t}\|\tilde{\rho}\|_{L^{2}}^{2}+n_{\infty}\|\tilde{\rho}\|_{L^{2}}^{2} \leq & \frac{n_{\infty}}{2}\|\tilde{\rho}\|_{L^{2}}^{2}+\frac{n_{\infty}}{2}\|\nabla \cdot \tilde{u}\|_{L^{2}}^{2}+\frac{1}{2}\left\|\nabla \cdot u_{1}\right\|_{L^{\infty}} \int_{\mathbb{R}^{3}}|\tilde{\rho}|^{2} d x \\
& +\left\|\rho_{2}\right\|_{L^{\infty}} \int_{\mathbb{R}^{3}}\left(|\nabla \cdot \tilde{u}|^{2}+|\tilde{\rho}|^{2}\right) d x+\left\|\nabla \rho_{2}\right\|_{L^{\infty}} \int_{\mathbb{R}^{3}}\left(|\tilde{u}|^{2}+|\tilde{\rho}|^{2}\right) d x \\
& +\left\|\left[\rho_{1}+\rho_{2}\right]\right\|_{L^{\infty}} \int_{\mathbb{R}^{3}}|\tilde{\rho}|^{2} d x \tag{3.16}
\end{align*}
$$

Next, we establish the energy estimates for $\tilde{u}$. By multiplying $\tilde{u}$ to both sides of the second equation of (3.15) and integrating in $x$, we have

$$
\begin{aligned}
& \int_{\mathbb{R}^{3}} \tilde{u} \cdot \partial_{t} \tilde{u} d x+\int_{\mathbb{R}^{3}} \tilde{u} \cdot\left(u_{1} \cdot \nabla \tilde{u}\right) d x-\delta \int_{\mathbb{R}^{3}} \tilde{u} \cdot \Delta \tilde{u} d x \\
&=-\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}} \int_{\mathbb{R}^{3}} \tilde{u} \cdot \nabla u_{2} d x+\int_{\mathbb{R}^{3}} \tilde{u} \cdot \nabla \tilde{\rho} d x \\
&+\int_{\mathbb{R}^{3}} \tilde{u} \cdot\left(\frac{p^{\prime}\left(\rho_{1}+n_{\infty}\right)}{\rho_{1}+n_{\infty}}-\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}}\right) \nabla \tilde{\rho}+\int_{\mathbb{R}^{3}} \tilde{u} \cdot \nabla\left(\left(c_{1,1}+c_{1,2}\right) \tilde{c_{1}}\right) d x \\
&-\int_{\mathbb{R}^{3}} \tilde{u} \cdot \nabla\left(\left(c_{2,1}+c_{2,2}\right) \tilde{c_{2}}\right) d x-\int_{\mathbb{R}^{3}} \tilde{u} \cdot\left(\frac{p^{\prime}\left(\rho_{1}+n_{\infty}\right)}{\rho_{1}+n_{\infty}} d x-\frac{p^{\prime}\left(\rho_{2}+n_{\infty}\right)}{\rho_{2}+n_{\infty}}\right) \nabla \rho_{2} d x .
\end{aligned}
$$

By using integration by parts and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\frac{1}{2} \frac{d}{d t}\|\tilde{u}\|_{L^{2}}^{2}+\delta\|\nabla \cdot \tilde{u}\|_{L^{2}}^{2} \leq & \left\|\nabla \cdot u_{1}\right\|_{L^{\infty}}\|\tilde{u}\|_{L^{2}}^{2}+\left\|\nabla \cdot u_{2}\right\|_{L^{\infty}}\|\tilde{u}\|_{L^{2}}^{2}+\frac{p^{\prime}\left(n_{\infty}\right)}{2 n_{\infty}}\|\nabla \cdot \tilde{u}\|_{L^{2}}^{2}+\frac{p^{\prime}\left(n_{\infty}\right)}{2 n_{\infty}}\|\tilde{\rho}\|_{L^{2}}^{2} \\
& +\left\|\rho_{1}\right\|_{L^{\infty}}\left(\|\nabla \cdot \tilde{u}\|_{L^{2}}^{2}+\|\tilde{\rho}\|_{L^{2}}^{2}\right)+\left\|\nabla \rho_{1}\right\|_{L^{\infty}}\left(\|\tilde{u}\|_{L^{2}}^{2}+\|\tilde{\rho}\|_{L^{2}}^{2}\right) \\
& \left.+\left\|c_{1,1}+c_{1,2}\right\|_{L^{\infty}}\|\nabla \cdot \tilde{u}\|_{L^{2}}^{2}+\left\|\tilde{c}_{1}\right\|_{L^{2}}^{2}\right) \\
& +\left\|c_{2,1}+c_{2,2}\right\|_{L^{\infty}}\left(\|\nabla \cdot \tilde{u}\|_{L^{2}}^{2}+\left\|\tilde{c}_{2}\right\|_{L^{2}}^{2}\right)+\left\|\nabla \rho_{2}\right\|_{L^{\infty}}\left(\|\tilde{u}\|_{L^{2}}^{2}+\|\tilde{\rho}\|_{L^{2}}^{2}\right) .
\end{aligned}
$$

Since $L^{\infty}$ norms of $\rho_{i}, u_{i}, c_{1, i}, c_{2, i}$ where $i=1,2$ are bounded, we have

$$
\begin{equation*}
\frac{1}{2} \frac{d}{d t}\|\tilde{u}\|_{L^{2}}^{2}+\frac{\delta}{2}\|\nabla \cdot \tilde{u}\|_{L^{2}}^{2} \leq C\|\tilde{u}\|_{L^{2}}^{2}+C\|\tilde{\rho}\|_{L^{2}}^{2}+C\left\|\tilde{c}_{1}\right\|_{L^{2}}^{2}+C\left\|\tilde{c}_{2}\right\|_{L^{2}}^{2} . \tag{3.17}
\end{equation*}
$$

We have a similar way to estimate $\tilde{c}_{1}$ and $\tilde{c}_{2}$ as follows:

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t}\left\|\tilde{c}_{1}\right\|_{L^{2}}^{2}+\left\|\nabla \tilde{c}_{1}\right\|_{L^{2}}^{2}+a_{12}\left\|\tilde{c}_{1}\right\|_{L^{2}}^{2} \leq a_{11}\left\|\rho_{1}\right\|_{L^{\infty}}\left\|\tilde{c}_{1}\right\|_{L^{2}}^{2}+\frac{a_{11}}{2}\left\|c_{1,2}\right\|_{L^{\infty}}\left(\|\tilde{\rho}\|_{L^{2}}^{2}+\left\|\tilde{c}_{1}\right\|_{L^{2}}^{2}\right)  \tag{3.18}\\
& \frac{1}{2} \frac{d}{d t}\left\|\tilde{c}_{2}\right\|_{L^{2}}^{2}+\left\|\nabla \tilde{c}_{2}\right\|_{L^{2}}^{2}+a_{22}\left\|\tilde{c}_{2}\right\|_{L^{2}}^{2} \leq a_{21}\left\|\rho_{1}\right\|_{L^{\infty}}\left\|\tilde{c}_{2}\right\|_{L^{2}}^{2}+\frac{a_{21}}{2}\left\|c_{2,2}\right\|_{L^{\infty}}\left(\|\tilde{\rho}\|_{L^{2}}^{2}+\left\|\tilde{c}_{2}\right\|_{L^{2}}^{2}\right) . \tag{3.19}
\end{align*}
$$

By taking a linear combination of all estimates, we obtain

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t}\left(\|\tilde{\rho}\|_{L^{2}}^{2}+\|\tilde{u}\|_{L^{2}}^{2}+\left\|\tilde{c}_{1}\right\|_{L^{2}}^{2}+\left\|\tilde{c}_{2}\right\|_{L^{2}}^{2}\right)+\lambda_{1}\left(\|\nabla \cdot \tilde{u}\|_{L^{2}}^{2}+\left\|\tilde{\nabla} c_{1}\right\|_{L^{2}}^{2}+\left\|\tilde{\nabla} c_{2}\right\|_{L^{2}}^{2}\right) \\
& \quad+\lambda_{2}\left(\|\tilde{\rho}\|_{L^{2}}^{2}+\left\|\tilde{c}_{1}\right\|_{L^{2}}^{2}+\left\|\tilde{c}_{2}\right\|_{L^{2}}^{2}\right) \leq C\left(\|\tilde{\rho}\|_{L^{2}}^{2}+\|\tilde{u}\|_{L^{2}}^{2}+\left\|\tilde{c}_{1}\right\|_{L^{2}}^{2}+\left\|\tilde{c}_{2}\right\|_{L^{2}}^{2}\right) . \tag{3.20}
\end{align*}
$$

The Gronwall's inequality implies

$$
\begin{align*}
& \sup _{0 \leq t \leq T_{1}}\left(\|\tilde{\rho}\|_{L^{2}}^{2}+\|\tilde{u}\|_{L^{2}}^{2}+\left\|\tilde{c}_{1}\right\|_{L^{2}}^{2}+\left\|\tilde{c}_{2}\right\|_{L^{2}}^{2}\right) \\
& \leq e^{c T_{1}}\left(\|\tilde{\rho}(0)\|_{L^{2}}^{2}+\|\tilde{u}(0)\|_{L^{2}}^{2}+\left\|\tilde{c}_{1}(0)\right\|_{L^{2}}^{2}+\left\|\tilde{c}_{2}(0)\right\|_{L^{2}}^{2}\right) \tag{3.21}
\end{align*}
$$

Since the initial data of ( $\left.\tilde{\rho}, \tilde{u}, \tilde{c}_{1}, \tilde{c}_{2}\right)$ are all zero for $T>0$, that implies the uniqueness of the local solution.

### 3.2 A priori estimates

In this subsection, we provide some estimates for the solutions for any $t>0$. We use the energy method to obtain uniform-in-time a priori estimates for smooth solutions to Cauchy problems (2.2)-(2.3).
Lemma 3.2 (A priori estimates). Let $U(t)=\left(\rho, u, c_{1}, c_{2}\right) \in C\left([0, T] ; H^{N}\left(\mathbb{R}^{3}\right)\right.$ be the smooth solution to the Cauchy problem (2.2)-(2.3) for $T>0$ with

$$
\begin{equation*}
\sup _{0 \leq t \leq T}\left\|\left(\rho, u, c_{1}, c_{2}\right)(t)\right\|_{N} \leq \epsilon \tag{3.22}
\end{equation*}
$$

for $0<\epsilon \leq 1$. Then, there are $\epsilon_{0}>0, C_{0}>0, \lambda_{1}>0$ and $\lambda_{2}>0$ such that for any $\epsilon \leq \epsilon_{0}$,

$$
\begin{equation*}
\left\|\left[\rho, u, c_{1}, c_{2}\right]\right\|_{H^{N}}^{2}+\lambda_{1} \int_{0}^{t}\left\|\nabla\left[u, c_{1}, c_{2}\right]\right\|_{H^{N}}^{2}+\lambda_{2} \int_{0}^{t}\left\|\left[\rho, c_{1}, c_{2}\right]\right\|_{H^{N}}^{2} \leq C_{0}\left\|\left[\rho_{0}, u_{0}, c_{1,0}, c_{2,0}\right]\right\|_{H^{N}}^{2} \tag{3.23}
\end{equation*}
$$

holds for any $t \in[0, T]$.
Proof. First, we find the zero-order estimates. For the estimate of $\rho$, multiplying $\rho$ to both sides of the first equation of (2.2) and taking integrations in $x \in \mathbb{R}^{3}$, we obtain

$$
\int_{\mathbb{R}^{3}} \rho \rho_{t} \mathrm{~d} x+n_{\infty} \int_{\mathbb{R}^{3}} \rho \nabla \cdot u \mathrm{~d} x+n_{\infty} \int_{\mathbb{R}^{3}}|\rho|^{2} \mathrm{~d} x=-\int_{\mathbb{R}^{3}} \rho \nabla \cdot(\rho u) \mathrm{d} x-\int_{\mathbb{R}^{3}} \rho \rho^{2} \mathrm{~d} x .
$$

Using integration by parts and the Cauchy-Schwarz inequality, we have

$$
\begin{align*}
& \frac{1}{2} \int_{\mathbb{R}^{3}}\left(\rho^{2}\right)_{t} \mathrm{~d} x+n_{\infty} \int_{\mathbb{R}^{3}}|\rho|^{2} \mathrm{~d} x+n_{\infty} \int_{\mathbb{R}^{3}} \rho \nabla \cdot u \mathrm{~d} x \\
& \quad \leq \frac{1}{2} \sup _{x}|\nabla u| \int_{\mathbb{R}^{3}}|\rho|^{2} \mathrm{~d} x+\sup _{x}|\rho| \int_{\mathbb{R}^{3}}|\rho|^{2} \mathrm{~d} x \\
& \quad \leq C\|\rho, u\|_{H^{N}} \int_{\mathbb{R}^{3}}|\rho|^{2} \mathrm{~d} x . \tag{3.24}
\end{align*}
$$

Now, we estimate $u$ by multiplying the second equation of (2.2) by $u$ and integrating over $\mathbb{R}^{3}$. Then, we have

$$
\begin{aligned}
& \int_{\mathbb{R}^{3}} u \cdot u_{t} \mathrm{~d} x+\int_{\mathbb{R}^{3}} u \cdot(u \cdot \nabla u) \mathrm{d} x-\delta \int_{\mathbb{R}^{3}} u \cdot \Delta u \mathrm{~d} x+\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}} \int_{\mathbb{R}^{3}} u \cdot \nabla \rho \mathrm{~d} x \\
& \quad=\int_{\mathbb{R}^{3}} u \cdot \nabla c_{1}^{2} \mathrm{~d} x-\int_{\mathbb{R}^{3}} u \cdot \nabla c_{2}^{2} \mathrm{~d} x-\int_{\mathbb{R}^{3}} u \cdot\left(\frac{p^{\prime}\left(\rho+n_{\infty}\right)}{\rho+n_{\infty}}-\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}}\right) \nabla \rho \mathrm{d} x .
\end{aligned}
$$

By using integration by parts and the Cauchy-Schwarz inequality, we have

$$
\begin{align*}
& \frac{1}{2} \int_{\mathbb{R}^{3}}\left(u^{2}\right)_{t} \mathrm{~d} x+\delta \int_{\mathbb{R}^{3}}|\nabla u|^{2} \mathrm{~d} x-\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}} \int_{\mathbb{R}^{3}} \rho \nabla \cdot u \mathrm{~d} x \\
& \quad \leq\|u\|_{H^{1}} \int_{\mathbb{R}^{3}}|\nabla u|^{2} \mathrm{~d} x+C\|u\|_{H^{N}} \int_{\mathbb{R}^{3}}\left(\left|c_{1}\right|^{2}+\left|c_{2}\right|^{2}+|\rho|^{2}\right) \mathrm{d} x . \tag{3.25}
\end{align*}
$$

For the estimates of $c_{1}$, we multiply $c_{1}$ to both sides of the equation of $c_{1}$ and integrate with respect to $x$, and we have

$$
\int_{\mathbb{R}^{3}} c_{1}\left(c_{1}\right)_{t} \mathrm{~d} x-\int_{\mathbb{R}^{3}} c_{1} \Delta c_{1} \mathrm{~d} x+\left(a_{12}-n_{\infty} a_{11}\right) \int_{\mathbb{R}^{3}}\left|c_{1}\right|^{2} \mathrm{~d} x \leq a_{11} \sup _{x}|\rho| \int_{\mathbb{R}^{3}}\left|c_{1}\right|^{2} \mathrm{~d} x .
$$

By using integration by parts, we have

$$
\begin{equation*}
\frac{1}{2} \int_{\mathbb{R}^{3}}\left(c_{1}^{2}\right)_{t} \mathrm{~d} x+\int_{\mathbb{R}^{3}}\left|\nabla c_{1}\right|^{2} \mathrm{~d} x+\left(a_{12}-n_{\infty} a_{11}\right) \int_{\mathbb{R}^{3}}\left|c_{1}\right|^{2} \mathrm{~d} x \leq a_{11}\|\rho\|_{H^{2}} \int_{\mathbb{R}^{3}}\left|c_{1}\right|^{2} \mathrm{~d} x . \tag{3.26}
\end{equation*}
$$

Similar to above, from the equation of $c_{2}$, we have

$$
\begin{equation*}
\frac{1}{2} \int_{\mathbb{R}^{3}}\left(c_{2}^{2}\right)_{t} \mathrm{~d} x+\int_{\mathbb{R}^{3}}\left|\nabla c_{2}\right|^{2} \mathrm{~d} x+\left(a_{22}-n_{\infty} a_{21}\right) \int_{\mathbb{R}^{3}}\left|c_{2}\right|^{2} \mathrm{~d} x \leq a_{21}\|\rho\|_{H^{2}} \int_{\mathbb{R}^{3}}\left|c_{2}\right|^{2} \mathrm{~d} x . \tag{3.27}
\end{equation*}
$$

Consider the linear combination $d_{1} \times(3.24)+(3.25)+(3.26)+(3.27)$, where $d_{1}=\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}^{2}}$. We see that as long as $\mathcal{E}_{N}^{\frac{1}{2}}(U)=\|U\|_{H^{N}}$ is small so that

$$
\begin{aligned}
& \left(a_{12}-n_{\infty} a_{11}\right)>a_{11} \mathcal{E}_{N}^{\frac{1}{2}}(U), \\
& \left(a_{22}-n_{\infty} a_{21}\right)>a_{21} \mathcal{E}_{N}^{\frac{1}{2}}(U)
\end{aligned}
$$

are satisfied, the linear combination yields

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t} \int_{\mathbb{R}^{3}}\left(d_{1}|\rho|^{2}+|u|^{2}+\left|c_{1}\right|^{2}+\left|c_{2}\right|^{2}\right) \mathrm{d} x+n_{\infty} \int_{\mathbb{R}^{3}}|\rho|^{2} \mathrm{~d} x+\delta \int_{\mathbb{R}^{3}}|\nabla u|^{2} \mathrm{~d} x \\
& \quad+\int_{\mathbb{R}^{3}}\left|\nabla c_{1}\right|^{2} \mathrm{~d} x+\int_{\mathbb{R}^{3}}\left|\nabla c_{2}\right|^{2} \mathrm{~d} x+\left(a_{12}-n_{\infty} a_{11}\right) \int_{\mathbb{R}^{3}}\left|c_{1}\right|^{2} \mathrm{~d} x+\left(a_{22}-n_{\infty} a_{21}\right) \int_{\mathbb{R}^{3}}\left|c_{2}\right|^{2} \mathrm{~d} x
\end{aligned}
$$

$$
\begin{equation*}
\leq 0 \tag{3.28}
\end{equation*}
$$

Now, we make estimates on the high-order derivatives of ( $\rho, u, c_{1}, c_{2}$ ). Take $\alpha$ with $1 \leq|\alpha| \leq N$. Applying $\partial^{\alpha}$ to the first equation of (2.2), multiplying by $\partial^{\alpha} \rho$ and then integrating in $x$, we have

$$
\begin{gathered}
\int_{\mathbb{R}^{3}} \partial^{\alpha} \rho \partial^{\alpha} \rho_{t} \mathrm{~d} x+n_{\infty} \int_{\mathbb{R}^{3}} \partial^{\alpha} \rho \partial^{\alpha} \nabla \cdot u \mathrm{~d} x+n_{\infty} \int_{\mathbb{R}^{3}} \partial^{\alpha} \rho \partial^{\alpha} \rho \mathrm{d} x \\
=-\int_{\mathbb{R}^{3}} \partial^{\alpha} \rho \partial^{\alpha} \nabla \cdot(\rho u) \mathrm{d} x-\int_{\mathbb{R}^{3}} \partial^{\alpha} \rho \partial^{\alpha} \rho^{2} \mathrm{~d} x .
\end{gathered}
$$

By using integration by parts and Cauchy-Schwarz inequality, we obtain

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t} \int_{\mathbb{R}^{3}}\left(\partial^{\alpha} \rho\right)^{2} \mathrm{~d} x+n_{\infty} \int_{\mathbb{R}^{3}}\left|\partial^{\alpha} \rho\right|^{2} \mathrm{~d} x+n_{\infty} \int_{\mathbb{R}^{3}} \partial^{\alpha} \rho \partial^{\alpha} \nabla \cdot u \mathrm{~d} x \\
& \quad=\int_{\mathbb{R}^{3}} \partial^{\alpha} \rho \sum_{\beta=0}^{\alpha} C_{\alpha}^{\beta} \partial^{\beta} \nabla \cdot u \partial^{\alpha-\beta} \rho \mathrm{d} x+\int_{\mathbb{R}^{3}} \partial^{\alpha} \rho \sum_{\beta=0}^{\alpha} C_{\alpha}^{\beta} \partial^{\beta} u \cdot \partial^{\alpha-\beta} \nabla \rho \mathrm{d} x-\int_{\mathbb{R}^{3}} \partial^{\alpha} \rho \partial^{\alpha} \rho^{2} \mathrm{~d} x \\
& \quad \leq C\|u\|_{H^{N}} \int_{\mathbb{R}^{3}}\left|\partial^{\alpha} \rho\right|^{2}+C\|\rho\|_{H^{N}} \int_{\mathbb{R}^{3}}\left|\partial^{\alpha} \rho\right|^{2}+\left|\partial^{\alpha} \nabla u\right|^{2} \mathrm{~d} x . \tag{3.29}
\end{align*}
$$

Similarly for $\partial^{\alpha} u$, what follows from $(2.2)_{2}$ is

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t} \int_{\mathbb{R}^{3}}\left(\partial^{\alpha} u\right)^{2} \mathrm{~d} x-\delta \int_{\mathbb{R}^{3}} \partial^{\alpha} u \cdot \partial^{\alpha} \Delta u \mathrm{~d} x+\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}} \int_{\mathbb{R}^{3}} \partial^{\alpha} u \cdot \partial^{\alpha} \nabla \rho \mathrm{d} x \\
& \quad=-\int_{\mathbb{R}^{3}} \partial^{\alpha} u \cdot \partial^{\alpha}(u \cdot \nabla u) \mathrm{d} x+\int_{\mathbb{R}^{3}} \partial^{\alpha} u \cdot \partial^{\alpha} \nabla c_{1}^{2} \mathrm{~d} x-\int_{\mathbb{R}^{3}} \partial^{\alpha} u \cdot \partial^{\alpha} \nabla c_{2}^{2} \mathrm{~d} x \\
& \quad-\int_{\mathbb{R}^{3}} \partial^{\alpha} u \cdot \partial^{\alpha}\left(\left(\frac{p^{\prime}\left(\rho+n_{\infty}\right)}{\rho+n_{\infty}}-\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}}\right) \nabla \rho\right) \mathrm{d} x .
\end{aligned}
$$

By using integration by parts and the Cauchy-Schwarz inequality, we have

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t} \int_{\mathbb{R}^{3}}\left(\partial^{\alpha} u\right)^{2} \mathrm{~d} x+\delta \int_{\mathbb{R}^{3}}\left|\partial^{\alpha} \nabla u\right|^{2} \mathrm{~d} x-\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}} \int_{\mathbb{R}^{3}} \partial^{\alpha} \nabla \cdot u \partial^{\alpha} \rho \mathrm{d} x \\
& \quad \leq C\|u\|_{H^{N}} \int_{\mathbb{R}^{3}}\left|\partial^{\alpha} u\right|^{2} \mathrm{~d} x+C\left\|c_{1}\right\|_{H^{N}} \int_{\mathbb{R}^{3}}\left(\left|\partial^{\alpha} u\right|^{2}+\left|\partial^{\alpha} \nabla c_{1}\right|^{2}\right) \mathrm{d} x \\
& \quad+C\left\|c_{2}\right\|_{H^{N}} \int_{\mathbb{R}^{3}}\left(\left|\partial^{\alpha} u\right|^{2}+\left|\partial^{\alpha} \nabla c_{2}\right|^{2}\right) \mathrm{d} x+C\|\rho\|_{H^{N}} \int_{\mathbb{R}^{3}}\left|\partial^{\alpha} u\right|^{2} \mathrm{~d} x+\left|\partial^{\alpha} \rho\right|^{2} \mathrm{~d} x \tag{3.30}
\end{align*}
$$

Similarly, we estimate $c_{1}, c_{2}$ as follows:

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t} \int_{\mathbb{R}^{3}}\left(\partial^{\alpha} c_{1}\right)^{2}+\int_{\mathbb{R}^{3}}\left|\nabla \partial^{\alpha} c_{1}\right|^{2} \mathrm{~d} s+\left(a_{12}-n_{\infty} a_{11}\right) \int_{\mathbb{R}^{3}}\left|\partial^{\alpha} c_{1}\right|^{2} \mathrm{~d} s \\
& \quad \leq C\|\rho\|_{H^{N}} \int_{\mathbb{R}^{3}}\left\|\left.\partial^{\alpha} c_{1}\right|^{2} \mathrm{~d} s+C\right\| c_{1} \|_{H^{N}} \int_{\mathbb{R}^{3}}\left(\left|\partial^{\alpha} c_{1}\right|^{2}+\left|\partial^{\alpha} \rho\right|^{2}\right) \mathrm{d} s \tag{3.31}
\end{align*}
$$

and

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t} \int_{\mathbb{R}^{3}}\left(\partial^{\alpha} c_{2}\right)^{2}+\int_{\mathbb{R}^{3}}\left|\nabla \partial^{\alpha} c_{2}\right|^{2} \mathrm{~d} s+\left(a_{22}-n_{\infty} a_{21}\right) \int_{\mathbb{R}^{3}}\left|\partial^{\alpha} c_{2}\right|^{2} \mathrm{~d} s \\
& \quad \leq C\|\rho\|_{H^{N}} \int_{\mathbb{R}^{3}}\left\|\left.\partial^{\alpha} c_{2}\right|^{2} \mathrm{~d} s+C\right\| c_{2} \|_{H^{N}} \int_{\mathbb{R}^{3}}\left(\left|\partial^{\alpha} c_{2}\right|^{2}+\left|\partial^{\alpha} \rho\right|^{2}\right) \mathrm{d} s \tag{3.32}
\end{align*}
$$

Then, after taking the summation over $1 \leq|\alpha| \leqslant N$ and the combination $(3.29) \times d_{1}+(3.30)+$ $(3.31)+(3.32)$, we obtain

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t} \sum_{1 \leq|\alpha| \leq N} C_{\alpha} \int_{\mathbb{R}^{3}}\left|\partial^{\alpha}\left(\rho, u, c_{1}, c_{2}\right)\right|^{2}+\lambda_{1} \sum_{1 \leq|\alpha| \leq N} \int_{\mathbb{R}^{3}}\left|\partial^{\alpha} \nabla\left(u, c_{1}, c_{2}\right)\right|^{2} \mathrm{~d} x \\
&+\lambda_{2} \sum_{1 \leq|\alpha| \leq N} \int_{\mathbb{R}^{3}}\left|\partial^{\alpha}\left(\rho, c_{1}, c_{2}\right)\right|^{2} \mathrm{~d} x \leq 0 \tag{3.33}
\end{align*}
$$

for some positive constants $C_{\alpha}, \lambda_{1}$ and $\lambda_{2}$. Therefore (3.23) follows from the further linear combination of (3.28) and (3.33) and the time integration over $[0, T]$. This completes the proof of Lemma 3.2.

Now, we are ready to present the proof of Proposition 2.1.
Proof of Proposition 2.1. Choose a positive constant $M=\min \left\{\epsilon_{0}, \epsilon_{1}\right\}$, where $\epsilon_{0}>0$ and $\epsilon_{1}>0$ are given in Lemma 3.1 and Lemma 3.2.

Let $U_{0} \in H^{N}\left(\mathbb{R}^{3}\right)$ satisfy $\left\|U_{0}\right\|_{H^{N}}<\frac{M}{2 \sqrt{C_{0}+1}}$. Now, let us define

$$
T=\left\{t \geq 0: \sup _{0 \leq s \leq t}\|U(s)\|_{H^{N}} \leq M\right\}
$$

Since $\left\|U_{0}\right\|_{H^{N}} \leq \frac{M}{2 \sqrt{C_{0}+1}} \leq \frac{M}{2}<M \leq \epsilon_{0}$, then $T>0$ holds from the local existence result. If $T$ is finite, from the definition of $T$, we have

$$
\begin{equation*}
\sup _{0 \leq s \leq t}\|U\|_{H^{N}}=M \tag{3.34}
\end{equation*}
$$

On the other hand, from a priori estimates, we have

$$
\sup _{0 \leq s \leq t}\|U(s)\|_{H^{N}} \leq \sqrt{C_{0}}\left\|U_{0}\right\|_{H^{N}} \leq \frac{M \sqrt{C_{0}}}{2 \sqrt{C_{0}+1}} \leq \frac{M}{2}
$$

which is a contradiction to (3.34). Therefore, $T=\infty$ holds. This implies that the local solution $U(t)$ obtained in Lemma 3.1 can be extended to infinity in time. Thus, we have a global solution $\left(\rho, u, c_{1}, c_{2}\right)(t) \in C\left([0, \infty) ; H^{N}\right)$. This completes the proof of Proposition 2.1.

## 4 Linearized homogeneous system

In this section, to study the time-decay property of solutions to the nonlinear system (2.2), we have to consider the following Cauchy problem arising from the system (2.2)-(2.3)

$$
\left\{\begin{array}{l}
\partial_{t} \rho+n_{\infty} \nabla \cdot u+n_{\infty} \rho=g_{1}  \tag{4.1}\\
\partial_{t} u-\delta \Delta u+\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}} \nabla \rho=g_{2} \\
\partial_{t} c_{1}-\Delta c_{1}+\left(a_{12}-a_{11}\right) c_{1}=g_{3} \\
\partial_{t} c_{2}-\Delta c_{2}+\left(a_{22}-a_{21}\right) c_{2}=g_{4}
\end{array}\right.
$$

with initial data

$$
\begin{equation*}
\left.\left(\rho, u, c_{1}, c_{2}\right)\right|_{t=0}=U_{0}=\left(\rho_{0}, u_{0}, c_{1,0}, c_{2,0}\right) . \tag{4.2}
\end{equation*}
$$

Here, the nonlinear source term takes the form

$$
\left\{\begin{array}{l}
g_{1}=-\nabla \cdot(\rho u)-\rho^{2}  \tag{4.3}\\
g_{2}=-u \cdot \nabla u+\nabla c_{1}^{2}-\nabla c_{2}^{2}-\left(\frac{p^{\prime}\left(\rho+n_{\infty}\right)}{\rho+n_{\infty}}-\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}}\right) \nabla \rho \\
g_{3}=a_{11} \rho c_{1} \\
g_{4}=a_{21} \rho c_{2}
\end{array}\right.
$$

To obtain the time-decay rates of the solution to the system (4.1) in the next section, we are concerned with the following Cauchy problem for the linearized homogenous system corresponding to (4.1)

$$
\left\{\begin{array}{l}
\partial_{t} \rho+n_{\infty} \nabla \cdot u+n_{\infty} \rho=0  \tag{4.4}\\
\partial_{t} u-\delta \Delta u+\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}} \nabla \rho=0 \\
\partial_{t} c_{1}-\Delta c_{1}+\left(a_{12}-a_{11}\right) c_{1}=0 \\
\partial_{t} c_{2}-\Delta c_{2}+\left(a_{22}-a_{21}\right) c_{2}=0 .
\end{array}\right.
$$

In this section, we always denote $U_{1}=[\rho, u]$ as the solution to the linearized homogeneous system

$$
\left\{\begin{array}{l}
\partial_{t} \rho+n_{\infty} \nabla \cdot u+n_{\infty} \rho=0  \tag{4.5}\\
\partial_{t} u-\delta \Delta u+\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}} \nabla \rho=0,
\end{array}\right.
$$

with the initial data $\left.U_{1}\right|_{t=0}=U_{1,0}=\left(\rho_{0}, u_{0}\right)$ in $\mathbb{R}^{3}$.

### 4.1 Representation of solutions

We first find the explicit representation of the Fourier transform of the solution $U_{1}=[\rho, u]$ for the system

$$
\left\{\begin{array}{l}
\rho_{t}+n_{\infty} \nabla \cdot u+n_{\infty} \rho=0  \tag{4.6}\\
u_{t}-\delta \Delta u+\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}} \nabla \rho=0,
\end{array}\right.
$$

with initial data $\left.U_{1}\right|_{t=0}=U_{1,0}=\left(\rho_{0}, u_{0}\right)$.
After taking the Fourier transform in $x$ for the first equation of (4.6), we have

$$
\begin{equation*}
\hat{\rho}_{t}+n_{\infty} i \xi \hat{\mathcal{u}}+n_{\infty} \hat{\rho}=0, \tag{4.7}
\end{equation*}
$$

with initial data $\left.\hat{\rho}\right|_{t=0}=\hat{\rho}_{0}$.

Similarly, by taking the Fourier transform for the second equation of (4.6), we get

$$
\begin{equation*}
\hat{u}_{t}+\delta|\xi|^{2} \hat{u}+\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}} i \xi \hat{\rho}=0 \tag{4.8}
\end{equation*}
$$

with initial data $\left.\hat{u}\right|_{t=0}=\hat{u}_{0}$.
Further, by taking the dot product of (4.8) with $\tilde{\xi}$, we have

$$
\begin{equation*}
\tilde{\xi} \cdot \hat{u}_{t}+\delta|\xi|^{2} \tilde{\xi} \cdot \hat{u}+i \frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}} \tilde{\xi} \cdot \xi \hat{\rho}=0 \tag{4.9}
\end{equation*}
$$

Here and in the sequel we set $\tilde{\xi}=\frac{\xi}{|\xi|}$ for $|\xi| \neq 0$.
Then, we have

$$
\left\{\begin{array}{l}
\hat{\rho}_{t}+i n_{\infty} \xi \cdot \hat{u}+n_{\infty} \hat{\rho}=0  \tag{4.10}\\
\tilde{\xi} \cdot \hat{u}_{t}+\delta|\xi|^{2} \tilde{\xi} \cdot \hat{u}+i \frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}} \tilde{\zeta} \cdot \xi \hat{\rho}=0
\end{array}\right.
$$

We can rewrite (4.10) as

$$
\begin{equation*}
\partial_{t} \hat{U}=A(\tilde{\xi}) \hat{U} \tag{4.11}
\end{equation*}
$$

with $\hat{U}(\xi, t)=(\hat{\rho}(\xi, t), \tilde{\xi} \cdot \hat{u}(\xi, t))^{T}$ and

$$
A(\xi)=\left[\begin{array}{cc}
-n_{\infty} & -i n_{\infty}|\xi| \\
-i \frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}}|\xi| & -\delta|\xi|^{2}
\end{array}\right]
$$

where $T$ denotes the transpose of a row vector. Then,

$$
\operatorname{det}(A-\lambda I)=\lambda^{2}+\left(\delta \xi^{2}+n_{\infty}\right) \lambda+\delta n_{\infty}|\xi|^{2}+p^{\prime}\left(n_{\infty}\right)|\xi|^{2}=0
$$

The eigenvalues of the system are as follows

$$
\begin{aligned}
& \left.\lambda_{1}=-\frac{1}{2}\left(\delta \xi^{2}+n_{\infty}\right)+\frac{1}{2} \sqrt{\left(\delta \xi^{2}+n_{\infty}\right)^{2}-4|\xi|^{2}\left(\delta n_{\infty}+p^{\prime}\left(n_{\infty}\right)\right.}\right) \\
& \left.\lambda_{2}=-\frac{1}{2}\left(\delta \xi^{2}+n_{\infty}\right)-\frac{1}{2} \sqrt{\left(\delta \xi^{2}+n_{\infty}\right)^{2}-4|\xi|^{2}\left(\delta n_{\infty}+p^{\prime}\left(n_{\infty}\right)\right.}\right)
\end{aligned}
$$

Therefore, the eigenvectors corresponding to the eigenvalues $\lambda$ of $A(\xi)$ that satisfy $(A-$ $\lambda I) X=0$ are

$$
v_{1}=\left[\begin{array}{c}
i n_{\infty}|\xi| \\
-\left(n_{\infty}+\lambda_{1}\right)
\end{array}\right]
$$

and

$$
v_{2}=\left[\begin{array}{c}
i n_{\infty}|\xi| \\
-\left(n_{\infty}+\lambda_{2}\right)
\end{array}\right]
$$

From the work above, one can define the general solution of (4.10) as

$$
\left[\begin{array}{c}
\hat{\rho}  \tag{4.12}\\
\tilde{\xi} \cdot \hat{u}
\end{array}\right]=\left[\begin{array}{cc}
i n_{\infty}|\xi| e^{\lambda_{1} t} & i n_{\infty}|\xi| e^{\lambda_{2} t} \\
-\left(n_{\infty}+\lambda_{1}\right) e^{\lambda_{1} t} & -\left(n_{\infty}+\lambda_{2}\right) e^{\lambda_{2} t}
\end{array}\right]\left[\begin{array}{l}
d_{1} \\
d_{2}
\end{array}\right]
$$

where $d_{1}, d_{2}$ satisfy

$$
\left[\begin{array}{c}
\left.\hat{\rho}\right|_{t=0} \\
\left.\tilde{\xi} \cdot \hat{u}\right|_{t=0}
\end{array}\right]=\left[\begin{array}{cc}
\operatorname{in}_{\infty}|\xi| & \operatorname{in}_{\infty}|\xi| \\
-\left(n_{\infty}+\lambda_{1}\right) & -\left(n_{\infty}+\lambda_{2}\right)
\end{array}\right]\left[\begin{array}{l}
d_{1} \\
d_{2}
\end{array}\right] .
$$

From this, we deduce that

$$
\left[\begin{array}{l}
d_{1}  \tag{4.13}\\
d_{2}
\end{array}\right]=\frac{1}{i n_{\infty}|\xi|\left(\lambda_{1}-\lambda_{2}\right)}\left[\begin{array}{cc}
-\left(n_{\infty}+\lambda_{2}\right) & -i n_{\infty}|\xi| \\
\left(n_{\infty}+\lambda_{1}\right) & i n_{\infty}|\xi|
\end{array}\right]\left[\begin{array}{c}
\hat{\rho}_{0} \\
\tilde{\xi} \cdot \hat{u}_{0}
\end{array}\right]
$$

Therefore, we have

$$
\left[\begin{array}{c}
\hat{\rho}  \tag{4.14}\\
\tilde{\xi} \cdot \hat{u}
\end{array}\right]=\frac{1}{i n_{\infty}|\xi|\left(\lambda_{1}-\lambda_{2}\right)}\left[\begin{array}{cc}
i n_{\infty}|\xi| e^{\lambda_{1} t} & i n_{\infty}|\xi| e^{\lambda_{2} t} \\
-\left(n_{\infty}+\lambda_{1}\right) e^{\lambda_{1} t} & -\left(n_{\infty}+\lambda_{2}\right) e^{\lambda_{2} t}
\end{array}\right]\left[\begin{array}{cc}
-\left(n_{\infty}+\lambda_{2}\right) & -i n_{\infty}|\xi| \\
\left(n_{\infty}+\lambda_{1}\right) & i n_{\infty}|\xi|
\end{array}\right]\left[\begin{array}{c}
\hat{\rho}_{0} \\
\tilde{\xi} \cdot \hat{u}_{0}
\end{array}\right] .
$$

It is straightforward to obtain

$$
\begin{equation*}
\hat{\rho}=\frac{\left(\lambda_{1}+n_{\infty}\right) e^{\lambda_{2} t}-\left(\lambda_{2}+n_{\infty}\right) e^{\lambda_{1} t}}{\left(\lambda_{1}-\lambda_{2}\right)} \hat{\rho}_{0}-i n_{\infty} \frac{e^{\lambda_{1} t}-e^{\lambda_{2} t}}{\left(\lambda_{1}-\lambda_{2}\right)} \xi \cdot \hat{u}_{0} \tag{4.15}
\end{equation*}
$$

and

$$
\begin{equation*}
\tilde{\xi} \cdot \hat{u}=\frac{\left(n_{\infty}+\lambda_{1}\right)\left(n_{\infty}+\lambda_{2}\right)}{i n_{\infty}|\tilde{\zeta}|}\left(\frac{e^{\lambda_{1} t}-e^{\lambda_{2} t}}{\lambda_{1}-\lambda_{2}}\right) \hat{\rho}_{0}+\frac{\left(\lambda_{1}+n_{\infty}\right) e^{\lambda_{1} t}-\left(\lambda_{2}+n_{\infty}\right) e^{\lambda_{2} t}}{\lambda_{1}-\lambda_{2}} \tilde{\xi} \cdot \hat{u}_{0} . \tag{4.16}
\end{equation*}
$$

Moreover, by taking the curl for the second equation of (4.6), we have

$$
\begin{equation*}
\nabla \times u_{t}-\delta \nabla \times \Delta u+\frac{p^{\prime}\left(n_{\infty}\right)}{n_{\infty}} \nabla \times \nabla \rho=0, \tag{4.17}
\end{equation*}
$$

since $\nabla \times \nabla \rho=0$ implies

$$
\partial_{t}(\nabla \times u)-\delta \nabla \times \Delta u=0 .
$$

Taking the Fourier transform in $x$ for the above equation, we have

$$
\begin{equation*}
\partial_{t}(\tilde{\xi} \times \hat{u})+\delta|\xi|^{2}(\tilde{\xi} \times \hat{u})=0 . \tag{4.18}
\end{equation*}
$$

Initial data is given as

$$
\begin{equation*}
\left.(\tilde{\xi} \times \hat{u})\right|_{t=0}=\tilde{\xi} \times \hat{u}_{0} . \tag{4.19}
\end{equation*}
$$

By solving the initial value problem (4.18) and (4.19), we have

$$
\begin{equation*}
\tilde{\xi} \times \hat{u}=e^{-\delta|\xi|^{2} t} \tilde{\xi} \times \hat{u}_{0} . \tag{4.20}
\end{equation*}
$$

For $t \geq 0$ and $\tilde{\xi} \in \mathbb{R}^{3}$ with $|\xi| \neq 0$, one has the decomposition $\hat{u}=\tilde{\xi} \tilde{\xi} \cdot \hat{u}-\tilde{\xi} \times(\tilde{\xi} \times \hat{u})$. It is straightforward to get

$$
\begin{align*}
\hat{u}= & \frac{\left(n_{\infty}+\lambda_{1}\right)\left(n_{\infty}+\lambda_{2}\right)}{i n_{\infty}|\tilde{\xi}|^{2}}\left(\frac{e^{\lambda_{1} t}-e^{\lambda_{2} t}}{\lambda_{1}-\lambda_{2}}\right) \xi \cdot \hat{\rho}_{0} \\
& +\left(\frac{\left(\lambda_{1}+n_{\infty}\right) e^{\lambda_{1} t}-\left(\lambda_{2}+n_{\infty}\right) e^{\lambda_{2} t}}{\lambda_{1}-\lambda_{2}}\right) \tilde{\xi} \tilde{\xi} \cdot \hat{u}_{0}-e^{-\delta|\tilde{\xi}|^{2} t} \tilde{\xi} \times\left(\tilde{\xi} \times \hat{u}_{0}\right) . \tag{4.21}
\end{align*}
$$

Then

$$
\begin{align*}
\hat{u}= & \frac{\left(n_{\infty}+\lambda_{1}\right)\left(n_{\infty}+\lambda_{2}\right)}{i n_{\infty}|\xi|}\left(\frac{e^{\lambda_{1} t}-e^{\lambda_{2} t}}{\lambda_{1}-\lambda_{2}}\right) \frac{\xi}{|\xi|} \hat{\rho}_{0} \\
& +\left(\frac{\left(\lambda_{1}+n_{\infty}\right) e^{\lambda_{1} t}-\left(\lambda_{2}+n_{\infty}\right) e^{\lambda_{2} t}}{\lambda_{1}-\lambda_{2}}\right) \frac{\xi \otimes \xi}{|\xi|^{2}} \hat{u}_{0}+e^{-\delta|\xi|^{2} t}\left(I_{3}-\frac{\xi \otimes \xi}{|\xi|^{2}}\right) \hat{u}_{0} . \tag{4.22}
\end{align*}
$$

After summarizing the above computations on the explicit representation of the Fourier transform of the solution $U_{1}=[\rho, u]$, we have

$$
\left[\begin{array}{l}
\hat{\rho}(\xi, t) \\
\hat{u}(\xi, t)
\end{array}\right]=\hat{G}(\xi, t)\left[\begin{array}{l}
\hat{\rho}(\xi, 0) \\
\hat{u}(\xi, 0)
\end{array}\right] .
$$

We can verify the exact expression of the Fourier transform $\hat{G}(\xi, t)$ of Green's function $G(\xi, t)=$ $e^{t B}$ as

$$
\left.\begin{array}{rl}
\hat{G}(\xi, t) & =\left[\begin{array}{ll}
\hat{G}_{11} & \hat{G}_{12} \\
\hat{G}_{21} & \hat{G}_{22}
\end{array}\right] \\
& =\left[\begin{array}{cc}
\frac{\left(\lambda_{1}+n_{\infty}\right) e^{\lambda_{2} t}-\left(\lambda_{2}+n_{\infty}\right) e^{\lambda_{1} t}}{\lambda_{1}-\lambda_{2}} & -i n_{\infty} \xi{\frac{\xi}{}{ }^{\lambda_{1} t}-e^{\lambda_{2} t}}_{\left(\lambda_{1}-\lambda_{2}\right)}^{\left(n_{\infty}+\lambda_{1}\right)\left(n_{\infty}+\lambda_{2}\right) \xi} \\
i n_{\infty}|\xi|^{2} & \left.\frac{e^{\lambda_{1} t}-e^{\lambda_{2} t}}{\lambda_{1}-\lambda_{2}}\right)
\end{array} \frac{\left(\lambda_{1}+n_{\infty}\right) e^{\lambda_{1} t}-\left(\lambda_{2}+n_{\infty}\right) e^{\lambda_{2} t}}{\lambda_{1}-\lambda_{2}} \frac{\xi \otimes \xi}{|\xi|^{2}}+e^{-\delta \xi^{2} t}\left(I_{3}-\frac{\xi \otimes \xi}{|\xi|^{2}}\right)\right. \tag{4.23}
\end{array}\right] . . ~ .
$$

## $4.2 \quad L^{2}-L^{q}$ time-decay property

In this subsection, we use (4.23) to obtain the refined $L^{2}-L^{q}$ time-decay property for

$$
U_{1}=(\rho, u)=e^{t B} U_{1,0}
$$

where $e^{t B}$ is the linear solution operator for $t \geq 0$. For this, we need to find the time-frequency pointwise estimate on $\hat{\rho}, \hat{u}$ in the following lemma.

Lemma 4.1. Let $U_{1}=[\rho, u]$ be the solution to the linear homogeneous system (4.6) with the initial data $\left.U_{1}\right|_{t=0}=\left(\rho_{0}, u_{0}\right)$. Then there exist constants $\epsilon>0, \lambda>0, C>0$ such that for all $t>0,|\xi| \leq \epsilon$,

$$
\begin{align*}
& |\hat{\rho}(\xi, t)| \leq C\left(|\xi|^{2} e^{-\lambda|\xi|^{2} t}+e^{-n_{\infty} \lambda t}\right)\left|\hat{\rho}_{0}(\xi)\right|+C\left(|\xi| e^{-\lambda|\xi|^{2} t}+|\xi| e^{-n_{\infty} \lambda t}\right)\left|\hat{u}_{0}(\xi)\right|,  \tag{4.24}\\
& \quad|\hat{u}(\xi, t)| \leq C|\xi|\left(e^{-\lambda|\xi|^{2} t}+e^{-n_{\infty} \lambda t}\right)\left|\hat{\rho}_{0}(\xi)\right|+C\left(e^{-\lambda|\xi|^{2} t}+|\xi|^{2} e^{-n_{\infty} \lambda t}\right)\left|\hat{u}_{0}(\xi)\right|, \tag{4.25}
\end{align*}
$$

and for all $t>0,|\xi| \geq \epsilon$,

$$
\begin{align*}
& |\hat{\rho}(\xi, t)| \leq C e^{-\lambda t}\left|\hat{\rho}_{0}(\xi), \hat{u}_{0}(\xi)\right|,  \tag{4.26}\\
& |\hat{u}(\xi, t)| \leq C e^{-\lambda t}\left|\hat{\rho}_{0}(\xi), \hat{u}_{0}(\xi)\right| . \tag{4.27}
\end{align*}
$$

Proof. In order to obtain the upper bound of $\hat{\rho}(\xi, t)$ and $\hat{u}(\xi, t)$, we have to estimate $\hat{G}_{11}, \hat{G}_{12}$, $\hat{G}_{21}$, and $\hat{G}_{22}$ in (4.23). To do so, we need to deal with the low frequency $|\xi| \leq \epsilon$ and high frequency $|\xi|>\epsilon$. By using the definition of the eigenvalue, we can analyze the eigenvalue for $|\xi| \rightarrow 0$ as

$$
\begin{aligned}
& \lambda_{1} \sim-O(1)|\xi|^{2}, \\
& \lambda_{2} \sim-n_{\infty}+O(1)|\xi|^{2} .
\end{aligned}
$$

On the other hand, we have the leading orders of the eigenvalue for $|\xi| \rightarrow \infty$ as

$$
\begin{aligned}
& \lambda_{1} \sim-O(1), \\
& \lambda_{2} \sim-\delta \xi^{2}+O(1) .
\end{aligned}
$$

Now, we can estimate $\hat{G}(\xi, t)$ as follows: For $|\xi| \leq \epsilon$,

$$
\begin{aligned}
\left|\hat{G}_{11}\right| & \leq C\left(|\xi|^{2} e^{-\lambda|\xi|^{2} t}+e^{-n_{\infty} \lambda t}\right), \\
\left|\hat{G}_{12}\right| & \leq|\xi|\left(e^{-\lambda|\xi|^{2} t}+e^{-n_{\infty} \lambda t}\right), \\
\left|\hat{G}_{21}\right| & \leq C|\xi|\left(e^{-\left.\lambda|\xi| \xi\right|^{2} t}+e^{-n_{\infty} \lambda t}\right), \\
\left|\hat{G}_{22}\right| & \leq C\left(e^{-\lambda|\xi|^{2} t}+|\xi|^{2} e^{-n_{\infty} \lambda t}\right)+C e^{-\delta|\xi|^{2} t}, \\
& \leq C\left(e^{-\lambda|\xi|^{2} t}+|\xi|^{2} e^{-n_{\infty} \lambda t}\right),
\end{aligned}
$$

and for $|\xi|>\epsilon$

$$
\begin{aligned}
& \left|\hat{G}_{11}\right| \leq C e^{-O(1) \lambda t} \leq C e^{-\lambda t}, \\
& \left|\hat{G}_{12}\right|=\left|\hat{G}_{21}\right| \leq C e^{-\lambda t}, \\
& \left|\hat{G}_{22}\right| \leq C e^{-\left.\delta|\xi|\right|^{2} t}+C e^{-O(1) t} \leq C e^{-\lambda t} .
\end{aligned}
$$

Since the real parts of the eigenvalues are negative except when $\bar{\xi}=0, \hat{G}$ decays exponentially when the eigenvalues coalesce.

Therefore, after plugging the above computations into (4.15) and (4.22), it holds that

$$
|\hat{\rho}(\xi, t)| \leq C\left(|\xi|^{2} e^{-\lambda|\xi|^{2} t}+e^{-n_{\infty} \lambda t}\right)\left|\hat{\rho}_{0}(\xi)\right|+C\left(|\xi| e^{-\lambda|\xi|^{2} t}+|\xi| e^{-n_{\infty} \lambda t}\right)\left|\hat{u}_{0}(\xi)\right|
$$

and

$$
|\hat{u}(\xi, t)| \leq C|\xi|\left(e^{-\lambda|\xi|^{2} t}+e^{-n_{\infty} \lambda t}\right)\left|\hat{\rho}_{0}(\xi)\right|+C\left(e^{-\lambda|\xi|^{2} t}+|\xi|^{2} e^{-n_{\infty} \lambda t}\right)\left|\hat{u}_{0}(\xi)\right|,
$$

for $|\xi| \leq \epsilon$. This proves (4.24) and (4.25). Finally, (4.26) and (4.27) can be proven in the completely same way as for (4.24) and (4.25). This completes the proof of Lemma 4.1.

Theorem 4.2. Let $2 \leq q \leq \infty$, and let $m \geq 0$ be an integer. Suppose that $U_{1}=e^{B t} U_{1,0}$ is the solution to the Cauchy problem (4.6) with the initial data $U_{1,0}=\left(\rho_{0}, u_{0}\right)$. Then $U_{1}=[\rho, u]$ satisfies the following time-decay property:

$$
\begin{align*}
& \left\|\nabla^{m} \rho(t)\right\|_{L^{q}} \leq C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{q}\right)-\frac{m+1}{2}}\left\|\rho_{0}, u_{0}\right\|_{L^{1}}+e^{-\lambda t}\left\|\nabla^{m+\left[3\left(\frac{1}{2}-\frac{1}{q}\right)\right]+}\left(\rho_{0}, u_{0}\right)\right\|_{L^{2}}  \tag{4.28}\\
& \left\|\nabla^{m} u(t)\right\|_{L^{9}} \leq C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{q}\right)-\frac{m}{2}}\left\|\rho_{0}, u_{0}\right\|_{L^{1}}+e^{-\lambda t}\left\|\nabla^{m+\left[3\left(\frac{1}{2}-\frac{1}{q}\right)\right]+}\left(\rho_{0}, u_{0}\right)\right\|_{L^{2}} \tag{4.29}
\end{align*}
$$

for any $t \geq 0$, where $C=C(m, q)$ and $\left[3\left(\frac{1}{2}-\frac{1}{q}\right)\right]_{+}$is defined as

$$
\left[3\left(\frac{1}{2}-\frac{1}{q}\right)\right]_{+}= \begin{cases}0 & \text { if } q=2  \tag{4.30}\\ {\left[3\left(\frac{1}{2}-\frac{1}{q}\right)\right]_{-}+1} & \text { if } q \neq 2\end{cases}
$$

where $[\cdot]_{-}$denotes the integer part of the argument.
Proof. Take $2 \leq q \leq \infty$ and an integer $m \geq 0$. Set $U_{1}=e^{B t} U_{1,0}$. From the Hausdorff-Young inequality,

$$
\begin{align*}
&\left\|\nabla^{m} \rho(t)\right\|_{L^{q}\left(\mathbb{R}_{x}^{3}\right)} \leq C\left\||\xi|^{m} \hat{\rho}(\xi, t)\right\|_{L^{\prime}\left(\mathbb{R}_{\xi}^{3}\right)} \\
& \quad \leq C\left\||\xi|^{m} \hat{\rho}(\xi, t)\right\|_{L^{q^{\prime}}(|\xi| \leq \epsilon)}+C\left\||\xi|^{m} \hat{\rho}(\xi, t)\right\|_{L^{q^{\prime}}(|\xi| \geq \epsilon)} \tag{4.31}
\end{align*}
$$

where $\frac{1}{q}+\frac{1}{q^{\prime}}=1$.
We estimate the first term of (4.31) by using (4.24), as follows:

$$
\begin{aligned}
\left\||\xi|^{m} \hat{\rho}(\xi, t)\right\|_{L^{\prime \prime}}^{q^{\prime}}(|\xi| \leq \epsilon) \leq & c \int_{|\xi| \leq \epsilon}\left[\left(|\xi|^{(m+2) q^{\prime}} e^{-\lambda q^{\prime}|\xi|^{2} t}+|\xi|^{m q^{\prime}} e^{-n_{\infty} \lambda q^{\prime} t}\right)\left|\hat{\rho}_{0}(\xi)\right| q^{q^{\prime}}\right. \\
& \left.+c\left(\left.|\xi|\right|^{m q^{\prime}+q^{\prime}} e^{-\lambda q^{\prime}|\xi|^{2} t}+|\xi|^{m q^{\prime}+q^{\prime}} e^{-n_{\infty} \lambda q^{\prime} t}\right)\left|\hat{u}_{0}(\xi)\right|^{\mid q^{\prime}}\right] d \xi \\
\leq & C \sup _{\xi}\left|\hat{\rho}_{0}\right|^{q q^{\prime}} \int_{|\xi| \leq \epsilon}\left(|\xi|^{(m+2) q^{\prime}} e^{-q^{\prime} \lambda|\xi|^{2}(1+t)+q^{\prime} \lambda|\xi|^{2}}+|\xi|^{\mid q^{\prime}} e^{-n_{\infty} \lambda \lambda q^{\prime} t}\right) d \xi \\
& +C \sup _{\hat{\xi}}\left|\hat{u}_{0}\right|^{q \prime} \int_{|\xi| \leq \epsilon}\left(|\xi|^{(m+1) q^{\prime}} e^{-\lambda q^{\prime}|\xi|^{2}(1+t)+\lambda q^{\prime}|\xi|^{2}}+|\xi|^{(m+1) q^{\prime}} e^{-n_{\infty} \lambda \lambda q^{\prime} t}\right) d \xi \\
\leq & C(1+t)^{-\frac{m q^{\prime}+2 q^{\prime}+3}{2}}\left\|\rho_{0}\right\|_{L^{1}}^{q^{\prime}}+C(1+t)^{-\frac{m q^{\prime}+q^{\prime}+3}{2}}\left\|u_{0}\right\|_{L^{1}}^{q^{\prime}} \\
& +C e^{-n_{\infty} \lambda \lambda^{\prime} t}\left\|\left[\rho_{0}, u_{0}\right]\right\|_{L^{1}}^{q^{\prime}} .
\end{aligned}
$$

Thus,

$$
\begin{align*}
\left\||\xi|^{m} \hat{\rho}(\xi, t)\right\|_{L^{q^{\prime}}(|\xi| \leq \epsilon)} \leq & C(1+t)^{-\frac{3}{2 q^{\prime}}-\frac{m+2}{2}}\left\|\rho_{0}\right\|_{L^{1}}+C(1+t)^{-\frac{3}{2 q^{\prime}}-\left(\frac{m+1}{2}\right)}\left\|u_{0}\right\|_{L^{1}} \\
& +C e^{-n_{\infty} \lambda t}\left\|\left[\rho_{0}, u_{0}\right]\right\|_{L^{1}} \\
\leq & C(1+t)^{-\frac{3}{2}\left[1-\frac{1}{q}\right]-\frac{m+1}{2}}\left\|\left[\rho_{0}, u_{0}\right]\right\|_{L^{1}} \tag{4.32}
\end{align*}
$$

Now, we estimate the second term of (4.31) from (4.26) as

$$
\left\||\xi|^{m} \hat{\rho}(\xi, t)\right\|_{q^{q^{\prime}}(|\xi| \geq \epsilon)} \leq C\left[\int_{|\xi| \geq \epsilon}|\xi|^{m q^{\prime}} e^{-q^{\prime} \lambda t}\left|\hat{\rho}_{0}(\xi), \hat{u}_{0}(\xi)\right|^{q^{\prime}} d \xi\right]^{\frac{1}{q^{\prime}}}
$$

Now, take $\epsilon_{1}>0$ which is small enough. By the Hölder inequality $\frac{1}{q^{\prime}}=\frac{1}{2}+\frac{2-q^{\prime}}{2 q^{\prime}}$, we have

$$
\begin{align*}
& \left\||\xi|^{m} \hat{\rho}(\xi, t)\right\|_{L^{q^{\prime}}(|\xi| \geq \epsilon)} \leq C\left[\int_{|\xi| \geq \epsilon}|\xi|^{-(3+\epsilon)\left(\frac{2-q^{\prime}}{2}\right)}|\xi|^{(3+\epsilon)\left(\frac{2-q^{\prime}}{2}\right)+m q^{\prime}} e^{-q^{\prime} \lambda t}\left|\hat{\rho}_{0}(\xi), \hat{u}_{0}(\xi)\right|^{q^{\prime}} d \xi\right]^{\frac{1}{q^{\prime}}} \\
& \quad \leq C e^{-\lambda t}\left[\int_{|\xi| \geq \epsilon}|\xi|^{-(3+\epsilon)} d \xi\right]^{\frac{2-q^{\prime}}{2 q^{\prime}}}\left[\int_{|\xi| \geq \epsilon}|\xi|^{\left((3+\epsilon)\left(\frac{2-q^{\prime}}{2}\right)+m q^{\prime}\right) \frac{2}{q^{\prime}}}\left|\hat{\rho}_{0}(\xi), \hat{u}_{0}(\xi)\right|^{q^{\prime}\left(\frac{2}{q^{\prime}}\right)} d \xi\right]^{\left(\frac{1}{q^{\prime}}\right)\left(\frac{q^{\prime}}{2}\right)} \\
& \quad \leq C e^{-\lambda t}\left\||\xi|^{-(3+\epsilon)}\right\|\left\|^{\frac{2-q^{\prime}}{2 q^{\prime}}}\right\||\xi|^{(3+\epsilon) \frac{2-q^{\prime}}{2 q^{\prime}}+m}\left[\hat{\rho}_{0}(\xi), \hat{u}_{0}(\xi)\right] \|_{L^{2}} \\
& \quad \leq C e^{-\lambda t}\left\|\nabla^{m+(3+\epsilon) \frac{2-q^{\prime}}{2 q^{\prime}}}\left[\rho_{0}, u_{0}\right]\right\|_{L^{2}} \\
& \quad \leq C e^{-\lambda t}\left\|\nabla^{m+3\left[\frac{1}{q^{\prime}}-\frac{1}{2}\right]+}\left[\rho_{0}, u_{0}\right]\right\|_{L^{2}} \\
& \quad \leq C e^{-\lambda t}\left\|\nabla^{m+3\left[\frac{1}{2}-\frac{1}{q}\right]+}\left[\rho_{0}, u_{0}\right]\right\|_{L^{2}} \tag{4.33}
\end{align*}
$$

after plugging (4.33) and (4.32) into (4.31) implies (4.28).
To prove (4.29), it similarly holds that

$$
\begin{align*}
& \left\|\nabla^{m} u(t)\right\|_{L^{q}\left(\mathbb{R}_{x}^{3}\right)} \leq C\left\||\xi|^{m} \hat{u}(\xi, t)\right\|_{L^{q^{\prime}}\left(\mathbb{R}_{\xi}^{3}\right)} \\
& \quad \leq C\left\||\xi|^{m} \hat{u}(\xi, t)\right\|_{L^{q^{\prime}}(|\xi| \leq \epsilon)}+C\left\||\xi|^{m} \hat{u}(\xi, t)\right\|_{L^{q^{\prime}}(|\xi| \geq \epsilon)}, \tag{4.34}
\end{align*}
$$

where from (4.25), the first term is

$$
\begin{aligned}
\left\||\xi|^{m} \hat{u}(\xi, t)\right\|_{L^{q^{\prime}}(|\xi| \leq \epsilon)}^{q^{\prime}} \leq & C \int_{|\xi| \leq \epsilon}\left(|\xi|^{m q^{\prime}+q^{\prime}}\left(e^{-q^{\prime} \lambda|\xi|^{2}(t+1)}+e^{-n_{\infty} \lambda q^{\prime} t}\right)\left|\hat{\rho}_{0}(\xi)\right|^{q \prime}\right) d \xi \\
& +C \int_{\xi \leq \epsilon}\left(|\xi|^{m q^{\prime}} e^{-\lambda q^{\prime}|\xi|^{2}(t+1)}+|\xi|^{(m+2) q^{\prime}} e^{-n_{\infty} \lambda q^{\prime} t}\right)\left|\hat{u}_{0}(\xi)\right|^{q^{\prime}} d \xi \\
\leq & C(1+t)^{-\frac{m q^{\prime}+q^{\prime}+3}{2}} \|\left[\rho_{0}\left\|_{L^{1}}^{q^{\prime}}+(1+t)^{-\frac{m q^{\prime}+3}{2}}\right\| u_{0} \|_{L^{1}}^{q^{\prime}}\right. \\
& +C e^{-n_{\infty} \lambda q^{\prime} t}\left\|\left[\rho_{0}, u_{0}\right]\right\|_{L^{1}}^{q \prime} .
\end{aligned}
$$

It follows that

$$
\begin{align*}
\left\||\xi|^{m} \hat{u}(\xi, t)\right\|_{L^{q^{\prime}}(|\xi| \leq \epsilon)} \leq & C(1+t)^{-\frac{3}{2 q^{\prime}}-\frac{m+1}{2}} \|\left[\rho_{0} \|_{L^{1}}\right. \\
& +(1+t)^{-\frac{3}{2 q^{\prime}}-\frac{m}{2}}\left\|u_{0}\right\|_{L^{1}}+C e^{-n_{\infty} \lambda t}\left\|\left[\rho_{0}, u_{0}\right]\right\|_{L^{1}} \\
\leq & C(1+t)^{-\frac{3}{2}\left[1-\frac{1}{q}\right]-\frac{m+1}{2}}\left\|\rho_{0}\right\|_{L^{1}}+(1+t)^{-\frac{3}{2}\left[1-\frac{1}{q}\right]-\frac{m}{2}}\left\|u_{0}\right\|_{L^{1}} \\
\leq & C(1+t)^{-\frac{3}{2}\left[1-\frac{1}{q}\right]-\frac{m}{2}}\left\|\left[\rho_{0}, u_{0}\right]\right\|_{L^{1}} \tag{4.35}
\end{align*}
$$

Similarly to obtaining (4.33), one has

$$
\begin{equation*}
\left\||\xi|^{m} \hat{u}(\xi, t)\right\|_{L^{q^{\prime}}(|\xi| \geq \epsilon)} \leq C e^{-\lambda t}\left\|\nabla^{m+3\left[\frac{1}{2}-\frac{1}{q}\right]_{+}}\left[\rho_{0}, u_{0}\right]\right\|_{L^{2}} . \tag{4.36}
\end{equation*}
$$

Thus, plugging (4.35) and (4.36) into (4.34) implies (4.29). This completes the proof of Theorem 4.2.

Corollary 4.3. Assume that $U_{1}=e^{B t} U_{1,0}$ is the solution to the Cauchy problem (4.6) with initial data $U_{1,0}=\left[\rho_{0}, u_{0}\right]$. Then $U_{1}=[\rho, u]$ satisfies the following:

$$
\begin{align*}
&\|\rho(t)\|_{L^{2}} \leq C(1+t)^{-\frac{5}{4}}\left\|\left[\rho_{0}, u_{0}\right]\right\|_{L^{1}}+e^{-\lambda t}\left\|\left[\rho_{0}, u_{0}\right]\right\|_{L^{2}},  \tag{4.37}\\
&\|u(t)\|_{L^{2}} \leq C(1+t)^{-\frac{3}{4}}\left\|\left[\rho_{0}, u_{0}\right]\right\|_{L^{1}}+e^{-\lambda t}\left\|\left[\rho_{0}, u_{0}\right]\right\|_{L^{2}},  \tag{4.38}\\
&\|\rho(t)\|_{L^{\infty}} \leq C(1+t)^{-2}\left\|\left[\rho_{0}, u_{0}\right]\right\|_{L^{1}}+e^{-\lambda t}\left\|\left[\rho_{0}, u_{0}\right]\right\|_{\dot{H}^{2}},  \tag{4.39}\\
&\|u(t)\|_{L^{\infty}} \leq C(1+t)^{-\frac{3}{2}}\left\|\left[\rho_{0}, u_{0}\right]\right\|_{L^{1}}+e^{-\lambda t}\left\|\left[\rho_{0}, u_{0}\right]\right\|_{\dot{H}^{2}} . \tag{4.40}
\end{align*}
$$

## 5 Time-decay rates for the nonlinear system

In this section, we will prove (2.5)-(2.7) in Proposition 2.2. The main idea is to introduce a general approach to combine the energy estimates and spectral analysis. We will apply the linear $L^{2}-L^{q}$ time-decay property of the linearized homogeneous system (4.4), studied in the previous section, to the nonlinear case. We need the mild form of the original nonlinear Cauchy problem (2.2). Throughout this section, we suppose that $U=\left[\rho, u, c_{1}, c_{2}\right]$ is the solution to the Cauchy problem (2.3) with initial data $U_{0}=\left(\rho_{0}, u_{0}, c_{1,0}, c_{2,0}\right)$.

Then, by Duhamel's principle, the solution $U=\left[\rho, u, c_{1}, c_{2}\right]$ can be formally written as

$$
\begin{equation*}
U(t)=e^{B t} U_{0}+\int_{0}^{t} e^{(t-s) B}\left[g_{1}, g_{2}, g_{3}, g_{4}\right] d s, \tag{5.1}
\end{equation*}
$$

where $e^{B t} U_{0}$ is the solution to the Cauchy problem (4.1) with initial data $U_{0}=\left(\rho_{0}, u_{0}, c_{1,0}, c_{2,0}\right)$. Here, the nonlinear source term takes the form (4.3).

### 5.1 Time rate for the energy functional and high-order energy functional

In this subsection, we will prove the time-decay rate for the energy functional $\|U(t)\|_{H^{N}}^{2}$ and the time-decay rate for the high-order energy functional $\|\nabla U(t)\|_{H^{N}}^{2}$. For that, we investigate the time-decay rates of solutions in Proposition 2.1 under extra conditions on the given initial data $U_{0}=\left[\rho_{0}, u_{0}, c_{1,0}, c_{2,0}\right]$. We define

$$
\begin{equation*}
\epsilon_{H^{N}}\left(U_{0}\right)=\left\|U_{0}\right\|_{H^{N}}+\left\|\left[\rho_{0}, u_{0}\right]\right\|_{L^{1}}, \tag{5.2}
\end{equation*}
$$

for an integer $N \geq 4$. We also define $\mathcal{E}_{N} U(t) \sim\left\|\left[\rho, u, c_{1}, c_{2}\right]\right\|_{H^{N}}^{2}$ as the energy functional and $\mathcal{D}_{N} U(t) \sim\left\|\left[\nabla\left(u, c_{1}, c_{2}\right)\right]\right\|_{H^{N}}^{2}, \mathcal{D}_{N}^{h} U(t) \sim\left\|\left[\rho, c_{1}, c_{2}\right]\right\|_{H^{N}}^{2}$ as the dissipation rates.

First, we start with this proposition for the energy functional and the high-order energy functional.

Proposition 5.1. Let $U=\left[\rho, u, c_{1}, c_{2}\right]$ be the solution to the Cauchy problem (2.2) with initial data $U_{0}=\left(\rho_{0}, u_{0}, c_{1,0}, c_{2,0}\right)$. If $\epsilon_{N+1}\left(U_{0}\right)>0$ is small enough, then the solution $U=\left[\rho, u, c_{1}, c_{2}\right]$ satisfies

$$
\begin{equation*}
\|U(t)\|_{H^{N}} \leq \epsilon_{N+1}\left(U_{0}\right)(1+t)^{\frac{-3}{4}} \tag{5.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\|\nabla U(t)\|_{H^{N}} \leq \epsilon_{N+1}\left(U_{0}\right)(1+t)^{\frac{-5}{4}} \tag{5.4}
\end{equation*}
$$

for any $t \geq 0$.
Proof. Suppose $\epsilon_{N+1}\left(U_{0}\right)$ is sufficiently small. From Proposition 2.1 the solution $U=\left[\rho, u, c_{1}, c_{2}\right]$ satisfies:

$$
\begin{equation*}
\frac{d}{d t} \mathcal{E}_{N}(U(t))+\lambda_{1} \mathcal{D}_{N}(U(t))+\lambda_{2} \mathcal{D}_{N}^{h}(U(t)) \leq 0 \tag{5.5}
\end{equation*}
$$

for $t \geq 0$.
Now, we proceed by making the time-weighted estimate and iteration for the inequality (5.5). Let $l \geq 0$. Multiplying (5.5) by $(1+t)^{l}$ and integrating over $[0, t]$ gives

$$
\begin{aligned}
(1+t & )^{l} \mathcal{E}_{N} U(t)+\lambda_{1} \int_{0}^{t}(1+s)^{l} \mathcal{D}_{N}(U(s)) d s+\lambda_{2} \int_{0}^{t}(1+s)^{l} \mathcal{D}_{N}^{h}(U(s)) d s \\
& \leq \mathcal{E}_{N}\left(U_{0}\right)+l \int_{0}^{t}(1+s)^{l-1} \mathcal{E}_{N} U(s) d s \\
& \leq \mathcal{E}_{N}\left(U_{0}\right)+C l \int_{0}^{t}(1+s)^{l-1}\left(\mathcal{D}_{N-1} U(s)+\mathcal{D}_{N}^{h}(U(s))+\|u(s)\|_{L^{2}}^{2}\right) d s
\end{aligned}
$$

where we have used

$$
\mathcal{E}_{N} U(t) \leq C \mathcal{D}_{N-1} U(t)+C \mathcal{D}_{N}^{h}(U(t))+\|u(t)\|_{L^{2}}^{2}
$$

Using (5.5) again, we have

$$
\mathcal{E}_{N+1}(U(t))+\lambda_{1} \int_{0}^{t} \mathcal{D}_{N+1}(U(t))+\lambda_{2} \int_{0}^{t} \mathcal{D}_{N+1}^{h}(U(t)) \leq \mathcal{E}_{N+1}\left(U_{0}\right)
$$

and

$$
\begin{aligned}
(1+ & t)^{l-1} \mathcal{E}_{N+1} U(t)+\lambda_{1} \int_{0}^{t}(1+s)^{l-1} \mathcal{D}_{N+1}(U(s)) d s+\lambda_{2} \int_{0}^{t}(1+s)^{l-1} \mathcal{D}_{N+1}^{h}(U(s)) d s \\
& \leq \mathcal{E}_{N+1}\left(U_{0}\right)+C(l-1) \int_{0}^{t}(1+s)^{l-2} \mathcal{E}_{N+1} U(s) d s \\
& \leq \mathcal{E}_{N+1}\left(U_{0}\right)+C(l-1) \int_{0}^{t}(1+s)^{l-2}\left(\mathcal{D}_{N} U(s)+C \mathcal{D}_{N+1}^{h}(U(s))+\|u(s)\|_{L^{2}}^{2}\right) d s
\end{aligned}
$$

By iterating the above estimates for $1<l<2$, we have

$$
\begin{align*}
& (1+t)^{l} \mathcal{E}_{N} U(t)+\lambda_{1} \int_{0}^{t}(1+s)^{l} \mathcal{D}_{N}(U(s)) d s+\lambda_{2} \int_{0}^{t}(1+s)^{l} \mathcal{D}_{N}^{h}(U(s)) d s \\
& \quad \leq \mathcal{E}_{N+1}\left(U_{0}\right)+C \int_{0}^{t}(1+s)^{l-1}\|u(s)\|_{L^{2}}^{2} d s \tag{5.6}
\end{align*}
$$

To estimate the integral term on the right-hand side of (5.6), let us define

$$
\mathcal{E}_{N, \infty}(U(t))=\sup _{0 \leq s \leq T}(1+t)^{\frac{3}{2}} \mathcal{E}_{N} U(t)
$$

Now, we estimate the integral term on the right-hand side of (5.6) by applying the linear estimate on $u$ in (4.38) to the mild form (5.1), giving us

$$
\begin{align*}
\|u(t)\|_{L^{2}} \leq & C(1+t)^{\frac{-3}{4}}\left\|\rho_{0}, u_{0}\right\|_{L^{1}}+C e^{-\lambda t}\left\|\rho_{0}, u_{0}\right\|_{L^{2}} \\
& \left.+C \int_{0}^{t}(1+t-s)^{\frac{-3}{4}}\left\|g_{1}, g_{2}\right\|_{L^{1}} d s+C \int_{0}^{t} e^{-\lambda(t-s)}\right)\left\|g_{1}, g_{2}\right\|_{L^{2}} d s . \tag{5.7}
\end{align*}
$$

Recall the definitions (4.3) of $g_{1}$ and $g_{2}$. It is direct to check that for any $0 \leq s \leq t$,

$$
\left\|g_{1}(s), g_{2}(s)\right\|_{L^{1} \cap L^{2}} \leq C \mathcal{E}_{N} U(t) \leq C(1+s)^{\frac{-3}{2}} \mathcal{E}_{N, \infty} U(t)
$$

where

$$
\mathcal{E}_{N, \infty}(U(t))=\sup _{0 \leq s \leq T}(1+t)^{\frac{3}{2}} \mathcal{E}_{N} U(t) .
$$

Putting the above inequalities into (5.7), gives

$$
\begin{equation*}
\|u(t)\|_{L^{2}} \leq C(1+t)^{\frac{-3}{4}}\left(\left\|\rho_{0}, u_{0}\right\|_{L^{1} \cap L^{2}}+\mathcal{E}_{N, \infty} U(t)\right) . \tag{5.8}
\end{equation*}
$$

Next, we prove the uniform-in-time boundedness of $\mathcal{E}_{N, \infty} U(t)$ which yields the time-decay rates of the energy functional $\mathcal{E}_{N} U(t)$. In fact, by taking $l=\frac{3}{2}+\epsilon$ in (5.6) where $\epsilon>0$ is sufficiently small, it follows that

$$
\begin{aligned}
& (1+t)^{\frac{3}{2}+\epsilon} \mathcal{E}_{N} U(t)+\lambda_{1} \int_{0}^{t}(1+s)^{\frac{3}{2}+\epsilon} \mathcal{D}_{N}(U(s)) d s+\lambda_{2} \int_{0}^{t}(1+s)^{\frac{3}{2}+\epsilon} \mathcal{D}_{N}^{h}(U(s)) d s \\
& \quad \leq \mathcal{E}_{N+1}\left(U_{0}\right)+C \int_{0}^{t}(1+s)^{\frac{1}{2}+\epsilon}\|u(s)\|_{L^{2}}^{2} d s .
\end{aligned}
$$

Here, using (5.10) and the fact that $\mathcal{E}_{N, \infty}(U(t))$ is non-decreasing in $t$, it further holds that

$$
\left.\int_{0}^{t}(1+s)^{\frac{1}{2}+\epsilon}\|u(t)\|_{L^{2}}^{2} d s \leq C(1+t)^{\epsilon}\left(\mathcal{E}_{N, \infty}^{2} U(t)\right)+\left\|\rho_{0}, u_{0}\right\|_{L^{1} \cap L^{2}}^{2}\right) .
$$

Therefore, it follows that

$$
\begin{aligned}
& (1+t)^{\frac{3}{2}+\epsilon} \mathcal{E}_{N} U(t)+\lambda_{1} \int_{0}^{t}(1+s)^{\frac{3}{2}+\epsilon} \mathcal{D}_{N}(U(s)) d s+\lambda_{2} \int_{0}^{t}(1+s)^{\frac{3}{2}+\epsilon} \mathcal{D}_{N}^{h}(U(s)) d s \\
& \left.\quad \leq \mathcal{E}_{N+1}\left(U_{0}\right)+C(1+t)^{\epsilon}\left(\mathcal{E}_{N, \infty}^{2} U(t)\right)+\left\|\rho_{0}, u_{0}\right\|_{L^{1} \cap L^{2}}^{2}\right)
\end{aligned}
$$

which implies

$$
(1+t)^{\frac{3}{2}} \mathcal{E}_{N} U(t) \leq C\left(\mathcal{E}_{N+1}\left(U_{0}\right)+\left\|\rho_{0}, u_{0}\right\|_{L^{1}}^{2}+\mathcal{E}_{N, \infty}^{2} U(t)\right)
$$

and thus

$$
\mathcal{E}_{N, \infty} U(t) \leq C\left(\epsilon_{N+1}^{2}\left(U_{0}\right)+\mathcal{E}_{N, \infty}^{2} U(t)\right) .
$$

Since $\epsilon_{N+1}\left(U_{0}\right)>0$ is sufficiently small, it holds that $\left.\mathcal{E}_{N, \infty} U(t)\right) \leq C \epsilon_{N+1}^{2}\left(U_{0}\right)$ for any $t \geq 0$, which gives $\|U(s)\|_{H^{N}} \leq C\left(\mathcal{E}_{N} U(t)\right)^{\frac{1}{2}} \leq C \epsilon_{N+1}\left(U_{0}\right)(1+t)^{-\frac{3}{4}}$. This proves (5.3).

Now, we estimate the high-order energy functional. By comparing the definitions of $\mathcal{E}_{N} U(t), \mathcal{D}_{N} U(t)$ and $\mathcal{D}_{N}^{h} U(t)$, it follows from (5.5) that we have

$$
\frac{d}{d t}\|\nabla U(t)\|_{H^{N}}^{2}+\lambda\|\nabla U(t)\|_{H^{N}}^{2} \leq C\|\nabla u(t)\|_{L^{2}}^{2}
$$

which implies

$$
\begin{equation*}
\|\nabla U(t)\|_{H^{N}}^{2} \leq e^{-\lambda t}\left\|\nabla U_{0}\right\|_{H^{N}}^{2}+C \int_{0}^{t} e^{-\lambda(t-s)}\|\nabla u(s)\|_{L^{2}}^{2} d s \tag{5.9}
\end{equation*}
$$

for any $t \geq 0$.
Similarly to obtaining (5.8), we estimate the time integral term on the (r.h.s.) of the above inequality. One can apply the linear estimate (4.29) to the mild form (5.1) so that

$$
\begin{align*}
& \|\nabla u(t)\|_{L^{2}} \leq C(1+t)^{\frac{-5}{4}}\left\|\rho_{0}, u_{0}\right\|_{L^{1}}+C e^{-\lambda t}\left\|\left[\rho_{0}, u_{0}\right]\right\|_{\dot{H}^{1}} \\
& \quad+C \int_{0}^{t}(1+t-s)^{\frac{-5}{4}}\left\|\left[g_{1}(s), g_{2}(s)\right]\right\|_{L^{1}} d s+C \int_{0}^{t} e^{-\lambda(t-s)}\left\|\left[g_{1}(s), g_{2}(s)\right]\right\|_{\dot{H}^{1}} d s . \tag{5.10}
\end{align*}
$$

Recall the definition (4.3) of $g_{1}$ and $g_{2}$. It is straightforward to check that for any $0 \leq s \leq t$,

$$
\left\|\left[g_{1}(s), g_{2}(s)\right]\right\|_{L^{1} \cap \dot{H}^{1}} \leq C \mathcal{E}_{N} U(s) \leq C \epsilon_{N+1}^{2}\left(U_{0}\right)(1+s)^{\frac{-3}{2}}
$$

Putting this into (5.10) gives

$$
\begin{equation*}
\|\nabla u(t)\|_{L^{2}} \leq C \epsilon_{N+1}\left(U_{0}\right)(1+t)^{\frac{-5}{4}} . \tag{5.11}
\end{equation*}
$$

Then, by using (5.11) in (5.9), we have

$$
\|\nabla U(t)\|_{H^{N}}^{2} \leq e^{-\lambda t}\left\|\nabla U_{0}\right\|_{H^{N}}^{2}+C \epsilon_{N+1}^{2}\left(U_{0}\right)(1+t)^{\frac{-5}{2}}
$$

which implies (5.4). The proof of Proposition 5.1 is complete.

### 5.2 Time-decay rate in $L^{q}$

In this subsection, we will prove Proposition 2.2 for time-decay rates in $L^{q}$ with $2 \leq q \leq \infty$ corresponding to (1.4)-(1.6) in Theorem 1.1. For $N \geq 4$, Proposition 5.1 shows that if $\epsilon_{N+1}\left(U_{0}\right)$ is small enough,

$$
\begin{equation*}
\|U(s)\|_{H^{N}} \leq C \epsilon_{N+1}\left(U_{0}\right)(1+t)^{-\frac{3}{4}} \tag{5.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\|\nabla U(t)\|_{H^{N}} \leq C \epsilon_{N+1}\left(U_{0}\right)(1+t)^{\frac{-5}{4}} \tag{5.13}
\end{equation*}
$$

Now, let us establish the estimates on $u, \rho$ as follows.
Estimate on $\|u(t)\|_{L^{q}}$. For the $L^{2}$ rate, it is easy to see from (5.8) and (5.12) that

$$
\|u(t)\|_{L^{2}} \leq C \epsilon_{N+1}\left(U_{0}\right)(1+t)^{\frac{-3}{4}} \leq C(1+t)^{\frac{-3}{4}} .
$$

For the $L^{\infty}$ rate, by applying the $L^{\infty}$ linear estimate on $u$ in (4.40) to the mild form (5.1), we have

$$
\begin{align*}
\|u(t)\|_{L^{\infty}} \leq & C(1+t)^{\frac{-3}{2}}\left\|\rho_{0}, u_{0}\right\|_{L^{1}}+C e^{-\lambda t}\left\|\nabla^{2}\left[\rho_{0}, u_{0}\right]\right\|_{L^{2}} \\
& +C \int_{0}^{t}(1+t-s)^{\frac{-3}{2}} \|\left[\left[g_{1}(s), g_{2}(s)\right]\left\|_{L^{1}} d s+C \int_{0}^{t} e^{-\lambda(t-s)}\right\| \nabla^{2}\left[g_{1}(s), g_{2}(s)\right] \|_{L^{2}} d s\right. \\
\leq & C(1+t)^{\frac{-3}{2}}\left\|\rho_{0}, u_{0}\right\|_{L^{1} \cap \dot{H}^{2}}+C \int_{0}^{t}(1+t-s)^{\frac{-3}{2}}\left\|\left[g_{1}(s), g_{2}(s)\right]\right\|_{L^{1} \cap \dot{H}^{2}} d s . \tag{5.14}
\end{align*}
$$

Since by (5.12) and (5.13)

$$
\left\|\left[g_{1}(s), g_{2}(s)\right]\right\|_{L^{1} \cap H^{2}} \leq C\|\nabla U(t)\|_{H^{N}}\|U(s)\|_{H^{N}} \leq C \epsilon_{N+1}^{2}\left(U_{0}\right)(1+s)^{-2}
$$

it follows that

$$
\|u(t)\|_{L^{\infty}} \leq C \epsilon_{N+1}\left(U_{0}\right)(1+t)^{\frac{-3}{2}} .
$$

Then, by $L^{2}-L^{\infty}$ interpolation,

$$
\begin{equation*}
\|u\|_{L^{q}} \leq C \epsilon_{N+1}\left(U_{0}\right)(1+t)^{\frac{-3}{2}+\frac{3}{2 q}} \tag{5.15}
\end{equation*}
$$

for $2 \leq q \leq \infty$.
Estimate on $\|\rho(t)\|_{L^{g}}$. For the $L^{2}$ rate, utilizing the $L^{2}$ estimate on $\rho$ in (4.37) to (5.1), we have

$$
\begin{align*}
\|\rho(t)\|_{L^{2}} \leq & C(1+t)^{\frac{-5}{4}}\left\|\rho_{0}, u_{0}\right\|_{L^{1}}+C e^{-\lambda t}\left\|\rho_{0}, u_{0}\right\|_{L^{2}}+C \int_{0}^{t}(1+t-s)^{\frac{-5}{4}}\left\|g_{1}, g_{2}\right\|_{L^{1}} d s \\
& +C \int_{0}^{t} e^{-\lambda(t-s)}\left\|g_{1}(s), g_{2}(s)\right\|_{L^{2}} d s . \tag{5.16}
\end{align*}
$$

Due to (5.12),

$$
\left\|g_{1}(s), g_{2}^{*}(s)\right\|_{L^{1} \cap L^{2}} \leq C\|U(s)\|_{H^{N}}^{2} \leq C \epsilon_{N+1}^{2}\left(U_{0}\right)(1+t)^{\frac{-3}{2}} .
$$

Then (5.16) implies the slower decay estimate

$$
\begin{equation*}
\|\rho(t)\|_{L^{2}} \leq C \epsilon_{N+1}\left(U_{0}\right)(1+t)^{\frac{-5}{4}} \leq C(1+t)^{\frac{-5}{4}} . \tag{5.17}
\end{equation*}
$$

For the $L^{\infty}$ rate, utilizing the $L^{\infty}$ estimate on $\rho$ in (4.39) to (5.1), we have

$$
\begin{equation*}
\|\rho(t)\|_{L^{\infty}} \leq(1+t)^{-2}\left\|\rho_{0}, u_{0}\right\|_{L^{1} \cap \dot{H}^{2}}+C \int_{0}^{t}(1+t-s)^{-2}\left\|\left[g_{1}(s), g_{2}(s)\right]\right\|_{L^{1} \cap \dot{H}^{2}} d s . \tag{5.18}
\end{equation*}
$$

Since by (5.12) and (5.13)

$$
\left\|\left[g_{1}(s), g_{2}(s)\right]\right\|_{L^{1} \cap \dot{H}^{2}} \leq C\|\nabla U(t)\|_{H^{N}}\|U(s)\|_{H^{N}} \leq C \epsilon_{N+1}^{2}\left(U_{0}\right)(1+s)^{-2},
$$

which yields from (5.18) that

$$
\|\rho(t)\|_{L^{\infty}} \leq C \epsilon_{N+1}\left(U_{0}\right)(1+s)^{-2} .
$$

Therefore, by $L^{2}-L^{\infty}$ interpolation,

$$
\begin{equation*}
\|\rho(t)\|_{L^{9}} \leq C \epsilon_{N+1}\left(U_{0}\right)(1+s)^{-2+\frac{3}{2 q}} \tag{5.1}
\end{equation*}
$$

for $2 \leq q \leq \infty$.
Next, we estimate the time-decay rate of $\left[c_{1}, c_{2}\right]$. We start with the estimate on $\left\|c_{1}(t)\right\|_{L^{q}}$. For the $L^{2}$ rate,

$$
\begin{align*}
\left\|c_{1}\right\|_{L^{2}} & \leq C\left\|\hat{c}_{1}\right\|_{L^{2}(\xi)}  \tag{5.20}\\
& \leq C\left[\int_{\xi} e^{-2\left(|\xi|^{2}+\left(a_{12}-a_{11} n_{\infty}\right)\right) t}\left|\hat{c}_{0}\right|^{2} d \xi\right]^{\frac{1}{2}}+a_{11} \int_{0}^{t}\left[\int_{\tilde{\xi}}\left[e^{-2\left(|\xi|^{2}+\left(a_{12}-a_{11} n_{\infty}\right)\right)(t-s)}\left|\rho \hat{c}_{1}\right|^{2} d \xi\right]^{\frac{1}{2}} d s\right. \\
& \leq e^{-\left(a_{12}-a_{11} n_{\infty}\right) t}\left[\int_{\xi} e^{-2|\xi|^{2}(t)}\left|\hat{c}_{0}\right|^{2} d \xi\right]^{\frac{1}{2}}+C \int_{0}^{t} e^{-\left(a_{12}-a_{11} n_{\infty}\right)(t-s)}\left[\int_{\xi} e^{-2|\xi|^{2}(t-s+1)}\left|\rho \hat{c}_{1}\right|^{2} d \xi\right]^{\frac{1}{2}} d s \\
& \leq C e^{-\left(\left(a_{12}-a_{11} n_{\infty}\right) t\right.}\left\|\hat{c}_{0}\right\|_{L^{2}}+C \int_{0}^{t} e^{-\left(a_{12}-a_{11} n_{\infty}\right)(t-s)} \sup _{\tilde{\xi}} e^{-\left||\xi|^{2}(t-s+1)\right.}\left\|\rho c_{1}(s)\right\|_{L^{2} 2} \tag{5.21}
\end{align*}
$$

Due to (5.12),

$$
\left\|\rho c_{1}(s)\right\|_{L^{2}} \leq C\|U(s)\|_{N}^{2} \leq C \epsilon_{N+1}^{2}\left(U_{0}\right)(1+t)^{\frac{-3}{2}} .
$$

Then (5.20) implies the slower decay estimate

$$
\begin{equation*}
\left\|c_{1}\right\|_{L^{2}} \leq C \epsilon_{N+1}\left(U_{0}\right)(1+t)^{\frac{-3}{2}} \tag{5.22}
\end{equation*}
$$

Similarly, we have

$$
\begin{equation*}
\left\|c_{2}\right\|_{L^{2}} \leq C \epsilon_{N+1}\left(U_{0}\right)(1+t)^{\frac{-3}{2}} \tag{5.23}
\end{equation*}
$$

For $L^{\infty}$ rate, from the Hausdorff-Young inequality and the Hölder inequality, we have

$$
\begin{align*}
\left\|c_{1}\right\|_{L^{\infty}} \leq & C\left\|\hat{c}_{1}\right\|_{L^{1}} \leq C \int_{\xi \leq \epsilon} e^{-\left(|\xi|^{2}+\left(a_{12}-a_{11} n_{\infty}\right)\right) t}\left|\hat{c}_{1,0}\right| d \xi \\
& +C \int_{0}^{t} \int_{\xi \leq \epsilon} e^{-\left(|\xi|^{2}+\left(a_{12}-a_{11} n_{\infty}\right)\right)(t-s)}\left|\rho \hat{c}_{1}\right| d \xi d s \\
& +C \int_{|\xi| \geq \epsilon} e^{\left.-\left(a_{12}-a_{11} n_{\infty}\right)\right) t}\left|\hat{c}_{1,0}\right| d \xi+C \int_{0}^{t} \int_{|\xi| \geq \epsilon} e^{\left.-\left(a_{12}-a_{11} n_{\infty}\right)\right)(t-s)}\left|\rho \hat{c}_{1}\right| d \xi d s \\
\leq & C e^{-\left(a_{12}-a_{11} n_{\infty}\right) t}(1+t)^{\frac{-3}{2}}\left\|c_{0}\right\|_{L^{1}}+C \int_{0}^{t} e^{-\left(a_{12}-a_{11} n_{\infty}\right)(t-s)}\left\|\rho \hat{c}_{1}(s)\right\|_{L^{1}} \\
& +C e^{\left.-\left(a_{12}-a_{11} n_{\infty}\right)\right) t}\left[\int_{|\xi| \geq \epsilon}|\xi|^{-4} d \xi\right]^{\frac{1}{2}}\left[\int_{|\xi| \geq \epsilon}|\xi|^{4}\left|\hat{c}_{1,0}\right|^{2} d \xi\right]^{\frac{1}{2}} \\
& +C \int_{0}^{t} e^{\left.-\left(a_{12}-a_{11} n_{\infty}\right)\right)(t-s)}\left[\int_{|\xi| \geq \epsilon}|\xi|^{-4} d \xi\right]^{\frac{1}{2}}\left[\int_{|\xi| \geq \epsilon}|\xi|^{4}\left|\rho \hat{c}_{1}\right|^{2} d \xi\right]^{\frac{1}{2}} d s \\
\leq & C e^{-\left(a_{12}-a_{11} n_{\infty}\right) t}(1+t)^{\frac{-3}{2}}\left\|c_{0}\right\|_{L^{1}}+C \int_{0}^{t} e^{-\left(a_{12}-a_{11} n_{\infty}\right)(t-s)}\left\|\rho c_{1}(s)\right\|_{L^{1}} d s \\
& +C e^{\left.-\left(a_{12}-a_{11} n_{\infty}\right)\right) t}\left\|\nabla^{2} c_{0}\right\|_{L^{2}}+C \int_{0}^{t} e^{-\left(a_{12}-a_{11} n_{\infty}\right)(t-s)}\left\|\nabla^{2}\left(\rho c_{1}(s)\right)\right\|_{L^{2}} d s \tag{5.24}
\end{align*}
$$

Since by (5.12)

$$
\left\|\rho c_{1}(s)\right\|_{L^{1} \cap \dot{H}^{2}} \leq C\|U(s)\|_{N}^{2} \leq C \epsilon_{N+1}^{2}\left(U_{0}\right)(1+t)^{\frac{-3}{2}}
$$

Then, (5.24) implies the slower decay estimate

$$
\begin{equation*}
\left\|c_{1}\right\|_{L^{\infty}} \leq C \epsilon_{N+1}\left(U_{0}\right)(1+t)^{\frac{-3}{2}} \tag{5.25}
\end{equation*}
$$

Similarly, we have

$$
\begin{equation*}
\left\|c_{2}\right\|_{L^{\infty}} \leq C \epsilon_{N+1}\left(U_{0}\right)(1+t)^{\frac{-3}{2}} \tag{5.26}
\end{equation*}
$$

So, by $L^{2}-L^{\infty}$ interpolation,

$$
\begin{equation*}
\left\|c_{1}, c_{2}\right\|_{L^{q}} \leq C \epsilon_{N+1}\left(U_{0}\right)(1+t)^{\frac{-3}{2}} \tag{5.27}
\end{equation*}
$$

for $2 \leq q \leq \infty$.
This completes the proof of Proposition 2.2 and hence Theorem 1.1.

## 6 Conclusion

We have studied a chemotaxis model where a compressible fluid model for cells and a diffusive Lotka-Volterra model for chemoattractants and repellents are used. The previous results for chemotaxis are mostly extensions of the Keller and Segel model or in the case of fluid dynamical models, the incompressible fluid models for the cells are used. We showed the
existence of global solutions and their asymptotic behavior in three dimensions with the initial data as a small perturbation of the constant state ( $n_{\infty}, 0,0,0$ ). Our method is based on the basic energy estimates used for the a priori estimates and the iterative method in solving the Cauchy problem (1.1). Moreover, we have also shown the decay estimates of solutions to the Cauchy problem (1.1) in $\mathbb{R}^{3}$, in which the detailed analysis of Green's functions of the linear system is combined with the refined energy estimates with the help of Duhamel's principle. We proved the decay property of solutions as time goes to infinity. Our results are complementary to Ambrosi, Bussolino and Preziosi [2], where the modeling aspects such as qualitative analysis and numerical simulations of the compressible fluid model for cells with chemoattractants are examined for vasculogenesis.

## Acknowledgements

The authors are grateful to the reviewer for helpful comments and suggestions, with which we were able to improve the quality of the paper.

## References

[1] J. Adler, Chemotaxis in bacteria, Science 153(1966), 708-716. https://doi.org/10.1126/ science.153.3737.708
[2] D. Ambrosi, F. Bussolino, L. Preziosi, A review of vasculogenesis models, J. Theor. Med. 6(2005), 1-19. https://doi.org/10.1080/1027366042000327098; MR2157640; Zbl 1167.92004
[3] M. Chae, K. Kang, J. Lee, Existence of smooth solutions to chemotaxis-fluid equations, Discrete Contin. Dyn. Syst. 33(2013), 2271-2297. https://doi.org/10.3934/dcds.2013. 33.2271; MR3007686; Zbl 1277.35276
[4] M. A. J. Chaplain, A. M. Stuart, A model mechanism for the chemotactic response of endothelial cells to tumour angiogenesis factor, IMA J. Math. Appl. Med. Biol. 10(1993), 149-168. Zbl 0783.92019
[5] L. Corrias, B. Perthame, H. Zaag, Global solutions of some chemotaxis and angiogenesis systems in high space dimensions, Milan J. Math. 72(2004), 1-28. https: //doi.org/10.1007/s00032-003-0026-x; MR2099126
[6] R. J. Duan, Global smooth flows for the compressible Euler-Maxwell system: Relaxation case, J. Hyperbolic Differ. Equ. 8(2011), 375-413. https://doi.org/10.1142/ S0219891611002421; MR2812147; Zbl 1292.76080
[7] R. J. Duan, Q. Liu, C. Zhu, The Cauchy problem on the compressible two-fluids EulerMaxwell equations, SIAM J. Math. Anal. 44(2012), 102-133. https://doi.org/10.1137/ 110838406; MR2888282; Zbl 1236.35116
[8] R. Duan, A. Lorz, P. Markowich, Global solutions to the coupled chemotaxis-fluid equations, Comm. Partial Differential Equations 35(2010), 1635-1673. https://doi.org/10. 1080/03605302.2010.497199; MR2754058; Zbl 1275.35005
[9] S. Gala, Q. Liu, M. A. Ragusa, A new regularity criterion for the nematic liquid crystal flows, Appl. Anal. 91(2012), 1741-1747. https://doi.org/10.1080/00036811.2011. 581233; MR2968649; Zbl 1253.35120
[10] Z. Goodarzi, A. Razani, M. R. Мокhtarzadeh, A periodic solution of the coupled matrix Riccati differential equations, Miskolc Math. Notes 20(2019), 887-898. MR4048957; Zbl 1449.34065
[11] E. F. Keller, L. A. Segel, A model for chemotaxis, J. Theor. Biol. 30(1971), 225-234. https : //doi.org/10.1016/0022-5193(71)90050-6
[12] E. F. Keller, L. A. Segel, Initiation of slide mold aggregation viewed as an instability, J. Theor. Biol. 26(1970), 399-415. https://doi.org/10.1016/0022-5193(70) 90092-5
[13] E. Lankeit, J. Lankeit, Classical solutions to a logistic chemotaxis model with singular sensitivity and signal absorption, Nonlinear Anal. Real World Appl. 46(2019), 421-445. https://doi.org/10.1016/j.nonrwa.2018.09.012; MR3887138; Zbl 1414.35239
[14] J. Liu, Z. Wang, Classical solutions and steady states of an attraction-repulsion chemotaxis in one dimension, J. Biol. Dyn. 6(2012), 1-19. https://doi.org/10.1080/17513758. 2011.571722; MR2928369; Zbl 1310.35005
[15] M. Luca, A. Chavez-Ross, L. Edelstein-Keshet, A. Mogilner, Chemotactic signaling, microglia, and Alzheimer's disease senile plaques: is there a connection?, Bull. Math. Biol. 65(2003), 693-730. https://doi.org/10.1016/S0092-8240 (03)00030-2; Zbl 1334.92077
[16] A. Matsumura and T. Nishida, The initial value problem for the equations of motion of viscous and heat conductive gases, J. Math. Kyoto Univ. 20(1980), 67-104. https://doi. org/10.1215/kjm/1250522322; MR564670; Zbl 0429.76040
[17] M. R. Мokhtarzadeh, M. R. Pournaki, A. Razani, An existence-uniqueness theorem for a class of boundary value problems, Fixed Point Theory 13(2012), 583-592. MR3024341; Zbl 1286.34038
[18] M. R. Мokhtarzadeh, M. R. Pournaki, A. Razani, A note on periodic solutions of Riccati equations, Nonlinear Dynam. 62(2010) 119-125. https://doi.org/10.1007/ s11071-010-9703-9; MR2736981; Zbl 1209.34049
[19] A. D. Rodríguez, L. C. F. Ferreira, E. J. Villamizar-Roa, Global existence for an attraction-repulsion chemotaxis fluid model with logistic source, Discrete Contin. Dyn. Syst. Ser. B 23(2018), 557-571. https://doi.org/10.3934/dcdsb.2018180; MR3927437; Zbl 1406.35428
[20] Z. Tan, J. Zhou, Global existence and time decay estimate of solutions to the KellerSegel system, Math Meth Appl. Sci. 42(2019), 375-402. https://doi.org/10.1002/mma. 5352; MR3905793; Zbl 1407.35107
[21] Y. Wang, Boundedness in a three-dimensional attraction-repulsion chemotaxis system with nonlinear diffusion and logistic source, Electron. J. Differential Equations 2016, No. 176, 1-21. MR3522231; Zbl 1342.35148

# Oscillatory solutions of Emden-Fowler type differential equation 

Miroslav Bartušek ${ }^{1}$, Zuzana Došlá ${ }^{\boxtimes 1}$ and Mauro Marini ${ }^{2}$<br>${ }^{1}$ Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, Brno, CZ-61137, Czech Republic<br>${ }^{2}$ Department of Mathematics and Computer Science, University of Florence, via S. Marta 3, Florence, I-50139, Italy

Received 22 March 2021, appeared 26 July 2021
Communicated by Josef Diblík


#### Abstract

The paper deals with the coexistence between the oscillatory dynamics and the nonoscillatory one for a generalized super-linear Emden-Fowler differential equation. In particular, the coexistence of infinitely many oscillatory solutions with unbounded positive solutions are proved. The asymptotics of the unbounded positive solutions are described as well.


Keywords: second order nonlinear differential equation, oscillatory solution, globally positive solution, intermediate solutions.
2020 Mathematics Subject Classification: Primary 34C10, Secondary 34C15.

## 1 Introduction

In the paper we investigate the second order nonlinear differential equation

$$
\begin{equation*}
x^{\prime \prime}+b(t) t^{-\gamma}|x|^{\beta} \operatorname{sgn} x=0, \quad t \in[1, \infty) \tag{1.1}
\end{equation*}
$$

where the function $b \in A C[1, \infty)$ is positive on $[1, \infty)$ and bounded away from zero, i.e.,

$$
\inf _{t \in[1, \infty)} b(t)=b_{0}>0
$$

and the constants $\beta$ and $\gamma$ are positive and satisfy

$$
\beta>1, \quad \gamma=\frac{\beta+3}{2} .
$$

Equation (1.1) is the so-called generalized super-linear Emden-Fowler differential equation; it is widely studied in the literature, see, e.g., $[16,20,26]$ and references therein. Equation

[^17](1.1) arises also in the study for searching spherically symmetric solutions of the nonlinear elliptic equation
$$
\operatorname{div}(r(\mathbf{x}) \nabla u)+q(\mathbf{x}) F(u)=0,
$$
where $r$ and $q$ are smooth functions defined on $\mathbb{R}^{d}, d \geq 2, r$ is positive, $F \in C(\mathbb{R})$. The search for radially symmetric solutions outside of a ball of radius $R$ leads to the equation
\[

$$
\begin{equation*}
\left(t^{d-1} r(t) u^{\prime}\right)^{\prime}+t^{d-1} q(t) F(u)=0, \quad t \geq R, \tag{1.2}
\end{equation*}
$$

\]

where $t=|\mathbf{x}|$. In the special case $r(t)=t^{1-d}, q(t)=b(t) t^{1-\gamma-d}$ for $t \geq 1$ and $F(u)=|u|^{\beta} \operatorname{sgn} u$, we get (1.1).

By a solution of (1.1) we mean a function $x$, defined on some interval of positive measure contained on $[1, \infty)$, satisfying (1.1). Further, $x$ is said to be proper if it is defined on some interval $\left[t_{x}, \infty\right), t_{x} \geq 1$, and $\sup _{t \in[\tau, \infty)}|x(t)|>0$ for any $\tau \geq t_{x}$. In other words, a proper solution of (1.1) is a solution that is continuable to infinity and different from the trivial solution in any neighborhood of infinity. Since $\beta>1$, the initial value problem associated to (1.1) has a unique local solution, that is a solution $x$ such that $x(\bar{t})=x_{0}, x^{\prime}(\bar{t})=x_{1}$, defined in a suitable neighborhood of $\bar{t} \in\left[t_{x}, \infty\right)$ for arbitrary numbers $x_{0}, x_{1}$. Moreover, in view of the assumptions on the function $b$, any nontrivial local solution of (1.1) is a proper solution, see, e.g., [16, Theorem 17.1] or [26, Section 3]. Observe that, if $b(t)>0$ but $b \notin A C[1, \infty)$, then equation (1.1) with uncontinuable to infinity solutions may exist, see, e.g., [10,15].

As usual, a proper solution $x$ of (1.1) is said to be nonoscillatory if $x$ is different from zero for any large $t$ and oscillatory otherwise. Clearly, in view of the positiveness of $b$, any eventually positive solution $x$ of (1.1) is increasing for any large $t$. Thus, nonoscillatory solutions $x$ of (1.1) can be $a$-priori divided into three classes. More precisely, $x$ is called a subdominant solution if

$$
\lim _{t \rightarrow \infty} x(t)=\ell_{x}, \quad 0<\ell_{x}<\infty,
$$

or intermediate solution if

$$
\lim _{t \rightarrow \infty} x(t)=\infty, \quad \lim _{t \rightarrow \infty} x^{\prime}(t)=0,
$$

or dominant solution if

$$
\lim _{t \rightarrow \infty} x(t)=\infty, \quad \lim _{t \rightarrow \infty} x^{\prime}(t)=\ell_{x}, \quad 0<\ell_{x}<\infty,
$$

see, e.g., [11, 18, 24, 25].
In the literature great attention has been devoted to the existence of unbounded solutions which are dominant solutions, sometimes called asymptotically linear solutions. However, unbounded nonoscillatory solutions, which are not asymptotically linear solutions, are very difficult to treat. Indeed, as far we know, until now no general necessary and sufficient conditions for existence of intermediate solutions of (1.1) are known; this fact mainly is due to the lack of sharp upper and lower bounds for intermediate solutions, see, e.g., [1, page 241], [13, page 3], [18, page 2].

For the special case of (1.1) with $b(t)=1 / 4$, that is for the equation

$$
\begin{equation*}
x^{\prime \prime}+\frac{1}{4} t^{-\gamma}|x|^{\beta} \operatorname{sgn} x=0, \quad t \in[1, \infty) \tag{1.3}
\end{equation*}
$$

the above three types of nonoscillatory solutions cannot simultaneously coexist, as Moore and Nehari proved in [21]. The problem of this triple coexistence has been solved in a negative way for the more general equation

$$
\left(a(t)\left|x^{\prime}\right|^{\alpha} \operatorname{sgn} x^{\prime}\right)^{\prime}+b(t)|x|^{\beta} \operatorname{sgn} x=0,
$$

where $a$ is a positive continuous function on $[1, \infty)$ and $\alpha$ is a positive constant, $\alpha \neq \beta$, in [14,22] and [5,7], according to $\alpha>\beta$ or $\alpha<\beta$, respectively.

A much more subtle question concerns the possible coexistence between oscillatory solutions and nonoscillatory solutions. The particular equation (1.3), as it is shown in [21], has both oscillatory solutions and nonoscillatory solutions. These nonoscillatory solutions are either subdominant solutions or intermediate solutions and both types exist. Moreover, intermediate solutions of (1.3) intersect the intermediate solution $\sqrt{t}$ infinitely many times.

Many efforts have been made to obtain the existence of at least one oscillatory solution for more general equations than (1.3). A classical approach is due to Jasný [12] and Kurzweil [17], see also [16, Theorem 18.4.], and is based on certain properties of an auxiliary energy-type function. In particular, in $[12,17]$ it is proved that, if the function $b$ is nondecreasing for large $t$, then any proper solution $x$ of (1.1), with $x\left(t_{1}\right)=0$ and $\left|x^{\prime}\left(t_{1}\right)\right|$ sufficiently large, $t_{1} \geq 1$, is oscillatory. The sharpness of this monotonicity condition follows from a Skhalyakho-Kiguradze result, see e.g., [20, Theorem 14.3.], where it is shown that if the function $t^{\varepsilon} b(t)$ is nonincreasing for any large $t$ and some $\varepsilon>0$, then every proper solution of (1.1) is nonoscillatory.

Roughly speaking, in view of the above quoted results by Jasný, Kurzweil and Kiguradze, equation (1.3) can be considered as the border equation between oscillation and nonoscillation.

Our aim here is to study how the quoted results in [21] for (1.3) can be extended to the perturbed equation (1.1).

Since $b \in A C[1, \infty)$, there exists the derivative of $b$ almost everywhere on $[1, \infty)$. Thus, under the additional assumption

$$
\begin{equation*}
\int_{1}^{\infty}\left|b^{\prime}(t)\right| d t<\infty, \tag{1.4}
\end{equation*}
$$

we will study the existence of at least one oscillatory solution to (1.1) and its coexistence with intermediate solutions. Observe that in view of (1.4), the function $b$ is of bounded variation on $[1, \infty)$, but $b$ could not be monotone for large $t$.

Our main results are the following.
Theorem 1.1. Assume (1.4) holds. Then (1.1) has infinitely many oscillatory solutions.
Theorem 1.2. Assume (1.4) holds. Equation (1.1) has infinitely many intermediate solutions $x$ defined on $[1, \infty)$ such that

$$
\begin{equation*}
C_{0} t^{1 / 2} \leq x(t) \leq C_{1} t^{1 / 2} \quad \text { for large } t \tag{1.5}
\end{equation*}
$$

where $C_{0}$ is a suitable positive constant which does not depends on the choice of $x$, and

$$
C_{1}=\left(\frac{\beta+1}{8 b_{0}}\right)^{1 /(\beta-1)}
$$

Moreover, intermediate solutions intersect the function

$$
\left(\frac{1}{4 b(t)}\right)^{1 /(\beta-1)} \sqrt{t}
$$

infinitely many times.
Corollary 1.3. Assume (1.4) holds. Equation (1.1) admits simultaneously infinitely many oscillatory solutions, subdominant solutions, and intermediate solutions.

For equation (1.1), Theorem 1.2 extends analogues results in [6, Theorem 2.1] and [3, Theorem 3.1], where $b$ is required to be nonincreasing for $t \geq 1$. Recently, the existence of intermediate solutions of (1.1) has been considered in [23,24]. More precisely, in these papers, the existence problem is reduced, by means of an ingenious change of variables, to the solvability of a system of two integral equations on the half-line $[1, \infty)$. Moreover, an asymptotic formula for these solutions is presented, too. Observe that asymptotic forms of intermediate solutions of (1.1) are given also in [13], where the existence problem is not studied. Hence, Theorem 1.2 extends also these quoted results in $[13,23,24]$.

## 2 Preliminaries

We start by recalling the following asymptotic property of nonoscillatory solutions of (1.1).
Lemma 2.1. Any nonoscillatory solution $x$ of (1.1) satisfies $\lim _{t \rightarrow \infty} x^{\prime}(t)=0$. Consequently, $x$ is either subdominant solution or intermediate solution.

Proof. Since

$$
\beta-\gamma=\frac{\beta-3}{2}>-1,
$$

and $b$ is bounded away from zero, we obtain

$$
\int_{1}^{\infty} t^{\beta-\gamma} b(t) d t=\infty .
$$

Hence, in view of [8, Theorem 1], equation (1.1) does not have nonoscillatory solutions $x$ such that $\lim _{t \rightarrow \infty} x^{\prime}(t) \neq 0$.

The approach for proving our main results is based on the following lemma.
Lemma 2.2. The change of variable

$$
\begin{equation*}
x(t)=t^{1 / 2} u(s), \quad s=\log t, \quad t \in[1, \infty) \tag{2.1}
\end{equation*}
$$

transforms equation (1.1) into equation

$$
\begin{equation*}
\ddot{u}-\frac{u}{4}+b\left(e^{s}\right)|u(s)|^{\beta} \operatorname{sgn} u(s)=0, \quad s \in[0, \infty), \tag{2.2}
\end{equation*}
$$

where ". "denotes the derivative with respect to the variable s.
Proof. We have

$$
\begin{aligned}
x^{\prime}(t) & =\frac{1}{2 t^{1 / 2}} u(s)+t^{1 / 2} \dot{u}(s) \frac{1}{t}=\frac{1}{t^{1 / 2}}\left(\frac{u(s)}{2}+\dot{u}(s)\right) \\
x^{\prime \prime}(t) & =-\frac{1}{2 t^{3 / 2}}\left(\frac{u(s)}{2}+\dot{u}(s)\right)+\frac{1}{t^{1 / 2}}\left(\frac{\dot{u}(s)}{2}+\ddot{u}(s)\right) \frac{1}{t} \\
& =\frac{1}{t^{3 / 2}}\left(-\frac{u(s)}{4}+\ddot{u}(s)\right) .
\end{aligned}
$$

Substituting into (1.1) we get (2.2).
Lemma 2.3. All the solutions of (2.2) are defined on $[0, \infty)$. Moreover, any solution $u$ of (2.2) such that $u(S)=0, \dot{u}(S)=0$ at some $S \geq 0$, satisfies $u(s) \equiv 0$ for $s \geq 0$.

Proof. The continuability at infinity follows from the same property for (1.1), see, e.g., [16, Theorem 17.1]. Another approach employs an idea of Conti [4] and uses two Lyapunov functions, see [9, Theorem 3.1.] and [27, Appendix A]. The second statement follows, e.g., from [19, Lemma 1.1.] and Lemma 2.2.

Set, for $u \geq 0$,

$$
\begin{equation*}
Q(u)=-u^{2}+\frac{8 b_{0}}{\beta+1} u^{\beta+1} \tag{2.3}
\end{equation*}
$$

and

$$
\begin{equation*}
A_{0}=\left(\frac{1}{4 b_{0}}\right)^{\frac{1}{\beta-1}}, \quad A=\left(\frac{\beta+1}{8 b_{0}}\right)^{\frac{1}{\beta-1}} . \tag{2.4}
\end{equation*}
$$

Since $\beta>1$, we have $A_{0}<A$. The following holds.
Lemma 2.4. The function $Q$ satisfies

$$
Q(0)=Q(A)=0, \quad Q\left(A_{0}\right)=-A_{0}^{2} \frac{\beta-1}{\beta+1}
$$

Moreover, $Q$ is decreasing on $\left[0, A_{0}\right)$ and increasing on $\left(A_{0}, A\right]$.
Proof. Since $8 b_{0} A^{\beta+1} /(\beta+1)=A^{2}$, we obtain

$$
Q(A)=-A^{2}+\frac{8 b_{0}}{\beta+1} A^{\beta+1}=0
$$

From $d Q / d u=2 u\left(-1+4 b_{0} u^{\beta-1}\right)$ we get $d Q / d u=0$ for $u=A_{0}, d Q / d u<0$ for $u \in\left(0, A_{0}\right)$, and $d Q / d u>0$ for $u \in\left(A_{0}, A\right)$. This gives the assertion.

Lemma 2.5. Let $u$ be a solution of (2.2). For fixed $\bar{s} \in[0, \infty)$, the solution $u$ satisfies for $s \in[0, \infty)$

$$
\begin{align*}
4 \dot{u}^{2}(s)+Q(|u(s)|)= & 4 \dot{u}^{2}(\bar{s})+Q(|u(\bar{s})|)+\frac{8}{\beta+1}\left(b_{0}-b\left(e^{s}\right)\right)|u(s)|^{\beta+1} \\
& -\frac{8}{\beta+1}\left(b_{0}-b\left(e^{\bar{s}}\right)\right)|u(\bar{s})|^{\beta+1}+\frac{8}{\beta+1} \int_{\bar{s}}^{s} b^{\prime}\left(e^{\sigma}\right) e^{\sigma}|u(\sigma)|^{\beta+1} d \sigma \tag{2.5}
\end{align*}
$$

Proof. Multiplying equation (2.2) by $8 \dot{u}$, we get

$$
8 \ddot{u} \dot{u}-2 \dot{u} u+8 b_{0}|u|^{\beta} \dot{u} \operatorname{sgn} u=8\left(b_{0}-b\left(e^{s}\right)\right)|u|^{\beta} \dot{u} \operatorname{sgn} u .
$$

Integrating this equality on $[\bar{s}, s]$ we obtain

$$
4 \dot{u}^{2}(s)+Q(|u(s)|)=4 \dot{u}^{2}(\bar{s})+Q(|u(\bar{s})|)+8 \int_{\bar{s}}^{s}\left(b_{0}-b\left(e^{\sigma}\right)\right)|u(\sigma)|^{\beta} \dot{u}(\sigma) \operatorname{sgn} u(\sigma) d \sigma
$$

Hence (2.5) follows by integrating by parts.
Lemma 2.6. Let $0<b_{1} \leq b_{0}$ and $T \geq 1$ be such that $b(t) \geq b_{1}$ on $[T, \infty)$. Let $x$ be a nonoscillatory solution of (1.1) such that $x(t) \neq 0$ on $[T, \infty)$ and $u$ be given by (2.1) with $s_{0}=\log T$. Then we have for $t \geq T$

$$
\begin{equation*}
|x(t)| \leq K t^{1 / 2} \quad \text { with } K=\left(\frac{\beta+1}{4 b_{1}}\right)^{\frac{1}{\beta-1}} \tag{2.6}
\end{equation*}
$$

and

$$
\begin{equation*}
|u(s)| \leq K \quad \text { for } s \geq s_{0} . \tag{2.7}
\end{equation*}
$$

Moreover, set $b_{2}=\sup _{t \geq T} b(t)$. Then we have for $t \geq T$

$$
\begin{equation*}
\left|x^{\prime}(t)\right| \leq K_{1} t^{-1 / 2}, \quad \text { with } K_{1}=2 K^{\beta} b_{2} \tag{2.8}
\end{equation*}
$$

and

$$
\begin{equation*}
|\dot{u}(s)| \leq K_{1}+K / 2 \quad \text { for } s \geq s_{0} . \tag{2.9}
\end{equation*}
$$

Proof. Let $x$ be nonoscillatory solution of (1.1) such that

$$
x(t)>0, \quad x^{\prime}(t)>0 \quad \text { for } t \geq T .
$$

Using Lemma 2.1, we have $\lim _{t \rightarrow \infty} x^{\prime}(t)=0$. Integrating (1.1) on $[t, \infty), t \geq T$, we get

$$
x^{\prime}(t)=\int_{t}^{\infty} b(\tau) \tau^{-\gamma} x^{\beta}(\tau) d \tau \geq b_{1} x^{\beta}(t) \int_{t}^{\infty} \sigma^{-\frac{\beta+3}{2}} d \sigma=C t^{-\frac{\beta+1}{2}} x^{\beta}(t)
$$

with $C=\frac{2}{\beta+1} b_{1}$. Hence,

$$
\frac{x^{\prime}(t)}{x^{\beta}(t)} \geq C t^{-\frac{\beta+1}{2}}
$$

or

$$
\frac{x^{-\beta+1}(t)}{\beta-1} \geq \frac{2 C}{\beta-1} t^{-\frac{\beta-1}{2}} .
$$

Thus, we have for $t \geq T$

$$
x(t) \leq\left(\frac{1}{2 C}\right)^{\frac{1}{\beta-1}} t^{1 / 2}=K t^{1 / 2} .
$$

Since $b(t) \leq b_{2}<\infty$ on [T, $\infty$ ), integrating (1.1) and using (2.6) we obtain for $t \geq T$

$$
x^{\prime}(t)=\int_{t}^{\infty} b(\tau) \tau^{-\gamma} x^{\beta}(\tau) d \tau \leq b_{2} K^{\beta} \int_{t}^{\infty} \tau^{-3 / 2} d \tau=2 K^{\beta} b_{2} t^{-1 / 2}=K_{1} t^{-1 / 2}
$$

Thus, (2.8) holds and using the transformation (2.1), the estimations for $u$ and $\dot{u}$ follow.
Lemma 2.7. Equation (2.2) has two types of nonoscillatory solutions. Namely:
Type (a): solution u satisfies for large s

$$
\begin{equation*}
0<|u(s)| \leq D e^{-s / 2} \tag{2.10}
\end{equation*}
$$

where $|u|$ is decreasing and $D>0$ is a suitable constant.
Type (b): solution $u$ intersects the function

$$
\begin{equation*}
Z(s)=\left(\frac{1}{4 b\left(e^{s}\right)}\right)^{\frac{1}{\beta-1}} \tag{2.11}
\end{equation*}
$$

infinitely many times, i.e., there exists a sequence $\left\{s_{n}\right\}_{n=1}^{\infty}, \lim _{n} s_{n}=\infty$, such that $\left|u\left(s_{n}\right)\right|=Z\left(s_{n}\right)$.

Proof. First, observe that the function $Z$ in (2.11) satisfies

$$
\begin{equation*}
\lim _{s \rightarrow \infty} Z(s)=\left(\frac{1}{4 b_{0}}\right)^{\frac{1}{\beta-1}}=A_{0} . \tag{2.12}
\end{equation*}
$$

Let $u$ be a nonoscillatory solution of (2.2) and, for sake of simplicity, assume

$$
\begin{equation*}
u(s)>0 \quad \text { for } s \geq S \geq 0, \tag{2.13}
\end{equation*}
$$

where $S$ is chosen such that for any $s \geq S$

$$
b\left(e^{s}\right) \geq b_{0} / 2 .
$$

According to (2.7), we get for $s \geq S$

$$
\begin{equation*}
0<u(s) \leq K, \tag{2.14}
\end{equation*}
$$

where $K$ is given by (2.6) with $b_{1}=b_{0} / 2$.
Then, from (2.2), we get the following:

$$
\begin{array}{lll}
\ddot{u}(s)>0 & \text { if and only if } & u(s)<Z(s) \\
\ddot{u}(s)<0 & \text { if and only if } & u(s)>Z(s)  \tag{2.15}\\
\ddot{u}(s)=0 & \text { if and only if } & u(s)=Z(s) .
\end{array}
$$

Since $A_{0}<K$, from (2.14) and (2.15), a-priori, only one of the following possibilities holds:
(i) $A_{0}<\lim _{s \rightarrow \infty} u(s) \leq K, \ddot{u}(s)<0$ for large $s$;
(ii) $0 \leq \lim _{s \rightarrow \infty} u(s) \leq A_{0}, \ddot{u}(s)>0$ for large $s$;
(iii) $u$ intersects infinitely many times the function $Z$.

Observe that in case (iii), the solution $u$ is of Type (b) and the corresponding solution $x$ of (1.1) satisfies $\lim _{t \rightarrow \infty} x(t)=\infty, \lim _{t \rightarrow \infty} x^{\prime}(t)=0$. Thus, $x$ is an intermediate solution of (1.1).

To prove the lemma, it is sufficient to prove that in cases (i) and (ii), the solution $u$ is of Type (a).
Case (i). Since $\lim _{s \rightarrow \infty} u(s)=B>A_{0}$, we get from (2.2)

$$
\begin{aligned}
\lim _{s \rightarrow \infty} \ddot{u}(s) & =\lim _{s \rightarrow \infty}\left[\frac{u(s)}{4}-b\left(e^{s}\right) u^{\beta}(s)\right]=\frac{B}{4}-B^{\beta} b_{0} \\
& =\frac{B}{4}\left(1-4 b_{0} B^{\beta-1}\right)<\frac{B}{4}\left(1-4 b_{0} A_{0}^{\beta-1}\right)=0 .
\end{aligned}
$$

Hence, $\lim _{s \rightarrow \infty} \dot{u}(s)=\lim _{s \rightarrow \infty} u(s)=-\infty$, which is a contradiction with the positiveness of the constant $B$. Thus, the case (i) cannot occur.

Case (ii). If $0<B=\lim _{s \rightarrow \infty} u(s)<1$, reasoning in a similar way as in case (i), we get a contradiction. Now suppose $\lim _{s \rightarrow \infty} u(s)=0$. According to (2.12), there exists $S_{1} \geq S$ such that for $s \geq S_{1}$,

$$
\begin{equation*}
u(s)<Z(s), \quad 0<u(s) \leq\left(\frac{\beta+1}{24 b_{0}}\right)^{1 /(\beta-1)}, \quad b\left(e^{s}\right) \leq \frac{3}{2} b_{0} . \tag{2.16}
\end{equation*}
$$

From this and (2.15) we obtain $\ddot{u}(s)>0$. Thus, we have for $s \in\left[S_{1}, \infty\right)$

$$
\begin{equation*}
\dot{u}(s)<0 \quad \text { and } \quad \lim _{s \rightarrow \infty} \dot{u}(s)=0 \tag{2.17}
\end{equation*}
$$

Let $S_{1} \leq s<\bar{s}$. Multiplying (2.2) by $8 \dot{u}$ and integrating on $[s, \bar{s}]$ we get

$$
4 \dot{u}^{2}(\bar{s})-u^{2}(\bar{s})=4 \dot{u}^{2}(s)-u^{2}(s)-8 \int_{s}^{\bar{s}} b\left(e^{\sigma}\right) u^{\beta}(\sigma) \dot{u}(\sigma) d \sigma
$$

From this, (2.16) and (2.17), as $\bar{s}$ tends to infinity, we have

$$
4 \dot{u}^{2}(s)-u^{2}(s)-8 \int_{s}^{\infty} b\left(e^{\sigma}\right) u^{\beta}(\sigma) \dot{u}(\sigma) d \sigma=0
$$

and

$$
\begin{aligned}
\frac{4 \dot{u}^{2}(s)}{u^{2}(s)} & =1+\frac{8}{u^{2}(s)} \int_{s}^{\infty} b\left(e^{\sigma}\right) u^{\beta}(\sigma) \dot{u}(\sigma) d \sigma \geq 1+\frac{12 b_{0}}{u^{2}(s)} \int_{s}^{\infty} u^{\beta}(\sigma) \dot{u}(\sigma) d \sigma \\
& =1-\frac{12 b_{0}}{\beta+1} u^{\beta-1}(s)>0
\end{aligned}
$$

Since $\dot{u}(s)<0$, we obtain

$$
\begin{equation*}
\frac{\dot{u}(s)}{u(s)} \leq-\frac{1}{2} \sqrt{1-\frac{12 b_{0}}{\beta+1} u^{\beta-1}(s)} \leq-\frac{1}{2}\left(1-\frac{12 b_{0}}{\beta+1} u^{\beta-1}(s)\right) \tag{2.18}
\end{equation*}
$$

Using the estimation for $u$ in (2.16), we get for $s \geq S_{1}$

$$
\frac{\dot{u}(s)}{u(s)} \leq-\frac{1}{4}
$$

or

$$
u(s) \leq u\left(S_{1}\right) e^{\left(-s+S_{1}\right) / 4}
$$

Applying this estimation to the inequality (2.18), we have for $s \geq S_{1}$

$$
\frac{\dot{u}(s)}{u(s)} \leq-\frac{1}{2}+\frac{6 b_{0}}{\beta+1} u^{\beta-1}\left(S_{1}\right) e^{-(\beta-1)\left(s-S_{1}\right) / 4}
$$

or

$$
\log \frac{u(s)}{u\left(S_{1}\right)} \leq-\frac{1}{2}\left(s-S_{1}\right)+\frac{24 b_{0}}{\beta^{2}-1} u^{\beta-1}\left(S_{1}\right) e^{(\beta-1) S_{1} / 4} e^{-(\beta-1) s / 4} \leq-\frac{s}{2}+C
$$

where

$$
C=\frac{1}{2} S_{1}+\frac{24 b_{0} u^{\beta-1}\left(S_{1}\right) e^{(\beta-1) S_{1} / 4}}{\beta^{2}-1}
$$

Therefore, setting $K_{2}=u\left(S_{1}\right) e^{C}$, we obtain

$$
u(s) \leq K_{2} e^{-s / 2}
$$

and in view of (2.17), $u$ is of Type (a).
Remark 2.8. Solutions $u$ of Type (a) in Lemma 2.7 correspond, via the transformation (2.1), to subdominant solutions of equation (1.1) because

$$
x(t)=t^{1 / 2} u(s) \leq t^{1 / 2} K_{2} e^{-s / 2}=K_{2}
$$

while solutions $u$ of Type (b) correspond to intermediate solutions of (1.1).

## 3 Proof of Theorem 1.1

Proof of Theorem 1.1. Consider equation (2.2) and the function $Q$ given by (2.3). In view of (1.4), there exists $s_{0} \geq 0$ such that for $s \geq s_{0}$

$$
\begin{equation*}
\int_{s_{0}}^{\infty}\left|b^{\prime}\left(e^{\sigma}\right)\right| e^{\sigma} d \sigma \leq \frac{b_{0}}{8}, \quad\left|b_{0}-b\left(e^{s}\right)\right| \leq \frac{b_{0}}{8} \tag{3.1}
\end{equation*}
$$

Let $u$ be a solution of (2.2) such that

$$
\begin{equation*}
u\left(s_{0}\right)=0, \quad \dot{u}\left(s_{0}\right)=d>0 \tag{3.2}
\end{equation*}
$$

where

$$
\begin{equation*}
d>\sqrt{3} K_{3}, \quad K_{3}=\left(\frac{9}{4} b_{0} K^{\beta}+\frac{K}{2}\right) \tag{3.3}
\end{equation*}
$$

and $K$ is given by (2.6) with $b_{1}=7 b_{0} / 8$, i.e.,

$$
K=\left(\frac{2(\beta+1)}{7 b_{0}}\right)^{1 /(\beta-1)}
$$

Let us prove that $u$ is oscillatory. By contradiction, suppose that there exists $s_{2} \geq s_{0}$ such that

$$
\begin{equation*}
u\left(s_{2}\right)=0, \quad u(s) \neq 0 \quad \text { for } s>s_{2} \tag{3.4}
\end{equation*}
$$

Applying Lemma 2.6 with $b_{1}=7 b_{0} / 8, b_{2}=9 b_{0} / 8$, we have for $s \geq s_{2}$

$$
|u(s)| \leq K
$$

Using (2.9), we obtain for $s \geq s_{2}$

$$
\begin{equation*}
|\dot{u}(s)| \leq 2 K^{\beta} b_{2}+\frac{K}{2}=\frac{9}{2} b_{0} K^{\beta}+\frac{K}{2}=K_{3} . \tag{3.5}
\end{equation*}
$$

If $s_{2}=s_{0}$, inequality (3.5) contradicts (3.2) and (3.3). Thus, suppose that $s_{0}<s_{2}$. From (3.2) and (3.4), there exists $s_{1}, s_{0}<s_{1}<s_{2}$, such that

$$
\left|u\left(s_{1}\right)\right|=\max _{s_{0} \leq s \leq s_{2}}|u(s)| .
$$

Obviously, $\dot{u}\left(s_{1}\right)=0$. Put

$$
B=(\beta+1) /\left(4 b_{0}\right)
$$

and consider two cases:

$$
\text { (i) }\left|u\left(s_{1}\right)\right|<B^{1 /(\beta-1)}, \quad \text { (ii) }\left|u\left(s_{1}\right)\right| \geq B^{1 /(\beta-1)}
$$

Assume case (i) holds. Applying Lemma 2.5 with $\bar{s}=s_{0}, s=s_{2}$, using (3.1), (3.2), and (3.4), we get

$$
\begin{aligned}
4 \dot{u}^{2}\left(s_{2}\right) & =4 d^{2}+\frac{8}{\beta+1} \int_{s_{0}}^{s_{2}} b^{\prime}\left(e^{\sigma}\right) e^{\sigma}|u(\sigma)|^{\beta+1} d \sigma \\
& \geq 4 d^{2}-\frac{2}{B b_{0}} B^{\frac{\beta+1}{\beta-1}} \int_{s_{0}}^{\infty}\left|b^{\prime}\left(e^{\sigma}\right)\right| e^{\sigma} d \sigma \geq 4 d^{2}-\frac{1}{4} B^{\frac{2}{\beta-1}} \\
& \geq 4 d^{2}-\left(\frac{K}{2}\right)^{2} \geq 4 d^{2}-\left(K_{3}\right)^{2} \geq 4 d^{2}-\frac{d^{2}}{3}=\frac{11}{3} d^{2}
\end{aligned}
$$

Therefore,

$$
\left|\dot{u}\left(s_{2}\right)\right| \geq \sqrt{\frac{11}{12}} d \geq \sqrt{\frac{33}{12}} K_{3}
$$

which contradicts (3.5).
Assume case (ii) holds. We have

$$
\frac{\left|u\left(s_{1}\right)\right|^{\beta+1}}{B} \geq u^{2}\left(s_{1}\right) .
$$

Thus,

$$
\begin{equation*}
\frac{Q\left(\left|u\left(s_{1}\right)\right|\right)}{2}=\frac{1}{B}\left|u\left(s_{1}\right)\right|^{\beta+1}-\frac{u^{2}\left(s_{1}\right)}{2} \geq \frac{1}{2 B}\left|u\left(s_{1}\right)\right|^{\beta+1} . \tag{3.6}
\end{equation*}
$$

From here, applying Lemma 2.5 with $\bar{s}=s_{0}, s=s_{1}$, using (3.1) and $\dot{u}\left(s_{1}\right)=0$, we get

$$
\begin{aligned}
Q\left(\left|u\left(s_{1}\right)\right|\right) & =4 d^{2}+\frac{8}{\beta+1}\left(b_{0}-b\left(e^{s_{1}}\right)\right)\left|u\left(s_{1}\right)\right|^{\beta+1}+\frac{8}{\beta+1} \int_{s_{0}}^{s_{1}} b^{\prime}\left(e^{\sigma}\right) e^{\sigma}|u(\sigma)|^{\beta+1} d \sigma \\
& \geq 4 d^{2}-\frac{2 b_{0}}{\beta+1}\left|u\left(s_{1}\right)\right|^{\beta+1} \geq 4 d^{2}-\frac{1}{2 B}\left|u\left(s_{1}\right)\right|^{\beta+1} \geq 4 d^{2}-\frac{Q\left(\left|u\left(s_{1}\right)\right|\right)}{2} .
\end{aligned}
$$

Thus,

$$
\begin{equation*}
Q\left(\left|u\left(s_{1}\right)\right|\right) \geq \frac{8}{3} d^{2} \tag{3.7}
\end{equation*}
$$

Applying Lemma 2.5 with $\bar{s}=s_{1}, s=s_{2}$, using (3.1), (3.6) and (3.7), we have

$$
\begin{aligned}
4 \dot{u}^{2}\left(s_{2}\right) & =Q\left(\left|u\left(s_{1}\right)\right|\right)-\frac{8}{\beta+1}\left(b_{0}-b\left(e^{s_{1}}\right)\right)\left|u\left(s_{1}\right)\right|^{\beta+1}+\frac{8}{\beta+1} \int_{s_{1}}^{s_{2}} b^{\prime}\left(e^{\sigma}\right) e^{\sigma}|u(\sigma)|^{\beta+1} d \sigma \\
& \geq Q\left(\left|u\left(s_{1}\right)\right|\right)-\frac{2 b_{0}}{\beta+1}\left|u\left(s_{1}\right)\right|^{\beta+1} \\
& \geq Q\left(\left|u\left(s_{1}\right)\right|\right)-\frac{1}{2 B}\left|u\left(s_{1}\right)\right|^{\beta+1} \geq \frac{Q\left(\left|u\left(s_{1}\right)\right|\right)}{2} \geq \frac{4}{3} d^{2} .
\end{aligned}
$$

From this and (3.3), we obtain

$$
\left|\dot{u}\left(s_{2}\right)\right| \geq \frac{d}{\sqrt{3}}>K_{3},
$$

which contradicts (3.5).
Thus, the solution $u$ satisfying the initial condition (3.2) is defined on $\left[s_{0}, \infty\right)$ and is oscillatory. According to Lemma 2.3, the solution $u$ can be extended to $[0, \infty)$. Moreover, since $s_{0}$ does not depend on the value $d$, equation (2.2) has infinitely many oscillatory solutions and, in virtue of the transformation (2.1), the same occurs for equation (1.1).

## 4 Proof of Theorem 1.2

Proof of Theorem 1.2. Let $\delta$ be a constant such that

$$
|\delta|<\frac{1}{2}\left(\frac{1}{4 b_{0}}\right)^{\frac{1}{\beta-1}} \sqrt{\frac{\beta-1}{2(\beta+1)}}
$$

and put

$$
\varepsilon=\frac{1}{24} b_{0}(\beta-1)\left(\frac{2}{\beta+1}\right)^{(\beta+1) /(\beta-1)}
$$

Let $T \geq 1$ be such that

$$
\begin{equation*}
\int_{T}^{\infty}\left|b^{\prime}(t)\right| d t \leq \varepsilon, \quad\left|b_{0}-b(t)\right| \leq \varepsilon \quad \text { for } t \geq T \tag{4.1}
\end{equation*}
$$

For $s_{0}=\log T$, we have

$$
\begin{equation*}
\int_{s_{0}}^{\infty}\left|b^{\prime}\left(e^{\sigma}\right)\right| e^{\sigma} d \sigma=\int_{T}^{\infty}\left|b^{\prime}(t)\right| d t \leq \varepsilon, \quad\left|b_{0}-b\left(e^{s}\right)\right| \leq \varepsilon \quad \text { for } s \geq s_{0} \tag{4.2}
\end{equation*}
$$

Now, consider the solution $u$ of (2.2) with

$$
\begin{equation*}
u\left(s_{0}\right)=A_{0}, \quad \dot{u}\left(s_{0}\right)=\delta \tag{4.3}
\end{equation*}
$$

where $A_{0}$ is given by (2.4). By Lemma 2.4 we get

$$
\begin{equation*}
Q\left(u\left(s_{0}\right)\right)=-\frac{\beta-1}{\beta+1}\left(\frac{1}{4 b_{0}}\right)^{\frac{2}{\beta-1}} \tag{4.4}
\end{equation*}
$$

and there exists $u_{0}, 0<u_{0}<A_{0}$, such that

$$
\begin{equation*}
Q\left(u_{0}\right)=-\frac{Q\left(u\left(s_{0}\right)\right)}{4}=-\frac{\beta-1}{4(\beta+1)}\left(\frac{1}{4 b_{0}}\right)^{\frac{2}{\beta-1}} \tag{4.5}
\end{equation*}
$$

We want to prove that the solution $u$ of (2.2) with (4.3) satisfies for $s \geq s_{0}$

$$
\begin{equation*}
0<u_{0} \leq u(s) \leq A \tag{4.6}
\end{equation*}
$$

where $A$ is given in (2.4). Note that (4.6) is satisfied for $s=s_{0}$ and

$$
\begin{equation*}
u_{0}<u\left(s_{0}\right)=A_{0}<A \tag{4.7}
\end{equation*}
$$

Step 1. We claim that if there exists $s_{1}>s_{0}$ such that

$$
\begin{equation*}
u\left(s_{1}\right)=u_{0}, \quad u(s)>u_{0} \quad \text { for } s \in\left[s_{0}, s_{1}\right) \tag{4.8}
\end{equation*}
$$

then

$$
\begin{equation*}
u(s) \leq A \quad \text { on }\left[s_{0}, s_{1}\right] \tag{4.9}
\end{equation*}
$$

Since $u\left(s_{1}\right)=u_{0}$, from (4.7) we get $u\left(s_{1}\right)<A$. By contradiction, suppose that there exists $s_{2}, s_{0}<s_{2}<s_{1}$, such that

$$
\begin{equation*}
u\left(s_{2}\right)=A, \quad u(s)<A \quad \text { for } s \in\left[s_{0}, s_{2}\right) \tag{4.10}
\end{equation*}
$$

Using Lemma 2.4, we have

$$
\begin{equation*}
Q\left(u\left(s_{2}\right)\right)=0 \tag{4.11}
\end{equation*}
$$

According to (4.3) and (4.8), we can use Lemma 2.5 for $\bar{s}=s_{0}, s=s_{2}$ and this together with (2.5), (4.4) and (4.11) imply

$$
\begin{aligned}
4 \dot{u}^{2}\left(s_{2}\right)= & 4 \dot{u}^{2}\left(s_{2}\right)+Q\left(u\left(s_{2}\right)\right) \\
= & 4 \dot{u}^{2}\left(s_{2}\right)+Q\left(u\left(s_{0}\right)\right)+\frac{8}{\beta+1}\left(b_{0}-b\left(e^{s_{2}}\right)\right) u^{\beta+1}\left(s_{2}\right) \\
& -\frac{8}{\beta+1}\left(b_{0}-b\left(e^{s_{0}}\right)\right) u^{\beta+1}\left(s_{0}\right)+\frac{8}{\beta+1} \int_{s_{0}}^{s_{2}} b^{\prime}\left(e^{\sigma}\right) e^{\sigma}|u(\sigma)|^{\beta+1} d \sigma
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
4 \dot{u}^{2}\left(s_{2}\right) \leq & 4 \delta^{2}-\frac{\beta-1}{\beta+1}\left(\frac{1}{4 b_{0}}\right)^{\frac{2}{\beta-1}}+\frac{8}{\beta+1}\left|b_{0}-b\left(e^{s_{2}}\right)\right| u^{\beta+1}\left(s_{2}\right) \\
& +\frac{8}{\beta+1}\left|b_{0}-b\left(e^{s_{0}}\right)\right| A_{0}^{\beta+1}+\frac{8}{\beta+1} A^{\beta+1} \int_{s_{0}}^{s_{2}}\left|b^{\prime}\left(e^{\sigma}\right)\right| e^{\sigma} d \sigma .
\end{aligned}
$$

From this, (2.4), (4.2), and (4.7), we have

$$
\begin{equation*}
4 \dot{u}^{2}\left(s_{2}\right) \leq 4 \delta^{2}-\frac{\beta-1}{\beta+1}\left(\frac{1}{4 b_{0}}\right)^{\frac{2}{\beta-1}}+\frac{24}{\beta+1} \varepsilon A^{\beta+1} \tag{4.12}
\end{equation*}
$$

Since

$$
\begin{equation*}
4 \delta^{2}<\frac{\beta-1}{2(\beta+1)}\left(\frac{1}{4 b_{0}}\right)^{\frac{2}{\beta-1}} \tag{4.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{24}{\beta+1} \varepsilon A^{\beta+1}=\frac{\beta-1}{4(\beta+1)}\left(\frac{1}{4 b_{0}}\right)^{\frac{2}{\beta-1}} \tag{4.14}
\end{equation*}
$$

the inequality (4.12) implies

$$
4 \dot{u}^{2}\left(s_{2}\right) \leq-\frac{\beta-1}{4(\beta+1)}\left(\frac{1}{4 b_{0}}\right)^{\frac{2}{\beta-1}}<0
$$

and this contradiction proves Step 1.
Step 2. Now, we prove that

$$
\begin{equation*}
u(s)>u_{0}>0 \quad \text { for } s \geq s_{0} . \tag{4.15}
\end{equation*}
$$

As claimed, (4.15) holds for $s=s_{0}$. By contradiction, assume that (4.8) is valid and $s_{1}>s_{0}$ exists such that $u\left(s_{1}\right)=u_{0}$ and $u(s)>u_{0}$ on $\left[s_{0}, s_{1}\right)$. Hence, in view of (4.8) and (4.9) we obtain

$$
\begin{equation*}
0<u_{0} \leq u(s) \leq A \quad \text { for } s \in\left[s_{0}, s_{1}\right] . \tag{4.16}
\end{equation*}
$$

Using this inequality and Lemma 2.5 with $\bar{s}=s_{0}$ and $s=s_{1}$, we have

$$
\begin{aligned}
4 \dot{u}^{2}\left(s_{1}\right)+Q\left(u\left(s_{1}\right)\right)= & 4 \dot{u}^{2}\left(s_{0}\right)+Q\left(u\left(s_{0}\right)\right)+\frac{8}{\beta+1}\left(b_{0}-b\left(e^{s_{1}}\right)\right) u^{\beta+1}\left(s_{1}\right) \\
& -\frac{8}{\beta+1}\left(b_{0}-b\left(e^{s_{0}}\right)\right) u^{\beta+1}\left(s_{0}\right)+\frac{8}{\beta+1} \int_{s_{0}}^{s_{1}} b^{\prime}\left(e^{\sigma}\right) e^{\sigma} u^{\beta+1}(\sigma) d \sigma \\
\leq & 4 \dot{u}^{2}\left(s_{0}\right)+Q\left(u\left(s_{0}\right)\right)+\frac{8}{\beta+1}\left|b_{0}-b\left(e^{s_{0}}\right)\right| A^{\beta+1} \\
& +\frac{8}{\beta+1}\left|b_{0}-b\left(e^{s_{1}}\right)\right| A^{\beta+1}+\frac{8 A^{\beta+1}}{\beta+1} \int_{s_{0}}^{s_{1}}\left|b^{\prime}\left(e^{\sigma}\right)\right| e^{\sigma} d \sigma .
\end{aligned}
$$

From this, (4.2), (4.4) and (4.5) we have

$$
4 \dot{u}^{2}\left(s_{1}\right)-\frac{\beta-1}{4(\beta+1)}\left(\frac{1}{4 b_{0}}\right)^{\frac{2}{\beta-1}} \leq 4 \delta^{2}-\frac{\beta-1}{\beta+1}\left(\frac{1}{4 b_{0}}\right)^{\frac{2}{\beta-1}}+\frac{24}{\beta+1} \varepsilon A^{\beta+1} .
$$

Hence, in view of (4.13) and (4.14), we get

$$
4 \dot{u}^{2}\left(s_{1}\right)<0,
$$

which is a contradiction. This proves the validity of (4.15).
From here, using the transformation (2.1) and Remark 2.8, we obtain that the corresponding solution $x$ of (1.1) is an intermediate solution.

In a similar way, we prove that the second inequality of (4.6) is valid for $s \geq s_{0}$; the details are left to the reader.

Thus, from the inequality (4.15) and Lemma 2.7, the solution $u$ intersects the function $Z(s)$, given by (2.11), infinitely many times. Using the transformation (2.1), the final statement of Theorem 1.2 follows.

Proof of Corollary 1.3. By using a similar argument to the one presented in [11, Theorem 4.3.], equation (1.1) has infinitely many subdominant solutions. Thus, the assertion follows from Theorems 1.1 and 1.2.

## 5 Case $b$ nondecreasing

The assumption (1.4) is fulfilled if, in addition, the function $b$ is either nondecreasing and bounded or nonincreasing and bounded away from zero.

If $b$ is nondecreasing, then intermediate solutions $x$ are globally positive, that is $x(t) \neq 0$ on the whole interval $[1, \infty)$. Moreover, any solution with a zero is oscillatory. These properties follow from the following.

Theorem 5.1. Let $b^{\prime}(t) \geq 0$ for $t \in[1, \infty)$ and $\lim _{t \rightarrow \infty} b(t)=b_{0}, b_{0}>0$. Then
(i) Equation (1.1) has infinitely many intermediate solutions.
(ii) Any eventually positive solution $x$ is globally positive on $[1, \infty)$ and satisfies (2.6) and (2.8).
(iii) For any $a \geq 1$ every solution of (1.1) with the initial condition

$$
x(a)=0 \quad \text { or } \quad|x(a)|>K \sqrt{a} \quad \text { or } \quad\left|x^{\prime}(a)\right|>K_{1} a^{-1 / 2}
$$

where

$$
K=\left(\frac{\beta+1}{4 b(a)}\right)^{1 /(\beta-1)}, \quad K_{1}=2 b_{0} K^{\beta},
$$

is oscillatory.
Proof. Claim (i) follows from Theorem 1.2.
Claim (ii). Let $u$ be the solution of (2.2), which is obtained from $x$ by the change of variable (2.1). For proving that $x$ is globally positive, it is sufficient to show that $u(s)>0$ on $[0, \infty)$. By contradiction, suppose that there exists $s_{0}$ such that

$$
\begin{equation*}
u\left(s_{0}\right)=0, \quad u(s)>0 \quad \text { on }\left(s_{0}, \infty\right) . \tag{5.1}
\end{equation*}
$$

According to Lemma 2.7, we obtain $\liminf _{s \rightarrow \infty} u(s)=\bar{u}$, where $\bar{u} \in\left[0, A_{0}\right]$. Moreover, either $\lim _{s \rightarrow \infty} u(s)=0$, or $u$ is an intermediate solution of (2.2).

For these solutions, let $\left\{s_{n}\right\}$ be a sequence such that $\lim _{n} s_{n}=\infty, s_{1}>s_{0}$,

$$
\begin{equation*}
\lim _{n} u\left(s_{n}\right)=\bar{u}, \quad \lim _{n} \dot{u}\left(s_{n}\right)=0, \tag{5.2}
\end{equation*}
$$

and

$$
\begin{equation*}
0<u\left(s_{n}\right) \leq \frac{1}{2}\left(A+A_{0}\right), \quad n \in \mathbb{N}, \tag{5.3}
\end{equation*}
$$

where $A>A_{0}$ is given by (2.4). The sequence $\left\{s_{n}\right\}$ may be defined in the following way, according to whether $u$ is either of Type (a) or of Type (b).

Let $u$ be of Type (a). Then $\lim _{s \rightarrow \infty} u(s)=0$ and by Lemma 2.7 , we obtain $\dot{u}(s)<0$ for large $s$. Then any sequence $\left\{s_{n}\right\}$ tending to infinity satisfies (5.2) and (5.3).

Let $u$ be of Type (b). Then by Lemma 2.7 , the solution $u$ intersects the function $Z$, for which $\lim _{s \rightarrow \infty} Z(s)=A_{0}, Z$ is decreasing and $\lim _{s \rightarrow \infty} \dot{Z}(s)=0$. Thus, $\bar{u} \in\left[0, A_{0}\right]$. Now, consider two cases:
(i) $\bar{u} \in\left[0, A_{0}\right)$,
(ii) $\bar{u}=A_{0}$.

In the first case the sequence $\left\{s_{n}\right\}$ can be choosen as points at which $u$ has a local minimum. In the second case, if $u$ has a local minimum, then $\left\{s_{n}\right\}$ can be defined as in the first case; if $u$ does not have local minima, i.e., $u$ is nonincreasing to $A_{0}$, we choose $\left\{s_{n}\right\}$ as

$$
\begin{align*}
u\left(s_{n}\right) & =Z\left(s_{n}\right),  \tag{5.4}\\
u(s) & <Z(s) \quad \text { in a left neighborhood of } s_{n} .
\end{align*}
$$

Indeed, the first relation in (5.2) follows from $\lim _{s \rightarrow \infty} Z(s)=A_{0}$. Since $\lim _{s \rightarrow \infty} \dot{Z}(s)=0,0>$ $\dot{u}\left(s_{n}\right) \geq \dot{Z}\left(s_{n}\right)$ and $\lim _{n} \dot{Z}\left(s_{n}\right)=0$, the second relation in (5.2) follows. Thus, $\lim _{n \rightarrow \infty} \dot{u}(s)=0$.

From here and Lemma 2.4, we obtain

$$
\begin{equation*}
Q\left(s_{n}\right)<0, \quad n \in \mathbb{N} . \tag{5.5}
\end{equation*}
$$

By Lemma 2.3 and (5.1) we have

$$
\begin{equation*}
\dot{u}\left(s_{0}\right)>0 . \tag{5.6}
\end{equation*}
$$

Thus, applying Lemma 2.5 for $\bar{s}=s_{0}$ and $s=s_{n}$, from (5.1) we obtain

$$
\begin{aligned}
4 \dot{u}^{2}\left(s_{n}\right)+Q\left(u\left(s_{n}\right)\right)= & 4 \dot{u}^{2}\left(s_{0}\right)+\frac{8}{\beta+1}\left(b_{0}-b\left(e^{s_{n}}\right)\right) u^{\beta+1}\left(s_{n}\right) \\
& +\frac{8}{\beta+1} \int_{s_{0}}^{s_{n}} b^{\prime}\left(e^{\sigma}\right) e^{\sigma} u^{\beta+1}(\sigma) d \sigma \geq 4 \dot{u}^{2}\left(s_{0}\right) .
\end{aligned}
$$

Therefore, from (5.2) and (5.6) we get

$$
\liminf _{n \rightarrow \infty} Q\left(s_{n}\right) \geq 4 \liminf _{n \rightarrow \infty} \dot{u}^{2}\left(s_{n}\right)+4 \dot{u}^{2}\left(s_{0}\right)=4 \dot{u}^{2}\left(s_{0}\right)>0,
$$

which contradicts (5.5). Hence, $u$ is positive for any $s \geq 0$.
The estimations (2.6), (2.8) follow from Lemma 2.6 and Claim (ii) is proved.
It remains to prove Claim (iii). If $x(a)=0$, the assertion follows from (ii). Otherwise, using Lemma 2.6 with $T=a$, every nonoscillatory solution of (1.1) satisfies (2.6), (2.8) for $t \geq a$. Therefore, every solution $x$ of (1.1) with the initial condition $|x(a)|>K \sqrt{a}$ or $\left|x^{\prime}(a)\right|>K_{1} a^{-1 / 2}$ must be oscillatory, and the proof is now complete.

If $b$ is nondecreasing, it would be interesting to give conditions for the existence of intermediate solutions of (1.1) in case $b$ is unbounded. For example, the equation

$$
x^{\prime \prime}+\frac{15}{64} t^{-11 / 4} x^{3}=0
$$

has an intermediate solution $x(t)=t^{3 / 8}$, whereby $\gamma=3$ and $b(t)=t^{1 / 4}$.

## Acknowledgements

The authors thank to anonymous referee for his/her valuable comments.
The research of the first and second authors has been supported by the grant GA20-11846S of the Czech Science Foundation. The third author was partially supported by Gnampa, National Institute for Advanced Mathematics (INdAM).

## References

[1] R. P. Agarwal, S. R. Grace, D. O'Regan, Oscillation theory for second order linear, halflinear, superlinear and sublinear dynamic equations, Kluwer Acad. Publ., Dordrecht, 2002. https://doi.org/10.1007/978-94-017-2515-6; MR2091751
[2] M. Bartušek, Singular solutions for the differential equation with $p$-Laplacian, Arch. Math. (Brno) 41(2005), 123-128. MR2142148
[3] M. Bartušek, Z. Došlá, M. Marini, Unbounded solutions for differential equations with $p$-Laplacian and mixed nonlinearities, Georgian Math. J. 24(2017), 15-28. https:// doi.org/10.1515/gmj-2016-0082; MR3607237
[4] R. Conti, Sulla prolungabilità delle soluzioni di un sistema di equazioni differenziali ordinarie, Boll. Un. Mat. It. Ser. 3 11(1956), 510-514. MR83628
[5] Z. Došlá, M. Marini, On super-linear Emden-Fowler type differential equations, J. Math. Anal. Appl. 416(2014), 497-510. https://doi.org/10.1016/j.jmaa.2014.02.052; MR3188719
[6] Z. Došlá, M. Marini, Monotonicity conditions in oscillation to superlinear differential equations. Electron. J. Qual. Theory Differ. Equ. 2016, No. 54, 1-13. https://doi.org/10. 14232/ejqtde.2016.1.54; MR3533264
[7] Z. Došlá, M. Marini, A coexistence problem for nonoscillatory solutions to EmdenFowler type differential equations, Enlight. Pure Appl. Math. 2(2016), 87-104. https:// core.ac.uk/download/pdf/301572318.pdf
[8] Á. Elbert, T. Kusano, Oscillation and non-oscillation theorems for a class of second order quasilinear differential equations, Acta Math. Hungar. 56(1990), 325-336. https: //doi.org/10.1007/BF01903849; MR1111319
[9] T. Hara, T. Yoneyama, J. Sugie, Continuation results for differential equations by two Liapunov functions, Annali Mat. Pura Appl. 133(1983), 79-92. https ://doi. org/10.1007/ BF01766012; MR0725020
[10] S. T. Hastings, Boundary value problems in one differential equation with a discontinuity, J. Differential Equations 1(1965), 346-369. https://doi.org/10.1016/00220396 (65) 90013-6; MR0180723
[11] H. Hoshino, R. Imabayashi, T. Kusano, T. Tanigawa, On second-order half-linear oscillations, Adv. Math. Sci. Appl. 8(1998), 199-216. MR1623342
[12] M. Jasný, On the existence of an oscillating solution of the non-linear differential equation of the second order $y^{\prime \prime}+f(x) y^{2 n-1}=0, f(x)>0$ (in Russian), Časopis Pěst. Mat. 85(1960), 78-83. https://doi.org/10.21136/CPM.1960.108129; MR0142840
[13] K. Kamo, H. Usami, Asymptotic forms of weakly increasing positive solutions for quasilinear ordinary differential equations, Electronic J. Differential Equations 2007, No. 126, 1-12. MR2349954
[14] K. Камо, H. Usami, Characterization of slowly decaying positive solutions of secondorder quasilinear ordinary differential equations with sub-homogeneity, Bull. Lond. Math. Soc. 42(2010), 420-428. https://doi.org/10.1112/blms/bdq004; MR2651937
[15] I. T. Kiguradze, A note on the oscillation of solutions of the equation $u^{\prime \prime}+$ $a(t)|u|^{n} \operatorname{sgn} u=0$ (in Russian), Časopis Pěst. Mat. 92(1967), 343-350. https://doi.org/ 10.21136/CPM.1967.108395; MR0221012
[16] I. T. Kiguradze, A. Chanturia, Asymptotic properties of solutions of nonautonomous ordinary differential equations, Kluwer Acad. Publ., Dordrecht, 1993. https://doi.org/10.1007/ 978-94-011-1808-8; MR1220223
[17] J. Kurzweil, A note on oscillatory solutions of the equation $y^{\prime \prime}+f(x) y^{2 n-1}=0$ (in Russian), Časopis Pěst. Mat. 85(1960), 357-358. https ://doi. org/10.21136/CPM.1960.117339; MR0126025
[18] T. Kusano, J. V. Manojlović, J. Miloŝević, Intermediate solutions of second order quasilinear ordinary differential equations in the framework of regular variation, Appl. Math. Comput. 219(2013), 8178-8191. https://doi.org/10.1016/j.amc.2013.02. 007; MR3037526
[19] J. D. Mirzov, On some analogs of Sturm's and Kneser's theorems for nonlinear systems. J. Math. Anal. Appl. 53 (1976), 418-425. https://doi.org/10.1016/0022-247X (76) 90120-7; MR0402184
[20] J. D. Mirzov, Asymptotic properties of solutions of the systems of nonlinear nonautonomous ordinary differential equations (in Russian), Maikop, Adygeja Publ., 1993. English translation: Folia, Mathematics, Vol. 14, Masaryk University, Brno, 2004. MR2144761
[21] A. R. Moore, Z. Nehari, Nonoscillation theorems for a class of nonlinear differential equations. Trans. Amer. Math. Soc. 93(1959), 30-52. https://doi.org/10.1090/S0002-9947-1959-0111897-8; MR0111897
[22] M. Naito, On the asymptotic behavior of nonoscillatory solutions of second order quasilinear ordinary differential equations, J. Math. Anal. Appl. 381(2011), 315-327. https: //doi.org/10.1016/j.jmaa.2011.04.006; MR2796212
[23] M. Naito, A note on the existence of slowly growing positive solutions to second order quasilinear ordinary differential equations, Mem. Differential Equations Math. Phys. 57(2012), 95-108. MR3089214
[24] M. Naito, A remark on the existence of slowly growing positive solutions to second order super-linear ordinary differential equations, NoDEA Nonlinear Differential Equations Appl. 20(2013), 1759-1769. https://doi.org/10.1007/s00030-013-0229-y; MR3128693
[25] J. WANG, Oscillation and nonoscillation theorems for a class of second order quasilinear functional differential equations, Hiroshima Math. J. 27(1997), 449-466. https://doi .org/ 10.32917/hmj/1206126963; MR1482952
[26] J. S. W. Wong, On the generalized Emden-Fowler equation, SIAM Rev. 17(1975), 339-360. https://doi.org/10.1137/1017036; MR0367368
[27] N. ҮАмаока, Oscillation criteria for second order damped nonlinear differential equations with p-Laplacian, J. Math. Anal. Appl. 325(2007), 932-948. https://doi.org/10. 1016/j.jmaa.2006.02.021; MR2270061

# Bifurcation from zero or infinity in nonlinearizable Sturm-Liouville problems with indefinite weight 

Ziyatkhan S. Aliyev ${ }^{\boxtimes 1,2}$ and Leyla V. Nasirova ${ }^{3}$<br>${ }^{1}$ Department of Mathematical Analysis, Baku State University, Z. Khalilov Str. 23, Baku, AZ1148, Azerbaijan<br>${ }^{2}$ Department of Differential Equations, Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, B. Vahabzadeh Str. 9, Baku, AZ1141, Azerbaijan<br>${ }^{3}$ Department of Mathematical Analysis, Sumgait State University, Baku Str. 1, Sumgait, AZ5008, Azerbaijan

Received 27 January 2021, appeared 28 July 2021
Communicated by Alberto Cabada


#### Abstract

In this paper, we consider bifurcation from zero or infinity of nontrivial solutions of the nonlinear Sturm-Liouville problem with indefinite weight. This problem is mainly important because of it is related with a selection-migration model in genetic population. We show the existence of four families of unbounded continua of nontrivial solutions to this problem bifurcating from intervals of the line of trivial solutions or the line $\mathbb{R} \times\{\infty\}$ (these intervals are called bifurcation intervals). Moreover, these global continua have the usual nodal properties in some neighborhoods of bifurcation intervals.


Keywords: nonlinear Sturm-Liouville problem, indefinite weight, population genetics, selection-migration model, bifurcation point, bifurcation interval, global continua.
2020 Mathematics Subject Classification: 34B15, 34B24, 34C10, 34C23, 34L15, 45C05, 47J10, 47J15.

## 1 Introduction

We consider the following nonlinear Sturm-Liouville eigenvalue problem

$$
\begin{gather*}
(\ell(u))(x) \equiv-\left(p(x) u^{\prime}(x)\right)^{\prime}+q(x) u(x)=\lambda \rho(x) u(x)+h\left(x, u(x), u^{\prime}(x), \lambda\right), \quad x \in(0,1), \\
\alpha_{0} u(0)-\beta_{0} u^{\prime}(0)=0, \quad \alpha_{1} u(1)+\beta_{1} u^{\prime}(1)=0, \tag{1.1}
\end{gather*}
$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, $p \in C^{1}([0,1] ;(0,+\infty)), q \in C([0,1] ;[0,+\infty))$, and $\rho \in$ $C([0,1] ; \mathbb{R})$ such that there exist $\varsigma, \xi \in[0,1]$ for which $\rho(\varsigma) \rho(\xi)<0$, and $\alpha_{i}, \beta_{i}, i=0,1$, are real

[^18]constants such that $\left|\alpha_{i}\right|+\left|\beta_{i}\right|>0$ and $\alpha_{i} \beta_{i} \geq 0, i=0,1$. The function $h$ has the representation $h=f+g$, where the functions $f, g \in C\left([0,1] \times \mathbb{R}^{3} ; \mathbb{R}\right)$ and satisfy the conditions
\[

$$
\begin{equation*}
u f(x, u, s, \lambda) \leq 0, \quad u g(x, u, s, \lambda) \leq 0 ; \tag{1.3}
\end{equation*}
$$

\]

there exists a constant $M>0$ such that

$$
\begin{equation*}
\left|\frac{f(x, u, s, \lambda)}{u}\right| \leq M, \quad(x, u, s, \lambda) \in[0,1] \times \mathbb{R}^{2} \times \mathbb{R}, \quad u \neq 0 . \tag{1.4}
\end{equation*}
$$

Moreover, at various points in the paper, we will impose one or the other or both of the following conditions on the function $g$ :

$$
\begin{equation*}
g(x, u, s, \lambda)=o(|u|+|s|), \quad \text { as }|u|+|s| \rightarrow \infty, \tag{1.5}
\end{equation*}
$$

or

$$
\begin{equation*}
g(x, u, s, \lambda)=o(|u|+|s|), \quad \text { as }|u|+|s| \rightarrow 0, \tag{1.6}
\end{equation*}
$$

uniformly for $x \in[0,1]$ and $\lambda \in \Lambda$, for any bounded interval $\Lambda \subset \mathbb{R}$.
Nonlinear eigenvalue problems of the type (1.1), (1.2) have been intensively studied recently, as they arise from selection-migration models in population genetics (see for example $[2,3,9,10,14,16,18]$ and references therein). Note that population genetics is one of the important branches of biology, which studies the genetic structure and evolution of populations. It has close ties to ecology, demography, epidemiology, phylogeny, genomics, and molecular evolution. Population genetics is mainly used in human genetics and medicine, as well as in animal and plant breeding. In the case of $p(x) \equiv 1$ and $h\left(x, u(x), u^{\prime}(x), \lambda\right)=$ $\lambda \rho(x)[u(x)-m(u(x))]$, where $m(u)=u(1-u)\left[h_{0}(1-u)+\left(1-h_{0}\right) u\right]$ and $h_{0} \in(0,1)$, Eq. (1.1) is an one-dimensional reaction-diffusion equation, the interval $[0,1]$ refers to the habitat of a species, the boundary conditions (1.2) for $\beta_{0}=\beta_{1}=0$ means that no individuals cross the boundary of the habitat. Moreover, the weight function $\rho(x)$ represents either the selective strength of the environment on genes, or the intrinsic growth rate of the species at location $x$, and the real parameter $\lambda$ corresponds to the reciprocal of the diffusion coefficient (see [10, 14, 18]).

If condition (1.6) is satisfied, then we consider bifurcation from $u=0$, i.e. bifurcation from the line of trivial solutions $\mathcal{R}_{0}=\mathbb{R} \times\{0\}$. In the case when $\rho(x)>0$ for $x \in[0,1]$ the global bifurcation of solutions of the nonlinear eigenvalue problem (1.1), (1.2) under conditions (1.4) and (1.6) (but without the conditions $\alpha_{i} \beta_{i} \geq 0, i=0,1$ ) was considered in [ $\left.8,11,12,22,23,25,26\right]$. These papers it was shown the existence of two families of unbounded continua of nontrivial solutions in $\mathbb{R} \times C^{1}[0,1]$, possessing the usual nodal properties and bifurcating from points and intervals of the line $\mathcal{R}_{0}$ corresponding to the eigenvalues of the linear problem obtained from (1.1), (1.2) by setting $h \equiv 0$. Similar results in nonlinear eigenvalue problems for ordinary differential equations of fourth order were established in the paper [1].

If condition (1.5) is satisfied, then problem (1.1), (1.2) for $f \equiv 0$ is asymptotically linear (see [17]), and, therefore, we must investigate the bifurcation from infinity, that is, the existence of non-trivial solutions to this problem with large norms. Note that in the case when $\rho(x)>0$ for $x \in[0,1]$ the global bifurcation from infinity of nontrivial solutions of problem (1.1), (1.2) under conditions (1.4) and (1.5) (again without the conditions $\alpha_{i} \beta_{i} \geq 0, i=0,1$ ) was studied in $[11,12,22,24,25,27,28]$, where in particular, it was shown that there are two families of global continua of nontrivial solutions of this problem bifurcating from points and intervals of the set $\mathcal{R}_{\infty}=\mathbb{R} \times\{\infty\}$ corresponding to the eigenvalues of the linear problem (1.1), (1.2)
with $h \equiv 0$ and having usual nodal properties in some neighborhoods these points and intervals. Moreover, it was also established that these continua either contain other asymptotic bifurcation points and intervals, or intersect the line $\mathcal{R}_{0}$, or have an unbounded projection onto $\mathcal{R}_{0}$. Similar global results for fourth order nonlinear eigenvalue problems were obtained in the paper [6].

The problem (1.1), (1.2) in cases (i) $f \equiv 0$, and (ii) $g \equiv 0$ and $f$ satisfies condition (1.4) for any $(x, u, s, \lambda) \in[0,1] \times \mathbb{R}^{2} \times \mathbb{R}$ such that $u \neq 0$ and $|u|+|s| \leq \tau_{0}$, where $\tau_{0}>0$ is some constant, was considered in $[7,21]$. These papers prove the existence of four families of unbounded continua of solutions having the usual nodal properties and bifurcating from points and intervals of the line of trivial solutions corresponding to the positive and negative eigenvalues of linear problem (1.1), (1.2) with $h \equiv 0$.

The purpose of this paper is to study the location of bifurcation intervals in $\mathcal{R}_{0}$ and $\mathcal{R}_{\infty}$, and the structure of global continua of nontrivial solutions of problem (1.1), (1.2) emanating from these bifurcation intervals.

In Section 2, we present the main properties of the eigenvalues and eigenfunctions of the linear problem (1.1), (1.2) with $h \equiv 0$. Here we introduce classes of functions in $\mathbb{R} \times C^{1}[0,1]$ with a fixed oscillation counter and also possessing other properties of the eigenfunctions of this linear problem. Here we consider problem (1.1), (1.2) under conditions (1.3), (1.4) and (1.6). Then we find the bifurcation intervals of the line of trivial solutions with respect to the above-mentioned oscillation classes and establish that the connected components of solutions emanating from bifurcation intervals are contained in the corresponding oscillation classes, and are unbounded in $\mathbb{R} \times C^{1}[0,1]$. In Section 3 and 4 we consider problem (1.1), (1.2) under conditions (1.3)- (1.5). In Section 3 developing the approximation technique from [8], we prove the existence of nontrivial solutions of problem (1.1), (1.2) with large norms contained in the classes with a fixed oscillation count. Moreover, we find intervals containing asymptotically bifurcation points of problem (1.1), (1.2) with respect to these classes. Note that the approximation equation introduced here is more natural than those introduced in [24,25]. It is important to note that the solutions of problem (1.1), (1.2) from the global continuum emanating from the bifurcation interval of the set $\mathcal{R}_{\infty}$ with respect to a certain class of a fixed oscillation count and located outside a some neighborhood of this interval may not be included in this oscillation class. In Section 4, we present and prove the main result of this paper, namely, we show that there are four classes of unbounded continua of solutions of problem (1.1), (1.2) emanating from asymptotically bifurcation intervals which have usual nodal properties in a some neighborhoods of these intervals and for each of them one of the following statements holds: either contain other asymptotic bifurcation intervals, or intersect the line $\mathcal{R}_{0}$, or have an unbounded projection onto $\mathcal{R}_{0}$. Similar results in nonlinear eigenvalue problems for ordinary differential equations of fourth order and semi-linear elliptic partial differential equations with indefinite weight in the classes of positive and negative functions were obtained in recent papers [2-5]. In Section 5 we consider problem (1.1), (1.2) under both conditions (1.5) and (1.6). Here we manage to show that the continua emanating from asymptotically bifurcation intervals are contained in the corresponding oscillation classes and therefore they do not intersect other asymptotically bifurcation intervals. In Section 6 we consider problem (1.1), (1.2) in the case when the weight function $\rho(x) \geq 0$ for $x \in[0,1]$. Note that in this case linear problem obtained from (1.1), (1.2) by setting $h \equiv 0$ has only one sequence of positive simple eigenvalues, and consequently, in this case problem (1.1), (1.2) has two families of global connected components emanating from bifurcation intervals of $\mathcal{R}_{0}$ or $\mathcal{R}_{\infty}$, and having the properties of global continua from Sections 2 and 3-4, respectively.

Note that similar results was obtained in $[11,12]$ in the case of a special form of the nonlinear term $f$.

## 2 Preliminary

By (b.c.) we denote the set of functions satisfying the boundary conditions (1.2).
It is known [15, Ch. 10, §10.61] that the spectrum of the linear eigenvalue problem

$$
\left\{\begin{array}{l}
(\ell(u))(x)=\lambda \rho(x) u(x), \quad x \in(0,1),  \tag{2.1}\\
u \in(\text { b.c. }),
\end{array}\right.
$$

obtained from (1.1), (1.2) by setting $h \equiv 0$ consists of two sequences of real and simple eigenvalues

$$
0<\lambda_{1}^{+}<\lambda_{2}^{+}<\cdots<\lambda_{k}^{+} \mapsto+\infty \quad \text { and } \quad 0>\lambda_{1}^{-}>\lambda_{2}^{-}>\cdots>\lambda_{k}^{-} \mapsto-\infty ;
$$

for each $k \in \mathbb{N}$ the eigenfunctions $u_{k}^{+}$and $u_{k}^{-}$corresponding to the eigenvalues $\lambda_{k}^{+}$and $\lambda_{k}^{-}$, respectively, have exactly $k-1$ simple nodal zeros in $(0,1)$ (by a nodal zero, we mean that the function changes sign at the zero, and at a simple nodal zero, the derivative of the function is nonzero). Moreover, according to [21, formula (2.10)] for each $k \in \mathbb{N}$ the eigenfunctions $u_{k}^{+}$ and $u_{k}^{-}$satisfy the following relations

$$
\begin{equation*}
\int_{0}^{1} \rho(x)\left(u_{k}^{+}(x)\right)^{2} d x>0 \quad \text { and } \quad \int_{0}^{1} \rho(x)\left(u_{k}^{-}(x)\right)^{2} d x<0 \tag{2.2}
\end{equation*}
$$

respectively.
Let $E=C^{1}[0,1] \cap$ (b.c.) be a Banach space with the usual norm $\|u\|_{1}=\|u\|_{\infty}+\left\|u^{\prime}\right\|_{\infty}$, where $\|u\|_{\infty}=\max _{x \in[0,1]}|u(x)|$.

From now on $\sigma$ ( $v$ respectively) will denote either + or - .
For each $k \in \mathbb{N}$, each $\sigma$ and each $v$ we denote by $\mathcal{S}_{k, \sigma}^{v}$ the set of functions $u \in E$ satisfying the following conditions:
(i) $u$ has exactly $k-1$ simple nodal zeros in the interval $(0,1)$;
(ii) $\sigma \int_{0}^{1} \rho(x) u^{2}(x) d x>0$;
(iii) $v u(x)$ is positive in a deleted neighborhood of the point $x=0$.

It follows from definition of the set $\mathcal{S}_{k, \sigma}^{v}, k \in \mathbb{N}$, that this set is open in $E$ for each $\sigma$ and each $v$. Note that for any $(k, \sigma, v) \neq\left(k^{\prime}, \sigma^{\prime}, v^{\prime}\right)$ the following relation holds:

$$
\mathcal{S}_{k, \sigma}^{v} \cap \mathcal{S}_{k^{\prime}, \sigma^{\prime}}^{v^{\prime}}=\varnothing .
$$

Moreover, if $u \in \partial \mathcal{S}_{k, \sigma}^{v}$, then either
(i) $\int_{0}^{1} \rho(x) u^{2}(x) d x=0$, or
(ii) there exists $x_{0} \in[0,1]$ that such $u\left(x_{0}\right)=u^{\prime}\left(x_{0}\right)=0$.

Remark 2.1. It follows from the above arguments that

$$
u_{k}^{\sigma} \in \mathcal{S}_{k, \sigma}=\mathcal{S}_{k, \sigma}^{+} \cup \mathcal{S}_{k, \sigma^{\prime}}^{-} \quad k \in \mathbb{N} .
$$

Remark 2.2. If $(\lambda, u) \in \mathbb{R} \times E$ be a nontrivial solution of problem (1.1), (1.2), then

$$
\lambda \int_{0}^{1} \rho(x) u^{2}(x) d x \neq 0
$$

Indeed, multiplying both sides of (1.1) by $u(x)$, integrating the obtained equality in the range from 0 to 1 , using the formula for integration by parts, and taking into account conditions (1.2), (1.3), we get

$$
\begin{align*}
& \int_{0}^{1}\left\{p(x) u^{\prime 2}(x)+q(x) u^{2}(x)\right\} d x+N[u]=\lambda \int_{0}^{1} \rho(x) u^{2}(x) d x \\
& \quad+\int_{0}^{1} f\left(x, u(x), u^{\prime}(x), \lambda\right) u(x) d x+\int_{0}^{1} g\left(x, u(x), u^{\prime}(x), \lambda\right) u(x) d x \tag{2.3}
\end{align*}
$$

where $N[u]=-\left.\left(p(x) u^{\prime}(x) u(x)\right)\right|_{x=0} ^{x=1}$. Since the conditions $\left|\alpha_{i}\right|+|\beta|>0$ and $\alpha_{i} \beta_{i} \geq 0, i=0,1$, are satisfied it follows that $N[u] \geq 0$ for any function $u \in E$. Consequently, the left hand side of (2.3) is positive, and if $\lambda \int_{0}^{1} \rho(x) u^{2}(x) d x=0$, then by conditions (1.3) the right hand side of this relation is non-positive, a contradiction.

Let $\left\{\lambda_{k, M}^{+}\right\}_{k=1}^{\infty}$ and $\left\{\lambda_{k, M}^{-}\right\}_{k=1}^{\infty}$ be sequences of positive and negative eigenvalues, respectively, of the following spectral problem

$$
\left\{\begin{array}{l}
(\ell(u))(x)+M u(x)=\lambda \rho(x) u(x), \quad x \in(0,1)  \tag{2.4}\\
u \in(b . c .)
\end{array}\right.
$$

which are simple (see [15, Ch. 10, §10.61]).
We introduce the notations:

$$
\begin{aligned}
& \mathcal{I}_{k}^{+}=\left[\lambda_{k}^{+}, \lambda_{k, M}^{+}\right], \quad I_{k}^{-}=\left[\lambda_{k, M}^{-}, \lambda_{k}^{-}\right] \\
& \mathbb{R}^{+}=(0,+\infty), \mathbb{R}^{-}=(-\infty, 0), \quad \mathcal{R}_{0}^{+}=\mathbb{R}^{+} \times\{0\}, \quad \mathcal{R}_{0}^{-}=\mathbb{R}^{-} \times\{0\}
\end{aligned}
$$

and

$$
\mathcal{R}_{\infty}^{+}=\mathbb{R}^{+} \times\{\infty\}, \quad \mathcal{R}_{\infty}^{-}=\mathbb{R}^{-} \times\{\infty\}
$$

By $\mathcal{D}$ we denote the set of nontrivial solutions of the nonlinear eigenvalue problem (1.1), (1.2). For any $\lambda \in \mathbb{R}$, we say that a subset $\mathcal{C} \subset \mathcal{D}$ meets $(\lambda, 0)$ (respectively $(\lambda, \infty)$ ) if there exists a sequence $\left\{\left(\lambda_{n}, u_{n}\right)\right\}_{n=1}^{\infty} \subset \mathcal{C}$ such that $\lambda_{n} \rightarrow \lambda$ and $\left\|u_{n}\right\|_{1} \rightarrow 0$ (respectively $\left\|u_{n}\right\|_{1} \rightarrow$ $+\infty)$ as $n \rightarrow+\infty$. Furthermore, we will say that $\mathcal{C} \subset \mathcal{D}$ meets $(\lambda, 0)$ (respectively $(\lambda, \infty)$ ) with respect to the set $\mathbb{R} \times \mathcal{S}_{k, \sigma}^{v}$, if the sequence $\left\{\left(\lambda_{n}, u_{n}\right)\right\}_{n=1}^{\infty}$ can be chosen so that $u_{n} \in \mathcal{S}_{k, \sigma}^{v}$ for all $n \in \mathbb{N}$. Moreover, we say $(\lambda, 0)$ (respectively $(\lambda, \infty)$ ) is a bifurcation point of problem (1.1), (1.2) with respect to the set $\mathbb{R} \times \mathcal{S}_{k, \sigma}^{v}$ if there exists a sequence $\left\{\left(\lambda_{n}, u_{n}\right)\right\}_{n=1}^{\infty} \subset \mathcal{D} \cap\left(\mathbb{R} \times \mathcal{S}_{k, \sigma}^{v}\right)$ such that $\lambda_{n} \rightarrow \lambda$ and $\left\|u_{n}\right\|_{1} \rightarrow 0$ (respectively $\left\|u_{n}\right\|_{1} \rightarrow+\infty$ ) as $n \rightarrow+\infty$. If $I \subset \mathbb{R}$ is a bounded interval we say that $\mathcal{C}$ meets $I \times\{0\}(I \times\{\infty\}$ respectively) if $C$ meets $(\lambda, 0)$ (respectively $(\lambda, \infty)$ ) for some $\lambda \in I$. Furthermore, we will say that $\mathcal{C}$ meets $I \times\{0\}(I \times\{\infty\}$ respectively) with respect to the set $\mathbb{R} \times \mathcal{S}_{k, \sigma^{\prime}}^{v}$ if $C$ meets $(\lambda, 0)$ (respectively $(\lambda, \infty)$ ), $\lambda \in I$, with respect to the set $\mathbb{R} \times \mathcal{S}_{k, \sigma}^{v}$ (see $[1,6,25]$ ).

Now we consider problem (1.1), (1.2) under the conditions (1.3), (1.4) and (1.6). Following the corresponding reasoning given in $[7,19,21]$ (see also $[1,8]$ ) and using Remark 2.2 we are convinced that the following results hold for this problem.

Lemma 2.3. Let $(\lambda, u) \in \mathbb{R} \times E$ be a solution of problem (1.1), (1.2) such that $u \in \partial \mathcal{S}_{k, \sigma^{\prime}}^{v} k \in \mathbb{N}$, $\sigma, v \in\{+,-\}$. Then $u \equiv 0$.

Lemma 2.4. For each $k \in \mathbb{N}$, each $\sigma$ and each $v$ the set $\mathcal{B}_{k, \sigma}^{v}$ of bifurcation points (from zero) of problem (1.1), (1.2) with respect to the set $\mathbb{R}^{\sigma} \times \mathcal{S}_{k, \sigma}^{v}$ is nonempty. Furthermore, $\mathcal{B}_{k, \sigma}^{\nu} \subset \mathcal{I}_{k}^{\sigma} \times\{0\}$.

For each $k \in \mathbb{N}$, each $\sigma$ and each $v$ by $\mathcal{D}_{k, \sigma}^{v, *}$ we denote the union of all the components of $\mathcal{D}$ which meet $\mathcal{I}_{k}^{\sigma} \times\{0\}$ with respect to the set $\mathbb{R}^{\sigma} \times \mathcal{S}_{k, \sigma}^{\nu}$ ([25, Theorem 3.2] and Lemma 2.4 implies that $\mathcal{D}_{k, \sigma}^{v, *} \neq \varnothing$ ). The set $\mathcal{D}_{k, \sigma}^{v, *}$ may not be connected in $\mathbb{R}^{\sigma} \times E$, but joining the interval $\mathcal{I}_{k}^{\sigma} \times\{0\}$ to this set gives a connected set $\mathcal{D}_{k, \sigma}^{v}=\mathcal{D}_{k, \sigma}^{v, *} \cup\left(\mathcal{I}_{k}^{\sigma} \times\{0\}\right)$.

Remark 2.5. By Lemma 2.4 it follows from Remark 2.2 that $\mathcal{D}_{k, \sigma}^{\nu} \subset \mathbb{R}^{\sigma} \times E$.
Theorem 2.6. For each $k \in \mathbb{N}$, each $\sigma$ and each $v$ the connected set $\mathcal{D}_{k, \sigma}^{v}$ which contains $I_{k}^{\sigma} \times\{0\}$ lies in $\left(\mathbb{R}^{\sigma} \times \mathcal{S}_{k, \sigma}^{v}\right) \cup\left(\mathcal{I}_{k}^{\sigma} \times\{0\}\right)$ and is unbounded in $\mathbb{R} \times E$.

## 3 The existence of asymptotic bifurcation points of problem (1.1), (1.2) with respect to the set $S_{k, \sigma}^{\nu}$

In the next two sections, we will consider problem (1.1), (1.2) under the conditions (1.3)-(1.5).
To study the structure of the set of asymptotic bifurcation points of problem (1.1), (1.2), we introduce the following modified nonlinear eigenvalue problem

$$
\left\{\begin{array}{l}
\ell(u)=\lambda \rho(x) u+\frac{f\left(x,|u|^{\varepsilon} u, u^{\prime}, \lambda\right)}{\left(1+|u|+\left|u^{\prime}\right|\right)^{\varepsilon \varepsilon}}+g\left(x, u, u^{\prime}, \lambda\right), \quad x \in(0,1),  \tag{3.1}\\
u \in(\text { b.c. }),
\end{array}\right.
$$

where $\varepsilon \in(0,1]$. It is seen that this problem in a sense approximates problem (1.1), (1.2) for $\varepsilon$ near 0 . Note that approximations similar to this one were previously used in $[6,25]$.

Since $f \in C\left([0,1] \times \mathbb{R}^{3} ; \mathbb{R}\right)$ it follows that $\frac{\left|f\left(x,|u|^{\mathcal{\varepsilon}} u, s, \lambda\right)\right|}{\left(1+|u|+|s|^{2 \varepsilon}\right.} \in C\left([0,1] \times \mathbb{R}^{3} ; \mathbb{R}\right)$ for any $\varepsilon \in(0,1]$. Moreover, by condition (1.4) we get

$$
\frac{\left|f\left(x,|u|^{\varepsilon} u, s, \lambda\right)\right|}{(1+|u|+|s|)^{2 \varepsilon}(|u|+|s|)} \leq \frac{M|u|^{\varepsilon}|u|}{(|u|+|s|)^{2 \varepsilon}(|u|+|s|)} \leq \frac{M}{(|u|+|s|)^{\varepsilon}},
$$

whence implies that for each bounded interval $\Lambda \subset \mathbb{R}$

$$
\frac{f\left(x,|u|^{\varepsilon} u, s, \lambda\right)}{(1+|u|+|s|)^{2 \varepsilon}}=o(|u|+|s|) \quad \text { as }|u|+|s| \rightarrow \infty,
$$

uniformly for $(x, \lambda) \in[0,1] \times \Lambda$. Then, by conditions (1.5), it follows from [24, Theorem 2.4] that for each $k \in \mathbb{N}$, each $\sigma$ and each $v$ there exist a neighborhood $\mathcal{P}_{k, \sigma}^{v}$ of the point $\left(\lambda_{k}^{\sigma}, \infty\right)$ and a continuum $\mathfrak{D}_{k, \sigma, \varepsilon}^{\nu} \subset \mathbb{R}^{\sigma} \times E$ of the set of solutions of problem (3.1) bifurcating from ( $\lambda_{k}^{\sigma}, \infty$ ) such that
(i)
$\left(\mathfrak{D}_{k, \sigma, \varepsilon}^{v} \cap \mathcal{P}_{k, \sigma}^{v}\right) \subset \mathbb{R} \times \mathcal{S}_{k, \sigma}^{v} ;$
(ii) either $\mathfrak{D}_{k, \sigma, \varepsilon}^{v} \backslash \mathcal{P}_{k, \sigma}^{v}$ is bounded in $\mathbb{R} \times E$, and in this case $\mathfrak{D}_{k, \sigma, \varepsilon}^{v} \backslash \mathcal{P}_{k, \sigma}^{v}$ meets $\mathcal{R}_{0}^{\sigma}$, or $\mathfrak{D}_{k, \sigma, \varepsilon}^{v} \backslash \mathcal{P}_{k, \sigma}^{v}$ is unbounded, and if in this case $\mathfrak{D}_{k, \sigma, \varepsilon}^{v} \backslash \mathcal{P}_{k, \sigma}^{v}$ has a bounded projection on $\mathcal{R}_{0}^{\sigma}$, then this set meets $\left(\lambda_{k^{\prime}}^{\sigma}, \infty\right)$ with respect to $\mathcal{S}_{k^{\prime}, \sigma}^{\prime^{\prime}}$ for some $\left(k^{\prime}, v^{\prime}\right) \neq(k, v)$.

Lemma 3.1. For each $k \in \mathbb{N}$, each $\sigma$, each $v$ and any sufficiently large $R>0$ there exists a solution $\left(\lambda_{k, \sigma, R^{\prime}}^{v} u_{k, \sigma, R}^{v}\right)$ of problem (1.1), (1.2) such that $\lambda_{k, \sigma, R}^{v} \in \mathbb{R}^{\sigma}, u_{k, \sigma, R}^{v} \in \mathcal{S}_{k, \sigma^{\prime}}^{v}$ and $\left\|u_{k, \sigma, R}^{v}\right\|_{1}=R$.

Proof. Let $R$ be a sufficiently large positive number. Property (i) of the set $\mathfrak{D}_{k, \sigma, \varepsilon}^{v}$ implies that for any $\varepsilon \in(0,1)$ there exists a solution $\left(\lambda_{k, \sigma, R, \varepsilon}^{v}, u_{k, \sigma, R, \varepsilon}^{v}\right)$ of problem (3.1) such that

$$
\lambda_{k, \sigma, R, \varepsilon}^{v} \in \mathbb{R}^{\sigma}, \quad u_{k, \sigma, R, \varepsilon}^{v} \in \mathcal{S}_{k, \sigma^{\prime}}^{v} \quad\left\|u_{k, \sigma, R, \varepsilon}^{v}\right\|_{1}=R
$$

Then it follows from (3.1) that $\left(\lambda_{k, \sigma, R, \varepsilon}^{v}, u_{k, \sigma, R, \varepsilon}^{v}\right)$ solves the following problem

$$
\left\{\begin{array}{l}
(\ell(u))(x)+\varphi_{k, \sigma, R, \varepsilon}^{v}(x) u(x)=\lambda \rho(x) u(x)+g\left(x, u(x), u^{\prime}(x), \lambda\right), \quad x \in(0,1)  \tag{3.2}\\
u \in(b . c .)
\end{array}\right.
$$

where

$$
\varphi_{k, \sigma, R, \varepsilon}^{v}(x)= \begin{cases}-\frac{f\left(x,\left|u_{k, \sigma, R, \varepsilon}^{v}(x)\right|^{\varepsilon} u_{k, \sigma, R, \varepsilon}^{v}(x),\left(u_{k, \sigma, R, \varepsilon}^{v}\right)^{\prime}(x), \lambda_{k, \sigma, R, \varepsilon}^{v}\right)}{u_{k, \sigma, R, \varepsilon}^{v}(x)\left(1+\left|u_{k, \sigma, R, \varepsilon}^{v}(x)\right|+\left|\left(u_{k, \sigma, R, \varepsilon}^{v}\right)^{\prime}(x)\right|\right)^{2 \varepsilon}} & \text { if } u_{k, \sigma, R, \varepsilon}^{v}(x) \neq 0,  \tag{3.3}\\ 0 & \text { if } u_{k, \sigma, R, \varepsilon}^{v}(x)=0 .\end{cases}
$$

In view of conditions (1.3) and (1.4), by (3.3) we have

$$
\begin{align*}
\varphi_{k, \sigma, R, \varepsilon}^{v}(x) & \geq 0 \quad \text { and } \\
\left|\varphi_{k, \sigma, R, \varepsilon}^{v}(x)\right| & \leq \frac{M\left|u_{k, \sigma, R, \varepsilon}^{v}(x)\right|^{\varepsilon}}{\left(1+\left|u_{k, \sigma, R, \varepsilon}^{v}(x)\right|+\left|\left(u_{k, \sigma, R, \varepsilon}^{v}\right)^{\prime}(x)\right|\right)^{2 \varepsilon}}  \tag{3.4}\\
& \leq \frac{M}{\left(1+\left|u_{k, \sigma, R, \varepsilon}^{v}(x)\right|+\left|\left(u_{k, \sigma, R, \varepsilon}^{v}\right)^{\prime}(x)\right|\right)^{\varepsilon}} \leq M \quad \text { for } x \in[0,1]
\end{align*}
$$

Since $C[0,1]$ is dense in $L_{1}[0,1]$ and the function $u_{k, \sigma, R, \varepsilon}^{v}$ has a finite number of zeros in $(0,1)$, by relation (3.4), it follows from [15, Ch. $10, \S 10 \cdot 61$ ] that the eigenvalues of problem

$$
\left\{\begin{array}{l}
(\ell(u))(x)+\varphi_{k, \sigma, R, \varepsilon}^{v}(x) u(x)=\lambda \rho(x) u(x), \quad x \in(0,1)  \tag{3.5}\\
u \in(\text { b.c. })
\end{array}\right.
$$

are real, simple and form a positive infinitely increasing and negative infinitely decreasing sequences $\left\{\lambda_{k, \sigma, R, \varepsilon}^{v,+}\right\}_{k=1}^{\infty}$ and $\left\{\lambda_{k, \sigma, R, \varepsilon}^{v,-}\right\}_{k=1}^{\infty}$ respectively. In this case, for each $k \in \mathbb{N}$ the function $u_{k, \sigma, R, \varepsilon}^{v,+}\left(u_{k, \sigma, R, \varepsilon}^{v,-}\right.$ respectively) corresponding to the eigenvalue $\lambda_{k, \sigma, R, \varepsilon}^{v,+}\left(\lambda_{k, \sigma, R, \varepsilon}^{v,-}\right.$ respectively) has $k-1$ simple nodal zeros in the interval ( 0,1 ). Moreover, by [21, Lemma 2.2], the following relations hold:

$$
\begin{equation*}
\lambda_{k}^{+} \leq \lambda_{k,+, R, \varepsilon}^{v,+} \leq \lambda_{k, M}^{+} \quad \text { and } \quad \lambda_{k, M}^{-} \leq \lambda_{k,-, R, \varepsilon}^{v,-} \leq \lambda_{k}^{-}, \quad k \in \mathbb{N} \tag{3.6}
\end{equation*}
$$

Hence it follows from (3.6) that

$$
\begin{equation*}
\lambda_{k, \sigma, R, \varepsilon}^{v, \sigma} \in \mathcal{I}_{k}^{\sigma}, \quad k \in \mathbb{N} \tag{3.7}
\end{equation*}
$$

Let

$$
\mathcal{I}_{k}^{+}(\delta)=\left[\lambda_{k}^{+}-\delta, \lambda_{k, M}^{+}+\delta\right], \quad \mathcal{I}_{k}^{-}(\delta)=\left[\lambda_{k, M}^{-}-\delta, \lambda_{k}^{-}+\delta\right]
$$

where $\delta$ is a positive number.

By [17, Ch. 4, §3, Theorem 3.1] for each $k \in \mathbb{N}$, each $\sigma$ and each $v$ the point $\left(\lambda_{k, \sigma, R, \varepsilon^{\prime}}^{v, \sigma} \infty\right)$ is an unique asymptotic bifurcation point of the nonlinear eigenvalue problem (3.2) with respect to the set $\mathbb{R}^{\sigma} \times \mathcal{S}_{k, \sigma}^{v}$. Then for any sufficiently large $R>0$ there exists a sufficiently small $\tau_{R, \varepsilon}$ such that

$$
\lambda_{k, \sigma, R, \varepsilon}^{v} \in\left[\lambda_{k, \sigma, R, \varepsilon}^{v, \sigma}-\tau_{R, \varepsilon}, \lambda_{k, \sigma, R, \varepsilon}^{v, \sigma}+\tau_{R, \varepsilon}\right] \subseteq \mathcal{I}_{k}^{\sigma}\left(\tau_{R, \varepsilon}\right) \subset \mathbb{R}^{\sigma} .
$$

Let $\tau_{0}=\sup _{R, \varepsilon} \tau_{R, \varepsilon}$. Hence it follows from last relation that

$$
\begin{equation*}
\lambda_{k, \sigma, R, \varepsilon}^{v} \in \mathcal{I}_{k}^{\sigma}\left(\tau_{0}\right) \subset \mathbb{R}^{\sigma} . \tag{3.8}
\end{equation*}
$$

Since $\left\|u_{k, \sigma, R, \varepsilon}^{v}\right\|_{1}=R$ and $f, g \in C\left([0,1] \times \mathbb{R}^{3} ; \mathbb{R}\right)$, by relation (3.8), it follows from (3.1) that

$$
\left\|u_{k, \sigma, R, \varepsilon}^{v}\right\|_{2} \leq \text { const, }
$$

where $\|u\|_{2}=\|u\|_{\infty}+\left\|u^{\prime}\right\|_{\infty}+\left\|u^{\prime \prime}\right\|_{\infty}$. Then by the Arzelà-Ascoli theorem the set $\left\{u_{k, \sigma, R, \varepsilon}^{v}\right\}_{\varepsilon \in(0,1]}$ is precompact in $E$. Hence we can choose the sequence $\left\{\varepsilon_{n}\right\}_{n=1}^{\infty} \subset(0,1)$ converging to 0 as $n \rightarrow \infty$ such that

$$
\left(\lambda_{k, \sigma, R, \varepsilon_{n}}^{v} u_{k, \sigma, R, \varepsilon_{n}}^{v}\right) \rightarrow\left(\lambda_{k, \sigma, R}^{v}, u_{k, \sigma, R}^{v}\right) \quad \text { as } n \rightarrow \infty \text { in } \mathbb{R} \times E,
$$

and by (3.1) the sequence $\left\{\left(\lambda_{k, \sigma, R, \varepsilon_{n}}^{v}, u_{k, \sigma, R, \varepsilon_{n}}^{v}\right)\right\}_{n=1}^{\infty}$ is convergent in $\mathbb{R} \times C^{2}[0,1]$. Putting $\left(\lambda_{k, \sigma, R, \varepsilon_{n}}^{v}, u_{k, \sigma, R, \varepsilon_{n}}^{v}\right)$ instead of $(\lambda, u)$ in (3.1) and passing to the limit (as $n \rightarrow \infty$ ) in this relation we obtain that ( $\lambda_{k, \sigma, R}^{v}, u_{k, \sigma, R}^{v}$ ) is a solution of problem (1.1), (1.2). It is obvious that ( $\lambda_{k, \sigma, R}^{v}, u_{k, \sigma, R}^{v}$ ) has the following properties

$$
\lambda_{k, \sigma, R}^{v} \in \mathcal{I}_{k}^{\sigma}\left(\tau_{0}\right), \quad\left\|u_{k, \sigma, R}^{v}\right\|_{1}=R, \quad \text { and } \quad u_{k, \sigma, R}^{v} \in \overline{\mathcal{S}_{k, \sigma}^{v}}=\mathcal{S}_{k, \sigma}^{v} \cup \partial \mathcal{S}_{k, \sigma}^{v} .
$$

If $u_{k, \sigma, R}^{v} \in \partial \mathcal{S}_{k, \sigma^{\prime}}^{v}$ then either
(i) $\int_{0}^{1} \rho(x)\left(u_{k, \sigma, R}^{v}(x)\right)^{2} d x=0$, or
(ii) there exists $x_{0} \in[0,1]$ that such $u_{k, \sigma, R}^{v}\left(x_{0}\right)=\left(u_{k, \sigma, R}^{v}\right)^{\prime}\left(x_{0}\right)=0$.

Since $\left\|u_{k, \sigma, R}^{v}\right\|_{1}=R$ it follows from Remark 2.2 that $\int_{0}^{1} \rho(x)\left(u_{k, \sigma, R}^{v}(x)\right)^{2} d x \neq 0$.
Now let there exists $x_{0} \in[0,1]$ such that $u_{k, \sigma, R}^{v}\left(x_{0}\right)=\left(u_{k, \sigma, R}^{v}\right)^{\prime}\left(x_{0}\right)=0$. By (1.1), (1.2) we have

$$
\begin{align*}
\ell\left(u_{k, \sigma, R}^{v}\right)=\lambda_{k, \sigma, R}^{v} \rho(x) u_{k, \sigma, R}^{v} & +f\left(x, u_{k, \sigma, R}^{v}\left(u_{k, \sigma, R}^{v}\right)^{\prime}, \lambda_{k, \sigma, R}^{v}\right) \\
& +g\left(x, u_{k, \sigma, R}^{v}\left(u_{k, \sigma, R}^{v}\right)^{\prime}, \lambda_{k, \sigma, R}^{v}\right), \quad x \in(0,1), \quad u_{k, \sigma, R}^{v} \in(b . c .) . \tag{3.9}
\end{align*}
$$

Dividing both sides of (3.9) by $\left\|u_{k, \sigma, R}^{v}\right\|_{1}$ and setting $v_{k, \sigma, R}^{v}=\frac{u_{k, \sigma, R}^{\nu}}{\left\|u_{k, \sigma, R}^{k}\right\|_{1}}$ we get

$$
\begin{align*}
& \ell\left(v_{k, \sigma, R}^{v}\right)=\lambda_{k, \sigma, R}^{v} \rho(x) v_{k, \sigma, R}^{v}+\frac{f\left(x, u_{k, \sigma, R}^{v}\left(u_{k, \sigma, R}^{v}\right)^{\prime}, \lambda_{k, \sigma, R}^{v}\right)}{\left\|u_{k, \sigma, R}^{v}\right\|_{1}} \\
&+\frac{g\left(x, u_{k, \sigma, R}^{v},\left(u_{k, \sigma, R}^{v}\right)^{\prime}, \lambda_{k, \sigma, R}^{v}\right)}{\left\|u_{k, \sigma, R}^{v}\right\|_{1}}, \quad x \in(0,1) . \tag{3.10}
\end{align*}
$$

In view of (1.4) we have

$$
\begin{equation*}
\left|\frac{f\left(x, u_{k, \sigma, R}^{v}\left(u_{k, \sigma, R}^{v}\right)^{\prime}, \lambda_{k, \sigma, R}^{v}\right)}{\left\|u_{k, \sigma, R}^{v}\right\|_{1}}\right| \leq M\left|v_{k, \sigma, R}^{v}\right| . \tag{3.11}
\end{equation*}
$$

By virtue of condition (1.5) for any sufficiently small fixed $\epsilon>0$ there exists a sufficiently large $\delta_{\epsilon}>0$ such that

$$
\begin{equation*}
|g(x, u, s, \lambda)|<\varepsilon(|u|+|s|) / 2 \quad \text { for any }(x, u, s, \lambda) \in[0,1] \times \mathbb{R}^{2} \times \Lambda,|u|+|s|>\delta_{\epsilon}, \tag{3.12}
\end{equation*}
$$

where $\Lambda \in \mathbb{R}^{\sigma}$ is a bounded interval. In other hand, since $g \in C\left([0,1] \times \mathbb{R}^{3} ; \mathbb{R}\right)$ it follows that there exists a positive number $K_{\epsilon}$ such that

$$
\begin{equation*}
|g(x, u, s, \lambda)| \leq K_{\epsilon} \quad \text { for any }(x, u, s, \lambda) \in[0,1] \times \mathbb{R}^{2} \times \Lambda,|u|+|s| \leq \delta_{\epsilon} . \tag{3.13}
\end{equation*}
$$

We choose $R$ large enough to satisfy the relations

$$
R>\delta_{\epsilon} \quad \text { and } \quad K_{\epsilon}<R \epsilon / 2
$$

Then by (3.12) and (3.13) we have

$$
\begin{align*}
&\left|\frac{g\left(x, u_{k, \sigma, R}^{v}(x)\left(u_{k, \sigma, R}^{v}\right)^{\prime}(x), \lambda_{k, \sigma, R}^{v}\right)}{\left\|u_{k, \sigma, R}^{v}\right\|_{1}}\right| \\
& \quad= \frac{1}{R}\left|g\left(x, u_{k, \sigma, R}^{v}(x)\left(u_{k, \sigma, R}^{v}\right)^{\prime}(x), \lambda_{k, \sigma, R}^{v}\right)\right| \\
& \leq \frac{1}{R}\left\{\begin{array}{l}
\left\{x \in[0,1]:\left|u_{k, \sigma, R}^{v}(x)\right|+\left|\left(u_{k, \sigma, R}^{v}\right)^{\prime}(x)\right| \leq \delta_{\epsilon}\right\}
\end{array}\left|g\left(x, u_{k, \sigma, R}^{v}(x),\left(u_{k, \sigma, R}^{v}\right)^{\prime}(x), \lambda_{k, \sigma, R}^{v}\right)\right|\right.  \tag{3.14}\\
&\left.+\max _{\left\{x \in[0,1]:\left|u_{k, \sigma, R}^{v}(x)\right|+\left|\left(u_{k, \sigma, R}^{v}\right)^{\prime}(x)\right|>\delta_{\epsilon}\right\}}\left|g\left(x, u_{k, \sigma, R}^{v}(x),\left(u_{k, \sigma, R}^{v}\right)^{\prime}(x), \lambda_{k, \sigma, R}^{v}\right)\right|\right\} \\
& \leq \frac{1}{R}\left\{K_{\epsilon}+\epsilon R / 2\right\}=\frac{K_{\epsilon}}{R}+\epsilon / 2<\epsilon / 2+\epsilon / 2=\epsilon .
\end{align*}
$$

Taking into account (3.11) and (3.14), from (3.10) we obtain

$$
\begin{aligned}
p_{0}\left|\left(v_{k, \sigma, R}^{v}\right)^{\prime \prime}(x)\right| & \leq\left|p(x)\left(v_{k, \sigma, R}^{v}\right)^{\prime \prime}(x)\right| \\
& \leq\left(\left|\lambda_{k, \sigma, R}^{v}\right||\rho(x)|+|q(x)|+M\right)\left|v_{k, \sigma, R}^{v}(x)\right|+\left|p^{\prime}(x)\right|\left|\left(v_{k, \sigma, R}^{v}\right)^{\prime}(x)\right|+\epsilon
\end{aligned}
$$

which implies that

$$
\begin{equation*}
\left|\left(v_{k, \sigma, R}^{v}\right)^{\prime \prime}(x)\right| \leq c_{0}\left(\left|v_{k, \sigma, R}^{v}(x)\right|+\left|\left(v_{k, \sigma, R}^{v}\right)^{\prime}(x)\right|\right)+\frac{\epsilon}{p_{0}}, \tag{3.15}
\end{equation*}
$$

where

$$
c_{0}=\frac{1}{p_{0}} \max _{x \in[0,1]}\left\{\max \left\{\lambda_{k, M^{\prime}}^{+}\left|\lambda_{k, M}^{-}\right|\right\}|\rho(x)|+|q(x)|+M,\left|p^{\prime}(x)\right|\right\}, \quad p_{0}=\min _{x \in[0,1]} p(x) .
$$

Let $w_{k, \sigma, R}^{v}=\binom{v_{k, \sigma, R}^{v}}{\left(v_{k, \sigma, R}^{v}\right)^{\prime}} \in \mathbb{R}^{2}$ with the norm that given by

$$
\left|w_{k, \sigma, R}^{v}\right|_{2}=\left|v_{k, \sigma, R}^{v}\right|+\left|\left(v_{k, \sigma, R}^{v}\right)^{\prime}\right| .
$$

Then it follows from (3.15) that

$$
\left|\left(w_{k, \sigma, R}^{v}\right)^{\prime}(x)\right|_{2} \leq c_{1}\left|w_{k, \sigma, R}^{v}(x)\right|_{2}+\frac{\epsilon}{p_{0}}
$$

where $c_{1}=c_{0}+1$. Integrating the last relation in the range from $x_{0}$ to $x$ we have

$$
\begin{equation*}
\left.\left|\int_{x_{0}}^{x}\right|\left(w_{k, \sigma, R}^{v}\right)^{\prime}(t)\right|_{2} d t\left|\leq c_{1}\right| \int_{x_{0}}^{x}\left|w_{k, \sigma, R}^{v}(t)\right|_{2} d t \left\lvert\,+\frac{\epsilon}{p_{0}} .\right. \tag{3.16}
\end{equation*}
$$

Using the relation $v_{k, \sigma, R}^{v}\left(x_{0}\right)=\left(v_{k, \sigma, R}^{v}\right)^{\prime}\left(x_{0}\right)=0$ and inequality (3.16) we get

$$
\left|w_{k, \sigma, R}^{v}(x)\right|_{2}=\left.\left|\int_{x_{0}}^{x}\right|\left(w_{k, \sigma, R}^{v}\right)^{\prime}(t)\right|_{2} d t\left|\leq c_{1}\right| \int_{x_{0}}^{x}\left|w_{k, \sigma, R}^{v}(t)\right|_{2} d t \left\lvert\,+\frac{\epsilon}{p_{0}}\right.,
$$

whence, with regard the Gronwall's inequality, we get

$$
\begin{equation*}
\left|w_{k, \sigma, R}^{\nu}(x)\right|_{2} \leq \frac{\epsilon}{p_{0}} e^{c_{1}\left|x-x_{0}\right|} \leq \frac{\epsilon}{p_{0}} e^{c_{1}}<1, \quad x \in[0,1], \tag{3.17}
\end{equation*}
$$

(in advance we could choose $\epsilon$ so small enough that the inequality $\epsilon<\frac{p_{0}}{e^{c_{1}}}$ holds). Then it follows from (3.17) that $\left\|v_{k, \sigma, R}^{v}\right\|_{1}<1$ which contradicts the condition $\left\|v_{k, \sigma, R}^{v}\right\|_{1}=1$. Therefore, we have $u_{k, \sigma, R}^{v} \in \mathcal{S}_{k, \sigma}^{v}$.

Corollary 3.2. For each $k \in \mathbb{N}$, each $\sigma$ and each $v$ there exists a sufficiently large positive number $R_{k, \sigma}^{v}$ such that for any $R \geq R_{k, \sigma}^{v}$ problem (1.1), (1.2) has a solution $(\lambda, u)$ which satisfies the following properties:

$$
\lambda \in \mathcal{I}_{k}^{\sigma}\left(\tau_{0}\right), \quad u \in \mathcal{S}_{k, \sigma}^{v} \quad \text { and } \quad\|u\|_{1}=R .
$$

Recall that $(\lambda, \infty), \lambda \in \mathbb{R}^{\sigma}$, is an asymptotic bifurcation point of problem (1.1), (1.2) with respect to the set $\mathbb{R}^{\sigma} \times \mathcal{S}_{k, \sigma}^{v}, k \in \mathbb{N}$, if for any sufficiently small $r>0$ there exists a solution $\left(\lambda_{k, \sigma, r}^{v} u_{k, \sigma, r}^{v}\right) \in \mathbb{R}^{\sigma} \times E$ such that

$$
\left|\lambda_{k, \sigma, r}^{v}-\lambda\right|<r, \quad\left\|u_{k, \sigma, r}^{v}\right\|_{1}>r^{-1} \quad \text { and } \quad u_{k, \sigma, r}^{v} \in \mathcal{S}_{k, \sigma}^{v} .
$$

Remark 3.3. We add the points $\{(\lambda, \infty): \lambda \in \mathbb{R}\}$ to the space $\mathbb{R} \times E$ and define an appropriate topology on the resulting set.

For each $k \in \mathbb{N}$, each $\sigma$ and each $v$ by $\mathfrak{B}_{k, \sigma}^{v}, k \in \mathbb{N}$, we denote the set of asymptotic bifurcation of (1.1), (1.2) with respect to $\mathbb{R}^{\sigma} \times \mathcal{S}_{k, \sigma}^{v}$.

The following result immediately follows from Lemma 3.1 and Corollary 3.2.
Corollary 3.4. For each $k \in \mathbb{N}$, each $\sigma$ and each $v$ the set $\mathfrak{B}_{k, \sigma}^{v}$ is nonempty. Furthermore, $\mathfrak{B}_{k, \sigma}^{v} \subset$ $\mathcal{I}_{k}^{\sigma} \times\{\infty\}$.

## 4 Structures of global continua emanating from the asymptotic bifurcation points of problem (1.1), (1.2)

Let conditions (1.3)-(1.5) hold.
For each $k \in \mathbb{N}$, each $\sigma$ and each $v$ we define the set $\mathfrak{D}_{k, \sigma}^{v, *}$ as the union of all the components of $\mathcal{D}$ bifurcating from $\mathcal{I}_{k}^{\sigma} \times\{\infty\}$ with respect to the set $\mathbb{R} \times \mathcal{S}_{k, \sigma}^{v}$. It follows from Corollary 3.4 that the set $\mathfrak{D}_{k, \sigma}^{\nu, *}$ is nonempty. This set may not be connected in $\mathbb{R} \times E$, but the set $\mathfrak{D}_{k, \sigma}^{\nu}=$ $\mathfrak{D}_{k, \sigma}^{\nu, *} \cup\left(\mathcal{I}_{k}^{\sigma} \times\{\infty\}\right)$ will be connected in this space (see Remark 3.3).

The main result of this paper is the following theorem.
Theorem 4.1. For each $k \in \mathbb{N}$, each $\sigma$ and each $v$ the set $\mathfrak{D}_{k, \sigma}^{\nu}$ is contained in $\mathbb{R}^{\sigma} \times E$ and for this set at least one of the following statements holds:
(i) there exists $\left(k^{\prime}, v^{\prime}\right) \neq(k, v)$ such that $\mathfrak{D}_{k, \sigma}^{v}$ meets $\mathcal{I}_{k^{\prime}}^{\sigma} \times\{\infty\}$ with respect to the set $\mathbb{R}^{\sigma} \times \mathcal{S}_{k^{\prime}, \sigma^{\prime}}^{v^{\prime}}$;
(ii) there exists $\lambda \in \mathbb{R}^{\sigma}$ such that $\mathfrak{D}_{k, \sigma}^{v}$ meets $\mathcal{R}_{0}^{\sigma}$ at the point $(\lambda, 0)$;
(iii) the projection $\mathcal{P}_{\mathcal{R}_{0}^{\sigma}}\left(\mathfrak{D}_{k, \sigma}^{v}\right)$ of $\mathfrak{D}_{k, \sigma}^{v}$ on $\mathcal{R}_{0}^{\sigma}$ is unbounded.

Proof. Let $(\lambda, u) \in \mathcal{D}$ and $v=\frac{u}{\|u\|_{1}^{2}}$. Then we have $\|v\|_{1}=\frac{1}{\|u\|_{1}}$ and $u=\frac{v}{\|v\|_{1}^{2}}$.
Dividing both sides of (1.1), (1.2) by $\|u\|_{1}^{2}$ we obtain

$$
\left\{\begin{array}{l}
(\ell(v))(x)=\lambda \rho(x) v(x)+\frac{f\left(x, u(x), u^{\prime}(x), \lambda\right)}{\|u\|_{1}^{2}}+\frac{g\left(x, u(x), u^{\prime}(x), \lambda\right)}{\|u\|_{1}^{2}}, \quad x \in(0,1)  \tag{4.1}\\
v \in(\text { b.c. })
\end{array}\right.
$$

We set

$$
\hat{f}\left(x, v(x), v^{\prime}(x), \lambda\right)= \begin{cases}\|v\|_{1}^{2} f\left(x, \frac{v(x)}{\|v\|_{1}^{2}} \frac{v^{\prime}(x)}{\|v\|_{1}^{2}}, \lambda\right) & \text { if } v(x) \neq 0  \tag{4.2}\\ 0 & \text { if } v(x)=0\end{cases}
$$

and

$$
\hat{g}\left(x, v(x), v^{\prime}(x), \lambda\right)= \begin{cases}\|v\|_{1}^{2} g\left(x, \frac{v(x)}{\|v\|_{1}^{2}} \frac{v^{\prime}(x)}{\|v\|_{1}^{\|_{1}}}, \lambda\right) & \text { if } v(x) \neq 0,  \tag{4.3}\\ 0 & \text { if } v(x)=0 .\end{cases}
$$

By conditions (1.4) for any $x \in[0,1]$ with $v(x) \neq 0$, and $\lambda \in \mathbb{R}$ the following estimates hold:

$$
\begin{align*}
\frac{\left|\hat{f}\left(x, v(x), v^{\prime}(x), \lambda\right)\right|}{|v(x)|} & =\frac{\|v\|_{1}^{2}\left|f\left(x, \frac{v(x)}{\|v\|_{1}^{2}}, \frac{v^{\prime}(x)}{\|v\|_{1}^{2}}, \lambda\right)\right|}{|v(x)|}=\frac{\left|f\left(x, \frac{v(x)}{\|v\|_{1}^{2}} \frac{v^{\prime}(x)}{\|v\|_{1}^{2}}, \lambda\right)\right|}{\left|\frac{v(x)}{\|v\|_{1}^{2}}\right|}  \tag{4.4}\\
& =\frac{\left|f\left(x, u(x), u^{\prime}(x), \lambda\right)\right|}{|u(x)|} \leq M .
\end{align*}
$$

We choose $\delta_{\epsilon, 1}>\delta_{\epsilon}$ so that (see Section 3)

$$
\frac{K_{\epsilon}}{\delta_{\epsilon, 1}}<\frac{\epsilon}{2}
$$

Let $(\lambda, u) \in \mathbb{R} \times E$ such that $\lambda \in \Lambda$ and $\|u\|_{1}>\delta_{\epsilon, 1}$, where $\Lambda \subset \mathbb{R}$ is any fixed bounded interval. Then for any $x \in[0,1]$ we have

$$
\begin{align*}
\mid g(x, & \left.u(x) u^{\prime}(x), \lambda\right) \mid \\
\leq & \left\{\max _{\left\{x \in[0,1]:|u(x)|+\left|u^{\prime}(x)\right| \leq \delta_{\epsilon}\right\}}\left|g\left(x, u(x), u^{\prime}(x), \lambda\right)\right|\right. \\
& \left.+\max _{\left\{x \in[0,1]:|u(x)|+\left|u^{\prime}(x)\right|>\delta_{\varepsilon}\right\}}\left|g\left(x, u(x), u^{\prime}(x), \lambda\right)\right|\right\}  \tag{4.5}\\
\leq & K_{\epsilon}+\frac{\epsilon}{2}\left\{|u(x)|+\left|u^{\prime}(x)\right|\right\} \leq \frac{\epsilon}{2} \delta_{\epsilon, 1}+\frac{\epsilon}{2}\|u\|_{1} \leq \frac{\epsilon}{2}\|u\|_{1}+\frac{\epsilon}{2}\|u\|_{1}=\epsilon\|u\|_{1} .
\end{align*}
$$

By (4.5) for any $x \in[0,1], v \in E$ with $0<\|v\|_{1}<\frac{1}{\delta_{\varepsilon, 1}}$, and $\lambda \in \Lambda$ we get

$$
\begin{aligned}
\left|\hat{g}\left(x, v(x), v^{\prime}(x), \lambda\right)\right| & =\|v\|_{1}^{2}\left|g\left(x, \frac{v(x)}{\|v\|_{1}^{2}}, \frac{v^{\prime}(x)}{\|v\|_{1}^{2}}, \lambda\right)\right|=\frac{\|v\|_{1}}{\|u\|_{1}}\left|g\left(x, \frac{v(x)}{\|v\|_{1}^{2}} \frac{v^{\prime}(x)}{\|v\|_{1}^{2}}, \lambda\right)\right| \\
& =\|v\|_{1} \cdot \frac{1}{\|u\|_{1}}\left|g\left(x, u(x), u^{\prime}(x), \lambda\right)\right| \leq \epsilon\|v\|_{1},
\end{aligned}
$$

which shows that

$$
\begin{equation*}
\left|\hat{g}\left(x, v(x), v^{\prime}(x), \lambda\right)\right|_{\infty}=o\left(\|v\|_{1}\right) \quad \text { as }\|v\|_{1} \rightarrow 0, \tag{4.6}
\end{equation*}
$$

uniformly for $(x, \lambda) \in[0,1] \times \Lambda$.
By (4.2) and (4.3) it follows from (4.1) that ( $\lambda, v$ ) solves the nonlinear eigenvalue problem

$$
\left\{\begin{array}{l}
(\ell(v))(x)=\lambda \rho(x) v(x)+\hat{f}\left(x, v(x), v^{\prime}(x), \lambda\right)+\hat{g}\left(x, v(x), v^{\prime}(x), \lambda\right), \quad x \in(0,1),  \tag{4.7}\\
v \in(\text { b.c. }) .
\end{array}\right.
$$

Conditions (4.4) and (4.6) show that the inversion

$$
\begin{equation*}
(\lambda, u) \rightarrow\left(\lambda, \frac{u}{\|u\|_{1}^{2}}\right)=(\lambda, v) \tag{4.8}
\end{equation*}
$$

transforms "bifurcation at infinity" problem (1.1), (1.2) into "bifurcation from zero" problem (4.7).

It follows from Lemma 2.4 that the set of bifurcation points of problem (4.7) with respect to the set $\mathbb{R}^{\sigma} \times \mathcal{S}_{k, \sigma}^{v}$ is nonempty; moreover, if $(\lambda, 0)$ is a bifurcation point of this problem with respect to $\mathbb{R}^{\sigma} \times \mathcal{S}_{k, \sigma}^{\nu}$, then $\lambda \in \mathcal{I}_{k}^{\sigma}$.

Let $\hat{\mathcal{D}}$ be the set of nontrivial solutions of problem (4.7). It is obvious that inversion (4.8) transforms $\mathcal{D}$ into $\hat{\mathcal{D}}$. By $\hat{\mathcal{D}}_{k, \sigma}^{\nu, *}$ we denote the union of all the components of $\hat{\mathcal{D}}$ that meet $\mathcal{I}_{k}^{\sigma} \times\{0\}$ by the set $\mathbb{R}^{\sigma} \times \mathcal{S}_{k, \sigma}^{v}$, and let $\hat{\mathcal{D}}_{k, \sigma}^{v}=\hat{\mathcal{D}}_{k, \sigma}^{v, *} \cup\left(\mathcal{I}_{k}^{\sigma} \times\{0\}\right)$. Then the set $\mathfrak{D}_{k, \sigma}^{v}$ is the inverse image of the set $\hat{\mathcal{D}}_{k, \sigma}^{\nu}$ under inversion (4.8).

Let $\mathcal{Q}_{k, \sigma}^{v}$ be some neighborhood of the set $\mathcal{I}_{k}^{\sigma}\left(\tau_{0}\right) \times\left(E \backslash B_{R_{k, \sigma}^{v}}\right)$, where $B_{R_{k, \sigma}^{\nu}}$ is a ball in $\mathbb{R} \times E$ with center 0 and radius $R_{k, \sigma}^{v}$. Note that inversion (4.8) transforms the set $\mathcal{I}_{k}^{\sigma}\left(\tau_{0}\right) \times\left(E \backslash B_{R_{k, \sigma}}\right)$ into the set $\mathcal{I}_{k}^{\sigma}\left(\tau_{0}\right) \times B_{\frac{1}{R_{k, \sigma}, \sigma}}$, and the set $\mathcal{Q}_{k, \sigma}^{v}$ into the set $\mathcal{T}_{k, \sigma}^{v}$ which is some neighborhood of $\mathcal{I}_{k}^{\sigma}\left(\tau_{0}\right) \times B_{\frac{1}{R_{k, \sigma}^{\nu}}}$. Then it follows from Corollary 3.2 that $\left(\hat{\mathcal{D}}_{k, \sigma}^{v} \cap\left(\mathcal{I}_{k}^{\sigma}\left(\tau_{0}\right) \times B_{\frac{1}{R_{k, \sigma}^{\nu}}}\right)\right) \subset\left(\mathbb{R}^{\sigma} \times\right.$ $\left.\mathcal{S}_{k, \sigma}^{\nu}\right) \cup\left(\mathcal{I}_{k}^{\sigma}\left(\tau_{0}\right) \times\{0\}\right)$.

Remark 4.2. If $(\lambda, u) \in(\mathbb{R} \times E) \backslash\left(\bigcup_{k \in \mathbb{N}, \sigma, v} \mathcal{Q}_{k, \sigma}^{v}\right)$ is a solution of (1.1), (1.2) such that $u \in \partial \mathcal{S}_{k, \sigma}^{v}$, then it is seen from the proof of Lemma 3.1 that, in contrast to Lemma 2.3, the relation $u \equiv 0$ may not hold. Consequently, if $(\lambda, u) \in \mathbb{R}^{\sigma} \times E$ is a solution of (1.1), (1.2) outside $\bigcup_{k \in \mathbb{N}, \sigma, v} \mathcal{Q}_{k, \sigma}^{v}$ such that $u \in \partial \mathcal{S}_{k, \sigma}^{v}$, then the relation $u \equiv 0$ may not hold.

Due to Remark 4.2, it need not be the case that $\left(\hat{\mathcal{D}}_{k, \sigma}^{\nu} \backslash \mathcal{T}_{k, \sigma}^{\nu}\right) \subset \mathbb{R}^{\sigma} \times \mathcal{S}_{k, \sigma}^{\nu}$ (see for example [24, Remark 2.12]). Hence the assertions of Theorem 2.6 will not hold for $\hat{D}_{k, \sigma}^{v}$. However, using Remark 2.2, the above arguments and the techniques of [6,25], and combining it with the global results from [13] and [23], we can show that for each $k \in \mathbb{N}$, each $\sigma$ and each $v$ the set $\hat{\mathcal{D}}_{k, \sigma}^{v}$ lies in $\mathbb{R}^{\sigma} \times E$ and for this set at least one of the following hold:
(a) there exists $\left(k^{\prime}, v^{\prime}\right) \neq(k, v)$ such that $\hat{\mathcal{D}}_{k, \sigma}^{v}$ meets $\mathcal{I}_{k^{\prime}}^{\sigma} \times\{0\}$ with respect to $\mathbb{R}^{\sigma} \times \mathcal{S}_{k^{\prime}, \sigma}^{v^{\prime}}$;
(b) $\hat{\mathcal{D}}_{k, \sigma}^{\nu}$ is unbounded in $\mathbb{R} \times E$.

Now alternative (i) of Theorem 4.1 for $\mathfrak{D}_{k, \sigma}^{v}$ is obtained from the alternative (a) for $\hat{\mathcal{D}}_{k, \sigma}^{v}$ by the inversion (4.8). The alternatives (ii) and (iii) of this theorem for $\mathfrak{D}_{k, \sigma}^{v}$ correspond, via the inversion (4.8), to the various ways of the alternative (b) in which $\hat{\mathcal{D}}_{k, \sigma}^{v}$ can be unbounded.

Now suppose that along with the constant $M>0$ there exists a sufficiently large constant $\chi>0$ such that

$$
\begin{equation*}
\left|\frac{f(x, u, s, \lambda)}{u}\right| \leq M, \quad(x, u, s, \lambda) \in[0,1] \times \mathbb{R}^{3}, u \neq 0,|u|+|s| \geq \chi . \tag{4.9}
\end{equation*}
$$

Then the following result holds.
Lemma 4.3. Let conditions (1.3), (1.5) and (4.9) hold. Then there are functions $f_{1}, g_{1} \in C([0,1] \times$ $\left.\mathbb{R}^{3} ; \mathbb{R}\right)$ such that $f_{1}$ satisfies condition (1.4) for any $(x, u, s, \lambda) \in[0,1] \times \mathbb{R}^{3}$ with $u \neq 0$ and $g_{1}$ satisfies the conditions (1.5) uniformly for $(x, \lambda) \in[0,1] \times \Lambda$, and the function $h$ can also have a representation $h=f_{1}+g_{1}$.

The proof of this lemma is similar to that of [6, Lemma 5.1].
Remark 4.4. Let conditions (1.3), (1.5) and (4.9) hold. Then, in this case, for problem (1.1), (1.2), Theorem 4.1 again holds.

## 5 Global bifurcation of solutions of problem (1.1), (1.2) under both conditions (1.5) and (1.6)

In the case when both conditions (1.5) and (1.6) are satisfied, Theorems 2.6 and 4.1 can be improved as follows.

Theorem 5.1. Let both conditions (1.5) and (1.6) be satisfied. Then for each $k \in \mathbb{N}$, each $\sigma$ and each $v,\left(\mathfrak{D}_{k, \sigma}^{\nu} \backslash\left(\mathcal{I}_{k}^{\sigma} \times\{\infty\}\right)\right) \subset \mathbb{R}^{\sigma} \times \mathcal{S}_{k, \sigma^{\prime}}^{v}$ and alternative (i) of Theorem 4.1 cannot hold. Moreover, if $\mathfrak{D}_{k, \sigma}^{v}$ meets $\mathcal{R}_{0}^{\sigma}$ for some $\lambda \in \mathbb{R}^{\sigma}$, then $\lambda \in \mathcal{I}_{k}^{\sigma}$. Similarly, if $\mathcal{D}_{k, \sigma}^{v}$ meets $\mathcal{R}_{\infty}^{\sigma}$ for some $\lambda \in \mathbb{R}^{\sigma}$, then $\lambda \in \mathcal{I}_{k}^{\sigma}$.

Proof. If (1.6) holds, then by Lemma 2.3 we have

$$
\mathcal{D} \cap\left(\mathbb{R}^{\sigma} \times \partial \mathcal{S}_{k, \sigma}^{v}\right)=\varnothing,
$$

whence implies that the sets

$$
\mathcal{D} \cap\left(\mathbb{R}^{\sigma} \times \mathcal{S}_{k, \sigma}^{v}\right) \quad \text { and } \quad \mathcal{D} \backslash\left(\mathbb{R}^{\sigma} \times \mathcal{S}_{k, \sigma}^{v}\right)
$$

are mutually separated in $\mathbb{R} \times E$. Hence, in view of [29, Corollary 26.6], every component of $\mathcal{D}$ must be a subset of $\mathcal{D} \cap\left(\mathbb{R}^{\sigma} \times \mathcal{S}_{k, \sigma}^{v}\right)$ or $\mathcal{D} \backslash\left(\mathbb{R}^{\sigma} \times \mathcal{S}_{k, \sigma}^{v}\right)$. Recall that $\mathfrak{D}_{k, \sigma}^{v, *}=\mathfrak{D}_{k, \sigma}^{v} \backslash\left(\mathcal{I}_{k}^{\sigma} \times\{\infty\}\right)$ is the union of all components of the set $\mathcal{D}$ which intersect the set $\mathbb{R}^{\sigma} \times \mathcal{S}_{k, \sigma}^{v}$. Therefore, each of these components must be contained in $\mathbb{R}^{\sigma} \times \mathcal{S}_{k, \sigma}^{v}$, and consequently, $\mathfrak{D}_{k, \sigma}^{\nu, *} \subset \mathbb{R}^{\sigma} \times S_{k, \sigma}^{v}$. Then, by virtue of Theorem 4.1, its alternative (i) will not hold.

Now let $\mathfrak{D}_{k, \sigma}^{v}$ meets $\mathcal{R}_{0}^{\sigma}$ for some $\lambda \in \mathbb{R}^{\sigma}$. Then it follows from Lemma 2.4 that $\lambda \in \mathcal{I}_{k}^{\sigma}$. Similarly, if $\mathcal{D}_{k, \sigma}^{v}$ meets $\mathcal{R}_{\infty}^{\sigma}$ for some $\lambda \in \mathbb{R}^{\sigma}$, then Corollary 3.4 implies that $\lambda \in \mathcal{I}_{k}^{\sigma}$.

## 6 Bifurcation of problem (1.1), (1.2) in the case $\rho(x) \geq 0$

In this section we consider problem (1.1), (1.2) in the case when weight function $\rho(x) \geq 0$ on $[0,1]$ and $\rho(x) \not \equiv 0$ on any subinterval of $[0,1]$. Then it follows from [15, Ch. $10, \S 10 \cdot 6$ and
10.61] that the spectrum of the linear spectral problem consists of one sequence of positive and simple eigenvalues

$$
\lambda_{1}<\lambda_{2}<\cdots<\lambda_{k} \mapsto+\infty ;
$$

for each $k \in \mathbb{N}$ the eigenfunctions $u_{k}$ corresponding to the eigenvalues $\lambda_{k}$ has exactly $k-1$ simple nodal zeros in ( 0,1 ). Moreover, it follows from (2.3) with $f \equiv 0$ and $g \equiv 0$ that for each $k \in \mathbb{N}$ the eigenfunction $u_{k}(x)$ satisfies the condition

$$
\int_{0}^{1} \rho(x) u_{k}^{2}(x) d x>0
$$

By following the arguments in Sections 2-4 in this case we can justify the following results.
Theorem 6.1. Let the condition (1.6) holds. Then for each $k \in \mathbb{N}$ and each $v$ there exists a connected component $\mathcal{D}_{k}^{v}$ of the closure of the set of nontrivial solutions of (1.1), (1.2) which contains $\mathcal{I}_{k} \times\{0\}$ lies in $\left(\mathbb{R}^{+} \times \mathcal{S}_{k,+}^{v}\right) \cup\left(\mathcal{I}_{k} \times\{0\}\right)$ and is unbounded in $\mathbb{R} \times E$, where $\mathcal{I}_{k}=\left[\lambda_{k}, \lambda_{k, M}\right], k \in \mathbb{N}$, and $\lambda_{k, M}$ is the $k$ th eigenvalue of problem (2.4).

Theorem 6.2. Let the condition (1.5) holds. Then for each $k \in \mathbb{N}$ and each $v$ there exists a connected component $\mathfrak{D}_{k}^{v}$ of the closure of the set of nontrivial solutions of (1.1), (1.2) which contains $I_{k} \times\{\infty\}$ is contained in $\mathbb{R}^{+} \times E$ and for this set at least one of the following statements holds:
(i) there exists $\left(k^{\prime}, v^{\prime}\right) \neq(k, v)$ such that $\mathfrak{D}_{k}^{v}$ meets $\mathcal{I}_{k^{\prime}} \times\{\infty\}$ with respect to the set $\mathbb{R} \times \mathcal{S}_{k^{\prime},+}^{\nu^{\prime}}$;
(ii) there exists $\lambda \in \mathbb{R}^{+}$such that $\mathfrak{D}_{k}^{v}$ meets $\mathcal{R}_{0}^{+}$at the point $(\lambda, 0)$;
(iii) the projection $\mathcal{P}_{\mathcal{R}_{0}^{+}}\left(\mathfrak{D}_{k}^{v}\right)$ of $\mathfrak{D}_{k}^{\nu}$ on $\mathcal{R}_{0}^{+}$is unbounded.

Theorem 6.3. Let both conditions (1.5) and (1.6) be satisfied. Then for each $k \in \mathbb{N}$ and each $v$, $\left(\mathfrak{D}_{k}^{v} \backslash\left(\mathcal{I}_{k} \times\{\infty\}\right)\right) \subset \mathbb{R}^{+} \times \mathcal{S}_{k,+}^{v}$, and alternative (i) of Theorem 6.2 cannot hold. Moreover, if $\mathfrak{D}_{k}^{v}$ meets $\mathcal{R}_{0}^{+}$for some $\lambda \in \mathbb{R}^{+}$, then $\lambda \in \mathcal{I}_{k}$. Similarly, if $\mathcal{D}_{k}^{v}$ meets $\mathcal{R}_{\infty}^{+}$for some $\lambda \in \mathbb{R}^{+}$, then $\lambda \in \mathcal{I}_{k}$.

## Acknowledgements

The authors are grateful to the referee for valuable comments that contributed to a significant improvement in the text of the article.

## References

[1] Z. S. Aliev, On the global bifurcation of solutions of some nonlinear eigenvalue problems for ordinary differential equations of fourth order, Sb. Math. 207(2016), No. 12, 3-29. https://doi.org/10.1070/SM8369; MR3588983; Zbl 1371.34038
[2] Z. S. Aliyev, Sh. M. Hasanova, Global bifurcation of positive solutions of semi-linear elliptic partial differential equations with indefinite weight, Z. Anal. Anwend. 38(2019), No. 1, 1-15. https://doi.org/10.4171/ZAA/1625; Zbl 1412.35087
[3] Z. S. Aliyev, Sh. M. Hasanova, Global bifurcation of positive solutions from zero in nonlinearizable elliptic problems with indefinite weight, J. Math. Anal. Appl. 491(2020), No. 1, 1-11. https://doi.org/10.1016/j.jmaa.2020.124252; Zbl 1451.35021
[4] Z. S. Aliyev, R.A. Huseynova, Bifurcation in nonlinearizable eigenvalue problems for ordinary differential equations of fourth order with indefinite weight, Electron. J. Qual. Theory Differ. Equ. 2017, No. 92, 1-12. https://doi.org/10.14232/ejqtde.2017.1.92; Zbl 1413.34144
[5] Z. S. Aliyev, R.A. Huseynova, Global bifurcation from infinity in some nonlinearizable eigenvalue problems with indefinite weight, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. 44(2018), No. 1, 123-134. Zbl 1402.34021
[6] Z. S. Aliyev, N. A. Mustafayeva, Bifurcation of solutions from infinity for certain nonlinear eigenvalue problems of fourth-order ordinary differential equations, Electron. J. Differential Equations 2018, No. 98, 1-19. MR3831844; Zbl 06866720
[7] Z. S. Aliyev, L. V. Ashurova (Nasirova), Bifurcation of positive and negative solutions of nonlinearizable Sturm-Liouville problems with indefinite weight, Miskolc Math. Notes 21(2020), No. 1, 19-29. https://doi.org/10.18514/MMN.2020.2876; Zbl 07254879
[8] H. Berestycki, On some nonlinear Sturm-Liouville problems, J. Diffential Equations 26(1977), No. 3, 375-390. https://doi.org/10.1016/0022-0396(77) 90086-9; MR481230; Zbl 0331.34020
[9] K. J. Brown, Local and global bifurcation results for a semilinear boundary value problem, J. Differential Equations 239(2007), No. 2, 296-310. https://doi.org/doi.org/10. 1016/j.jde.2007.05.013; MR2344274; Zbl 1331.35129
[10] R. S. Cantrell, C. Cosner, Spatial ecology via reaction-diffusion equations, Wiley, Chichester, 2003. https://doi.org/10.1002/0470871296; MR2191264; Zbl 1087.92058
[11] G. Dai, Global bifurcation from intervals for Sturm-Liouville problems which are not linearizable, Electron. J. Qual. Theory Differ. Equ. 2013, No. 65, 1-7. https://doi.org/10. 14232/ejqtde.2013.1.65; Zbl 1340.34110
[12] G. Dai, R. Ma, Bifurcation from intervals for Sturm-Liouville problems and its applications, Electron. J. Differential Equations 2014, No. 3, 1-10. MR3159412; Zbl 1292.34027
[13] E. N. Dancer, On the structure of solutions of nonlinear eigenvalue problems, Indiana Univ. Math. J. 23(1974), No. 3, 1069-1076. https://doi.org/10.1512/iumj.1974.23. 23087; MR348567; Zbl 0276.47051
[14] W. H. Fleming, A selection-migration model in population genetics, J. Math. Biol. 2(1975), No. 3, 219-233. https://doi.org/10.1007/BF00277151; MR403720; Zbl 0325.92009
[15] E. L. Ince, Ordinary differential equations, Dover, New York, 1926.
[16] B. Ko, K. J. Brown, The existence of positive solutions for a class of indefinite weight semilinear elliptic boundary value problems, Nonlinear Anal. 3(2000), No. 5, 587-597. https://doi.org/10.1016/S0362-546X(98)00223-5; MR 1727270; Zbl 0945.35036
[17] M. A. Krasnoselskin, Topological methods in the theory of nonlinear integral equations, The Macmillan Co., New York, 1964. MR0159197; Zbl 0111.30303
[18] Y. Lou, E. Yanagida, Minimization of the principal eigenvalue for an elliptic boundary value problem with indefinite weight, and applications to population dynamics, Japan J. Ind. Appl. Math. 23(2006), No. 3, 275-292. https://doi.org/10.1007/BF03167595; MR2281509; Zbl 1185.35059
[19] A. P. Makhmudov, Z. S. Aliev, Global bifurcation of solutions of certain nonlinearizable eigenvalue problems, Differential Equations 25(1989), No. 3, 71-76. MR986400; Zbl 0684.35014
[20] T. Nagylaki, Y. Lou, The dynamics of migration-selection models, in: A. Friedman (Ed.), Tutorials in mathematical biosciences. IV, Lecture Notes in Mathematics, Vol. 1922, Springer, Berlin, 2008, pp. 117-170. https://doi.org/10.1007/978-3-540-74331-6_4; MR2392286; Zbl 1300.92059
[21] L. V. Nasirova, Global bifurcation from intervals of solutions of nonlinear SturmLiouville problems with indefinite weight. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Math. 39(2019), No. 4, 148-154. Zbl 1444.34040
[22] J. Przybycin, Bifurcation from infinity for the special class of nonlinear differential equations, J. Differential Equations 65(1986), No. 2, 235-239. https://doi.org/10.1016/ 0022-0396(86)90035-5; Zbl 0599.34013
[23] P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, J. Funct. Anal. 7(1971), No. 3, 487-513. https://doi.org/10.1016/0022-1236(71)90030-9; MR0301587; Zbl 0212.16504
[24] P. H. Rabinowitz, On bifurcation from infinity, J. Differential Equations 14(1973), No. 3, 462-475. https://doi.org/10.1016/0022-0396(73) 90061-2; MR328705; Zbl 0272.35017
[25] B. P. Rynne, Bifurcation from zero or infinity in Sturm-Liouville problems which are not linearizable, J. Math. Anal. Appl. 228(1998), No. 1, 141-156. https://doi.org/10.1006/ jmaa.1998.6122; MR1659893; Zbl 0918.34028
[26] K. Schmitt, H. L. Smith, On eigenvalue problems for nondifferentiable mappings, J. Differential Equations 33(1979), No. 3, 294-319. https ://doi. org/10.1016/0022-0396(79) 90067-6; MR543701; Zbl 0389.34019
[27] C. A. Stuart, Solutions of large norm for non-linear Sturm-Liouville problems, Quart. J. Math. Oxford Ser. (2) 24(1973), No. 2, 129-139. https://doi.org/10.1093/qmath/24.1. 129
[28] J. F. Toland, Asymptotic linearity and nonlinear eigenvalue problems, Quart. J. Math. Oxford Ser. (2) 24(1973), No. 2, 241-250. https://doi.org/10.1093/qmath/24.1.241; Zbl 0256.47049
[29] S. Willard, General topology, Addison-Wesley, Reading, MA, 1970. MR0264581

# Multiplicity of positive solutions for a class of nonlocal problem involving critical exponent 

Xiaotao Qian ${ }^{\boxtimes}$<br>Department of Basic Teaching and Research, Yango University, Fuzhou 350015, P. R. China

Received 17 January 2021, appeared 30 July 2021
Communicated by Dimitri Mugnai

Abstract. In this paper, we study the following critical nonlocal problem

$$
\begin{cases}-\left(a-\lambda b \int_{\Omega}|\nabla u|^{2} d x\right) \Delta u=\lambda|u|^{p-2} u+Q(x)|u|^{2} u, & x \in \Omega \\ u=0, & x \in \partial \Omega\end{cases}
$$

where $a>0, b \geq 0,2<p<4, \lambda>0$ is a parameter, $\Omega$ is a smooth bounded domain in $\mathbb{R}^{4}$ and $Q(x) \in C(\bar{\Omega})$ is a nonnegative function. By virtue of variational methods and delicate estimates, we prove that problem admits $k$ positive solutions for $\lambda>0$ sufficiently small, provided that the maximum of $Q(x)$ is achieved at $k$ interior points in $\Omega$.
Keywords: nonlocal problem, variational methods, critical nonlinearity, multiple positive solutions.

2020 Mathematics Subject Classification: 35B33, 35J75.

## 1 Introduction

In this paper, we concern with the multiplicity of positive solutions to the nonlocal problem

$$
\begin{cases}-\left(a-\lambda b \int_{\Omega}|\nabla u|^{2} d x\right) \Delta u=\lambda|u|^{p-2} u+Q(x)|u|^{2} u, & x \in \Omega  \tag{1.1}\\ u=0, & x \in \partial \Omega\end{cases}
$$

where $a>0, b \geq 0,2<p<4, \lambda>0$ is a parameter, $\Omega$ is a smooth bounded domain in $\mathbb{R}^{4}$ (2* $=4$ is the critical exponent in dimension four) and $Q(x) \in C(\bar{\Omega})$ is a nonnegative function satisfying:
$\left(Q_{1}\right)$ There exist $k$ different points $x^{1}, x^{2}, \ldots, x^{k} \in \Omega$ such that $Q\left(x^{j}\right)$ are strict local maximums and satisfy

$$
Q\left(x^{j}\right)=Q_{M}=\max \{Q(x): x \in \bar{\Omega}\}>0, \quad j=1,2, \ldots, k ;
$$

[^19]$\left(Q_{2}\right) Q_{M}-Q(x)=O\left(\left|x-x^{j}\right|^{2}\right)$ for $x$ near $x^{j}, j=1,2, \ldots, k$.
In the past decade, the following Kirchhoff type problem involving critical growth on a bounded domain $\Omega \subset \mathbb{R}^{N}$
\[

$$
\begin{cases}-\left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \Delta u=g(x, u)+K(x)|u|^{2^{*}-2} u, & x \in \Omega  \tag{1.2}\\ u=0, & x \in \partial \Omega\end{cases}
$$
\]

has attracted considerable attention, where $a, b>0$ are constants, $2^{*}=2 N /(N-2)$ with $N \geq 3$ and $K(x)$ is a nonnegative continuous function. Kirchhoff type problem is often viewed as nonlocal due to the presence of the term $b \int_{\Omega}|\nabla u|^{2} d x$ which implies that such problem is no longer pointwise identity. It is commonly known that Kirchhoff type problem has a mechanical and biological motivation, see $[1,8]$. Under different hypotheses on $g(x, u)$ and $K(x)$, there are many interesting results of positive solutions to (1.2) by using variational methods, see e.g. [6,7,15]. In particular, Fan [6] showed how the topology of the maximum set of $K(x)$ affects the number of positive solutions to (1.2) via Ljusternik-Schnirelmann category theory when $N=3$ and $f(x, u)=f(x) u^{q}$ with $f(x) \in L^{\frac{6}{6-q}}(\Omega)$ and $3<q<5$. There are also several existence results for (1.2) in the whole space $\mathbb{R}^{N}$, see $[5,11,12]$ and the references therein.

In (1.2), if we replace $a+b \int_{\Omega}|\nabla u|^{2} d x$ by $a-b \int_{\Omega}|\nabla u|^{2} d x$, it turns to be a new nonlocal one. This kind of nonlocal problem presents some interesting difficulties different from Kirchhoff type problem. Such nonlocal problem with subcritical growth

$$
\begin{cases}-\left(a-b \int_{\Omega}|\nabla u|^{2} d x\right) \Delta u=f_{\lambda}(x)|u|^{p-2} u, & x \in \Omega  \tag{1.3}\\ u=0, & x \in \partial \Omega\end{cases}
$$

has been studied by some researchers, where $f_{\lambda}(x) \in L^{\frac{2^{*}}{2^{*}-p}}(\Omega)$ and $\Omega \subset \mathbb{R}^{N}$ is a bounded domain. If $f_{\lambda}(x) \equiv 1$ and $2<p<2^{*}$, Yin and Liu [23] obtained two nontrivial solutions to (1.3); Qian [18] proved the existence and asymptotic behavior of ground state sign-changing solutions for (1.3); Wang et al. [22] proved that (1.3) has infinitely many sign-changing solutions. For $1 \leq p<2^{*}$, Duan et al. [4] established the existence of multiple positive solutions to (1.3). In [10], the multiplicity result of positive solutions to (1.3) was obtained for $0<p<1$. When $f_{\lambda}(x)$ has indefinite sign, Lei et al. [9] and Qian and Chao [16] proved the existence of positive solution to (1.3) for $1<p<2$ and $3<p<6$, respectively. For more results about (1.3) with general nonlinearities and its variants on unbounded domain, we refer the interested readers to $[19,20,24]$. To the best of our knowledge, there is little result for (1.3) when $f(x, u)$ exhibits a critical exponent. Only Wang et al. [21] investigated the existence of two positive solutions for the following problem involving critical exponent

$$
\left\{\begin{array}{l}
-\left(a-b \int_{\mathbb{R}^{4}}|\nabla u|^{2} d x\right) \Delta u=\lambda g(x)+|u|^{2} u, x \in \mathbb{R}^{4}, \\
u \in \mathcal{D}^{1,2}\left(\mathbb{R}^{4}\right),
\end{array}\right.
$$

under the assumptions $\lambda>0$ is sufficiently small and $g(x) \in L^{4 / 3}\left(\mathbb{R}^{4}\right)$ is a nonnegative function.

When $a=1, b=0, \mathbb{R}^{4}$ and $Q(x)|u|^{2} u$ are replaced by $\mathbb{R}^{N}$ and $Q(x)|u|^{2^{*}-2} u$, respectively, (1.1) is reduced to the following local one

$$
\begin{cases}-\Delta u=\lambda|u|^{p-2} u+Q(x)|u|^{2^{*}-2} u, & x \in \Omega  \tag{1.4}\\ u=0, & x \in \partial \Omega\end{cases}
$$

which does not depend on the nonlocal term $\int_{\Omega}|\nabla u|^{2} d x$ any more. The study by Cao and Noussair [3] is the first to investigate the effect of the shape of the graph of $Q(x)$ on the number of positive solutions to (1.4) with $p=2$. More precisely, they proved that for small enough $\lambda>0$, (1.4) has $k$ positive solutions if the maximum of $Q(x)$ is achieved at exactly $k$ different points of $\Omega$, by applying Nehari manifold method. Liao et al. [13] extended the result of [3] in the sense that a more wider range of $p$ is covered. In [17], Qian and Chen got a similar but more complicated result for (1.4) with an additional fast increasing weight.

Motivated by the idea of $[3,6,21]$, it is natural and interesting to ask: can we apply the shape of the graph of $Q(x)$ to prove the multiplicity of positive solutions for the critical nonlocal problem (1.1) as in Kirchhoff problem (1.2)? In the present paper, we will give a positive answer to this question.

Our main results can be stated as follows.
Theorem 1.1. Assume that $a>0, b \geq 0,2<p<4$ and $\Omega$ is a smooth bounded domain in $\mathbb{R}^{4}$. If the conditions $\left(Q_{1}\right)$ and $\left(Q_{2}\right)$ hold, then there exists $\Lambda_{0}>0$, such that for each $\lambda \in\left(0, \Lambda_{0}\right),(1.1)$ has at least $k$ positive solutions.

Since the result of Theorem 1.1 still holds for $b=0$, then we obtain the following corollary related to the multiplicity result of positive solutions for a semilinear problem with critical exponent.

Corollary 1.2. Assume that $a>0,2<p<4$ and $\Omega$ is a smooth bounded domain in $\mathbb{R}^{4}$. If the conditions $\left(Q_{1}\right)$ and $\left(Q_{2}\right)$ hold, then there exists $\Lambda_{1}>0$, such that for each $\lambda \in\left(0, \Lambda_{1}\right)$, the problem

$$
\begin{cases}-a \Delta u=\lambda|u|^{p-2} u+Q(x)|u|^{2} u, & x \in \Omega  \tag{1.5}\\ u=0, & x \in \partial \Omega\end{cases}
$$

has at least $k$ positive solutions.
Associated with (1.1), we define the functional $I_{\lambda}$ on $H_{0}^{1}(\Omega)$ by

$$
I_{\lambda}(u)=\frac{a}{2}\|u\|^{2}-\frac{\lambda b}{4}\|u\|^{4}-\frac{\lambda}{p} \int_{\Omega}|u|^{p} d x-\frac{1}{4} \int_{\Omega} Q(x)|u|^{4} d x
$$

where $\|u\|^{2}=\int_{\Omega}|\nabla u|^{2} d x$. Then $I_{\lambda} \in C^{1}\left(H_{0}^{1}(\Omega), \mathbb{R}\right)$. Moreover, there exists a one to one correspondence between the critical points of $I_{\lambda}$ on $H_{0}^{1}(\Omega)$ and the weak solutions of (1.1). Here, we say that $u$ is a weak solution of (1.1), if $u \in H_{0}^{1}(\Omega)$ and for all $v \in H_{0}^{1}(\Omega)$, there holds

$$
\left(a-\lambda b\|u\|^{2}\right) \int_{\Omega} \nabla u \nabla v d x-\lambda \int_{\Omega}|u|^{p-2} u v d x-\int_{\Omega} Q(x)|u|^{2} u v d x=0 .
$$

The proof of Theorem 1.1 is based on variational methods. Since (1.1) has a negative nonlocal term, the approaches used in [6] to deal with Kirchhoff problem do not work here. Indeed, we shall apply the ideas introduced by Cao and Noussair [3]. However, in the present paper, there are some new difficulties caused by the competing effect of the nonlocal term
with the nonlinear terms and the non-compactness due to the critical exponent. To overcome these difficulties, we need to add the factor $\lambda$ of $|u|^{p-2} u$ to the nonlocal term $-b \int_{\Omega}|\nabla u|^{2} d x$ in problem (1.1). This modification will play an important role in our arguments (see Lemma 2.2 below). Moreover, inspired by [21], we consider our problem in dimension 4 and make some delicate estimates in order to get the compactness condition. We also point out that it is not clear whether the multiplicity result in Theorem 1.1 still holds for critical problem (1.1) in other dimension, from which it follows that the critical exponent $2^{*}$ is no longer equal to 4 .

In Section 2, we present some lemmas which will be used to prove Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.1.

## 2 Notations and preliminaries

Throughout the paper, for simplicity we write $\int u$ instead of $\int_{\Omega} u(x) d x . H_{0}^{1}(\Omega)$ and $L^{r}(\Omega)$ are the usual Sobolev spaces equipped with the standard norms $\|u\|$ and $|u|_{r}$, respectively. $\mathcal{D}^{1,2}\left(\mathbb{R}^{4}\right)=\left\{u \in L^{4}\left(\mathbb{R}^{4}\right): \nabla u \in L^{2}\left(\mathbb{R}^{4}\right)\right\}$. Denote by $B_{r}(x)$ the ball centered at $x$ with radius $r>0$. Let $\bar{B}_{r}(x)$ and $\partial B_{r}(x)$ denote the closure and the boundary of $B_{r}(x)$, respectively. We use $\rightarrow(-)$ to denote the strong (weak) convergence. $O\left(\varepsilon^{t}\right)$ denotes $\left|O\left(\varepsilon^{t}\right)\right| / \varepsilon^{t} \leq C$ as $\varepsilon \rightarrow 0$, and $o\left(\varepsilon^{t}\right)$ denotes $\left|o\left(\varepsilon^{t}\right)\right| / \varepsilon^{t} \rightarrow 0$ as $\varepsilon \rightarrow 0$. $C$ and $C_{i}$ denote various positive constants whose exact values are not essential. Let $S$ be the best constant of the Sobolev embedding $H_{0}^{1}(\Omega) \hookrightarrow L^{4}(\Omega)$, that is,

$$
S=\inf _{u \in H_{0}^{1}(\Omega) \backslash\{0\}} \frac{\int|\nabla u|^{2}}{\left(\int|u|^{4}\right)^{1 / 2}} .
$$

The Nehari manifold corresponding to $I_{\lambda}$ is defined by

$$
\mathcal{M}_{\lambda}=\left\{u \in H_{0}^{1}(\Omega) \backslash\{0\}:\left\langle I_{\lambda}^{\prime}(u), u\right\rangle=0\right\} .
$$

By the condition $\left(Q_{1}\right)$, we can take $\eta>0$ sufficiently small such that $B_{2 \eta}\left(x^{j}\right) \subset \Omega$ are disjoint and $Q(x)<Q\left(x^{j}\right)$ for $x \in \bar{B}_{2 \eta}\left(x^{j}\right) \backslash\left\{x^{j}\right\}, j=1,2, \ldots, k$. Following the argument of [3], we define a barycenter map $\beta: H_{0}^{1}(\Omega) \backslash\{0\} \rightarrow \mathbb{R}^{4}$ by setting

$$
\beta(u)=\frac{\int x|u|^{4}}{\int|u|^{4}} .
$$

With the help of the map above, we will first separate the Nehari manifold $\mathcal{M}_{\lambda}$, then study minimization problems of $I_{\lambda}$ on its proper subset. We point out that, a key role of $\beta$ is to insure that the minimizers of the considered minimization problems are distinct.

For $j=1,2, \ldots, k$, we consider the following subsets of $\mathcal{M}_{\lambda}$,

$$
\mathcal{M}_{\lambda}^{j}=\left\{u \in \mathcal{M}_{\lambda}: \beta(u) \in B_{\eta}\left(x^{j}\right)\right\} \quad \text { and } \quad \mathcal{O}_{\lambda}^{j}=\left\{u \in \mathcal{M}_{\lambda}: \beta(u) \in \partial B_{\eta}\left(x^{j}\right)\right\} .
$$

Correspondingly, study the following minimization problems

$$
m_{\lambda}^{j}=\inf _{u \in \mathcal{M}_{\lambda}^{j}} I_{\lambda}(u) \text { and } \widetilde{m}_{\lambda}^{j}=\inf _{u \in \mathcal{O}_{\lambda}^{j}} I_{\lambda}(u) \text {. }
$$

For all $\varepsilon>0$ and $x_{0} \in \mathbb{R}^{4}$, we define

$$
U_{\varepsilon, x_{0}}=\frac{(8)^{1 / 2} \varepsilon}{\left(\varepsilon^{2}+\left|x-x_{0}\right|^{2}\right)^{\prime}}
$$

which solves $-\Delta u=|u|^{2} u$ in $\mathbb{R}^{4}$. For $j=1,2, \ldots, k$ fixed, define a cut off function $\varphi_{j} \in C_{0}^{\infty}\left(\mathbb{R}^{4}\right)$ such that $0 \leq \varphi_{j} \leq 1, \varphi_{j}(x)=1$ for $\left|x-x^{j}\right|<\rho$ and $\varphi_{j}(x)=0$ for $\left|x-x^{j}\right| \geq 2 \rho$ with $0<\rho<\eta / 2$. Let $u_{\varepsilon, j}=\varphi_{j}\left(x-x^{j}\right) U_{\varepsilon, x^{j}}(x)$. By [2], we have for $2<p<4$,

$$
\begin{aligned}
\left\|u_{\varepsilon, j}\right\|^{2} & =S^{2}+O\left(\varepsilon^{2}\right) \\
\left|u_{\varepsilon, j}\right|_{4}^{4} & =S^{2}+O\left(\varepsilon^{4}\right) \\
\left|u_{\varepsilon, j}\right|_{p}^{p} & =O\left(\varepsilon^{4-p}\right) .
\end{aligned}
$$

Lemma 2.1. For $j=1,2, \ldots, k$ and $\lambda>0$, we have

$$
\begin{equation*}
m_{\lambda}^{j}<\frac{a^{2} S^{2}}{4\left(\lambda b S^{2}+Q_{M}\right)} \tag{2.1}
\end{equation*}
$$

Proof. It is easy to see that there exists a unique $t_{\varepsilon}>0$ such that $t_{\varepsilon} u_{\varepsilon, j} \in \mathcal{M}_{\lambda}$ and $I_{\lambda}\left(t_{\varepsilon} u_{\varepsilon, j}\right)=$ $\sup _{t>0} I_{\lambda}\left(t u_{\varepsilon, j}\right)$. By the symmetry of $u_{\varepsilon, j}$ about $x^{j}$, we further obtain $t_{\varepsilon} u_{\varepsilon, j} \in \mathcal{M}_{\lambda}^{j}$. Thus, to complete the proof of lemma, it suffices to prove that

$$
\begin{equation*}
\sup _{t>0} I_{\lambda}\left(t u_{\varepsilon, j}\right)<\frac{a^{2} S^{2}}{4\left(\lambda b S^{2}+Q_{M}\right)} . \tag{2.2}
\end{equation*}
$$

At this point, we can suppose that $t_{\varepsilon} \geq C_{1}>0$ for any $\varepsilon>0$ small. Otherwise, there is a sequence $\varepsilon_{n} \rightarrow 0^{+}$such that $t_{\varepsilon_{n}} \rightarrow 0$. By the continuity of $I_{\lambda}$ and the boundedness of $\left\{u_{\varepsilon_{n}, j}\right\}$,

$$
\sup _{t>0} I_{\lambda}\left(t u_{\varepsilon_{n}, j}\right)=I_{\lambda}\left(t_{\varepsilon_{n}} u_{\varepsilon_{n}, j}\right) \rightarrow 0<\frac{a^{2} S^{2}}{4\left(\lambda b S^{2}+Q_{M}\right)^{\prime}},
$$

that is, the proof is complete. Similarly, we also suppose that $t_{\varepsilon} \leq C_{2}$ for some positive constant $C_{2}$ and any $\varepsilon>0$ small.

To proceed, set

$$
h(t)=\frac{a t^{2}}{2}\left\|u_{\varepsilon, j}\right\|^{2}-\frac{\lambda b t^{4}}{4}\left\|u_{\varepsilon, j}\right\|^{4}-\frac{t^{4}}{4} \int Q_{M}\left|u_{\varepsilon, j}\right|^{4} .
$$

We easily see that $h(t)$ achieves its maximum at

$$
\begin{aligned}
t_{\max } & =\left(\frac{a\left\|u_{\varepsilon, j}\right\|^{2}}{\lambda b\left\|u_{\varepsilon, j}\right\|^{4}+Q_{M}\left|u_{\varepsilon, j}\right|_{4}^{4}}\right)^{1 / 2} \\
& =\left(\frac{a S^{2}+O\left(\varepsilon^{2}\right)}{\lambda b S^{4}+Q_{M} S^{2}+O\left(\varepsilon^{2}\right)}\right)^{1 / 2} \\
& =\left(\frac{a S^{2}}{\lambda b S^{4}+Q_{M} S^{2}}\right)^{1 / 2}+O\left(\varepsilon^{2}\right)
\end{aligned}
$$

with

$$
\begin{equation*}
h\left(t_{\max }\right)=\frac{a^{2} S^{2}}{4\left(\lambda b S^{2}+Q_{M}\right)}+O\left(\varepsilon^{2}\right) \tag{2.3}
\end{equation*}
$$

Using condition $\left(Q_{2}\right)$, we also have

$$
\begin{equation*}
\int\left(Q_{M}-Q(x)\right)\left|u_{\varepsilon, j}\right|^{4}=O\left(\varepsilon^{2}\right) \tag{2.4}
\end{equation*}
$$

By (2.3) and (2.4),

$$
\begin{aligned}
\sup _{t>0} I_{\lambda}\left(t u_{\varepsilon, j}\right) & =I_{\lambda}\left(t_{\varepsilon} u_{\varepsilon, j}\right) \\
& =h\left(t_{\varepsilon}\right)+\frac{t_{\varepsilon}^{4}}{4} \int\left(Q_{M}-Q(x)\right)\left|u_{\varepsilon, j}\right|^{4}-\frac{\lambda}{p} t_{\varepsilon}^{p} \int\left|u_{\varepsilon, j}\right|^{p} \\
& \leq h\left(t_{\max }\right)+\frac{C_{2}^{4}}{4} \int\left(Q_{M}-Q(x)\right)\left|u_{\varepsilon, j}\right|^{4}-\frac{\lambda}{p} C_{1}^{p} \int\left|u_{\varepsilon, j}\right|^{p} \\
& =\frac{a^{2} S^{2}}{4\left(\lambda b S^{2}+Q_{M}\right)}+O\left(\varepsilon^{2}\right)-O\left(\varepsilon^{4-p}\right) .
\end{aligned}
$$

Since $2<p<4$, (2.2) holds for $\varepsilon>0$ small enough. This ends the proof.
Lemma 2.2. Assume that condition ( $Q_{1}$ ) holds. Then there exists $\Lambda_{0}>0$ such that

$$
\widetilde{m}_{\lambda}^{j}>\frac{a^{2} S^{2}}{4 Q_{M}}
$$

for $j=1,2, \ldots, k$, and $\lambda \in\left(0, \Lambda_{0}\right)$.
Proof. Let us argue by contradiction and suppose that there exist sequences $\lambda_{n} \rightarrow 0$, and $\left\{u_{n}\right\} \subset \mathcal{O}_{\lambda_{n}}^{j}$ satisfying

$$
I_{\lambda_{n}}\left(u_{n}\right) \rightarrow c \leq \frac{a^{2} S^{2}}{4 Q_{M}},
$$

and

$$
\begin{equation*}
a \int\left|\nabla u_{n}\right|^{2}-\lambda_{n} b\left(\int\left|\nabla u_{n}\right|^{2}\right)^{2}=\lambda_{n} \int\left|u_{n}\right|^{p}+\int Q(x)\left|u_{n}\right|^{4} . \tag{2.5}
\end{equation*}
$$

By $\left\{u_{n}\right\} \subset \mathcal{O}_{\lambda_{n}}^{j}$, one has for $n$ large,

$$
\begin{aligned}
c+1 & \geq I_{\lambda_{n}}\left(u_{n}\right)-\frac{1}{p}\left\langle I_{\lambda_{n}}^{\prime}\left(u_{n}\right), u_{n}\right\rangle \\
& =a\left(\frac{1}{2}-\frac{1}{p}\right)\left\|u_{n}\right\|^{2}+\lambda_{n} b\left(\frac{1}{p}-\frac{1}{4}\right)\left\|u_{n}\right\|^{4}+\left(\frac{1}{p}-\frac{1}{4}\right) \int Q(x)\left|u_{n}\right|^{4} \\
& \geq a\left(\frac{1}{2}-\frac{1}{p}\right)\left\|u_{n}\right\|^{2}
\end{aligned}
$$

which implies that $\left\{u_{n}\right\}$ is bounded in $H_{0}^{1}(\Omega)$. Using (2.5) and Sobolev embedding, we also have

$$
a\left\|u_{n}\right\|^{2}=\lambda_{n} b\left\|u_{n}\right\|^{4}+\lambda_{n}\left|u_{n}\right|_{p}^{p}+\int Q(x)\left|u_{n}\right|^{4} \leq \lambda_{n} b\left\|u_{n}\right\|^{4}+\lambda_{n} C\left\|u_{n}\right\|^{p}+Q_{M} S^{-2}\left\|u_{n}\right\|^{4}
$$

from which we infer that

$$
\left\|u_{n}\right\| \geq C_{3}>0 .
$$

Noting that $\lambda_{n} \rightarrow 0$, we then deduce from (2.5) that there is a constant $C_{4}>0$ such that

$$
\int Q(x)\left|u_{n}\right|^{4} \geq C_{4}>0
$$

for all $n \in \mathbb{N}$. Thus, we are able to choose $t_{n}>0$ such that $v_{n}=t_{n} u_{n}$ satisfies

$$
\begin{equation*}
a \int\left|\nabla v_{n}\right|^{2}=\int Q_{M}\left|v_{n}\right|^{4} \tag{2.6}
\end{equation*}
$$

This and Sobolev inequality give that $\frac{a S^{2}}{Q_{M}} \leq\left\|v_{n}\right\|^{2}$. Moreover,

$$
t_{n}=\left(\frac{\int Q(x)\left|u_{n}\right|^{4}+\lambda_{n} b\left(\int\left|\nabla u_{n}\right|^{2}\right)^{2}+\lambda_{n} \int\left|u_{n}\right|^{p}}{\int Q_{M}\left|u_{n}\right|^{4}}\right)^{1 / 2}
$$

It follows that $\left\{t_{n}\right\}$ is uniformly bounded. Then, we can assume $\lim _{n \rightarrow \infty} t_{n}=t_{0}$. By $Q(x) \leq$ $Q_{M}, \lambda_{n} \rightarrow 0$ and the boundedness of $\left\{u_{n}\right\}$, we see that $t_{0} \leq 1$. We show next that the case $t_{0} \leq 1$ leads to a contradiction. Since for $t_{0} \leq 1$, we have

$$
\begin{aligned}
\frac{a^{2} S^{2}}{4 Q_{M}} \leq & \lim _{n \rightarrow \infty} \frac{1}{4} a \int\left|\nabla v_{n}\right|^{2}=\lim _{n \rightarrow \infty} \frac{1}{4} a t_{n}^{2} \int\left|\nabla u_{n}\right|^{2} \\
= & \lim _{n \rightarrow \infty} t_{n}^{2}\left[\left(\frac{1}{2}-\frac{1}{4}\right)\left(a \int\left|\nabla u_{n}\right|^{2}-\lambda_{n} b\left(\int\left|\nabla u_{n}\right|^{2}\right)^{2}-\lambda_{n} \int\left|u_{n}\right|^{p}\right)\right. \\
& \left.\quad+\lambda_{n} b\left(\frac{1}{2}-\frac{1}{4}\right)\left(\int\left|\nabla u_{n}\right|^{2}\right)^{2}+\lambda_{n}\left(\frac{1}{2}-\frac{1}{p}\right) \int\left|u_{n}\right|^{p}\right] \\
= & \lim _{n \rightarrow \infty} t_{n}^{2} I_{\lambda_{n}}\left(u_{n}\right)=t_{0}^{2} c \leq c \leq \frac{a^{2} S^{2}}{4 Q_{M}},
\end{aligned}
$$

then it follows that

$$
\begin{equation*}
c=\frac{a^{2} S^{2}}{4 Q_{M}} \quad \text { and } \quad \lim _{n \rightarrow \infty} \int\left|\nabla v_{n}\right|^{2}=\frac{a S^{2}}{Q_{M}} . \tag{2.7}
\end{equation*}
$$

Let $w_{n}=v_{n} /\left|v_{n}\right|_{4}$, then $\left|w_{n}\right|_{4}=1$. Moreover, by (2.6) and (2.7),

$$
\lim _{n \rightarrow \infty} \int\left|\nabla w_{n}\right|^{2}=\lim _{n \rightarrow \infty} \frac{\left\|v_{n}\right\|^{2}}{\left|v_{n}\right|_{4}^{2}}=\lim _{n \rightarrow \infty} \frac{\left\|v_{n}\right\|^{2}}{\left(a\left\|v_{n}\right\|^{2} / Q_{M}\right)^{1 / 2}}=S
$$

namely, $\left\{w_{n}\right\}$ is a minimizing sequence for $S$. According to [14], we can find a point $y_{0} \in \bar{\Omega}$ such that

$$
\begin{equation*}
\left|\nabla w_{n}\right|^{2} \rightharpoonup d \mu=S \delta_{y_{0}} \quad \text { and } \quad\left|w_{n}\right|_{4} \rightharpoonup d v=\delta_{y_{0}} \tag{2.8}
\end{equation*}
$$

with the above convergence holding weakly in the sense of measure, where $\delta_{y_{0}}$ is a Dirac mass at $y_{0}$. Then

$$
\beta\left(u_{n}\right)=\frac{\int x\left|u_{n}\right|^{4}}{\int\left|u_{n}\right|^{4}}=\frac{\int x\left|v_{n}\right|^{4}}{\int\left|v_{n}\right|^{4}}=\frac{\int x\left|w_{n}\right|^{4}}{\int\left|w_{n}\right|^{4}} \rightarrow y_{0}, \quad \text { as } n \rightarrow \infty .
$$

This together with $\beta\left(u_{n}\right) \in \partial B_{\eta}\left(x^{j}\right)$ imply that $y_{0} \in \partial B_{\eta}\left(x^{j}\right)$. Thus, from (2.6) and (2.8), we conclude that

$$
\begin{aligned}
\lim _{n \rightarrow \infty} I_{\lambda_{n}}\left(u_{n}\right)= & \lim _{n \rightarrow \infty} t_{n}^{2}\left[\left(\frac{1}{2}-\frac{1}{4}\right)\left(a \int\left|\nabla u_{n}\right|^{2}-\lambda_{n} b\left(\int\left|\nabla u_{n}\right|^{2}\right)^{2}-\lambda_{n} \int\left|u_{n}\right|^{p}\right)\right. \\
& \left.+\lambda_{n} b\left(\frac{1}{2}-\frac{1}{4}\right)\left(\int\left|\nabla u_{n}\right|^{2}\right)^{2}+\lambda_{n}\left(\frac{1}{2}-\frac{1}{p}\right) \int\left|u_{n}\right|^{p}\right] \\
\leq & \lim _{n \rightarrow \infty} \frac{1}{4} \int Q(x)\left|u_{n}\right|^{4} \\
= & \lim _{n \rightarrow \infty} \frac{1}{4} \int Q(x)\left|v_{n}\right|^{4} \\
= & \frac{Q\left(y_{0}\right)}{4 Q_{M}} \lim _{n \rightarrow \infty} \int Q_{M}\left|v_{n}\right|^{4} \\
= & \frac{Q\left(y_{0}\right)}{4 Q_{M}} \lim _{n \rightarrow \infty} a \int\left|\nabla v_{n}\right|^{2} \\
= & \frac{Q\left(y_{0}\right)}{4 Q_{M}} \frac{a^{2} S^{2}}{Q_{M}}<\frac{a^{2} S^{2}}{4 Q_{M}},
\end{aligned}
$$

which contradicts with (2.7). This completes the proof.
Lemma 2.3. For any $u \in \mathcal{M}_{\lambda}^{j}$, there exist $\rho>0$ and a differential function $g=g(w)$ defined for $w \in H_{0}^{1}(\Omega), w \in B_{\rho}(0)$ satisfying that

$$
g(0)=1, \quad g(w)(u-w) \in \mathcal{M}_{\lambda}^{j}
$$

and

$$
\left\langle g^{\prime}(0), \phi\right\rangle=\frac{\left(2 a-4 \lambda b\|u\|^{2}\right) \int \nabla u \nabla \phi-\lambda p \int|u|^{p-2} u \phi-4 \int Q(x)|u|^{2} u \phi}{a\|u\|^{2}-3 \lambda b\|u\|^{4}-\lambda(p-1) \int|u|^{p}-3 \int Q(x)|u|^{4}} .
$$

Proof. Define $F: \mathbb{R}^{+} \times H_{0}^{1}(\Omega) \rightarrow \mathbb{R}$ by

$$
F(t, w)=a t\|u-w\|^{2}-\lambda b t^{3}\|u-w\|^{4}-\lambda t^{p-1} \int|u-w|^{p}-t^{3} \int Q(x)|u-w|^{4} .
$$

By $u \in \mathcal{M}_{\lambda}^{j}$, we get $F(1,0)=0$ and

$$
\begin{aligned}
F_{t}(1,0) & =a\|u\|^{2}-3 \lambda b\|u\|^{4}-\lambda(p-1) \int|u|^{p}-3 \int Q(x)|u|^{4} \\
& =a(2-p)\|u\|^{2}-\lambda b(4-p)\|u\|^{4}-(4-p) \int Q(x)|u|^{4} \\
& <0 .
\end{aligned}
$$

Thus, we can use the implicit function theorem for $F$ at the point $(1,0)$ and obtain $\bar{\rho}>0$ and a functional $g=g(w)>0$ defined for $w \in H_{0}^{1}(\Omega),\|w\|<\bar{\rho}$ satisfying that

$$
g(0)=1, \quad g(w)(u-w) \in \mathcal{M}_{\lambda}, \quad \forall w \in H_{0}^{1}(\Omega),\|w\|<\bar{\rho}
$$

By the continuity of the maps $g$ and $\beta$, we can further take $\rho>0$ possibly smaller $(\rho<\bar{\rho})$ such that

$$
\beta(g(w)(u-w)) \in B_{\eta}\left(x^{j}\right), \quad \forall w \in H_{0}^{1}(\Omega),\|w\|<\rho,
$$

which means that $g(w)(u-w) \in \mathcal{M}_{\lambda}^{j}$.
Moreover, we also have for all $\phi \in H_{0}^{1}(\Omega), r>0$,

$$
\begin{aligned}
F(1,0 & +r \phi)-F(1,0) \\
= & a\|u-r \phi\|^{2}-\lambda b\|u-r \phi\|^{4}-\lambda \int|u-r \phi|^{p}-\int Q(x)|u-r \phi|^{4} \\
& -a\|u\|^{2}+\lambda b\|u\|^{4}+\lambda \int|u|^{p}+\int Q(x)|u|^{4} \\
= & -a \int\left(2 r \nabla u \nabla \phi-r^{2}|\nabla \phi|^{2}\right) \\
& +\lambda b\left[2 \int|\nabla u|^{2} \int\left(2 r \nabla u \nabla \phi-r^{2}|\nabla \phi|^{2}\right)-\left(\int\left(2 r \nabla u \nabla \phi-r^{2}|\nabla \phi|^{2}\right)\right)^{2}\right] \\
& -\lambda \int\left(|u-r \phi|^{p}-|u|^{p}\right)-\int Q(x)\left(|u-r \phi|^{4}-|u|^{4}\right) .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left.\left\langle F_{w}, \phi\right\rangle\right|_{t=1, w=0} & =\lim _{r \rightarrow 0} \frac{F(1,0+r \phi)-F(1,0)}{r} \\
& =-\left(2 a-4 \lambda b\|u\|^{2}\right) \int \nabla u \nabla \phi+p \lambda \int|u|^{p-2} u \phi+4 \int Q(x)|u|^{2} u \phi .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left\langle g^{\prime}(0), \phi\right\rangle & =-\left.\frac{\left\langle F_{w}, \phi\right\rangle}{F_{t}}\right|_{t=1, w=0} \\
& =\frac{\left(2 a-4 \lambda b\|u\|^{2}\right) \int \nabla u \nabla \phi-\lambda p \int|u|^{p-2} u \phi-4 \int Q(x)|u|^{2} u \phi}{a\|u\|^{2}-3 \lambda b\|u\|^{4}-\lambda(p-1) \int|u|^{p}-3 \int Q(x)|u|^{4}} .
\end{aligned}
$$

The proof is completed.
Lemma 2.4. There exist $\Lambda_{0}>0$ and a sequence $\left\{u_{n}\right\} \subset \mathcal{M}_{\lambda}^{j}$ such that

$$
u_{n} \geq 0, \quad I_{\lambda}\left(u_{n}\right) \rightarrow m_{\lambda}^{j}, \quad I_{\lambda}^{\prime}\left(u_{n}\right) \rightarrow 0,
$$

for $j=1,2, \ldots, k$, and $\lambda \in\left(0, \Lambda_{0}\right)$.
Proof. Note that $\overline{\mathcal{M}}_{\lambda}^{j}=\mathcal{M}_{\lambda}^{j} \cup \mathcal{O}_{\lambda}^{j}$ and $\mathcal{O}_{\lambda}^{j}$ is the boundary of $\overline{\mathcal{M}}_{\lambda}^{j}$. In view of Lemmas 2.1 and 2.2, we know that there exists $\Lambda_{0}>0$ such that

$$
m_{\lambda}^{j}<\widetilde{m}_{\lambda}^{j}
$$

for $\lambda \in\left(0, \Lambda_{0}\right), j=1,2, \ldots, k$. This implies that

$$
m_{\lambda}^{j}=\inf \left\{I_{\lambda}(u): u \in \overline{\mathcal{M}}_{\lambda}^{j}\right\} .
$$

Then, for each $j=1,2, \ldots, k$, we can apply Ekeland's variational principle to construct a minimizing sequence $\left\{u_{n}\right\} \subset \mathcal{M}_{\lambda}^{j}$ satisfying the following properties :
(i) $\lim _{n \rightarrow \infty} I_{\lambda}\left(u_{n}\right)=m_{\lambda}^{j}$,
(ii) $I_{\lambda}\left(u_{n}\right) \leq I_{\lambda}(w)+\frac{1}{n}\left\|w-u_{n}\right\|, \quad$ for each $w \in \overline{\mathcal{M}}_{\lambda}^{j}$.

Since $I_{\lambda}(|u|)=I_{\lambda}(u)$, we may assume $u_{n} \geq 0$. Using Lemma 2.3 with $u=u_{n}$, we get $\rho_{n}>0$, a differential function $g_{n}(w)$ defined for $w \in H_{0}^{1}(\Omega), w \in B_{\rho_{n}}(0)$ such that $g_{n}(w)\left(u_{n}-w\right) \in \mathcal{M}_{\lambda}^{j}$. Let $0<\delta<\rho_{n}$ and let $w_{\delta}=\delta u$ with $\|u\|=1$. Fix $n$ and set $z_{\delta}=g_{n}\left(w_{\delta}\right)\left(u_{n}-w_{\delta}\right)$. By $z_{\delta} \in \mathcal{M}_{\lambda}^{j}$ and the property (ii), one has

$$
I_{\lambda}\left(z_{\delta}\right)-I_{\lambda}\left(u_{n}\right) \geq-\frac{1}{n}\left\|z_{\delta}-u_{n}\right\| .
$$

Then, by mean value theorem

$$
\left\langle I_{\lambda}^{\prime}\left(u_{n}\right), z_{\delta}-u_{n}\right\rangle+o\left(\left\|z_{\delta}-u_{n}\right\|\right) \geq-\frac{1}{n}\left\|z_{\delta}-u_{n}\right\| .
$$

Thus,

$$
\left\langle I_{\lambda}^{\prime}\left(u_{n}\right),\left(u_{n}-w_{\delta}\right)+\left(g_{n}\left(w_{\delta}\right)-1\right)\left(u_{n}-w_{\delta}\right)-u_{n}\right\rangle \geq-\frac{1}{n}\left\|z_{\delta}-u_{n}\right\|+o\left(\left\|z_{\delta}-u_{n}\right\|\right)
$$

which yields that

$$
-\delta\left\langle I_{\lambda}^{\prime}\left(u_{n}\right), u\right\rangle+\left(g_{n}\left(w_{\delta}\right)-1\right)\left\langle I_{\lambda}^{\prime}\left(u_{n}\right), u_{n}-w_{\delta}\right\rangle \geq-\frac{1}{n}\left\|z_{\delta}-u_{n}\right\|+o\left(\left\|z_{\delta}-u_{n}\right\|\right) .
$$

Combining this with $\left\langle I_{\lambda}^{\prime}\left(z_{\delta}\right), g_{n}\left(w_{\delta}\right)\left(u_{n}-w_{\delta}\right)\right\rangle=0$, we obtain

$$
\begin{equation*}
\left\langle I_{\lambda}^{\prime}\left(u_{n}\right), u\right\rangle \leq \frac{1}{n} \frac{\left\|z_{\delta}-u_{n}\right\|}{\delta}+\frac{o\left(\left\|z_{\delta}-u_{n}\right\|\right)}{\delta}+\frac{g_{n}\left(w_{\delta}\right)-1}{\delta}\left\langle I_{\lambda}^{\prime}\left(u_{n}\right), u_{n}-w_{\delta}\right\rangle . \tag{2.9}
\end{equation*}
$$

By Lemma 2.3 and the boundedness of $\left\{u_{n}\right\}$, we easily see that

$$
\left\|z_{\delta}-u_{n}^{j}\right\|=\left\|\left(g_{n}\left(w_{\delta}\right)-1\right)\left(u_{n}^{j}-w_{\delta}\right)-w_{\delta}\right\| \leq\left|g_{n}\left(w_{\delta}\right)-1\right| C_{5}+\delta
$$

and

$$
\lim _{\delta \rightarrow 0} \frac{\left|g_{n}\left(w_{\delta}\right)-1\right|}{\delta}=\left\langle g_{n}^{\prime}(0), u\right\rangle \leq\left\|g_{n}^{\prime}(0)\right\| \leq C_{6} .
$$

Therefore, for fixed $n$, we can conclude by passing $\delta \rightarrow 0$ in (2.9) that

$$
\left\langle I_{\lambda}^{\prime}\left(u_{n}\right), u\right\rangle \leq \frac{C}{n},
$$

which implies that $I_{\lambda}^{\prime}\left(u_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$, and Lemma 2.4 is proved.
Lemma 2.5. For all $\lambda>0$, if $\left\{u_{n}\right\} \subset \mathcal{M}_{\lambda}$ is a sequence satisfying

$$
I_{\lambda}\left(u_{n}\right) \rightarrow c<\frac{a^{2} S^{2}}{4\left(\lambda b S^{2}+Q_{M}\right)} \quad \text { and } \quad I_{\lambda}^{\prime}\left(u_{n}\right) \rightarrow 0
$$

as $n \rightarrow \infty$, then $\left\{u_{n}\right\}$ has a convergent subsequence.

Proof. As in the proof of Lemma 2.2, it is easy to verify that $\left\{u_{n}\right\}$ is bounded in $H_{0}^{1}(\Omega)$. Hence, we may assume that for some $u_{*} \in H_{0}^{1}(\Omega)$,

$$
\begin{array}{ll}
u_{n} \rightharpoonup u_{*} & \text { in } H_{0}^{1}(\Omega) \\
u_{n} \rightarrow u_{*} & \text { in } L^{r}(\Omega) \\
u_{n} \rightarrow u_{*} & \text { a.e. in } \Omega
\end{array}
$$

Denote $v_{n}=u_{n}-u_{*}$ and we claim that $\left\|v_{n}\right\| \rightarrow 0$. If not, there is a subsequence (still denoted by $\left\{v_{n}\right\}$ ) such that $\left\|v_{n}\right\| \rightarrow L$ with $L>0$. By $\left\langle I_{\lambda}^{\prime}\left(u_{n}\right), u_{*}\right\rangle=o(1)$ and the weak convergence of $u_{n}$, we see that

$$
\begin{equation*}
0=a\left\|u_{*}\right\|^{2}-\lambda b\left(L^{2}+\left\|u_{*}\right\|^{2}\right)\left\|u_{*}\right\|^{2}-\lambda \int\left|u_{*}\right|^{p}-\int Q(x)\left|u_{*}\right|^{4} \tag{2.10}
\end{equation*}
$$

Moreover, by $\left\langle I_{\lambda}^{\prime}\left(u_{n}\right), u_{n}\right\rangle=0$, we can apply the Brézis-Lieb Lemma to get

$$
\begin{align*}
0= & a\left(\left\|v_{n}\right\|^{2}+\left\|u_{*}\right\|^{2}\right)-\lambda b\left(\left\|v_{n}\right\|^{4}+2\left\|v_{n}\right\|^{2}\left\|u_{*}\right\|^{2}+\left\|u_{*}\right\|^{4}\right) \\
& -\lambda \int\left|u_{*}\right|^{p}-\int Q(x)\left|v_{n}\right|^{4}-\int Q(x)\left|u_{*}\right|^{4}+o(1) \tag{2.11}
\end{align*}
$$

Combining (2.10) and (2.11), we have

$$
\begin{equation*}
o(1)=a\left\|v_{n}\right\|^{2}-\lambda b\left\|v_{n}\right\|^{4}-\lambda b\left\|v_{n}\right\|^{2}\left\|u_{*}\right\|^{2}-\int Q(x)\left|v_{n}\right|^{4} \tag{2.12}
\end{equation*}
$$

and consequently,

$$
a\left\|v_{n}\right\|^{2}-\lambda b\left\|v_{n}\right\|^{4}-\lambda b\left\|v_{n}\right\|^{2}\left\|u_{*}\right\|^{2}=\int Q(x)\left|v_{n}\right|^{4}+o(1) \leq Q_{M} S^{-2}\left\|v_{n}\right\|^{4}+o(1)
$$

Passing the limit as $n \rightarrow \infty$, we obtain that

$$
\begin{equation*}
L^{2} \geq \frac{S^{2}\left(a-\lambda b\left\|u_{*}\right\|^{2}\right)}{\lambda b S^{2}+Q_{M}} \geq 0 \tag{2.13}
\end{equation*}
$$

By (2.10) and (2.13), we have

$$
\begin{align*}
I_{\lambda}\left(u_{*}\right) & =\frac{a}{2}\left\|u_{*}\right\|^{2}-\frac{\lambda b}{4}\left\|u_{*}\right\|^{4}-\frac{\lambda}{p} \int\left|u_{*}\right|^{p}-\frac{1}{4} \int Q(x)\left|u_{*}\right|^{4} \\
& =\frac{\lambda b}{4}\left\|u_{*}\right\|^{4}+\frac{\lambda b}{2} L^{2}\left\|u_{*}\right\|^{2}+\lambda\left(\frac{1}{2}-\frac{1}{p}\right) \int\left|u_{*}\right|^{p}+\frac{1}{4} \int Q(x)\left|u_{*}\right|^{4} \\
& \geq \frac{\lambda b}{4}\left\|u_{*}\right\|^{4}+\frac{\lambda b}{2} \frac{S^{2}\left(a-\lambda b\left\|u_{*}\right\|^{2}\right)}{\lambda b S^{2}+Q_{M}}\left\|u_{*}\right\|^{2}  \tag{2.14}\\
& =\frac{\lambda b\left(\lambda b S^{2}+Q_{M}\right)\left\|u_{*}\right\|^{4}}{4\left(\lambda b S^{2}+Q_{M}\right)}+\frac{\lambda a b S^{2}\left\|u_{*}\right\|^{2}}{2\left(\lambda b S^{2}+Q_{M}\right)}-\frac{\lambda^{2} b^{2} S^{2}\left\|u_{*}\right\|^{4}}{2\left(\lambda b S^{2}+Q_{M}\right)} \\
& \geq \frac{\lambda a b S^{2}\left\|u_{*}\right\|^{2}}{2\left(\lambda b S^{2}+Q_{M}\right)}-\frac{\lambda^{2} b^{2} S^{2}\left\|u_{*}\right\|^{4}}{4\left(\lambda b S^{2}+Q_{M}\right)}
\end{align*}
$$

Furthermore, using (2.12)-(2.14), we deduce that

$$
\begin{aligned}
c+o(1)= & I_{\lambda}\left(u_{n}\right) \\
= & \frac{a}{2}\left\|u_{n}\right\|^{2}-\frac{\lambda b}{4}\left\|u_{n}\right\|^{4}-\frac{\lambda}{p} \int\left|u_{n}\right|^{p}-\frac{1}{4} \int Q(x)\left|u_{n}\right|^{4} \\
= & \frac{a}{2}\left\|u_{*}\right\|^{2}-\frac{\lambda b}{4}\left\|u_{*}\right\|^{4}-\frac{\lambda}{p} \int\left|u_{*}\right|^{p}-\frac{1}{4} \int Q(x)\left|u_{*}\right|^{4} \\
& +\frac{a}{2}\left\|v_{n}\right\|^{2}-\frac{\lambda b}{4}\left\|v_{n}\right\|^{4}-\frac{\lambda b}{2}\left\|v_{n}\right\|^{2}\left\|u_{*}\right\|^{2}-\frac{1}{4} \int Q(x)\left|v_{n}\right|^{4}+o(1) \\
= & I\left(u_{*}\right)+\frac{a}{4}\left\|v_{n}\right\|^{2}-\frac{\lambda b}{4}\left\|v_{n}\right\|^{2}\left\|u_{*}\right\|^{2}+o(1) \\
= & I\left(u_{*}\right)+\frac{a-\lambda b\left\|u_{*}\right\|^{2}}{4} L^{2}+o(1) \\
\geq & I\left(u_{*}\right)+\frac{a^{2} S^{2}}{4\left(\lambda b S^{2}+Q_{M}\right)}-\frac{\lambda a b S^{2}\left\|u_{*}\right\|^{2}}{2\left(\lambda b S^{2}+Q_{M}\right)}+\frac{\lambda^{2} b^{2} S^{2}\left\|u_{*}\right\|^{4}}{4\left(\lambda b S^{2}+Q_{M}\right)}+o(1) \\
\geq & \frac{a^{2} S^{2}}{4\left(\lambda b S^{2}+Q_{M}\right)}+o(1)
\end{aligned}
$$

a contradiction to the assumption $c<\frac{a^{2} S^{2}}{4\left(\lambda b S^{2}+Q_{M}\right)}$. Therefore, the claim holds, namely, $u_{n} \rightarrow u_{*}$ in $H_{0}^{1}(\Omega)$. This completes the proof of Lemma 2.5.

## 3 Proof of Theorem 1.1

Proof of Theorem 1.1. By Lemma 2.4, we know that there exists $\Lambda_{0}$ such that for each $\lambda \in$ $\left(0, \Lambda_{0}\right)$ and $j=1,2, \ldots, k$, there is a minimizing sequence $\left\{u_{n}^{j}\right\} \subset \mathcal{M}_{\lambda}^{j}$ satisfying $u_{n}^{j} \geq 0$, $I_{\lambda}\left(u_{n}^{j}\right) \rightarrow m_{\lambda}^{j}$ and $I_{\lambda}^{\prime}\left(u_{n}^{j}\right) \rightarrow 0$. From Lemmas 2.1 and 2.5 , it follows that $u_{n}^{j} \rightarrow u^{j}$ and $u^{j} \geq 0$ is a weak solution of (1.1). Furthermore, standard elliptic regularity argument and strong maximum principle imply that $u^{j}$ is a positive solution. Finally, $u^{j}, j=1,2, \ldots, k$, are different positive solutions since $\beta\left(u^{j}\right) \in B_{\eta}\left(x^{j}\right)$ and $B_{\eta}\left(x^{j}\right)$ are disjoint. The proof is completed.

## Acknowledgements

The author would like to thank the referee for careful reading of this paper and for the helpful comments. This research was supported by the National Natural Science Foundation of China (11871152).

## References

[1] C. O. Alves, F. J. S. A. Corrêa, T. F. Ma, Positive solutions for a quasilinear elliptic equation of Kirchhoff type, Comput. Math. Appl. 49(2005), No. 1, 85-93. https://doi. org/10.1016/j.camwa.2005.01.008; MR2123187; Zbl 1130.35045
[2] H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic problems involving critical Sobolev exponent, Comm. Pure Appl. Math. 36(1983), No. 4, 437-477. https: //doi.org/10.1002/cpa.3160360405; MR0709644; Zbl 0541.35029
[3] D. M. Cao, E. S. Noussair, Multiple positive and nodal solutions for semilinear elliptic problems with critical exponents, Indiana Univ. Math. J. 44(1995), No. 4, 1249-1271. https: //doi.org/10.1512/iumj.1995.44.2027; MR1386768; Zbl 0849.35030
[4] Y. Duan, X. Sun, H. Y. Li, Existence and multiplicity of positive solutions for a nonlocal problem, J. Nonlinear Sci. Appl. 10(2017), No. 11, 6056-6061. https://doi .org/10. 22436/ jnsa.010.11.40; MR3738822; Zbl 1412.35021
[5] Y. Duan, X. Sun, J.F. Liao, Multiplicity of positive solutions for a class of critical Sobolev exponent problems involving Kirchhoff-type nonlocal term, Comput. Math. Appl. 75(2018), No. 12, 4427-4437. https://doi.org/10.1016/j.camwa.2018.03.041; MR3800181; Zbl 1421.35106
[6] H. N. Fan, Multiple positive solutions for a class of Kirchhoff type problems involving critical Sobolev exponents, J. Math. Anal. Appl. 431(2015), No. 1, 150-168. https://doi. org/10.1016/j.jmaa.2015.05.053; MR3357580; Zbl 1319.35050
[7] G. Figueiredo, Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument, J. Math. Anal. Appl. 401(2013), No. 2, 706-713. https: //doi.org/10.1016/j.jmaa.2012.12.053; MR3018020; Zbl 1307.35110
[8] G. Kirchнoff, Mechanik, Teubner, Leipzig, 1883.
[9] C. Y. Lei, J. F. Liao, H. M. Suo, Multiple positive solutions for nonlocal problems involving a sign-changing potential, Electron. J. Differential Equations 2017, No. 9, 1-8. MR3609137; Zbl 1357.35107
[10] C. Y. Lei, C. M. Chu, H. M. Suo, Positive solutions for a nonlocal problem with singularity, Electron. J. Differential Equations 2017, No. 85, 1-9. MR3651882; Zbl 1370.35086
[11] G. B. Li, H.Y. Ye, Existence of positive solutions for nonlinear Kirchhoff type problems in $\mathbb{R}^{3}$ with critical Sobolev exponent, Math. Method. Appl. Sci. 37(2014), No. 16, 2570-2584. https://doi.org/10.1002/mma.3000; MR3271105; Zbl 1303.35009
[12] S. H. Liang, J. H. Zhang, Existence of solutions for Kirchhoff type problems with critical nonlinearity in $\mathbb{R}^{3}$, Nonlinear Anal. Real World Appl. 17(2014), 126-136. https: //doi. org/ 10.1016/j.nonrwa.2013.10.011; MR3158465; Zbl 1302.35031
[13] J. F. Liao, J. Liu, P. Zhang, C. L. Tang, Existence and multiplicity of positive solutions for a class of elliptic equations involving critical Sobolev exponents, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 110(2016), No. 2, 483-501. https://doi.org/10. 1007/s13398-015-0244-4; MR3534502; Zbl 1372.35104
[14] P. L. Lions, The concentration-compactness principle in the calculus of variations: the limit case II, Rev. Mat. Iberoamericana 1(1985), No. 2, 45-121. https ://doi .org/10.4171/ RMI/12; MR0850686; Zbl 0704.49006
[15] D. Naimen, The critical problem of Kirchhoff type elliptic equations in dimension four, J. Differential Equations 257(2014), No. 4, 1168-1193. https://doi.org/10.1016/j.jde. 2014.05.002; MR3210026; Zbl 1301.35022
[16] X. T. Qian, W. Chao, Existence of positive solutions for nonlocal problems with indefinite nonlinearity, Bound. Value Probl. 2020, Paper No. 40, 13 pp. https://doi.org/10.1186/ s13661-020-01343-2; MR4069873
[17] X. T. Qian, J. Q. Chen, Multiple positive and sign-changing solutions of an elliptic equation with fast increasing weight and critical growth, J. Math. Anal. Appl. 465(2018), No. 2, 1186-1208. https://doi.org/10.1016/j.jmaa. 2018.05.058; MR3809352; Zbl 1455.35069
[18] X.T. Qian, Ground state sign-changing solutions for a class of nonlocal problem, J. Math. Anal. Appl. 495(2021), No. 2, 1-15. https://doi.org/10.1016/j.jmaa.2020.124753; MR4182964; Zbl 1459.35165
[19] X. T. Qian, W. Chao, Existence and concentration of ground state solutions for a class of nonlocal problem in $\mathbb{R}^{N}$, Nonlinear Anal. 203(2021), 112170, 15 pp . https://doi.org/10. 1016/j.na.2020.112170; MR4164555; Zbl 1459.35166
[20] Z. Y. Tang, Z. Q. Ou, Infinitely many solutions for a nonlocal problem, J. Appl. Anal. Comput. 10(2020), No. 5, 1912-1917. https://doi.org/10.11948/20190286; MR4147814; Zbl 07331967
[21] Y. Wang, H. M. Suo, C. Y. Lei, Multiple positive solutions for a nonlocal problem involving critical exponent, Electron. J. Differential Equations 2017, No. 275, 1-11. MR3723548; Zbl 1386.35007
[22] Y. Wang, X. Yang, Infinitely many solutions for a new Kirchhoff-type equation with subcritical exponent, Appl. Anal., published online: 25 May 2020. https://doi.org/10. 1080/00036811.2020.1767288
[23] G. S. Yin, J. S. Liu, Existence and multiplicity of nontrivial solutions for a nonlocal problem, Bound. Value Probl. 2015, 2015:26, 1-7. https://doi.org/10.1186/s13661-015-0284-x; MR3311505; Zbl 1332.35099
[24] J. Zhang, Z. Y. Zhang, Existence of nontrivial solution for a nonlocal problem with subcritical nonlinearity, Adv. Difference Equations 2018, Paper No. 359, 1-8. https://doi. org/10.1186/s13662-018-1823-4; MR3863087; Zbl 1448.35203

Electronic Journal of Qualitative Theory of Differential Equations

# Uniqueness and monotonicity of solutions for fractional equations with a gradient term 

Pengyan Wang ${ }^{\boxtimes}$<br>School of Mathematics and Statistics, Xinyang Normal University, Xinyang, 464000, P. R. China

Received 4 May 2021, appeared 30 July 2021
Communicated by Patrizia Pucci


#### Abstract

In this paper, we consider the following fractional equation with a gradient term $$
(-\Delta)^{s} u(x)=f(x, u(x), \nabla u(x)),
$$ in a bounded domain and the upper half space. Firstly, we prove the monotonicity and uniqueness of solutions to the fractional equation in a bounded domain by the sliding method. In order to obtain maximum principle on unbounded domain, we need to estimate the singular integrals define the fractional Laplacians along a sequence of approximate maximum points by using a generalized average inequality. Then we prove monotonicity and uniqueness of solutions to fractional equation in $\mathbb{R}_{+}^{n}$ by the sliding method. In order to solve the difficulties caused by the gradient term, some new techniques are developed. The paper may be considered as an extension of Berestycki and Nirenberg [J. Geom. Phys. 5(1988), 237-275].


Keywords: fractional equation with gradient term, monotonicity, uniqueness, sliding method.

2020 Mathematics Subject Classification: 35R11, 35A09, 35B06, 35B09.

## 1 Introduction

During the last decades, fractional Laplacian has attracted more and more attention due to its various applications. The methods to study the fractional Laplacian are the extension method [6], moving planes method in integral form [11], the method of moving sphere [26] and direct methods of moving planes [9,22] etc. Recently, to study the monotonicity of the solution, Liu [28], Wu and Chen [35,36] introduced a direct sliding method for fractional Laplacian and fractional $p$-Laplacian. Berestycki and Nirenberg [3-5] first developed the sliding method, which was used to establish qualitative properties of solutions for nonlinear elliptic equations involving the regular Laplacian such as monotonicity, nonexistence and uniqueness etc. The essential ingredients are different forms of maximum principles. The main idea lies in comparing values of the solution to the equation at two different points, between which one point is obtained from the other by sliding the domain in a given direction, and then the domain

[^20]is slide back to a critical position. While in the method of moving planes, one point is the reflection of the other.

Inspired by the above article, in this article, we show the monotonicity, antisymmetry and uniqueness of solutions for the following fractional equation with a gradient term

$$
\begin{equation*}
(-\Delta)^{s} u(x)=f(x, u(x), \nabla u(x)), \tag{1.1}
\end{equation*}
$$

where $\nabla u$ denotes the gradient of $u$, the fractional Laplacian $(-\Delta)^{s}$ with $0<s<1$ is given by

$$
\begin{aligned}
(-\Delta)^{s} u(x) & =C_{n, s} \text { P.V. } \int_{\mathbb{R}^{n}} \frac{u(x)-u(y)}{|x-y|^{n+2 s}} d y \\
& =C_{n, s} \lim _{\epsilon \rightarrow 0} \int_{\mathbb{R}^{n} \backslash B_{\epsilon}(x)} \frac{u(x)-u(y)}{|x-y|^{n+2 s}} d y .
\end{aligned}
$$

Define

$$
\mathcal{L}_{2 s}=\left\{u: u \in L_{l o c}^{1}\left(\mathbb{R}^{n}\right), \int_{\mathbb{R}^{n}} \frac{|u(x)|}{1+|x|^{n+2 s}} d x<\infty\right\},
$$

then it is easy to see that for $u \in C_{\text {loc }}^{1,1}\left(\mathbb{R}^{n}\right) \cap \mathcal{L}_{2 s},(-\Delta)^{s} u$ is well defined.
When $s=1$, [3] derived the monotonicity, symmetry and uniqueness of (1.1) in a finite cylinder and a bounded domain which is convex in the $x_{1}$ direction by the sliding method. In the case $s=1, f(x, u, \nabla u)=f(u)$, Gidas, Ni, Nirenberg [21] obtained monotonicity and symmetry for positive solutions of (1.1), vanishing on the boundary, using the maximum principle and the method of moving planes; in $[2,5]$, Berestycki, Cafferelli and Nirenberg considered the monotonicity and uniqueness of solution for (1.1) by the sliding method. Recently, in the case $0<s<1$, Chen, Li and Li [9] investigated the semilinear equation in the whole space with $f(x, u, \nabla u)=u^{p}, 1<p \leq \frac{n+2 s}{n-2 s}$, developed a direct method of moving planes for the fractional Laplacian and showed that the nonnegative solution of (1.1) is radially symmetric and monotone decreasing about some point in the critical case $p=\frac{n+2 s}{n-2 s}$ and nonexistence of positive solutions in the subcritical case $1<p<\frac{n+2 s}{n-2 s}$; Dipierro, Soave and Valdinoci [16] proved symmetry, monotonicity and rigidity results to (1.1) in an unbounded domain with the epigraph property.

The purpose of the present paper is to extend the results in [3] to the fractional equation. On the one hand, we extent the case $s=1$ in [3] to the fractional case $0<s<1$, and extend bounded domain to $\mathbb{R}_{+}^{n}$. On the other hand, the nonlinear term $f(x, u, \nabla u)$ has a broader form containing nonlinear term $f(u)$ and $f(x, u)$.

In order to solve the difficulty that the nonlinear term at the right side of (1.1) contain the gradient term, in the bounded domain when deriving the contradiction for the minimum point of the function $w^{\tau}(x)$ (see Section 2 below for definition), for the first time, we use the technique of finding the minimum value of the function $w^{\tau}(x)$ for the variables $\tau$ and $x$ at the same time. This is different from the previous sliding process which only finds the minimum value of the variable $x$ for the fixed $\tau$. In the whole space, we estimate the singular integrals defining the fractional Laplacian along a sequence of approximate maximum, and the estimating is for $\tau$ and the sequence of approximate maximum at the same time.

In order to apply the sliding method, we give the exterior condition on $u$. Let $u(x)=$ $\varphi(x), x \in \Omega^{c}$, and assume that
(C) for any three points $x=\left(x^{\prime}, x_{n}\right), y=\left(x^{\prime}, y_{n}\right)$ and $z=\left(x^{\prime}, z_{n}\right)$ lying on a segment parallel to the $x_{n}$ axis, $y_{n}<x_{n}<z_{n}$, with $y, z \in \Omega^{c}$, we have

$$
\begin{equation*}
\varphi(y)<u(x)<\varphi(z), \quad \text { if } x \in \Omega \tag{1.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\varphi(y) \leq \varphi(x) \leq \varphi(z), \quad \text { if } x \in \Omega^{c} . \tag{1.3}
\end{equation*}
$$

Remark 1.1. The same monotonicity conditions (1.2) and (1.3) (with $\Omega^{c}$ replaced by $\partial \Omega$ ) were assumed in [4,5,35].

The main result of this paper is
Theorem 1.2. Suppose that $u \in C_{\text {loc }}^{1,1}(\Omega) \cap C(\bar{\Omega})$ satisfies (C) and is a solution of equation

$$
\begin{cases}(-\Delta)^{s} u(x)=f(x, u, \nabla u), & x \in \Omega  \tag{1.4}\\ u(x)=\varphi(x), & x \in \Omega^{c}\end{cases}
$$

where $\Omega$ is a bounded domain which is convex in $x_{n}$ direction. Assume that $f$ is continuous in all variables, locally Lipschitz continuous in $(u, \nabla u)$ and is nondecreasing in $x_{n}$. Then $u$ is strictly monotone increasing with respect to $x_{n}$ in $\Omega$, i.e., for any $\tau>0$,

$$
u\left(x^{\prime}, x_{n}+\tau\right)>u\left(x^{\prime}, x_{n}\right), \quad \text { for all }\left(x^{\prime}, x_{n}\right),\left(x^{\prime}, x_{n}+\tau\right) \in \Omega .
$$

Furthermore, the solution of (1.4) is unique.
Remark 1.3. Theorem (1.2) includes the result of Theorem 2 in [35], and we also prove the uniqueness of solutions in bounded domain. If $\Omega$ is the finite cylinder $\mathcal{C}=\left\{x=\left(x^{\prime}, x_{n}\right) \in\right.$ $\left.\mathbb{R}^{n}| | x_{n} \mid<l, x^{\prime} \in \omega\right\}$, where $l>0$ and $\omega$ is a bounded domain in $\mathbb{R}^{n-1}$ with smooth boundary, the result of Theorem 1.2 still holds.

Remark 1.4. The conditions in Theorem 1.2 and Theorem 1 of [14] are different. Neither implies the other. Cheng, Huang and Li [14] studied the positive solution $u$ and obtained that $u$ is strictly increasing in the half of $\Omega$ in $x_{n}$-direction with $x_{n}<0$ by the method of moving planes, but the solution we study can be negative and is strictly increasing with respect to $x_{n}$ in the whole domain $\Omega$ by the sliding method.

We also have a new antisymmetry result for the equation (1.4) if the bounded domain $\Omega$ is symmetric about $x_{n}=0$.

Corollary 1.5 (Antisymmetry). Assume that the conditions of Theorem 1.2 are satisfied and in addition that $\varphi$ is odd in $x_{n}$ on $\Omega^{c}$. If $f(x, u, \nabla u)$ is odd in $\left(x_{n}, u, \nabla_{x^{\prime}} u\right)$. Then $u$ is odd, i.e. antisymmetric in $x_{n}$ :

$$
u\left(x^{\prime},-x_{n}\right)=-u\left(x^{\prime}, x_{n}\right), \quad \forall x \in \Omega .
$$

This follows from the fact that $\bar{u}=-u\left(x^{\prime},-x_{n}\right)$ is a solution satisfying the same conditions, and so is $u$.

For the unbounded domain, we give the following result on $\mathbb{R}_{+}^{n}$.
Theorem 1.6. Suppose that $u \in C_{\text {loc }}^{1,1}\left(\mathbb{R}_{+}^{n}\right) \cap \mathcal{L}_{2 s}\left(\mathbb{R}^{n}\right) \cap C\left(\overline{\mathbb{R}_{+}^{n}}\right)$ is a solution of

$$
\begin{cases}(-\Delta)^{s} u(x)=f(u, \nabla u), & x \in \mathbb{R}_{+}^{n},  \tag{1.5}\\ 0<u(x) \leq \mu, & x \in \mathbb{R}_{+}^{n}, \\ u(x)=0, & x \notin \mathbb{R}_{+}^{n},\end{cases}
$$

and

$$
\begin{equation*}
\lim _{x_{n} \rightarrow+\infty} u\left(x^{\prime}, x_{n}\right)=\mu, \quad \text { uniformly for all } x^{\prime} \in \mathbb{R}^{n-1} . \tag{1.6}
\end{equation*}
$$

Assume that $f$ is bounded, continuous in all variables and nonincreasing in $u \in[\mu-\delta, \mu]$ for some $\delta>0$. Then $u$ is strictly monotone increasing in $x_{n}$ direction, and moreover it depends on $x_{n}$ only.

Furthermore, the solution of (1.5) is unique.
Theorem 1.6 is closely related to the following well-known De Giorgi conjecture [19].
Conjecture (De Giorgi [19]). If $u$ is a solution of

$$
-\Delta u=u-u^{3},
$$

such that

$$
\lim _{x_{n} \rightarrow \pm \infty} u\left(x^{\prime}, x_{n}\right)= \pm 1, \quad \text { for all } x^{\prime} \in \mathbb{R}^{n-1},
$$

and

$$
|u(x)| \leq 1, \quad x \in \mathbb{R}^{n}, \quad \frac{\partial u}{\partial x_{n}}>0 .
$$

Then there exists a vector $\mu \in \mathbb{R}^{n-1}$ and a function $u_{1}: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$
u\left(x^{\prime}, x_{n}\right)=u_{1}\left(\mu x^{\prime}+x_{n}\right) \quad \text { in } \mathbb{R}^{n} .
$$

The other symmetry, uniqueness and monotonicity results on local and nonlocal equations, we also refer readers to [ $1,18,24,25$ ] for semilinear elliptic equations, [ $9,13,17,23,30,31]$ for fractional equations, $[34,38]$ for weighted fractional equation, $[14,37]$ for fractional equations with a gradient term, [27] for integral system with negative exponents, [12] for weighted Hardy-sobolev type system, $[8,32,33]$ for fully nonlinear nonlocal equations with gradient term, $[7,15,29]$ for fractional $p$-Laplace equation, and references therein.

The paper is organized as follows. In Section 2 we prove Theorem 1.2 via the sliding method. In Section 3, we first establish a maximum principle in the unbounded domain, then uniqueness and monotonicity for the fractional equation with a gradient term on $\mathbb{R}_{+}^{n}$ are obtained.

## 2 The proof of Theorem 1.2

For convenience, we list some notations used frequently. For $\tau \in \mathbb{R}$, denote $x=\left(x^{\prime}, x_{n}\right)$, $x^{\prime}=\left(x_{1}, \cdots, x_{n-1}\right) \in \mathbb{R}^{n-1}$. Set

$$
u^{\tau}(x)=u\left(x^{\prime}, x_{n}+\tau\right), \quad w^{\tau}(x)=u^{\tau}(x)-u(x) .
$$

Proof of Theorem 1.2. For $\tau>0$, it is defined on the set $\Omega^{\tau}=\Omega-\tau e_{n}$ which is obtained from $\Omega$ by sliding it downward a distance $\tau$ parallel to the $x_{n}$ axis, where $e_{n}=(0, \ldots, 0,1)$. Set

$$
D^{\tau}:=\Omega^{\tau} \cap \Omega, \quad \tilde{\tau}=\sup \left\{\tau \mid \tau>0, D^{\tau} \neq \varnothing\right\}
$$

and

$$
w^{\tau}(x)=u^{\tau}(x)-u(x), \quad x \in D^{\tau} .
$$

We mainly divide the following two steps to prove that $u$ is strictly increased in the $x_{n}$ direction, i.e.

$$
\begin{equation*}
w^{\tau}(x)>0, x \in D^{\tau}, \quad \text { for any } 0<\tau<\tilde{\tau} . \tag{2.1}
\end{equation*}
$$

Step 1. For $\tau$ sufficiently close to $\tilde{\tau}$ i.e., $D^{\tau}$ is narrow, we claim that there exists $\delta>0$ small enough such that

$$
\begin{equation*}
w^{\tau}(x) \geq 0, \quad \forall x \in D^{\tau}, \forall \tau \in(\tilde{\tau}-\delta, \tilde{\tau}) \tag{2.2}
\end{equation*}
$$

Otherwise, we set

$$
A_{0}=\min _{\substack{x \in \bar{D}^{\tau} \\ \tilde{\tau}-\delta<\tau<\tilde{\tau}}} w^{\tau}(x)<0
$$

From condition (C), $A_{0}$ can be obtained for some $\left(\tau^{0}, x^{0}\right) \in\left\{(\tau, x) \mid(\tau, x) \in(\tilde{\tau}-\delta, \tilde{\tau}) \times D^{\tau}\right\}$. Noticing that $w^{\tau^{0}}(x) \geq 0, x \in \partial D^{\tau^{0}}$, we arrive at $x^{0} \in D^{\tau^{0}}$. So $w^{\tau^{0}}\left(x^{0}\right)=A_{0}$. Since $\left(\tau^{0}, x^{0}\right)$ is a minimizing point, we have $\nabla w^{\tau^{0}}\left(x^{0}\right)=0$, i.e., $\nabla u^{\tau^{0}}\left(x^{0}\right)=\nabla u\left(x^{0}\right)$. Since $u^{\tau^{0}}$ satisfies the same equation (1.4) in $\Omega^{\tau^{0}}$ as $u$ does in $\Omega$, and $f$ is nondecreasing in $x_{n}$, so we have

$$
\begin{align*}
(-\Delta)^{s} w^{\tau^{0}}\left(x^{0}\right) & =f\left(\left(x^{0}\right)^{\prime}, x_{n}^{0}+\tau^{0}, u^{\tau^{0}}\left(x^{0}\right), \nabla u^{\tau^{0}}\left(x^{0}\right)\right)-f\left(x^{0}, u\left(x^{0}\right), \nabla u\left(x^{0}\right)\right) \\
& \geq f\left(x^{0}, u^{\tau^{0}}\left(x^{0}\right), \nabla u^{\tau^{0}}\left(x^{0}\right)\right)-f\left(x^{0}, u\left(x^{0}\right), \nabla u\left(x^{0}\right)\right) \\
& =f\left(x^{0}, u^{\tau^{0}}\left(x^{0}\right), \nabla u\left(x^{0}\right)\right)-f\left(x^{0}, u\left(x^{0}\right), \nabla u\left(x^{0}\right)\right)  \tag{2.3}\\
& =-c^{\tau^{0}}\left(x^{0}\right) w^{\tau^{0}}\left(x^{0}\right)
\end{align*}
$$

where $-c^{\tau^{0}}\left(x^{0}\right)=\frac{f\left(x^{0}, u^{\tau^{0}}\left(x^{0}\right), \nabla u\left(x^{0}\right)\right)-f\left(x^{0}, u\left(x^{0}\right), \nabla u\left(x^{0}\right)\right)}{u^{\tau^{0}}\left(x^{0}\right)-u\left(x^{0}\right)}$ is a $L^{\infty}$ function satisfying

$$
\left|c^{\tau^{0}}\left(x^{0}\right)\right| \leq C, \quad \forall x^{0} \in D^{\tau^{0}}
$$

Hence

$$
(-\Delta)^{s} w^{\tau^{0}}\left(x^{0}\right)+c^{\tau^{0}}\left(x^{0}\right) w^{\tau^{0}}\left(x^{0}\right) \geq 0
$$

On the other hand, we obtain

$$
\begin{align*}
& (-\Delta)^{s} w^{\tau^{0}}\left(x^{0}\right)+c^{\tau^{0}}\left(x^{0}\right) w^{\tau^{0}}\left(x^{0}\right) \\
& \quad=C_{n, s} P . V \cdot \int_{\mathbb{R}^{n}} \frac{w^{\tau^{0}}\left(x^{0}\right)-w^{\tau^{0}}(y)}{\left|x^{0}-y\right|^{n+2 s}} d y+c^{\tau^{0}}\left(x^{0}\right) w^{\tau^{0}}\left(x^{0}\right) \\
& \quad \leq C_{n, s} w^{\tau^{0}}\left(x^{0}\right) \int_{\left(D^{\tau^{0}}\right)^{c}} \frac{1}{\left|x^{0}-y\right|^{n+2 s}} d y+\inf _{D^{\tau^{0}}} c^{\tau^{0}}(x) w^{\tau^{0}}\left(x^{0}\right)  \tag{2.4}\\
& \quad \leq w^{\tau^{0}}\left(x^{0}\right)\left(\frac{C_{1}}{d_{n}^{2 s}}-C\right) \\
& \quad<0
\end{align*}
$$

where $d_{n}$ denotes the width of $D^{\tau^{0}}$ in the $x_{n}$ direction and $D^{\tau^{0}}$ is narrow. This is a contradiction.

Therefore we derive (2.2) is true for $\tau$ sufficiently close to $\tilde{\tau}$.
Step 2. The inequality (2.2) provides a starting point, from which we can carry out the sliding. Now we decrease $\tau$ as long as (2.2) holds to its limiting position. Define

$$
\tau_{0}=\inf \left\{\tau \mid w^{\tau}(x) \geq 0, x \in D^{\tau}, 0<\tau<\tilde{\tau}\right\}
$$

We will prove

$$
\tau_{0}=0
$$

Otherwise, assume $\tau_{0}>0$, we show that the domain $\Omega$ can be slided upward a little bit more and we still have

$$
\begin{equation*}
w^{\tau}(x) \geq 0, \quad x \in D^{\tau}, \quad \text { for any } \tau_{0}-\varepsilon<\tau \leq \tau_{0} \tag{2.5}
\end{equation*}
$$

which contradicts the definition of $\tau_{0}$.
Since $w^{\tau_{0}}(x)>0, x \in \Omega \cap \partial D^{\tau_{0}}$ by condition $(C)$ and $w^{\tau_{0}}(x) \geq 0, x \in D^{\tau_{0}}$, then

$$
w^{\tau_{0}}(x) \not \equiv 0, \quad x \in D^{\tau_{0}}
$$

If there exists a point $\tilde{x} \in D^{\tau_{0}}$ such that $w^{\tau_{0}}(\tilde{x})=0$, then $\tilde{x}$ is the minimum point. So we have $\nabla w^{\tau_{0}}(\tilde{x})=0$ and

$$
(-\Delta)^{s} w^{\tau_{0}}(\tilde{x})=C_{n, s} P . V \cdot \int_{\mathbb{R}^{n}} \frac{w^{\tau_{0}}(\tilde{x})-w^{\tau_{0}}(y)}{|\tilde{x}-y|^{n+2 s}} d y<0
$$

which contradicts to

$$
\begin{aligned}
(-\Delta)^{s} w^{\tau_{0}}(\tilde{x}) & =f\left(\tilde{x}^{\prime}, \tilde{x}_{n}+\tau_{0}, u^{\tau_{0}}(\tilde{x}), \nabla u^{\tau_{0}}(\tilde{x})\right)-f(\tilde{x}, u(\tilde{x}), \nabla u(\tilde{x})) \\
& \geq f\left(\tilde{x}, u^{\tau_{0}}(\tilde{x}), \nabla u^{\tau_{0}}(\tilde{x})\right)-f(\tilde{x}, u(\tilde{x}), \nabla u(\tilde{x})) \\
& =0
\end{aligned}
$$

Hence,

$$
\begin{equation*}
w^{\tau_{0}}(x)>0, \quad x \in D^{\tau_{0}} \tag{2.6}
\end{equation*}
$$

Next we will prove (2.5). Suppose (2.5) is not true, one has

$$
A_{1}=\min _{\substack{x \in D^{\tau} \\ \tau_{0}-\varepsilon<\tau<\tau_{0}}} w^{\tau}(x)<0
$$

The minimum $A_{1}$ can be obtained for some $\mu \in\left(\tau_{0}-\varepsilon, \tau_{0}\right), \bar{x} \in D^{\mu}$ where $w^{\mu}(\bar{x})=A_{1}$ by condition (C). We carve out of $D^{\tau_{0}}$ a closed set $K \subset D^{\tau_{0}}$ such that $D^{\tau_{0}} \backslash K$ is narrow. According to (2.6),

$$
w^{\tau_{0}}(x) \geq C_{0}>0, \quad x \in K
$$

From the continuity of $w^{\tau}$ in $\tau$, we have for $\operatorname{small} \varepsilon>0$,

$$
\begin{equation*}
w^{\mu}(x) \geq 0, \quad x \in K \tag{2.7}
\end{equation*}
$$

From (C), it follows

$$
w^{\mu}(x) \geq 0, \quad x \in\left(D^{\mu}\right)^{c}
$$

So $\bar{x} \in D^{\mu} \backslash K$ and $\nabla w^{\mu}(\bar{x})=0$. Since $D^{\tau_{0}} \subset D^{\mu}$ and small $\varepsilon$, we obtain that $D^{\mu} \backslash K$ is a narrow domain. Similar to (2.3), we have

$$
(-\Delta)^{s} w^{\mu}(\bar{x})+c(\bar{x}) w^{\mu}(\bar{x}) \geq 0
$$

Similar to (2.4) and narrow domain $D^{\mu} \backslash K$, we have

$$
(-\Delta)^{s} w^{\mu}(\bar{x})+c(\bar{x}) w^{\mu}(\bar{x})<0
$$

This is a contradiction. Hence we derive (2.5), which contradicts to the definition of $\tau_{0}$. So $\tau_{0}=0$. Therefore, we have shown that

$$
\begin{equation*}
w^{\tau}(x) \geq 0, \quad x \in D^{\tau}, \quad \text { for any } 0<\tau<\tilde{\tau} \tag{2.8}
\end{equation*}
$$

Next we prove (2.1). Since

$$
w^{\tau}(x) \not \equiv 0, \quad x \in D^{\tau}, \quad \text { for any } 0<\tau<\tilde{\tau},
$$

if there exists a point $x^{0}$ for some $\tau_{1} \in(0, \tilde{\tau})$ such that $w^{\tau_{1}}\left(x^{0}\right)=0$, then $x^{0}$ is the minimum point and

$$
(-\Delta)^{s} w^{\tau_{1}}\left(x^{0}\right)=C_{n, s} P . V \cdot \int_{\mathbb{R}^{n}} \frac{w^{\tau_{1}}\left(x^{0}\right)-w^{\tau_{1}}(y)}{\left|x^{0}-y\right|^{n+2 s}} d y<0 .
$$

This contradicts to

$$
(-\Delta)^{s} w^{\tau_{1}}\left(x^{0}\right)=f\left(\left(x^{0}\right)^{\prime}, x_{n}^{0}+\tau_{1}, u^{\tau_{1}}\left(x^{0}\right), \nabla u^{\tau_{1}}\left(x^{0}\right)\right)-f\left(x^{0}, u\left(x^{0}\right), \nabla u\left(x^{0}\right)\right) \geq 0 .
$$

Therefore, we arrive at (2.1).
Now we prove uniqueness. If $\underline{u}$ is another solution satisfying the same conditions, the same argument as before but replace $w^{\tau}=u^{\tau}-u$ with $w^{\tau}=\underline{u}^{\tau}-u$. Similarly to (2.8), we have $\underline{u}^{\tau}(x) \geq u$ in $D^{\tau}$ for any $0<\tau<\tilde{\tau}$. Hence, $\underline{u} \geq u$. Interchanging the roles of $u$ and $\underline{u}$, we find the opposite inequality. Therefore, $\underline{u}=u$.

This completes the proof of Theorem 1.2.

## 3 The uniqueness and monotonicity of solution on $\mathbb{R}_{+}^{n}$

In the section, we will prove Theorem 1.6. We first establish a maximum principle in the unbounded domain for the fractional equation with a gradient term.
Lemma 3.1 (Maximum principle). Let $D$ be an open set in $\mathbb{R}^{n}$, possibly unbounded and disconnected, suppose that

$$
\varliminf_{k \rightarrow \infty} \frac{\left|D^{c} \cap\left(B_{2^{k+1}}(q) \backslash B_{2^{k}}(q)\right)\right|}{\left|\left(B_{2^{k+1}}(q) \backslash B_{2^{k}}(q)\right)\right|}>0
$$

where $q$ is any point in $D$. Let $w \in C_{l o c}^{1,1}(D) \cap \mathcal{L}_{2 s}$ be bounded from above and satisfy

$$
\begin{cases}(-\Delta)^{s} w(x)+c(x) w(x)+\sum_{j=1}^{n} b_{j}(x) w_{j}(x) \leq 0, & x \in D  \tag{3.1}\\ w(x) \leq 0, & x \in \mathbb{R}^{n} \backslash D\end{cases}
$$

for some nonnegation function $c(x)$. Then

$$
w(x) \leq 0, x \in D
$$

Furthermore, we have

$$
\begin{equation*}
\text { either } w(x)<0 \text { in } D \text { or } w(x) \equiv 0 \text { in } \mathbb{R}^{n} . \tag{3.2}
\end{equation*}
$$

Remark 3.2. The proof of Lemma 3.1 is different from Theorem 3 in [35]. Here we mainly use the following generalized average inequality.
Lemma 3.3 ([35] A generalized average inequality). Suppose that $w \in C_{\text {loc }}^{1,1}\left(\mathbb{R}^{n}\right) \cap \mathcal{L}_{2 s}$ and $\bar{x}$ is a maximum point of $w$ in $\mathbb{R}^{n}$. Then for any $r>0$, we have

$$
\frac{C_{0}}{C_{n, s}} r^{2 s}(-\Delta)^{s} w(\bar{x})+C_{0} \int_{B_{r}^{c}(\bar{x})} \frac{r^{2 s}}{|\bar{x}-y|^{n+2 s}} w(y) d y \geq w(\bar{x}),
$$

where $C_{0}$ satisfies

$$
C_{0} \int_{B_{r}^{c}(\bar{x})} \frac{r^{2 s}}{|\bar{x}-y|^{n+2 s}} d y=1 .
$$

Proof of Lemma 3.1. Suppose on the contrary, there is some point $x$ such that $w(x)>0$ in $D$, then

$$
\begin{equation*}
0<A:=\sup _{x \in \mathbb{R}^{n}} w(x)<\infty \tag{3.3}
\end{equation*}
$$

There exists a sequence $\left\{x^{k}\right\} \subset D$ such that

$$
\begin{equation*}
w\left(x^{k}\right) \rightarrow A>0, \quad \text { as } k \rightarrow \infty \tag{3.4}
\end{equation*}
$$

Let

$$
\eta(x)= \begin{cases}c e^{\frac{1}{|x|^{2}-1}}, & |x|<1  \tag{3.5}\\ 0, & |x| \geq 1\end{cases}
$$

where $c>0$ is a constant, taking $c=e$ such that $\eta(0)=\max _{\mathbb{R}^{n}} \eta(x)=1$.
Set

$$
\begin{equation*}
\psi_{k}(x)=\eta\left(x-x^{k}\right) \tag{3.6}
\end{equation*}
$$

From (3.4), there exists a sequence $\left\{\varepsilon_{k}\right\}$ with $\varepsilon_{k}>0$ such that

$$
w\left(x^{k}\right)+\varepsilon_{k} \psi_{k}\left(x^{k}\right) \geq A
$$

Since $w(x) \leq 0, x \in \mathbb{R}^{n} \backslash D$, it follows from (3.4) that $x^{k}$ is away from $\partial D$. Without loss of generality, we may assume that $\operatorname{dist}\left(x^{k}, \partial D\right)=2$. So $B_{1}\left(x^{k}\right) \subset D$. Since for any $x \in$ $D \backslash B_{1}\left(x^{k}\right), w(x) \leq A$ and $\psi_{k}(x)=0$, hence

$$
w\left(x^{k}\right)+\varepsilon_{k} \psi_{k}\left(x^{k}\right) \geq w(x)+\varepsilon_{k} \psi_{k}(x), \quad \text { for any } x \in \mathbb{R}^{n} \backslash B_{1}\left(x^{k}\right)
$$

It follows that there exists a point $\bar{x}^{k} \in B_{1}\left(x^{k}\right)$ such that

$$
\begin{equation*}
w\left(\bar{x}^{k}\right)+\varepsilon_{k} \psi_{k}\left(\bar{x}^{k}\right)=\max _{\mathbb{R}^{n}}\left(w(x)+\varepsilon_{k} \psi_{k}(x)\right)>A . \tag{3.7}
\end{equation*}
$$

So $\left(w\left(\bar{x}^{k}\right)+\varepsilon_{k} \psi_{k}\left(\bar{x}^{k}\right)\right)_{j}=0$ and

$$
\begin{equation*}
w_{j}\left(\bar{x}^{k}\right) \rightarrow 0, \quad \text { as } k \rightarrow \infty . \tag{3.8}
\end{equation*}
$$

For $w+\varepsilon_{k} \psi_{k}$, using Lemma 3.3, we obtain

$$
\left(w+\varepsilon_{k} \psi_{k}\right)\left(\bar{x}^{k}\right) \leq C_{1}(-\Delta)^{s}\left(w+\varepsilon_{k} \psi_{k}\right)\left(\bar{x}^{k}\right)+C_{2} \int_{B_{2}^{c}\left(\bar{x}^{k}\right)} \frac{\left(w+\varepsilon_{k} \psi_{k}\right)(y)}{\left|\bar{x}^{k}-y\right|^{n+2 s}} d y
$$

Let $\varepsilon_{k} \rightarrow 0$, by the first inequality of (3.1), it implies that

$$
\begin{align*}
w\left(\bar{x}^{k}\right) & \leq C_{1}(-\Delta)^{s} w\left(\bar{x}^{k}\right)+C_{2} \int_{B_{2}^{c}\left(\bar{x}^{k}\right)} \frac{w(y)}{\left|\bar{x}^{k}-y\right|^{n+2 s}} d y \\
& \leq-c\left(\bar{x}^{k}\right) w\left(\bar{x}^{k}\right)-\sum_{j=1}^{n} b_{j}(x) w_{j}\left(\bar{x}^{k}\right)+C_{2} \int_{B_{2}^{c}\left(\bar{x}^{k}\right)} \frac{w(y)}{\left|\bar{x}^{k}-y\right|^{n+2 s}} d y \tag{3.9}
\end{align*}
$$

Letting $k \rightarrow \infty$, combining (3.4), (3.8), (3.9) and nonnegative function $c(x)$, we arrive at

$$
0<(c(x)+1) A \leftarrow\left(c\left(\bar{x}^{k}\right)+1\right) w\left(\bar{x}^{k}\right) \leq C_{2} \int_{B_{2}^{c}\left(\bar{x}^{k}\right)} \frac{w(y)}{\left|\bar{x}^{k}-y\right|^{n+2 s}} d y
$$

this is impossible because of (3.3) and the second inequality of (3.1).

Based on above result, if $w=0$ at some point $x_{0} \in D$, then $x_{0}$ is a maximum point of $w$ in $D$. And we still have $w_{j}=0$ in the maximum point. If $w \not \equiv 0$ in $\mathbb{R}^{n}$, then we have

$$
(-\Delta)^{s} w\left(x_{0}\right)+c\left(x_{0}\right) w\left(x_{0}\right)+\sum_{j=1}^{n} b_{j}\left(x_{0}\right) w_{j}\left(x_{0}\right)=C_{n, s} P \cdot V \cdot \int_{\mathbb{R}^{n}} \frac{-w(y)}{\left|x_{0}-y\right|^{n+2 s}} d y>0 .
$$

This is a contradiction with (3.1). So we have either $w<0$ in $D$ or $w \equiv 0$ in $\mathbb{R}^{n}$.
This completes the proof Lemma 3.1.
We also need the following lemma.
Lemma 3.4 ([9], Maximum principle). Let $\Gamma$ be a bounded domain in $\mathbb{R}^{n}$. Assume that $u \in$ $C_{\text {loc }}^{1,1}(\Gamma) \cap \mathcal{L}_{2 s}$ and $u$ be lower semi-continuous on $\bar{\Gamma}$, and satisfy

$$
\begin{cases}(-\Delta)^{s} u(x) \geq 0, & x \in \Gamma \\ u(x) \geq 0, & x \in \mathbb{R}^{n} \backslash \Gamma .\end{cases}
$$

Then

$$
u(x) \geq 0, x \in \Gamma .
$$

If $u(x)=0$ at some point $x \in \Gamma$, then

$$
u(x)=0 \quad \text { almost everywhere in } \mathbb{R}^{n} .
$$

Proof of Theorem 1.6. Define $\mathbb{R}_{+}^{n}=\left\{x=\left(x_{1}, \ldots, x_{n}\right) \mid x_{n}>0\right\}$. Let

$$
u^{\tau}(x)=u\left(x^{\prime}, x_{n}+\tau\right) \quad \text { and } \quad U^{\tau}(x)=u(x)-u^{\tau}(x) .
$$

Outline of the proof: We will use the sliding method to prove the monotonicity and uniqueness of $u$ and divide the proof into three steps.

In Step 1, we will show that for $\tau$ sufficiently large, we have $U^{\tau}(x) \leq 0, x \in \mathbb{R}^{n}$. Especially, since $u \rightarrow \mu$ uniformly as $x_{n} \rightarrow+\infty$, for $\delta>0$, there exists a $M_{0}>0$ such that for $x_{n} \geq M_{0}$, $u \in[\mu-\delta, \mu]$ and $f$ is nondecreasing in $u \in[\mu-\delta, \mu]$. Hence we will show that

$$
\begin{equation*}
U^{\tau}(x) \leq 0, \quad x \in \mathbb{R}^{n}, \forall \tau \geq M_{0} \tag{3.10}
\end{equation*}
$$

This provides the starting point for the sliding method. Then in Step 2, we decrease $\tau$ continuously as long as (3.10) holds to its limiting position. Define

$$
\begin{equation*}
\tau_{0}:=\inf \left\{\tau \mid U^{\tau}(x) \leq 0, x \in \mathbb{R}^{n}, 0<\tau<M_{0}\right\} . \tag{3.11}
\end{equation*}
$$

We first will show that $\tau_{0}=0$. Then we deduce that the solution $u$ must be strictly monotone increasing in $x_{n}$. In Step 3, we obtain that the solution $u$ depends on $x_{n}$. Finally we will prove the uniqueness.

Now we show the details in the three steps.
Step 1. Since $u(x)=0, x \in \mathbb{R}^{n} \backslash \mathbb{R}_{+}^{n}$, it yields that

$$
U^{\tau}(x) \leq 0, \quad \forall x \in \mathbb{R}^{n} \backslash \mathbb{R}_{+}^{n} .
$$

For $\tau \geq M_{0}$, suppose (3.10) is violated, there exists a constant $A>0$ such that

$$
\begin{equation*}
\sup _{x \in \mathbb{R}_{+}^{n}} U^{\tau}(x)=A, \tag{3.12}
\end{equation*}
$$

hence for some $\tau_{1} \geq M_{0}$ there exists a sequence $\left\{x^{k}\right\} \subset \mathbb{R}_{+}^{n}$ such that

$$
\begin{equation*}
U^{\tau_{1}}\left(x^{k}\right) \rightarrow A, \quad \text { as } k \rightarrow \infty . \tag{3.13}
\end{equation*}
$$

We will apply Lemma 3.1 to function $U^{\tau_{1}}(x)-\frac{A}{2}$.
Since $\tau_{1} \geq M_{0}$, we have $u^{\tau_{1}}(x) \in[\mu-\delta, \mu]$. Let

$$
D=\left\{x \in \mathbb{R}^{n} \left\lvert\, U^{\tau_{1}}(x)-\frac{A}{2}>0\right.\right\} .
$$

For $x \in D$, we have $u(x) \geq u^{\tau_{1}} \geq \mu-\delta$. From equation (1.5), $U^{\tau_{1}}(x)$ satisfied

$$
\begin{aligned}
(-\Delta)^{s} U^{\tau_{1}}(x) & =f(u, \nabla u)-f\left(u^{\tau_{1}}, \nabla u^{\tau_{1}}\right) \\
& :=-b_{j}(x)\left(U^{\tau_{1}}\right)_{j}(x)-c(x) U^{\tau_{1}}(x),
\end{aligned}
$$

where $c(x)=-\frac{f(u, \nabla u)-f\left(u^{\tau_{1}}, \nabla u\right)}{u-u^{\tau_{1}}} \leq 0$ by the monotonicity of $f$.
Hence $U^{\tau_{1}}(x)-\frac{A}{2}$ satisfies

$$
\begin{cases}(-\Delta)^{s} U^{\tau_{1}}(x)+b_{j}(x)\left(U^{\tau_{1}}\right)_{j}(x)+c(x)\left(U^{\tau_{1}}(x)-\frac{A}{2}\right)=0, & x \in D, \\ U^{\tau_{1}}(x)-\frac{A}{2} \leq 0, & x \in \mathbb{R}^{n} \backslash D .\end{cases}
$$

By Lemma 3.1, we derive

$$
U^{\tau_{1}}(x)-\frac{A}{2} \leq 0, x \in \mathbb{R}^{n}
$$

which contradicts (3.13). Hence we obtain (3.10) and finish the proof of Step 1.
We also give an alternative proof which is an application of the general average inequality (Lemma 3.3), and this idea can be applied to other problems.

For $\tau \geq M_{0}$, if (3.10) is violated, we have (3.13). Obviously, $U^{\tau_{1}}(x) \leq 0, x \in \partial \mathbb{R}_{+}^{n}$. So by (3.13) we have $x^{k}$ is away from $\partial \mathbb{R}_{+}^{n}$, without loss of generality, assume $\operatorname{dist}\left(x^{k}, \partial \mathbb{R}_{+}^{n}\right)>2$. Thus there exists $0<\varepsilon_{k} \rightarrow 0, \bar{x}^{k} \in B_{1}\left(x^{k}\right)$ such that

$$
U^{\tau_{1}}\left(\bar{x}^{k}\right)+\varepsilon_{k} \psi_{k}\left(\bar{x}^{k}\right)=\max _{\mathbb{R}^{n}}\left(U^{\tau_{1}}(x)+\varepsilon_{k} \psi_{k}(x)\right) \geq A,
$$

where $\psi_{k}\left(\bar{x}^{k}\right)$ is as stated in (3.6). So $\nabla\left(U^{\tau_{1}}\left(\bar{x}^{k}\right)+\varepsilon_{k} \psi_{k}\left(\bar{x}^{k}\right)\right)=0$ and

$$
\begin{equation*}
\nabla U^{\tau_{1}}\left(\bar{x}^{k}\right) \rightarrow 0, \quad \text { as } k \rightarrow \infty . \tag{3.14}
\end{equation*}
$$

Since

$$
\left[U^{\tau_{1}}+\varepsilon_{k} \psi_{k}\right]\left(\bar{x}^{k}\right) \geq\left[U^{\tau_{1}}+\varepsilon_{k} \psi_{k}\right]\left(x^{k}\right)
$$

and $\psi_{k}\left(\bar{x}^{k}\right) \leq \psi_{k}\left(x^{k}\right)$, we obtain

$$
\begin{equation*}
U^{\tau_{1}}\left(\bar{x}^{k}\right) \geq U^{\tau_{1}}\left(x^{k}\right) . \tag{3.15}
\end{equation*}
$$

Hence for $\tau_{1} \geq M_{0}$,

$$
u\left(\bar{x}^{k}\right) \geq u^{\tau_{1}}\left(\bar{x}^{k}\right) \geq \mu-\delta .
$$

This means $u\left(\bar{x}^{k}\right), u^{\tau_{1}}\left(\bar{x}^{k}\right)$ are all in the nondecreasing interval of $f$. So

$$
\begin{align*}
& f\left(u\left(\bar{x}^{k}\right), \nabla u\left(\bar{x}^{k}\right)\right)-f\left(u^{\tau_{1}}\left(\bar{x}^{k}\right), \nabla u^{\tau_{1}}\left(\bar{x}^{k}\right)\right) \\
& \quad=f\left(u, \nabla u\left(\bar{x}^{k}\right)\right)-f\left(u, \nabla u^{\tau_{1}}\left(\bar{x}^{k}\right)\right)+f\left(u, \nabla u^{\tau_{1}}\left(\bar{x}^{k}\right)\right)-f\left(u^{\tau_{1}}, \nabla u^{\tau_{1}}\left(\bar{x}^{k}\right)\right)  \tag{3.16}\\
& \quad \leq f\left(u\left(\bar{x}^{k}\right), \nabla u\left(\bar{x}^{k}\right)\right)-f\left(u\left(\bar{x}^{k}\right), \nabla u^{\tau_{1}}\left(\bar{x}^{k}\right)\right) .
\end{align*}
$$

Using Lemma 3.3 to the function $U^{\tau_{1}}+\varepsilon_{k} \psi_{k}$ at $\bar{x}^{k}$, we obtain

$$
\left(U^{\tau_{1}}+\varepsilon_{k} \psi_{k}\right)\left(\bar{x}^{k}\right) \leq C_{1}(-\Delta)^{s}\left(U^{\tau_{1}}+\varepsilon_{k} \psi_{k}\right)\left(\bar{x}^{k}\right)+C_{2} \int_{B_{2}^{c}\left(\bar{x}^{k}\right)} \frac{\left(U^{\tau_{1}}+\varepsilon_{k} \psi_{k}\right)(y)}{\left|\bar{x}^{k}-y\right|^{n+2 s}} d y .
$$

Let $\varepsilon_{k} \rightarrow 0$, by the equation (1.5), it implies that

$$
\begin{align*}
U^{\tau_{1}}\left(\bar{x}^{k}\right) & \leq C_{1}(-\Delta)^{s} U^{\tau_{1}}\left(\bar{x}^{k}\right)+C_{2} \int_{B_{2}^{c}\left(\bar{x}^{k}\right)} \frac{U^{\tau_{1}}(y)}{\left|\bar{x}^{k}-y\right|^{n+2 s}} d y  \tag{3.17}\\
& =C_{1}\left[f\left(u, \nabla u\left(\bar{x}^{k}\right)\right)-f\left(u, \nabla u^{\tau_{1}}\left(\bar{x}^{k}\right)\right)\right]+C_{2} \int_{B_{2}^{c}\left(\bar{x}^{k}\right)} \frac{U^{\tau_{1}}(y)}{\left|\bar{x}^{k}-y\right|^{n+2 s}} d y .
\end{align*}
$$

From (3.13) and (3.15), we have

$$
\begin{equation*}
U^{\tau_{1}}\left(\bar{x}^{k}\right) \rightarrow A>0, \quad \text { as } k \rightarrow \infty . \tag{3.18}
\end{equation*}
$$

Letting $k \rightarrow \infty$, combining (3.14), (3.17) and (3.18), we arrive at

$$
0<A \leftarrow U^{\tau_{1}}\left(\bar{x}^{k}\right) \leq C_{2} \int_{B_{2}^{c}\left(x^{k}\right)} \frac{U^{\tau_{1}}(y)}{\left|\bar{x}^{k}-y\right|^{n+2 s}} d y,
$$

this is impossible because of (3.12) and $U^{\tau_{1}}(y) \leq 0, y \in \mathbb{R}^{n} \backslash \mathbb{R}_{+}^{n}$.
Hence (3.10) is correct and we have finished the proof of Step 1.
Step 2. Firstly, we will check that

$$
\begin{equation*}
\tau_{0}=0, \tag{3.19}
\end{equation*}
$$

where $\tau_{0}$ as defined in (3.11). In fact, suppose on the contrary $\tau_{0}>0$, then $\tau_{0}$ can be decreased a little bit. To be more rigorously, there exists a $\epsilon>0$ such that for any $\tau \in\left(\tau_{0}-\epsilon, \tau_{0}\right]$, one has

$$
\begin{equation*}
U^{\tau}(x) \leq 0, \quad \text { for any } x \in \mathbb{R}_{+}^{n} . \tag{3.20}
\end{equation*}
$$

This is a contradiction with the definition of $\tau_{0}$. Hence (3.19) is correct. In the sequel, we will prove (3.20).

To do so, we just need to prove

$$
\begin{equation*}
\sup _{\mathbb{R}^{n-1} \times\left(0, M_{0}+1\right]} U^{\tau}(x)<0, \quad \forall \tau \in\left(\tau_{0}-\epsilon, \tau_{0}\right] \tag{3.21}
\end{equation*}
$$

and

$$
\begin{equation*}
\sup _{\mathbb{R}^{n-1} \times\left(M_{0}+1,+\infty\right)} U^{\tau}(x) \leq 0, \quad \forall \tau \in\left(\tau_{0}-\epsilon, \tau_{0}\right] . \tag{3.22}
\end{equation*}
$$

In order to prove (3.21) we need to show that

$$
\begin{equation*}
\sup _{\mathbb{R}^{n-1 \times\left(0, M_{0}+1\right]}} U^{\tau_{0}}(x)<0 . \tag{3.23}
\end{equation*}
$$

If not, then

$$
\sup _{\mathbb{R}^{n-1} \times\left(0, M_{0}+1\right]} U^{\tau_{0}}(x)=0
$$

So there exists a sequence $\left\{x^{k}\right\} \subset \mathbb{R}^{n-1} \times\left(0, M_{0}+1\right]$ such that

$$
\begin{equation*}
U^{\tau_{0}}\left(x^{k}\right) \rightarrow 0, \quad \text { as } k \rightarrow \infty . \tag{3.24}
\end{equation*}
$$

We first show that $x^{k}$ is away from the boundary $\partial \mathbb{R}_{+}^{n}$. Suppose that $z$ be a point on $\partial \mathbb{R}_{+}^{n}$. Denote $r_{z}:=\operatorname{dist}\left(z+\tau_{0} e_{n}, \partial \mathbb{R}_{+}^{n}\right), e_{n}=(0, \ldots, 0,1)$. For each fixed $\tau_{0}>0$, we have

$$
\inf _{x \in \partial \mathbb{R}_{+}^{n}} \operatorname{dist}\left(z+\tau_{0} e_{n}, \partial \mathbb{R}_{+}^{n}\right):=r_{0}>0 .
$$

For every point $z$ on $\partial \mathbb{R}_{+}^{n}$, there exists a ball $B_{r_{z}}\left(z+\tau_{0} e_{n}\right) \subset \mathbb{R}_{+}^{n}$ with radius of $r_{z}$ centered at $z+\tau_{0} e_{n}$. For simplicity of notation, we use $B$ instead of $B_{r_{z}}\left(z+\tau_{0} e_{n}\right)$.

Let

$$
E=\left\{x \in \mathbb{R}_{+}^{n} \mid \operatorname{dist}\left(x, \partial \mathbb{R}_{+}^{n}\right) \geq 2\right\} .
$$

We construct a subsolution

$$
\bar{u}(x)=u_{E}(x)+\varepsilon \Phi(x), \quad x \in B
$$

where $\Phi(x)=\left(1-|x|^{2}\right)^{s}, u_{E}:=u \cdot \chi_{E}$ and $\chi_{E}$ is define as

$$
\chi_{E}(x)= \begin{cases}1, & x \in E \\ 0, & x \in \mathbb{R}^{n} \backslash E .\end{cases}
$$

By $(-\Delta)^{s} \Phi(x)=C[20]$, for $x \in B$ it yields

$$
\begin{aligned}
(-\Delta)^{s} \bar{u}(x) & =(-\Delta)^{s}\left(u_{E}+\varepsilon \Phi\right)(x) \\
& =\varepsilon(-\Delta)^{s} \Phi(x)+(-\Delta)^{s} u_{E}(x) \\
& \leq \varepsilon C-\varepsilon_{1} C_{n, s} \int_{E} \frac{1}{|x-y|^{n+2 s}} d y \\
& \leq \varepsilon C-\varepsilon_{1} C C_{n, s} .
\end{aligned}
$$

We can choose $\varepsilon \leq \varepsilon_{1} C_{n, s} C C^{-1}:=\varepsilon_{0}$ such that $(-\Delta)^{s} \underline{u}(x) \leq 0, x \in B$. Then fixing $\varepsilon=\frac{\varepsilon_{0}}{2}$, combining $u(x) \geq \underline{u}(x), x \in B^{c}$ and Lemma 3.4, we derive

$$
u^{\tau_{0}}(z)=u\left(z+\tau_{0} e_{n}\right) \geq \underline{u}\left(z+\tau_{0} e_{n}\right) \geq \frac{\varepsilon_{0}}{2} \Phi\left(z+\tau_{0} e_{n}\right) \geq C_{\tau_{0}}>0, \quad \forall z \in \partial \mathbb{R}_{+}^{n} .
$$

Then, we infer that

$$
\begin{equation*}
U^{\tau_{0}}(z)=u^{\tau_{0}}(z)>C_{\tau_{0}}>0, \quad \forall z \in \partial \mathbb{R}_{+}^{n} . \tag{3.25}
\end{equation*}
$$

By (3.24) and (3.25), we obtain that $x^{k}$ is away from the boundary $\partial \mathbb{R}_{+}^{n}$. Without loss of generality, we may assume $B_{1}\left(x^{k}\right) \subset \mathbb{R}_{+}^{n}$. Similar to the argument as Lemma 3.1, let $\psi(x)=$ $\eta\left(x-x^{k}\right)$, where $\eta$ is as stated in (3.5), $x^{k}$ satisfies $\operatorname{dist}\left(x^{k}, \partial \mathbb{R}_{+}^{n}\right) \geq 2$ and $B_{1}\left(x^{k}\right) \subset \mathbb{R}_{+}^{n}$. Then there exists a sequence $\varepsilon_{k} \rightarrow 0$ such that

$$
U^{\tau_{0}}\left(x^{k}\right)+\varepsilon_{k} \psi\left(x^{k}\right)>0 .
$$

Since for $x \in \mathbb{R}_{+}^{n} \backslash B_{1}\left(x^{k}\right)$, noting that $U^{\tau_{0}}(x) \leq 0$ and $\psi(x)=0$, we have

$$
U^{\tau_{0}}\left(x^{k}\right)+\varepsilon_{k} \psi\left(x^{k}\right)>U^{\tau_{0}}(x)+\varepsilon_{k} \psi(x), \quad \text { for any } x \in \mathbb{R}^{n} \backslash B_{1}\left(x^{k}\right) .
$$

Then there exists $\bar{x}^{k} \in B_{1}\left(x^{k}\right)$ such that

$$
\begin{equation*}
U^{\tau_{0}}\left(\bar{x}^{k}\right)+\varepsilon_{k} \psi\left(\bar{x}^{k}\right)=\max _{\mathbb{R}^{n}}\left(U^{\tau_{0}}(x)+\varepsilon_{k} \psi(x)\right)>0 . \tag{3.26}
\end{equation*}
$$

It can be seen from

$$
U^{\tau_{0}}\left(\bar{x}^{k}\right)+\varepsilon_{k} \psi\left(\bar{x}^{k}\right) \geq U^{\tau_{0}}\left(x^{k}\right)+\varepsilon_{k} \psi\left(x^{k}\right),
$$

and $\psi\left(\bar{x}^{k}\right) \leq \psi\left(x^{k}\right)$ that

$$
0>U^{\tau_{0}}\left(\bar{x}^{k}\right) \geq U^{\tau_{0}}\left(x^{k}\right)+\varepsilon_{k} \psi\left(x^{k}\right)-\varepsilon_{k} \psi\left(\bar{x}^{k}\right) \geq U^{\tau_{0}}\left(x^{k}\right) \rightarrow 0, \quad \text { as } k \rightarrow \infty .
$$

Hence

$$
U^{\tau_{0}}\left(\bar{x}^{k}\right) \rightarrow 0, \quad \text { as } k \rightarrow \infty .
$$

Since $f$ is continuous, we have

$$
\begin{equation*}
f\left(u\left(\bar{x}^{k}\right), \nabla u\left(\bar{x}^{k}\right)\right)-f\left(u^{\tau_{0}}\left(\bar{x}^{k}\right), \nabla u^{\tau_{0}}\left(\bar{x}^{k}\right)\right) \rightarrow 0, \quad \text { as } k \rightarrow \infty . \tag{3.27}
\end{equation*}
$$

On one hand, we have

$$
\begin{align*}
(-\Delta)^{s}\left(U^{\tau_{0}}+\varepsilon_{k} \psi\right)\left(\bar{x}^{k}\right) & =(-\Delta)^{s} U^{\tau_{0}}\left(\bar{x}^{k}\right)+(-\Delta)^{s}\left(\varepsilon_{k} \psi\right)\left(\bar{x}^{k}\right) \\
& =f\left(u\left(\bar{x}^{k}\right), \nabla u\left(\bar{x}^{k}\right)\right)-f\left(u^{\tau_{0}}\left(\bar{x}^{k}\right), \nabla u^{\tau_{0}}\left(\bar{x}^{k}\right)\right)+\varepsilon_{k}(-\Delta)^{s} \psi\left(\bar{x}^{k}\right) . \tag{3.28}
\end{align*}
$$

On the other hand,

$$
\begin{align*}
(-\Delta)^{s}\left(U^{\tau_{0}}+\varepsilon_{k} \psi\right)\left(\bar{x}^{k}\right) & =C_{n, s} P \cdot V \cdot \int_{\mathbb{R}^{n}} \frac{U^{\tau_{0}}\left(\bar{x}^{k}\right)+\varepsilon_{k} \psi\left(\bar{x}^{k}\right)-U^{\tau_{0}}(y)-\varepsilon_{k} \psi(y)}{\left|\bar{x}^{k}-y\right|^{n+2 s}} d y \\
& \geq C \int_{B_{2}^{c}\left(x^{k}\right)} \frac{\left|U^{\tau_{0}}(y)\right|}{\left|x^{k}-y\right|^{n+2 s}} d y  \tag{3.29}\\
& =C \int_{B_{2}^{c}(0)} \frac{\left|U^{\tau_{0}}\left(z+x^{k}\right)\right|}{|z|^{n+2 s}} d z .
\end{align*}
$$

Denote

$$
u_{k}(x)=u\left(x+x^{k}\right) \quad \text { and } \quad U_{k}^{\tau_{0}}(x)=U^{\tau_{0}}\left(x+x^{k}\right) .
$$

Since $f$ is bounded, one can derive (see [10]) that $u(x)$ is at least uniformly Hölder continuous, so $u(x)$ is uniformly continuous, by the Arzelà-Ascoli theorem, up to extraction of a subsequence, one has

$$
u_{k}(x) \rightarrow u_{\infty}(x), \quad x \in \mathbb{R}_{+}^{n}, \quad \text { as } k \rightarrow \infty .
$$

Combining (3.27), (3.28) and (3.29), letting $k \rightarrow \infty$, we obtain

$$
U_{k}^{\tau_{0}}(x) \rightarrow 0, \quad x \in B_{2}^{c}(0), \quad \text { uniformly, as } k \rightarrow \infty
$$

Therefore,

$$
\begin{equation*}
U_{k}^{\tau_{0}}(x) \rightarrow u_{\infty}(x)-u_{\infty}^{\tau_{0}}(x) \equiv 0, \quad x \in B_{2}^{c}(0) . \tag{3.30}
\end{equation*}
$$

Recall that $u>0$ in $\mathbb{R}_{+}^{n}$ while $u(x) \equiv 0, x \in \mathbb{R}^{n} \backslash \mathbb{R}_{+}^{n}$. Since $x^{k} \in \mathbb{R}^{n-1} \times\left(0, M_{0}\right]$, there exists $x^{0}$ such that $u_{\infty}\left(x^{0}\right)=0$, then by (3.30),

$$
\begin{align*}
0 & =u_{\infty}\left(x^{0}\right)=u_{\infty}^{\tau_{0}}\left(x^{0}\right)=u_{\infty}\left(\left(x^{0}\right)^{\prime}, x_{n}^{0}+\tau_{0}\right)=u_{\infty}^{\tau_{0}}\left(\left(x^{0}\right)^{\prime}, x_{n}^{0}+\tau_{0}\right) \\
& =u_{\infty}\left(\left(x^{0}\right)^{\prime}, x_{n}^{0}+2 \tau_{0}\right)=\cdots=u_{\infty}\left(\left(x^{0}\right)^{\prime}, x_{n}^{0}+k \tau_{0}\right) . \tag{3.31}
\end{align*}
$$

We obtain from (1.6) that

$$
\lim _{x_{n} \rightarrow+\infty} u_{\infty}(x)=\mu>0, \quad \text { uniformly in } x^{\prime}=\left(x_{1}, \ldots, x_{n-1}\right),
$$

that is

$$
u_{\infty}\left(\left(x^{0}\right)^{\prime}, x_{n}^{0}+k \tau_{0}\right) \rightarrow \mu, \quad \text { as } k \rightarrow \infty .
$$

This is a contradiction with (3.31). Hence (3.23) is correct. Now (3.23) implies immediately that (3.21) holds by the continuity of $U^{\tau}(x)$ with respect to $\tau$.

Next we prove (3.22). Otherwise, there exists a constant $A>0$ such that

$$
\sup _{x \in \mathbb{R}^{n-1} \times\left(M_{0}+1,+\infty\right)} U^{\tau}(x)=A>0, \quad \forall \tau \in\left(\tau_{0}-\epsilon, \tau_{0}\right] .
$$

Then for some $\tau_{2} \in\left(M_{0}+1,+\infty\right)$ there exists a sequence $\left\{x^{k}\right\} \subset \mathbb{R}^{n-1} \times\left(M_{0}+1,+\infty\right)$ such that

$$
\begin{equation*}
U^{\tau_{2}}\left(x^{k}\right) \rightarrow A, \quad \text { as } k \rightarrow \infty . \tag{3.32}
\end{equation*}
$$

Since $u=0$ in $\mathbb{R}^{n} \backslash \mathbb{R}_{+}^{n}$, it follows that

$$
U^{\tau_{2}}(x) \leq 0, \quad \text { for any } x \in \mathbb{R}^{n} \backslash \mathbb{R}_{+}^{n} .
$$

Denote $x^{k}=\left(x_{1}^{k}, \ldots, x_{n}^{k}\right)$. Since $U^{\tau_{2}}\left(x^{k}\right)=u\left(x^{k}\right)-u^{\tau_{2}}\left(x^{k}\right) \rightarrow 0$ as $x_{n}^{k} \rightarrow+\infty$, then there exists $M_{0}>0$ such that

$$
\left|x_{n}^{k}\right| \leq M_{0}
$$

Set $\psi_{k}(x)=\eta\left(x-x^{k}\right)$, where $\eta$ is as stated in (3.5). From (3.32), there exists a sequence $\left\{\varepsilon_{k}\right\}$, with $\varepsilon_{k} \rightarrow 0$ such that

$$
U^{\tau_{2}}\left(x^{k}\right)+\varepsilon_{k} \psi_{k}\left(x^{k}\right)>A .
$$

Since for any $x \in \mathbb{R}_{+}^{n} \backslash B_{1}\left(x^{k}\right), U^{\tau_{2}}(x) \leq A$ and $\psi_{k}(x)=0$, hence

$$
U^{\tau_{2}}\left(x^{k}\right)+\varepsilon_{k} \psi_{k}\left(x^{k}\right)>U^{\tau_{2}}(x)+\varepsilon_{k} \psi_{k}(x), \quad \text { for any } x \in \mathbb{R}_{+}^{n} \backslash B_{1}\left(x^{k}\right) .
$$

It follows that there exists a point $\bar{x}^{k} \in B_{1}\left(x^{k}\right)$ i.e. $\bar{x}^{k} \in \mathbb{R}^{n-1} \times\left(M_{0},+\infty\right)$ such that

$$
\begin{equation*}
U^{\tau_{2}}\left(\bar{x}^{k}\right)+\varepsilon_{k} \psi_{k}\left(\bar{x}^{k}\right)=\max _{\mathbb{R}^{n}}\left(U^{\tau_{2}}(x)+\varepsilon_{k} \psi_{k}(x)\right)>A . \tag{3.33}
\end{equation*}
$$

On one hand, by the monotonicity of $f$, we obtain

$$
\begin{align*}
(-\Delta)^{s}\left(U^{\tau_{2}}+\varepsilon_{k} \psi_{k}\right)\left(\bar{x}^{k}\right) & =f\left(u\left(\bar{x}^{k}\right), \nabla u\left(\bar{x}^{k}\right)\right)-f\left(u^{\tau_{2}}\left(\bar{x}^{k}\right), \nabla u^{\tau_{2}}\left(\bar{x}^{k}\right)\right)+\varepsilon_{k}(-\Delta)^{s} \psi_{k}\left(\bar{x}^{k}\right) \\
& \leq f\left(u\left(\bar{x}^{k}\right), \nabla u\left(\bar{x}^{k}\right)\right)-f\left(u\left(\bar{x}^{k}\right), \nabla u^{\tau_{2}}\left(\bar{x}^{k}\right)\right)+\varepsilon_{k}(-\Delta)^{s} \psi_{k}\left(\bar{x}^{k}\right) . \tag{3.34}
\end{align*}
$$

On the other hand,

$$
\begin{align*}
(-\Delta)^{s}\left(U^{\tau_{2}}+\varepsilon_{k} \psi_{k}\right)\left(\bar{x}^{k}\right) & =C_{n, s} P \cdot V \cdot \int_{\mathbb{R}^{n}} \frac{U^{\tau_{2}}\left(\bar{x}^{k}\right)+\varepsilon_{k} \psi_{k}\left(\bar{x}^{k}\right)-\left(U^{\tau_{2}}(y)+\varepsilon_{k} \psi_{k}(y)\right)}{\left|\bar{x}^{k}-y\right|^{n+2 s}} d y \\
& \geq C \int_{D_{M}} \frac{A-\frac{A}{2}}{\left|\bar{x}^{k}-y\right|^{n+2 s}} d y  \tag{3.35}\\
& \geq \frac{C A}{2} \int_{D_{M}} \frac{1}{\left|\bar{x}^{k}-y\right|^{n+2 s}} d y \\
& \geq C A \frac{1}{\left[\operatorname{dist}\left(\bar{x}^{k}, D_{M}\right)\right]^{2 s}},
\end{align*}
$$

where $M>M_{0}$ and $D_{M}=\left\{\left|x_{n}\right| \geq M\right\}$, in which $U^{\tau}(y)+\varepsilon_{k} \psi_{k}(y) \leq \frac{A}{2}$.

Therefore we obtain

$$
\begin{align*}
0 & <c \leq C A \frac{1}{\left[\operatorname{dist}\left(\bar{x}^{k}, D_{M}\right)\right]^{2 s}}  \tag{3.36}\\
& \leq f\left(u\left(\bar{x}^{k}\right), \nabla u\left(\bar{x}^{k}\right)\right)-f\left(u\left(\bar{x}^{k}\right), \nabla u^{\tau_{2}}\left(\bar{x}^{k}\right)\right)+\varepsilon_{k}(-\Delta)^{s} \psi_{k}\left(\bar{x}^{k}\right),
\end{align*}
$$

from (3.33), so $\nabla\left(U^{\tau_{2}}\left(\bar{x}^{k}\right)+\varepsilon_{k} \psi_{k}\left(\bar{x}^{k}\right)\right)=0$, i.e. $\nabla U^{\tau_{2}}\left(\bar{x}^{k}\right) \rightarrow 0$ as $k \rightarrow \infty$. Let $k \rightarrow \infty$, then the right-hand side of (3.36) is less than or equal to 0 , this is impossible. So (3.22) is true, which contradicts to the definition of $\tau_{0}$. Therefore, $\tau_{0}=0$, we arrive at (3.20).

Secondly, we will show that $u$ is strictly increasing with respect to $x_{n}$ and $u(x)$ depends on $x_{n}$ only. We already have

$$
\begin{equation*}
U^{\tau}(x) \leq 0, \quad x \in \mathbb{R}_{+}^{n}, \quad \forall \tau>0 . \tag{3.37}
\end{equation*}
$$

Now we claim that

$$
\begin{equation*}
U^{\tau}(x)<0, \quad x \in \mathbb{R}_{+}^{n}, \quad \forall \tau>0 . \tag{3.38}
\end{equation*}
$$

Otherwise, from (3.37) for some $\tau_{1}>0$ there exists $x^{0} \in \mathbb{R}_{+}^{n}$ such that $U^{\tau_{1}}\left(x^{0}\right)=0$, then $x^{0}$ is the maximum point of $U^{\tau_{1}}$ in $\mathbb{R}_{+}^{n}$. On one hand, since $\nabla U^{\tau_{1}}\left(x^{0}\right)=0$ we have

$$
(-\Delta)^{s} U^{\tau_{1}}\left(x^{0}\right)=f\left(u\left(x^{0}\right), \nabla u\left(x^{0}\right)\right)-f\left(u^{\tau_{1}}\left(x^{0}\right), \nabla u^{\tau_{1}}\left(x^{0}\right)\right) \leq 0 .
$$

On the other hand,

$$
(-\Delta)^{s} U^{\tau_{1}}\left(x^{0}\right)=C_{n, s} P . V . \int_{\mathbb{R}^{n}} \frac{-U^{\tau_{1}}(y)}{\left|x^{0}-y\right|^{n+2 s}} d y>0,
$$

where the last inequality holds due to $U^{\tau_{1}}(y) \not \equiv 0$ in $\mathbb{R}^{n}$.
This is a contradiction. Hence (3.38) must be true.
Step 3. We will claim that $u(x)$ depends on $x_{n}$ only and uniqueness. In fact, it can be seen from the above process that the argument still holds if we replace $u^{\tau}(x)$ by $u(x+\tau v)$, where $v=\left(v_{1}, \ldots, v_{n}\right)$ with $v_{n}>0$ being an arbitrary vector pointing upward. Applying the similar arguments as in Steps 1 and 2, we can derive that, for each of such $v$,

$$
u(x+\tau v)>u(x), \quad \forall \tau>0, x \in \mathbb{R}_{+}^{n} .
$$

Letting $v_{n} \rightarrow 0$, from the continuity of $u$, we deduce that for arbitrary $v$ with $v_{n}=0$,

$$
u(x+\tau v) \geq u(x)
$$

By replacing $v$ by $-v$, we obtain that

$$
u(x+\tau v)=u(x)
$$

for arbitrary $v$ with $v=0$. It implies that $u$ is independent of $x^{\prime}$, hence $u(x)=u\left(x_{n}\right)$.
Finally we prove the uniqueness. Assume that $u$ and $v$ are two bounded solutions of (1.5). For $\tau \geq 0$, denote

$$
\tilde{U}^{\tau}(x)=v(x)-u^{\tau}(x) .
$$

We first show that for $\tau$ sufficiently large,

$$
\begin{equation*}
\tilde{U}^{\tau}(x) \leq 0, \quad x \in \mathbb{R}_{+}^{n} . \tag{3.39}
\end{equation*}
$$

The proof of (3.39) is completely similar to the proof of (3.10), so we omit the details. Note that (3.39) provides a starting point from which we can decrease $\tau$ continuously as long as (3.39) holds.

We show that

$$
\begin{equation*}
\tilde{U}^{\tau}(x) \leq 0, \quad \forall \tau \geq 0, \forall x \in \mathbb{R}_{+}^{n} . \tag{3.40}
\end{equation*}
$$

Define

$$
\tau_{0}:=\inf \left\{\tau>0 \mid \tilde{U}^{\tau}(x) \leq 0, \forall x \in \mathbb{R}_{+}^{n}, \quad 0<\tau<M_{0}\right\} .
$$

Let us prove that

$$
\begin{equation*}
\tau_{0}=0 \tag{3.41}
\end{equation*}
$$

Suppose on the contrary $\tau_{0}>0$. Similarly to the argument of monotonicity in Step 2, one can deduce that

$$
\begin{equation*}
v_{\infty}(x) \equiv u_{\infty}^{\tau_{0}}(x), \quad \forall x \in \mathbb{R}^{n} \backslash B_{2}(0) \tag{3.42}
\end{equation*}
$$

and

$$
\begin{equation*}
u^{\tau_{0}}(z) \geq C_{\tau_{0}}>0, \quad \forall z \in \partial \mathbb{R}_{+}^{n} \tag{3.43}
\end{equation*}
$$

Obviously, this property is preserved under translation. Let

$$
\mathbb{R}_{+k}^{n}=\left\{x \mid x+x^{k} \in \mathbb{R}_{+}^{n}\right\} \quad \text { and } \quad \mathbb{R}_{+\infty}^{n}=\lim _{k \rightarrow \infty} \mathbb{R}_{+k}^{n}
$$

Taking a point $x^{0} \in \partial \mathbb{R}_{+\infty}^{n}$, we deduce from (3.43) that

$$
u_{\infty}^{\tau_{0}}\left(x^{0}\right)>0, \quad \text { but } \quad v_{\infty}\left(x^{0}\right)=0 .
$$

This contradicts (3.42). Hence we have $\tau_{0}=0$. This verifies that (3.40) is correct, and implies that $v(x) \leq u(x)$. Interchanging $u$ and $v$, we obtain $u(x) \leq v(x)$. Therefore, we have $u \equiv v$. This yields the uniqueness.

The proof of Theorem 1.6 is completed.

## Acknowledgements

The author would like to thank Professor Wenxiong Chen for suggesting the problem and all useful discussions. The author appreciates the hospitality of Professor Wenxiong Chen. This work was completed when the author was visiting the Department of Mathematical Science, Yeshiva University.

## References

[1] F. Altomare, S. Milella, G. Musceo, Multiplicative perturbations of the Laplacian and related approximation problems, J. Evol. Equ. 11(2011), 771-792. https://doi.org/10. 1007/s00028-011-0110-6
[2] H. Berestycki, L. Caffarelli, L. Nirenberg, Monotonicity for elliptic equations in an unbounded Lipschitz domain, Comm. Pure Appl. Math. 50(1997), 1089-1111. https ://doi. org/10.1002/(SICI) 1097-0312(199711)50:11<1089: :AID-CPA2>3.0.CO;2-6
[3] H. Berestycki, L. Nirenberg, Monotonicity, symmetry and antisymmetry of solutions of semilinear elliptic equations, J. Geom. Phys. 5(1988), 237-275. https://doi.org/10.1016/ 0393-0440 (88) 90006-X
[4] H. Berestycki, L. Nirenberg, Some qualitative properties of solutions of semilinear elliptic equations in cylindrical domains, in: Analysis, et cetera, , Academic Press, Boston, MA, 1990, pp. 115-164. https://doi.org/10.1016/B978-0-12-574249-8.50011-0
[5] H. Berestycki, L. Nirenberg, On the method of moving planes and the sliding method, Bol. Soc. Brasil. Mat. (N.S.) 22(1991), 1-37. https://doi.org/10.1007/bf01244896
[6] L .Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32(2007), 1245-1260. https://doi .org/10.1080/ 03605300600987306
[7] L. Cao, X. Wang, Z. Dai, Radial symmetry and monotonicity of solutions to a system involving fractional $p$-Laplacian in a ball, Adv. Math. Phys. 2018, Art. ID 1565731, 6 pp. https://doi.org/10.1155/2018/1565731
[8] W. Chen, C. Li, G. Li, Maximum principles for a fully nonlinear fractional order equation and symmetry of solutions, Calc. Var. Partial Differential Equations 56(2017), Art. 29, 18 pp. https://doi.org/10.1007/s00526-017-1110-3
[9] W. Chen, C. Li, Y. Li, A direct method of moving planes for the fractional Laplacian, Adv. Math. 308(2017), 404-437. https://doi.org/10.1016/j.aim.2016.11.038
[10] W. Chen, Y. Li, P. Ma, The fractional Laplacian, World Scientific Publishing Company, 2019. https://doi.org/10.1142/10550
[11] W. Chen, C. Li, B. Ou, Classification of solutions for an integral equation, Commun. Pure Appl. Math. 59(2006), 330-343. https://doi.org/10.1002/cpa. 20116
[12] L. Chen, Z. Liu, G. Lu, Symmetry and regularity of solutions to the weighted HardySobolev type system, Adv. Nonlinear Stud. 16(2016), 1-13. https://doi.org/10.1515/ ans-2015-5005
[13] T. Cheng, Monotonicity and symmetry of solutions to fractional Laplacian equation, Discrete Contin. Dyn. Syst. 37(2017), 3587-3599. https://doi. org/10.3934/dcds. 2017154
[14] T. Cheng, G. Huang, C. Li, The maximum principles for fractional Laplacian equations and their applications, Commun. Contemp. Math. 19(2017), 1750018, 12 pp. https://doi. org/10.1142/S0219199717500183
[15] W. Dai, Z. Liu, P. Wang, Monotonicity and symmetry of solutions for fractional p-Laplace equation, Commun. Contemp. Math., published online, 2150005, 2021. https://doi.org/ 10.1142/S021919972150005X
[16] S. Dipierro, N. Soave, E. Valdinoci, On fractional elliptic equations in Lipschitz sets and epigraphs: regularity, monotonicity and rigidity results, Math. Ann. 369 (2017), 12831326. https://doi.org/10.1007/s00208-016-1487-x
[17] M. M. Fall, E. Valdinoci, Uniqueness and nondegeneracy of positive solutions of $(-\Delta)^{s} u+u=u^{p}$ in $\mathbb{R}^{N}$ when $s$ is close to 1, Commun. Math. Phys. 32(2014), 383-404. https://doi.org/10.1007/s00220-014-1919-y
[18] A. Farina, Symmetry for solutions of semilinear elliptic equations in $R^{N}$ and related conjectures, Ricerche Mat. 48(1999), suppl., 129-154, Papers in memory of Ennio De Giorgi. MR1765681
[19] E. De Giorgi, Convergence problems for functionals and operators, in: Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), Pitagora, Bologna, 1979, pp. 131-188. MR533166
[20] R. K. Getoor, First passage times for symmetric stable processes in space, Trans. Amer. Math. Soc. 101(1961), 75-90. https: //doi.org/10.2307/1993412
[21] B. Gidas, W. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68(1979), 209-243. https://doi.org/10.1007/BF01221125
[22] S. Jarohs, T. Weth, Symmetry via antisymmetric maximum principles in nonlocal problems of variable order, Ann. Mat. Pura Appl. 195(2016), 273-291. https://doi.org/10. 1007/s10231-014-0462-y
[23] T. Jin, J. Xiong, A fractional Yemabe flow and some applications, J. Reine Angew. Math. 696(2014), 187-223. https://doi.org/10.1515/crelle-2012-0110
[24] C. Li, Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on bounded domains, Comm. Partial Differential Equations 16(1991), 491-526. https://doi. org/10.1080/03605309108820766
[25] C. Li, Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on unbounded domains, Comm. Partial Differential Equations 16(1991), 585-615. https : //doi. org/10.1080/03605309108820770
[26] Y. Li, M. Zhu, Uniqueness theorems through the method of moving spheres, Duke Math. J. 80 (1995), 383-418. https://doi.org/10.1215/S0012-7094-95-08016-8
[27] Z. Liu, Symmetry and monotonicity of positive solutions for an integral system with negative exponents, Pacific J. Math. 300(2019), 419-430. https://doi.org/10.2140/pjm. 2019.300.419
[28] Z. Liv, Maximum principles and monotonicity of solutions for fractional $p$-equations in unbounded domains, J. Differential Equations 270(2021), 1043-1078. https: //doi . org/10. 1016/j.jde.2020.09.001
[29] L. Ma, Z. Zhang, Symmetry of positive solutions for Choquard equations with fractional p-Laplacian, Nonlinear Anal. 182(2019), 248-262. https://doi.org/10.1016/j.na. 2018. 12.015
[30] P. Ma, X. Shang, J. Zhang, Symmetry and nonexistence of positive solutions for fractional Choquard equations, Pacific J. Math. 304(2020), 143-167. https://doi.org/10. 2140/pjm. 2020.304 .143
[31] Y. Sire, E. Valdinoci, Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result, J. Funct. Anal. 256(2009), 1842-1864. https : //doi.org/10.1016/j.jfa.2009.01.020
[32] P. Wang, P. Niu, Symmetric properties of positive solutions for fully nonlinear nonlocal system, Nonlinear Anal. 187(2019), 134-146. https://doi.org/10.1016/j.na. 2019. 04.002
[33] P. Wang, L. Chen, P. Niu, Symmetric properties for Choquard equations involving fully nonlinear nonlocal operator, Bull. Braz. Math. Soc. (N.S.), 2021. https://doi.org/10. 1007/s00574-020-00234-5
[34] P. Wang, Y. Wang, Positive solutions for a weighted fractional system, Acta Math. Sci. Ser. B (Engl. Ed.) 38(2018), 935-949. https://doi.org/10.3969/j.issn.0252-9602.2018.03. 015
[35] L. Wu, W. Chen, Monotonicity of solutions for fractional equations with De Giorgi type nonlinearities (in Chinese), Sci. Sin. Math., 2020. https://doi.org/10.1360/SCM-20190668
[36] L. Wu, W. Chen, The sliding methods for the fractional p-Laplacian, Adv. Math. 361(2020), 106933. https://doi.org/10.1016/j.aim.2019.106933
[37] Z. Wu, H. Xu, Symmetry properties in systems of fractional Laplacian equations, Discrete Contin. Dyn. Syst. 39(2019), 1559-1571. https ://doi. org/10.3934/dcds. 2019068
[38] F. Zeng, Symmetric properties for system involving uniformly elliptic nonlocal operators, Mediterr. J. Math. 17(2020), No. 3, Art. 79, 17 pp. https://doi.org/10.1007/s00009-020-01514-6

# Ground state solution for a class of supercritical nonlocal equations with variable exponent 

Xiaojing Feng ${ }^{\boxtimes}$<br>School of Mathematical Sciences, Shanxi University, Taiyuan 030006, Shanxi, P.R. China

Received 1 October 2020, appeared 3 August 2021
Communicated by Petru Jebelean


#### Abstract

In this paper, we obtain the existence of positive critical point with least energy for a class of functionals involving nonlocal and supercritical variable exponent nonlinearities by applying the variational method and approximation techniques. We apply our results to the supercritical Schrödinger-Poisson type systems and supercritical Kirchhoff type equations with variable exponent, respectively.


Keywords: Schrödinger-Poisson type system, Kirchhoff type equations, supercritical exponent, variational method.
2020 Mathematics Subject Classification: 35J20, 35J60.

## 1 Introduction and main results

We divide this section into two parts. In the first part, we present a critical point theory of abstract functional inspired by the article of Marcos do Ó, Ruf and Ubilla [21]. The second part is devoted to introduce its applications to a class of Schrödinger-Poisson type systems and a class of Kirchhoff type equations.

### 1.1 Abstract critical point theory

In the pioneering article [8], Brézis and Nirenberg considered the existence of solution to the following nonlinear elliptic equation

$$
\begin{cases}-\Delta u=u^{5}+f(x, u), & \text { in } \Omega,  \tag{1.1}\\ u>0, & \text { in } \Omega, \\ u=0, & \text { on } \partial \Omega,\end{cases}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^{3}$. If $f(x, u)=0$ and $\Omega$ is star shaped, a well-known nonexistence result of Pohozaev [26] asserts that (1.1) has no solution. But the lower-order terms perturbation can reverse this situation. Brézis and Nirenberg [8] proved the existence of solutions to (1.1) under the assumptions on the lower-order perturbation term $f(x, u)$. On

[^21]the other hand, the topology and the shape of the domain can affect the existence of solution for (1.1) with $f(x, u)=0$. For example, Coron [12] used a variational approach to prove that (1.1) is solvable if $\Omega$ exhibits a small hole. Rey [27] established existence of multiple solutions if $\Omega$ exhibits several small holes. As $\Omega$ is an annulus, Kazdan and Warner [17] observed that there exists a solution to (1.1) without any constraint by critical exponent.

It is worth noticing that there are also a few papers concerning on the supercritical equations except adding lower-order perturbation terms or changing the topology of region $\Omega$. The papers in $[10,21]$ considered the following nonlinear supercritical elliptic problem

$$
\begin{cases}-\Delta u=|u|^{4+|x|^{\alpha}} u, & \text { in } B,  \tag{1.2}\\ u=0, & \text { on } \partial B,\end{cases}
$$

where $B \subset \mathbb{R}^{3}$ is the unit ball and $0<\alpha<1$. By using the mountain pass lemma and approximation techniques, a radial positive solution for (1.2) is obtained by Marcos do Ó, Ruf and Ubilla in [21]. Cao, Li and Liu [10] considered the existence of infinitely many nodal solutions to (1.2) by looking for a minimizer of a constrained minimization problem in a special space.

Let $H$ be the subspace of $H_{0}^{1}(B)$ consisting of radially symmetric functions. From [21], we know that (1.2) possesses a variational structure, its solutions can be found as critical points of the functional

$$
I_{0}(u)=\frac{1}{2} \int_{B}|\nabla u|^{2}-\int_{B} \frac{1}{6+|x|^{\alpha}}|u|^{6+|x|^{\alpha}}, \quad u \in H .
$$

The solutions to this kind of supercritical elliptic equations involving nonlocal nonlinearities can be found to look for the critical points of a suitable perturbation of $I_{0}$,

$$
J(u)=\frac{1}{2} \int_{B}|\nabla u|^{2}+\lambda R(u)-\int_{B} \frac{1}{6+|x|^{\alpha}}|u|^{6+|x|^{\alpha}}, \quad u \in H,
$$

where $\lambda \in \mathbb{R}$ and $R \in C(H, \mathbb{R})$. In order to obtain the nontrivial critical point of $J$, we need to consider the approximation functional $I: H \rightarrow \mathbb{R}$ associated to $J$ given by

$$
I(u)=\frac{1}{2} \int_{B}|\nabla u|^{2}+\lambda R(u)-\frac{1}{6} \int_{B}|u|^{6} .
$$

In this paper, we are interested in researching the least energy critical point of $J$, the following assumptions are needed:
(i) $R \in C^{1}\left(H, \mathbb{R}^{+}\right)$with $\mathbb{R}^{+}=[0,+\infty)$;
(ii) there exist $C, q>0$ such that for $t>0$,

$$
R(t u)=t^{q} R(u), \quad R(u) \leq C\|u\|^{q}, \quad \forall u \in H ;
$$

(iii) $q R(u)=\left\langle R^{\prime}(u) u\right\rangle, u \in H$;
(iv) if $\left\{u_{n}\right\}$ is a $(P S)_{c}$ sequence of $J$ for some $c>0$ and $u_{n} \rightharpoonup u$ weakly in $H$ as $n \rightarrow \infty$, then $J^{\prime}(u)=0$.

Inspired by above papers, the main purpose of this paper is to consider the existence of ground state for the functional $J$. Our main result reads as follows.

Theorem 1.1. Assume that $\lambda>0,2<q<6$ or $\lambda<0, q>6$ and the assumptions (i)-(iv) hold. Then the functional J possesses a $(P S)_{c}$ sequence with some $c>0$. Moreover if the functional I satisfies the $(P S)_{c}$ condition, then $J$ admits a nontrivial critical point.

Theorem 1.2. Suppose that the assumptions of Theorem 1.1 are satisfied. If $R$ is even and weakly lower semicontinuous, then the functional J possesses a least energy critical point.

Remark 1.3. The variable exponent function $p(x)=6+|x|^{\alpha}$ has a strictly supercritical growth except the origin and a critical growth in the origin. Hence, the functional $J$ can be regarded as the supercritical perturbation of the functional $I$.

Remark 1.4. In each case of $\lambda>0,0<q<6$ or $\lambda<0, q>6$, we can show that $J$ possesses the mountain pass structure. Hence, a minimax level for the functional $J$ can be constructed. It is important to verify that this level lies below the non-compactness level of the functional $I$. It is worthwhile pointing out that the term $R$ affects the non-compactness level of the functional $I$. In most cases, it is difficult to calculate the level of the non-compactness level accurately.

Remark 1.5. Since the method of proving (iv) is different when $R$ is different, the condition $(i v)$ is needed. The weak lower semicontinuity of $R$ guarantees the existence of a ground state for functional $J$.

Remark 1.6. Relatively speaking, the condition (iv) is easy to get for some functional $J$ involving nonlocal nonlinearities. It is obvious to see from (iv) that $u$ is a critical point of the functional $J$. Hence, we just need to show that $u$ is nontrivial.

As an application, we apply the case of $\lambda<0$ to a class of Schrödinger-Poisson type systems and the case of $\lambda>0$ to a class of Kirchhoff type equations, respectively.

### 1.2 Applications to two nonlocal problems

As a first application, we consider the existence of nontrivial solution to the supercritical Schrödinger-Poisson type systems with variable exponent

$$
\begin{cases}-\Delta u-\phi|u|^{3} u=|u|^{4+|x|^{\alpha}} u & \text { in } B,  \tag{1.3}\\ -\Delta \phi=|u|^{5} & \text { in } B, \\ u=\phi=0 & \text { on } \partial B,\end{cases}
$$

where $B \subset \mathbb{R}^{3}$ is the unit ball and $0<\alpha<1$. The Schrödinger-Poisson system as a model describing the interaction of a charge particle with an electromagnetic field arises in many mathematical physics context (we refer to [7] for more details on the physical aspects). There are a few references which investigated the well-known Schrödinger-Poisson system with nonlocal critical growth in a bounded domain (see e.g. [3-5]). Azzollini, d'Avenia [3] considered the following problem involving the nonlocal critical growth

$$
\begin{cases}-\Delta u-\phi|u|^{3} u=\lambda u & \text { in } B,  \tag{1.4}\\ -\Delta \phi=|u|^{5} & \text { in } B, \\ u=\phi=0 & \text { on } \partial B .\end{cases}
$$

They proved the existence of positive solution depending on the value of $\lambda$ and (1.4) has no solution for $\lambda \leq 0$ via Pohozaev's identity. Later, Azzollini, d'Avenia and Vaira [5] improved
the results in [3]. They proved existence and nonexistence results of positive solutions for (1.4) when $\lambda$ is in proper region. By applying the variational arguments and the cut-off function technique, Azzollini, d'Avenia and Luisi [4] studied the following generalized SchrödingerPoisson system

$$
\begin{cases}-\Delta u+\varepsilon q \phi f(u)=\eta|u|^{p-1} u & \text { in } \Omega \\ -\Delta \phi=2 q F(u) & \text { in } \Omega \\ u=\phi=0 & \text { on } \partial \Omega\end{cases}
$$

where $\Omega \subset \mathbb{R}^{3}$ is a bounded domain with smooth boundary $\partial \Omega, 1<p<5, q>0, \varepsilon, \eta= \pm 1$, $f \in C(\mathbb{R}, \mathbb{R}), F(s)=\int_{0}^{s} f(t) d t$. In the case where $f$ is critical growth, they obtained the existence and nonexistence results.

In the recent years, there have been a lot of researches dealing with the SchrödingerPoisson systems

$$
\begin{cases}-\Delta u+\phi u=f(x, u) & \text { in } \Omega  \tag{1.5}\\ -\Delta \phi=u^{2} & \text { in } \Omega \\ u=\phi=0 & \text { on } \partial \Omega\end{cases}
$$

When $f(x, u)=|u|^{p-1} u$ with $p \in(1,5)$, Ruiz and Siciliano [29] considered the existence, nonexistence and multiplicity results by using variational methods. Alves and Souto [2] studied system (1.5) when $f$ has a subcritical growth. They obtained the existence of least energy nodal solution by using variational methods. Ba and He [6] proved the existence of ground state solution for system (1.5) with a general 4-superlinear nonlinearity $f$ by the aid of the Nehari manifold. Pisani and Siciliano [25] proved the existence of infinitely many solutions of (1.5) by means of variational methods. In [1], Almuaalemi, Chen and Khoutir obtained the existence of nontrivial solutions for (1.5) when $f$ has a critical growth via variational methods.

Motivated by above papers, by applying Theorems 1.1 and 1.2 , we obtain the existence of positive ground state solution for system (1.3) with both nonlinearity supercritical growth and nonlocal critical growth. From the technical point of view, there are two difficulties to prove our result. Firstly, the supercritical nonlinearity in the system sets an obstacle since the bounded ( $P S$ ) sequence could not converge. Secondly, due to the system has two critical terms, it is difficult to estimate the critical level of mountain pass. In order to overcome these difficulties, by employing the ideas of [21], we first estimate the critical level of the mountain pass for the functional corresponding to (1.3) via approximation techniques and then show that the level is below the non-compactness level of the functional. Finally, the existence of positive ground state solution is obtained by applying the Nehari manifold method and regularity theory. Hence, we have the following result:

Theorem 1.7. System (1.3) possesses at least a positive ground state solution.
Remark 1.8. By the Pohozaev's identity used in [3], we can deduce that (1.3) has no nontrivial solution if $|x|^{\alpha}=0$. Hence, our result is interesting phenomena due to the nonlinearity $|u|^{4+|x|^{\alpha}} u$ has supercritical growth everywhere in $B$ except in the origin and critical growth in the origin.

Next, as the second application, we consider the following Kirchhoff type equations:

$$
\begin{cases}-\left(1+b \int_{B}|\nabla u|^{2} d x\right) \Delta u=|u|^{4+|x|^{\alpha}} u, & \text { in } B  \tag{1.6}\\ u=0, & \text { on } \partial B\end{cases}
$$

where $b>0,0<\alpha<1$. This kind of equation is related to the stationary analogue of the equation

$$
\rho \frac{\partial^{2} u}{\partial t^{2}}-\left(\frac{\rho_{0}}{h}+\frac{E}{2 l} \int_{0}^{L}\left|\frac{\partial u}{\partial x}\right|^{2} d x\right) \frac{\partial^{2} u}{\partial x^{2}}=0
$$

presented by Kirchhoff in [18]. The equation extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. The solvability of the Kirchhoff type equations has been well studied in a general dimension by many authors after Lions [20] introduced an abstract framework to this problem. By using new analytical skills and non-Nehari manifold method, Tang and Cheng [31] obtained the ground state sign-changing solutions for a class of Kirchhoff type problems in bounded domains. In [11], Chen, Zhang and Tang considered the existence and non-existence results for Kirchhoff-type problems with convolution nonlinearity based on variational and some new analytical techniques. There are also many papers devoted to the existence and multiplicity of solutions for the following critical Kirchhoff type equations with subcritical disturbance

$$
\begin{cases}-\left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \Delta u=f(x, u)+u^{5} & \text { in } \Omega  \tag{1.7}\\ u>0 & \text { in } \Omega \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

where $a, b$ are positive constants. By using concentration-compactness principle and variational method, Naimen in [22] obtained the existence and multiplicity of (1.7) with $f(x, u)=$ $\lambda u$. Xie, Wu and Tang [34] derived the existence and multiplicity of solutions to (1.7) via variational method by discussing the sign of $a$ and $b$ and adding different conditions on $f$. By controlling concentrating Palais-Smale sequences, Naimen and Shibata [23] proved the existence of two positive solutions for (1.7) with $f(x, u)=u^{q}, 1 \leq q<5$.

In particular, there are some papers considered the equations with critical and supercritical growth by adding the smallness of the coefficient in front of critical and supercritical which is used to overcome the difficulty provoked by supercritical growth. By combining an appropriate method of truncation function with Moser's iteration technique, Corrêa and Figueiredo $[13,14]$ considered the existence of positive solution for a class of $p$-Kirchhoff type equations and Kirchhoff type equations with supercritical growth, respectively.

Motivated by the above fact, we study the existence of positive ground state solution for (1.6) with variable exponential perturbation by using the similar method introduced by Marcos do Ó, Ruf and Ubilla in [21]. The result reads as follows.

Theorem 1.9. The equation (1.6) possesses at least a positive ground state solution.
Remark 1.10. Recall that in [22], if $|x|^{\alpha}=0$, (1.6) has no nontrivial solution by Pohozaev's identity. Hence, our result is interesting phenomena for this kind of Kirchhoff type equations due to the nonlinearity $|u|^{4+|x|^{\alpha}} u$ has supercritical growth everywhere in $B$ except the origin and critical growth in the origin.

Remark 1.11. Throughout the paper we denote by $C>0$ various positive constants which may vary from line to line and are not essential to the problem.

The paper is organized as follows: in Section 2, some notations and preliminary results are presented. We obtain the existence of nontrivial critical point to the functional $J$ in Section 3. By using Nehari manifold method, the least energy critical point of the functional $J$ is derived
in Section 4. Sections 5 and 6 are devoted to show that the Theorems 1.1 and 1.2 can be applied to the nonlinear Schrödinger-Poisson type systems and the Kirchhoff type equations, respectively.

## 2 Preliminary

In this Section, we will give some notations and lemmas which will be used throughout this paper. Let $B \subset \mathbb{R}^{3}$ denote the unit ball, $H=H_{0, \text { rad }}^{1}(B)=\left\{u \in H_{0}^{1}(B): u(x)=u(|x|)\right\}$ be the Sobolev space of radial functions, with respect to the norm

$$
\|u\|=\left(\int_{B}|\nabla u|^{2}\right)^{1 / 2}
$$

Let $C_{+}(\bar{B})=\{h: h \in C(\bar{B}), h(x)>1, x \in \bar{B}\}$. For any $h \in C_{+}(\bar{B})$, we denote

$$
h^{+}=\sup _{x \in B} h(x), \quad h^{-}=\inf _{x \in B} h(x)
$$

Then for each $p \in C_{+}(\bar{B})$, the variable exponent function space $L^{p(x)}(B)$ is defined as follows

$$
L^{p(x)}(B)=\left\{u: u \text { is a measurable function in } B \text { such that } \int_{B}|u(x)|^{p(x)} d x<\infty\right\}
$$

with the norm defined by

$$
\|u\|_{L^{p(x)}}=\inf \left\{\lambda>0, \int_{B}\left|\frac{u}{\lambda}\right|^{p(x)} \leq 1\right\} .
$$

We denote by $L^{p^{\prime}(x)}(B)$ the conjugate space of $L^{p(x)}(B)$, where $1 / p(x)+1 / p^{\prime}(x)=1$. For any $u \in L^{p(x)}(B)$ and $v \in L^{p^{\prime}(x)}(B)$, there holds the Hölder type inequality

$$
\left|\int_{B} u v\right| \leq\left(\frac{1}{p^{-}}+\frac{1}{p^{\prime-}}\right)\|u\|_{L^{p(x)}}\|v\|_{L^{p^{\prime}(x)}} .
$$

Lemma 2.1 ([15]). Set $\rho(u)=\int_{B}|u(x)|^{p(x)}$. For $u \in L^{p(x)}(B)$, we have
(1) $\|u\|_{L^{p(x)}}<1(=1 ;>1) \Leftrightarrow \rho(u)<1(=1 ;>1)$;
(2) If $\|u\|_{L^{p(x)}}>1$, then $\|u\|_{L^{p(x)}}^{p^{-}} \leq \rho(u) \leq\|u\|_{L^{p(x)}}^{p^{+}}$;
(3) If $\|u\|_{L^{p(x)}}<1$, then $\|u\|_{L^{p(x)}}^{p^{+}} \leq \rho(u) \leq\|u\|_{L^{p(x)}}^{p^{-}}$.

Lemma 2.2 ([21]). Let $q(x)=6+\beta|x|^{\alpha}, x \in B$ and $\alpha, \beta>0$. The following embedding is continuous:

$$
H \hookrightarrow L^{q(x)}(B)
$$

It is easy to check by $(i)$, Lemma 2.2 and Hölder type inequality that $J$ is well defined on $H$ and $J \in C^{1}(H, \mathbb{R})$, and

$$
\left\langle J^{\prime}(u), v\right\rangle=\int_{B} \nabla \cdot u \nabla v+\lambda\left\langle R^{\prime}(u), v\right\rangle-\int_{B}|u|^{4+|x|^{\alpha}} u v, \quad u, v \in H .
$$

In the following we define the best embedding constant $S$ by

$$
\begin{equation*}
S=\inf _{u \in H \backslash\{0\}} \frac{\int_{B}|\nabla u|^{2}}{\left(\int_{B}|u|^{6}\right)^{\frac{1}{3}}} . \tag{2.1}
\end{equation*}
$$

Let $\chi \in C_{0}^{\infty}(B)$ be a cut-off function with $\chi=1$ on $B_{1 / 2}(0)$ and $\eta \in[0,1]$ on $B$. Let us define the function

$$
U_{\varepsilon}(x)=\left(3 \varepsilon^{2}\right)^{1 / 4}\left(\varepsilon^{2}+|x|^{2}\right)^{-1 / 2}, \quad \varepsilon>0
$$

which satisfies the equation

$$
-\Delta u=u^{5} \quad \text { on } \mathbb{R}^{3}
$$

Then define $u_{\varepsilon}=\chi(x) U_{\varepsilon}(x)$, the following estimates can be deduced via standard arguments as $\varepsilon \rightarrow 0^{+}$(see [33]),

$$
\begin{equation*}
\int_{B}\left|\nabla u_{\varepsilon}\right|^{2}=S^{\frac{3}{2}}+O(\varepsilon), \quad \int_{B} u_{\varepsilon}^{6}=S^{\frac{3}{2}}+O\left(\varepsilon^{3}\right) \tag{2.2}
\end{equation*}
$$

## 3 The nontrivial critical point

In this section, we first show that the functional $J$ possesses the mountain pass structure under the assumption $\lambda<0, q>6$ or $\lambda>0,0<q<6$, respectively. And hence $J$ has a $(P S)_{c}$ sequence $\left\{u_{n}\right\}$ with some $c>0$. Then we prove that $\left\{u_{n}\right\}$ is bounded and is also a $(P S)_{c}$ sequence of $I$, which is a key in the existence of nontrivial critical point.

Lemma 3.1. Assume that $\lambda<0, q>2$ and the assumptions (i) and (ii) hold.
(a) There exist $\rho_{1}>0, \eta_{1}>0$ such that $\inf \left\{J(u): u \in H\right.$, with $\left.\|u\|=\rho_{1}\right\}>\eta_{1}$.
(b) There exists $e_{1} \in H$ with $\left\|e_{1}\right\|>\rho_{1}$ such that $J\left(e_{1}\right)<0$.

Proof. (a) For $\rho_{1}>0$, let

$$
\Sigma_{\rho_{1}}=\left\{u \in H:\|u\| \leq \rho_{1}\right\}
$$

We deduce, from the Sobolev inequality and Lemma 2.1, that for $u \in \partial \Sigma_{\rho_{1}}$ and $C>0$,

$$
\begin{aligned}
J(u) & =\frac{1}{2}\|u\|^{2}+\lambda R(u)-\int_{B} \frac{1}{6+|x|^{\alpha}}|u|^{6+|x|^{\alpha}} \\
& \geq \frac{1}{2}\|u\|^{2}+C \lambda\|u\|^{q}-C\left(\|u\|^{6}+\|u\|^{7}\right) \\
& =\frac{1}{2} \rho_{1}^{2}+C \lambda \rho_{1}^{q}-C \rho_{1}^{6}-C \rho_{1}^{7} .
\end{aligned}
$$

Hence, by letting $\rho_{1}>0$ small enough, it is easy to see that there is $\eta_{1}>0$ such that (a) holds.
(b) By [21, Lemma 3.1], we know that there exists a constant $C>0$ such that for $\varepsilon>0$ small,

$$
\begin{align*}
\int_{B}\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}} & \geq \int_{B}\left|u_{\varepsilon}\right|^{6}+C|\log \varepsilon| \varepsilon^{\alpha}+O(\varepsilon)  \tag{3.1}\\
& =S^{3 / 2}+C|\log \varepsilon| \varepsilon^{\alpha}+O(\varepsilon)
\end{align*}
$$

This together with (2.2) implies that for $t \geq 1$ and $\varepsilon>0$ small enough,

$$
\begin{aligned}
J\left(t u_{\varepsilon}\right) & =\frac{t^{2}}{2}\left\|u_{\varepsilon}\right\|^{2}+\lambda t^{q} R\left(u_{\varepsilon}\right)-\int_{B} \frac{t^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}} \\
& \leq \frac{t^{2}}{2}\left\|u_{\varepsilon}\right\|^{2}-\frac{t^{6}}{7} \int_{B}\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}} \\
& \leq S^{3 / 2} t^{2}-\frac{S^{3 / 2}}{14} t^{6} \rightarrow-\infty
\end{aligned}
$$

as $t \rightarrow+\infty$. Let $T>0$ and define a path $\tilde{h}:[0,1] \rightarrow H$ by $\tilde{h}(t)=t T u_{\varepsilon}$. For $T>0$ large enough, we have

$$
\int_{B}|\nabla \tilde{h}(1)|^{2}>\rho_{1}^{2}, \quad J(\tilde{h}(1))<0
$$

By taking $e_{1}=\tilde{h}(1)$, then $(b)$ is valid. The proof is completed.
Lemma 3.2. Assume that $\lambda>0,0<q<6$ and the assumptions (i) and (ii) hold.
(a) There exist $\rho_{2}>0, \eta_{2}>0$ such that $\inf \left\{J(u): u \in H\right.$, with $\left.\|u\|=\rho_{2}\right\}>\eta_{2}$.
(b) There exists $e_{2} \in H$ with $\left\|e_{2}\right\|>\rho$ such that $J\left(e_{2}\right)<0$.

Proof. (a) Let us define

$$
\Sigma_{\rho_{2}}=\left\{u \in H:\|u\| \leq \rho_{2}\right\}, \quad \rho_{2}>0 .
$$

It follows from the Sobolev inequality and Lemma 2.1 that for $u \in \partial \Sigma_{\rho_{2}}$ and $C>0$,

$$
\begin{aligned}
J(u) & =\frac{1}{2}\|u\|^{2}+\lambda R(u)-\int_{B} \frac{1}{6+|x|^{\alpha}}|u|^{6+|x|^{\alpha}} \\
& \geq \frac{1}{2}\|u\|^{2}-C\left(\|u\|^{6}+\|u\|^{7}\right) \\
& =\frac{1}{2} \rho_{2}^{2}-C \rho_{2}^{6}-C \rho_{2}^{7} .
\end{aligned}
$$

Hence, by letting $\rho_{2}>0$ small enough, it is easy to see that there is $\eta_{2}>0$ such that (a) holds.
(b) By using (2.2) and (3.1) again, we have for $t \geq 1$ and $\varepsilon>0$ small enough,

$$
\begin{aligned}
J\left(t u_{\varepsilon}\right) & =\frac{t^{2}}{2}\left\|u_{\varepsilon}\right\|^{2}+\lambda t^{q} R\left(u_{\varepsilon}\right)-\int_{B} \frac{t^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}} \\
& \leq \frac{t^{2}}{2}\left\|u_{\varepsilon}\right\|^{2}+C \lambda t^{q}\left\|u_{\varepsilon}\right\|^{q}-\frac{t^{6}}{7} \int_{B}\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}} \\
& \leq S^{3 / 2} t^{2}+2 C \lambda S^{3 q / 4} t^{q}-\frac{t^{6}}{14} S^{3 / 2} \rightarrow-\infty
\end{aligned}
$$

as $t \rightarrow+\infty$. Let $T>0$ and define a path $\hat{h}:[0,1] \rightarrow H$ by $\hat{h}(t)=t T u_{\varepsilon}$. For $T>0$ large enough, we have

$$
\int_{B}|\nabla \hat{h}(1)|^{2}>\rho_{2}^{2}, \quad J(\hat{h}(1))<0 .
$$

By taking $e_{2}=\hat{h}(1)$, we proof $(b)$. The proof is completed.

From Lemmas 3.1 and 3.2, we know that the functional $J$ possesses the mountain pass geometry. Then there is a $(P S)_{c}$ sequence $\left\{u_{n}\right\} \subset H$ for $J$ with the property that

$$
J\left(u_{n}\right) \rightarrow c, \quad\left\|J^{\prime}\left(u_{n}\right)\right\|_{H^{-1}} \rightarrow 0, \quad n \rightarrow \infty,
$$

where $c$ is given by

$$
\begin{equation*}
c=\inf _{\gamma \in \Gamma} \max _{t \in[0,1]} J(\gamma(t)), \tag{3.2}
\end{equation*}
$$

and $\Gamma=\{\gamma \in C([0,1], H): \gamma(0)=0, J(\gamma(1))<0\}$.
Lemma 3.3. Assume that $\lambda<0, q>6$ or $\lambda>0,0<q<6$ and the assumption (iii) holds. If $\left\{u_{n}\right\} \subset H$ is a $(P S)_{c}$ sequence for $J$ with $c>0$, then $\left\{u_{n}\right\}$ is bounded in $H$.

Proof. For $n$ large enough, it is easy to deduce from (iii) that

$$
\begin{aligned}
c+1 & \geq J\left(u_{n}\right)-\frac{1}{6}\left\langle J^{\prime}\left(u_{n}\right), u_{n}\right\rangle \\
& =\frac{1}{3}\left\|u_{n}\right\|^{2}+\lambda\left(\frac{1}{q}-\frac{1}{6}\right)\left\langle R^{\prime}\left(u_{n}\right), u_{n}\right\rangle+\int_{B}\left(\frac{1}{6}-\frac{1}{6+|x|^{\alpha}}\right)\left|u_{n}\right|^{6+|x|^{\alpha}} \\
& \geq \frac{1}{3}\left\|u_{n}\right\|^{2}
\end{aligned}
$$

which implies that $\left\{u_{n}\right\}$ is bounded in $H$. The proof is completed.
Lemma 3.4 ([21]). Assume that $u \in H$. Then

$$
|u(r)| \leq r^{-1 / 2}\|u\|, \quad r>0 .
$$

Proof of Theorem 1.1. By using Lemmas 3.1 and 3.2 respectively, there exists a sequence $\left\{u_{n}\right\} \subset$ $H$ satisfying $J\left(u_{n}\right) \rightarrow c, J^{\prime}\left(u_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$, where $c$ is given in (3.2). By Lemma 3.3, $\left\{u_{n}\right\}$ is a bounded sequence in $H$. Passing to a subsequence if necessary, we may assume that there exists $u \in H$ such that

$$
u_{n} \rightharpoonup u \text { in } H, \text { and } u_{n}(x) \rightarrow u(x), \text { a.e. } x \in B .
$$

If $u \neq 0$, then $u$ is a nontrivial critical point of the functional $J$ follows from the assumption (iv). In what follows, we will deal with the case of $u=0$ and show that this is impossible. In fact, since $H_{r}^{1}\left(B \backslash B_{\delta}\right) \hookrightarrow \hookrightarrow L^{p}\left(B \backslash B_{\delta}\right)$, for $\delta \in(0,1)$ and $p \geq 1$, there holds

$$
\begin{equation*}
\int_{\delta}^{1}\left|u_{n}\right|^{6+r^{\alpha}} r^{2} \rightarrow 0, \text { as } n \rightarrow \infty \tag{3.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{\delta}^{1}\left|u_{n}\right|^{6} r^{2} \rightarrow 0, \text { as } n \rightarrow \infty . \tag{3.4}
\end{equation*}
$$

In the following, we will show that $\left\{u_{n}\right\}$ is also a $(P S)_{c}$ sequence of $I$. Hence, it is sufficient to prove
(a) $J\left(u_{n}\right)=I\left(u_{n}\right)+o(1)$;
(b) $\left\langle J^{\prime}\left(u_{n}\right), v\right\rangle=\left\langle I^{\prime}\left(u_{n}\right), v\right\rangle+o(1)\|v\|, v \in H$.

We first claim that (a) is valid, indeed we only need to estimate

$$
\begin{align*}
& \int_{B}\left(\frac{1}{6}\left|u_{n}\right|^{6}-\frac{1}{6+|x|^{\alpha}}\left|u_{n}\right|^{6+|x|^{\alpha}}\right) \\
& \quad=\int_{B}\left(\frac{1}{6}\left|u_{n}\right|^{6}-\frac{1}{6+|x|^{\alpha}}\left|u_{n}\right|^{6}\right)+\int_{B}\left(\frac{1}{6+|x|^{\alpha}}\left|u_{n}\right|^{6}-\frac{1}{6+|x|^{\alpha}}\left|u_{n}\right|^{6+|x|^{\alpha}}\right) . \tag{3.5}
\end{align*}
$$

For any $\varepsilon>0$, there exist $\delta>0$ and $n_{1} \in \mathbb{N}$ such that for any $n \geq n_{1}$, we have, by (3.4),

$$
\begin{align*}
\int_{B}\left(\frac{1}{6}\left|u_{n}\right|^{6}-\frac{1}{6+|x|^{\alpha}}\left|u_{n}\right|^{6}\right) & \leq \frac{\omega}{36} \int_{0}^{1}\left|u_{n}\right|^{6} r^{2+\alpha} \\
& =\frac{\omega}{36} \int_{0}^{\delta}\left|u_{n}\right|^{6} r^{2+\alpha}+\frac{\omega}{36} \int_{\delta}^{1}\left|u_{n}\right|^{6} r^{2+\alpha}  \tag{3.6}\\
& \leq \frac{\left\|u_{n}\right\|^{6}}{36 \alpha} \omega \delta^{\alpha}+\frac{\omega}{36} \int_{\delta}^{1}\left|u_{n}\right|^{6} r^{2} \leq \frac{\varepsilon}{2},
\end{align*}
$$

where $\omega$ is the surface area of the unit sphere in $\mathbb{R}^{3}$. Similarly, for above $\varepsilon>0$, there exist $\delta_{1}>0$ small enough and $n_{2} \in \mathbb{N}$ such that for any $n \geq n_{2}$, it follows from (3.3) and (3.4) that

$$
\begin{align*}
&\left|\int_{B}\left(\frac{1}{6+|x|^{\alpha}}\left|u_{n}\right|^{6}-\frac{1}{6+|x|^{\alpha}}\left|u_{n}\right|^{6+|x|^{\alpha}}\right)\right| \\
& \leq \left.\left.\frac{\omega}{6} \int_{\left[0, \delta_{1}\right] \cap\left\{\left|u_{n}\right|>1\right\}}\left|u_{n}\right|^{6}| | u_{n}\right|^{r^{\alpha}}-\left.\left.1\left|r^{2}+\frac{\omega}{6} \int_{\left[0, \delta_{1}\right] \cap\left\{\left|u_{n}\right| \leq 1\right\}}\right| u_{n}\right|^{6}| | u_{n}\right|^{r^{\alpha}}-1 \right\rvert\, r^{2} \\
& \left.+\left.\frac{\omega}{6}\left|\int_{\delta_{1}}^{1}\right| u_{n}\right|^{6}\left(\left|u_{n}\right|^{\alpha^{\alpha}}-1\right) r^{2} \right\rvert\, \\
& \leq \left.\frac{\omega}{6} \int_{0}^{\delta_{1}}\left|u_{n}\right|^{6} r^{2}\left|\exp \left[-\frac{r^{\alpha}}{2} \log (C r)\right]-1\right|+\frac{\omega}{18} \delta_{1}^{3}+\left.\frac{\omega}{6}\left|\int_{\delta_{1}}^{1}\right| u_{n}\right|^{6}\left(\left|u_{n}\right|^{\alpha^{\alpha}}-1\right) r^{2} \right\rvert\,  \tag{3.7}\\
& \left.\leq C \omega \int_{0}^{\delta_{1}}\left|u_{n}\right|^{6} r^{2} r^{\alpha}|\log C r|+\frac{\omega}{18} \delta_{1}^{3}+\left.\frac{\omega}{6}\left|\int_{\delta_{1}}^{1}\right| u_{n}\right|^{6}\left(\left|u_{n}\right|^{r^{\alpha}}-1\right) r^{2} \right\rvert\, \\
& \leq \left.C_{1} \omega \delta_{1}^{\alpha}\left|\log C \delta_{1}\right|+\frac{\omega}{18} \delta_{1}^{3}+\left.\frac{\omega}{6}\left|\int_{\delta_{1}}^{1}\right| u_{n}\right|^{6}\left(\left|u_{n}\right|^{r^{\alpha}}-1\right) r^{2} \right\rvert\, \leq \frac{\varepsilon}{2} .
\end{align*}
$$

Hence, combining (3.5), (3.6) and (3.7), we have for above $\varepsilon>0$, there exists $n_{0}=\max \left\{n_{1}, n_{2}\right\}$, such that for any $n \geq n_{0}$,

$$
\left|\int_{B}\left(\frac{1}{6}\left|u_{n}\right|^{6}-\frac{1}{6+|x|^{\alpha}}\left|u_{n}\right|^{6+|x|^{\alpha}}\right)\right| \leq \varepsilon,
$$

which implies that (a) is true.
Secondly, we will devoted to verify that (b) is correct. In fact, by Lemma 3.4, for $0<\eta<1$
small enough and $v \in H$,

$$
\begin{aligned}
& \left.\left|\int_{0}^{\eta}\right| u_{n}\right|^{5}|v|\left(\left|u_{n}\right|^{r^{\alpha}}-1\right) r^{2} \mid \\
& \quad \leq\left.\left|\int_{[0, \eta] \cap\left\{\left|u_{n}\right|>1\right\}}\right| u_{n}\right|^{5}|v|\left(\left|u_{n}\right|^{r^{\alpha}}-1\right) r^{2}\left|+\left|\int_{[0, \eta] \cap \cap\left\{\left|u_{n}\right| \leq 1\right\}}\right| u_{n}\right|^{5}|v|\left(\left|u_{n}\right|^{r^{\alpha}}-1\right) r^{2} \mid \\
& \quad \leq \int_{0}^{\eta}\left|u_{n}\right|^{5}|v|\left|(C r)^{-r^{\alpha} / 2}-1\right| r^{2}+C \eta^{3 / 2}\|v\| \\
& \quad \leq \int_{0}^{\eta}\left|u_{n}\right|^{5}|v|\left|\exp \left(r^{\alpha} / 2 \log (C r)^{-1}\right)-1\right| r^{2}+C \eta^{3 / 2}\|v\| \\
& \quad \leq C \int_{0}^{\eta}\left|u_{n}\right|^{5}|v| r^{\alpha}|\log (C r)| r^{2}+C \eta^{3 / 2}\|v\| \\
& \quad \leq C \eta^{\alpha}|\log (C \eta)| \int_{0}^{1}\left|u_{n}\right|^{5}|v| r^{2}+C \eta^{3 / 2}\|v\| \\
& \quad \leq C \eta^{\alpha}|\log (C \eta)|\left\|u_{n}\right\|^{5}\|v\|+C \eta^{3 / 2}\|v\| .
\end{aligned}
$$

Hence, for any $\varepsilon>0$, there exists $\eta=\eta(\varepsilon)>0$ sufficiently small such that

$$
C \eta^{\alpha}|\log (C \eta)|\left\|u_{n}\right\|^{5}\|v\|+C \eta^{3 / 2}\|v\|<\frac{\varepsilon}{3}\|v\|,
$$

and then

$$
\begin{equation*}
\left.\left|\int_{0}^{\eta}\right| u_{n}\right|^{5}|v|\left(\left|u_{n}\right|^{r^{\alpha}}-1\right) r^{2} \left\lvert\,<\frac{\varepsilon}{3}\|v\| .\right. \tag{3.8}
\end{equation*}
$$

On the other hand, it follows that for above $\varepsilon>0$, there exists $n_{1} \in \mathbb{N}$ such that for $n>n_{1}$,

$$
\begin{equation*}
\int_{\eta}^{1}\left|u_{n}\right|^{5+r^{\alpha}}|v| r^{2} \leq \mathrm{C}\left(\int_{\eta}^{1}\left|u_{n}\right|^{6+r^{\alpha}} r^{2}\right)^{5 / 7}\|v\| \leq \frac{\varepsilon}{3}\|v\| . \tag{3.9}
\end{equation*}
$$

Similarly, we have for above $\varepsilon>0$, there exists $n_{2} \in \mathbb{N}$ such that for $n>n_{2}$,

$$
\begin{equation*}
\int_{\eta}^{1}\left|u_{n}\right|^{5}|v| r^{2} \leq C\left(\int_{\eta}^{1}\left|u_{n}\right|^{6} r^{2}\right)^{5 / 6}\|v\| \leq \frac{\varepsilon}{3}\|v\| . \tag{3.10}
\end{equation*}
$$

Combining (3.8), (3.9) and (3.10), we obtain for $\varepsilon>0$, there exists $n_{0}=\max \left\{n_{1}, n_{2}\right\}$ such that for $n>n_{0}$,

$$
\begin{aligned}
& \left.\left|\int_{0}^{1}\right| u_{n}\right|^{4+r^{\alpha}} u_{n} v r^{2}-\int_{0}^{1}\left|u_{n}\right|^{4} u_{n} v r^{2} \mid \\
& \quad \leq\left.\int_{0}^{1}\left|u_{n}\right|^{5}|v|| | u_{n}\right|^{r^{\alpha}}-1 \mid r^{2} \\
& \quad \leq\left.\int_{0}^{\eta}\left|u_{n}\right|^{5}|v|| | u_{n}\right|^{r^{\alpha}}-\left.1\left|r^{2}+\int_{\eta}^{1}\right| u_{n}\right|^{5}|v| r^{2}+\int_{\eta}^{1}\left|u_{n}\right|^{5}|v|\left|u_{n}\right|^{r^{\alpha}} r^{2} \leq \varepsilon\|v\|, \quad v \in H,
\end{aligned}
$$

which ensures that $(b)$ is valid. Thereby, it is obvious that $\left\{u_{n}\right\}$ is also a $(P S)_{c}$ sequence for the functional $I$. Recall that $I$ satisfies $(P S)_{c}$ condition, we have that $u_{n} \rightarrow u=0$ strongly in $H$, which is a contradiction to $I\left(u_{n}\right) \rightarrow c>0$. The proof is completed.

## 4 The least energy critical point

In this section, we will use the Nehari manifold method to show the existence of nontrivial nonnegative ground state of the functional $J$. In order to obtain the ground state, we need the Nehari manifold associated with $J$ given by

$$
\mathcal{N}=\left\{u \in H \backslash\{0\}:\left\langle J^{\prime}(u), u\right\rangle=0\right\} .
$$

Lemma 4.1. Assume that $\lambda<0, q>2$ or $\lambda>0,2<q<6$ and the assumptions (i)-(ii) hold. Then, for each $u \in H \backslash\{0\}$, there exists a unique $t(u)>0$ such that $t(u) u \in \mathcal{N}$. Moreover, $J(t(u) u)=\max _{t \geq 0} J(t u)$.
Proof. (a) Let $u \in H \backslash\{0\}$ be fixed. For convenience, we define the function $h(t)=J(t u)$ for $t>0$. Note that $h^{\prime}(t)=\left\langle J^{\prime}(t u), u\right\rangle=0$ if and only if $t u \in \mathcal{N}$. By simple calculation, we see that when $\lambda<0, q>2$

$$
\begin{aligned}
h^{\prime}(t) & =t\|u\|^{2}+\lambda q t^{q-1} R(u)-\int_{B} t^{5+|x|^{\alpha}}|u|^{6+|x|^{\alpha}} \\
& =t\left(\|u\|^{2}+\lambda t^{q-2} R(u)-\int_{B} t^{4+|x|^{\alpha}}|u|^{6+|x|^{\alpha}}\right) \\
& =t \xi(t) .
\end{aligned}
$$

It is obvious that $\xi$ is a non-increasing function for $t>0$ and $\lim _{t \rightarrow 0^{+}} \xi(t)=\|u\|^{2}>0$, $\lim _{t \rightarrow \infty} \xi(t)=-\infty$. Hence, there exists a unique $t(u)>0$ such that $h^{\prime}(t(u))=0$ and $t(u) u \in$ $\mathcal{N}$. Moreover, $J(t(u) u)=\max _{t \geq 0} J(t u)$.
(b) By simple calculation, we see that for $\lambda>0,2<q<6$,

$$
\begin{aligned}
h^{\prime}(t) & =t\|u\|^{2}+\lambda q t^{q-1} R(u)-\int_{B} t^{5+|x|^{\alpha}}|u|^{6+|x|^{\alpha}} \\
& =t^{q-1}\left(\frac{1}{t^{q-2}}\|u\|^{2}+\lambda q R(u)-\int_{B} t^{6-q+|x|^{\alpha}}|u|^{6+|x|^{\alpha}}\right) \\
& =t^{q-1} \xi(t) .
\end{aligned}
$$

It is easy to see that $\xi$ is a non-increasing for $t>0$ and $\lim _{t \rightarrow 0^{+}} \xi(t)=\infty, \lim _{t \rightarrow \infty} \xi(t)=-\infty$. Hence, there exists a unique $t(u)>0$ such that $h^{\prime}(t(u))=0$ and $t(u) u \in \mathcal{N}$. In addition, $J(t(u) u)=\max _{t \geq 0} J(t u)$. The proof is completed.

Lemma 4.2. Assume that $\lambda<0, q>6$ or $\lambda>0,2<q<6$ and the assumptions (i)-(iii) hold. Then $J$ is bounded from below on $\mathcal{N}$.
Proof. For $u \in \mathcal{N}$, it follows from (i) and (ii) that

$$
\begin{aligned}
\|u\|^{2} & =-\lambda q R(u)+\int_{B}|u|^{6+|x|^{\alpha}} \\
& \leq C\left(\|u\|^{6}+\|u\|^{7}+\|u\|^{q}\right)
\end{aligned}
$$

which implies that there exists a positive constant $C$ such that $\|u\| \geq C$. On the other hand, we have

$$
\begin{aligned}
J(u) & =J(u)-\frac{1}{6}\left\langle J^{\prime}(u), u\right\rangle \\
& =\frac{1}{3}\|u\|^{2}+\lambda\left(\frac{1}{q}-\frac{1}{6}\right)\left\langle R^{\prime}(u), u\right\rangle+\int_{B}\left(\frac{1}{6}-\frac{1}{6+|x|^{\alpha}}\right)|u|^{6+|x|^{\alpha}} \\
& \geq \frac{1}{3}\|u\|^{2}, u \in \mathcal{N} .
\end{aligned}
$$

Hence, $J$ is bounded below. The proof is completed.
By Lemmas 4.1 and 4.2, we can define

$$
c^{*}=\inf _{u \in \mathcal{N}} J(u), \quad c^{* *}=\inf _{u \in H \backslash\{0\}} \max _{t \geq 0} J(t u) .
$$

Lemma 4.3. Assume that $\lambda<0, q>6$ or $\lambda>0,2<q<6$ and the assumptions (i)-(iii) hold. Then $c=c^{*}=c^{* *}$.

Proof. It follows from Lemma 4.1 that $c^{*}=c^{* *}$. In the following, we will show that $c=c^{*}$. Indeed, let $u \in \mathcal{N}$, by Lemmas 3.1 and 3.2 there exists some $t_{0}>1$ such that $J\left(t_{0} u\right)<0$. Thus, $J(u)=\max _{t>0} J(t u) \geq \max _{t \in[0,1]} J\left(t t_{0} u\right) \geq c$, which leads to $c^{*} \geq c$.

On the other hand, we find for $u \in H$ that

$$
\begin{align*}
J(u)-\frac{1}{6}\left\langle J^{\prime}(u), u\right\rangle & =\frac{1}{3}\|u\|^{2}+\lambda\left(\frac{1}{q}-\frac{1}{6}\right)\left\langle R^{\prime}(u), u\right\rangle+\int_{B}\left(\frac{1}{6}-\frac{1}{6+|x|^{\alpha}}\right)|u|^{6+|x|^{\alpha}}  \tag{4.1}\\
& \geq \frac{1}{3}\|u\|^{2} \geq 0
\end{align*}
$$

Let $\gamma \in \Gamma$, then it follows from (4.1) that $\left\langle J^{\prime}(\gamma(1)), \gamma(1)\right\rangle \leq 6 J(\gamma(1))<0$. Let us define $t_{1}=\inf \left\{t \in[0,1):\left\langle J^{\prime}(\gamma(s)), \gamma(s)\right\rangle<0, s \in(t, 1]\right\}$. Then $\left\langle J^{\prime}\left(\gamma\left(t_{1}\right)\right), \gamma\left(t_{1}\right)\right\rangle=0$ and $\gamma(s) \neq 0$ for all $s \in\left(t_{1}, 1\right]$. We now show that $\gamma\left(t_{1}\right) \neq 0$. Otherwise, $\gamma\left(t_{1}\right)=0$ then Lemma 3.1 implies that $\left\langle J^{\prime}(\gamma(s)), \gamma(s)\right\rangle>0$ as $s \rightarrow t_{1}^{+}$, thus there exists $\delta>0$ such that $t_{1}+\delta<1$ and $\left\langle J^{\prime}\left(\gamma\left(t_{1}+\right.\right.\right.$ $\left.\delta)), \gamma\left(t_{1}+\delta\right)\right\rangle>0$. Note that the definition of $t_{1}$, there holds $\left\langle J^{\prime}\left(\gamma\left(t_{1}+\delta\right)\right), \gamma\left(t_{1}+\delta\right)\right\rangle<0$. This comes to a contradiction. Thus, we conclude that $\gamma\left(t_{1}\right) \in \mathcal{N}$ and $c \geq c^{*}$. The proof is completed.

The following lemma can be also obtained by Implicit Function Theorem or by the Lusternik Theorem. We give the other proof by applying the Lagrange multiplier method.

Lemma 4.4. Assume that $\lambda<0, q>6$ or $\lambda>0,2<q<6$ and the assumptions (i)-(iii) hold. If $c^{*}$ is attained at some $u \in \mathcal{N}$, then $u$ is a critical point of J in $H$.
Proof. Let $G(u)=\left\langle J^{\prime}(u), u\right\rangle$, then $G \in C^{1}(H, \mathbb{R})$. By Lemma 4.1, $\mathcal{N} \neq \varnothing$. We claim that $0 \notin \partial \mathcal{N}$. In fact,

$$
\begin{aligned}
G(u) & =\|u\|^{2}+\lambda R^{\prime}(u) u-\int_{B}|u|^{6+|x|^{\alpha}} \\
& \geq \frac{1}{2}\|u\|^{2}-C\left(\|u\|^{6}+\|u\|^{7}\right)>0
\end{aligned}
$$

for any $u \in H$ with $\|u\|$ small. Note that for any $u \in \mathcal{N}$

$$
\begin{align*}
\left\langle G^{\prime}(u), u\right\rangle & =\left\langle G^{\prime}(u), u\right\rangle-6 G(u) \\
& =-4\|u\|^{2}+\lambda q(q-6) R(u)-\int_{B}|x|^{\alpha}|u|^{6+|x|^{\alpha}}<0 . \tag{4.2}
\end{align*}
$$

Hence, $G^{\prime}(u) \neq 0$ for any $u \in \mathcal{N}$. Then the implicit function theorem implies that $\mathcal{N}$ is a $C^{1}$ manifold. Recall that $u$ is minimizer of $J$ on $u \in \mathcal{N}$. Then by the Lagrange multiplier method, there exists $\lambda \in \mathbb{R}$ such that

$$
\begin{equation*}
J^{\prime}(u)=\lambda G^{\prime}(u) . \tag{4.3}
\end{equation*}
$$

Combining (4.2) and (4.3), we can find $J^{\prime}(u)=0$. The proof is completed.

Proof of Theorem 1.2. Recall that Theorem 1.1 shows that $u \in \mathcal{N}$ and hence $J(u) \geq c^{*}$. Then by applying Lemma 4.3, Fatou's lemma and weak semicontinuity of the norm, we derive

$$
\begin{aligned}
c^{*} & =\liminf _{n \rightarrow \infty}\left[J\left(u_{n}\right)-\frac{1}{6}\left\langle J^{\prime}\left(u_{n}\right), u_{n}\right\rangle\right] \\
& =\liminf _{n \rightarrow \infty}\left[\frac{1}{3}\left\|u_{n}\right\|^{2}+\lambda q\left(\frac{1}{q}-\frac{1}{6}\right) R\left(u_{n}\right)+\int_{B}\left(\frac{1}{6}-\frac{1}{6+|x|^{\alpha}}\right)\left|u_{n}\right|^{6+|x|^{\alpha}}\right] \\
& \geq \frac{1}{3}\|u\|^{2}+\lambda q\left(\frac{1}{q}-\frac{1}{6}\right) R(u)+\int_{B}\left(\frac{1}{6}-\frac{1}{6+|x|^{\alpha}}\right)|u|^{6+|x|^{\alpha}} \\
& =J(u)-\frac{1}{6}\left\langle J^{\prime}(u), u\right\rangle=J(u) .
\end{aligned}
$$

This shows that $J(u)=c^{*}$. It is easy to see that $J(|u|)=J(u)=c^{*}$. Thus, Lemma 4.4 implies that $|u|$ is a ground state of $J$. The proof is completed.

## 5 The Schrödinger-Poisson type system

This section is devoted to apply the Theorems 1.1 and 1.2 to a class of Schrödinger-Poisson type system. We first estimate the critical level of mountain pass of the functional $\tilde{J}$ associated to (1.3) and then show that the critical level of mountain pass is below the non-compactness level of $\tilde{J}$. Secondly, we are devoted to verify that the ( $P S$ ) sequence of the functional $\tilde{J}$ is also the one of the approximation functional associated to $\tilde{J}$ by using approximation techniques. Finally, by using the regularity theory, the positive ground state solution of (1.3) is obtained. We establish the following lemmas, which guarantee that the conditions in the Theorems 1.1 and 1.2 are valid.

We observe that by [3], for given $u \in H$, there exists a unique solution $\phi=\phi_{u} \in H$ satisfying $-\Delta \phi_{u}=|u|^{5}$ in $B, u=0$ on $\partial B$ in a weak sense and it has the following properties.

Lemma 5.1 ([5]). For every fixed $u \in H$, we have
(i) $\phi_{u} \geq 0$ a.e. in $B$;
(ii) $\phi_{t u}=t^{5} \phi_{u}$ for all $t>0$;
(iii) $\left\|\phi_{u}\right\| \leq S^{-3}\|u\|^{5}$ and

$$
\begin{equation*}
\int_{B} \phi_{u}|u|^{5} \leq S^{-6}\|u\|^{10}, \tag{5.1}
\end{equation*}
$$

where $S$ is defined in (2.1);
(iv) if $u_{n} \rightharpoonup u$ in $H$, then, up to a subsequence, $\phi_{u_{n}} \rightharpoonup \phi_{u}$ in $H$.

Moreover, (1.3) is variational and its solutions are the critical points of the functional defined in $H$ by

$$
\tilde{J}(u)=\frac{1}{2} \int_{B}|\nabla u|^{2}-\frac{1}{10} \int_{B} \phi_{u}|u|^{5}-\int_{B} \frac{1}{6+|x|^{\alpha}}|u|^{6+|x|^{\alpha}} .
$$

It is easy to check by Lemmas 2.2 and 5.1 that $\tilde{J}$ is well defined on $H$ and $\tilde{J} \in C^{1}(H, \mathbb{R})$, and

$$
\left\langle\tilde{J}^{\prime}(u), v\right\rangle=\int_{B} \nabla u \nabla v-\int_{B} \phi_{u}|u|^{3} u v-\int_{B}|u|^{4+|x|^{\alpha}} u v, \quad u, v \in H .
$$

Lemma 5.2. Let $\alpha_{1}, \beta_{1}, \gamma_{1}>0$ and define $f_{1}:[0, \infty) \rightarrow \mathbb{R}$ as

$$
f_{1}(t)=\frac{\alpha_{1}}{2} t^{2}-\frac{\beta_{1}}{10} t^{10}-\frac{\gamma_{1}}{6} t^{6}
$$

Then

$$
\sup _{t \in[0, \infty)} f_{1}(t)=\left(\frac{\sqrt{\gamma_{1}^{2}+4 \alpha_{1} \beta_{1}}-\gamma_{1}}{2 \beta_{1}}\right)^{1 / 2} \frac{12 \alpha_{1} \beta_{1}+\gamma_{1}^{2}-\gamma_{1} \sqrt{\gamma_{1}^{2}+4 \alpha_{1} \beta_{1}}}{30 \beta_{1}} .
$$

Proof. For $t \geq 0$, we have

$$
f_{1}^{\prime}(t)=\alpha_{1} t-\beta_{1} t^{9}-\gamma_{1} t^{5}=t\left(\alpha_{1}-\beta_{1} t^{8}-\gamma_{1} t^{4}\right) .
$$

Set $h(t)=\alpha_{1}-\beta_{1} t^{8}-\gamma_{1} t^{4}=0$, we write at

$$
t^{4}=\frac{\sqrt{\gamma_{1}^{2}+4 \alpha_{1} \beta_{1}}-\gamma_{1}}{2 \beta_{1}}
$$

Substituting it into $f_{1}(t)$, the result is obtained. The proof is completed.
Lemma 5.3. Let

$$
g_{1}(t)=\frac{t^{2}}{2}\left\|u_{\varepsilon}\right\|^{2}-\frac{t^{10}}{10} \int_{B} \phi_{u_{\varepsilon}}\left|u_{\varepsilon}\right|^{5}-\frac{t^{6}}{6} \int_{B}\left|u_{\varepsilon}\right|^{6},
$$

then we have, as $\varepsilon \rightarrow 0^{+}$,

$$
\sup _{t \geq 0} g_{1}(t) \leq \frac{13-\sqrt{5}}{30}\left(\frac{\sqrt{5}-1}{2}\right)^{1 / 2} S^{3 / 2}+O(\varepsilon)=: \Lambda+O(\varepsilon)
$$

Proof. Since $-\Delta \phi_{u_{\varepsilon}}=\left|u_{\varepsilon}\right|^{5}$, we have

$$
\begin{aligned}
\int_{B}\left|u_{\varepsilon}\right|^{6} & =\int_{B} \nabla \phi_{u_{\varepsilon}} \nabla\left|u_{\varepsilon}\right| \\
& \leq \frac{1}{2} \int_{B}|\nabla| u_{\varepsilon}| |^{2}+\frac{1}{2} \int_{B}\left|\nabla \phi_{u_{\varepsilon}}\right|^{2} \\
& =\frac{1}{2} \int_{B} \phi_{u_{\varepsilon}}\left|u_{\varepsilon}\right|^{5}+\frac{1}{2} \int_{B}\left|\nabla u_{\varepsilon}\right|^{2} .
\end{aligned}
$$

Then thanks to (2.2) we derive that, for $\varepsilon>0$ sufficiently small,

$$
\begin{aligned}
\int_{B} \phi_{u_{\varepsilon}}\left|u_{\varepsilon}\right|^{5} & \geq 2 \int_{B}\left|u_{\varepsilon}\right|^{6}-\int_{B}\left|\nabla u_{\varepsilon}\right|^{2} \\
& =S^{\frac{3}{2}}+O(\varepsilon) .
\end{aligned}
$$

This together with Lemma 5.2 and the estimate (2.2) implies that

$$
\begin{aligned}
g_{1}(t) & =\frac{t^{2}}{2}\left\|u_{\varepsilon}\right\|^{2}-\frac{t^{10}}{10} \int_{B} \phi_{u_{\varepsilon}}\left|u_{\varepsilon}\right|^{5}-\frac{t^{6}}{6} \int_{B}\left|u_{\varepsilon}\right|^{6} \\
& \leq \frac{t^{2}}{2}\left(S^{3 / 2}+O(\varepsilon)\right)-\frac{t^{10}}{10}\left(S^{3 / 2}+O(\varepsilon)\right)-\frac{t^{6}}{6}\left(S^{3 / 2}+O(\varepsilon)\right) \\
& \leq \frac{13-\sqrt{5}}{30}\left(\frac{\sqrt{5}-1}{2}\right)^{1 / 2} S^{3 / 2}+O(\varepsilon),
\end{aligned}
$$

for $\varepsilon>0$ sufficiently small. The proof is completed.

From Lemma 3.1, we know that the functional $\tilde{J}$ possesses the mountain pass geometry. Then there is a $(P S)_{c_{1}}$ sequence $\left\{u_{n}\right\} \subset H$ for $\tilde{J}$ with the property that

$$
\tilde{J}\left(u_{n}\right) \rightarrow c_{1}, \quad\left\|\tilde{J}^{\prime}\left(u_{n}\right)\right\|_{H^{-1}} \rightarrow 0, \quad n \rightarrow \infty,
$$

where $c_{1}$ is given by

$$
\begin{equation*}
c_{1}=\inf _{\gamma \in \tilde{\Gamma}} \max _{t \in[0,1]} \tilde{J}(\gamma(t)), \tag{5.2}
\end{equation*}
$$

and $\tilde{\Gamma}=\{\gamma \in C([0,1], H): \gamma(0)=0, \tilde{J}(\gamma(1))<0\}$.
In the following we give an estimate of the upper bound of the critical level $c_{1}$ by using above two lemmas.

Lemma 5.4. Let $c_{1}$ be defined by (5.2), then $0<c_{1}<\Lambda$.
Proof. It follows from (3.1) that, for $\varepsilon$ small enough,

$$
\begin{aligned}
\tilde{J}\left(t u_{\varepsilon}\right) & =\frac{t^{2}}{2} \int_{B}\left|\nabla u_{\varepsilon}\right|^{2}-\frac{t^{10}}{10} \int_{B} \phi_{u_{\varepsilon}}\left|u_{\varepsilon}\right|^{5}-\int_{B} \frac{t^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}} \\
& \leq \frac{t^{2}}{2}\left\|u_{\varepsilon}\right\|^{2}-\frac{t^{6}}{7} \int_{B}\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}} \\
& \leq S^{3 / 2} t^{2}-\frac{S^{3 / 2}}{14} t^{6}:=\varphi(t) .
\end{aligned}
$$

Thus, there exists $R_{1}>0$ sufficiently large which is independent of $\varepsilon$, such that $\varphi\left(R_{1}\right)=0$ and $\tilde{J}\left(R_{1} u_{\varepsilon}\right) \leq 0$ for $\varepsilon$ small enough. Hence, we can find $0<t_{\varepsilon}<R_{1}$ satisfying

$$
0<\eta_{1} \leq c_{1} \leq \max _{t \in\left[0, R_{1}\right]} \tilde{J}\left(t u_{\varepsilon}\right)=\tilde{J}\left(t_{\varepsilon} u_{\varepsilon}\right) .
$$

Since $\left.\frac{d}{d t} \tilde{J}\left(t u_{\varepsilon}\right)\right|_{t=t_{\varepsilon}}=0$, we have

$$
t_{\varepsilon}\left\|u_{\varepsilon}\right\|^{2}=t_{\varepsilon}^{9} \int_{B} \phi_{u_{\varepsilon}}\left|u_{\varepsilon}\right|^{5}+\int_{B} t_{\varepsilon}^{5+|x|^{\alpha}}|u|^{6+|x|^{\alpha}}
$$

Hence we deduce from (2.2) that

$$
\begin{align*}
S^{\frac{3}{2}}+O(\varepsilon) & =t_{\varepsilon}^{8} \int_{B} \phi_{u_{\varepsilon}}\left|u_{\varepsilon}\right|^{5}+t_{\varepsilon}^{4} \int_{B}\left|u_{\varepsilon}\right|^{6}+t_{\varepsilon}^{4} \int_{B}\left(t_{\varepsilon}^{|x| \alpha^{\alpha}}\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}}-\left|u_{\varepsilon}\right|^{6}\right) \\
& =t_{\varepsilon}^{8} \int_{B} \phi_{u_{\varepsilon}}\left|u_{\varepsilon}\right|^{5}+t_{\varepsilon}^{4}\left[S^{\frac{3}{2}}+O\left(\varepsilon^{3}\right)+A_{\varepsilon}\right]  \tag{5.3}\\
& =t_{\varepsilon}^{8} \int_{B} \phi_{u_{\varepsilon}}\left|u_{\varepsilon}\right|^{5}+t_{\varepsilon}^{4}\left[S^{\frac{3}{2}}+O\left(\varepsilon^{3}\right)+O\left(\varepsilon^{\alpha}|\log \varepsilon|\right)+O\left(\varepsilon^{3 / 2}\right)\right],
\end{align*}
$$

where $A_{\varepsilon}=O\left(\varepsilon^{\alpha}|\log \varepsilon|\right)+O\left(\varepsilon^{3 / 2}\right)$ is given in [21]. For convenience, we set $A=S^{\frac{3}{2}}+O(\varepsilon)$, $B=\int_{B} \phi_{u_{\varepsilon}}\left|\mathcal{u}_{\varepsilon}\right|^{5}$ and $C=S^{\frac{3}{2}}+O\left(\varepsilon^{3}\right)+O\left(\varepsilon^{\alpha}|\log \varepsilon|\right)+O\left(\varepsilon^{3 / 2}\right)$. Thus, (5.3) can be rewritten as $A=B t_{\varepsilon}^{8}+C t_{\varepsilon}^{4}$. It is easy to see that for $\varepsilon$ small,

$$
t_{\varepsilon}^{4}=\frac{\sqrt{C^{2}+4 A B}-C}{2 B}=\frac{\sqrt{5 S^{3}+O(\varepsilon)+O\left(\varepsilon^{\alpha}|\log \varepsilon|\right)}-S^{\frac{3}{2}}-O\left(\varepsilon^{\alpha}|\log \varepsilon|\right)-O\left(\varepsilon^{3 / 2}\right)}{2 S^{3 / 2}+O(\varepsilon)}
$$

Thereby, for $\varepsilon$ small enough, there holds

$$
\begin{equation*}
(\sqrt{5}-1) / 4<t_{\varepsilon}^{2}<4 / 5 \tag{5.4}
\end{equation*}
$$

In what follows, we will estimate the term

$$
\begin{align*}
& \int_{B} \frac{t_{\varepsilon}^{6}}{6}\left|u_{\varepsilon}\right|^{6}-\int_{B} \frac{t_{\varepsilon}^{6+|x|^{\alpha}}}{6+\mid x x^{\alpha}}\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}} \\
&= \int_{B}\left(\frac{t_{\varepsilon}^{6}}{6}-\frac{t_{\varepsilon}^{6}}{6+|x|^{\alpha}}\right)\left|u_{\varepsilon}\right|^{6}+\int_{B} \frac{t_{\varepsilon}^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\left(\left|u_{\varepsilon}\right|^{6}-\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}}\right)  \tag{5.5}\\
& \quad+\int_{B}\left(\frac{t_{\varepsilon}^{6}}{6+|x|^{\alpha}}-\frac{t_{\varepsilon}^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\right)\left|u_{\varepsilon}\right|^{6} \\
&= I+I I+I I I .
\end{align*}
$$

By [21, page 16] and (5.4), we can find

$$
\begin{equation*}
I=\int_{B}\left(\frac{t_{\varepsilon}^{6}}{6}-\frac{t_{\varepsilon}^{6}}{6+|x|^{\alpha}}\right)\left|u_{\varepsilon}\right|^{6} \leq C \varepsilon^{\alpha} \tag{5.6}
\end{equation*}
$$

and

$$
\begin{equation*}
I I=\int_{B} \frac{t_{\varepsilon}^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\left(\left|u_{\varepsilon}\right|^{6}-\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}}\right) \leq-C \varepsilon^{\alpha}|\log \varepsilon| . \tag{5.7}
\end{equation*}
$$

It follows from (5.4) again that

$$
\begin{align*}
I I I & =\int_{B}\left(\frac{t_{\varepsilon}^{6}}{6+|x|^{\alpha}}-\frac{t_{\varepsilon}^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\right)\left|u_{\varepsilon}\right|^{6} \leq C \int_{B}\left(1-t_{\varepsilon}^{|x|^{\alpha}}\right)\left|u_{\varepsilon}\right|^{6} \\
& =C \int_{B}\left(1-\exp \left(|x|^{\alpha} \log t_{\varepsilon}\right)\right)\left|u_{\varepsilon}\right|^{6} \leq C \int_{B}|x|^{\alpha \alpha}\left|u_{\varepsilon}\right|^{6}  \tag{5.8}\\
& \leq C \omega \int_{0}^{\varepsilon} r^{\alpha} \varepsilon^{-3} r^{2}+C \omega \int_{\varepsilon}^{1} r^{\alpha} \varepsilon^{3} r^{-4} \\
& \leq C \varepsilon^{\alpha}+C\left(\varepsilon^{\alpha}-\varepsilon^{3}\right) \leq C \varepsilon^{\alpha} .
\end{align*}
$$

Combining (5.5)-(5.8) and using Lemma 5.3, we derive

$$
\begin{align*}
\tilde{J}\left(t_{\varepsilon} u_{\varepsilon}\right) & =\frac{t_{\varepsilon}^{2}}{2}\left\|u_{\varepsilon}\right\|^{2}-\frac{t_{\varepsilon}^{10}}{10} \int_{B} \phi_{u_{\varepsilon}}\left|u_{\varepsilon}\right|^{5}-\int_{B} \frac{t_{\varepsilon}^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}} \\
& =\frac{t_{\varepsilon}^{2}}{2}\left\|u_{\varepsilon}\right\|^{2}-\frac{t_{\varepsilon}^{10}}{10} \int_{B} \phi_{u_{\varepsilon}}\left|u_{\varepsilon}\right|^{5}-\int_{B} \frac{t_{\varepsilon}^{6}}{6}\left|u_{\varepsilon}\right|^{6}+\int_{B} \frac{t_{\varepsilon}^{6}}{6}\left|u_{\varepsilon}\right|^{6}-\int_{B} \frac{t_{\varepsilon}^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}} \\
& \leq \sup _{t \geq 0} g_{1}(t)+\int_{B} \frac{t_{\varepsilon}^{6}}{6}\left|u_{\varepsilon}\right|^{6}-\int_{B} \frac{t_{\varepsilon}^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\left|u_{\varepsilon}\right|^{6+\left.|x|\right|^{\alpha}}  \tag{5.9}\\
& \leq \frac{13-\sqrt{5}}{30}\left(\frac{\sqrt{5}-1}{2}\right)^{1 / 2} S^{3 / 2}+O(\varepsilon)+C \varepsilon^{\alpha}-C \varepsilon^{\alpha}|\log \varepsilon| .
\end{align*}
$$

By choosing $\varepsilon>0$ small enough, we derive by (5.9),

$$
0<\eta_{1} \leq c_{1} \leq \tilde{J}\left(t_{\varepsilon} u_{\varepsilon}\right)<\Lambda .
$$

The proof is finished.
Lemma 5.5. If $\left\{u_{n}\right\}$ is $a(P S)_{c_{1}}$ sequence of $\tilde{J}$, then there exists $u \in H$ such that, up to a subsequence, $u_{n} \rightharpoonup u$ and $\tilde{J}^{\prime}(u)=0$.

Proof. From Lemma 3.3 we see that $\left\{u_{n}\right\}$ is bounded in $H$. Then, up to a subsequence, we can assume that $\left\{u_{n}\right\}$ converges to $u$ weakly in $H$ and $u_{n} \rightarrow u$ a.e. in $B$. By taking $\varphi \in C_{0}^{\infty}(B)$, we find

$$
\left\langle\tilde{J}^{\prime}\left(u_{n}\right), \varphi\right\rangle=\int_{B} \nabla u_{n} \nabla \varphi-\int_{B} \phi_{u_{n}}\left|u_{n}\right|^{3} u_{n} \varphi-\int_{B}\left|u_{n}\right|^{4+|x|^{\alpha}} u_{n} \varphi .
$$

It follows from Lemma 5.1 that $\phi_{u_{n}} \rightharpoonup \phi_{u}$ in $H$, which implies $\phi_{u_{n}} \rightharpoonup \phi_{u}$ in $L^{6}(B)$. Then

$$
\begin{equation*}
\int_{B}\left(\phi_{u_{n}}-\phi_{u}\right)|u|^{3} u \varphi \rightarrow 0, \quad n \rightarrow \infty . \tag{5.10}
\end{equation*}
$$

Since $u_{n} \rightarrow u$ a.e. in $B$ and

$$
\int_{B}\left|\phi_{u_{n}}\left(\left|u_{n}\right|^{3} u_{n}-|u|^{3} u\right)\right|^{\frac{6}{5}} \leq C\left(\left.\left|\phi_{u_{n}}{ }_{6}^{\frac{6}{5}}\right| u_{n}\right|_{6} ^{\frac{24}{5}}+\left|\phi_{u_{n}}\right|_{6}^{\frac{6}{5}}|u|_{6}^{\frac{24}{5}}\right) \leq C,
$$

we have $\phi_{u_{n}}\left(\left|u_{n}\right|^{3} u_{n}-|u|^{3} u\right) \rightharpoonup 0$ in $L^{\frac{6}{5}}(B)$ and thus

$$
\int_{B} \phi_{u_{n}}\left(\left|u_{n}\right|^{3} u_{n}-|u|^{3} u\right) \varphi \rightarrow 0, \quad n \rightarrow \infty
$$

which together with (5.10) ensures that

$$
\begin{equation*}
\int_{B} \phi_{u_{n}}\left|u_{n}\right|^{3} u_{n} \varphi \rightarrow \int_{B} \phi_{u}|u|^{3} u \varphi, \quad n \rightarrow \infty . \tag{5.11}
\end{equation*}
$$

For any measurable subset $Q \subset B$, we have

$$
\begin{aligned}
\left|\int_{Q}\left(\left|u_{n}\right|^{4+|x|^{\alpha}} u_{n}-|u|^{4+|x|^{\alpha}} u\right) \varphi\right| & \leq \int_{Q}\left(\left|u_{n}\right|^{5+|x|^{\alpha}}+|u|^{5+|x|^{\alpha}}\right)|\varphi| \\
& \leq\left\|\left|u_{n}\right|^{5+|x|^{\alpha}}+|u|^{5+|x|^{\alpha}}\right\|_{L^{p(\cdot)-1}}(Q)
\end{aligned}\|\varphi\|_{L^{p(\cdot)}(Q)},
$$

where $p(x)=6+|x|^{\alpha}$. Hence, Vitali's theorem (see [28]) implies

$$
\begin{equation*}
\int_{B}\left|u_{n}\right|^{4+|x|^{\alpha}} u_{n} \varphi \rightarrow \int_{B}|u|^{4+|x|^{\alpha}} u \varphi, \quad \text { as } n \rightarrow \infty . \tag{5.12}
\end{equation*}
$$

Combining (5.10), (5.11) and (5.12), there holds

$$
\left\langle\tilde{J}^{\prime}(u), \varphi\right\rangle=\lim _{n \rightarrow \infty}\left\langle\tilde{J}^{\prime}\left(u_{n}\right), \varphi\right\rangle=0 .
$$

Therefore, by density, we derive that $\tilde{J}^{\prime}(u)=0$. The proof is completed.
In order to obtain the nontrivial solution of (1.3), we need define the approximation functional $\tilde{I}: H \rightarrow \mathbb{R}$ associated to $\tilde{J}$ given by

$$
\tilde{I}(u)=\frac{1}{2}\|u\|^{2}-\frac{1}{10} \int_{B} \phi_{u}|u|^{5}-\frac{1}{6} \int_{B}|u|^{6} .
$$

Lemma 5.6. The functional Ĩ satisfies the $(P S)_{c_{1}}$ condition with $c_{1} \in(0, \Lambda)$.

Proof. Suppose that $\left\{u_{n}\right\}$ is a $(P S)_{c_{1}}$ sequence of $\tilde{I}$ for $c_{1} \in(0, \Lambda)$, i.e.

$$
\tilde{I}\left(u_{n}\right) \rightarrow c_{1}, \quad \tilde{I}^{\prime}\left(u_{n}\right) \rightarrow 0 \quad \text { as } n \rightarrow \infty
$$

Similarly to Lemma 3.3, it is easy to see that $\left\{u_{n}\right\}$ is bounded in $H$. Going if necessary to a subsequence, we can find $u \in H$ such that $u_{n} \rightharpoonup u$ in $H$. By the same argument used in Lemma 5.5, we deduce that $\tilde{I}^{\prime}(u)=0$, hence

$$
\begin{align*}
\tilde{I}(u) & =\tilde{I}(u)-\frac{1}{6}\left\langle\tilde{I}^{\prime}(u), u\right\rangle \\
& =\frac{1}{3}\|u\|^{2}+\frac{1}{15} \int_{B} \phi_{u}|u|^{5} \geq 0 \tag{5.13}
\end{align*}
$$

Now, let $v_{n}=u_{n}-u$, it is obvious to see that

$$
\left\|u_{n}\right\|^{2}=\left\|v_{n}\right\|^{2}+\|u\|^{2}+o(1) .
$$

From Brézis-Lieb Lemma in $[9,19]$, we have

$$
\int_{B}\left|u_{n}\right|^{6} d x=\int_{B}\left|v_{n}\right|^{6} d x+\int_{B}|u|^{6} d x+o(1)
$$

and

$$
\int_{B} \phi_{u_{n}}\left|u_{n}\right|^{5}=\int_{B} \phi_{v_{n}}\left|v_{n}\right|^{5}+\int_{B} \phi_{u}|u|^{5}+o(1)
$$

These three equalities imply that

$$
\begin{align*}
c_{1}-\tilde{I}(u)= & \tilde{I}\left(u_{n}\right)-\tilde{I}(u)+o(1) \\
= & \frac{1}{2}\left\|u_{n}\right\|^{2}-\frac{1}{2}\|u\|^{2}-\frac{1}{10} \int_{B} \phi_{u_{n}}\left|u_{n}\right|^{5}+\frac{1}{10} \int_{B} \phi_{u}|u|^{5} \\
& -\frac{1}{6} \int_{B}\left|u_{n}\right|^{6}+\frac{1}{6} \int_{B}|u|^{6}+o(1)  \tag{5.14}\\
= & \frac{1}{2}\left\|v_{n}\right\|^{2}-\frac{1}{10} \int_{B} \phi_{v_{n}}\left|v_{n}\right|^{5}-\frac{1}{6} \int_{B}\left|v_{n}\right|^{6}+o(1),
\end{align*}
$$

and similarly

$$
\begin{align*}
o(1) & =\left\langle\tilde{I}^{\prime}\left(u_{n}\right), u_{n}\right\rangle-\left\langle\tilde{I}^{\prime}(u), u\right\rangle \\
& =\left\|u_{n}\right\|^{2}-\|u\|^{2}-\int_{B} \phi_{u_{n}}\left|u_{n}\right|^{5}+\int_{B} \phi_{u}|u|^{5}-\int_{B}\left|u_{n}\right|^{6}+\int_{B}|u|^{6}  \tag{5.15}\\
& =\left\|v_{n}\right\|^{2}-\int_{B} \phi_{v_{n}}\left|v_{n}\right|^{5}-\int_{B}\left|v_{n}\right|^{6}+o(1) .
\end{align*}
$$

We will show that $\left\|v_{n}\right\| \rightarrow 0$. Otherwise, there exists a subsequence still denoted by $\left\{v_{n}\right\}$ such that $\left\|v_{n}\right\|^{2} \rightarrow l>0$. For convenience, let $a_{n}=\int_{B} \phi_{v_{n}}\left|v_{n}\right|^{5}$ and $b_{n}=\int_{B}\left|v_{n}\right|^{6}$. Without loss of generality, we may assume $a_{n} \rightarrow a_{1}$ and $b_{n} \rightarrow b_{1}$, as $n \rightarrow \infty$. Notice that

$$
\begin{aligned}
\int_{B}\left|v_{n}\right|^{6} & =\int_{B} \nabla \phi_{v_{n}} \nabla\left|v_{n}\right| \\
& \leq \frac{\varepsilon^{2}}{2} \int_{B}|\nabla| v_{n}| |^{2}+\frac{1}{2 \varepsilon^{2}} \int_{B}\left|\nabla \phi_{v_{n}}\right|^{2} \\
& =\frac{1}{2 \varepsilon^{2}} \int_{B} \phi_{v_{n}}\left|v_{n}\right|^{5}+\frac{\varepsilon^{2}}{2} \int_{B}\left|\nabla v_{n}\right|^{2}
\end{aligned}
$$

then as $n \rightarrow \infty$ passing to the limit, we conclude that

$$
b_{1} \leq \frac{1}{2 \varepsilon^{2}} a_{1}+\frac{\varepsilon^{2}}{2} l .
$$

Taking $\varepsilon^{2}=\frac{\sqrt{5}-1}{2}$, and combining with (5.15) leads to

$$
a_{1} \geq \frac{3-\sqrt{5}}{2} l
$$

from which we get by (5.13), (5.14) and (5.15) that

$$
\begin{equation*}
c_{1} \geq c_{1}-\tilde{I}(u)=\frac{2}{5} a_{1}+\frac{1}{3} b_{1}+o(1)=\frac{1}{3} l+\frac{1}{15} a_{1}+o(1) \geq \frac{13-\sqrt{5}}{30} l+o(1) \tag{5.16}
\end{equation*}
$$

On the other hand, (5.1) and (5.15) yield

$$
l \leq S^{-6} l^{5}+S^{-3} l^{3}
$$

Therefore we get $l^{2} \geq \frac{-1+\sqrt{5}}{2} S^{3}$. This together with (5.16) implies that $c_{1} \geq \Lambda$, which will come to a contradiction. Therefore $v_{n} \rightarrow 0$ strongly in $H$, or equivalently, $u_{n} \rightarrow u$ in $H$ as $n \rightarrow \infty$. The proof is completed.

Lemma 5.7 ([30]). Let $\Omega$ be a domain in $\mathbb{R}^{3}$ and $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Caratéodory function such that for almost every $x \in \Omega$, there holds

$$
|g(x, u)| \leq a(x)(1+|u|) .
$$

If $0 \leq a \in L^{\frac{3}{2}}(\Omega)$ and $u \in H_{0}^{1}(\Omega)$ is a weak solution of equation $-\Delta u=g(\cdot, u)$ in $\Omega$. Then, $u \in L^{p}(\Omega)$ for all $p<\infty$.

Proof of Theorem 1.7. The Lemmas 5.4, 5.5 and Theorem 1.2 imply that (1.3) admits a nonnegative nontrivial ground state solution $u \in H$, which satisfies the following equation in weak sense

$$
-\Delta u=\phi_{u}|u|^{3} u+u^{5+|x|^{\alpha}} \quad \text { in } B .
$$

Let us define

$$
\tilde{g}(u(x))=\phi_{u}|u|^{3} u+u^{5+|x|^{\alpha}}, \quad x \in B .
$$

Then thanks to Lemma 2.2, we have $\int_{B} u^{6+\frac{3}{2}|x|^{\alpha}} \leq C$. The fact $\phi_{u} \in D^{1,2}(B)$ that implies $\phi_{u} \in L^{6}(B)$. On the other hand, it is easy to see that $\left|\phi_{u}\right|^{\frac{3}{2}} \in L^{4}(B)$ and $|u|^{\frac{9}{2}} \in L^{\frac{4}{3}}(B)$. Thus we derive from the Hölder inequality that $\phi_{u}|u|^{3} \in L^{\frac{3}{2}}(B)$, which implies

$$
a=\frac{\tilde{g}(u)}{1+|u|} \in L^{\frac{3}{2}}(B) .
$$

Thereby, we deduce immediately from Lemma 5.7 that $u \in L^{q}(B)$ for any $1<q<\infty$. Hence, there holds $\tilde{g}(u) \in L^{q}(B)$ for any $1<q<\infty$. Now, arguing by the Calderón-Zygmund inequality and $L^{p}$ estimate given in $[16,30]$, we derive $u \in W^{2, q}(B)$, whence also $u \in C^{1, \alpha_{1}}(B)$ by Sobolev embedding theorem for any $0<\alpha_{1}<1$. Moreover, the Harnack inequality [32] implies $u(x)>0$ for all $x \in B$. The proof is completed.

## 6 The Kirchhoff type equation

In this section, we obtain the existence of positive ground state solution of (1.6) by using Theorem 1.2 with $\lambda=1, q=4$. Similarly to Section 4 , we first estimate the level of mountain critical of the functional $\hat{J}$ corresponding to (1.6) and show that the critical level is below the non-compactness level of $\hat{J}$ by using approximation techniques. Then we are devoted to verify that the $(P S)$ sequence of the functional $\hat{J}$ is also the one of the approximation functional associated to $\hat{J}$. Finally, by the regularity theory of the elliptic equation, the positive ground state solution of (1.6) is obtained. In order to find the weak solutions to (1.6) and it is natural to consider the energy functional on $H$ :

$$
\hat{J}(u)=\frac{1}{2}\|u\|^{2}+\frac{b}{4}\|u\|^{4}-\int_{B} \frac{1}{6+|x|^{\alpha}}|u|^{6+|x|^{\alpha}} .
$$

Then we have from Lemma 2.2 that $\hat{J}$ is well defined on $H$ and is of $C^{1}$, and

$$
\left(\hat{J}^{\prime}(u), v\right)=\left(1+b\|u\|^{2}\right) \int_{B} \nabla u \nabla v-\int_{B}|u|^{4+|x|^{\alpha}} u v, \quad u, v \in H .
$$

It is standard to verify that the weak solutions of (1.6) correspond to the critical points of the functional $\hat{J}$.

Lemma 6.1. Let $\alpha_{2}, \beta_{2}, \gamma_{2}>0$ and define $f_{2}:[0, \infty) \rightarrow \mathbb{R}$ as

$$
f_{2}(t)=\frac{\alpha_{2}}{2} t^{2}+\frac{\beta_{2}}{4} t^{4}-\frac{\gamma_{2}}{6} t^{6} .
$$

Then

$$
\sup _{t \in[0, \infty)} f_{2}(t)=\frac{6 \alpha_{2} \beta_{2} \gamma_{2}+\beta_{2}^{3}+4 \alpha_{2} \gamma_{2} \sqrt{\beta_{2}^{2}+4 \alpha_{2} \gamma_{2}}+\beta_{2}^{2} \sqrt{\beta_{2}^{2}+4 \alpha_{2} \gamma_{2}}}{24 \gamma_{2}^{2}} .
$$

Proof. For $t \geq 0$, we have

$$
f_{2}^{\prime}(t)=\alpha_{2} t+\beta_{2} t^{3}-\gamma_{2} t^{5}=t\left(\alpha_{2}+\beta_{2} t^{2}-\gamma_{2} t^{4}\right) .
$$

Let $\alpha_{2}+\beta_{2} t^{2}-\gamma_{2} t^{4}=0$, we write at

$$
t^{2}=\frac{\sqrt{\beta_{2}^{2}+4 \alpha_{2} \gamma_{2}}+\beta_{2}}{2 \gamma_{2}}
$$

Substituting it into $f_{2}(t)$, the result is valid. The proof is completed.
Lemma 6.2. Let

$$
g_{2}(t)=\frac{t^{2}}{2}\left\|u_{\varepsilon}\right\|^{2}+\frac{b t^{4}}{4}\left\|u_{\varepsilon}\right\|^{4}-\frac{t^{6}}{6} \int_{\mathbb{R}^{3}}\left|u_{\varepsilon}\right|^{6},
$$

then we have, as $\varepsilon \rightarrow 0^{+}$,

$$
\sup _{t \geq 0} g_{2}(t) \leq \Lambda_{1}+O(\varepsilon)
$$

where $\Lambda_{1}=\frac{b}{4} S^{3}+\frac{b^{3}}{24} S^{6}+\frac{1}{6} S \sqrt{S^{4} b^{2}+4 S}+\frac{b^{2}}{24} S^{4} \sqrt{S^{4} b^{2}+4 S}$.

Proof. It follows from Lemma 6.1 and the estimate (2.2) that

$$
\begin{aligned}
g_{2}(t) & =\frac{t^{2}}{2}\left\|u_{\varepsilon}\right\|^{2}+\frac{b t^{4}}{4}\left\|u_{\varepsilon}\right\|^{4}-\frac{t^{6}}{6} \int_{\mathbb{R}^{3}}\left|u_{\varepsilon}\right|^{6} \\
& =\frac{t^{2}}{2}\left(S^{3 / 2}+O(\varepsilon)\right)+\frac{b t^{4}}{4}\left(S^{3}+O(\varepsilon)\right)-\frac{t^{6}}{6}\left(S^{3 / 2}+O\left(\varepsilon^{3}\right)\right) \\
& \leq \frac{b}{4} S^{3}+\frac{b^{3}}{24} S^{6}+\frac{1}{6} S \sqrt{S^{4} b^{2}+4 S}+\frac{b^{2}}{24} S^{4} \sqrt{S^{4} b^{2}+4 S}+O(\varepsilon),
\end{aligned}
$$

for $\varepsilon>0$ sufficiently small. The proof is completed.
From Lemma 3.2, we know that the functional $\hat{J}$ possesses the mountain pass geometry. Then there is a $(P S)_{c_{2}}$ sequence $\left\{u_{n}\right\} \subset H$ for $\hat{J}$ with the property that

$$
\hat{J}\left(u_{n}\right) \rightarrow c_{2}, \quad\left\|\hat{J}^{\prime}\left(u_{n}\right)\right\|_{H^{-1}} \rightarrow 0, \quad n \rightarrow \infty,
$$

where $c_{2}$ is given by

$$
c_{2}=\inf _{\hat{\gamma} \in \Gamma} \max _{t \in[0,1]} \hat{J}(\gamma(t)),
$$

and $\hat{\Gamma}=\{\gamma \in C([0,1], H): \gamma(0)=0, \hat{J}(\gamma(1))<0\}$.
In the following we give an estimate of the upper bound of the critical level $c_{2}$ by using above two lemmas.

Lemma 6.3. There holds $0<c_{2}<\Lambda_{1}$.
Proof. Similar to Lemma 5.4, there exists $R_{2}>0$ sufficiently large, such that $\hat{J}\left(R_{2} u_{\varepsilon}\right) \leq 0$ for $\varepsilon$ small enough, hence, we can find $0<t_{\varepsilon}<R_{2}$ satisfying

$$
0<\eta_{2} \leq c_{2} \leq \max _{t \in\left[0, R_{2}\right]} \hat{J}\left(t u_{\varepsilon}\right)=\hat{J}\left(t_{\varepsilon} u_{\varepsilon}\right)
$$

Since $\left.\frac{d}{d t} \hat{J}\left(t u_{\varepsilon}\right)\right|_{t=t_{\varepsilon}}=0$, we have

$$
t_{\varepsilon}\left\|u_{\varepsilon}\right\|^{2}+b t_{\varepsilon}^{3}\left\|u_{\varepsilon}\right\|^{4}=\int_{B} t_{\varepsilon}^{5+|x|^{\alpha}}|u|^{6+|x|^{\alpha}} .
$$

Hence we deduce from (2.2) that

$$
\begin{align*}
S^{\frac{3}{2}}+O(\varepsilon)+b t_{\varepsilon}^{2}\left(S^{3}+O(\varepsilon)\right) & =t_{\varepsilon}^{4} \int_{B}\left|u_{\varepsilon}\right|^{6}+t_{\varepsilon}^{4} \int_{B}\left(t_{\varepsilon}^{|x| \alpha}\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}}-\left|u_{\varepsilon}\right|^{6}\right) \\
& =t_{\varepsilon}^{4}\left[S^{\frac{3}{2}}+O\left(\varepsilon^{3}\right)+A_{\varepsilon}\right]  \tag{6.1}\\
& =t_{\varepsilon}^{4}\left[S^{\frac{3}{2}}+O\left(\varepsilon^{3}\right)+O\left(\varepsilon^{\alpha}|\log \varepsilon|\right)+O\left(\varepsilon^{3 / 2}\right)\right],
\end{align*}
$$

where $A_{\varepsilon}=O\left(\varepsilon^{\alpha}|\log \varepsilon|\right)+O\left(\varepsilon^{3 / 2}\right)$ is given in [21, page 14]. For convenience, we set $A=$ $S^{\frac{3}{2}}+O(\varepsilon), B=b\left(S^{3}+O(\varepsilon)\right)$ and $C=S^{\frac{3}{2}}+O\left(\varepsilon^{3}\right)+O\left(\varepsilon^{\alpha}|\log \varepsilon|\right)+O\left(\varepsilon^{3 / 2}\right)$. Thus, (6.1) can be rewritten as $A+B t_{\varepsilon}^{2}=C t_{\varepsilon}^{4}$. It is easy to see that

$$
t_{\varepsilon}^{2}=\frac{B+\sqrt{B^{2}+4 A C}}{2 C}=\frac{b S^{3}+O(\varepsilon)+\sqrt{b^{2} S^{6}+4 S^{3}+O(\varepsilon)+O\left(\varepsilon^{\alpha}|\log \varepsilon|\right)}}{2 S^{3 / 2}+O\left(\varepsilon^{3 / 2}\right)+O\left(\varepsilon^{\alpha}|\log \varepsilon|\right)} .
$$

Thereby, $t_{\varepsilon}^{2}>1$ for $\varepsilon$ small enough, which implies

$$
\begin{align*}
& \int_{B} \frac{t_{\varepsilon}^{6}}{6}\left|u_{\varepsilon}\right|^{6}-\int_{B} \frac{t_{\varepsilon}^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}} \\
&= \int_{B}\left(\frac{t_{\varepsilon}^{6}}{6}-\frac{t_{\varepsilon}^{6}}{6+|x|^{\alpha}}\right)\left|u_{\varepsilon}\right|^{6}+\int_{B} \frac{t_{\varepsilon}^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\left(\left|u_{\varepsilon}\right|^{6}-\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}}\right) \\
&+\int_{B}\left(\frac{t_{\varepsilon}^{6}}{6+|x|^{\alpha}}-\frac{t_{\varepsilon}^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\right)\left|u_{\varepsilon}\right|^{6}  \tag{6.2}\\
& \leq \int_{B}\left(\frac{t_{\varepsilon}^{6}}{6}-\frac{t_{\varepsilon}^{6}}{6+|x|^{\alpha}}\right)\left|u_{\varepsilon}\right|^{6}+\int_{B} \frac{t_{\varepsilon}^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\left(\left|u_{\varepsilon}\right|^{6}-\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}}\right) .
\end{align*}
$$

By [21, page 16 ] and using the fact that $t_{\varepsilon}<R_{2}$, we have

$$
\begin{equation*}
\int_{B}\left(\frac{t_{\varepsilon}^{6}}{6}-\frac{t_{\varepsilon}^{6}}{6+|x|^{\alpha}}\right)\left|u_{\varepsilon}\right|^{6} \leq C \varepsilon^{\alpha} \tag{6.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{B} \frac{t_{\varepsilon}^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\left(\left|u_{\varepsilon}\right|^{6}-\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}}\right) \leq-C \varepsilon^{\alpha}|\log \varepsilon| . \tag{6.4}
\end{equation*}
$$

Combining (6.3), (6.4) with (6.2) and using Lemma 6.2, we derive

$$
\begin{align*}
\hat{J}\left(t_{\varepsilon} u_{\varepsilon}\right) & =\frac{t_{\varepsilon}^{2}}{2}\left\|u_{\varepsilon}\right\|^{2}+\frac{b t_{\varepsilon}^{4}}{4}\left\|u_{\varepsilon}\right\|^{4}-\int_{B} \frac{t_{\varepsilon}^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}} \\
& =\frac{t_{\varepsilon}^{2}}{2}\left\|u_{\varepsilon}\right\|^{2}+\frac{b t_{\varepsilon}^{4}}{4}\left\|u_{\varepsilon}\right\|^{4}-\int_{B} \frac{t_{\varepsilon}^{6}}{6}\left|u_{\varepsilon}\right|^{6}+\int_{B} \frac{t_{\varepsilon}^{6}}{6}\left|u_{\varepsilon}\right|^{6}-\int_{B} \frac{t_{\varepsilon}^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}}  \tag{6.5}\\
& =\sup _{t \geq 0} g_{2}(t)+\int_{B} \frac{t_{\varepsilon}^{6}}{6}\left|u_{\varepsilon}\right|^{6}-\int_{B} \frac{t_{\varepsilon}^{6+|x|^{\alpha}}}{6+\mid x x^{\alpha}}\left|u_{\varepsilon}\right|^{6+|x|^{\alpha}} \\
& \leq \frac{b}{4} S^{3}+\frac{b^{3}}{24} S^{6}+\frac{1}{6} S \sqrt{S^{4} b^{2}+4 S}+\frac{b^{2}}{24} S^{4} \sqrt{S^{4} b^{2}+4 S}+O(\varepsilon)+C \varepsilon^{\alpha}-C \varepsilon^{\alpha}|\log \varepsilon| .
\end{align*}
$$

By choosing $\varepsilon>0$ small enough, we derive by (6.5),

$$
0<\eta_{2} \leq c_{2} \leq \hat{J}\left(t_{\varepsilon} u_{\varepsilon}\right)<\Lambda_{1} .
$$

The proof is completed.
Lemma 6.4. If $\left\{u_{n}\right\}$ is $a(P S)_{c_{2}}$ sequence of $\hat{\jmath}$, then there exists $u \in H$ such that, up to a subsequence, $u_{n} \rightharpoonup u$ and $\hat{J}^{\prime}(u)=0$.

Proof. By Lemma 3.3, $\left\{u_{n}\right\}$ is bounded in $H$ and hence, going if necessary to a subsequence, we may assume that $u_{n} \rightharpoonup u$ in $H$. Let $A>0$ be such that $\int_{B}\left|\nabla u_{n}\right|^{2} \rightarrow A^{2}$. If $u=0$, it is easy to see that $\hat{J}^{\prime}(u)=0$. If $u \neq 0$, then by the weakly lower semi-continuity of the norm, $\int_{B}|\nabla u|^{2} \leq A^{2}$. In the sequel, we will claim that $\int_{B}|\nabla u|^{2}=A^{2}$. In fact, if it is false, then $\int_{B}|\nabla u|^{2}<A^{2}$. For any measurable subset $Q \subset B$, we have for $v \in H$,

$$
\begin{aligned}
\left|\int_{Q}\left(\left|u_{n}\right|^{4+|x|^{\alpha}} u_{n}-|u|^{4+|x|^{\alpha}} u\right) v\right| & \leq \int_{Q}\left(\left|u_{n}\right|^{5+|x|^{\alpha}}+|u|^{5+|x|^{\alpha}}\right)|v| \\
& \leq\left\|\left|u_{n}\right|^{5+|x|^{\alpha}}+|u|^{5+|x|^{\alpha}}\right\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(Q)}\|v\|_{L^{p(\cdot)}(Q)}
\end{aligned}
$$

where $p(x)=6+|x|^{\alpha}$. Hence, the Vitali theorem (see [28]) leads to

$$
\int_{B}\left|u_{n}\right|^{4+|x|^{\alpha}} u_{n} v \rightarrow \int_{B}|u|^{4+|x|^{\alpha}} u v, \quad \text { as } n \rightarrow \infty .
$$

This together with the fact that $\hat{j}^{\prime}\left(u_{n}\right) \rightarrow 0$ ensures that

$$
\begin{equation*}
\left(1+A^{2} b\right) \int_{B} \nabla u \nabla v=\int_{B}|u|^{4+|x|^{\alpha}} u v, \quad v \in H . \tag{6.6}
\end{equation*}
$$

By taking $v=u$ in (6.6), there holds $\left\langle\hat{J}^{\prime}(u), u\right\rangle<0$. Similarly to the proof of Lemma 3.1, we have $\left\langle\hat{J}^{\prime}(t u), t u\right\rangle>0$ for small $t>0$. Thus, there exists a $t_{u} \in(0,1)$ such that $\hat{J}\left(t_{u} u\right)=$ $\max _{t \geq 0} \hat{J}(t u)$ and $\left\langle\hat{J}^{\prime}\left(t_{u} u\right), t_{u} u\right\rangle=0$. Then, we deduce by the weak lower semicontinuity of the norm and Fatou's lemma that

$$
\begin{aligned}
c_{2} & \leq \hat{J}\left(t_{u} u\right)-\frac{1}{6}\left\langle\hat{J}^{\prime}\left(t_{u} u\right), t_{u} u\right\rangle \\
& =\frac{t_{u}^{2}}{3}\|u\|^{2}+\frac{t_{u}^{4} b}{12}\|u\|^{4}+\int_{B}\left(\frac{t_{u}^{6+|x|^{\alpha}}}{6}-\frac{t_{u}^{6+|x|^{\alpha}}}{6+|x|^{\alpha}}\right)|u|^{6+|x|^{\alpha}} \\
& <\frac{1}{3}\|u\|^{2}+\frac{b}{12}\|u\|^{4}+\int_{B}\left(\frac{1}{6}-\frac{1}{6+|x|^{\alpha}}\right)|u|^{6+|x|^{\alpha}} \\
& \leq \liminf _{n \rightarrow \infty}\left(\frac{1}{3}\left\|u_{n}\right\|^{2}+\frac{b}{12}\left\|u_{n}\right\|^{4}+\int_{B}\left(\frac{1}{6}-\frac{1}{6+|x|^{\alpha}}\right)\left|u_{n}\right|^{6+|x|^{\alpha}}\right) \\
& =\liminf _{n \rightarrow \infty}\left(J\left(u_{n}\right)-\frac{1}{6}\left\langle J^{\prime}\left(u_{n}\right), u_{n}\right\rangle\right)=c_{2},
\end{aligned}
$$

which is impossible. Thus, $\int_{B}|\nabla u|^{2}=A^{2}$ and $\hat{J}^{\prime}(u)=0$. The proof is completed.
In order to obtain the nontrivial solution of (1.6), we need define the functional $\hat{I}: H \rightarrow \mathbb{R}$ by

$$
\hat{I}(u)=\frac{1}{2}\|u\|^{2}+\frac{b}{4}\|u\|^{4}-\frac{1}{6} \int_{B}|u|^{6} .
$$

Lemma 6.5. Assume that $0<c_{2}<\Lambda_{1}$. The functional I satisfies the (PS $)_{c_{2}}$ condition.
Proof. Suppose that $\left\{u_{n}\right\}$ is a $(P S)_{c_{2}}$ sequence for $c_{2} \in\left(0, \Lambda_{1}\right)$, i.e.

$$
\hat{I}\left(u_{n}\right) \rightarrow c_{2}, \quad \hat{I}^{\prime}\left(u_{n}\right) \rightarrow 0 \quad \text { as } n \rightarrow \infty .
$$

By repeating the arguments used in Lemma 3.3, it is easy to show that $\left\{u_{n}\right\}$ is bounded in $H$. Then passing to a subsequence, we can find $u \in H$ such that $u_{n} \rightharpoonup u$ in $H$. Now, let $v_{n}=u_{n}-u$, we claim that $\left\|v_{n}\right\| \rightarrow 0$. In fact, we use an argument of contradiction and suppose that there exists a subsequence still denoted by $\left\{v_{n}\right\}$ such that $\left\|v_{n}\right\| \rightarrow \tilde{l}>0$. It is easy to verify that

$$
\begin{equation*}
\left\|u_{n}\right\|^{2}=\left\|v_{n}\right\|^{2}+\|u\|^{2}+o(1) \tag{6.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|u_{n}\right\|^{4}=\left\|v_{n}\right\|^{4}+\|u\|^{4}+2\left\|v_{n}\right\|^{2}\|u\|^{2}+o(1) . \tag{6.8}
\end{equation*}
$$

From the Brezis-Lieb lemma in [9], we have

$$
\begin{equation*}
\int_{B}\left|u_{n}\right|^{6} d x=\int_{B}\left|v_{n}\right|^{6} d x+\int_{B}|u|^{6} d x+o(1) . \tag{6.9}
\end{equation*}
$$

Recall that $\hat{I}^{\prime}\left(u_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$, there holds by (6.7),

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\langle\hat{I}^{\prime}\left(u_{n}\right), u\right\rangle=\|u\|^{2}+b \hat{l}^{2}\|u\|^{2}+b\|u\|^{4}-\int_{B}|u|^{6} d x=0, \tag{6.10}
\end{equation*}
$$

which yields

$$
\begin{align*}
\hat{I}(u) & =\hat{I}(u)-\frac{1}{4}\left(\|u\|^{2}+b l^{2}\|u\|^{2}+b\|u\|^{4}-\int_{B}|u|^{6} d x\right) \\
& =\frac{1}{2}\|u\|^{2}+\frac{1}{12} \int_{B}|u|^{6} d x-\frac{b}{4} \tilde{l}^{2}\|u\|^{2}  \tag{6.11}\\
& \geq-\frac{b}{4} \tilde{l}^{2}\|u\|^{2} .
\end{align*}
$$

On the other hand, combining (6.7), (6.8) with (6.9) leads to

$$
\begin{align*}
\hat{I}\left(u_{n}\right) & -\hat{I}(u)+o(1) \\
& =\frac{1}{2}\left\|u_{n}\right\|^{2}-\frac{1}{2}\|u\|^{2}+\frac{b}{4}\left\|u_{n}\right\|^{4}-\frac{b}{4}\|u\|^{4}-\frac{1}{6} \int_{B}\left|u_{n}\right|^{6}+\frac{1}{6} \int_{B}|u|^{6}+o(1)  \tag{6.12}\\
& =\frac{1}{2}\left\|v_{n}\right\|^{2}+\frac{b}{4}\left\|v_{n}\right\|^{4}+\frac{b}{2}\left\|v_{n}\right\|^{2}\|u\|^{2}-\frac{1}{6} \int_{B}\left|v_{n}\right|^{6}+o(1) .
\end{align*}
$$

Similarly, by using (6.10) again, we deduce

$$
\begin{align*}
o(1) & =\left\langle\hat{I}^{\prime}\left(u_{n}\right), u_{n}\right\rangle-\left(\|u\|^{2}+b \hat{l}^{2}\|u\|^{2}+b\|u\|^{4}-\int_{B}|u|^{6} d x\right) \\
& =\left\|u_{n}\right\|^{2}-\|u\|^{2}+b\left\|u_{n}\right\|^{4}-b\|u\|^{4}-b \hat{l}^{2}\|u\|^{2}-\int_{B}\left|u_{n}\right|^{6}+\int_{B}|u|^{6}  \tag{6.13}\\
& =\left\|v_{n}\right\|^{2}+b\left\|v_{n}\right\|^{4}+b\left\|v_{n}\right\|^{2}\|u\|^{2}-\int_{B}\left|v_{n}\right|^{6}+o(1) .
\end{align*}
$$

Then, taking the limit on the both sides in (6.13) as $n \rightarrow \infty$, we find $\tilde{l}^{2}+b \tilde{l}^{4}+b \tilde{l}^{2}\|u\|^{2} \leq S^{-3} \tilde{l}^{6}$, which implies that

$$
\begin{equation*}
\tilde{l}^{2} \geq \frac{S^{3} b+S \sqrt{S^{4} b^{2}+4\left(1+b\|u\|^{2}\right) S}}{2} \tag{6.14}
\end{equation*}
$$

It follows from (6.12) and (6.13) that

$$
\hat{I}(u)=\hat{I}\left(u_{n}\right)-\left(\frac{1}{2}-\frac{1}{6}\right)\left\|v_{n}\right\|^{2}-\left(\frac{1}{4}-\frac{1}{6}\right) b\left\|v_{n}\right\|^{4}-\left(\frac{1}{2}-\frac{1}{6}\right) b\left\|v_{n}\right\|^{2}\|u\|^{2}+o(1) .
$$

This together with (6.14) ensures that

$$
\begin{aligned}
\hat{I}(u)= & c_{2}-\left(\frac{1}{3} \tilde{l}^{2}+\frac{1}{12} b \tilde{l}^{4}+\frac{1}{3} b \tilde{l}^{2}\|u\|^{2}\right) \\
\leq & c_{2}-\frac{b}{4} S^{3}-\frac{1}{24} b^{3} S^{6}-\frac{S}{6} \sqrt{b^{2} S^{4}+4\left(1+b\|u\|^{2}\right) S} \\
& -\frac{b^{2} S^{4}}{24} \sqrt{b^{2} S^{4}+4\left(1+b\|u\|^{2}\right) S} \\
& -\frac{1}{24}\left(3 b^{2} S^{3}+S \sqrt{b^{2} S^{4}+4\left(1+b\|u\|^{2}\right) S}\right)\|u\|^{2}-\frac{b}{4} \tilde{l}^{2}\|u\|^{2} \\
\leq & c_{2}-\left(\frac{b}{4} S^{3}+\frac{b^{3}}{24} S^{6}+\frac{S}{6} \sqrt{b^{2} S^{4}+4 S}+\frac{b^{2}}{24} S^{4} \sqrt{b^{2} S^{4}+4 S}\right)-\frac{b}{4} \tilde{l}^{2}\|u\|^{2} \\
\leq & c_{2}-\Lambda-\frac{b}{4} \tilde{l}^{2}\|u\|^{2}<-\frac{b}{4} \tilde{l}^{2}\|u\|^{2},
\end{aligned}
$$

which contradicts to (6.11). Therefore $v_{n} \rightarrow 0$ strongly in $H$, or equivalently, $u_{n} \rightarrow u$ in $H$ as $n \rightarrow \infty$. The proof is completed.

Proof of Theorem 1.9. By Lemmas 6.4, 6.5, we know that the assumptions in Theorem 1.2 are valid. Hence, (1.6) possesses a nonnegative nontrivial ground state solution $u \in H$, which satisfies the following equation in weak sense

$$
-\left(1+b \int_{B}|\nabla u|^{2}\right) \Delta u=u^{5+|x|^{\alpha}} \quad \text { in } B .
$$

Let us define

$$
\hat{g}(u(x))=\frac{u^{5+|x|^{\alpha}}}{1+b \int_{B}|\nabla u|^{2}}, \quad x \in B .
$$

It follows from Lemma 2.2 that $\int_{B} u^{6+\frac{3}{2}|x|^{\alpha}} \leq C$, which implies

$$
a=\frac{\hat{g}(u)}{1+|u|} \in L^{\frac{3}{2}}(B) .
$$

Hence, we deduce immediately from Lemma 5.7 that $u \in L^{q}(B)$ for any $1<q<\infty$. Then, there holds $\hat{g}(u) \in L^{q}(B)$ for any $1<q<\infty$. By the Calderón-Zygmund inequality and $L^{p}$ estimate given in [16,30], we derive $u \in W^{2, q}(B)$, whence also $u \in C^{1, \alpha_{2}}(B)$ by Sobolev embedding theorem for any $0<\alpha_{2}<1$. Moreover, the Harnack inequality [32] implies $u(x)>0$ for all $x \in B$. The proof is completed.

## 7 Acknowledgement

This work was partially supported by National Natural Science Foundation of China (Grant Nos. 12071266, 12026217, 12026218) and Shanxi Scholarship Council of China (Grant No. 2020-005). The author would like to express their sincere gratitude to anonymous referees for his/her constructive comments for improving the quality of this paper.

## References

[1] B. Almuallemi, H. Chen, S. Khoutir, Existence of nontrivial solutions for SchrödingerPoisson systems with critical exponent on bounded domains, Bull. Malays. Math. Sci. Soc. 42(2019), 1675-1686. https://doi.org/10.1007/s40840-017-0570-0; MR3963850; Zbl 1421.35141
[2] C. O. Alves, M. A. S. Souto, Existence of least energy nodal solution for a SchrödingerPoisson system in bounded domains, Z. Angew. Math. Phys. 65(2014), 1153-1166. https : //doi.org/10.1007/s00033-013-0376-3; MR3279523; Zbl 1308.35262
[3] A. Azzollini, P. d’Avenia, On a system involving a critically growing nonlinearity, J. Math. Anal. Appl. 387(2012), 433-438. https://doi.org/10.1016/j.jmaa.2011.09.012; MR2845762; Zbl 1229.35060
[4] A. Azzollini, P. d'Avenia, V. Luisi, Generalized Schrödinger-Poisson type systems, Coттип. Pure Appl. Anal. 12(2013), 867-879. https://doi.org/10.3934/cpaa.2013.12. 867; MR2982795; Zbl 1270.35227
[5] A. Azzollini, P. d'Avenia, G. Vaira, Generalized Schrödinger-Newton system in dimension $N \geq$ 3: Critical case, J. Math. Anal. Appl. 449(2017), 531-552. https://doi. org/ 10.1016/j.jmaa.2016.12.008; MR3595217; Zbl 1373.35120
[6] Z. BA, X. He, Solutions for a class of Schrödinger-Poisson system in bounded domains, J. Appl. Math. Comput. 51(2016), 287-297. https://doi.org/10.1007/s12190-015-0905-7; MR3490972; Zbl 1342.35094
[7] V. Benci, D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations, Topol. Methods Nonlinear Anal. 11(1998), 283-293. https://doi .org/10.12775/TMNA. 1998. 019; MR1659454; Zbl 0926.35125
[8] H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. 36(1983), 437-477. https://doi.org/10. 1002/cpa.3160360405; MR0709644; Zbl 0541.35029
[9] H. Brézis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88(1983), 486-490. https ://doi .org/10.2307/ 2044999; MR0699419; Zbl 0526.46037
[10] D. Cao, S. Li, Z. Liu, Nodal solutions for a supercritical semilinear problem with variable exponent, Calc. Var. Partial Differential Equations 57(2018), No. 2, Paper No. 38, 19 pp. https://doi.org/10.1007/s00526-018-1305-2; MR3763423; Zbl 1392.35156
[11] S. Chen, B. Zhang, X. Tang, Existence and non-existence results for Kirchhoff-type problems with convolution nonlinearity, Adv. Nonlinear Anal. 9(2020), 148-167. https: //doi.org/10.1515/anona-2018-0147; MR3935867; Zbl 1421.35100
[12] J. M. Coron, Topologie et cas limite des injections de Sobolev (in French) [Topology and limit case of Sobolev embeddings], C. R. Acad. Sci. Paris Ser. I 299(1984), 209-212. MR0762722; Zbl 0569.35032
[13] F. J. S. A. Corrêa, G. Figueiredo, On an elliptic equation of $p$-Kirchhoff type via variational methods, Bull. Aust. Math. Soc. 74(2006), 263-277. https://doi.org/10.1017/ S000497270003570X; MR2260494; Zbl 1108.45005
[14] F. J. S. A. Corrêa, G .M. Figueiredo, On the existence of positive solutions for an elliptic equation of Kirchhof-type via Moser iteration method, Bound. Value Probl. 2006, Art. ID 79679, 10 pp. https://doi.org/10.1155/bvp/2006/79679; MR2251799
[15] X. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m, p(x)}(\Omega)$, J. Math. Anal. Appl. 263(2001), 424-446. https://doi.org/10.1006/jmaa.2000.7617; MR1866056
[16] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, Reprint of the 1998 ed., Classics in Mathematics, Springer-Verlag, Berlin, 2001. https://doi.org/ 10.1007/978-3-642-61798-0; MR1814364
[17] J. Kazdan, F. Warner, Remarks on some quasilinear elliptic equations, Comm. Pure Appl. Math. 28(1975), 567-597. https://doi.org/10.1002/cpa.3160280502; MR0477445; Zbl 0325.35038
[18] G. Кirchноғf, Mechanik, Teubner, Leipzig, 1883.
[19] F. Li, Y. Li, J. Shi, Existence of positive solutions to Schrödinger-Poisson type systems with critical exponent, Commun. Contemp. Math. 16(2014), No. 6, 1450036, 28 pp. https: //doi.org/10.1142/S0219199714500369; MR3277956; Zbl 1309.35025
[20] J. Lions, On some questions in boundary value problems of mathematical physics, in: G. M. de la Penha, L. A. Medeiros (Eds.), Contemporary developments in continuum mechanics and partial differential equations (Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977), North-Holland Math. Stud., Vol. 30, North-Holland, Amsterdam, 1978, pp. 284-346. MR519648
[21] J. Marcos do Ó, B. Ruf, P. Ubilla, On supercritical Sobolev type inequalities and related elliptic equations, Calc. Var. Partial Differential Equations 55(2016), No. 4, Art. 83, 18 pp. https://doi.org/10.1007/s00526-016-1015-6; MR3514752; Zbl 1356.35106
[22] D. Naimen, On the Brezis-Nirenberg problem with a Kirchhoff type Perturbation, Adv. Nonlinear Stud. 15(2015), No. 1, 135-156. https://doi.org/10.1515/ans-2015-0107; MR3299386; Zbl 1317.35080
[23] D. Naimen, M. Shibata, Two positive solutions for the Kirchhoff type elliptic problem with critical nonlinearity in high dimension, Nonlinear Anal. 186(2019), 187-208. https: //doi.org/10.1016/j.na.2019.02.003; MR3987393; Zbl 1421.35116
[24] W. M. Ni, A nonlinear Dirichlet problem on the unit ball and its applications, Indiana Univ. Math. J. 31(1982), 801-807. https://doi.org/10.1512/iumj.1982.31.31056; MR674869; Zbl 0515.35033
[25] L. Pisani, G. Siciliano, Note on a Schrödinger-Poisson system in a bounded domain, Appl. Math. Lett. 21(2008), 521-528. https://doi.org/10.1016/j.aml.2007.06. 005; MR2402846; Zbl 1158.35424
[26] S. I. Pohozaev, Eigenfunctions of the equation $\Delta u+\lambda f(u)=0$, Soviet Math. Doklady 6(1965), 1408-1411. MR0192184; Zbl 0141.30202
[27] O. Rey, Sur un problème variationnel non compact: l'effet de petits trous dans le domaine (in French) [On a variational problem with lack of compactness: the effect of small holes in the domain], C. R. Acad. Sci. Paris Ser. I Math. 308(1989), 349-352. MR992090; Zbl 0686.35047
[28] W. Rudin, Real and complex analysis, McGraw-Hill Science, New York, 1986. MR924157
[29] D. Ruiz, G. Siciliano, A note on the Schrödinger-Poisson-Salter equation on bounded domain, Adv. Nonlinear Stud. 8(2008), 179-190. https://doi.org/10.1515/ans-20080106; MR2378870; Zbl 1160.35020
[30] M. Struwe, Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems, Second Edition, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 30, Springer-Verlag, 1996. https://doi.org/10.1007/978-3-662-03212-1; MR1411681
[31] X. H. Tang, B. T. Cheng, Ground state sign-changing solutions for Kirchhoff type problems in bounded domains, J. Differential Equations 261(2016), 2384-2402. https: //doi.org/10.1016/j.jde.2016.04.032; MR3505194; Zbl 1343.35085
[32] N. S. Trudinger, On Harnack type inequality and their applications to quasilinear elliptic equations, Comm. Pure Appl. Math. 20(1967), 721-747. https://doi.org/10.1002/cpa. 3160200406; MR226198
[33] M. Willem, Minimax theorems, Birkhäuser, 1996. https://doi.org/10.1007/978-1-4612-4146-1; MR1400007
[34] Q. Xie, X. Wu, C. Tang, Existence and multiplicity of solutions for Kirchhoff type problem with critical exponent, Commun. Pure Appl. Anal. 12(2013), 2773-2786. https://doi. org/ 10.3934/сраа. 2013.12.2773; MR3060908; Zbl 1264.65206

# Existence of positive solutions for a fractional compartment system 

Lingju Kong ${ }^{1}$ and Min Wang ${ }^{\boxtimes 2}$<br>${ }^{1}$ Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA<br>${ }^{2}$ Department of Mathematics, Kennesaw State University, Marietta, GA 30060, USA

Received 4 May 2021, appeared 5 August 2021
Communicated by Nickolai Kosmatov


#### Abstract

In this article, we investigate the existence of positive solutions of a boundary value problem for a system of fractional differential equations. The resilience of a fractional compartment system is also studied to demonstrate the application of the result.


Keywords: boundary value problem, Green's function, positive solution, fractional compartment model.
2020 Mathematics Subject Classification: 34B18, 34B15, 34B60.

## 1 Introduction

In this paper, we consider the boundary value problem (BVP) consisting of a system of $n$ fractional order compartment models

$$
\begin{equation*}
u_{i}^{\prime}+a_{i} D_{0+}^{\alpha_{i}} u_{i}=f_{i}\left(u_{1} \ldots, u_{n}, t\right), \quad 0<t<1 \tag{1.1}
\end{equation*}
$$

and the boundary conditions (BCs)

$$
\begin{equation*}
u_{i}(0)=b_{i} u_{i}(1), \quad i=1, \ldots, n \tag{1.2}
\end{equation*}
$$

where $0<\alpha_{i}<1$ and $D_{0+}^{\alpha_{i}} u_{i}$ denotes the $\alpha_{i}$-th left Riemann-Liouville fractional derivative of $u_{i}$ defined by

$$
\left(D_{0+}^{\alpha_{i}} u_{i}\right)(t)=\frac{1}{\Gamma\left(1-\alpha_{i}\right)} \frac{d}{d t} \int_{0}^{t}(t-s)^{-\alpha_{i}} u_{i}(s) d s
$$

provided the right-hand side exists with $\Gamma$ being the Gamma function. We further assume that for any $i=1, \ldots, n$,
(H1) $a_{i}>0, b_{i}>0, f_{i} \in C\left(\mathbb{R}^{n} \times[0,1]\right)$, and $f_{i}(0, \ldots, 0, t) \not \equiv 0$ on $[0,1]$.

[^22]Fractional differential equations have been an active research area for decades and attracted extensive attention from scholars in both applied and theoretic fields. Due to the superior capability of capturing long term memory and/or long range interaction, fractional models have been successfully developed to investigate problems on fractal porous media, social media networks, epidemiology, finance, control, etc. Those models were further generalized and studied both analytically and numerically. The reader is referred to [1-11,13-16] and references therein for some recent advances.

This paper is mainly motivated by the study of a fractional compartment system for a bike share system. In [7], the station inventory, i.e., the number of bikes at a station, is modeled by

$$
\begin{equation*}
y_{i}^{\prime}=q_{i}(t)-\omega_{i}(t) y_{i}-\Theta_{i}(t) c_{i}^{-\beta_{i}} D_{0+}^{1-\beta_{i}}\left(\frac{y_{i}}{\Theta_{i}}\right), \quad t>0, i=1, \ldots, n . \tag{1.3}
\end{equation*}
$$

The resilience of station inventory, i.e., the capability that the station inventory will restore to certain level without extra interference, was further studied in $[10,16]$ by converting the resilience of Eq. (1.3) to a special case of BVP (1.1), (1.2) with $n=1$ (the scalar case). Intuitively, it is more sensible to investigate BVP (1.1), (1.2) with $n>1$ as the interactions among multiple stations are inevitable. From the practical perspective, we are particularly interested in finding conditions that guarantee the existence of positive solutions of BVP (1.1), (1.2). However, the extension from scalar to system is not trivial and it will require new auxiliary results to study the existence of positive solutions of the resulting system.

In this paper, a framework consisting of an appropriate Banach space and the associated operator will be proposed so that the fixed point theory can be applied to study the existence of positive solutions of BVP (1.1), (1.2). This framework will also be applicable to other fixed point theorems. Our result will be further applied to establish the sufficient conditions for the resilience of a fractional bike share inventory model. These conditions will provide guidance for the development of operational policy. Therefore, our work will make contributions in both theoretic and application aspects.

The paper is organized as follows: After this introduction, the main theoretic result and its proof will be presented in Section 2. The resilience of a bike share model will then be considered in Section 3 to demonstrate the application of our result.

## 2 Main results

We first introduce some needed notations and definitions. For any $x=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}^{n}$, let $\|x\|_{1}=\sum_{i=1}^{n}\left|x_{i}\right|$ and

$$
\begin{equation*}
K_{r}=\left\{x \in \mathbb{R}^{n}:\|x\|_{1} \leq r, x_{i} \geq 0, i=1, \ldots, n\right\} . \tag{2.1}
\end{equation*}
$$

For any $u=\left(u_{1}, \ldots, u_{n}\right) \in \prod_{i=1}^{n} C[0,1]$, let $\|u\|=\max _{t \in[0,1]} \sum_{i=1}^{n}\left|u_{i}(t)\right|$. By a solution of BVP (1.1), (1.2), we mean a vector-valued function $u \in \prod_{i=1}^{n} C[0,1]$ that satisfies (1.1) and (1.2). Furthermore, $u$ is said to be a positive solution of BVP (1.1), (1.2) if $u_{i}(t) \geq 0, i=1, \ldots, n$, and $\|u\|>0$.

Let $E_{\alpha}(t)$ be the Mittag-Leffler function defined by

$$
E_{\alpha}(t)=\sum_{n=0}^{\infty} \frac{t^{n}}{\Gamma(n \alpha+1)}
$$

and $\Lambda_{i}(t)$ be defined by

$$
\begin{equation*}
\Lambda_{i}(t)=E_{1-\alpha_{i}}\left(-a_{i} t^{1-\alpha_{i}}\right), \quad i=1, \ldots, n . \tag{2.2}
\end{equation*}
$$

Throughout this paper, we assume
(H2) $b_{i} \Lambda_{i}(1)<1, i=1, \ldots, n$.
Define

$$
\begin{equation*}
\overline{G_{i}}=\max _{t \in[0,1]}\left\{\frac{b_{i} \Lambda_{i}(t)}{1-b_{i} \Lambda_{i}(1)}, \frac{b_{i} \Lambda_{i}(t) \Lambda_{i}(1-t)}{1-b_{i} \Lambda_{i}(1)}+1\right\} \tag{2.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\underline{G_{i}}=\min _{t \in[0,1]}\left\{\frac{\Lambda_{i}(t)}{1-b_{i} \Lambda_{i}(1)}, \frac{b_{i} \Lambda_{i}(t) \Lambda_{i}(1-t)}{1-b_{i} \Lambda_{i}(1)}\right\}, \quad i=1, \ldots, n . \tag{2.4}
\end{equation*}
$$

Then we have the following result.
Theorem 2.1. Let $K_{r}$ and $\overline{G_{i}}, i=1, \ldots, n$, be defined in (2.1) and (2.3), respectively. Assume that (H1) and (H2) hold and that there exist $r>0$ and $\eta_{i}>0, i=1, \ldots, n$, such that
(a) $\sum_{i=1}^{n} \bar{G}_{i} \eta_{i} \leq r$; and
(b) for any $t \in[0,1]$ and $x \in K_{r}, 0 \leq f_{i}(x, t) \leq \eta_{i}, i=1, \ldots, n$.

Then BVP (1.1), (1.2) has at least one positive solution $u$ with $\|u\| \leq r$.

The following lemma plays an important role in the proof of Theorem 2.1.
Lemma 2.2. Assume (H2) holds. For $i=1, \ldots, n$, let $\Lambda_{i}, \overline{G_{i}}, \underline{G_{i}}$ be defined by (2.2), (2.3), (2.4), respectively, and

$$
G_{i}(t, s)= \begin{cases}\frac{b_{i} \Lambda_{i}(t) \Lambda_{i}(1-s)}{1-b_{i} \Lambda_{i}(1)}+\Lambda_{i}(t-s), & 0 \leq s \leq t \leq 1  \tag{2.5}\\ \frac{b_{i} \Lambda_{i}(t) \Lambda_{i}(1-s)}{1-b_{i} \Lambda_{i}(1)}, & 0 \leq t<s \leq 1\end{cases}
$$

Then $G_{i}(t, s)$ is the Green's function for the scalar BVP

$$
\begin{aligned}
u_{i}^{\prime}+a_{i} D_{0+}^{\alpha_{i}} u_{i} & =0, \quad 0<t<1 \\
u_{i}(0) & =b_{i} u(1)
\end{aligned}
$$

and satisfies

$$
\begin{equation*}
0<\underline{G_{i}} \leq G_{i} \leq \overline{G_{i}}, \quad i=1, \ldots, n . \tag{2.6}
\end{equation*}
$$

Proof. Let $\mathbb{R}_{+}:=[0, \infty)$. By [7, Lemma 3.1], for any $h \in C\left(\mathbb{R}_{+}, \mathbb{R}\right)$ and $i=1, \ldots, n$, the equation

$$
u_{i}^{\prime}+a_{i} D_{0+}^{\alpha_{i}} u_{i}=h(t), \quad t>0
$$

has a unique solution given by

$$
u_{i}(t)=\int_{0}^{t} \Lambda_{i}(t-s) h(s) d s+u_{i}(0) \Lambda_{i}(t)
$$

Then BC (1.2) implies

$$
u_{i}(0)=\int_{0}^{1} \frac{b_{i} \Lambda_{i}(1-s)}{1-b_{i} \Lambda_{i}(1)} h(s) d s .
$$

Hence by (2.5) we have

$$
\begin{align*}
u_{i}(t) & =\int_{0}^{t} \Lambda_{i}(t-s) h(s) d s+\left(\int_{0}^{1} \frac{b_{i} \Lambda_{i}(1-s)}{1-b_{i} \Lambda_{i}(1)} h(s) d s\right) \Lambda_{i}(t)  \tag{2.7}\\
& =\int_{0}^{1} G_{i}(t, s) h(s) d s
\end{align*}
$$

It is notable that when $\alpha_{i} \in(0,1)$, we have $\Lambda_{i}^{\prime}(t) \leq 0$ on $(0, \infty), \Lambda_{i}(0)=1, \lim _{t \rightarrow \infty} \Lambda_{i}(t)=0$, and $0<\Lambda_{i}(t)<1$; see for example [12]. Then by (2.5), for any $t \in[0,1]$,

$$
\frac{\partial G_{i}}{\partial s} \geq 0 \quad \text { on }(0, t) \cup(t, 1) .
$$

Hence

$$
\begin{aligned}
\frac{\Lambda_{i}(t)}{1-b_{i} \Lambda_{i}(1)} & \leq G_{i}(t, s) \leq \frac{b_{i} \Lambda_{i}(t) \Lambda_{i}(1-t)}{1-b_{i} \Lambda_{i}(1)}+1, \quad 0 \leq s \leq t \\
\frac{b_{i} \Lambda_{i}(t) \Lambda_{i}(1-t)}{1-b_{i} \Lambda_{i}(1)} & \leq G_{i}(t, s) \leq \frac{b_{i} \Lambda_{i}(t)}{1-b_{i} \Lambda_{i}(1)}, \quad t<s \leq 1 .
\end{aligned}
$$

Therefore, (2.6) holds.
Remark 2.3. It is clear that $G_{i}$ defined by (2.5) is discontinuous at $t=s$. However, by (2.7), $u_{i}$ is continuous on $[0,1]$ when $h \in C[0,1], i=1, \ldots, n$.

With Lemma 2.2, we are able to construct a needed operator on an appropriate Banach space. In the sequel, we choose the Banach space $X=\prod_{i=1}^{n} C[0,1]$ with the norm $\|u\|=$ $\max _{t \in[0,1]} \sum_{i=1}^{n}\left|u_{i}(t)\right|$, where $u=\left(u_{1}(t), \ldots, u_{n}(t)\right) \in X$. Define an operator $T: X \rightarrow X$ by

$$
\begin{equation*}
(T u)_{i}(t)=\int_{0}^{1} G_{i}(t, s) f_{i}\left(u_{1}(s), \ldots, u_{n}(s), s\right) d s, \quad t \in[0,1], i=1, \ldots, n \tag{2.8}
\end{equation*}
$$

where $G_{i}$ is defined by (2.5). By Lemma 2.2, it is easy to see that $u$ is a solution of BVP (1.1), (1.2) if and only if $u$ is a fixed point of $T$.

Proof of Theorem 2.1. First of all, it is obvious that $(0, \ldots, 0)$ is not a fixed point of $T$. By Remark 2.3, we have $T(X) \subset X$. We need to prove that $T: X \rightarrow X$ is a compact operator. For any $u, v \in X, t \in[0,1]$, and $i=1, \ldots, n$,

$$
\begin{aligned}
\left|(T u)_{i}(t)-(T v)_{i}(t)\right| & =\left|\int_{0}^{1} G_{i}(t, s) f_{i}\left(u_{1}(s), \ldots, u_{n}(s), s\right) d s-\int_{0}^{1} G_{i}(t, s) f_{i}\left(v_{1}(s), \ldots, v_{n}(s), s\right) d s\right| \\
& \leq \overline{G_{i}} \max _{s \in[0,1]}\left|f_{i}\left(u_{1}(s), \ldots, u_{n}(s), s\right)-f_{i}\left(v_{1}(s), \ldots, v_{n}(s), s\right)\right| .
\end{aligned}
$$

Hence $T$ is continuous by the continuity of $f_{i}, i=1, \ldots, n$.

Let $\Omega=\{u \in X:\|u\| \leq B\}$. For any $u \in \Omega, t \in[0,1]$, and $i=1, \ldots, n$,

$$
\begin{aligned}
\left|(T u)_{i}(t)\right| & =\left|\int_{0}^{1} G_{i}(t, s) f_{i}\left(u_{1}(s), \ldots, u_{n}(s), s\right) d s\right| \\
& \leq \overline{G_{i}} \max _{v \in \Omega, s \in[0,1]}\left|f_{i}\left(v_{1}(s), \ldots, v_{n}(s), s\right)\right| .
\end{aligned}
$$

Hence $T$ is uniformly bounded. For any $0 \leq t_{1}<t_{2} \leq 1$, by (2.7),

$$
\begin{aligned}
&\left|(T u)_{i}\left(t_{1}\right)-(T u)_{i}\left(t_{2}\right)\right| \\
&=\left|\int_{0}^{1} G_{i}\left(t_{1}, s\right) f_{i}\left(u_{1}(s), \ldots, u_{n}(s), s\right) d s-\int_{0}^{1} G_{i}\left(t_{2}, s\right) f_{i}\left(u_{1}(s), \ldots, u_{n}(s), s\right) d s\right| \\
&= \mid \int_{0}^{t_{1}} \Lambda_{i}\left(t_{1}-s\right) f_{i}\left(u_{1}(s), \ldots, u_{n}(s), s\right) d s \\
&+\left(\int_{0}^{1} \frac{b_{i} \Lambda_{i}(1-s)}{1-b_{i} \Lambda_{i}(1)} f_{i}\left(u_{1}(s), \ldots, u_{n}(s), s\right) d s\right) \Lambda_{i}\left(t_{1}\right) \\
&-\int_{0}^{t_{2}} \Lambda_{i}\left(t_{2}-s\right) f_{i}\left(u_{1}(s), \ldots, u_{n}(s), s\right) d s \\
& \left.-\left(\int_{0}^{1} \frac{b_{i} \Lambda_{i}(1-s)}{1-b_{i} \Lambda_{i}(1)} f_{i}\left(u_{1}(s), \ldots, u_{n}(s), s\right) d s\right) \Lambda_{i}\left(t_{2}\right) \right\rvert\, \\
& \leq \int_{0}^{t_{1}}\left|\Lambda_{i}\left(t_{1}-s\right)-\Lambda_{i}\left(t_{2}-s\right)\right|\left|f_{i}\left(u_{1}(s), \ldots, u_{n}(s), s\right)\right| d s \\
&+\int_{t_{1}}^{t_{2}}\left|\Lambda_{i}\left(t_{2}-s\right)\right|\left|f_{i}\left(u_{1}(s), \ldots, u_{n}(s), s\right)\right| d s \\
&+\int_{0}^{1}\left|\frac{b_{i} \Lambda_{i}(1-s)}{1-b_{i} \Lambda_{i}(1)}\right|\left|f_{i}\left(u_{1}(s), \ldots, u_{n}(s), s\right)\right| d s\left|\Lambda_{i}\left(t_{1}\right)-\Lambda_{i}\left(t_{2}\right)\right| \\
& \leq \max _{v \in, s \in[0,1]}\left(\left|f_{i}\left(v_{1}(s), \ldots, v_{n}(s), s\right)\right|\left|\Lambda_{i}\left(t_{1}-s\right)-\Lambda_{i}\left(t_{2}-s\right)\right|\right) \\
&+\left|t_{1}-t_{2}\right| \max _{v \in \Omega, s \in[0,1]}\left|f_{i}\left(v_{1}(s), \ldots, v_{n}(s), s\right)\right| \\
&+\left|\Lambda_{i}\left(t_{1}\right)-\Lambda_{i}\left(t_{2}\right)\right| \\
& \max _{v \in \Omega, s \in[0,1]}\left(\left|\frac{b_{i} \Lambda_{i}(1-s)}{1-b_{i} \Lambda_{i}(1)}\right|\left|f_{i}\left(v_{1}(s), \ldots, v_{n}(s), s\right)\right|\right) .
\end{aligned}
$$

Then $T$ is equicontinuous on $\Omega$ since $\Lambda_{i}$ is uniformly continuous on [ 0,1 ]. By Arzelà-Ascoli Theorem, we can prove $T$ is a compact operator.

Let $K_{r}$ be defined by (2.1) and $K \subset X$ be defined by

$$
K=\left\{u \in X: u(t) \in K_{r}, t \in[0,1]\right\} .
$$

It is easy to see that $K$ is a nonempty, closed, bounded, and convex subset of $X$. We claim that $T(K) \subset K$.

In fact, by (2.8), for any $u \in K$ and $i=1, \ldots, n$,

$$
\begin{aligned}
\left|(T u)_{i}(t)\right| & =\left|\int_{0}^{1} G_{i}(t, s) f_{i}\left(u_{1}(s), \ldots, u_{n}(s), s\right) d s\right| \\
& \leq \int_{0}^{1} \overline{G_{i}} f_{i}\left(u_{1}(s), \ldots, u_{n}(s), s\right) d s \leq \int_{0}^{1} \overline{G_{i}} \eta_{i} d s \leq \overline{G_{i}} \eta_{i} .
\end{aligned}
$$

Then we have

$$
\sum_{i=1}^{n}\left|(T u)_{i}(t)\right| \leq \sum_{i=1}^{n} \overline{G_{i}} \eta_{i} \leq r .
$$

So $\|T u\| \leq r$. Moreover, it is easy to see that $(T u)_{i}(t) \geq 0$ on $[0,1], i=1, \ldots, n$. Hence $T(K) \subset K$.

Therefore by the Schauder Fixed-Point Theorem [17, Theorem 2.A], $T$ has a fixed point $u \in K$.

Remark 2.4. It is notable that the Banach space $(X,\|\cdot\|)$ and the operator $T$ defined by (2.8) form a general framework to study the existence of solutions for BVP (1.1), (1.2). Other fixed point theorems can also be applied to obtain more results on the existence and/or uniqueness of solution or positive solutions, see for example [4,9,15,17].

## 3 Resilience of a bike share inventory model

In this section, we consider the resilience of a bike share inventory model involving multiple stations. We first revisit the inventory model proposed in [7]. Let $y_{i}(t)$ be the inventory at time $t$ at Station $i, i=1, \ldots, n$. Then $y_{i}$ satisfies

$$
\begin{equation*}
y_{i}^{\prime}=q_{i}(t)-\omega_{i}(t) y_{i}-\Theta_{i}(t) c_{i}^{-\beta_{i}} D_{0+}^{1-\beta_{i}}\left(\frac{y_{i}}{\Theta_{i}}\right), \quad t>0, i=1, \ldots, n . \tag{3.1}
\end{equation*}
$$

For $i=1, \ldots, n$,

- $q_{i}(t)$ represents the arrival flux at a station;
- $\omega_{i}(t) y_{i}$ represents a Markov removal process that is independent of the history;
- $\beta_{i} \in(0,1)$ is a parameter relating to the bike waiting time distribution at a station; and
- $\Theta_{i}(t) c_{i}^{-\beta_{i}} D_{0+}^{1-\beta_{i}}\left(\frac{y_{i}}{\Theta_{i}}\right)$ represents a non-Markov removal process that relates to the bike waiting time at a station with

$$
\Theta_{i}(t)=\exp \left(-\int_{0}^{t} \omega(s) d s\right) .
$$

All the terms above are nonnegative. The reader is referred to [7] for the details of the terms.
To reflect the interactions among stations, we will extend Eq. (3.1) by modifying the arrival flux term $q_{i}$. Assume the total number of bikes in the entire bike share system is a constant $Y$. Clearly $\left(Y-\sum_{j=1}^{n} y_{j}\right)$ represents the total number of bikes in use at time $t$. Let $p_{i}\left(y_{i}, t\right) \in[0,1]$ be the return rate of in-use bikes to Station $i$ at time $t$ with

$$
\begin{equation*}
p_{i}\left(y_{i}, t\right) \geq 0, \quad \sum_{i=1}^{n} p_{i}\left(y_{i}, t\right) \leq 1, \quad t>0, i=1, \ldots, n \tag{3.2}
\end{equation*}
$$

Then the inventory $y_{i}$ satisfies

$$
\begin{equation*}
y_{i}^{\prime}=\left(Y-\sum_{j=1}^{n} y_{j}\right) p_{i}\left(y_{i}, t\right)-\omega_{i}(t) y_{i}-\Theta_{i}(t) c_{i}^{-\beta_{i}} D_{0+}^{1-\beta_{i}}\left(\frac{y_{i}}{\Theta_{i}}\right), \quad t>0, i=1, \ldots, n . \tag{3.3}
\end{equation*}
$$

If the inventory will restore at some time $\tau_{1}>0$, then $y_{i}$ must satisfy

$$
\begin{equation*}
y_{i}(0)=y_{i}\left(\tau_{1}\right), \quad i=1, \ldots, n \tag{3.4}
\end{equation*}
$$

Therefore, the resilience problem can be described by BVP (3.3), (3.4).

Remark 3.1. Since Eq. (3.3) models the rate of changes of $y_{i}$ at Station $i$, we assume the units of both $p_{i}$ and $\omega_{i}$ in (3.3) are 1 / [unit of time] so that the units on both sides of the equation are consistent.

The following result is obtained by applying Theorem 2.1.

Theorem 3.2. Let $K_{r}$ and $\overline{G_{i}}$ be defined by (2.1) and (2.3), respectively. If for any $x=\left(x_{1}, \ldots, x_{n}\right) \in$ $K_{Y}$ and $t \in[0,1]$, the return rates $p_{i}, i=1, \ldots, n$, satisfy

$$
\begin{equation*}
\sum_{i=1}^{n} \overline{G_{i}} \frac{\tau_{1} p_{i}\left(\Theta_{i}\left(\tau_{1} t\right) x_{i}, t\right)}{\Theta_{i}\left(\tau_{1} t\right)} \leq 1 \tag{3.5}
\end{equation*}
$$

then BVP (3.3), (3.4) has at least one positive solution $y$ with $\|y\| \leq Y$.
Proof. By an idea similar to [7], i.e., making a change of variables and rescaling $\left[0, \tau_{1}\right]$ to $[0,1]$, BVP (3.3), (3.4) can be converted to BVP (1.1), (1.2) with $u_{i}(t)=y_{i}\left(\tau_{1} t\right) / \Theta_{i}\left(\tau_{1} t\right), \alpha_{i}=1-\beta_{i}$, $a_{i}=c_{i}^{-\beta_{i}}, b_{i}=\Theta_{i}\left(\tau_{1}\right)$, and

$$
\begin{equation*}
f_{i}\left(u_{1}, \ldots, u_{n}, t\right)=\frac{\tau_{1} p_{i}\left(\Theta_{i}\left(\tau_{1} t\right) u_{i}, t\right)}{\Theta_{i}\left(\tau_{1} t\right)}\left(Y-\sum_{j=1}^{n}\left(\Theta_{j}\left(\tau_{1} t\right) u_{j}\right)\right), \quad i=1, \ldots, n \tag{3.6}
\end{equation*}
$$

Let $K_{Y}$ be defined by (2.1) with $r=Y$. By (3.2) and (3.6), it is easy to see that for any $x \in K_{Y}$ and $i=1, \ldots, n$, we have $f_{i}(x, t) \geq 0$ and

$$
\begin{aligned}
f_{i}\left(x_{1}, \ldots, x_{n}, t\right) & =\frac{\tau_{1} p_{i}\left(\Theta_{i}\left(\tau_{1} t\right) x_{i}, t\right)}{\Theta_{i}\left(\tau_{1} t\right)}\left(Y-\sum_{j=1}^{n}\left(\Theta_{j}\left(\tau_{1} t\right) x_{j}\right)\right) \\
& \leq \frac{\tau_{1} p_{i}\left(\Theta_{i}\left(\tau_{1} t\right) x_{i}, t\right)}{\Theta_{i}\left(\tau_{1} t\right)} \Upsilon
\end{aligned}
$$

Therefore, all the conditions of Theorem 2.1 are satisfied. The conclusion then follows immediately from Theorem 2.1.

Remark 3.3. Based on our assumption, the return rates $p_{i}, i=1, \ldots, n$, depend on both time $t$ and current station inventory. Theorem 3.2 shows that it is feasible to manage the station inventory by adjusting the return rates based on real-time status at each station. Therefore, new operational policies may be developed based on Theorem 3.2 by monitoring the return rates so that (3.5) is satisfied all the time.

## Acknowledgement

M. Wang's research in this paper is supported by the National Science Foundation under Grant No. 1830489.

## References

[1] A. Alsaedi, B. Ahmad, M. Alblewi, S. K. Ntouyas, Existence results for nonlinear fractional-order multi-term integro-multipoint boundary value problems, AIMS Math. 6(2021), 3319-3338. https://doi.org/10.3934/math. 2021199; MR4209586
[2] C. N. Angstmann, B. I. Henry, A. V. McGann, A fractional order recovery SIR model from a stochastic process, Bull. Math. Biol. 78(2016), 468-499. https://doi.org/10.1007/ s11538-016-0151-7; MR3485267
[3] C. N. Angstmann, B. I. Henry, A. V. McGann, A fractional-order infectivity and recovery SIR model, Fractal Fract. 1(2017), No. 1, Article no. 11. https://doi.org/10.3390/ fractalfract1010011
[4] A. Cabada, O. K. Wanassi, Existence results for nonlinear fractional problems with nonhomogeneous integral boundary conditions, Mathematics 8(2020), No. 2, Article no. 255. https://doi.org/10.3390/math8020255
[5] K. Diethelm, N. J. Ford, A. D. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations, Nonlinear Dyn. 29(2002), 3-22. https://doi. org/10.1023/A:1016592219341; MR1926466
[6] K. Diethelm, N. J. Ford, A. D. Freed, Detailed error analysis for a fractional Adams method, Numer. Algorithms 36(2004), 31-52. https://doi.org/10.1023/B: NUMA. 0000027736 . 85078. be; MR2063572
[7] J. R. Graef, S. S. Ho, L. Kong, M. Wang, A fractional differential equation model for bike share systems, J. Nonlinear Funct. Anal. 2019, Article ID 23, 1-14. https://doi.org/ 10.23952/jnfa.2019.23
[8] J. R. Graef, L. Kong, A. Ledoan, M. Wang, Stability analysis of a fractional online social network model, Math. Comput. Simulat. 178(2020), 625-645. https://doi.org/10.1016/ j.matcom.2020.07.012; MR4129090
[9] J. R. Graef, L. Kong, M. Wang, Existence and uniqueness of solutions for a fractional boundary value problem on a graph, Fract. Calc. Appl. Anal. 17(2014), 499-510. https: //doi.org/10.2478/s13540-014-0182-4; MR3181068
[10] K. Lam, M. Wang, Existence of solutions of a fractional compartment model with periodic boundary condition, Commun. Appl. Anal. 23(2019), 125-136. https://doi. org/10. 12732/caa.v23i1. 9
[11] K. Lan, Compactness of Riemann-Liouville fractional integral operators, Electron. J. Qual. Theory Differ. Equ. 2020, No. 84, 1-15. https://doi.org/10.14232/ejqtde.2020.1.84; MR4208491
[12] G. D. Lin, On the Mittag-Leffler distributions, J. Stat. Plann. Inference 74(1998), No. 1, 1-9. https://doi.org/10.1016/S0378-3758(98)00096-2; MR1665117
[13] I. Podlubny, Fractional differential equations, Academic Press, Inc., San Diego, CA, 1999. MR1658022
[14] V. Tarasov, Fractional dynamics. Applications of fractional calculus to dynamics of particles, fields and media, Springer-Verlag, Berlin-Heidelberg, 2010. https://doi.org/10.1007/ 978-3-642-14003-7; MR2796453
[15] A. Tudorache, R. Luca, Positive solutions for a system of Riemann-Liouville fractional boundary value problems with p-Laplacian operators, Adv. Difference Equ. 2020, Paper No. 292, 30 pp. https://doi.org/10.1186/s13662-020-02750-6; MR4111776
[16] M. Wang, On the resilience of a fractional compartment model, Appl. Anal., published online, 2020. https://doi.org/10.1080/00036811.2020.1712370
[17] E. Zeidler, Nonlinear functional analysis and its applications. I. Fixed-point theorems, Springer-Verlag, New York, 1986. https://doi.org/10.1007/978-1-4612-4838-5; MR816732

Electronic Journal of Qualitative Theory of Differential Equations

# Bistable equation with discontinuous density dependent diffusion with degenerations and singularities 

Dedicated to the memory of Professor Josef Daněček, our friend and mentor

Pavel Drábek ${ }^{\boxtimes}$ and Michaela Zahradníková<br>Department of Mathematics, Faculty of Applied Sciences, University of West Bohemia, Univerzitní 8, 30100 Plzeň, Czech Republic

Received 17 May 2021, appeared 8 September 2021
Communicated by Sergei Trofimchuk


#### Abstract

In this article we introduce rather general notion of the stationary solution of the bistable equation which allows to treat discontinuous density dependent diffusion term with singularities and degenerations, as well as degenerate or non-Lipschitz balanced bistable reaction term. We prove the existence of new-type solutions which do not occur in case of the "classical" setting of the bistable equation. In the case of the power-type behavior of the diffusion and bistable reaction terms near the equilibria we provide detailed asymptotic analysis of the corresponding solutions and illustrate the lack of smoothness due to the discontinuous diffusion.


Keywords: density dependent diffusion, bistable balanced nonlinearity, asymptotic behavior, discontinuous diffusion, degenerate and singular diffusion, degenerate nonLipschitz reaction.

2020 Mathematics Subject Classification: 35Q92, 35K92, 34C60, 34A12.

## 1 Introduction

Let us consider the bistable equation

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}}+g(u) \tag{1.1}
\end{equation*}
$$

in $\mathbb{R}$, where the reaction term $g:[0,1] \rightarrow \mathbb{R}$ is continuous and there exists $s_{*} \in(0,1)$ such that

$$
g(0)=g\left(s_{*}\right)=g(1)=0, \quad g(s)<0 \text { for } s \in\left(0, s_{*}\right), \quad g(s)>0 \text { for } s \in\left(s_{*}, 1\right) .
$$

Equation (1.1) appears in many mathematical models in population dynamics, genetics, combustion or nerve propagation, see e.g. [1,2] and references therein.

[^23]This kind of reaction is called bistable, cf. [3,7-9]. We distinguish between two different cases of bistable reactions which lead to different type of solutions to (1.1). Namely, when

$$
\begin{equation*}
\int_{0}^{1} g(s) \mathrm{d} s=0 \tag{1.2}
\end{equation*}
$$

we say that $g$ is balanced bistable nonlinearity while in case

$$
\int_{0}^{1} g(s) \mathrm{d} s \neq 0
$$

the bistable nonlinearity $g$ is called unbalanced. In the former case the equation (1.1) possesses (time independent) stationary solutions which connect constant equilibria $u_{0} \equiv 0$ and $u_{1} \equiv 1$, i.e., solutions $u=u(x)$ of (1.1) satisfying

$$
\begin{equation*}
\lim _{x \rightarrow-\infty} u(x)=0 \quad \text { and } \quad \lim _{x \rightarrow+\infty} u(x)=1 \tag{1.3}
\end{equation*}
$$

or

$$
\begin{equation*}
\lim _{x \rightarrow-\infty} u(x)=1 \quad \text { and } \quad \lim _{x \rightarrow+\infty} u(x)=0 \tag{1.4}
\end{equation*}
$$

On the other hand, the latter case leads to the (time dependent) nonstationary travelling wave solutions connecting $u_{0}$ and $u_{1}$, see e.g. [6,10].

The stationary solutions of (1.1) satisfying (1.3) or (1.4) can be found in the closed form for special reaction terms. For example, for

$$
g(s)=s(1-s)\left(s-\frac{1}{2}\right)
$$

we get stationary solution of (1.1), (1.3) in the following form

$$
u(x)=\frac{1}{2} \tanh \left(\frac{x}{2 \sqrt{2}}\right)+\frac{1}{2},
$$

cf. [4]. Then solution $u=u(x) \in(0,1), x \in \mathbb{R}$, is a strictly increasing function which approaches equilibria $u_{0}$ and $u_{1}$ at an exponential rate:

$$
\begin{equation*}
u(x) \sim \mathrm{e}^{x} \text { as } x \rightarrow-\infty \quad \text { and } \quad 1-u(x) \sim \mathrm{e}^{-x} \quad \text { as } x \rightarrow+\infty . \tag{1.5}
\end{equation*}
$$

If we consider the quasilinear bistable equation

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x}\right)+g(u), \tag{1.6}
\end{equation*}
$$

where $p>1$ and $g$ is balanced bistable nonlinearity then the structure of stationary solutions to (1.6), (1.3) or (1.6), (1.4) may be considerably different as shown in [4]. For example, if

$$
g(s)=s^{\alpha}(1-s)^{\alpha}\left(s-\frac{1}{2}\right), \quad s \in(0,1), \quad \alpha>0
$$

we distinguish between the following two qualitatively different cases:
Case 1: $\alpha+1 \geq p$,
Case 2: $\alpha+1<p$.

In Case 1 solution $u=u(x)$ of (1.6), (1.3) is again a strictly increasing continuously differentiable function which assumes values in $(0,1)$. However, (1.5) holds only in the case $\alpha+1=p$. In the case $\alpha+1>p$ we have

$$
u(x) \sim|x|^{\frac{p}{p-(\alpha+1)}} \quad \text { as } x \rightarrow-\infty \quad \text { and } \quad 1-u(x) \sim|x|^{\frac{p}{p-(\alpha+1)}} \quad \text { as } x \rightarrow+\infty .
$$

In Case 2 there exist real numbers $x_{0}<x_{1}$ such that for all $x \in\left(x_{0}, x_{1}\right)$ we have $u(x) \in$ $(0,1), u$ is strictly increasing continuously differentiable, $u(x)=0$ for all $x \in\left(-\infty, x_{0}\right]$ and $u(x)=1$ for all $x \in\left[x_{1},+\infty\right)$. Moreover,

$$
u(x) \sim\left(x-x_{0}\right)^{\frac{p}{p-(\alpha+1)}} \quad \text { as } x \rightarrow x_{0}+\quad \text { and } \quad 1-u(x) \sim\left(x_{1}-x\right)^{\frac{p}{p-(\alpha+1)}} \quad \text { as } x \rightarrow x_{1}-.
$$

Our ambition in this paper is to study similar properties for the quasilinear bistable equation with density dependent diffusion coefficient $d=d(s)$

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(d(u)\left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x}\right)+g(u), \tag{1.7}
\end{equation*}
$$

where the properties of $d=d(s)$ are specified in the next section.

## 2 Preliminaries

Let $p>1, g:[0,1] \rightarrow \mathbb{R}, g \in C[0,1]$ be such that $g(0)=g\left(s_{*}\right)=g(1)=0$ for $s_{*} \in(0,1)$ and

$$
g(s)<0, s \in\left(0, s_{*}\right), \quad g(s)>0, s \in\left(s_{*}, 1\right) .
$$

The diffusion coefficient $d:[0,1] \rightarrow \mathbb{R}$ is supposed to be a nonnegative lower semicontinuous function and $d>0$ in $(0,1)$. There exist $0=s_{0}<s_{1}<s_{2}<\cdots<s_{n}<s_{n+1}=1$ such that $\left.d\right|_{\left(s_{i}, s_{i+1}\right)} \in C\left(s_{i}, s_{i+1}\right), i=0, \ldots, n$, and $d$ has discontinuity of the first kind (finite jump) at $s_{i}$, $i=1, \ldots, n$.

For $p=2$ and $d(s) \equiv 1$ in $[0,1]$ equation (1.7) reduces to the bistable equation (1.1) with constant diffusion coefficient and bistable reaction term $g$. In this paper we deal with diffusion which allows for singularities and for degenerations both at 0 and/or 1 . We also consider $d$ to be a discontinuous function. Last but not least, reaction term $g$ can degenerate in 0 and/or in 1. In particular, we admit $g^{\prime}(0)=0$ and/or $g^{\prime}(1)=0$, as well as $g^{\prime}(0)=-\infty$ and/or $g^{\prime}(1)=-\infty$. This in turn yields that our solution is not a $C^{1}$-function in $\mathbb{R}$ and it does not satisfy the equation pointwise in the classical sense. For this purpose we have to employ the first integral of the second order differential equation. Since our primary interest in this paper is the investigation of stationary solutions to (1.7) which are monotone (i.e., nonincreasing or nondecreasing) between the equilibria 0 and 1 , we provide rather general definition of monotone solutions to the second order ODE

$$
\begin{equation*}
\left(d(u)\left|u^{\prime}\right|^{p-2} u^{\prime}\right)^{\prime}+g(u)=0, \tag{2.1}
\end{equation*}
$$

where, for the sake of simplicity, we write $(\cdot)^{\prime}$ instead of $\frac{d}{d x}(\cdot)$.
Let $u: \mathbb{R} \rightarrow[0,1]$ be a monotone continuous function. We denote

$$
M_{u}:=\left\{x \in \mathbb{R}: u(x)=s_{i}, i=1,2, \ldots, n\right\}, \quad N_{u}:=\{x \in \mathbb{R}: u(x)=0 \text { or } u(x)=1\} .
$$

Then $M_{u}$ and $N_{u}$ are closed sets, $M_{u}$ is a union of a finite number of points or intervals,

$$
N_{u}=\left(-\infty, x_{0}\right] \cup\left[x_{1},+\infty\right),
$$

where $-\infty \leq x_{0}<x_{1} \leq+\infty$ and we use the convention $\left(-\infty, x_{0}\right]=\varnothing$ if $x_{0}=-\infty$ and $\left[x_{1},+\infty\right)=\varnothing$ if $x_{1}=+\infty$.

Definition 2.1. A monotone continuous function $u: \mathbb{R} \rightarrow[0,1]$ is a solution of equation (2.1) if
(a) For any $x \notin M_{u} \cup N_{u}$ there exists finite derivative $u^{\prime}(x)$ and for any $x \in \operatorname{int} M_{u} \cup \operatorname{int} N_{u}$ we have $u^{\prime}(x)=0$.
(b) For any $x \in \partial M_{u}$ there exist finite one sided derivatives $u^{\prime}(x-), u^{\prime}(x+)$ and

$$
L(x):=\left|u^{\prime}(x-)\right|^{p-2} u^{\prime}(x-) \lim _{y \rightarrow x-} d(u(y))=\left|u^{\prime}(x+)\right|^{p-2} u^{\prime}(x+) \lim _{y \rightarrow x+} d(u(y)) .
$$

(c) Function $v: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$
v(x):= \begin{cases}d(u(x))\left|u^{\prime}(x)\right|^{p-2} u^{\prime}(x), & x \notin M_{u} \cup N_{u}, \\ 0, & x \in N_{u} \cup \operatorname{int} M_{u}, \\ L(x), & x \in \partial M_{u}\end{cases}
$$

is continuous and for any $x, y \in \mathbb{R}$

$$
\begin{equation*}
v(y)-v(x)+\int_{x}^{y} g(u(\xi)) \mathrm{d} \xi=0 . \tag{2.2}
\end{equation*}
$$

Moreover, $\lim _{x \rightarrow \pm \infty} v(x)=0$ if either $\lim _{x \rightarrow-\infty} u(x)=0$ and $\lim _{x \rightarrow+\infty} u(x)=1$ or $\lim _{x \rightarrow-\infty} u(x)=1$ and $\lim _{x \rightarrow+\infty} u(x)=0$.
Remark 2.2. Constant functions

$$
u_{0}(x)=0, \quad u_{*}(x)=s_{*}, \quad u_{1}(x)=1, \quad x \in \mathbb{R},
$$

are solutions of (2.1). It follows from the properties of $d$ and $g$ that those are the only constant solutions of (2.1) and they are called equilibria.
Remark 2.3. If we set $y=x+h, h \neq 0$ in (2.2), multiply both sides of (2.2) by $\frac{1}{h}$ and pass to the limit for $h \rightarrow 0$, we obtain that $v$ is continuously differentiable and the equation

$$
\begin{equation*}
v^{\prime}(x)+g(u(x))=0 \tag{2.3}
\end{equation*}
$$

holds for all $x \in \mathbb{R}$.
Remark 2.4. Let $u$ be a solution of (2.1) in the sense of Definition 2.1. If $M_{u} \neq \varnothing$, i.e., $d$ is not continuous in ( 0,1 ), then $M_{u}=\partial M_{u}$, int $M_{u}=\varnothing$ unless $s_{i}=s_{*}$ for some $i=1,2, \ldots, n$. In this case $u$ can be constant on some interval $(a, b),-\infty \leq a<b \leq+\infty$, and equal to $s_{*}$. The equation (2.1) would then be satisfied pointwise for all $x \in(a, b)$ and ( $a, b) \subset \operatorname{int} M_{u}$. Furthermore, it follows from the continuity of $v$ that if $a>-\infty$ or $b<+\infty$ we have $u^{\prime}(a)=$ $u^{\prime}(b)=0$ because $d\left(s_{*}\right)>0$. Also note that for $x \in \partial N_{u}$ one sided derivatives $u^{\prime}(x-), u^{\prime}(x+)$ exist but one of them can be infinite.

If $u$ is strictly monotone between 0 and 1 then $M_{u}=\left\{\xi_{1}, \xi_{2}, \ldots, \xi_{n}\right\}$ where $u\left(\xi_{i}\right)=s_{i}$, $i=1,2, \ldots, n$.

Remark 2.5. Let $p=2, d \equiv 1$ and $g \in C^{1}[0,1]$. Let $u=u(x)$ be a solution in the sense of Definition 2.1. Then $M_{u}=\varnothing$ if $u$ is not a constant, $N_{u}=\varnothing$, and (2.1) holds pointwise, i.e., $u \in C^{2}(\mathbb{R})$ and it is a classical solution, cf. [1], [2] or [6].

## 3 Existence results

We are concerned with the existence of solutions of the equation (2.1) which satisfy the "boundary conditions"

$$
\begin{equation*}
\lim _{x \rightarrow-\infty} u(x)=0 \text { and } \lim _{x \rightarrow+\infty} u(x)=1 \tag{3.1}
\end{equation*}
$$

Remark 3.1. Let $u$ be a solution of the BVP (2.1), (3.1). Passing to the limit for $x \rightarrow-\infty$ in (2.2) and writing $x$ in place of $y$, we derive that for arbitrary $x \in \mathbb{R}$ we have

$$
\begin{equation*}
v(x)+\int_{-\infty}^{x} g(u(\xi)) \mathrm{d} \xi=0 . \tag{3.2}
\end{equation*}
$$

Theorem 3.2. Let $d$ and $g$ be as in Section 2 and recall that $p>1$. Then the BVP (2.1), (3.1) has a nondecreasing solution if and only if

$$
\begin{equation*}
\int_{0}^{1}(d(s))^{\frac{1}{p-1}} g(s) \mathrm{d} s=0 \tag{3.3}
\end{equation*}
$$

If (3.3) holds then there is a unique solution $u=u(x)$ of (2.1), (3.1) such that the following conditions hold (see Figure 3.1):
(i) there exist $-\infty \leq x_{0}<0<x_{1} \leq+\infty$ such that $u(x)=0$ for $x \leq x_{0}, u(x)=1$ for $x \geq x_{1}$ and $0<u(x)<1$ for $x \in\left(x_{0}, x_{1}\right)$;
(ii) $u$ is strictly increasing in $\left(x_{0}, x_{1}\right), u(0)=s_{*}$;
(iii) for $i=1,2, \ldots, n$ let $\xi_{i} \in \mathbb{R}$ be such that $u\left(\xi_{i}\right)=s_{i}, \xi_{0}=x_{0}$ and $\xi_{n+1}=x_{1}$. Then $u$ is $a$ piecewise $C^{1}$-function in the sense that $u$ is continuous,

$$
\left.u\right|_{\left(\xi_{i}, \xi_{i+1}\right)} \in C^{1}\left(\xi_{i}, \xi_{i+1}\right), \quad i=0,1, \ldots, n,
$$

and the limits $u^{\prime}\left(\xi_{i}-\right):=\lim _{x \rightarrow \tilde{\zeta}_{i}-} u^{\prime}(x), u^{\prime}\left(\xi_{i}+\right):=\lim _{x \rightarrow \xi_{i}+} u^{\prime}(x)$ exist finite for all $i=$ $1,2, \ldots, n$;
(iv) for any $i=1,2, \ldots, n$, the following transition condition holds:

$$
\left(u^{\prime}\left(\xi_{i}-\right)\right)^{p-1} \lim _{s \rightarrow s_{i}-} d(s)=\left(u^{\prime}\left(\xi_{i}+\right)\right)^{p-1} \lim _{s \rightarrow s_{i}+} d(s) .
$$



Figure 3.1: Increasing solutions

Proof. Necessity of (3.3). Let $u=u(x)$ be a nondecreasing solution of the BVP (2.1), (3.1) such that $u(0)=s_{*}$. Since the equation is autonomous this condition is just a normalization of a solution. It follows from (3.1) that

$$
-\infty \leq x_{0}:=\inf \{x \in \mathbb{R}: u(x)>0\}<0
$$

is well defined. By (3.2) and continuity of $v$ we have

$$
0<x_{1}:=\sup \left\{x \in \mathbb{R}: v(y)>0 \text { for all } y \in\left(x_{0}, x\right)\right\} \leq+\infty .
$$

Since $d(s)>0, s \in(0,1)$, it follows from the definition of $v(x)$ that $u$ is a strictly increasing function in $\left(x_{0}, x_{1}\right)$ and therefore the following limit

$$
\bar{u}\left(x_{1}\right):=\lim _{x \rightarrow x_{1}-} u(x)
$$

is well defined. If $x_{1}=+\infty$ then by the second condition in (3.1) it must be $\bar{u}\left(x_{1}\right)=1$. On the other hand, if $x_{1}<+\infty$, we have $\bar{u}\left(x_{1}\right)=u\left(x_{1}\right), v\left(x_{1}\right)=0$ and $s_{*}<u\left(x_{1}\right) \leq 1$. We rule out the case $u\left(x_{1}\right)<1$. Indeed, $v\left(x_{1}\right)=0$ implies $u^{\prime}\left(x_{1}-\right)=u^{\prime}\left(x_{1}+\right)=u^{\prime}\left(x_{1}\right)=0$. From $s_{*}<u\left(x_{1}\right)$ and (2.3) we deduce $v^{\prime}\left(x_{1}\right)=-g\left(u\left(x_{1}\right)\right)<0$. Therefore, there exists $\delta>0$ such that for all $x \in\left(x_{1}, x_{1}+\delta\right)$ we have $v(x)<0$ and hence also $u^{\prime}(x-)<0$ and $u^{\prime}(x+)<0$. This contradicts our assumption that $u$ is nondecreasing.

We proved that $u\left(x_{1}\right)=1$, i.e., $u=u(x)$ is strictly increasing and maps $\left(x_{0}, x_{1}\right)$ onto $(0,1)$. Let $\xi_{i} \in\left(x_{0}, x_{1}\right)$ be such that

$$
u\left(\xi_{i}\right)=s_{i}, \quad i=1,2, \ldots, n, \quad \xi_{0}=x_{0}, \quad \xi_{n+1}=x_{1} .
$$

Then $u$ is continuous in $\left(x_{0}, x_{1}\right)$ and piecewise $C^{1}$-function in the sense that

$$
\left.u\right|_{\left(\xi_{i}, \xi_{i+1}\right)} \in C^{1}\left(\xi_{i}, \xi_{i+1}\right), \quad u^{\prime}(x)>0, \quad x \in\left(\xi_{i}, \xi_{i+1}\right), \quad i=0,1, \ldots, n,
$$

and the limits $\lim _{x \rightarrow \xi_{i}-} u^{\prime}(x), \lim _{x \rightarrow \xi_{i}+} u^{\prime}(x), i=1,2, \ldots, n$, exist finite. Hence there exists continuous strictly increasing inverse function $u^{-1}:(0,1) \rightarrow\left(x_{0}, x_{1}\right), x=u^{-1}(u)$, such that

$$
\left.u^{-1}\right|_{\left(s_{i}, s_{i+1}\right)} \in C^{1}\left(s_{i}, s_{i+1}\right), \quad i=0,1, \ldots, n,
$$

and the limits

$$
\lim _{u \rightarrow s_{i}-} \frac{\mathrm{d}}{\mathrm{~d} u} u^{-1}(u), \quad \lim _{u \rightarrow s_{i}+} \frac{\mathrm{d}}{\mathrm{~d} u} u^{-1}(u)
$$

exist finite, $i=1,2, \ldots, n$. We employ the change of variables as indicated in [5, p. 174]. Set

$$
w(u)=v\left(u^{-1}(u)\right), \quad u \in(0,1) .
$$

Then $w$ is piecewise $C^{1}$-function in $(0,1)$,

$$
\left.w\right|_{\left(s_{i}, s_{i+1}\right)} \in C^{1}\left(s_{i}, s_{i+1}\right), \quad i=0,1, \ldots, n
$$

with finite limits $\lim _{u \rightarrow s_{i}-} w^{\prime}(u), \lim _{u \rightarrow s_{i}+} w^{\prime}(u), i=1,2, \ldots, n$. For any $x \in\left(\xi_{i}, \xi_{i+1}\right)$ and $u \in\left(s_{i}, s_{i+1}\right), i=0,1, \ldots, n$,

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} x} v(x)=\frac{\mathrm{d}}{\mathrm{~d} x} w(u(x))=\frac{\mathrm{d} w}{\mathrm{~d} u}(u(x)) u^{\prime}(x) . \tag{3.4}
\end{equation*}
$$

From $v(x)=d(u(x))\left(u^{\prime}(x)\right)^{p-1}$ we deduce

$$
\begin{equation*}
u^{\prime}(x)=\left(\frac{v(x)}{d(u(x))}\right)^{p^{\prime}-1}, \quad p^{\prime}=\frac{p}{p-1} . \tag{3.5}
\end{equation*}
$$

It follows from (3.4), (3.5) that

$$
\frac{\mathrm{d} v}{\mathrm{~d} x}=\frac{\mathrm{d} w}{\mathrm{~d} u}(u(x))\left(\frac{v(x)}{d(u(x))}\right)^{p^{\prime}-1}=\frac{\mathrm{d} w}{\mathrm{~d} u}(u)\left(\frac{w(u)}{d(u)}\right)^{p^{\prime}-1} .
$$

Therefore, the equation

$$
v^{\prime}(x)+g(u(x))=0, \quad x \in\left(\xi_{i}, \xi_{i+1}\right),
$$

transforms to

$$
\frac{\mathrm{d} w}{\mathrm{~d} u}\left(\frac{w(u)}{d(u)}\right)^{p^{\prime}-1}+g(u)=0, \quad u \in\left(s_{i}, s_{i+1}\right)
$$

$i=0,1, \ldots, n$, or equivalently,

$$
\begin{align*}
& (w(u))^{p^{\prime}-1} \frac{\mathrm{~d} w}{\mathrm{~d} u}+(d(u))^{p^{\prime}-1} g(u)=0,  \tag{3.6}\\
& \frac{1}{p^{\prime}} \frac{\mathrm{d}}{\mathrm{~d} u}(w(u))^{p^{\prime}}+(d(u))^{p^{\prime}-1} g(u)=0 . \tag{3.7}
\end{align*}
$$

The last equality holds in $(0,1)$ except the points $s_{1}, s_{2}, \ldots, s_{n}$ and $w$ is continuous in $(0,1)$. Set

$$
\begin{gathered}
f(s):=-(d(s))^{\frac{1}{p-1}} g(s), \quad s \in(0,1), \\
F(s):=\int_{0}^{s} f(\sigma) \mathrm{d} \sigma .
\end{gathered}
$$

Integrating (3.7) over the interval $(0, u)$ we arrive at

$$
(w(u))^{p^{\prime}}=p^{\prime} F(u)+(w(0+))^{p^{\prime}}, \quad u \in(0,1) .
$$

Clearly, $F(0)=0$, and

$$
\begin{equation*}
\lim _{u \rightarrow 0+} w(u)=\lim _{x \rightarrow x_{0}+} v(x)=0 \tag{3.8}
\end{equation*}
$$

by the definition of a solution. Therefore we have

$$
\begin{equation*}
w(u)=\left(p^{\prime} F(u)\right)^{\frac{1}{p^{\prime}}}, \quad u \in(0,1) . \tag{3.9}
\end{equation*}
$$

By the definition of a solution we must also have

$$
\begin{equation*}
\lim _{u \rightarrow 1-} w(u)=\lim _{x \rightarrow x_{1}-} v(x)=0 . \tag{3.10}
\end{equation*}
$$

But (3.9) and (3.10) imply $F(1)=0$, i.e., (3.3) must hold. Therefore, (3.3) is a necessary condition.

Sufficiency of (3.3). Let (3.3) hold. Then $w=w(u)$ given by (3.9) satisfies (3.6)-(3.10) above. For $u \in(0,1)$ set

$$
x(u)=\int_{s_{*}}^{u}\left(\frac{d(s)}{w(s)}\right)^{\frac{1}{p-1}} \mathrm{~d} s .
$$

The function $x=x(u)$ is strictly increasing and maps the interval $(0,1)$ onto $\left(x_{0}, x_{1}\right)$ where $-\infty \leq x_{0}<0<x_{1} \leq+\infty$. Let $u:\left(x_{0}, x_{1}\right) \rightarrow(0,1)$ be an inverse function. Then $u(0)=s_{*}, u$ is strictly increasing and

$$
\lim _{x \rightarrow x_{0}^{+}} u(x)=0, \quad \lim _{x \rightarrow x_{1}-} u(x)=1 .
$$

Let $x \in\left(\xi_{i}, \xi_{i+1}\right), i=0,1, \ldots, n$, where $u\left(\xi_{i}\right)=s_{i}, i=0,1, \ldots, n+1$. Then

$$
\frac{\mathrm{d} u(x)}{\mathrm{d} x}=\frac{1}{\frac{\mathrm{~d} x(u)}{\mathrm{d} u}}=\left(\frac{w(u(x))}{d(u(x))}\right)^{\frac{1}{p-1}}, \quad u(x) \in\left(s_{i}, s_{i+1}\right),
$$

i.e., $u \in C^{1}\left(\mathfrak{\xi}_{i}, \xi_{i+1}\right), u^{\prime}(x)>0$ and

$$
\begin{align*}
d(u(x))\left(\frac{\mathrm{d} u(x)}{\mathrm{d} x}\right)^{p-1} & =w(u(x))=: v(x)  \tag{3.11}\\
\frac{\mathrm{d}}{\mathrm{~d} x}\left[d(u(x))\left(\frac{\mathrm{d} u(x)}{\mathrm{d} x}\right)^{p-1}\right] & =\frac{\mathrm{d}}{\mathrm{~d} x} w(u(x))=\frac{\mathrm{d} w}{\mathrm{~d} u} \frac{\mathrm{~d} u(x)}{\mathrm{d} x} . \tag{3.12}
\end{align*}
$$

From (3.6), (3.11) we deduce

$$
\begin{aligned}
\frac{\mathrm{d} w}{\mathrm{~d} u} & =-(w(u))^{-\left(p^{\prime}-1\right)}(d(u))^{p^{\prime}-1} g(u) \\
& =-(d(u(x)))^{-\left(p^{\prime}-1\right)}\left(\frac{\mathrm{d} u(x)}{\mathrm{d} x}\right)^{-(p-1)\left(p^{\prime}-1\right)}(d(u(x)))^{p^{\prime}-1} g(u(x)) \\
& =-\left(\frac{\mathrm{d} u(x)}{\mathrm{d} x}\right)^{-1} g(u(x)) .
\end{aligned}
$$

Substituting this to (3.12), we get

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left[d(u(x))\left(\frac{\mathrm{d} u(x)}{\mathrm{d} x}\right)^{p-1}\right]=-g(u(x)), \quad x \in\left(\xi_{i}, \xi_{i+1}\right) .
$$

It follows from (3.8), (3.10) and (3.11) that

$$
\lim _{x \rightarrow x_{0}+} d(u(x))\left(\frac{\mathrm{d} u(x)}{\mathrm{d} x}\right)^{p-1}=\lim _{x \rightarrow x_{1}-} d(u(x))\left(\frac{\mathrm{d} u(x)}{\mathrm{d} x}\right)^{p-1}=0
$$

and the following one-sided limits are finite

$$
\begin{equation*}
\lim _{x \rightarrow \xi_{i}-} d(u(x))\left(\frac{\mathrm{d} u(x)}{\mathrm{d} x}\right)^{p-1}=\lim _{x \rightarrow \tilde{\zeta}_{i}+} d(u(x))\left(\frac{\mathrm{d} u(x)}{\mathrm{d} x}\right)^{p-1} \tag{3.13}
\end{equation*}
$$

$i=1,2, \ldots, n$. Since $u=u(x)$ is monotone increasing function, we have

$$
\begin{equation*}
\lim _{x \rightarrow \tilde{\xi}_{i}-} d(u(x))=\lim _{s \rightarrow s_{i}-} d(s) \quad \text { and } \quad \lim _{x \rightarrow \tilde{\xi}_{i}+} d(u(x))=\lim _{s \rightarrow s_{i}+} d(s) . \tag{3.14}
\end{equation*}
$$

Transition condition (iv) now follows from (3.13), (3.14).
Therefore, if for $x_{0}>-\infty$ we set $u(x)=0, x \in\left(-\infty, x_{0}\right]$ and for $x_{1}<+\infty$ we set $u(x)=$ $1, x \in\left[x_{1},+\infty\right)$, then $u=u(x), x \in \mathbb{R}$, is a nondecreasing solution of the BVP (2.1), (3.1) and it has the properties listed in the statement of Theorem 3.2. This proves the sufficiency of (3.3).

Remark 3.3. The condition (3.3) substitutes the balanced bistable nonlinearity condition (1.2) in case of density dependent diffusion. It follows from Theorem 3.2 that it is not only the reaction term but rather mutual interaction between the density dependent diffusion coefficient and reaction which decides about the existence and/or nonexistence of nonconstant stationary solutions of the generalized version of the bistable equation (1.6).

Remark 3.4. Let us replace the boundary conditions (3.1) by "opposite" ones:

$$
\begin{equation*}
\lim _{x \rightarrow-\infty} u(x)=1 \text { and } \lim _{x \rightarrow+\infty} u(x)=0 \tag{3.15}
\end{equation*}
$$

If $u$ is a solution of the BVP (2.1), (3.15) then passing to the limit for $y \rightarrow+\infty$ in (2.2) we arrive at

$$
\begin{equation*}
v(x)-\int_{x}^{+\infty} g(u(\xi)) \mathrm{d} \xi=0 \tag{3.16}
\end{equation*}
$$

for arbitrary $x \in \mathbb{R}$. Modifying the proof of Theorem 3.2 and using (3.16) instead of (3.2), we show that (3.3) is a necessary and sufficient condition for the existence of nonincreasing solution of the BVP (2.1), (3.15). If (3.3) holds then there is a unique solution $u=u(x)$ of (2.1), (3.15) satisfying analogue of (i)-(iv). In particular, it is strictly decreasing in $\left(x_{0}, x_{1}\right), u(x)=1$ for $x \in\left(-\infty, x_{0}\right]$ if $x_{0}>-\infty$ and $u(x)=0$ for $x \in\left[x_{1},+\infty\right)$ if $x_{1}<+\infty$, see Figure 3.2.


Figure 3.2: Decreasing solutions

Remark 3.5. It follows from the proof of Theorem 3.2 that

$$
\begin{align*}
& x_{0}=x(0)=\int_{s_{*}}^{0}\left(\frac{d(s)}{w(s)}\right)^{\frac{1}{p-1}} \mathrm{~d} s=\left(\frac{1}{p^{\prime}}\right)^{\frac{1}{p}} \int_{s_{*}}^{0} \frac{(d(s))^{\frac{1}{p-1}}}{-\int_{0}^{s}(d(\sigma))^{\frac{1}{p-1}} g(\sigma) \mathrm{d} \sigma} \mathrm{~d} s,  \tag{3.17}\\
& x_{1}=x(1)=\int_{s_{*}}^{1}\left(\frac{d(s)}{w(s)}\right)^{\frac{1}{p-1}} \mathrm{~d} s=\left(\frac{1}{p^{\prime}}\right)^{\frac{1}{p}} \int_{s_{*}}^{1} \frac{(d(s))^{\frac{1}{p-1}}}{-\int_{0}^{s}(d(\sigma))^{\frac{1}{p-1}} g(\sigma) \mathrm{d} \sigma} \mathrm{~d} s . \tag{3.18}
\end{align*}
$$

Therefore, the fact that $x_{0}$ and $x_{1}$ are finite or infinite depends on the asymptotic behavior of the diffusion coefficient $d=d(s)$ and reaction term $g=g(s)$ near the equilibria 0 and 1 . The detailed discussion of different configurations between $d$ and $g$ which lead to $x_{0}$ and/or $x_{1}$ finite or infinite is presented in the next section.

Remark 3.6. Since the equation (2.1) is autonomous, if $u=u(x)$ is a solution to (2.1), (3.1) then given any $\tilde{\xi} \in \mathbb{R}$ fixed, $\tilde{u}=\tilde{u}(x):=u(x-\xi)$ is also a solution of (2.1), (3.1). Of course, if $x_{0}$ and/or $x_{1}$ are finite, then corresponding $\tilde{x}_{0}$ and $\tilde{x}_{1}$ associated with $\tilde{u}$ satisfy $\tilde{x}_{0}=x_{0}+\xi$ and $\tilde{x}_{1}=x_{1}+\xi$. Obviously, the same applies to (2.1), (3.15). If $x_{0}=-\infty$ and $x_{1}=+\infty$ and (3.3) holds, all possible solutions of (2.1), (3.1) are strictly increasing in $(-\infty,+\infty)$ and satisfy (i)-(iv) of Theorem 3.2, where $u(0)=s_{*}$ is replaced by $u(\xi)=s_{*}, \xi \in \mathbb{R}$. On the
other hand, if $x_{0} \in \mathbb{R}$ and/or $x_{1} \in \mathbb{R}$, then the set of possible solutions of (2.1), (3.1) is much richer than in the previous case. Indeed, we have plenty of possibilities how to define also a nonmonotone solution of (2.1), (3.1) (or (2.1), (3.15)). For example, if both $x_{0}$ and $x_{1}$ associated with strictly increasing solution $u=u(x)$ from Theorem 3.2 are finite then the same holds for corresponding $\hat{x}_{0}$ and $\hat{x}_{1}$ associated with the strictly decreasing solution from Remark 3.4. Having in mind the translation invariance of solutions mentioned above, we may choose $u_{1}$ and $\hat{u}$ such that $x_{1}<\hat{x}_{0}$. If we define $u(x)=0, x \in\left(-\infty, x_{0}\right], u(x)=u_{1}(x), x \in\left(x_{0}, x_{1}\right)$, $u(x)=1, x \in\left[x_{1}, \hat{x}_{0}\right], u(x)=\hat{u}(x), x \in\left(\hat{x}_{0}, \hat{x}_{1}\right), u(x)=0, x \in\left[\hat{x}_{1},+\infty\right)$, we get solution of (2.1) satisfying the boundary conditions

$$
\begin{equation*}
\lim _{x \rightarrow-\infty} u(x)=\lim _{x \rightarrow+\infty} u(x)=0 . \tag{3.19}
\end{equation*}
$$

Now, if $\tilde{u}_{1}=\tilde{u}_{1}(x)$ is a translation of $u_{1}$ such that $\tilde{x}_{0}>\hat{x}_{1}$, we can extend the previous function $u$ as $u(x)=0, x \in\left[\hat{x}_{1}, \tilde{x}_{0}\right], u(x)=\tilde{u}(x), x \in\left(\tilde{x}_{0}, \tilde{x}_{1}\right), u(x)=1, x \in\left[\tilde{x}_{1},+\infty\right)$ to get a nonmonotone solution of (2.1), (3.1), see Figure 3.3. It is obvious that by suitably modifying the above construction we may construct continuum of solutions not only of (2.1), (3.1) but also of (2.1), (3.19). Of course, the same approach leads to the continuum of solutions of (2.1), (3.15) and of (2.1), (3.20), respectively, where

$$
\begin{equation*}
\lim _{x \rightarrow-\infty} u(x)=\lim _{x \rightarrow+\infty} u(x)=1 . \tag{3.20}
\end{equation*}
$$



Figure 3.3: Nonmonotone solutions

## 4 Qualitative properties of solutions

In this section we study the qualitative properties of the solutions from Theorem 3.2. In particular, we focus on two issues. Our primary concern is to provide detailed classification of the asymptotic behavior of the stationary solution $u=u(x)$ as $x \rightarrow-\infty$ and $x \rightarrow+\infty$ and to show how it is affected by the behavior of the diffusion coefficient $d$ and reaction $g$ near the equilibria 0 and 1 . However, we also want to study the impact of the discontinuity of $d=d(s)$ on the lack of smoothness of the solution $u=u(x)$. The role of the transition condition at the points where $u$ assumes values where the discontinuity of $d$ occurs will be illustrated.

In order to simplify the expressions arising throughout this section we will use the following notation:

$$
h_{1}(t) \sim h_{2}(t) \text { as } t \rightarrow t_{0} \quad \text { if and only if } \quad \lim _{t \rightarrow t_{0}} \frac{h_{1}(t)}{h_{2}(t)} \in(0,+\infty) .
$$

We start with the asymptotic analysis of

$$
x(u)=x\left(s_{*}\right)+\left(\frac{1}{p^{\prime}}\right)^{\frac{1}{p}} \int_{s_{*}}^{u} \frac{(d(s))^{\frac{1}{p-1}}}{\left(-\int_{0}^{s}(d(\sigma))^{\frac{1}{p-1}} g(\sigma) \mathrm{d} \sigma\right)^{\frac{1}{p}}} \mathrm{~d} s
$$

for $u \rightarrow 0+$. Let us assume that $g(s) \sim-s^{\alpha}, d(s) \sim s^{\beta}$ as $s \rightarrow 0+$ for some $\alpha>0, \beta \in \mathbb{R}$. Then formally we get

$$
-\int_{0}^{s}(d(\sigma))^{\frac{1}{p-1}} g(\sigma) \mathrm{d} \sigma \sim \int_{0}^{s} \sigma^{\alpha+\frac{\beta}{p-1}} \mathrm{~d} \sigma \sim s^{\alpha+\frac{\beta}{p-1}+1} \quad \text { as } \quad s \rightarrow 0+.
$$

Since we assume that $s \mapsto(d(s))^{\frac{1}{p-1}} g(s)$ is integrable in $(0,1)$, we have to assume

$$
\begin{equation*}
\alpha+\frac{\beta}{p-1}>-1 \tag{4.1}
\end{equation*}
$$

Then for $u \rightarrow 0+$ we can write

$$
\begin{equation*}
x(u) \sim \int_{s_{*}}^{u} \frac{(d(s))^{\frac{1}{p-1}}}{\left(-\int_{0}^{s}(d(\sigma))^{\frac{1}{p-1}} g(\sigma) \mathrm{d} \sigma\right)^{\frac{1}{p}}} \mathrm{~d} s \sim \int_{s_{*}}^{u} s^{\frac{\beta}{p-1}-\frac{\alpha}{p}-\frac{\beta}{p(p-1)}-\frac{1}{p}} \mathrm{~d} s=\int_{s_{*}}^{u} s^{\frac{\beta-\alpha-1}{p}} \mathrm{~d} s . \tag{4.2}
\end{equation*}
$$

Convergence or divergence of the integral

$$
I:=\int_{0}^{s_{*}} s^{\frac{\beta-\alpha-1}{p}} \mathrm{~d} s
$$

leads to the following primary distinction between two qualitatively different cases:
Case 1: $I=+\infty$ if $\alpha-\beta \geq p-1$,
Case 2: $I<+\infty$ if $\alpha-\beta<p-1$.
Case 1. Let $\alpha-\beta=p-1$. Then (4.2) implies that $x(u) \sim \ln u$ as $u \rightarrow 0+$ and performing the change of variables yields the asymptotics for $u=u(x)$ :

$$
u(x) \sim \mathrm{e}^{x} \rightarrow 0+\text { for } x \rightarrow-\infty
$$

For $\alpha-\beta>p-1$ we have by (4.2) that $x(u) \sim-u^{\frac{\beta-\alpha-1}{p}+1}=-u^{\frac{p-1-(\alpha-\beta)}{p}} \rightarrow-\infty$ as $u \rightarrow 0+$ and applying the inverse function we obtain

$$
u(x) \sim|x|^{\frac{p}{p-1-(\alpha-\beta)}} \rightarrow 0+\quad \text { for } x \rightarrow-\infty
$$

In both cases $x_{0}$ defined by (3.17) is equal to $-\infty$ and solution $u=u(x)$ approaches zero at either an exponential or power rate.

Remark 4.1. It is interesting to observe that $x_{0}=-\infty$ occurs even in the case when the diffusion coefficient degenerates or has a singularity if this fact is compensated by a proper degeneration of the reaction term $g$.

Possible values of parameters $\alpha, \beta$ for which Case 1 occurs for different values of $p$ are shown in Figures 4.1, 4.2, 4.3 where condition (4.1) is taken into account.
Case 2. Let $\alpha-\beta<p-1$. Then $I<+\infty$ and hence from (3.17) we deduce $x(0)=x_{0}>-\infty$. Moreover,

$$
I \rightarrow+\infty \quad \text { as } \quad \frac{\beta-\alpha-1}{p} \rightarrow-1+
$$

i.e., we have $x_{0} \rightarrow-\infty$ as $p-1-(\alpha-\beta) \rightarrow 0+$. More precisely, for $u \rightarrow 0+$ we have

$$
x(u)-x_{0} \sim u^{\frac{\beta-\alpha-1}{p}+1}=u^{\frac{p-1-(\alpha-\beta)}{p}} .
$$



Figure 4.1: $p=\frac{3}{2}$


Figure 4.2: $p=2$


Figure 4.3: $p=3$

Depending on the shape of $x(u)$ we further distinguish among three cases:
a) $\left.\frac{\mathrm{d} x}{\mathrm{~d} u}\right|_{u=0+} \sim u^{\frac{\beta-\alpha-1}{p}} \rightarrow+\infty \quad$ for $u \rightarrow 0+\quad$ if $\alpha-\beta>-1$,
b) $\left.\frac{\mathrm{d} x}{\mathrm{~d} u}\right|_{u=0+} \sim u^{0} \rightarrow k>0 \quad$ for $u \rightarrow 0+\quad$ if $\alpha-\beta=-1$,
c) $\left.\frac{\mathrm{d} x}{\mathrm{~d} u}\right|_{u=0+} \sim u^{\frac{\beta-\alpha-1}{p}} \rightarrow 0_{+} \quad$ for $u \rightarrow 0+\quad$ if $\alpha-\beta<-1$.

An inverse point of view gives us the asymptotics of $u=u(x)$ for $x \rightarrow x_{0}$ :

$$
u(x) \sim\left(x-x_{0}\right)^{\frac{p}{p-1-(\alpha-\beta)}} .
$$

As for the derivatives, we have
a) $\left.\frac{\mathrm{d} u}{\mathrm{~d} x}\right|_{x=x_{0}+} \sim\left(x-x_{0}\right)^{\frac{\alpha-\beta+1}{p-1-(\alpha-\beta)}} \rightarrow 0 \quad$ for $x \rightarrow x_{0}+\quad$ if $\alpha-\beta>-1$,
b) $\left.\frac{\mathrm{d} u}{\mathrm{~d} x}\right|_{x=x_{0}+} \sim\left(x-x_{0}\right)^{0} \rightarrow k>0 \quad$ for $x \rightarrow x_{0}+\quad$ if $\alpha-\beta=-1$,
c) $\left.\quad \frac{\mathrm{d} u}{\mathrm{~d} x}\right|_{x=x_{0}+} \sim\left(x-x_{0}\right)^{\frac{\alpha-\beta+1}{p-1-(\alpha-\beta)}} \rightarrow+\infty \quad$ for $x \rightarrow x_{0}+\quad$ if $\alpha-\beta<-1$.

Remark 4.2. We observe that only in case a) the solution $u=u(x)$ is smooth in the neighborhood of $x_{0}$ since $u(x)=0$ for $x \in\left(-\infty, x_{0}\right]$. In the other two cases we only get continuous solutions instead of smooth ones as a consequence of allowing for the diffusion term $d=d(s)$ to degenerate as $s \rightarrow 0+$. The asymptotic behavior of such solutions near the point $x_{0}$ is illustrated in Figures 4.4, 4.5, 4.6.


Figure 4.4: Case a)


Figure 4.5: Case b)


Figure 4.6: Case c)

Values of $\alpha, \beta$ for which these cases occur are for different values of $p$ depicted in Figures 4.7, 4.8, 4.9. Areas corresponding to cases a) - c) are shown in respective colors as in Figures 4.4, 4.5, 4.6.

Proceeding similarly for $u \rightarrow 1$ - and assuming $g(s) \sim(1-s)^{\gamma}, d(s) \sim(1-s)^{\delta}$ as $s \rightarrow 1-$ for some $\gamma>0, \delta \in \mathbb{R}$ satisfying the analogue of condition (4.1):

$$
\gamma+\frac{\delta}{p-1}>-1
$$

we get the following asymptotics:
Case 1: $\gamma-\delta \geq p-1$. Then $x_{1}=+\infty$ by (3.18) and we distinguish between two cases. Either

$$
u(x) \sim 1-\mathrm{e}^{-x} \rightarrow 1-\text { for } x \rightarrow+\infty
$$



Figure 4.7: $p=\frac{3}{2}$


Figure 4.8: $p=2$


Figure 4.9: $p=3$
if $\gamma-\delta=p-1$, or else

$$
u(x) \sim 1-|x|^{\frac{p}{p-1-(\gamma-\delta)}} \rightarrow 1-
$$

if $\gamma-\delta>p-1$.
Case 2: $\gamma-\delta<p-1$. Then $x_{1}<+\infty$ by (3.18) and

$$
u(x) \sim 1-\left(x_{1}-x\right)^{\frac{p}{p-1-(\gamma-\beta)}} \rightarrow 1-\quad \text { for } x \rightarrow x_{1}-.
$$

As for the one-sided derivatives of $u$ at $x_{1}$ we have
a) $\left.\frac{\mathrm{d} u}{\mathrm{~d} x}\right|_{x=x_{1}-} \sim\left(x_{1}-x\right)^{\frac{\gamma-\delta+1}{p-1-(\gamma-\delta)}} \rightarrow 0 \quad$ for $x \rightarrow x_{1}-\quad$ if $\gamma-\delta>-1$,
b) $\left.\frac{\mathrm{d} u}{\mathrm{~d} x}\right|_{x=x_{1}-} \sim\left(x_{1}-x\right)^{0} \rightarrow k>0 \quad$ for $x \rightarrow x_{1}-\quad$ if $\gamma-\delta=-1$,
c) $\left.\frac{\mathrm{d} u}{\mathrm{~d} x}\right|_{x=x_{1}-} \sim\left(x_{1}-x\right)^{\frac{\gamma-\delta+1}{p-1-(\gamma-\delta)}} \rightarrow+\infty \quad$ for $x \rightarrow x_{1}-\quad$ if $\gamma-\delta<-1$.

Remark 4.3. While all the illustrative pictures in Section 3 do not reflect the effect of the discontinuity of $d$, finally, we want to focus on how the solution $u=u(x)$ is affected by
discontinuous diffusion coefficient $d=d(s)$. Let us assume for simplicity that $d$ only has one point of discontinuity $s_{1} \in(0,1)$ and it is smooth and bounded in $\left(0, s_{1}\right)$ and $\left(s_{1}, 1\right)$. Then $M_{u}=\left\{\xi_{1}\right\}$ and it follows from Theorem 3.2, (iv), that the jump of $d$ at $s_{1}$ must be compensated by the proper "opposite" jump of $u^{\prime}$ at $\xi_{1}$, see Figure 4.10, namely

$$
\left|u^{\prime}\left(\xi_{1}-\right)\right|^{p-2} u^{\prime}\left(\xi_{1}-\right) \lim _{s \rightarrow s_{1}-} d(s)=\left|u^{\prime}\left(\xi_{1}+\right)\right|^{p-2} u^{\prime}\left(\xi_{1}+\right) \lim _{s \rightarrow s_{1}+} d(s) .
$$



Figure 4.10: Profile of solution $u=u(x)$ for $d$ discontinuous at $s_{1}$

## 5 Final discussions

Let us consider the initial value problem for the quasilinear bistable equation

$$
\left\{\begin{array}{l}
\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(d(u(x, t))\left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x}\right)+g(u(x, t)), \quad x \in \mathbb{R}, t>0  \tag{5.1}\\
u(x, 0)=\varphi(x), \quad x \in \mathbb{R}
\end{array}\right.
$$

Here, $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $d$ and $g$ are as in Section 2 and (3.3) (balanced bistable condition) holds. If $\varphi=\varphi(x)$ satisfies the hypothesis

$$
\limsup _{x \rightarrow-\infty} \varphi(x)<s_{*} \text { and } \quad \liminf _{x \rightarrow+\infty} \varphi(x)>s_{*}
$$

then one would expect that there exists $\xi \in \mathbb{R}$ such that the solution $u=u(x, t)$ of (5.1) satisfies

$$
\lim _{t \rightarrow+\infty} u(x, t)=u(x-\xi), \quad x \in \mathbb{R},
$$

where $u=u(x)$ is a solution given by Theorem 3.2, see Figure 5.1.


Figure 5.1: Convergence to a stationary solution
It is maybe too ambitious to prove this fact if $d$ is a discontinuous function. However, an affirmative answer to this question, even for $d$ continuous or smooth, would be an interesting result. Even reliable numerical simulation of the asymptotic behavior of the solution $u=$ $u(x, t)$ of the initial value problem (5.1) for $t \rightarrow+\infty$ might be of great help.

## Acknowledgement

Michaela Zahradníková was supported by the project SGS-2019-010 of the University of West Bohemia in Pilsen.

## References

[1] D. G. Aronson, H. F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, in: Partial differential equations and related topics (Program, Tulane Univ., New Orleans, La., 1974), Springer, Berlin, 1975, pp. 5-49. https://doi.org/10.1007/BFb0070595; MR0427837
[2] D. G. Aronson, H. F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, Adv. in Math. 30(1978), No. 1, 33-76. https://doi.org/10.1016/ 0001-8708(78)90130-5; MR511740
[3] J. Carr, R. L. Pego, Metastable patterns in solutions of $u_{t}=\epsilon^{2} u_{x x}-f(u)$, Comm. Pure Appl. Math. 42(1989), No. 5, 523-576. https://doi.org/10.1002/cpa.3160420502; MR997567
[4] P. Drábek, New-type solutions for the modified Fischer-Kolmogorov equation, Abstr. Appl. Anal. (2011), 1-7. https://doi.org/10.1155/2011/247619; MR2802829
[5] R. Enguiça, A. Gavioli, L. Sanchez, A class of singular first order differential equations with applications in reaction-diffusion, Discrete Contin. Dyn. Syst. 33(2013), No. 1, 173191. https://doi.org/10.3934/dcds.2013.33.173; MR2972953
[6] P. C. Fife, J. B. McLeod, The approach of solutions of nonlinear diffusion equations to travelling front solutions, Arch. Rational Mech. Anal. 65(1977), No. 4, 335-361. https : //doi.org/10.1007/BF00250432; MR442480
[7] G. Fusco, J. K. Hale, Slow-motion manifolds, dormant instability, and singular perturbations, J. Dynam. Differential Equations 1(1989), No. 1, 75-94. https ://doi. org/10.1007/ BF01048791; MR1010961
[8] J. Nagumo, S. Yoshizawa, S. Аrimoto, Bistable transmission lines, IEEE Trans. Circ. Theor. 12(1965), No. 3, 400-412. https://doi.org/10.1109/TCT.1965.1082476
[9] D. E. Strier, D. H. Zanette, H. S. Wio, Wave fronts in a bistable reaction-diffusion system with density-dependent diffusivity, Physica A 226(1996), No. 3, 310-323. https: //doi.org/10.1016/0378-4371(95)00397-5
[10] A. I. Volpert, V. A. Volpert, V. A. Volpert, Traveling wave solutions of parabolic systems, Translations of Mathematical Monographs, Vol. 140, American Mathematical Society, Providence, RI, 1994. https ://doi.org/10.1090/mmono/140; MR1297766

# Existence and exact multiplicity of positive periodic solutions to forced non-autonomous Duffing type differential equations 

Jirí Šremr ${ }^{\boxtimes}$<br>Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology, Technická 2, 61669 Brno, Czech Republic

Received 31 March 2021, appeared 8 September 2021
Communicated by Alberto Cabada


#### Abstract

The paper studies the existence, exact multiplicity, and a structure of the set of positive solutions to the periodic problem


$$
u^{\prime \prime}=p(t) u+q(t, u) u+f(t) ; \quad u(0)=u(\omega), u^{\prime}(0)=u^{\prime}(\omega)
$$

where $p, f \in L([0, \omega])$ and $q:[0, \omega] \times \mathbb{R} \rightarrow \mathbb{R}$ is Carathéodory function. Obtained general results are applied to the forced non-autonomous Duffing equation

$$
u^{\prime \prime}=p(t) u+h(t)|u|^{\lambda} \operatorname{sgn} u+f(t)
$$

with $\lambda>1$ and a non-negative $h \in L([0, \omega])$. We allow the coefficient $p$ and the forcing term $f$ to change their signs.
Keywords: positive periodic solution, second-order differential equation, Duffing equation, existence, uniqueness, multiplicity.

2020 Mathematics Subject Classification: 34C25, 34B18.

## 1 Introduction

On an interval $[0, \omega]$, we consider the periodic problem

$$
\begin{align*}
& u^{\prime \prime}=p(t) u+q(t, u) u+f(t)  \tag{1.1}\\
& u(0)=u(\omega), u^{\prime}(0)=u^{\prime}(\omega) \tag{1.2}
\end{align*}
$$

where $p, f \in L([0, \omega])$ and $q:[0, \omega] \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function. By a solution to problem (1.1), (1.2), as usual, we understand a function $u:[0, \omega] \rightarrow \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies (1.1) almost everywhere, and meets periodic conditions (1.2). A periodic boundary value problem for differential equations of different types has been extensively studied in the literature. To make the list of references shorter,

[^24]the reader is referred to the well-known monographs $[2,3]$ for a historical background and an extensive list of relevant references.

In this paper, we study the existence and multiplicity of positive solutions to problem (1.1), (1.2). Since we are interested in a Duffing type equation, which is originally characterized by a super-linear non-linearity, we write a non-linear term in the form $q(t, u) u$. We continue our previous studies presented in [8], where problem (1.1), (1.2) with $f(t) \equiv 0$ is considered. We have shown, among other things, that, if the function $q$ is non-negative, then for the existence of a positive solution to (1.1), (1.2) with $f(t) \equiv 0$, it is necessary that $p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$ (see Definitions 2.2 and 2.3). Therefore, we restrict ourselves to the case of (1.1), in which the "linear part" satisfies $p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$.

A particular case of (1.1) is the non-autonomous Duffing equation

$$
\begin{equation*}
u^{\prime \prime}=p(t) u+h(t)|u|^{\lambda} \operatorname{sgn} u+f(t), \tag{1.3}
\end{equation*}
$$

with $p, h, f \in L([0, \omega])$ and $\lambda>1$, that is frequently studied in the literature (not only for ODEs), because arises in mathematical modelling in mechanics (mainly with $\lambda=3$ ). Such an equation (with constant coefficients $p, h$ ) is the central topic of the monograph [1] by Duffing published in 1918 and still bears his name (see also [5]). Let us show, as a motivation, what happens in the autonomous case. If $p(t):=-a$, then $p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$ if and only if $a>0$ (see Remark 2.4). Therefore, consider the equation

$$
\begin{equation*}
x^{\prime \prime}=-a x+b|x|^{\lambda} \operatorname{sgn} x+c, \tag{1.4}
\end{equation*}
$$

where $a>0$ and $b, c \in \mathbb{R}$. In this paper, we are interested in the equation (1.3) with a nonnegative $h$ and, thus, we assume that $b>0$ in (1.4). By direct calculation, the phase portraits of (1.4) can be elaborated depending on the choice of $c$, which leads to the following proposition.

Proposition 1.1. Let $\lambda>1$ and $a, b>0$. Then, the following conclusions hold:
(1) If $c \leq 0$, then equation (1.4) has a unique positive equilibrium (saddle) and no other positive periodic solutions occur.
(2) If $0<c<\frac{(\lambda-1) a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation (1.4) possesses exactly two positive equilibria $x_{1}>x_{2}$ ( $x_{1}$ is a saddle and $x_{2}$ is a center), a unique negative equilibrium $x_{3}$ (saddle), and non-constant (both positive and sign-changing) periodic solutions with different periods. Moreover, all non-constant periodic solutions are smaller then $x_{1}$ and oscillate around $x_{2}$.
(3) If $c=\frac{(\lambda-1) a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation (1.4) has a unique positive equilibrium (cusp), a unique negative equilibrium (saddle), and no other periodic solutions occur.
(4) If $c>\frac{(\lambda-1) a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation (1.4) has a unique negative equilibrium (saddle) and no other periodic solutions occur.

In [4], the authors study the stability and exact multiplicity of solutions to the periodic problem

$$
\begin{equation*}
x^{\prime \prime}+c x^{\prime}+a x-x^{3}=d(t) ; \quad u(0)=u(\omega), u^{\prime}(0)=u^{\prime}(\omega), \tag{1.5}
\end{equation*}
$$

where $c>0,0<a<\frac{\pi^{2}}{\omega^{2}}+\frac{c^{2}}{4}$, and $d:[0, \omega] \rightarrow \mathbb{R}$ is a positive continuous function. It follows from the proof of Theorem 1.1 in [4] that all the conclusions of Theorem 1.1 remain true, except of the asymptotic stability, even in the case of $c=0$. Therefore, [4, Theorem 1.1] yields

Proposition 1.2. Let $0<a<\frac{\pi^{2}}{\omega^{2}}$ and $d_{0}:=\frac{2 a}{3} \sqrt{\frac{a}{3}}$. Then, the following conclusions hold:
(1) Problem (1.5), with $c=0$, has a unique solution that is negative if $d(t)>d_{0}$ for $t \in[0, \omega]$.
(2) Problem (1.5), with $c=0$, has exactly three ordered solutions if $0<d(t)<d_{0}$ for $t \in[0, \omega]$. Moreover, the minimal solution is negative and the other two solutions are positive.

In Section 3, we generalize some conclusions of Propositions 1.1 and 1.2. We use a technique developed in [8] and determine a well-ordered pair of positive lower and upper functions, which allows us to establish general results guaranteeing the existence and exact multiplicity of positive solutions to (1.1), (1.2) as well as to provide some properties of the set of all positive solutions to (1.1), (1.2). The obtained results and their consequences for (1.3), (1.2) will be compared with the conclusions of Propositions 1.1 and 1.2 (see Remarks 3.18, 3.20, $3.21,3.23,3.28$, and 3.35).

It is worth mentioning that, in contrast to [4], our results cover also the case of a signchanging coefficient $p$ and a sign-changing forcing term $f$.

## 2 Notation and definitions

The following notation is used throughout the paper:
$-\mathbb{R}$ is the set of real numbers. For $x \in \mathbb{R}$, we put $[x]_{+}=\frac{1}{2}(|x|+x)$ and $[x]_{-}=\frac{1}{2}(|x|-x)$.

- $C(I)$ denotes the set of continuous real functions defined on the interval $I \subseteq \mathbb{R}$. For $u \in C([a, b])$, we put $\|u\|_{C}=\max \{|u(t)|: t \in[a, b]\}$.
- $A C^{1}([a, b])$ is the set of functions $u:[a, b] \rightarrow \mathbb{R}$ which are absolutely continuous together with their first derivatives.
$-A C_{\ell}([a, b])\left(\right.$ resp. $\left.A C_{u}([a, b])\right)$ is the set of absolutely continuous functions $u:[a, b] \rightarrow \mathbb{R}$ such that $u^{\prime}$ admits the representation $u^{\prime}(t)=\gamma(t)+\sigma(t)$ for a.e. $t \in[a, b]$, where $\gamma:[a, b] \rightarrow \mathbb{R}$ is absolutely continuous and $\sigma:[a, b] \rightarrow \mathbb{R}$ is a non-decreasing (resp. non-increasing) function whose derivative is equal to zero almost everywhere on $[a, b]$.
- $L([a, b])$ is the Banach space of Lebesgue integrable functions $p:[a, b] \rightarrow \mathbb{R}$ equipped with the norm $\|p\|_{L}=\int_{a}^{b}|p(s)| \mathrm{d} s$. The symbol Int $A$ stands for the interior of the set $A \subset L([a, b])$.

Definition 2.1. Let $I \subseteq \mathbb{R}$. A function $f:[a, b] \times I \rightarrow \mathbb{R}$ is said to be Carathéodory function if
(a) the function $f(\cdot, x):[a, b] \rightarrow \mathbb{R}$ is measurable for every $x \in I$,
(b) the function $f(t, \cdot): I \rightarrow \mathbb{R}$ is continuous for almost every $t \in[0, \omega]$,
(c) for any $r>0$, there exists $q_{r} \in L([a, b])$ such that $|f(t, x)| \leq q_{r}(t)$ for a.e. $t \in[a, b]$ and all $x \in I,|x| \leq r$.

Definition 2.2 ([6, Definitions 0.1 and 15.1, Propositions 15.2 and 15.4]). We say that a function $p \in L([0, \omega])$ belongs to the set $\mathcal{V}^{+}(\omega)$ (resp. $\left.\mathcal{V}^{-}(\omega)\right)$ if, for any function $u \in A C^{1}([0, \omega])$ satisfying

$$
u^{\prime \prime}(t) \geq p(t) u(t) \quad \text { for a.e. } t \in[0, \omega], \quad u(0)=u(\omega), \quad u^{\prime}(0) \geq u^{\prime}(\omega)
$$

the inequality

$$
u(t) \geq 0 \quad \text { for } t \in[0, \omega] \quad(\text { resp. } u(t) \leq 0 \quad \text { for } t \in[0, \omega])
$$

holds.
Definition 2.3 ([6, Definition 0.2]). We say that a function $p \in L([0, \omega])$ belongs to the set $\mathcal{V}_{0}(\omega)$ if the problem

$$
\begin{equation*}
u^{\prime \prime}=p(t) u ; \quad u(0)=u(\omega), u^{\prime}(0)=u^{\prime}(\omega) \tag{2.1}
\end{equation*}
$$

has a positive solution.
Remark 2.4. Let $\omega>0$. If $p(t):=p_{0}$ for $t \in[0, \omega]$, then one can show by direct calculation that:
$\triangleright p \in \mathcal{V}^{-}(\omega)$ if and only if $p_{0}>0$,
$\triangleright p \in \mathcal{V}_{0}(\omega)$ if and only if $p_{0}=0$,
$\triangleright p \in \mathcal{V}^{+}(\omega)$ if and only if $p_{0} \in\left[-\frac{\pi^{2}}{\omega^{2}}, 0[\right.$,
$\triangleright p \in \operatorname{Int} \mathcal{V}^{+}(\omega)$ if and only if $\left.p_{0} \in\right]-\frac{\pi^{2}}{\omega^{2}}, 0[$.
If the function $p \in L([0, \omega])$ is not constant, efficient conditions for $p$ to belong to each of the sets $\mathcal{V}^{+}(\omega)$ and $\mathcal{V}^{-}(\omega)$ are provided in [6].

Remark 2.5. It is well known that, if the homogeneous problem (2.1) has only the trivial solution, then, for any $f \in L([0, \omega])$, the problem

$$
\begin{equation*}
u^{\prime \prime}=p(t) u+f(t) ; \quad u(0)=u(\omega), u^{\prime}(0)=u^{\prime}(\omega) \tag{2.2}
\end{equation*}
$$

possesses a unique solution $u$ and this solution satisfies

$$
|u(t)| \leq \Delta(p) \int_{0}^{\omega}|f(s)| \mathrm{d} s \quad \text { for } t \in[0, \omega]
$$

where $\Delta(p)$, depending only on $p$, denotes a norm of the Green's operator of problem (2.1). Clearly, $\Delta(p)>0$.

Assume that $p \in \operatorname{Int} \mathcal{V}^{+}(\omega)$. Extend the function $p$ periodically to the whole real axis denoting it by the same symbol. It is proved in [6, Section 6] that, for any $a \in \mathbb{R}$, the problem

$$
u^{\prime \prime}=p(t) u ; \quad u(a)=1, u(a+\omega)=1
$$

has a unique solution $u_{a}$ and $u_{a}(t)>0$ for $t \in[0, \omega]$. We put

$$
\begin{equation*}
\Gamma(p):=\sup \left\{\left\|u_{a}\right\|_{C}: a \in[0, \omega]\right\} \mathrm{e}^{\int_{0}^{\omega}[p(s)]+\mathrm{ds}} \tag{2.3}
\end{equation*}
$$

It is clear that $\Gamma(p) \geq 1$.
Remark 2.6. If $p \in \mathcal{V}^{+}(\omega)$, then the number $\Delta(p)$ defined in Remark 2.5 can be estimated, for example, by using a maximal value of the Green's function of problem (2.1) (see, e.g., [9]). On the other hand, assuming $p \in \operatorname{Int} \mathcal{V}^{+}(\omega)$, some estimates of the number $\Gamma(p)$ given by (2.3) are provided in $[6$, Section 6].

For instance, if $p(t):=p_{0}$ for $t \in[0, \omega]$ and $p_{0} \in\left[-\frac{\pi^{2}}{\omega^{2}}, 0\left[\right.\right.$, resp. $\left.p_{0} \in\right]-\frac{\pi^{2}}{\omega^{2}}, 0[$, then

$$
\Delta(p) \leq\left(2 \sqrt{\left|p_{0}\right|} \sin \frac{\omega \sqrt{\left|p_{0}\right|}}{2}\right)^{-1}, \quad \text { resp. } \quad \Gamma(p)=\left(\cos \frac{\omega \sqrt{\left|p_{0}\right|}}{2}\right)^{-1}
$$

## 3 Main results

This section contains formulations of all the main results of the paper. Their proofs are presented in detail in Section 5.

### 3.1 Existence theorems

Let us introduce the hypothesis

$$
\left.\begin{array}{l}
q(t, x) \geq q_{0}(t, x) \text { for a.e. } t \in[0, \omega] \text { and all } x \geq x_{0},  \tag{1}\\
x_{0} \geq 0, q_{0}:[0, \omega] \times\left[x_{0},+\infty[\rightarrow \mathbb{R} \text { is a Carathéodory function, }\right. \\
q_{0}(t, \cdot):\left[x_{0},+\infty[\rightarrow \mathbb{R} \text { is non-decreasing for a.e. } t \in[0, \omega] .\right.
\end{array}\right\}
$$

Theorem 3.1. Let hypothesis $\left(H_{1}\right)$ be fulfilled, and there exist $R>x_{0}$ such that $p+q_{0}(\cdot, R) \in \mathcal{V}^{-}(\omega)$. Let, moreover, there exist a positive function $\alpha \in A C_{\ell}([0, \omega])$ satisfying

$$
\begin{gather*}
\alpha(0)=\alpha(\omega), \quad \alpha^{\prime}(0) \geq \alpha^{\prime}(\omega),  \tag{3.1}\\
\alpha^{\prime \prime}(t) \geq p(t) \alpha(t)+q(t, \alpha(t)) \alpha(t)+f(t) \quad \text { for a.e. } t \in[0, \omega] . \tag{3.2}
\end{gather*}
$$

Then, problem (1.1), (1.2) has a positive solution $u$ satisfying

$$
\begin{equation*}
u(t) \geq \alpha(t) \quad \text { for } t \in[0, \omega] \text {. } \tag{3.3}
\end{equation*}
$$

We now provide an effective condition guaranteeing the existence of the function $\alpha$ in Theorem 3.1.

Corollary 3.2. Let $p+[q(\cdot, 0)]_{+} \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$, hypothesis $\left(H_{1}\right)$ be fulfilled, and

$$
\begin{equation*}
\lim _{x \rightarrow+\infty} \int_{E} q_{0}(s, x) \mathrm{d} s=+\infty \quad \text { for every } E \subseteq[0, \omega] \text {, meas } E>0 \tag{3.4}
\end{equation*}
$$

Let, moreover,

$$
\begin{equation*}
\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s<\sup \left\{\frac{r}{\Delta\left(p+q^{*}(\cdot, r)\right)}: r>0, p+q^{*}(\cdot, r) \in \mathcal{V}^{+}(\omega)\right\}, \tag{3.5}
\end{equation*}
$$

where $\Delta$ is defined in Remark 2.5 and

$$
\begin{equation*}
q^{*}(t, \varrho):=\max \left\{[q(t, x)]_{+}: x \in[0, \varrho]\right\} \quad \text { for a.e. } t \in[0, \omega] \text { and all } \varrho \geq 0 . \tag{3.6}
\end{equation*}
$$

Then, problem (1.1), (1.2) has at least one positive solution.
Remark 3.3. In Corollary 3.2, $q^{*}$ is obviously a Carathéodory function satisfying $q^{*}(t, 0) \equiv$ $[q(t, 0)]_{+}$. By Lemma 4.15, it follows from hypothesis (3.4) that there exists $R>x_{0}$ such that $p+q_{0}(\cdot, R) \in \mathcal{V}^{-}(\omega)$. Moreover, $q^{*}(t, R) \geq q_{0}(t, R)$ for a. e. $t \in[0, \omega]$ and, therefore, Lemma 4.12 yields $p+q^{*}(\cdot, R) \in \mathcal{V}^{-}(\omega)$. Since $p+q^{*}(\cdot, 0) \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$, by virtue of Lemma 4.11 (with $\ell(t, x):=p(t)+q^{*}(t, x)$ ), there exists $\left.r \in\right] 0, R\left[\right.$ such that $p+q^{*}(\cdot, r) \in$ $\mathcal{V}^{+}(\omega)$ and, thus, hypothesis (3.5) of Corollary 3.2 is consistent.

Remark 3.4. If the supremum on the right-hand side of (3.5) is achieved at some $r_{0}>0$, then the strict inequality (3.5) in Corollary 3.2 (as well as Corollary 3.7) can be weakened to the non-strict one (see the end of the proof of Corollary 3.2).

Remark 3.5. By Lemma 4.1, the hypothesis $p+[q(\cdot, 0)]_{+} \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$ of Corollary 3.2 is satisfied provided that

$$
\int_{0}^{\omega}\left(p(s)+[q(s, 0)]_{+}\right) \mathrm{d} s \leq 0, \quad p(t)+[q(t, 0)]_{+} \not \equiv 0
$$

Remark 3.6. If

$$
\begin{equation*}
f(t) \leq 0 \quad \text { for a. e. } t \in[0, \omega] \tag{3.7}
\end{equation*}
$$

then condition (3.5) is obviously satisfied.
Assuming $p \in \mathcal{V}^{+}(\omega)$, hypothesis (3.4) of Corollary 3.2 can be weakened to

$$
\begin{equation*}
\lim _{x \rightarrow+\infty} \int_{0}^{\omega} q_{0}(s, x) \mathrm{d} s=+\infty \tag{3.8}
\end{equation*}
$$

Moreover, in such a case, another type of condition on $[f]_{+}$can be provided instead of (3.5).
Corollary 3.7. Let $p \in \mathcal{V}^{+}(\omega), q(t, 0) \equiv 0$, hypothesis $\left(H_{1}\right)$ be fulfilled, (3.8) hold, and there exist $x_{1}>x_{0}$ such that

$$
\begin{equation*}
q_{0}\left(t, x_{1}\right) \geq 0 \quad \text { for a.e. } t \in[0, \omega] \tag{3.9}
\end{equation*}
$$

Let, moreover, either (3.5) hold or $[f(t)]_{+} \not \equiv 0$ and

$$
\begin{equation*}
\Delta(p)<\sup \left\{\frac{r}{\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s+r \int_{0}^{\omega} q^{*}(s, r) \mathrm{d} s}: r>0\right\} \tag{3.10}
\end{equation*}
$$

where $\Delta$ is defined in Remark 2.5 and $q^{*}$ is given by (3.6). Then, problem (1.1), (1.2) has at least one positive solution.

Remark 3.8. If the supremum on the right-hand side of (3.10) is achieved at some $r_{0}>0$, then the strict inequality (3.10) in Corollary 3.7 can be weakened to the non-strict one (see the end of the proof of Corollary 3.7).

It follows from Remark 2.6 that, in some particular cases, the number $\Delta$ defined in $\operatorname{Re}-$ mark 2.5 can be estimated from above and, thus, the effective conditions guaranteeing the validity of (3.5) and (3.10) can be found. In Section 3.3, we will provide such conditions for the Duffing equation (1.3).

### 3.2 Uniqueness and multiplicity theorems

Proposition 1.1 (1) implies that, if $a, b>0$ and $c \leq 0$, then, for any $\omega>0$, equation (1.4) possesses a unique positive $\omega$-periodic solution. Now we show that, under a certain monotonicity condition on $q$, a positive solution in Theorem 3.1 is unique provided that the function $f$ is non-positive. Moreover, we generalize the ideas used in the proof of [4, Theorem 1.1] and, thus, we obtain some conditions on the forcing term $f$ leading to the exact multiplicity of positive solutions to problem (1.1), (1.2).

Theorem 3.9. Assume that $p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega), q(t, 0) \equiv 0$, (3.7) holds, and

$$
\left.\begin{array}{l}
\text { for every } d>c>0 \text { and } e>0 \text {, there exists } h_{c d e} \in L([0, \omega]) \text { such that } \\
h_{\text {cde }}(t)>0 \text { for a.e. } t \in[0, \omega] \text {, }  \tag{2}\\
q(t, x+e)-q(t, x) \geq h_{c d e}(t) \text { for a.e. } t \in[0, \omega] \text { and all } x \in[c, d] .
\end{array}\right\}
$$

Then, problem (1.1), (1.2) has at most one positive solution.

Combining Corollary 3.2 and Theorem 3.9, we get
Corollary 3.10. Let $p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega), q(t, 0) \equiv 0$, hypotheses $\left(H_{1}\right)$ and $\left(H_{2}\right)$ be fulfilled, and conditions (3.4) and (3.7) hold. Then, problem (1.1), (1.2) has a unique positive solution.

In the next theorem, we assume that the non-linearity $q(t, u) u$ in (1.1) is "locally uniformly strictly concave/convex" in the sense of hypothesis $\left(H_{3}^{\ell}\right)$.

Proposition 3.11. Assume that $p, f \in L([0, \omega]), \ell \in\{1,2\}$, and

$$
\begin{align*}
& \text { for every } d_{1}>c_{1}>0, d_{2}>c_{2}>0, d_{3}>c_{3}>0 \text { there exists } \\
& h^{*} \in L([0, \omega]), h^{*}(t) \geq 0 \text { for a.e. } t \in[0, \omega], h^{*}(t) \not \equiv 0, \\
& (-1)^{\ell}\left[\frac{q\left(t, x_{3}\right) x_{3}-q\left(t, x_{2}\right) x_{2}}{x_{3}-x_{2}}-\frac{q\left(t, x_{2}\right) x_{2}-q\left(t, x_{1}\right) x_{1}}{x_{2}-x_{1}}\right] \geq h^{*}(t)  \tag{3}\\
& \text { for a.e. } t \in[0, \omega] \text { and all } c_{1} \leq x_{1} \leq d_{1}, x_{1}+c_{2} \leq x_{2} \leq x_{1}+d_{2}, \\
& x_{2}+c_{3} \leq x_{3} \leq x_{2}+d_{3},
\end{align*}
$$

Then, there are no three solutions $u_{1}, u_{2}, u_{3}$ to problem (1.1), (1.2) satisfying

$$
\begin{equation*}
u_{3}(t)>u_{2}(t)>u_{1}(t)>0 \quad \text { for } t \in[0, \omega] . \tag{3.11}
\end{equation*}
$$

Remark 3.12. Let $q(t, x):=h(t) \varphi(x)$, where $h \in L([0, \omega])$ and $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Then, $q$ satisfies hypothesis $\left(H_{3}^{1}\right)$ (resp. $\left(H_{3}^{2}\right)$ provided that $h(t) \geq 0$ for a.e. $t \in[0, \omega]$, $h(t) \not \equiv 0$, and the function $x \mapsto \varphi(x) x$ is strictly concave (resp. convex) on $] 0,+\infty[$.

If $p \in \mathcal{V}^{+}(\omega)$, then hypothesis $\left(H_{2}\right)$ of Theorem 3.9 can be weakened to $\left(H_{2}^{\prime}\right)$. Moreover, one can show some other properties of solutions to problem (1.1), (1.2) in such a case. Introduce the hypothesis:

$$
\left.\begin{array}{l}
\text { For every } d>c>0 \text { and } e>0 \text {, there exists } h_{c d e} \in L([0, \omega]) \text { such that } \\
h_{c d e}(t) \geq 0 \text { for a.e. } t \in[0, \omega], h_{c d e}(t) \not \equiv 0,  \tag{2}\\
q(t, x+e)-q(t, x) \geq h_{c d e}(t) \text { for a.e. } t \in[0, \omega] \text { and all } x \in[c, d] .
\end{array}\right\}
$$

Theorem 3.13. Let $p \in \mathcal{V}^{+}(\omega)$. Then, the following conclusions hold:
(1) If $q$ satisfies hypothesis $\left(H_{2}^{\prime}\right)$,

$$
\begin{equation*}
q(t, 0) \geq 0 \quad \text { for a.e. } t \in[0, \omega] \tag{3.12}
\end{equation*}
$$

and $u, v$ are distinct positive solutions to problem (1.1), (1.2), then

$$
\begin{equation*}
u(t) \neq v(t) \quad \text { for } t \in[0, \omega] . \tag{3.13}
\end{equation*}
$$

(2) If (3.7) and (3.12) hold and $q$ satisfies hypothesis $\left(H_{2}^{\prime}\right)$, then problem (1.1), (1.2) has at most one positive solution.
(3) If $\ell \in\{1,2\}$, (3.12) holds and $q$ satisfies hypotheses $\left(H_{2}^{\prime}\right)$ and $\left(H_{3}^{\ell}\right)$, then problem (1.1), (1.2) has at most two positive solutions.
(4) If

$$
\begin{align*}
& q(t, x) \geq 0 \quad \text { for a.e. } t \in[0, \omega] \text { and all } x \in \mathbb{R}  \tag{3.14}\\
& f(t) \geq 0 \quad \text { for a.e. } t \in[0, \omega], \quad f(t) \not \equiv 0 \tag{3.15}
\end{align*}
$$

then every solution to (1.1), (1.2) is either positive or negative.

Combining Corollary 3.7 and Theorem 3.13 (2), we get
Corollary 3.14. Let $p \in \mathcal{V}^{+}(\omega), q(t, 0) \equiv 0$, hypotheses $\left(H_{1}\right)$ and $\left(H_{2}^{\prime}\right)$ be fulfilled, there exist $x_{1}>x_{0}$ such that (3.9) holds, and conditions (3.7) and (3.8) be satisfied. Then, problem (1.1), (1.2) has a unique positive solution.

### 3.3 Consequences for the non-autonomous Duffing equation (1.3)

We now apply the above general results for the non-autonomous Duffing equation (1.3) and compare the obtained results with those stated in Propositions 1.1 and 1.2. In this section, we assume that the function $h$ in (1.3) is non-negative. However, the properties of the given periodic problem differ in the following two cases: $h(t)>0$ a.e. on $[0, \omega]$ and $h(t) \geq 0$ a.e. on $[0, \omega], h(t) \not \equiv 0$. Such phenomenon does not occur in the autonomous case of (1.3) (i. e., in (1.4)).

Theorem 3.15. Let $\lambda>1, p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$, and

$$
\begin{equation*}
h(t)>0 \quad \text { for a.e. } t \in[0, \omega] . \tag{3.16}
\end{equation*}
$$

Then, the following conclusions hold:
(1) There are no three solutions $u_{1}, u_{2}, u_{3}$ to problem (1.3), (1.2) satisfying (3.11).
(2) Assume that there exists a positive function $\alpha \in A C_{\ell}([0, \omega])$ such that (3.1) holds and

$$
\begin{equation*}
\alpha^{\prime \prime}(t) \geq p(t) \alpha(t)+h(t) \alpha^{\lambda}(t)+f(t) \quad \text { for a.e. } t \in[0, \omega] . \tag{3.17}
\end{equation*}
$$

Then, problem (1.3), (1.2) has a positive solution $u^{*}$ satisfying

$$
\begin{equation*}
u^{*}(t) \geq \alpha(t) \quad \text { for } t \in[0, \omega] \tag{3.18}
\end{equation*}
$$

such that every solution $u$ to problem (1.3), (1.2) satisfies

$$
\begin{equation*}
\text { either } u(t)<u^{*}(t) \quad \text { for } t \in[0, \omega], \quad \text { or } \quad u(t) \equiv u^{*}(t) . \tag{3.19}
\end{equation*}
$$

Moreover, for any couple of distinct positive solutions $u_{1}, u_{2}$ to (1.3), (1.2) satisfying

$$
\begin{equation*}
u_{1}(t) \not \equiv u^{*}(t), \quad u_{2}(t) \not \equiv u^{*}(t), \tag{3.20}
\end{equation*}
$$

the conditions

$$
\begin{align*}
& \min \left\{u_{1}(t)-u_{2}(t): t \in[0, \omega]\right\}<0, \\
& \max \left\{u_{1}(t)-u_{2}(t): t \in[0, \omega]\right\}>0 \tag{3.21}
\end{align*}
$$

hold.
(3) If (3.7) holds, then problem (1.3), (1.2) has a unique positive solution.

Now we provide a sufficient condition guaranteeing the existence of the function $\alpha$ in Theorem 3.15(2)

Corollary 3.16. Let $\lambda>1, p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$, $h$ satisfy (3.16), and

$$
\begin{equation*}
\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s<\sup \left\{\frac{r}{\Delta\left(p+r^{\lambda-1} h\right)}: r>0, p+r^{\lambda-1} h \in \mathcal{V}^{+}(\omega)\right\}, \tag{3.22}
\end{equation*}
$$

where $\Delta$ is defined in Remark 2.5. Then, there exists a positive function $\alpha \in A C^{1}([0, \omega])$ satisfying (3.17) and

$$
\begin{equation*}
\alpha(0)=\alpha(\omega), \quad \alpha^{\prime}(0)=\alpha^{\prime}(\omega), \tag{3.23}
\end{equation*}
$$

and, thus, the conclusions of Theorem 3.15 (2) hold.
Remark 3.17. It follows from the proof of Corollary 3.16 and Remark 3.4 that, if the supremum on the right-hand side of (3.22) is achieved at some $r_{0}>0$, then the strict inequality (3.22) can be weakened to the non-strict one.

Remark 3.18. Observe that Theorem 3.15 (and Corollary 3.16) extends the conclusions of Proposition 1.1 for the non-autonomous Duffing equation (1.3). Indeed, let $\omega>0$ and

$$
\begin{equation*}
p(t):=-a, \quad h(t):=b \quad \text { for } t \in[0, \omega], \tag{3.24}
\end{equation*}
$$

where $a, b>0$. Then, $p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$ (see Remark 2.4) and the function $h$ satisfies (3.16). We emphasize, in particular, the conclusion of Corollary 3.16, which claims: If the forcing term $f$ satisfies the integral-type condition (3.22), then problem (1.3), (1.2) has a maximal solution $u^{*}$ that is positive. Moreover, every two positive solutions to problem (1.3), (1.2) (different from $u^{*}$ ) must intersect each other; compare it with Proposition 1.1 (2).

As we have mentioned in Remark 2.6, in the case of constant functions, the number $\Delta$ defined in Remark 2.5 can be estimated from above. Therefore, for the problem

$$
\begin{equation*}
u^{\prime \prime}=-a u+b|u|^{\lambda} \operatorname{sgn} u+f(t) ; \quad u(0)=u(\omega), u^{\prime}(0)=u^{\prime}(\omega) \tag{3.25}
\end{equation*}
$$

with $a, b>0, \lambda>1$, and $f \in L([0, \omega])$, Corollary 3.16 yields the following corollary.
Corollary 3.19. Let $\lambda>1, a, b>0$, and

$$
\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s \leq \begin{cases}\frac{2 \omega}{\pi} \frac{(\lambda-1) a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}} & \text { if } a<\frac{\lambda}{\lambda-1}\left(\frac{\pi}{\omega}\right)^{2},  \tag{3.26}\\ \frac{2 \pi}{\omega}\left[\frac{1}{b}\left(a-\frac{\pi^{2}}{\omega^{2}}\right)\right]^{\frac{1}{\lambda-1}} & \text { if } a \geq \frac{\lambda}{\lambda-1}\left(\frac{\pi}{\omega}\right)^{2} .\end{cases}
$$

Then, problem (3.25) has at least one positive solution.
Remark 3.20. Observe that, if $f(t) \equiv c$ and $0<a \leq \frac{\lambda}{\lambda-1}\left(\frac{\pi}{\omega}\right)^{2}$, then (3.26) reads as

$$
\begin{equation*}
c \leq \frac{2}{\pi} \frac{(\lambda-1) a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}} . \tag{3.27}
\end{equation*}
$$

The right-hand side of (3.27) is, up to the factor $\frac{2}{\pi}$, the number appearing in Proposition 1.1. Since condition (3.27) was derived from the integral-type condition (3.26) concerning nonconstant forcing terms, it is not surprising that it can be improved in the autonomous case.
Remark 3.21. Let $f \in L([0, \omega])$ be such that

$$
[f(t)]_{+} \leq f_{0} \quad \text { for a.e. } t \in[0, \omega],
$$

where

$$
f_{0}:= \begin{cases}\frac{2}{\pi} \frac{2 a}{3} \sqrt{\frac{\pi}{3}} & \text { if } a<\frac{3}{2}\left(\frac{\pi}{\omega}\right)^{2} \\ \frac{2 \pi}{\omega^{2}} \sqrt{a-\frac{\pi^{2}}{\omega^{2}}} & \text { if } a \geq \frac{3}{2}\left(\frac{\pi}{\omega}\right)^{2} .\end{cases}
$$

Then, condition (3.26), with $b=1$ and $\lambda=3$, holds and, thus, Corollary 3.19 guarantees the existence of a positive solution to problem (1.5), with $c=0$ and $d(t) \equiv f(t)$. Therefore, Corollary 3.19 complements the conclusions of Proposition 1.2 for the case of $a \geq \frac{\pi^{2}}{\omega^{2}}$ and a sign-changing forcing term $d$.

From Theorem 3.15 (3), we get the following generalization of Proposition 1.1 (1) for the Duffing equation with the constant coefficients and a non-constant forcing term.

Corollary 3.22. Let $\lambda>1, a, b>0$, and (3.7) hold. Then, problem (3.25) has a unique positive solution.

Remark 3.23. Corollary 3.22 complements the conclusions of Proposition 1.2 by the existence and uniqueness of a negative solution to problem (1.5), with $c=0$, provided that $a>0$ and the forcing term $d$ is non-negative.

We have shown in [8, Example 2.8] that, if $f(t) \equiv 0$, then hypothesis (3.16) in the above statements is optimal and cannot be weakened to

$$
\begin{equation*}
h(t) \geq 0 \quad \text { for a. e. } t \in[0, \omega], \quad h(t) \not \equiv 0 . \tag{3.28}
\end{equation*}
$$

However, this weaker assumption on $h$ can be considered instead of (3.16) under a stronger assumption on $p$, namely, $p \in \mathcal{V}^{+}(\omega)$. Moreover, one can show the exact multiplicity of solutions to problem (1.3), (1.2) in such a case. We first introduce the following definition.

Definition 3.24 ([6, Definition 16.1]). Let $p, f \in L([0, \omega])$. We say that the pair $(p, f)$ belongs to the set $\mathcal{U}(\omega)$ if problem (2.2) has a unique solution which is positive.

Theorem 3.25. Let $\lambda>1, p \in \mathcal{V}^{+}(\omega)$, and (3.28) be fulfilled. Then, the following conclusions hold:
(1) Problem (1.3), (1.2) has at most two positive solutions.
(2) Assume that (3.22) holds, where $\Delta$ is defined in Remark 2.5. Then, problem (1.3), (1.2) has either one or two positive solutions.
(3) Assume that there exists a positive function $\alpha \in A C_{\ell}([0, \omega])$ satisfying (3.1) and (3.17). Then, problem (1.3), (1.2) has a positive solution $u^{*}$ satisfying (3.18) such that, for every solution $u$ to problem (1.3), (1.2), condition (3.19) holds.
(4) Assume that $(p, f) \in \mathcal{U}(\omega)$ and there exist functions $\alpha_{1} \in A C_{\ell}([0, \omega])$ and $\alpha_{2} \in A C^{1}([0, \omega])$ such that

$$
\begin{gather*}
0<\alpha_{2}(t)<\alpha_{1}(t) \quad \text { for } t \in[0, \omega],  \tag{3.29}\\
\alpha_{k}(0)=\alpha_{k}(\omega), \quad \alpha_{k}^{\prime}(0) \geq \alpha_{k}^{\prime}(\omega) \quad \text { for } k=1,2,  \tag{3.30}\\
\alpha_{k}^{\prime \prime}(t) \geq p(t) \alpha_{k}(t)+h(t) \alpha_{k}^{\lambda}(t)+f(t) \quad \text { for a.e. } t \in[0, \omega], k=1,2 . \tag{3.31}
\end{gather*}
$$

Then, problem (1.3), (1.2) possesses exactly two positive solutions $u_{1}, u_{2}$ and these solutions satisfy

$$
\begin{equation*}
u_{1}(t)>u_{2}(t)>0 \quad \text { for } t \in[0, \omega] . \tag{3.32}
\end{equation*}
$$

Moreover, for every solution $u$ to problem (1.3), (1.2) different from $u_{1}$, the condition

$$
\begin{equation*}
u(t)<u_{1}(t) \text { for } t \in[0, \omega] \tag{3.33}
\end{equation*}
$$

holds.
(5) If (3.7) holds, then problem (1.3), (1.2) has a unique positive solution.

Remark 3.26. It follows from Lemma 4.3 that, if $p \in \operatorname{Int} \mathcal{V}^{+}(\omega)$, then the inclusion $(p, f) \in$ $\mathcal{U}(\omega)$ holds for every function $f \in L([0, \omega])$ satisfying $f(t) \not \equiv 0$ and

$$
\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s \geq \Gamma(p) \int_{0}^{\omega}[f(s)]_{-} \mathrm{d} s,
$$

where $\Gamma$ is given by (2.3).
On the other hand, if $p \in \mathcal{V}^{+}(\omega)$ and $f$ satisfies (3.15), then $(p, f) \in \mathcal{U}(\omega)$ as well (see Lemma 4.2).

Remark 3.27. It follows from the proof of Theorem 3.25 that the solution $u_{1}$ in the conclusion of Theorem $3.25(4)$ satisfies $u_{1}(t) \geq \alpha_{1}(t)$ for $t \in[0, \omega]$ and the solution $u_{2}$ is such that $u_{2}\left(t_{0}\right) \leq \alpha_{2}\left(t_{0}\right)$ for some $t_{0} \in[0, \omega]$.

Remark 3.28. Let $\omega>0$ and the functions $p, h$ be defined by (3.24), where $0<a \leq \frac{\pi^{2}}{\omega^{2}}$ and $b>0$. Then, $p \in \mathcal{V}^{+}(\omega)$ (see Remark 2.4) and the function $h$ satisfies (3.28). Therefore, it follows from Theorem 3.25 (1) that, for any $c \in \mathbb{R}$, equation (1.4) has at most two positive $\omega$ periodic solutions. Consequently, if $0<c \leq \frac{(\lambda-1) a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$ and $u_{0}$ be a non-constant positive periodic solution appearing in conclusion (2) of Proposition 1.1, then the minimal period $T$ of the solution $u_{0}$ satisfies

$$
T>\frac{\pi}{\sqrt{a}} .
$$

Now we provide sufficient conditions guaranteeing the existence of the functions $\alpha$ and $\alpha_{1}, \alpha_{2}$ in Theorem 3.25(3,4).

Corollary 3.29. Let $\lambda>1, p \in \mathcal{V}^{+}(\omega)$, and $h$ satisfy (3.28). Then, the following conclusions hold:
(1) If

$$
\begin{equation*}
0<\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s \leq \frac{\lambda-1}{\lambda[\Delta(p)]^{\frac{\lambda}{\lambda-1}}\left[\lambda \int_{0}^{\omega} h(s) \mathrm{d} s\right]^{\frac{1}{\lambda-1}}}, \tag{3.34}
\end{equation*}
$$

where $\Delta$ is defined in Remark 2.5, then there exists a positive function $\alpha \in A C^{1}([0, \omega])$ satisfying (3.17) and (3.23) and, thus, the conclusion of Theorem 3.25 (3) holds.
(2) If $(p, f) \in \mathcal{U}(\omega)$ and

$$
\begin{equation*}
0<\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s<\frac{\lambda-1}{\lambda[\Delta(p)]^{\frac{\lambda}{\lambda-1}}\left[\lambda \int_{0}^{\omega} h(s) \mathrm{d} s\right]^{\frac{1}{\lambda-1}}} \tag{3.35}
\end{equation*}
$$

where $\Delta$ is defined in Remark 2.5, then there exists functions $\alpha_{1}, \alpha_{2} \in A C^{1}([0, \omega])$ satisfying (3.29), (3.31), and

$$
\begin{equation*}
\alpha_{k}(0)=\alpha_{k}(\omega), \quad \alpha_{k}^{\prime}(0)=\alpha_{k}^{\prime}(\omega) \quad \text { for } k=1,2 \tag{3.36}
\end{equation*}
$$

and, thus, the conclusions of Theorem 3.25 (4) hold.
For the constant coefficient $p$ in (1.3), we derive the following corollary.
Corollary 3.30. Let $\left.\lambda>1, a \in] 0, \frac{\pi^{2}}{\omega^{2}}\right]$, (3.28) hold, and

$$
\begin{equation*}
0<\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s \leq \frac{\lambda-1}{\lambda} \frac{\left[2 \sqrt{a} \sin \frac{\omega \sqrt{a}}{2}\right]^{\frac{\lambda}{\lambda-1}}}{\left[\lambda \int_{0}^{\omega} h(s) \mathrm{d} s\right]^{\frac{1}{\lambda-1}}} . \tag{3.37}
\end{equation*}
$$

Then, the problem

$$
\begin{equation*}
u^{\prime \prime}=-a u+h(t)|u|^{\lambda} \operatorname{sgn} u+f(t) ; \quad u(0)=u(\omega), u^{\prime}(0)=u^{\prime}(\omega) \tag{3.38}
\end{equation*}
$$

has either one or two solutions.
Corollary 3.31. Let $\left.\lambda>1, a \in] 0, \frac{\pi^{2}}{\omega^{2}}\right]$, and conditions (3.7) and (3.28) hold. Then, problem (3.38) has a unique positive solution.

Theorem 3.25 (2) and Corollary 3.29 say, among other things, that, if the forcing term $f$ is such that $\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s$ is "small enough", then problem (1.3), (1.2) has at least one positive solution. The next theorem confirms that hypotheses of such a kind cannot be omitted. More precisely, Theorem 3.32 below claims that, if $f$ is such that $\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s$ is "large enough", then problem (1.3), (1.2) has no positive solution.

Theorem 3.32. Let $\lambda>1, p \in \operatorname{Int} \mathcal{V}^{+}(\omega)$, condition (3.28) hold, $f(t) \not \equiv 0$, and

$$
\begin{equation*}
\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s-\Gamma(p) \int_{0}^{\omega}[f(s)]_{-} \mathrm{d} s \geq \frac{\lambda-1}{\lambda} \frac{\left|\Gamma(p) \int_{0}^{\omega}[p(s)]_{-} \mathrm{d} s-\int_{0}^{\omega}[p(s)]_{+} \mathrm{d} s\right|^{\frac{\lambda}{\lambda-1}}}{\left[\lambda \int_{0}^{\omega} h(s) \mathrm{d} s\right]^{\frac{1}{\lambda-1}}} . \tag{3.39}
\end{equation*}
$$

where $\Gamma$ is given by (2.3). Then, problem (1.3), (1.2) has no non-negative solution.
If the forcing term $f$ is non-negative, then the conclusions of Corollary 3.29 (2) and Theorem 3.32 can be extended as follows.

Theorem 3.33. Let $\lambda>1$ and conditions (3.15) and (3.28) be fulfilled. Then, the following conclusions hold:
(1) Assume that $p \in \mathcal{V}^{+}(\omega)$ and

$$
\begin{equation*}
\int_{0}^{\omega} f(s) \mathrm{d} s<\frac{\lambda-1}{\lambda} \frac{[\Delta(p)]^{-\frac{\lambda}{\lambda-1}}}{\left[\lambda \int_{0}^{\omega} h(s) \mathrm{d} s\right]^{\frac{1}{\lambda-1}}}, \tag{3.40}
\end{equation*}
$$

where $\Delta$ is defined in Remark 2.5. Then, problem (1.3), (1.2) possesses exactly three solutions $u_{1}$, $u_{2}, u_{3}$ and these solutions satisfy

$$
\begin{equation*}
u_{1}(t)>u_{2}(t)>0, \quad u_{3}(t)<0 \quad \text { for } t \in[0, \omega] . \tag{3.41}
\end{equation*}
$$

(2) Assume that $p \in \operatorname{Int} \mathcal{V}^{+}(\omega)$ and

$$
\begin{equation*}
\int_{0}^{\omega} f(s) \mathrm{d} s \geq \frac{\lambda-1}{\lambda} \frac{\left[\Gamma(p) \int_{0}^{\omega}[p(s)]_{-} \mathrm{d} s-\int_{0}^{\omega}[p(s)]_{+} \mathrm{d} s\right]^{\frac{\lambda}{\lambda-1}}}{\left[\lambda \int_{0}^{\omega} h(s) \mathrm{d} s\right]^{\frac{1}{\lambda-1}}} \tag{3.42}
\end{equation*}
$$

where $\Gamma$ is given by (2.3). Then, problem (1.3), (1.2) has a unique solution $u_{0}$ and this solution is negative.

Remark 3.34. If $\omega>0$ and $p(t):=-a$ for $t \in[0, \omega]$, with $a \in] 0, \frac{\pi^{2}}{\omega^{2}}\left[\right.$, then $p \in \operatorname{Int} \mathcal{V}^{+}(\omega)$ (see Remark 2.4) and, for any $h, f \in L([0, \omega])$ satisfying (3.15) and (3.28), conditions (3.40) and (3.42) are satisfied provided that

$$
\int_{0}^{\omega} f(s) \mathrm{d} s<\frac{\lambda-1}{\lambda}\left[2 \sqrt{a} \sin \frac{\omega \sqrt{a}}{2}\right]^{\frac{\lambda}{\lambda-1}} \frac{1}{\left[\lambda \int_{0}^{\omega} h(s) \mathrm{d} s\right]^{\frac{1}{\lambda-1}}}
$$

and

$$
\begin{equation*}
\int_{0}^{\omega} f(s) \mathrm{d} s \geq \frac{\lambda-1}{\lambda}\left[\frac{\omega a}{\cos \frac{\omega \sqrt{a}}{2}}\right]^{\frac{\lambda}{\lambda-1}} \frac{1}{\left[\lambda \int_{0}^{\omega} h(s) \mathrm{d} s\right]^{\frac{1}{\lambda-1}}} \tag{3.43}
\end{equation*}
$$

(see Remark 2.6). If, moreover, $h(t):=b$ for $t \in[0, \omega]$, with $b>0$, then (3.43) reads as

$$
\frac{1}{\omega} \int_{0}^{\omega} f(s) \mathrm{d} s \geq\left[\frac{1}{\cos \frac{\omega \sqrt{a}}{2}}\right]^{\frac{\lambda}{\lambda-1}} \frac{(\lambda-1) a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}
$$

compare this condition with those in Proposition 1.1 (4).
Remark 3.35. Theorem 3.33 extends the conclusions of Proposition 1.2 for the non-autonomous Duffing equation (1.3). Indeed, let $\omega>0$ and the functions $p, h$ be defined by (3.24), where $0<a \leq \frac{\pi^{2}}{\omega^{2}}$ and $b=1$. Then, $p \in \mathcal{V}^{+}(\omega)$ (see Remark 2.4) and the function $h$ satisfies (3.28). As opposed to Proposition 1.2, where point conditions on the forcing term $d$ are obtained, Theorem 3.33 (1) provides the integral-type conditions. This confirms conjecture (1) formulated by authors of [4] on p. 3930 - the graph of a forcing term may cross the line $y=\frac{2 a}{3} \sqrt{\frac{a}{3}}$ mentioned therein.

## 4 Auxiliary statements

We first recall some results stated in $[6,8]$.
Lemma 4.1 ([6, Proposition 10.8, Remark 0.7]). If $p \in \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$, then either $\int_{0}^{\omega} p(s) \mathrm{d} s>0$ or $p(t) \equiv 0$.

Lemma 4.2. Let $g \in \mathcal{V}^{+}(\omega)$. Then, for any non-negative function $\ell \in L([0, \omega])$, the problem

$$
\begin{equation*}
u^{\prime \prime}=g(t) u+\ell(t) ; \quad u(0)=u(\omega), u^{\prime}(0)=u^{\prime}(\omega) \tag{4.1}
\end{equation*}
$$

has a unique solution $u$ and this solution satisfies

$$
0 \leq u(t) \leq \Delta(g) \int_{0}^{\omega} \ell(s) \mathrm{d} s \quad \text { for } t \in[0, \omega]
$$

where $\Delta$ is defined in Remark 2.5. Moreover, if $\ell(t) \not \equiv 0$, then the solution $u$ is positive.
Proof. The conclusions of the lemma follow from Definition 2.2, Remark 2.5, and [6, Remark 9.2].

Lemma $4.3\left(\left[6\right.\right.$, Theorem 16.4]). Let $g \in \operatorname{Int} \mathcal{V}^{+}(\omega)$ and $\ell \in L([0, \omega])$ be such that $\ell(t) \not \equiv 0$ and

$$
\int_{0}^{\omega}[\ell(s)]_{+} \mathrm{d} s \geq \Gamma(g) \int_{0}^{\omega}[\ell(s)]_{-} \mathrm{d} s
$$

where $\Gamma$ is given by (2.3). Then,

$$
\Gamma(g) \int_{0}^{\omega}[g(s)]_{-} \mathrm{d} s>\int_{0}^{\omega}[g(s)]_{+} \mathrm{d} s
$$

and problem (4.1) has a unique solution $u$, which satisfies

$$
u(t)>v\left(\int_{0}^{\omega}[\ell(s)]_{+} \mathrm{d} s-\Gamma(g) \int_{0}^{\omega}[\ell(s)]_{-} \mathrm{d} s\right) \quad \text { for } t \in[0, \omega]
$$

where

$$
v:=\left(\Gamma(g) \int_{0}^{\omega}[g(s)]_{-} \mathrm{d} s-\int_{0}^{\omega}[g(s)]_{+} \mathrm{d} s\right)^{-1}
$$

Lemma 4.4 ([6, Theorem 16.2]). Let $g \in \mathcal{V}^{-}(\omega)$. Then, there exists $v_{0}>0$ such that, for any non-positive function $\ell \in L([0, \omega])$, problem (4.1) has a unique solution $u$ and this solution satisfies

$$
u(t) \geq v_{0} \int_{0}^{\omega}|\ell(s)| \mathrm{d} s \quad \text { for } t \in[0, \omega] .
$$

Lemma 4.5 ([6, Proposition 10.2]). The set $\mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$ is closed in $L([0, \omega])$.
Definition 4.6 ([6, Definition 0.4$])$. We say that a function $p \in L([0, \omega])$ belongs to the set $\mathcal{D}(\omega)$ if the problem

$$
u^{\prime \prime}=\widetilde{p}(t) u ; \quad u(a)=0, u(b)=0
$$

has no non-trivial solution for any $a, b \in \mathbb{R}$ satisfying $0<b-a<\omega$, where $\widetilde{p}$ is the $\omega$-periodic extension of $p$ to the whole real axis.

Lemma 4.7. $\mathcal{D}(\omega)=\mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \cup \mathcal{V}^{+}(\omega)$ and $\operatorname{Int} \mathcal{D}(\omega)=\mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \cup \operatorname{Int} \mathcal{V}^{+}(\omega)$.
Proof. It follows from Propositions 2.1, 10.5, and 10.6 stated in [6].
Lemma 4.8 ([6, Proposition 2.5]). Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be an $\omega$-periodic function such that $g \in \mathcal{D}(\omega)$. Then, for any $a, b \in \mathbb{R}$ and $w \in A C^{1}([a, b])$ satisfying $0<b-a<\omega$ and

$$
w^{\prime \prime}(t) \geq g(t) w(t) \quad \text { for a.e. } t \in[a, b], \quad w(a) \leq 0, \quad w(b) \leq 0,
$$

the inequality $w(t) \leq 0$ holds for $t \in[a, b]$.
Lemma 4.9 ([8, Lemma 3.10]). Let $p \in \mathcal{D}(\omega)$ and $\ell \in L([0, \omega])$ be such that

$$
\begin{equation*}
\ell(t) \geq 0 \quad \text { for a.e. } t \in[0, \omega], \quad \ell(t) \not \equiv 0 . \tag{4.2}
\end{equation*}
$$

Then, $p+\ell \in \operatorname{Int} \mathcal{D}(\omega)$.
Lemma 4.10 ([6, Lemma 2.7]). Let $g \in \mathcal{D}(\omega), \ell \in L([0, \omega])$ be a function satisfying (4.2), and $u$ be a solution to problem (4.1). Then, the function $u$ is either positive or negative.

Lemma 4.11. Let $\ell:[0, \omega] \times\left[\lambda_{1}, \lambda_{2}\right] \rightarrow \mathbb{R}$ be a Carathéodory function such that

$$
\begin{equation*}
\ell\left(\cdot, \lambda_{1}\right) \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega), \quad \ell\left(\cdot, \lambda_{2}\right) \in \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \tag{4.3}
\end{equation*}
$$

Then, there exists $r \in] \lambda_{1}, \lambda_{2}\left[\right.$ such that $\ell(\cdot, r) \in \operatorname{Int} \mathcal{V}^{+}(\omega)$.
Proof. Let

$$
\begin{equation*}
A:=\left\{\lambda \in\left[\lambda_{1}, \lambda_{2}\right]: \ell(\cdot, x) \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \text { for } x \in\left[\lambda_{1}, \lambda\right]\right\} . \tag{4.4}
\end{equation*}
$$

In view of (4.3), it is clear that $A \neq \varnothing$. Put

$$
\begin{equation*}
\lambda^{*}:=\sup A . \tag{4.5}
\end{equation*}
$$

Since the set $\mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$ is closed (see Lemma 4.5), it follows from (4.4) and (4.5) that $\lambda^{*}>\lambda_{1}$ and

$$
\begin{equation*}
\ell(\cdot, x) \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \quad \text { for } x \in\left[\lambda_{1}, \lambda^{*}[.\right. \tag{4.6}
\end{equation*}
$$

We first show that

$$
\begin{equation*}
\ell\left(\cdot, \lambda^{*}\right) \in \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \tag{4.7}
\end{equation*}
$$

Indeed, suppose on the contrary that $\ell\left(\cdot, \lambda^{*}\right) \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$. Then, hypothesis (4.3) yields $\lambda^{*}<\lambda_{2}$. Since the set $\mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$ is closed (see Lemma 4.5), there exists $\varepsilon>0$ such that $\ell(\cdot, x) \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$ for $x \in\left[\lambda^{*}-\varepsilon, \lambda^{*}+\varepsilon\right]$. However, this condition, together with (4.4) and (4.6), implies that $\lambda^{*}+\varepsilon \in A$, which contradicts (4.5).

Now, in view of (4.7), it follows from Lemma 4.7 that $\ell\left(\cdot, \lambda^{*}\right) \in \operatorname{Int} \mathcal{D}(\omega)$. Therefore, there exists $\eta \in] 0, \lambda^{*}-\lambda_{1}\left[\right.$ such that $\ell\left(\cdot, \lambda^{*}-\eta\right) \in \operatorname{Int} \mathcal{D}(\omega)$. By Lemma 4.7 , we get $\ell\left(\cdot, \lambda^{*}-\right.$ $\eta) \in \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \cup \operatorname{Int} \mathcal{V}^{+}(\omega)$ and, thus, condition (4.6) yields $\ell\left(\cdot, \lambda^{*}-\eta\right) \in \operatorname{Int} \mathcal{V}^{+}(\omega)$. Consequently, the conclusion of the lemma holds with $r:=\lambda^{*}-\eta$.

Lemma 4.12 ([6, Remark 8.5]). Let $p \in \mathcal{V}^{-}(\omega)$. Then, for any $g \in L([0, \omega])$ satisfying $g(t) \geq p(t)$ for a.e. $t \in[0, \omega]$, the inclusion $g \in \mathcal{V}^{-}(\omega)$ holds.

Lemma 4.13 ([6, Remark 8.4]). Let $p \in \mathcal{V}_{0}(\omega)$. Then, for any $g \in L([0, \omega])$ satisfying $g(t) \geq p(t)$ for a.e. $t \in[0, \omega]$ and $g(t) \not \equiv p(t)$, the inclusion $g \in \mathcal{V}^{-}(\omega)$ holds.

Lemma 4.14 ([8, Proposition 3.16]). Let $g \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$ and $\ell:[0, \omega] \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying $\ell(t, 0) \equiv 0$. Then, for any $c>0$, there exists a function $\alpha \in$ $A C^{1}([0, \omega])$ such that (3.23) holds and

$$
\begin{gathered}
\alpha^{\prime \prime}(t) \geq g(t) \alpha(t)+\ell(t, \alpha(t)) \alpha(t) \text { for a.e. } t \in[0, \omega], \\
0<\alpha(t) \leq c \quad \text { for } t \in[0, \omega] .
\end{gathered}
$$

Lemma 4.15. Let $p \in L([0, \omega]), x_{0} \geq 0$, and $q_{0}:[0, \omega] \times\left[x_{0},+\infty[\rightarrow \mathbb{R}\right.$ be a Carathéodory function such that

$$
\begin{equation*}
\text { the function } q_{0}(t, \cdot):\left[x_{0},+\infty[\rightarrow \mathbb{R} \text { is non-decreasing for a.e. } t \in[0, \omega]\right. \tag{4.8}
\end{equation*}
$$

and (3.4) holds. Then, there exists $K>x_{0}$ such that $p+q_{0}(\cdot, x) \in \mathcal{V}^{-}(\omega)$ for $x \geq K$.
Proof. It follows from [8, Proposition 3.13] with

$$
f(t, x):= \begin{cases}q_{0}(t, x) & \text { if } x>x_{0},  \tag{4.9}\\ q_{0}\left(t, x_{0}\right) & \text { if } x_{0} \geq x>0 .\end{cases}
$$

Lemma 4.16. Let $p \in \mathcal{V}^{+}(\omega), x_{0} \geq 0, q_{0}:[0, \omega] \times\left[x_{0},+\infty[\rightarrow \mathbb{R}\right.$ be a Carathéodory function satisfying (3.8) and (4.8), and there exist $x_{1}>x_{0}$ such that (3.9) holds. Then, there exists $K>x_{0}$ such that $p+q_{0}(\cdot, x) \in \mathcal{V}^{-}(\omega)$ for $x \geq K$.
Proof. It follows from [8, Proposition 3.14] with $f$ given by (4.9).
Now we recall a classical results concerning the solvability of the periodic problem

$$
\begin{equation*}
u^{\prime \prime}=g(t, u) ; \quad u(0)=u(\omega), u^{\prime}(0)=u^{\prime}(\omega), \tag{4.10}
\end{equation*}
$$

where $g:[0, \omega] \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function (see, e.g., [3]).
Lemma 4.17. Let there exist functions $\alpha \in A C_{\ell}([a, b])$ and $\beta \in A C_{u}([a, b])$ satisfying

$$
\begin{array}{ll} 
& \alpha(t) \leq \beta(t) \quad \text { for } t \in[a, b], \\
\alpha^{\prime \prime}(t) \geq g(t, \alpha(t)) & \text { for a.e. } t \in[a, b], \quad \alpha(0)=\alpha(\omega), \quad \alpha^{\prime}(0) \geq \alpha^{\prime}(\omega), \\
\beta^{\prime \prime}(t) \leq g(t, \beta(t)) \quad \text { for a.e. } t \in[a, b], \quad \beta(0)=\beta(\omega), \quad \beta^{\prime}(0) \leq \beta^{\prime}(\omega) . \tag{4.13}
\end{array}
$$

Then, problem (4.10) has at least one solution $u$ such that

$$
\begin{equation*}
\alpha(t) \leq u(t) \leq \beta(t) \quad \text { for } t \in[a, b] . \tag{4.14}
\end{equation*}
$$

The next existence result is also known.
Lemma 4.18 ( $\left[7\right.$, Theorem 1.1 and Remark 1.2]). Let there exist $p_{0} \in \operatorname{Int} \mathcal{D}(\omega)$ and a Carathóodory function $z:[0, \omega] \times[0,+\infty[\rightarrow[0,+\infty[$ such that

$$
g(t, x) \operatorname{sgn} x \geq p_{0}(t)|x|-z(t,|x|) \quad \text { for a.e. } t \in[0, \omega] \text { and all } x \in \mathbb{R}
$$

and

$$
\lim _{x \rightarrow+\infty} \frac{1}{x} \int_{0}^{\omega} z(s, x) \mathrm{d} s=0
$$

Let, moreover, there exist functions $\alpha \in A C_{\ell}([0, \omega])$ and $\beta \in A C_{u}([0, \omega])$ satisfying (4.12) and (4.13). Then, problem (4.10) has a solution u such that

$$
\min \left\{\alpha\left(t_{u}\right), \beta\left(t_{u}\right)\right\} \leq u\left(t_{u}\right) \leq \max \left\{\alpha\left(t_{u}\right), \beta\left(t_{u}\right)\right\} \quad \text { for some } t_{u} \in[0, \omega] .
$$

The following three propositions concern the existence of the functions $\alpha, \beta$ appearing in Lemmas 4.17 and 4.18 (with $g(t, x):=p(t) x+q(t, x) x+f(t)$ ), which are usually referred to as lower and upper functions of problem (1.1), (1.2).

Proposition 4.19. Let $p, f \in L([0, \omega]), q:[0, \omega] \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function, and there exist $r_{0}>0$ such that $p+q^{*}\left(\cdot, r_{0}\right) \in \mathcal{V}^{+}(\omega)$ and

$$
\begin{equation*}
0<\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s \leq \frac{r_{0}}{\Delta\left(p+q^{*}\left(\cdot, r_{0}\right)\right)}, \tag{4.15}
\end{equation*}
$$

where $\Delta$ is defined in Remark 2.5 and $q^{*}$ is given by (3.6). Then, there exists a function $\alpha \in A C^{1}([0, \omega])$ satisfying (3.2), (3.23), and

$$
\begin{equation*}
0<\alpha(t) \leq r_{0} \quad \text { for } t \in[0, \omega] . \tag{4.16}
\end{equation*}
$$

Moreover, if both inequalities in (4.15) are strict, then there exists $A>0$ such that

$$
\begin{equation*}
\alpha^{\prime \prime}(t) \geq p(t) \alpha(t)+q(t, \alpha(t)) \alpha(t)+f(t)+A \quad \text { for a.e. } t \in[0, \omega] . \tag{4.17}
\end{equation*}
$$

Proof. Hypothesis (4.15) implies that there exists $\varepsilon \geq 0$ such that

$$
\begin{equation*}
0<\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s \leq \frac{r_{0}-\varepsilon}{\Delta\left(p+q^{*}\left(\cdot, r_{0}\right)\right)} \tag{4.18}
\end{equation*}
$$

and $\varepsilon>0$ if both inequalities in (4.15) are strict.
Since we assume that $p+q^{*}\left(\cdot, r_{0}\right) \in \mathcal{V}^{+}(\omega)$ and $[f(t)]_{+} \not \equiv 0$, it follows from Lemma 4.2 (with $g(t):=p(t)+q^{*}\left(t, r_{0}\right)$ and $\ell(t):=[f(t)]_{+}+\frac{\varepsilon}{\omega \Delta\left(p+q^{*}\left(\cdot, r_{0}\right)\right)}$ ) that the problem

$$
\alpha^{\prime \prime}=\left(p(t)+q^{*}\left(t, r_{0}\right)\right) \alpha+[f(t)]_{+}+\frac{\varepsilon}{\omega \Delta\left(p+q^{*}\left(\cdot, r_{0}\right)\right)} ; \quad \alpha(0)=\alpha(\omega), \quad \alpha^{\prime}(0)=\alpha^{\prime}(\omega)
$$

has a unique solution $\alpha$ and this solution satisfies

$$
0<\alpha(t) \leq \varepsilon+\Delta\left(p+q^{*}\left(\cdot, r_{0}\right)\right) \int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s \quad \text { for } t \in[0, \omega] .
$$

Therefore, in view of (4.18), conditions (3.23) and (4.16) hold. Moreover, (3.6) implies that

$$
\begin{equation*}
\text { the function } q^{*}(t, \cdot):[0,+\infty[\rightarrow \mathbb{R} \text { is non-decreasing for a. e. } t \in[0, \omega] \tag{4.19}
\end{equation*}
$$

and, thus, the function $\alpha$ satisfies

$$
\begin{aligned}
\alpha^{\prime \prime}(t) & \geq p(t) \alpha(t)+q^{*}(t, \alpha(t)) \alpha(t)+[f(t)]++\frac{\varepsilon}{\omega \Delta\left(p+q^{*}\left(\cdot, r_{0}\right)\right)} \\
& \geq p(t) \alpha(t)+q(t, \alpha(t)) \alpha(t)+f(t)+\frac{\varepsilon}{\omega \Delta\left(p+q^{*}\left(\cdot, r_{0}\right)\right)} \\
& \geq p(t) \alpha(t)+q(t, \alpha(t)) \alpha(t)+f(t) \quad \text { for a. e. } t \in[0, \omega],
\end{aligned}
$$

i. e., (3.2) holds. Furthermore, if both inequalities in (4.15) are strict, then $\varepsilon>0$ and, therefore, condition (4.17) is fulfilled with $A:=\frac{\varepsilon}{\omega \Delta\left(p+q^{*}\left(\cdot r_{0}\right)\right)}$.
Proposition 4.20. Let $p \in \mathcal{V}^{+}(\omega), f \in L([0, \omega])$, and $q:[0, \omega] \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function. Let, moreover, $[f(t)]_{+} \not \equiv 0$ and there exist $r_{0}>0$ such that

$$
\begin{equation*}
\Delta(p) \leq \frac{r_{0}}{\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s+r_{0} \int_{0}^{\omega} q^{*}\left(s, r_{0}\right) \mathrm{d} s} \tag{4.20}
\end{equation*}
$$

where $\Delta$ is defined in Remark 2.5 and $q^{*}$ is given by (3.6). Then, there exists a function $\alpha \in A C^{1}([0, \omega])$ satisfying(3.2), (3.23), (4.16), and

$$
\begin{equation*}
\alpha^{\prime \prime}(t) \geq p(t) \alpha(t)+q(t, \alpha(t)) \alpha(t)+[f(t)]_{+} \quad \text { for a.e. } t \in[0, \omega] . \tag{4.21}
\end{equation*}
$$

Moreover, if inequality (4.20) is strict, then there exists $A>0$ such that a satisfies (4.17).
Proof. Hypothesis (4.20) implies that there exists $\varepsilon \geq 0$ such that

$$
\begin{equation*}
\Delta(p) \leq \frac{r_{0}-\varepsilon}{\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s+r_{0} \int_{0}^{\omega} q^{*}\left(s, r_{0}\right) \mathrm{d} s} \tag{4.22}
\end{equation*}
$$

and $\varepsilon>0$ if inequality (4.20) is strict.
It follows from Lemma $4.2\left(\right.$ with $g(t):=p(t)$ and $\left.\ell(t):=r_{0} q^{*}\left(t, r_{0}\right)+[f(t)]_{+}+\frac{\varepsilon}{\omega \Delta(p)}\right)$ that the problem

$$
\alpha^{\prime \prime}=p(t) \alpha+r_{0} q^{*}\left(t, r_{0}\right)+[f(t)]_{+}+\frac{\varepsilon}{\omega \Delta(p)} ; \quad \alpha(0)=\alpha(\omega), \quad \alpha^{\prime}(0)=\alpha^{\prime}(\omega)
$$

has a unique solution $\alpha$ and this solution satisfies

$$
0<\alpha(t) \leq \varepsilon+\Delta(p)\left(r_{0} \int_{0}^{\omega} q^{*}\left(s, r_{0}\right) \mathrm{d} s+\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s\right) \quad \text { for } t \in[0, \omega] .
$$

Therefore, in view of (4.22), conditions (3.23) and (4.16) hold. Moreover, (3.6) yields (4.19) and, thus, the function $\alpha$ satisfies

$$
\begin{aligned}
\alpha^{\prime \prime}(t) & \geq p(t) \alpha(t)+q^{*}(t, \alpha(t)) \alpha(t)+[f(t)]_{+}+\frac{\varepsilon}{\omega \Delta(p)} \\
& \geq p(t) \alpha(t)+q(t, \alpha(t)) \alpha(t)+[f(t)]_{+}+\frac{\varepsilon}{\omega \Delta(p)} \\
& \geq p(t) \alpha(t)+q(t, \alpha(t)) \alpha(t)+f(t)+\frac{\varepsilon}{\omega \Delta(p)} \\
& \geq p(t) \alpha(t)+q(t, \alpha(t)) \alpha(t)+f(t) \quad \text { for a.e. } t \in[0, \omega],
\end{aligned}
$$

i. e., (3.2) and (4.21) hold. Furthermore, if inequality (4.20) is strict, then $\varepsilon>0$ and, therefore, condition (4.17) is fulfilled with $A:=\frac{\varepsilon}{\omega \Delta(p)}$.

Proposition 4.21. Let $p, f \in L(0, \omega), q:[0, \omega] \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying hypothesis $\left(H_{1}\right)$, and there exist $R>x_{0}$ such that $p+q_{0}(\cdot, R) \in \mathcal{V}^{-}(\omega)$. Then, for any $c>0$, there exist $B>0$ and a function $\beta \in A C^{1}([0, \omega])$ such that

$$
\begin{gather*}
\beta^{\prime \prime}(t) \leq p(t) \beta(t)+q(t, \beta(t)) \beta(t)+f(t)-B \quad \text { for a.e. } t \in[0, \omega],  \tag{4.23}\\
\beta(0)=\beta(\omega), \quad \beta^{\prime}(0)=\beta^{\prime}(\omega),  \tag{4.24}\\
\beta(t) \geq c \quad \text { for } t \in[0, \omega] . \tag{4.25}
\end{gather*}
$$

Proof. Let $v_{0}>0$ be the number appearing in the conclusion of Lemma 4.4 (with $g(t):=$ $\left.p(t)+q_{0}(t, R)\right)$ and let $c>0$ be arbitrary. Then, it follows from Lemma 4.4 that the problem

$$
\beta^{\prime \prime}=\left(p(t)+q_{0}(t, R)\right) \beta-[f(t)]_{-}-\frac{\max \{c, R\}}{v_{0} \omega} ; \quad \beta(0)=\beta(\omega), \quad \beta^{\prime}(0)=\beta^{\prime}(\omega)
$$

has a unique solution $\beta$ and this solution satisfies

$$
\beta(t) \geq \max \{c, R\} \quad \text { for } t \in[0, \omega] .
$$

Obviously, (4.24) and (4.25) hold. Since $\beta(t) \geq R>x_{0}$ for $t \in[0, \omega]$, by hypothesis ( $H_{1}$ ), we get

$$
\begin{aligned}
\beta^{\prime \prime}(t) & \leq p(t) \beta(t)+q_{0}(t, \beta(t)) \beta(t)-[f(t)]_{-}-\frac{\max \{c, R\}}{v_{0} \omega} \\
& \leq p(t) \beta(t)+q(t, \beta(t)) \beta(t)+f(t)-\frac{\max \{c, R\}}{v_{0} \omega} \text { for a.e. } t \in[0, \omega]
\end{aligned}
$$

i. e., (4.23) is fulfilled with $B:=\frac{\max \{c, R\}}{v_{0} \omega}$.

The following lemma concerning problem (1.3), (1.2) we use in the proof of Theorem 3.15.
Lemma 4.22. Let $u_{1}, u_{2}$ be solutions to problem (1.3), (1.2) such that

$$
\begin{equation*}
u_{2}(t) \geq u_{1}(t) \quad \text { for } t \in[0, \omega], \quad u_{2}(t) \not \equiv u_{1}(t) . \tag{4.26}
\end{equation*}
$$

Then,

$$
\begin{equation*}
u_{2}(t)>u_{1}(t) \text { for } t \in[0, \omega] \text {. } \tag{4.27}
\end{equation*}
$$

Proof. Suppose on the contrary that (4.27) does not hold. Then, there exists $t_{0} \in[0, \omega]$ such that

$$
\begin{equation*}
u_{2}\left(t_{0}\right)=u_{1}\left(t_{0}\right) . \tag{4.28}
\end{equation*}
$$

Extend the functions $p, h, f, u_{1}, u_{2}$ periodically to the whole real axis denoting them by the same symbols. Then, in view of (4.26) and (4.28), we get

$$
\begin{equation*}
u_{2}^{\prime}\left(t_{0}\right)=u_{1}^{\prime}\left(t_{0}\right) . \tag{4.29}
\end{equation*}
$$

Since the function $x \mapsto|x|^{\lambda} \operatorname{sgn} x$ is Lipschitz on every compact interval, for any $c_{1}, c_{2} \in \mathbb{R}$, the Cauchy problem

$$
u^{\prime \prime}=p(t) u+h(t)|u|^{\lambda} \operatorname{sgn} u+f(t) ; \quad u\left(t_{0}\right)=c_{1}, \quad u^{\prime}\left(t_{0}\right)=c_{2}
$$

is uniquely solvable. Therefore, (4.28) and (4.29) yield $u_{2}(t) \equiv u_{1}(t)$, which contradicts (4.26).

We finally provide a technical lemma, which we use in the proof of Theorem 3.32.
Lemma 4.23. Let $Q \geq 1$ and $f, g \in L([0, \omega])$ be such that

$$
\begin{equation*}
g(t) \geq 0 \quad \text { for a.e. } t \in[0, \omega] . \tag{4.30}
\end{equation*}
$$

Then,

$$
\begin{align*}
& \int_{0}^{\omega}[g(s)+f(s)]_{+} \mathrm{d} s-\varrho \int_{0}^{\omega}[g(s)+f(s)]_{-} \mathrm{d} s  \tag{4.31}\\
& \quad \geq \int_{0}^{\omega} g(s) \mathrm{d} s+\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s-\varrho \int_{0}^{\omega}[f(s)]_{-} \mathrm{d} s .
\end{align*}
$$

Proof. Put

$$
A^{+}:=\{t \in[0, \omega]: g(t)+f(t) \geq 0\}, \quad A^{-}:=\{t \in[0, \omega]: g(t)+f(t)<0\} .
$$

Then, by (4.30) and the hypothesis $\varrho \geq 1$, we get

$$
\begin{aligned}
\int_{0}^{\omega}[g(s)+f(s)]_{+} \mathrm{d} s & =\int_{A^{+}} g(s) \mathrm{d} s+\int_{A^{+}}[f(s)]_{+} \mathrm{d} s-\int_{A^{+}}[f(s)]_{-} \mathrm{d} s \\
& \geq \int_{A^{+}} g(s) \mathrm{d} s+\int_{A^{+}}[f(s)]_{+} \mathrm{d} s-\varrho \int_{A^{+}}[f(s)]_{-} \mathrm{d} s
\end{aligned}
$$

and

$$
\begin{aligned}
\varrho \int_{0}^{\omega}[g(s)+f(s)]_{-} \mathrm{d} s & =\varrho\left(-\int_{A^{-}} g(s) \mathrm{d} s-\int_{A^{-}}[f(s)]_{+} \mathrm{d} s+\int_{A^{-}}[f(s)]_{-} \mathrm{d} s\right) \\
& \leq-\int_{A^{-}} g(s) \mathrm{d} s-\int_{A^{-}}[f(s)]_{+} \mathrm{d} s+\varrho \int_{A^{-}}[f(s)]_{-} \mathrm{d} s,
\end{aligned}
$$

which yields (4.31).

## 5 Proofs of main results

Proof of Theorem 3.1. Let $\alpha \in A C_{\ell}([0, \omega])$ be a positive function such that (3.1) and (3.2) hold. It follows from Proposition 4.21 that there exists a function $\beta \in A C^{1}([0, \omega])$ satisfying (4.11), (4.24), and

$$
\begin{equation*}
\beta^{\prime \prime}(t) \leq p(t) \beta(t)+q(t, \beta(t)) \beta(t)+f(t) \quad \text { for a.e. } t \in[0, \omega] . \tag{5.1}
\end{equation*}
$$

Consequently, all the hypotheses of Lemma 4.17 (with $g(t, x):=p(t) x+q(t, x) x+f(t)$ ) are fulfilled and, thus, problem (1.1), (1.2) has a positive solution $u$ such that (4.14) holds.

Proof of Corollary 3.2. By Lemma 4.15 (with $p(t)+[q(t, 0)]_{+}$and $q_{0}(t, x)-[q(t, 0)]_{+}$instead of $p(t)$ and $q_{0}(t, x)$ ), there exists $R>x_{0}$ such that the inclusion $p+q_{0}(\cdot, R) \in \mathcal{V}^{-}(\omega)$ holds. It follows from (3.5) that there exists $r_{0}>0$ such that $p+q^{*}\left(\cdot, r_{0}\right) \in \mathcal{V}^{+}(\omega)$ and

$$
\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s<\frac{r_{0}}{\Delta\left(p+q^{*}\left(\cdot, r_{0}\right)\right)}
$$

If $[f(t)]_{+} \not \equiv 0$, then, by Proposition 4.19, there exists a function $\alpha \in A C^{1}([0, \omega])$ satisfying (3.2), (3.23), and (4.16).

Assume that $[f(t)]_{+} \equiv 0$. In view of (3.6) and the hypothesis $p+[q(\cdot, 0)]_{+} \notin \mathcal{V}^{-}(\omega) \cup$ $\mathcal{V}_{0}(\omega)$, it follows from Lemma 4.14 (with $g(t):=p(t)+[q(t, 0)]_{+}, \ell(t, x):=q^{*}(t,|x|)-$
$[q(t, 0)]_{+}$, and $\left.c:=r_{0}\right)$ that there exists a function $\alpha \in A C^{1}([0, \omega])$ satisfying (3.23), (4.16), and

$$
\alpha^{\prime \prime}(t) \geq\left(p(t)+[q(t, 0)]_{+}\right) \alpha(t)+\left(q^{*}(t,|\alpha(t)|)-[q(t, 0)]_{+}\right) \alpha(t) \quad \text { for a.e. } t \in[0, \omega] .
$$

Since $\alpha$ is positive and $f(t) \leq 0$ for a.e. $t \in[0, \omega]$, the latter inequality yields (3.2).
Consequently, the conclusion of the corollary follows from Theorem 3.1.
We finally prove the assertion stated in Remark 3.4. Assume that a supremum on the right-hand side of (3.5) is achieved at some $r_{0}>0$ and

$$
\begin{equation*}
\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s=\frac{r_{0}}{\Delta\left(p+q^{*}\left(\cdot, r_{0}\right)\right)} . \tag{5.2}
\end{equation*}
$$

Then, by Proposition 4.19, there exist a function $\alpha \in A C^{1}([0, \omega])$ satisfying (3.2), (3.23), and (4.16). Therefore, it follows from Theorem 3.1 that problem (1.1), (1.2) has at least one positive solution.

Proof of Corollary 3.7. By Lemma 4.16, there exists $R>x_{0}$ such that the inclusion $p+q_{0}(\cdot, R) \in$ $\mathcal{V}^{-}(\omega)$ holds.

First assume that (3.5) is fulfilled, where $\Delta$ is defined in Remark 2.5 and $q^{*}$ is given by (3.6). In much the same way as in the proof of Corollary 3.2, we show that there exists a function $\alpha \in A C^{1}([0, \omega])$ satisfying (3.2), (3.23), and (4.16).

Now assume that $[f(t)]_{+} \not \equiv 0$ and (3.10) holds, where $\Delta$ is defined in Remark 2.5 and $q^{*}$ is given by (3.6). It follows from (3.10) that there exists $r_{0}>0$ such that

$$
\Delta(p)<\frac{r_{0}}{\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s+r_{0} \int_{0}^{\omega} q^{*}\left(s, r_{0}\right) \mathrm{d} s}
$$

and, thus, Proposition 4.20 guarantees the existence of a function $\alpha \in A C^{1}([0, \omega])$ such that (3.2), (3.23), and (4.16) hold.

Consequently, in both cases (3.5) and (3.10), the conclusion of the corollary follows from Theorem 3.1.

We finally prove the assertions stated in Remarks 3.4 and 3.8. First assume that a supremum on the right-hand side of (3.5) is achieved at some $r_{0}>0$ and (5.2) holds. Then, by Proposition 4.19, there exist a function $\alpha \in A C^{1}([0, \omega])$ satisfying (3.2), (3.23), and (4.16). Therefore, it follows from Theorem 3.1 that problem (1.1), (1.2) has at least one positive solution.

Now assume that $[f(t)]_{+} \not \equiv 0$, a supremum on the right-hand side of (3.10) is achieved at some $r_{0}>0$, and

$$
\Delta(p)=\frac{r_{0}}{\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s+r_{0} \int_{0}^{\omega} q^{*}\left(s, r_{0}\right) \mathrm{d} s} .
$$

Then, by Proposition 4.20, there exist a function $\alpha \in A C^{1}([0, \omega])$ satisfying (3.2), (3.23), and (4.16). Therefore, it follows from Theorem 3.1 that problem (1.1), (1.2) has at least one positive solution.

Proof of Theorem 3.9. It follows from hypothesis $\left(H_{2}\right)$ that

$$
\begin{equation*}
\text { the function } q(t, \cdot):[0,+\infty[\rightarrow \mathbb{R} \text { is non-decreasing for a. e. } t \in[0, \omega] \text {. } \tag{5.3}
\end{equation*}
$$

Suppose on the contrary that $u, w$ are positive solutions to problem (1.1), (1.2) satisfying

$$
\max \{u(t)-w(t): t \in[0, \omega]\}>0
$$

Put

$$
\beta_{0}(t):=\min \{u(t), w(t)\} \quad \text { for } t \in[0, \omega] .
$$

It is not difficult to verify that $\beta_{0} \in A C_{u}([0, \omega])$,

$$
\begin{gather*}
\beta_{0}^{\prime \prime}(t)=p(t) \beta_{0}(t)+q\left(t, \beta_{0}(t)\right) \beta_{0}(t)+f(t) \quad \text { for a.e. } t \in[0, \omega],  \tag{5.4}\\
\beta_{0}(0)=\beta_{0}(\omega), \quad \beta_{0}^{\prime}(0) \leq \beta_{0}^{\prime}(\omega),  \tag{5.5}\\
\beta_{0}(t) \leq u(t) \quad \text { for } t \in[0, \omega], \quad \beta_{0}(t) \not \equiv u(t) . \tag{5.6}
\end{gather*}
$$

By Lemma 4.14 (with $g(t):=p(t)$ and $\ell(t, x):=q(t, x)$ ), there exists a function $\alpha \in A C^{1}([0, \omega])$ such that (3.23) holds,

$$
\begin{gather*}
\alpha^{\prime \prime}(t) \geq p(t) \alpha(t)+q(t, \alpha(t)) \alpha(t) \quad \text { for a. e. } t \in[0, \omega], \\
\alpha(t) \leq \beta_{0}(t) \quad \text { for } t \in[0, \omega] . \tag{5.7}
\end{gather*}
$$

In view of hypothesis (3.7), it is clear that the function $\alpha$ satisfies also (3.2). Therefore, by virtue of (3.2), (3.23), (5.4), (5.5), and (5.7), all the hypotheses of Lemma 4.17 (with $g(t, x):=$ $p(t) x+q(t, x) x+f(t)$ and $\left.\beta(t):=\beta_{0}(t)\right)$ that there exists a solution $v$ to problem (1.1), (1.2) such that

$$
\alpha(t) \leq v(t) \leq \beta_{0}(t) \quad \text { for } t \in[0, \omega] .
$$

However, the latter condition and (5.6) imply that there exist $t_{1}, t_{2} \in[0, \omega]$ such that $t_{1}<t_{2}$ and

$$
\begin{equation*}
u(t) \geq v(t)>0 \quad \text { for } t \in[0, \omega], \quad u(t)>v(t) \quad \text { for } t \in\left[t_{1}, t_{2}\right] . \tag{5.8}
\end{equation*}
$$

Consequently, there exist $v_{*}, v^{*}, e_{0}>0$ such that

$$
\begin{equation*}
u(t) \geq v(t)+e_{0}, \quad v^{*} \geq v(t) \geq v_{*} \quad \text { for } t \in\left[t_{1}, t_{2}\right] \tag{5.9}
\end{equation*}
$$

and, thus, in view of (5.3), (5.8), (5.9), and ( $H_{2}$ ), we get

$$
\begin{equation*}
q(t, u(t)) \geq q(t, v(t)) \quad \text { for a. e. } t \in[0, \omega] \tag{5.10}
\end{equation*}
$$

and

$$
\begin{equation*}
q(t, u(t))-q(t, v(t)) \geq q\left(t, v(t)+e_{0}\right)-q(t, v(t)) \geq h_{v_{*} v^{*} e_{0}}(t) \tag{5.11}
\end{equation*}
$$

for a.e. $t \in\left[t_{1}, t_{2}\right]$. It follows immediately from (1.1) that $u$ and $v$ are solutions to the equations

$$
\begin{align*}
& z^{\prime \prime}=\left(p(t)+q(t, v(t))+\frac{f(t)}{v(t)}\right) z+[q(t, u(t))-q(t, v(t))] u(t)-\frac{f(t)}{v(t)}[u(t)-v(t)] \\
& z^{\prime \prime}=\left(p(t)+q(t, v(t))+\frac{f(t)}{v(t)}\right) z \tag{5.12}
\end{align*}
$$

respectively. Therefore, by virtue of (3.7), (5.8), (5.10), and (5.11), third Fredholm's theorem yields the contradiction

$$
\begin{aligned}
0 & =\int_{0}^{\omega}\left([q(t, u(t))-q(t, v(t))] u(t)-\frac{f(t)}{v(t)}[u(t)-v(t)]\right) v(t) \mathrm{d} t \\
& \geq \int_{t_{1}}^{t_{2}}[q(t, u(t))-q(t, v(t))] u(t) v(t) \mathrm{d} t \geq v_{*}^{2} \int_{t_{1}}^{t_{2}} h_{v_{*} v^{*} e_{0}}(t) \mathrm{d} t>0 .
\end{aligned}
$$

Proof of Proposition 3.11. Suppose on the contrary that $u_{1}, u_{2}, u_{3}$ are solutions to problem (1.1), (1.2) satisfying (3.11). It is clear that there exist $d_{1}>c_{1}>0, d_{2}>c_{2}>0$, and $d_{3}>c_{3}>0$ such that

$$
\begin{aligned}
c_{1} \leq u_{1}(t) \leq d_{1} & \text { for } t \in[0, \omega], \\
c_{k} \leq u_{k}(t)-u_{k-1}(t) \leq d_{k} & \text { for } t \in[0, \omega], k=2,3 .
\end{aligned}
$$

Put

$$
\varphi_{k}(t):=\frac{q\left(t, u_{k+1}(t)\right) u_{k+1}(t)-q\left(t, u_{k}(t)\right) u_{k}(t)}{u_{k+1}(t)-u_{k}(t)} \quad \text { for a.e. } t \in[0, \omega], k=1,2 .
$$

It follows from hypothesis $\left(H_{3}^{\ell}\right)$ with $\left(x_{1}:=u_{1}(t), x_{2}:=u_{2}(t)\right.$, and $\left.x_{3}:=u_{3}(t)\right)$ that

$$
\begin{equation*}
(-1)^{\ell}\left[\varphi_{2}(t)-\varphi_{1}(t)\right] \geq h^{*}(t) \geq 0 \quad \text { for a.e. } t \in[0, \omega], \quad h^{*}(t) \not \equiv 0 . \tag{5.13}
\end{equation*}
$$

Now let $z_{k}(t):=u_{k+1}(t)-u_{k}(t)$ for $t \in[0, \omega], k=1,2$. Then, (1.1) yields

$$
z_{k}^{\prime \prime}(t)=p(t) z_{k}(t)+q\left(t, u_{k+1}(t)\right) u_{k+1}(t)-q\left(t, u_{k}(t)\right) u_{k}(t)=\left(p(t)+\varphi_{k}(t)\right) z_{k}(t)
$$

for a.e. $t \in[0, \omega], k=1,2$, and, in view of (3.11), we get

$$
z_{1}(t)>0, \quad z_{2}(t)>0 \quad \text { for } t \in[0, \omega] .
$$

Therefore, by Definition 2.3, we get

$$
\begin{equation*}
p+\varphi_{1} \in \mathcal{V}_{0}(\omega), \quad p+\varphi_{2} \in \mathcal{V}_{0}(\omega) \tag{5.14}
\end{equation*}
$$

On the other hand, (5.13) yields

$$
p(t)+\varphi_{\ell}(t) \geq p(t)+\varphi_{3-\ell}(t) \quad \text { for a. e. } t \in[0, \omega]
$$

and

$$
p(t)+\varphi_{\ell}(t) \not \equiv p(t)+\varphi_{3-\ell}(t)
$$

which, by virtue of Lemma 4.13, contradicts (5.14).
Proof of Theorem 3.13. Conclusion (1): Assume that ( $H_{2}^{\prime}$ ) and (3.12) hold and $u, v$ are positive solutions to problem (1.1), (1.2) such that

$$
\begin{equation*}
\max \{u(t)-v(t): t \in[0, \omega]\}>0 . \tag{5.15}
\end{equation*}
$$

It follows from hypothesis $\left(H_{2}^{\prime}\right)$ that (5.3) is fulfilled which, together with (3.12), yields

$$
\begin{equation*}
q(t, x) \geq 0 \quad \text { for a.e. } t \in[0, \omega] \text { and all } x \geq 0 . \tag{5.16}
\end{equation*}
$$

Suppose on the contrary that (3.13) does not hold. Then, either

$$
\begin{align*}
& \qquad u(t) \geq v(t) \quad \text { for } t \in[0, \omega], \quad u(t) \not \equiv v(t),  \tag{5.17}\\
& \text { there exists } t_{0} \in[0, \omega] \text { such that } u\left(t_{0}\right)=v\left(t_{0}\right), \tag{5.18}
\end{align*}
$$

or

$$
\begin{equation*}
\min \{u(t)-v(t): t \in[0, \omega]\}<0 . \tag{5.19}
\end{equation*}
$$

First assume that (5.17) and (5.18) are satisfied. Then, in view of (5.16), condition (5.3) yields

$$
q(t, u(t)) u(t) \geq q(t, v(t)) u(t) \geq q(t, v(t)) v(t) \quad \text { for a.e. } t \in[0, \omega] \text {. }
$$

Put $z(t):=u(t)-v(t)$ for $t \in[0, \omega]$. The function $z$ is a solution to the linear periodic problem

$$
z^{\prime \prime}=p(t) z+q(t, u(t)) u(t)-q(t, v(t)) v(t) ; \quad z(0)=z(\omega), z^{\prime}(0)=z^{\prime}(\omega) .
$$

If $q(t, u(t)) u(t) \not \equiv q(t, v(t)) v(t)$, then, in view of Lemma 4.2 (with $g(t):=p(t)$ and $\ell(t):=$ $q(t, u(t)) u(t)-q(t, v(t)) v(t))$, we get $z(t)>0$ for $t \in[0, \omega]$, which contradicts (5.18). On the other hand, if $q(t, u(t)) u(t) \equiv q(t, v(t)) v(t)$, then Lemma $4.2($ with $g(t):=p(t)$ and $\ell(t):=0)$ yields $z(t) \equiv 0$, which is in contradiction with (5.17).

Now assume that (5.19) holds. Extend the functions $u, v, p, f, q(\cdot, x)$ periodically to the whole real axis denoting them by the same symbols. Then, in view of (5.15) and (5.19), there exist $a, b \in \mathbb{R}$ such that $0<b-a<\omega$ and

$$
\begin{equation*}
u(t)>v(t) \quad \text { for } t \in] a, b[, \quad u(a)=v(a), \quad u(b)=v(b) . \tag{5.20}
\end{equation*}
$$

Put $w(t):=u(t)-v(t)$ for $t \in[a, b]$. By virtue of (5.3), (5.16), and (5.20), it follows from (1.1) that

$$
\begin{aligned}
w^{\prime \prime}(t) & =p(t) w(t)+[q(t, u(t))-q(t, v(t))] u(t)+q(t, v(t))[u(t)-v(t)] \\
& \geq p(t) w(t) \quad \text { for a.e. } t \in[a, b] .
\end{aligned}
$$

Since $w(a)=0$ and $w(b)=0$, by Lemma 4.7 and Lemma 4.8 (with $g(t):=p(t)$ ), we get $w(t) \leq 0$ for $t \in[a, b]$, which is in contradiction with (5.20).

Conclusion (2): Assume that (3.7), (3.12), and ( $H_{2}^{\prime}$ ) are fulfilled. It follows from hypothesis $\left(H_{2}^{\prime}\right)$ that (5.3) holds.

Suppose on the contrary that $u, v$ are positive solutions to problem (1.1), (1.2) satisfying (5.15). Then, the above-proved conclusion (1) yields

$$
\begin{equation*}
u(t)>v(t) \quad \text { for } t \in[0, \omega] \tag{5.21}
\end{equation*}
$$

and, thus, there exist $v_{*}, v^{*}, e_{0}>0$ such that

$$
u(t) \geq v(t)+e_{0}, \quad v^{*} \geq v(t) \geq v_{*} \quad \text { for } t \in[0, \omega] .
$$

Therefore, by using (5.3) and ( $H_{2}^{\prime}$ ), we get

$$
\begin{equation*}
q(t, u(t))-q(t, v(t)) \geq q\left(t, v(t)+e_{0}\right)-q(t, v(t)) \geq h_{v * v} v^{*} e_{0}(t) \tag{5.22}
\end{equation*}
$$

for a. e. $t \in[0, \omega]$. It follows immediately from (1.1) that $u$ and $v$ are solutions to equations (5.12) and, thus, by virtue of (3.7), (5.21), and (5.22), third Fredholm's theorem yields the contradiction

$$
\begin{aligned}
0 & =\int_{0}^{\omega}\left([q(t, u(t))-q(t, v(t))] u(t)-\frac{f(t)}{v(t)}[u(t)-v(t)]\right) v(t) \mathrm{d} t \\
& \geq \int_{0}^{\omega}[q(t, u(t))-q(t, v(t))] u(t) v(t) \mathrm{d} t \geq v_{*}^{2} \int_{0}^{\omega} h_{v_{*} v^{*} e_{0}}(t) \mathrm{d} t>0 .
\end{aligned}
$$

Conclusion (3): Assume that $\ell \in\{1,2\}$, condition (3.12) holds, and hypotheses ( $H_{2}^{\prime}$ ) and $\left(H_{3}^{\ell}\right)$ are fulfilled.

Suppose on the contrary that $u_{1}, u_{2}, u_{3}$ are mutually distinct positive solutions to problem (1.1), (1.2). Then, the above-proved conclusion (1) implies that we can assume without loss of generality that $u_{1}, u_{2}, u_{3}$ satisfy (3.11), which is in contradiction with the conclusion of Proposition 3.11.

Conclusion (4): Assume that (3.14) and (3.15) hold and let $u$ be a solution to problem (1.1), (1.2). Then, $u$ is a solution to the linear periodic problem

$$
z^{\prime \prime}=(p(t)+q(t, u(t))) z+f(t) ; \quad z(0)=z(\omega), z^{\prime}(0)=z^{\prime}(\omega) .
$$

By virtue of (3.14), Lemmas 4.7 and 4.9 yield $p+q(\cdot, u(\cdot)) \in \mathcal{D}(\omega)$. Since $f$ satisfies (3.15), by Lemma 4.10 (with $g(t):=p(t)+q(t, u(t))$ and $\ell(t):=f(t)$ ), we conclude that the function $u$ is either positive or negative.

Proof of Theorem 3.15. Put

$$
\begin{equation*}
q(t, x):=h(t)|x|^{\lambda-1} \quad \text { for a.e. } t \in[0, \omega] \text { and all } x \in \mathbb{R} . \tag{5.23}
\end{equation*}
$$

In view of (3.16), it is clear that $q$ is a Carathéodory function satisfying hypothesis $\left(H_{1}\right)$ with $q_{0}(t, x):=h(t) x^{\lambda-1}$ and $x_{0}:=0$. Moreover, $q(t, 0) \equiv 0$, condition (3.4) holds and hypothesis $\left(H_{2}\right)$ is fulfilled. Furthermore, since the function $x \mapsto x^{\lambda}$ is strictly convex on $] 0,+\infty[$, one can show that $q$ satisfies also hypothesis $\left(H_{0}^{2}\right)$.

Conclusion (1): It follows immediately from Proposition 3.11.
Conclusion (2): Let $\alpha \in A C_{\ell}([0, \omega])$ be a positive function satisfying (3.1) and (3.17). By Lemma 4.15, there exists $R>0$ such that

$$
\begin{equation*}
p+q_{0}(\cdot, R) \in \mathcal{V}^{-}(\omega) \tag{5.24}
\end{equation*}
$$

Consequently, all the hypotheses of Theorem 3.1 are fulfilled and, thus, problem (1.3), (1.2) has a positive solution $u_{0}$ such that

$$
\begin{equation*}
u_{0}(t) \geq \alpha(t) \quad \text { for } t \in[0, \omega] . \tag{5.25}
\end{equation*}
$$

We now determine a solution $u^{*}$ to problem (1.3), (1.2) satisfying (3.18) such that, for any solution $u$ to problem (1.3), (1.2), condition (3.19) is fulfilled.

First assume that problem (1.3), (1.2) has a unique positive solution. Put $u^{*}:=u_{0}$. In view of (5.25), it is clear that (3.18) holds. We show that every solution $u$ to problem (1.3), (1.2) satisfies (3.19). Suppose on the contrary that $u$ is a solution to (1.3), (1.2) such that (3.19) does not hold. Lemma 4.22 implies that, if $u(t) \leq u^{*}(t)$ for $t \in[0, \omega]$ and $u(t) \not \equiv u^{*}(t)$, then $u(t)<u^{*}(t)$ for $t \in[0, \omega]$. Therefore, $u$ satisfies

$$
\begin{equation*}
\max \left\{u(t)-u^{*}(t): t \in[0, \omega]\right\}>0 . \tag{5.26}
\end{equation*}
$$

Put

$$
\alpha_{0}(t):=\max \left\{u(t), u^{*}(t)\right\} \quad \text { for } t \in[0, \omega] .
$$

It is not difficult to verify that $\alpha_{0} \in A C_{\ell}([0, \omega])$,

$$
\begin{gather*}
\alpha_{0}^{\prime \prime}(t)=p(t) \alpha_{0}(t)+q\left(t, \alpha_{0}(t)\right) \alpha_{0}(t)+f(t) \quad \text { for a.e. } t \in[0, \omega],  \tag{5.27}\\
\alpha_{0}(0)=\alpha_{0}(\omega), \quad \alpha_{0}^{\prime}(0) \geq \alpha_{0}^{\prime}(\omega),  \tag{5.28}\\
\alpha_{0}(t) \geq u^{*}(t) \quad \text { for } t \in[0, \omega], \quad \alpha_{0}(t) \not \equiv u^{*}(t) . \tag{5.29}
\end{gather*}
$$

In view of (5.24), Proposition 4.21 implies that there exists $\beta \in A C^{1}([0, \omega])$ satisfying (4.24), (5.1), and

$$
\beta(t) \geq \alpha_{0}(t) \quad \text { for } t \in[0, \omega] .
$$

Therefore, by virtue of (4.24), (5.1), (5.27), and (5.28), it follows from Lemma 4.17 (with $g(t, x):=p(t) x+q(t, x) x+f(t)$ and $\left.\alpha(t):=\alpha_{0}(t)\right)$ that there exists a solution $\tilde{u}$ to problem (1.3), (1.2) such that

$$
\alpha_{0}(t) \leq \tilde{u}(t) \leq \beta(t) \quad \text { for } t \in[0, \omega] .
$$

However, in view of (5.29), the latter condition yields

$$
\tilde{u}(t) \geq u^{*}(t) \quad \text { for } t \in[0, \omega], \quad \tilde{u}(t) \not \equiv u^{*}(t)
$$

Consequently, by Lemma 4.22, we get

$$
\begin{equation*}
\tilde{u}(t)>u^{*}(t)>0 \quad \text { for } t \in[0, \omega], \tag{5.30}
\end{equation*}
$$

which contradicts our assumption that problem (1.3), (1.2) has a unique positive solution.
Now assume that problem (1.3), (1.2) has at least two positive solutions. Then, there exists a positive solution $v$ to problem (1.3), (1.2) different from $u_{0}$. We can assume without loss of generality that

$$
\begin{equation*}
\min \left\{v(t)-u_{0}(t): t \in[0, \omega]\right\}<0 \tag{5.31}
\end{equation*}
$$

We first determine a positive solution $u^{*}$ to problem (1.3), (1.2) satisfying (3.18) and

$$
\begin{equation*}
u^{*}(t)>v(t) \quad \text { for } t \in[0, \omega] . \tag{5.32}
\end{equation*}
$$

It is clear that either

$$
\begin{equation*}
\max \left\{v(t)-u_{0}(t): t \in[0, \omega]\right\} \leq 0 \tag{5.33}
\end{equation*}
$$

or

$$
\begin{equation*}
\max \left\{v(t)-u_{0}(t): t \in[0, \omega]\right\}>0 \tag{5.34}
\end{equation*}
$$

Let (5.33) hold. Then, $v(t) \leq u_{0}(t)$ for $t \in[0, \omega]$ and, in view of (5.31), Lemma 4.22 yields $v(t)<u_{0}(t)$ for $t \in[0, \omega]$. We put $u^{*}:=u_{0}$ and, in view of (5.25), we conclude immediately that (3.18) and (5.32) are satisfied.

Let (5.34) hold. Put

$$
\alpha_{0}(t):=\max \left\{v(t), u_{0}(t)\right\} \quad \text { for } t \in[0, \omega] .
$$

In much the same way as above, we determine a solution $u^{*}$ to problem (1.3), (1.2) such that

$$
u^{*}(t) \geq \max \left\{v(t), u_{0}(t)\right\} \quad \text { for } t \in[0, \omega] .
$$

By virtue of (5.25) and (5.31), the solution $u^{*}$ satisfies (3.18) and

$$
u^{*}(t) \geq v(t) \quad \text { for } t \in[0, \omega], \quad u^{*}(t) \not \equiv v(t) .
$$

Therefore, in view Lemma 4.22, (5.32) holds.
Hence, in both cases (5.33) and (5.34), we have determined a solution $u^{*}$ to problem (1.3), (1.2) satisfying (3.18) and (5.32). Now we show that every solution $u$ to (1.3), (1.2) satisfies (3.19). Suppose on the contrary that $u$ is a solution to problem (1.3), (1.2) such that (3.19) does not hold. Lemma 4.22 implies that, if $u(t) \leq u^{*}(t)$ for $t \in[0, \omega]$ and $u(t) \not \equiv u^{*}(t)$, then
$u(t)<u^{*}(t)$ for $t \in[0, \omega]$. Therefore, $u$ satisfies (5.26). In much the same way as above, we determine a solution $\tilde{u}$ to problem (1.3), (1.2) such that (5.30) holds. Hence, conditions (5.30) and (5.32) yields

$$
\tilde{u}(t)>u^{*}(t)>v(t)>0 \quad \text { for } t \in[0, \omega],
$$

which contradicts the above-proved conclusion (1).
It remains to show that, for any couple of distinct positive solutions $u_{1}, u_{2}$ to problem (1.3), (1.2) satisfying (3.20), conditions (3.21) hold. Assume that $u_{1}, u_{2}$ are distinct positive solutions to (1.3), (1.2) satisfying (3.20). We have proved above that

$$
\begin{equation*}
u_{1}(t)<u^{*}(t), \quad u_{2}(t)<u^{*}(t) \quad \text { for } t \in[0, \omega] . \tag{5.35}
\end{equation*}
$$

Suppose on the contrary that (3.21) does not hold, i.e., there exists $k \in\{1,2\}$ such that

$$
u_{k}(t) \leq u_{3-k}(t) \quad \text { for } t \in[0, \omega], \quad u_{k}(t) \not \equiv u_{3-k}(t) .
$$

Then, Lemma 4.22, together with (5.35), yields

$$
0<u_{k}(t)<u_{3-k}(t)<u^{*}(t) \quad \text { for } t \in[0, \omega],
$$

which contradicts the above-proved conclusion (1).
Conclusion (3): Assume that (3.7) holds. Then, the existence and uniqueness of a positive solution to problem (1.3), (1.2) follows from Corollary 3.10.

Proof of Corollary 3.16. Let the function $q$ be defined by formula (5.23). In view of (3.16), it is clear that $q$ is a Carathéodory function satisfying hypothesis $\left(H_{1}\right)$ with $q_{0}(t, x):=h(t) x^{\lambda-1}$ and $x_{0}:=0$. Moreover, $q(t, 0) \equiv 0$ and condition (3.4) holds. According to (3.22), inequality (3.5) is obviously satisfied, because we have $q^{*}(t, \varrho)=h(t) \varrho^{\lambda-1}$. Therefore, by Corollary 3.2, problem (1.3), (1.2) has a positive solution $u_{0}$ and, thus, all the hypotheses of Theorem 3.15 (2) (with $\left.\alpha(t):=u_{0}(t)\right)$ are fulfilled.

Proof of Corollary 3.19. Put

$$
\begin{equation*}
p(t):=-a, \quad h(t):=b \quad \text { for } t \in[0, \omega] . \tag{5.36}
\end{equation*}
$$

It is clear that (3.16) holds and, by Remark 2.4, we get $p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$.
Let us show that condition (3.22) is satisfied, where $\Delta$ is defined in Remark 2.5. It follows from Remark 2.4 that

$$
p+r^{\lambda-1} h \in \mathcal{V}^{+}(\omega) \quad \text { if and only if } \quad-\frac{\pi^{2}}{\omega^{2}} \leq-a+b r^{\lambda-1}<0
$$

Moreover, by Remark 2.6, we get

$$
\Delta\left(p+r^{\lambda-1} h\right) \leq\left(2 \sqrt{a-b r^{\lambda-1}} \sin \frac{\omega \sqrt{a-b r^{\lambda-1}}}{2}\right)^{-1}
$$

for $r>0,-\frac{\pi^{2}}{\omega^{2}} \leq-a+b r^{\lambda-1}<0$. It is easy to see that $\sin x>\frac{2}{\pi} x$ for $\left.x \in\right] 0, \frac{\pi}{2}[$ and, thus,

$$
\begin{equation*}
\frac{1}{\Delta\left(p+r^{\lambda-1} h\right)}>\frac{2 \omega}{\pi}\left(a-b r^{\lambda-1}\right) \text { for } r>0,-\frac{\pi^{2}}{\omega^{2}}<-a+b r^{\lambda-1}<0 \tag{5.37}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{1}{\Delta\left(p+r^{\lambda-1} h\right)} \geq \frac{2 \omega}{\pi}\left(a-b r^{\lambda-1}\right) \quad \text { for } r>0,-\frac{\pi^{2}}{\omega^{2}}=-a+b r^{\lambda-1} \tag{5.38}
\end{equation*}
$$

Put

$$
\varphi(r):=a r-b r^{\lambda} \quad \text { for } 0 \leq r \leq\left(\frac{a}{b}\right)^{\frac{1}{\lambda-1}}
$$

By direct calculation, we show that

$$
\max \left\{\varphi(r): 0 \leq r \leq\left(\frac{a}{b}\right)^{\frac{1}{\lambda-1}}\right\}=\varphi\left(r^{*}\right), \quad \varphi^{\prime}(r)<0 \quad \text { for } r^{*}<r \leq\left(\frac{a}{b}\right)^{\frac{1}{\lambda-1}}
$$

where $r^{*}:=\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$.
If $a<\frac{\lambda}{\lambda-1}\left(\frac{\pi}{\omega}\right)^{2}$, then either $a \leq \frac{\pi^{2}}{\omega^{2}}$ or $a>\frac{\pi^{2}}{\omega^{2}},\left[\frac{1}{b}\left(a-\frac{\pi^{2}}{\omega^{2}}\right)\right]^{\frac{1}{\lambda-1}}<r^{*}$. Hence, we get

$$
\begin{equation*}
\max \left\{\varphi(r): r>0, \frac{1}{b}\left(a-\frac{\pi^{2}}{\omega^{2}}\right) \leq r^{\lambda-1}<\frac{a}{b}\right\}=\varphi\left(r^{*}\right) \tag{5.39}
\end{equation*}
$$

and, moreover, (5.37) yields

$$
\begin{equation*}
\frac{2 \omega}{\pi} \varphi\left(r^{*}\right)<\frac{r^{*}}{\Delta\left(p+\left(r^{*}\right)^{\lambda-1} h\right)} \tag{5.40}
\end{equation*}
$$

If $a \geq \frac{\lambda}{\lambda-1}\left(\frac{\pi}{\omega}\right)^{2}$, then $r^{*} \leq\left[\frac{1}{b}\left(a-\frac{\pi^{2}}{\omega^{2}}\right)\right]^{\frac{1}{\lambda-1}}$ and, thus,

$$
\begin{equation*}
\max \left\{\varphi(r): r>0, \frac{1}{b}\left(a-\frac{\pi^{2}}{\omega^{2}}\right) \leq r^{\lambda-1}<\frac{a}{b}\right\}=\varphi\left(r_{0}\right) \tag{5.41}
\end{equation*}
$$

where $r_{0}:=\left[\frac{1}{b}\left(a-\frac{\pi^{2}}{\omega^{2}}\right)\right]^{\frac{1}{\lambda-1}}$. Moreover, (5.38) implies

$$
\begin{equation*}
\frac{2 \omega}{\pi} \varphi\left(r_{0}\right) \leq \frac{r_{0}}{\Delta\left(p+r_{0}^{\lambda-1} h\right)} \tag{5.42}
\end{equation*}
$$

Therefore, from (3.26), (5.37), (5.38), (5.39), and (5.41), we conclude that the function $f$ satisfies

$$
\begin{align*}
\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s & \leq \frac{2 \omega}{\pi} \max \left\{\varphi(r): r>0, \frac{1}{b}\left(a-\frac{\pi^{2}}{\omega^{2}}\right) \leq r^{\lambda-1}<\frac{a}{b}\right\} \\
& \leq \sup \left\{\frac{r}{\Delta\left(p+r^{\lambda-1} h\right)}: r>0, p+r^{\lambda-1} h \in \mathcal{V}^{+}(\omega)\right\} \tag{5.43}
\end{align*}
$$

Furthermore, it follows from (5.40) and (5.42) that, if (5.43) holds in the form of equalities, then $a \geq \frac{\lambda}{\lambda-1}\left(\frac{\pi}{\omega}\right)^{2}$ and a supremum on the right-hand side of (5.43) is achieved at $r_{0}$. Consequently, taking into account Remark 3.17, all the hypotheses of Corollary 3.16 are fulfilled and, thus, problem (3.25) has at least one positive solution.

Proof of Corollary 3.22. Let the functions $p$ and $h$ be defined by (5.36). Then, (3.16) holds and, by Remark 2.4 , we get $p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega)$. Consequently, the conclusion of the corollary follows from Theorem 3.15 (3).

Proof of Theorem 3.25. Let the function $q$ be defined by formula (5.23). In view of (3.28), it is clear that $q$ is a Carathéodory function satisfying hypothesis $\left(H_{1}\right)$ with $q_{0}(t, x):=h(t) x^{\lambda-1}$ and $x_{0}:=0$. Moreover, hypothesis $\left(H_{2}^{\prime}\right)$ holds, $q(t, 0) \equiv 0$, and conditions (3.8), (3.9) with $x_{1}:=1$, and (3.14) are fulfilled. Furthermore, since the function $x \mapsto x^{\lambda}$ is strictly convex on $] 0,+\infty[$, one can show that $q$ satisfies hypothesis $\left(H_{0}^{2}\right)$.

Conclusion (1): It follows from Theorem 3.13 (3) with $\ell:=2$.
Conclusion (2): Assume that (3.22) holds, where $\Delta$ is defined in Remark 2.5. Then, inequality (3.5) is obviously satisfied, because we have $q^{*}(t, \varrho)=h(t) \varrho^{\lambda-1}$. Consequently, all the hypotheses of Corollary 3.7 are fulfilled and, thus, problem (1.3), (1.2) has at least one positive solution.

On the other hand, in view of the above-proved conclusion (1), problem (1.3), (1.2) has at most two positive solutions.

Conclusion (3): Let $\alpha \in A C_{\ell}([0, \omega])$ be a positive function satisfying (3.1) and (3.17). According to Lemma 4.16, there exists $R>0$ such that (5.24) holds. Consequently, all the hypotheses of Theorem 3.1 are fulfilled and, thus, problem (1.3), (1.2) has a positive solution $u$ satisfying (3.3). By the above-proved conclusion (1), problem (1.3), (1.2) has either one or two positive solutions.

If (1.3), (1.2) has a unique positive solution $u_{0}$, then we put $u^{*}:=u_{0}$. If (1.3), (1.2) has exactly two positive solutions $u_{1}, u_{2}$, then it follows from Theorem 3.13(1) that $u_{1}(t) \neq u_{2}(t)$ for $t \in[0, \omega]$, and we put

$$
u^{*}(t):=\max \left\{u_{1}(t), u_{2}(t)\right\} \quad \text { for } t \in[0, \omega] .
$$

It is clear that, in both these cases, $u^{*}$ satisfies (3.18).
We now show that every solution $u$ to problem (1.3), (1.2) satisfies (3.19). Suppose on the contrary that $u$ is a solution to problem (1.3), (1.2) such that (3.19) does not hold. Lemma 4.22 implies that, if $u(t) \leq u^{*}(t)$ for $t \in[0, \omega]$ and $u(t) \not \equiv u^{*}(t)$, then $u(t)<u^{*}(t)$ for $t \in[0, \omega]$. Therefore, $u$ satisfies (5.26). Put

$$
\alpha_{0}(t):=\max \left\{u(t), u^{*}(t)\right\} \quad \text { for } t \in[0, \omega] .
$$

Since we have proved that (5.24) holds for some $R>0$, in much the same way as in the proof of Theorem 3.15 (2), we determine a solution $\tilde{u}$ to problem (1.3), (1.2) satisfying (5.30), which is in contradiction with the definition of $u^{*}$.

Conclusion (4): Let $\alpha_{1} \in A C_{\ell}([0, \omega])$ and $\alpha_{2} \in A C^{1}([0, \omega])$ be such that (3.29), (3.30), and (3.31) hold. According to Lemma 4.16, there exists $R>0$ such that (5.24) holds. Consequently, it follows from Proposition 4.21 that there exists a function $\beta \in A C^{1}([0, \omega])$ satisfying (4.24), (5.1), and

$$
\beta(t) \geq \alpha_{1}(t) \quad \text { for } t \in[0, \omega] .
$$

Therefore, by virtue of (3.30), (3.31), (4.24), and (5.1), all the hypotheses of Lemma 4.17 (with $g(t, x):=p(t) x+q(t, x) x+f(t))$ are fulfilled and, thus, problem (1.3), (1.2) has a solution $u_{1}$ such that

$$
\begin{equation*}
\alpha_{1}(t) \leq u_{1}(t) \leq \beta(t) \quad \text { for } t \in[a, b] . \tag{5.44}
\end{equation*}
$$

We further determine a solution $u_{2}$ to problem (1.3), (1.2) satisfying (3.32). It follows from the hypothesis $(p, f) \in \mathcal{U}(\omega)$ (see Definition 3.24) that the problem

$$
\begin{equation*}
v^{\prime \prime}=p(t) v+f(t) ; \quad v(0)=v(\omega), v^{\prime}(0)=v^{\prime}(\omega) \tag{5.45}
\end{equation*}
$$

has a unique solution $v$, which is positive. Since $h$ satisfies (3.28) and $\alpha_{2}$ is positive, it follows from (3.30), (3.31), and (5.45) that

$$
\begin{gather*}
v^{\prime \prime}(t) \leq p(t) v(t)+h(t) v^{\lambda}(t)+f(t) \quad \text { for a.e. } t \in[0, \omega],  \tag{5.46}\\
\alpha_{2}(0)-v(0)=\alpha_{2}(\omega)-v(\omega), \quad \alpha_{2}^{\prime}(0)-v^{\prime}(0) \geq \alpha_{2}^{\prime}(\omega)-v^{\prime}(\omega) \quad \text { for } k=1,2,
\end{gather*}
$$

and

$$
\left(\alpha_{2}(t)-v(t)\right)^{\prime \prime} \geq p(t)\left(\alpha_{2}(t)-v(t)\right) \quad \text { for a. e. } t \in[0, \omega] .
$$

Therefore, by the hypothesis $p \in \mathcal{V}^{+}(\omega)$, the latter inequality yields

$$
\begin{equation*}
v(t) \leq \alpha_{2}(t) \quad \text { for } t \in[0, \omega] . \tag{5.47}
\end{equation*}
$$

Put

$$
\begin{equation*}
\delta:=\min \{v(t): t \in[0, \omega]\} \tag{5.48}
\end{equation*}
$$

and consider the periodic problem

$$
\begin{equation*}
u^{\prime \prime}=p(t) u+h(t)[\chi(u)]^{\lambda-1} u+f(t) ; \quad u(0)=u(\omega), u^{\prime}(0)=u^{\prime}(\omega), \tag{5.49}
\end{equation*}
$$

where

$$
\chi(x)= \begin{cases}x & \text { for } x \geq \delta  \tag{5.50}\\ \delta & \text { for } x<\delta\end{cases}
$$

In view of (3.28), it is clear that $\chi(x) \geq \delta$ for $x \in \mathbb{R}$ and

$$
\begin{aligned}
&(p(t) x\left.+h(t)[\chi(x)]^{\lambda-1} x+f(t)\right) \operatorname{sgn} x \\
& \geq\left(p(t)+\delta^{\lambda-1} h(t)\right)|x|-|f(t)| \quad \text { for a.e. } t \in[0, \omega] \text { and all } x \in \mathbb{R}
\end{aligned}
$$

By Lemmas 4.7 and 4.9, we get $p+\delta^{\lambda-1} h \in \operatorname{Int} \mathcal{D}(\omega)$, because $\delta>0$ and $h$ satisfies (3.28). Therefore, in view of (3.30), (3.31), (5.45), (5.46), (5.47), and (5.50), all the hypotheses of Lemma 4.18 (with $g(t, x):=p(t) x+[\chi(x)]^{\lambda-1} x+f(t), p_{0}(t):=p(t)+\delta^{\lambda-1} h(t), z(t, x):=$ $|f(t)|, \alpha(t):=\alpha_{2}(t)$, and $\left.\beta(t):=v(t)\right)$ are fulfilled and, thus, problem (5.49) possesses a solution $u_{2}$ such that

$$
\begin{equation*}
v\left(t_{0}\right) \leq u_{2}\left(t_{0}\right) \leq \alpha_{2}\left(t_{0}\right) \quad \text { for some } t_{0} \in[0, \omega] . \tag{5.51}
\end{equation*}
$$

Let $z(t):=u_{2}(t)-v(t)$ for $t \in[0, \omega]$. It is clear that $z$ is a solution to the linear problem

$$
\begin{gathered}
z^{\prime \prime}=\left(p(t)+\delta^{\lambda-1} h(t)\right) z+h(t)\left(\left[\chi\left(u_{2}(t)\right)\right]^{\lambda-1}-\delta^{\lambda-1}\right) u_{2}(t)+\delta^{\lambda-1} h(t) v(t) \\
z(0)=z(\omega), \quad z^{\prime}(0)=z^{\prime}(\omega)
\end{gathered}
$$

and, by virtue of (3.28), (5.50), and the condition $\delta>0$, we get

$$
\begin{gathered}
h(t)\left(\left[\chi\left(u_{2}(t)\right)\right]^{\lambda-1}-\delta^{\lambda-1}\right) u_{2}(t)+\delta^{\lambda-1} h(t) v(t) \geq 0 \quad \text { for a.e. } t \in[0, \omega] \\
h(t)\left(\left[\chi\left(u_{2}(t)\right)\right]^{\lambda-1}-\delta^{\lambda-1}\right) u_{2}(t)+\delta^{\lambda-1} h(t) v(t) \not \equiv 0 .
\end{gathered}
$$

Therefore, in view of the inclusion $p+\delta^{\lambda-1} h \in \operatorname{Int} \mathcal{D}(\omega)$ and condition (5.51), it follows from Lemma 4.10 (with $g(t):=p(t)+\delta^{\lambda-1} h(t)$ and $\ell(t):=h(t)\left(\left[\chi\left(u_{2}(t)\right)\right]^{\lambda-1}-\delta^{\lambda-1}\right) u_{2}(t)+$ $\left.\delta^{\lambda-1} h(t) v(t)\right)$ that $z(t)>0$ for $t \in[0, \omega]$, i. e.,

$$
u_{2}(t)>v(t) \quad \text { for } t \in[0, \omega] .
$$

Consequently, (5.48) yields $u_{2}(t)>\delta$ for $t \in[0, \omega]$, which, in view of (5.50), yields $\chi\left(u_{2}(t)\right)=$ $u_{2}(t)$ for $t \in[0, \omega]$ and, thus, $u_{2}$ is a positive solution to problem (1.3), (1.2). Moreover, (3.29), (5.44), and (5.51) yield $u_{1}\left(t_{0}\right)>u_{2}\left(t_{0}\right)$ for some $t_{0} \in[0, \omega]$. Therefore, by Theorem 3.13 (1), we conclude that the solutions $u_{1}, u_{2}$ satisfy (3.32). Furthermore, the above-proved conclusion (1) implies that problem (1.3), (1.2) has exactly two positive solutions.

Finally, let $u$ be a solution to problem (1.3), (1.2) different from $u_{1}$. Then, it follows from the above-proved conclusion (3) that $u$ satisfies (3.33).

Conclusion (5): Assume that (3.7) holds. Then, the existence and uniqueness of a positive solution to problem (1.1), (1.2) follow from Corollary 3.14.

Proof of Corollary 3.29. Let the function $q$ be defined by formula (5.23).
Conclusion (1): Assume that (3.34) holds, where $\Delta$ is defined in Remark 2.5. Observe that the function $q^{*}$ given by (3.6) is of the form $q^{*}(t, \varrho)=h(t) \varrho^{\lambda-1}$. Put

$$
H:=\int_{0}^{\omega} h(s) \mathrm{d} s, \quad F:=\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s .
$$

Since $[f(t)]_{+} \not \equiv 0$, by direct calculation, we get

$$
\begin{aligned}
\sup \left\{\frac{r}{\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s+r \int_{0}^{\omega} q^{*}(s, r) \mathrm{d} s}: r>0\right\} & =\sup \left\{\frac{r}{F+H r^{\lambda}}: r>0\right\} \\
& =\frac{(\lambda-1)^{\frac{\lambda-1}{\lambda}}}{\lambda} F^{-\frac{\lambda-1}{\lambda}} H^{-\frac{1}{\lambda}}
\end{aligned}
$$

and this supremum is achieved at $r_{0}:=\left[\frac{F}{(\lambda-1) H}\right]^{\frac{1}{\lambda}}$. Therefore, (3.34) yields

$$
\Delta(p) \leq \frac{r_{0}}{\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s+r_{0} \int_{0}^{\omega} q^{*}\left(s, r_{0}\right) \mathrm{d} s}
$$

and, thus, Proposition 4.20 guarantees that there exists a positive function $\alpha \in A C^{1}([0, \omega])$ satisfying (3.17) and (3.23). Consequently, all the hypotheses of Theorem 3.25 (3) are fulfilled.

Conclusion (2): Assume that (3.35) holds, where $\Delta$ is defined in Remark 2.5. Then, there exits $\varepsilon>1$ such that

$$
0<\int_{0}^{\omega}[\varepsilon f(s)]_{+} \mathrm{d} s \leq \frac{\lambda-1}{\lambda[\Delta(p)]^{\frac{\lambda}{\lambda-1}}\left[\lambda \int_{0}^{\omega} h(s) \mathrm{d} s\right]^{\frac{1}{\lambda-1}}}
$$

In much the same way as in the proof of conclusion (1), we show that there exists $r_{0}>0$ such that

$$
\Delta(p) \leq \frac{r_{0}}{\int_{0}^{\omega}[\varepsilon f(s)]_{+} \mathrm{d} s+r_{0} \int_{0}^{\omega} q^{*}\left(s, r_{0}\right) \mathrm{d} s}
$$

By Proposition 4.20 (with $[\varepsilon f]_{+}$instead of $[f]_{+}$), there exists a positive function $\alpha_{1} \in A C^{1}([0, \omega])$ such that

$$
\begin{gather*}
\alpha_{1}(0)=\alpha_{1}(\omega), \quad \alpha_{1}^{\prime}(0)=\alpha_{1}^{\prime}(\omega),  \tag{5.52}\\
\alpha_{1}^{\prime \prime}(t) \geq p(t) \alpha_{1}(t)+q\left(t, \alpha_{1}(t)\right) \alpha_{1}(t)+\varepsilon[f(t)]_{+} \quad \text { for a.e. } t \in[0, \omega] . \tag{5.53}
\end{gather*}
$$

Since $\varepsilon>1$, the function $\alpha_{1}$ satisfies

$$
\left.\alpha_{1}^{\prime \prime}(t) \geq p(t) \alpha_{1}(t)+h(t) \alpha_{1}^{\lambda}(t)\right)+f(t) \quad \text { for a.e. } t \in[0, \omega] .
$$

Put $\alpha_{2}(t):=\frac{1}{\varepsilon} \alpha_{1}(t)$ for $t \in[0, \omega]$. Then, (3.29) holds and from (5.52) and (5.53), we get

$$
\alpha_{2}(0)=\alpha_{2}(\omega), \quad \alpha_{2}^{\prime}(0)=\alpha_{2}^{\prime}(\omega)
$$

and

$$
\begin{aligned}
\alpha_{2}^{\prime \prime}(t) & \geq p(t) \alpha_{2}(t)+q\left(t, \varepsilon \alpha_{2}(t)\right) \alpha_{2}(t)+[f(t)]_{+} \\
& =p(t) \alpha_{2}(t)+\varepsilon^{\lambda-1} h(t) \alpha_{2}^{\lambda}(t)+[f(t)]_{+} \\
& \geq p(t) \alpha_{2}(t)+h(t) \alpha_{2}^{\lambda}(t)+f(t) \quad \text { for a.e. } t \in[0, \omega],
\end{aligned}
$$

because $\varepsilon>1$ and $h$ satisfies (3.28). Consequently, $\alpha_{1}, \alpha_{2}$ satisfy (3.29), (3.31), and (3.36) and, thus, all the hypotheses of Theorem 3.25 (4) are fulfilled.

Proof of Corollary 3.30. Put

$$
\begin{equation*}
p(t):=-a \quad \text { for } t \in[0, \omega] . \tag{5.54}
\end{equation*}
$$

By Remarks 2.4 and 2.6 , we get $p \in \mathcal{V}^{+}(\omega)$ and

$$
\Delta(p) \leq\left(2 \sqrt{a} \sin \frac{\omega \sqrt{a}}{2}\right)^{-1}
$$

Consequently, hypothesis (3.37) yields (3.34) and, thus, problem (3.38) has either one or two positive solutions as follows from Corollary 3.29 (1) and Theorem $3.25(1,3)$.

Proof of Corollary 3.31. Let the function $p$ be defined by (5.54). By Remark 2.4, we get $p \in$ $\mathcal{V}^{+}(\omega)$ and, thus, Theorem 3.25 (5) implies that problem (3.38) has a unique positive solution.

Proof of Theorem 3.32. Suppose on the contrary that $u$ is a non-negative solution to problem (1.3), (1.2). In view of (3.28) and (3.39), it follows from Lemma 4.23 (with $g(t):=h(t) u^{\lambda}(t)$ and $\varrho:=\Gamma(p))$ that

$$
\begin{align*}
& \int_{0}^{\omega}\left[h(s) u^{\lambda}(s)+f(s)\right]_{+} \mathrm{d} s-\Gamma(p) \int_{0}^{\omega}\left[h(s) u^{\lambda}(s)+f(s)\right]_{-} \mathrm{d} s \\
& \quad \geq \int_{0}^{\omega} h(s) u^{\lambda}(s) \mathrm{d} s+\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s-\Gamma(p) \int_{0}^{\omega}[f(s)]_{-} \mathrm{d} s \geq 0 . \tag{5.55}
\end{align*}
$$

Assuming $h(t) u^{\lambda}(t)+f(t) \equiv 0$, we conclude easily that $u$ is a solution to problem (2.1) which, together with the hypothesis $p \in \operatorname{Int} \mathcal{V}^{+}(\omega)$, yields $u(t) \equiv 0$. However, this is in contradiction with the hypothesis $f(t) \not \equiv 0$. Therefore, $h(t) u^{\lambda}(t)+f(t) \not \equiv 0$ and, thus, from Lemma 4.3 (with $g(t):=p(t)$ and $\ell(t):=h(t) u^{\lambda}(t)+f(t)$ ), we get

$$
\begin{equation*}
\Gamma(p) \int_{0}^{\omega}[p(s)]_{-} \mathrm{d} s>\int_{0}^{\omega}[p(s)]_{+} \mathrm{d} s \tag{5.56}
\end{equation*}
$$

and

$$
\begin{array}{rl}
u(t)>v & v \int_{0}^{\omega}\left[h(s) u^{\lambda}(s)+f(s)\right]_{+} \mathrm{d} s  \tag{5.57}\\
& \left.-\Gamma(p) \int_{0}^{\omega}\left[h(s) u^{\lambda}(s)+f(s)\right]_{-} \mathrm{d} s\right) \quad \text { for } t \in[0, \omega]
\end{array}
$$

where

$$
v:=\left(\Gamma(p) \int_{0}^{\omega}[p(s)]_{-} \mathrm{d} s-\int_{0}^{\omega}[p(s)(s)]_{+} \mathrm{d} s\right)^{-1} .
$$

The latter condition, together with (5.55) and (5.56) yields

$$
\begin{equation*}
m>0 \tag{5.58}
\end{equation*}
$$

where $m:=\min \{u(t): t \in[0, \omega]\}$. Put

$$
H:=\int_{0}^{\omega} h(s) \mathrm{d} s, \quad F:=\int_{0}^{\omega}[f(s)]_{+} \mathrm{d} s-\Gamma(p) \int_{0}^{\omega}[f(s)]_{-} \mathrm{d} s .
$$

Then, it follows (3.28), (3.39), and (5.56) that $H>0$ and $F>0$. Moreover, (5.55) and (5.57) lead to the inequality

$$
\begin{equation*}
m>v H m^{\lambda}+v F . \tag{5.59}
\end{equation*}
$$

Put

$$
\varphi(x):=-x+v H x^{\lambda}+v F \quad \text { for } x>0 .
$$

One can show by direct calculation that

$$
\inf \{\varphi(x): x>0\}=v F-\frac{\lambda-1}{\lambda}\left(\frac{1}{\lambda v H}\right)^{\frac{1}{\lambda-1}}
$$

and, thus, hypothesis (3.39) implies that $\inf \{\varphi(x): x>0\} \geq 0$. Hence,

$$
-x+v H x^{\lambda}+v F \geq 0 \quad \text { for } x>0,
$$

which, in view of (5.58), contradicts (5.59).
Proof of Theorem 3.33. Let the function $q$ be defined by formula (5.23). In view of (3.28), it is clear that $q$ is a Carathéodory function satisfying (3.14).

Conclusion (1): Assume that $p \in \mathcal{V}^{+}(\omega)$ and (3.40) holds, where $\Delta$ is defined in Remark 2.5. Since $f$ satisfies (3.15), the inclusion $(p, f) \in \mathcal{U}(\omega)$ holds (see Remark 3.26) and condition (3.35) is fulfilled. Therefore, it follows from Corollary 3.29 (2) and Theorem 3.25 (4) that problem (1.3), (1.2) has exactly two positive solutions $u_{1}, u_{2}$ and these solutions satisfy (3.32).

Since $u$ is a negative solution to problem (1.3), (1.2) if and only if the function $-u$ is a positive solution to the problem

$$
\begin{equation*}
z^{\prime \prime}=p(t) z+h(t)|z|^{\lambda} \operatorname{sgn} z-f(t) ; \quad z(0)=z(\omega), \quad z^{\prime}(0)=z^{\prime}(\omega), \tag{5.60}
\end{equation*}
$$

it follows from Theorem 3.25 (5) that problem (1.3), (1.2) possesses a unique negative solution $u_{3}$.

Finally, by Theorem $3.13(4)$, we conclude that problem (1.3), (1.2) has exactly three solutions $u_{1}, u_{2}, u_{3}$ and these solutions satisfy (3.41).

Conclusion (2): Assume that $p \in \operatorname{Int} \mathcal{V}^{+}(\omega)$ and (3.42) holds, where $\Gamma$ is given by (2.3). Since $u$ is a negative solution to problem (1.3), (1.2) if and only if the function $-u$ is a positive solution to problem (5.60), it follows from Theorem 3.25 (5) that problem (1.3), (1.2) possesses a unique negative solution $u_{0}$. Moreover, Theorem 3.32 implies that problem (1.3), (1.2) has no positive solution.

Therefore, by Theorem 3.13 (4), we conclude that problem (1.3), (1.2) has exactly one solutions $u_{0}$ and this solution is negative.

## Acknowledgements

The research has been supported by the internal grant FSI-S-20-6187 of FME BUT.

## References

[1] G. Duffing, Erzwungen Schwingungen bei veränderlicher Eigenfrequenz und ihre technische Bedeutung, Vieweg Heft 41/42, Vieweg, Braunschweig, 1918. https://doi.org/10.1002/ zamm. 19210010109
[2] A. Fonda, Playing around resonance. An invitation to the search of periodic solutions for second order ordinary differential equations, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser/Springer, Cham, 2016. https://doi.org/10.1007/978-3-319-47090-0; MR3585909; Zbl 1364.34004
[3] P. Habets, C. De Coster, Two-point boundary value problems: lower and upper solutions, Mathematics in Science and Engineering, Vol. 205, Elsevier B.V., Amsterdam, 2006. MR2225284; Zbl 1330.34009
[4] H. Chen, Y. Li, Stability and exact multiplicity of periodic solutions of Duffing equations with cubic nonlinearities, Proc. Amer. Math. Soc. 135(2007), No. 12, 3925-3932. https: //doi.org/10.1090/S0002-9939-07-09024-7; MR2341942; Zbl 1166.34313
[5] I. Kovacic, M. J. Brennan (Eds.), The Duffing equation. Nonlinear oscillators and their behaviour, John Wiley \& Sons, Ltd., Publication, Hoboken, NJ, 2011. https://doi. org/10. 1002/9780470977859 MR2866747; Zbl 1220.34002
[6] A. Lomtatidze, Theorems on differential inequalities and periodic boundary value problem for second-order ordinary differential equations, Mem. Differential Equations Math. Phys. 67(2016), 1-129. MR3472904; Zbl 1352.34033
[7] A. Lomtatidze, On periodic boundary value problem for second order ordinary differential equations, Commun. Contemp. Math. 22(2020), No. 6, 1950049. https://doi.org/ 10.1142/S0219199719500494; MR4130252; Zbl 7237218
[8] A. Lomtatidze, J. Šremr, On periodic solutions to second-order Duffing type equations, Nonlinear Anal. Real World Appl. 40(2018), 215-242. https ://doi. org/10.1016/j .nonrwa. 2017.09.001; MR3718982; Zbl 1396.34024
[9] P. J. Torres, Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem, J. Differential Equations 190(2003), No. 2, 643-662. https://doi.org/10.1016/S0022-0396(02)00152-3; MR1970045; Zbl 1032.34040

# Subharmonic bouncing solutions of generalized Lazer-Solimini equation 

Jan Tomeček ${ }^{\boxtimes}$ and Věra Krajščáková<br>Department of Mathematical Analysis and Applications of Mathematics, Faculty of Science, Palacký University, 17. listopadu 12, Olomouc, 771 46, Czechia

Received 21 April 2021, appeared 8 September 2021
Communicated by Gennaro Infante


#### Abstract

The paper deals with the singular differential equation $x^{\prime \prime}+g(x)=p(t)$, with $g$ having a weak singularity at $x=0$ and $2 \pi$-periodic function $p$. For any positive integers $m$ and $n$, the coexistence of $2 m \pi$-periodic bouncing solutions having $n$ impacts with the singularity and a classical positive periodic solution is proven.


Keywords: subharmonic solution, elastic impact, nonnegative solution, impulsive differential equation, generalized Lazer-Solimini equation, coexistence of solutions, weak singularity.
2020 Mathematics Subject Classification: 34A37, 34B18, 34C25.

## 1 Introduction

Investigation of Lazer-Solimini equation dates back to the year 1987 when the authors Lazer and Solimini published their existence results for the equation

$$
x^{\prime \prime} \pm \frac{1}{x^{\alpha}}=p(t), \quad \alpha>0, p \text { is a } 2 \pi \text {-periodic function, }
$$

where they found necessary and sufficient conditions for the existence of periodic solution, see [3].

Later, many authors (e.g. see [1,2,8] or see an overview of the results in [10]) obtained existence results for equation with a generalized singular term

$$
\begin{equation*}
x^{\prime \prime}+g(x)=p(t) \tag{1.1}
\end{equation*}
$$

where $g:(0, \infty) \rightarrow \mathbb{R}$ has various types of singularity at $x=0$. Two types of this singularity are distinguished:

- attractive, i.e. $\lim _{x \rightarrow 0+} g(x)=+\infty$, vs. repulsive, i.e. $\lim _{x \rightarrow 0+} g(x)=-\infty$
and

[^25]- weak, i.e. $\int_{0}^{1} g(x) \mathrm{d} x \in \mathbb{R}$ vs. strong, i.e. $\int_{0}^{1} g(x) \mathrm{d} x= \pm \infty$.

It is a well-known result, that under the assumptions that $g$ is positive, nonincreasing and continuous on $(0, \infty), \lim _{x \rightarrow \infty} g(x)=0$ and $p$ is a $2 \pi$-periodic and continuous on $\mathbb{R}$, then the necessary and sufficient condition for the existence of a classical positive $2 \pi$-periodic solution of Eq. (1.1) is the assumption

$$
\bar{p}:=\frac{1}{2 \pi} \int_{0}^{2 \pi} p(t) \mathrm{d} t>0
$$

where $2 \pi$ is the period of the function $p$-see e.g. $[1,8]$ (the necessity can be immediately seen by integrating Eq. (1.1) over the interval $[0,2 \pi]$ ).

Otherwise, i.e. if $\bar{p} \leq 0$, the Eq. (1.1) can be understood as an impact oscillator having a singularity at the obstacle. Therefore one can investigate another type of solution - so called bouncing solution - e.g. see [5]. It is a generalized solution of Eq. (1.1) in the sense that

- such function is a solution of Eq. (1.1) only on certain open intervals where it is positive,
- it satisfies certain impulsive conditions at those instants where the solution reaches zero - see Definition 2.2.

The problems of the existence of such solutions were investigated using Poincaré-Birkhoff Twist Map Theorem for an area preserving homeomorphism of an annulus, e.g. see [4-7,9].

In particular, in 2004, Qian and Torres [6] investigated Eq. (1.1) with an attractive weak singularity for the case $\bar{p}<0$, i.e. if no classical solution exists. They found sufficient conditions for the existence of periodic and subharmonic solutions with prescribed number of bounces in each period. They suggested a possible existence of this type of solution even in the case when the classical solution exists, i.e. a classical solution would coexist with a bouncing one. In [9], this question was partially answered. Sufficient conditions ensuring the existence of at least two $2 \pi$-periodic bouncing solutions with one bounce in each period were given.

The purpose of this paper is to extend the results of [9] and find sufficient conditions guaranteeing the existence of the subharmonic solutions with prescribed number of bounces in each period. The proofs in [9] are based on the investigation of the area-preserving homeomorphism $T$ which has been constructed just for one bounce in the period. But $T$ looses some needed properties (e.g. the monotonicity of its first component $T_{1}$ ) if the construction of $T$ is extended for more bounces in the period, and so the approach of [9] cannot be directly used. Therefore the proofs in this paper are based on the combination of the results obtained in [6] and [9].

The paper is organized as follows. In Section 2 we give necessary definitions of a classical and bouncing solution, the main result (Theorem 2.3) together with a consequent result for Lazer-Solimini equation (Corollary 2.4) and an example. In the third section we prove a slight modification of the existence theorem from [6] in order to apply it to the properly constructed auxiliary equation (3.4). In Section 4, the estimations of bouncing solutions of the auxiliary problem are given and subsequently the proof of the main result is finished.

## 2 Problem formulation and main results

We investigate the differential equation of the second order (1.1) under the following assumptions:

$$
\begin{equation*}
g \text { is locally Lipschitz continuous function, positive and nonincreasing on }(0, \infty) \tag{2.1}
\end{equation*}
$$

$$
\begin{gather*}
\lim _{x \rightarrow 0+} g(x)=\infty, \quad \int_{0}^{1} g(x) \mathrm{d} x<\infty,  \tag{2.2}\\
p \text { is continuous function, } 2 \pi \text {-periodic on } \mathbb{R},  \tag{2.3}\\
\frac{1}{2 \pi} \int_{0}^{2 \pi} p(s) \mathrm{d} s=: \bar{p}>g(\infty):=\lim _{x \rightarrow \infty} g(x) . \tag{2.4}
\end{gather*}
$$

Let us precisely define the types of solutions of Eq. (1.1) used in this article. To emphasize the concept of bouncing solution, we start with a classical solution.

Definition 2.1. We say that $x$ is a (classical) solution of Eq. (1.1) on an interval $J \subset \mathbb{R}$ iff $x$ is a positive, twice continuously differentiable function on $J$, and $x$ satisfies the differential equation (1.1) on $J$.

Definition 2.2. We say that $x: \mathbb{R} \rightarrow \mathbb{R}$ is a bouncing solution of Eq. (1.1) iff there exists a doubly infinite sequence $\left\{t_{i}\right\}_{i \in \mathbb{Z}}, t_{i}<t_{i+1}, i \in \mathbb{Z}$ such that
(i) $x\left(t_{i}\right)=0$,
(ii) $x^{\prime}\left(t_{i}+\right)=-x^{\prime}\left(t_{i}-\right)$,
(iii) $x$ is a classical solution of Eq. (1.1) on $\left(t_{i}, t_{i+1}\right)$.

We call $t_{i}$ the bounces of the solution $x$.
We can see that a bouncing solution consists of several maximal classical solutions separated by bounces.

To state the main result of the paper, we introduce the notation

$$
p_{\max }=\max _{s \in \mathbb{R}} p(s), \quad p_{\min }=\min _{s \in \mathbb{R}} p(s),
$$

and denote by $K$ a positive constant satisfying

$$
\begin{equation*}
g(K)>p_{\max } . \tag{2.5}
\end{equation*}
$$

The existence of such $K$ follows from assumptions (2.1)-(2.3).
Theorem 2.3 (Main result: Coexistence of bouncing and classical periodic solutions). Let (2.1)(2.4) hold and let

$$
\begin{equation*}
\left(\frac{K}{m}\right)^{2}+2 \pi^{2} p_{\min } K \geq 2 \pi^{2} \int_{0}^{K} g(x) \mathrm{d} x \tag{2.6}
\end{equation*}
$$

where $K$ fulfills (2.5), $m \in \mathbb{N}$. Then
(i) there exists a classical solution of Eq. (1.1) greater than K,
(ii) there exist at least two $2 m \pi$-periodic bouncing solutions of Eq. (1.1) with one bounce in each period such that their maximal values are lower than $K$,
(iii) for any $n \in \mathbb{N}, n>1$ there exist at least one $2 m \pi$-periodic bouncing solution of Eq. (1.1) with exactly $n$ bounces in each period, which has the maximum value lower than $K$.

We give even more effective sufficient conditions for the existence of solutions of LazerSolimini equation

$$
\begin{equation*}
x^{\prime \prime}+x^{-\alpha}=p(t), \tag{2.7}
\end{equation*}
$$

with $\alpha \in(0,1)$.

Corollary 2.4. Let $p: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous, $2 \pi$-periodic function, $\bar{p}>0, \alpha \in(0,1)$ and $m \in \mathbb{N}$ be such that

$$
\begin{equation*}
m<\frac{1}{\pi} \sqrt{\frac{1-\alpha}{2 p_{\max }^{\frac{1}{\alpha}}\left(p_{\max }-(1-\alpha) p_{\min }\right)}} \tag{2.8}
\end{equation*}
$$

Then
(i) there exists a classical solution of Eq. (2.7) greater or equal to $p_{\max }^{-\frac{1}{a}}$,
(ii) there exist at least two $2 m \pi$-periodic bouncing solutions of Eq. (2.7) with one bounce in each period such that their maximal values are lower than $p_{\text {max }}^{-\frac{1}{a}}$
(iii) for any $n \in \mathbb{N}, n>1$ there exist at least one $2 m \pi$-periodic bouncing solution of Eq. (2.7) with exactly $n$ bounces in each period, which has the maximum value lower than $p_{\max }^{-\frac{1}{\alpha}}$.

Proof. We apply Theorem 2.3 on Eq. (2.7). The assumptions (2.1)-(2.4) are trivially satisfied for $g(x)=x^{-\alpha}, x>0$. It remains to find a positive $K$ satisfying (2.5) and (2.6). These conditions are satisfied iff

$$
\begin{equation*}
K^{-\alpha}>p_{\max } \tag{2.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\frac{K}{m}\right)^{2}+2 \pi^{2} p_{\min } K \geq 2 \pi^{2} \frac{K^{1-\alpha}}{1-\alpha} \tag{2.10}
\end{equation*}
$$

The inequality (2.9) is equivalent to $K<p_{\max }^{-\frac{1}{\alpha}}$. And the inequality (2.10) can be written in the form

$$
\omega(K):=\frac{K}{2 \pi^{2} m^{2}}+p_{\min }-\frac{K^{-\alpha}}{1-\alpha} \geq 0
$$

where $\omega: \mathbb{R} \rightarrow \mathbb{R}$ is continuous on $(0, \infty), \lim _{K \rightarrow 0+} \omega(K)=-\infty, \lim _{K \rightarrow \infty} \omega(K)=\infty$ and $\omega^{\prime}(K)>0$ for each $K \in(0, \infty)$. Therefore there exists a unique $K_{0}>0$ such that $\omega\left(K_{0}\right)=0$. Since $\omega$ is strictly increasing, $K$ satisfies (2.9) and (2.10) iff $K \in\left[K_{0}, p_{\max }^{-\frac{1}{\alpha}}\right.$ ). From (2.8) we get $\omega\left(p_{\max }^{-\frac{1}{\alpha}}\right)>0$, which implies that the interval $\left[K_{0}, p_{\max }^{-\frac{1}{\alpha}}\right)$ is nonempty. Let us choose some $K \in\left[K_{0}, p_{\max }^{-\frac{1}{\alpha}}\right)$. According to Theorem 2.3 (i) there exists a classical solution $x$ of Eq. (2.7) greater than $K$. If $\min _{t \in \mathbb{R}} x(t)<p_{\text {max }}^{-\frac{1}{\alpha}}$, then there exists $t_{0} \in \mathbb{R}$ such that $x\left(t_{0}\right)<p_{\text {max }}^{-\frac{1}{\alpha}}$ $x^{\prime}\left(t_{0}\right)=0$ and $x^{\prime \prime}\left(t_{0}\right) \geq 0$. In view of (2.7) we get

$$
x^{\prime \prime}\left(t_{0}\right)=-\left(x\left(t_{0}\right)\right)^{-\alpha}+p\left(t_{0}\right)<-p\left(t_{0}\right)+p\left(t_{0}\right)=0
$$

which is a contradiction. Therefore the classical solution is bounded from below by $p_{\text {maxx }}^{-\frac{1}{x}}$.
The assertions (ii) and (iii) follow directly from Theorem 2.3 (ii), (iii) and from the inequality $K<p_{\text {max }}^{-\frac{1}{\alpha}}$.

The feasibility of the obtained result is illustrated in the following example.
Example 2.5. We consider Lazer-Solimini equation (2.7), where $\alpha \in(0,1), m \in \mathbb{N}$ and $p(t)=$ $b \sin t+c$ with $b, c>0$. Then $p_{\max }=b+c$ and $p_{\min }=c-b$, and so the condition (2.8) can be written as

$$
\begin{equation*}
m<\frac{1}{\pi} \sqrt{\frac{1-\alpha}{2(b+c)^{\frac{1}{\alpha}}(2 b+\alpha(c-b))}} \tag{2.11}
\end{equation*}
$$

For instance, the condition (2.11) is valid for these values of the parameters:

- $\alpha=0.5, b=0.01, c=0.11, m \leq 3$, or
- $\alpha=0.1, b=0.01, c=0.51, m \leq 30$.

Remark 2.6. Let us note that the assumptions of Theorem 2.3 always fail to be satisfied for high $m$. Indeed, let the assumptions of Theorem 2.3 hold for each $m \in \mathbb{N}$ with $K=K_{m}$ in (2.5) and (2.6), i.e.

$$
\begin{equation*}
g\left(K_{m}\right)>p_{\max } \tag{2.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\frac{K_{m}}{m}\right)^{2}+2 \pi^{2} p_{\min } K_{m} \geq 2 \pi^{2} \int_{0}^{K_{m}} g(x) \mathrm{d} x \tag{2.13}
\end{equation*}
$$

for each $m \in \mathbb{N}$. From (2.1), (2.2) and (2.4) it follows that there exists $\bar{K}>0$ such that $g(\bar{K})=p_{\max }$ and according to (2.12) and (2.1) also $\bar{K}>K_{m}$ for every $m \in \mathbb{N}$, i.e. $\left\{K_{m}\right\}$ is bounded. On the other hand, from (2.1) and (2.12) we get

$$
\int_{0}^{K_{m}} g(x) \mathrm{d} x \geq \int_{0}^{K_{m}} g\left(K_{m}\right) \mathrm{d} x=g\left(K_{m}\right) K_{m}>p_{\max } K_{m}
$$

This estimate together with (2.13) yields an inequality

$$
\left(\frac{K_{m}}{m}\right)^{2}+2 \pi^{2} p_{\min } K_{m}>2 \pi^{2} p_{\max } K_{m}
$$

which gives

$$
\left(\frac{K_{m}}{m}\right)^{2}>2 \pi^{2}\left(p_{\max }-p_{\min }\right) K_{m}
$$

and finally

$$
K_{m}>2 \pi^{2}\left(p_{\max }-p_{\min }\right) m^{2}
$$

for every $m \in \mathbb{N}$. The last inequality contradicts the boundedness of $\left\{K_{m}\right\}$. Therefore, the (non)existence of subharmonic solutions of arbitrary period is still an open problem.

## 3 Auxiliary equation

First, let us state the main result from [6], which will be used here as the main existence principle.

Theorem 3.1 (see [6, Theorem 1.2]). Let us assume that

$$
\left.\begin{array}{l}
g:(0, \infty) \rightarrow(0, \infty) \text { is locally Lipschitz continuous function, }  \tag{3.1}\\
\text { there exists } \varepsilon>0 \text { such that } g \text { is strictly decreasing on }(0, \varepsilon),
\end{array}\right\}
$$

$g$ satisfies (2.2), $p$ fulfills (2.3) and

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{0}^{2 \pi} p(s) \mathrm{d} s=: \bar{p}<0=g(\infty):=\lim _{x \rightarrow \infty} g(x) \tag{3.2}
\end{equation*}
$$

Then, for any $m \in \mathbb{N}$, there exist at least two $2 m \pi$-periodic bouncing solutions of Eq. (1.1) with one bounce in each period. Moreover, for any $n, m \in \mathbb{N}, n \geq 2$, there exists at least one $2 m \pi$-periodic bouncing solution of Eq. (1.1) with $n$ bounces in each period.

In the current paper we will use this result under slightly different assumptions. More precisely, we replace assumption (3.1) by (2.1) and assumption (3.2) by

$$
\begin{equation*}
\bar{p}<0, \quad 0 \leq g(\infty):=\lim _{x \rightarrow \infty} g(x) \tag{3.3}
\end{equation*}
$$

Theorem 3.2. Let us assume that (2.1), (2.2), (2.3) and (3.3) hold. Then the assertions of Theorem 3.1 remain valid.

Proof. Decreasing character of $g$ in (3.1) is used in the paper [6] only to prove uniqueness in the singular IVP (1.1), $x\left(t_{0}\right)=0, x^{\prime}\left(t_{0}\right)=y_{0}>0$, see [6, Remark 2.4]. Since this uniqueness was already proved in [9] under the assumptions (2.1)-(2.3), the replacement of (3.1) by (2.1) in Theorem 3.2 is correct.

In [6], only the positivity of the function $g$ is used, not the fact $g(\infty)=0$. Therefore the replacement of (3.2) by (3.3) is also correct.

Now, we introduce the auxiliary equation

$$
\begin{equation*}
x^{\prime \prime}+f(x)=p(t) \tag{3.4}
\end{equation*}
$$

where $f:(0, \infty) \rightarrow(0, \infty)$ is defined by

$$
f(x)= \begin{cases}g(x) & \text { if } x \in(0, K]  \tag{3.5}\\ g(K) & \text { if } x>K\end{cases}
$$

with $g, p$ and $K$ satisfying (2.1), (2.2), (2.3) and (2.5). From (2.5) it follows that there exists $\varepsilon>0$ such that

$$
g(K)-p_{\max }>\varepsilon
$$

and therefore

$$
\begin{equation*}
f(x)-p_{\max }>\varepsilon \tag{3.6}
\end{equation*}
$$

for each $x>0$.
Theorem 3.3. Let us assume that (2.1), (2.2), (2.3) and (3.3) hold and let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by (3.5). Then the assertions of Theorem 3.2 are valid for Eq. (3.4).

Proof. Let us consider the differential equation

$$
\begin{equation*}
x^{\prime \prime}+h(x)=r(t) \tag{3.7}
\end{equation*}
$$

with

$$
\begin{equation*}
h(x)=f(x)-p_{\max }-\frac{\varepsilon}{2}, \quad x>0 \tag{3.8}
\end{equation*}
$$

and

$$
\begin{equation*}
r(t)=p(t)-p_{\max }-\frac{\varepsilon}{2}, \quad t \in \mathbb{R} \tag{3.9}
\end{equation*}
$$

By (2.3), (3.8) and (3.9), we see that $r$ is a continuous $2 \pi$-periodic function, so $r$ fulfills condition (2.3).

By (2.1) and (3.6) we see that $h$ is locally Lipschitz continuous, positive and nonincreasing on $(0, \infty)$ which means that $h$ fulfills conditions (2.1).

Using (2.2), (3.6), (3.8) and (3.9), we get

$$
\bar{r}=\bar{p}-p_{\max }-\frac{\varepsilon}{2}<0
$$

and

$$
\lim _{x \rightarrow \infty} h(x)=g(K)-p_{\max }-\frac{\varepsilon}{2} \geq \varepsilon-\frac{\varepsilon}{2}>0
$$

Therefore also conditions (2.2) and (3.3) are satisfied. From Theorem 3.2 we get that assertions of Theorem 3.1 are valid for Eq. (3.7).

Note that Eq. (3.7) is equivalent to Eq. (3.4). Indeed Eq. (3.7) is obtained from Eq. (3.4) by subtracting the expression $p_{\max }+\varepsilon / 2$ from both sides. Therefore the assertions of Theorem 3.1 remains valid also for Eq. (3.4).

## 4 Bounds of bouncing solutions

By Theorem 3.3 there exist at least two $2 m \pi$-periodic bouncing solutions of Eq. (3.4) with one bounce in each period and existence of at least one $2 m \pi$-periodic bouncing solution of Eq. (3.4) with $n(n>1)$ bounces in each period. It remains to prove that all these solutions are bounded from above by the constant $K$ and therefore they are also bouncing solutions of Eq. (1.1). This is the main purpose of this section.

To achieve this goal we use several auxiliary results from [9], namely Lemma 4.1, 4.2 and 4.3 from Section 4 of that paper. Here, we assume that (2.1)-(2.4) are satisfied - these are the same assumption as in [9, Section 4].

Let us consider an initial value problem (3.4),

$$
\begin{equation*}
x\left(t_{0}\right)=0, \quad x^{\prime}\left(t_{0}+\right)=y_{0}, \tag{4.1}
\end{equation*}
$$

where $t_{0} \in \mathbb{R}, y_{0}>0$.
Lemma 4.1 (see [9, Lemma 8]).
(a) Let $t_{0} \in \mathbb{R}, y_{0}>0$. Then there exists a finite $t_{1}>t_{0}$ and a unique maximal solution $x$ of IVP (3.4), (4.1) on $\left(t_{0}, t_{1}\right)$ such that $x\left(t_{1}-\right)=0$. Moreover there exists $a \in\left(t_{0}, t_{1}\right)$ such that

$$
x^{\prime}(a)=0, \quad x^{\prime}>0 \text { on }\left(t_{0}, a\right), \quad x^{\prime}<0 \text { on }\left(a, t_{1}\right), \quad x^{\prime}\left(t_{1}-\right)<0 .
$$

(b) Let $t_{1} \in \mathbb{R}, y_{1}>0$. Then there exists a finite $t_{0}<t_{1}$ and a unique maximal solution $x$ of TVP (3.4), $x\left(t_{1}\right)=0, x^{\prime}\left(t_{1}-\right)=-y_{1}$ on $\left(t_{0}, t_{1}\right)$ such that $x\left(t_{0}+\right)=0$. Moreover there exists $a \in\left(t_{0}, t_{1}\right)$ such that

$$
x^{\prime}(a)=0, \quad x^{\prime}>0 \text { on }\left(t_{0}, a\right), \quad x^{\prime}<0 \text { on }\left(a, t_{1}\right), \quad x^{\prime}\left(t_{0}+\right)>0 .
$$

Further we need some estimates. First we define several useful functions

$$
\begin{equation*}
F(x)=\int_{0}^{x} f(s) \mathrm{d} s, \quad \alpha(x)=F(x)-p_{\max } x, \quad \beta(x)=F(x)-p_{\min } x, \quad x \in[0, \infty) . \tag{4.2}
\end{equation*}
$$

Finally, we will need the following assertions from [9].
Lemma 4.2 (see [9, Lemma 10]). Let $x$ be a maximal solution of Eq. (3.4) on the interval $\left(t_{0}, t_{1}\right)$. Then

$$
\begin{align*}
\sqrt{2 \alpha\left(x_{\max }\right)} & \leq x^{\prime}\left(t_{0}+\right) \leq \sqrt{2 \beta\left(x_{\max }\right)}  \tag{4.3}\\
-\sqrt{2 \beta\left(x_{\max }\right)} & \leq x^{\prime}\left(t_{1}-\right) \leq-\sqrt{2 \alpha\left(x_{\max }\right)} \tag{4.4}
\end{align*}
$$

$$
\begin{gather*}
\beta^{-1}\left(\frac{y_{0}^{2}}{2}\right) \leq x_{\max } \leq \alpha^{-1}\left(\frac{y_{0}^{2}}{2}\right)  \tag{4.5}\\
t_{1}-t_{0} \leq \frac{2 y_{0}}{\varepsilon}  \tag{4.6}\\
\forall \eta \in\left(0, x_{\max }\right): t_{1}-t_{0}>\sqrt{\frac{2\left(x_{\max }-\eta\right)}{f(\eta)-p_{\min }}} \tag{4.7}
\end{gather*}
$$

where $\alpha, \beta$ are from (4.2), $x(a):=x_{\max }:=\max \left\{x(t): t \in\left(t_{0}, t_{1}\right)\right\}$, and $\varepsilon$ is from (3.6).
Lemma 4.3 (see [9, Lemma 13]). There exists a continuous $2 \pi$-periodic function $\psi: \mathbb{R} \rightarrow \mathbb{R}$, $\psi(\mathbb{R}) \subset[\sqrt{2 \alpha(K)}, \sqrt{2 \beta(K)}]$ such that the solution $x$ of IVP (3.4), (4.1) with $y_{0}=\psi\left(t_{0}\right)$ has its maximum value $x_{\max }$ equal to $K$, for each $t_{0} \in \mathbb{R}$.

The following lemma is a generalization of [9, Lemma 11].
Lemma 4.4. Let $x, \tilde{x}$ be two different maximal classical solutions of Eq. (3.4) defined on the intervals $\left(t_{0}, t_{1}\right),\left(\tilde{t}_{0}, \tilde{t}_{1}\right)$, respectively. If $\left(t_{0}, t_{1}\right) \subset\left(\tilde{t}_{0}, \tilde{t}_{1}\right)$, then

$$
0<x(t)<\tilde{x}(t), \quad t \in\left(t_{0}, t_{1}\right)
$$

Proof. Let us prove the lemma by contradiction. Let the assumptions be satisfied and there exists $\tau \in\left(t_{0}, t_{1}\right)$ such that $x(\tau) \geq \tilde{x}(\tau)$. We put $v(t)=x(t)-\tilde{x}(t), t \in\left(t_{0}, t_{1}\right)$. Then $v\left(t_{0}+\right) \leq 0, v\left(t_{1}-\right) \leq 0$ and $v(\tau) \geq 0$. From the continuity of $v$ it follows that there exists an interval $\left(\tau_{0}, \tau_{1}\right) \subset\left(t_{0}, t_{1}\right)$ such that $v\left(\tau_{0}+\right)=v\left(\tau_{1}-\right)=0$ and $v(t) \geq 0$ for $t \in\left(\tau_{0}, \tau_{1}\right)$. This implies $v^{\prime}\left(\tau_{0}+\right) \geq 0$. There are two possibilities:

CASE A. If $v^{\prime}\left(\tau_{0}+\right)=0$, then $x$ and $\tilde{x}$ would be solutions of the same IVP and according to the uniqueness (see Lemma 4.1), we get $x=\tilde{x}$, which is a contradiction.

Case B. Let $v^{\prime}\left(\tau_{0}+\right)>0$. From the Mean Value Theorem we get that there exists $\xi \in\left(\tau_{0}, \tau_{1}\right)$ such that $v^{\prime}(\xi)=v\left(\tau_{1}-\right)-v\left(\tau_{0}+\right)=0$. Since $x$ and $\tilde{x}$ are solutions of Eq. (3.4) on $\left(\tau_{0}, \tau_{1}\right)$ and $f$ is decreasing, we get

$$
v^{\prime \prime}(t)=x^{\prime \prime}(t)-\tilde{x}^{\prime \prime}(t)=-f(x(t))+f(\tilde{x}(t)) \geq 0
$$

for $t \in\left(\tau_{0}, \tau_{1}\right)$. Integrating this inequality over the interval $\left(\tau_{0}, \xi\right)$, we get $v^{\prime}(\xi) \geq v^{\prime}\left(\tau_{0}+\right)>0$, which is also a contradiction.

The next lemma is a very slight generalization of [9, Lemma 14].
Lemma 4.5. Let $x$ be a maximal solution of IVP (3.4), (4.1) with $y_{0}=\psi\left(t_{0}\right)$ defined on $\left(t_{0}, t_{1}\right)$. If

$$
\begin{equation*}
\left(\frac{K}{m}\right)^{2}+2 \pi^{2} p_{\min } K \geq 2 \pi^{2} F(K) \tag{4.8}
\end{equation*}
$$

then

$$
\begin{equation*}
t_{1}-t_{0}>2 m \pi \tag{4.9}
\end{equation*}
$$

Proof. Let us consider linear functions

$$
q(t)=\left(t-t_{0}\right) \sqrt{2 \beta(K)}, \quad t \in \mathbb{R}
$$



Figure 4.1: The solution $x$ of IVP and auxiliary functions $q$ and $r$ from the proof of Lemma 4.5.
and

$$
r(t)=\left(t_{1}-t\right) \sqrt{2 \beta(K)}, \quad t \in \mathbb{R} .
$$

The graph of function $q$ passes through some point $(\hat{t}, K)$, where $K=q(\hat{t})=\left(\hat{t}-t_{0}\right) \sqrt{2 \beta(K)}$. Consequently,

$$
\hat{t}-t_{0}=\frac{K}{\sqrt{2 \beta(K)}} .
$$

Similarly, the graph of function $r$ passes through some point $(\tilde{t}, K)$, so

$$
t_{1}-\tilde{t}=\frac{K}{\sqrt{2 \beta(K)}}
$$

The solution $x$ is concave on $\left(t_{0}, t_{1}\right)$ and from Lemma 4.3 we obtain that $x$ has its maximum value equal to $K$. Denote $x(a)=K, a \in\left(t_{0}, t_{1}\right)$. Therefore $\hat{t} \in\left(t_{0}, a\right)$ and $\tilde{t} \in\left(a, t_{1}\right)$, see Figure 4.1. From (4.2) and assumption (4.8) we have

$$
\begin{equation*}
K \geq m \pi \sqrt{2 \beta(K)} \tag{4.10}
\end{equation*}
$$

Finally we obtain

$$
t_{1}-t_{0}=t_{1}-a+a-t_{0}>t_{1}-\tilde{t}+\hat{t}-t_{0}=\frac{2 K}{\sqrt{2 \beta(K)}} \geq \frac{2 m \pi \sqrt{2 \beta(K)}}{\sqrt{2 \beta(K)}}=2 m \pi,
$$

where the last inequality follows from (4.10).
Finally, in the next lemma we get the upper bound of bouncing solutions.
Lemma 4.6. Let $x$ be $2 \pi m$-periodic bouncing solution of Eq. (3.4) with $n$ bounces in each period, $m, n \in \mathbb{N}$. Then $x(t)<K$ for each $t \in \mathbb{R}$.

Proof. Let $t_{0} \in \mathbb{R}$ be such that $x\left(t_{0}\right)=0$. Then there exist bounces $t_{1}, \ldots, t_{n} \in \mathbb{R}$ such that $t_{0}<t_{1}<\cdots<t_{n}=t_{0}+2 \pi m$. Let $\tilde{x}$ be a maximal classical solution of IVP (3.4), (4.1) with $y_{0}=\psi\left(t_{0}\right)$ defined on the interval $\left(t_{0}, \tilde{f}_{1}\right)$. According to Lemma 4.5, $\tilde{t}_{1}>t_{0}+2 \pi m=t_{n}$. Then for $i=0, \ldots, n-1$ we have $\left(t_{i}, t_{i+1}\right) \subset\left(t_{0}, \tilde{t}_{1}\right)$, which by Lemma 4.4 implies that $x(t)<\tilde{x}(t) \leq$ $K$ for each $t \in\left(t_{i}, t_{i+1}\right)$. This proves that $x$ is lower than $K$ on the interval $\left[t_{0}, t_{0}+2 \pi m\right]$ and the rest follows from $2 \pi m$-periodicity.

Now, we are ready to prove the main theorem of this paper.
Proof of Theorem 2.3. Let (2.1)-(2.6) be satisfied. Case (i) is proved in [9]. Let us prove the cases (ii) and (iii). Due to Theorem 3.3 for any $m \in \mathbb{N}$, there exists at least two $2 m \pi$-periodic bouncing solutions of Eq. (3.4) with one bounce in each period and for any $n, m \in \mathbb{N}, n \geq 2$, there exists at least one $2 m \pi$-periodic bouncing solution of Eq. (3.4) with $n$ bounces in each period. By Lemma 4.6 every bouncing solution of Eq. (3.4) is lower than K. According to (3.5), these functions are also bouncing solutions of Eq. (1.1).

## Acknowledgements

The authors are indebted to prof. Irena Rachůnková for her invaluable help. This work was supported by Palacký University in Olomouc (grant no. IGA_PrF_2021_008).

## References

[1] P. Habets, L. Sanchez, Periodic solution of some Liénard equations with singularities, Proc. Am. Math. Soc. 176(1990), 1135-1044. https://doi.org/10.2307/2048134; MR1009991; Zbl 0695.34036
[2] R. Hakl, P. J. Torres, M. Zamora, Periodic solutions to singular second order differential equations: The repulsive case, Topol. Methods Nonlinear Anal. 39(2012), 199-220. MR2985878; Zbl 1279.34038
[3] A. Lazer, S. Solimini, On periodic solutions of nonlinear differential equations with singularities, Proc. Am. Math. Soc. 1(1987), 109-114. https://doi.org/10.2307/2046279; MR866438; Zbl 0616.34033
[4] R. Ortega, Asymmetric oscillators and twist mappings, J. Lond. Math. Soc. 53(1996), 325342. https://doi.org/10.1112/j1ms/53.2.325; MR1373064; Zbl 0860.34017
[5] R. Ortega, Linear motions in a periodically forced Kepler problem, Port. Math. 68(2011), No. 2, 149-176. https://doi.org/10.4171/PM/1885; MR2849852; Zbl 1235.34136
[6] D. Qian, P. J. Torres, Bouncing solutions of an equation with attractive singularity, Proc. Roy. Soc. Edinburgh Sect. A 134(2004), No. 1, 201-213. https://doi.org/10.1017/ S0308210500003164; MR2039912; Zbl 1062.34047
[7] D. Qian, P. J. Torres, Periodic motions of linear impact oscillators via the successor map, SIAM J. Math. Anal. 36(2005), No. 6, 1707-1725. https://doi.org/10.1137/ S003614100343771X; MR2178218; Zbl 1092.34019
[8] I. Rachůnková, M. Tvrdý, I. Vrкoč, Existence of nonnegative and nonpositive solutions for second order periodic boundary value problems, J. Differential Equations 176(2001), 445-469. https://doi.org/10.1006/jdeq. 2000.3995; MR1866282; Zbl 1004.34008
[9] J. Tomeček, I. Rachůnková, J. Burkotová, J. Stryja, Coexistence of bouncing and classical periodic solutions of generalized Lazer-Solimini equation, Nonlinear Anal. 196(2020), 1-24. https://doi.org/10.1016/j.na.2020.111783; MR4064869; Zbl 1441.34052
[10] P. J. Torres, Mathematical models with singularities. A zoo of singular creatures, Atlantis Briefs in Differential Equations, Vol. 3, Atlantis Press, Amsterdam-Paris-Beijing, 2015. https: //doi.org/10.2991/978-94-6239-106-2; Zbl 1305.00097

# On existence and asymptotic behavior of solutions of elliptic equations with nearly critical exponent and singular coefficients 

Shiyu Li ${ }^{1}$, Gongming Wei ${ }^{\boxtimes 1}$ and Xueliang Duan ${ }^{2}$<br>${ }^{1}$ College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China<br>${ }^{2}$ School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China

Received 5 August 2020, appeared 8 September 2021
Communicated by Dimitri Mugnai

Abstract. In this paper we study the existence and asymptotic behavior of solutions of

$$
-\Delta u=\mu \frac{u}{|x|^{2}}+|x|^{\alpha} u^{p(\alpha)-1-\varepsilon}, \quad u>0 \text { in } B_{R}(0)
$$

with Dirichlet boundary condition. Here, $-2<\alpha<0, p(\alpha)=\frac{2(N+\alpha)}{N-2}, 0<\varepsilon<p(\alpha)-1$ and $p(\alpha)-1-\varepsilon$ is a nearly critical exponent. We combine variational arguments with the moving plane method to prove the existence of a positive radial solution. Moreover, the asymptotic behaviour of the solutions, as $\varepsilon \rightarrow 0$, is studied by using ODE techniques.
Keywords: asymptotic behavior, critical Sobolev exponent, Hardy exponents, singular coefficient.
2020 Mathematics Subject Classification: 35A15, 35A24, 35B40.

## 1 Introduction

In this paper, we consider the following elliptic problem:

$$
\begin{cases}-\Delta u=\mu \frac{u}{|x|^{2}}+|x|^{\alpha} u^{p(\alpha)-1-\varepsilon}, & x \in \Omega  \tag{1.1}\\ u>0, & x \in \Omega \\ u=0, & x \in \partial \Omega\end{cases}
$$

where $\Omega$ is a ball $B_{R}(0)$ in $\mathbb{R}^{N}(N \geq 3),-2<\alpha<0, p(\alpha)=\frac{2(N+\alpha)}{N-2}, 0<\varepsilon<p(\alpha)-1$, $0 \leq \mu<\bar{\mu}=\left(\frac{N-2}{2}\right)^{2}$.

The equation in problem (1.1) is the Euler-Lagrange equation of the energy functional $E: H_{0}^{1}(\Omega) \rightarrow \mathbb{R}$ defined by

$$
E(u)=\frac{1}{2}\left(\int_{\Omega}|\nabla u|^{2}-\mu \frac{u^{2}}{|x|^{2}}\right)-\frac{1}{p(\alpha)-\varepsilon} \int_{\Omega}|x|^{\alpha} u^{p(\alpha)-\varepsilon}, \quad \forall u \in H_{0}^{1}(\Omega) .
$$

[^26]It is known that critical points of functional $E(u)$ correspond to solutions of (1.1).
We denote

$$
\|u\| \triangleq\left(\int_{\Omega}|\nabla u|^{2}-\mu \frac{u^{2}}{|x|^{2}}\right)^{\frac{1}{2}}, \quad \forall u \in H_{0}^{1}(\Omega) .
$$

Let us recall the Sobolev-Hardy inequality (see Lemma 2.1 in this paper), which using the fact $0 \leq \mu<\bar{\mu}$ implies that $\|u\|$ is equivalent to the norm of $H_{0}^{1}(\Omega)$.

In the case $\mu=0$ and $\alpha=0$, a prototype of problem (1.1) is

$$
\begin{cases}-\Delta u=u^{2^{*}-1-\varepsilon}, & x \in \Omega  \tag{1.2}\\ u>0, & x \in \Omega \\ u=0, & x \in \partial \Omega\end{cases}
$$

When $\varepsilon=0$, it is well known that the solution of problem (1.2) is bounded in the neighborhood of the origin. Gidas, Ni and Nirenberg [17] proved that all the solutions with reasonable behavior at infinity, namely

$$
\begin{equation*}
u=O\left(|x|^{2-N}\right), \tag{1.3}
\end{equation*}
$$

are radially symmetric about some point. So, the form of the solutions may be assumed as

$$
u(x)=\frac{\left[N(N-2) \lambda^{2}\right]^{\frac{N-2}{4}}}{\left(\lambda^{2}+\left|x-x_{0}\right|^{2}\right)^{\frac{N}{2}-1}}
$$

for some $\lambda>0$ and $x_{0} \in \mathbb{R}^{N}$.
Later in [7, Corollay 8.2] and [9, Theorem 2.1], the growth assumption (1.3) was removed, which implies that, for positive $C^{2}$ solutions of problem (1.2), we have the same result.

When $\varepsilon>0$, Atkinson and Peletier [2] used ODE arguments to obtain exact asymptotic estimates of the radially symmetric solution of problem (1.2) as $\varepsilon \rightarrow 0$. The following are their principal results

$$
\lim _{\varepsilon \rightarrow 0} \varepsilon u^{2}(0, \varepsilon)=\frac{4}{N-2}\{N(N-2)\}^{\frac{N-2}{2}} \frac{\Gamma(N)}{\left[\Gamma\left(\frac{N}{2}\right)\right]^{2}} \frac{1}{R^{N-2}}
$$

and for $x \neq 0$

$$
\lim _{\varepsilon \rightarrow 0} \varepsilon^{-\frac{1}{2}} u(x, \varepsilon)=\frac{1}{2} N^{\frac{N-2}{4}}(N-2)^{\frac{N}{4}} R^{\frac{R-2}{2}} \frac{\Gamma\left(\frac{N}{2}\right)}{[\Gamma(N)]^{\frac{1}{2}}}\left(\frac{1}{|x|^{N-2}}-\frac{1}{R^{N-2}}\right) .
$$

In the case $\mu=0$ and $\alpha>0$, problem (1.1) is known as the Hénon equation

$$
\begin{cases}-\Delta u=|x|^{\alpha} u^{p-1}, & x \in \Omega,  \tag{1.4}\\ u>0, & x \in \Omega, \\ u=0, & x \in \partial \Omega,\end{cases}
$$

where $p \in\left(2,2^{*}\right)$. Equation (1.4) was proposed by Hénon when he studied rotating stellar structures and readers can refer to Ni [24], Smets [26] and Cao-Peng [11]. Among these works, for equations with critical, supercritical and slightly subcritical growth, the existence and multiplicity of non-radial solutions, the symmetry and asymptotic behavior of ground states were studied by variational method (for $p \rightarrow \frac{2 N}{N-2}$ or $\alpha \rightarrow \infty$ ).

In the case $0 \leq \mu<\bar{\mu}=\left(\frac{N-2}{2}\right)^{2}$ and $\alpha=0$, problem (1.1) can be written as

$$
\begin{cases}-\Delta u=\mu \frac{u}{|x|^{2}}+u^{2^{*}-1-\varepsilon}, & x \in \Omega  \tag{1.5}\\ u>0, & x \in \Omega \\ u=0, & x \in \partial \Omega\end{cases}
$$

By using Moser iteration and a generalized comparison principle, Cao and Peng [10] proved $u(x) \in H_{0}^{1}(\Omega)$ satisfying

$$
\left\{\begin{array}{l}
u(x)|x|^{v} \geq C_{1}, \quad \forall x \in \Omega^{\prime} \subset \subset \Omega \\
u(x)|x|^{v} \leq C_{2}, \quad \forall x \in \Omega
\end{array}\right.
$$

where $C_{1}$ and $C_{2}$ are two positive constants, $v=\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}$. When $\Omega=B_{R}, u(x)$ is radially symmetric. Hence, they converted (1.5) to ODE and obtained the following:

$$
\lim _{\varepsilon \rightarrow 0} \lim _{|x| \rightarrow 0} \varepsilon u_{\varepsilon}^{2}|x|^{2 v}=4(2 \sqrt{\bar{\mu}-\mu})^{N-1} N^{\frac{N-2}{2}}(N-2)^{-\frac{N+2}{2}} \frac{\Gamma(N)}{\left[\Gamma\left(\frac{N}{2}\right)\right]^{2}} \frac{1}{R^{2 \sqrt{\bar{\mu}-\mu}}}
$$

and for $x \neq 0$

$$
\left.\begin{array}{rl}
\lim _{\varepsilon \rightarrow 0} \varepsilon^{-\frac{1}{2}} u_{\varepsilon}(x)= & \frac{1}{2 \sqrt{2}}(2 \sqrt{\bar{\mu}-\mu})^{\frac{N-3}{2}} N^{\frac{N-2}{4}}(N-2)^{-\frac{N-6}{4}} R \sqrt{\bar{\mu}-\mu} \\
{\left[\Gamma\left(\frac{N}{2}\right)\right.} \\
& \times\left(\frac{1}{\mid x(N)]^{\frac{1}{2}}}\right. \\
& \frac{1}{\sqrt{\mu}+\sqrt{\bar{\mu}-\mu}}-\frac{1}{|x|^{\sqrt{\mu}}-\sqrt{\bar{\mu}-\mu}}|R|^{2 \sqrt{\bar{\mu}-\mu}}
\end{array}\right) .
$$

Motivated by the previous works and remark 4.2 in [10], we first prove the existence and radial symmetry of positive solution of (1.1). Then we focus on the asymptotic behavior of the solutions of problem (1.1) as $\varepsilon \rightarrow 0$.

To state our main results, for convenience, we set $p=p(\alpha)-1-\varepsilon, v=\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}}-\mu$, $\Omega=B_{R}=\left\{x \in \mathbb{R}^{N}:|x|<R\right\}, R>0$. We denote by $u_{\varepsilon}(x)$ the solution of (1.1) and $\Gamma(x)$ is the Gamma function.

Theorem 1.1. Suppose that $-2<\alpha<0,0 \leq \mu<\bar{\mu}, 0<\varepsilon<p(\alpha)-1$. Then problem (1.1) has $a$ radially symmetric solution in $H_{0}^{1}(\Omega)$.

For the proof of this Theorem 1.1, we first obtain a solution by the Mountain Pass Lemma. Then, by moving plane method for elliptic equations with variable coefficients in [14], we can prove that the posotive solution is radially symmetric. For problem (1.2), the solution satisfies Gidas-Ni-Nirenberg Theorem in [17] and hence all solutions of (1.2) are radial symmetric. However, here we cannot use Gidas-Ni-Nirenberg theorem directly since problem (1.1) includes the hardy term $\mu \frac{u}{|x|^{2}}$ and singular coefficient $|x|^{\alpha}$. Luckily, through a transformation of the original solution $u_{\varepsilon}(x)$, the new equation satisfied by the new solution $v(x)$ satisfies the conditions of a Corollary in [14] and we obtain the result. To be more precise, set

$$
v(x)=|x|^{-\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}} u_{\varepsilon}(x)
$$

using Moser iteration and a generalized comparison principle introduced by Merle and Peletier [22], we prove that $v \in L^{\infty}(\Omega)$ and is bounded from below and above. Thus we obtain that the precise singularity of $u_{\varepsilon}(x)$ at the origin is like $|x|^{-} \sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}$. Then applying

Lemma 2.5 in Section 2 to this new equation, we deduce that $v(x)$ is radially symmetric and satisfies the following ODE:

$$
\begin{cases}v^{\prime \prime}+\frac{N-1-2 v}{r} v^{\prime}+\frac{1}{r^{(p(\alpha)-2-\varepsilon) v-\alpha}} v^{p(\alpha)-1-\varepsilon}=0, & \text { for } 0<r<R,  \tag{1.6}\\ v(r)>0, & \text { for } 0<r<R, \\ v(R)=0 . & \end{cases}
$$

Because (1.6) is still singular at the origin, we can use the well-known shooting argument introduced by Atkinson and Peletier [2] to convert (1.6) to the following ODE:

$$
\left\{\begin{array}{l}
y^{\prime \prime}(t)=-t^{-k(\alpha, \varepsilon) y^{p(\alpha)-1-\varepsilon}}  \tag{1.7}\\
y(t)>0, \text { for } T<t<\infty \\
y(T)=0
\end{array}\right.
$$

where $k(\alpha, \varepsilon)=\frac{2 m+\alpha}{m-1}-\frac{(p(\alpha)-2-\varepsilon) v}{m-1}, m=1+2 \sqrt{\bar{\mu}-\mu}=N-2 v-1, T=\left(\frac{m-1}{R}\right)^{m-1}, p(\alpha)-1=$ $2 k(\alpha, \varepsilon)-3-\frac{2 v \varepsilon}{m-1}$.

Till now, study on behaviors and precise properties of the original solution $u_{\varepsilon}(x)$ can be reduced to deal with (1.7). Based on this, we have
Theorem 1.2. Let $u_{\varepsilon}(x) \in H_{0}^{1}(\Omega)$ be a solution of problem (1.1). Then

$$
\lim _{\varepsilon \rightarrow 0} \lim _{|x| \rightarrow 0} \varepsilon u_{\varepsilon}^{2}|x|^{2 v}=2(\alpha+2)(2 \sqrt{\bar{\mu}-\mu})^{\frac{2 N+\alpha-2}{\alpha+2}}(N+\alpha)^{\frac{N-2}{\alpha+2}}(N-2)^{-\frac{2 \alpha+N+2}{\alpha+2}} \frac{\Gamma\left(\frac{2(N+2)}{\alpha+2}\right)}{\left[\Gamma\left(\frac{N+\alpha}{\alpha+2}\right)\right]^{2}} \frac{1}{R^{2 \sqrt{\bar{\mu}-\mu}}} .
$$

Theorem 1.3. Let $u_{\varepsilon}(x) \in H_{0}^{1}(\Omega)$ be a solution of problem (1.1). Then, for every $x \neq 0$,

$$
\left.\begin{array}{rl}
\lim _{\varepsilon \rightarrow 0} \varepsilon^{-\frac{1}{2}} u_{\varepsilon}(x)= & \frac{1}{2}(\alpha+2)^{-\frac{1}{2}}(2 \sqrt{\bar{\mu}-\mu})^{\frac{2 N-\alpha-6}{2 \alpha+4}}(N+\alpha)^{\frac{N-2}{2 \alpha+4}}(N-2)^{\frac{2 \alpha-N+6}{2 \alpha+4}} R \sqrt{\bar{\mu}-\mu} \frac{\Gamma\left(\frac{N+\alpha}{\alpha+2}\right)}{\left[\Gamma\left(\frac{2(N+\alpha)}{\alpha+2}\right)\right]^{\frac{1}{2}}} \\
& \times\left(\frac{1}{|x|^{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}}-\frac{1}{|x|^{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}|R|^{2 \sqrt{\bar{\mu}-\mu}}\right.
\end{array}\right) .
$$

## Notations:

- $C, C_{i}, i=0,1,2, \ldots$ denote positive constants, which may vary from line to line;
- $\|\cdot\|$ and $\|\cdot\|_{L^{q}}$ denote the usual norms of the spaces $H_{0}^{1}(\Omega)$ and $L^{q}(\Omega)$, respectively, $\Omega \in \mathbb{R}^{N}$;
- Some of the notations that will appear in the following paragraphs:

$$
\begin{array}{rlrl}
m & =1+2 \sqrt{\bar{\mu}-\mu}=N-2 v-1, & T & =\left(\frac{m-1}{R}\right)^{m-1}, \\
k & =k(\alpha, \varepsilon)=\frac{2 m+\alpha}{m-1}-\frac{(p(\alpha)-2-\varepsilon) v}{m-1}, k_{1}(\alpha, \varepsilon) & =(k-1)^{\frac{1}{k-2}}, \\
k_{2}(\alpha, \varepsilon) & =\frac{k-1}{k-2}, & T_{\alpha, \varepsilon} & =\frac{\gamma^{\frac{p(\alpha)-2-\varepsilon}{k-2}}}{k_{1}(\alpha, \varepsilon)}=\frac{\gamma^{2-\frac{m-1-2 v}{(m-1)(k-2)}} k_{1}(\alpha, \varepsilon)}{k^{2}}, \\
\tau(\alpha, \varepsilon) & =\left(\frac{t}{T_{\alpha, \varepsilon}}\right)^{k-2}, & \varphi(\alpha, \varepsilon) & =\frac{m-1+2 v}{(m-1)(k-2)} \varepsilon, \\
C_{\alpha, \beta, \varepsilon} & =\frac{\beta}{\left(1+\beta^{k-2}\right)^{\frac{1}{k-2}}}, & d_{\alpha, \beta, \varepsilon} & =\frac{\left(1-C_{\alpha, \beta, \varepsilon}\right)(1+2 v /(m-1))}{C_{\alpha, \beta, \varepsilon}^{2+(1+2 v /(m-1)) \varepsilon}} .
\end{array}
$$

## 2 Preliminary results and existence of solution

In this section, we shall provide some preliminaries which will be used in the sequel and prove the existence of solution to problem (1.1).

Lemma 2.1 (see [16, Lemma 3.1 and 3.2]). Suppose $-2<\alpha<0,2 \leq q \leq p(\alpha)$, and $0 \leq \mu<\bar{\mu}$. Then
(i) (Hardy inequality)

$$
\int_{\Omega} \frac{u^{2}}{|x|^{2}} \leq \frac{1}{\bar{\mu}} \int_{\Omega}|\nabla u|^{2}, \quad \forall u \in H_{0}^{1}(\Omega)
$$

(ii) (Sobolev-Hardy inequality)
there exists a constant $C>0$ such that

$$
\left(\int_{\Omega}|x|^{\alpha}|u|^{q}\right)^{\frac{1}{q}} \leq C\|u\|, \quad \forall u \in H_{0}^{1}(\Omega) ;
$$

(iii) the map $u \mapsto|x|^{\frac{\alpha}{q}} u$ from $H_{0}^{1}(\Omega)$ into $L^{q}(\Omega)$ is compact for $q<p(\alpha)$.

Lemma 2.2 (see [5, Theorem 2.2]). Let J be a $C^{1}$ function on a Banach space X. Suppose there exists a neighborhood $U$ of 0 in $X$ and a constant $\rho$ such that $J(u) \geq \rho$ for every $u$ in the boundary of $U$,

$$
J(0)<\rho \text { and } J(v)<\rho \text { for some } v \notin U .
$$

Set

$$
c=\inf _{g \in \Gamma} \max _{\omega \in g} J(\omega) \geq \rho,
$$

where $\Gamma=\{g \in C([0,1], X): g(0)=0, g(1)=v, J(g(1))<\rho\}$.
Conclusion: there is a sequence $\left\{u_{n}\right\}$ in $X$ such that $J\left(u_{n}\right) \rightarrow c$ and $J^{\prime}\left(u_{n}\right) \rightarrow 0$ in $X^{*}$.
Lemma 2.3 (The Caffarelli-Kohn-Nirenberg inequalities, see [8] and [12]). For all $u \in C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$,

$$
\left(\int_{\mathbb{R}^{N}}|x|^{-b q}\left|u^{q}\right|\right)^{\frac{p}{q}} \leq C_{a, b} \int_{\mathbb{R}^{N}}|x|^{-a p}|D u|^{p} d x
$$

where (i) for $n>p$,

$$
-\infty<a<\frac{n-p}{p}, \quad 0 \leq b-a \leq 1, \quad \text { and } \quad q=\frac{n p}{n-p+p(b-a)}
$$

and (ii) for $n \leq p$,

$$
-\infty<a<\frac{n-p}{p}, \quad \frac{p-n}{p} \leq b-a \leq 1, \quad \text { and } \quad q=\frac{n p}{n-p+p(b-a)} .
$$

Lemma 2.4 (see [25, page 4]). Suppose $V$ is a reflexive Banach space with norm $\|\cdot\|$, and let $M \subset V$ be a weakly closed subset of $V$. Suppose $E: M \rightarrow \mathbb{R} \cup\{+\infty\}$ is coercive and (sequentially) weakly lower semi-continuous on $M$ with respect to $V$, that is, suppose the following conditions are fulfilled:
(1) (coercive) $E(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty, u \in M$.
(2) (W.S.L.S.C) For any $u \in M$, any sequence $\left\{u_{m}\right\}$ in $M$ such that $u_{m} \rightharpoondown u$ weakly in $V$ there holds:

$$
E(u) \leq \liminf _{m \rightarrow \infty} E\left(u_{m}\right) .
$$

Then $E$ is bounded from below on $M$ and attains its infimum in $M$.
Lemma 2.5 (see [14, Corollary 1.6]). Let $u$ be a bounded $C^{2}\left(B_{R} \backslash\{0\}\right) \cap C^{1}\left(\overline{B_{R}} \backslash\{0\}\right)$ solution of

$$
\begin{cases}\partial_{i}\left(|x|^{b} \partial_{i} u\right)+K|x|^{a} u^{q}=0, & x \in B_{R} \backslash\{0\} \\ u>0, & x \in B_{R} \backslash\{0\} \\ u=0, & x \in \partial B_{R} \backslash\{0\}\end{cases}
$$

where $K$ is a positive constant. Then $u$ is radially symmetric in $B_{R}$ provided $q \geq 1, b\left(\frac{1}{2} b+N-2\right) \leq 0$ and $\frac{1}{2} b \geq \frac{a}{q}$.

Proof. When $b<0$, we have $|x|^{b}$ is singular at origin.
It's clear that

$$
\frac{S(x)}{|x|^{b}}-\frac{S\left(x^{\lambda}\right)}{\left|x^{\lambda}\right|^{b}}=\frac{1}{2} b\left(\frac{1}{2} b+N-2\right)\left(|x|^{b-2}-\left|x^{\lambda}\right|^{b-2}\right) \geq 0
$$

and

$$
K\left(\frac{|x|^{b}}{\left|x^{\lambda}\right|^{b}}\right)^{\frac{1}{2}}\left|x^{\lambda}\right|^{a} u^{q}-K|x|^{a}\left[\left(\frac{|x|^{b}}{\left|x^{\lambda}\right|^{b}}\right)^{\frac{1}{2}} u\right]^{q}=K|x|^{\frac{1}{2} b}\left|x^{\lambda}\right|^{a-\frac{1}{2} b}\left[1-\left(\frac{\left|x^{\lambda}\right|}{|x|}\right)^{\frac{1}{2} b q-a}\right] u^{q} \geq 0
$$

where $S(x)=\frac{1}{2}\left(\Delta|x|^{b}-\left.\left.\frac{1}{2|x|^{b}}|\nabla| x\right|^{b}\right|^{2}\right)$.
From [14], we have

$$
h_{\lambda}(x)=\left(\frac{\left|x^{\lambda}\right|^{b}}{|x|^{b}}\right)^{\frac{1}{2}} u\left(x^{\lambda}\right)-u(x)
$$

where $x^{\lambda}=\left(2 \lambda-x_{1}, x_{2}, \ldots, x_{N}\right)$.
By Lemma 4.2 in [14], we can obtain $u$ has a positive lower bound near the origin. Hence, we can get the estimate of $h_{\lambda}(x)$ near the origin. Furthermore, if $x^{\lambda}=0$, we have $h_{\lambda}(x)=\infty$.

Now, we consider the case of $u \in C^{2}\left(B_{1} \backslash\{0\}\right) \cap C^{1}\left(\overline{B_{1}} \backslash\{0\}\right)$ in Proposition 1.3 of [14]. Analogically, we can also obtain $u\left(x_{1}, x_{2}, \ldots, x_{N}\right) \leq u\left(-x_{1}, x_{2}, \ldots, x_{N}\right)$ for $x_{1} \in(-1,0)$ and $x_{1} \in(0,1)$. Hence, $u$ is symmetric in $x_{1}$. By Lemma 1.1 in [14], the above analysis and scaling transformation, $u(x)$ is radially symmetric in $B_{R}$.

Next, we shall prove the existence of solution to the problem (1.1). To start with, we prove the existence of nonnegative solution to the following Dirichlet problem:

$$
\begin{cases}-\Delta u=\mu \frac{u}{|x|^{2}}+|x|^{\alpha}|u|^{p(\alpha)-2-\varepsilon} u, & x \in \Omega  \tag{2.1}\\ u=0, & x \in \partial \Omega\end{cases}
$$

where $\Omega$ is a ball in $\mathbb{R}^{N}(N \geq 3)$ centered at the origin, $-2<\alpha<0, p(\alpha)=\frac{2(N+\alpha)}{N-2}, 0<\varepsilon<$ $p(\alpha)-1,0 \leq \mu<\bar{\mu}=\left(\frac{N-2}{2}\right)^{2}$.

The energy functional corresponding to problem (2.1) is

$$
J(u)=\frac{1}{2}\|u\|^{2}-\frac{1}{p(\alpha)-\varepsilon} \int_{\Omega}|x|^{\alpha}|u|^{p(\alpha)-\varepsilon}, \quad u \in H_{0}^{1}(\Omega)
$$

Lemma 2.6. The function $J$ satisfies $(P S)_{c}$ condition for every $c \in \mathbb{R}$.

Proof. Take $c \in \mathbb{R}$ and assume that $\left\{u_{n}\right\}$ is a Palais-Smale sequence at level $c$, namely such that

$$
J\left(u_{n}\right) \rightarrow c \quad \text { and } \quad J^{\prime}\left(u_{n}\right) \rightarrow 0 \quad\left(\text { in } H^{-1}(\Omega)\right) .
$$

This implies that there is a constant $M>0$ such that

$$
\begin{equation*}
\left|J\left(u_{n}\right)\right| \leq M \tag{2.2}
\end{equation*}
$$

From $J^{\prime}\left(u_{n}\right) \rightarrow 0$, we obtain

$$
\begin{equation*}
o(1)\left\|u_{n}\right\|=\left\langle J^{\prime}\left(u_{n}\right), u_{n}\right\rangle=\left\|u_{n}\right\|^{2}-\int_{\Omega}|x|^{\alpha}\left|u_{n}\right|^{p(\alpha)-\varepsilon} \tag{2.3}
\end{equation*}
$$

Calculating (2.2) $-\frac{1}{p(\alpha)-\varepsilon}(2.3)$, we have

$$
\begin{aligned}
M+o(1)\left\|u_{n}\right\| \geq & \frac{1}{2}\left\|u_{n}\right\|^{2}-\frac{1}{p(\alpha)-\varepsilon} \int_{\Omega}|x|^{\alpha}\left|u_{n}\right|^{p(\alpha)-\varepsilon} \\
& -\frac{1}{p(\alpha)-\varepsilon}\left\|u_{n}\right\|^{2}+\frac{1}{p(\alpha)-\varepsilon} \int_{\Omega}|x|^{\alpha}\left|u_{n}\right|^{p(\alpha)-\varepsilon} \\
= & \left(\frac{1}{2}-\frac{1}{p(\alpha)-\varepsilon}\right)\left\|u_{n}\right\|^{2},
\end{aligned}
$$

which implies the boundedness of $\left\{u_{n}\right\}$. By usual arguments, we can assume that up to a subsequence, there exists $u \in H_{0}^{1}(\Omega)$ such that

- $u_{n} \rightharpoonup u$ in $H_{0}^{1}(\Omega)$;
- $|x|^{\frac{\alpha}{p(\alpha)-\varepsilon}} u_{n} \rightarrow|x|^{\frac{\alpha}{p(\alpha)-\varepsilon}} u$ in $L^{p(\alpha)-\varepsilon}(\Omega)$;
-     - $u_{n} \rightarrow u$ for almost every $x \in \Omega$.

We now show that the convergence of $u_{n}$ to $u$ is strong.
First of all, from the above convergence properties, we obtain

$$
\left\|u_{n}|x|^{\frac{\alpha}{p(\alpha)-\varepsilon}}-u|x|^{\frac{\alpha}{p(\alpha)-\varepsilon}}\right\|_{L^{p(\alpha)-\varepsilon}(\Omega)} \rightarrow 0, \quad n \rightarrow \infty .
$$

As $J^{\prime}\left(u_{n}\right) \rightarrow 0$ and $u_{n} \rightarrow u$, we also have $\left\langle J^{\prime}\left(u_{n}\right), u_{n}-u\right\rangle \rightarrow 0$ and obviously $\left\langle J^{\prime}(u), u_{n}-u\right\rangle \rightarrow 0$. Then, as $n \rightarrow \infty$, on the one hand,

$$
\left\langle J^{\prime}\left(u_{n}\right)-J^{\prime}(u), u_{n}-u\right\rangle \leq\left|\left\langle J^{\prime}\left(u_{n}\right), u_{n}-u\right\rangle\right|+\left|\left\langle J^{\prime}(u), u_{n}-u\right\rangle\right|=o(1) .
$$

On the other hand,

$$
\begin{aligned}
& \left\langle J^{\prime}\left(u_{n}\right)-J^{\prime}(u), u_{n}-u\right\rangle \\
& \quad=\int_{\Omega}\left|\nabla u_{n}-\nabla u\right|^{2}-\int_{\Omega} \mu \frac{\left|u_{n}-u\right|^{2}}{|x|^{2}}-\int_{\Omega}|x|^{\alpha}\left(\left|u_{n}\right|^{p(\alpha)-2-\varepsilon} u_{n}-|u|^{p(\alpha)-2-\varepsilon} u\right)\left(u_{n}-u\right) \\
& \quad=\left\|u_{n}-u\right\|^{2}-\int_{\Omega}|x|^{\alpha}\left(\left|u_{n}\right|^{p(\alpha)-2-\varepsilon} u_{n}-|u|^{p(\alpha)-2-\varepsilon} u\right)\left(u_{n}-u\right) .
\end{aligned}
$$

We claim $\int_{\Omega}|x|^{\alpha}\left(\left|u_{n}\right|^{p(\alpha)-2-\varepsilon} u_{n}-|u|^{p(\alpha)-2-\varepsilon} u\right)\left(u_{n}-u\right) \rightarrow 0$.

Indeed, by Hölder's inequality,

$$
\begin{align*}
& \int_{\Omega}|x|^{\alpha}\left|u_{n}\right|^{p(\alpha)-2-\varepsilon} u_{n}\left(u_{n}-u\right) \\
& \leq \int_{\Omega}|x|^{\alpha}\left|u_{n}\right|^{p(\alpha)-1-\varepsilon}\left|u_{n}-u\right| \\
&=\int_{\Omega}|x|^{\alpha \cdot \frac{p(\alpha)-\varepsilon-\varepsilon}{p(\alpha)-\varepsilon}}\left|u_{n}\right|^{p(\alpha)-1-\varepsilon}|x|^{\alpha \cdot \frac{1}{p(\alpha)-\varepsilon}}\left|u_{n}-u\right| \\
& \leq\left[\int_{\Omega}\left(|x|^{\alpha \cdot \frac{p(\alpha)-1-\varepsilon}{p(\alpha)-\varepsilon}}\left|u_{n}\right|^{p(\alpha)-1-\varepsilon}\right)^{\frac{p(\alpha)-\varepsilon}{p(\alpha)-1-\varepsilon}}\right]^{\frac{p(\alpha)-1-\varepsilon}{p(\alpha)-\varepsilon}}\left[\int_{\Omega}\left(|x|^{\left.\alpha \cdot \frac{1}{p(\alpha)-\varepsilon} \right\rvert\,}\left|u_{n}-u\right|^{p(\alpha)-\varepsilon}\right]^{\frac{1}{p(\alpha)-\varepsilon}}\right.  \tag{2.4}\\
&=\left(\int_{\Omega}|x|^{\alpha}\left|u_{n}\right|^{p(\alpha)-\varepsilon}\right)^{\frac{p(\alpha)-1-\varepsilon}{p(\alpha)-\varepsilon}}\left\|\left.|x|\right|^{\frac{\alpha}{p(\alpha)-\varepsilon}}\left|u_{n}-u\right|\right\|_{L^{p(\alpha)-\varepsilon}(\Omega)} \\
& \leq C\left\|u_{n}\right\|^{p(\alpha)-1-\varepsilon}\left\|u_{n}|x|^{\frac{\alpha}{p(\alpha)-\varepsilon}}-u|x|^{\frac{\alpha}{p(\alpha)-\varepsilon}}\right\|_{L^{p(\alpha)-\varepsilon}(\Omega)} \\
& \quad=o(1) .
\end{align*}
$$

By (2.4), similar calculation also gives

$$
\begin{equation*}
\int_{\Omega}|x|^{\alpha}|u|^{p(\alpha)-2-\varepsilon} u\left(u_{n}-u\right)=o(1) . \tag{2.5}
\end{equation*}
$$

From the above analysis, we obtain

$$
o(1)=\left\langle J^{\prime}\left(u_{n}\right)-J^{\prime}(u), u_{n}-u\right\rangle=\left\|u_{n}-u\right\|^{2}+o(1),
$$

which implies $u_{n} \rightarrow u$ in $H_{0}^{1}(\Omega)$ and proves that $J$ satisfies $(P S)_{c}$ condition for every $c \in \mathbb{R}$.
Lemma 2.7. The function $J$ admits $a(P S)_{c}$ sequence in the cone of nonnegative function at the level

$$
c=\inf _{g \in \Gamma} \max _{t \in[0,1]} J(g(t)),
$$

where $\Gamma=\left\{g \in C\left([0,1], H_{0}^{1}(\Omega)\right): g(0)=0, J(g(1))<0\right\}$.
Proof. We next prove that $J$ satisfies all the hypotheses of the mountain pass lemma. Obviously, $J(0)=0$.

From the Sobolev-Hardy inequality, we obtain

$$
\begin{aligned}
J(u) & =\frac{1}{2}\|u\|^{2}-\frac{1}{p(\alpha)-\varepsilon} \int_{\Omega}|x|^{\alpha}|u|^{p(\alpha)-1-\varepsilon} u \\
& \geq \frac{1}{2}\|u\|^{2}-C_{1}\|u\|^{p(\alpha)-\varepsilon} .
\end{aligned}
$$

For any $\alpha$, we can choose $\varepsilon$ small enough such that $p(\alpha)-\varepsilon>2$. From the above analysis, there exist $\rho, e>0$ such that $J(u) \geq \rho, \forall u \in\left\{u \in H_{0}^{1}(\Omega):\|u\|=e\right\}$. Furthermore, for any $u \in H_{0}^{1}(\Omega)$,

$$
J(t u)=\frac{t^{2}}{2}\|u\|^{2}-\frac{t^{p(\alpha)-\varepsilon}}{p(\alpha)-\varepsilon} \int_{\Omega}|x|^{\alpha}|u|^{p(\alpha)-1-\varepsilon} u .
$$

We obtain $J(t u) \rightarrow-\infty$ as $t \rightarrow \infty$. Hence, we can choose $t_{0}>0$ such that $J\left(t_{0} u\right)<0$. Therefore, by Lemma 2.2, we infer that $J$ admits a $(P S)_{c}$ sequence at level $c$, such sequence may be chosen in the set of nonnegative functions because $J(|u|) \leq J(u)$ for all $u \in H_{0}^{1}(\Omega)$.

By Lemma 2.6, 2.7 and mountain pass lemma, we get a nonnegative solution $u \in H_{0}^{1}(\Omega)$ for (1.1), this solution is positive by the maximum principle.

## 3 Estimate of the singularity

First, we fix $p=p(\alpha)-1-\varepsilon>0$ in problem (1.1) and study the singularity and radial symmetry of the solution $u_{\varepsilon}(x) \in H_{0}^{1}(\Omega)$. By standard elliptic regularity theory, $u_{\varepsilon}(x) \in$ $C^{2}(\Omega \backslash\{0\}) \cap C^{1}(\bar{\Omega} \backslash\{0\})$. Hence the singular point of $u_{\varepsilon}(x)$ should be the origin.

Suppose that $u_{\varepsilon}(x) \in H_{0}^{1}(\Omega)$ satisfies problem (1.1).
Let $v(x)=|x|^{v} u_{\varepsilon}(x)$, then

$$
\begin{gathered}
-\Delta u=\left(-v^{2}-2 v+N v\right)|x|^{-v-2} v(x)+2 v|x|^{-v-2} x \nabla v(x)-|x|^{-v} \Delta v(x) \\
\mu \frac{u}{|x|^{2}}+|x|^{\alpha} u^{p(\alpha)-1-\varepsilon}=\mu|x|^{-v-2} v(x)+|x|^{\alpha-(p(\alpha)-1-\varepsilon) v} v(x)^{p(\alpha)-1-\varepsilon}
\end{gathered}
$$

From equation in (1.1),

$$
\begin{aligned}
\left(-v^{2}-2 v+N v\right)|x|^{-v-2} v(x)+2 v|x|^{-v-2} x & \nabla v(x)-|x|^{-v} \Delta v(x) \\
& =\mu|x|^{-v-2} v(x)+|x|^{\alpha-(p(\alpha)-1-\varepsilon) v} v(x)^{p(\alpha)-1-\varepsilon}
\end{aligned}
$$

Multiply both sides of the above equation by $|x|^{-v}$, then we get

$$
\left[-v^{2}+(N-2) v\right]|x|^{-2 v-2} v(x)-\operatorname{div}\left(|x|^{-2 v} \nabla v(x)\right)=\mu|x|^{-2 v-2} v(x)+|x|^{\alpha-(p(\alpha)-\varepsilon) v} v(x)^{p(\alpha)-1-\varepsilon} .
$$

For $v=\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}$, we obtain

$$
\begin{cases}-\operatorname{div}\left(|x|^{-2 v} \nabla v\right)=|x|^{-(p(\alpha)-\varepsilon) v+\alpha} v p(\alpha)-1-\varepsilon, & x \in \Omega  \tag{3.1}\\ v>0, & x \in \Omega \\ v=0, & x \in \partial \Omega\end{cases}
$$

By the regularity theory of elliptic equations, $v_{\varepsilon}(x) \in C^{2}(\Omega \backslash\{0\}) \cap C^{1}(\bar{\Omega} \backslash\{0\})$. Moreover, we have

## Lemma 3.1.

(i) $v(x) \in H_{0}^{1}\left(\Omega,|x|^{-2 v}\right)$.
(ii) $v(x)$ is bounded in $\Omega$.

Proof. (i) For any $u(x) \in H_{0}^{1}(\Omega)$ satisfying problem (1.1), by Hardy inequality, we have

$$
\begin{aligned}
\int_{\Omega}|x|^{-2 v}|\nabla v|^{2} & =\left.\int_{\Omega}|x|^{-2 v}| | x\right|^{v} \nabla u+\left.v|x|^{v-2} u x\right|^{2} \\
& \leq 2\left(\int_{\Omega}|\nabla u|^{2}+v^{2} \int_{\Omega} \frac{u^{2}}{|x|^{2}}\right) \\
& \leq C .
\end{aligned}
$$

Hence, we claim $v(x)=|x|^{v} u(x) \in H_{0}^{1}\left(\Omega,|x|^{-2 v}\right)$.
(ii) From Caffarelli-Kohn-Nirenberg inequality mentioned in Lemma 2.3, we have

$$
\begin{equation*}
\left(\int_{\Omega}|x|^{m_{1}}|\nabla u|^{2}\right)^{\frac{1}{2}} \geq C_{m_{1}, n_{1}}\left(\int_{\Omega}|x|^{n_{1}}|u|^{p\left(m_{1}, n_{1}\right)}\right)^{\frac{1}{p\left(m_{1}, n_{1}\right)}}, \quad \forall u \in H_{0}^{1}\left(\Omega,|x|^{m_{1}}\right) \tag{3.2}
\end{equation*}
$$

where

$$
m_{1}=-2 v, \quad n_{1}=-(p(\alpha)-\varepsilon) v+\alpha, \quad p\left(m_{1}, n_{1}\right)=p(\alpha)+\frac{\varepsilon v}{\sqrt{\bar{\mu}-\mu}}
$$

Note that

$$
\int_{\Omega}|x|^{m_{1}} \nabla v \cdot \nabla \varphi=\int_{\Omega}|x|^{n_{1}} v^{p} \varphi, \quad \forall \varphi \in H_{0}^{1}\left(\Omega,|x|^{m_{1}}\right) .
$$

For $s, l>1$, define $v_{l}(x)=\min \{v(x), l\}$. Taking $\varphi=v \cdot v_{l}^{2(s-1)} \in H_{0}^{1}\left(\Omega,|x|^{m_{1}}\right)$ in the above equation, we have

$$
\int_{\Omega}|x|^{m_{1}}|\nabla v|^{2} v_{l}^{2(s-1)}+2(s-1) \int_{\Omega}|x|^{m_{1}}\left|\nabla v_{l}\right|^{2} v_{l}^{2(s-1)}=\int_{\Omega}|x|^{n_{1}} v^{p+1} v_{l}^{2(s-1)} .
$$

Hence,

$$
\begin{align*}
& \left(\int_{\Omega}|x|^{n_{1}}\left(v \cdot v_{l}^{s-1}\right)^{p\left(m_{1}, n_{1}\right)}\right)^{\frac{2}{p\left(m_{1}, n_{1}\right)}} \\
& \quad \leq C_{m_{1}, n_{1}}^{-2} \int_{\Omega}|x|^{m_{1}}\left|\nabla\left(v \cdot v_{l}^{s-1}\right)\right|^{2}  \tag{3.3}\\
& \quad \leq 2 C_{m_{1}, n_{1}}^{-2}\left((s-1)^{2} \int_{\Omega}|x|^{m_{1}}\left|\nabla v_{l}\right|^{2} v_{l}^{2(s-1)}+\int_{\Omega}|x|^{m_{1}}|\nabla v|^{2} v_{l}^{2(s-1)}\right) \\
& \quad \leq 2 C_{m_{1}, n_{1}}^{-2} \int_{\Omega}|x|^{n_{1}} v^{p+2 s-1} .
\end{align*}
$$

From (3.3) and Levi's theorem, we see that $v \in L^{p+2 s-1}\left(\Omega,|x|^{n_{1}}\right)$ implies $v \in L^{s p\left(m_{1}, n_{1}\right)}\left(\Omega,|x|^{n_{1}}\right)$. For $j=0,1,2, \ldots$, by induction we define

$$
\begin{align*}
& \left\{\begin{array}{l}
p-1+2 s_{0}=p\left(m_{1}, n_{1}\right), \\
p-1+2 s_{j+1}=p\left(m_{1}, n_{1}\right) s_{j},
\end{array}\right.  \tag{3.4}\\
& \left\{\begin{array}{l}
M_{0}=\left(C \cdot C_{m_{1}, n_{1}}^{-2}\right)^{\frac{p\left(m_{1}, n_{1}\right)}{2}}, \\
M_{j+1}=\left(2 C_{m_{1}, n_{1}}^{-2} s_{j} M_{j}\right)^{\frac{p\left(m_{1}, n_{1}\right)}{2}},
\end{array}\right. \tag{3.5}
\end{align*}
$$

where $C$ is a fixed number such that $\int_{\Omega}|x|^{m_{1}}|\nabla v|^{2} \leq C$.
From (3.4), we see that

$$
s_{j}=\frac{\left(2^{-1} p\left(m_{1}, n_{1}\right)\right)^{j+1}\left(p\left(m_{1}, n_{1}\right)-p-1\right)+p-1}{p\left(m_{1}, n_{1}\right)-2} .
$$

From (3.5), similar to the computation in [21], we can see that

$$
\exists d>0 \text { and } d \text { is independent of } j \text {, such that } M_{j} \leq e^{d s_{j-1}} .
$$

Since $2<p+1<p\left(m_{1}, n_{1}\right)$, it follows that $s_{j}>1$ for all $j \geq 0, s_{j} \rightarrow+\infty$ as $j \rightarrow+\infty$.
By (3.3), (3.4) and (3.5),

$$
\begin{aligned}
\int_{\Omega}|x|^{n_{1}} v^{p+2 s_{1}-1} & \leq\left(2 C_{m_{1}, n_{1}}^{-2} s_{0}\right)^{\frac{p\left(m_{1}, n_{1}\right)}{2}}\left(\int_{\Omega}|x|^{n_{1}} v^{p+2 s_{0}-1}\right)^{\frac{p\left(m_{1}, n_{1}\right)}{2}} \\
& \leq\left(2 C_{m_{1}, n_{1}}^{-2} s_{0}\right)^{\frac{p\left(m_{1}, n_{1}\right)}{2}}\left(C^{\frac{p\left(m_{1}, n_{1}\right)}{2}} C_{m_{1}, n_{1}}^{-p\left(m_{1}, n_{1}\right)}\right)^{\frac{p\left(m_{1}, n_{1}\right)}{2}} \\
& \leq\left(2 C_{m_{1}, n_{1}}^{-2} s_{0} M_{0}\right)^{\frac{p\left(m_{1}, n_{1}\right)}{2}} \\
& \leq M_{1} .
\end{aligned}
$$

Similarly,

$$
\int_{\Omega}|x|^{n_{1} v^{p+2 s_{j}-1}} \leq M_{j} .
$$

Hence, by $p+2 s_{j+1}-1=p\left(m_{1}, n_{1}\right) s_{j}$, denoting $C\left(\Omega, n_{1}\right)=\max _{x \in \Omega}|x|^{-n_{1}}$, we obtain

$$
\begin{aligned}
|v|_{L^{p\left(m_{1}, n_{1}\right) s_{j}}(\Omega)} & \leq\left(\int_{\Omega}|v|^{p\left(m_{1}, n_{1}\right) s_{j}}|x|^{n_{1}} \cdot|x|^{-n_{1}}\right)^{\frac{1}{p\left(m_{1}, n_{1}\right) s_{j}}} \\
& \leq C\left(\Omega, n_{1}\right)^{\frac{1}{p\left(m_{1}, n_{1}\right) s_{j}}}|v|_{L^{p\left(m_{1}, n_{1}\right) s_{j}}}^{\frac{1}{p\left(m_{1}, s_{1}\right) s_{j}}}\left(\Omega,|x|^{n_{1}}\right) \\
& \leq C\left(\Omega, n_{1}\right)^{\frac{1}{p\left(m_{1}, n_{1}\right) s_{j}}} M_{j+1}^{\frac{1}{p\left(m_{1}, n_{1}\right) s_{j}}} \\
& \leq C\left(\Omega, n_{1}\right)^{\frac{1}{p\left(m_{1}, n_{1}\right) s_{j}}} e^{\frac{d}{p\left(m_{1}, n_{1}\right)}} .
\end{aligned}
$$

Taking limit on each side of the above inequality and using $s_{j} \rightarrow+\infty$, as $j \rightarrow+\infty$, we have

$$
|v|_{L^{\infty}(\Omega)} \leq e^{\frac{d}{p\left(m_{1}, m_{1}\right)}},
$$

which implies the conclusion.
From Lemma 3.1, we can see that $v(x)=|x|^{\nu} u(x)$ is bounded form above in $\Omega$. For the lower bound of $v(x)=|x|^{v} u(x)$, we have

Lemma 3.2. Suppose that $u(x) \in H_{0}^{1}(\Omega)$ satisfies problem (1.1) and $0 \leq \mu<\bar{\mu}$, then for any $B_{\rho} \subset \subset \Omega$ there exists a $C(\rho)>0$, such that

$$
u(x) \geq C(\rho)|x|^{-v}, \quad \forall x \in B_{\rho} \subset \subset \Omega .
$$

Proof. Let $f(x)=\min \left\{|x|^{\alpha} u^{p(\alpha)-1-\varepsilon}(x), l\right\}$ with $l>0$, then $f \in L^{\infty}(\Omega)$.
Let $u_{1} \geq 0$ and $u_{1} \in H_{0}^{1}(\Omega)$ be the solution of the following linear problem

$$
\begin{cases}-\Delta u_{1}=\mu \frac{u_{1}}{|x|^{2}}+f, & x \in \Omega,  \tag{3.6}\\ u_{1}=0, & x \in \partial \Omega .\end{cases}
$$

Set $U=u-u_{1}$, then $U \in H_{0}^{1}(\Omega)$ and $U$ satisfies the following problem

$$
\begin{cases}-\Delta U=\mu \frac{U}{|x|^{2}}+g, & x \in \Omega  \tag{3.7}\\ U=0, & x \in \partial \Omega\end{cases}
$$

where $g \geq 0$ and $0 \leq \mu<\bar{\mu}=\left(\frac{N-2}{2}\right)^{2}$.
From Lemma 2.4, there exist solutions for problem (3.6) and (3.7). From the Hardy inequality and the comparison principle proved in [15], we obtain that $u$ is a super-solution of problem (3.6) and $0 \leq u_{1} \leq u$. Actually we can prove this as follows. Multiplying $U^{-}:=\max \{0,-U(x)\}$ on both side of equation in (3.7) and integrating by parts, we have

$$
-\int_{\Omega}\left|\nabla U^{-}\right|^{2}=-\int_{\Omega} \mu \frac{\left(U^{-}\right)^{2}}{|x|^{2}}+\int_{\Omega} g U^{-} .
$$

It follows that $U^{-}=0$ in $\Omega$ and hence $U \geq 0$.

By Lemma 3.1, there exists a constant $C_{1}>0$ such that $0 \leq u_{1}(x) \leq u(x) \leq C_{1}|x|^{-v}$. So it suffices to prove the result for $u_{1}$.

Since $u_{1} \not \equiv 0, u_{1} \geq 0$ and $-\Delta u_{1} \geq 0$ in $\Omega$, there exists $\delta>0$ such that for sufficiently small $\rho>0$ it holds that $u_{1} \geq \delta$ for $\forall x \in B_{2 \rho}$. Choose $C(\rho) \geq 0$ satisfying $C(\rho)|x|^{-v} \leq \delta$ for $|x|=\rho$ and set $\omega=\left(u_{1}-C|x|^{-v}\right)^{-}$. By $\left.\left.\int_{B_{\rho}}|\nabla| x\right|^{-v}\right|^{2}<\infty$ and $u_{1} \in H_{0}^{1}\left(B_{\rho}\right)$, we have $\omega \in H_{0}^{1}\left(B_{\rho}\right)$.

From (3.6) and the fact that $|x|^{-v}$ is the solution of equation $-\Delta u-\mu \frac{u}{|x|^{2}}=0$, the linear combination of $u_{1}$ and $|x|^{-v}$ is the solution of $-\Delta u=\mu \frac{u}{|x|^{2}}+f$. Hence,

$$
-\Delta\left(u_{1}-C|x|^{-v}\right)=\mu \frac{\left(u_{1}-C|x|^{-v}\right)}{|x|^{2}}+f .
$$

Multiply $\omega$ on both side of the above equation and integrate by part, we obtain

$$
-\int_{B_{\rho}}|\nabla \omega|^{2}+\int_{B_{\rho}} \mu \frac{\omega^{2}}{|x|^{2}}=\int_{B_{\rho}} f \omega \geq 0 .
$$

Since $0 \leq \mu<\bar{\mu}$, it follows that $\omega=0$.
Another proof of $\omega=0$ : It only need to prove that $-\int_{B_{\rho}}|\nabla \omega|^{2}+\int_{B_{\rho}} \mu \frac{\omega^{2}}{|x|^{2}} \omega \geq 0$. Otherwise,

$$
\begin{aligned}
0 & >-\int_{B_{\rho}}|\nabla \omega|^{2}+\int_{B_{\rho}} \mu \frac{\omega^{2}}{|x|^{2}} \\
& =\int_{B_{\rho}} \nabla\left(u_{1}-C|x|^{-v}\right) \cdot \nabla \omega-\int_{B_{\rho}} \frac{\mu}{|x|^{2}}\left(u_{1}-C|x|^{-v}\right) \omega \\
& =\int_{B_{\rho}} f \omega-C\left(\int_{B_{\rho}} \nabla|x|^{-v} \cdot \nabla \omega-\int_{B_{\rho}} \frac{\mu}{|x|^{2}}|x|^{-v} \omega\right) \\
& =\int_{B_{\rho}} f \omega+\frac{C v}{\rho^{v+1}} \int_{\partial B_{\rho}} \omega \\
& >\frac{C v}{\rho^{v+1}} \int_{\partial B_{\rho}} \omega \\
& \geq 0 .
\end{aligned}
$$

This is a contradiction and we are done.
Proposition 3.3. Suppose that $u(x) \in H_{0}^{1}(\Omega)$ satisfies problem (1.1) and $0 \leq \mu<\bar{\mu}$. Then for any $\Omega^{\prime} \subset \subset \Omega$ there exists two positive constants $C_{1}$ and $C_{2}$, such that

$$
\left\{\begin{array}{l}
u(x)|x|^{v} \geq C_{1}, \quad \forall x \in \Omega^{\prime} \subset \subset \Omega .  \tag{3.8}\\
u(x)|x|^{v} \leq C_{2}, \quad \forall x \in \Omega .
\end{array}\right.
$$

Next, we use Lemma 2.5 and Proposition 3.3 to prove that the solution is radially symmetric with $\Omega=B_{R}$.
Theorem 3.4. Suppose that $-2<\alpha<0$ and $p(\alpha)=\frac{2(N+\alpha)}{N-2}$. Then the solution of problem (1.1) is radially symmetric.

Proof. Using the previous notations, we only need to show that $v(x)$ is radially symmetric in $\Omega$. By the regularity theory of elliptic equations, we have $v(x) \in C^{2}\left(B_{R} \backslash\{0\}\right) \cap C^{1}\left(\overline{B_{R}} \backslash\{0\}\right)$. Next, we have to prove that $v(x)$ satisfies Lemma 2.5. From (3.1), we obtain

$$
\partial_{i}\left(|x|^{-2 v} \partial_{i} v\right)+|x|^{-(p(\alpha)-\varepsilon) v+\alpha} v^{p(\alpha)-1-\varepsilon}=0 .
$$

Hence, $v(x)$ satisfies Lemma 2.5 when $b=-2 v, a=-(p(\alpha)-\varepsilon) v+\alpha, q=p(\alpha)-1-\varepsilon$ and $K=1$.

## 4 Some basic estimates

Set $r=|x|$. Let $v(r)=|x|^{v} u_{\varepsilon}(x)$. Then $v(r)$ satisfies

$$
\left\{\begin{array}{l}
v^{\prime \prime}+\frac{N-1-2 v}{r} v^{\prime}+\frac{1}{r^{(p(\alpha)-2-\varepsilon) v-\alpha}} v^{p(\alpha)-1-\varepsilon}=0,  \tag{4.1}\\
v(r)>0, \text { for } 0<r<R, \\
v(R)=0 .
\end{array}\right.
$$

Let $t=\left(\frac{N-2 v-2}{r}\right)^{N-2 v-2}$ and $y(t)=(N-2 v-2)^{-g(\alpha, \varepsilon)} v(r)$, where $g(\alpha, \varepsilon)=\frac{(p(\alpha)-2-\varepsilon) v-\alpha}{p(\alpha)-2-\varepsilon}$. Then problem (4.1) can be rewritten as

$$
\left\{\begin{align*}
y^{\prime \prime}(t) & =-t^{-k(\alpha, \varepsilon)} y^{p(\alpha)-1-\varepsilon}  \tag{4.2}\\
y(t) & >0, \text { for } T<t<\infty \\
y(T) & =0
\end{align*}\right.
$$

where $k(\alpha, \varepsilon)=\frac{2 m+\alpha}{m-1}-\frac{(p(\alpha)-2-\varepsilon) v}{m-1}, m=1+2 \sqrt{\bar{\mu}-\mu}=N-2 v-1, T=\left(\frac{m-1}{R}\right)^{m-1}, p(\alpha)-$ $1=2 k(\alpha, \varepsilon)-3-\frac{2 v \varepsilon}{m-1}$.

In order to simplify the expression, we will always replace $k(\alpha, \varepsilon)$ with $k$ in the sequel.
First we give
Lemma 4.1. Let $y(t)$ be a solution of problem (4.2), then there exists a positive number $\gamma<\infty$ such that

$$
\lim _{t \rightarrow \infty} y^{\prime}(t)=0 \quad \text { and } \quad \lim _{t \rightarrow \infty} y(t)=\gamma
$$

Proof. By Proposition 3.3, it is obvious that $y(t)$ is bounded in $[T, \infty)$. From (4.2), we know $y^{\prime \prime}(t)<0$ for all $t>T$, so $y^{\prime}(t)$ decreases strictly in $t \in(T, \infty)$. Hence

$$
y^{\prime}(t) \rightarrow c \quad \text { x0as } t \rightarrow+\infty .
$$

If $c>0$, we can deduce $y(t) \rightarrow+\infty$ when $t \rightarrow+\infty$. Similarly, when $c<0$, we have $y(t) \rightarrow-\infty$ when $t \rightarrow+\infty$. However, the boundedness of $y(t)$ leads to the contradiction. Hence, $\lim _{t \rightarrow \infty} y^{\prime}(t)=0$ and $\lim _{t \rightarrow \infty} y(t)=\gamma$.

## Remark 4.2.

(i) From Lemma 4.1, if we define $v(0)=\lim _{r \rightarrow 0} v(r)=(N-2 v-2)^{g(\alpha, \varepsilon)} \gamma$, then $v(r) \in$ $C[0, R]$. Furthermore, $v^{\prime}(r)<0$ for all $r \in(0, R]$.
(ii) $y^{\prime}(t)>0$ for all $t>T$ and $y^{\prime}(t) \sim \frac{1}{k-1} 1^{1-k} \gamma^{p(x)-1-\varepsilon}$ as $t \rightarrow \infty$.

Next, we consider

$$
\left\{\begin{array}{l}
y^{\prime \prime}(t)+t^{-k} y^{p(\alpha)-1-\varepsilon}=0, \quad t<\infty  \tag{4.3}\\
\lim _{t \rightarrow \infty} y(t)=\gamma
\end{array}\right.
$$

where $\gamma>0$.
Since $k>2$, it follows from [2] that problem (4.3) has a unique solution which will be denoted by $y(t, \gamma)$ for every $\gamma>0$. Define

$$
\begin{equation*}
T(\gamma)=\inf \{t>0: y(t, \gamma)>0 \text { on }(t, \infty)\} \tag{4.4}
\end{equation*}
$$

From Lemma 4.1, we have $\lim _{t \rightarrow \infty} w(s)=1$, where $w(s)=\frac{y(t)}{\gamma}$. Hence,

$$
T(1)=\inf \{s>0: w(s, 1)>0 \text { on }(s, \infty)\} .
$$

Set $t=\gamma^{\frac{p(x)-2-\varepsilon}{k-2}} s$, then

$$
w^{\prime \prime}(s)=-s^{-k} w^{p(\alpha)-1-\varepsilon}(s) .
$$

So, we have

$$
T(\gamma)=\gamma^{\frac{p(\alpha)-2-\varepsilon}{k-2}} T(1) .
$$

By Lemma 5.1 in Section 5, $T(1)>0$. Thus for every $\gamma>0, T(\gamma)>0$.
Hence, for any $T>0$ and given $\varepsilon>0$ small, there exists a unique $\gamma$ such that problem (4.3) has a solution $y(t, \gamma)$ such that $\gamma>0, T(\gamma)>0$.

Remark 4.3. From the above analysis, when $\Omega$ is a ball centered at the origin, we conclude that the solution to problem (1.1) is unique.

Now we give an upper and lower bound for $y(t, \gamma)$.
Lemma 4.4. Suppose $\varepsilon>0$, then

$$
\begin{equation*}
y(t, \gamma)<z(t, \gamma), \quad \text { for } T(\gamma) \leq t<\infty \tag{4.5}
\end{equation*}
$$

where

$$
z(t, \gamma)=\gamma\left(1+\frac{1}{k-1} \frac{\gamma^{p(\alpha)-2-\varepsilon}}{t^{k-2}}\right)^{-\frac{1}{k-2}}
$$

Proof. Since

$$
\left(y^{\prime} t^{k-1} y^{1-k}\right)^{\prime}=-(k-1) t^{k-2} y^{-k} H_{1}(t),
$$

where

$$
H_{1}(t)=t\left(y^{\prime}\right)^{2}-y y^{\prime}+\frac{1}{k-1} t^{1-k} y^{p+1}
$$

and $y^{\prime}(t) \sim \frac{1}{k-1} t^{1-k} \gamma^{p(\alpha)-1-\varepsilon}$ (see Remark 4.2), we have $\lim _{t \rightarrow \infty} H_{1}(t)=0$.
By $H_{1}^{\prime}(t)=\frac{1}{k-1} t^{1-k} y^{\prime}(t)(p-2 k+3) y^{p}$, we have

$$
H_{1}^{\prime}(t)<0 \quad \text { for } \forall t \in[T, \infty) .
$$

Hence $H_{1}(t)$ decreases strictly on $[T, \infty)$. In combination with $\lim _{t \rightarrow \infty} H_{1}(t)=0$, we can obtain $H_{1}(t)>0$ on $(T, \infty)$ which implies $\left(y^{\prime} t^{k-1} y^{1-k}\right)^{\prime}<0$.

Integrating $\left(y^{\prime} t^{k-1} y^{1-k}\right)^{\prime}<0$ from $t>T$ to $t=\infty$, we deduce

$$
y^{1-k} y^{\prime}(t)>\frac{1}{k-1} \gamma^{p-k+1} t^{1-k}, \quad \text { for } T<t<\infty .
$$

Integrating the above equation again from $t>T$ to $t=\infty$, we deduce

$$
y^{2-k}(t)>\frac{1}{k-1} \gamma^{p-k+1} t^{2-k}+\gamma^{2-k}, \quad \text { for } T<t<\infty,
$$

which implies the conclusion.

Remark 4.5. The function $z(t, \gamma)$ is the solution of the following problem

$$
\left\{\begin{array}{l}
z^{\prime \prime}(t)+t^{-k} \gamma^{-\left(\frac{2 v}{m-1}+1\right) \varepsilon} z^{2 k-3}=0, \quad 0<t<\infty  \tag{4.6}\\
\lim _{t \rightarrow \infty} z(t, \gamma)=\gamma
\end{array}\right.
$$

In the sequel, $z(t, \gamma)$ plays an important role.
Set
where $k_{1}(\alpha, \varepsilon)=(k-1)^{\frac{1}{k-2}}$.
Then for any $\beta>0$, direct computation gives

$$
\begin{equation*}
z\left(\beta T_{\alpha, \varepsilon}, \gamma\right)=C_{\alpha, \beta, \varepsilon} \gamma, \tag{4.8}
\end{equation*}
$$

where $C_{\alpha, \beta, \varepsilon}=\frac{\beta}{\left(1+\beta^{k-2}\right)^{\frac{1}{k-2}}}$.
Lemma 4.6. Let $\beta>0$ and $\varepsilon>0$, then for every $t \geq \beta T_{\alpha, \varepsilon}$,

$$
y(t, \gamma) \geq z(t, \gamma)\left(1-d_{\alpha, \beta, \varepsilon} \varepsilon\right)
$$

where

$$
d_{\alpha, \beta, \varepsilon}=\frac{\left(1-C_{\alpha, \beta, \varepsilon}\right)(1+2 v /(m-1))}{C_{\alpha, \beta, \varepsilon}^{2+(1+2 v /(m-1)) \varepsilon}}
$$

Proof. Integrating (4.3) twice, we have

$$
y(t, \gamma)=\gamma-\int_{t}^{\infty}(s-t) s^{-k} y^{2 k-3-(1+2 v /(m-1)) \varepsilon}(s, \gamma) d s
$$

Hence, by Lemma 4.4, we obtain

$$
y(t, \gamma)>\gamma-\int_{t}^{\infty}(s-t) s^{-k} z^{2 k-3-(1+2 v /(m-1)) \varepsilon}(s, \gamma) d s
$$

Similarly, integrate (4.6) for $z$ twice, then

$$
z(t, \gamma)=\gamma-\int_{t}^{\infty}(s-t) s^{-k} \gamma^{-(1+2 v /(m-1)) \varepsilon} z^{2 k-3}(s, \gamma) d s
$$

Hence

$$
\begin{equation*}
y(t, \gamma)>z(t, \gamma)-\int_{t}^{\infty}(s-t) s^{-k} z^{2 k-3}(s, \gamma)\left(z^{-(1+2 v /(m-1)) \varepsilon}-\gamma^{-(1+2 v /(m-1)) \varepsilon}\right) d s \tag{4.9}
\end{equation*}
$$

By the mean value theorem, we deduce

$$
\left|z^{-(1+2 v /(m-1)) \varepsilon}-\gamma^{-(1+2 v /(m-1)) \varepsilon}\right|=(1+2 v /(m-1)) \varepsilon \theta^{-1-(1+2 v /(m-1)) \theta}|z(s, \gamma)-\gamma|
$$

where $z(s, \gamma) \leq \theta \leq \gamma$.
Hence, using (4.8), if $\beta T_{\alpha, \varepsilon} \leq t<\infty$, we have

$$
\begin{aligned}
& \left|z^{-(1+2 v /(m-1)) \varepsilon}-\gamma^{-(1+2 v /(m-1)) \varepsilon}\right| \\
& \quad \leq(1+2 v /(m-1)) \varepsilon\left(C_{\alpha, \beta, \varepsilon} \varepsilon\right)^{-1-(1+2 v /(m-1)) \varepsilon} \gamma \\
& \quad \leq(1+2 v /(m-1)) \varepsilon C_{\alpha, \beta, \varepsilon}^{-1-(1+2 v /(m-1)) \varepsilon} \gamma^{-(1+2 v /(m-1)) \varepsilon} .
\end{aligned}
$$

Using this bound in (4.9), if $t \geq \beta T_{\alpha, \varepsilon}$,

$$
\begin{align*}
y(t, \gamma)> & z(t, \gamma)-(1+2 v /(m-1)) \varepsilon C_{\alpha, \beta, \varepsilon}^{-1-(1+2 v /(m-1)) \varepsilon} \\
& \times \int_{t}^{\infty}(s-t) s^{-k} \gamma^{-(1+2 v /(m-1)) \varepsilon} z^{2 k-3}(s, \gamma) d s  \tag{4.10}\\
= & z(t, \gamma)+(1+2 v /(m-1)) \varepsilon C_{\alpha, \beta, \varepsilon}^{-1-(1+2 v /(m-1)) \varepsilon}(z(t, \gamma)-\gamma) .
\end{align*}
$$

On the other hand, by (4.8), if $t \geq \beta T_{\alpha, \varepsilon}$,

$$
\gamma=C_{\alpha, \beta, z}^{-1} z\left(\beta T_{\alpha, \varepsilon}, \gamma\right) \leq C_{\alpha, \beta, z}^{-1} z(t, \gamma) .
$$

So we can deduce from (4.10) that

$$
\begin{aligned}
y(t, \gamma) & >z(t, \gamma)\left(1+\frac{(1+2 v /(m-1))}{C_{\alpha, \beta, \varepsilon}^{1+(1+2 v /(m-1)) \varepsilon}}\left(1-\frac{1}{C_{\alpha, \beta, \varepsilon}}\right) \varepsilon\right) \\
& =z(t, \gamma)\left(1-d_{\alpha, \beta, \varepsilon} \varepsilon\right)
\end{aligned}
$$

which is the bound we want to prove.
Now we return to problem (4.2). We fix $T$ and denote the solution by $y(t)$. Then

$$
\gamma(\varepsilon)=\lim _{t \rightarrow \infty} y(t)
$$

and $\gamma(\varepsilon)$ depends on $\varepsilon$. The next lemma tells us the asymptotic behavior of $\gamma(\varepsilon)$ as $\varepsilon \rightarrow 0$.

## Lemma 4.7.

$$
\lim _{\varepsilon \rightarrow 0} \gamma(\varepsilon)=\infty
$$

Proof. By contradiction, we can assume there exists a sequence $\left\{\varepsilon_{n}\right\}, \varepsilon_{n} \rightarrow 0$ as $n \rightarrow \infty$, and a number $M>0$ such that $\gamma\left(\varepsilon_{n}\right) \leq M$ for all $n$. Then we can choose a number $\beta_{1}>0$ satisfying

$$
\beta_{1} T_{\alpha, \varepsilon_{n}}=\beta_{1} k_{1}^{-1} \gamma\left(\varepsilon_{n}\right)^{2-\frac{m-1+2 \nu}{(m-1)(k-2)}} \leq \beta_{1}\left(\frac{n-2}{n+\alpha}\right)^{\frac{n-2}{\alpha+2}} M^{2}+o(1) \leq T \quad \text { for large } n .
$$

So by Lemma 4.6, we have

$$
z\left(t, \gamma\left(\varepsilon_{n}\right)\right)\left(1-d_{\alpha, \beta, \varepsilon_{n}} \varepsilon_{n}\right)<z\left(T, \gamma\left(\varepsilon_{n}\right)\right)\left(1-d_{\alpha, \beta, \varepsilon_{n}} \varepsilon_{n}\right) \leq y\left(T, \gamma\left(\varepsilon_{n}\right)\right)=0,
$$

for $0<t<\beta_{1} T_{\alpha, \varepsilon_{n}}$ and large $n$, which is impossible.
Finally, we give two formulae to use later. Define incomplete Beta function

$$
B(\varsigma, P, Q)=\int_{\varsigma}^{\infty} x^{P-1}(1+x)^{-P-Q} d x
$$

where $P$ and $Q$ are positive parameters. It is well-known that

$$
\begin{equation*}
B(0, P, Q)=\frac{\Gamma(P) \Gamma(Q)}{\Gamma(P+Q)} \tag{4.11}
\end{equation*}
$$

Lemma 4.8. Suppose $k>2, p=2 k-3-\left(1+\frac{2 v}{m-1}\right) \varepsilon$ and $\varepsilon$ small. Then

$$
\begin{equation*}
\int_{t}^{\infty} s^{-k} z^{p}(s, \gamma) d s=k_{1}(\alpha, \varepsilon) k_{2}(\alpha, \varepsilon) \gamma^{-1+\varphi(\alpha, \varepsilon)} B\left(\tau(\alpha, \varepsilon), 1-\varphi(\alpha, \varepsilon), k_{2}(\alpha, \varepsilon)\right) \tag{i}
\end{equation*}
$$

$$
\begin{equation*}
\int_{t}^{\infty} s^{-k} z^{p+1}(s, \gamma) d s=k_{1}(\alpha, \varepsilon) k_{2}(\alpha, \varepsilon) \gamma^{\varphi(\alpha, \varepsilon)} B\left(\tau(\alpha, \varepsilon), k_{2}(\alpha, \varepsilon)-\varphi(\alpha, \varepsilon), k_{2}(\alpha, \varepsilon)\right), \tag{ii}
\end{equation*}
$$

where $\varphi(\alpha, \varepsilon)=\frac{m-1+2 v}{(m-1)(k-2)} \varepsilon, k_{1}(\alpha, \varepsilon)=(k-1)^{\frac{1}{k-2}}, k_{2}(\alpha, \varepsilon)=\frac{k-1}{k-2}, \tau(\alpha, \varepsilon)=\left(\frac{t}{T_{\alpha, \varepsilon}}\right)^{k-2}$.

Proof. (i) Insert the expression

$$
z(t, \gamma)=\gamma\left(1+\frac{1}{k-1} \frac{\gamma^{p(\alpha)-2-\varepsilon}}{t^{k-2}}\right)^{-\frac{1}{k-2}}
$$

into the integral

$$
\begin{align*}
\int_{t}^{\infty} s^{-k} z^{p}(s, \gamma) d s & =\gamma^{p} \int_{t}^{\infty} s^{p-k}\left(s^{k-2}+\frac{1}{k-1} \gamma^{p-1}\right)^{-\frac{p}{k-2}} d s  \tag{4.12}\\
& =\gamma^{p} \int_{t}^{\infty} s^{p-k}\left(s^{k-2}+T_{\alpha, \varepsilon}^{k-2}\right)^{-\frac{p}{k-2}} d s
\end{align*}
$$

and by routine calculus, we can get the result as follows.
By making the change of variable $x=\left(\frac{s}{T_{\alpha, e}}\right)^{k-2}$, we can write (4.12) as

$$
\begin{aligned}
\int_{t}^{\infty} s^{-k} z^{p}(s, \gamma) d s & =\gamma^{p} \int_{t}^{\infty} s^{p-k}\left(s^{k-2}+T_{\alpha, \varepsilon}^{k-2}\right)^{-\frac{p}{k-2}} d s \\
& =\frac{\gamma^{p}}{k-2} T_{\alpha, \varepsilon}^{1-k} \int_{\left(\frac{t}{T_{\alpha, \varepsilon}}\right)}^{\infty} x^{k-2} x^{P-1}(1+x)^{-P-Q} d x,
\end{aligned}
$$

where $P=\frac{p-k-1}{k-2}$ and $Q=\frac{k-1}{k-2}$.
Since

$$
\frac{\gamma^{p}}{k-2} T_{\alpha, \varepsilon}^{1-k}=k_{1}(\alpha, \varepsilon) k_{2}(\alpha, \varepsilon) \gamma^{-1+\varphi(\alpha, \varepsilon)},
$$

we have

$$
\int_{t}^{\infty} s^{-k} z^{p}(s, \gamma) d s=k_{1}(\alpha, \varepsilon) k_{2}(\alpha, \varepsilon) \gamma^{-1+\varphi(\alpha, \varepsilon)} B\left(\tau(\alpha, \varepsilon), 1-\varphi(\alpha, \varepsilon), k_{2}(\alpha, \varepsilon)\right),
$$

where $\varphi(\alpha, \varepsilon)=\frac{m-1+2 v}{(m-1)(k-2)} \varepsilon, k_{1}(\alpha, \varepsilon)=(k-1)^{\frac{1}{k-2}}, k_{2}(\alpha, \varepsilon)=\frac{k-1}{k-2}, \tau(\alpha, \varepsilon)=\left(\frac{t}{T_{\alpha, \varepsilon}}\right)^{k-2}$.
(ii) In a similar way as in (i).

We end this section by giving

$$
\begin{align*}
& \lim _{\varepsilon \rightarrow 0} k=\frac{2 N-2+\alpha}{N-2} \triangleq k_{0}, \quad \lim _{\varepsilon \rightarrow 0} k_{1}(\alpha, \varepsilon)=\left(\frac{N+\alpha}{N-2}\right)^{\frac{N-2}{\alpha+2}} \triangleq k_{1}, \\
& \lim _{\varepsilon \rightarrow 0} k_{2}(\alpha, \varepsilon)=\frac{N+\alpha}{\alpha+2} \triangleq k_{2}, \quad \lim _{\varepsilon \rightarrow 0} C_{\alpha, \beta, \varepsilon}=\frac{\beta}{\left(1+\beta^{\frac{\alpha+2}{N-2}}\right)^{\frac{N-2}{\alpha+2}} \triangleq C_{\alpha, \beta}}  \tag{4.13}\\
& \lim _{\varepsilon \rightarrow 0} d_{\alpha, \beta, \varepsilon}=\frac{\left(1-c_{\alpha, \beta}\right)(1+2 v /(m-1))}{c_{\alpha, \beta}^{2}} \triangleq d_{\alpha, \beta} \quad \quad \lim _{\varepsilon \rightarrow 0} \varphi(\alpha, \varepsilon)=\lim _{\varepsilon \rightarrow 0} \tau(\alpha, \varepsilon)=0 .
\end{align*}
$$

## 5 Proof of the main results

Note that if $u_{\varepsilon}(x)$ is a solution of problem (1.1) when $\Omega=B_{R}$, then from the previous analysis, we know that

$$
\lim _{|x| \rightarrow 0} u_{\varepsilon}(x)|x|^{\nu}=(N-2 v-2)^{g(\alpha, \varepsilon)} \gamma(\varepsilon),
$$

where $g(\alpha, \varepsilon)=\frac{(p(\alpha)-2-\varepsilon) v-\alpha}{p(\alpha)-2-\varepsilon}$ and $R=(m-1) T^{-1 /(m-1)}$. Thus we need to understand how $\gamma(\varepsilon)$ tends to infinity as $\varepsilon \rightarrow 0$.

We define the following Pohozaev functional introduced from [1] and [22],

$$
\begin{equation*}
H(t)=t y^{\prime 2}-y y^{\prime}+2 t^{1-k} \frac{y^{p+1}}{p+1^{\prime}} \tag{5.1}
\end{equation*}
$$

where

$$
p=2 k-3-\left(1+\frac{2 v}{m-1}\right) \varepsilon=p(\alpha)-1-\varepsilon .
$$

If $y(t)$ solves problem (4.3), then

$$
\begin{equation*}
H^{\prime}(t)=-\frac{(1+2 v /(m-1)) \varepsilon}{p+1} t^{-k} y^{p+1} \tag{5.2}
\end{equation*}
$$

and $y^{\prime}(t)=O\left(t^{1-k}\right)$ as $t \rightarrow \infty$ (see Remark 4.2). Hence

$$
\lim _{t \rightarrow \infty} H(t)=0 .
$$

Since $H(T)=T y^{\prime 2}(T)$, integrating (5.2) from $t>T$ to $t=\infty$, we obtain

$$
\begin{equation*}
T y^{\prime 2}(T)=\frac{(1+2 v /(m-1)) \varepsilon}{p+1} \int_{T}^{\infty} t^{-k} y^{p+1}(t) d t . \tag{5.3}
\end{equation*}
$$

This equation is crucial for us to obtain the desired results.
Lemma 5.1. Let $T(\gamma)$ be defined as (4.4), then $T(1)>0$.
Proof. By Lemma 4.2, $y(t, 1) \leq z(t, 1)$ for $t \geq T(1)$. Suppose in contrast that $T(1)=0$, then

$$
y^{\prime}(0,1) \leq z^{\prime}(0,1)=k_{1}^{k-1}(\alpha, \varepsilon) .
$$

So

$$
y(t, 1) \leq k_{1}^{k-1}(\alpha, \varepsilon) t, \quad t \geq 0
$$

which means $H(0)=0$.
On the other hand, combination of (5.2) and the fact $\lim _{t \rightarrow \infty} H(t)=0$ yields $H(t)>0$ for $T(1) \leq t<\infty$. This is a contradiction and our conclusion follows.

Lemma 5.2. $\lim _{\varepsilon \rightarrow 0} \gamma^{1-\varphi(\alpha, \varepsilon)} y^{\prime}(T)=k_{1}$, where $\gamma=\gamma(\varepsilon)$.
Proof. Integrating equation (4.2) over ( $T, \infty$ ), we derive

$$
\begin{equation*}
y^{\prime}(T)=\int_{T}^{\infty} t^{-k} y^{p}(t) d t<\int_{T}^{\infty} t^{-k} z^{p}(t) d t . \tag{5.4}
\end{equation*}
$$

Hence, by Lemma 4.8 (i) and Lemma 4.4, as $\varepsilon \rightarrow 0$,

$$
\gamma^{1-\varphi(\alpha, \varepsilon)} y^{\prime}(T) \leq k_{1}(\alpha, \varepsilon) k_{2}(\alpha, \varepsilon) B\left(\left(\frac{T}{T_{\alpha, \varepsilon}}\right)^{k-2}, 1-\varphi(\alpha, \varepsilon), k_{2}(\alpha, \varepsilon)\right) \rightarrow k_{1} k_{2} B\left(0,1, k_{2}\right) .
$$

By (4.11) and the fact that $\Gamma(x+1)=x \Gamma(x)$, we deduce

$$
k_{1} k_{2} B\left(0,1, k_{2}\right)=\frac{k_{1} k_{2}}{k_{2}}=k_{1} .
$$

Therefore

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \sup \gamma^{1-\varphi(\alpha, \varepsilon)} y^{\prime}(T) \leq k_{1} \tag{5.5}
\end{equation*}
$$

Next, we shall show that for any $\delta>0$,

$$
\begin{equation*}
\liminf _{\varepsilon \rightarrow 0} \gamma^{1-\varphi(\alpha, \varepsilon)} y^{\prime}(T) \geq k_{1}-\delta \tag{5.6}
\end{equation*}
$$

which completes the proof of this lemma.
For a given $\beta>0$, by (4.2) and Lemma 4.7, we can choose $\varepsilon>0$ so small that $\beta T_{\alpha, \varepsilon}>T$. Thus (5.4) can be written as

$$
\begin{align*}
\gamma^{1-\varphi(\alpha, \varepsilon)} y^{\prime}(T) & =\gamma^{1-\varphi(\alpha, \varepsilon)} \int_{T}^{\beta T_{\alpha, \varepsilon}} t^{-k} y^{p}(t) d t+\gamma^{1-\varphi(\alpha, \varepsilon)} \int_{\beta T_{\alpha, \varepsilon}}^{\infty} t^{-k} y^{p}(t) d t  \tag{5.7}\\
& =G_{1}(\alpha, \beta, \varepsilon)+G_{2}(\alpha, \beta, \varepsilon)
\end{align*}
$$

Because $z(t) \leq\left(\frac{\gamma}{T_{\alpha, \varepsilon}}\right) t$ for all $t>0$, using Lemma 4.4, we have

$$
\begin{align*}
G_{1}(\alpha, \beta, \varepsilon) & \leq \gamma^{1-\varphi(\alpha, \varepsilon)}\left(\frac{\gamma}{T_{\alpha, \varepsilon}}\right)^{p} \int_{T}^{\beta T_{\alpha, \varepsilon}} t^{p-k} d t \\
& <\gamma^{1-\varphi(\alpha, \varepsilon)}\left(\frac{\gamma}{T_{\alpha, \varepsilon}}\right)^{p} \frac{\left(\beta T_{\alpha, \varepsilon}\right)^{p-k+1}}{p-k+1}  \tag{5.8}\\
& =\frac{k_{1}^{k-1}(\alpha, \varepsilon)}{(k-2)(1-\varphi(\alpha, \varepsilon))} \beta^{(k-2)(1-\varphi(\alpha, \varepsilon))}
\end{align*}
$$

On the other hand, by Lemma 4.3 and (i) of Lemma 4.8 , for $\varepsilon>0$ small,

$$
\begin{align*}
G_{2}(\alpha, \beta, \varepsilon) & >\gamma^{1-\varphi(\alpha, \varepsilon)}\left(1-d_{\alpha, \beta, \varepsilon} \varepsilon\right)^{p} \int_{\beta T_{\alpha, \varepsilon}}^{\infty} t^{-k} z^{p}(t) d t  \tag{5.9}\\
& =\left(1-d_{\alpha, \beta, \varepsilon} \varepsilon\right)^{p} k_{1}(\alpha, \varepsilon) k_{2}(\alpha, \varepsilon) B\left(\beta^{k-2}, 1-\varphi(\alpha, \varepsilon), k_{2}(\alpha, \varepsilon)\right)
\end{align*}
$$

Combining (5.7), (5.8) and (5.9), we derive

$$
\lim _{\varepsilon \rightarrow 0} \inf \gamma^{1-\varphi(\alpha, \varepsilon)} y^{\prime}(T) \geq k_{1} k_{2} B\left(\beta^{k_{0}-2}, 1, k_{2}\right)-L_{1} \beta^{k_{0}-2}
$$

where $L_{1}=\lim _{\varepsilon \rightarrow 0} \frac{k_{1}^{k-1}(\alpha, \varepsilon)}{(k-2)(1-\varphi(\alpha, \varepsilon))}$.
Hence, given any $\delta>0$, we can choose $\beta>0$ such that (5.6) holds. This completes the proof.

Lemma 5.3. $\lim _{\varepsilon \rightarrow 0} \gamma^{1-\varphi(\alpha, \varepsilon)} \int_{T}^{\infty} t^{-k} y^{p+1}(t, \gamma(\varepsilon)) d t=k_{1} k_{2}\left[\Gamma\left(k_{2}\right)\right]^{2} / \Gamma\left(2 k_{2}\right)$, where $\gamma=\gamma(\varepsilon)$.
Proof. By Lemma 4.4 and Lemma 4.8(ii), as $\varepsilon \rightarrow 0$, we deduce

$$
\begin{align*}
\gamma^{-\varphi(\alpha, \varepsilon)} \int_{T}^{\infty} t^{-k} y^{p+1}(t, \gamma) d t & \leq \gamma^{-\varphi(\alpha, \varepsilon)} \int_{T}^{\infty} t^{-k} z^{p+1}(t, \gamma) d t \\
& =k_{1}(\alpha, \varepsilon) k_{2}(\alpha, \varepsilon) B\left(\left(\frac{T}{T_{\alpha, \varepsilon}}\right)^{k-2}, k_{2}(\alpha, \varepsilon)-\varphi(\alpha, \varepsilon), k_{2}(\alpha, \varepsilon)\right)  \tag{5.10}\\
& \rightarrow k_{1} k_{2} B\left(0, k_{2}, k_{2}\right) \\
& =k_{1} k_{2}\left[\Gamma\left(k_{2}\right)\right]^{2} / \Gamma\left(2 k_{2}\right) .
\end{align*}
$$

Hence,

$$
\lim _{\varepsilon \rightarrow 0} \sup \gamma^{-\varphi(\alpha, \varepsilon)} \int_{T}^{\infty} t^{-k} y^{p+1}(t, \gamma) d t \leq k_{1} k_{2}\left[\Gamma\left(k_{2}\right)\right]^{2} / \Gamma\left(2 k_{2}\right) .
$$

Next, we shall show that for any $\delta>0$,

$$
\lim _{\varepsilon \rightarrow 0} \inf \gamma^{-\varphi(\alpha, \varepsilon)} \int_{T}^{\infty} t^{-k} y^{p+1}(t, \gamma) d t \geq k_{1} k_{2}\left[\Gamma\left(k_{2}\right)\right]^{2} / \Gamma\left(2 k_{2}\right)-\delta .
$$

which completes the proof of this lemma.
For a given $\beta>0$, by (4.2) and Lemma 4.7, we can choose a sufficiently small $\varepsilon$ such that $\beta T_{\alpha, \varepsilon}>T$. Thus (5.10) can be written as

$$
\begin{align*}
\gamma^{-\varphi(\alpha, \varepsilon)} & \int_{T}^{\infty} t^{-k} y^{p+1}(t, \gamma) d t \\
& =\gamma^{-\varphi(\alpha, \varepsilon)} \int_{T}^{\beta T_{\alpha, \varepsilon}} t^{-k} y^{p+1}(t, \gamma) d t+\gamma^{-\varphi(\alpha, \varepsilon)} \int_{\beta T_{\alpha, \varepsilon}}^{\infty} t^{-k} y^{p+1}(t, \gamma) d t  \tag{5.11}\\
& =G_{3}(\alpha, \beta, \varepsilon)+G_{4}(\alpha, \beta, \varepsilon) .
\end{align*}
$$

Because $z(t) \leq\left(\frac{\gamma}{T_{\alpha, \ell}}\right) t$ for all $t>0$, using Lemma 4.4, we have

$$
\begin{align*}
G_{3}(\alpha, \beta, \varepsilon) & <\gamma^{-\varphi(\alpha, \varepsilon)} \int_{T}^{\beta T_{\alpha, \varepsilon}} t^{-k} z^{p+1}(t, \gamma) d t \\
& \leq \gamma^{-\varphi(\alpha, \varepsilon)} \int_{T}^{\beta T_{\alpha, \varepsilon}} t^{-k}\left(\frac{\gamma}{T_{\alpha, \varepsilon}} t\right)^{p+1} d t  \tag{5.12}\\
& =\frac{k_{1}(\alpha, \varepsilon)}{k-1-\varepsilon} \beta^{k-1-\varepsilon} .
\end{align*}
$$

On the other hand, by Lemma 4.3 and (ii) of Lemma 4.8 , for $\varepsilon>0$ small,

$$
\begin{align*}
G_{4}(\alpha, \beta, \varepsilon) & >\gamma^{-\varphi(\alpha, \varepsilon)}\left(1-d_{\alpha, \beta, \varepsilon} \varepsilon\right)^{p+1} \int_{\beta T_{\alpha, \varepsilon}}^{\infty} t^{-k} z^{p+1}(t, \gamma) d t  \tag{5.13}\\
& =\left(1-d_{\alpha, \beta, \varepsilon} \varepsilon\right)^{p+1} k_{1}(\alpha, \varepsilon) k_{2}(\alpha, \varepsilon) B\left(\beta^{k-2}, k_{2}(\alpha, \varepsilon)-\varphi(\alpha, \varepsilon), k_{2}(\alpha, \varepsilon)\right)
\end{align*}
$$

Combining (5.11), (5.12) and (5.13), we derive

$$
\begin{equation*}
\liminf _{\varepsilon \rightarrow 0} \inf ^{-\varphi(\alpha, \varepsilon)} \int_{T}^{\infty} t^{-k} y^{p+1}(t, \gamma) d t \geq k_{1} k_{2} B\left(\beta^{k_{0}-2}, k_{2}, k_{2}\right)-L_{2} \beta^{k_{0}-1} . \tag{5.14}
\end{equation*}
$$

where $L_{2}=\lim _{\varepsilon \rightarrow 0} \frac{k_{1}(\alpha, \varepsilon)}{k-1-\varepsilon}$.
Hence, given any $\delta>0$, we can choose $\beta>0$ such that this conclusion is tenable.
Now we are ready to analyze the behavior of $\gamma(\varepsilon)$ as $\varepsilon \rightarrow 0$.
Theorem 5.4. Let $y(t)$ be the solution of problem (4.2) and denote

$$
\gamma(\varepsilon)=\lim _{t \rightarrow \infty} y(t) .
$$

Then

$$
\lim _{\varepsilon \rightarrow 0} \varepsilon \gamma^{2}(\varepsilon)=\frac{4(N+\alpha) \sqrt{\mu}-\mu}{(N-2)^{2}} \frac{k_{1}}{k_{2}} \frac{\Gamma\left(2 k_{2}\right)}{\left[\Gamma\left(k_{2}\right)\right]^{2}} T,
$$

where $k_{1}$ and $k_{2}$ are defined by (4.13).

Proof. Noting (5.3), we have

$$
\begin{equation*}
\left(1+\frac{2 v}{m-1}\right) \varepsilon \gamma^{2-\varphi(\alpha, \varepsilon)}=(p+1) T \frac{\left[\gamma^{1-\varphi(\alpha, \varepsilon)} y^{\prime}(T)\right]^{2}}{\gamma^{-\varphi(\alpha, \varepsilon)} \int_{T}^{\infty} t^{-k} y^{p+1}(t) d t} \tag{5.15}
\end{equation*}
$$

From (5.15), Lemma 5.2 and Lemma 5.3, we have

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \varepsilon \gamma^{2-\varphi(\alpha, \varepsilon)}(\varepsilon)=\frac{4(N+\alpha) \sqrt{\bar{\mu}-\mu}}{(N-2)^{2}} \frac{k_{1}}{k_{2}} \frac{\Gamma\left(2 k_{2}\right)}{\left[\Gamma\left(k_{2}\right)\right]^{2}} T . \tag{5.16}
\end{equation*}
$$

The exponent $2-\varphi(\alpha, \varepsilon)$ in (5.16) may be replaced by 2 , because

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \gamma(\varepsilon)^{\varphi(\alpha, \varepsilon)}=1 \tag{5.17}
\end{equation*}
$$

To see this, note that (5.16) implies that

$$
\gamma(\varepsilon)^{2-\varphi(\alpha, \varepsilon)}<\frac{C}{\varepsilon}
$$

for small $\varepsilon$ and some constant $C$. Therefore

$$
\ln \gamma(\varepsilon)^{\varphi(\alpha, \varepsilon)}=\varphi(\alpha, \varepsilon) \ln \gamma(\varepsilon)<\frac{\varphi(\alpha, \varepsilon)}{2-\varphi(\alpha, \varepsilon)} \ln \frac{C}{\varepsilon}
$$

This means that

$$
\ln \gamma(\varepsilon)^{\varphi(\alpha, \varepsilon)} \rightarrow 0 \quad \text { as } \varepsilon \rightarrow 0
$$

and (5.17) follows.
Proof of Theorem 1.2. If $y(t)$ is the solution of (4.2), then

$$
v(x)=(N-2 v-2)^{g(\alpha, \varepsilon)} y\left((m-1)^{m-1}|x|^{1-m}\right)
$$

is the solution of problem (3.1) in $B_{R}$ with $R=(m-1) T^{-1 /(m-1)}$ and

$$
u_{\varepsilon}(x)=|x|^{-v} v(x)=(N-2 v-2)^{g(\alpha, \varepsilon)}|x|^{-v} y\left((m-1)^{m-1}|x|^{1-m}\right)
$$

Therefore, Theorem 5.4 yields

$$
\begin{aligned}
\lim _{\varepsilon \rightarrow 0} \lim _{|x| \rightarrow 0} \varepsilon u_{\varepsilon}^{2}|x|^{2 v} & =\lim _{\varepsilon \rightarrow 0}(N-2 v-2)^{2 g(\alpha, \varepsilon)} \varepsilon \gamma^{2}(\varepsilon) \\
& =2(\alpha+2)(2 \sqrt{\bar{\mu}-\mu})^{\frac{2 N+\alpha-2}{\alpha+2}}(N+\alpha)^{\frac{N-2}{\alpha+2}}(N-2)^{-\frac{2 \alpha+N+2}{\alpha+2}} \frac{\Gamma\left(\frac{2(N+2)}{\alpha+2}\right)}{\left[\Gamma\left(\frac{N+\alpha}{\alpha+2}\right)\right]^{2}} \frac{1}{R^{2 \sqrt{\bar{\mu}-\mu}}}
\end{aligned}
$$

which is the content of Theorem 1.2.
Before proving Theorem 1.3, we first give two lemmas.
As a first observation, we note from Lemma 4.4 that

$$
y(t, \gamma)<z(t, \gamma)<k_{1} t \gamma^{-1+\varphi(\alpha, \varepsilon)} \quad \text { for } t>T
$$

Hence, by Theorem 5.4, for every fixed $t>T$, we have

$$
y(t, \gamma(\varepsilon))=O\left(\varepsilon^{\frac{1}{2}}\right) \quad \text { as } \varepsilon \rightarrow 0
$$

If we allow $t$ to tend to infinity as $\varepsilon \rightarrow 0$, we obtain the following upper bound.

Lemma 5.5. For every $M>0$ and $\xi \in\left(0, \frac{1}{2}\right)$,

$$
\limsup _{\varepsilon \rightarrow 0}\left\{y(t, \gamma(\varepsilon)): T<t<M \varepsilon^{-\xi}\right\}=0
$$

To obtain information about the limiting form of $y(t, \gamma(\varepsilon))$ as $\varepsilon \rightarrow 0$, we are led by Lemma 5.2 to multiply $y$ as the weight factor $\gamma^{1-\varphi(\alpha, \varepsilon)}$, because

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \gamma^{1-\varphi(\alpha, \varepsilon)} y^{\prime}(T)=k_{1} \tag{5.18}
\end{equation*}
$$

In the next lemma, we show that (5.18) continues to be true for values of $t>T$, provided

$$
t=O\left(\gamma^{\sigma}\right)
$$

where $\sigma$ may be any number less than 2 .
Lemma 5.6. Let $M>0$ and $0<\sigma<2$. Then

$$
\limsup _{\varepsilon \rightarrow 0}\left\{\left|\gamma^{1-\varphi(\alpha, \varepsilon)} y^{\prime}(t)-k_{1}\right|: T<t<M \gamma^{\sigma}\right\}=0
$$

Proof. By Lemma 5.2, and the concavity of $y$,

$$
\begin{equation*}
\limsup _{\varepsilon \rightarrow 0} \gamma^{1-\varphi(\alpha, \varepsilon)} y(t, \gamma) \leq k_{1}, \quad \forall t \geq T \tag{5.19}
\end{equation*}
$$

To get a lower bound on $y^{\prime}$, we also use the concave property of $y$. For $\forall t \geq T$ and for $t_{0}>t$, we have

$$
y^{\prime}\left(t_{0}\right)>\frac{y\left(t_{0}\right)-y(t)}{t_{0}-t}>\frac{1}{t_{0}}\left\{y\left(t_{0}\right)-y(t)\right\}
$$

Hence, by Lemma 4.4,

$$
\begin{equation*}
\gamma^{1-\varphi(\alpha, \varepsilon)} y^{\prime}(t)>\frac{\gamma^{1-\varphi(\alpha, \varepsilon)} y\left(t_{0}\right)}{t_{0}}-\frac{\gamma^{1-\varphi(\alpha, \varepsilon)} z(t)}{t} \cdot \frac{t}{t_{0}} \tag{5.20}
\end{equation*}
$$

We assume that $t=O\left(\gamma^{\sigma}\right)$ and $0<\sigma<2$, so it is possible for us to substitute $\beta T_{\alpha, \varepsilon}, \beta>0$ for $t_{0}$. Hence, for $\gamma \rightarrow \infty$,

$$
\begin{equation*}
\frac{t}{\beta T_{\alpha, \varepsilon}} \rightarrow 0 \tag{5.21}
\end{equation*}
$$

By Lemma 4.3,

$$
\begin{align*}
\frac{\gamma^{1-\varphi(\alpha, \varepsilon)} y\left(\beta T_{\alpha, \varepsilon}\right)}{\beta T_{\alpha, \varepsilon}} & \geq \frac{\gamma^{1-\varphi(\alpha, \varepsilon)} z\left(\beta T_{\alpha, \varepsilon}\right)}{\beta T_{\alpha, \varepsilon}}\left(1-d_{\alpha, \beta, \varepsilon} \varepsilon\right)  \tag{5.22}\\
& =k_{1}\left(1+\beta^{k-2}\right)^{-\frac{1}{k-2}}\left(1-d_{\alpha, \beta, \varepsilon} \varepsilon\right)
\end{align*}
$$

Thus, from (5.20)-(5.22), we conclude that

$$
\liminf _{\varepsilon \rightarrow 0} \gamma^{1-\varphi(\alpha, \varepsilon)} y(t, \gamma) \geq k_{1}-\delta(\beta)
$$

where $\delta(\beta) \rightarrow 0$ as $\beta \rightarrow 0$. Because we can choose $\beta$ small enough, this means

$$
\begin{equation*}
\liminf _{\varepsilon \rightarrow 0} \gamma^{1-\varphi(\alpha, \varepsilon)} y(t, \gamma) \geq k_{1} \tag{5.23}
\end{equation*}
$$

By (5.19) and (5.22), we obtain the desired result.

Proof of Theorem 1.3. By the concavity of $y(t)$, we deduce

$$
\begin{equation*}
y^{\prime}(t) \leq \frac{y(t)-y(T)}{t-T} \leq y^{\prime}(T), \quad t \geq T \tag{5.24}
\end{equation*}
$$

So, there exists a $\theta \in[T, t]$ such that

$$
\begin{equation*}
y(t)=y^{\prime}(\theta)(t-T) \tag{5.25}
\end{equation*}
$$

Combining Lemma 5.5, (5.25) and noting that $\lim _{\varepsilon \rightarrow 0} \gamma(\varepsilon)^{\varphi(\alpha, \varepsilon)}=1$, we obtain

$$
\begin{align*}
\lim _{\varepsilon \rightarrow 0} \varepsilon^{-\frac{1}{2}} y(t) & =\lim _{\varepsilon \rightarrow 0} \varepsilon^{-\frac{1}{2}} \gamma^{-1+\varphi(\alpha, \varepsilon)} \lim _{\varepsilon \rightarrow 0} \gamma^{1-\varphi(\alpha, \varepsilon)} y(t) \\
& =\left[A\left(k_{1}, k_{2}, T\right)\right]^{-\frac{1}{2}} \lim _{\varepsilon \rightarrow 0} \gamma^{1-\varphi(\alpha, \varepsilon)} y^{\prime}(\theta)(t-T)  \tag{5.26}\\
& =k_{1}\left[A\left(k_{1}, k_{2}, T\right)\right]^{-\frac{1}{2}}(t-T)
\end{align*}
$$

where $A\left(k_{1}, k_{2}, T\right)=\frac{4(N+\alpha) \sqrt{\bar{\mu}-\mu}}{(N-2)^{2}} \frac{k_{1}}{k_{2}} \frac{\Gamma\left(2 k_{2}\right)}{\left[\Gamma\left(k_{2}\right)\right]^{2}} T$,
and the convergence is uniform on bounded intervals.
For the solution $u_{\varepsilon}(x)$ of problem (1.1), (5.26) means that as $\varepsilon \rightarrow 0$

$$
\begin{aligned}
\lim _{\varepsilon \rightarrow 0} \varepsilon^{-\frac{1}{2}} u_{\varepsilon(x)}= & \lim _{\varepsilon \rightarrow 0}|x|^{-v}(N-2 v-2)^{g(\alpha, \varepsilon)} k_{1}\left[A\left(k_{1}, k_{2}, T\right)\right]^{-\frac{1}{2}}(t-T) \\
= & \frac{1}{2}(\alpha+2)^{-\frac{1}{2}}(2 \sqrt{\bar{\mu}}-\mu)^{\frac{2 N-\alpha-6}{2 \alpha+4}}(N+\alpha)^{\frac{N-2}{2 \alpha+4}}(N-2)^{\frac{2 \alpha-N+6}{2 \alpha+4}} R \sqrt{\bar{\mu}-\mu} \frac{\Gamma\left(\frac{N+\alpha}{\alpha+2}\right)}{\left[\Gamma\left(\frac{2(N+\alpha)}{\alpha+2}\right)\right]^{\frac{1}{2}}} \\
& \times\left(\frac{1}{|x|^{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}}-\frac{1}{|x|^{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}|R|^{2 \sqrt{\bar{\mu}-\mu}}}\right)
\end{aligned}
$$

Hence, we obtain the desired result.

## References

[1] F. V. Atkinson, H. Brezis, L. A. Peletier, Nodal solutions of elliptic equations with critical Sobolev exponents, J. Differential Equations 85(1990), No. 1, 151-170. https://doi . org/10.1016/0022-0396(90) 90093-5; MR1052332; Zbl 0702.35099
[2] F. V. Atkinson, L. A. Peletier, Elliptic equations with nearly critical growth, J. Differential Equations 70(1987), No. 3, 349-365. https://doi.org/10.1016/0022-0396(87)90156-2; MR0915493; Zbl 0657.35058
[3] A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14(1973), No. 4, 349-381. https://doi.org/10.1016/0022-1236(73)90051-7; MR0370183; Zbl 0273.49063
[4] M. Bhakta, D. Mukherjee, S. Santra, Profile of solutions for nonlocal equations with critical and supercritical nonlinearities, Commun. Contemp. Math. 21(2019), No. 1, 1750099, 35 pp. https://doi.org/10.1142/s0219199717500997; MR3904641; Zbl 1406.35459
[5] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical sobolev exponents, Commun. Pure Appl. Math. 36(983), No. 4, 437-477. https: //doi.org/10.1002/cpa.3160360405; MR0709644; Zbl 0541.35029
[6] M. Bhakta, S. Santra, On singular equations with critical and supercritical exponents, J. Differential Equations 263(2017), No. 5, 2886-2953. https://doi.org/10.1016/j.jde. 2017.04.018; MR3655804; Zbl 1373.35103
[7] L. A. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical growth, Comm. Pure Appl. Math. 42(1989), No. 3, 271-297. https://doi.org/10.1002/cpa.3160420304; MR0982351; Zbl 0702.35085
[8] L. Cafarelli, R. Kohn, L. Nirenberg, First order interpolation inequality with weights, Compos. Math. 53(1984), No. 3, 259-275. MR0768824; Zbl 0563.46024
[9] W. Chen, C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J. 63(1991), No. 3, 615-622. https://doi.org/10.1215/s0012-7094-91-06325-8; MR1121147; Zbl 0768.35025
[10] D. Cao, S. Peng, Asymptotic behavior for elliptic problems with singular coefficient and nearly critical Sobolev growth, Ann. Mat. Pura Appl. 185(2006), No. 2, 189-205. https: //doi.org/10.1007/s10231-005-0150-z; MR2214132; Zbl 1232.35058
[11] D. Cao, S. Peng, The asymptotic behaviour of the ground state solutions for Hénon equation, J. Math. Anal. Appl. 278(2003), No. 1, 1-17. https://doi.org/10.1016/S0022247X (02) 00292-5; MR1963460; Zbl 1086.35036
[12] F. Catrina, Z. Q. Wang, On the Caffarelli-Kohn-Nirenberg inequalities: Sharp constants, existence (and nonexistence), and symmetry of extremal functions, Commun. Pure Appl. Math. 54(2001), No. 2, 229-258. https://doi.org/10.1002/1097-0312 (200102) 54 : 2<229::aid-cpa4>3.0.co;2-i; MR1794994; Zbl 1072.35506
[13] D. CaO, S. Yan, Infinitely many solutions for an elliptic problem involving critical Sobolev growth and Hardy potential, Calc. Var. Partial Differential Equations 38(2010), No. 3-4, 471501. https://doi.org/10.1007/s00526-009-0295-5; MR2647129; Zbl 1194.35161
[14] K. S. Chou, C. W. Chu, On the best constant for a weighted Sobolev-Hardy inequality, J. London Math. Soc. (2) 48(1993), No. 1, 137-151. https://doi.org/10.1112/j1ms/s248.1.137; MR1223899; Zbl 0739.26013
[15] L. Dupaigne, A nonlinear elliptic PDE with the inverse-square potential, J. Anal. Math. 86(2002), No. 1, 359-398. https://doi.org/10.1007/bf02786656; MR1894489; Zbl 1034.35043
[16] N. Ghoussoub, C. Yuan, Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents, Trans. Amer. Math. Soc. 352(2000), No. 12, 5703-5743. https://doi.org/10.1090/s0002-9947-00-02560-5; MR1695021; Zbl 0956.35056
[17] B. Gidas, W. M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68(1979), No. 3, 209-243. https://doi.org/10.1007/ bf01221125; MR0544879; Zbl 0425.35020
[18] J. P. García Azorero, I. Peral Alonso, Hardy inequalities and some critical elliptic and parabolic problems, J. Differential Equations 144(1998), No. 2, 441-476. https://doi. org/ 10.1006/jdeq. 1997.3375; MR1616905; Zbl 0918.35052
[19] M. HÉNON, Numerical experiments on the stability of spherical stellar systems, Astronom. Astrophys. Lib. 62(1974), No. 24, 229-238. https://doi.org/10.1007/978-94-010-98779_37
[20] D. Kang, S. Peng, Positive solutions for singular critical elliptic problems, Appl. Math. Lett. 17(2004), No. 4, 411-416. https ://doi.org/10.1016/s0893-9659(04) 90082-1; MR2045745; Zbl 1133.35358
[21] C. S. Lin, W. M. Ni, I. Takagi, Large amplitude stationary solutions to a chemotaxis system, J. Differential Equations 72(1988), No. 1, 1-27. https://doi.org/10.1016/00220396 (88) 90147-7; MR0929196; Zbl 0676.35030
[22] F. Merle, L. A. Peletier, Asymptotic behaviour of positive solutions of elliptic equations with critical and supercritical growth I. The radial case, Arch. Ration. Mech. Anal. 112(1990), No. 1, 1-19. https://doi.org/10.1007/bf00431720; MR1073623; Zbl 0719.35004
[23] F. Merle, L. A. Peletier, Asymptotic behaviour of positive solutions of elliptic equations with critical and supercritical growth II. The nonradial case, J. Funct. Anal. 105(1992), No. 1, 1-41. https://doi.org/10.1016/0022-1236(92) 90070-y; MR1156668; Zbl 0771.35008
[24] W. M. Ni, A nonlinear Dirichlet problem on the unit ball and its applications, Indiana Univ. Math. J. 31(1982), No. 6, 801-807. MR0674869; Zbl 0515.35033
[25] M. Struwe, Variational methods, 4th ed., Springer, Berlin, 2008. https://doi.org/10. 1007/978-3-540-74013-1; MR2431434; Zbl 1284.49004
[26] D. Smets, M. Willem, J. Su, Non-radial ground states for the Hénon equation, Commun. Contemp. Math. 4(2002), No. 3, 467-480. https://doi.org/10.1142/s0219199702000725; MR1918755; Zbl 1160.35415

# Existence and multiplicity of positive solutions for singular $\phi$-Laplacian superlinear problems with nonlinear boundary conditions 

Dang Dinh Hai ${ }^{\boxtimes}$ and Xiao Wang<br>Mississippi State University, Mississippi State, MS 39762, USA<br>Received 3 March 2021, appeared 8 September 2021<br>Communicated by Paul Eloe


#### Abstract

We prove the existence and multiplicity of positive solutions to the singular $\phi$-Laplacian BVP $$
\left\{\begin{array}{l} -\left(r(t) \phi\left(u^{\prime}\right)\right)^{\prime}=\lambda g(t)\left(f(u)-\frac{a}{u^{u}}\right), \quad t \in(0,1), \\ u(0)=0, u^{\prime}(1)+H(u(1))=0 \end{array}\right.
$$ for a certain range of the parameter $\lambda>0$, where $a>0, \alpha \in(0,1), \phi$ is an odd, increasing and convex homeomorphism on $\mathbb{R}$, and $f$ is $\phi$-superlinear at $\infty$.


Keywords: $\phi$-Laplacian, infinite semipositone, positive solutions.
2020 Mathematics Subject Classification: 34B15, 34B16.

## 1 Introduction

Consider the one-dimensional $\phi$-Laplacian problem

$$
\left\{\begin{array}{l}
-\left(r(t) \phi\left(u^{\prime}\right)\right)^{\prime}=\lambda g(t)\left(f(u)-\frac{a}{u^{a}}\right), t \in(0,1),  \tag{1.1}\\
u(0)=0, u^{\prime}(1)+H(u(1))=0,
\end{array}\right.
$$

where $a>0, \alpha \in(0,1), \lambda$ is a positive parameter, and the following conditions are assumed:
(A1) $r:[0,1] \rightarrow(0, \infty)$ is continuous and nondecreasing.
(A2) $H:[0, \infty) \rightarrow[0, \infty)$ is continuous and nondecreasing with $H(0)=0$.
(A3) $g:(0,1) \rightarrow(0, \infty)$ is continuous with $g / p^{\alpha} \in L^{1}(0,1)$, where $p(t)=\min (t, 1-t)$.
(A4) $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is an odd, increasing homeomorphism such that $\phi$ is convex on $[0, \infty)$ and

$$
\limsup _{x \rightarrow \infty} \frac{\phi(\sigma x)}{\phi(x)}<\infty
$$

for all $\sigma>0$.

[^27](A5) $f:[0, \infty) \rightarrow[0, \infty)$ is continuous, nondecreasing, and
$$
\lim _{x \rightarrow \infty} \frac{f(x)}{\phi(x)}=\infty .
$$
(A6) There exist constants $m>0$ and $\beta \in(0,1)$ such that
$$
f(x) \geq\left(\frac{x}{m}\right)^{\beta} f(m)
$$
for $x \in[0, m]$.
By a positive solution of (1.1), we mean a function $u \in C^{1}[0,1]$ with $u>0$ on $(0,1]$ and $r(t) \phi\left(u^{\prime}\right)$ absolutely continuous on $[0,1]$ that satisfies (1.1).

Our main result is

## Theorem 1.1.

(i) Let (A1)-(A6) hold. Then there exist a positive number K and an interval $I \subset(0, \infty)$ such that if $f(m) \geq K$ then problem (1.1) has at least two positive solutions for $\lambda \in I$.
(ii) Let (A1)-(A5) hold. Then there exists a positive number $\lambda_{0}>0$ such that for $\lambda<\lambda_{0}$, problem (1.1) has a positive solution $u_{\lambda}$ with $u_{\lambda}(t) \rightarrow \infty$ as $\lambda \rightarrow 0^{+}$uniformly on compact subsets of $(0,1]$.

Example 1.2. Let $r$ satisfy (A1), $a, \alpha, \gamma, \delta>0$ with $\alpha+\gamma<1$, and $\phi(x)=\sum_{i=1}^{n} a_{i}|x|^{p_{i}-2} x$, where $a_{i}>0, p_{i} \geq 2$ for $i=1, \ldots, n$, and $H(z)=z^{\delta}$. Consider the BVP

$$
\left\{\begin{array}{l}
-\left(r(t) \phi\left(u^{\prime}\right)\right)^{\prime}=\frac{\lambda}{t^{\eta}}\left(f(u)-\frac{a}{u^{a}}\right), t \in(0,1),  \tag{1.2}\\
u(0)=0, u^{\prime}(1)+(u(1))^{\delta}=0 .
\end{array}\right.
$$

Let $q>\max _{1 \leq i \leq n}\left(p_{i}-1\right)$ Then
(i) By Theorem 1.1 (i) with $m=1$, problem (1.2) with

$$
f(u)=\left\{\begin{array}{l}
K u^{\beta}, 0 \leq u \leq 1, \\
K u^{q}, u>1,
\end{array}\right.
$$

where $0<\beta<1$, has two positive solutions for $\lambda$ in a certain range of $(0, \infty)$, provided that $K$ is large enough.
(ii) By Theorem 1.1 (ii), problem (1.2) with $f(u)=u^{q} e^{-\frac{b}{1+u}}$, where $b \geq 0$, has a large positive solution for $\lambda>0$ small.

A problem of the form (1.1) occurs in the study of positive radial solutions to the $p$ Laplacian problems in an exterior domain

$$
\left\{\begin{array}{l}
-\Delta_{p} u=\lambda K(|x|) f(u) \text { in } \Omega=\left\{x \in \mathbb{R}^{N}:|x|>r_{0}>0\right\}, \\
\frac{\partial u}{\partial n}+\tilde{c}(u) u=0 \text { on }|x|=r_{0}, \\
u(x) \rightarrow 0 \text { as }|x| \rightarrow \infty,
\end{array}\right.
$$

where $\Delta_{p} u=\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right), p>1, N>p, \tilde{c}:[0, \infty) \rightarrow[0, \infty), n$ denotes the outer unit normal vector on $\partial \Omega$, as it reduced (see [10]) via the Kevin transformation $r=|x|$, $t=\left(r / r_{0}\right)^{\frac{p-N}{p-1}}$ to the ODE problem

$$
\left\{\begin{array}{l}
-\left(\phi\left(u^{\prime}\right)\right)^{\prime}=\lambda h(t) f(u), t \in(0,1) \\
u(0)=0, u^{\prime}(1)+c(u(1)) u(1)=0
\end{array}\right.
$$

where $\phi(z)=|z|^{p-2} z, h(t)=\left(\frac{p-1}{N-p} r_{0}\right)^{p} t^{\frac{p(N-1)}{p-N}} K\left(r_{0} t^{\frac{p-1}{p-N}}\right)$, and $c(s)=\frac{p-1}{N-p} r_{0} \tilde{c}(s)$. It also arises in the study of radial solutions for the $(p, q)$-Laplacian problem in an annulus i.e.

$$
\left\{\begin{array}{l}
-\Delta_{p} u-\Delta_{q} u=f(|x|, u), \quad a<|x|<b \\
u=0 \text { on }|x|=a \\
\frac{\partial u}{\partial n}-H(u)=0 \text { on }|x|=b
\end{array}\right.
$$

where $p>q>1$, which stems from a variety of applied areas (see e.g. [2,3]). We are motivated by related results on the existence and multiplicity of positive radial solutions for the system

$$
\left\{\begin{array}{l}
-\Delta u_{i}=\lambda K_{i}(|x|) f_{i}\left(u_{j}\right) \text { in } \Omega=\left\{x \in \mathbb{R}^{N}: r_{0}<|x|<r_{1}\right\} \\
d_{i} \frac{\partial u_{i}}{\partial n}+\tilde{c}_{i}\left(u_{i}\right) u_{i}=0 \text { on }|x|=r_{0} \\
u_{i}=0 \text { on }|x|=r_{1}
\end{array}\right.
$$

in [6,7], where $d_{i}=0, \tilde{c}_{i} \equiv 1, r_{1}<\infty$ in [6] and $d_{i} \geq 0, \tilde{c}_{i}>0, r_{1}=\infty$ in [7], as well as its extension to $p$-Laplacian systems in [12]. The results in [6,7,12] have been obtained under the assumptions that the reaction terms satisfy a combined superlinear at $\infty$ and are allowed to have semipositone structures at 0 i.e. $f_{i}\left(0^{+}\right) \in[-\infty, 0)$. Searching for positive solutions in the semipositone case is known to be challenging due to the lack of the maximum principle. Our main result here on the one hand allows the $p$-Laplacian operator to be replaced by a general homeomorphism on $\mathbb{R}$, and on the other hand permits nonlinear boundary conditions that can not be linearized e.g. $u^{\prime}(1)+\sqrt[3]{u(1)}=0$, which are not allowed in $[6,7,12]$. We obtain the existence of a large positive solution to (1.1) for $\lambda>0$ small when $f$ is merely $\phi$-superlinear at $\infty$, and the existence of two positive solutions for $\lambda$ in a certain interval in $(0, \infty)$ if in addition $f$ satisfies a concavity condition on $[0, m]$ for some $m>0$ and $f(m)$ is large enough. It is worth noting that problem (1.1) is of infinite semipositone nature as $\lim _{u \rightarrow 0^{+}}\left(f(u)-\frac{a}{u^{\alpha}}\right)=-\infty$. Our approach is based on a Krasnoselskii's fixed point theorem in a Banach space.

We refer to $[4,5,8,9,11]$ for results in the PDE case related to (1.1), where $[9,11]$ are of particular relevance to this study. In [9], an overview of recent developments on elliptic variational problems with functional satisfying nonstandard growth of $(p, q)$-type is provided. Related existence results for positive solutions of the Brézis-Nirenberg type critical semipositone problem

$$
\left\{\begin{array}{l}
-\Delta_{p} u=\lambda u^{p-1}+u^{p^{*}-1}-\mu \text { in } \Omega \\
u=0 \text { on } \partial \Omega
\end{array}\right.
$$

can be found in [11], where $\Omega$ is a bounded domain in $\mathbb{R}^{N}, p^{*}=\frac{N p}{N-p}$, and $\lambda, \mu$ are positive parameters.

## 2 Preliminary results

For the rest of the paper, we define $r_{0}=\min _{t \in[0,1]} r(t), r_{1}=\max _{t \in[0,1]} r(t)$, and $H(z)=H(0)$ for $z<0$. The norm in $L^{p}(0,1)$ will be denoted by $\|\cdot\|_{p}$.

We first recall the following fixed point result of Krasnoselskii type in a Banach space (see e.g. [1, Theorem 12.3]).

Lemma A. Let $E$ be a Banach space and $T: E \rightarrow E$ be a completely continuous operator. Suppose there exist $h \in E, h \neq 0$ and positive constants $r, R$ with $r \neq R$ such that
(a) If $y \in E$ satisfies $y=\theta T y, \theta \in(0,1]$ then $\|y\| \neq r$,
(b) If $y \in E$ satisfies $y=T y+\xi h, \xi \geq 0$ then $\|y\| \neq R$.

Then $T$ has a fixed point $y \in E$ with $\min (r, R)<\|y\|<\max (r, R)$.

## Lemma 2.1.

(i) $\left|\phi^{-1}(x)-\phi^{-1}(y)\right| \leq 2 \phi^{-1}(|x-y|)$ for all $x, y \in \mathbb{R}$.
(ii) $\phi^{-1}(x-y) \geq \phi^{-1}(x)-2 \phi^{-1}(y)$ for all $x, y \in \mathbb{R}$ with $y \geq 0$.

Proof. (i) Without loss of generality, we need only to consider two cases.
Case 1. $x \geq y \geq 0$.
Since $\phi^{-1}$ is concave on $[0, \infty)$,

$$
\phi^{-1}(x-y)+\phi^{-1}(y) \geq \phi^{-1}(x)
$$

which implies

$$
\phi^{-1}(x)-\phi^{-1}(y) \leq \phi^{-1}(x-y) \leq 2 \phi^{-1}(x-y)
$$

i.e. (i) holds.

Case 2. $x \geq 0 \geq y$.
Then $\phi^{-1}(x) \leq \phi^{-1}(x-y)$ and $-\phi^{-1}(y)=\phi^{-1}(-y) \leq \phi^{-1}(x-y)$, from which (i) follows.
(ii) Since $y \geq 0$, it follows from (i) that

$$
\phi^{-1}(x-y)-\phi^{-1}(x) \geq-2 \phi^{-1}(|y|)=-2 \phi^{-1}(y)
$$

i.e. (ii) holds.

Lemma 2.2. Let $h \in L^{1}(0,1)$ and $u \in C^{1}[0,1]$ satisfy

$$
\left\{\begin{array}{l}
\left(r(t) \phi\left(u^{\prime}\right)\right)^{\prime} \leq h \quad \text { on }(0,1)  \tag{2.1}\\
u(0) \geq 0, u^{\prime}(1)+H(u(1)) \geq 0
\end{array}\right.
$$

Suppose $\|u\|_{\infty}>\phi^{-1}\left(\|h\|_{1} / r_{0}\right)$. Then $u(1) \geq 0$.
Proof. Suppose on the contrary that $u(1)<0$. Then the boundary condition at 1 implies that $u^{\prime}(1) \geq 0$. Let $\tau \in[0,1]$ be such that $\|u\|_{\infty}=|u(\tau)|$. Integrating the inequality in (2.1) on $[t, 1]$ we get

$$
r(t) \phi\left(u^{\prime}(t)\right)=r(1) \phi\left(u^{\prime}(1)\right)-\int_{t}^{1}\left(r(s) \phi\left(u^{\prime}\right)\right)^{\prime} d s \geq-\|h\|_{1}
$$

whence

$$
\begin{equation*}
u^{\prime}(t) \geq-\phi^{-1}\left(\frac{\|h\|_{1}}{r(t)}\right) \geq-\phi^{-1}\left(\frac{\|h\|_{1}}{r_{0}}\right) \tag{2.2}
\end{equation*}
$$

for $t \in[0,1]$. Next, integrating (2.2) on $[0, \tau]$ and $[\tau, 1]$ give

$$
\begin{equation*}
u(\tau) \geq u(0)-\phi^{-1}\left(\frac{\|h\|_{1}}{r_{0}}\right) \geq-\phi^{-1}\left(\frac{\|h\|_{1}}{r_{0}}\right) \tag{2.3}
\end{equation*}
$$

and

$$
\begin{equation*}
-u(\tau) \geq u(1)-u(\tau) \geq-\phi^{-1}\left(\frac{\|h\|_{1}}{r_{0}}\right) \tag{2.4}
\end{equation*}
$$

respectively. Combining (2.3) and (2.4), we deduce that

$$
\|u\|_{\infty} \leq \phi^{-1}\left(\frac{\|h\|_{1}}{r_{0}}\right)
$$

a contradiction. Thus $u(1) \geq 0$.
Lemma 2.3. Let $h \in L^{1}(0,1)$ with $h \geq 0$ and $u \in C^{1}[0,1]$ satisfy

$$
\left\{\begin{array}{l}
\left(r(t) \phi\left(u^{\prime}\right)\right)^{\prime} \leq h \quad \text { on }(0,1) \\
u(0) \geq 0, \quad u(1) \geq 0
\end{array}\right.
$$

Then
(i) $u(t) \geq\left(u(1)-2 \phi^{-1}\left(\frac{\|h\|_{1}}{r_{0}}\right)\right) t$ for $t \in[0,1]$.
(ii) $u(t) \geq\left(\|u\|_{\infty}-4 \phi^{-1}\left(\frac{\|h\|_{1}}{r_{0}}\right)\right) p(t)$ for $t \in[0,1]$. In particular,

$$
u(t) \geq \frac{1}{5}\|u\|_{\infty} p(t)
$$

for $t \in[0,1]$, provided that $\|u\|_{\infty} \geq 5 \phi^{-1}\left(\frac{\|h\|_{1}}{r_{0}}\right)$.

## Proof. Define

$$
w(t)=\phi^{-1}\left(\phi\left(u^{\prime}(t)\right)+\frac{1}{r(t)} \int_{t}^{1} h\right)-u^{\prime}(t), \quad z(t)=\int_{0}^{t} w
$$

for $t \in[0,1]$. Then $w, z \geq 0$ on $[0,1]$ and in view of Lemma 2.1 (i),

$$
w(t) \leq 2 \phi^{-1}\left(\frac{1}{r(t)} \int_{t}^{1} h\right) \leq 2 \phi^{-1}\left(\frac{\|h\|_{1}}{r_{0}}\right)
$$

which implies

$$
z(t) \leq 2 \phi^{-1}\left(\frac{\|h\|_{1}}{r_{0}}\right) t
$$

for $t \in[0,1]$. Since

$$
\left(r(t) \phi\left(u^{\prime}+z^{\prime}\right)\right)^{\prime}=\left(r(t) \phi\left(u^{\prime}+w\right)\right)^{\prime}=\left(r(t) \phi\left(u^{\prime}\right)\right)^{\prime}-h \leq 0 \quad \text { on }(0,1),
$$

$r(t) \phi\left(u^{\prime}+z^{\prime}\right)$ is nonincreasing on $[0,1]$. This, together with (A1), gives the concavity of $u+z$ on $[0,1]$. Since $(u+z)(0) \geq 0$, we obtain

$$
(u+z)(t) \geq t(u+z)(1) \geq t u(1),
$$

whence

$$
u(t) \geq t u(1)-z(t) \geq t\left(u(1)-2 \phi^{-1}\left(\frac{\|h\|_{1}}{r_{0}}\right)\right)
$$

i.e. (i) holds. Since $(u+z)(1) \geq 0$, we deduce from the concavity of $u+z$ on $[0,1]$ that

$$
u(t)+z(t) \geq\|u+z\|_{\infty} p(t)
$$

for $t \in[0,1]$. Consequently,

$$
u(t) \geq\left(\|u\|_{\infty}-\|z\|_{\infty}\right) p(t)-z(t)
$$

for $t \in[0,1]$, which implies

$$
\begin{equation*}
u(t) \geq\left(\|u\|_{\infty}-4 \phi^{-1}\left(\frac{\|h\|_{1}}{r_{0}}\right)\right) t \tag{2.5}
\end{equation*}
$$

for $t \in[0,1 / 2]$. Similarly, by defining

$$
w_{0}(t)=\phi^{-1}\left(\phi\left(u^{\prime}(t)\right)-\frac{1}{r(t)} \int_{0}^{t} h\right)-u^{\prime}(t), \quad z_{0}(t)=\int_{t}^{1} w_{0}
$$

for $t \in[0,1]$, and using $w_{0}, z_{0} \leq 0$ on $[0,1]$,

$$
\left|z_{0}(t)\right| \leq 2 \phi^{-1}\left(\frac{\|h\|_{1}}{r_{0}}\right)(1-t)
$$

for $t \in[0,1]$, together with

$$
\left\{\begin{array}{l}
\left(r(t) \phi\left(u^{\prime}-z_{0}^{\prime}\right)\right)^{\prime}=\left(r(t) \phi\left(u^{\prime}+w_{0}\right)\right)^{\prime}=\left(r(t) \phi\left(u^{\prime}\right)\right)^{\prime}-h \leq 0 \text { on }(0,1), \\
\left(u-z_{0}\right)(0) \geq 0, \quad\left(u-z_{0}\right)(1) \geq 0,
\end{array}\right.
$$

we obtain as above that

$$
\begin{equation*}
u(t) \geq\left(\|u\|_{\infty}-4 \phi^{-1}\left(\frac{\|h\|_{1}}{r_{0}}\right)\right)(1-t) \tag{2.6}
\end{equation*}
$$

for $t \in[1 / 2,1]$. Combining (2.5) and (2.6), we obtain

$$
u(t) \geq\left(\|u\|_{\infty}-4 \phi^{-1}\left(\frac{\|h\|_{1}}{r_{0}}\right)\right) p(t)
$$

for $t \in[0,1]$. In particular, $u(t) \geq \frac{1}{5}\|u\|_{\infty} p(t)$ if $\|u\|_{\infty} \geq 5 \phi^{-1}\left(\|h\|_{1} / r_{0}\right)$, which completes the proof.

## 3 Proof of the main result

Let $E=C[0,1]$ be equipped with $\|\cdot\|_{\infty}$.

Proof of Theorem 1.1. (i) Since $\phi$ is convex on $[0, \infty)$ and $\phi(0)=0$, it follows that

$$
\lim _{x \rightarrow 0^{+}} \frac{\phi(x)}{x^{\beta}}=0
$$

Hence there exists $\gamma \in(0, m)$ such that

$$
\begin{equation*}
\frac{\phi(8 \gamma)}{(\gamma / 5)^{\beta}}<\frac{r_{0} g_{0} \phi(m)}{64 r_{1}(4 m)^{\beta}}\left(\int_{0}^{1} \frac{g}{p^{\alpha}}\right)^{-1} \tag{3.1}
\end{equation*}
$$

where $g_{0}=\inf _{[1 / 4,3 / 4]} g$. Suppose $f(m)>K$, where

$$
K=\max \left\{\frac{a}{(\gamma / 5)^{\alpha}}, \frac{a \phi(m)}{4(\gamma / 5)^{\alpha} \phi(\gamma / 5)^{2}}, \frac{16 a r_{1}(4 m)^{\beta}}{g_{0} r_{0}(\gamma / 5)^{\alpha+\beta}}\left(\int_{0}^{1} \frac{g}{p^{\alpha}}\right)\right\}
$$

Let

$$
I=\left(\frac{16 r_{1} \phi(8 \gamma)}{g_{0} f(\gamma / 20)}, \frac{r_{0} \phi(m)}{4 f(m)}\left(\int_{0}^{1} \frac{g}{p^{\alpha}}\right)^{-1}\right)
$$

Then $I \neq \varnothing$. Indeed, it follows from (A6) that

$$
f\left(\frac{\gamma}{20}\right) \geq\left(\frac{\gamma}{20 m}\right)^{\beta} f(m)=\left(\frac{\gamma}{5}\right)^{\beta} \frac{f(m)}{(4 m)^{\beta}}
$$

which, together with (3.1), implies

$$
\frac{16 r_{1} \phi(8 \gamma)}{g_{0} f(\gamma / 20)} \leq \frac{16 r_{1}(4 m)^{\beta}}{g_{0} f(m)}\left(\frac{\phi(8 \gamma)}{(\gamma / 5)^{\beta}}\right)<\frac{r_{0} \phi(m)}{4 f(m)}\left(\int_{0}^{1} \frac{g}{p^{\alpha}}\right)^{-1}
$$

We shall verify that (1.1) has at least two positive solutions for $\lambda \in I$. For $\lambda \in I$ and $v \in E$, define $T_{\lambda} v=u$, where $u$ is the solution of

$$
\left\{\begin{array}{l}
-\left(r(t) \phi\left(u^{\prime}\right)\right)^{\prime}=\lambda g(t)\left(f(\tilde{v})-\frac{a}{\tilde{v}^{\alpha}}\right), 0<t<1 \\
u(0)=0, u^{\prime}(1)+H(u(1))=0
\end{array}\right.
$$

where $\tilde{v}(t)=\max (v(t), \gamma p(t) / 5)$. Note that $u$ is given by

$$
\begin{equation*}
u(t)=\int_{0}^{t} \phi^{-1}\left(\frac{C-\lambda \int_{0}^{s} g(z)\left(f\left(\tilde{v}-\frac{a}{\tilde{v}^{\alpha}}\right) d z\right.}{r(s)}\right) d s \tag{3.2}
\end{equation*}
$$

for $t \in(0,1)$, where $C$ is the unique number such that $u^{\prime}(1)+H(u(1))=0$ i.e.

$$
\phi^{-1}\left(\frac{C-\lambda \int_{0}^{1} g(z)\left(f(\tilde{v})-\frac{a}{\tilde{v}^{\alpha}}\right) d z}{r(1)}\right)+H\left(\int_{0}^{1} \phi^{-1}\left(\frac{C-\lambda \int_{0}^{s} g(z)\left(f(\tilde{v})-\frac{a}{\tilde{v}^{\alpha}}\right) d z}{r(s)}\right) d s\right)=0
$$

Note that

$$
|C| \leq \lambda \int_{0}^{1} g(t)\left|f(\tilde{v})-\frac{a}{\tilde{v}^{\alpha}}\right| d t
$$

from which (3.2) gives

$$
\begin{equation*}
|u|_{C^{1}} \leq \phi^{-1}\left(\frac{2 \lambda \int_{0}^{1} g(t)\left|f(\tilde{v})-\frac{a}{\tilde{v}^{\alpha}}\right| d t}{r_{0}}\right) \leq \phi^{-1}\left(\frac{2 \lambda \int_{0}^{1} g(t)\left(f(\tilde{v})+\frac{a}{(\gamma / 5)^{\alpha} p^{\alpha}(t)} d t\right)}{r_{0}}\right) \tag{3.3}
\end{equation*}
$$

where $|u|_{C^{1}}=\max \left(\|u\|_{\infty},\left\|u^{\prime}\right\|_{\infty}\right)$. From this and standard arguments, it follows that $T_{\lambda}: E \rightarrow$ $E$ is a completely continuous operator. Next, we verify the two conditions of Lemma A.
(a) Let $u \in E$ satisfy $u=\theta T_{\lambda} u$ for some $\theta \in(0,1]$. Then $\|u\|_{\infty} \neq m$.

Suppose on the contrary that $\|u\|_{\infty}=m$. Then $\|\tilde{u}\|_{\infty} \leq m$. Since $u / \theta=T_{\lambda} u$, we deduce from (3.3) and the assumption $f(m)>\frac{a}{(\gamma / 5)^{\alpha}}$ that

$$
\begin{aligned}
\left\|\frac{u}{\theta}\right\|_{\infty} & \leq \phi^{-1}\left(\frac{2 \lambda}{r_{0}} \int_{0}^{1} g(t)\left(f(m)+\frac{a}{(\gamma / 5)^{\alpha} p^{\alpha}(t)}\right) d t\right) \\
& \leq \phi^{-1}\left(\frac{2 \lambda}{r_{0}} \int_{0}^{1} g(t)\left(f(m)+\frac{f(m)}{p^{\alpha}(t)}\right) d t\right) \leq \phi^{-1}\left(\frac{4 \lambda f(m)}{r_{0}} \int_{0}^{1} \frac{g}{p^{\alpha}}\right) .
\end{aligned}
$$

Hence

$$
m \leq \phi^{-1}\left(\frac{4 \lambda f(m)}{r_{0}} \int_{0}^{1} \frac{g}{p^{\alpha}}\right)
$$

which implies $\lambda \geq \frac{r_{0} \phi(m)}{4 f(m)}\left(\int_{0}^{1} \frac{g}{p^{\alpha}}\right)^{-1}$, a contradiction with $\lambda \in I$. Thus $\|u\|_{\infty} \neq m$.
(b) Let $u \in E$ satisfy $u=T_{\lambda} u+\xi$ for some $\xi \geq 0$. Then $\|u\|_{\infty} \notin\{\gamma, R\}$ for $R \gg 1$.

Since $u-\xi=T_{\lambda} u$, $u$ satisfies

$$
\left\{\begin{array}{l}
-\left(r(t) \phi\left(u^{\prime}\right)\right)^{\prime}=\lambda g(t)\left(f(\tilde{u})-\frac{a}{\tilde{u}^{\alpha}}\right), 0<t<1 \\
u(0)=\xi \geq 0, u^{\prime}(1)+H(u(1))=H(u(1))-H(u(1)-\xi) \geq 0
\end{array}\right.
$$

Note that

$$
\lambda g(t)\left(f\left(\tilde{u}-\frac{a}{\tilde{u}^{\alpha}}\right) \geq-\frac{\lambda a g(t)}{(\gamma / 5)^{\alpha} p^{\alpha}(t)}=-h_{\lambda}(t)\right.
$$

where $h_{\lambda}(t)=\frac{\lambda a}{(\gamma / 5)^{\alpha}}\left(\frac{g(t)}{p^{\alpha}(t)}\right)$. Since $f(m)>\frac{a \phi(m)}{4(\gamma / 5)^{\alpha} \phi(\gamma / 5)}$ and $\lambda \in I$, we get

$$
\lambda<\frac{r_{0} \phi(m)}{4 f(m)}\left(\int_{0}^{1} \frac{g}{p^{\alpha}}\right)^{-1} \leq \frac{r_{0}(\gamma / 5)^{\alpha} \phi(\gamma / 5)}{a}\left(\int_{0}^{1} \frac{g}{p^{\alpha}}\right)^{-1}
$$

This implies $\phi(\gamma / 5)>\frac{\lambda a}{r_{0}(\gamma / 5)^{\alpha}}\left(\int_{0}^{1} \frac{g}{p^{\alpha}}\right)$, i.e.

$$
\begin{equation*}
\gamma>5 \phi^{-1}\left(\frac{\lambda a}{r_{0}(\gamma / 5)^{\alpha}}\left(\int_{0}^{1} \frac{g}{p^{\alpha}}\right)\right)=5 \phi^{-1}\left(\frac{\left\|h_{\lambda}\right\|_{1}}{r_{0}}\right) . \tag{3.4}
\end{equation*}
$$

Suppose $\|u\|_{\infty} \in\{\gamma, R\}$ with $R>\gamma$. Since $\|u\|_{\infty} \geq \gamma>5 \phi^{-1}\left(\frac{\left\|h_{\lambda}\right\|_{1}}{r_{0}}\right)$ in view of (3.4), it follows from Lemma 2.3 (ii) that

$$
\begin{equation*}
u(t) \geq \frac{1}{5}\|u\|_{\infty} p(t) \quad \text { for } t \in[0,1] \tag{3.5}
\end{equation*}
$$

In particular, $u(t) \geq(\gamma / 5) p(t)$ i.e. $\tilde{u} \equiv u$, and $u(t) \geq\|u\|_{\infty} / 20$ for $t \in[1 / 4,3 / 4]$. Consequently, $u$ satisfies

$$
\left\{\begin{array}{l}
-\left(r(t) \phi\left(u^{\prime}\right)\right)^{\prime} \geq \lambda g(t)\left(f\left(\frac{\|u\|_{\infty}}{20}\right)-\frac{a}{(\gamma / 5)^{\alpha} p^{\alpha}(t)}\right), \frac{1}{4}<t<\frac{3}{4} \\
u(1 / 4) \geq 0, u(3 / 4) \geq 0
\end{array}\right.
$$

By the comparison principle, $u \geq v$ on $[1 / 4,3 / 4]$, where $v$ is the solution of

$$
\left\{\begin{array}{l}
-\left(r(t) \phi\left(v^{\prime}\right)\right)^{\prime}=\lambda g(t)\left(f\left(\frac{\|u\|_{\infty}}{20}\right)-\frac{a}{(\gamma / 5)^{\alpha} p^{\alpha}(t)}\right), \frac{1}{4}<t<\frac{3}{4} \\
v(1 / 4)=0, v(3 / 4)=0
\end{array}\right.
$$

Let $t_{0} \in(1 / 4,3 / 4)$ be such that $v^{\prime}\left(t_{0}\right)=0$. Then upon integrating, we obtain

$$
\begin{equation*}
v^{\prime}(t)=\phi^{-1}\left(\frac{\lambda}{r(t)} \int_{t}^{t_{0}} g(s)\left(f\left(\frac{\|u\|_{\infty}}{20}\right)-\frac{a}{(\gamma / 5)^{\alpha} p^{\alpha}(s)}\right) d s\right) \tag{3.6}
\end{equation*}
$$

for $t \in[1 / 4,3 / 4]$. We shall distinguish two cases.
Case 1. $t_{0}>1 / 2$.
Integrating (3.6) on $[1 / 4,3 / 8]$ gives

$$
\begin{align*}
v(3 / 8) & =\int_{1 / 4}^{3 / 8} \phi^{-1}\left(\frac{\lambda}{r(t)} \int_{t}^{t_{0}} g(s)\left(f\left(\frac{\|u\|_{\infty}}{20}\right)-\frac{a}{(\gamma / 5)^{\alpha} p^{\alpha}(s)}\right) d s\right) d t \\
& \geq \int_{1 / 4}^{3 / 8} \phi^{-1}\left(\frac{\lambda}{r(t)}\left(\int_{3 / 8}^{1 / 2} g(s) f\left(\frac{\|u\|_{\infty}}{20}\right) d s-\frac{a}{(\gamma / 5)^{\alpha}} \int_{0}^{1} \frac{g}{p^{\alpha}}\right) d t\right) \\
& \geq \frac{1}{8} \phi^{-1}\left(\lambda\left(\frac{g_{0}}{8 r_{1}} f\left(\frac{\|u\|_{\infty}}{20}\right)-\frac{a}{(\gamma / 5)^{\alpha} r_{0}} \int_{0}^{1} \frac{g}{p^{\alpha}}\right)\right) . \tag{3.7}
\end{align*}
$$

Since

$$
f(m)>\frac{16 a r_{1}(4 m)^{\beta}}{g_{0} r_{0}(\gamma / 5)^{\alpha+\beta}}\left(\int_{0}^{1} \frac{g}{p^{\alpha}}\right),
$$

it follows from (A6) that

$$
\frac{g_{0}}{8 r_{1}} f\left(\frac{\|u\|_{\infty}}{20}\right) \geq \frac{g_{0}}{8 r_{1}} f\left(\frac{\gamma}{20}\right) \geq \frac{g_{0}}{8 r_{1}}\left(\frac{\gamma}{20 m}\right)^{\beta} f(m) \geq \frac{2 a}{(\gamma / 5)^{\alpha} r_{0}} \int_{0}^{1} \frac{g}{p^{\alpha}}
$$

Hence (3.7) gives

$$
v(3 / 8) \geq \frac{1}{8} \phi^{-1}\left(\frac{\lambda g_{0}}{16 r_{1}} f\left(\frac{\|u\|_{\infty}}{20}\right)\right)
$$

which implies

$$
\begin{equation*}
\|u\|_{\infty} \geq \frac{1}{8} \phi^{-1}\left(\frac{\lambda g_{0}}{16 r_{1}} f\left(\frac{\|u\|_{\infty}}{20}\right)\right) \tag{3.8}
\end{equation*}
$$

Case 2. $t_{0} \leq 1 / 2$.
Integrating (3.6) on $[5 / 8,3 / 4]$ gives

$$
\begin{align*}
v(5 / 8) & =\int_{5 / 8}^{3 / 4} \phi^{-1}\left(\frac{\lambda}{r(t)} \int_{t_{0}}^{t} g(s)\left(f\left(\frac{\|u\|_{\infty}}{20}\right)-\frac{a}{(\gamma / 5)^{\alpha} p^{\alpha}(s)}\right) d s\right) d t \\
& \geq \int_{5 / 8}^{3 / 4} \phi^{-1}\left(\frac{\lambda}{r(t)}\left(\int_{1 / 2}^{5 / 8} g(s) f\left(\frac{\|u\|_{\infty}}{20}\right) d s-\frac{a}{(\gamma / 5)^{\alpha}} \int_{0}^{1} \frac{g}{p^{\alpha}}\right) d t\right)  \tag{3.9}\\
& \geq \frac{1}{8} \phi^{-1}\left(\lambda\left(\frac{g_{0}}{8 r_{1}} f\left(\frac{\|u\|_{\infty}}{20}\right)-\frac{a}{(\gamma / 5)^{\alpha} r_{0}} \int_{0}^{1} \frac{g}{p^{\alpha}}\right)\right) \\
& \geq \frac{1}{8} \phi^{-1}\left(\frac{\lambda g_{0}}{16 r_{1}} f\left(\frac{\|u\|_{\infty}}{20}\right)\right),
\end{align*}
$$

i.e. (3.8) holds. Thus (3.8) holds in either case. If $\|u\|_{\infty}=\gamma$ then (3.8) gives

$$
\gamma \geq \frac{1}{8} \phi^{-1}\left(\frac{\lambda g_{0}}{16 r_{1}} f\left(\frac{\gamma}{20}\right)\right),
$$

which implies

$$
\lambda \leq \frac{16 r_{1} \phi(8 \gamma)}{g_{0} f(\gamma / 20)}
$$

a contradiction with $\lambda \in I$. Thus $\|u\|_{\infty} \neq \gamma$.
Since

$$
\frac{f\left(\frac{\|u\|_{\infty}}{20}\right)}{\phi\left(8\|u\|_{\infty}\right)} \leq \frac{16 r_{1}}{\lambda g_{0}}
$$

in view of (3.8) and $\lim _{z \rightarrow \infty} \frac{f(z / 20)}{\phi(8 z)}=\infty$ in view of (A4) and (A6), it follows that $\|u\|_{\infty} \neq R$ for $R \gg 1$.

By Lemma A, $T_{\lambda}$ has two fixed points $u_{i, \lambda}, i=1,2$, such that $\gamma<\left\|u_{1, \lambda}\right\|_{\infty}<m, m<$ $\left\|u_{2, \lambda}\right\|_{\infty}<R$. Since $\left\|u_{i, \lambda}\right\|_{\infty} \geq \gamma$, it follows from (3.5) with $\xi=0$ that $u_{i, \lambda}(t) \geq \frac{\gamma}{5} p(t)$ for $t \in[0,1]$ i.e. $\tilde{u}_{i, \lambda}=u_{i, \lambda}$ on $[0,1]$ for $i=1,2$. Hence $u_{i, \lambda}, i=1,2$, are positive solutions of (1.1).
(ii) We shall modify the above proof. Let $\lambda>0$ satisfy

$$
\phi^{-1}\left(\frac{2 \lambda}{r_{0}} \int_{0}^{1} g(t)\left(f(5)+\frac{a}{p^{\alpha}(t)}\right) d t\right)<5 .
$$

For $v \in E$, define $S_{\lambda} v=u$, where $u$ is the solution of

$$
\left\{\begin{array}{c}
-\left(r(t) \phi\left(u^{\prime}\right)\right)^{\prime}=\lambda g(t)\left(f(\tilde{v})-\frac{a}{\tilde{v}^{\star}}\right), 0<t<1, \\
u(0)=0, u^{\prime}(1)+H(u(1))=0,
\end{array}\right.
$$

where $\tilde{v}(t)=\max (v(t), p(t))$. ( $S_{\lambda}$ is $T_{\lambda}$ in part (i) with $\gamma=5$.) Then $S_{\lambda}: E \rightarrow E$ is completely continuous. We verify that
(c) Let $u \in E$ satisfy $u=\theta T_{\lambda} u$ for some $\theta \in(0,1]$. Then $\|u\|_{\infty} \neq 5$.

Suppose $\|u\|_{\infty}=5$. Then, as in part (a) above, we get

$$
5=\|u\|_{\infty} \leq \phi^{-1}\left(\frac{2 \lambda}{r_{0}} \int_{0}^{1} g(t)\left(f(5)+\frac{a}{p^{\alpha}(t)}\right) d t\right)<5,
$$

a contradiction with the choice of $\lambda$. Thus $\|u\|_{\infty} \neq 5$.
(d) Let $u \in E$ satisfy $u=T_{\lambda} u+\xi$ for some $\xi \geq 0$. Then $\|u\|_{\infty} \neq R$ for $R \gg 1$.

Suppose $\|u\|_{\infty}=R$. Using the same arguments as in part (b) above with $\gamma=5$ and note that for $R$ large

$$
R=\|u\|_{\infty}>5 \phi^{-1}\left(\frac{\left\|h_{\lambda}\right\|_{1}}{r_{0}}\right),
$$

where $h_{\lambda}(t)=\lambda a\left(\frac{g(t)}{p^{\alpha}(t)}\right)$. Hence

$$
\begin{equation*}
u(t) \geq \frac{\|u\|_{\infty}}{5} p(t) \geq p(t) \quad \text { for } t \in[0,1] \tag{3.10}
\end{equation*}
$$

i.e. $\tilde{u}=u$ on $[0,1]$. As in (3.7) and (3.9) above, we obtain

$$
\|u\|_{\infty} \geq \frac{1}{8} \phi^{-1}\left(\lambda\left(\frac{g_{0}}{8 r_{1}} f\left(\frac{\|u\|_{\infty}}{20}\right)-\frac{a}{r_{0}} \int_{0}^{1} \frac{g}{p^{\alpha}}\right)\right)
$$

i.e.

$$
\begin{equation*}
\frac{\frac{g_{0}}{8 r_{1}} f\left(\frac{\|u\|_{\infty}}{20}\right)-\frac{a}{r_{0}} \int_{0}^{1} \frac{g}{p^{\alpha}}}{\phi\left(8\|u\|_{\infty}\right)} \leq \frac{1}{\lambda} . \tag{3.11}
\end{equation*}
$$

Since the left side of (3.11) tends to $\infty$ as $\|u\|_{\infty}$ goes to $\infty$, we deduce that $\|u\|_{\infty}<R$ for $R \gg 1$, which proves (d). By Lemma A, $S_{\lambda}$ has a fixed point $u_{\lambda}$ with $\left\|u_{\lambda}\right\|_{\infty}>5$, which together with (3.10) with $\xi=0$ imply that $\tilde{u}_{\lambda}=u_{\lambda}$ on $[0,1]$, i.e. $u_{\lambda}$ is a positive solution of (1.1). We show next that $\left\|u_{\lambda}\right\|_{\infty} \rightarrow \infty$ as $\lambda \rightarrow 0^{+}$. Since $H\left(u_{\lambda}(1)\right) \geq H(0)=0$, it follows from the boundary condition of $u_{\lambda}$ at 1 that $u_{\lambda}^{\prime}(1) \leq 0$. Hence $u_{\lambda}$ satisfies

$$
\left\{\begin{array}{l}
-\left(r(t) \phi\left(u_{\lambda}^{\prime}\right)\right)^{\prime} \leq \lambda g(t) f\left(\left\|u_{\lambda}\right\|_{\infty}\right), \quad 0<t<1, \\
u_{\lambda}(0)=0, u_{\lambda}^{\prime}(1) \leq 0 .
\end{array}\right.
$$

By the comparison principle, $u_{\lambda} \leq v_{\lambda}$ on $[0,1]$, where $v_{\lambda}$ is the solution of

$$
\left\{\begin{array}{l}
-\left(r(t) \phi\left(v_{\lambda}^{\prime}\right)\right)^{\prime}=\lambda g(t) f\left(\left\|u_{\lambda}\right\|_{\infty}\right), \quad 0<t<1,  \tag{3.12}\\
v_{\lambda}(0)=0, v_{\lambda}^{\prime}(1)=0 .
\end{array}\right.
$$

Note that

$$
v_{\lambda}(t)=\int_{0}^{t} \phi^{-1}\left(\frac{\lambda f\left(\left\|u_{\lambda}\right\|_{\infty}\right)}{r(s)} \int_{s}^{1} g\right) d s \leq \phi^{-1}\left(\frac{\lambda\|g\|_{1} f\left(\left\|u_{\lambda}\right\|_{\infty}\right)}{r_{0}}\right)
$$

for $t \in[0,1]$. Hence

$$
\left\|u_{\lambda}\right\|_{\infty} \leq \phi^{-1}\left(\frac{\lambda\|g\|_{1} f\left(\left\|u_{\lambda}\right\|_{\infty}\right)}{r_{0}}\right),
$$

i.e.

$$
\frac{f\left(\left\|u_{\lambda}\right\|_{\infty}\right)}{\phi\left(\left\|u_{\lambda}\right\|_{\infty}\right)} \geq \frac{r_{0}}{\lambda\|g\|_{1}} .
$$

Consequently,

$$
\lim _{\lambda \rightarrow 0^{+}} \frac{f\left(\left\|u_{\lambda}\right\|_{\infty}\right)}{\phi\left(\left\|u_{\lambda}\right\|_{\infty}\right)}=\infty
$$

and since $\left\|u_{\lambda}\right\|_{\infty}>5$, it follows from (A5) that $\lim _{\lambda \rightarrow 0^{+}}\left\|u_{\lambda}\right\|_{\infty}=\infty$.
Next, we show that $u_{\lambda}(1) \rightarrow \infty$ as $\lambda \rightarrow 0^{+}$. Since $u_{\lambda}^{\prime}(1) \leq 0$, it follows upon integrating the equation in (1.1) and using Lemma 2.1 (ii) that

$$
\begin{aligned}
u_{\lambda}(1) & =\int_{0}^{1} \phi^{-1}\left(\frac{\lambda \int_{t}^{1} g(s)\left(f\left(u_{\lambda}\right)-\frac{a}{u_{\lambda}^{\alpha}}\right) d s-r(1) \phi\left(\left|u_{\lambda}^{\prime}(1)\right|\right)}{r(t)}\right) d t \\
& \geq \int_{0}^{1} \phi^{-1}\left(\frac{\lambda \int_{t}^{1} g(s)\left(f\left(u_{\lambda}\right)-\frac{a}{u_{\lambda}^{\alpha}}\right) d s}{r(t)}\right) d t-2 \int_{0}^{1} \phi^{-1}\left(\frac{r(1) \phi\left(\left|u_{\lambda}^{\prime}(1)\right|\right)}{r(t)}\right) d t .
\end{aligned}
$$

Thus

$$
\begin{equation*}
u_{\lambda}(1)+2 \int_{0}^{1} \phi^{-1}\left(\frac{r(1) \phi\left(H\left(u_{\lambda}(1)\right)\right)}{r(t)}\right) d t \geq \int_{0}^{1} \phi^{-1}\left(\frac{\lambda \int_{t}^{1} g(s)\left(f\left(u_{\lambda}\right)-\frac{a}{u_{\lambda}^{\alpha}}\right) d s}{r(t)}\right) d t . \tag{3.13}
\end{equation*}
$$

Since

$$
\begin{aligned}
& \int_{0}^{1} \phi^{-1}\left(\frac{\lambda \int_{t}^{1} g(s)\left(f\left(u_{\lambda}\right)-\frac{a}{u_{\lambda}^{\alpha}}\right) d s}{r(t)}\right) d t \geq \int_{0}^{1} \phi^{-1}\left(\frac{\lambda \int_{t}^{3 / 4} g(s) f\left(u_{\lambda}\right) d s}{r_{1}}-\frac{\lambda a}{r_{0}} \int_{0}^{1} \frac{g}{p^{\alpha}}\right) d t \\
& \quad \geq \int_{0}^{1 / 2} \phi^{-1}\left(\frac{\lambda \int_{t}^{3 / 4} g(s) f\left(u_{\lambda}\right) d s}{r_{1}}-\frac{\lambda a}{r_{0}} \int_{0}^{1} \frac{g}{p^{\alpha}}\right) d t-\frac{1}{2} \phi^{-1}\left(\frac{\lambda a}{r_{0}} \int_{0}^{1} \frac{g}{p^{\alpha}}\right) \\
& \quad \geq \frac{1}{2} \phi^{-1}\left(\frac{\lambda\left(\int_{1 / 2}^{3 / 4} g\right) f\left(\frac{\left\|u_{\lambda}\right\|_{\infty}}{4}\right)}{r_{1}}-\frac{\lambda a}{r_{0}} \int_{0}^{1} \frac{g}{p^{\alpha}}\right)-\frac{1}{2} \phi^{-1}\left(\frac{\lambda a}{r_{0}} \int_{0}^{1} \frac{g}{p^{\alpha}}\right) \rightarrow \infty \quad \text { as } \lambda \rightarrow \infty,
\end{aligned}
$$

we deduce from (3.13) that $u_{\lambda}(1) \rightarrow \infty$ as $\lambda \rightarrow \infty$. Since $u_{\lambda}$ satisfies

$$
\left(r(t) \phi\left(u_{\lambda}^{\prime}\right)\right)^{\prime} \leq h_{\lambda} \quad \text { on }(0,1),
$$

where $h_{\lambda}(t)=\lambda a\left(g(t) / p^{\alpha}(t)\right)$, Lemma 2.3 (i) gives

$$
u_{\lambda}(t) \geq\left(u_{\lambda}(1)-2 \phi^{-1}\left(\frac{\left\|h_{\lambda}\right\|_{1}}{r_{0}}\right)\right) t
$$

for $t \in[0,1]$. Consequently, $u_{\lambda}(t) \rightarrow \infty$ as $\lambda \rightarrow 0^{+}$uniformly on compact subsets of $(0,1]$, which completes the proof of Theorem 1.1.

## References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18(1976), 620-709. https://doi .org/10.1137/1018114; MR0415432
[2] V. Benci, D. Fortunato, L. Pisani, Soliton like solutions of a Lorentz invariant equation in dimension 3, Rev. Math. Phys. 10(1998), 315-344. https://doi.org/10.1142/ S0129055X98000100; MR1626832
[3] L. Cherfiltsm, Y. Il'yasov, On the stationary solutions of generalized reaction diffusion equations with $p \& q$-Laplacian, Commun. Pure Appl. Anal. 3 (2005), 9-22. https://doi. org/10.3934/cpaa.2005.4.9; MR2126276
[4] P. Drábek, J. Hernández, Existence and uniqueness of positive solutions for some quasilinear elliptic problems, Nonlinear Anal. 44(2001), 189-204. https://doi.org/10.1016/ S0362-546X (99) 00258-8; MR1816658
[5] P. Drábek, A. Kufner, F. Nicolosi, Quasilinear elliptic equations with degenerations and singularities, De Gruyter Series in Nonlinear Analysis and Applications, Vol. 5, Walter de Gruyter \& Co., Berlin, 1997. https://doi .org/10.1515/9783110804775; MR1460729
[6] D. D. Hai, R. Shivaji, Positive solutions for semi-positone systems in an annulus, Rocky Mountain J. Math. 29(1999), No. 4, 1285-1299. https://doi.org/10.1216/rmjm/ 1181070408; MR1743372
[7] D. D. Hai, R. Shivaji, Existence and multiplicity of positive radial solutions for singular superlinear elliptic systems in the exterior of a ball, J. Differential Equations 266(2019), 2232-2243. https://doi.org/10.1016/j.jde.2018.08.027; MR3906247
[8] P. L. Lions, On the existence of positive solutions of semilinear elliptic equations, SIAM Rev. 24(1982), 441-467. https://doi.org/10.1137/1024101; MR0678562
[9] G. Mingione, V. Rădulescu, Recent developments in problems with nondtandard growth and nonuniform ellipticity, J. Math. Anal. Anal. 501(2021), No. 1, Paper No. 125197, 41 pp. https://doi.org/10.1016/j.jmaa. 2021.125197; MR4258810
[10] Q. Morris, R. Shivaji, I. Sim, Existence of positive radial solutions for a superlinear semipositone $p$-Laplacian problem on the exterior of a ball, Proc. Roy. Soc. Edinburgh Sect. A 148(2018), 409-428. https://doi.org/10.1017/S0308210517000452; MR3782023
[11] K. Perera, R. Shivaji, I. Sim, A class of semipositone $p$-Laplacian problems with a critical growth reaction term, Adv. Nonlinear Anal. 9(2020), No. 1, 516-525. https://doi. org/10. 1515/anona-2020-0012; MR3969151
[12] B. Son, P. WANG, Analysis of positive radial solutions for singular superlinear $p$-Laplacian systems on the exterior of a ball, Nonlinear Anal. 192(2020), 111657, 15 pp. https://doi. org/10.1016/j.na.2019.111657; MR4021191

Electronic Journal of Qualitative Theory of Differential Equations

# On nonlocal problems for semilinear second order differential inclusions without compactness 

Tiziana Cardinali ${ }^{\boxtimes}$ and Giulia Duricchi<br>Department of Mathematics and Computer Science, University of Perugia, 1, via Vanvitelli, Perugia 06132, Italy

Received 7 May 2021, appeared 8 September 2021
Communicated by Gabriele Bonanno


#### Abstract

Existence of mild solutions for a nonlocal abstract problem driven by a semilinear second order differential inclusion is studied in Banach spaces in the lack of compactness both on the fundamental system generated by the linear part and on the nonlinear multivalued term. The method used for proving our existence theorems is based on the combination of a fixed point theorem and a selection theorem developed by ourselves with an approach that uses De Blasi measure of noncompactness and the weak topology. As application of our existence result we present the study of the controllability of a problem guided by a wave equation.


Keywords: nonlocal abstract problem, semilinear second order differential inclusion, fundamental system, De Blasi measure of noncompactness, controllability problem, wave equation.

2020 Mathematics Subject Classification: 34A60, 34G25, 34B15.

## 1 Introduction

Let us consider the nonlocal abstract problem controlled by a semilinear second order differential inclusion

$$
\left\{\begin{array}{l}
x^{\prime \prime}(t) \in A(t) x(t)+F(t, x(t)), t \in J=[0,1]  \tag{P}\\
x(0)=g(x) \\
x^{\prime}(0)=h(x)
\end{array}\right.
$$

where $g, h: \mathcal{C}(J ; X) \rightarrow X$ are suitable functions, without compactness conditions both on the multimap $F$ and on the fundamental system generated by the family $\{A(t)\}_{t \in J}$.

The concept of nonlocal initial condition was introduced to extend the classical theory of initial value problems by Byszewski in [3]. This notion is more appropriate then the classical one to describe natural phenomena because it allows us to consider additional informations. Nonlocal problems has been widely studied because of their applications in different fields to

[^28]applied science (see $[8,10,33]$ and the reference cited therein). For instance, in [10] the author described the diffusion phenomenon of a small amount of gas in a transparent tube by using a first order differential equation and the following
$$
g(u)=\sum_{i=0}^{p} c_{i} u\left(t_{i}\right),
$$
where $c_{i}$ is given constant and $t_{i}$ is a fixed instant of time, $i=0,1, \ldots, p$.
On the other hand, there exists an extensive literature concerning abstract second order equations in the autonomous case starting with the initial research works of Kato [19], [20] and [21] (see, e.g. [12, 23, 27,28,30]), while the theory dealing with non-autonomous second order abstract equations/inclusions has only recently been studied by using a concept of fundamental Cauchy operator generated by the family $\{A(t)\}_{t \in J}$, introduced by Kozak in [24].

On this subject we recall Henríquez [15], Henríquez, Poblete and Pozo [16] for second order differential equations; Cardinali and Gentili [5], Cardinali and De Angelis [4] for second order differential inclusions. In all these papers the existence of mild solutions is studied with topological techniques based on fixed point theorems for a suitable solution operator and requesting strong compactness conditions, which are usually not satisfied in an infinite dimensional framework.

Our purpose is to obtain existence results in the lack of this compactness both on the semigroup generated by the linear part and on the nonlinear multivalued term. To achieve this goal we use De Blasi measure of noncompactness and the weak topology. This approach is present in [2], but with the aim of studying the existence of mild solutions for a problem controlled by a semilinear first order differential inclusion.

Moreover the techniques for non-autonomous second order differential equations/inclusions developed in [24] and [5] play a key role in the proof of our existence results.

This paper is organized as follows. After introducing in Section 2 some notations and some preliminary results, in Section 3 we present the problem setting. Section 4 is devoted to obtain some properties of the fundamental Cauchy operator, a new version of a selection theorem proved in [2] (see Theorem 4.2) and, by using the classic Glicksberg Theorem, a variant of the fixed point theorem introduced in [2] for $x_{0}$-unpreserving multimaps (see Theorem 4.3) and its version in Banach spaces (see Corollary 4.4).

In Section 5 we deal with the existence of mild solutions for the nonlocal abstract problem controlled by a semilinear second order differential inclusion in Banach not necessarily reflexive spaces; we end this section by presenting also an new existence theorem in the context of reflexive spaces, omitting some assumption required in the previous result on the multimap $F$ and on the functions $g$ and $h$ (the reflexivity doesn't imply these hypotheses removed). Finally, in Section 6, we apply our abstract existence theorem in reflexive Banach spaces to study controllability of a Cauchy problem guided by the following wave equation

$$
\frac{\partial^{2} w}{\partial t^{2}}(t, \xi)=\frac{\partial^{2} w}{\partial \xi^{2}}(t, \xi)+b(t) \frac{\partial w}{\partial \xi}(t, \xi)+T(t) w(t, \cdot)(\xi)+u(t, \xi) .
$$

(see Theorem 6.1).

## 2 Preliminaries

In this paper $X$ is a Banach space with the norm $\|\cdot\|_{X}$ and $\mathcal{P}(X)$ is the family of nonempty subsets of $X$. Moreover we will use the following notations:

$$
\begin{aligned}
\mathcal{P}_{b}(X) & =\{H \in \mathcal{P}(X): H \text { bounded }\}, \\
\mathcal{P}_{c}(X) & =\{H \in \mathcal{P}(X): H \text { convex }\}, \\
\mathcal{P}_{w k}(X) & =\{H \in \mathcal{P}(X): H \text { weakly compact }\}, \ldots
\end{aligned}
$$

Further, we recall that a Banach space $X$ is said to be weakly compactly generated (WCG, for short) if there exists a weakly compact subset $K$ of $X$ such that $X=\overline{\operatorname{span}}\{K\}$ (see [14])

Remark 2.1. Let us note that every separable space is weakly compact generated as well as the reflexive ones (see [14]).

Moreover, we recall that (see [26, Theorem 1.12.15]) a Banach space $X$ is separable if and only if it is compactly generated.

Moreover, we denote as $X^{*}$ the dual space of $X$.
Now, if $\tau_{w}$ is the weak topology on $X$ and $\left(A_{n}\right)_{n}, A_{n} \in \mathcal{P}(X)$, we set (see [17, Definition 7.1.3])

$$
\begin{equation*}
w-\limsup _{n \rightarrow+\infty} A_{n}=\left\{x \in X: \exists\left(x_{n_{k}}\right)_{k}, x_{n_{k}} \in A_{n_{k}}, n_{k}<n_{k+1}, x_{n_{k}} \rightharpoonup x\right\} \tag{2.1}
\end{equation*}
$$

Then, we denote by $\bar{B}_{X}(0, n)$ the closed ball centered at the origin and of radius $n$ of $X$, and for a set $A \subset X$, the symbol $\bar{A}^{w}$ denotes the weak closure of $A$. We take for granted that a bounded subset $A$ of a reflexive space $X$ is relatively weakly compact. Moreover we recall that a subset $C$ of a Banach space $X$ is called relatively weakly sequentially compact if any sequences of points in $C$ has a subsequence weakly convergent to a point in $X$ (see [26]).

In the sequel, on the interval $J$ we consider the usual Lebesgue measure $\mu$ and we denote by $\mathcal{C}(J ; X)$ the space consisting of all continuous functions from $J$ to $X$ provided with the norm $\|\cdot\|_{\infty}$ of uniform convergence.

A function $f: J \rightarrow X$ is said weakly sequentially continuous if for every sequence $\left(x_{n}\right)_{n}$, $x_{n} \rightharpoonup x$, then $f\left(x_{n}\right) \rightharpoonup f(x)$. Moreover $f$ is said to be B-measurable if there is a sequence of simple functions $\left(s_{n}\right)_{n}$ which converges to $f$ almost everywhere in $J$ (see [11, Definition 3.10.1 (a)]).

It easy to see that Theorem 4 of [22] can be rewritten in the following way.
Theorem 2.2. Let $\left(f_{n}\right)_{n}$ and $g$ be respectively a sequence and a function in $\mathcal{C}(J ; X)$. Then $f_{n} \rightharpoonup g$ if and only if $\left(f_{n}-g\right)_{n}$ is uniformly bounded and $f_{n}(t) \rightharpoonup g(t)$, for every $t \in J$.

Moreover, we call by $L^{1}(J ; X)$ the space of all $X$ - valued Bochner integrable functions on $J$ with norm $\|u\|_{L^{1}(J ; X)}=\int_{0}^{a}\|u(t)\|_{X} d t$ and $L_{+}^{1}(J)=\left\{f \in L^{1}(J ; \mathbb{R}): f(t) \geq 0\right.$, a.e. $\left.t \in J\right\}$. If $X=\mathbb{R}$ we put $\|\cdot\|_{1}=\|\cdot\|_{L^{1}(J ; \mathbb{R})}$.

A set $A \subset L^{1}(J ; X)$ has the property of equi-absolute continuity of the integral if for every $\varepsilon>0$ there exists $\delta_{\varepsilon}>0$ such that, for every $E \in \mathcal{M}(J), \mu(E)<\delta_{\varepsilon}$, we have

$$
\int_{E}\|f(t)\|_{X} d t<\varepsilon
$$

whenever $f \in A$.

Remark 2.3. We observe that if $A \subset L^{1}(J ; X)$ is integrably bounded, i.e. there exists $v \in L_{+}^{1}(J)$ such that

$$
\|f(t)\|_{X} \leq v(t), \quad \text { a.e. } t \in J, \forall f \in A,
$$

then the set $A$ has the property of equi-absolute continuity of the integral.
Now we give Theorem 4.4.2 of [31] that we will use in Section 5 for the suitable pre-ideal regular Lebesgue-Bochner space $L^{2}(\mathbb{T}, \mathbb{C})$ (see [31, pp. 8,9,48]).

Theorem 2.4. An abstract function $x: J \rightarrow X$, where $X$ is a pre-ideal regular space on $\mathbb{R}$, is $B$ measurable if and only if there exists a measurable function $y: J \times \mathbb{R} \rightarrow X$, such that $x(t)=y(t, \cdot)$.

A multimap $F: X \rightarrow \mathcal{P}(Y)$, where $Y$ is a topological space:

- is upper semicontinuous at point $\bar{x} \in X$ if, for every open $W \subset Y$ such that $F(\bar{x}) \subset W$, there exists a neighborhood $V(\bar{x})$ of $\bar{x}$ with the property that $F(V(\bar{x})) \subset W$,
- is upper semicontinuous (u.s.c. for short) if it is upper semicontinuous at every point $x \in X$,
- is compact if its range $F(X)$ is relatively compact in $Y$, i.e. $\overline{F(X)}$ is compact in $Y$,
- is locally compact if every point $x \in X$ there exists a neighborhood $V(x)$ such that the restriction of $F$ to $V(x)$ is compact,
- has closed graph if the set graph $F=\{(x, y) \in X \times Y: y \in F(x)\}$ is closed in $X \times Y$,
- if $Y$ is a linear topological space, $F$ has ( $s-w$ )sequentially closed graph [weakly sequentially closed graph $]$ if for every $\left(x_{n}\right)_{n}, x_{n} \in X, x_{n} \rightarrow x\left[x_{n} \rightharpoonup x\right]$ and for every $\left(y_{n}\right)_{n}, y_{n} \in F\left(x_{n}\right)$, $y_{n} \rightharpoonup y$, we have $y \in F(x)$.

Next, we recall that, if $K$ is a subset of $X, F: K \rightarrow \mathcal{P}(X)$ is a multimap and $x_{0} \in K$, a closed convex set $M_{0} \subset K$ is ( $x_{0}, F$ )-fundamental, if $x_{0} \in M_{0}$ and $F\left(M_{0}\right) \subset M_{0}$ (see [2, p. 620]).

In this setting we recall the following result which allows to characterize the smallest $\left(x_{0}, F\right)$-fundamental set (see [2, Theorem 3.1])

Proposition 2.5. Let $X$ be a locally convex Hausdorff space, $K \subset X, x_{0} \in K$. Let $F: K \rightarrow \mathcal{P}(X)$ be a multimap such that
i) $\overline{c o}\left(F(K) \cup\left\{x_{0}\right\}\right) \subset K$.

Then

1) $\mathcal{F}=\left\{H: H\right.$ is $\left(x_{0}, F\right)$ - fundamental set $\} \neq \varnothing$;
2) put $M_{0}=\bigcap_{H \in \mathcal{F}} H$, we have $M_{0} \in \mathcal{F}$ and $M_{0}=\overline{c o}\left(F\left(M_{0}\right) \cup\left\{x_{0}\right\}\right)$.

Theorem 2.6 ([2, Theorem 4.4] (Containment Theorem)). Let $X$ a Banach space and $G_{n}, G: J \rightarrow$ $\mathcal{P}(X)$ be such that

ג) a.e. $t \in J$, for every $\left(u_{n}\right)_{n}, u_{n} \in G_{n}(t)$, there exists a subsequence $\left(u_{n_{k}}\right)_{k}$ of $\left(u_{n}\right)_{n}$ and $u \in G(t)$ such that $u_{n_{k}} \rightharpoonup u$;
$\alpha \alpha$ ) there exists a sequence $\left(y_{n}\right)_{n}, y_{n}: J \rightarrow X$, having the property of equi-absolute continuity of the integral, such that $y_{n} \in G_{n}(t)$, a.e. $t \in J$, for all $n \in \mathbb{N}$.

Then there exists a subsequence $\left(y_{n_{k}}\right)_{k}$ of $\left(y_{n}\right)_{n}$ such that $y_{n_{k}} \rightharpoonup y$ in $L^{1}(J ; X)$ and, moreover, $y(t) \in$ $\overline{c o} G(t)$, a.e. $t \in J$.

Now, a function $\varphi: \mathcal{P}_{b}(X) \rightarrow \mathbb{R}_{0}^{+}$is said to be a Sadovskij functional in $X$ if it satisfies $\varphi(\overline{c o}(\Omega))=\varphi(\Omega)$, for every $\Omega \in \mathcal{P}_{b}(X)$ (see [1]).

Definition 2.7 ([6, Definition 4.1]). A function $\omega: \mathcal{P}_{b}(X) \rightarrow \mathbb{R}_{0}^{+}$is said to be a measure of weak noncompactness (MwNC, for short) if the following properties are satisfied:
$\left.\omega_{1}\right) \omega$ is a Sadowskii functional;
$\left.\omega_{2}\right) \omega(\Omega)=0_{n}$ if and only if $\bar{\Omega}^{w}$ is weakly compact (i.e. $\omega$ is regular).
Further, a MwNC $\omega: \mathcal{P}_{b}(X) \rightarrow \mathbb{R}_{0}^{+}$is said to be:
monotone if $\Omega_{1}, \Omega_{2} \in \mathcal{P}_{b}(X): \Omega_{1} \subset \Omega_{2}$ implies $\omega\left(\Omega_{1}\right) \leq \omega\left(\Omega_{2}\right)$;
nonsingular if $\omega(\{x\} \cup \Omega)=\omega(\Omega)$, for every $x \in X, \Omega \in \mathcal{P}_{b}(X)$;
$x_{0}$-stable if, fixed $x_{0} \in X, \omega\left(\left\{x_{0}\right\} \cup \Omega\right)=\omega(\Omega), \Omega \in \mathcal{P}_{b}(X)$;
invariant under closure if $\omega(\bar{\Omega})=\omega(\Omega), \Omega \in \mathcal{P}_{b}(X)$;
invariant with respect to the union with compact set if $\omega(\Omega \cup C)=\omega(\Omega)$, for every relatively compact set $C \subset X$ and $\Omega \in \mathcal{P}_{b}(X)$.

Remark 2.8. In particular in [9] De Blasi introduces the function $\beta: \mathcal{P}_{b}(X) \rightarrow \mathbb{R}_{0}^{+}$so defined

$$
\beta(\Omega)=\inf \left\{\varepsilon \in \left[0, \infty\left[: \text { there exists } C \subset X \text { weakly compact : } \Omega \subseteq C+B_{X}(0, \varepsilon)\right\}\right.\right.
$$

and he proves that $\beta$ is a regular Sadowskii functional. Then $\beta$ is MwNC, named in literature De Blasi measure of weak noncompactness.

We recall that $\beta$ has all the properties mentioned before and it is also algebraically subadditive, i.e. $\beta\left(\sum_{k=1}^{n} M_{k}\right) \leq \sum_{k=1}^{n} \beta\left(M_{k}\right)$, where $M_{k} \in \mathcal{P}_{b}(X), k=1, \ldots, n$. Moreover, for every bounded linear operator $L: X \rightarrow X$ the following property holds ([18], p.35)

$$
\beta(L(\Omega)) \leq\|L\| \beta(\Omega), \quad \text { for every } \Omega \in \mathcal{P}_{b}(X),
$$

where $\|L\|$ denotes the norm of the operator $L$.
We recall the following interesting result for MwNC .
Proposition 2.9 ([25, Theorem 2.8 and Remark 2.7 (b)] or [2, Theorem 2.7]). Let ( $\Omega, \Sigma, \mu$ ) be a finite positive measure space and $X$ be a weakly compactly generated Banach space. Then for every countable family $C$ having the property of equi-absolute continuity of the integral of functions $x: \Omega \rightarrow X$, the function $\beta(C(\cdot))$ is measurable and

$$
\beta\left(\left\{\int_{\Omega} x(s) d s: x \in C\right\}\right) \leq \int_{\Omega} \beta(C(s)) d s,
$$

where $\beta$ is a MwNC.
We recall a Sadowskii functional that we will use in the following.

Definition 2.10 ([2, Definition 3.9]). Let $X$ a Banach space, $N \in \mathbb{R}$, and $M$ a bounded subspace of $\mathcal{C}([a, b] ; X)$.

We use the notation $M(t)=\{x(t): x \in M\}$ and define

$$
\begin{equation*}
\beta_{N}(M)=\sup _{\substack{C \subset M \\ \text { countable }}} \sup _{t \in[a, b]} \beta(C(t)) e^{-N t}, \tag{2.2}
\end{equation*}
$$

where $\beta$ is the De Blasi MwNC.
Remark 2.11. We recall that the Sadowskii functional $\beta_{N}$ is $x_{0}$-stable and monotone (see [2, Proposition 3.10]) and $\beta_{N}$ has the two following properties
(I) $\beta_{N}$ is algebraically subadditive;
(II) $M \subset \mathcal{C}([a, b] ; X)$ is relatively weakly compact $\Rightarrow \beta_{N}(M)=0$.

We note that (I) holds since $\beta$ is algebraically subadditive while (II) is true taking into account of the regularity of $\beta$.

## 3 Problem setting

First of all, on the linear part of the second order differential inclusion, presented in the nonlocal problem ( P ), we assume the following property:
(A) $\{A(t)\}_{t \in J}$ is a family of bounded linear operators $A(t): D(A) \rightarrow X$, where $D(A)$, independent on $t \in J$, is a subset dense in $X$, such that, for each $x \in D(A)$, the function $t \mapsto A(t) x$ is continuous on $J$ and generating a fundamental system $\{S(t, s)\}_{t, s \in J}$, and $F$ is a suitable $X$-valued multimap defined in $J \times X$.

In the following we recall the concept of fundamental system introduced by Kozak in [24] and recently used in [4], [5] and [16].

Definition 3.1. A family $\{S(t, s)\}_{t, s \in J}$ of bounded linear operators $S(t, s): X \rightarrow X$ is called a fundamental system generated by the family $\{A(t)\}_{t \in J}$ if

S1. for each $x \in X, S(\cdot, \cdot) x: J \times J \rightarrow X$ is a $\mathcal{C}^{1}$-function and
a. for each $t \in J, S(t, t) x=0$, for every $x \in X$;
b. for each $t, s \in J$ and for each $x \in X,\left.\frac{\partial}{\partial t} S(t, s)\right|_{t=s} x=x$ and

$$
\left.\frac{\partial}{\partial s} S(t, s)\right|_{t=s} x=-x ;
$$

S2. for all $t, s \in J, x \in D(A)$, then $S(t, s) x \in D(A)$, the map $S(\cdot, \cdot) x: J \times J \rightarrow X$ is of class $\mathcal{C}^{2}$ and

$$
\begin{aligned}
& \mathbf{a}^{\prime} \cdot \frac{\partial^{2}}{\partial t^{2}} S(t, s) x=A(t) S(t, s) x ; \\
& \mathrm{b}^{\prime} \cdot \frac{\partial^{2}}{\partial s^{2}} S(t, s) x=S(t, s) A(s) x ; \\
& \text { c }\left.^{\prime} \cdot \frac{\partial^{2}}{\partial \partial s s} S(t, s)\right|_{t=s} x=0 ;
\end{aligned}
$$

S3. for all $t, s \in J, x \in D(A)$, then $\frac{\partial}{\partial s} S(t, s) x \in D(A)$. Moreover, there exist $\frac{\partial^{3}}{\partial t^{2} \partial s} S(t, s) x$, $\frac{\partial^{3}}{\partial s^{2} \partial t} S(t, s) x$ and

$$
\begin{aligned}
& \mathrm{a}^{\prime \prime} \cdot \frac{\partial^{3}}{\partial t^{2} \partial s} S(t, s) x=A(t) \frac{\partial}{\partial s} S(t, s) x \\
& \mathrm{~b}^{\prime \prime} \cdot \frac{\partial^{3}}{\partial s^{2} \partial t} S(t, s) x=\frac{\partial}{\partial t} S(t, s) A(s) x
\end{aligned}
$$

and, for all $x \in D(A)$, the function $(t, s) \mapsto A(t) \frac{\partial}{\partial s} S(t, s) x$ is continuous in $J \times J$.
Moreover, a map $S: J \times J \rightarrow \mathcal{L}(X)$, where $\mathcal{L}(X)$ is the space of all bounded linear operators in $X$ with the norm $\|\cdot\|_{\mathcal{L}(X)}$, is said to be a fundamental operator if the family $\{S(t, s)\}_{t, s \in J}$ is a fundamental system.

To abbreviate the notation we use, for each $(t, s) \in J \times J$, the linear cosine operator

$$
C(t, s)=-\frac{\partial}{\partial s} S(t, s): X \rightarrow X .
$$

Remark 3.2. We recall that, by using Banach-Steinhaus Theorem, the fundamental system $\{S(t, s)\}_{t, s \in J}$ satisfies the following properties (see [5]): there exist $K, K^{*}>0$ such that

$$
\begin{aligned}
& \text { p1. }\|C(t, s)\|_{\mathcal{L}(X)} \leq K,(t, s) \in J \times J \\
& \text { p2. }\|S(t, s)\|_{\mathcal{L}(X)} \leq K|t-s|,(t, s) \in J \times J \\
& \text { p3. }\|S(t, s)\|_{\mathcal{L}(X)} \leq K a,(t, s) \in J \times J \\
& \text { p4. }\left\|S\left(t_{2}, s\right)-S\left(t_{1}, s\right)\right\|_{\mathcal{L}(X)} \leq K^{*}\left|t_{2}-t_{1}\right|, t_{1}, t_{2}, s \in J .
\end{aligned}
$$

Further we denote with $G_{S}: L^{1}(J ; X) \rightarrow \mathcal{C}(J ; X)$ the fundamental Cauchy operator, introduced in [5], defined by

$$
G_{s} f(t)=\int_{0}^{t} S(t, s) f(s) d s, \quad t \in J, f \in L^{1}(J ; X)
$$

It is easy to see that, by using Theorem 1.3.5 of [18] and the properties p3., p4. and S1., the operator $G_{S}$ is well posed.

We investigate the existence of mild solutions for the nonlocal problem ( P ) (see [5, Definition 2.2])

Definition 3.3. A continuous function $u: J \rightarrow X$ is a mild solution for $(\mathrm{P})$ if

$$
u(t)=C(t, 0) g(u)+S(t, 0) h(u)+\int_{0}^{t} S(t, \xi) f(\xi) d \xi, \quad t \in J,
$$

where $f \in S_{F(\cdot, u(\cdot))}^{1}=\left\{f \in L^{1}(J ; X): f(t) \in F(t, u(t))\right.$, a.e. $\left.t \in J\right\}$.

## 4 Auxiliary results

First of all we describe some properties of the fundamental Cauchy operator by the following
Proposition 4.1. If $\{S(t, s)\}_{(t, s) \in J \times J}$ is the fundamental system, then the fundamental Cauchy operator $G_{S}: L^{1}(J ; X) \rightarrow \mathcal{C}(J ; X)$ is linear, bounded, weakly continuous and weakly sequentially continuous.

Proof. Clearly $G_{S}$ is a bounded and linear operator. Hence we can deduce that $G_{S}$ is weakly continuous.

Now we prove that $G_{S}$ is also weakly sequentially continuous.

Fixed $t \in J$ and $e^{\prime} \in X^{*}$, let us consider the map $H_{t}: L^{1}(J ; X) \rightarrow \mathbb{R}$, where $H_{t}(g)=$ $e^{\prime}\left(G_{S} g(t)\right)$, for every $g \in L^{1}(J ; X)$.

Obviously $H_{t}$ is a linear and continuous functional. Fixed a sequence $\left(f_{n}\right)_{n}, f_{n} \in L^{1}(J ; X)$ such that $f_{n} \rightharpoonup f$, by using the properties of the weak convergence, we have $e^{\prime}\left(G_{s} f_{n}(t)\right) \rightarrow$ $e^{\prime}\left(G_{S} f(t)\right)$. Then, by the arbitrariness of $e^{\prime} \in X^{*}$, we have

$$
G_{S} f_{n}(t) \rightharpoonup G_{S} f(t), \forall t \in J
$$

Moreover we can say that the sequence $\left(G_{S}\left(f_{n}-f\right)\right)_{n}$ is uniformly bounded in $\mathcal{C}(J ; X)$. Indeed, by using p3. and the weak convergence of $\left(f_{n}\right)_{n}$, we can write

$$
\begin{aligned}
\left\|G_{S} f_{n}-G_{S} f\right\|_{\mathcal{C}(J ; X)} & =\sup _{t \in J}\left\|\int_{0}^{t} S(t, \xi)\left(f_{n}(\xi)-f(\xi)\right) d \xi\right\|_{X} \\
& \leq K\left(\left\|f_{n}\right\|_{L^{1}(J ; X)}+\|f\|_{L^{1}(J ; X)}\right) \leq K\left(Q+\|f\|_{L^{1}(J ; X)}\right)
\end{aligned}
$$

where $Q$ is a positive constant such that $\left\|f_{n}\right\|_{L^{1}(J ; X)} \leq Q$, for every $n \in \mathbb{N}$. Therefore $\left(G_{S} f_{n}-G_{S} f\right)_{n}$ satisfies all the hypotheses of Theorem 2.2 , so we have

$$
G_{s} f_{n} \rightharpoonup G_{S} f
$$

Now, let us introduce the following result, that will play a key role in the proof of our existence theorem. Let us note that the analogous Proposition 4.5 of [2] is not able to work in the proof of our existence theorem because the hypothesis $d$ ) is weaker of the assumption (d) required in Proposition 4.5 of [2].

Theorem 4.2. Let $M$ be a metric space, $X$ a Banach space and $F: J \times M \rightarrow \mathcal{P}(X)$ a multimap having the following properties:
a) for a.e. $t \in J$, for every $x \in M$, the set $F(t, x)$ is closed and convex ;
b) for every $x \in M$, the multimap $F(\cdot, x)$ has a $B$-measurable selection;
c) for a.e. $t \in J$ the multimap $F(t, \cdot): M \rightarrow \mathcal{P}(X)$ has a (s-w)sequentially closed graph in $M \times X$;
d) for almost all $t \in J$ and every convergent sequence $\left(x_{n}\right)_{n}$ in $M$ the set $\bigcup_{n} F\left(t, x_{n}\right)$ is relatively weakly compact;
e) there exists $\varphi: J \rightarrow[0, \infty): \varphi \in L_{+}^{1}(J)$ such that

$$
\sup _{z \in F(t, M)}\|z\| \leq \varphi(t), \quad \text { a.e. } t \in J .
$$

Then, for every B-measurable $u: J \rightarrow M$, there is a B-measurable y : $J \rightarrow X$ with $y(t) \in F(t, u(t))$ for a.e. $t \in J$.

Proof. First of all we note that hypothesis b) implies
b) ${ }_{w}$ for every $s: J \rightarrow M$ simple function, the multimap $F(\cdot, s(\cdot))$ has a B-measurable selection.

Next fix $u: J \rightarrow M$ a B-measurable function, then there exists a sequence $\left(u_{p}\right)_{p}, u_{p}: J \rightarrow M$ simple function, such that

$$
\begin{equation*}
u_{p}(t) \rightarrow u(t), \quad \text { a.e. } t \in J \tag{4.1}
\end{equation*}
$$

Using $b)_{w}$, for every $p \in \mathbb{N}$, in correspondence of the simple function $u_{p}$, there exists a Bmeasurable function $y_{p}: J \rightarrow X$ such that

$$
\begin{equation*}
y_{p}(t) \in F\left(t, u_{p}(t)\right), \quad \text { a.e. } t \in J \tag{4.2}
\end{equation*}
$$

Now, let us consider $A=\left\{y_{p}, p \in \mathbb{N}\right\}$, subset of $L^{1}(J ; X)$ (see e)).
First of all we note that, if $N$ is the null measure set for which $a), c), d), e),(4.1)$ and (4.2) hold, we can write (see (4.2))

$$
\begin{equation*}
A(t)=\left\{y_{p}(t), p \in \mathbb{N}\right\} \subset{\overline{\bigcup_{p \in \mathbb{N}}} \bar{N}\left(t, u_{p}(t)\right)}^{w}, \quad t \in J \backslash N \tag{4.3}
\end{equation*}
$$

where the set ${\overline{\bigcup_{p \in \mathbb{N}} F\left(t, u_{p}(t)\right)}}^{w}$ is weakly compact. Therefore the set $A(t)$ is relatively weakly compact.

Now, by using hypothesis e) we can say that $A$ is bounded in $L^{1}(J ; X)$. Indeed, put $r=$ $\|\varphi\|_{1}$, we have

$$
\left\|y_{p}\right\|_{L^{1}(J ; X)} \leq r, \quad \forall p \in \mathbb{N}
$$

Moreover, by recalling that $\varphi \in L_{+}^{1}(J)$, we can say that, for every $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ : for every $H \in \mathcal{M}(J), \mu(H)<\delta(\varepsilon)$ then

$$
\left|\int_{H} y_{p}(t) d t\right| \leq \int_{H}\left\|y_{p}(t)\right\|_{X} d t \leq \int_{H} \varphi(t) d t \leq \varepsilon, \quad \forall p \in \mathbb{N}
$$

i.e., $A$ has the property of equi-absolute continuity of the integral.

Since, as we have showed, the set $A$ satisfies all the hypotheses of [29, Corollary 9], we can conclude that $A$ is relatively weakly compact in $L^{1}(J ; X)$. Therefore there exists $\left(y_{p_{k}}\right)_{k} \subset\left(y_{p}\right)_{p}$ such that $y_{p_{k}} \rightharpoonup y, y \in L^{1}(J ; X)$.

Now, we can apply [[17], Proposition 7.3.9] to the multimap $G: J \rightarrow \mathcal{P}_{w k}(X)$, defined by $G(s)=B_{s}, \forall s \in J$, where $B_{s}={\overline{\bigcup_{p \in \mathbb{N}} F\left(s, u_{p}(s)\right)}}^{w}$, and to the sequence $\left(y_{p_{k}}\right)_{k}$ of $L^{1}(J ; X)$. It is possible since (see (4.3)) $y_{p_{k}}(t) \in B_{t}, t \in J \backslash N, \forall p_{k}$. Hence we can conclude that, for the fixed $t \in J \backslash N$, we have (see (2.1))

$$
\begin{equation*}
y(t) \in \overline{c o} w-\limsup _{k \rightarrow \infty}\left\{y_{p_{k}}(t)\right\}_{k} . \tag{4.4}
\end{equation*}
$$

Then, by (4.2), we can say

$$
\begin{equation*}
\overline{c o} w-\limsup _{k \rightarrow \infty}\left\{y_{p_{k}}(t)\right\}_{k} \subset \overline{c o} w-\limsup _{k \rightarrow \infty} F\left(t, u_{p_{k}}(t)\right) \tag{4.5}
\end{equation*}
$$

Finally, we will prove that (see hypothesis $a$ ) and (4.1))

$$
\begin{equation*}
\overline{c o} w-\limsup _{k \rightarrow \infty} F\left(t, u_{p_{k}}(t)\right) \subset F(t, u(t)) \tag{4.6}
\end{equation*}
$$

Let us fix $z \in \overline{c o} w-\lim \sup _{p_{k} \rightarrow \infty} F\left(t, u_{p_{k}}(t)\right)$, then there exists $z_{p_{k_{q}}} \in F\left(t, u_{p_{k_{q}}}(t)\right)$ such that

$$
z_{p_{k_{q}}} \rightharpoonup z
$$

in $X$, where $\left(p_{k_{q}}\right)_{q \in \mathbb{N}}$ is an increasing sequence. Moreover, by (4.1) we know that

$$
u_{p_{k_{q}}}(t) \rightarrow u(t) .
$$

Therefore, since $t \notin N$, hypothesis $c)$ implies that $z \in F(t, u(t))$. For the arbitrariness of $z$ we can conclude that (4.6) is true.

Thanks to (4.4), (4.5), (4.6), finally we can say that the map $y \in L^{1}(J ; X)$ satisfies $y(t) \in$ $F(t, u(t))$ a.e. $t \in J$, so the thesis holds.

Now, by using the concept of smallest $\left(x_{0}, T\right)$-fundamental set (see 2 ) of Proposition 2.5), taking into account of Proposition 2.5 and the classical Glicksberg Fixed Point Theorem of [13] we deduce a variant of Theorem 3.7 of [2] proved by Benedetti-Väth for $x_{0}$-unpreserving multimaps $T$.

Theorem 4.3. Let $X$ be a locally convex Hausdorff space, $K \subset X, x_{0} \in K$ and $T: K \rightarrow \mathcal{P}(X)$ a multimap such that
i) $\overline{c o}\left(T(K) \cup\left\{x_{0}\right\}\right) \subset K$;
ii) $T(x)$ is convex, for every $x \in M_{0}$;
iii) $M_{0}$ is compact;
iv) $T_{\mid M_{0}}$ has closed graph,
where $M_{0}$ is the smallest ( $x_{0}, T$ )-fundamental set.
Then there exists at least one fixed point for $T$, i.e. there exists $\bar{x} \in M_{0}: \bar{x} \in T(\bar{x})$.
Proof. First of all, since $M_{0}$ is a $\left(x_{0}, T\right)$-fundamental set, we know that $M_{0}$ is convex and $T\left(M_{0}\right) \subset M_{0}$. Moreover by $\left.i i i\right) M_{0}$ is also compact.

Therefore, taking into account of $i i$ ) and $i v)$, the multimap $T_{\mid M_{0}}: M_{0} \rightarrow \mathcal{P}\left(M_{0}\right)$ has convex values and closed graph. So we are in a position to apply the Glicksberg Theorem to the multimap $T_{\mid M_{0}}$, then there exists $\bar{x} \in M_{0}$ such that $\bar{x} \in T(\bar{x})$.

When we deal with the weak topology in a Banach space, we can replace equivalently the hypothesis about closed graph by a sequentially closed graph (see [2], Corollary 3.2), so we have the following:

Corollary 4.4. Let $X$ be a Banach space, $K \subset X, x_{0} \in K$ and $T: K \rightarrow \mathcal{P}(X)$ be a multimap such that
i) $\overline{c o}\left(T(K) \cup\left\{x_{0}\right\}\right) \subset K$;
ii) $T(x)$ convex, for every $x \in M_{0}$;
iii) $M_{0}$ is weakly compact;
iv) $T_{\mid M_{0}}$ has weakly sequentially closed graph,
where $M_{0}$ is the smallest ( $x_{0}, T$ )-fundamental set.
Then there exists at least one fixed point for $T$.

## 5 Existence result

In this section we assume the following hypotheses on the multimap $F: J \times X \rightarrow \mathcal{P}(X)$
F1. for every $(t, x) \in J \times X$, the set $F(t, x)$ is convex;
F2. for every $x \in X, F(\cdot, x): J \rightarrow X$ admits a $B$-measurable selection;
F3. for a.e. $t \in J, F(t, \cdot): X \rightarrow X$ has a weakly sequentially closed graph;
F4. there exists $\left(\varphi_{n}\right)_{n}, \varphi_{n} \in L_{+}^{1}(J)$, such that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \frac{\int_{0}^{1} \varphi_{n}(\tilde{\xi}) d \xi}{n}<\frac{1}{K} \tag{5.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|F\left(t, \bar{B}_{X}(0, n)\right)\right\| \leq \varphi_{n}(t), \text { a.e. } t \in J, n \in \mathbb{N}, \tag{5.2}
\end{equation*}
$$

where $K$ is the constant presented in Remark 3.2;
and the two properties related to functions $g, h: \mathcal{C}(J ; X) \rightarrow X$
gh1. $g, h$ are weakly sequentially continuous;
gh2. for every countable, bounded $H \subset \mathcal{C}(J ; X)$, the sets $g(H)$ and $h(H)$ are relatively compact.

Now we state the main result of the paper.
Theorem 5.1. Let $X$ be a weakly compactly generated Banach space and $\{A(t)\}_{t \in J}$ a family of operators which satisfies the property ( $A$ ).

Let $F: J \times X \rightarrow \mathcal{P}(X)$ be a multimap satisfying $F 1, F 2, F 3, F 4$ and the following hypothesis
F5. there exists $H \subset J, \mu(H)=0$, such that, for all $n \in \mathbb{N}$, there exists $v_{n} \in L_{+}^{1}(J)$ with the property

$$
\beta\left(C_{1}\right) \leq v_{n}(t) \beta\left(C_{0}\right), t \in J \backslash H
$$

for all countable $C_{0} \subseteq \bar{B}_{X}(0, n), C_{1} \subseteq F\left(t, C_{0}\right)$, where $\beta$ is the De Blasi measure of weak noncompactness.

Let $g, h: \mathcal{C}(J ; X) \rightarrow X$ be two functions satisfying gh1, gh2 and having the following properties gh3. g, h are bounded;
gh4. for every bounded and closed subset $M$ of $\mathcal{C}(J ; X)$, the sets

$$
C(\cdot, 0) g(M) \text { and } S(\cdot, 0) h(M)
$$

are relatively weakly compact in $\mathcal{C}(J ; X)$.
Then there exists at least one mild solution for the nonlocal problem (P).

Proof. First of all we prove that

$$
\begin{equation*}
F(t, x) \text { is closed, for a.e. } t \in J \text { and for every } x \in X . \tag{5.3}
\end{equation*}
$$

Denoted by $N$ a null measure set such that $F 3$. and $F 5$. hold in $J \backslash N$, we fix $t \in J \backslash N$ and $x \in X$. Put $C_{0}=\{x\}$ and $C_{1}=\left\{y_{n}: n \in \mathbb{N}\right\}$, where $y_{n} \in F(t, x), \forall n \in \mathbb{N}$. Being $C_{0} \subset \bar{B}_{X}(0, p)$, for a suitable $p \in \mathbb{N}$, and $C_{1} \subset F\left(t, C_{0}\right)$, by $F 5$. we have $\beta\left(C_{1}\right) \leq v_{p}(t) \beta\left(C_{0}\right)=0$. Therefore $C_{1}$ is relatively w-compact and so by Eberlein-Šmulian Theorem we can say that there exists $\left(y_{n_{k}}\right)_{k}$, $y_{n_{k}} \rightharpoonup y$. Then F3. implies that $y \in F(t, x)$. So we have that $F(t, x)$ is w-sequentially compact and, invoking again the Eberlein-Šmulian Theorem, $F(t, x)$ is w-compact. Therefore, by using Theorem 3 of [32], in order to establish the closeness of the convex set $F(t, x)$ it is sufficient to observe that $F(t, x)$ is w-sequentially closed by virtue of hypothesis $F 3$. too.

Now, we consider the integral multioperator $T: \mathcal{C}(J ; X) \rightarrow \mathcal{P}_{c}(\mathcal{C}(J ; X))$ defined, for every $u \in \mathcal{C}(J ; X)$, as

$$
\begin{align*}
& T u=\{y \in \mathcal{C}(J ; X): y(t)=C(t, 0) g(u)+S(t, 0) h(u) \\
&\left.+\int_{0}^{t} S(t, \xi) f(\xi) d \xi, t \in J, f \in S_{F(\cdot, u(\cdot))}^{1}\right\} \tag{5.4}
\end{align*}
$$

where

$$
\begin{equation*}
S_{F(\cdot, u(\cdot))}^{1}=\left\{f \in L^{1}(J ; X): f(t) \in F(t, u(t)) \text { a.e. } t \in J\right\} . \tag{5.5}
\end{equation*}
$$

Note that, for all $u \in \mathcal{C}(J ; X), T u \neq \varnothing$. Indeed, put

$$
\begin{equation*}
M_{u}=\bar{B}_{X}\left(0, n_{u}\right), \tag{5.6}
\end{equation*}
$$

where $n_{u} \in \mathbb{N}:\|u(t)\|_{X} \leq n_{u}$, for all $t \in J$, we note that the multimap $F_{J \times M_{u}}$ satisfies all the hypotheses of Theorem 4.2, by considering on $M_{u}$ the metric $d$ induced by that on $X$.

First of all F1., (5.3) and F2. imply respectively $a$ ) and $b$ ) of Theorem 4.2 for the restriction $F_{\mid J \times M_{u}}$.

By F3. we have that $F_{\mid J \times M_{u}}$ has the property $c$ ) of Theorem 4.2.
Moreover, fixed $t \in J \backslash H$ (where $H$ is presented in $F 5$.), if $\left(u_{n}\right)_{n}, u_{n} \in M_{u}, u_{n} \rightarrow v$ in $\left(M_{u}, d\right)$, we can consider the countable set $\tilde{C}_{0}=\left\{u_{n}: n \in \mathbb{N}\right\} \subset M_{u}$ and the set $\tilde{C}_{1}=$ $\cup_{n} F\left(t, u_{n}\right) \subset F\left(t, \tilde{C}_{0}\right)$ and by $F 5$. we can write

$$
\beta\left(\tilde{C}_{1}\right) \leq v_{n}(t) \beta\left(\tilde{C}_{0}\right)=0,
$$

hence $\beta\left(\tilde{C}_{1}\right)=0$, i.e. the set $\tilde{C}_{1}$ is relatively w-compact for the regularity of the De Blasi MwNC . Therefore also $d$ ) of Theorem 4.2 holds.

Finally, for $n_{u} \in \mathbb{N}$ presented in (5.6), by $F 4$. we can say that there exists $\varphi_{n_{u}} \in L_{+}^{1}(J)$ such that

$$
\left\|F\left(t, M_{u}\right)\right\| \leq \varphi_{n_{u}}(t), \quad \text { a.e. } t \in J
$$

and so also $e$ ) of Theorem 4.2 is satisfied. Therefore we can conclude that there exists a Bselection $f_{u}$ of the multimap $F(\cdot, u(\cdot))$, i.e. $S_{F(\cdot, u(\cdot))}^{1}$ is nonempty. Then the map $y_{u}$ defined by

$$
y_{u}(t)=C(t, 0) g(u)+S(t, 0) h(u)+G_{S} f_{u}, \quad t \in J,
$$

is such that $y_{u} \in T u$, i.e. $T u \neq \varnothing$.
Moreover $T$ takes convex values thanks the convexity of the values of $F$.

From now on we proceed by steps.
Step 1. The multioperator $T$ has a weakly sequentially closed graph.
Let $\left(q_{n}\right)_{n}$ and $\left(x_{n}\right)_{n}$ be two sequences in $\mathcal{C}(J ; X)$ such that

$$
\begin{equation*}
x_{n} \in T q_{n}, \quad \forall n \in \mathbb{N} \tag{5.7}
\end{equation*}
$$

and there exist $q, x \in \mathcal{C}(J ; X)$ such that

$$
\begin{equation*}
q_{n} \rightharpoonup q, \quad x_{n} \rightharpoonup x ; \tag{5.8}
\end{equation*}
$$

we have to show that $x \in T q$.
First of all we recall that, by the properties of the convergence $q_{n} \rightharpoonup q$, there exists $\bar{n} \in \mathbb{N}$ such that

$$
\begin{equation*}
\left\|q_{n}\right\|_{\mathcal{C}(J ; X)} \leq \bar{n}, \quad \forall n \in \mathbb{N} . \tag{5.9}
\end{equation*}
$$

Moreover, for every $t \in J$, the weak convergence of the sequence $\left(q_{n}\right)_{n}$ to $q$ implies also that

$$
\begin{equation*}
q_{n}(t) \rightharpoonup q(t) . \tag{5.10}
\end{equation*}
$$

Then by (5.7), for every $n \in \mathbb{N}$, there exists (see (5.5))

$$
\begin{equation*}
f_{n} \in S_{F\left(\cdot, q_{n}(\cdot)\right)}^{1} \tag{5.11}
\end{equation*}
$$

such that (see (5.4))

$$
x_{n}(t)=C(t, 0) g\left(q_{n}\right)+S(t, 0) h\left(q_{n}\right)+\int_{0}^{t} S(t, \xi) f_{n}(\xi) d \xi, \quad t \in J .
$$

Now we want to prove that the multimaps $G_{n}: J \rightarrow \mathcal{P}(X), n \in \mathbb{N}$ and $G: J \rightarrow \mathcal{P}(X)$ respectively defined by

$$
\begin{align*}
G_{n}(t) & =F\left(t, q_{n}(t)\right), & & t \in J,  \tag{5.12}\\
G(t) & =F(t, q(t)), & & t \in J \tag{5.13}
\end{align*}
$$

satisfy all the hypotheses of the Containment Theorem. To this aim we consider the null measure set $N$ for which F3. and F5. hold. Let us fix $t \in J \backslash N$, we consider a sequence $\left(u_{n}\right)_{n}$ such that

$$
\begin{equation*}
u_{n} \in G_{n}(t), \forall n \in \mathbb{N} . \tag{5.14}
\end{equation*}
$$

Now, we define a countable set of $X$

$$
\begin{equation*}
C_{0}=\left\{q_{n}(t): n \in \mathbb{N}\right\} . \tag{5.15}
\end{equation*}
$$

It is evident that $C_{0} \subset \bar{B}_{X}(0, \bar{n})$ (see (5.9)). Then, put $C_{1}=\left\{u_{n}: n \in \mathbb{N}\right\}$ we have that (see (5.14), (5.12) and (5.15))

$$
C_{1} \subset F\left(t,\left\{q_{n}(t)\right\}_{n}\right)=F\left(t, C_{0}\right) .
$$

Now, in correspondence of $\bar{n} \in \mathbb{N}$ chosen in (5.9), by virtue of $F 5$. there exists $v_{\bar{n}} \in L_{+}^{1}(J)$ such that

$$
\begin{equation*}
\beta\left(C_{1}\right) \leq v_{\bar{n}}(t) \beta\left(C_{0}\right) . \tag{5.16}
\end{equation*}
$$

Taking account of (5.10) we can say that the set $C_{0}$ is relatively weakly compact and so, for the regularity of $\beta, \beta\left(C_{0}\right)=0$. By virtue of the Eberlein-Šmulian Theorem, by (5.16) we deduce
that $C_{1}$ is relatively weakly sequentially compact, i.e. there exist $\left(u_{n_{k}}\right)_{k} \subset\left(u_{n}\right)_{n}$ and $u \in X$ such that $u_{n_{k}} \rightharpoonup u$. Now by (5.10), (5.14) and (5.12), thanks to F3., we have $u \in G(t)$. Moreover, being the sequence $\left(f_{n}\right)_{n}$ integrably bounded (see (5.11) and (5.9)), it has the property of equi-absolute continuity of the integral (also named uniformly integrability) and, obviously $f_{n}(t) \in G_{n}(t)$, a.e. $t \in J$ (see (5.12)).

Therefore, applying the Containment Theorem to the multimaps $G_{n}, G: J \rightarrow \mathcal{P}(X), n \in \mathbb{N}$, (see (5.12) and (5.13)), we can say that there exists $\left(f_{n_{k}}\right)_{k} \subset\left(f_{n}\right)_{n}$ such that

$$
f_{n_{k}} \rightharpoonup f \text { in } L^{1}(J ; X),
$$

where (see (5.13), F1. and (5.3))

$$
f(t) \in \overline{c o} G(t)=\overline{c o} F(t, q(t))=F(t, q(t)), \quad \text { a.e. } t \in J .
$$

Hence, we can conclude that

$$
\begin{equation*}
f \in S_{F(\cdot, q(\cdot))}^{1} \tag{5.17}
\end{equation*}
$$

By using the weak continuity of the Cauchy operator $G_{S}$ (see Proposition 4.1) we have $G_{S} f_{n_{k}} \rightharpoonup$ $G_{S} f$. Then, for every fixed $t \in J$ we have

$$
\begin{equation*}
G_{S} f_{n_{k}}(t) \rightharpoonup G_{S} f(t), \tag{5.18}
\end{equation*}
$$

and by hypothesis $g h 1$. and taking into account of the linearity and continuity of $S(t, 0)$ and $C(t, 0)$ we have

$$
C(t, 0) g\left(q_{n_{k}}\right) \rightharpoonup C(t, 0) g(q) \quad \text { and } \quad S(t, 0) h\left(q_{n_{k}}\right) \rightharpoonup S(t, 0) h(q) .
$$

So, by using (5.7) and (5.18), we can write

$$
\begin{equation*}
x_{n_{k}}(t) \rightharpoonup C(t, 0) g(q)+S(t, 0) h(q)+\int_{0}^{t} S(t, \xi) f(\tilde{\xi}) d \xi=: \tilde{x}(t) \tag{5.19}
\end{equation*}
$$

On the other hand, by (5.8)), we know that $x_{n_{k}} \rightharpoonup x$ in $\mathcal{C}(J ; X)$, hence $x_{n_{k}}(t) \rightharpoonup x(t)$, for all $t \in J$. From the uniqueness of the limit we have

$$
\begin{equation*}
x(t)=\tilde{x}(t), \quad t \in J . \tag{5.20}
\end{equation*}
$$

Finally, from (5.20), (5.19), (5.17) and (5.4) we deduce that $x \in T q$. Therefore we can conclude that $T$ has a weakly sequentially closed graph.
Step 2. There exists a subset of $\mathcal{C}(J ; X)$ which is invariant under the action of the operator $T$.
We will show that exists $\bar{p} \in \mathbb{N}$ such that the operator $T$ maps the ball $\bar{B}_{\mathcal{C}(J ; X)}(0, \bar{p})$ into itself.

Assume by contradiction that, for every $n \in \mathbb{N}$, there exists $q_{n} \in \mathcal{C}(J ; X)$, with $\left\|q_{n}\right\|_{\mathcal{C}(J ; X)} \leq$ $n$, such that there exists $x_{q_{n}} \in T q_{n},\left\|x_{q_{n}}\right\|_{\mathcal{C}(J ; X)}>n$.

Since $\left\|x_{q_{n}}\right\|_{\mathcal{C}(J ; X)}>n$, there exists $t_{n} \in J$ such that $\left\|x_{q_{n}}\left(t_{n}\right)\right\|_{X} \geq n$. Now, taking into account the p1. and p3. of Remark 3.2 we can write

$$
\begin{aligned}
n & \leq\left\|x_{q_{n}}\left(t_{n}\right)\right\|_{X} \leq\left\|C\left(t_{n}, 0\right) g\left(q_{n}\right)\right\|_{X}+\left\|S\left(t_{n}, 0\right) h\left(q_{n}\right)\right\|_{X}+\int_{0}^{t_{n}}\left\|S\left(t_{n}, \xi\right) f_{q_{n}}(\xi)\right\|_{X} d \xi \\
& \leq\left\|C\left(t_{n}, 0\right)\right\|_{\mathcal{L}(X)}\left\|g\left(q_{n}\right)\right\|_{X}+\left\|S\left(t_{n}, 0\right)\right\|_{\mathcal{L}(X)}\left\|h\left(q_{n}\right)\right\|_{X}+\int_{0}^{t_{n}}\left\|S\left(t_{n}, \xi\right)\right\|_{\mathcal{L}(X)}\left\|f_{q_{n}}(\xi)\right\|_{X} d \xi \\
& \leq K Q+K Q+K \int_{0}^{1}\left\|f_{q_{n}}(\xi)\right\|_{X} d \xi
\end{aligned}
$$

where $Q>0$ is such that $\|g(u)\|_{X} \leq Q,\|h(u)\|_{X} \leq Q$, for every $u \in \mathcal{C}(J ; X)$ (see gh3.) and $f_{q_{n}} \in S_{F\left(,, q_{n}(\cdot)\right)}^{1}$. Next, since $\left\|q_{n}\right\|_{\mathcal{C}(J ; X)}=\sup _{t \in J}\left\|q_{n}(t)\right\|_{X} \leq n$, there exists (see (5.2) of hypothesis F4.) a function $\varphi_{n} \in L_{+}^{1}(J)$ such that

$$
\left\|f_{q_{n}}(t)\right\|_{X} \leq \varphi_{n}(t), \quad \text { a.e. } t \in J,
$$

then we deduce

$$
\begin{equation*}
n \leq\left\|x_{q_{n}}\left(t_{n}\right)\right\|_{X} \leq 2 K Q+K \int_{0}^{1} \varphi_{n}(\xi) d \xi . \tag{5.21}
\end{equation*}
$$

Therefore, since (5.21) is true for every $n \in \mathbb{N}$, we have

$$
1 \leq \frac{2 K Q}{n}+\frac{K \int_{0}^{1} \varphi_{n}(\xi) d \xi}{n}, \quad \forall n \in \mathbb{N} .
$$

Hence, passing to the superior limit, by (5.1) we obtain the following contradiction

$$
1 \leq \limsup _{n \rightarrow \infty}\left(\frac{2 K Q}{n}+\frac{K \int_{0}^{1} \varphi_{n}(\xi) d \xi}{n}\right) \leq \limsup _{n \rightarrow \infty} \frac{K \int_{0}^{1} \varphi_{n}(\xi) d \xi}{n}<1
$$

Therefore we can conclude that there exists $\bar{p} \in \mathbb{N}$ such that $\bar{B}_{\mathcal{C}(J ; X)}(0, \bar{p})$ is invariant under the action of the operator $T$.
Step 3. There exists the smallest $(0, T)$-fundamental set which is weakly compact.
First of all, fixed $\bar{p}$ as in Step2., put $x_{0}=0$ and $K=\bar{B}_{\mathcal{C}(J ; X)}(0, \bar{p})$. We know that $K$ is a subset of the locally convex Hausdorff space $\mathcal{C}(J ; X)$ equipped with the weak topology. Since $T(K) \subset K$, we have $\overline{c o}(T(K) \cup\{0\}) \subset K$.

Therefore by Proposition 2.5, we can say that there exists the smallest ( $0, T$ )-fundamental set $M_{0}$ such that

$$
\begin{equation*}
M_{0} \subset \bar{B}_{\mathcal{C}(J ; X)}(0, \bar{p})=K, \tag{5.22}
\end{equation*}
$$

and

$$
\begin{equation*}
M_{0}=\overline{c o}\left(T\left(M_{0}\right) \cup\{0\}\right) . \tag{5.23}
\end{equation*}
$$

Now, we will prove that $M_{0}$ is weakly compact.
We consider the Sadovskij functional $\beta_{N}$, defined in (2.2), where $N \in \mathbb{R}^{+}$. Being $\beta_{N}$ 0 -stable (where 0 denotes the null function), we can write (see (5.23))

$$
\begin{equation*}
\beta_{N}\left(T\left(M_{0}\right)\right)=\beta_{N}\left(M_{0}\right), \tag{5.24}
\end{equation*}
$$

hence, since $\beta_{N}$ satisfies (I) and (II) of Remark 2.11, (5.24), (5.4) and gh4. imply

$$
\begin{align*}
\beta_{N}\left(M_{0}\right)= & \beta_{N}\left(\left\{C(\cdot, 0) g(u)+S(\cdot, 0) h(u)+G_{S} f: f \in S_{F(\cdot, u(\cdot))}^{1}, u \in M_{0}\right\}\right) \\
\leq & \beta_{N}\left(C(\cdot, 0) g\left(M_{0}\right)\right)+\beta_{N}\left(S(\cdot, 0) h\left(M_{0}\right)\right)+\beta_{N}\left(\left\{G_{S} f: f \in S_{F(\cdot, u(\cdot))}^{1}, u \in M_{0}\right\}\right) \\
= & \beta_{N}\left(\left\{G_{S} f: f \in S_{F(\cdot, u(\cdot)),}^{1} u \in M_{0}\right\}\right) \\
= & \sup _{\substack{C \subset S_{F\left(, M_{0}(\cdot)\right)}^{1} \\
\\
\\
\\
\text { countable }}} \beta\left(\left\{\int_{0}^{t} S(t, \xi) f(\xi) d \xi: f \in C\right\}\right) e^{-N t} . \tag{5.25}
\end{align*}
$$

Now, fixed $t \in J$ and a countable set $C \subset S_{F\left(\cdot, M_{0}(\cdot)\right)}^{1}$, we define

$$
C_{t}^{C}=\{S(t, \cdot) f(\cdot): f \in C\} .
$$

By using F4. and p3. of Remark 3.2 we can say that the countable set is integrably bounded, so it has the property of equi-absolute continuity of the integral. Now, since $X$ is a weakly compact generated Banach space, we are in the position to apply Proposition 2.9 of the Preliminaries to the countable set $C_{t}^{C}$, so we have

$$
\begin{equation*}
\beta\left(\left\{\int_{0}^{t} S(t, \xi) f(\xi) d \xi: f \in C\right\}\right) \leq \int_{0}^{t} \beta\left(C_{t}^{C}\right) d \xi, \quad t \in J \tag{5.26}
\end{equation*}
$$

so, by using (5.25), (5.26) and p3. of Remark $3.2(a=1)$, we can write

$$
\begin{align*}
& \beta_{N}\left(M_{0}\right) \leq \sup _{\substack{C \subset S_{F\left(, M_{0}(\cdot)\right)}^{1} \\
C \text { countable }}} \sup _{t \in J}\left(\int_{0}^{t} \beta\left(C_{t}^{C}\right) d \xi\right) e^{-N t} \\
& \leq \sup _{\substack{C \subset S_{F\left(,, M_{0}(\cdot)\right)}^{1} \\
C \text { countable }}} \sup _{t \in J}\left(\int_{0}^{t}\|S(t, \xi)\|_{\mathcal{L}(X)} \beta(C(\xi)) d \xi\right) e^{-N t} \\
& \leq \sup _{\substack{C \subset S_{F\left(, M_{0}(\cdot)\right.}^{1} \\
C \text { countable }}} \sup _{t \in J}\left(K \int_{0}^{t} \beta(C(\xi)) d \xi\right) e^{-N t} . \tag{5.27}
\end{align*}
$$

Further let us note that for every $f \in S_{F\left(\cdot, M_{0}(\cdot)\right)}^{1}$ we can consider, by the Axiom of Choice, a continuous map $q_{f} \in M_{0}$ such that $f(\xi) \in F\left(\xi, q_{f}(\xi)\right)$ a.e. $\xi \in J$. So the set $C_{0}^{C}=\left\{q_{f} \in M_{0}\right.$ : $f \in C\}$ is countable too. Now, taking into account of the numerability of $C$, there exists a null measure set $V \subset J: H \subset V$, where $H$ is the null measure set defined in $F 5$., such that

$$
f(\xi) \in F\left(\xi, q_{f}(\xi)\right) \text {, for every } \xi \in J \backslash V, f \in C \text {, }
$$

where $q_{f} \in C_{0}^{C}$.
Hence, fixed $\xi \in J \backslash V$, we observe that $C_{0}^{C}(\xi) \subset M_{0}(\xi) \subset \bar{B}_{X}(0, \bar{p})$ (see (5.22)) and $C(\xi) \subset$ $F\left(\xi, C_{0}^{C}(\xi)\right)$. By hypothesis $F 5$. we can write

$$
\beta(C(\xi)) \leq v_{\bar{p}}(\xi) \beta\left(C_{0}^{C}(\xi)\right)
$$

The above considerations allow to claim that, for every countable set $C \subset S_{F\left(\cdot, M_{0}(\cdot)\right)}^{1}$, there exists a countable subset $C_{0}^{C} \subset M_{0} \subset \bar{B}_{X}(0, \bar{p})$ such that

$$
\begin{equation*}
\beta(C(\xi)) \leq v_{\bar{p}}(\xi) \beta\left(C_{0}^{C}(\xi)\right) \leq v_{\bar{p}}(\xi) \sup _{\substack{C_{0} \subset M_{0} \\ C_{0} \text { countable }}} \beta\left(C_{0}(\xi)\right), \quad \text { a.e. } \xi \in J . \tag{5.28}
\end{equation*}
$$

Therefore, taking into account of (5.28), by (5.27) we deduce

$$
\begin{align*}
\beta_{N}\left(M_{0}\right) \leq & \sup _{\substack{C \subset S_{F\left(, M_{0}(\cdot)\right)}^{1} \\
C \text { countable }}} \sup _{t \in J}\left(K \int_{0}^{t} \beta(C(\xi)) d \xi\right) e^{-N t} \\
& \leq \sup _{\substack{C \subset S_{\left(,, M_{0}(\cdot)\right)}^{1} \\
C \text { countable }}} \sup _{t \in J}\left(K \int_{0}^{t} v_{\bar{p}}(\xi) \sup _{\substack{C_{0} \subset M_{0} \\
C_{0} \text { countable }}} \beta\left(C_{0}(\xi)\right) d \xi\right) e^{-N t} \\
& \leq \sup _{t \in J}\left(K \int_{0}^{t} e^{-N(t-\xi)} v_{\bar{p}}(\xi) \sup _{\substack{C_{0} \subset M_{0} \\
C_{0} \text { countable }}} \sup _{\tilde{\xi} \in J} e^{-N \xi} \beta\left(C_{0}(\xi)\right) d \xi\right) \\
= & \beta_{N}\left(M_{0}\right) \sup _{t \in J} \int_{0}^{t} K e^{-N(t-\xi)} v_{\bar{p}}(\xi) d \xi \tag{5.29}
\end{align*}
$$

By virtue of [7, Lemma 3.1] we can say that there exists $H \in \mathbb{N}$ such that

$$
\begin{equation*}
\sup _{t \in J} \int_{0}^{t} K e^{-H(t-\xi)} v_{\bar{p}}(\xi) d \xi<1 . \tag{5.30}
\end{equation*}
$$

Now, if we assume that $\beta_{H}\left(M_{0}\right)>0$, and we consider in (5.29) the constant $H$ characterized as in (5.30), we have the following contradiction

$$
\beta_{H}\left(M_{0}\right) \leq \beta_{H}\left(M_{0}\right) \sup _{t \in J} \int_{0}^{t} K e^{-H(t-\xi)} v_{\bar{p}}(\xi) d \xi<\beta_{H}\left(M_{0}\right)
$$

Therefore we conclude that this fact

$$
\begin{equation*}
\beta_{H}\left(M_{0}\right)=0 \tag{5.31}
\end{equation*}
$$

is true.
By definition of $\beta_{H}\left(M_{0}\right)$, first of all, we have that, for every $t \in J$, the set $M_{0}(t)$ is relatively weakly sequentially compact. Indeed, fixed $t \in J$ and a sequence $\left(q_{n}(t)\right)_{n}$ in $M_{0}(t)$, we consider the countable set $\tilde{C}(t)=\left\{q_{n}(t): n \in \mathbb{N}\right\}$. By (5.31) we can say that $\beta(\tilde{C}(t))=0$, so we deduce that $\tilde{C}(t)$ is relatively weakly compact. By the Eberlein-Šmulian Theorem we have that the set $\tilde{C}(t)$ is relatively weakly sequentially compact, i.e. there exists a subsequence $\left(q_{n_{k}}(t)\right)_{k}$ of $\left(q_{n}(t)\right)_{n}$ such that $q_{n_{k}}(t) \rightharpoonup q(t) \in X$. Therefore, by the arbitrariness of $\left(q_{n}(t)\right)_{n}$ we can conclude that $M_{0}(t)$ is relatively weakly sequentially compact.

Now, we show that also the set $S_{F\left(\cdot, M_{0}(\cdot)\right)}^{1}$ is relatively weakly compact in $L^{1}(J ; X)$.
To this aim we note that $S_{F\left(,, M_{0}(\cdot)\right)}^{1}$ is integrably bounded. Indeed, for every $f \in S_{F\left(\cdot, M_{0}(\cdot)\right)}^{1}$, taking into account of $M_{0} \subset \bar{B}_{\mathcal{C}(J ; X)}(0, \bar{p})$ (see (5.22)), we can write

$$
f(t) \in F\left(t, M_{0}(t)\right) \subset F\left(t, \bar{B}_{X}(0, \bar{p})\right), \quad \text { a.e. } t \in J .
$$

So, by F4. there exists $\varphi_{\bar{p}} \in L_{+}^{1}(J)$ such that

$$
\begin{equation*}
\|f(t)\|_{X} \leq \varphi_{\bar{p}}(t), \quad \text { a.e. } t \in J, \text { for every } f \in S_{F\left(\cdot, M_{0}(\cdot)\right)}^{1} \tag{5.32}
\end{equation*}
$$

Therefore $S_{F\left(\cdot, M_{0}(\cdot)\right)}^{1}$ is integrably bounded and then $S_{F\left(\cdot, M_{0}(\cdot)\right)}^{1}$ has the property of equi-absolute continuity of the integral (see Remark 2.3).

Moreover, by (5.32) we also deduce that $S_{F\left(,, M_{0}(\cdot)\right)}^{1}$ is bounded in $L^{1}(J ; X)$.
Now, we show that $S_{F\left(t, M_{0}(t)\right)}^{1}$ is relatively weakly compact in $X$, for a.e. $t \in J$.
Let us fix $t \in J \backslash H^{*}$, where $H^{*}$ is the null measure set for which F4. and F5. hold. First of all, we note that $S_{F\left(t, M_{0}(t)\right)}^{1}$ is norm bounded in $X$ by the constant $\varphi_{\bar{p}}(t)$. Indeed we have

$$
\|x\|_{X} \leq\left\|F\left(t, M_{0}(t)\right)\right\| \leq\left\|F\left(t, \bar{B}_{X}(0, \bar{p})\right)\right\| \leq \varphi_{\bar{p}}(t)
$$

for every $x \in S_{F\left(t, M_{0}(t)\right)}^{1}$.
Next, let us fix a sequence $\left(y_{n}\right)_{n}$, where $y_{n} \in S_{F\left(t, M_{0}(t)\right)}^{1}, n \in \mathbb{N}$. Then there exists a sequence $\left(f_{n}\right)_{n} \subset S_{F\left(, M_{0}(\cdot)\right)}^{1}$ such that

$$
y_{n}=f_{n}(t) \in F\left(t, M_{0}(t)\right) ;
$$

let us note that, for every $n \in \mathbb{N}$, there exists $q_{n} \in M_{0}(t)$ such that

$$
\begin{equation*}
y_{n} \in F\left(t, q_{n}\right) . \tag{5.33}
\end{equation*}
$$

Now, by considering the two countable sets $C_{0}=\left\{q_{n}: n \in \mathbb{N}\right\}$ and $C_{1}=\left\{y_{n}: n \in \mathbb{N}\right\}$ we have (see (5.22) and (5.33))

$$
C_{0} \subset \bar{B}_{X}(0, \bar{p}) \quad \text { and } \quad C_{1} \subset F\left(t, C_{0}\right)
$$

So, by F5. and recalling that $M_{0}(t)$ is relatively weakly compact we can write

$$
0 \leq \beta\left(C_{1}\right) \leq v_{\bar{p}}(t) \beta\left(C_{0}\right) \leq v_{\bar{p}}(t) \beta\left(M_{0}(t)\right)=0,
$$

so $\beta\left(C_{1}\right)=0$, i.e. $C_{1}$ is relatively weakly compact. Hence there exists a subsequence $\left(y_{n_{k}}\right)_{k} \subset$ $\left(y_{n}\right)_{n}$ such that $\left(y_{n_{k}}\right)_{k}$ is weakly convergent.

By the arbitrariness of $\left(y_{n}\right)_{n}$ in $S_{F\left(t, M_{0}(t)\right)}^{1}$ and taking into account the Eberlein-Šmulian Theorem, we can claim that $S_{F\left(t, M_{0}(t)\right)}^{1}$ is relatively weakly compact.

So, we are in the position to apply [29, Corollary 9], hence $S_{F\left(\cdot, M_{0}(\cdot)\right)}^{1}$ is relatively weakly compact in $L^{1}(J ; X)$.

Next we are able to prove that $T\left(M_{0}\right)$ is relatively weakly compact in $\mathcal{C}(J ; X)$.
To this aim we fix a sequence $\left(x_{n}\right)_{n}, x_{n} \in T\left(M_{0}\right)$. Then there exists $\left(p_{n}\right)_{n}, p_{n} \in M_{0}$ such that, for every $n \in \mathbb{N}, x_{n} \in T p_{n}$, hence

$$
\begin{equation*}
x_{n}(t)=C(t, 0) g\left(p_{n}\right)+S(t, 0) h\left(p_{n}\right)+\int_{0}^{t} S(t, \xi) f_{n}(\xi) d \xi, t \in J, \tag{5.34}
\end{equation*}
$$

where $f_{n} \in S_{F\left(\cdot, p_{n}(\cdot)\right)}^{1} \subset S_{F\left(\cdot, M_{0}(\cdot)\right)}^{1}$.
By the relative weak sequential compactness of $S_{F\left(\cdot, M_{0}(\cdot)\right)}^{1}$ in $L^{1}(J ; X)$ we can find a subsequence $\left(f_{n_{k}}\right)_{n_{k}}$ of $\left(f_{n}\right)_{n}$ such that $f_{n} \rightharpoonup f$ in $L^{1}(J ; X)$. By using Proposition 4.1, we have

$$
\begin{equation*}
G_{S} f_{n_{k}} \rightharpoonup G_{S} f \tag{5.35}
\end{equation*}
$$

Moreover, thanks to hypothesis $g h 2$., since $\left\{p_{n_{k}}: k \in \mathbb{N}\right\} \subset M_{0}$ is countable and bounded in $\mathcal{C}(J ; X)$ (see (5.22)), there exists a subsequence of $\left(p_{n_{k}}\right)_{k}$, w.l.o.g we name also $\left(p_{n_{k}}\right)_{k}$, such that

$$
\begin{equation*}
g\left(p_{n_{k}}\right) \rightarrow x \quad \text { and } \quad h\left(p_{n_{k}}\right) \rightarrow y \quad \text { in } X . \tag{5.36}
\end{equation*}
$$

Now, let us consider the subsequence $\left(x_{n_{k}}\right)_{k}$ of $\left(x_{n}\right)_{n}$ (see (5.34)).
For every linear and continuous functional $e^{\prime}: \mathcal{C}(J ; X) \rightarrow \mathbb{R}$, we can write

$$
e^{\prime}\left(x_{n_{k}}\right)=e^{\prime}\left(C(\cdot, 0) g\left(p_{n_{k}}\right)\right)+e^{\prime}\left(S(\cdot, 0) h\left(p_{n_{k}}\right)\right)+e^{\prime}\left(G_{S} f_{n_{k}}\right), \quad \forall n_{k} .
$$

Taking into account (5.35) and (5.36), passing to the limit for $k \rightarrow+\infty$, we obtain

$$
\lim _{k \rightarrow \infty} e^{\prime}\left(x_{n_{k}}\right)=e^{\prime}(C(\cdot, 0) x)+e^{\prime}(S(\cdot, 0) y)+e^{\prime}\left(G_{S} f\right)=e^{\prime}\left(C(\cdot, 0) x+S(\cdot, 0) y+G_{S} f\right)
$$

By definition of weak convergence we have

$$
x_{n_{k}} \rightharpoonup C(\cdot, 0) x+S(\cdot, 0) y+G_{S} f=: \bar{x}
$$

where $\bar{x} \in \mathcal{C}(J ; X)$, which means that $T\left(M_{0}\right)$ is relatively weakly sequentially compact and so, using again the Eberlein-Šmulian Theorem we can claim that $T\left(M_{0}\right)$ is relatively weakly compact.

Finally, recalling (5.23), we can conclude $M_{0}$ is weakly compact.
Step 4. Existence of a fixed point for $T$.

Finally we are in the position to apply Corollary 4.4 to the multimap $T_{\mid M_{0}}$. Hence the multioperator $T$ has a fixed point in $M_{0}$, i.e. there exists $x \in M_{0}$ such that

$$
x(t)=C(t, 0) g(x)+S(t, 0) h(x)+\int_{0}^{t} S(t, \xi) f(\xi) d \xi, \quad t \in J
$$

where $f \in S_{F(, x x \cdot)}^{1}$. Of course, $x$ is a mild solution for (P).
An immediate consequence of Theorem 5.1 is the following existence result for Cauchy problems.

Corollary 5.2. Let $X$ be a weakly compactly generated Banach space and $x_{0}, x_{1} \in X$. Under the assumptions (A), F1.-F5. of Theorem 5.1, there exists at least one mild solution for the Cauchy problem

$$
\left\{\begin{array}{l}
x^{\prime \prime}(t) \in A(t) x(t)+F(t, x(t)), t \in J  \tag{PC}\\
x(0)=x_{0} \\
x^{\prime}(0)=x_{1} .
\end{array}\right.
$$

Now, we propose the following existence result for reflexive Banach spaces. We note that in this proposition the assumptions F5. gh3. and gh4. of Theorem 5.1 are omitted. Let us note that the lack of these hypotheses implies that this result is a new one with respect to Theorem 5.1 since the reflexivity doesn't imply that these assumptions hold. For this reason it is necessary to modify in some points the proof of the previous existence result.

Theorem 5.3. Let $X$ be a reflexive Banach space and $\{A(t)\}_{t \in J}$ a family of operators which satisfies the property ( $\mathbf{A}$ ).

Let $F: J \times X \rightarrow \mathcal{P}(X)$ be a multimap satisfying F1, F2, F3, F4 and $g, h: \mathcal{C}(J ; X) \rightarrow X$ be two functions having the properties gh1 and gh2.

Then there exists at least one mild solution for the nonlocal problem ( P ).
Proof. First we note that if $N \subset J$ is null measure set such that (5.2) and F3. hold, fixed $t \in J \backslash N$ and $x \in X$, by (5.2) we deduce the boundedness of the set $F(t, x)$, therefore the reflexivity of the space $X$ imply the relative weak compactness of $F(t, x)$. Moreover, by F3. we have that the set $F(t, x)$ is weakly sequentially closed, so invoking Theorem 3 of [32] and $F 1$. we can claim that

$$
\begin{equation*}
F(t, x) \text { is closed, for a.e. } t \in J \text { and for every } x \in X . \tag{5.37}
\end{equation*}
$$

Let us consider the integral multioperator $T: \mathcal{C}(J ; X) \rightarrow \mathcal{P}_{c}(\mathcal{C}(J ; X))$ defined in (5.4) and (5.5).

First of all we have to prove that the multioperator $T$ is well defined, i.e. it has nonempty and convex values.

Let $\bar{u} \in \mathcal{C}(J ; X)$, by using the uniform continuity of $\bar{u}$ in $J$ we can construct a sequence $\left(u_{n}\right)_{n}, u_{n}: J \rightarrow X$, of step functions such that

$$
\begin{equation*}
\sup _{t \in J}\left\|u_{n}(t)-\bar{u}(t)\right\|_{X} \rightarrow 0, \quad \text { for } n \rightarrow \infty, \tag{5.38}
\end{equation*}
$$

then, for every $n \in \mathbb{N}$, by virtue of $F 2$., there exists a B-measurable function $f_{n}: J \rightarrow X$ such that

$$
\begin{equation*}
f_{n}(t) \in F\left(t, u_{n}(t)\right), \quad \text { a.e. } t \in J . \tag{5.39}
\end{equation*}
$$

Moreover, by (5.38), there exists $N_{\bar{u}} \in \mathbb{R}^{+}$such that

$$
\begin{equation*}
u_{n}(t), \bar{u}(t) \in \bar{B}_{X}\left(0, N_{\bar{u}}\right):=M_{\bar{u}}, \quad \text { for every } t \in J, n \in \mathbb{N}, \tag{5.40}
\end{equation*}
$$

so, by hypothesis $F 4$. and (5.39), we can claim

$$
\begin{equation*}
\left\|f_{n}(t)\right\|_{X} \leq\left\|F\left(t, \bar{B}_{X}\left(0, N_{\bar{u}}\right)\right)\right\| \leq \varphi_{N_{\bar{u}}}(t), \quad \text { a.e. } t \in J, \forall n \in \mathbb{N}, \tag{5.41}
\end{equation*}
$$

where $\varphi_{N_{\bar{u}}} \in L_{+}^{1}(J)$.
Therefore, since $f_{n}$ is B-measurable, by (5.41) we can deduce that $f_{n} \in L^{1}(J ; X)$, for every $n \in \mathbb{N}$.

Now, taking into account of (5.41), the set $A_{\bar{u}}=\left\{f_{n}: n \in \mathbb{N}\right\}$ is bounded in $L^{1}(J ; X)$ and it has the property of equi-absolute continuity of the integral. Moreover, by (5.41), $A_{\bar{u}}(t) \subset$ $\bar{B}_{X}\left(0, \varphi_{N_{\bar{u}}}(t)\right)$, a.e. $t \in J$. According to the reflexivity of the space and by [29, Corollary 9] we can conclude that the set $A_{\bar{u}}$ is relatively weakly compact in $L^{1}(J ; X)$. Therefore, there exists $\left(f_{n_{k}}\right)_{k}$, subsequence of $\left(f_{n}\right)_{n}$, such that

$$
f_{n_{k}} \rightharpoonup f_{\bar{u}} \in L^{1}(J ; X)
$$

Now, in order to obtain that $f_{\bar{u}} \in S_{F(\cdot \bar{u}(\cdot))}^{1}$, we want to prove that $f_{\bar{u}}(t) \in F(t, \bar{u}(t))$, a.e. $t \in J$.
Since $f_{n_{k}} \rightharpoonup f_{\bar{u}}$, by using Mazur's convexity theorem, there exists a sequence $\left(\tilde{f}_{n_{k}}\right)_{k}$ made up of convex combinations of the $f_{n_{k}}$ 's such that $\tilde{f}_{n_{k}} \rightarrow f_{\bar{u}}$ in $L^{1}(J ; X)$ and, up to a subsequence,

$$
\begin{equation*}
\tilde{f}_{n_{k p}}(t) \rightarrow f_{\bar{u}}(t), \quad \text { a.e. } t \in J . \tag{5.42}
\end{equation*}
$$

Now, put $H^{*}$ the null measure set for which hypothesis F3., (5.37), (5.39), (5.41) and (5.42) hold, by using respectively (5.37) and (5.41) we have

$$
\begin{gather*}
F(t, x) \text { is weakly closed, for every } x \in X, t \in J \backslash H^{*},  \tag{5.43}\\
\sup _{x \in \bar{B}_{X}\left(0, N_{\bar{u}}\right)}\|F(t, x)\| \leq \varphi_{N_{\bar{u}}}(t) \text {, for every } t \in J \backslash H^{*} . \tag{5.44}
\end{gather*}
$$

Next, we want to prove that fixed $\bar{t} \in J \backslash H^{*}$, the multimap $F_{\mid \bar{B}_{X}\left(0, N_{\bar{U}}\right)}\left(\bar{t}_{,} \cdot\right)$ is weakly upper semicontinuous and, in order to do that, we will show that all the hypotheses of [18, Theorem 1.1.5] are satisfied.

For every $x \in \bar{B}_{X}\left(0, N_{\bar{u}}\right)$ from (5.44), we can write $F_{\mid \bar{B}_{X}\left(0, N_{\bar{u}}\right)}(\bar{t}, x) \subset \bar{B}_{X}\left(0, \varphi_{N_{\bar{u}}}(\bar{t})\right)$, therefore, by the reflexivity of $X$, we can say (see (5.43)) that the set $F_{\mid \bar{B}_{X}\left(0, N_{\bar{U}}\right)}(\bar{t}, x)$ is weakly compact and the multimap $F_{\mid \bar{B}_{X}\left(0, N_{\bar{U}}\right)}(\bar{t}, \cdot)$ is weakly compact. Hence, recalling hypothesis F3., by [2, Corollary 3.2] we have that $F_{\mid \bar{B}_{X}\left(0, N_{\bar{u}}\right)}\left(\bar{t}_{,} \cdot\right)$ is a weakly closed multimap. Since all the hypotheses of [18, Theorem 1.1.5] are satisfied, $F_{\mid \bar{B}_{X}\left(0, N_{\bar{U}}\right)}\left(\bar{t}_{,} \cdot\right)$ is also weakly upper semicontinuous. Hence we can conclude that $F_{\bar{B}_{\underline{X}}\left(0, N_{\bar{u}}\right)}(t, \cdot)$ is weakly upper semicontinuous, for every $t \in J \backslash H^{*}$.

Now, let us fix again $\bar{t} \in J \backslash H^{*}$ and assume that absurdly $f_{\bar{u}}(\bar{t}) \notin F_{\mid \bar{B}_{X}\left(0, N_{\bar{u}}\right)}(\bar{t}, \bar{u}(\bar{t}))$.
We note that, thanks to F1. and (5.37), all the hypotheses of the Hahn-Banach Theorem are satisfied, so there exists a weakly open convex set $V \supset F_{\bar{B}_{X}\left(0, N_{\bar{u}}\right)}(\bar{t}, \bar{u}(\bar{t}))$ satisfying

$$
\begin{equation*}
f_{\bar{u}}(\bar{t}) \notin \bar{V}=\bar{V}^{w} . \tag{5.45}
\end{equation*}
$$

Taking into account of the weak upper semicontinuity of $F_{\mid \bar{B}_{X}\left(0, N_{\bar{U}}\right)}\left(\bar{t}_{,} \cdot\right)$, there exists a weak neighborhood $W_{\bar{u}(\bar{t})}$ of $\bar{u}(\bar{t})$ such that $\left.F_{\bar{B}_{X}\left(0, N_{\bar{u}}\right)}\left(\bar{t}, W_{\bar{u}(\bar{t})}\right)\right) \subset V$. Therefore

$$
\begin{equation*}
F(\bar{t}, x) \subset V, \quad \text { for every } x \in W_{\bar{u}(\bar{t})} \cap \bar{B}_{X}\left(0, N_{\bar{u}}\right) . \tag{5.46}
\end{equation*}
$$

Now, by (5.38) the subsequence $\left(u_{n_{k_{p}}}(\bar{t})\right)_{p}$, indexed as in (5.42), satisfies $u_{n_{k_{p}}}(\bar{t}) \rightharpoonup \bar{u}(\bar{t})$, so there exists $\bar{n} \in \mathbb{N}$ such that, for every $n_{k_{p}}>\bar{n}$ we have $u_{n_{k p}}(\bar{t}) \in W_{\bar{u}(\bar{t})}$, hence by (5.40) $u_{n_{k_{p}}}(\bar{t}) \in W_{\bar{u}(\bar{t})} \cap \bar{B}_{X}\left(0, N_{\bar{u}}\right)$.

Further, by (5.39) and (5.46) we deduce that $f_{n_{k p}}(\bar{t}) \in V$, for every $n_{k_{p}}>\bar{n}$.
Now, the convexity of $V$ implies that $\tilde{f}_{n_{k_{p}}}(\bar{t}) \in V$, for every $n_{k_{p}}>\bar{n}$ and, by the convergence of $\left(\tilde{f}_{n_{k p}}(\bar{t})\right)_{k}$ to $f_{\bar{u}}(\bar{t})$, we arrive to the contradictory conclusion $f_{\bar{u}}(\bar{t}) \in \bar{V}^{w}$ (see (5.45)). So we can conclude that $f_{\bar{u}}(t) \in F(t, \bar{u}(t))$ a.e. $t \in J$.

By recalling (5.5) and the fact that $f_{\bar{u}} \in L^{1}(J ; X)$ we finally obtain that $f_{\bar{u}} \in S_{F(\cdot \bar{u}(\cdot))}^{1}$, i.e. $S_{F(\cdot, \bar{\mu}(\cdot))}^{1} \neq \varnothing$.

Now, we consider the function $y_{\bar{u}}: J \rightarrow X$ defined by

$$
y_{\bar{u}}(t)=C(t, 0) g(\bar{u})+S(t, 0) h(\bar{u})+\int_{0}^{t} S(t, \xi) f_{\bar{u}}(\xi) d \xi, \quad t \in J .
$$

It is easy to prove that $y_{\bar{u}}$ is well posed and continuous in $J$, so $y_{\bar{u}} \in T \bar{u}$, i.e. $T \bar{u} \neq \varnothing$. Clearly, $T \bar{u}$ is convex.

We can conclude that the integral multioperator $T$ assumes values in $\mathcal{P}_{c}(\mathcal{C}(J ; X))$.
Form now on we proceed by steps.
Step 1. The multioperator $T$ has a weakly sequentially closed graph.
As in Step 1 of Theorem 5.1 we fix two sequences $\left(q_{n}\right)_{n}$ and $\left(x_{n}\right)_{n}$ in $\mathcal{C}(J ; X)$ with the properties (5.7) and (5.8).

Using analogous considerations of Step 1 of Theorem 5.1 we can say that (5.9) and (5.10) hold, so for every $n \in \mathbb{N}$, by (5.7) there exists (see (5.5))

$$
\begin{equation*}
f_{n} \in S_{F\left(, q_{n}(\cdot)\right)}^{1} \tag{5.47}
\end{equation*}
$$

such that (see (5.4))

$$
x_{n}(t)=C(t, 0) g\left(q_{n}\right)+S(t, 0) h\left(q_{n}\right)+\int_{0}^{t} S(t, \xi) f_{n}(\xi) d \xi, \quad t \in J
$$

Now we want to prove that, put $A=\left\{f_{n}: n \in \mathbb{N}\right\}$ (see (5.47)), $A$ satisfies all the hypotheses of [29, Corollary 9]. Obviously $A$ is a subset of $L^{1}(J ; X)$.

Moreover, by (5.47) and (5.9) we deduce

$$
\begin{equation*}
f_{n}(t) \in F\left(t, q_{n}(t)\right) \subset F\left(t, \bar{B}_{X}(0, \bar{n})\right), \quad \text { a.e. } t \in J, \forall n \in \mathbb{N} . \tag{5.48}
\end{equation*}
$$

Now, put $H$ the null measure set for which F4. and (5.48) hold, we have that there exists $\varphi_{\bar{n}} \in L_{+}^{1}(J)$ such that (see (5.2))

$$
\begin{equation*}
\left\|f_{n}(t)\right\|_{X} \leq \varphi_{\bar{n}}(t), \quad \forall t \in J \backslash H, \forall n \in \mathbb{N} \tag{5.49}
\end{equation*}
$$

that implies

$$
\begin{equation*}
\left\|f_{n}\right\|_{L^{1}(J ; X)} \leq\left\|\varphi_{\bar{n}}\right\|_{1}, \quad \forall n \in \mathbb{N} \tag{5.50}
\end{equation*}
$$

i.e. the set $A$ is bounded in $L^{1}(J ; X)$. Then, by (5.50) we also say that $A$ has the property of equi-absolute continuity of the integral (see Remark 2.3).

Now, by using (5.50) and the reflexivity of $X$ we can also say that $A(t)$ is relatively weakly compact a.e. $t \in J$. Hence, since also (5.49) is true, thanks to [29, Corollary 9], we can conclude
that $A$ is relatively weakly compact in $L^{1}(J ; X)$. So there exists a subsequence $\left(f_{n_{k}}\right)_{k}$ of $\left(f_{n}\right)_{n}$ such that $f_{n_{k}} \rightharpoonup f \in L^{1}(J ; X)$, then by using again the Mazur's convexity theorem and analogous arguments presented in the previous part of the proof we can claim that $f(t) \in F(t, q(t))$, a.e. $t \in J$. Therefore we can say that (see (5.5)) $f \in S_{F(, q(\cdot))}^{1}$.

Now, by using gh1. and the same technique of the final part of Step $\mathbf{1}$ of Theorem 5.1 we can obtain that $x \in T q$.

Therefore we can conclude that $T$ has a weakly sequentially closed graph.
Step 2. There exists a subset of $\mathcal{C}(J ; X)$ which is invariant under the action of the operator $T$.
We omit this step of the proof, since it is identical to Step 2 of the proof of Theorem 5.1. So, we can say that there exists $\bar{p} \in \mathbb{N}$ such that $\bar{B}_{\mathcal{C}(J ; X)}(0, \bar{p})$ is invariant under the action of the operator $T$.

Step 3. There exists the smallest $(0, T)$-fundamental set which is weakly compact.
First of all, fixed $\bar{p}$ as in Step 2, by Proposition 2.5, put $x_{0}=0$ and $K=\bar{B}_{\mathcal{C}(J ; X)}(0, \bar{p})$ a subset of the locally convex Hausdorff space $\mathcal{C}(J ; X)$ equipped with the weak topology, we can say that there exists

$$
\begin{equation*}
M_{0} \subset \bar{B}_{\mathcal{C}(j ; X)}(0, \bar{p})=K \tag{5.51}
\end{equation*}
$$

such that

$$
\begin{equation*}
M_{0}=\overline{c o}\left(T\left(M_{0}\right) \cup\{0\}\right) \tag{5.52}
\end{equation*}
$$

Now, we will prove that $M_{0}$ is weakly compact. To this end we establish that the set $T\left(M_{0}\right)$ is relatively weakly compact.

Let $\left(q_{n}\right)_{n}$ be a sequence in $M_{0}$ and $\left(x_{n}\right)_{n}$ be a sequence in $\mathcal{C}(J ; X)$ such that $x_{n} \in T q_{n}$, for every $n \in \mathbb{N}$. Now, by definition of the multioperator $T$, there exists a sequence $\left(f_{n}\right)_{n}$, $f_{n} \in S_{F\left(\cdot, q_{n}(\cdot)\right)}^{1}$, such that

$$
x_{n}(t)=C(t, 0) g\left(q_{n}\right)+S(t, 0) h\left(q_{n}\right)+\int_{0}^{t} S(t, \xi) f_{n}(\xi) d \xi, \quad t \in J
$$

Next, put $A=\left\{f_{n}: n \in \mathbb{N}\right\}$, reasoning as in Step 1 of this proof, we can show that $A$ is bounded in $L^{1}(J ; X)$, it has the property of equi-absolute continuity of the integral and, by using the reflexivity of $X$, we can say that $A(t)$ is relatively weakly compact, for a.e. $t \in J$. Therefore, thanks again to [29, Corollary 9] we can say that $A$ is relatively weakly compact in $L^{1}(J ; X)$, so there exists $\left(f_{n_{k}}\right)_{k}$ subsequence of $\left(f_{n}\right)_{n}$ such that $f_{n_{k}} \rightharpoonup f \in L^{1}(J ; X)$.

Now, by the weak sequential continuity of $G_{S}$ (see Proposition 4.1), we can write

$$
\begin{equation*}
G_{S} f_{n_{k}} \rightharpoonup G_{S} f . \tag{5.53}
\end{equation*}
$$

Moreover, thanks to hypothesis $g h 2$. , since $\left\{q_{n_{k}}: k \in \mathbb{N}\right\} \subset M_{0}$ is countable and bounded (see (5.51)), there exists a subsequence of $\left(q_{n_{k}}\right)_{k}$ w.l.o.g. named again $\left(q_{n_{k}}\right)_{k}$, such that

$$
\begin{equation*}
g\left(q_{n_{k}}\right)_{k} \rightarrow x \quad \text { and } \quad h\left(q_{n_{k}}\right)_{k} \rightarrow y \quad \text { in } X . \tag{5.54}
\end{equation*}
$$

Now, by (5.53) and (5.54) the subsequence $\left(x_{n_{k}}\right)_{k}$ of $\left(x_{n}\right)_{n}$ weakly converges to $\bar{x}=C(\cdot, 0) x+$ $S(\cdot, 0) y+G_{S} f \in \mathcal{C}(J ; X)$. Therefore $T\left(M_{0}\right)$ is relatively weakly compact and, invoking (5.52), $M_{0}$ is weakly compact.

Finally, reasoning as in Step 4 of Theorem 5.1 we can conclude that there exists at least one mild solution for (P).

We can immediately formulate the following consequence of Theorem 5.3 for Cauchy problems.

Corollary 5.4. Let $X$ be a reflexive Banach space and $x_{0}, x_{1} \in X$. Under the assumptions (A), F1.-F4. of Theorem 5.3, there exists at least one mild solution for the Cauchy problem (PC).

Remark 5.5. Let us note that, if $J=[0, a]$, all the results of Sections 4 and 5 hold too. In particular, Theorem 5.1, Theorem 5.3 and their respectively corollaries continue to be true if we modify (5.1) by

$$
\limsup _{n \rightarrow \infty} \frac{\int_{0}^{a} \varphi_{n}(\xi) d \xi}{n}<\frac{1}{K a} .
$$

## 6 An application

In this section we apply the theory developed in Section 5 to study the following controllability problem

$$
\left\{\begin{array}{l}
\frac{\partial^{2} w}{\partial t^{2}}(t, \xi)=\frac{\partial^{2} w}{\partial \xi^{2}}(t, \xi)+b(t) \frac{\partial w}{\partial \xi}(t, \xi)+T(t) w(t, \cdot)(\xi)+u(t, \xi)  \tag{6.1}\\
w(t, 0)=w(t, 2 \pi), \quad t \in J \\
\frac{\partial w}{\partial \xi}(t, 0)=\frac{\partial w}{\partial \xi}(t, 2 \pi), \quad t \in J \\
w(0, \xi)=x_{0}, \quad \xi \in \mathbb{R} \\
\frac{\partial w}{\partial t}(0, \xi)=x_{1}, \quad \xi \in \mathbb{R} \\
\|u(t, \xi)\| \mathbf{C} \in\left[f_{1}\left(t, \xi, \int_{0}^{2 \pi} k_{1}(t, \theta) w(t, \theta) d \theta\right), f_{2}\left(t, \xi, \int_{0}^{2 \pi} k_{2}(t, \theta) w(t, \theta) d \theta\right)\right]
\end{array}\right.
$$

where $x_{0}, x_{1} \in \mathbb{C}$ and $b: J \rightarrow \mathbb{R}, k_{i}: J \times \mathbb{R} \rightarrow \mathbb{R}, f_{i}: J \times \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{R}_{0}^{+}, i=1,2,\{T(t)\}_{t \in J}$ is a suitable family of operators.

First of all, as in [16], we will use the identification between functions defined on the quotient group $\mathbb{T}=\mathbb{R} / 2 \pi \mathbb{Z}$ with values in $\mathbb{C}$ and $2 \pi$-periodic functions from $\mathbb{R}$ to $\mathbb{C}$. In order to model the problem above in an abstract form we consider the space $X=L^{2}(\mathbb{T}, \mathbb{C})$, i.e. the space of all functions $x: \mathbb{R} \rightarrow \mathbb{C}, 2 \pi$-periodic and 2-integrable in $[0,2 \pi]$, endowed with the usual norm $\|\cdot\|_{L^{2}(\mathbb{T}, \mathbb{C})}$. Moreover we denote by $H^{1}(\mathbb{T}, \mathbb{C})$ and by $H^{2}(\mathbb{T}, \mathbb{C})$ respectively the following Sobolev subspaces of $L^{2}(\mathbb{T}, \mathbb{C})$

$$
\begin{aligned}
& H^{1}(\mathbb{T}, \mathbb{C})=\left\{x \in L^{2}(\mathbb{T}, \mathbb{C}): \frac{d x}{d \xi^{z}} \in L^{2}(\mathbb{T}, \mathbb{C})\right\} \\
& H^{2}(\mathbb{T}, \mathbb{C})=\left\{x \in L^{2}(\mathbb{T}, \mathbb{C}): \frac{d x}{d \tilde{\zeta}^{\prime}}, \frac{d^{2} x}{d \tilde{\zeta}^{2}} \in L^{2}(\mathbb{T}, \mathbb{C})\right\}
\end{aligned}
$$

where $\frac{d x}{d \xi^{\prime}}, \frac{d^{2} x}{d \xi^{2}}$ denote the weak derivatives.
Further we consider the operator $A_{0}: D\left(A_{0}\right)=H^{2}(\mathbb{T}, \mathbb{C}) \rightarrow L^{2}(\mathbb{T}, \mathbb{C})$ defined by

$$
A_{0} x=\frac{d^{2} x}{d \xi^{2}}, \quad x \in H^{2}(\mathbb{T}, \mathbb{C})
$$

and we assume that the operator $A_{0}$ is the infinitesimal generator of a strongly continuous cosine family $\left\{C_{0}(t)\right\}_{t \in \mathbb{R}}$, where $C_{0}(t): L^{2}(\mathbb{T}, \mathbb{C}) \rightarrow L^{2}(\mathbb{T}, \mathbb{C})$, for every $t \in \mathbb{R}$ (see [16]).

Then we fix the function $P: J \rightarrow \mathcal{L}\left(H^{1}(\mathbb{T}, \mathbb{C}), L^{2}(\mathbb{T}, \mathbb{C})\right)$ defined as

$$
P(t) x=b(t) \frac{d x}{d \xi^{\prime}}, \quad t \in J, x \in H^{1}(\mathbb{T}, \mathbb{C})
$$

where $b: J \rightarrow \mathbb{R}$ is of class $\mathcal{C}^{1}$ on $J$.
Now we are able to define the family $\{A(t): t \in J\}$ where, for every $t \in J, A(t): D(A)=$ $H^{2}(\mathbb{T}, \mathbb{C}) \rightarrow L^{2}(\mathbb{T}, \mathbb{C})$ is an operator defined as

$$
A(t):=A_{0}+P(t), \quad t \in J .
$$

Let us note that, as the Authors of [16] say (see Lemma 4.1), the family $\{A(t): t \in J\}$ generates a fundamental system $\{S(t, s)\}_{(t, s) \in J \times J}$. In the sequel, we denote with $K$ the constant, linked to $\{S(t, s)\}_{(t, s) \in J \times J}$, satisfying the properties of Remark 3.2.

In what follows we revise functions $w, u: J \times \mathbb{R} \rightarrow \mathbb{C}$ such that $w(t, \cdot) \in H^{2}(\mathbb{T}, \mathbb{C})$ and $u(t, \cdot) \in L^{2}(\mathbb{T}, \mathbb{C})$, for every $t \in J$, as the maps $x: J \rightarrow H^{2}(\mathbb{T}, \mathbb{C}), v: J \rightarrow L^{2}(\mathbb{T}, \mathbb{C})$, respectively, are defined by

$$
\begin{array}{ll}
x(t)(\xi)=w(t, \xi), & t \in J, \xi \in \mathbb{R} \\
v(t)(\xi)=u(t, \xi), & \\
t \in J, \xi \in \mathbb{R} .
\end{array}
$$

Moreover we construct, by using the family $\{T(t)\}_{t \in J}$ and the functions $f_{1}, f_{2}$, a suitable multimap $F$ such that we can rewrite the problem (6.1) in the abstract form

$$
\left\{\begin{array}{l}
\frac{d^{2} x}{d t^{2}}(t) \in A_{0} x(t)+P(t) x(t)+F(t, x(t))=A(t) x(t)+F(t, x(t)), t \in J  \tag{6.2}\\
x(0)=\tilde{x_{0}} \\
\frac{d x}{d t}(0)=\tilde{x_{1}}
\end{array}\right.
$$

where $\tilde{x_{0}}, \tilde{x_{1}}: \mathbb{R} \rightarrow \mathbb{C}$ are functions of $L^{2}(\mathbb{T}, \mathbb{C})$ respectively defined $\tilde{x_{0}}(\xi)=x_{0}, \tilde{x_{1}}(\xi)=x_{1}$, for every $\xi \in \mathbb{R}$.

Let us note that, since we settle for proving the existence of a mild solution (therefore the existence of derivatives is not necessary) it is sufficient to consider that $w(t, \cdot) \in L^{2}(\mathbb{T}, \mathbb{C})$ instead of $w(t, \cdot) \in H^{2}(\mathbb{T}, \mathbb{C})$.

Hence, in order to apply our Corollary 5.4 we consider $X=L^{2}(\mathbb{T}, \mathbb{C})$ and we assume the following properties on the family of operators $\{T(t)\}_{t \in J}$ and the functions $k_{i}, f_{i}, i=1,2$
(T) for every $t \in J, T(t): L^{2}(\mathbb{T}, \mathbb{C}) \rightarrow L^{2}(\mathbb{T}, \mathbb{C})$ is linear, bounded and, for every $y \in$ $L^{2}(\mathbb{T}, \mathbb{C}), T(\cdot) y$ is B-measurable and $\|T(\cdot)\|_{\mathcal{L}\left(L^{2}(\mathbb{T}, \mathbb{C})\right)} \in L_{+}^{1}(J) ;$
(k) $k_{i}(t, \cdot) \in L^{2}(\mathbb{T})$, for every $t \in J, i=1,2$;
(f1) $f_{1}, f_{2}: J \times \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{R}_{0}^{+}$are $2 \pi$ - periodic functions with respect to the second variable, such that
(1) for every $(t, \xi, w) \in J \times \mathbb{R} \times \mathbb{C}, f_{1}(t, \xi, w) \leq f_{2}(t, \xi, w)$;
(2) for each $y \in L^{2}(\mathbb{T}, \mathbb{C})$ there exists a B-measurable map $z_{y}: J \times \mathbb{R} \rightarrow \mathbb{C}$ such that

$$
\left\|z_{y}(t, \xi)\right\|_{\mathrm{C}} \in\left[f_{1}\left(t, \xi, \int_{0}^{2 \pi} k_{1}(t, \theta) y(\theta) d \theta\right), f_{2}\left(t, \xi, \int_{0}^{2 \pi} k_{2}(t, \theta) y(\theta) d \theta\right)\right],
$$

for every $t \in J$ and for a.e. $\xi \in \mathbb{R}$;
(f2) for a.e. $t \in J$ and for a.e. $\xi \in \mathbb{R}, f_{1}(t, \xi, \cdot)$ is lower semicontinuous and $f_{2}(t, \xi, \cdot)$ is upper semicontinuous, i.e.

$$
f_{1}(t, \xi, s) \leq \liminf _{w \rightarrow s} f_{1}(t, \xi, w) \quad f_{2}(t, \xi, s) \geq \limsup _{w \rightarrow s} f_{2}(t, \xi, w),
$$

for every $s \in \mathbb{C}$;
(f3) there exists $\varphi \in L_{+}^{1}(J), K \int_{0}^{1}\left(\varphi(\theta)+\|T(\theta)\|_{\mathcal{L}\left(L^{2}(\mathbb{T}, \mathrm{C})\right.}\right) d \theta<1$, such that, for every $t \in J$ and each $r>0$, there exists $\psi_{t, r} \in L_{+}^{2}([0,2 \pi])$, with

$$
\sup _{\|s\|_{C} \leq r\left\|k_{2}(t,)\right\|_{L^{2}(\mathbb{T})}} f_{2}(t, \xi, s) \leq \psi_{t, r}(\xi) \text {, a.e. } \xi \in[0,2 \pi] \text {, }
$$

such that

$$
\begin{equation*}
\left\|\psi_{t, r}\right\|_{L^{2}([0,2 \pi])} \leq r \varphi(t) \tag{6.3}
\end{equation*}
$$

Now we define the function $g: J \times L^{2}(\mathbb{T}, \mathbb{C}) \rightarrow L^{2}(\mathbb{T}, \mathbb{C})$ such that

$$
\begin{equation*}
g(t, y)(\xi)=(T(t) y)(\xi), \xi \in \mathbb{R},(t, y) \in J \times L^{2}(\mathbb{T}, \mathbb{C}) . \tag{6.4}
\end{equation*}
$$

Recalling that $T(t)$ assumes values in $L^{2}(\mathbb{T}, \mathbb{C})$, we have that $g$ is obviously well posed.
Next we consider the multimap $U: J \times L^{2}(\mathbb{T}, \mathbb{C}) \rightarrow \mathcal{P}\left(L^{2}(\mathbb{T}, \mathbb{C})\right)$, defined for every $t \in J$ and $y \in L^{2}(\mathbb{T}, \mathbb{C})$ by

$$
\begin{align*}
U(t, y)=\{ & z \in L^{2}(\mathbb{T}, \mathbb{C}): \\
& f_{1}\left(t, \xi, \int_{0}^{2 \pi} k_{1}(t, \theta) y(\theta) d \theta\right) \leq\|z(\xi)\|_{\mathbb{C}} \leq f_{2}\left(t, \xi, \int_{0}^{2 \pi} k_{2}(t, \theta) y(\theta) d \theta\right), \\
& \text { a.e. } \xi \in \mathbb{R}\} . \tag{6.5}
\end{align*}
$$

Let us show that the multimap $U$ assumes non empty values.
First of all, fixed $y \in L^{2}(\mathbb{T}, \mathbb{C})$, we consider the B-measurable map $z_{y}: J \times \mathbb{R} \rightarrow \mathbb{C}$ characterized in (2) of (f1). Fixed $t \in J$, by the B-measurability of $z_{y}$, we can claim that $z_{y}(t, \cdot)$ is also B-measurable. Moreover $\left\|z_{y}(t, \cdot)\right\|_{\mathrm{C}} \in L^{2}(\mathbb{T})$, indeed taking into account of (f1)(2) we have

$$
\begin{equation*}
\left\|z_{y}(t, \xi)\right\|_{\mathbf{C}} \leq f_{2}\left(t, \xi \int_{0}^{2 \pi} k_{2}(t, \theta) y(\theta) d \theta\right), \quad \text { a.e. } \xi \in \mathbb{R} \tag{6.6}
\end{equation*}
$$

Now, by hypothesis (k), we have $\left\|\int_{0}^{2 \pi} k_{2}(t, \theta) y(\theta) d \theta\right\|_{\mathbb{C}} \leq\left\|k_{2}(t, \cdot)\right\|_{L^{2}(\mathbb{T})}\|y\|_{L^{2}(\mathbb{T}, \mathbb{C})}$, therefore, put $r=\|y\|_{L^{2}(\mathbb{T}, \mathrm{C})}$, by (f3) and (6.6) there exists $\psi_{t, r} \in L_{+}^{2}([0,2 \pi])$ such that $\left\|z_{y}(t, \xi)\right\|_{\mathrm{C}} \leq$ $\psi_{t, r}(\xi)$, a.e. $\xi \in \mathbb{R}$. Hence $\left\|z_{y}(t, \cdot)\right\|_{\mathcal{C}} \in L^{2}(\mathbb{T})$ and so

$$
\begin{equation*}
z_{y}(t, \cdot) \in L^{2}(\mathbb{T}, \mathbb{C}) . \tag{6.7}
\end{equation*}
$$

Finally, using again hypothesis (f1)(2) and by (6.7) we conclude that $z_{y}(t, \cdot) \in U(t, y)$ (see (6.5)), so $U(t, y)$ is non empty.

We are in the position to define the multimap $F: J \times L^{2}(\mathbb{T}, \mathbb{C}) \rightarrow \mathcal{P}\left(L^{2}(\mathbb{T}, \mathbb{C})\right)$ as (see (6.4) and (6.5))

$$
\begin{equation*}
F(t, y)=\{T(t) y+v, v \in U(t, y)\}, \quad t \in J, y \in L^{2}(\mathbb{T}, \mathbb{C}) . \tag{6.8}
\end{equation*}
$$

Since the operator $T(t)$ assumes values in $L^{2}(\mathbb{T}, \mathbb{C}), F$ is obviously well defined.
Now we want to show that we can apply Corollary 5.4 to the problem (6.2).
First of all we note that $X=L^{2}(\mathbb{T}, \mathbb{C})$ is obviously a reflexive Banach space. Moreover hypothesis (A) is clearly true because of our construction of the family $\{A(t): t \in J\}$.

Now, let us show that hypotheses F1.- F4. are satisfied.
First of all, since $U$ has convex values, we can say that $F$ takes convex values too, i.e. F1. of our Corollary 5.4 holds.

Next, we prove that, fixed $y \in L^{2}(\mathbb{T}, \mathbb{C})$, the multimap $F(\cdot, y)$ has a B-selection.
By the previous arguments we can say that the function $z_{y}: J \times \mathbb{R} \rightarrow \mathbb{C}$ characterized in (2) of (f1) is such that, for every $t \in J$,

$$
\begin{equation*}
z_{y}(t, \cdot) \in U(t, y) . \tag{6.9}
\end{equation*}
$$

Now, taking into account that the function $z_{y}: J \times \mathbb{R} \rightarrow \mathbb{C}$ is B-measurable with $z_{y}(t, \cdot) \in$ $L^{2}(\mathbb{T}, \mathbb{C})$, for every $t \in J$, from Theorem 2.4 , we can say that the following abstract function

$$
\begin{equation*}
\hat{z}_{y}: J \rightarrow L^{2}(\mathbb{T}, \mathbb{C}) \text {, defined by } \hat{z}_{y}(t)=z_{y}(t, \cdot) \text {, is B-measurable. } \tag{6.10}
\end{equation*}
$$

Next, we define $p_{y}: J \rightarrow L^{2}(\mathbb{T}, \mathbb{C})$ as

$$
p_{y}(t)=T(t) y+\hat{z}_{y}(t), \quad t \in J .
$$

By using (6.10) and hypothesis (T) we have that $p_{y}$ is obviously well posed and B-measurable.
Moreover, as a consequence of (6.9) and (6.8) we can write that $p_{y}(t) \in F(t, y)$, for every $t \in J$.

Therefore, for every $y \in L^{2}(\mathbb{T}, \mathbb{C})$, $p_{y}$ is a B-selection of $F(\cdot, y)$, i.e. hypothesis $F 2$. of our Corollary 5.4 holds.

Now, let us show that also hypothesis F3. is satisfied.
Let $N \subset J$ be the null measure set for which hypothesis (f2) holds, $t \in J \backslash N$ and $\left(y_{n}\right)_{n}$ and $\left(q_{n}\right)_{n}$ be two sequences in $L^{2}(\mathbb{T}, \mathbb{C})$ such that $y_{n} \rightharpoonup y, y \in L^{2}(\mathbb{T}, \mathbb{C}), q_{n} \rightharpoonup q, q \in L^{2}(\mathbb{T}, \mathbb{C})$ and $q_{n} \in F\left(t, y_{n}\right), \forall n \in \mathbb{N}$, i.e. $q_{n}=T(t) y_{n}+v_{n}$, where $v_{n} \in U\left(t, y_{n}\right)$, for every $n \in \mathbb{N}$.

Now, if we consider

$$
\begin{equation*}
v_{n}=q_{n}-T(t) y_{n}, \quad \forall n \in \mathbb{N} \tag{6.11}
\end{equation*}
$$

taking into account of hypothesis (T) and the weak convergence of $\left(y_{n}\right)_{n}$ and $\left(q_{n}\right)_{n}$ we have

$$
\begin{equation*}
v_{n} \rightharpoonup q-T(t) y=: v, \tag{6.12}
\end{equation*}
$$

i.e. $q=T(t) y+v$, where $v \in L^{2}(\mathbb{T}, \mathbb{C})$.

Further, from (6.12), for every $\xi \in \mathbb{R}$, we can write

$$
\begin{equation*}
v_{n}(\xi) \rightharpoonup v(\xi) . \tag{6.13}
\end{equation*}
$$

In order to prove that $q \in F(t, y)$, we establish that $v \in U(t, y)$ (see (6.8)).
First of all, in the sequel we consider for $i=1,2$ and for every $t \in J$, the linear and bounded operator $l_{i}^{t}: L^{2}(\mathbb{T}, \mathbb{C}) \rightarrow \mathbb{C}$ defined by

$$
l_{i}^{t}(y)=\int_{0}^{2 \pi} k_{i}(t, \theta) y(\theta) d \theta
$$

for every $y \in L^{2}(\mathbb{T}, \mathbb{C})$.
Taking into account of the weak convergence of $\left(y_{n}\right)_{n}$ we have $\lim _{n \rightarrow \infty} l_{i}^{t}\left(y_{n}\right)=l_{i}^{t}(y)$, $i=1,2$.

Now, we consider the multimaps $G_{n}^{t}: \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}), n \in \mathbb{N}$, and $G^{t}: \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ respectively defined by

$$
\begin{align*}
G_{n}^{t}(\xi) & =\left[f_{1}\left(t, \xi, l_{1}^{t}\left(y_{n}\right)\right), f_{2}\left(t, \xi, l_{2}^{t}\left(y_{n}\right)\right)\right]  \tag{6.14}\\
G^{t}(\xi) & =\left[f_{1}\left(t, \xi, l_{1}^{t}(y)\right), f_{2}\left(t, \xi, l_{2}^{t}(y)\right)\right],
\end{align*}
$$

for every $\xi \in \mathbb{R}$.
Moreover, let us fix a null measure set $H_{t} \subset \mathbb{R}$ for which hypotheses (f2) and (f3) and (see (6.11) and (6.5)) $\left\|v_{n}(\xi)\right\|_{\mathrm{C}} \in G_{n}^{t}(\xi)$ hold, for every $\xi \in \mathbb{R} \backslash H_{t}$.

Let us note that, in order to apply the Containment Theorem (see Theorem 2.6), since $f_{1}, f_{2}$ are $2 \pi$-period functions with respect to the second variable, we can assume without loss of generality that $G_{n}^{t}$ and $G^{t}$ are defined on $[0,2 \pi]$.

Now, fixed $\xi \in[0,2 \pi] \backslash\left(H_{t} \cap[0,2 \pi]\right)$, we consider an arbitrary sequence $\left(u_{n}\right)_{n}$ such that $u_{n} \in G_{n}^{t}(\xi)$, for all $n \in \mathbb{N}$, i.e.

$$
\begin{equation*}
f_{1}\left(t, \xi, l_{1}^{t}\left(y_{n}\right)\right) \leq u_{n} \leq f_{2}\left(t, \xi, l_{2}^{t}\left(y_{n}\right)\right), \quad \forall n \in \mathbb{N} . \tag{6.15}
\end{equation*}
$$

Next, by the strong convergence of $\left(l_{2}^{t}\left(y_{n}\right)\right)_{n}$, there exists $\hat{r}>0$ such that $\left\|l_{2}^{t}\left(y_{n}\right)\right\|_{C} \leq$ $\hat{r}, \forall n \in \mathbb{N}$. Hence, taking into account that $f_{1}$ is a nonnegative function, fixed $r>0$ such that $r\left\|k_{2}(t, \cdot)\right\|_{L^{2}(\mathbb{T})}=\hat{r}$, by hypothesis (f3) there exists $\psi_{t, r} \in L_{+}^{2}([0,2 \pi])$ such that

$$
G_{n}^{t}(\xi) \subset\left[0, \psi_{t, r}(\xi)\right], \quad \forall n \in \mathbb{N},
$$

so, we can say that there exists a subsequence $\left(u_{n_{k}}\right)_{k}$ of $\left(u_{n}\right)_{n}$ such that

$$
\begin{equation*}
u_{n_{k}} \rightarrow u . \tag{6.16}
\end{equation*}
$$

Now, taking into account of (6.15), by using (f2) and (6.16), passing to the limit we obtain

$$
f_{1}\left(t, \xi, l_{1}^{t}(y)\right) \leq u \leq f_{2}\left(t, \xi, l_{2}^{t}(y)\right) .
$$

i.e. $u \in G^{t}(\xi)$ (see (6.14)). So hypothesis $\alpha$ ) of the Containment Theorem holds.

Next, let $\left(\hat{y}_{n}\right)_{n}$ be a sequence such that, for all $n \in \mathbb{N}, \hat{y}_{n}:[0,2 \pi] \rightarrow \mathbb{R}$ is defined by

$$
\hat{y}_{n}(\xi)=\left\|v_{n}(\xi)\right\|_{c}, \quad \xi \in[0,2 \pi],
$$

where $v_{n}$ is a function presented in (6.11).
First of all, fixed $\xi \in[0,2 \pi] \backslash\left(H_{t} \cap[0,2 \pi]\right)$, we know that $\hat{y}_{n}(\xi)=\left\|v_{n}(\xi)\right\|_{C} \in G_{n}^{t}(\xi)$, for all $n \in \mathbb{N}$. Moreover by the same arguments above presented we have

$$
\left|\hat{y}_{n}(\xi)\right| \leq \psi_{t, r}(\xi), \quad \forall n \in \mathbb{N},
$$

Being $\psi_{t, r} \in L_{+}^{2}([0,2 \pi]) \subset L_{+}^{1}([0,2 \pi])$, we can say that $\left(\hat{y}_{n}\right)_{n}$ is an integrably bounded sequence and so it has also the property of equi-absolute continuity of the integral (see Remark 2.3), i.e. hypothesis $\alpha \alpha$ ) of Theorem 2.6 is true.

Now, since all the hypotheses of the Containment Theorem hold, there exists a subsequence $\left(\hat{y}_{n_{k}}\right)_{k}$ of $\left(\hat{y}_{n}\right)_{n}$ such that

$$
\begin{equation*}
\hat{y}_{n_{k}} \rightharpoonup \hat{y} \quad \text { in } L_{+}^{1}([0,2 \pi]) \tag{6.17}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{y}(\xi) \in \overline{c_{0}} G^{t}(\xi)=G^{t}(\xi), \quad \text { a.e. } \xi \in[0,2 \pi] . \tag{6.18}
\end{equation*}
$$

Next, since the strong and weak topologies are the same in $\mathbb{R}$ and $\mathbb{C}$, taking into account of (6.17) and (6.13) respectively, we can write

$$
\begin{equation*}
\hat{y}_{n_{k}}(\xi)=\left\|v_{n_{k}}(\xi)\right\|_{\mathrm{C}} \rightarrow \hat{y}(\xi), \tag{6.19}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|v_{n_{k}}(\xi)\right\|_{\mathrm{C}} \rightarrow\|v(\xi)\|_{\mathrm{C}}, \tag{6.20}
\end{equation*}
$$

for a.e. $\xi \in[0,2 \pi]$.
Finally, by using (6.19), (6.20) and the uniqueness of the limit we have (see (6.18))

$$
\|v(\xi)\|_{C} \in G^{t}(\xi) \text {, a.e. } \xi \in[0,2 \pi] .
$$

In the same way, by applying the Containment Theorem to the restrictions $G_{n \mid[2 k \pi, 2(k+1) \pi]}^{t}$ and $G_{[2 k \pi, 2(k+1) \pi]}^{t}$ we have

$$
\|v(\xi)\|_{\mathrm{c}} \in\left[f_{1}\left(t, \xi, l_{1}^{t}(y)\right), f_{2}\left(t, \xi, l_{2}^{t}(y)\right)\right],
$$

a.e. $\xi \in \mathbb{R}$.

In conclusion, by recalling that $v \in L^{2}(\mathbb{T}, \mathbb{C})$ (see (6.12)), we can claim that $v \in U(t, y)$, a.e. $t \in J$. So also F3. of Corollary 5.4 holds.

Now we will prove that hypothesis $F 4$. is true. First of all, for every $n \in \mathbb{N}$, let us fix $y \in \bar{B}_{L^{2}(\mathbb{T}, \mathrm{C})}(0, n), t \in J$. Now, fixed $q \in F(t, y)$, there exists $v \in U(t, y)$ such that $q=T(t) y+v$ (see (6.8)) and we have

$$
\begin{equation*}
\|q\|_{L^{2}(\mathbb{T}, \mathbb{C})}=\|T(t) y+v\|_{L^{2}(\mathbb{T}, \mathrm{C})} \leq\|T(t) y\|_{L^{2}(\mathbb{T}, \mathrm{C})}+\|v\|_{L^{2}(\mathbb{T}, \mathrm{C})} . \tag{6.21}
\end{equation*}
$$

By using analogous arguments of the previous part of the proof, (k) and (f3) imply

$$
\|v(\xi)\|_{\mathrm{C}} \leq f_{2}\left(t, \xi, l_{2}^{t}(y)\right) \leq \psi_{t, n}(\xi),
$$

for a.e. $\xi \in[0,2 \pi]$, where $\psi_{t, n} \in L_{+}^{2}([0,2 \pi])$.
Therefore, by using (6.3) of (f3) we have

$$
\begin{equation*}
\|v\|_{L^{2}(\mathbb{T}, \mathrm{C})} \leq\left\|\psi_{t, r}\right\|_{L_{+}^{2}([0,2 \pi])} \leq n \varphi(t) . \tag{6.22}
\end{equation*}
$$

Then, by using (6.21) and (6.22) we are in the position to claim the following inequality

$$
\|q\|_{L^{2}(\mathbb{T}, \mathbb{C})} \leq n\left(\varphi(t)+\|T(t)\|_{\mathcal{L}\left(L^{2}(\mathbb{T}, \mathbb{C})\right)}\right) .
$$

Therefore, by the arbitrariness of $y \in \bar{B}_{L^{2}(\mathbb{T}, \mathrm{C})}(0, n)$ we deduce

$$
\|F(t, y)\| \leq n\left(\varphi(t)+\|T(t)\|_{\mathcal{L}\left(L^{2}(\mathbb{T}, \mathcal{C})\right)}\right)=: \varphi_{n}(t),
$$

where $\varphi_{n} \in L_{+}^{1}(J)$, since $\varphi \in L_{+}^{1}(J)$ and $\|T(\cdot)\|_{\mathcal{L}\left(L^{2}(\mathbb{T}, \mathrm{C})\right)} \in L_{+}^{1}(J)$ (see (T)).
Finally we note that (see hypothesis (f3))

$$
\limsup _{n \rightarrow \infty} \frac{K \int_{0}^{1} \varphi_{n}(\theta) d \theta}{n}=K \int_{0}^{1}\left(\varphi(\theta)+\|T(\theta)\|_{\mathcal{L}\left(L^{2}(\mathbb{T}, \mathrm{C})\right)}\right) d \theta<1,
$$

so also F4. of Corollary 5.4 holds.
By means of the arguments above presented, we are in a position to apply the Cauchy version of our Theorem 5.3. Then we can deduce that there exists a continuous function $\hat{x}: J \rightarrow L^{2}(\mathbb{T}, \mathbb{C})$ that is a mild solution for (6.2), i.e.

$$
\hat{x}(t)=C(t, 0) x_{0}+S(t, 0) x_{1}+\int_{0}^{t} S(t, \theta) \hat{q}(\theta) d \theta, \quad t \in J,
$$

where

$$
\begin{equation*}
\hat{q} \in S_{F(, \hat{x}(\cdot))}^{1}=\left\{q \in L^{1}\left(J ; L^{2}(\mathbb{T}, \mathbb{C})\right): q(t) \in F(t, \hat{x}(t)) \text { a.e. } t \in J\right\} \text {. } \tag{6.23}
\end{equation*}
$$

Therefore, since a.e. $t \in J, \hat{q}(t) \in F(t, \hat{x}(t))$, there exists $v_{\hat{x}}(t) \in U(t, \hat{x}(t))$ (see (6.8)) such that

$$
\begin{equation*}
v_{\hat{x}}(t)=\hat{q}(t)-T(t) \hat{x}(t), \quad \text { a.e. } t \in J . \tag{6.24}
\end{equation*}
$$

Hence we can consider the map $v_{\hat{x}}: J \rightarrow L^{2}(\mathbb{T}, \mathbb{C})$ defined as in (6.24) which is B-measurable, since $\hat{q}(\cdot)$ and $T(\cdot) \hat{x}(\cdot)$ are B-measurable (see respectively (6.23) and (T)).

At this point, by considering functions $w: J \times \mathbb{R} \rightarrow \mathbb{C}$ and $u: J \times \mathbb{R} \rightarrow \mathbb{C}$ respectively defined by

$$
\begin{aligned}
w(t, \xi) & =\hat{x}(t)(\xi), & & t \in J, \xi \in \mathbb{R} \\
u(t, \xi) & =v_{\hat{x}}(t)(\xi), & & t \in J, \xi \in \mathbb{R},
\end{aligned}
$$

which are $2 \pi$-periodic with respect to the second variable and 2 -integrable in $[0,2 \pi]$, we can conclude that $\{w, u\}$ is an admissible mild-pair for problem (6.1).

Finally we are able to enunciate the following result.
Theorem 6.1. In the framework above described, there exists an admissible mild-pair $\{w, u\}$ for problem (6.1), i.e. $w, u: J \times \mathbb{R} \rightarrow \mathbb{C}$ satisfying the following properties
(w1) for every $t \in J, w(t, \cdot)$ is $2 \pi$-periodic and 2-integrable on $[0,2 \pi]$;
(w2) for every $\xi \in \mathbb{R}, w(\cdot, \xi)$ is continuous on $J$;
(w3) $w(0, \xi)=x_{0}$, for every $\xi \in \mathbb{R}$;
(w4) for every $\xi \in \mathbb{R}$ such that $w(\cdot, \xi)$ is derivable at 0 , we have $\frac{\partial w}{\partial t}(0, \xi)=x_{1}$;
(u1) for every $t \in J, u(t, \cdot)$ is $2 \pi$-periodic and 2-integrable on $[0,2 \pi]$;
(u2) for every $\xi \in \mathbb{R}, u(\cdot, \xi)$ is $B$-measurable and such that
$\|u(t, \xi)\|_{\mathbf{C}} \in\left[f_{1}\left(t, \xi, \int_{0}^{2 \pi} k_{1}(t, \theta) w(t, \theta) d \theta\right), f_{2}\left(t, \xi, \int_{0}^{2 \pi} k_{2}(t, \theta) w(t, \theta) d \theta\right)\right]$,
a.e. $t \in J$ and for every $\xi \in \mathbb{R}$.

Moreover, $w, u$ are such that

$$
w(t, \xi)=C(t, 0) x_{0}+S(t, 0) x_{1}+\int_{0}^{t} S(t, \theta) q(\theta, \xi) d \theta, \quad t \in J, \xi \in \mathbb{R}
$$

where $q: J \times \mathbb{R} \rightarrow \mathbb{C}$ is defined by

$$
q(t, \xi)=T(t) w(t, \cdot)(\xi)+u(t, \xi), \quad t \in J, \xi \in \mathbb{R} .
$$

## Acknowledgements

This research is carried out within the national group GNAMPA of INdAM. The first author is partially supported by the Department of Mathematics and Computer Science of the University of Perugia (Italy) and by the projects "Metodi della Teoria dell'Approssimazione, Analisi Reale, Analisi Nonlineare e loro applicazioni" and "Integrazione, Approssimazione, Analisi Nonlineare e loro Applicazioni", funded by 2018 and 2019 basic research fund of the University of Perugia, and by a 2020 GNAMPA-INdAM Project "Processi evolutivi con memoria descrivibili tramite equazioni integro-differenziali".

Finally, we thank the Referee for his suggestions that have led to an improvement in the drafting of the paper.

## References

[1] J. Appell, Measure of noncompactness, condensing operators and fixed points: an application-oriented survey, Fixed Point Theory 6(2005), No. 2, 157-229. Zbl 1102.47041
[2] I. Benedetti, M. Vëth, Semilinear inclusion with nonlocal conditions without compactness in non-reflexive spaces, Topol. Methods Nonlinear Anal. 48(2016), 613-636. https: //doi.org/10.12775/TMNA.2016.061
[3] L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, J. Math. Anal. Appl. 162(1991), No. 2, 494-505. https : //doi.org/10.1016/0022-247X (91) 90164-U
[4] T. Cardinali, E. De Angelis, Non-autonomous second order differential inclusions subject to a perturbation with a stabilizing effect, Results Math. 78(2021), No. 8. https: //doi.org/10.1007/s00025-020-01305-1
[5] T. Cardinali, S. Gentili, An existence theorem for a non-autonomous second order nonlocal multivalued problem, Stud. Univ. Babes-Bolyai Math. 62(2017), 1-19. https:// doi.org/10.24193/subbmath. 2017.0008
[6] T. Cardinali, P. Rubbioni, Multivalued fixed point theorems in terms of weak topology and measure of weak noncompactness, J. Math. Anal. Appl. 405(2013), 409-415. https: //doi.org/10.1016/j.jmaa.2013.03.045
[7] T. Cardinali, P. Rubbioni, Hereditary evolution processes under impulsive effects, Mediterr. J. Math. 18(2021), No. 91. https://doi.org/10.1007/s00009-021-01730-8
[8] M. Chandrasekaran, Nonlocal Cauchy problem for quasilinear integro-differential equations in Banach spaces, Electron. J. Differential Equations 2007, No. 33, 1-6. Zbl 1113.45015
[9] F. S. De Blasi, On a property of the unit sphere in a Banach space, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 21(1977), 259-262. Zbl 0365.46015
[10] K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, J. Math. Anal. Appl. 21(1993), 630-637. https://doi.org/10.1006/ jmaa. 1993.1373
[11] Z. Denkowski, S. Migorski, N. S. Papageorgiou, An introduction to nonlinear analisys: theory, Kluwer Academic Publishers, New York, 2003. MR2024162
[12] H. O. Fattorini, Second order linear differential equations in Banach spaces, North-Holland Publishing Co., Amsterdam, 1985. MR0797071
[13] I. L. Glicksberg, A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points, Proc. Amer. Math. Soc. 3(1952), No. 1, 170-174. https://doi.org/10.2307/2032478
[14] G. Godefroy, S. Troyanski, J. H. M. Whitfield, V. Zizler, Smoothness in weakly compactly generated Banach spaces, J. Funct. Anal. 52(1983), 344-352. https://doi. org/10. 1016/0022-1236(83) 90073-3
[15] H. R. Henríquez, Existence of solutions of non autonomous second order differential equations with infinite delay, Nonlinear Anal. 74(2011), No. 10, 3333-3352. https://doi. org/10.1016/j.na.2011.02.010
[16] H. R. Henríquez, V. Poblete, J. C. Pozo, Mild solutions of nonautonomous second order problems with nonlocal initial conditions, J. Math. Anal. Appl. 412(2014), 1064-1083. https://doi.org/10.1016/j.jmaa.2013.10.086
[17] S. Hu, N. S. Papageorgiou, Handbook of multivalued analysis. Volume I: Theory, Kluwer Academic Publishers, 1997. MR1485775
[18] M. Kamenskir, V. ОbuкhovskiI, P. Zecca, Condensing multivalued maps and semilinear differential inclusions in Banach spaces, Walter de Gruyter, New York, 2001. https://doi. org/10.1515/9783110870893
[19] T. Кato, Integration of the equation of evolution in a Banach space, J. Math. Soc. Jpn. 5(1953), 208-234. https://doi.org/10.2969/jmsj/00520208
[20] T. Kato, On linear differential equations in Banach spaces, Comm. Pure Appl. Math. 9(1956), 479-486. https://doi.org/10.1002/cpa.3160090319
[21] T. Kato, Linear evolution equations of "hyperbolic" type, J. Fac. Sci. Univ. Tokyo, Sect. I 17(1970), 241-258. https://doi.org/10.2969/jmsj/02540648
[22] S. S. Khurana, J. Vielma, Weak sequential convergence and weak compactness in space of vector-valued continuous functions, J. Math. Anal. Appl. 195(1995), 251-260. https: //doi.org/10.1006/jmaa.1995.1353
[23] J. Kisynski, On cosine operator functions and one-parameter groups of operators, Studia Math. 44(1972), 93-105. Zbl 0232.47045
[24] M. Kozaк, A fundamental solution of a second-order differential equation in Banach space, Univ. Iagel. Acta Math. 32(1995), 275-289. MR1345144
[25] M. Kunze, G. Schlüchtermann, Strongly generated Banach spaces and measure of noncompactness, Math. Nachr. 191(1998), 197-214. https://doi.org/10.1002/mana. 19981910110
[26] R. E. Megginson, An introduction to Banach space theory, Springer-Verlag, New York, Berlin, Heidelberg, 1998. https://doi.org/10.1007/978-1-4612-0603-3
[27] H. Serizawa, M. Watanabe, Time-dependent perturbation for cosine families in Banach spaces, Houston J. Math. 12(1986), 579-586. https://doi.org/10.1.1.637.6393
[28] C. C. Travis, G. F. Webb, Compactness, regularity and uniform continuity properties of strongly continuous cosine family, Houston J. Math. 3(1977), 555-567. https://doi.org/ 10.1007/BF00992842
[29] A. Ülger, Weak compactness in $L^{1}(\mu, X)$, Proc. Amer. Math. Soc. 113(1991), 143-149. https://doi.org/10.2307/2048450
[30] V. V. Vasilev, S. I. Piskarev, Differential equations in Banach spaces, II, Theory of cosine operator functions, J. Math. Sci. (N.Y.) 122(2004), 3055-3174. https ://doi.org/10.1023/ B: JOTH. 0000029697.92324 .47
[31] M. Väth, Ideal spaces, Lect. Notes in Math., Springer, Berlin, Heidelberg 1997. https: //doi.org/10.1007/BFb0093548
[32] H. Vogt, An Eberlein-Šmulian type result for weak* topology, Arch. Math. (Basel) 95(2010), 31-34. https://doi .org/10.1007/s00013-010-0128-y
[33] X. Xue, Existence of solutions for semilinear nonlocal Cauchy problems in Banach spaces, Electron. J. Differential Equations 2005, No. 64, 1-7. https ://doi.org/10.1016/j.na. 2005. 05.019

# Hyers-Ulam stability for a partial difference equation 

Konstantinos Konstantinidis, Garyfalos Papaschinopoulos ${ }^{\boxtimes}$ and Christos J. Schinas<br>Democritus University of Thrace, School of Engineering 67100 Xanthi, Greece

Received 23 June 2021, appeared 9 September 2021
Communicated by Stevo Stević


#### Abstract

Under the exponential trichotomy condition we study the Hyers-Ulam stability for the linear partial difference equation: $$
x_{n+1, m}=A_{n} x_{n, m}+B_{n, m} x_{n, m+1}+f\left(x_{n, m}\right), \quad n, m \in \mathbb{Z}
$$ where $A_{n}$ is a $k \times k$ matrix whose elements are sequences of $n, B_{n, m}$ is a $k \times k$ matrix whose elements are double sequences of $m, n$ and $f: \mathbb{R}^{k} \rightarrow \mathbb{R}^{k}$ is a vector function. We also investigate the Hyers-Ulam stability in the case where the matrices $A_{n}, B_{n, m}$ and the vector function $f=f_{n, m}$ are constant. Keywords: partial difference equations, Hyers-Ulam stability, exponential dichotomy. 2020 Mathematics Subject Classification: 39A14.


## 1 Introduction

Partial difference equations is an area which deals with difference equations with several variables. Some classical results in the area can be found, for example, in books [4,9,12,14]. Despite the fact that the study of partial difference equations is pretty much complicated, both theoretically and technically, there are some investigations on solvability, stability and other topics related to the equations (see, for example, [5-7,10, 13, 15, 25, 29, 31, $34,40,41]$ and the related references therein). Many partial difference equations are obtained from some problems in combinatorics, probability, discrete mathematics and other related areas of mathematics and science (see, for example, $[11,22,43]$ ).

In [8] the authors studied the so-called $\mu$-exponentially weighted shadowing property of the equation

$$
x_{m+1}=L_{m} x_{m}+f_{m}\left(x_{m}\right), \quad m \in \mathbb{Z},
$$

where $L_{m}$ is a sequence of linear operators, $f_{m}$ is a sequence of nonlinear operators $m \in \mathbb{Z}$ assuming that the linear equation

$$
x_{m+1}=L_{m} x_{m}
$$

has an exponential dichotomy and the sequence $f_{m}$ is uniformly Lipschitz continuous.

[^29]Inspired by the above work, as well as some applications of solvability methods for difference equations, here we investigate Hyers-Ulam stability for the nonhomogenous linear partial difference equation of the form:

$$
\begin{equation*}
x_{n+1, m}=A_{n} x_{n, m}+B_{n, m} x_{n, m+1}+f\left(x_{n, m}\right), \quad n, m \in \mathbb{Z}, \tag{1.1}
\end{equation*}
$$

where $A_{n}$ is a $k \times k$ invertible matrix whose elements are sequences of $n, B_{n, m}$ is a $k \times k$ matrix whose elements are double sequences of $m, n$ and $f: \mathbb{R}^{k} \rightarrow \mathbb{R}^{k}$ is a vector function.

In what follows we denote by $|\cdot|$ any convenient norm either of a vector or of a matrix. We say that the linear difference equation

$$
\begin{equation*}
x_{v+1}=C_{v} x_{v}, \quad v \in \mathbb{Z}, \ldots \tag{1.2}
\end{equation*}
$$

where $C_{v}$ is an invertible matrix has an exponential trichotomy (see [16,17]) if there exist constants $K>0,0<p<1$ and projections $P_{1}, P_{2}, P_{3}\left(P_{i}^{2}=P_{i}, i=1,2,3\right), P_{1}+P_{2}+P_{3}=1$ such that

$$
\begin{array}{ll}
\left|X_{v} P_{1} X_{s}^{-1}\right| \leq K p^{v-s}, & v \geq s, s, v \in \mathbb{Z} \\
\left|X_{v} P_{2} X_{s}^{-1}\right| \leq K p^{s-v,} & s \geq v, s, v \in \mathbb{Z} \\
\left|X_{v} P_{3} X_{s}^{-1}\right| \leq K p^{v-s}, & v \geq s \geq 0  \tag{1.3}\\
\left|X_{v} P_{3} X_{s}^{-1}\right| \leq K p^{s-v}, & 0 \geq s \geq v
\end{array}
$$

where $X_{v}$ is a fundamental matrix solution of (1.2) given by

$$
X_{v}= \begin{cases}\left(\prod_{s=0}^{v-1} C_{v-s-1}\right) C, & v \geq 0 \\ \left(\prod_{s=v}^{-1} C_{s}^{-1}\right) C, & v \leq 0\end{cases}
$$

and $C$ is a constant matrix. We regard that $X_{0}=C$.
For the readers' convenience we give a simple example concerning exponential trichotomy for a linear difference equation. Consider equation (1.2) where

$$
C_{v}=\left[\begin{array}{ccc}
1 / 2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & c_{v}
\end{array}\right], \quad c_{v}= \begin{cases}1 / 2, & v \geq 0 \\
2, & v<0\end{cases}
$$

Then if we take $C=I_{3}, I_{3}$ the $3 \times 3$ indentity matrix, we get

$$
X_{v}=\left[\begin{array}{ccc}
(1 / 2)^{v} & 0 & 0 \\
0 & 2^{v} & 0 \\
0 & 0 & d_{v}
\end{array}\right], \quad d_{v}= \begin{cases}(1 / 2)^{v}, & v \geq 0 \\
2^{v}, & v \leq 0\end{cases}
$$

If we take the projections

$$
P_{1}=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right], \quad P_{2}=\left[\begin{array}{lll}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{array}\right], \quad P_{3}=\left[\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

we have $P_{1}+P_{2}+P_{3}$ and (1.3) hold with $K=1$ and $p=1 / 2$.

Moreover, we give some details concerning the form of operator $T$ given in Proposition 2.2. In Proposition 1 of [16] the author proved that if equation (1.2) has an exponential trichotomy (1.3) then the inhomogenous ordinary difference equation

$$
x_{v+1}=A_{v} x_{v}+f_{v}, \quad v \in \mathbb{Z},
$$

$f_{v}: \mathbb{Z} \rightarrow \mathbb{R}^{k},\left|f_{v}\right| \leq M, v \in \mathbb{Z}$ where $M$ is a positive constant, has at least bounded solution $y_{v}$ given by

$$
\begin{array}{ll}
y_{v}=\sum_{s=-\infty}^{-1} X_{v} P_{1} X_{s+1}^{-1} f_{s}+\sum_{s=0}^{v-1} X_{v}\left(I-P_{2}\right) X_{s+1}^{-1} f_{s}-\sum_{s=v}^{\infty} X_{v} P_{2} X_{s+1}^{-1} f_{s}, & v \geq 0,  \tag{1.4}\\
y_{v}=\sum_{s=-\infty}^{v-1} X_{v} P_{1} X_{s+1}^{-1} f_{s}-\sum_{s=v}^{-1} X_{v}\left(I-P_{1}\right) X_{s+1}^{-1} f_{s}-\sum_{s=0}^{\infty} X_{v} P_{2} X_{s+1}^{-1} f_{s}, & v \leq 0 .
\end{array}
$$

According to [21] we say that (1.1) has the Hyers-Ulam stability if for any $\epsilon>0$ there exists a $\delta>0$ such that if $y_{n, m}$ satisfies either

$$
\begin{equation*}
\left|y_{n+1, m}-A_{n} y_{n, m}-B_{n, m} y_{n, m+1}-f\left(y_{n, m}\right)\right|<\delta \tag{1.5}
\end{equation*}
$$

or

$$
\begin{equation*}
\left|y_{n, m+1}+B_{n, m}^{-1} A_{n} y_{n, m}-B_{n, m}^{-1} y_{n+1, m}+B_{n, m}^{-1} f\left(y_{n, m}\right)\right|<\delta \tag{1.6}
\end{equation*}
$$

then there exists a solution $x_{n, m}$ of (1.1) such that

$$
\begin{equation*}
\left|x_{n, m}-y_{n, m}\right|<\epsilon, \quad n, m \in \mathbb{Z} . \tag{1.7}
\end{equation*}
$$

Now in this paper assuming that equation (1.2) where $C_{n}=A_{n}$ has an exponential trichotomy then, under some assumptions on the matrices $A_{n}, B_{n, m}$ and the function $f$, we prove that (1.1) has the Hyers-Ulam stability. In addition, if $B_{n, m}=A_{n} D_{m}, A_{n}, D_{m}$ are invertible matrices and the equation (1.2) where $C_{m}=-D_{m}^{-1}$ has an exponential trichotomy, then, under some assumptions on the matrices $A_{n}, D_{m}$ and the function $f$, we prove that equation (1.1) has also the Hyers-Ulam stability. Finally we study the Hyers-Ulam stability in the case where the matrices $A_{n}, B_{n, m}$ are constants, that is $A_{n}=A, B_{n, m}=B$ and the function $f=f_{n, m}$ is independent on $x$ that is $f_{n, m}: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}^{k}$.

Roughly speaking the stability of Hyers-Ulam means that for any approximate solution of equation (1.1) there exists a solution of (1.1) which is near the approximate solution. Since this is very important there exists an increasing interest in studying this stability. Therefore there are many papers which deal with this subject (see $[1,3,8,21]$ and the related references therein).

In what follows we denote

$$
l^{\infty}=l^{\infty}\left(\mathbb{Z}^{2}\right)
$$

the space of all double sequences $\left(z_{n, m}\right) \subset \mathbb{R}^{k}$ which are bounded.
In the study we will essentially use a method related to the solvability of the linear difference equation of first order, by which the studied difference equations are transformed to some difference equations of 'integral' type, for which it is easier to apply methods from nonlinear functional analysis. Here it is applied the contraction principle. It should be mentioned that recently appeared many papers on difference equations and systems of difference equations which have been solved by transforming them to some linear solvable ones (see, for example, $[2,20,26-28,30,32,33,35-39,42]$ and the related references therein).

Finally it should be mentioned that, there is a plenty of papers dealing with solvability or invariants for difference equations (see, for example, [18, 19, 23, 24, 28, $30,35,36,38,39]$ ).

## 2 Main results

Firstly we give a proposition which concerns the existence and uniqueness of the solutions of (1.1).

## Proposition 2.1.

(i) For a given sequence $c_{m}$ there exists a unique solution $x_{n, m}$ of (1.1) such that $x_{0, m}=c_{m}, m \in \mathbb{Z}$. Moreover, $x_{n, m}$ satisfies the following relations

$$
x_{n, m}= \begin{cases}X_{n} X_{0}^{-1} c_{m}+\sum_{s=0}^{n-1} X_{n} X_{s+1}^{-1}\left(B_{s, m} x_{s, m+1}+f\left(x_{s, m}\right)\right), & n \geq 0, m \in \mathbb{Z}  \tag{2.1}\\ X_{n} X_{0}^{-1} c_{m}-\sum_{s=n}^{-1} X_{n} X_{s+1}^{-1}\left(B_{s, m} x_{s, m+1}+f\left(x_{s, m}\right)\right), & n \leq 0, m \in \mathbb{Z}\end{cases}
$$

where $X_{n}$ is a fundamental matrix solution of (1.2) with $C_{n}=A_{n}$.
(ii) Suppose that $B_{n, m}=A_{n} D_{m}$ and $A_{n}, B_{m}$ are invertible matrices. Then there exists a unique solution of (1.1) such that $x_{n, 0}=d_{n}, n \in \mathbb{Z}$ where $d_{n}$ is given sequence. In addition if

$$
R\left(x_{n, s}\right)=D_{s}^{-1} A_{n}^{-1} x_{n+1, s}-D_{s}^{-1} A_{n}^{-1} f\left(x_{n, s}\right),
$$

$x_{n, m}$ satisfies the following equalities

$$
x_{n, m}= \begin{cases}X_{m} X_{0}^{-1} d_{n}+\sum_{s=0}^{m-1} X_{m} X_{s+1}^{-1} R\left(x_{n, s}\right), & m \geq 0, n \in \mathbb{Z}  \tag{2.2}\\ X_{m} X_{0}^{-1} d_{n}-\sum_{s=m}^{-1} X_{m} X_{s+1}^{-1} R\left(x_{n, s}\right), & m \leq 0, n \in \mathbb{Z}\end{cases}
$$

where $X_{m}$ is a fundamental matrix solution of (1.2) with $C_{m}=-D_{m}^{-1}$.
From (1.1) and using the constant variation formula for a fixed $m$ we can prove (2.1).
Since from (1.1) we have

$$
\begin{align*}
x_{n, m+1} & =-B_{n, m}^{-1} A_{n} x_{n, m}+B_{n, m}^{-1} x_{n+1, m}-B_{n, m}^{-1} f\left(x_{n, m}\right) \\
& =-D_{m}^{-1} x_{n, m}+D_{m}^{-1} A_{n}^{-1} x_{n+1, m}-D_{m}^{-1} A_{n}^{-1} f\left(x_{n, m}\right), \tag{2.3}
\end{align*}
$$

using the constant variation formula for a fixed $n$ we can easily get (2.2).
We prove the Hyers-Ulam stability in the case where equation (1.2) with $C_{n}=A_{n}$ or $C_{m}=-D_{m}^{-1}, B_{n, m}=A_{n} D_{m}$ has an exponential trichotomy.

Proposition 2.2. The following statements are true:
(i) Suppose that (1.2) with $C_{n}=A_{n}$ has an exponential trichotomy (1.3), that there exists a positive number $M$ such that

$$
\begin{equation*}
\left|B_{n, m}\right| \leq M, \quad n, m \in \mathbb{Z}, \tag{2.4}
\end{equation*}
$$

and that $f: \mathbb{R}^{k} \rightarrow \mathbb{R}^{k}$ is a vector function such that for all $x, y \in \mathbb{R}^{k}$

$$
\begin{equation*}
|f(x)-f(y)| \leq L|x-y|, \tag{2.5}
\end{equation*}
$$

where $L$ is a positive constant. Then if

$$
\begin{equation*}
(M+L) \frac{2 K(p+1)}{1-p}<1 \tag{2.6}
\end{equation*}
$$

equation (1.1) has the Hyers-Ulam stability.
(ii) Suppose that $B_{n, m}=A_{n} D_{m}, n, m \in \mathbb{Z}$ where $A_{n}, D_{m}$ are invertible matrices for any $m \in \mathbb{Z}$, that equation (1.2) where $C_{m}=-D_{m}^{-1}$ has an exponential trichotomy (1.3), that there exists a positive number $M$ such that

$$
\begin{equation*}
\left|D_{m}^{-1} A_{n}^{-1}\right| \leq M, \quad n, m \in \mathbb{Z} \tag{2.7}
\end{equation*}
$$

and that (2.5) is true. Then if

$$
\begin{equation*}
M(1+L) \frac{2 K(p+1)}{1-p}<1 \tag{2.8}
\end{equation*}
$$

equation (1.1) has the Hyers-Ulam stability.
Proof. (i) Let $\epsilon$ be an arbitrary positive number and $\delta$ be a positive number such that

$$
\begin{equation*}
\delta<\frac{1-p-2 K(1+p)(M+L)}{2 K(1+p)} \epsilon . \tag{2.9}
\end{equation*}
$$

Suppose that $y_{n, m}$ is a double sequence such that (1.5) is satisfied. Let

$$
\begin{equation*}
H\left(z_{n, m}\right)=-y_{n+1, m}+A_{n} y_{n, m}+B_{n, m}\left(y_{n, m+1}+z_{n, m+1}\right)+f\left(y_{n, m}+z_{n, m}\right) . \tag{2.10}
\end{equation*}
$$

Inspired by (1.4) we define the operator $T$ on $l^{\infty}$ as follows: If $z_{n, m} \in l^{\infty}$ then we set

$$
\begin{align*}
T z_{n, m}= & \sum_{s=-\infty}^{-1} X_{n} P_{1} X_{s+1}^{-1} H\left(z_{s, m}\right)+\sum_{s=0}^{n-1} X_{n}\left(I-P_{2}\right) X_{s+1}^{-1} H\left(z_{s, m}\right) \\
& -\sum_{s=n}^{\infty} X_{n} P_{2} X_{s+1}^{-1} H\left(z_{s, m}\right), \quad n \geq 0, m \in \mathbb{Z} .  \tag{2.11}\\
T z_{n, m}= & \sum_{s=-\infty}^{n-1} X_{n} P_{1} X_{s+1}^{-1} H\left(z_{s, m}\right)-\sum_{s=n}^{-1} X_{n}\left(I-P_{1}\right) X_{s+1}^{-1} H\left(z_{s, m}\right) \\
& -\sum_{s=0}^{\infty} X_{n} P_{2} X_{s+1}^{-1} H\left(z_{s, m}\right), \quad n \leq 0, m \in \mathbb{Z} .
\end{align*}
$$

We prove that $T\left(l^{\infty}\right) \subseteq l^{\infty}$. Let

$$
|z|_{\infty}=\sup \left\{\left|z_{n, m}\right|, n, m, \in \mathbb{Z}\right\} .
$$

From (2.10) we obtain

$$
\begin{align*}
H\left(z_{n, m}\right)= & -y_{n+1, m}+A_{n} y_{n, m}+B_{n, m} y_{n, m+1}+f\left(y_{n, m}\right)  \tag{2.12}\\
& +B_{n, m} z_{n, m+1}+f\left(y_{n, m}+z_{n, m}\right)-f\left(y_{n, m}\right) .
\end{align*}
$$

Then from (1.5), (2.4), (2.5) and (2.12) we have

$$
\begin{equation*}
\left|H\left(z_{n, m}\right)\right| \leq \delta+(M+L)|z|_{\infty} . \tag{2.13}
\end{equation*}
$$

Therefore from (1.3), (2.13) and since $I-P_{2}=P_{1}+P_{3}$ for $n \geq 0, m \in \mathbb{Z}$ we get

$$
\begin{align*}
\left|T z_{n, m}\right| & \leq\left(\sum_{s=-\infty}^{-1} K p^{n-s-1}+2 \sum_{s=0}^{n-1} K p^{n-s-1}+\sum_{s=n}^{\infty} K p^{-n+s+1}\right)\left(\delta+(M+L)|z|_{\infty}\right)  \tag{2.14}\\
& \leq \frac{2 K(1+p)}{1-p}\left(\delta+(M+L)|z|_{\infty}\right) .
\end{align*}
$$

Furthermore from (1.3), (2.13) and since $I-P_{1}=P_{2}+P_{3}$ for $n \leq 0, m \in \mathbb{Z}$ we have

$$
\begin{align*}
\left|T z_{n, m}\right| & \leq\left(\sum_{s=-\infty}^{n-1} K p^{n-s-1}+2 \sum_{s=n}^{-1} K p^{-n+s+1}+\sum_{s=0}^{\infty} K p^{-n+s+1}\right)\left(\delta+(M+L)|z|_{\infty}\right)  \tag{2.15}\\
& \leq \frac{2 K(1+p)}{1-p}\left(\delta+(M+L)|z|_{\infty}\right) .
\end{align*}
$$

Relations (2.14) and (2.15) imply that $T\left(l^{\infty}\right) \subseteq l^{\infty}$. We prove now that $T$ is a contraction on the space $S$. Let $z_{n, m}, w_{n, m} \in l^{\infty}$. Using (2.10) we get for $n, m \in \mathbb{Z}$

$$
H\left(z_{n, m}\right)-H\left(w_{n, m}\right)=B_{n, m}\left(z_{n, m+1}-w_{n, m+1}\right)+f\left(y_{n, m}+z_{n, m}\right)-f\left(y_{n, m}+w_{n, m}\right)
$$

and so from (2.4), (2.5) we have

$$
\begin{equation*}
\left|H\left(z_{n, m}\right)-H\left(w_{n, m}\right)\right| \leq(M+L)|z-w|_{\infty}, \quad n, m \in \mathbb{Z} . \tag{2.16}
\end{equation*}
$$

From (2.11) we have for $n \geq 0, m \in \mathbb{Z}$

$$
\begin{aligned}
T z_{n, m}-T w_{n, m}= & \sum_{s=-\infty}^{-1} X_{n} P_{1} X_{s+1}^{-1}\left(H\left(z_{s, m}\right)-H\left(w_{s, m}\right)\right)+\sum_{s=0}^{n-1} X_{n}\left(I-P_{2}\right) X_{s+1}^{-1}\left(H\left(z_{s, m}\right)-H\left(w_{s, m}\right)\right) \\
& -\sum_{s=n}^{\infty} X_{n} P_{2} X_{s+1}^{-1}\left(H\left(z_{s, m}\right)-H\left(w_{s, m}\right)\right) .
\end{aligned}
$$

Then relations (1.3) and (2.16) for $n \geq 0$ and $m \in \mathbb{Z}$ imply that

$$
\begin{align*}
\left|T z_{n, m}-T w_{n, m}\right| & \leq\left(\sum_{s=-\infty}^{-1} K p^{n-s-1}+2 \sum_{s=0}^{n-1} K p^{n-s-1}+\sum_{s=n}^{\infty} K p^{-n+s+1}\right)(M+L)|z-w|_{\infty}  \tag{2.17}\\
& \leq \frac{2 K(1+p)}{1-p}(M+L)|z-w|_{\infty}
\end{align*}
$$

Moreover from (1.3) and (2.16) for $n \leq 0$ and $m \in \mathbb{Z}$ we get

$$
\begin{align*}
\left|T z_{n, m}-T w_{n, m}\right| & \leq\left(\sum_{s=-\infty}^{n-1} K p^{n-s-1}+\sum_{s=n}^{-1} K p^{-n+s+1}+\sum_{s=0}^{\infty} K p^{-n+s+1}\right)(M+L)|z-w|_{\infty}  \tag{2.18}\\
& \leq \frac{2 K(1+p)}{1-p}(M+L)|z-w|_{\infty}
\end{align*}
$$

So, from (2.6), (2.17) and (2.18) $T$ is a contraction on the complete metric space $l^{\infty}$. Hence there exists a unique $z_{n, m} \in l^{\infty}$ such that

$$
\begin{equation*}
T z_{n, m}=z_{n, m}, \quad n, m \in \mathbb{Z} . \tag{2.19}
\end{equation*}
$$

From (2.10), (2.11) and (2.19) we obtain for $n \geq 0, m \in \mathbb{Z}$

$$
\begin{align*}
z_{n, m}= & \sum_{s=-\infty}^{-1} X_{n} P_{1} X_{s+1}^{-1} H\left(z_{s, m}\right)+\sum_{s=0}^{n-1} X_{n}\left(I-P_{2}\right) X_{s+1}^{-1} H\left(z_{s, m}\right) \\
& +\sum_{s=0}^{n-1} X_{n} P_{2} X_{s+1}^{-1} H\left(z_{s, m}\right)-\sum_{s=0}^{n-1} X_{n} P_{2} X_{s+1}^{-1} H\left(z_{s, m}\right)-\sum_{s=n}^{\infty} X_{n} P_{2} X_{s+1}^{-1} H\left(z_{s, m}\right) \\
= & \sum_{s=-\infty}^{-1} X_{n} P_{1} X_{s+1}^{-1} H\left(z_{s, m}\right)+\sum_{s=0}^{n-1} X_{n} X_{s+1}^{-1} H\left(z_{s, m}\right)-\sum_{s=0}^{\infty} X_{n} P_{2} X_{s+1}^{-1} H\left(z_{s, m}\right)  \tag{2.20}\\
= & X_{n} X_{0}^{-1} \sum_{s=-\infty}^{-1} X_{0} P_{1} X_{s+1}^{-1} H\left(z_{s, m}\right)+\sum_{s=0}^{n-1} X_{n} X_{s+1}^{-1}\left(-y_{s+1, m}+A_{s} y_{s, m}\right) \\
& +\sum_{s=0}^{n-1} X_{n} X_{s+1}^{-1}\left(B_{s, m}\left(y_{s, m+1}+z_{s, m+1}\right)+f\left(y_{s, m}+z_{s, m}\right)\right) \\
& -X_{n} X_{0}^{-1} \sum_{s=0}^{\infty} X_{0} P_{2} X_{s+1}^{-1} H\left(z_{s, m}\right) .
\end{align*}
$$

Then for $n=0$ we get

$$
\begin{equation*}
z_{0, m}=\sum_{s=-\infty}^{-1} X_{0} P_{1} X_{s+1}^{-1} H\left(z_{s, m}\right)-\sum_{s=0}^{\infty} X_{0} P_{2} X_{s+1}^{-1} H\left(z_{s, m}\right) \tag{2.21}
\end{equation*}
$$

We claim that

$$
\begin{equation*}
y_{n, m}=X_{n} X_{0}^{-1} y_{0, m}+\sum_{s=0}^{n-1} X_{n} X_{s+1}^{-1}\left(y_{s+1, m}-A_{s} y_{s, m}\right), \quad n \geq 0, m \in \mathbb{Z} \tag{2.22}
\end{equation*}
$$

It is obvious that (2.22) is true for $n=0$. Suppose that (2.22) holds for a fixed $n$. Then

$$
\begin{aligned}
& X_{n+1} X_{0}^{-1} y_{0, m}+\sum_{s=0}^{n} X_{n+1} X_{s+1}^{-1}\left(y_{s+1, m}-A_{s} y_{s, m}\right) \\
& \quad=A_{n} X_{n} X_{0}^{-1} y_{0, m}+y_{n+1, m}-A_{n} y_{n, m}+A_{n} \sum_{s=0}^{n-1} X_{n} X_{s+1}^{-1}\left(y_{s+1, m}-A_{s} y_{s, m}\right) \\
& \quad=A_{n} y_{n, m}+y_{n+1, m}-A_{n} y_{n, m}=y_{n+1, m} .
\end{aligned}
$$

Therefore (2.22) is true for every $n$. Using (2.20), (2.21) and (2.22) we obtain for $n \geq 0, m \in \mathbb{Z}$.

$$
z_{n, m}+y_{n, m}=X_{n} X_{0}^{-1}\left(y_{0, m}+z_{0, m}\right)+\sum_{s=0}^{n-1} X_{n} X_{s+1}^{-1}\left(B_{s, m}\left(y_{s, m+1}+z_{s, m+1}\right)+f\left(y_{s, m}+z_{s, m}\right)\right)
$$

Then if $x_{n, m}=z_{n, m}+y_{n, m}$ from (2.1) we have that $x_{n, m}, n \geq 0, m \in \mathbb{Z}$ is a solution of (1.1). So, from (2.9), (2.14) and (2.19) we have

$$
|x-y|_{\infty}=|z|_{\infty} \leq \frac{2 K(1+p) \delta}{1-p-2 K(p+1)(M+L)}<\epsilon .
$$

In addition from (2.11) and (2.19) we have for $n \leq 0, m \in \mathbb{Z}$

$$
\begin{align*}
z_{n, m}= & \sum_{s=-\infty}^{n-1} X_{n} P_{1} X_{s+1}^{-1} H\left(z_{s, m}\right)-\sum_{s=n}^{-1} X_{n}\left(I-P_{1}\right) X_{s+1}^{-1} H\left(z_{s, m}\right) \\
& -\sum_{s=n}^{-1} X_{n} P_{1} X_{s+1}^{-1} H\left(z_{s, m}\right)+\sum_{s=n}^{-1} X_{n} P_{1} X_{s+1}^{-1} H\left(z_{s, m}\right)-\sum_{s=0}^{\infty} X_{n} P_{2} X_{s+1}^{-1} H\left(z_{s, m}\right) \\
= & \sum_{s=-\infty}^{-1} X_{n} P_{1} X_{s+1}^{-1} H\left(z_{s, m}\right)-\sum_{s=n}^{-1} X_{n} X_{s+1}^{-1} H\left(z_{s, m}\right)-\sum_{s=0}^{\infty} X_{n} P_{2} X_{s+1}^{-1} H\left(z_{s, m}\right)  \tag{2.23}\\
= & \left.X_{n} X_{0}^{-1} \sum_{s=-\infty}^{-1} X_{0} P_{1} X_{s+1}^{-1} H\left(z_{s, m}\right)-\sum_{s=n}^{-1} X_{n} X_{s+1}^{-1}\left(-y_{s+1, m}+A_{s} y_{s, m}\right)\right) \\
& -\sum_{s=n}^{-1} X_{n} X_{s+1}^{-1}\left(B_{s, m}\left(y_{s, m+1}+z_{s, m+1}\right)+f\left(y_{s, m}+z_{s, m}\right)\right) \\
& -X_{n} X_{0}^{-1} \sum_{s=0}^{\infty} X_{0} P_{2} X_{s+1}^{-1} H\left(z_{s, m}\right) .
\end{align*}
$$

So, for $n=0$ we get (2.21). Moreover, arguing as in (2.22) we can show that

$$
\begin{equation*}
y_{n, m}=X_{n} X_{0}^{-1} y_{0, m}-\sum_{s=n}^{-1} X_{n} X_{s+1}^{-1}\left(y_{s+1, m}-A_{s} y_{s, m}\right), \quad n \leq 0, m \in \mathbb{Z} . \tag{2.24}
\end{equation*}
$$

Therefore from (2.1), (2.23) and (2.24) we can prove that $x_{n, m}=y_{n, m}+z_{n, m}$ is a solution of (1.1). Using (2.9), (2.15) the proof of (i) is completed.
(ii) Let $\epsilon$ be an arbitrary positive number and $\delta$ be a positive number such that

$$
\begin{equation*}
\delta<\frac{1-p-2 K M(1+p)(1+L)}{2 K(1+p)} \epsilon . \tag{2.25}
\end{equation*}
$$

Suppose that $y_{n, m}$ is a double sequence such that (1.6) is satisfied. Then using (2.2), (2.3), (2.5), (2.7), (2.8), (2.25) and arguing as in the case (i) we can prove (ii).

In what follows we study the Hyers-Ulam stability for the equation

$$
\begin{equation*}
x_{n+1, m}=A x_{n, m}+B x_{n, m+1}+f_{n, m}, \quad n, m \in \mathbb{N} \tag{2.26}
\end{equation*}
$$

where $A, B$ are $k \times k$ are constant matrices and $f_{n, m}: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}^{k}$ is a double sequence. Firstly we give a formula for the solutions of (2.26).

Let $x_{n, m}$ be a double sequence. Then we define the operators $E_{1}, E_{2}$ as follows:

$$
E_{1} x_{n, m}=x_{n+1, m}, \quad E_{2} x_{n, m}=x_{n, m+1} .
$$

Proposition 2.3. Consider the partial difference equations (2.26). Then the following statements are true:
(i) There exists a unique solution $x_{n, m}$ of (2.26) with $x_{0, m}=c_{m}, c_{m}$ is a given sequence. Moreover $x_{n, m}$ is given by

$$
\begin{equation*}
x_{n, m}=\left(A+B E_{2}\right)^{n} c_{m}+\sum_{s=0}^{n-1}\left(A+B E_{2}\right)^{n-s-1} f_{s, m} . \tag{2.27}
\end{equation*}
$$

(ii) Let $B$ be an invertible matrix. There exists a unique solution $x_{n, m}$ of (2.26) where $x_{n, 0}=d_{n}, d_{n}$ is a given sequence. Furthermore $x_{n, m}$ is given by

$$
\begin{equation*}
x_{n, m}=\left(-B^{-1} A+B^{-1} E_{1}\right)^{m} d_{n}+\sum_{s=0}^{m-1}\left(-B^{-1} A+B^{-1} E_{1}\right)^{m-s-1}\left(-B^{-1}\right) f_{n, s} . \tag{2.28}
\end{equation*}
$$

Proof. (i) From (2.26) we get

$$
\begin{equation*}
x_{n+1, m}=A x_{n, m}+B E_{2} x_{n, m}+f_{n, m}=\left(A+B E_{2}\right) x_{n, m}+f_{n, m}, \quad n, m \in \mathbb{N} . \tag{2.29}
\end{equation*}
$$

Then from (2.29), for a fixed $m \in \mathbb{N}$ by the constant variation formula we get (2.27). So, the proof of part (i) is completed.
(ii) From (2.26) we get for a fixed $n \in \mathbb{N}$

$$
\begin{aligned}
x_{n, m+1} & =B^{-1} x_{n+1, m}-B^{-1} A x_{n, m}-B^{-1} f_{n, m} \\
& =\left(B^{-1} E_{1}-B^{-1} A\right) x_{n, m}-B^{-1} f_{n, m} .
\end{aligned}
$$

Then by the constant variation formula we take (2.28). This completes the proof of the proposition.

Proposition 2.4. Suppose that $A, B$ are $k \times k$ matrices. Suppose that either

$$
\begin{equation*}
|A|+|B|<1 \tag{2.30}
\end{equation*}
$$

or if $B$ is invertible and

$$
\begin{equation*}
\left|B^{-1}\right|+\left|B^{-1} A\right|<1 . \tag{2.31}
\end{equation*}
$$

Then equation (2.26) has the Hyers-Ulam stability.
Proof. Suppose firstly that (2.30) is satisfied. Let $\epsilon$ be an arbitrary number and $\delta=$ $\epsilon(1-(|A|+|B|))$. Let $y_{n, m}$ be a double sequence such that

$$
\begin{equation*}
\left|y_{n+1, m}-A y_{n, m}-B y_{n, m+1}-f_{n, m}\right|<\delta . \tag{2.32}
\end{equation*}
$$

We set

$$
y_{n+1, m}-A y_{n, m}-B y_{n, m+1}-f_{n, m}=Q_{n, m} .
$$

Then, from (2.32), it is obvious that

$$
\begin{equation*}
\left|Q_{n, m}\right|<\delta, \quad n, m \in \mathbb{N} . \tag{2.33}
\end{equation*}
$$

Arguing as in the case (i) of Proposition 2.3 we obtain

$$
\begin{equation*}
y_{n, m}=\left(A+B E_{2}\right)^{n} y_{0, m}+\sum_{s=0}^{n-1}\left(A+B E_{2}\right)^{n-s-1}\left(f_{s, m}+Q_{s, m}\right) . \tag{2.34}
\end{equation*}
$$

Let $x_{n, m}$ be a solution of (2.26) with $x_{0, m}=y_{0, m}$. Then from (2.27) and (2.34) we have

$$
\begin{equation*}
x_{n, m}-y_{n, m}=-\sum_{s=0}^{n-1}\left(A+B E_{2}\right)^{n-s-1} Q_{s, m} . \tag{2.35}
\end{equation*}
$$

Relations (2.30), (2.33) and (2.35) imply that

$$
\begin{aligned}
\left|x_{n, m}-y_{n, m}\right| & \leq \sum_{s=0}^{n-1}\left(|A|+|B| E_{2}\right)^{n-s-1}\left|Q_{s, m}\right| \\
& =\sum_{s=0}^{n-1} \sum_{k=0}^{n-s-1} \frac{(n-s-1)!}{k!(n-s-1-k)!}|A|^{n-s-1-k}|B|^{k} E_{2}^{k}\left|Q_{s, m}\right| \\
& =\sum_{s=0}^{n-1} \sum_{k=0}^{n-s-1} \frac{(n-s-1)!}{k!(n-s-1-k)!}|A|^{n-s-1-k}|B|^{k}\left|Q_{s, m+k}\right| \\
& <\delta \sum_{s=0}^{n-1}(|A|+|B|)^{n-s-1} \leq \frac{\delta}{1-(|A|+|B|)}=\epsilon .
\end{aligned}
$$

This completes the proof of case (i).
Suppose that (2.31) is fulfilled. Let $\epsilon$ be a positive number and $\delta=\epsilon\left(1-\left(\left|B^{-1}\right|+\left|B^{-1} A\right|\right)\right)$. Let $y_{n, m}$ be a double sequence such that

$$
\begin{equation*}
\left|y_{n, m+1}+B^{-1} A y_{n, m}-B^{-1} y_{n+1, m}+B^{-1} f_{n, m}\right|<\delta . \tag{2.36}
\end{equation*}
$$

We set

$$
y_{n, m+1}+B^{-1} A y_{n, m}-B^{-1} y_{n+1, m}+B^{-1} f_{n, m}=\widehat{Q}_{n, m} .
$$

Then using the same argument as in the case (ii) of Proposition 2.3 we get,

$$
\begin{equation*}
y_{n, m}=\left(-B^{-1} A+B^{-1} E_{1}\right)^{m} y_{n, 0}+\sum_{s=0}^{m-1}\left(-B^{-1} A+B^{-1} E_{1}\right)^{m-s-1}\left(\widehat{Q}_{n, s}-B^{-1} f_{n, s}\right) . \tag{2.37}
\end{equation*}
$$

Let $x_{n, m}$ be a solution of (2.26) with $x_{n, 0}=y_{n, 0}$. Then from (2.28) and (2.37) we obtain

$$
x_{n, m}-y_{n, m}=-\sum_{s=0}^{m-1}\left(-B^{-1} A+B^{-1} E_{1}\right)^{m-s-1} \widehat{Q}_{n, s} .
$$

Hence from (2.31) and (2.33) we get

$$
\begin{aligned}
\left|x_{n, m}-y_{n, m}\right| & \leq \sum_{s=0}^{m-1}\left(\left|B^{-1} A\right|+\left|B^{-1}\right| E_{1}\right)^{m-s-1}\left|\widehat{Q}_{n, s}\right| \\
& =\sum_{s=0}^{m-1} \sum_{k=0}^{m-s-1} \frac{(m-s-1)!}{k!(m-s-1-k)!}\left|B^{-1} A\right|^{m-s-1-k}\left|B^{-1}\right|^{k} E_{1}^{k}\left|\widehat{Q}_{n, s}\right| \\
& =\sum_{s=0}^{m-1} \sum_{k=0}^{m-s-1} \frac{(m-s-1)!}{k!(m-s-1-k)!}\left|B^{-1} A\right|^{m-s-1-k}\left|B^{-1}\right| k\left|\widehat{Q}_{n+k, s}\right| \\
& =\delta \sum_{s=0}^{m-1}\left(\left|B^{-1}\right|+\left|B^{-1} A\right|\right)^{m-s-1}<\frac{\delta}{1-\left(\left|B^{-1}\right|+\left|B^{-1} A\right|\right)}=\epsilon .
\end{aligned}
$$

This completes the proof of the proposition.

## Acknowledgements

The authors would like to thank the referees for their helpful suggestions.

## References

[1] D. Barbu, C. Buse, A. Tabassum, Hyers-Ulam stability and discrete dichotomy, J. Math. Anal. Appl. 423(2015), 1738-1752. https://doi.org/10.1016/j.jmaa.2014.10.082; MR3278225
[2] L. Berg, S. Stević, On some systems of difference equations, Appl. Math. Comput. 218(2011), 1713-1718. https://doi.org/10.1016/j.amc.2011.06.050; MR2831394
[3] C. Buse, D. O. Regan, O. Saierli, A. Tabassum, Hyers-Ulam stability and discrete dichotomy for difference periodic systems, Bull. Sci. Math. 140(2016), 908-934. https: //doi.org/10.1016/j.bulsci.2016.03.010; MR3569197
[4] S. S. Cheng, Partial difference equations, Advances in Discrete Mathematics and Applications, Vol. 3, Taylor and Francis, London and New York, 2003. https://doi.org/10. 1201/9780367801052; MR2193620
[5] S. S. Cheng, L. Y. Hsieh, Z. T. Chao, Discrete Lyapunov inequality conditions for partial difference equations, Hokkaido Math. J. 19(1990), 229-239. https://doi.org/10.14492/ hokmj/1381517357; MR1059167
[6] S. S. Cheng, G. H. Lin, Green's function and stability of a linear partial difference scheme, Comput. Math. Appl. 35(1998), No. 5, 27-41. https://doi.org/10.1016/S0898-1221(98) 00003-0; MR1612273
[7] S. S. Cheng, Y. F. Lu, General solutions of a three-level partial difference equation, Comput. Math. Appl. 38(1999), No. 7-8, 65-79. https://doi.org/10.1016/S0898-1221(99) 00239-4; MR1713163
[8] D. Dragičević, M. Pituk, Shadowing for nonautonomous difference equations with infinite delay, Appl. Math. Lett. 120(2021), 107284. https://doi.org/10.1016/j.aml. 2021. 107284; MR4244602
[9] C. Jordan, Calculus of finite differences, Chelsea Publishing Company, New York, 1965. MR0183987
[10] K. Konstaninidis, G. Papaschinopoulos, C. J. Schinas, Asymptotic behaviour of the solutions of systems of partial linear homogeneous and nonhomogeneous difference equations, Math. Meth. Appl. Sci. 43(2020), No. 7, 3925-3935. https ://doi.org/10.1002/mma. 6163; MR4085597
[11] V. A. Krechmar, A problem book in algebra, Mir Publishers, Moscow, 1974.
[12] H. Levy, F. Lessman, Finite difference equations, Dover Publications, Inc., New York, 1992. MR1217083
[13] Y. Z. Lin, S. S. Cheng, Stability criteria for two partial difference equations, Comput. Math. Appl. 32(1996), No. 7, 87-103. https://doi.org/10.1016/0898-1221(96)00158-7; MR1418716
[14] R. E. Mickens, Difference equations. Theory and applications, Van Nostrand Reinhold Co., New York, 1990. MR1158461
[15] A. Musielak, J. Popenda, On the hyperbolic partial difference equations and their oscillatory properties, Glas. Math. 33(1998), 209-221. MR1695527
[16] G. Papaschinopoulos, On exponential trichotomy of linear difference equations, Appl. Anal. 40(1991), 89-109. https://doi.org/10.1080/00036819108839996; MR1095407
[17] G. Papaschinopoulos, A characterization of exponential trichotomy via Lyapunov functions for difference equations, Math. Japon. 37(1992), No. 3, 555-562. MR1162469
[18] G. Papaschinopoulos, C. J. Schinas, Invariants for systems of two nonlinear difference equations, Differential Equations Dynam. Systems 7(1999), 181-196. MR1860787
[19] G. Papaschinopoulos, C. J. Schinas, Invariants and oscillation for systems of two nonlinear difference equations, Nonlinear Anal. 46(2001), 967-978. https: //doi. org/10. 1016/S0362-546X (00) 00146-2; MR1866733
[20] G. Papaschinopoulos, G. Stefanidou, Asymptotic behavior of the solutions of a class of rational difference equations, Int. J. Difference Equ. 5(2010), No. 2, 233-249. MR2771327
[21] D. Popa, Hyers-Ulam stability of the linear recurrence with constants coefficients, Adv. Difference Equ. 2005, No. 2, 101-107. https://doi.org/10.1155/ADE.2005.101; MR2197125
[22] J. Riordan, Combinatorial identities, John Wiley \& Sons Inc., New York, London, Sydney, 1968. MR0231725
[23] C. Schinas, Invariants for difference equations and systems of difference equations of rational form, J. Math. Anal. Appl. 216(1997), 164-179. https://doi.org/10.1006/jmaa. 1997.5667; MR1487258
[24] C. Schinas, Invariants for some difference equations, J. Math. Anal. Appl. 212(1997), 281291. https://doi.org/10.1006/jmaa.1997.5499; MR1460198
[25] A. Slavik and P. Stehlik, Explicit solutions to dynamic diffusion-type equations and their time integrals, Appl. Math. Comput. 234(2014), 486-505. https://doi.org/10.1016/ j.amc.2014.01.176; MR3190559
[26] S. Stević, On a third-order system of difference equations, Appl. Math. Comput. 218(2012), 7649-7654. https://doi.org/10.1016/j.amc.2012.01.034; MR2892731
[27] S. Stević, On the difference equation $x_{n}=x_{n-k} /\left(b+c x_{n-1} \cdots x_{n-k}\right)$, Appl. Math. Comput. 218(2012), 6291-6296. https://doi.org/10.1016/j.amc.2011.11.107; MR2879110
[28] S. Stević, Solutions of a max-type system of difference equations, Appl. Math. Comput. 218(2012), 9825-9830. https://doi.org/10.1016/j.amc.2012.03.057; MR2916163
[29] S. Stević, Note on the binomial partial difference equations, Electron. J. Qual. Theory Differ. Equ. 2015, No. 96, 1-11. https://doi.org/10.14232/ejqtde.2015.1.96; MR3438736
[30] S. Stević, Product-type system of difference equations of second-order solvable in closed form, Electron. J. Qual. Theory Differ. Equ. 2015, No. 56, 1-16. https://doi.org/10.14232/ ejqtde.2015.1.56; MR3407224
[31] S. Stević, Solvability of boundary value problems for a class of partial difference equations on the combinatorial domain, Adv. Difference Equ. 2016, Article No. 262, 10 pp. https://doi.org/10.1186/s13662-016-0987-z; MR3561916
[32] S. Stević, Solvable subclasses of a class of nonlinear second-order difference equations, Adv. Nonlinear Anal. 5(2016), No. 2, 147-165. https://doi.org/10.1515/anona-20150077; MR3510818
[33] S. Stević, Existence of a unique bounded solution to a linear second order difference equation and the linear first order difference equation, Adv. Difference Equ. 2017, Article No. 169, 13 pp. https://doi.org/10.1186/s13662-017-1227-x; MR3663764
[34] S. Stević, On an extension of a recurrent relation from combinatorics, Electron. J. Qual. Theory Differ. Equ. 2017, No. 84, 1-13. https://doi.org/10.14232/ejqtde.2017.1.84; MR3737099
[35] S. Stević, Representations of solutions to linear and bilinear difference equations and systems of bilinear difference equations, Adv. Difference Equ. 2018, Article No. 474, 21 pp. https://doi.org/10.1186/s13662-018-1930-2; MR3894606
[36] S. Stević, Solvability of a product-type system of difference equations with six parameters, Adv. Nonlinear Anal. 8(2019), No. 1, 29-51. https://doi.org/10.1515/anona-20160145; MR3918365
[37] S. Stević, J. Diblík, B. Iričanin, Z. Šmarda, On a third-order system of difference equations with variable coefficients, Abstr. Appl. Anal. 2012, Article ID 508523, 22 pp. https://doi.org/10.1155/2012/508523; MR2926886
[38] S. Stević, J. Diblík, B. Iričanin, Z. Šmarda, On some solvable difference equations and systems of difference equations, Abstr. Appl. Anal. 2012, Article ID 541761, 11 pp. https://doi.org/10.1155/2012/541761; MR2991014
[39] S. Stević, B. Iričanin, Z. Šmarda, Two-dimensional product-type system of difference equations solvable in closed form, Adv. Difference Equ. 2016, Article No. 253, 20 pp. https : //doi.org/10.1186/s13662-016-0980-6; MR3553954
[40] S. Stević, B. Iričanin, Z. Šmarda, Boundary value problems for some important classes of recurrent relations with two independent variables, Symmetry 9(2017), No. 12, Article No. 323, 16 pp. https://doi.org/10.3390/sym9120323
[41] S. Stević, B. Iričanin, Z. Šmarda, On a solvable symmetric and a cyclic system of partial difference equations, Filomat 32(2018), No. 6, 2043-2065. https://doi.org/10. 2298/FIL1806043S; MR3900909
[42] S. Stević, B. Iričanin, Z. Šmarda, On a symmetric bilinear system of difference equations, Appl. Math. Lett. 89(2019), 15-21. https://doi.org/10.1016/j.aml.2018.09.006; MR3886971
[43] S. V. Yablonskiy, Introduction to discrete mathematics, Mir Publishers, Moscow, Russia, 1989. MR1017897

# A nonzero solution for bounded selfadjoint operator equations and homoclinic orbits of Hamiltonian systems 

Mingliang Song ${ }^{\boxtimes 1}$ and Runzhen Li $^{2}$<br>${ }^{1}$ Mathematics and Information Technology School, Jiangsu Second Normal University, Nanjing, 210013, P. R. China.<br>${ }^{2}$ School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210097, P. R. China

> Received 20 November 2020, appeared 10 September 2021
> Communicated by Petru Jebelean


#### Abstract

We obtain an existence theorem of nonzero solution for a class of bounded selfadjoint operator equations. The main result contains as a special case the existence result of a nontrivial homoclinic orbit of a class of Hamiltonian systems by Coti Zelati, Ekeland and Séré. We also investigate the existence of nontrivial homoclinic orbit of indefinite second order systems as another application of the theorem.


Keywords: bounded selfadjoint operator equations, nonzero solution, homoclinic orbit, Hamiltonian systems, indefinite second order systems.
2020 Mathematics Subject Classification: 34A34, 34C37, 37N05, 58E05.

## 1 Introduction

In recent years several authors studied the existence of homoclinic orbits for first or second order Hamiltonian systems via variational methods and critical point theory, see for instance [2,4-6,9,12-16]. In particular, with the aid of a bounded self-adjoint linear operator and the dual action principle, Coti Zelati, Ekeland and Séré [4] obtained some existence theorems of nonzero homoclinic orbit for first order Hamiltonian systems

$$
\left\{\begin{array}{l}
x^{\prime}=J A x+J H^{\prime}(t, x) \\
x( \pm \infty)=0
\end{array}\right.
$$

via the Ambrosetti-Rabinowitz mountain-pass theorem and concentration compactness principle. Inspired by the ideas of [4], we consider the more generalized operator equation

$$
\begin{equation*}
L u-G^{\prime}(t, u)=0, \tag{1.1}
\end{equation*}
$$

where $L: L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right) \rightarrow W^{1, \beta}\left(\mathbb{R}, \mathbb{R}^{N}\right) \cap L^{\gamma}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ is a bounded linear operator for all $\gamma \geq \beta$ and for some $\beta \in(1,2)$ and $\int_{\mathbb{R}}((L u)(t), v(t)) d t=\int_{\mathbb{R}}((L v)(t), u(t)) d t$ for all $u, v \in L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$,

[^30]$G: \mathbb{R} \times \mathbb{R}^{N} \rightarrow \mathbb{R}$ and $G^{\prime}(t, u)$ denotes the gradient of $G$ with respect to $u$. $u=u(t) \in$ $L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ is called a solution of (1.1) if $(L u)(t)-G^{\prime}(t, u(t))=0$ a.e. $t \in \mathbb{R}$.

We need the following assumptions:
( $\mathrm{L}_{1}$ ) For any bounded $\left\{u_{n}\right\} \subset L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ and $R>0$, there exists a subsequence $\left\{u_{n_{j}}\right\}$ such that $L u_{n_{j}} \rightarrow w$ in $C\left([-R, R], \mathbb{R}^{N}\right)$.
$\left(\mathrm{L}_{2}\right)$ There exists $v_{0} \in L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ such that $\int_{-\infty}^{+\infty}\left(L v_{0}, v_{0}\right) d t>0$.
$\left(\mathrm{L}_{3}\right)(L u(\cdot+T))(t)=(L u)(t+T)$ for all $t \in \mathbb{R}$, where $T>0$ is a constant.
$\left(\mathrm{L}_{4}\right)|(L u)(t)| \leq c_{0} \int_{-\infty}^{+\infty} e^{-l|t-\tau|}|u(\tau)| d \tau$ for all $u \in L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$, where $c_{0}, l>0$ are two constants.
$\left(\mathrm{G}_{1}\right) G(t, \cdot)$ and $G^{\prime}(t, \cdot)$ are continuous for a.e. $t \in \mathbb{R}, G(\cdot, u)$ and $G^{\prime}(\cdot, u)$ are measurable for all $u \in \mathbb{R}^{N}, G(t, \cdot)$ is convex for all $t \in \mathbb{R}$ and $G^{* \prime}(t, \cdot)$ exists for a.e. $t \in \mathbb{R}$.
$\left(\mathrm{G}_{2}\right) G(t+T, u)=G(t, u)$ for all $t \in \mathbb{R}$.
$\left(G_{3}\right) c_{1}|u|^{\beta} \leq G(t, u) \leq c_{2}|u|^{\beta}$, where $c_{2} \geq c_{1}>0$ are two constants.
$\left(\mathrm{G}_{4}\right) \quad 0 \leq \frac{1}{\beta}\left(G^{\prime}(t, u), u\right) \leq G(t, u)$.
(G5) $\left|G^{\prime}(t, u)\right| \leq c_{3}|u|^{\beta-1}$, where $c_{3}>0$ is a constant.
Now we state our main result as follows.
Theorem 1.1. Assume $L$ and $G$ satisfy $\left(\mathrm{L}_{1}\right)-\left(\mathrm{L}_{4}\right)$ and $\left(\mathrm{G}_{1}\right)-\left(\mathrm{G}_{5}\right)$. Then (1.1) has a nonzero solution.
Remark 1.1. Although the equation (1.1) also appeared in the proof of Theorem 4.2 in [4], the bounded linear operator $L$ there equal (2.2) which comes from first order Hamiltonian systems. In this paper, $L$ discussed in (1.1) contains not only (2.2) but also (2.4) coming from indefinite second order Hamiltonian systems. In addition, introducing the condition ( $\mathrm{L}_{1}$ ) makes the proof of conclusion clearer and simpler.

The rest of this paper is organized as follows. In Section 2, we firstly establish a preliminary lemma, and then, we give two application examples for homoclinic orbit of Hamiltonian systems. In Section 3, we give the proof of our main result.

## 2 Preliminaries and examples

To complete the proof of Theorem 1.1, we need a lemma.
Lemma 2.1. Let $\frac{1}{\alpha}+\frac{1}{\beta}=1$.
(1) If $u \in L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ and $b>a>0$, then

$$
\left(\int_{|t| \geq b}\left(\int_{-a}^{a} e^{-l|t-\tau|}|u(\tau)| d \tau\right)^{\alpha} d t\right)^{\frac{1}{\alpha}} \leq 2(\alpha l)^{-\frac{2}{\alpha}} e^{-l(b-a)}\left(\int_{-a}^{a}|u(\tau)|^{\beta} d \tau\right)^{\frac{1}{\beta}} .
$$

(2) If $w, u \in L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ and $b \geq a>r \geq 0$, then

$$
\begin{aligned}
& \int_{|t| \geq b}|u(t)| \int_{a \geq|\tau| \geq r} e^{-l|t-\tau|}|w(\tau)| d \tau d t \\
& \leq 2(\alpha l)^{-\frac{2}{\alpha}} e^{-l(b-a)}\left(\int_{|t| \geq b}|u(t)|^{\beta} d t\right)^{\frac{1}{\beta}}\left(\int_{a \geq|\tau| \geq r}|w(\tau)|^{\beta} d \tau\right)^{\frac{1}{\beta}} .
\end{aligned}
$$

(3) If $w, u \in L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ and $b>a>r>0$, then

$$
\begin{aligned}
& \int_{a \leq|t| \leq b}|u(t)| \int_{|\tau| \leq a} e^{-l|t-\tau|}|w(\tau)| d \tau d t \\
& \leq 2(\alpha l)^{-\frac{2}{\alpha}}\|u\|_{L^{\beta}}\left[e^{-l(a-r)}\|w\|_{L^{\beta}}+\left(\int_{r \leq|\tau| \leq a}|w(\tau)|^{\beta} d \tau\right)^{\frac{1}{\beta}}\right] .
\end{aligned}
$$

Proof. For $u \in L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ and $b>a>0$, by some simple calculations, we have

$$
\begin{aligned}
& \left(\int_{|t| \geq b}\left(\int_{-a}^{a} e^{-l|t-\tau|}|u(\tau)| d \tau\right)^{\alpha} d t\right)^{\frac{1}{\alpha}} \\
& \quad \leq\left(\left(\int_{b}^{+\infty}+\int_{-\infty}^{-b}\right) \int_{-a}^{a} e^{-\alpha l|t-\tau|} d \tau d t\right)^{\frac{1}{\alpha}}\left(\int_{-a}^{a}|u(\tau)|^{\beta} d \tau\right)^{\frac{1}{\beta}} \\
& \quad=2^{\frac{1}{\alpha}}(\alpha l)^{-\frac{2}{\alpha}}\left(1-e^{-2 \alpha a l}\right)^{\frac{1}{\alpha}} e^{-l(b-a)}\left(\int_{-a}^{a}|u(\tau)|^{\beta} d \tau\right)^{\frac{1}{\beta}} \\
& \quad \leq 2(\alpha l)^{-\frac{2}{\alpha}} e^{-l(b-a)}\left(\int_{-a}^{a}|u(\tau)|^{\beta} d \tau\right)^{\frac{1}{\beta}},
\end{aligned}
$$

which implies that (1) holds. The same arguments also prove that (2) holds.
By (2), we have

$$
\begin{aligned}
& \int_{a \leq|t| \leq b}|u(t)| \int_{|\tau| \leq a} e^{-l|t-\tau|}|w(\tau)| d \tau d t \\
& \quad=\int_{a \leq|t| \leq b}|u(t)|\left(\int_{|\tau| \leq r}+\int_{r \leq|\tau| \leq a}\right) e^{-l|t-\tau|}|w(\tau)| d \tau d t \\
& \quad \leq 2(\alpha l)^{-\frac{2}{\alpha}}\|u\|_{L^{\beta}}\left[e^{-l(a-r)}\|w\|_{L^{\beta}}+\left(\int_{r \leq|\tau| \leq a}|w(\tau)|^{\beta} d \tau\right)^{\frac{1}{\beta}}\right] .
\end{aligned}
$$

This shows that (3) holds.
Next, we return to applications to homoclinic orbit of Hamiltonian systems. For systematic researches of homoclinic orbit of Hamiltonian systems, we refer to the excellent papers [2,4-6,9,12-16] and references therein.

As the first example we consider

$$
\left\{\begin{array}{l}
x^{\prime}=J A x+J H^{\prime}(t, x)  \tag{2.1}\\
x( \pm \infty)=0
\end{array}\right.
$$

where $J=\left(\begin{array}{cc}0 & I_{n} \\ -I_{n} & 0\end{array}\right)$ is the standard symplectic matrix in $\mathbb{R}^{2 N}, A$ is a $2 N \times 2 N$ symmetric matrix and all the eigenvalues of $J A$ have non-zero real part, $H(t, x)$ satisfies
$\left(\mathrm{H}_{1}\right) H \in C\left(\mathbb{R} \times \mathbb{R}^{2 N}, \mathbb{R}\right), H^{\prime} \in C\left(\mathbb{R} \times \mathbb{R}^{2 N}, \mathbb{R}^{2 N}\right)$ and $H(t, \cdot)$ is strictly convex;
$\left(\mathrm{H}_{2}\right) H(t+T, x)=H(t, x)$ for some $T>0$;
$\left(H_{3}\right) k_{1}|x|^{\alpha} \leq H(t, x) \leq k_{2}|x|^{\alpha}$ for some $\alpha>2$ and $0<k_{1} \leq k_{2}$;
( $\left.\mathrm{H}_{4}\right) H(t, x) \leq \frac{1}{\alpha}\left(H^{\prime}(t, x), x\right)$.
As in [4], define $G(t, u)=\sup _{x \in \mathbb{R}^{2 N}}\{(u, x)-H(t, x)\}$ and $G$ satisfies $\left(G_{1}\right)-\left(G_{5}\right)$.
Define $L: L^{\beta}\left(\mathbb{R}, \mathbb{R}^{2 N}\right) \rightarrow W^{1, \beta}\left(\mathbb{R}, \mathbb{R}^{N}\right) \cap L^{\alpha}\left(\mathbb{R}, \mathbb{R}^{2 N}\right)$ by $z=L u$ satisfies

$$
-J z^{\prime}-A z=u, z( \pm \infty)=0
$$

Then

$$
\begin{equation*}
z(t)=\int_{-\infty}^{t} e^{E(t-\tau)} P_{s} J u(\tau) d \tau-\int_{t}^{+\infty} e^{E(t-\tau)} P_{u} J u(\tau) d \tau \tag{2.2}
\end{equation*}
$$

where $E=J A, \mathbb{R}^{2 N}=E_{u} \oplus E_{s}$ and $P_{s}$ and $P_{u}$ are the projections onto $E_{s}$ and $E_{u}$ respectively satisfying $\left|e^{t E} P_{s} \xi\right| \leq k e^{-b t}|\xi|$ for $t \geq 0$ and $\left|e^{t E} P_{u} \xi\right| \leq k e^{b t}|\xi|$ for $t \leq 0, \xi \in \mathbb{R}^{2 N}$ and some $b, k>0$. So

$$
\begin{aligned}
|(L u)(t)| & \leq \int_{-\infty}^{t} k e^{-b(t-\tau)}|u(\tau)| d \tau+\int_{t}^{+\infty} k e^{b(t-\tau)}|u(\tau)| d \tau \\
& =k \int_{-\infty}^{+\infty} e^{-b|t-\tau|}|u(\tau)| d \tau,
\end{aligned}
$$

which implies that ( $\mathrm{L}_{4}$ ) holds. From Lemma 2.1 of [4], we know that $L: L^{\beta}\left(\mathbb{R}, \mathbb{R}^{2 N}\right) \rightarrow$ $W^{1, \beta}\left(\mathbb{R}, \mathbb{R}^{2 N}\right) \cap L^{\gamma}\left(\mathbb{R}, \mathbb{R}^{2 N}\right)$ is a bounded linear operator for $\gamma \geq \beta, \beta \in(1,2)$ and

$$
\int_{\mathbb{R}}((L u)(t), v(t)) d t=\int_{\mathbb{R}}((L v)(t), u(t)) d t
$$

for all $u, v \in L^{\beta}\left(\mathbb{R}, \mathbb{R}^{2 N}\right)$.
By $z^{\prime}(t)=J u(t)+E z(t)$ for all $t \in \mathbb{R}$, we have

$$
\begin{aligned}
\left|z\left(t_{1}\right)-z\left(t_{2}\right)\right| & =\left|\int_{t_{1}}^{t_{2}}(J u(t)+E z(t)) d t\right| \\
& \leq\left|t_{2}-t_{1}\right|^{\frac{1}{\alpha}}\|u\|_{L^{\beta}}+M_{0}\left|t_{2}-t_{1}\right|\|z\|_{\infty}
\end{aligned}
$$

where $M_{0}>0$, which implies that $\left(\mathrm{L}_{1}\right)$ holds. Note that the proof of (b) of Lemma 4.1 in [4], we see that there exists $v_{0} \in L^{\beta}\left(\mathbb{R}, \mathbb{R}^{2 N}\right)$ such that $\left(\mathrm{L}_{2}\right)$ holds. The validity of $\left(\mathrm{L}_{3}\right)$ is obvious.

Moreover, $G^{*}(t, x)=H(t, x)$ and a solution $u \in L^{\beta}\left(\mathbb{R}, \mathbb{R}^{2 N}\right) \backslash\{0\}$ of $L u-G^{\prime}(t, u)=0$ corresponds to a nonzero solution $x=L u$ of

$$
\left\{\begin{array}{l}
-J x^{\prime}-A x=H^{\prime}(t, x) \\
x( \pm \infty)=0
\end{array}\right.
$$

Therefore, we have the following corollary.
Corollary 2.2 ([4, Theorem 4.2]). Assume $H$ satisfies $\left(\mathrm{H}_{1}\right)-\left(\mathrm{H}_{4}\right)$. Then (2.1) has a nonzero solution, i.e., the Hamiltonian system

$$
-J x^{\prime}-A x=H^{\prime}(t, x)
$$

has at least one nontrivial homoclinic orbit.

Remark 2.1. The above corollary was essentially [4, Theorem 4.2] by Coti Zelati, Ekeland and Séré using the Ekeland variational principle and concentration compactness principle, and the equation (1.1) also appeared in the proof the theorem already.

As a second example we consider

$$
\left\{\begin{array}{l}
D x^{\prime \prime}-B x=V^{\prime}(t, x)  \tag{2.3}\\
x( \pm \infty)=0
\end{array}\right.
$$

where $D, B$ are $N \times N$ symmetric matrix, $( \pm \sigma(D)) \cap(0,+\infty) \neq \varnothing, D$ is invertible, $D^{-1} B=Q^{2}$ with $Q$ being a $N \times N$ matrix and all the eigenvalues of $Q$ have positive real part, $V: \mathbb{R} \times$ $\mathbb{R}^{N} \rightarrow \mathbb{R}$ and $V^{\prime}(t, x)$ denotes the gradient of $V$ with respect to $x$. The system was called indefinite second order system in [3].

Let

$$
\left\{\begin{array}{l}
D x^{\prime \prime}-B x=u, \\
x( \pm \infty)=0
\end{array}\right.
$$

Then

$$
\left\{\begin{array}{l}
x^{\prime \prime}-D^{-1} B x=x^{\prime \prime}-Q^{2} x=D^{-1} u \\
x( \pm \infty)=0
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
{\left[e^{t Q}\left(x^{\prime}-Q x\right)\right]^{\prime}=e^{t Q} D^{-1} u} \\
x( \pm \infty)=0
\end{array}\right.
$$

Assume $x^{\prime}(-\infty)=0$ (and this will be verified later). Then

$$
x^{\prime}-Q x=e^{-t Q} \int_{-\infty}^{t} e^{\tau Q} D^{-1} u(\tau) d \tau
$$

and

$$
\left(e^{-t Q} x\right)^{\prime}=e^{-2 t Q} \int_{-\infty}^{t} e^{\tau Q} D^{-1} u(\tau) d \tau
$$

So, we have

$$
\begin{aligned}
x & =-e^{t Q} \int_{t}^{+\infty} e^{-2 s Q}\left(\int_{-\infty}^{s} e^{\tau Q} D^{-1} u(\tau) d \tau\right) d s \\
& =-\frac{Q^{-1}}{2} e^{t Q} \int_{-\infty}^{t} e^{-2 t Q} e^{\tau Q} D^{-1} u(\tau) d \tau-\frac{Q^{-1}}{2} e^{t Q} \int_{t}^{+\infty} e^{-\tau Q} D^{-1} u(\tau) d \tau \\
& =-\frac{Q^{-1}}{2} \int_{-\infty}^{+\infty} e^{-|t-\tau| Q} D^{-1} u(\tau) d \tau .
\end{aligned}
$$

For $u \in L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$, set

$$
\begin{equation*}
x=L u=-\frac{Q^{-1}}{2} \int_{-\infty}^{+\infty} e^{-|t-\tau| Q} D^{-1} u(\tau) d \tau \tag{2.4}
\end{equation*}
$$

We claim that

$$
x=L u \in W^{1, \beta}\left(\mathbb{R}, \mathbb{R}^{N}\right) \bigcap L^{\gamma}\left(\mathbb{R}, \mathbb{R}^{N}\right)
$$

for $\gamma \geq \beta, \beta \in(1,2)$ and $u \in L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$. In fact, from all the eigenvalues of $Q$ have positive real part, we know that there exist $\lambda_{0}>0$ and $c_{4}>0$ such that $\left|e^{-|t| Q} \mathfrak{\xi}\right| \leq c_{4} e^{-\lambda_{0}|t|}|\xi|$ for $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^{N}$. By $\int_{-\infty}^{+\infty} e^{-\eta|t|} d t=\frac{2}{\eta}$, we have

$$
e^{-\lambda_{0}|t|} \in L^{\eta}(\mathbb{R}, \mathbb{R}) \quad \text { and } \quad\left\|e^{-\lambda_{0}|t|}\right\|_{L^{\eta}}^{\eta}=\frac{2}{\lambda_{0} \eta} \forall \eta \geq 1
$$

Using the convolution inequality, we have

$$
\begin{align*}
\left(\int_{-\infty}^{+\infty}|L u|^{r} d t\right)^{\frac{1}{r}} & \leq \frac{c_{4}\left\|Q^{-1}\right\| \cdot\left\|D^{-1}\right\|}{2}\left(\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} e^{-\lambda_{0}|t-\tau|}|u(\tau)| d \tau\right)^{r} d t\right)^{\frac{1}{r}} \\
& \leq \frac{c_{4}\left\|Q^{-1}\right\| \cdot\left\|D^{-1}\right\|}{2}\left\|e^{-\lambda_{0}|t|}\right\|_{L^{p}} \cdot\|u\|_{L^{\beta}} \tag{2.5}
\end{align*}
$$

for $\frac{1}{r}=\frac{1}{p}+\frac{1}{\beta}-1$ and $r, p \geq 1$, which shows that $L u \in L^{r}\left(\mathbb{R}, \mathbb{R}^{N}\right) \forall r \in[\beta,+\infty]$. Similarly, from the equation

$$
\begin{equation*}
x^{\prime}=Q x+\int_{-\infty}^{t} e^{-(t-\tau) Q} D^{-1} u(\tau) d \tau \tag{2.6}
\end{equation*}
$$

it is easy to see that $L u \in W^{1, \beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$. Moreover, by (2.5), we can also see that $L: L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right) \rightarrow$ $W^{1, \beta}\left(\mathbb{R}, \mathbb{R}^{N}\right) \cap L^{\gamma}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ is a bounded linear operator for $\gamma \geq \beta$. This implies $x( \pm \infty)=0$ and $x^{\prime}(-\infty)=0$ via the above equation.

Let $x=L u$ and $y=L v$. Then

$$
\begin{aligned}
\int_{\mathbb{R}}((L u)(t), v(t)) d t & \left.=\int_{-\infty}^{+\infty}\left(x, D y^{\prime \prime}-B y\right)\right) d t \\
& \left.=\int_{-\infty}^{+\infty}\left(D x^{\prime \prime}-B x\right), y\right) d t \\
& =\int_{\mathbb{R}}(u(t),(L v)(t)) d t
\end{aligned}
$$

for all $u, v \in L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$, which implies that $L: L^{\beta} \rightarrow L^{\alpha}$ is self-adjoint.
By (2.6) for all $t_{1}, t_{2} \in \mathbb{R}$, we have

$$
\begin{aligned}
\left|x\left(t_{1}\right)-x\left(t_{2}\right)\right| & =\left|\int_{t_{1}}^{t_{2}}\left(Q x+\int_{-\infty}^{t} e^{-(t-\tau) Q} D^{-1} u(\tau) d \tau\right) d t\right| \\
& \leq\|Q\| \cdot\|x\|_{\infty} \cdot\left|t_{2}-t_{1}\right|+c_{4}\left(\lambda_{0} \alpha\right)^{\frac{-1}{\alpha}}\left\|D^{-1}\right\| \cdot\|u\|_{L^{\beta}} \cdot\left|t_{2}-t_{1}\right|
\end{aligned}
$$

which implies that $\left(L_{1}\right)$ holds.
Since $( \pm \sigma(D)) \bigcap(0,+\infty) \neq \varnothing$, we know that there exist $\lambda_{1}<0$ and $\xi_{0} \in \mathbb{R}^{N} \backslash\{0\}$ such that $\left|\xi_{0}\right|=1$ and $D \xi_{0}=\lambda_{1} \xi_{0}$. Let

$$
x_{0}(t)= \begin{cases}\xi_{0} \sin k t, & t \in[0,2 m \pi] \\ \xi_{0}\left[\frac{k}{\pi^{2}}(t-2 m \pi-\pi)^{3}+\frac{k}{\pi}(t-2 m \pi-\pi)^{2}\right], & t \in[2 m \pi, 2 m \pi+\pi] \\ 0, & t \geq 2 m \pi+\pi \\ -x_{0}(-t), & t<0\end{cases}
$$

where $k, m \in \mathbf{N} \backslash\{0\}$. Then

$$
\left\{\begin{array}{l}
D x_{0}^{\prime \prime}-B x_{0}=v_{0} \\
x_{0}( \pm \infty)=0
\end{array}\right.
$$

and

$$
\begin{aligned}
\int_{-\infty}^{+\infty}\left(L v_{0}, v_{0}\right) d t & =2 \int_{0}^{+\infty}\left(D x_{0}^{\prime \prime}(t)-B x_{0}(t), x_{0}(t)\right) d t \\
& =2\left(\int_{0}^{2 m \pi}+\int_{2 m \pi}^{2 m \pi+\pi}\right)\left(D x_{0}^{\prime \prime}(t)-B x_{0}(t), x_{0}(t) d t\right. \\
& \geq 2\left(\int_{0}^{2 m \pi}+\int_{2 m \pi}^{2 m \pi+\pi}\right)\left[-\lambda_{1}\left|x_{0}^{\prime}(t)\right|^{2}-\|B\| \cdot\left|x_{0}(t)\right|^{2}\right] d t \\
& =-2 \lambda_{1}\left(k^{2} m \pi+\frac{2}{15} k^{2} \pi\right)-2\|B\| \cdot\left(m \pi+\frac{k^{2} \pi^{3}}{105}\right) \\
& >-2 \lambda_{1} m k^{2}-2 \pi\|B\| \cdot\left(m+k^{2}\right) \\
& >0
\end{aligned}
$$

provided $m=k^{2}$ and $k^{2}>\frac{2 \pi\|B\|}{-\lambda_{1}}$. This shows that there exists $v_{0} \in L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ such that $\left(L_{2}\right)$ holds. The validity of $\left(\mathrm{L}_{3}\right)$ and $\left(\mathrm{L}_{4}\right)$ are obvious.

Further, assume $V$ satisfies $\left(\mathrm{H}_{1}\right)-\left(\mathrm{H}_{4}\right)$ with $H(t, x)$ replaced with $V(t, x)$ and $2 N$ replaced with $N$. Define $V^{*}(t, u)=\sup _{x \in \mathbb{R}^{N}}\{(u, x)-V(t, x)\}$. Then $V^{*}(t, u)$ satisfies $\left(\mathrm{G}_{1}\right)-\left(\mathrm{G}_{5}\right)$ with $G(t, u)$ replaced with $V^{*}(t, u)$. By the Legendre reciprocity formula

$$
V^{*^{\prime}}(t, u)=x \Leftrightarrow u=V^{\prime}(t, x),
$$

we see that (2.3) is equivalent to

$$
\begin{equation*}
L u-V^{* \prime}(t, u)=0, \quad u \in L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right) \tag{2.7}
\end{equation*}
$$

Therefore, we have the following result from Theorem 1.1.
Corollary 2.3. Assume $V$ satisfies $\left(\mathrm{H}_{1}\right)-\left(\mathrm{H}_{4}\right)$ with $H(t, x)$ replaced with $V(t, x)$ and $2 N$ replaced with $N$. Then (2.3) has a nonzero solution.

## 3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. The method comes from [4] with some modifications.
Proof of Theorem 1.1. We define the functional I on $L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ by

$$
\begin{equation*}
I(u)=\int_{\mathbb{R}} G(t, u) d t-\frac{1}{2} \int_{\mathbb{R}}(L u, u) d t \tag{3.1}
\end{equation*}
$$

for all $u \in L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$. From $\left(G_{3}\right)$, we have

$$
0 \leq \int_{\mathbb{R}} G(t, u) d t \leq c_{2} \int_{\mathbb{R}}|u|^{\beta} d t<+\infty .
$$

Noticing that $L u \in L^{\alpha}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ and $L$ is a bounded linear operator, then $\int_{\mathbb{R}}(L u, u) d t$ is well defined. Since $G(t, \cdot)$ and $G^{\prime}(t, \cdot)$ are continuous for a.e. $t \in \mathbb{R}$, from $\left(G_{5}\right)$, we know that the functional $I$ is a $C^{1}$ functional. Moreover, a solution of (1.1) correspond to a critical point of the functional $I$.

Next, we take five steps to prove the existence of the critical point of the functional $I$.
Step 1. There exists a sequence $\left\{u_{n}\right\} \subset L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ such that $I\left(u_{n}\right) \rightarrow c>0$ and $I^{\prime}\left(u_{n}\right) \rightarrow 0$.

By $\left(\mathrm{L}_{2}\right)$ and $\left(\mathrm{G}_{3}\right)$, for $v_{0} \in L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right), \beta \in(1,2)$ and $s>0$ we have

$$
\begin{aligned}
I\left(s v_{0}\right) & =\int_{\mathbb{R}} G\left(t, s v_{0}\right) d t-\frac{s^{2}}{2} \int_{\mathbb{R}}\left(L v_{0}, v_{0}\right) d t \\
& \leq c_{2} s^{\beta} \int_{\mathbb{R}}\left|v_{0}\right|^{\beta} d t-\frac{s^{2}}{2} \int_{\mathbb{R}}\left(L v_{0}, v_{0}\right) d t \\
& \rightarrow-\infty \text { as } s \rightarrow+\infty
\end{aligned}
$$

which shows there is $s_{0}>0$ such that $I\left(s_{0} v_{0}\right)<0$. Set $u_{0}=s_{0} v_{0}$ and define

$$
c=\inf _{\gamma \in \Gamma} \sup _{u \in \gamma([0,1])} I(u)
$$

where $\Gamma=\left\{\gamma \in C\left([0,1], L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)\right) \mid \gamma(0)=0, \gamma(1)=u_{0}\right\}$.
By $\left(G_{3}\right)$, we have

$$
\begin{aligned}
I(u) & \geq c_{1} \int_{\mathbb{R}}|u|^{\beta} d t-\frac{1}{2} \int_{\mathbb{R}}(L u, u) d t \\
& \geq c_{1}\|u\|_{L^{\beta}}^{\beta}-\frac{M}{2}\|u\|_{L^{\beta}}^{2}
\end{aligned}
$$

where $M>0$ and $\|L u\|_{L^{\alpha}} \leq M\|u\|_{L^{\beta}}$. Since $\beta \in(1,2)$, there exists $r \in\left(0,\left\|u_{0}\right\|_{L^{\beta}}\right)$ such that $c_{1} r^{\beta}-\frac{M}{2} r^{2}>0$. So $\sup _{u \in \gamma([0,1])} I(u) \geq c_{1} r^{\beta}-\frac{M}{2} r^{2}>0$ and $c>0$. By [7, Theorem V.1.6], the result follows.
Step 2. We prove that the sequence $\left\{u_{n}\right\} \subset L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ is bounded and there exist $\delta_{2}>\delta_{1}>0$ such that $\left\|u_{n}\right\|_{L^{\beta}} \in\left[\delta_{1}, \delta_{2}\right]$.

Clearly,

$$
\left\langle I^{\prime}\left(u_{n}\right), u_{n}\right\rangle=\int_{\mathbb{R}}\left(G^{\prime}\left(t, u_{n}\right), u_{n}\right) d t-\int_{\mathbb{R}}\left(L u_{n}, u_{n}\right) d t
$$

Using $\left(G_{3}\right)$ and $\left(G_{4}\right)$, we have

$$
\begin{aligned}
I\left(u_{n}\right)+\frac{1}{2}\left\|I^{\prime}\left(u_{n}\right)\right\|_{L^{\alpha}} \cdot\left\|u_{n}\right\|_{L^{\beta}} & \geq I\left(u_{n}\right)-\frac{1}{2}\left\langle I^{\prime}\left(u_{n}\right), u_{n}\right\rangle \\
& =\int_{\mathbb{R}} G\left(t, u_{n}\right) d t-\frac{1}{2} \int_{\mathbb{R}}\left(G^{\prime}\left(t, u_{n}\right), u_{n}\right) d t \\
& \geq\left(1-\frac{\beta}{2}\right) \int_{\mathbb{R}} G\left(t, u_{n}\right) d t \\
& \geq\left(1-\frac{\beta}{2}\right) c_{1}\left\|u_{n}\right\|_{L^{\beta}}^{\beta} .
\end{aligned}
$$

Since $c_{1}>0,1<\beta<2, I\left(u_{n}\right) \rightarrow c>0$ and $\left\|I^{\prime}\left(u_{n}\right)\right\|_{L^{\alpha}} \rightarrow 0$, we deduce that $\left\{u_{n}\right\}$ is bounded in $L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$.

Again, from (3.1) and $\left(\mathrm{G}_{3}\right)$, we have

$$
\begin{aligned}
I\left(u_{n}\right) & \leq c_{2} \int_{\mathbb{R}}\left|u_{n}\right|^{\beta} d t-\frac{1}{2} \int_{\mathbb{R}}\left(L u_{n}, u_{n}\right) d t \\
& \leq c_{2}\left\|u_{n}\right\|_{L^{\beta}}^{\beta}+\frac{M}{2}\left\|u_{n}\right\|_{L^{\beta}}^{2} .
\end{aligned}
$$

If there is a subsequence $\left\{u_{n_{k}}\right\}$ such that $\left\|u_{n_{k}}\right\|_{L^{\beta}} \rightarrow 0$, then

$$
I\left(u_{n_{k}}\right) \leq c_{2}\left\|u_{n_{k}}\right\|_{L^{\beta}}^{\beta}+\frac{M}{2}\left\|u_{n_{k}}\right\|_{L^{\beta}}^{2} \rightarrow 0 \Rightarrow c \leq 0
$$

which contradicts $c>0$.

Set $\rho_{n}(t)=\frac{\left|u_{n}(t)\right|^{\beta}}{\left\|u_{n}\right\|_{L^{\beta}}^{\beta}}$. Then $\int_{-\infty}^{+\infty} \rho_{n}(t) d t=1$. By [4, page 145 , Lemma] (also see $[10,11]$ ), we have three possibilities:
(i) vanishing

$$
\sup _{y \in \mathbb{R}} \int_{y-R}^{y+R} \rho_{n}(t) d t \rightarrow 0 \text { as } n \rightarrow \infty \forall R>0
$$

(ii) concentration

$$
\exists y_{n} \in \mathbb{R}: \forall \varepsilon>0 \exists R>0: \int_{y_{n}-R}^{y_{n}+R} \rho_{n}(t) d t \geq 1-\varepsilon \forall n ;
$$

(iii) dichotomy
$\exists y_{n} \in \mathbb{R}, \exists \lambda \in(0,1), \exists R_{n}^{1}, R_{n}^{2} \in \mathbb{R}$ such that
(a) $R_{n}^{1}, R_{n}^{2} \rightarrow+\infty, \frac{R_{n}^{1}}{R_{n}^{2}} \rightarrow 0$;
(b) $\int_{y_{n}-R_{n}^{1}}^{y_{n}+R_{n}^{1}} \rho_{n}(t) d t \rightarrow \lambda$ as $n \rightarrow \infty$;
(c) $\forall \varepsilon>0 \exists R>0$ such that $\int_{y_{n}-R}^{y_{n}+R} \rho_{n}(t) d t \geq \lambda-\varepsilon \forall n$;
(d) $\int_{y_{n}-R_{n}^{2}}^{y_{n}+R_{n}^{2}} \rho_{n}(t) d t \rightarrow \lambda$ as $n \rightarrow \infty$.

Step 3. Vanishing cannot occur.
Otherwise, there exists a nonnegative sequence $\varepsilon_{n} \rightarrow 0$ such that

$$
\int_{s-1}^{s+1}\left|u_{n}(t)\right|^{\beta} d t \leq \varepsilon_{n}\left\|u_{n}\right\|_{L^{\beta}}^{\beta} \quad \forall s \in \mathbb{R}
$$

By ( $\mathrm{L}_{4}$ ), we have

$$
\begin{aligned}
\left|\left(L u_{n}\right)(t)\right| \leq & c_{0} \int_{-\infty}^{+\infty} e^{-l|t-\tau|}\left|u_{n}(\tau)\right| d \tau \\
= & c_{0} \int_{t}^{+\infty} e^{-l|t-\tau|}\left|u_{n}(\tau)\right| d \tau+c_{0} \int_{-\infty}^{t} e^{-l|t-\tau|}\left|u_{n}(\tau)\right| d \tau \\
\leq & c_{0} e^{l t} \sum_{k=0}^{+\infty}\left(\int_{t+k}^{t+k+1} e^{-\alpha l \tau} d \tau\right)^{\frac{1}{\alpha}}\left(\int_{t+k}^{t+k+1}\left|u_{n}(\tau)\right|^{\beta} d \tau\right)^{\frac{1}{\beta}} \\
& +c_{0} e^{-l t} \sum_{k=0}^{+\infty}\left(\int_{t-k-1}^{t-k} e^{\alpha l \tau} d \tau\right)^{\frac{1}{\alpha}}\left(\int_{t-k-1}^{t-k}\left|u_{n}(\tau)\right|^{\beta} d \tau\right)^{\frac{1}{\beta}} \\
\leq & 2 c_{0} \varepsilon_{n}^{\frac{1}{\beta}}\left\|u_{n}\right\|_{L^{\beta}}\left(\frac{1-e^{-\alpha l}}{\alpha l}\right)^{\frac{1}{\alpha}} \cdot \frac{1}{1-e^{-l}} \rightarrow 0
\end{aligned}
$$

as $n \rightarrow \infty$ uniformly for $t \in \mathbb{R}$, which implies that $\left\|L u_{n}\right\|_{\infty} \rightarrow 0$.
From $L: L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right) \rightarrow W^{1, \beta}\left(\mathbb{R}, \mathbb{R}^{N}\right) \bigcap L^{\gamma}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ is a bounded linear operator for $\gamma \geq \beta$, we obtain $\left\|L u_{n}\right\|_{L^{\beta}} \leq c_{5}\left\|u_{n}\right\|_{L^{\beta}}$, where $c_{5}>0$. Since

$$
\left\|L u_{n}\right\|_{L^{\alpha}}^{\alpha}=\int_{\mathbb{R}}\left|L u_{n}\right|^{\alpha} d t \leq\left\|L u_{n}\right\|_{\infty}^{\alpha-\beta} \int_{\mathbb{R}}\left|L u_{n}\right|^{\beta} d t \leq c_{5}\left\|u_{n}\right\|_{L^{\beta}}\left\|L u_{n}\right\|_{\infty}^{\alpha-\beta},
$$

we have $\left\|L u_{n}\right\|_{L^{\alpha}} \rightarrow 0$. By $\left(G_{3}\right)$ and the convexity of $G(t, \cdot), G(t, 0) \equiv 0$ and $G\left(t, u_{n}\right) \leq$ ( $\left.G^{\prime}\left(t, u_{n}\right), u_{n}\right)$. So

$$
\int_{\mathbb{R}}\left|u_{n}\right|^{\beta} d t \leq \frac{1}{c_{1}} \int_{\mathbb{R}}\left(G^{\prime}\left(t, u_{n}\right), u_{n}\right) d t \leq \frac{1}{c_{1}}\left\|G^{\prime}\left(t, u_{n}\right)\right\|_{L^{*}} \cdot\left\|u_{n}\right\|_{L^{\beta}} \rightarrow 0,
$$

since $G^{\prime}\left(t, u_{n}\right)=L u_{n}+I^{\prime}\left(u_{n}\right) \rightarrow 0$ in $L^{\alpha}\left(\mathbb{R}, \mathbb{R}^{N}\right)$. This is a contradiction to $\left\|u_{n}\right\|_{L^{\beta}} \geq \delta_{1}>0$.
Step 4. Concentration implies the existence of a nontrivial solution of (1.1).
If concentration occurs, we set

$$
w_{n}(t)=u_{n}\left(t+y_{n}\right), \quad v_{n}(t)=\frac{w_{n}(t)}{\left\|w_{n}\right\|_{L^{\beta}}} .
$$

Then $\int_{\mathbb{R}}\left|v_{n}(t)\right|^{\beta} d t=1$ and for every $\varepsilon_{1}>0$ there exists $R>0$ such that

$$
\begin{equation*}
1-\varepsilon_{1} \leq \int_{-R}^{R}\left|v_{n}(t)\right|^{\beta} d t \leq 1 . \tag{3.2}
\end{equation*}
$$

We claim there is $\bar{z}$ and a subsequence denoted also by itself such that

$$
\begin{equation*}
L v_{n} \rightarrow \bar{z} \quad \text { in } L^{\alpha}\left(\mathbb{R}, \mathbb{R}^{N}\right) . \tag{3.3}
\end{equation*}
$$

In fact it suffices to show that for every $\varepsilon>0$ there exist $z_{\varepsilon} \in L^{\alpha}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ and subsequence $v_{n_{j}}$ such that

$$
\left\|L v_{n_{j}}-z_{\varepsilon}\right\|_{L^{x}} \leq \varepsilon
$$

Let $v_{n}^{(1)}(t)=v_{n}(t) \chi_{[-R, R]}(t)$ and $v_{n}^{(2)}(t)=v_{n}(t)-v_{n}^{(1)}(t)$. By $\left(\mathrm{L}_{1}\right)$, for every $t_{0}>0$ there exist $\left\{v_{n_{j}}^{(1)}\right\}$ and $u_{\varepsilon}^{(1)} \in C\left(\left[-t_{0}, t_{0}\right], \mathbb{R}^{N}\right)$ such that $L v_{n_{j}}^{(1)} \rightarrow u_{\varepsilon}^{(1)}$ in $C\left(\left[-t_{0}, t_{0}\right], \mathbb{R}^{N}\right)$. Define $u_{\varepsilon}(t)=u_{\varepsilon}^{(1)}(t)$ for $t \in\left[-t_{0}, t_{0}\right]$ and $u_{\varepsilon}(t)=0$ otherwise. Then

$$
\left\|L v_{n_{j}}-u_{\varepsilon}\right\|_{L^{\alpha}} \leq\left\|L v_{n_{j}}^{(2)}\right\|_{L^{\alpha}}+\left\|L v_{n_{j}}^{(1)}-u_{\varepsilon}\right\|_{L^{\alpha}} \leq M \varepsilon_{1}^{\frac{1}{b}}+\left\|L v_{n_{j}}^{(1)}-u_{\varepsilon}\right\|_{L^{\alpha}}
$$

and

$$
\begin{aligned}
& \left(\int_{-\infty}^{+\infty}\left|L v_{n_{j}}^{(1)}-u_{\varepsilon}\right|^{\alpha} d t\right)^{\frac{1}{\alpha}} \\
& \quad \leq\left(\int_{|t| \geq t_{0}}\left|L v_{n_{j}}^{(1)}\right|^{\alpha} d t+2 t_{0}\left\|L v_{n_{j}}^{(1)}-u_{\varepsilon}\right\|_{C\left[-t_{0}, t_{0}\right]}^{\alpha}\right)^{\frac{1}{\alpha}} \\
& \quad \leq c_{0}\left(\int_{|t| \geq t_{0}}\left(\int_{-R}^{R} e^{-l|t-\tau|}\left|v_{n_{j}}^{(1)}(\tau)\right| d \tau\right)^{\alpha} d t\right)^{\frac{1}{\alpha}}+\left(2 t_{0}\right)^{\frac{1}{\alpha}}\left\|L v_{n_{j}}^{(1)}-u_{\varepsilon}\right\|_{C\left[-t_{0}, t_{0}\right]} \\
& \quad \leq 2 c_{0}(\alpha l)^{-\frac{2}{\alpha}} e^{-l\left(t_{0}-R\right)}\left(\int_{-R}^{R}\left|v_{n_{j}}^{(1)}(\tau)\right|^{\beta} d \tau\right)^{\frac{1}{\beta}}+\left(2 t_{0}\right)^{\frac{1}{\alpha}}\left\|L v_{n_{j}}^{(1)}-u_{\varepsilon}\right\|_{C\left[-t_{0}, t_{0}\right]} \\
& \quad \leq 2 c_{0}(\alpha l)^{-\frac{2}{\alpha}} e^{-l\left(t_{0}-R\right)}+\left(2 t_{0}\right)^{\frac{1}{\alpha}}\left\|L v_{n_{j}}^{(1)}-u_{\varepsilon}\right\|_{C\left[-t_{0}, t_{0}\right]}
\end{aligned}
$$

via (1) of Lemma 2.1 and $\int_{\mathbb{R}}\left|v_{n}(t)\right|^{\beta} d t=1$, where $t_{0}>R$.
For any $\varepsilon>0$, there is $\varepsilon_{1}>0$ such that $M \varepsilon_{1}^{\frac{1}{\beta}} \leq \frac{\varepsilon}{3}$, and there exists $R=R\left(\varepsilon_{1}\right)>0$ such that (3.2) is satisfied. For the above $R>0$, there exists $t_{0}>R$ such that

$$
2 c_{0}(\alpha l)^{-\frac{2}{\alpha}} e^{-l\left(t_{0}-R\right)} \leq \frac{\varepsilon}{3} .
$$

Then we can choose subsequence $v_{n_{j}}$ such that

$$
\left(2 t_{0}\right)^{\frac{1}{\alpha}}\left\|L v_{n_{j}}^{(1)}-u_{\varepsilon}\right\|_{\mathcal{C}\left[-t_{0}, t_{0}\right]} \leq \frac{\varepsilon}{3}
$$

via $\left(L_{1}\right)$. It follows that

$$
\left\|L v_{n_{j}}-u_{\varepsilon}\right\|_{L^{a}} \leq \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon .
$$

From (3.3) and the boundedness of $\left\|w_{n}\right\|_{L^{\beta}}$, there exists $z \in L^{\alpha}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ such that $L w_{n} \rightarrow$ $z$ in $L^{\alpha}\left(\mathbb{R}, \mathbb{R}^{N}\right)$. We assume $\frac{y_{n}}{T} \in \mathbf{Z}$. It follows that $I\left(w_{n}\right)=I\left(u_{n}\right)$ and that $I^{\prime}\left(w_{n}\right)(t)=$ $I^{\prime}\left(u_{n}\right)\left(t+y_{n}\right)$, and $I\left(w_{n}\right) \rightarrow c, I^{\prime}\left(w_{n}\right) \rightarrow 0$ in $L^{\alpha}\left(\mathbb{R}, \mathbb{R}^{N}\right)$. Then

$$
z_{n}(t)=G^{\prime}\left(t, w_{n}\right)=I^{\prime}\left(w_{n}\right)(t)+\left(L w_{n}\right)(t) \rightarrow z \quad \text { in } L^{\alpha}\left(\mathbb{R}, \mathbb{R}^{N}\right)
$$

We have $w_{n}=G^{* \prime}\left(t, z_{n}\right) \rightarrow G^{* \prime}(t, z)=w$ on $L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$. Taking limit on both sides of

$$
G^{\prime}\left(t, w_{n}\right)-L w_{n}=I^{\prime}\left(w_{n}\right),
$$

we have $G^{\prime}(t, w)-L w=0$, i.e., $u=w$ is a nontrivial solution of (1.1).
Step 5. Dichotomy also leads to a nontrivial solution of (1.1).
If dichotomy occurs, we set

$$
\begin{aligned}
w_{n}(t) & =u_{n}\left(t+y_{n}\right) \\
w_{n}^{(1)}(t) & =w_{n}(t) \chi_{\left[-R_{n}^{1}, R_{n}^{1}\right]}(t) \\
w_{n}^{(2)}(t) & =w_{n}(t)\left(1-\chi_{\left[-R_{n}^{2}, R_{n}^{2}\right]}(t)\right) \\
w_{n}^{(3)}(t) & =w_{n}(t)-w_{n}^{(1)}(t)-w_{n}^{(2)}(t) \\
v_{n}^{(1)}(t) & =\frac{w_{n}^{(1)}(t)}{\left\|w_{n}^{(1)}\right\|_{L^{\beta}}}
\end{aligned}
$$

By (b) of the dichotomy, we have

$$
\int_{-\infty}^{+\infty} \frac{\left|w_{n}^{(1)}(t)\right|^{\beta}}{\left\|w_{n}\right\|_{L^{\beta}}^{\beta}} d t=\int_{-R_{n}^{1}}^{R_{n}^{1}} \frac{\left|w_{n}(t)\right|^{\beta}}{\left\|w_{n}\right\|_{L^{\beta}}^{\beta}} d t \rightarrow \lambda>0 .
$$

From $\delta_{2} \geq\left\|w_{n}\right\|_{L^{\beta}}=\left\|u_{n}\right\|_{L^{\beta}} \geq \delta_{1}$, we can see that there exists $\delta_{3}>0$ such that $\left\|w_{n}^{(1)}\right\|_{L^{\beta}}>\delta_{3}$. By Step 4 and $\left(\mathrm{L}_{1}\right), w_{n}^{(1)}(t) \rightarrow z$ in $L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$ and $\|z\|_{L^{\beta}} \geq \delta_{3}$. We will show that $I^{\prime}\left(w_{n}^{(1)}\right) \rightarrow 0$, and hence $I^{\prime}(z)=0$, that is, $u=z$ is a nontrivial solution of (1.1). In fact, for any $u \in$ $L^{\beta}\left(\mathbb{R}, \mathbb{R}^{N}\right)$, as the splitting of $w_{n}, u=u^{(1)}+u^{(2)}+u^{(3)}$, and

$$
\begin{aligned}
\left\langle I^{\prime}\left(w_{n}^{(1)}\right), u\right\rangle & =\int_{-\infty}^{+\infty}\left(G^{\prime}\left(t, w_{n}^{(1)}\right), u^{(1)}\right) d t-\int_{-\infty}^{+\infty}\left(L w_{n}^{(1)}, u\right) d t \\
& =\left\langle I^{\prime}\left(w_{n}\right), u^{(1)}\right\rangle-\int_{-\infty}^{+\infty}\left(L w_{n}^{(1)}, u^{(2)}+u^{(3)}\right) d t+\int_{-\infty}^{+\infty}\left(L\left(w_{n}^{(2)}+w_{n}^{(3)}\right), u^{(1)}\right) d t .
\end{aligned}
$$

In the following we assume $\|u\|_{L^{\beta}} \leq 1$ and the limits will be taken as $n \rightarrow+\infty$. From (b) and (d) of the dichotomy, we have

$$
\left\|w_{n}^{(3)}\right\|_{L^{\beta}}^{\beta}=\int_{-\infty}^{+\infty}\left|w_{n}^{(3)}\right|^{\beta} d t=\int_{|t| \leq R_{n}^{2}}\left|w_{n}\right|^{\beta} d t-\int_{|t| \leq R_{n}^{1}}\left|w_{n}^{(1)}\right|^{\beta} d t \rightarrow 0,
$$

which shows that

$$
\begin{equation*}
\left|\int_{-\infty}^{+\infty}\left(L w_{n}^{(3)}, u^{(1)}\right) d t\right| \leq M\left\|w_{n}^{(3)}\right\|_{L^{\beta}}\left\|u^{(1)}\right\|_{L^{\beta}} \leq M\left\|w_{n}^{(3)}\right\|_{L^{\beta}} \rightarrow 0 \tag{3.4}
\end{equation*}
$$

Using ( $\mathrm{L}_{4}$ ), (2) of Lemma 2.1 and (a) of the dichotomy, we have

$$
\begin{align*}
\left|\int_{-\infty}^{+\infty}\left(L w_{n}^{(2)}, u^{(1)}\right) d t\right| & \leq c_{0} \int_{-R_{n}^{1}}^{R_{n}^{1}}|u(t)| \int_{|\tau| \geq R_{n}^{2}} e^{-l|t-\tau|}\left|w_{n}(\tau)\right| d \tau d t \\
& \leq 2 c_{0}(\alpha l)^{-\frac{2}{\alpha}} e^{-l\left(R_{n}^{2}-R_{n}^{1}\right)}\|u\|_{L^{\beta}}\left\|w_{n}\right\|_{L^{\beta}} \\
& \leq 2 c_{0}(\alpha l)^{-\frac{2}{\alpha}} e^{-l\left(R_{n}^{2}-R_{n}^{1}\right)} \delta_{2} \rightarrow 0 \tag{3.5}
\end{align*}
$$

and

$$
\begin{align*}
\left|\int_{-\infty}^{+\infty}\left(L w_{n}^{(1)}, u^{(2)}\right) d t\right| & =\left|\int_{-\infty}^{+\infty}\left(L u^{(2)}, w_{n}^{(1)}\right) d t\right| \\
& \leq 2 c_{0}(\alpha l)^{-\frac{2}{\alpha}} e^{-l\left(R_{n}^{2}-R_{n}^{1}\right)} \delta_{2} \rightarrow 0 . \tag{3.6}
\end{align*}
$$

By (c) of the dichotomy, we have that for any $\varepsilon_{1}>0$ there is $R>0$ such that $\int_{-R}^{R} \frac{\left|w_{n}(t)\right|^{\beta}}{\left\|w_{n}\right\|_{L^{\beta}}^{\beta}} d t \geq$ $\lambda-\varepsilon_{1}$. Using (b) of the dichotomy, we obtain $\int_{R \leq|\tau| \leq R_{n}^{1}}\left|w_{n}(\tau)\right|^{\beta} d \tau \leq \varepsilon_{1}\left\|w_{n}\right\|_{L^{\beta}}^{\beta}$. By ( $\mathrm{L}_{4}$ ), (3) of Lemma 2.1 and (a) of the dichotomy, we have

$$
\begin{align*}
\left|\int_{-\infty}^{+\infty}\left(L w_{n}^{(1)}, u^{(3)}\right) d t\right| & \leq c_{0} \int_{R_{n}^{1} \leq|t| \leq R_{n}^{2}}|u(t)| \int_{|\tau| \leq R_{n}^{1}} e^{-l|t-\tau|}\left|w_{n}(\tau)\right| d \tau d t \\
& \leq 2 c_{0}(\alpha l)^{-\frac{2}{\alpha}}\|u\|_{L^{\beta}}\left[e^{-l\left(R_{n}^{1}-R\right)}\left\|w_{n}\right\|_{L^{\beta}}+\left(\int_{R \leq|\tau| \leq R_{n}^{1}}\left|w_{n}(\tau)\right|^{\beta} d \tau\right)^{\frac{1}{\beta}}\right] \\
& \leq 2 c_{0}(\alpha l)^{-\frac{2}{\alpha}}\|u\|_{L^{\beta}}\left\|w_{n}\right\|_{L^{\beta}}\left(e^{-l\left(R_{n}^{1}-R\right)}+\varepsilon_{1}^{\frac{1}{\beta}}\right) \\
& \leq 2 c_{0}(\alpha l)^{-\frac{2}{\alpha}} \delta_{2}\left(e^{-l\left(R_{n}^{1}-R\right)}+\varepsilon_{1}^{\frac{1}{\beta}}\right) \\
& \rightarrow 2 c_{0}(\alpha l)^{-\frac{2}{\alpha}} \delta_{2} \varepsilon_{1}^{\frac{1}{\beta}} \tag{3.7}
\end{align*}
$$

Noticing $I^{\prime}\left(w_{n}\right) \rightarrow 0$, from (3.4)-(3.7), for any $\epsilon>0$ choosing $\varepsilon_{1}>0$ satisfying $2 c_{0}(\alpha l)^{-\frac{2}{\alpha}} \delta_{2} \varepsilon_{1}^{\frac{1}{\beta}} \leq \epsilon$, we find that $\lim \sup _{n \rightarrow+\infty}\left\|I^{\prime}\left(w_{n}^{(1)}\right)\right\|_{L^{\beta}} \leq \epsilon$ and hence $I^{\prime}\left(w_{n}^{(1)}\right) \rightarrow 0$. The proof is complete.

## Acknowledgements

This research was supported by the National Natural Science Foundation of China (Grant No. 12071219).

## References

[1] R. A. Adams, J. J. F. Fournier, Sobolev spaces, Elsevier, 2003. MR2424078
[2] P. Caldiroli, P. Montecchiari, Homoclinic orbits for second order Hamiltonian systems with potential changing sign, Commun. Appl. Nonlinear Anal. 1(1994), 97-129. MR1280118
[3] A. Capietto, F. Dalbono, A. Portaluri, A multiplicity result for a class of strongly indefinite asymptotically linear second order systems, Nonlinear Anal. 72(2010), 28742890. https://doi.org/10.1016/j.na.2009.11.032; MR2580145
[4] V. Coti Zelati, I. Ekeland, E. Séré, A variational approach to homoclinic orbits in Hamiltonian systems, Math. Ann. 288(1990), 133-160. https://doi.org/10.1007/ BF01444526; MR1070929
[5] V. Coti Zelati, P. H. Rabinowitz, Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials, J. Amer. Math. Soc. 4(1991), No. 4, 693-727. https://doi.org/10.2307/2939286; MR1119200
[6] Y. Ding, M. Willem, Homoclinic orbits of a Hamiltonian system, Z. Angew. Math. Phys. 50(1999), No. 5, 759-778. https://doi.org/10.1007/s000330050177; MR1721793
[7] I. Ekeland, Convexity methods in Hamiltonian mechanics, Springer-Verlag, Berlin, 1990. https://doi.org/10.1007/978-3-642-74331-3; MR1051888
[8] I. Ekeland, R. Temam, Convex analysis and variational problems, North Holland, Amsterdam, 1976. MR0463994
[9] H. Hofer, K. Wysocki, First order elliptic systems and the existence of homoclinic orbits in Hamiltonian systems, Math. Ann. 288(1990), No. 1, 483-503. https://doi.org/10. 1007/BF01444543; MR1079873
[10] P. L. Lions, Solutions of Hartree-Fock equations for Coulomb systems, Comm. Math. Phys. 109(1987), No. 1, 33-97.
[11] P. L. Lions, The concentration-compactness principle in the calculus of variations, Rev. Mat. Iberoamericana 1(1985), 145-201. https ://doi. org/10.4171/RMI/6; MR834360
[12] P. Montecchiari, P. H. Rabinowitz, Solutions of mountain pass type for double well potential systems, Calc. Var. Partial Differential Equations 57(2018), No. 5, Paper No. 114, 31 pp. https://doi.org/10.1007/s00526-018-1400-4; MR3829548
[13] W. Omana, M. Willem, Homoclinic orbits for a class of Hamiltonian systems, Differential Integral Equations 5(1992), No. 5, 33-38. MR1171983
[14] P. H. Rabinowitz, Homoclinic orbits for a class of Hamiltonian systems, Proc. Roy. Soc. Edinburgh Sect. A 114(1990), 33-38. https://doi.org/10.1017/S0308210500024240; MR1051605
[15] P. H. Rabinowitz, Homoclinic and heteroclinic orbits for a class of Hamiltonian systems, Calc. Var. Partial Differential Equations 1(1993), 1-36. https://doi.org/10.1007/ BF02163262; MR1261715
[16] E. Séré, Existence of infinitely many homoclinic orbits in Hamiltonian systems, Math. Z. 209(1992), 27-42. https://doi.org/10.1007/BF02570817; MR1143210

# Limit cycles of planar discontinuous piecewise linear Hamiltonian systems without equilibria separated by reducible cubics 

Rebiha Benterki ${ }^{1}$, Johana Jimenez ${ }^{\boxtimes 2}$ and Jaume Llibre ${ }^{3}$<br>${ }^{1}$ Département de Mathématiques, Université Mohamed El Bachir El Ibrahimi, Bordj Bou Arréridj 34000, El Anasser, Algeria<br>${ }^{2}$ Universidade Federal do Oeste da Bahia, 46470000 Bom Jesus da Lapa, Bahia, Brazil<br>${ }^{3}$ Departament de Matematiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Received 16 January 2021, appeared 10 September 2021
Communicated by Gabriele Villari


#### Abstract

Due to their applications to many physical phenomena during these last decades the interest for studying the discontinuous piecewise differential systems has increased strongly. The limit cycles play a main role in the study of any planar differential system, but to determine the maximum number of limits cycles that a class of planar differential systems can have is one of the main problems in the qualitative theory of the planar differential systems. Thus in general to provide a sharp upper bound for the number of crossing limit cycles that a given class of piecewise linear differential system can have is a very difficult problem. In this paper we characterize the existence and the number of limit cycles for the piecewise linear differential systems formed by linear Hamiltonian systems without equilibria and separated by a reducible cubic curve, formed either by an ellipse and a straight line, or by a parabola and a straight line parallel to the tangent at the vertex of the parabola. Hence we have solved the extended 16th Hilbert problem to this class of piecewise differential systems.


Keywords: limit cycles, discontinuous piecewise linear Hamiltonian systems, reducible cubic curves.
2020 Mathematics Subject Classification: 34C29, 34C25, 47H11.

## 1 Introduction and statement of the main results

Andronov, Vitt and Khaikin [1] started around 1920's the study of the piecewise differential systems mainly motivated for their applications to some mechanical systems, and nowadays these systems still continue to receive the attention of many researchers. Thus these differential systems are widely used to model processes appearing in mechanics, electronics, economy, etc., see for instance the books [8] and [28], and the survey [25], as well as the hundreds of references cited there.

[^31]A limit cycle is a periodic orbit of the differential system isolated in the set of all periodic orbits of the system. Limit cycles are important in the study of the differential systems. Thus limit cycles have played and are playing a main role for explaining physical phenomena, see for instance the limit cycle of van der Pol equation [26,27], or the one of the BelousovZhavotinskii model [3,29], etc.

The extended 16th Hilbert problem, that is, to find an upper bound for the maximum number of limit cycles that a given class of differential systems can exhibit, is in general an unsolved problem. Only for very few classes of differential system this problem has been solved. For the class of discontinuous piecewise differential systems here studied, we can obtain its solution by using the first integrals provided by the Hamiltonians of the systems which form the discontinuous piecewise differential systems. For the statement of the classical 16th Hilbert problem see [16,18,21].

Of course in order that a discontinuous piecewise differential system be defined on the discontinuous line, which separates the different differential systems forming the discontinuous piecewise differential system, we follow the rules of Filippov, see [11].

The discontinuous piecewise differential systems formed by linear differential systems can exhibit two kinds of limit cycles, the crossing and the sliding limit cycles, the first are the ones which only contain isolated points of the line of discontinuity, and the second the ones which contains arcs of the line of discontinuity. Here we only study the crossing limit cycles.

The simplest class of discontinuous piecewise differential systems are the planar ones formed by two pieces separated by a straight line having a linear differential system in each piece. Several authors have tried to determine the maximum number of crossing limit cycles for this class of discontinuous piecewise differential systems. Thus, in one of the first papers dedicated to this problem, Giannakopoulos and Pliete [14] in 2001, showed the existence of discontinuous piecewise linear differential systems with two crossing limit cycles. Then, in 2010 Han and Zhang [15] found other discontinuous piecewise linear differential systems with two crossing limit cycles and they conjectured that the maximum number of crossing limit cycles for discontinuous piecewise linear differential systems with two pieces separated by a straight line is two. But in 2012 Huan and Yang [17] provided numerical evidence of the existence of three crossing limit cycles in this class of discontinuous piecewise linear differential systems. In 2012, Llibre and Ponce [24] inspired by the numerical example of Huan and Yang, proved for the first time that there are discontinuous piecewise linear differential systems with two pieces separated by a straight line having three crossing limit cycles. Later on, other authors obtained also three crossing limit cycles for discontinuous piecewise linear differential systems with two pieces separated by a straight line, see Braga and Mello [9] in 2013, Buzzi, Pessoa and Torregrosa [10] in 2013, Liping Li [22] in 2014, Freire, Ponce and Torres [13] in 2014, and Llibre, Novaes and Teixeira [23] in 2015. But proving that discontinuous piecewise linear differential systems separated by a straight line have at most three crossing limit cycles is an open problem.

Recently, in $[4,6,7,19,20]$ the authors have studied the extended 16th Hilbert problem to discontinuous piecewise linear differential centers separated by either conics, or cubics. However for the discontinuous piecewise linear Hamiltonian systems without equilibrium points, it was proven in [12] that such systems separated by two parallel straight lines can have at most one crossing limit cycle. In [5] it was proven that there is an example of two crossing limit cycles when these systems are separated by three parallel straight lines, and they can also have two crossing limit cycles if the curve of separation is a parabola, and three crossing limit cycles if the curve of separation is either an ellipse or a hyperbola. In [2] the
authors provided the maximum number of crossing limit cycles when the curve of separation of these systems is an irreducible cubic.

In this paper we give the solution of the extended 16th Hilbert problem for discontinuous piecewise linear differential Hamiltonian systems without equilibrium points separated by two different reducible cubic curves, formed either by an ellipse and a straight line, or by a parabola and a straight line parallel to the tangent at the vertex of the parabola. More precisely, we provide the maximum number of crossing limit cycles for these systems, when these limit cycles intersected with the cubic of separation in four points.

Note that if a crossing limit cycle of a discontinuous piecewise linear differential Hamiltonian systems without equilibrium points intersects in two points the discontinuity line formed either by an ellipse and a straight line, or by a parabola and a straight line parallel to the tangent at the vertex of the parabola, this crossing limit cycle must intersect in two points either the straight line, or the ellipse or the parabola, and these types of crossing limit cycles already have been studied in $[5,12]$, as we have mention previously. For this reason in this paper we study the crossing limit cycles with intersect in four points the reducible cubic formed by either by an ellipse and a straight line, or by a parabola and a straight line parallel to the tangent at the vertex of the parabola.

Doing an affine change if the reducible cubic is formed by an ellipse and a straight line we can transform it into the reducible cubic

$$
\Gamma_{k}=\left\{(x, y) \in \mathbb{R}^{2}:(x-k)\left(x^{2}+y^{2}-1\right)=0, k \geq 0\right\}
$$

formed by the circle $x^{2}+y^{2}=1$ and the straight line $x=k$ with $k \geq 0$. In a similar way if the reducible cubic is formed by a parabola and a straight line parallel to the tangent at the vertex of the parabola we can transform it into the reducible cubic

$$
\Sigma_{k}=\left\{(x, y) \in \mathbb{R}^{2}:(y-k)\left(y-x^{2}\right)=0, k \in \mathbb{R}\right\},
$$

formed by the parabola $y=x^{2}$ and the straight line $y=k$ with $k \in \mathbb{R}$
First in Subsection 1.1 we shall consider the piecewise linear Hamiltonian systems without equilibrium points separated by the reducible cubic $\Gamma_{k}$, and after in Subsection 1.2 we shall consider the piecewise linear Hamiltonian systems without equilibrium points separated by the reducible cubic $\Sigma_{k}$.

The next result is proved in [12].
Lemma 1.1. An arbitrary linear differential Hamiltonian system in $\mathbb{R}^{2}$ without equilibrium points can be written as

$$
\dot{x}=-\lambda b x+b y+\mu, \quad \dot{y}=-\lambda^{2} b x+\lambda b y+\sigma,
$$

where $\sigma \neq \lambda \mu$ and $b \neq 0$. The Hamiltonian function of this Hamiltonian system is

$$
\begin{equation*}
H(x, y)=-\frac{1}{2} \lambda^{2} b x^{2}+\lambda b x y-\frac{b}{2} y^{2}+\sigma x-\mu y . \tag{1.1}
\end{equation*}
$$

Of course $H(x, y)$ is a first integral of the Hamiltonian system.

### 1.1 The line of discontinuity is a circle and a straight line

We denote by $\mathcal{C}_{1}$ the class of planar discontinuous piecewise linear Hamiltonian systems without equilibrium points separated by $\Gamma_{k}$ with $k>1$. Let $\mathcal{C}_{2}$ be the class of planar discontinuous
piecewise linear Hamiltonian systems without equilibrium points separated by $\Gamma_{k}$ with $k=1$. For these two classes we get the following three zones

$$
\begin{align*}
& Z^{1}=\left\{(x, y) \in \mathbb{R}^{2}: x^{2}+y^{2}<1\right\}, \\
& Z^{2}=\left\{(x, y) \in \mathbb{R}^{2}: x^{2}+y^{2}>1 \text { and } x<k\right\},  \tag{1.2}\\
& Z^{3}=\left\{(x, y) \in \mathbb{R}^{2}: x^{2}+y^{2}>1 \text { and } x>k\right\} .
\end{align*}
$$

Now we denote by $\mathcal{C}_{3}$ the class of piecewise linear Hamiltonian systems without equilibrium points separated by $\Gamma_{k}$ with $0 \leq k<1$. In this case $\Gamma_{k}$ separate the plane into four zones

$$
\begin{align*}
& Z^{1}=\left\{(x, y) \in \mathbb{R}^{2}: x^{2}+y^{2}>1 \text { and } x>k\right\}, \\
& Z^{2}=\left\{(x, y) \in \mathbb{R}^{2}: x^{2}+y^{2}>1 \text { and } x<k\right\}, \\
& Z^{3}=\left\{(x, y) \in \mathbb{R}^{2}: x^{2}+y^{2}<1 \text { and } x<k\right\},  \tag{1.3}\\
& Z^{4}=\left\{(x, y) \in \mathbb{R}^{2}: x^{2}+y^{2}<1 \text { and } x>k\right\} .
\end{align*}
$$


(a)

(b)

Figure 1.1: (a) The three zones for the class $\mathcal{C}_{1}$. (b) The four zones for the class $\mathcal{C}_{3}$.
We have three different configurations of crossing limit cycles for the class $\mathcal{C}_{3}$. The first one which will be denoted by Conf 1, here we have the limit cycles formed by four pieces of orbits, such that in each zone of (1.3) we have one piece of orbit of each of the four Hamiltonian systems considered, see Figure 1.4a.

The second configuration of limit cycles denoted by Conf 2, where we have the limit cycles formed by pieces of orbits belonging to the three zones either $Z^{1}, Z^{2}$ and $Z^{4}$, or $Z^{1}, Z^{2}$ and $Z^{3}$. We are going to consider only the three zones $Z^{1}, Z^{2}$ and $Z^{4}$, because by a similar analysis we obtain the crossing limit cycles intersecting the three zones $Z^{1}, Z^{2}$ and $Z^{3}$, for this configuration, see Figure 1.4b.

Finally the third configuration namely Conf $\mathbf{3}$ where we have limit cycles formed by pieces of orbits belonging to the three zones either $Z^{1}, Z^{3}$ and $Z^{4}$, or $Z^{2}, Z^{3}$ and $Z^{4}$. For the same reason as in the second configuration, we are going to consider only the three zones $Z^{1}, Z^{3}$ and $Z^{4}$, see Figure 1.4c.

We notice that we can obtain two new configurations by combining the three previous ones, such as Conf 1 and Conf 2, Conf 1 and Conf 3 . Note that we cannot have the configuration Conf 2 and Conf 3, and Conf 1, Conf 2 and Conf 3.

Our main result on the crossing limit cycles of the discontinuous piecewise linear Hamiltonian systems without equilibria when the discontinuity line is formed by a circle and a straight line is the following one.

Theorem 1.2. The following statements hold for the discontinuous piecewise linear Hamiltonian systems without equilibria when the discontinuity line is formed by a circle and a straight line. The maximum number of crossing limit cycles intersecting the cubic of separation in four points for the class
(i) $\mathcal{C}_{1}$ or $\mathcal{C}_{2}$ is three and this maximum is reached in Example 1 for the class $\mathcal{C}_{1}$ and in Example 2 for the class $\mathcal{C}_{2}$, see Figures 1.3a and 1.3b, respectively;
(ii) $\mathcal{C}_{3}$ with Conf 1 is three and this maximum is reached in Example 3, see Figure 1.4a;
(iii) $\mathcal{C}_{3}$ with Conf 2 is three and this maximum is reached in Example 4, see Figure 1.4b;
(iv) $\mathcal{C}_{3}$ with Conf 3 is three and this maximum is reached in Example 5, see Figure 1.4c;
(v) $\mathcal{C}_{3}$ with Conf 1 and Conf 2 simultaneously is six and this maximum is reached in Example 6, see Figure 1.5;
(vi) $\mathcal{C}_{3}$ with Conf 1 and Conf 3 simultaneously is six and this maximum is reached in Example 7, see Figure 1.6.

Theorem 1.2 is proved in Section 2.
1.2 The line of discontinuity is a parabola and a straight line parallel to the tangent at the vertex of the parabola


Figure 1.2: (a) Three zones for the class $\mathcal{C}_{k^{-}}$. (b) Four zones for the class $\mathcal{C}_{0}$. (c) Five zones for the class $\mathcal{C}_{k^{+}}$.

Let $\mathcal{C}_{\Sigma_{k^{-}}}$be the class of discontinuous piecewise linear Hamiltonian systems without equilibria separated by $\Sigma_{k^{-}}$with $k<0$. In this case we have following three zones in the plane

$$
\begin{aligned}
& Z_{\Sigma_{k^{-}}}^{1}=\left\{(x, y) \in \mathbb{R}^{2}: y>x^{2}\right\}, \\
& Z_{\Sigma_{k^{-}}}^{2}=\left\{(x, y) \in \mathbb{R}^{2}: y<x^{2} \text { and } y>k\right\}, \\
& Z_{\Sigma_{k^{-}}}^{3}=\left\{(x, y) \in \mathbb{R}^{2}: y<x^{2} \text { and } y<k\right\},
\end{aligned}
$$

see Figure 1.2a. Let $\mathcal{C}_{\Sigma_{0}}$ be the class of discontinuous piecewise linear Hamiltonian systems without equilibria separated by $\Sigma_{k}$ with $k=0$. When the discontinuity curve is $\Sigma_{0}$ we have
following four zones in the plane

$$
\begin{aligned}
& Z_{\Sigma_{0}}^{1}=\left\{(x, y) \in \mathbb{R}^{2}: y>x^{2}\right\}, \\
& Z_{\Sigma_{0}}^{2}=\left\{(x, y) \in \mathbb{R}^{2}: y<x^{2} \text { and } y>0, x<0\right\}, \\
& Z_{\Sigma_{0}}^{3}=\left\{(x, y) \in \mathbb{R}^{2}: y<x^{2} \text { and } y<0\right\}, \\
& Z_{\Sigma_{0}}^{4}=\left\{(x, y) \in \mathbb{R}^{2}: y<x^{2} \text { and } y>0, x>0\right\},
\end{aligned}
$$

see Figure 1.2b. In this class we have two configurations of crossing limit cycles, first crossing limit cycles with Conf 4 which are constituted by pieces of orbits of the four Hamiltonian systems considered, see Figure 3.2a. Second crossing limit cycles with Conf 5 which intersect only three zones, in this case we have two options, first we have the case where the crossing limit cycles are formed by parts of orbits of the Hamiltonian systems in the zones $Z_{\Sigma_{0}}^{1}, Z_{\Sigma_{0}}^{3}$ and $Z_{\Sigma_{0}}^{4}$ and second the crossing limit cycles that intersect only the three zones $Z_{\Sigma_{0}}^{1}, Z_{\Sigma_{0}}^{2}$ and $Z_{\Sigma_{0}}^{3}$, without loss of generality we can consider the first case because the study of the second is the same, see Figure 3.2b. Here we observe that it is not possible to have crossing limit cycles with Conf 5 that satisfy those two cases simultaneously, because the orbits of the Hamiltonian system in the zone $Z_{\Sigma_{0}}^{3}$ would not be nested. In statement (ii) of Theorem 1.3 we study the discontinuous piecewise linear Hamiltonian systems without equilibria in $\mathcal{C}_{\Sigma_{0}}$ which have crossing limit cycles with Conf 4 and Conf 5 separately, and in statement (iii) of Theorem 1.3 we study the case when the crossing limit cycles with Conf 4 and Conf 5 appear simultaneously.

Let $\mathcal{E}_{\Sigma^{+}}$be the class of discontinuous piecewise linear Hamiltonian systems without equilibria separated by $\Sigma_{k}$ with $k>0$, in this case we have the following five zones in the plane

$$
\begin{aligned}
& Z_{\Sigma_{k+}}^{1}=\left\{(x, y) \in \mathbb{R}^{2}: y>x^{2} \text { and } y>k\right\}, \\
& Z_{\Sigma_{k^{+}}}^{2}=\left\{(x, y) \in \mathbb{R}^{2}: y<x^{2} \text { and } y>k, x<-\sqrt{k}\right\}, \\
& Z_{\Sigma_{k^{+}}}^{3}=\left\{(x, y) \in \mathbb{R}^{2}: y>x^{2} \text { and } y<k\right\}, \\
& Z_{\Sigma_{k^{+}}}^{4}=\left\{(x, y) \in \mathbb{R}^{2}: y<x^{2} \text { and } y>k, x>\sqrt{k}\right\}, \\
& Z_{\Sigma_{k^{+}}}^{5}=\left\{(x, y) \in \mathbb{R}^{2}: x^{2}<y<k\right\},
\end{aligned}
$$

see Figure 1.2c. In this class we have six different configurations of crossing limit cycles.
First we have crossing limit cycles such that are formed by pieces of orbits of the four Hamiltonian systems in the zones $Z_{\Sigma_{k^{+}}}^{1}, Z_{\Sigma_{k^{+}}}^{5}, Z_{\Sigma_{k^{+}}}^{3}$ and $Z_{\Sigma_{k^{+}}}^{4}$, or crossing limit cycles formed by pieces of orbits of the four Hamiltonian systems in the zones $Z_{\Sigma_{k^{+}}}^{1}, Z_{\Sigma_{k^{+}}}^{2}, Z_{\Sigma_{k^{+}}}^{3}$ and $Z_{\Sigma_{k^{+}}}^{5}$, namely crossing limit cycles with Conf $\mathbf{6}^{+}$and crossing limit cycles with Conf $6^{-}$, respectively, see Figure 3.5. In statement (ii) of Theorem 1.3 we study the crossing limit cycles with Conf $6^{+}$because the study for the case of crossing limit cycles with Conf $6^{-}$is the same. Second we have crossing limit cycles with Conf 7 , which intersect the three zones $Z_{\Sigma_{k^{+}}}^{1}, Z_{\Sigma_{k^{+}}}^{5}$ and $Z_{\Sigma_{k^{+}}}^{3}$, see Figure 3.3b. Third we have the crossing limit cycles with Conf 8, which intersect the zones $Z_{\Sigma_{k^{+}}}^{1}, Z_{\Sigma_{k^{+}}}^{2}, Z_{\Sigma_{k^{+}}}^{3}$ and $Z_{\Sigma_{k^{+}}}^{4}$, see Figure 3.3c. And finally we have the crossing limit cycles formed by pieces of orbits of the three Hamiltonian systems in the zones $Z_{\Sigma_{k^{+}}}^{1}, Z_{\Sigma_{k^{+}}}^{3}$ and $Z_{\Sigma_{k^{+}}}^{4}$, or crossing limit cycles formed by pieces of orbits of the three Hamiltonian systems in the zones $Z_{\Sigma_{k+}+}^{1}, Z_{\Sigma_{k+}}^{2}$ and $Z_{\Sigma_{k^{+}}}^{3}$, namely crossing limit cycles with Conf $9^{+}$and crossing limit cycles with Conf $\mathbf{9}^{-}$, respectively, see Figure 3.3d. Without loss of generality in statement (ii) of Theorem 1.3 we study the crossing limit cycles with Conf $9^{+}$because the study by
the crossing limit cycles with Conf $9^{-}$is the same. We observe that there are no crossing limit cycles that intersect the five zones $Z_{\Sigma_{k^{+}}}^{i}$ for $i=1,2,3,4,5$. Then in statement (ii) of Theorem 1.3 we study the crossing limit cycles with Conf $6^{+}$, Conf 7 , Conf 8 and Conf $9^{+}$ separately. In statements (iii)-(ix) of Theorem 1.3 we study the discontinuous piecewise linear Hamiltonian systems without equilibria in the class $\mathcal{C}_{\Sigma_{k}+}$ which have crossing limit cycles with two configurations simultaneously. Finally in statements (x)-(xii) we study the discontinuous piecewise linear Hamiltonian systems without equilibria in the class $\mathcal{C}_{\Sigma_{k^{+}}}$which have crossing limit cycles with three different configurations simultaneously.

Our main result on the crossing limit cycles of the discontinuous piecewise linear Hamiltonian systems without equilibria when the discontinuity curve is formed by a parabola and a straight line parallel to the tangent at the vertex of the parabola is the following one.

Theorem 1.3. The following statements hold for the discontinuous piecewise linear Hamiltonian systems without equilibria when the discontinuity line is formed by a parabola and a straight line parallel to the tangent at the vertex of the parabola. The maximum number of crossing limit cycles intersecting the cubic of separation in four points for the class
(i) $\mathcal{C}_{\Sigma_{k^{-}}}$is three and this maximum is reached, see Figure 3.1;
(ii) $\mathcal{C}_{\Sigma_{0}}$ or $\mathcal{C}_{\Sigma_{k}+}$ with either Conf 4 , or Conf 5 , or Conf $\mathbf{6}^{+}$, or Conf 7 , or Conf $\mathbf{8}$, or Conf $\mathbf{9}^{+}$is three, respectively, see Figures 3.2a-3.3d;
(iii) $\mathcal{C}_{\Sigma_{k^{+}}}$with Conf 4 and Conf 5 simultaneously is six, see Figure 3.4;
(iv) $\mathcal{C}_{\Sigma_{k^{+}}}$with Conf $\mathbf{6}^{+}$and Conf $\mathbf{6}^{-}$simultaneously is six, see Figure 3.5;
(v) $\mathcal{C}_{\Sigma_{k^{+}}}$with Conf $\mathbf{6}^{-}$and Conf 7 simultaneously is six, see Figure 3.6a;
(vi) $\mathcal{C}_{\Sigma^{+}}$with Conf $\mathbf{6}^{+}$and Conf $\mathbf{8}$ simultaneously is six, see Figure 3.6b;
(vii) $\mathcal{C}_{\Sigma_{k^{+}}}$with Conf $\mathbf{6}^{+}$and Conf $\mathbf{9}^{+}$simultaneously is six, see Figure 3.7;
(viii) $\mathcal{C}_{\Sigma_{k^{+}}}$with Conf 7 and Conf 8 simultaneously is six, see Figure 3.8;
(ix) $\mathcal{C}_{\Sigma_{k^{+}}}$with Conf $\mathbf{8}$ and Conf $\mathbf{9}^{+}$simultaneously is six, see Figure 3.9;
(x) $\mathcal{C}_{\Sigma_{k^{+}}}$with Conf $6^{-}$, Conf 7 and Conf 8 simultaneously is nine, see Figure 3.10;
(xi) $\mathcal{C}_{\Sigma_{k^{+}}}$with Conf $\mathbf{6}^{+}$, Conf $\mathbf{8}$ and Conf $\mathbf{9}^{+}$simultaneously is nine, see Figure 3.11;
(xii) $\mathcal{C}_{\Sigma_{k^{+}}}$with Conf $\mathbf{6}^{-}$, Conf $\mathbf{6}^{+}$and Conf 8 simultaneously is six with 2 (resp. 3) limit cycles with Conf $6^{-}, 3$ (resp. 2) limit cycles with Conf $\mathbf{6}^{+}$and 1 limit cycle with Conf 8 , Figure 3.12 (resp. 3.13).

Theorem 1.3 is proved in Section 3.

## 2 Proof of Theorem 1.2

Proof of statement $(i)$ of Theorem 1.2. We have to prove that the maximum number of crossing limit cycles of the class $\mathcal{C}_{1}$ intersecting the curve $\Gamma_{k}$ in four points is three. In a similar way we should prove the statement for the classes $\mathcal{C}_{2}$ and $\mathcal{C}_{3}$.


Figure 1.3: (a) The three limit cycles of the discontinuous piecewise differential system (2.3). (b) The three limit cycles of the discontinuous piecewise differential system (2.4).

By Lemma 1.1 we can consider the discontinuous piecewise linear Hamiltonian systems

$$
\begin{equation*}
\dot{x}=-\lambda_{i} b_{i} x+b_{i} y+\mu_{i}, \quad \dot{y}=-\lambda_{i}^{2} b_{i} x+\lambda_{i} b_{i} y+\sigma_{i}, \text { in the zone } Z_{i}, \text { with } i=1,2,3 . \tag{2.1}
\end{equation*}
$$

with $b_{i} \neq 0$ and $\sigma_{i} \neq \lambda_{i} \mu_{i}$, and the three zones $Z_{i}$ are defined in (1.2). Their corresponding Hamiltonian first integrals are as (1.1)

$$
H_{i}(x, y)=-\left(\lambda_{i}^{2} b_{i} / 2\right) x^{2}+\lambda_{i} b_{i} x y-\left(b_{i} / 2\right) y^{2}+\sigma_{i} x-\mu_{i} y, \text { with } i=1,2,3 .
$$

In order to have a crossing limit cycle which intersects $\Gamma_{k}$ in the points $A_{i}=\left(x_{i}, y_{i}\right), B_{i}=$ $\left(z_{i}, w_{i}\right), C_{i}=\left(k, f_{i}\right)$ and $D_{i}=\left(k, h_{i}\right)$, where $k>1, A_{i}$ and $B_{i}$ are points on the circle $x^{2}+y^{2}-$ $1=0$, these points must satisfy the following system

$$
\begin{align*}
e_{1}=H_{1}\left(x_{i}, y_{i}\right)-H_{1}\left(z_{i}, w_{i}\right) & =0, \\
e_{2}=H_{2}\left(x_{i}, y_{i}\right)-H_{2}\left(k, f_{i}\right) & =0, \\
e_{3}=H_{2}\left(z_{i}, w_{i}\right)-H_{2}\left(k, h_{i}\right) & =0,  \tag{2.2}\\
e_{4}=H_{3}\left(k, f_{i}\right)-H_{3}\left(k, h_{i}\right) & =0, \\
x_{i}^{2}+y_{i}^{2}-1 & =0, \\
z_{i}^{2}+w_{i}^{2}-1 & =0 .
\end{align*}
$$

We suppose that the discontinuous piecewise linear differential system (2.1) has four limit cycles. For this we must suppose that system (2.2) has four real solutions, namely ( $A_{i}, B_{i}, C_{i}, D_{i}$ ), $i=1,2,3,4$. The points $A_{i}$ and $B_{i}$ can take the form $A_{i}=\left(\cos r_{i}, \sin r_{i}\right), B_{i}=\left(\cos s_{i}, \sin s_{i}\right)$. Then by solving $e_{1}=0$ for the parameter $\sigma_{1}$ and $e_{4}=0$ for $\mu_{3}$, we get

$$
\begin{aligned}
\sigma_{1}= & \frac{1}{2\left(\cos r_{1}-\cos s_{1}\right)}\left(b _ { 1 } \operatorname { s i n } ( r _ { 1 } - s _ { 1 } ) \left(-\left(\lambda_{1}^{2}-1\right) \sin \left(r_{1}+s_{1}\right)-2 \lambda_{1}\right.\right. \\
& \left.\left.\cos \left(r_{1}+s_{1}\right)\right)+2 \mu_{1}\left(\sin r_{1}-\sin s_{1}\right)\right),
\end{aligned}
$$



Figure 1.4: (a) The three limit cycles of Conf 1 of the discontinuous piecewise differential system (2.5). (b) The three limit cycles of Conf 2 of the discontinuous piecewise differential system (2.7). (c) The three limit cycles of Conf 3 of the discontinuous piecewise differential system (2.9).
and $\mu_{3}=\frac{b_{3}}{2}\left(f_{1}+h_{1}-2 k \lambda_{3}\right)$, respectively.
Now we consider the second real solution of (2.2) for $i=2$, and we fix the three points $A_{2}=\left(\cos r_{2}, \sin r_{2}\right), B_{2}=\left(\cos s_{2}, \sin s_{2}\right)$ and $\left(k, f_{2}\right)$, so by solving $e_{1}=0$ for $\mu_{1}$ and $e_{4}=0$ for $h_{2}$, we obtain

$$
\begin{aligned}
\mu_{1}= & \frac{1}{4\left(\cos \left(\frac{1}{2}\left(r_{1}-2 r_{2}+s_{1}\right)\right)-\cos \left(\frac{1}{2}\left(r_{1}+s_{1}-2 s_{2}\right)\right)\right)}\left(b_{1} \csc \left(\frac{r_{1}-s_{1}}{2}\right)\right. \\
& \left(-\lambda_{1} \cos r_{1} \sin \left(2 r_{2}\right)+\cos r_{2} \sin \left(r_{1}-s_{1}\right)\left(\left(\lambda_{1}^{2}-1\right) \sin \left(r_{1}+s_{1}\right)+2 \lambda_{1}\right.\right. \\
& \left.\cos \left(r_{1}+s_{1}\right)\right)-\left(\lambda_{1}^{2}-1\right) \cos r_{1} \sin \left(r_{2}-s_{2}\right) \sin \left(r_{2}+s_{2}\right)+\lambda_{1}^{2}\left(-\cos s_{2}\right) \\
& \sin \left(r_{1}-s_{1}\right) \sin \left(r_{1}+s_{1}\right)+\cos s_{2} \sin \left(r_{1}-s_{1}\right) \sin \left(r_{1}+s_{1}\right)-\lambda_{1} \sin \left(2 r_{1}\right) \\
& \cos s_{2}+\lambda_{1} \cos r_{1} \sin \left(2 s_{2}\right)-\lambda_{1}^{2} \cos ^{2} r_{2} \cos s_{1}+\cos s_{1}\left(\lambda_{1} \sin \left(2 r_{2}\right)\right. \\
& \left.\left.\left.-\sin ^{2} r_{2}+\left(\sin s_{2}-\lambda_{1} \cos s_{2}\right)^{2}\right)+\lambda_{1} \sin \left(2 s_{1}\right) \cos s_{2}\right)\right),
\end{aligned}
$$



Figure 1.5: Three limit cycles of Conf 1 and three limit cycles of Conf 2 for the class of the discontinuous piecewise differential system (2.10).


Figure 1.6: Three limit cycles of Conf $\mathbf{1}$ and three limit cycles of Conf $\mathbf{3}$ for the class of the discontinuous piecewise differential system (2.11).
and $h_{2}=f_{1}-f_{2}+h_{1}$.
Likewise, the points $A_{3}=\left(\cos r_{3}, \sin r_{3}\right), B_{3}=\left(\cos s_{3}, \sin s_{3}\right),\left(k, f_{3}\right)$ and $\left(k, h_{3}\right)$ are solution of (2.2), we fix $A_{3}, B_{3}$ and $\left(k, f_{3}\right)$, then by solving equation $e_{4}=0$ for $h_{3}$ and $e_{1}=0$ for $\lambda_{1}$ we have $h_{3}=f_{1}-f_{3}+h_{1}$ and we get the two values $\lambda_{1}^{1,2}=\left(A \pm 2 \sqrt{2} \sin \left(\frac{1}{2}\left(r_{1}-r_{2}+\right.\right.\right.$ $\left.\left.\left.s_{1}-s_{2}\right)\right) \sqrt{B}\right) / C$ given in the appendix.

Finally, if we fix the three points $A_{4}=\left(\cos r_{4}, \sin r_{4}\right), B_{4}=\left(\cos s_{4}, \sin s_{4}\right)$, and $\left(k, f_{4}\right)$, then from the equation $e_{4}=0$ and $e_{1}=0$ we have that $h_{4}=f_{1}-f_{4}+h_{1}$ and $b_{1}=0$ which is a contradiction to the assumptions. Therefore we have proved that the maximum number of crossing limit cycles for the class $\mathcal{C}_{1}$ intersecting the curve $\Gamma_{k}$ in four points is three.

Now we shall provide differential systems of class $\mathcal{C}_{1}, \mathcal{C}_{2}$ and $\mathcal{C}_{3}$ separated by $\Gamma_{k}$ with three limit cycles.

We will explain the method for constructing an example of three crossing limit cycles intersecting $\Gamma_{k}$ in four points, and by a similar way we build the remaining examples.

Example 1: Three crossing limit cycles for the class $\mathcal{C}_{1}$. Here we consider the three zones
defined in (1.2) for $k=2.5$. We consider the Hamiltonian systems

$$
\begin{align*}
& \dot{x}=-0.02 . . x+0.2 . . y+0.316667 . ., \dot{y}=0.02 . . y-0.002 . . x \text { in } Z^{1}, \\
& \dot{x}=10.8 . . x+18 y-3, \dot{y}=-6.48 . . x-10.8 . . y \text { in } Z^{2},  \tag{2.3}\\
& \dot{x}=5 . x-3 y-1.38889 \ldots, \dot{y}=8.33333 . . x-5 y \text { in } Z^{3} .
\end{align*}
$$

The first integrals of the linear Hamiltonian systems (2.3) are

$$
\begin{aligned}
& H_{1}(x, y)=-0.001 . . x^{2}+0.02 \ldots x y-0.1 . . y^{2}-0.316667 . . y \\
& H_{2}(x, y)=-3.24 . . x^{2}-10.8 . . x y-9 y^{2}+3 y \\
& H_{3}(x, y)=4.16667 . . x^{2}-5 x y+1.5 y^{2}+1.38889 . . y
\end{aligned}
$$

respectively.
The discontinuous piecewise linear differential system formed by the linear Hamiltonian systems (2.3) has exactly three crossing limit cycles, because the system of equations (2.2) has the three real solutions $S_{i}=\left(x_{i}, y_{i}, z_{i}, w_{i}, f_{i}, h_{i}\right)$ for $i=1,2,3$, where

$$
\begin{aligned}
& S_{1}=(0.244811 . .,-0.969571 . ., 0.767202 . ., 0.641406 . .,-2.05982 . .,-0.60685 . .), \\
& S_{2}=(0.390566 . .,-0.920575 ., 0.912879 . ., 0.40823 . .,-1.8861 . .,-0.780563 . .), \\
& S_{3}=(0.535321 . .,-0.844649 . ., 0.979509 . ., 0.201401 . .,-1.62201 . ., 1.04466 . .) .
\end{aligned}
$$

Then these three limit cycles are drawn in 1.3a.
Example 2: Three crossing limit cycles for the class $\mathcal{C}_{2}$. We consider the three zones defined in (1.2) with $k=1$. We consider the Hamiltonian systems

$$
\begin{align*}
& \dot{x}=15 x-3 y-11.25 . ., \dot{y}=75 . x-15 . y+22.5 \text { in } Z^{1}, \\
& \dot{x}=4 x+20 y-3, \dot{y}=-0.8 x-4 y+6 \text { in } Z^{2},  \tag{2.4}\\
& \dot{x}=-0.4 x+4 y+0.6, \dot{y}=-0.04 x+0.4 y-1 \text { in } Z^{3} .
\end{align*}
$$

The first integrals of the Hamiltonian systems (2.4) are

$$
\begin{aligned}
& H_{1}(x, y)=37.5 . . x^{2}-15 x y+22.5 . . x+\frac{3 y^{2}}{2}+11.25 . . y \\
& H_{2}(x, y)=-0.4 x^{2}-4 x y+6 x-10 y^{2}+3 y \\
& H_{3}(x, y)=-0.02 . . x^{2}+0.4 \ldots x y-x-2 y^{2}-0.6 . . y
\end{aligned}
$$

respectively.
The discontinuous piecewise linear Hamiltonian system (2.4) has exactly three crossing limit cycles, because the system of equations (2.2) has the three real solutions $S_{i}=$ $\left(x_{i}, y_{i}, z_{i}, w_{i}, f_{i}, h_{i}\right)$ for $i=1,2,3$, where

$$
\begin{aligned}
& S_{1}=(0.559983 . ., 0.828504 . ., 0.619895 . .,-0.784685 . ., 0.878709 . .,-0.978709 . .), \\
& S_{2}=(0.755607 . ., 0.655025 . ., 0.754335 . .,-0.65649 . ., 0.7,-0.8), \\
& S_{3}=(0.903742 . ., 0.881627 . .,-0.471947 . ., 0.428077 . ., 0.462348 . .,-0.562348 . .) .
\end{aligned}
$$

These solutions provide three crossing limit cycles of the piecewise linear differential Hamiltonian system (2.2), which are illustrate in Figure 1.3b. This completes the proof of statement (i).

To complete the proof of statements (ii)-(iv) of Theorem 1.2 we shall provide discontinuous piecewise linear Hamiltonian systems without equilibrium points separated by the cubic curve $\Gamma_{k}$ with three limit cycles for the class $\mathcal{C}_{3}$ of Conf 1; Conf 2; Conf 3.

Example 3: Three crossing limit cycles of Conf 1 for the class $\mathcal{C}_{3}$. For this class we consider the four zones defined in (1.3). We consider the Hamiltonian systems

$$
\begin{align*}
& \dot{x}=-6.8 . . x+4 y-2, \dot{y}=-11.56 x+6.8 y-2 \text { in } Z^{1}, \\
& \dot{x}=1.06216 \ldots x+2 y-1.28925 . ., \dot{y}=-0.564089 . . x-1.06216 . . y+3.92358 . . \text { in } Z^{2}, \\
& \dot{x}=-4 x+2 y-2.8 . ., \dot{y}=-8 x+4 y-1 \text { in } Z^{3},  \tag{2.5}\\
& \dot{x}=121.33 . . x+3 y+508.239 . ., \dot{y}=-4907.01 . . x-121.33 . . y+611.017 . . \text { in } Z^{4} .
\end{align*}
$$

The linear Hamiltonian systems in (2.5) have the first integrals

$$
\begin{aligned}
& H_{1}(x, y)=-5.78 . . x^{2}+6.8 \ldots x y-2 x-2 y^{2}+2 y, \\
& H_{2}(x, y)=-0.282045 . . x^{2}-1.06216 \ldots x y+3.92358 . . x-y^{2}+1.28925 . . y, \\
& H_{3}(x, y)=-4 x^{2}+4 x y-x-y^{2}+2.8 . . y \\
& H_{4}(x, y)=-2453.5 . . x^{2}-121.33 \ldots x y+611.017 . . x-1.5 y^{2}-508.239 . . y,
\end{aligned}
$$

respectively.
The discontinuous piecewise linear Hamiltonian system (2.5) has exactly three crossing limit cycles intersecting $\Gamma_{k}$ in the points $A_{i}=\left(x_{i}, y_{i}\right), B_{i}=\left(z_{i}, w_{i}\right), C_{i}=\left(k, f_{i}\right)$ and $D_{4}=\left(k, h_{i}\right)$ for $i=1,2,3$, where $A_{i}$ and $B_{i}$ are points on the circle $x^{2}+y^{2}-1=0$, because the system of equations

$$
\begin{align*}
H_{1}\left(x_{i}, y_{i}\right)-H_{1}\left(k, f_{i}\right) & =0, \\
H_{2}\left(z_{i}, w_{i}\right)-H_{2}\left(k, f_{i}\right) & =0, \\
H_{3}\left(z_{i}, w_{i}\right)-H_{3}\left(k, h_{i}\right) & =0,  \tag{2.6}\\
H_{4}\left(x_{i}, y_{i}\right)-H_{4}\left(k, h_{i}\right) & =0, \\
x_{i}^{2}+y_{i}^{2}-1 & =0, \\
z_{i}^{2}+w_{i}^{2}-1 & =0,
\end{align*}
$$

with $k=0$, has only three real solutions $S_{i}=\left(x_{i}, y, i, z_{i}, w_{i}, f_{i}, h_{i}\right)$ for $i=1,2,3$, where

$$
\begin{aligned}
& S_{1}=(0.859402 . ., 0.5113 . .,-0.573716 . ., 0.819054 . ., 3.12047 . .,-0.724745 . .), \\
& S_{2}=(0.795991 . ., 0.605309 . .,-0.403541 . ., 0.914962 . ., 2.8 . .,-0.5 . .), \\
& S_{3}=(0.708174 . ., 0.706038 . .,-0.208691 . ., 0.977982 . ., 2.3798 . .,-0.207107 . .) .
\end{aligned}
$$

These three limit cycles are drawn in Figure 1.4a. This completes the proof of statement (ii).
Example 4: Three crossing limit cycles of Conf 2 for the class $\mathcal{C}_{3}$. In (1.3), we work only with the three zones $Z^{1}, Z^{2}$ and $Z^{4}$, with $k=0$, and we consider the Hamiltonian systems

$$
\begin{align*}
& \dot{x}=19-18 x-3 y, \dot{y}=-68+108 x+18 y \text { in } Z^{1}, \\
& \dot{x}=-3.88389 x-2 y+5.99641 . ., \dot{y}=7.54231 . . x+3.88389 . . y-7.99048 . . \text { in } Z^{2},  \tag{2.7}\\
& \dot{x}=6+2 x-2 y, \dot{y}=-2+2 x-2 y \text { in } Z^{4} .
\end{align*}
$$

The first integrals of the Hamiltonian systems (2.7) are

$$
\begin{aligned}
& H_{1}(x, y)=54 x^{2}+18 x y-68 x+\frac{3 y^{2}}{2}-19 y, \\
& H_{2}(x, y)=3.77115 . . x^{2}+3.88389 \ldots x y-7.99048 \ldots x+y^{2}-5.99641 . . y, \\
& H_{4}(x, y)=x^{2}-2 x y-2 x+y^{2}-6 y,
\end{aligned}
$$

respectively
The discontinuous piecewise linear differential system formed by the linear Hamiltonian systems (2.7) has exactly three crossing limit cycles, because the system of equations

$$
\begin{align*}
H_{1}\left(x_{i}, y_{i}\right)-H_{1}\left(k, f_{i}\right) & =0, \\
H_{1}\left(z_{i}, w_{i}\right)-H_{1}\left(k, h_{i}\right) & =0, \\
H_{2}\left(k, h_{i}\right)-H_{2}\left(k, f_{i}\right) & =0,  \tag{2.8}\\
H_{4}\left(x_{i}, y_{i}\right)-H_{4}\left(z_{i}, w_{i}\right) & =0, \\
x_{i}^{2}+y_{i}^{2}-1 & =0, \\
z_{i}^{2}+w_{i}^{2}-1 & =0,
\end{align*}
$$

for $k=0$ has the three real solutions $S_{i}=\left(x_{i}, y_{i}, z_{i}, w_{i}, f_{i}, h_{i}\right)$ for $i=1,2,3$, where

$$
\begin{aligned}
& S_{1}=(0.597407 . ., 0.801938 . ., 0.29046 . ., 0.956887 . ., 4.80282 . ., 1.19718 . .), \\
& S_{2}=(0.736107 . ., 0.676866 . ., 0.161682 . ., 0.986843 . ., 4.86511 . ., 1.13489 . .), \\
& S_{3}=(0.831057 . ., 0.556188 . ., 0.0773343 . ., 0.997005 . ., 4.92764 . ., 1.07236 . .) .
\end{aligned}
$$

These three limit cycles are drawn in Figure 1.4b. This completes the proof of statement (iii).
Example 5: Three crossing limit cycles of Conf 3 for the class $\mathcal{C}_{3}$. Here we consider the three zones $Z^{1}, Z^{3}$ and $Z^{4}$ defined in (1.3) with $k=0$.

$$
\begin{align*}
\dot{x}= & -\frac{43 x}{2}+43 y+6, \dot{y}=-\frac{43 x}{4}+\frac{43 y}{2}-2, \text { in } Z^{1}, \\
\dot{x}= & -5.01788 . . x+10 y+1.37209 \ldots, \dot{y}=-2.51792 \ldots x+5.01788 . . y  \tag{2.9}\\
& -0.356396 . ., \text { in } Z^{4}, \\
\dot{x}= & -5.2 \ldots x+13 y+1.78427 . ., \dot{y}=-2.08 . . x+5.2 . . y+7, \text { in } Z^{3} .
\end{align*}
$$

The first integrals of the Hamiltonian systems (2.9) are

$$
\begin{aligned}
& H_{1}(x, y)=-\frac{43 x^{2}}{8}+\frac{43 x y}{2}-2 x-\frac{43 y^{2}}{2}-6 y \\
& H_{2}(x, y)=-1.25896 . . x^{2}+5.01788 . . x y-0.356396 . . x-5 y^{2}-1.37209 . . y \\
& H_{3}(x, y)=-1.04 \ldots x^{2}+5.2 x y+7 x-\frac{13 y^{2}}{2}-1.78427 . . y
\end{aligned}
$$

respectively.
The discontinuous piecewise linear differential system formed by the linear Hamiltonian systems (2.9) has exactly three crossing limit cycles, because the system of equations (2.2) has the solutions $S_{i}=\left(x_{i}, y_{i}, z_{i}, w_{i}, f_{i}, h_{i}\right)$ for $i=1,2,3$, where

$$
\begin{aligned}
& S_{1}=(0.92178 . .,-0.387712 . ., 0.478499 . ., 0.878088 . .,-0.974503 . ., 0.7), \\
& S_{2}=(0.988715 . .,-0.149808 . ., 0.647429 . ., 0.762126 . .,-0.819428 . ., 0.544924 . .), \\
& S_{3}=(0.980618 . ., 0.195928 . ., 0.855019 . ., 0.518597 .,-0.616986 . ., 0.342483 . .) .
\end{aligned}
$$

These three limit cycles are drawn in Figure 1.4c. This completes the proof of statement (iv).

Proof of statement (v) of Theorem 1.2. In order to have limit cycles with Conf 1 and Conf 2 simultaneously, the intersection points of the limit cycles of Conf 1 with $\Gamma_{k}$ must satisfy system (2.6) with $k=0$, and the points of intersection of the limit cycles with Conf 2 with $\Gamma_{k}$ must satisfy system (2.8). In statement (ii) and (iii) of Theorem 1.2 we proved that the maximum number of limit cycles with Conf $\mathbf{1}$ and Conf $\mathbf{2}$ is three, then we know that the upper bound of maximum number of limit cycles with both configurations is six.

Example 6: Six crossing limit cycles for the class $\mathcal{C}_{3}$, with three limit cycles of Conf 1 and three limit cycles of Conf 2. Here we consider the four zones defined in (1.3).

$$
\begin{align*}
& \dot{x}=-3.37125 . . x-y+3.95604 . ., \dot{y}=11.3653 . . x+3.37125 . . y-11.7972 . . \text { in } Z^{1}, \\
& \dot{x}=-0.121473 . . x-\frac{1}{2} y+2.02017 . ., \dot{y}=0.0295115 . . x+0.121473 . . y-0.684232 . . \text { in } Z^{2},  \tag{2.10}\\
& \dot{x}=0.328515 . . x+y+3, \dot{y}=-0.107922 . . x-0.328515 . . y-1.29868 . . \text { in } Z^{3}, \\
& \dot{x}=-9.2 x-2.3 y+17, \dot{y}=36.8 x+9.2 y-56 \text { in } Z^{4} .
\end{align*}
$$

The first integrals of the Hamiltonian systems (2.10) are

$$
\begin{aligned}
& H_{1}(x, y)=5.68265 . . x^{2}+3.37125 . . x y-11.7972 . . x+\frac{y^{2}}{2}-3.95604 . . y \\
& H_{2}(x, y)=0.0147557 . . x^{2}+0.121473 . . x y-0.684232 \ldots x+\frac{1}{4} y^{2}-2.02017 . . y \\
& H_{3}(x, y)=-0.0539609 x^{2}-0.328515 x y-1.29868 x-\frac{y^{2}}{2}-3 y \\
& H_{4}(x, y)=18.4 x^{2}+9.2 x y-56 x+1.15 y^{2}-17 y
\end{aligned}
$$

respectively.
For the discontinuous piecewise differential system (2.11), system (2.6) with $k=0$, has the three real solutions

$$
\begin{aligned}
& S_{1}=(0.224513 . .,-0.974471 . .,-0.98,0.198997,8.21167 . .,-0.231664 . .), \\
& S_{2}=(0.359928 . .,-0.93298 . .,-0.812094 . ., 0.583526 . ., 7.77944 . ., 0.239163 . .), \\
& S_{2}=(0.503738 . .,-0.863856 . .,-0.41,0.912086 . ., 7.31697 . ., 0.743252 . .) .
\end{aligned}
$$

and system (2.8), has the three real solutions

$$
\begin{aligned}
& S_{1}=(0.65827 . .,-0.752782 . ., 0.093398 . ., 0.995629 . ., 6.82398 . ., 1.25669 . .), \\
& S_{2}=(0.825187 . .,-0.56486 . ., 0.309897 . ., 0.95077 . ., 6.31504 . ., 1.76563 . .), \\
& S_{2}=(0.986374 . .,-0.164516 . ., 0.630863 . ., 0.775894 . ., 5.87164 . ., 2.20904 . .) .
\end{aligned}
$$

These six limit cycles are presented in Figure 1.5. This completes the proof of statement (v).

Proof of statement (vi) of Theorem 1.2. To get limit cycles with Conf $\mathbf{1}$ and Conf $\mathbf{3}$ simultaneously, the points of intersection of the limit cycles with Conf 1 and Conf 3 with $\Gamma_{k}$ must satisfy system (2.6) and (2.2), respectively, with $k=0$. In statement (ii) and (iv) of Theorem 1.2 we showed that the maximum number of limit cycles with Conf $\mathbf{1}$ and Conf 3 is three, then we know that the upper bound of maximum number of limit cycles with both configurations is six.

Example 7: Six crossing limit cycles for the class $\mathcal{C}_{3}$, with three limit cycles of Conf 1 and three others of Conf 3. Here we consider the four zones defined in (1.3) with $k=0$ with the following Hamiltonian systems

$$
\begin{align*}
& \dot{x}=-8.8 x+22 y-3, \dot{y}=-3.52 x+8.8 y-4 \text { in } Z^{1} \\
& \dot{x}=30.9637 . . x+30 y+0.9 . ., \dot{y}=-31.9584 . . x-30.9637 . . y+24.1071 . . \text { in } Z^{2}, \\
& \dot{x}=0.713131 . . x+0.9 y-0.162525 . ., \dot{y}=-0.565063 . . x-0.713131 . . y+0.620587 . . \text { in } Z^{3},  \tag{2.11}\\
& \dot{x}=-8.37872 . . x+22 y-3.97708 . ., \dot{y}=-3.19104 . . x+8.37872 . . y-3.05205 . . \text { in } Z^{4} .
\end{align*}
$$

The first integrals of the Hamiltonian systems (2.11) are

$$
\begin{aligned}
& H_{1}(x, y)=-1.76 x^{2}+8.8 x y-4 x-11 y^{2}+3 y \\
& H_{2}(x, y)=-13.0028 . . x^{2}-27.9315 . . x y+24.0252 . . x-15 y^{2}-0.9 y \\
& H_{3}(x, y)=-0.282531 . . x^{2}-0.713131 . . x y+0.620587 . . x-0.45 . . y^{2}+0.162525 . . y \\
& H_{4}(x, y)=-1.59552 . . x^{2}+8.37872 . . x y-3.05205 . . x-11 y^{2}+3.97708 . . y
\end{aligned}
$$

respectively.
For the discontinuous piecewise differential system (2.11), system (2.6) with $k=0$, has the three real solutions

$$
\begin{aligned}
& S_{1}=(0.859956 . .,-0.510369 . ., 0.89,0.45596 . ., 1.232 . .,-0.895261 . .) \\
& S_{2}=(0.925727 . .,-0.378193 . .,-0.818732 . ., 0.574176 . ., 1.14562 . .,-0.79916 . .), \\
& S_{3}=(0.969836 . .,-0.243758 . .,-0.7,0.714143 . ., 1.05112 . .,-0.694334 . .),
\end{aligned}
$$

and system (2.2), has the three real solutions

$$
\begin{aligned}
& S_{1}=(0.995048 . .,-0.0993944 . ., 0.167496 . ., 0.985873 . ., 0.937707 . .,-0.576541 . .), \\
& S_{2}=(0.997733 . ., 0.0672986 . ., 0.41691 . ., 0.908948 . ., 0.799221 . .,-0.438055 . .), \\
& S_{3}=(0.954489 . ., 0.298247 . ., 0.659704 . ., 0.751525 . .0 .621163 . .,-0.259997 . .) .
\end{aligned}
$$

These six limit cycles are drawn in 1.6. This completes the proof of statement (vi).

## 3 Proof of Theorem 1.3

We will prove the statement (i). For the other statements the proof is completely analogous.
Proof of statement (i) of Theorem 1.3. From Lemma 1.1 we can consider an arbitrary piecewise linear differential Hamiltonian system in $\mathcal{C}_{\Sigma_{k^{-}}}$formed by the following three linear Hamiltonian systems without equilibrium points

$$
\begin{equation*}
\dot{x}=-\lambda_{i} b_{i} x+b_{i} y+\mu_{i}, \quad \dot{y}=-\lambda_{i}^{2} b_{i} x+\lambda_{i} b_{i} y+\sigma_{i} \text { in } Z_{\Sigma_{k^{-}}}^{i} \tag{3.1}
\end{equation*}
$$

for $i=1,2,3$, where $\sigma_{i} \neq \lambda_{i} \mu_{i}$ and $b_{i} \neq 0$. The Hamiltonian functions associated to these systems are

$$
H_{i}(x, y)=-\frac{1}{2} \lambda_{i}^{2} b_{i} x^{2}+\lambda_{i} b_{i} x y-\frac{b_{i}}{2} y^{2}+\sigma_{i} x-\mu_{i} y, \text { in } Z_{\Sigma_{k^{-}}}^{i} \text { for } i=1,2,3
$$



Figure 3.1: Three limit cycles of system (3.3) intersecting $\Sigma_{-1}$.

In order to have a limit cycle which intersects $\Sigma_{k^{-}}$in four different points $\left(x_{1}, x_{1}^{2}\right),\left(x_{2}, x_{2}^{2}\right)$, $\left(x_{3}, k\right)$ and $\left(x_{4}, k\right)$ with $k<0$, these points must satisfy the system

$$
\begin{align*}
H_{1}\left(x_{1}, x_{1}^{2}\right)-H_{1}\left(x_{2}, x_{2}^{2}\right) & =0 \\
H_{2}\left(x_{2}, x_{2}^{2}\right)-H_{2}\left(x_{3}, k\right) & =0  \tag{3.2}\\
H_{3}\left(x_{3}, k\right)-H_{3}\left(x_{4}, k\right) & =0, \\
H_{2}\left(x_{4}, k\right)-H_{2}\left(x_{1}, x_{1}^{2}\right) & =0, \quad k<0 .
\end{align*}
$$

Assume that the discontinuous piecewise linear differential system (3.1) has four limit cycles. For this we must suppose that system (3.2) has four real solutions, namely $\left(x_{1}^{(i)}, x_{2}^{(i)}, x_{3}^{(i)}, x_{4}^{(i)}\right)$, with $i=1,2,3,4$. Firstly we consider that $\left(x_{1}^{(1)}, x_{2}^{(1)}, x_{3}^{(1)}, x_{4}^{(1)}\right)$ satisfies system (3.2). From the first equation, and by assuming that $x_{1}^{(1)}+x_{2}^{(1)} \neq 0$, we obtain the expression

$$
\mu_{1}=\left(2 \sigma_{1}-b_{1}\left(x_{1}^{(1)}+x_{2}^{(1)}-\lambda_{1}\right)\left(\left(x_{1}^{(1)}\right)^{2}+\left(x_{2}^{(1)}\right)^{2}-\left(x_{1}^{(1)}+x_{2}^{(1)}\right) \lambda_{1}\right)\right) /\left(2\left(x_{1}^{(1)}+x_{2}^{(1)}\right)\right) .
$$

By the second equation we get $\mu_{2}$,

$$
\begin{aligned}
\mu_{2}= & \left(-b_{2}\left(x_{2}^{(1)}\right)^{2}+b_{2}\left(x_{2}^{(1)}\right)^{4}-2 b_{2}\left(x_{2}^{(1)}\right)^{3} \lambda_{2}+2 b_{2} k x_{3}^{(1)} \lambda_{2}+b_{2}\left(x_{2}^{(1)}\right)^{2} \lambda_{2}^{2}\right. \\
& \left.-b_{2}\left(x_{3}^{(1)}\right)^{2} \lambda_{2}^{2}-2 x_{2}^{(1)} \sigma_{2}+2 x_{3}^{(1)} \sigma_{2}\right) / 2\left(k-\left(x_{2}^{(1)}\right)^{2}\right) .
\end{aligned}
$$

We observed that $k-\left(x_{2}^{(1)}\right)^{2}<0$, since $k<0$.
Solving the third equation we have the parameter $\sigma_{3}$,

$$
\sigma_{3}=b_{3} \lambda_{3}\left(-2 k+\left(x_{3}^{(1)}+x_{4}^{(1)}\right) \lambda_{3}\right) / 2 .
$$

By the fourth equation we obtain

$$
\begin{aligned}
\sigma_{2}= & \left(-b_{2} k^{3}+b_{2} k^{2}\left(x_{1}^{(1)}\right)^{2}+b_{2} k\left(x_{2}^{(1)}\right)^{4}-b_{2}\left(x_{1}^{(1)}\right)^{2}\left(x_{2}^{(1)}\right)^{4}-2 b_{2} k\left(x_{2}^{(1)}\right)^{3} \lambda_{2}+2 b_{2}\left(x_{1}^{(1)}\right)^{2}\left(x_{2}^{(1)}\right)^{3} \lambda_{2}\right. \\
& +2 b_{2} k^{2} x_{3}^{(1)} \lambda_{2}-2 b_{2} k\left(x_{1}^{(1)}\right)^{2} x_{3}^{(1)} \lambda_{2}+b_{2} k\left(x_{2}^{(1)}\right)^{2} \lambda_{2}^{2}-b_{2}\left(x_{1}^{(1)}\right)^{2}\left(x_{2}^{(1)}\right)^{2} \lambda_{2}^{2}-b_{2} k\left(x_{3}^{(1)}\right)^{2} \lambda_{2}^{2} \\
& +b_{2}\left(x_{1}^{(1)}\right)^{2}\left(x_{3}^{(1)}\right)^{2} \lambda_{2}^{2}+\left(k-\left(x_{2}^{(1)}\right)^{2}\right) b_{2}\left(k-\left(x_{1}^{(1)}\right)^{2}+x_{1}^{(1)} \lambda_{2}-x_{4}^{(1)} \lambda_{2}\right)\left(k+\left(x_{1}^{(1)}\right)^{2}\right. \\
& \left.\left.-\left(x_{1}^{(1)}+x_{4}^{(1)}\right) \lambda_{2}\right)\right) / 2\left(\left(x_{2}^{(1)}-x_{3}^{(1)}\right)\left(k-\left(x_{1}^{(1)}\right)^{2}\right)+\left(x_{4}^{(1)}-x_{1}^{(1)}\right)\left(k-\left(x_{2}^{(1)}\right)^{2}\right)\right),
\end{aligned}
$$

considering $\left(x_{2}^{(1)}-x_{3}^{(1)}\right)\left(k-\left(x_{1}^{(1)}\right)^{2}\right)+\left(x_{4}^{(1)}-x_{1}^{(1)}\right)\left(k-\left(x_{2}^{(1)}\right)^{2}\right) \neq 0$.
Now we suppose the second solution of system (3.2), we fixed the points $\left(x_{2}^{(2)}, x_{3}^{(2)}\right)$, then by the second equation we obtain the parameter $\lambda_{2}$, then we have determined the values of the parameters $\mu_{2}, \sigma_{2}$ and $\lambda_{2}$ of the Hamiltonian function $H_{2}$ in the zone $Z_{\Sigma_{-}}^{2}$. By solving the third equation we get that $x_{4}^{(2)}=x_{3}^{(1)}+x_{4}^{(1)}-x_{3}^{(2)}$, solving the fourth equation we get the point $x_{1}^{(2)}$ which depends of parameters $\mu_{2}, \sigma_{2}, \lambda_{2}$ and $b_{2}$, moreover the parameter $\lambda_{2}$ depends of parameters $\mu_{2}, \sigma_{2}$ and $b_{2}$, therefore we write $x_{1}^{(2)}$ depends of $\lambda_{2}$, this is $x_{1}^{(2)}=x_{1}^{(2)}\left(\lambda_{2}\right)$. With these points $\left(x_{1}^{(2)}, x_{2}^{(2)}, x_{3}^{(2)}, x_{4}^{(2)}\right)$ and solving first equation we obtain the parameter $\sigma_{1}$

$$
\begin{aligned}
\sigma_{1}= & -b_{1}\left(\left(x_{1}^{(1)}+x_{2}^{(1)}\right)\left(x_{2}^{(2)}+x_{1}^{(2)}\right)\left(-\left(x_{1}^{(1)}\right)^{2}-\left(x_{2}^{(1)}\right)^{2}+\left(x_{2}^{(2)}\right)^{2}+\left(x_{1}^{(2)}\right)^{2}\right)+2\left(x _ { 2 } ^ { ( 2 ) } \left(\left(x_{1}^{(1)}\right)^{2}\right.\right.\right. \\
& \left.+x_{1}^{(1)} x_{2}^{(1)}+\left(x_{2}^{(1)}\right)^{2}-\left(x_{1}^{(1)}+x_{2}^{(1)}\right) x_{2}^{(2)}\right)+\left(\left(x_{1}^{(1)}\right)^{2}+x_{1}^{(1)} x_{2}^{(1)}+\left(x_{2}^{(1)}\right)^{2}-\left(x_{1}^{(1)}\right.\right. \\
& \left.\left.\left.\left.+\left(x_{2}^{(1)}\right)\right) x_{2}^{(2)}\right) x_{1}^{(2)}-\left(x_{1}^{(1)}+x_{2}^{(1)}\right)\left(x_{1}^{(2)}\right)^{2}\right) \lambda_{1}\right) /\left(2\left(-x_{1}^{(1)}-x_{2}^{(1)}+x_{2}^{(2)}+x_{1}^{(2)}\right)\right)
\end{aligned}
$$

considering $\left(-x_{1}^{(1)}-x_{2}^{(1)}+x_{2}^{(2)}+x_{1}^{(2)}\right) \neq 0$.
Likewise, we consider the third solution, and we fixed the point $x_{2}^{(3)}$. Then by the second equation we obtain the point $x_{3}^{(3)}$ which depends of parameter $\lambda_{2}$, solving the third equation we get that $x_{4}^{(3)}=x_{3}^{(1)}+x_{4}^{(1)}-x_{3}^{(3)}$ and by fourth equation we obtain the point $x_{1}^{(3)}$ which depends of parameter $\lambda_{2}$, finally with these points $\left(x_{1}^{(3)}, x_{2}^{(3)}, x_{3}^{(3)}, x_{3}^{(1)}+x_{4}^{(1)}-x_{3}^{(3)}\right)$ and by the first equation we obtain $\lambda_{1}=A / B$ with $B \neq 0$, where

$$
\begin{aligned}
A= & \left(\left(x_{1}^{(1)}\right)^{3}\left(x_{1}^{(2)}+x_{2}^{(2)}-x_{1}^{(3)}-x_{2}^{(3)}\right)+\left(x_{1}^{(1)}\right)^{2} x_{2}^{(1)}\left(x_{1}^{(2)}+x_{2}^{(2)}-x_{1}^{(3)}-x_{2}^{(3)}\right)+\left(x_{2}^{(1)}\right)^{3}\left(x_{1}^{(2)}\right.\right. \\
& \left.+x_{2}^{(2)}-x_{1}^{(3)}-x_{2}^{(3)}\right)+\left(x_{1}^{(2)}+x_{2}^{(2)}\right)\left(x_{1}^{(3)}+x_{2}^{(3)}\right)\left(\left(x_{1}^{(2)}\right)^{2}+\left(x_{2}^{(2)}\right)^{2}-\left(x_{1}^{(3)}\right)^{2}-\left(x_{2}^{(3)}\right)^{2}\right) \\
& +x_{2}^{(1)}\left(-\left(x_{1}^{(2)}\right)^{3}-\left(x_{1}^{(2)}\right)^{2} x_{2}^{(2)}-x_{1}^{(2)}\left(x_{2}^{(2)}\right)^{2}-\left(x_{2}^{(2)}\right)^{3}+\left(x_{1}^{(3)}\right)^{3}+\left(x_{1}^{(3)}\right)^{2} x_{2}^{(3)}\right. \\
& \left.+x_{1}^{(3)}\left(x_{2}^{(3)}\right)^{2}+\left(x_{2}^{(3)}\right)^{3}\right)+x_{1}^{(1)}\left(-\left(x_{1}^{(2)}\right)^{3}-\left(x_{1}^{(2)}\right)^{2} x_{2}^{(2)}-x_{1}^{(2)}\left(x_{2}^{(2)}\right)^{2}-\left(x_{2}^{(2)}\right)^{3}+\left(x_{1}^{(3)}\right)^{3}\right. \\
& \left.\left.+\left(x_{2}^{(1)}\right)^{2}\left(x_{1}^{(2)}+x_{2}^{(2)}-x_{1}^{(3)}-x_{2}^{(3)}\right)+\left(x_{1}^{(3)}\right)^{2} x_{2}^{(3)}+x_{1}^{(3)}\left(x_{2}^{(3)}\right)^{2}+\left(x_{2}^{(3)}\right)^{3}\right)\right), \\
B= & 2\left(\left(x_{1}^{(2)}\right)^{2} x_{1}^{(3)}+x_{1}^{(2)} x_{2}^{(2)} x_{1}^{(3)}+\left(x_{2}^{(2)}\right)^{2} x_{1}^{(3)}-x_{1}^{(2)}\left(x_{1}^{(3)}\right)^{2}-x_{2}^{(2)}\left(x_{1}^{(3)}\right)^{2}+\left(x_{1}^{(1)}\right)^{2}\left(x_{1}^{(2)}\right.\right. \\
& \left.+x_{2}^{(2)}-x_{1}^{(3)}-x_{2}^{(3)}\right)+\left(x_{2}^{(1)}\right)^{2}\left(x_{1}^{(2)}+x_{2}^{(2)}-x_{1}^{(3)}-x_{2}^{(3)}\right)+\left(x_{1}^{(2)}\right)^{2} x_{2}^{(3)}+x_{1}^{(2)} x_{2}^{(2)} x_{2}^{(3)} \\
& +\left(x_{2}^{(2)}\right)^{2} x_{2}^{(3)}-x_{1}^{(2)} x_{1}^{(3)} x_{2}^{(3)}-x_{2}^{(2)} x_{1}^{(3)} x_{2}^{(3)}-x_{1}^{(2)}\left(x_{2}^{(3)}\right)^{2}-x_{2}^{(2)}\left(x_{2}^{(3)}\right)^{2}+x_{2}^{(1)}\left(-\left(x_{1}^{(2)}\right)^{2}\right. \\
& \left.-x_{1}^{(2)} x_{2}^{(2)}-\left(x_{2}^{(2)}\right)^{2}+\left(x_{1}^{(3)}\right)^{2}+x_{1}^{(3)} x_{2}^{(3)}+\left(x_{2}^{(3)}\right)^{2}\right)+x_{1}^{(1)}\left(-\left(x_{1}^{(2)}\right)^{2}-x_{1}^{(2)} x_{2}^{(2)}-\left(x_{2}^{(2)}\right)^{2}\right. \\
& \left.\left.+\left(x_{1}^{(3)}\right)^{2}+x_{2}^{(1)}\left(x_{1}^{(2)}+x_{2}^{(2)}-x_{1}^{(3)}-x_{2}^{(3)}\right)+x_{1}^{(3)} x_{2}^{(3)}+\left(x_{2}^{(3)}\right)^{2}\right)\right) .
\end{aligned}
$$

We observed that we have determined the values of the parameters $\mu_{1}, \sigma_{1}$ and $\lambda_{1}$ of the Hamiltonian function $H_{1}$ in the zone $Z_{\Sigma_{-}}^{1}$.

By a similar way, we consider the fourth solution, and we fixed the point $x_{2}^{(4)}$, then by the second equation we obtain the point $x_{3}^{(4)}$, solving the third equation we get that $x_{4}^{(4)}=x_{3}^{(1)}+$ $x_{4}^{(1)}-x_{3}^{(4)}$, by the fourth equation we get the point $x_{1}^{(4)}$. With these points $\left(x_{1}^{(4)}, x_{2}^{(4)}, x_{3}^{(4)}, x_{3}^{(1)}+\right.$ $\left.x_{4}^{(1)}-x_{3}^{(4)}\right)$ from the first equation we have that $b_{1}=0$ which is a contradiction, because from Lemma $1.1 b_{i} \neq 0$ for $i=1,2,3$. Therefore the maximum number of limit cycles in this case is three.

Now we prove that this upper bound is attached. We have that the unique restriction of value $k$ is that the denominator in the expressions of $\sigma_{2}$ is different from zero. We observed that it is possible to choose values to the points $x_{1}^{(1)}, x_{2}^{(1)}, x_{3}^{(1)}$ and $x_{4}^{(1)}$ such that
$k \neq\left(\left(x_{1}^{(1)}\right)^{2}\left(x_{2}^{(2)}-x_{3}^{(1)}\right)+\left(x_{1}^{(1)}\right)^{2}\left(x_{2}^{(2)}-x_{3}^{(1)}\right)\right) /\left(x_{2}^{(1)}-x_{3}^{(1)}+x_{4}^{(1)}-x_{1}^{(1)}\right)$, for instance if we consider $x_{2}^{(1)}<0, x_{1}^{(1)}>-x_{2}^{(1)}, x_{1}^{(1)}<x_{4}^{(1)}<x_{1}^{(1)}-x_{2}^{(1)}$ and $x_{3}^{(1)}<\left(\left(x_{1}^{(1)}\right)^{2} x_{2}^{(1)}-x_{1}^{(1)}\left(x_{2}^{(1)}\right)^{2}+\right.$ $\left.\left(x_{2}^{(1)}\right)^{2} x_{4}^{(1)}\right) /\left(x_{1}^{(1)}\right)^{2}$ we have that the expression $\left(\left(x_{1}^{(1)}\right)^{2}\left(x_{2}^{(2)}-x_{3}^{(1)}\right)+\left(x_{1}^{(1)}\right)^{2}\left(x_{2}^{(2)}-x_{3}^{(1)}\right)\right) /$ $\left(x_{2}^{(1)}-x_{3}^{(1)}+x_{4}^{(1)}-x_{1}^{(1)}\right)$ is always positive therefore it is different of value of $k$, since that $k<0$. Then we can consider without loss of generality that $k=-1$. We consider the discontinuous piecewise linear differential system defined by the following three linear Hamiltonian systems

$$
\begin{align*}
& \dot{x}=-24.293899 . .-0.692634 \ldots x+\frac{3}{2} y, \dot{y}=-19.232427 . .-0.319828 . . x+0.692634 . . y, \\
& \dot{x}=-378.204351 . .+62.383901 . . x-4 y, \dot{y}=916.621187 . .+972.937795 . . x-62.383901 . . y,  \tag{3.3}\\
& \dot{x}=\frac{9}{10}-\frac{7}{2} x-\frac{35}{4} y, \dot{y}=\frac{7}{2}+\frac{7}{5} x+\frac{7}{2} y,
\end{align*}
$$

in the zones $Z_{\Sigma_{-1}}^{1}, Z_{\Sigma_{-1}}^{2}$ and $Z_{\Sigma_{-1}}^{3}$, respectively. Then for system (3.3), we have that system (3.2) has three real solutions $\left(x_{1}^{(i)}, x_{2}^{(i)}, x_{3}^{(i)}, x_{4}^{(i)}\right), i=1,2,3$, namely

$$
\left(3,-2,-\frac{7}{2}, \frac{7}{2}\right), \quad\left(2.625658 . .,-\frac{17}{10 . .,}-\frac{31}{10}, \frac{31}{10}\right), \quad\left(\frac{14}{5},-1.843412 . .,-3.287307 . ., 3.287307 . .\right) .
$$

These three real solutions provide the three limit cycles intersecting $\Sigma_{-1}$ shown in Figure 3.1. This completes the proof of statement (i).


Proof of statement (ii) of Theorem 1.3. The proof in this statement is similar to the proof of statement (i). For each configuration of limit cycles that intersect $\Sigma_{k}$ with $k \geq 0$ we have that the upper bound of limit cycles is three. In what follows we show examples of piecewise linear differential system in $\mathcal{C}_{\Sigma_{0}}$ with three limit cycles with Conf 4 and Conf 5, respectively. And piecewise linear differential system in $\mathcal{C}_{\Sigma_{k+}}$ with three limit cycles with Conf $6^{+}$, Conf 7, Conf 8 and Conf $9^{+}$, respectively.
Crossing limit cycles with Conf 4: In order to have a limit cycle with Conf 4 which intersects $\Sigma_{0}$ in four different points $\left(x_{1}, x_{1}^{2}\right),\left(x_{2}, x_{2}^{2}\right),\left(x_{3}, 0\right)$ and $\left(x_{4}, 0\right)$, these points must satisfy system
(3.2) with $k=0$. We consider the discontinuous piecewise linear differential system defined by the following four linear Hamiltonian systems

$$
\begin{align*}
\dot{x} & =\frac{17}{2}-\frac{207}{50} x-\frac{69}{10} y, \dot{y}=-\frac{53}{10}+\frac{621}{250} x+\frac{207}{50} y \\
\dot{x} & =48.069511 . .+11.825263 . . x-\frac{31}{5} y, \dot{y}=-7.155434 . .+22.554330 \ldots x-11.825263 . . y,  \tag{3.4}\\
\dot{x} & =\frac{9}{2}+\frac{156}{25} x-\frac{39}{10} y, \dot{y}=-\frac{13}{10}+9.984000 . . x-\frac{156}{25} y \\
\dot{x} & =17.727172 . .-7.176019 \ldots x-\frac{27}{5} y, \dot{y}=-1.428092 . .+9.536159 \ldots x+7.176019 . . y
\end{align*}
$$

in the zones $Z_{\Sigma_{0}}^{1}, ~ Z_{\Sigma_{0}}^{2}, Z_{\Sigma_{0}}^{3}$ and $Z_{\Sigma_{0}}^{4}$, respectively. For the discontinuous piecewise differential system (3.4), system (3.2) has three real solutions $\left(x_{1}^{(i)}, x_{2}^{(i)}, x_{3}^{(i)}, x_{4}^{(i)}\right), i=1,2,3$ given by

$$
\begin{aligned}
& (1.517382 . .,-2.102549 . .,-1.142394 . ., 1.402811 . .), \\
& (1.474836 . .,-2.058730 . .,-0.973819 . ., 1.234236 . .), \\
& (1.427170 . .,-2.00939 . .,-0.774355 . ., 1.034772 . .) .
\end{aligned}
$$

These solutions provide the three limit cycles with Conf 4 shown in Figure 3.2a. Crossing limit cycles with Conf 5: In order to have a limit cycle with Conf 5 which intersects $\Sigma_{0}$ in the four different points $\left(x_{1}, x_{1}^{2}\right),\left(x_{2}, x_{2}^{2}\right),\left(x_{3}, 0\right)$ and $\left(x_{4}, 0\right)$, they must satisfy

$$
\begin{align*}
H_{1}\left(x_{1}, x_{1}^{2}\right)-H_{1}\left(x_{2}, x_{2}^{2}\right) & =0 \\
H_{4}\left(x_{2}, x_{2}^{2}\right)-H_{4}\left(x_{3}, k\right) & =0  \tag{3.5}\\
H_{3}\left(x_{3}, k\right)-H_{3}\left(x_{4}, k\right) & =0 \\
H_{4}\left(x_{4}, k\right)-H_{4}\left(x_{1}, x_{1}^{2}\right) & =0, \quad \text { with } k=0 .
\end{align*}
$$

We consider the discontinuous piecewise linear differential system defined by the following three linear Hamiltonian systems

$$
\begin{align*}
\dot{x} & =-4.711119 . .+3.915394 . . x-\frac{3}{2} y, \dot{y}=-11.965988 . .+10.220210 . . x-3.915394 . . y, \\
\dot{x} & =\frac{9}{10}+\frac{27}{10} x-\frac{3}{2} y, \dot{y}=-5.022000 . .+\frac{243}{50} x-\frac{27}{10} y,  \tag{3.6}\\
\dot{x} & =-3.005265 . .+2.848936 . . x-\frac{11}{10} y, \dot{y}=-7.616106 . .+7.378583 \ldots x-2.848936 . . y,
\end{align*}
$$

in the zones $Z_{\Sigma_{0}}^{1}, Z_{\Sigma_{0}}^{3}$ and $Z_{\Sigma_{0}}^{4}$, respectively. For the discontinuous piecewise differential system (3.6), system (3.5) has three real solutions $\left(x_{1}^{(i)}, x_{2}^{(i)}, x_{3}^{(i)}, x_{4}^{(i)}\right), i=1,2,3$ given by

$$
\left(2, \frac{1}{2}, \frac{2}{5}, \frac{5}{3}\right),\left(\frac{93}{50}, 0.628914 . ., \frac{47}{100}, \frac{479}{300}\right),(1.696225 . ., 0.780317 . ., 0.534387 . ., 1.532279 . .) .
$$

These solutions provide the three limit cycles with Conf 5 shown in Figure 3.2b.
Crossing limit cycles with Conf $6^{+}$: In order to have a limit cycle with Conf $6^{+}$which intersects $\Sigma^{+}$in four different points $\left(x_{1}, x_{1}^{2}\right),\left(x_{2}, k\right),\left(x_{3}, x_{3}^{2}\right)$ and $\left(x_{4}, k\right)$, these points must satisfy

$$
\begin{align*}
& H_{1}\left(x_{1}, x_{1}^{2}\right)-H_{1}\left(x_{2}, k\right)=0 \\
& H_{5}\left(x_{2}, k\right)-H_{5}\left(x_{3}, x_{3}^{2}\right)=0  \tag{3.7}\\
& H_{3}\left(x_{3}, x_{3}^{2}\right)-H_{3}\left(x_{4}, k\right)=0 \\
& H_{4}\left(x_{4}, k\right)-H_{4}\left(x_{1}, x_{1}^{2}\right)=0, \text { for } k>0 .
\end{align*}
$$



Figure 3.3: (a) Three limit cycles with Conf $\mathbf{6}^{+}$of system (3.9). (b) Three limit cycles with Conf 7 of system (3.11). (c) Three limit cycles with Conf 8 of system (3.13). (d) Three limit cycles with Conf $\mathbf{9}^{+}$of system (3.14).

To have a limit cycle with Conf $6^{-}$which intersects $\Sigma^{+}$in four different points, these points must satisfy the system

$$
\begin{align*}
& H_{2}\left(x_{1}, x_{1}^{2}\right)-H_{2}\left(x_{2}, k\right)=0, \\
& H_{3}\left(x_{2}, k\right)-H_{3}\left(x_{3}, x_{3}^{2}\right)=0, \\
& H_{5}\left(x_{3}, x_{3}^{2}\right)-H_{5}\left(x_{4}, k\right)=0,  \tag{3.8}\\
& H_{1}\left(x_{4}, k\right)-H_{1}\left(x_{1}, x_{1}^{2}\right)=0, \quad \text { for } k>0 .
\end{align*}
$$

We provide an example of a piecewise linear differential system with three limit cycles with Conf $6^{+}$. We observed that the upper bound found does not depend of the value of the parameter $k>0$, then we can consider without loss of generality that $k=4$. We consider the discontinuous piecewise linear differential system defined by the following four linear

Hamiltonian systems

$$
\begin{align*}
\dot{x} & =-4.133710 . .+6.015251 . . x-\frac{3}{2} y, \dot{y}=-5.920817 . .+24.122170 . . x-6.015251 . . y \\
\dot{x} & =5.325742 . .+2.017936 . . x-\frac{17}{5} y, \dot{y}=4.036253 . .+1.197666 . . x-2.017936 . . y  \tag{3.9}\\
\dot{x} & =-8.981178 . .+3.946297 . . x-y, \dot{y}=-15.942643 . .+15.573265 . . x-3.946297 . . y \\
\dot{x} & =-2.454956 . .+4.664679 . . x+\frac{3}{2} y, \dot{y}=6.613677 . .-14.506158 . . x-4.664679 . . y
\end{align*}
$$

in the zones $Z_{\Sigma_{4}}^{1}, Z_{\Sigma_{4}}^{3}, Z_{\Sigma_{4}}^{4}$ and $Z_{\Sigma_{4}}^{5}$, respectively. For the discontinuous piecewise differential system (3.9), system (3.7) has three real solutions $\left(x_{1}^{(i)}, x_{2}^{(i)}, x_{3}^{(i)}, x_{4}^{(i)}\right), i=1,2,3$ given by

$$
\left(4,-\frac{2}{5}, \frac{1}{2}, 5\right), \quad\left(\frac{193}{50},-\frac{31}{100}, \frac{13}{20}, \frac{483}{100}\right), \quad\left(\frac{7}{2},-\frac{3}{25}, \frac{83}{100}, \frac{441}{100}\right) .
$$

These solutions provide the three limit cycles with Conf $\mathbf{6}^{+}$shown in Figure 3.3a.
Crossing limit cycles with Conf 7: In order to have a limit cycle with Conf 7 which intersects $\Sigma^{+}$in the four different points $\left(x_{1}, k\right),\left(x_{2}, k\right),\left(x_{3}, x_{3}^{2}\right)$ and $\left(x_{4}, x_{4}^{2}\right)$, they must satisfy the system

$$
\begin{align*}
H_{1}\left(x_{1}, k\right)-H_{1}\left(x_{2}, k\right) & =0, \\
H_{5}\left(x_{2}, k\right)-H_{5}\left(x_{3}, x_{3}^{2}\right) & =0, \\
H_{3}\left(x_{3}, x_{3}^{2}\right)-H_{3}\left(x_{4}, x_{4}^{2}\right) & =0,  \tag{3.10}\\
H_{5}\left(x_{4}, x_{4}^{2}\right)-H_{5}\left(x_{1}, k\right) & =0, \quad \text { with } k>0 .
\end{align*}
$$

We can suppose without loss of generality that $k=4$. We consider the discontinuous piecewise linear differential system defined by the following three linear Hamiltonian systems

$$
\begin{align*}
\dot{x} & =-2-6 x-\frac{3}{2} y, \dot{y}=-5.491482 . .+24 x+6 y, \\
\dot{x} & =22.645454 . .-36.659999 . . x-\frac{47}{5} y, \dot{y}=-35.463636 . .+142.973999 . . x+36.659999 . . y, \\
\dot{x} & =5.300000 . .-8.579999 . . x-\frac{11}{5} y, \dot{y}=-8.300000 . .+33.461999 . . x+8.579999 . . y, \tag{3.11}
\end{align*}
$$

in the zones $Z_{\Sigma_{3}}^{1}, Z_{\Sigma_{3}}^{3}$ and $Z_{\Sigma_{3}}^{5}$, respectively. For the discontinuous piecewise differential system (3.11), system (3.10) has three real solutions $\left(x_{1}^{(i)}, x_{2}^{(i)}, x_{3}^{(i)}, x_{4}^{(i)}\right), i=1,2,3$ given by

$$
\begin{aligned}
& (0.502842 . .,-1.545218 . .,-0.572025 . ., 0.848539 . .), \\
& (0.442709 . .,-1.485086 . .,-0.427227 . ., 0.781483 . .) \\
& (0.378567 . .,-1.420944 . .,-0.276975 . ., 0.700080 . .)
\end{aligned}
$$

These solutions provide the three limit cycles with Conf 7 shown in 3.3b.
Crossing limit cycles with Conf 8: In order to have a limit cycle with Conf 8 which intersects $\Sigma^{+}$in four different points $\left(x_{1}, x_{1}^{2}\right),\left(x_{2}, x_{2}^{2}\right),\left(x_{3}, k\right)$ and $\left(x_{4}, k\right)$, they must satisfy

$$
\begin{align*}
H_{1}\left(x_{1}, x_{1}^{2}\right)-H_{1}\left(x_{2}, x_{2}^{2}\right) & =0 \\
H_{2}\left(x_{2}, x_{2}^{2}\right)-H_{2}\left(x_{3}, k\right) & =0 \\
H_{3}\left(x_{3}, k\right)-H_{3}\left(x_{4}, k\right) & =0  \tag{3.12}\\
H_{4}\left(x_{4}, k\right)-H_{4}\left(x_{1}, x_{1}^{2}\right) & =0, \quad \text { with } k>0
\end{align*}
$$

We can consider without loss of generality that $k=2$. We consider the discontinuous piecewise linear differential system defined by the following four linear Hamiltonian systems

$$
\begin{align*}
& \dot{x}=\frac{9}{2}+\frac{19}{50} x-\frac{19}{10} y, \dot{y}=\frac{17}{10}+0.076000 . . x-\frac{19}{50} y \\
& \dot{x}=10.930108 . .+7.204668 . . x-\frac{11}{5} y, \dot{y}=99.090506 . .+23.594202 . x-7.204668 . . y, \\
& \dot{x}=-\frac{69}{2}-6.229999 . . x-\frac{89}{10} y, \dot{y}=-\frac{93}{10}+\frac{4361}{1000} x+6.229999 . . y, \\
& \dot{x}=32.954952 . .-16.575663 . . x-\frac{17}{5} y, \dot{y}=-277.274017 . .+80.809593 . . x+16.575663 . . y, \tag{3.13}
\end{align*}
$$

in the pieces $Z_{\Sigma_{2}}^{1}, Z_{\Sigma_{2}}^{2}, Z_{\Sigma_{2}}^{3}$ and $Z_{\Sigma_{2}}^{4}$, respectively. For the discontinuous piecewise differential system (3.13), system (3.12) has three real solutions $\left(x_{1}^{(i)}, x_{2}^{(i)}, x_{3}^{(i)}, x_{4}^{(i)}\right), i=1,2,3$ given by

$$
\begin{aligned}
& (2.514526 . .,-2.427396 . .,-6.114467 . ., 4.665258 . .), \\
& (2.449236 . .,-2.371782 . .,-6.028697 . ., 4.579488 . .) \\
& (2.374832 . .,-2.310077 . .,-5.941517 . ., 4.492308 . .)
\end{aligned}
$$

These solutions provide the three limit cycles with Conf 8 shown in Figure 3.3c.
Crossing limit cycles with Conf $9^{+}$: In order to have a limit cycle with Conf $9^{+}$which intersects $\Sigma^{+}$in the four different points $\left(x_{1}, x_{1}^{2}\right),\left(x_{2}, x_{2}^{2}\right),\left(x_{3}, k\right)$ and $\left(x_{4}, k\right)$, they must satisfy system (3.5) with $k>0$. Without loss of generality we can suppose that $k=4$. We consider the discontinuous piecewise linear differential system defined by the following three linear Hamiltonian systems

$$
\begin{align*}
\dot{x} & =-170.859539 . .+99.779168 . . x-15 y, \dot{y}=-1139.726782 . .+663.725497 . . x-99.779168 . . y \\
\dot{x} & =\frac{9}{10}+\frac{148}{5} x-4 y, \dot{y}=-779.664000 . .+219.040000 . . x-\frac{148}{5} y \\
\dot{x} & =116.632274 . .-30.946111 . . x-\frac{23}{10} y, \dot{y}=-1635.644521 . .+416.374692 . . x+30.946111 . . y \tag{3.14}
\end{align*}
$$

in the zones $Z_{\Sigma_{4}}^{1}, Z_{\Sigma_{4}}^{3}$, and $Z_{\Sigma_{4}}^{4}$, respectively. For the discontinuous piecewise differential system (3.14), system (3.5) with $k=4$, has three real solutions $\left(x_{1}^{(i)}, x_{2}^{(i)}, x_{3}^{(i)}, x_{4}^{(i)}\right), i=1,2,3$ given by

$$
\left(4,3, \frac{16}{5}, 5\right),\left(4.109491 . ., \frac{141}{50}, \frac{303}{100}, \frac{517}{100}\right),\left(\frac{47}{10}, 2.053733 . ., 2.068270 . ., 6.131729 . .\right)
$$

These solutions provide the three limit cycles with Conf $9^{+}$shown in Figure 3.3d. This completes the proof of statement (ii).

Proof of statement (iii) of Theorem 1.3. In order to have limit cycles with Conf 4 and Conf 5 simultaneously, the points of intersection of the limit cycles with Conf 4 with $\Sigma_{0}$ must satisfy system (3.2) with $k=0$, and the points of intersection of the limit cycles with Conf 5 with $\Sigma_{0}$ must satisfy system (3.5). In statement (ii) we proved that the maximum number of limit cycles with Conf 4 and Conf 5 is three, then we have that the upper bound of maximum number of limit cycles with both configurations is six. We provide an example of a piecewise linear


Figure 3.4: Three limit cycles with Conf 4, and three limit cycles with Conf 5 of system (3.15) simultaneously.
differential system in $\mathcal{C}_{\Sigma_{0}}$ such that have six limit cycles with three limit cycles with Conf 4 and Conf 5 , respectively. This is the upper bound is reached. We consider the discontinuous piecewise linear differential system defined by the following four linear Hamiltonian systems

$$
\begin{align*}
\dot{x} & =-154.076990 . .-28.017658 . . x+15 y, \dot{y}=-387.181918 . .-52.332611 . . x+28.017658 . . y, \\
\dot{x} & =-0.400024 . .+0.532848 . . x-\frac{3}{10} y, \dot{y}=0.058205 . .+0.946425 . . x-0.532848 . . y, \\
\dot{x} & =\frac{9}{10}+\frac{28}{5} x-4 y, \dot{y}=-8.101333 . .+7.839999 . . x-\frac{28}{5} y, \\
\dot{x} & =-3.005265 . .+2.848936 . . x-\frac{11}{10} y, \dot{y}=-7.616106 . .+7.378583 . . x-2.848936 . . y, \tag{3.15}
\end{align*}
$$

in the zones $Z_{\Sigma_{0}}^{1}$, $Z_{\Sigma_{0}}^{2}$, $Z_{\Sigma_{0}}^{3}$ and $Z_{\Sigma_{0}}^{4}$, respectively. For the discontinuous piecewise differential system (3.15), system (3.2) with $k=0$, has the following three real solutions

$$
\begin{gathered}
\left(\frac{27}{10},-\frac{1}{10},-0.166825 . ., 2.233491 . .\right), \quad\left(\frac{69}{25},-0.147032 . .,-0.232725 . ., 2.299392 . .\right), \\
\left(\frac{14}{5},-0.177898 . .,-0.278200 . ., 2.344866 . .\right),
\end{gathered}
$$

and system (3.5) with $k=0$, has the three real solutions

$$
\left(2, \frac{1}{2}, \frac{2}{5}, \frac{5}{3}\right),\left(\frac{93}{50}, 0.628914 . ., \frac{47}{100}, \frac{479}{300}\right),(1.393438 . ., 1.075216 . ., 0.604474 . ., 1.462192 . .) .
$$

These solutions provide the three limit cycles with Conf 4 and Conf 5 shown in Figure 3.4. This completes the proof of statement (iii).

Proof of statement (iv) of Theorem 1.3. In order to have limit cycles with Conf $\mathbf{6}^{-}$and Conf $\mathbf{6}^{+}$simultaneously, the points of intersection of the limit cycles with Conf $6^{+}$and $\Sigma_{k^{+}}$must satisfy system (3.7), and the points of intersection of the limit cycles with Conf $\mathbf{6}^{-}$and $\Sigma_{k^{+}}$ must satisfy system (3.8). In statement (ii) we proved that the maximum number of limit cycles with each configuration is three, then we have that the upper bound of maximum number of limit cycles with both configurations is six. We provide an example of a piecewise linear


Figure 3.5: Three limit cycles with Conf $\mathbf{6}^{-}$and Conf $\mathbf{6}^{+}$of system (3.16).
differential system in $\mathcal{C}_{\Sigma_{k+}}$ such that have six limit cycles with three limit cycles with each configuration. This is the upper bound is reached. Without loss of generality we can suppose that $k=4$. We consider the discontinuous piecewise linear differential system defined by the following five linear Hamiltonian systems

$$
\begin{align*}
\dot{x}= & -23138.489410 . .+403.676452 . . x+\frac{9}{2} y, \dot{y}=2942.120325 . .-36212.150741 . . x \\
& -403.676452 . . y, \\
\dot{x}= & 4.276633 . .+1.873985 . . x-\frac{3}{10} y, \dot{y}=4.991226 . .+11.706072 . . x-1.873985 . . y, \\
\dot{x}= & 15.472057 . .-3.117904 . . x-\frac{17}{5} y, \dot{y}=-13.354567 . .+2.859213 . . x+3.117904 . . y,  \tag{3.16}\\
\dot{x}= & 48.158492 . .-6.082779 . . x-y, \dot{y}=-31.590984 . .+37.000210 . . x+6.082779 . . y, \\
\dot{x}= & -151.854124 . .-136.354901 . . x+\frac{3}{2} y, \dot{y}=-10611.949690 . .-12395.106180 . . x \\
& +136.354901 . . y,
\end{align*}
$$

in the pieces $Z_{\Sigma_{4}}^{1}, Z_{\Sigma_{4}}^{2}, Z_{\Sigma_{4}}^{3}, Z_{\Sigma_{4}}^{4}$ and $Z_{\Sigma_{4}}^{5}$, respectively. For the discontinuous piecewise differential system (3.16), (3.7), has the three real solutions

$$
\left(5, \frac{1}{2}, \frac{9}{20}, \frac{23}{5}\right), \quad\left(\frac{9}{2}, \frac{19}{20}, \frac{91}{100}, \frac{7}{2}\right), \quad\left(\frac{41}{10}, 1.196150 . ., 1.163297 . ., 2.719447 . .\right),
$$

and system (3.8), has the following three real solutions

$$
\begin{gathered}
\left(-\frac{18}{5},-\frac{9}{2},-\frac{49}{50},-1\right), \quad(-3,-3.411586 . .,-1.557354 . .,-1.546135 . .), \\
\left(-2.809209 . .,-\frac{31}{10},-1.671884 . .,-1.662653 . .\right) .
\end{gathered}
$$

These solutions provide the three limit cycles with Conf $\mathbf{6}^{-}$and Conf $\mathbf{6}^{+}$shown in Figure 3.5. This completes the proof of statement (iv).

Proof of statement (v) of Theorem 1.3. In order to have limit cycles with Conf $\mathbf{6}^{-}$and Conf 7 simultaneously, the points of intersection of the limit cycles with Conf $\mathbf{6}^{-}$and $\Sigma_{k^{+}}$must satisfy
system (3.8), and the points of intersection of the limit cycles with Conf 7 and $\Sigma_{k^{+}}$must satisfy system (3.10). In statement (ii) we proved that the maximum number of limit cycles with each configuration is three, then we have that the maximum number of limit cycles with both configurations is six. Moreover this upper bound is reached. Without loss of generality we can suppose that $k=3$. We consider the discontinuous piecewise linear differential system defined by the following four linear Hamiltonian systems

$$
\begin{align*}
\dot{x} & =-0.567977 . .-5.151614 . . x-\frac{3}{2} y, \dot{y}=-6.233588 . .+17.692757 . . x+5.151614 . . y, \\
\dot{x} & =11.250254 . .+0.637407 . . x-\frac{2}{5} y, \dot{y}=-35.085985 . .+1.015720 . . x-0.637407 . . y, \\
\dot{x} & =22.645454 . .-36.659999 . . x-\frac{47}{5} y, \dot{y}=-35.463636 . .+142.973999 . . x+36.659999 . . y, \\
\dot{x} & =5.300000 . .-8.579999 . . x-\frac{11}{5} y, \dot{y}=-8.300000 . .+33.461999 . . x+8.579999 . . y, \tag{3.17}
\end{align*}
$$

in the pieces $Z_{\Sigma_{3^{\prime}}}^{1} Z_{\Sigma_{3^{\prime}}}^{2}, Z_{\Sigma_{3}}^{3}$ and $Z_{\Sigma_{3^{\prime}}}^{5}$, respectively. For the discontinuous piecewise differential


Figure 3.6: (a) Three limit cycles with Conf $\mathbf{6}^{-}$and Conf 7 of system (3.17). (b) Three limit cycles with Conf $\mathbf{6}^{+}$and Conf 8 of system (3.18).
system (3.17), system (3.8), has the following three real solutions

$$
\begin{gathered}
\left(-\frac{41}{10},-\frac{5}{2}, 1.569412 . ., 1.457623 . .\right), \quad\left(-\frac{106}{25},-\frac{263}{100}, 1.647799 . ., 1.587623 . .\right), \\
\left(-\frac{199}{50},-2.401954 . ., 1.508473 . ., 1.359577 . .\right),
\end{gathered}
$$

and (3.10), has the three real solutions

$$
\begin{aligned}
& (0.502842 . .,-1.545218 . .,-0.572025 . ., 0.848539 . .), \\
& (0.442709 . .,-1.485086 . .,-0.427227 . ., 0.781483 . .), \\
& (0.378567 . .,-1.420944 . .,-0.276975 . ., 0.700080 . .) .
\end{aligned}
$$

These solutions provide the three limit cycles with Conf $\mathbf{6}^{-}$and Conf 7 shown in Figure 3.6a. This completes the proof of statement (v).

Proof of statement (vi) of Theorem 1.3. In order to have limit cycles with Conf $\mathbf{6}^{+}$and Conf 8 simultaneously, the points of intersection of the limit cycles with Conf $\mathbf{6}^{+}$and $\Sigma_{k^{+}}$must satisfy system (3.8), and the points of intersection of the limit cycles with Conf 8 and $\Sigma_{k^{+}}$must satisfy system (3.12). In statement (ii) we proved that the maximum number of limit cycles with each configuration is three, then we have that the maximum number of limit cycles with both configurations is six. Moreover this upper bound is reached. Without loss of generality we can consider that $k=4$. We consider the discontinuous piecewise linear differential system defined by the following five linear Hamiltonian systems

$$
\begin{align*}
\dot{x} & =-0.325270 . .-1.247316 . . x-\frac{3}{2} y, \dot{y}=-35.808990 . .+1.037199 . . x+1.247316 . . y, \\
\dot{x} & =-13.295856 . .-8.394370 . . x-2 y, \dot{y}=47.825295 . .+35.232724 . . x+8.394370 . . y, \\
\dot{x} & =33.366737 . .-7.961636 . . x-\frac{17}{5} y, \dot{y}=-44.896945 . .+18.643428 . . x+7.961636 . . y, \\
\dot{x} & =65.056521 . .-17.010074 . . x-y, \dot{y}=-1052.5380642 . .+289.342621 . . x+17.010074 . . y, \\
\dot{x} & =74.167422 . .+12.003227 . . x+\frac{3}{2} y, \dot{y}=-187.420662 . .-96.051650 . . x-12.003227 . . y, \tag{3.18}
\end{align*}
$$

in the pieces $Z_{\Sigma_{4}}^{1}, Z_{\Sigma_{4}}^{2}, Z_{\Sigma_{4}}^{3}, Z_{\Sigma_{4}}^{4}$ and $Z_{\Sigma_{4}}^{5}$, respectively. For the discontinuous piecewise differential system (3.18), system (3.7), has the following three real solutions

$$
\left(\frac{7}{2},-\frac{6}{5}, \frac{2}{5}, \frac{19}{5}\right), \quad\left(\frac{71}{20},-\frac{143}{100}, \frac{31}{100}, \frac{389}{100}\right), \quad\left(\frac{343}{100},-0.893313 . ., 0.533583 . ., 3.654678 . .\right),
$$

and (3.12), has the three real solutions

$$
\begin{gathered}
\left(4,-3,-\frac{16}{5}, \frac{23}{5}\right), \quad\left(4.073407 . .,-3.179377 . .,-3.311999 . ., \frac{589}{125}\right), \\
\left(4.144187 . .,-3.341881 . .,-3.420000 . ., \frac{241}{50}\right) .
\end{gathered}
$$

These solutions provide the three limit cycles with Conf $\mathbf{6}^{+}$and Conf 8 shown in Figure 3.6b. This completes the proof of statement (vi).

Proof of statement (vii) of Theorem 1.3. In order to have limit cycles with Conf $\mathbf{6}^{+}$and Conf $\mathbf{9}^{+}$simultaneously, the points of intersection of the limit cycles with Conf $6^{+}$and $\Sigma_{k+}$ must satisfy system (3.7), and the points of intersection of the limit cycles with Conf $9^{+}$and $\Sigma_{k^{+}}$ must satisfy system (3.5) with $k>0$. In statement (ii) we proved that the maximum number of limit cycles with each configuration is three, then we have that the maximum number of limit cycles with both configurations is six. Moreover this upper bound is reached. We can suppose without loss of generality that $k=4$. We consider the discontinuous piecewise linear differential system defined by the following four linear Hamiltonian systems

$$
\begin{align*}
\dot{x} & =-17.085953 . .+9.977916 . . x-\frac{3}{2} y, \dot{y}=-113.972678 . .+66.372549 . . x-9.977916 . . y, \\
\dot{x} & =34.897550 . .-4.677048 . . x-\frac{7}{2} y, \dot{y}=-44.332934 . .+6.249936 \ldots x+4.677048 . . y, \\
\dot{x} & =65.922589 . .-17.491280 . . x-\frac{13}{10} y, \dot{y}=-924.494729 . .+235.342217 . . x+17.491280 . . y, \\
\dot{x} & =13.883036 . .+3.280745 . . x-\frac{9}{2} y, \dot{y}=-44.382913 . .+2.391842 . . x-3.280745 . . y, \tag{3.19}
\end{align*}
$$

in the pieces $Z_{\Sigma_{4}}^{1}, Z_{\Sigma_{4}}^{3}, Z_{\Sigma_{4}}^{4}$ and $Z_{\Sigma_{4}}^{5}$, respectively. For the discontinuous piecewise differential system (3.19), system (3.7), has the following three real solutions

$$
\begin{gathered}
\left(6,1.209968 . . \frac{7}{5}, 8.457532 . .\right), \quad\left(\frac{156}{25}, 1.006799 . . \frac{5}{4}, 8.915579 . .\right), \\
\left(\frac{117}{20}, 1.328327 . ., 1.486618 . ., 8.175706 . .\right),
\end{gathered}
$$

and (3.5), has the three real solutions

$$
\left(4,3, \frac{16}{5}, 5\right), \quad\left(4.109491 . ., \frac{141}{50}, \frac{303}{100}, \frac{517}{100}\right), \quad\left(\frac{47}{10}, 2.053733 . ., 2.068270 . ., 6.131729 . .\right) .
$$

These solutions provide the three limit cycles with Conf $\mathbf{6}^{+}$and $\operatorname{Conf} \mathbf{9}^{+}$shown in Figure 3.7. This completes the proof of statement (vii).


Figure 3.7: Three limit cycles with Conf $\mathbf{6}^{+}$and Conf $\mathbf{9}^{+}$of system (3.19).

Proof of statement (viii) of Theorem 1.3. In order to have limit cycles with Conf 7 and Conf 8 simultaneously, the points of intersection of the limit cycles with Conf 7 and $\Sigma_{k^{+}}$must satisfy system (3.10), and the points of intersection of the limit cycles with Conf 8 and $\Sigma_{k^{+}}$must satisfy system (3.12). In statement (ii) we proved that the maximum number of limit cycles with each configuration is three, then we have that the maximum number of limit cycles with both configurations is six. Moreover this upper bound is reached. Without loss of generality we can suppose that $k=3$. We consider the discontinuous piecewise linear differential system defined by the following five linear Hamiltonian systems

$$
\begin{align*}
\dot{x} & =-453.807220 . .-20.414445 . . x-\frac{3}{2} y, \dot{y}=83.559977 . .+277.833055 . . x+20.414445 . . y, \\
\dot{x} & =-29.218386 . .-5.465711 . . x+\frac{2}{5} y, \dot{y}=-414.702614 . .-74.685013 . . x+5.465711 . . y, \\
\dot{x} & =22.645454 . .-36.659999 . . x-\frac{47}{5} y, \dot{y}=-35.463636 . .+142.973999 . . x+36.659999 . . y, \\
\dot{x} & =3.918325 . .-3.744301 . . x+\frac{6}{5} y, \dot{y}=33.556264 . .-11.683162 . . x+3.744301 . . y, \\
\dot{x} & =5.300000 . .-8.579999 . . x-\frac{11}{5} y, \dot{y}=-8.300000 . .+33.461999 . . x+8.579999 . . y, \tag{3.20}
\end{align*}
$$

in the zones $Z_{\Sigma_{3}}^{1}, ~ Z_{\Sigma_{3}}^{2}, ~ Z_{\Sigma_{3}}^{3}, ~ Z_{\Sigma_{3}}^{4}$ and $Z_{\Sigma_{3}}^{5}$, respectively. For the discontinuous piecewise differential system (3.20), system (3.10), has the following three real solutions

$$
\begin{aligned}
& (0.502842 . .,-1.545218 . .,-0.572025 . ., 0.848539 . .), \\
& (0.442709 . .,-1.485086 . .,-0.427227 . ., 0.781483 . .), \\
& (0.378567 . .,-1.420944 . .,-0.276975 . ., 0.700080 . .),
\end{aligned}
$$

and (3.12), has the three real solutions

$$
\begin{gathered}
\left(\frac{84}{25},-\frac{79}{20},-4.562376 \ldots, \frac{88}{25}\right), \quad\left(\frac{3297}{1000},-\frac{387}{100},-4.492376 \ldots, \frac{69}{20}\right), \\
\left(\frac{34301}{10000},-4.039424 . .,-4.622376 . ., \frac{79}{50}\right) .
\end{gathered}
$$

These solutions provide the three limit cycles with Conf 7 and Conf 8 shown in Figure 3.8. This completes the proof of statement (viii).


Figure 3.8: Three limit cycles with Conf 7 and Conf 8 of system (3.20).

Proof of statement (ix) of Theorem 1.3. In order to have limit cycles with Conf 8 and Conf $\mathbf{9}^{+}$ simultaneously, the points of intersection of the limit cycles with Conf 8 and $\Sigma_{k+}$ must satisfy system (3.12), and the points of intersection of the limit cycles with Conf $9^{+}$and $\Sigma_{k^{+}}$must satisfy system (3.5) with $k>0$. In statement (ii) we proved that the maximum number of limit cycles with each configuration is three, then we have that the maximum number of limit cycles with both configurations is six. Moreover this upper bound is reached. Without loss of generality we can suppose that $k=4$. We consider the discontinuous piecewise linear differential system defined by the following four linear Hamiltonian systems

$$
\begin{align*}
& \dot{x}=-17.085953 . .+9.977916 . . x-\frac{3}{2} y, \dot{y}=-113.972678 . .+66.372549 . . x-9.977916 . . y, \\
& \dot{x}=-23.136372 . .+2.354826 . . x-\frac{3}{10} y, \dot{y}=81.642102 . .+18.484031 . . x-2.354826 . . y, \\
& \dot{x}=\frac{431}{10}+14.700000 . . x-\frac{7}{2} y, \dot{y}=-194.334000 . .+\frac{3087}{50} x-14.700000 . . y, \\
& \dot{x}=65.922589 . .-17.491280 . . x-\frac{13}{10} y, \dot{y}=-924.494729 . .+235.342217 . . x+17.491280 . . y, \tag{3.21}
\end{align*}
$$

in the zones $Z_{\Sigma_{4}}^{1}, Z_{\Sigma_{4}}^{2}, Z_{\Sigma_{4}}^{3}$ and $Z_{\Sigma_{4}}^{4}$, respectively. For the discontinuous piecewise differential system (3.21), system (3.12), has the following three real solutions

$$
\begin{gathered}
(9,-2.341532 . .,-6.604799 . ., 14.804799 . .), \quad\left(\frac{457}{50},-2.481840 . .,-6.933823 . ., 15.133823 . .\right), \\
\left(\frac{187}{20},-2.692262 . .,-7.432829 . ., 15.632829 . .\right),
\end{gathered}
$$

and (3.5), has the three real solutions

$$
\left(4,3, \frac{16}{5}, 5\right), \quad\left(4.109491 . ., \frac{141}{50}, \frac{303}{100}, \frac{517}{100}\right), \quad\left(\frac{47}{10}, 2.053733 . ., 2.068270 . ., 6.131729 . .\right)
$$

These solutions provide the three limit cycles with Conf 8 and Conf $9^{+}$shown in Figure 3.9. This completes the proof of statement (ix).


Figure 3.9: Three limit cycles with Conf 8 and Conf $\mathbf{9}^{+}$of system (3.21).


Figure 3.10: Three limit cycles with Conf $6^{-}$, Conf 7 and Conf 8 of system (3.22).

Proof of statement (x) of Theorem 1.3. In order to have limit cycles with Conf $\mathbf{6}^{-}$, Conf 7 and Conf 8 simultaneously, the points of intersection of the limit cycles with Conf $6^{-}$and $\Sigma_{k^{+}}$


Figure 3.11: Three limit cycles with Conf $\mathbf{6}^{+}$, Conf 8 and Conf $\mathbf{9}^{+}$of system (3.23).
must satisfy system (3.8), the points of intersection of the limit cycles with Conf 7 and $\Sigma_{k^{+}}$ must satisfy system (3.10) and the points of intersection of the limit cycles with Conf 8 and $\Sigma_{k^{+}}$must satisfy system (3.12). In statement (ii) we proved that the maximum number of limit cycles with each configuration is three, then we have that the maximum number of limit cycles with the three configurations simultaneously, is nine. Moreover this upper bound is reached. Without loss of generality we can suppose that $k=3$. We consider the discontinuous piecewise linear differential system defined by the following five linear Hamiltonian systems

$$
\begin{align*}
\dot{x} & =-0.567977 . .-5.151614 . . x-\frac{3}{2} y, \dot{y}=-6.233588 . . \\
\dot{x} & =11.250254 . .692757 . . x+5.151614 . . y, \\
\dot{x} & =22.645454 . .-36.659999 . . x-\frac{47}{5} y, \dot{y}=-35.463636 . .+142.973999 . . x+36.659999 . . y, \\
\dot{x} & =-13.170507 . .+3.348185 . . x+\frac{6}{5} y, \dot{y}=-35.085985 . .+1.015720 . . x-0.637407 . . y, \\
\dot{x} & =5.300000 . .-8.579999 . . x-\frac{11}{5} y, \dot{y}=-8.300000 . .+33.461999 \ldots x+8 . . .579999 . . y, \tag{3.22}
\end{align*}
$$

in the zones $Z_{\Sigma_{3}}^{1}, Z_{\Sigma_{3}}^{2}, Z_{\Sigma_{3}}^{3}, Z_{\Sigma_{3}}^{4}$ and $Z_{\Sigma_{3}}^{5}$, respectively. For the discontinuous piecewise differential system (3.22), system (3.8), has the following three real solutions

$$
\begin{gathered}
\left(-\frac{41}{10},-\frac{5}{2}, 1.569412 . ., 1.457623 . .\right), \quad\left(-\frac{106}{25},-\frac{263}{100}, 1.647799 . ., 1.587623 . .\right), \\
\left(-\frac{199}{50},-2.401954 . ., 1.508473 . ., 1.359577 . .\right),
\end{gathered}
$$

and (3.10), has the three real solutions

$$
\begin{aligned}
& (0.502842 . .,-1.545218 . .,-0.572025 . ., 0.848539 . .), \\
& (0.442709 . .,-1.485086 . .,-0.427227 . ., 0.781483 . .), \\
& (0.378567 . .,-1.420944 . .,-0.276975 . ., 0.700080 . .),
\end{aligned}
$$

and system (3.12), has the three real solutions

$$
\begin{gathered}
(1.847758 . .,-4.593279 . .,-3.042376 . ., 2), \quad\left(1.910216 . .,-4.699349 . .,-3.192376 . ., \frac{43}{20}\right), \\
\left(1.962805 . .,-4.784530 . .,-3.322376 . ., \frac{57}{25}\right) .
\end{gathered}
$$

These solutions provide the three limit cycles with Conf $6^{-}$, Conf 7 and Conf 8 shown in Figure 3.10. This completes the proof of statement (x).

Proof of statement (xi) of Theorem 1.3. In order to have limit cycles with Conf $\mathbf{6}^{+}$,Conf 8 and Conf $9^{+}$simultaneously, the points of intersection of the limit cycles with Conf $6^{+}$and $\Sigma_{k^{+}}$ must satisfy system (3.7), the points of intersection of the limit cycles with Conf 8 and $\Sigma_{k^{+}}$ must satisfy system (3.12) and the points of intersection of the limit cycles with Conf $9^{+}$and $\Sigma_{k^{+}}$must satisfy system (3.5) with $k>0$. In statement (ii) we proved that the maximum number of limit cycles with each configuration is three, then we have that the maximum number of limit cycles with the three configurations, is nine. Moreover this upper bound is reached. Without loss of generality we can suppose that $k=3$. We consider the discontinuous piecewise linear differential system defined by the following five linear Hamiltonian systems

$$
\begin{align*}
& \dot{x}=-17.085953 . .+9.977916 . . x-\frac{3}{2} y, \dot{y}=-113.972678 . .+66.372549 . . x-9.977916 . . y, \\
& \dot{x}=-2.306102 . .-0.633078 . . x-\frac{3}{10} y, \dot{y}=2.662449 . .+1.335961 . . x+0.633078 . . y, \\
& \dot{x}=34.897550 . .-4.677048 . . x-\frac{7}{2} y, \dot{y}=-44.332934 . .+6.249936 . . x+4.677048 . . y, \\
& \dot{x}=65.922589 . .-17.491280 . . x-\frac{13}{10} y, \dot{y}=-924.494729 . .+235.342217 . . x+17.491280 . . y, \\
& \dot{x}=13.883036 . .+3.280745 . . x-\frac{9}{2} y, \dot{y}=-44.382913 . .+2.391842 . . x-3.280745 . . y, \tag{3.23}
\end{align*}
$$

in the zones $Z_{\Sigma_{4}}^{1}, Z_{\Sigma_{4}}^{2}, Z_{\Sigma_{4}}^{3}, Z_{\Sigma_{4}}^{4}$ and $Z_{\Sigma_{4}}^{5}$, respectively. For the discontinuous piecewise differential system (3.23), system (3.7), has the following three real solutions

$$
\begin{gathered}
\left(6,1.209968 . ., \frac{7}{5}, 8.457532 . .\right), \quad\left(\frac{156}{25}, 1.006799 . ., \frac{5}{4}, 8.915579 . .\right), \\
\left(\frac{117}{20}, 1.328327 . ., 1.486618 . ., 8.175706 . .\right),
\end{gathered}
$$

and (3.12), has the three real solutions

$$
\begin{gathered}
(9,-2.341532 . .,-6.604799 . ., 14.804799 . .), \quad\left(\frac{93}{10},-2.642166 . .,-7.313423 . ., 15.513423 . .\right), \\
\left(\frac{943}{100},-2.772412 . .,-7.624652 . ., 15.824652 . .\right),
\end{gathered}
$$

and system (3.5), has the three real solutions

$$
\left(4,3, \frac{16}{5}, 5\right), \quad\left(4.109491 . ., \frac{141}{50}, \frac{303}{100}, \frac{517}{100}\right), \quad\left(\frac{47}{10}, 2.053733 . ., 2.068270 . ., 6.131729 . .\right)
$$

These solutions provide the three limit cycles with Conf $6^{+}$, Conf 8 and Conf $9^{+}$shown in 3.11. This completes the proof of statement (xi).

Proof of statement (xii) of Theorem 1.3. In order to have limit cycles with Conf $\mathbf{6}^{+}$, Conf $\mathbf{6}^{-}$ and Conf 8 simultaneously, the points of intersection of the limit cycles with Conf $6^{+}$and $\Sigma_{k^{+}}$must satisfy system (3.7), the points of intersection of the limit cycles with Conf $6^{-}$and


Figure 3.12: Three limit cycles with $\operatorname{Conf} \mathbf{6}^{+}$, two limit cycles with $\operatorname{Conf} 6^{-}$and one limit cycle with Conf 8 of system (3.24).


Figure 3.13: Two limit cycles with Conf $\mathbf{6}^{+}$, three limit cycles with Conf $\mathbf{6}^{-}$and one limit cycle with Conf 8 of system (3.25).
$\Sigma_{k^{+}}$must satisfy system (3.8) and the points of intersection of the limit cycles with Conf 8 and $\Sigma_{k^{+}}$must satisfy system (3.12). If we suppose that there is one solution for each system (3.7) and (3.8), then similar to statement (i) of Theorem 1.3, we obtain the value of the parameters $\gamma_{1}, \delta_{1}, \gamma_{2}, \gamma_{3}, \delta_{3}, \gamma_{4}, \gamma_{5}, \delta_{5}$.

Now we have two options, first we suppose that there is a solution of system (3.12), then we obtain the value of the parameters $\lambda_{1}, \delta_{2}, \lambda_{3}$ and $\delta_{4}$, therefore we have two options, first we can suppose that there is a second solution of system (3.12) then we obtain the value of the parameters $\lambda_{2}$ and $\lambda_{4}$, hence in the zones $Z_{\Sigma^{+}}^{1}, Z_{\Sigma^{+}}^{2}, Z_{\Sigma^{+}}^{3}, Z_{\Sigma^{+}}^{4}$ we only have the parameters $b_{1}, b_{2}, b_{3}, b_{4}$ as unknowns and in the zone $Z_{\Sigma^{+}}^{5}$ we have $\lambda_{5}, b_{5}$ as unknowns. Therefore we can obtain at most one solution either of system (3.7) or of system (3.8) and we cannot obtain more solutions of systems (3.7), (3.8) and (3.12), because we would have that $b_{i}=0$ for some $i=1,2,3,4,5$. Hence we would have five limit cycles with two (resp. one) limit cycles with Conf $6^{+}$, one (resp. two) limit cycle(s) with Conf $6^{-}$and two limit cycles with Conf 8 . Second we can suppose that there is a second solution for each system (3.7) and (3.8), then we obtain
the values of the parameters $\lambda_{2}, \lambda_{4}, \lambda_{5}$ hence we cannot obtain more solutions of systems (3.7), (3.8) and (3.12), because we would have that $b_{i}=0$ for some $i=1,2,3,4,5$. Therefore in this case we would obtain five limit cycles with two limit cycles with Conf $6^{+}$, two limit cycles with Conf $6^{-}$and one limit cycle with Conf 8.

Second, after considering the first solution of each system (3.7) and (3.8), we can suppose that there is a second solution for each system (3.7) and (3.8), then we obtain the values of $\lambda_{1}, \delta_{2}, \lambda_{3}$ and $\delta_{4}$. Then in the zones $Z_{\Sigma^{+}}^{1}, Z_{\Sigma^{+}}^{3}$ and $Z_{\Sigma^{+}}^{5}$ we only have the parameters $b_{1}, b_{3}$ and $b_{5}$ as unknowns and in the zones $Z_{\Sigma^{+}}^{2}$ and $Z_{\Sigma^{+}}^{4}$ we have the parameters $\lambda_{2}, b_{2}, \lambda_{4}, b_{4}$ unknowns. Hence can have two cases, first we can suppose that there is a solution of system (3.12), then we determine the value of parameters $\lambda_{2}$ and $\lambda_{4}$, hence we cannot to have more limit cycles because we would have that $b_{i}=0$ for some $i=1,2,3,4,5$. Therefore we would have five limit cycles with two limit cycles with Conf $6^{+}$, two limit cycles with Conf $6^{-}$and one limit cycle with Conf 8 . Second we can suppose that there is a third solution of system (3.7) (resp. (3.8)) and we obtain the value of parameter $\lambda_{4}$ (res. $\lambda_{2}$ ), then in the zone $Z_{\Sigma^{+}}^{4}$ (resp. $Z_{\Sigma^{+}}^{2}$ ) we only have the parameter $b_{4}$ (resp. $b_{2}$ ) as unknown and in the zone $Z_{\Sigma^{+}}^{2}$ (resp. $Z_{\Sigma^{+}}^{4}$ ) we have that the parameters $\lambda_{2}, b_{2}$ (res. $\lambda_{4}, b_{4}$ ) as unknowns. Now we suppose that there is one solution of system (3.12) and we obtain the parameter $\lambda_{2}\left(\right.$ res. $\lambda_{4}$ ). We observe that we cannot obtain more solutions of systems (3.7), (3.8) and (3.12), because we would have that $b_{i}=0$ for some $i=1,2,3,4,5$. Therefore we have at most six limit cycles with three (resp. two) limit cycles with Conf $6^{+}$, two (resp. three) limit cycles with Conf $6^{-}$and one limit cycle with Conf 8. We observe that these six limit cycles can be either three limit cycles with Conf $6^{+}$, two limit cycles with Conf $6^{-}$and one limit cycle with Conf 8 or two limit cycles with Conf $6^{+}$, three limit cycles with Conf $\mathbf{6}^{-}$and one limit cycle with Conf 8 . We shall give an example of each case.

We observe that without loss of generality we can suppose that $k=4$. We consider the discontinuous piecewise linear differential system defined by the following five linear Hamiltonian systems

$$
\begin{align*}
\dot{x}= & -23138.489410 . .+403.676452 . . x+\frac{9}{2} y, \dot{y}=2942.120325 . .-36212.150741 . . x \\
& -403.676452 . . y, \\
\dot{x}= & 1.812606 . .+1.308936 . . x-\frac{3}{10} y, \dot{y}=-25.828218 . .+5.711045 . . x-1.308936 . . y, \\
\dot{x}= & 15.472057 . .-3.117904 . . x-\frac{17}{5} y, \dot{y}=-13.354567 . .+2.859213 . . x+3.117904 . . y,  \tag{3.24}\\
\dot{x}= & 48.158492 . .-6.082779 . . x-y, \dot{y}=-31.590984 . .+37.000210 . . x+6.082779 . . y, \\
\dot{x}= & -151.854124 . .-136.354901 . . x+\frac{3}{2} y, \dot{y}=-10611.949690 . .-12395.106180 . . x \\
& +136.354901 . . y,
\end{align*}
$$

in the zones $Z_{\Sigma_{3}}^{1}, Z_{\Sigma_{3}}^{2}, Z_{\Sigma_{3}}^{3}, ~ Z_{\Sigma_{3}}^{4}$ and $Z_{\Sigma_{3}}^{5}$, respectively. For the discontinuous piecewise differential system (3.24), system (3.7) has the following three real solutions

$$
\left(5, \frac{1}{2}, \frac{9}{20}, \frac{23}{5}\right), \quad\left(\frac{9}{2}, \frac{19}{20}, \frac{91}{100}, \frac{7}{2}\right), \quad\left(\frac{41}{10}, 1.196150 . ., 1.163297 . ., 2.719447 . .\right) ;
$$

and (3.8) has the two real solutions

$$
\left(-\frac{18}{5},-\frac{9}{2},-\frac{49}{50},-1\right), \quad(-3,-3.411586 . .,-1.557354 . .,-1.546135 . .) ;
$$

and system (3.12) has the real solution

$$
\left(5.688640 . .,-4.154651 . .,-5.682382 . ., \frac{63}{10}\right) .
$$

These solutions provide the three limit cycles with Conf $\mathbf{6}^{+}$, the two limits cycles with Conf $6^{-}$and the limit cycle with Conf 8 shown in Figure 3.12.

Now we consider the discontinuous piecewise linear differential system defined by the following five linear Hamiltonian systems

$$
\begin{align*}
\dot{x}= & -23138.489410 . .+403.676452 . . x+\frac{9}{2} y, \dot{y}=2942.120325 . .-36212.150741 . . x \\
& -403.676452 . . y, \\
\dot{x}= & 4.276633 . .+1.873985 . . x-\frac{3}{10} y, \dot{y}=4.991226 . .+11.706072 . . x-1.873985 . . y, \\
\dot{x}= & 15.472057 . .-3.117904 \ldots x-\frac{17}{5} y, \dot{y}=-13.354567 . .+2.859213 . . x+3.117904 . . y,  \tag{3.25}\\
\dot{x}= & 293.931246 . .-44.804431 . . x-y, \dot{y}=-6905.938713 . .+2007.437106 . . x \\
& +44.804431 . . y, \\
\dot{x}= & -151.854124 . .-136.354901 . . x+\frac{3}{2} y, \dot{y}=-10611.949690 . .-12395.106180 \ldots x \\
& +136.354901 . . y,
\end{align*}
$$

in the zones $Z_{\Sigma_{3}}^{1}, Z_{\Sigma_{3}}^{2}, Z_{\Sigma_{3}}^{3}, Z_{\Sigma_{3}}^{4}$ and $Z_{\Sigma_{3}}^{5}$, respectively. For the discontinuous piecewise differential system (3.25), system (3.7) has the following two real solutions

$$
\left(5, \frac{1}{2}, \frac{9}{20}, \frac{23}{5}\right), \quad\left(\frac{9}{2}, \frac{19}{20}, \frac{91}{100}, \frac{7}{2}\right) ;
$$

and (3.8) has the three real solutions

$$
\begin{gathered}
\left(-\frac{18}{5},-\frac{9}{2},-\frac{49}{50},-1\right), \quad(-3,-3.411586 . .,-1.557354 . .,-1.546135 . .), \\
\left(-2.809209 . .,-\frac{31}{10},-1.671884 . .,-1.662653 . .\right) ;
\end{gathered}
$$

and system (3.12) has the real solution

$$
\left(\frac{667}{100},-4.419374 . .,-6.225080 . ., 6.842698 . .\right) \text {. }
$$

These solutions provide the two limit cycles with Conf $\mathbf{6}^{+}$, the three limits cycles with Conf $6^{-}$and the limit cycle with Conf 8 shown in Figure 3.13. This completes the proof of statement (xii).

## 4 Appendix

Here we provide the values $A, B$ and $C$

$$
\begin{aligned}
A= & \csc \left(\frac{r_{3}-s_{3}}{2}\right)\left(-\cos \left(\frac{1}{2}\left(3 r_{1}-r_{2}+2 r_{3}+s_{1}-s_{2}\right)\right)+\cos \left(\frac{1}{2}\left(r_{1}-r_{2}+4 r_{3}+s_{1}-s_{2}\right)\right)\right. \\
& +\cos \left(\frac{1}{2}\left(r_{1}-3 r_{2}-2 r_{3}+s_{1}-s_{2}\right)\right)-\cos \left(\frac{1}{2}\left(r_{1}-r_{2}-4 r_{3}+s_{1}-s_{2}\right)\right) \\
& -\cos \left(\frac{1}{2}\left(r_{1}-r_{2}+2 r_{3}+3 s_{1}-s_{2}\right)\right)-\cos \left(\frac{1}{2}\left(3 r_{1}+r_{2}-2 r_{3}+s_{1}+s_{2}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\cos \left(\frac{1}{2}\left(r_{1}+3 r_{2}-2 r_{3}+s_{1}+s_{2}\right)\right)-\cos \left(\frac{1}{2}\left(r_{1}+r_{2}-2 r_{3}+3 s_{1}+s_{2}\right)\right) \\
& +\cos \left(\frac{1}{2}\left(r_{1}+r_{2}-2 r_{3}+s_{1}+3 s_{2}\right)\right)+\cos \left(\frac{1}{2}\left(r_{1}-r_{2}-2 r_{3}+s_{1}-3 s_{2}\right)\right) \\
& +\cos \left(\frac{1}{2}\left(3 r_{1}-r_{2}+s_{1}-s_{2}+2 s_{3}\right)\right)+\cos \left(\frac{1}{2}\left(r_{1}-r_{2}+3 s_{1}-s_{2}+2 s_{3}\right)\right) \\
& -\cos \left(\frac{1}{2}\left(r_{1}-r_{2}+s_{1}-s_{2}+4 s_{3}\right)\right)-\cos \left(\frac{1}{2}\left(r_{1}-3 r_{2}+s_{1}-s_{2}-2 s_{3}\right)\right) \\
& +\cos \left(\frac{1}{2}\left(3 r_{1}+r_{2}+s_{1}+s_{2}-2 s_{3}\right)\right)-\cos \left(\frac{1}{2}\left(r_{1}+3 r_{2}+s_{1}+s_{2}-2 s_{3}\right)\right) \\
& +\cos \left(\frac{1}{2}\left(r_{1}+r_{2}+3 s_{1}+s_{2}-2 s_{3}\right)\right)-\cos \left(\frac{1}{2}\left(r_{1}+r_{2}+s_{1}+3 s_{2}-2 s_{3}\right)\right) \\
& -\cos \left(\frac{1}{2}\left(r_{1}-r_{2}+s_{1}-3 s_{2}-2 s_{3}\right)\right)+\cos \left(\frac{1}{2}\left(r_{1}-r_{2}+s_{1}-s_{2}-4 s_{3}\right)\right), \\
& B=-\left(\cos \left(r_{1}-r_{2}\right)+\cos \left(r_{1}-r_{3}\right)+\cos \left(r_{2}-r_{3}\right)-2 \cos \left(r_{1}-s_{1}\right)+\cos \left(r_{2}-s_{1}\right)\right. \\
& +\cos \left(r_{3}-s_{1}\right)+\cos \left(r_{1}-s_{2}\right)-2 \cos \left(r_{2}-s_{2}\right)+\cos \left(r_{3}-s_{2}\right)+\cos \left(s_{1}-s_{2}\right) \\
& +2 \cos \left(r_{1}-r_{2}+s_{1}-s_{2}\right)+\cos \left(2 r_{1}-2 r_{2}+s_{1}-s_{2}\right)-\cos \left(2 r_{1}-r_{2}-r_{3}+s_{1}-s_{2}\right) \\
& -\cos \left(r_{1}-2 r_{2}+r_{3}+s_{1}-s_{2}\right)+\cos \left(r_{1}-2 r_{2}+2 s_{1}-s_{2}\right)-\cos \left(r_{1}-r_{2}-r_{3}+2 s_{1}-s_{2}\right) \\
& +\cos \left(2 r_{1}-r_{2}+s_{1}-2 s_{2}\right)-\cos \left(r_{1}-r_{2}+r_{3}+s_{1}-2 s_{2}\right) \\
& +\cos \left(r_{1}-r_{2}+2 s_{1}-2 s_{2}\right)+\cos \left(r_{1}-s_{3}\right)+\cos \left(r_{2}-s_{3}\right)-2 \cos \left(r_{3}-s_{3}\right) \\
& +\cos \left(s_{1}-s_{3}\right)+2 \cos \left(r_{1}-r_{3}+s_{1}-s_{3}\right)-\cos \left(2 r_{1}-r_{2}-r_{3}+s_{1}-s_{3}\right) \\
& +\cos \left(2 r_{1}-2 r_{3}+s_{1}-s_{3}\right)-\cos \left(r_{1}+r_{2}-2 r_{3}+s_{1}-s_{3}\right)-\cos \left(r_{1}-r_{2}-r_{3}+2 s_{1}-s_{3}\right) \\
& +\cos \left(r_{1}-2 r_{3}+2 s_{1}-s_{3}\right)-\cos \left(2 r_{1}-r_{2}+s_{1}-s_{2}-s_{3}\right)-\cos \left(2 r_{1}-r_{3}+s_{1}-s_{2}-s_{3}\right) \\
& +\cos \left(r_{1}+r_{2}-r_{3}+s_{1}-s_{2}-s_{3}\right)+\cos \left(r_{1}-r_{2}+r_{3}+s_{1}-s_{2}-s_{3}\right) \\
& -\cos \left(r_{1}-r_{2}+2 s_{1}-s_{2}-s_{3}\right)-\cos \left(r_{1}-r_{3}+2 s_{1}-s_{2}-s_{3}\right)+\cos \left(s_{2}-s_{3}\right) \\
& +2 \cos \left(r_{2}-r_{3}+s_{2}-s_{3}\right)-\cos \left(r_{1}+r_{2}-2 r_{3}+s_{2}-s_{3}\right)+\cos \left(2 r_{2}-2 r_{3}+s_{2}-s_{3}\right) \\
& +\cos \left(r_{1}+r_{2}-r_{3}-s_{1}+s_{2}-s_{3}\right)-\cos \left(2 r_{2}-r_{3}-s_{1}+s_{2}-s_{3}\right) \\
& +\cos \left(r_{1}-r_{2}-r_{3}+s_{1}+s_{2}-s_{3}\right)-\cos \left(r_{1}-2 r_{3}+s_{1}+s_{2}-s_{3}\right) \\
& -\cos \left(r_{2}-2 r_{3}+s_{1}+s_{2}-s_{3}\right)+\cos \left(r_{2}-2 r_{3}+2 s_{2}-s_{3}\right)-\cos \left(r_{2}-r_{3}-s_{1}+2 s_{2}-s_{3}\right) \\
& -\cos \left(r_{1}-2 r_{2}+r_{3}-s_{2}+s_{3}\right)+\cos \left(r_{1}-r_{2}+r_{3}-s_{1}-s_{2}+s_{3}\right) \\
& -\cos \left(r_{1}-2 r_{2}+s_{1}-s_{2}+s_{3}\right)+\cos \left(r_{1}-r_{2}-r_{3}+s_{1}-s_{2}+s_{3}\right) \\
& -\cos \left(r_{1}-r_{2}+r_{3}-2 s_{2}+s_{3}\right)-\cos \left(r_{1}-r_{2}+s_{1}-2 s_{2}+s_{3}\right)+\cos \left(2 r_{1}-r_{3}+s_{1}-2 s_{3}\right) \\
& -\cos \left(r_{1}+r_{2}-r_{3}+s_{1}-2 s_{3}\right)+\cos \left(r_{1}-r_{3}+2 s_{1}-2 s_{3}\right)-\cos \left(r_{1}+r_{2}-r_{3}+s_{2}-2 s_{3}\right) \\
& +\cos \left(2 r_{2}-r_{3}+s_{2}-2 s_{3}\right)-\cos \left(r_{1}-r_{3}+s_{1}+s_{2}-2 s_{3}\right)-\cos \left(r_{2}-r_{3}+s_{1}+s_{2}-2 s_{3}\right) \\
& \left.+\cos \left(r_{2}-r_{3}+2 s_{2}-2 s_{3}\right)-6\right) \csc ^{2}\left(\frac{1}{2}\left(r_{1}-r_{2}+s_{1}-s_{2}\right)\right) \sin ^{2}\left(\frac{r_{3}-s_{3}}{2}\right) \text {, } \\
& C=2\left(-\cos \left(\frac{1}{2}\left(r_{1}-3 r_{2}-r_{3}+s_{1}-s_{2}-s_{3}\right)\right)+\cos \left(\frac{1}{2}\left(r_{1}-r_{2}-3 r_{3}+s_{1}-s_{2}-s_{3}\right)\right)\right. \\
& -\cos \left(\frac{1}{2}\left(3 r_{1}+r_{2}-r_{3}+s_{1}+s_{2}-s_{3}\right)\right)+\cos \left(\frac{1}{2}\left(r_{1}+3 r_{2}-r_{3}+s_{1}+s_{2}-s_{3}\right)\right) \\
& -\cos \left(\frac{1}{2}\left(r_{1}+r_{2}-r_{3}+3 s_{1}+s_{2}-s_{3}\right)\right)+\cos \left(\frac{1}{2}\left(r_{1}+r_{2}-r_{3}+s_{1}+3 s_{2}-s_{3}\right)\right) \\
& -\cos \left(\frac{1}{2}\left(r_{1}-r_{2}-r_{3}+s_{1}-3 s_{2}-s_{3}\right)\right)+\cos \left(\frac{1}{2}\left(3 r_{1}-r_{2}+r_{3}+s_{1}-s_{2}+s_{3}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
& -\cos \left(\frac{1}{2}\left(r_{1}-r_{2}+3 r_{3}+s_{1}-s_{2}+s_{3}\right)\right)+\cos \left(\frac{1}{2}\left(r_{1}-r_{2}+r_{3}+3 s_{1}-s_{2}+s_{3}\right)\right) \\
& \left.-\cos \left(\frac{1}{2}\left(r_{1}-r_{2}+r_{3}+s_{1}-s_{2}+3 s_{3}\right)\right)+\cos \left(\frac{1}{2}\left(r_{1}-r_{2}-r_{3}+s_{1}-s_{2}-3 s_{3}\right)\right)\right) .
\end{aligned}
$$

## Acknowledgements

The third author is supported by the Ministerio de Ciencia, Innovación y Universidades, Agencia Estatal de Investigación grants MTM2016-77278-P (FEDER) and PID2019-104658GB-I00 (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911.

## References

[1] A. Andronov, A. Vitt, S. Khaikin, Theory of oscillations, Pergamon Press, Oxford, 1996. https://doi.org/10.1002/zamm.19670470720; MR198734; Zbl 0188.56304
[2] A. Belfar, R. Benterki, J. Llibre, Limit cycles of planar discontinuous piecewise linear Hamiltonian systems without equilibrium points and separated by irreducible cubics, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms, accepted.
[3] B. P. Belousov, A periodic reaction and its mechanism (in Russian), A Collection of Short Papers on Radiation Medicine for 1958, Moscow: Med. Publ., 1959.
[4] R. Benterki, L. Damene, J. Llibre, The limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves II, Differ. Equ. Dyn. Syst., published online. https://doi.org/10.1007/s12591-021-00564-w
[5] R. Benterki, J. LLibre, Crossing limit cycles of planar piecewise linear Hamiltonian systems without equilibrium points, Mathematics 8(2020), No. 5, Article No. 755, 14 pp. https://doi.org/10.3390/math8050755
[6] R. Benterki, J. Llibre, The limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves I, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 28(2021), 153-192. MR4249577
[7] R. Benterki, J. Llibre, On the limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves III., submitted, 2021.
[8] M. di Bernardo, C. J. Budd, A. R. Champneys, P. Kowalczyk, Piecewise-smooth dynamical systems. Theory and applications, Applied Mathematical Sciences, Vol. 163, SpringerVerlag, London, 2008. https://doi.org/10.1007/978-1-84628-708-4; MR2368310; Zbl 1146.37003
[9] D. C. Braga, L. F. Mello, Limit cycles in a family of discontinuous piecewise linear differential systems with two zones in the plane, Nonlinear Dynam. 73(2013), 1283-1288. https://doi.org/10.1007/s11071-013-0862-3; MR3083780; Zbl 1281.34037
[10] C. Buzzi, C. Pessoa, J. Torregrosa, Piecewise linear perturbations of a linear center, Discrete Contin. Dyn. Syst. 33(2013), 3915-3936. https://doi.org/10.3934/dcds. 2013. 33.3915; MR3038046; Zbl 1312.37037
[11] A. F. Filippov, Differential equations with discontinuous right-hand sides, Mathematics and its Applications (Soviet Series), Vol. 18, Kluwer Academic Publishers Group, Dordrecht, 1988. https://doi.org/10.1007/978-94-015-7793-9; MR1028776; Zbl 1028776
[12] A. F. Fonseca, J. Llibre, L. F. Mello, Limit cycles in planar piecewise linear Hamiltonian systems with three zones without equilibrium points, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 30(2020), No. 11, 2050157, 8 pp. https://doi.org/10.3390/math8050755; MR4152369; Zbl 07258121
[13] E. Freire, E. Ponce, F. Torres, A general mechanism to generate three limit cycles in planar Filippov systems with two zones, Nonlinear Dynam. 78(2014), 251-263. https: //doi.org/10.1007/s11071-014-1437-7; MR3266440; Zbl 1314.37031
[14] F. Giannakopoulos, K. Pliete, Planar systems of piecewise linear differential equations with a line of discontinuity, Nonlinearity 14 (2001), 1611-1632. https ://doi . org/10.1088/ 0951-7715/14/6/311; MR1867095; Zbl 1003.34009
[15] M. Han, W. Zhang, On Hopf bifurcation in non-smooth planar systems, J. Differential Equations 248(2010), No. 9, 2399-2416. https://doi.org/10.1016/j.jde.2009.10.002; MR2595726; Zbl 1198.34059
[16] D. Hilbert, Mathematische probleme, Lecture, Second Internat. Congr. Math. (Paris, 1900), Nachr. Ges. Wiss. Göttingen Math. Phys. KL. (1900), 253-297; English transl., Bull. Amer. Math. Soc. 8(1902), 437-479; Bull. (New Series) Amer. Math. Soc. 37(2000), 407-436. https : //doi.org/10.1090/S0002-9904-1902-00923-3; MR1557926
[17] S. M. Huan, X. S. Yang, On the number of limit cycles in general planar piecewise linear systems, Discrete Contin. Dyn. Syst. 32(2012), No. 6, 2147-2164. https://doi.org/ 10.3934/dcds.2012.32.2147; MR2885803; Zbl 1248.34033
[18] Yu. Ilyashenko, Centennial history of Hilbert's 16th problem, Bull. (New Series) Amer. Math. Soc. 39(2002), 301-354. https://doi.org/10.1090/S0273-0979-02-00946-1; MR1898209; Zbl 1004.34017
[19] J. J. Jimenez, J. Llibre, J. C. Medrado, Crossing limit cycles for a class of piecewise linear differential centers separated by a conic, Electron. J. Differential Equations 2020, No. 41, 1-36. MR4098332; Zbl 07244060
[20] J. J. Jimenez, J. Llibre, J. C. Medrado, Crossing limit cycles for piecewise linear differential centers separated by a reducible cubic curve, Electron. J. Qual. Theory Differ. Equ. 2020, No. 19, 1-48. https://doi.org/10.14232/ejqtde.2020.1.19; MR4089921; Zbl 07254929
[21] J. Li, Hilbert's 16th problem and bifurcations of planar polynomial vector fields, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 13(2003), 47-106. https://doi.org/10.1142/ S0218127403006352; MR1965270; Zbl 1063.34026
[22] L. Li, Three crossing limit cycles in planar piecewise linear systems with saddle-focus type, Electron. J. Qual. Theory Differ. Equ. 2014, No. 70, 1-14. https://doi.org/10.14232/ ejqtde.2014.1.70; MR3304196; Zbl 1324.34025
[23] J. Llibre, D. D. Novaes, M. A. Teixeira, Maximum number of limit cycles for certain piecewise linear dynamical systems, Nonlinear Dyn. 82(2015), No. 3, 1159-1175. https: //doi.org/10.1007/s11071-015-2223-x; MR3412479; Zbl 1348.34065
[24] J. Llibre, E. Ponce, Three nested limit cycles in discontinuous piecewise linear differential systems with two zones, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms 19(2012), No. 3, 325-335. https://doi.org/10.1007/s11071-014-1437-7; MR2963277; Zbl 1268.34061
[25] O. Makarenkov, J. S. W. Lamb, Dynamics and bifurcations of nonsmooth systems: a survey, Phys. D 241(2012), 1826-1844. https://doi.org/10.1016/j.physd.2012.08.002; MR2994324
[26] B. van der Pol, A theory of the amplitude of free and forced triode vibrations, Radio Review (later Wireless World) 1(1920), 701-710.
[27] B. van der Pol, On relaxation-oscillations, London Edinburgh Philos. Mag. J. Sci. 2(1926), No. 11, 978-992. https://doi.org/10.1080/14786442608564127
[28] D. J. W. Simpson, Bifurcations in piecewise-smooth continuous systems, World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, Vol. 70, World Scientific, Singapore, 2010. https://doi. org/10.1142/7612; MR3524764; Zbl 05655816
[29] A. M. Zhabotinsky, Periodical oxidation of malonic acid in solution (a study of the Belousov reaction kinetics), Biofizika 9(1964), 306-311.

# Ground state sign-changing solutions and infinitely many solutions for fractional logarithmic Schrödinger equations in bounded domains 

Yonghui Tong ${ }^{1}$, Hui Guo ${ }^{\boxtimes 2}$ and Giovany M. Figueiredo ${ }^{3}$<br>${ }^{1}$ School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, P.R. China<br>${ }^{2}$ College of Mathematics and Computing Science, Hunan University of Science and Technology, Xiangtan, Hunan 411201, P.R. China<br>${ }^{3}$ Departamento de Matemática, Universidade de Brasilia-UNB, CEP:70910-900, Brasília-DF, Brazil

Received 7 June 2021, appeared 11 September 2021
Communicated by Patrizia Pucci


#### Abstract

We consider a class of fractional logarithmic Schrödinger equation in bounded domains. First, by means of the constraint variational method, quantitative deformation lemma and some new inequalities, the positive ground state solutions and ground state sign-changing solutions are obtained. These inequalities are derived from the special properties of fractional logarithmic equations and are critical for us to obtain our main results. Moreover, we show that the energy of any sign-changing solution is strictly larger than twice the ground state energy. Finally, we obtain that the equation has infinitely many nontrivial solutions. Our result complements the existing ones to fractional Schrödinger problems when the nonlinearity is sign-changing and satisfies neither the monotonicity condition nor Ambrosetti-Rabinowitz condition.


Keywords: logarithmic Schrödinger equation, fractional Laplacian, sign-changing solutions, non-Nehari method, infinitely many solutions.

2020 Mathematics Subject Classification: 35J20, 35R11, 35J65.

## 1 Introduction

In this paper, we consider the following fractional Schrödinger equation with logarithmic nonlinearity:

$$
\left\{\begin{array}{l}
(-\Delta)^{\alpha} u+V(x) u=|u|^{p-2} u \ln u^{2}, \quad x \in \Omega  \tag{1.1}\\
u=0, \quad x \in \mathbb{R}^{N} \backslash \Omega
\end{array}\right.
$$

where $\alpha \in(0,1), N>2 \alpha$ and $2<p<2_{\alpha}^{*}:=\frac{2 N}{N-2 \alpha},(-\Delta)^{\alpha}$ denotes the fractional Laplacian operator, $\Omega$ is a bounded domain with smooth boundary in $\mathbb{R}^{N}$ and $V: \Omega \mapsto \mathbb{R}$ satisfy
$\left(V_{1}\right) V \in \mathcal{C}(\Omega, \mathbb{R})$.

[^32]$\left(V_{2}\right) \inf \sigma\left((-\Delta)^{\alpha}+V(x)\right)>0$, where $\sigma\left((-\Delta)^{\alpha}+V\right)$ is the spectrum of the operator $(-\Delta)^{\alpha}+V$. The general form of problem (1.1) can be given by
\[

$$
\begin{equation*}
(-\Delta)^{\alpha} u+V(x) u=f(x, u), \quad \text { in } \mathbb{R}^{N}, \tag{1.2}
\end{equation*}
$$

\]

which arises in the study of standing waves to the time-dependent Schrödinger equation

$$
\begin{equation*}
i \frac{\partial \psi}{\partial t}=(-\Delta)^{\alpha} \psi+M(x) \psi-F(x, \psi), \tag{1.3}
\end{equation*}
$$

where $\psi: \mathbb{R}^{N} \times(0,+\infty) \mapsto \mathbb{R}$. This equation is of particular interest in fractional quantum mechanics for the study of particles on stochastic field modelled by Lévy processes. A path integral over the Lévy flights paths and a fractional Schrödinger equation of fractional quantum mechanics are formulated by Laskin [16] from the idea of Feynman and Hibbs path integrals. We call $\psi$ a standing waves solution if it possesses the form $\psi(x, t)=e^{i \omega t} u(x)$. Then $\psi$ is a standing waves solution for (1.3) if and only if $u$ solves (1.2) with $V(x)=M(x)-\omega$. Our goal is to study the case for logarithmic nonlinearity $F(x, \psi)=|\psi|^{p-2} \psi \log |\psi|^{2}$. Here, the fractional Laplacian operator $(-\Delta)^{\alpha}$ can be characterized as the singular integral (see, for example [11])

$$
\begin{equation*}
(-\Delta)^{\alpha} u(x)=C(N, \alpha) \text { P.V. } \int_{\mathbb{R}^{N}} \frac{u(x)-u(y)}{|x-y|^{N+2 \alpha}} \mathrm{~d} y, \tag{1.4}
\end{equation*}
$$

for all $x \in \mathbb{R}^{N}$, where $C(N, \alpha)$ is a normalization constant and P.V. stands for the principal value. When $u$ has sufficient regularity, the fractional Laplacian has a pointwise expression (see [11, Lemma 3.2])

$$
(-\Delta)^{\alpha} u(x)=-\frac{1}{2} C(N, \alpha) \int_{\mathbb{R}^{N}} \frac{u(x+y)+u(x-y)-2 u(x)}{|y|^{N+2 \alpha}} \mathrm{~d} y, \quad \forall x \in \mathbb{R}^{N} .
$$

Equation (1.1) and (1.2) admit applications related to quantum mechanics, phase transitions and minimal surfaces etc. (see [11] and the references therein). There are much attention by various scholars, especially on existence of ground state solution, multiple solutions, semiclassical states and the concentration behavior of positive solutions, see for example [3,9,20,24], and the references therein. When $\alpha=1$, Chen et al. [5] proved the existence of ground state sign-changing solutions of problem (1.2) with $f(x, u)=Q(x)|u|^{p-2} u \ln u^{2}$. When $p=2$, Pietro d'Avenia et al. [9] obtained the existence of infinitely many weak solutions of problem (1.1). If $\alpha=1$ and $p=2$, then the problem (1.1) reduces to the classical logarithmic Schrödinger equation

$$
\begin{equation*}
-\Delta u+V(x) u=u \ln u^{2} . \tag{1.5}
\end{equation*}
$$

More recently, many scholars focused on the problem (1.5), such as the existence of ground state solution, multiple solutions, semiclassical states and the concentration behavior of positive solutions, see for example $[1,2,8,18,25]$, and the references therein.

In 2014, Chang et al. [4] proved the existence of a nodal solution of (1.2) with $V(x)=0$ in bounded domain. They assume that the nonlinearity $f(x, t)$ satisfies the following AmbrosettiRabinowitz condition and monotonicity condition:
(AR) There exists $\mu \in\left(2,2_{\alpha}^{*}\right)$ such that

$$
0<\mu F(x, t) \leq t f(x, t)
$$

for a.e. $x \in \Omega$ and all $t \neq 0$, where $F(x, t)=\int_{0}^{t} f(x, \tau) d \tau$.
(NC) $t \mapsto f(x, t) /|t|$ is strictly increasing on $(-\infty, 0) \cup(0,+\infty)$ for every $x \in \Omega$.
F. G. Rodrigo, et al. [13] considered the existence of sign-changing solution for (1.2) with $V(x)=0$ and $f(x, u)=\lambda g(x, u)+|u|^{2 *} u$, where $g(x, u)$ satisfies the conditions (AR) and (NC). When $f(x, u)$ satisfies a monotonicity condition, Deng et al. [10] dealt with the least energy sign-changing solutions for fractional elliptic equations (1.2) in bounded domain. Ji [15] concerned with the existence of the least energy sign-changing solutions for a class of fractional Schrödinger-Poisson system when $f(x, t)$ satisfies the following monotonicity condition:
(F) $t \mapsto f(x, t) / t^{3}$ is strictly increasing on $(-\infty, 0) \cup(0,+\infty)$ for every $x \in \mathbb{R}^{3}$.

For more discussions on the existence of sign-changing solutions, we refer the readers to other references, such as $[6,7,14,22,23]$ and so on.

However, the logarithmic nonlinearity $f(x, u)=|u|^{p-2} u \ln u^{2}$ is sign-changing and satisfies neither the condition (AR) nor monotonicity condition (NC). In addition, the nonlocal operator brings some new difficulties, such as

$$
\int_{\mathbb{R}^{N}}\left|(-\Delta)^{\frac{\alpha}{2}} u(x)\right|^{2} \mathrm{~d} x \neq \int_{\mathbb{R}^{N}}\left|(-\Delta)^{\frac{\alpha}{2}} u^{+}(x)\right|^{2} \mathrm{~d} x+\int_{\mathbb{R}^{N}}\left|(-\Delta)^{\frac{\alpha}{2}} u^{-}(x)\right|^{2} \mathrm{~d} x,
$$

where

$$
u^{+}(x):=\max \{u(x), 0\} \quad \text { and } \quad u^{-}(x):=\min \{u(x), 0\} .
$$

But, most methods for local problem heavily rely on the decompositions

$$
\int_{\mathbb{R}^{N}}|\nabla u(x)|^{2} \mathrm{~d} x=\int_{\mathbb{R}^{N}}\left|\nabla u^{+}(x)\right|^{2} \mathrm{~d} x+\int_{\mathbb{R}^{N}}\left|\nabla u^{-}(x)\right|^{2} \mathrm{~d} x .
$$

Thus, these classic methods do not work for equation (1.1). Therefore, combining constraint variational method, quantitative deformation lemma, non-Nehari manifold method and some new energy inequalities, we will establish the existence of positive ground state solutions and ground state sign-changing solutions for (1.1). Finally, we analysis that the existence of infinitely many nontrivial solutions. To the best of our knowledge, there seem no results concerned with sign-changing solutions for fractional problem (1.1).

Before stating our main results, we introduce some useful results of fractional Sobolev spaces. For $0<\alpha<1$, the fractional Sobolev space is defined as

$$
H_{0}^{\alpha}(\Omega):=\left\{u \in L^{2}(\Omega):[u]_{\alpha}<\infty, u=0 \text { a.e. in } \mathbb{R}^{N} \backslash \Omega\right\},
$$

where the Gagliardo seminorm $[u]_{\alpha}$ is given by

$$
[u]_{\alpha}=\iint_{\mathbb{R}^{2 N}} \frac{|u(x)-u(y)|^{2}}{|x-y|^{N+2 \alpha}} \mathrm{~d} x \mathrm{~d} y .
$$

It is well known that $H_{0}^{\alpha}(\Omega)$ is a Hilbert space endowed with the standard inner product

$$
\langle u, v\rangle=\iint_{\mathbb{R}^{2 N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2 \alpha}} \mathrm{~d} x \mathrm{~d} y+\int_{\Omega} u(x) v(x) \mathrm{d} x,
$$

and the correspondent induced norm

$$
\begin{equation*}
\|u\|_{H_{0}^{\alpha}(\Omega)}=\sqrt{\langle u, u\rangle} . \tag{1.6}
\end{equation*}
$$

In light of the Propositions 3.4 and 3.6 in [11], we have

$$
\left\|(-\Delta)^{\frac{\alpha}{2}} u\right\|_{2}^{2}=\frac{1}{2} C(n, \alpha) \iint_{\mathbb{R}^{2 N}} \frac{|u(x)-u(y)|^{2}}{|x-y|^{N+2 \alpha}} \mathrm{~d} x \mathrm{~d} y,
$$

where $\hat{u}$ stands for the Fourier transform of $u, \xi \in \mathbb{R}^{N}$ and $C(n, \alpha)=\left(\int_{\mathbb{R}^{N}} \frac{1-\cos \xi_{1}}{|\xi|^{n+2 \alpha}} \mathrm{~d} x\right)^{-1}$. As a consequence, the norms on $H^{\alpha}(\Omega)$ defined below

$$
\begin{aligned}
& u \mapsto\left(\int_{\Omega} u(x)^{2} \mathrm{~d} x+\int_{\mathbb{R}^{N}}\left|(-\Delta)^{\frac{\alpha}{2}} u(x)\right|^{2} \mathrm{~d} x\right)^{\frac{1}{2}} \\
& u \mapsto\left(\int_{\Omega} u(x)^{2} \mathrm{~d} x+\iint_{\mathbb{R}^{N}} \frac{|u(x)-u(y)|^{2}}{|x-y|^{N+2 \alpha}} \mathrm{~d} x \mathrm{~d} y\right)^{\frac{1}{2}}
\end{aligned}
$$

are equivalent. To find solutions of (1.1), we will use a variational approach. Hence, we will associate a suitable functional to our problem. More precisely, the energy functional associated with problem (1.1) is given by $\Psi: H \mapsto \mathbb{R}$ defined as follows

$$
\begin{equation*}
\Psi(u):=\frac{1}{2} \int_{\mathbb{R}^{N}}\left(\left|(-\Delta)^{\frac{\alpha}{2}} u\right|^{2}+V(x) u^{2}\right) \mathrm{d} x+\frac{2}{p^{2}} \int_{\Omega}|u|^{p} \mathrm{~d} x-\frac{1}{p} \int_{\Omega}|u|^{p} \ln u^{2} \mathrm{~d} x . \tag{1.7}
\end{equation*}
$$

We define the suitable subspace of $H_{0}^{\alpha}(\Omega)$,

$$
H:=\left\{u \in H_{0}^{\alpha}(\Omega): \int_{\Omega} V(x) u^{2}<+\infty\right\} .
$$

In view of assumptions $\left(V_{1}\right)$ and $\left(V_{2}\right)$, it is not hard to check that $H$ is a Hilbert space endowed with the inner product

$$
\langle u, v\rangle_{H}=\int_{\mathbb{R}^{N}}(-\Delta)^{\frac{\alpha}{2}} u(-\Delta)^{\frac{\alpha}{2}} v \mathrm{~d} x+\int_{\Omega} V(x) u v \mathrm{~d} x
$$

and the induced norm $\|u\|^{2}=\langle u, u\rangle_{H}$, which is equivalent to $\|u\|_{H_{0}^{\alpha}(\Omega)}$.
The basic property of Sobolev space $H$ that we need is summarized in the following lemma.

Lemma 1.1 ([11]). The embedding $H \hookrightarrow L^{p}(\Omega)$ is compact for $p \in\left(2,2_{\alpha}^{*}\right)$.
Note that

$$
\lim _{t \rightarrow 0} \frac{t^{p-1} \ln t^{2}}{t}=0 \quad \text { and } \quad \lim _{t \rightarrow \infty} \frac{t^{p-1} \ln t^{2}}{t^{q-1}}=0
$$

where $q \in\left(p, 2_{\alpha}^{*}\right)$, thus, for any $\epsilon>0$, there exists $C_{\epsilon}>0$ such that

$$
\begin{equation*}
|t|^{p-1}\left|\ln t^{2}\right| \leq \epsilon|t|+C_{\epsilon}|t|^{q-1}, \quad \forall x \in \Omega, t \in \mathbb{R} \backslash\{0\} . \tag{1.8}
\end{equation*}
$$

By (1.8) and a standard argument, it is easy to check that $\Psi \in C^{1}(H, \mathbb{R})$ and

$$
\begin{equation*}
\left\langle\Psi^{\prime}(u), v\right\rangle=\int_{\mathbb{R}^{N}}(-\Delta)^{\frac{\alpha}{2}} u(-\Delta)^{\frac{\alpha}{2}} v \mathrm{~d} x+\int_{\Omega} V(x) u v \mathrm{~d} x-\int_{\Omega}|u|^{p-2} u v \ln u^{2} \mathrm{~d} x, \tag{1.9}
\end{equation*}
$$

for any $u, v \in H$.

Definition 1.2. We say that $u \in H$ is a weak solution of (1.1), if $u$ a critical point of the functional $\Psi$, that is

$$
\int_{\mathbb{R}^{N}}(-\Delta)^{\frac{\alpha}{2}} u(-\Delta)^{\frac{\alpha}{2}} v \mathrm{~d} x+\int_{\Omega} V(x) u v \mathrm{~d} x=\int_{\Omega}|u|^{p-2} u v \ln u^{2} \mathrm{~d} x,
$$

for all $v \in H$. Moreover, if $u \in H$ is a solution of (1.1) and $u^{ \pm} \neq 0$, then $u$ is called a sign-changing solution.

Definition 1.3. The $u \in H$ is called a classical solution of (1.1), if $(-\Delta)^{\alpha} u$ can be written as (1.4) and equation (1.1) is satisfied pointwise in $\Omega$.

Remark 1.4. Since $\left(u^{+}, u^{-}\right)_{\alpha}:=\int_{\mathbb{R}^{N}}(-\Delta)^{\frac{\alpha}{2}} u^{+}(-\Delta)^{\frac{\alpha}{2}} u^{-} \mathrm{d} x>0$ for $u^{ \pm} \neq 0$, it follows from a simple computation that

$$
\begin{equation*}
\Psi(u)=\Psi\left(u^{+}\right)+\Psi\left(u^{-}\right)+\left(u^{+}, u^{-}\right)_{\alpha}>\Psi\left(u^{+}\right)+\Psi\left(u^{-}\right), \tag{1.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\left\langle\Psi^{\prime}(u), u^{ \pm}\right\rangle=\left\langle\Psi^{\prime}\left(u^{ \pm}\right), u^{ \pm}\right\rangle+\left(u^{+}, u^{-}\right)_{\alpha}\right\rangle\left\langle\Psi^{\prime}\left(u^{ \pm}\right), u^{ \pm}\right\rangle \tag{1.11}
\end{equation*}
$$

Let

$$
c:=\inf _{u \in \mathcal{N}} \Psi(u) \text { and } m:=\inf _{u \in \mathcal{M}} \Psi(u)
$$

where

$$
\mathcal{N}:=\left\{u \in H \backslash\{0\} \mid\left\langle\Psi^{\prime}(u), u\right\rangle=0\right\},
$$

and

$$
\mathcal{M}:=\left\{u \in H, u^{ \pm} \neq 0 \mid\left\langle\Psi^{\prime}(u), u^{+}\right\rangle=\left\langle\Psi^{\prime}(u), u^{-}\right\rangle=0\right\} .
$$

The main result of this work can now be stated as follows.
Theorem 1.5. Assume that $\left(V_{1}\right)$ and $\left(V_{2}\right)$ hold. Then problem (1.1) possesses one positive ground state solution $\bar{u} \in \mathcal{N}$ such that $\Psi(\bar{u})=c:=\inf _{\mathcal{N}} \Psi(u)$.

Theorem 1.6. Assume that $\left(V_{1}\right)$ and $\left(V_{2}\right)$ hold. Then problem (1.1) has a ground state sign-changing solution $\tilde{u} \in \mathcal{M}$ such that $\Psi(\tilde{u})=m:=\inf _{\mathcal{M}}$. Moreover, $m>2 c$.

Theorem 1.6 indicates that the energy of any sign-changing solution of (1.1) is strictly larger than twice of the ground state energy. In terms of the results, Theorem 1.6 is a relatively new result for fractional equations. In terms of processing technology, we adopt some new technique inequalities derived by the variable transformation and the special concave properties of energy functional.

Theorem 1.7. Suppose that $\left(V_{1}\right)$ and $\left(V_{2}\right)$ hold. Then problem (1.1) possesses infinitely many nontrivial solutions.

The remaining of the paper is organized as follows: In Section 2, we present some preliminary results and we set up the variational framework to our problem. In Section 3 and 4, we prove our main result. Throughout this paper, the symbol $S$ denote unit sphere, the $C, C_{1}$, $C_{2}, \ldots$ represent several different positive constants.

## 2 Some preliminary results

In this section, we give some preliminary lemmas which are crucial for proving our results.
For a fixed function $u \in H$ with $u^{ \pm} \neq 0$. We define a continuous function $J:[0, \infty) \times$ $[0, \infty) \mapsto \mathbb{R}$ by

$$
\begin{align*}
J(s, t):= & \Psi\left(s^{\frac{1}{p}} u^{+}+t^{\frac{1}{p}} u^{-}\right) \\
= & \frac{1}{2}\left\|s^{\frac{1}{p}} u^{+}+t^{\frac{1}{p}} u^{-}\right\|^{2}+\frac{2}{p^{2}} \int_{\Omega}\left|s^{\frac{1}{p}} u^{+}+t^{\frac{1}{p}} u^{-}\right|^{p} \mathrm{~d} x  \tag{2.1}\\
& -\frac{1}{p} \int_{\Omega}\left|s^{\frac{1}{p}} u^{+}+t^{\frac{1}{p}} u^{-}\right|^{p} \ln \left(s^{\frac{1}{p}} u^{+}+t^{\frac{1}{p}} u^{-}\right)^{2} \mathrm{~d} x .
\end{align*}
$$

The following lemma is derived from the special properties of fractional logarithmic equations, which is critical to our results.
Lemma 2.1. The $J(s, t)$ defined in (2.1) is strictly concave in $(0,+\infty)^{2}$ and thus there exists a unique global maximum point in $(0,+\infty)^{2}$.

Proof. It follows from (2.1) that

$$
\begin{align*}
\frac{\partial J}{\partial s}(s, t)= & \frac{1}{p} s^{\frac{2}{p}-1}\left\|u^{+}\right\|^{2}+\frac{1}{p} s^{\frac{1}{p}-1} t^{\frac{1}{p}}\left(u^{+}, u^{-}\right)_{\alpha}-\frac{1}{p} \int_{\Omega}\left|u^{+}\right|^{p} \ln \left(u^{+}\right)^{2} \mathrm{~d} x \\
& -\frac{1}{p} \int_{\Omega}\left|u^{+}\right|^{p} \ln \left(s^{\frac{2}{p}}\right) \mathrm{d} x,  \tag{2.2}\\
\frac{\partial J}{\partial t}(s, t)= & \frac{1}{p} t^{\frac{2}{p}-1}\left\|u^{-}\right\|^{2}+\frac{1}{p} t^{\frac{1}{p}-1} s^{\frac{1}{p}}\left(u^{+}, u^{-}\right)_{\alpha}-\frac{1}{p} \int_{\Omega}\left|u^{-}\right|^{p} \ln \left(u^{-}\right)^{2} \mathrm{~d} x  \tag{2.3}\\
& -\frac{1}{p} \int_{\Omega}\left|u^{-}\right|^{p} \ln \left(t^{\frac{2}{p}}\right) \mathrm{d} x, \\
\frac{\partial^{2} J}{\partial s^{2}}(s, t)= & \frac{2-p}{p^{2}} s^{\frac{2}{p}-2}\left\|u^{+}\right\|^{2}+\frac{1-p}{p^{2}} s^{\frac{1}{p}-2} t^{\frac{1}{p}}\left(u^{+}, u^{-}\right)_{\alpha}-\frac{2}{p^{2} s} \int_{\Omega}\left|u^{+}\right|^{p} \mathrm{~d} x,  \tag{2.4}\\
\frac{\partial^{2} J}{\partial t^{2}}(s, t)= & \frac{2-p}{p^{2}} t^{\frac{2}{p}-2}\left\|u^{-}\right\|^{2}+\frac{1-p}{p^{2}} t^{\frac{1}{p}-2} s^{\frac{1}{p}}\left(u^{+}, u^{-}\right)_{\alpha}-\frac{2}{p^{2} t} \int_{\Omega}\left|u^{-}\right|^{p} \mathrm{~d} x \tag{2.5}
\end{align*}
$$

and

$$
\begin{equation*}
\frac{\partial^{2} J}{\partial s \partial t}(s, t)=\frac{\partial^{2} G}{\partial t \partial s}(s, t)=\frac{1}{p^{2}} s^{\frac{1}{p}-1} t^{\frac{1}{p}-1}\left(u^{+}, u^{-}\right)_{\alpha} . \tag{2.6}
\end{equation*}
$$

Therefore, the Hessian matrix $D^{2} J(s, t)$ is

$$
\begin{align*}
D^{2} J(s, t)= & \left(\begin{array}{cc}
\frac{\partial^{2} J}{\partial s^{2}} & \frac{\partial^{2} J}{\partial s^{2} t} \\
\frac{\partial^{2} J}{\partial t \partial s} & \frac{\partial^{2} J}{\partial t^{2}}
\end{array}\right)(s, t) \\
= & \frac{2-p}{p^{2}}\left(\begin{array}{cc}
s^{\frac{2}{p}-2}\left\|u^{+}\right\|^{2}+s^{\frac{1}{p}-2} t^{\frac{1}{p}}\left(u^{+}, u^{-}\right)_{\alpha} & 0 \\
0 & t^{\frac{2}{p}-2}\left\|u^{-}\right\|^{2}+t^{\frac{1}{p}-2} s^{\frac{1}{p}}\left(u^{+}, u^{-}\right)_{\alpha}
\end{array}\right) \\
& +\frac{1}{p^{2}}\left(u^{+}, u^{-}\right)_{\alpha}\left(\begin{array}{cc}
-s^{\frac{1}{p}-2} t^{\frac{1}{p}} & s^{\frac{1}{p}-1} t^{\frac{1}{p}-1} \\
s^{\frac{1}{p}-1} t^{\frac{1}{p}-1} & -t^{\frac{1}{p}-2} s^{\frac{1}{p}}
\end{array}\right)  \tag{2.7}\\
& +\frac{2}{p^{2}}\left(\begin{array}{cc}
-\frac{1}{s} \int_{\Omega}\left|u^{-}\right|^{p} \mathrm{~d} x & 0 \\
0 & -\frac{1}{t} \int_{\Omega}\left|u^{-}\right|^{p} \mathrm{~d} x
\end{array}\right) \\
= & J_{1}(s, t)+J_{2}(s, t)+J_{3}(s, t) .
\end{align*}
$$

Note that $2<p<2_{\alpha}^{*}$ and $\left(u^{+}, u^{-}\right)_{\alpha}>0$, it is not difficult to verify that $J_{1}(s, t), J_{2}(s, t)$ and $J_{3}(s, t)$ are negative definite matrices for $s, t>0$. Thus, $D^{2} J(s, t)$ is a negative definite matrix. Since $J(0,0)=0$ and

$$
J(s, t) \rightarrow-\infty \quad \text { as }|(s, t)| \rightarrow+\infty,
$$

which shows that $J(s, t)$ is strictly concave and there exists a unique global maximum point in $(0,+\infty)^{2}$. We complete the proof.

In view of Lemma 2.1, we have the following corollaries.
Corollary 2.2. Assume that $u \in \mathcal{M}$, then

$$
\begin{equation*}
\Psi\left(u^{+}+u^{-}\right)=\max _{\widetilde{\widetilde{s}}, \widetilde{t} \geq 0} \Psi\left(\widetilde{s}^{\frac{1}{p}} u^{+}+\widetilde{t}^{\frac{1}{p}} u^{-}\right)>\Psi\left(s^{\frac{1}{p}} u^{+}+t^{\frac{1}{p}} u^{-}\right) \tag{2.8}
\end{equation*}
$$

for any $s, t \geq 0$ and $(s, t) \neq(1,1)$.
Proof. Let $J:[0, \infty) \times[0, \infty) \rightarrow \mathbb{R}$ be defined in (2.1). Since $u \in \mathcal{M}$, then $\left\langle\Psi^{\prime}(u), u^{+}\right\rangle=$ $\left\langle\Psi^{\prime}(u), u^{-}\right\rangle=0$. This, combined with (2.2) and (2.3), implies that

$$
\frac{\partial J}{\partial s}(1,1)=0 \quad \text { and } \quad \frac{\partial J}{\partial t}(1,1)=0
$$

Then, by the strict concavity of $J$ in Lemma 2.1, (2.8) follows immediately, which is the desired conclusion.

Since $\left\langle\Psi^{\prime}(u), u^{+}\right\rangle=p \frac{\partial J}{\partial s}(1,1)$ and $\left\langle\Psi^{\prime}(u), u^{-}\right\rangle=p \frac{\partial J}{\partial t}(1,1)$, the following corollary can be directly derived from Lemma 2.1.

Corollary 2.3. If $u \in H$ with $u^{ \pm} \neq 0$, there exists a unique pair $\left(s_{u}, t_{u}\right) \in \mathbb{R}^{+} \times \mathbb{R}^{+}$such that $s_{u}^{\frac{1}{p}} u^{+}+t_{u}^{\frac{1}{p}} u^{-} \in \mathcal{M}$.

Corollary 2.4. Assume that $u \in \mathcal{N}$, then

$$
\begin{equation*}
\Psi(u)=\max _{t \geq 0} \Psi\left(t^{\frac{1}{p}} u\right)>\Psi\left(\overparen{t}^{\frac{1}{p}} u\right) \tag{2.9}
\end{equation*}
$$

for any $\tilde{t} \geq 0$ and $\tilde{t} \neq 1$.
Proof. By setting $s=t$ in (2.1), we can deduce similarly that

$$
\widetilde{J}(t)=\Psi\left(t^{\frac{1}{p}} u\right)
$$

is strictly concave in $(0,+\infty)$ and has a unique global maximum point. This, together with $u \in \mathcal{N}$, implies the desired conclusion.

The following corollary directly follows from the Corollary 2.4 and [19, Proposition 8].
Corollary 2.5. For any $u \in H \backslash\{0\}$, there exists a unique $t=t(u)>0$ such that $t u \in \mathcal{N}$. Moreover, the map $\hat{\pi}: H \backslash\{0\} \mapsto \mathcal{N}$ is continuous for $\hat{\pi}(u)=t(u) u$ and $\pi:=\left.\hat{\pi}\right|_{S}$ defines a homeomorphism between the unit sphere $S$ of $H$ with $\mathcal{N}$.

In view of Corollaries 2.2, 2.3, 2.4 and 2.5 , we have the following results.

Lemma 2.6. The following equalities hold true:

$$
\inf _{\mathcal{N}} \Psi(u)=: c=\inf _{u \in E, u \neq 0} \max _{t \geq 0} \Psi\left(t^{\frac{1}{p}} u\right)
$$

and

$$
\inf _{\mathcal{M}} \Psi(u)=: m=\inf _{u \in E, u^{ \pm} \neq 0} \max _{s, t \geq 0} \Psi\left(s^{\frac{1}{p}} u^{+}+t^{\frac{1}{p}} u^{-}\right) .
$$

Proof. We only prove the second equality because the other case is similar. On the one hand, it follows from Corollary 2.2 that

$$
\begin{equation*}
\inf _{u \in E, u^{ \pm} \neq 0} \max _{s, t \geq 0} \Psi\left(s^{\frac{1}{p}} u^{+}+t^{\frac{1}{p}} u^{-}\right) \leq \inf _{u \in \mathcal{M}} \max _{s, t \geq 0} \Psi\left(s^{\frac{1}{p}} u^{+}+t^{\frac{1}{p}} u^{-}\right)=\inf _{u \in \mathcal{M}} \Psi(u)=m \tag{2.10}
\end{equation*}
$$

On the other hand, for any $u \in H$ with $u^{ \pm} \neq 0$, by Corollary 2.3, we have

$$
\begin{equation*}
\max _{s, t \geq 0} \Psi\left(s^{\frac{1}{p}} u^{+}+t^{\frac{1}{p}} u^{-}\right) \geq \Psi\left(s_{u}^{\frac{1}{p}} u^{+}+t_{u}^{\frac{1}{p}} u^{-}\right) \geq \inf _{v \in \mathcal{M}} \Psi(v)=m \tag{2.11}
\end{equation*}
$$

Thus, the conclution directly follows from (2.10) and (2.11).
Proposition 2.7. For any $u \in \mathcal{M}$, there exists $\varrho>0$ such that $\left\|u^{ \pm}\right\|_{q} \geq \varrho$.
Proof. Since $u \subset \mathcal{M}$, we have $\left\langle\Psi^{\prime}(u), u^{ \pm}\right\rangle=0$, that is

$$
\int_{\mathbb{R}^{N}}(-\Delta)^{\frac{\alpha}{2}} u(-\Delta)^{\frac{\alpha}{2}} u^{ \pm} \mathrm{d} x+\int_{\Omega} V(x)\left|u^{ \pm}\right|^{2} \mathrm{~d} x=\int_{\Omega}\left|u^{ \pm}\right|^{p} \ln \left|u^{ \pm}\right|^{2} \mathrm{~d} x
$$

Then, by (1.8), $\left(u^{+}, u^{-}\right)_{\alpha}>0$ and the Sobolev inequality, we have

$$
\begin{aligned}
\left\|u^{ \pm}\right\|^{2} & \leq \int_{\Omega}\left|u^{ \pm}\right|^{p} \ln \left(u^{ \pm}\right)^{2} \mathrm{~d} x \\
& \leq \frac{1}{2}\left\|u^{ \pm}\right\|^{2}+C_{1}\left\|u^{ \pm}\right\|^{2}\left\|u^{ \pm}\right\|_{q}^{q-2}
\end{aligned}
$$

for some $C_{1}>0$ independent of $u$. Thus there exists a constant $\varrho>0$ such that $\left\|u^{ \pm}\right\|_{q} \geq \varrho$.
Proposition 2.8. For any $u \in \mathcal{N}$, there exists $\gamma>0$ such that $\|u\|_{q} \geq \gamma$.
Proof. By (1.8) and the Sobolev inequality, for any $u \in \mathcal{N}$, we deduce that

$$
\begin{aligned}
\|u\|^{2} & =\int_{\Omega}|u|^{p} \ln u^{2} \mathrm{~d} x \\
& \leq \frac{1}{2}\|u\|^{2}+C_{2}\|u\|^{2}\|u\|_{q}^{q-2}
\end{aligned}
$$

for some $C_{2}>0$ independent of $u$. Then there exists $\gamma>0$ such that $\|u\|_{q} \geq \gamma$.
Lemma 2.9. $c>0$ and $m>0$ can be achieved.
Proof. We only prove that $m>0$ and is achieved since the other case is similar. Let $\left\{u_{n}\right\} \in \mathcal{M}$ be such that $\Psi\left(u_{n}\right) \rightarrow m$. By (1.7) and (1.9), one has

$$
\begin{aligned}
m+o(1) & =\Psi\left(u_{n}\right)-\frac{1}{p}\left\langle\Psi^{\prime}\left(u_{n}\right), u_{n}\right\rangle \\
& =\left(\frac{1}{2}-\frac{1}{p}\right)\left\|u_{n}\right\|^{2}+\frac{2}{p^{2}} \int_{\Omega}\left|u_{n}\right|^{p} \mathrm{~d} x \\
& \geq\left(\frac{1}{2}-\frac{1}{p}\right)\left\|u_{n}\right\|^{2}
\end{aligned}
$$

This shows that $\left\{u_{n}\right\}$ is bounded. Thus, passing to a subsequence, we may assume that $u_{n}^{ \pm} \rightharpoonup \hat{u}^{ \pm}$weakly in $H$ and $u_{n}^{ \pm} \rightarrow \hat{u}^{ \pm}$strongly in $L^{s}(\Omega)$ for $2 \leq s<2_{\alpha}^{*}$. Since $\left\{u_{n}\right\} \subset \mathcal{M}$, then it follows from Proposition 2.7 that there exists a constant $\varrho>0$ such that $\left\|u_{n}^{ \pm}\right\|_{q} \geq \varrho$. By the compactness of the embedding $H \hookrightarrow L^{s}(\Omega)$ for $2 \leq s<2_{\alpha}^{*}$, we have

$$
\left\|\hat{u}^{ \pm}\right\|_{q}=\lim _{n \rightarrow \infty}\left\|u_{n}^{ \pm}\right\|_{q} \geq \varrho,
$$

which shows $\hat{u}^{ \pm} \neq 0$. By (1.8), (1.9), the Theorem A. 2 in [21], the weak semicontinuity of norm and the Lebesgue dominated convergence theorem, we have

$$
\begin{align*}
\left\|\hat{u}^{ \pm}\right\|^{2}+\int_{\mathbb{R}^{N}}(-\Delta)^{\frac{\alpha}{2}} \hat{u}^{\mp}(-\Delta)^{\frac{\alpha}{2}} \hat{u}^{ \pm} \mathrm{d} x & \leq \liminf _{n \rightarrow \infty}\left(\left\|u_{n}^{ \pm}\right\|^{2}+\int_{\mathbb{R}^{N}}(-\Delta)^{\frac{\alpha}{2}} u_{n}^{\mp}(-\Delta)^{\frac{\alpha}{2}} u_{n}^{ \pm} \mathrm{d} x\right) \\
& =\liminf _{n \rightarrow \infty} \int_{\Omega}\left|u_{n}^{ \pm}\right|^{p} \ln \left(u_{n}^{ \pm}\right)^{2} \mathrm{~d} x  \tag{2.12}\\
& =\int_{\Omega}\left|\hat{u}^{ \pm}\right|^{p} \ln \left(\hat{u}^{ \pm}\right)^{2} \mathrm{~d} x
\end{align*}
$$

which implies

$$
\begin{equation*}
\left\langle\Psi^{\prime}(\hat{u}), \hat{u}^{ \pm}\right\rangle \leq 0 . \tag{2.13}
\end{equation*}
$$

According to Corollary 2.3 , there exist $\hat{s}, \hat{t}>0$ such that $\hat{s}^{\frac{1}{p}} \hat{u}^{+}+\hat{t}^{\frac{1}{p}} \hat{u}^{-} \in \mathcal{M}$ and

$$
\begin{equation*}
\Psi\left(\hat{s}^{\frac{1}{p}} \hat{\mathcal{u}}^{+}+\hat{t}^{\frac{1}{p}} \hat{\mathcal{u}}^{-}\right) \geq m . \tag{2.14}
\end{equation*}
$$

By the concavity of $\hat{J}(s, t):=\Psi\left(s^{\frac{1}{p}} \hat{u}^{+}+t^{\frac{1}{p}} \hat{u}^{-}\right)$for $s, t \geq 0$ and the Taylor expansion, for some $\theta \in(0,1)$, we have

$$
\begin{align*}
\hat{J}(\hat{s}, \hat{t})= & \hat{J}(1,1)+\hat{J}_{s}^{\prime}(1,1)(\hat{s}-1)+\hat{J}_{t}^{\prime}(1,1)(\hat{t}-1) \\
& +\frac{1}{2!}((\hat{s}-1),(\hat{t}-1)) D^{2} \hat{J}(1+\theta(\hat{s}-1), 1+\theta(\hat{t}-1))((\hat{s}-1),(\hat{t}-1))^{\mathrm{T}}  \tag{2.15}\\
\leq & \hat{J}(1,1)+\hat{J}_{s}^{\prime}(1,1)(\hat{s}-1)+\hat{J}_{t}^{\prime}(1,1)(\hat{t}-1) .
\end{align*}
$$

That is

$$
\begin{equation*}
\Psi(\hat{u}) \geq \Psi\left(\hat{s}^{\frac{1}{p}} \hat{u}^{+}+\hat{t}^{\frac{1}{p}} \hat{u}^{-}\right)-\frac{1}{p}(\hat{s}-1)\left\langle\Psi^{\prime}(\hat{u}), \hat{u}^{+}\right\rangle-\frac{1}{p}(\hat{t}-1)\left\langle\Psi^{\prime}(\hat{u}), \hat{u}^{-}\right\rangle . \tag{2.16}
\end{equation*}
$$

Therefore, it follows from (1.7), (1.9), (2.12), (2.13), (2.14), (2.16), Lemma 2.1, Corollary 2.2 and the weak semicontinuity of norm that

$$
\begin{aligned}
m & =\lim _{n \rightarrow \infty}\left(\Psi\left(u_{n}\right)-\frac{1}{p}\left\langle\Psi^{\prime}\left(u_{n}\right), u_{n}\right\rangle\right) \\
& =\lim _{n \rightarrow \infty}\left(\left(\frac{1}{2}-\frac{1}{p}\right)\left\|u_{n}\right\|^{2}+\frac{2}{p^{2}} \int_{\Omega}\left|u_{n}\right|^{p} \mathrm{~d} x\right) \\
& \geq\left(\frac{1}{2}-\frac{1}{p}\right)\|\hat{u}\|^{2}+\frac{2}{p^{2}} \int_{\Omega}|\hat{\hat{p}}|^{p} \mathrm{~d} x \\
& =\Psi(\hat{u})-\frac{1}{p}\left\langle\Psi^{\prime}(\hat{u}), \hat{u}\right\rangle \\
& \geq \Psi\left(\hat{s}^{\frac{1}{p}} \hat{u}^{+}+\hat{t}^{\frac{1}{p}} \hat{u}^{-}\right)-\frac{\hat{s}}{p}\left\langle\Psi^{\prime}(\hat{u}), \hat{u}^{+}\right\rangle-\frac{\hat{t}}{p}\left\langle\Psi^{\prime}(\hat{u}), \hat{u}^{-}\right\rangle \\
& \geq m
\end{aligned}
$$

which implies that

$$
\begin{equation*}
\left\langle\Psi^{\prime}(\hat{u}), \hat{u}^{ \pm}\right\rangle=0 \quad \text { and } \quad \Psi(\hat{u})=m . \tag{2.17}
\end{equation*}
$$

Therefore, $\hat{u} \in \mathcal{M}$ and $\Psi(\hat{u})=m$. Since $\hat{u}^{ \pm} \neq 0$, then by (1.7), (1.9) and (2.17), we have

$$
\begin{aligned}
m=\Psi(\hat{u}) & =\frac{1}{2}\|\hat{u}\|^{2}+\frac{2}{p^{2}} \int_{\Omega}|\hat{u}|^{p} \mathrm{~d} x-\frac{1}{p} \int_{\Omega}|\hat{u}|^{p} \ln \hat{u}^{2} \mathrm{~d} x \\
& \geq \frac{1}{2}\|\hat{u}\|^{2}-\frac{1}{p} \int_{\Omega}|\hat{u}|^{p} \ln \hat{u}^{2} \mathrm{~d} x \\
& =\left(\frac{1}{2}-\frac{1}{p}\right)\left\|\hat{u}^{+}\right\|^{2}+\frac{1}{p}\left\langle\Psi^{\prime}(\hat{u}), \hat{u}\right\rangle \\
& \geq\left(\frac{1}{2}-\frac{1}{p}\right)\left\|\hat{u}^{+}\right\|^{2}+\left(\frac{1}{2}-\frac{1}{p}\right)\left\|\hat{u}^{-}\right\|^{2}+\frac{1}{p}\left\langle\Psi^{\prime}(\hat{u}), \hat{u}^{+}\right\rangle+\frac{1}{p}\left\langle\Psi^{\prime}(\hat{u}), \hat{u}^{-}\right\rangle \\
& >0 .
\end{aligned}
$$

That is $m>0$. The proof is completed.
Lemma 2.10. The minimizers of $\inf _{\mathcal{N}} \Psi(u)$ and $\inf _{\mathcal{M}} \Psi(u)$ are critical points of $\Psi$.
Proof. We prove it by contradiction. Assume that $\widetilde{u} \in \mathcal{M}, \Psi(\widetilde{u})=m$ and $\Psi^{\prime}(\widetilde{u}) \neq 0$. Then there exists $\delta>0, \mu>0$ such that $\left\|\Psi^{\prime}(v)\right\| \geq \mu$, for $\|v-\widetilde{u}\| \leq 3 \delta$. Let $D=\left(\frac{1}{2}, \frac{3}{2}\right) \times\left(\frac{1}{2}, \frac{3}{2}\right)$. By Lemma 2.1, we have

$$
\begin{equation*}
\beta:=\max _{s, t \in \partial D} \Psi\left(s^{\frac{1}{p}} \widetilde{\mathcal{u}}^{+}+t^{\frac{1}{\bar{p}}} \widetilde{\mathcal{u}}^{-}\right)<m . \tag{2.18}
\end{equation*}
$$

Applying the classical deformation [21, Lemma 2.3] with $\varepsilon:=\min \{(m-\beta) / 3, \mu \delta / 8\}$ and $S:=B_{\delta}(\widetilde{u})$, there exists a deformation $\eta \in \mathcal{C}([0,1] \times H, H)$ such that
(a) $\eta(1, u)=u$, if $u \notin \Psi^{-1}(m-2 \varepsilon, m+2 \varepsilon)$,
(b) $\eta\left(1, \Psi^{m+\varepsilon} \cap S\right) \subset \Psi^{m-\varepsilon}$,
(c) $\Psi(\eta(1, u)) \leq u, \forall u \in H$.

Corollary 2.2 implies that $\Psi\left(s^{\frac{1}{p}} \widetilde{u}^{+}+t^{\frac{1}{p}} \widetilde{u}^{-}\right) \leq \Psi(\widetilde{u})=m$, for $s>0, t>0$. Then it follows from (b) that

$$
\begin{equation*}
\Psi\left(\eta\left(1, s^{\frac{1}{\bar{p}}} \widetilde{u}^{+}+t^{\frac{1}{p}} \widetilde{\mathcal{u}}^{-}\right)\right) \leq m-\varepsilon, \tag{2.19}
\end{equation*}
$$

for $s>0, t>0$ and $|s-1|^{2}+|t-1|^{2}<\delta^{2} /\|\widetilde{u}\|^{2}$. Furthermore, using Lemma 2.1 and (c), we derive that

$$
\begin{equation*}
\Psi\left(\eta\left(1, s^{\frac{1}{p}} \widetilde{\mathcal{u}}^{+}+t^{\frac{1}{\bar{p}}} \widetilde{\mathcal{u}}^{-}\right)\right) \leq \Psi\left(s^{\frac{1}{p}} \widetilde{\mathcal{u}}^{+}+t^{\frac{1}{\bar{p}}} \widetilde{\mathcal{u}}^{-}\right)<\Psi(\widetilde{\mathfrak{u}})=m, \tag{2.20}
\end{equation*}
$$

for $s>0, t>0$ and $|s-1|^{2}+|t-1|^{2} \geq \delta^{2} /\|\widetilde{u}\|^{2}$. Thus, from (2.19) and (2.20), we obtain

$$
\max _{s, t \in \bar{D}} \Psi\left(\eta\left(1, s^{\frac{1}{p}} \widetilde{\mathcal{u}}^{+}+t^{\frac{1}{p}} \widetilde{u}^{-}\right)\right)<m
$$

Define $g(s, t)=s^{\frac{1}{\bar{p}}} \widetilde{u}^{+}+t^{\frac{1}{p}} \widetilde{u}^{-}$. To complete the proof it suffices to prove that

$$
\begin{equation*}
\eta(1, g(D)) \cap \mathcal{M} \neq \varnothing, \tag{2.21}
\end{equation*}
$$

which implies $\max _{s, t \in \bar{D}} \Psi\left(\eta\left(1, s^{\frac{1}{\bar{u}}} \widetilde{u}^{+}+t^{\frac{1}{p}} \widetilde{\mathcal{u}}^{-}\right)\right) \geq m$ and it contradicts (2.21). Let us define $\kappa(s, t):=\eta(1, g(s, t))$ and

$$
\phi(s, t):=\left(\frac{1}{p s}\left\langle\Psi^{\prime}(\kappa(s, t)),(\kappa(s, t))^{+}\right\rangle, \frac{1}{p t}\left\langle\Psi^{\prime}(\kappa(s, t)),(\kappa(s, t))^{-}\right\rangle\right) .
$$

Since $\left.\kappa(s, t)\right|_{\partial D}=g(s, t)$, we have

$$
\frac{1}{p s}\left\langle\Psi^{\prime}(g(s, t)), s^{\frac{1}{p}} u^{+}\right\rangle=J_{s}^{\prime}(s, t), \quad \text { on } \partial D,
$$

and

$$
\frac{1}{p t}\left\langle\Psi^{\prime}(g(s, t)), t^{\frac{1}{p}} u^{+}\right\rangle=J_{t}^{\prime}(s, t), \quad \text { on } \partial D .
$$

Therefore, by the homotopy invariance of Brouwer's degree, we can deduce from (2.7) that

$$
\begin{aligned}
\operatorname{deg}(\phi, D,(0,0)) & =\operatorname{deg}\left(\left(J_{s^{\prime}}^{\prime}, J_{t}^{\prime}\right), D,(0,0)\right) \\
& =\operatorname{sgn}\left(\operatorname{det}\binom{J_{s}^{\prime}}{J_{t}^{\prime}}(1,1)\right)=1,
\end{aligned}
$$

which implies that $\phi(\bar{s}, \bar{t})=0$ for some $(\bar{s}, \bar{t}) \in D$, that is $\kappa(\bar{s}, \bar{t})=\eta(1, g(\bar{s}, \bar{t})) \in \mathcal{M}$, which is a contradiction.

The proof of $\inf _{\mathcal{N}} \Psi(u)$ is critical points of $\Psi$ is similar to above argument and hence is omitted here.

## 3 Proof of Theorems 1.5 and 1.6

We first prove Theorem 1.5. According to 2.9 and 2.10, there exists $\bar{u} \in \mathcal{N}$ such that $\Psi(\bar{u})=c$ and $\Psi^{\prime}(\bar{u})=0$. Now, we only need to prove that $u$ is a positive solution of problem (1.1). Indeed, replacing $\Psi(u)$ with the functional

$$
\Psi^{+}(u): \left.=\frac{1}{2} \int_{\mathbb{R}^{N}}\left(\left|(-\Delta)^{\frac{\alpha}{2}} u\right|^{2}+V(x) u^{2}\right) \mathrm{d} x+\frac{2}{p^{2}} \int_{\Omega} \right\rvert\,\left(\left.u^{+}\right|^{p} \mathrm{~d} x-\frac{1}{p} \int_{\Omega}\left|u^{+}\right|^{p} \ln \left(u^{+}\right)^{2} \mathrm{~d} x .\right.
$$

In this way we can get a solution $u$ such that

$$
\begin{equation*}
(-\Delta)^{\alpha} u+V(x) u=\left|u^{+}\right|^{p-2} u^{+} \ln \left(u^{+}\right)^{2} \quad \text { in } \Omega . \tag{3.1}
\end{equation*}
$$

Testing equation (3.1) with $u^{-}$, we obtain

$$
\begin{equation*}
\int_{\mathbb{R}^{N}}(-\Delta)^{\frac{\alpha}{2}} u(-\Delta)^{\frac{\alpha}{2}} u^{-} \mathrm{d} x+\int_{\Omega} V(x)\left|u^{-}\right|^{2} \mathrm{~d} x=0 . \tag{3.2}
\end{equation*}
$$

On the other hand,

$$
\begin{align*}
\int_{\mathbb{R}^{N}}(-\Delta)^{\frac{\alpha}{2}} u(-\Delta)^{\frac{\alpha}{2}} u^{-} \mathrm{d} x & =\int_{\mathbb{R}^{N}}\left|(-\Delta)^{\frac{\alpha}{2}} u^{-}\right|^{2} \mathrm{~d} x+\int_{\mathbb{R}^{N}}(-\Delta)^{\frac{\alpha}{2}} u^{+}(-\Delta)^{\frac{\alpha}{2}} u^{-} \mathrm{d} x \\
& =\int_{\mathbb{R}^{N}}\left|(-\Delta)^{\frac{\alpha}{2}} u^{-}\right|^{2} \mathrm{~d} x+\left(u^{+}, u^{-}\right)_{\alpha} \geq 0 . \tag{3.3}
\end{align*}
$$

Thus, it follows from (3.2) and (3.3), we have $u^{-}=0$ and $u \geq 0$. Since $|t|^{p-1}\left|\ln t^{2}\right| \leq|t|+$ $C_{q}|t|^{q-1}, \forall x \in \Omega, t \in \mathbb{R} \backslash\{0\}$, for $q \in\left(2,2_{\alpha}^{*}\right)$, by the regularity theorem [12, Lemma 3.4],
we can obtain that $u \in C^{0, \mu}$ for some $\mu \in(0,1)$. Therefore, using the maximum principle [17, Proposition 2.17], we obtain $u \equiv 0$ in $\Omega$, a contradiction. Thus, $u$ is a positive solution of problem (1.1).

Finally, we prove Theorem 1.6. We conclude from Lemma 2.9 and Lemma 2.10 that problem (1.1) has a sign-changing solution $\tilde{u} \in \mathcal{M}$ such that $\Psi(\tilde{u})=m$ and $\Psi^{\prime}(\tilde{u})=0$. It remains to prove that $\Psi(\tilde{u})=m:=\inf _{\mathcal{M}} \Psi(u)>2 c$. Indeed, by (1.10), Corollary 2.2 and Lemma 2.6, we have

$$
\begin{aligned}
m=\Psi(\tilde{u}) & =\max _{s, t \geq 0} \Psi\left(s^{\frac{1}{p}} \tilde{u}^{+}+t^{\frac{1}{p}} \tilde{u}^{-}\right) \\
& >\max _{s \geq 0} \Psi\left(s^{\frac{1}{p}} \tilde{u}^{+}\right)+\max _{t \geq 0} \Psi\left(t^{\frac{1}{p}} \tilde{u}^{-}\right) \geq 2 c .
\end{aligned}
$$

The proof is completed.

## 4 Infinitely many solutions

In the following, we analysis the existence of infinitely many nontrivial solutions for problem (1.1).

Define $\hat{\varphi}: H \mapsto \mathbb{R}$ and $\varphi: S \mapsto \mathbb{R}$ by $\hat{\varphi}(u)=\Psi(\hat{\pi}(u))$ and $\varphi:=\left.\hat{\varphi}\right|_{S}$, respectively. Clearly, $\hat{\varphi}$ and $\varphi$ are even since $\Psi$ is even. It is not difficult to verify that $\varphi$ is bounded from below in $S$ and $\varphi$ satisfies the Palais-Smale condition on $S$. Hence, arguing as [19], the functional $\Psi$ has infinitely many critical points, which shows that (1.1) has infinitely many nontrivial solutions. The Theorem 1.7 is proved.

## Acknowledgements

The authors are deeply grateful to Professor Changfeng Gui for his kind support and useful comments. Hui Guo was supported by Natural Science Foundation of Hunan Province (Grant No. 2020JJ5151) and Scientific Research Fund of Hunan Provincial Education Department (Grant No. 19C0781) the National Natural Science Foundation of China (Grant No. 12001188).

## References

[1] C. O. Alves, D. C. M. Filho, G. M. Figueiredo, On concentration of solution to a Schrödinger logarithmic equation with deepening potential well, Math. Methods Appl. Sci. 42(2019), No. 14, 4862-4875. https ://doi.org/10.1002/mma.5699; MR3992943; Zbl 1426.35128
[2] C. O. Alves, C. Jı, Multiple positive solutions for a Schrödinger logarithmic equation, Discrete Contin. Dyn. Syst. 40(2020), No. 5, 2671-2685. https://doi.org/10.3934/dcds. 2020145; MR4097474; Zbl 1435.35028
[3] V. Ambrosio, Multiplicity and concentration results for a class of critical fractional Schrödinger-Poisson systems via penalization method, Commun. Contemp. Math. 22(2018), No. 1, 1850078, 45 pp. https://doi.org/10.1142/S0219199718500785; MR4064905; Zbl 1434.35270
[4] X. J. Chang, Z. Q. Wang, Nodal and multiple solutions of nonlinear problems involving the fractional Laplacian, J. Differential Equations 256(2014), No. 8, 2965-2992. https:// doi.org/10.1016/j.jde.2014.01.027; MR3199753; Zbl 1327.35397
[5] S. T. Chen, X. H. Tang, Ground state sign-changing solutions for elliptic equations with logarithmic nonlinearity, Acta. Math. Hung. 157(2019), No. 1, 27-38. https://doi.org/ 10.1142/S0219199718500785; MR3911157; Zbl 1438.35192
[6] S. T. Chen, X. H. Tang, F. F. Liao, Existence and asymptotic behavior of signchanging solutions for fractional Kirchhoff-type problems in low dimensions, NoDEA Nonlinear Differential Equations Appl. 25(2018), No. 5, 40-63. https://doi.org/10.1007/ s00030-018-0531-9; MR3841989; Zbl 1417.35032
[7] K. Cheng, Q. Gao, Sign-changing solutions for the stationary Kirchhoff problems involving the fractional Laplacian in $\mathbb{R}^{N}$, Acta. Mathe. Sci. 38(2018), No. 6, 1712-1730. https://doi.org/10.1016/S0252-9602(18) 30841-5; MR3868696; Zbl 1438.35139
[8] P. D'Avenia, E. Montefusco, M. Squassina, On the logarithmic Schrödinger equation, Commun. Contemp. Math. 16(2014), No. 2, 1350032, 15 pp. https://doi.org/10.1142/ S0219199713500326; MR3195154; Zbl 1292.35259
[9] P. d’Avenia, M. Squassina, M. Zenari, Fractional logarithmic Schrödinger equations, Math. Methods Appl. Sci. 38(2015), No. 18, 5207-5216. https://doi.org/10.1002/mma. 3449; MR3449663; Zbl 1334.35408
[10] Y. B. Deng, W. Shuai, Sign-changing solutions for non-local elliptic equations involving the fractional Laplacain, Adv. Differerential Equations 23(2018), No. 1, 109-134. MR3718170; Zbl 1386.35087
[11] D. N. Eleonora, P. Giampiero, V. Enrico, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136(2012), No. 5, 521-573. https://doi.org/10.1016/j.bulsci. 2011.12.004; MR2944369; Zbl 1252.46023
[12] P. Felmer, A. Quaas, J. G. Tan, Positive solutions of the nonlinear Schrödinger equation with fractional Laplacian, Proc. Roy. Soc. Edinburgh Sect. A 142(2012), No. 6, 1237-1262. https://doi.org/10.1017/S0308210511000746; MR3002595; Zbl 1290.35308
[13] R. F. Gabert, R. S. Rodrigues, Existence of sign-changing solution for a problem involving the fractional Laplacian with critical growth nonlinearities, Complex Var. Elliptic Equ. 65(2020), No. 2, 272-292. https://doi.org/10.1080/17476933.2019.1579208; MR4043951; Zbl 1436.49007
[14] H. Guo, T, Wang, A multiplicity result for a non-local critical problem, Taiwanese J. Math. 23(2019), No. 6, 1389-1421. https://doi.org/10.11650/tjm/181201; MR4033551; Zbl 1427.35027
[15] C. Ji, Ground state sign-changing solutions for a class of nonlinear fractional Schrödinger-Poisson system in $\mathbb{R}^{3}$, Ann. Mat. Pura Appl. (4) 198(2019), No. 5, 1563-1579. https://doi.org/10.1007/s10231-019-00831-2; MR4022109; Zbl 1430.35093
[16] N. Laskin, Fractional Schrödinger equation, Phys. Rev. E (3) 66(2002), No. 5, 056108, 7 pp. https://doi.org/10.1103/PhysRevE.66.056108; MR1948569
[17] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, Commun. Pure Appl. Math. 60(2007), No. 1, 67-112. https ://doi. org/10.1002/ cpa. 20153; MR2270163; Zbl 1141.49035
[18] M. Squassina, A. Szulkin, Multiple solutions to logarithmic Schrödinger equations with periodic potential, Calc. Var. Partial Differential Equations 54(2015), No. 1, 585-597. https: //doi.org/10.1007/s00526-014-0796-8; MR3385171; Zbl 1326.35358
[19] A. Szulkin, T. Weth, The method of Nehari manifold, in: Handbook of nonconvex analysis and applications, Int. Press, Somerville, MA, 2010, pp. 597-632,. MR2768820
[20] T. Wang, H. Guo, Infinitely many solutions for nonhomogeneous Choquard equations, Electron. J. Qual. Theory Differ. Equ. 2019, No. 24, 1-10. https://doi.org/10.14232/ ejqtde.2019.1.24; MR3938292; Zbl 1438.35196
[21] M. Willem, Minimax theorems, Progress in Nonlinear Differential Equations and Their Applications, Vol. 24, Birkhäuser Boston, Inc., Boston, MA, 1996. MR1400007; Zbl 0856.49001
[22] C. X. Ye, K. M. Teng, Ground state and sign-changing solutions for fractional Schrödinger-Poisson system with critical growth, Complex Var. Elliptic Equ. 65(2020), No. 8, 1360-1393. https://doi.org/10.1080/17476933.2019.1652278; MR4118692; Zbl 1452.35028
[23] Y. Y. Yu, F. K. Zhao, L. G. Zhao, Positive and sign-changing least energy solutions for a fractional Schrödinger-Poisson system with critical exponent, Appl. Anal. 99(2020), No. 13, 2229-2257. https://doi.org/10.1080/00036811.2018.1557325; MR4144117; Zbl 1448.35425
[24] J. J. Zhang, J. M. do Ó, S. Marco, Fractional Schrödinger-Poisson systems with a general subcritical or critical nonlinearity, Adv. Nonlinear Stud. 16(2016), No. 1, 15-30. https: //doi.org/10.1515/ans-2015-5024; MR3456743; Zbl 1334.35407
[25] C. X. Zhang, X. Zhang, Bound states for logarithmic Schrödinger equations with potentials unbounded below, Calc. Var. Partial Differential Equations 59(2020), No. 1, Paper No. 23, 31 pp. https://doi.org/10.1007/s00526-019-1677-y; MR4048332; Zbl 1433.35374

# A class of fourth-order elliptic equations with concave and convex nonlinearities in $\mathbb{R}^{N}$ 

Zijian Wu and Haibo Chen ${ }^{\boxtimes}$<br>School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, P. R. China

Received 15 October 2020, appeared 15 September 2021<br>Communicated by Roberto Livrea


#### Abstract

In this article, we study the multiplicity of solutions for a class of fourth-order elliptic equations with concave and convex nonlinearities in $\mathbb{R}^{N}$. Under the appropriate assumption, we prove that there are at least two solutions for the equation by Nehari manifold and Ekeland variational principle, one of which is the ground state solution.


Keywords: fourth-order elliptic equation, multiple solutions, Nehari manifold, Ekeland variational principle.
2020 Mathematics Subject Classification: 35J35, 35J60.

## 1 Introduction and main results

In this article, we consider the multiplicity results of solutions of the following fourth-order elliptic equation:

$$
\left\{\begin{array}{l}
\Delta^{2} u-\Delta u+u=f(x)|u|^{q-2} u+|u|^{p-2} u, \quad \text { in } \mathbb{R}^{N}  \tag{1.1}\\
u \in H^{2}\left(\mathbb{R}^{N}\right),
\end{array}\right.
$$

where $N>4,1<q<2<p<2_{*}\left(2_{*}=2 N /(N-4)\right)$, the weight function $f$ satisfies the following condition:
(F) $f \geq 0, f \in L^{r_{q}}\left(\mathbb{R}^{N}\right) \cap L^{\infty}\left(\mathbb{R}^{N}\right)$ where $r_{q}=\frac{r}{r-q}$ for some $r \in\left(2,2_{*}\right)$.

Associated with (1.1), we consider the $C^{1}$-functional $I_{f}$, for each $u \in H^{2}\left(\mathbb{R}^{N}\right)$,

$$
I_{f}(u)=\frac{1}{2}\|u\|^{2}-\frac{1}{q} \int_{\mathbb{R}^{N}} f(x)|u|^{q} d x-\frac{1}{p} \int_{\mathbb{R}^{N}}|u|^{p} d x,
$$

where $\|u\|=\left(\int_{\mathbb{R}^{N}}\left(|\Delta u|^{2}+|\nabla u|^{2}+u^{2}\right) d x\right)^{1 / 2}$ is the norm in $H^{2}\left(\mathbb{R}^{N}\right)$. It is well known that the solutions of (1.1) are the critical points of the energy functional $I_{f}$ [14].

[^33]In reality, elliptic equations with concave ang convex nonlinearities in bounded domains have been the focus of a great deal of research in recent years. Ambrosetti et al. [1], for example, considered the following equation:

$$
\begin{cases}-\Delta u=\lambda u^{q-1}+u^{p-1}, & \text { in } \Omega  \tag{1.2}\\ u>0, & \text { in } \Omega, \\ u \in H_{0}^{1}(\Omega), & \end{cases}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^{N}$ with $1<q<2<p<2^{*}\left(2^{*}=\frac{2 N}{N-2}\right.$ if $N \geq 3 ; 2^{*}=$ $\infty$ if $N=1,2$ ) and $\lambda>0$. They found that there is $\lambda_{0}>0$ such that (1.2) admits at least two positive solutions for $\lambda \in\left(0, \lambda_{0}\right)$, has a positive solution for $\lambda=\lambda_{0}$ and no positive solution exists for $\lambda>\lambda_{0}$. Actually, many scholars have also obtained the same results in the unit ball $B^{N}(0 ; 1)$, see $[2,6,10,13]$.

Furthermore, it is also an important subject to deal with elliptic equation with concaveconvex nonlinearities when a bounded domain $\Omega$ is replaced by $\mathbb{R}^{N}$. Wu [18] studied the concave-convex elliptic problem:

$$
\begin{cases}-\Delta u+u=f_{\lambda}(x) u^{q-1}+g_{\mu}(x) u^{p-1}, & \text { in } \mathbb{R}^{N}  \tag{1.3}\\ u>0, & \text { in } \mathbb{R}^{N} \\ u \in H^{1}\left(\mathbb{R}^{N}\right), & \end{cases}
$$

where $1<q<2<p<2^{*}\left(2^{*}=2 N /(N-2)\right.$ if $N \geq 3,2^{*}=\infty$ if $\left.N=1,2\right)$,

$$
f_{\lambda}=\lambda f_{+}+f_{-}\left(f_{ \pm}= \pm \max \{0, \pm f\} \neq 0\right)
$$

is sign-changing, $g_{\mu}=a+\mu b$ and the parameters $\lambda, \mu>0$. When the functions $f_{+}, f_{-}, a, b$ satisfy appropriate hypotheses, author obtained the multiplicity of positive solutions for the problem (1.3). Hsu and Lin [9] dealt with the existence and multiplicity of positive solutions for the following semilinear elliptic equation:

$$
\begin{cases}-\Delta u+u=\lambda a(x)|u|^{q-2} u+b(x)|u|^{p-2} u, & \text { in } \mathbb{R}^{N}  \tag{1.4}\\ u>0, & \text { in } \mathbb{R}^{N} \\ u \in H^{1}\left(\mathbb{R}^{N}\right), & \end{cases}
$$

where $a, b$ are measurable functions and meet the right conditions. They obtained the result of multiple solutions of the equation (1.4).

Inspired by the existing literature $[5,8,9,11,15,18-20]$, the main aim of this article is to study (1.1) involving concave-convex nonlinearities on the whole space $\mathbb{R}^{N}$. As far as we know, there are few articles dealing with this type of fourth-order elliptic equation (1.1) involving concave-convex nonlinearities. Using arguments similar to those used in [16], we will prove the existence of two nontrivial solutions by using Ekeland variational principle [7].

Let

$$
\sigma=\left(\frac{p-2}{p-q}\right)\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}} S_{p}^{\frac{p(2-q)}{2(p-2)}} S_{r}^{\frac{q}{2}}>0,
$$

where $S_{p}$ and $S_{r}$ are the best Sobolev constant. Now, we state the main result.
Theorem 1.1. Assume that (F) holds. If $|f|_{r_{q}} \in(0, \sigma)$, then (1.1) has at least two nontrivial solutions, one of which is the ground state solution.

This paper is organized as follows. In Section 2, we give some notations and preliminaries. In Section 3, we are concerned with the proof of Theorem 1.1.

## 2 Notations and preliminaries

We shall throughout use the Sobolev space $H^{2}\left(\mathbb{R}^{N}\right)$ with standard norm. The dual space of $H^{2}\left(\mathbb{R}^{N}\right)$ will be denoted by $H^{-2}\left(\mathbb{R}^{N}\right)$. $\langle\cdot, \cdot\rangle$ denotes the usual scalar product in $H^{2}\left(\mathbb{R}^{N}\right)$. $L^{r}\left(\mathbb{R}^{N}\right)$ is the usual Lebesgue space whose norms we denote by $|u|_{r}=\left(\int_{\mathbb{R}^{N}}|u|^{r} d x\right)^{1 / r}$ for $1 \leq p<\infty$. Moreover, we denote by $S_{r}$ the best Sobolev constant for the embedding of $H^{2}\left(\mathbb{R}^{N}\right)$ in $L^{r}\left(\mathbb{R}^{N}\right)$.

Now, we consider the Nehari minimization problem:

$$
\alpha_{f}=\inf \left\{I_{f}(u) \mid u \in \mathcal{N}_{f}\right\},
$$

where $\mathcal{N}_{f}=\left\{u \in H^{2}\left(\mathbb{R}^{N}\right) \backslash\{0\} \mid\left\langle I_{f}^{\prime}(u), u\right\rangle=0\right\}$. Define

$$
\psi_{f}(u)=\left\langle I_{f}^{\prime}(u), u\right\rangle=\|u\|^{2}-\int_{\mathbb{R}^{N}} f(x)|u|^{q} d x-\int_{\mathbb{R}^{N}}|u|^{p} d x .
$$

Then for $u \in \mathcal{N}_{f}$,

$$
\begin{aligned}
\left\langle\psi_{f}^{\prime}(u), u\right\rangle & =\left\langle\psi_{f}^{\prime}(u), u\right\rangle-\left\langle I_{f}^{\prime}(u), u\right\rangle \\
& =\|u\|^{2}-(q-1) \int_{\mathbb{R}^{N}} f(x)|u|^{q} d x-(p-1) \int_{\mathbb{R}^{N}}|u|^{p} d x .
\end{aligned}
$$

Similarly to the skill used in Tarantello [16], we split $\mathcal{N}_{f}$ into three parts:

$$
\begin{aligned}
\mathcal{N}_{f}^{+} & =\left\{u \in \mathcal{N}_{f} \mid\left\langle\psi_{f}^{\prime}(u), u\right\rangle>0\right\}, \\
\mathcal{N}_{f}^{0} & =\left\{u \in \mathcal{N}_{f} \mid\left\langle\psi_{f}^{\prime}(u), u\right\rangle=0\right\}, \\
\mathcal{N}_{f}^{-} & =\left\{u \in \mathcal{N}_{f} \mid\left\langle\psi_{f}^{\prime}(u), u\right\rangle<0\right\}
\end{aligned}
$$

and note that if $u \in \mathcal{N}_{f}$, that is, $\left\langle I_{f}^{\prime}(u), u\right\rangle=0$, then

$$
\begin{align*}
\left\langle\psi_{f}^{\prime}(u), u\right\rangle & =(2-p)\|u\|^{2}-(q-p) \int_{\mathbb{R}^{N}} f(x)|u|^{q} d x \\
& =(2-q)\|u\|^{2}-(p-q) \int_{\mathbb{R}^{N}}|u|^{p} d x . \tag{2.1}
\end{align*}
$$

Then, we have the following results.
Lemma 2.1. If $|f|_{r_{q}} \in(0, \sigma)$, then the submanifold $\mathcal{N}^{0}=\varnothing$.
Proof. Suppose the contrary. Then $\mathcal{N}_{f}^{0} \neq \varnothing$, i.e., there exist $u \in \mathcal{N}_{f}$ such that $\left\langle\psi_{f}^{\prime}(u), u\right\rangle=0$. Then for $u \in \mathcal{N}^{0}$ by (2.1) and Sobolev inequality, we have

$$
(2-q)\|u\|^{2}=(p-q) \int_{\mathbb{R}^{N}}|u|^{p} d x \leq(p-q) S_{p}^{-\frac{p}{2}}\|u\|^{p},
$$

and so

$$
\begin{equation*}
\|u\| \geq\left(\frac{(2-q) S_{p}^{\frac{p}{2}}}{p-q}\right)^{\frac{1}{p-2}} \tag{2.2}
\end{equation*}
$$

Similarly, using (2.1), Sobolev and Hölder inequalities, we have

$$
(p-2)\|u\|^{2}=(p-q) \int_{\mathbb{R}^{N}} f(x)|u|^{q} d x \leq(p-q)|f|_{r_{q}} S_{r}^{-\frac{q}{2}}\|u\|^{q},
$$

which implies that

$$
\begin{equation*}
\|u\| \leq\left(\frac{(p-q)|f|_{r_{q}}}{(p-2) S_{r}^{\frac{q}{2}}}\right)^{\frac{1}{2-q}} \tag{2.3}
\end{equation*}
$$

Combining (2.2) and (2.3) we deduce that

$$
|f|_{r_{q}} \geq\left(\frac{p-2}{p-q}\right)\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}} S_{p}^{\frac{p(2-q)}{(p-2)}} S_{r}^{\frac{q}{2}}=\sigma,
$$

which is a contradiction. This completes the proof.
Lemma 2.2. If $|f|_{r_{q}} \in(0, \sigma)$, then the set $\mathcal{N}_{f}^{-}$is closed in $H^{2}\left(\mathbb{R}^{N}\right)$.
Proof. Let $\left\{u_{n}\right\} \subset \mathcal{N}_{f}^{-}$such that $u_{n} \rightarrow u$ in $H^{2}\left(\mathbb{R}^{N}\right)$. In the following we show $u \in \mathcal{N}_{f}^{-}$. In fact, by $\left\langle I_{f}^{\prime}\left(u_{n}\right), u_{n}\right\rangle=0$ and

$$
\left\langle I_{f}^{\prime}\left(u_{n}\right), u_{n}\right\rangle-\left\langle I_{f}^{\prime}(u), u\right\rangle=\left\langle I_{f}^{\prime}\left(u_{n}\right)-I_{f}^{\prime}(u), u\right\rangle+\left\langle I_{f}^{\prime}\left(u_{n}\right), u_{n}-u\right\rangle \rightarrow 0 \quad \text { as } n \rightarrow \infty,
$$

we have $\left\langle I_{f}^{\prime}(u), u\right\rangle=0$. So $u \in \mathcal{N}_{f}$. For any $u \in \mathcal{N}_{f}^{-}$, that is, $\left\langle\psi_{f}^{\prime}(u), u\right\rangle<0$, from (2.1) we have

$$
(2-q)\|u\|^{2}<(p-q) \int_{\mathbb{R}^{N}}|u|^{p} d x \leq(p-q) S_{p}^{-\frac{p}{2}}\|u\|^{p}
$$

and so

$$
\|u\|>\left(\frac{(2-q) S_{p}^{\frac{p}{2}}}{p-q}\right)^{\frac{1}{p-2}}>0
$$

Hence $\mathcal{N}_{f}^{-}$is bounded away from 0 . Obviously, by (2.1), it follows that $\left\langle\psi_{f}^{\prime}\left(u_{n}\right), u_{n}\right\rangle \rightarrow$ $\left\langle\psi_{f}^{\prime}(u), u\right\rangle$ as $n \rightarrow+\infty$. From $\left\langle\psi_{f}^{\prime}\left(u_{n}\right), u_{n}\right\rangle<0$, we have $\left\langle\psi_{f}^{\prime}(u), u\right\rangle \leq 0$. By Lemma 2.1, for $|f|_{r_{q}} \in(0, \sigma), \mathcal{N}_{f}^{0}=\varnothing$, then $\left\langle\psi_{f}^{\prime}(u), u\right\rangle<0$. Thus we deduce $u \in \mathcal{N}_{f}^{-}$. This completes the proof.

Lemma 2.3. The energy functional $I_{f}$ is coercive and bounded below on $\mathcal{N}_{f}$.
Proof. For $u \in \mathcal{N}_{f}$, then, by Sobolev and Hölder inequalities,

$$
\begin{aligned}
I_{f}(u) & =I_{f}(u)-\frac{1}{p}\left\langle I_{f}^{\prime}(u), u\right\rangle \\
& =\frac{p-2}{2 p}\|u\|^{2}-\frac{p-q}{p q} \int_{\mathbb{R}^{N}} f(x)|u|^{q} d x \\
& \geq \frac{p-2}{2 p}\|u\|^{2}-\frac{p-q}{p q}|f|_{r_{q}} S_{r}^{-\frac{q}{2}}\|u\|^{q} .
\end{aligned}
$$

This completes the proof.
The following lemma shows that the minimizers on $\mathcal{N}_{f}$ are "usually" critical points for $I_{f}$. The details of the proof can be referred to Brown and Zhang [4].

Lemma 2.4. Suppose that $\widehat{u}$ is a local minimizer for $I_{f}$ on $\mathcal{N}_{f}$. Then, if $\widehat{u} \notin \mathcal{N}_{f}^{0}, \widehat{u}$ is a critical point of $I_{f}$.

For each $u \in H^{2}\left(\mathbb{R}^{N}\right) \backslash\{0\}$, we write

$$
t_{\max }:=\left(\frac{(2-q)\|u\|^{2}}{(p-q) \int_{\mathbb{R}^{N}}|u|^{p} d x}\right)^{\frac{1}{p-2}}>0
$$

Then, we have the following lemma.
Lemma 2.5. For each $u \in H^{2}\left(\mathbb{R}^{N}\right) \backslash\{0\}$ and $|f|_{r_{q}} \in(0, \sigma)$, we have
(i) there exist unique $0<t^{+}:=t^{+}(u)<t_{\max }<t^{-}:=t^{-}(u)$ such that $t^{+} u \in \mathcal{N}_{f}^{+}, t^{-} u \in \mathcal{N}_{f}^{-}$ and

$$
I_{f}\left(t^{+} u\right)=\inf _{t_{\max } \geq t \geq 0} I_{f}(t u), \quad I_{f}\left(t^{-} u\right)=\sup _{t \geq t_{\text {max }}} I_{f}(t u) .
$$

(ii) $t^{-}$is a continuous function for nonzero $u$.
(iii) $\mathcal{N}_{f}^{-}=\left\{u \in H^{2}\left(\mathbb{R}^{N}\right) \backslash\{0\} \left\lvert\, \frac{1}{\|u\|} t^{-}\left(\frac{u}{\|u\|}\right)=1\right.\right\}$.

Proof. (i) Fix $u \in H^{2}\left(\mathbb{R}^{N}\right) \backslash\{0\}$. Let

$$
s(t)=t^{2-q}\|u\|^{2}-t^{p-q} \int_{\mathbb{R}^{N}}|u|^{p} d x \quad \text { for } t \geq 0
$$

We have $s(0)=0, s(t) \rightarrow-\infty$ as $t \rightarrow \infty, s(t)$ is concave and achieves its maximum at $t_{\max }$. Moreover, for $|f|_{r_{q}} \in(0, \sigma)$,

$$
\begin{aligned}
s\left(t_{\max }\right) & =\left(\frac{(2-q)\|u\|^{2}}{(p-q) \int_{R^{N}}|u|^{p} d x}\right)^{\frac{2-q}{p-2}}\|u\|^{2}-\left(\frac{(2-q)\|u\|^{2}}{(p-q) \int_{R^{N}}|u|^{p} d x}\right)^{\frac{p-q}{p-2}} \int_{\mathbb{R}^{N}}|u|^{p} d x \\
& =\|u\|^{q}\left(\frac{\|u\|^{p}}{\int_{R^{N}}|u|^{p} d x}\right)^{\frac{2-q}{p-2}}\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}} \frac{p-2}{p-q} \\
& \geq\|u\|^{q}\left(\frac{\|u\|^{p}}{S_{p}^{-\frac{p}{2}}\|u\|^{p}}\right)^{\frac{2-q}{p-2}}\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}} \frac{p-2}{p-q} \\
& =\|u\|^{q}\left(\frac{(2-q) S_{p}^{\frac{p}{2}}}{p-q}\right)^{\frac{2-q}{p-2}} \frac{p-2}{p-q} \\
& >|f|_{r_{q}} S_{r}^{-\frac{q}{2}}\|u\|^{q} \\
& \geq \int_{\mathbb{R}^{N}} f(x)|u|^{q} d x>0 .
\end{aligned}
$$

Hence, there are unique $t^{+}$and $t^{-}$such that $0<t^{+}<t_{\max }<t^{-}$,

$$
s\left(t^{+}\right)=\int_{\mathbb{R}^{N}} f(x)|u|^{q} d x=s\left(t^{-}\right)
$$

and

$$
s^{\prime}\left(t^{+}\right)>0>s^{\prime}\left(t^{-}\right) .
$$

Note that

$$
\left\langle I_{f}^{\prime}(t u), t u\right\rangle=t^{q-1}\left(s(t)-\int_{\mathbb{R}^{N}} f(x)|u|^{q} d x\right)
$$

and

$$
\left\langle\psi_{f}^{\prime}(t u), t u\right\rangle=t^{q+1} s^{\prime}(t) \quad \text { for } t u \in \mathcal{N}_{f} .
$$

We have $t^{+} u \in \mathcal{N}_{f}^{+}, t^{-} u \in \mathcal{N}_{f}^{-}$, and $I_{f}\left(t^{-} u\right) \geq I_{f}(t u) \geq I_{f}\left(t^{+} u\right)$ for each $t \in\left[t^{+}, t^{-}\right]$and $I_{f}\left(t^{+} u\right) \geq I_{f}(t u)$ for each $t \in\left[0, t^{+}\right]$. Thus,

$$
I_{f}\left(t^{+} u\right)=\inf _{t_{\max } \geq t \geq 0} I_{f}(t u), \quad I_{f}\left(t^{-} u\right)=\sup _{t \geq t_{\max }} I_{f}(t u) .
$$

(ii) By the uniqueness of $t^{-}$and the external property of $t^{-}$, we have that $t^{-}$is a continuous function of $u \neq 0$.
(iii) For $u \in \mathcal{N}_{f}^{-}$, let $v=\frac{u}{\|u\|}$. By part (i), there is a unique $t^{-}(v)>0$ such that $t^{-}(v) v \in \mathcal{N}_{f}^{-}$, that is $t^{-}\left(\frac{u}{\|u\|}\right) \frac{u}{\|u\|} \in \mathcal{N}_{f}^{-}$. Since $u \in \mathcal{N}_{f}^{-}$, we have $t^{-}\left(\frac{u}{\|u\|}\right) \frac{u}{\|u\|}=1$, which implies

$$
\mathcal{N}_{f}^{-} \subset\left\{u \in H^{2}\left(\mathbb{R}^{N}\right) \backslash\{0\} \left\lvert\, \frac{1}{\|u\|} t^{-}\left(\frac{u}{\|u\|}\right)=1\right.\right\} .
$$

Conversely, let $u \in H^{2}\left(\mathbb{R}^{N}\right) \backslash\{0\}$ such that $\frac{1}{\|u\|} t^{-}\left(\frac{u}{\|u\|}\right)=1$. Then $t^{-}\left(\frac{u}{\|u\|}\right) \frac{u}{\|u\|} \in \mathcal{N}_{f}^{-}$. Thus,

$$
\mathcal{N}_{f}^{-}=\left\{u \in H^{2}\left(\mathbb{R}^{N}\right) \backslash\{0\} \left\lvert\, \frac{1}{\|u\|} t^{-}\left(\frac{u}{\|u\|}\right)=1\right.\right\} .
$$

This completes the proof.
By Lemma 2.1, for $|f|_{r_{q}} \in(0, \sigma)$ we write $\mathcal{N}_{f}=\mathcal{N}_{f}^{+} \cup \mathcal{N}_{f}^{-}$and define

$$
\alpha_{f}^{+}=\inf _{u \in \mathcal{N}_{f}^{+}} I_{f}(u), \quad \alpha_{f}^{-}=\inf _{u \in \mathcal{N}_{f}^{-}} I_{f}(u)
$$

Lemma 2.6. For $|f|_{r_{q}} \in(0, \sigma)$, we have $\alpha_{f} \leq \alpha_{f}^{+}<0$.
Proof. Let $u \in \mathcal{N}_{f}^{+}$. By (2.1) we have

$$
\int_{\mathbb{R}^{N}}|u|^{p} d x<\frac{2-q}{p-q}\|u\|^{2},
$$

and so

$$
\begin{aligned}
I_{f}(u) & =\left(\frac{1}{2}-\frac{1}{q}\right)\|u\|^{2}+\left(\frac{1}{q}-\frac{1}{p}\right) \int_{\mathbb{R}^{N}}|u|^{p} d x \\
& <\left[\left(\frac{1}{2}-\frac{1}{q}\right)+\left(\frac{1}{q}-\frac{1}{p}\right)\left(\frac{2-q}{p-q}\right)\right]\|u\|^{2} \\
& =-\frac{(p-2)(2-q)}{2 p q}\|u\|^{2}<0 .
\end{aligned}
$$

Therefore, $\alpha_{f} \leq \alpha_{f}^{+}<0$.

## 3 Proof of Theorem 1.1

First, we will use the idea of Ni and Takagi [12] to get the following lemmas.

Lemma 3.1. If $|f|_{r_{g}} \in(0, \sigma)$, then for every $u \in \mathcal{N}_{f}$, there exist $\epsilon>0$ and a differentiable function $g: B_{\epsilon}(0) \subset H^{2}\left(\mathbb{R}^{N}\right) \rightarrow \mathbb{R}^{+}:=(0,+\infty)$ such that

$$
g(0)=1, \quad g(\omega)(u-\omega) \in \mathcal{N}_{f}, \quad \forall \omega \in B_{\epsilon}(0)
$$

and

$$
\begin{equation*}
\left\langle g^{\prime}(0), v\right\rangle=\frac{2(u, v)-q \int_{\mathbb{R}^{N}} f(x)|u|^{q-2} u v d x-p \int_{\mathbb{R}^{N}}|u|^{p-2} u v d x}{\left\langle\psi_{f}^{\prime}(u), u\right\rangle} \tag{3.1}
\end{equation*}
$$

for all $v \in H^{2}\left(\mathbb{R}^{N}\right)$. Moreover, if $0<C_{1} \leq\|u\| \leq C_{2}$, then there exists $C>0$ such that

$$
\begin{equation*}
\left|\left\langle g^{\prime}(0), v\right\rangle\right| \leq C\|v\| . \tag{3.2}
\end{equation*}
$$

Proof. We define $F: \mathbb{R} \times H^{2}\left(\mathbb{R}^{N}\right) \rightarrow \mathbb{R}$ by

$$
F(t, \omega)=t\|u-\omega\|^{2}-t^{q-1} \int_{\mathbb{R}^{N}} f(x)|u-\omega|^{q} d x-t^{p-1} \int_{\mathbb{R}^{N}}|u-\omega|^{p} d x
$$

it is easy to see $F$ is differentiable. Since $F(1,0)=0$ and $F_{t}(1,0)=\left\langle\psi_{f}^{\prime}(u), u\right\rangle \neq 0$, we apply the implicit function theorem at point $(1,0)$ to get the existence of $\epsilon>0$ and differentiable function $g: B_{\epsilon}(0) \rightarrow \mathbb{R}^{+}$such that $g(0)=1$ and $F(g(\omega), \omega)=0$ for $\forall \omega \in B_{\epsilon}(0)$. Thus,

$$
g(\omega)(u-\omega) \in \mathcal{N}_{f}, \quad \forall \omega \in B_{\epsilon}(0)
$$

Also by the differentiability of the implicit function theorem, for all $v \in H^{2}\left(\mathbb{R}^{N}\right)$, we know that

$$
\left\langle g^{\prime}(0), v\right\rangle=-\frac{\left\langle F_{\omega}(1,0), v\right\rangle}{F_{t}(1,0)} .
$$

Note that

$$
-\left\langle F_{\omega}(1,0), v\right\rangle=2(u, v)-q \int_{\mathbb{R}^{N}} f(x)|u|^{q-2} u v d x-p \int_{\mathbb{R}^{N}}|u|^{p-2} u v d x
$$

and $F_{t}(1,0)=\left\langle\psi_{f}^{\prime}(u), u\right\rangle$. So (3.1) holds.
Moreover, by (3.1), $0<C_{1} \leq\|u\| \leq C_{2}$ and Hölder's inequality, we have

$$
\left|\left\langle g^{\prime}(0), v\right\rangle\right| \leq \frac{\widetilde{C}\|v\|}{\left\langle\psi_{f}^{\prime}(u), u\right\rangle}
$$

for some $\widetilde{C}>0$. To prove (3.2), therefore, we only need to show that $\left|\left\langle\psi_{f}^{\prime}(u), u\right\rangle\right|>d$ for some $d>0$. We argue by contradiction. Assume that there exists a sequence $\left\{u_{n}\right\} \in \mathcal{N}_{f}$, $C_{1} \leq\left\|u_{n}\right\| \leq C_{2}$, we have $\left\langle\psi_{f}^{\prime}\left(u_{n}\right), u_{n}\right\rangle=o_{n}(1)$. Then by (2.1) and Sobolev's inequality, we have

$$
\begin{aligned}
(2-q)\left\|u_{n}\right\|^{2} & =(p-q) \int_{\mathbb{R}^{N}}\left|u_{n}\right|^{p} d x+o_{n}(1) \\
& \leq(p-q) S_{p}^{-\frac{p}{2}}\left\|u_{n}\right\|^{p}+o_{n}(1)
\end{aligned}
$$

and so

$$
\begin{equation*}
\left\|u_{n}\right\| \geq\left(\frac{(2-q) S_{p}^{\frac{p}{2}}}{p-q}\right)^{\frac{1}{p-2}}+o_{n}(1) \tag{3.3}
\end{equation*}
$$

Similarly, using (2.1) and Hölder and Sobolev inequalities, we have

$$
\begin{aligned}
(p-2)\left\|u_{n}\right\|^{2} & =(p-q) \int_{\mathbb{R}^{N}} f(x)\left|u_{n}\right|^{q} d x+o_{n}(1) \\
& \leq(p-q)|f|_{r_{q}} S_{r}^{-\frac{q}{2}}\left\|u_{n}\right\|^{q}+o_{n}(1),
\end{aligned}
$$

which implies that

$$
\begin{equation*}
\left\|u_{n}\right\| \leq\left(\frac{(p-q)|f|_{r_{q}}}{(p-2) S_{r}^{\frac{q}{2}}}\right)^{\frac{1}{2-q}}+o_{n}(1) . \tag{3.4}
\end{equation*}
$$

Combining (3.3) and (3.4) as $n \rightarrow+\infty$, we deduce that

$$
|f|_{r_{q}} \geq\left(\frac{p-2}{p-q}\right)\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-q}} S_{p}^{\frac{p(2-q)}{2(p-2)}} S_{r}^{\frac{q}{2}}=\sigma,
$$

which is a contradiction. Thus if $0<C_{1} \leq\|u\| \leq C_{2}$, there exists $C>0$ such that

$$
\left|\left\langle g^{\prime}(0), v\right\rangle\right| \leq C\|v\| .
$$

This completes the proof.
Lemma 3.2. If $|f|_{r_{q}} \in(0, \sigma) \in(0, \sigma)$, then for every $u \in \mathcal{N}_{f}^{-}$, there exist $\epsilon>0$ and a differentiable function $g^{-}: B_{\epsilon}(0) \subset H^{2}\left(\mathbb{R}^{N}\right) \rightarrow \mathbb{R}^{+}$such that

$$
g^{-}(0)=1, \quad g^{-}(\omega)(u-\omega) \in \mathcal{N}_{f}^{-}, \quad \forall \omega \in B_{\epsilon}(0)
$$

and

$$
\begin{equation*}
\left\langle\left(g^{-}\right)^{\prime}(0), v\right\rangle=\frac{2(u, v)-q \int_{\mathbb{R}^{N}} f(x)|u|^{q-2} u v d x-p \int_{\mathbb{R}^{N}}|u|^{p-2} u v d x}{\left\langle\psi_{f}^{\prime}(u), u\right\rangle} \tag{3.5}
\end{equation*}
$$

for all $v \in H^{2}\left(\mathbb{R}^{N}\right)$. Moreover, if $0<C_{1} \leq\|u\| \leq C_{2}$, then there exists $C>0$ such that

$$
\begin{equation*}
\left|\left\langle\left(g^{-}\right)^{\prime}(0), v\right\rangle\right| \leq C\|v\| . \tag{3.6}
\end{equation*}
$$

Proof. Similar to the argument in Lemma 3.2, there exist $\epsilon>0$ and a differentiable function $g^{-}: B_{\epsilon}(0) \rightarrow \mathbb{R}^{+}$such that $g^{-}(0)=1$ and $g^{-}(\omega)(u-\omega) \in \mathcal{N}_{f}$ for all $\omega \in B_{\epsilon}(0)$. By $u \in \mathcal{N}_{f}^{-}$, we have

$$
\left\langle\psi_{f}^{\prime}(u), u\right\rangle=\|u\|^{2}-(q-1) \int_{\mathbb{R}^{N}} f(x)|u|^{q} d x-(p-1) \int_{\mathbb{R}^{N}}|u|^{p} d x<0 .
$$

Since $g^{-}(\omega)(u-\omega)$ is continuous with respect to $\omega$, when $\epsilon$ is small enough, we know for $\omega \in B_{\epsilon}(0)$
$\left\|g^{-}(\omega)(u-\omega)\right\|^{2}-(q-1) \int_{\mathbb{R}^{N}} f(x)\left|g^{-}(\omega)(u-\omega)\right|^{q} d x-(p-1) \int_{\mathbb{R}^{N}}\left|g^{-}(\omega)(u-\omega)\right|^{p} d x<0$.
Thus, $g^{-}(\omega)(u-\omega) \in \mathcal{N}_{f}^{-}, \forall \omega \in B_{\epsilon}(0)$. Moreover, the proof details of (3.5) and (3.6) are similar to Lemma 3.1.

Lemma 3.3. If $|f|_{r_{q}} \in(0, \sigma)$, then
(i) there exists a minimizing sequence $\left\{u_{n}\right\} \in \mathcal{N}_{f}$ such that

$$
\begin{aligned}
& I_{f}\left(u_{n}\right)=\alpha_{f}+o_{n}(1), \\
& I_{f}^{\prime}\left(u_{n}\right)=o_{n}(1) \quad \text { in } H^{-2}\left(\mathbb{R}^{N}\right) ;
\end{aligned}
$$

(ii) there exists a minimizing sequence $\left\{u_{n}\right\} \in \mathcal{N}_{f}^{-}$such that

$$
\begin{aligned}
& I_{f}\left(u_{n}\right)=\alpha_{f}^{-}+o_{n}(1), \\
& I_{f}^{\prime}\left(u_{n}\right)=o_{n}(1) \quad \text { in } H^{-2}\left(\mathbb{R}^{N}\right) .
\end{aligned}
$$

Proof. (i) By Lemma 2.3 and the Ekeland variational principle on $\mathcal{N}_{f}$, there exists a minimizing sequence $\left\{u_{n}\right\} \subset \mathcal{N}_{f}$ such that

$$
\begin{equation*}
\alpha_{f} \leq I_{f}\left(u_{n}\right)<\alpha_{f}+\frac{1}{n} \tag{3.7}
\end{equation*}
$$

and

$$
\begin{equation*}
I_{f}\left(u_{n}\right) \leq I_{f}(v)+\frac{1}{n}\left\|v-u_{n}\right\| \quad \text { for each } v \in \mathcal{N}_{f} . \tag{3.8}
\end{equation*}
$$

And we can show that there exists $C_{1}, C_{2}>0$ such that $0<C_{1} \leq\left\|u_{n}\right\| \leq C_{2}$. Indeed, if not, that is, $u_{n} \rightarrow 0$ in $H^{2}\left(\mathbb{R}^{N}\right)$, then $I_{f}\left(u_{n}\right)$ would converge to zero, which contradict with $I_{f}\left(u_{n}\right) \rightarrow \alpha_{f}<0$. Moreover, by Lemma 2.3 we know that $I_{f}(u)$ is coercive on $\mathcal{N}_{f},\left\{u_{n}\right\}$ is bounded in $\mathcal{N}_{f}$.

Now, we show that

$$
\left\|I_{f}^{\prime}\left(u_{n}\right)\right\|_{H^{-2}\left(\mathbb{R}^{N}\right)} \rightarrow 0 \quad \text { as } n \rightarrow \infty .
$$

Applying Lemma 3.1 with $u_{n}$ to obtain the functions $g_{n}(\omega): B_{\epsilon_{n}}(0) \rightarrow \mathbb{R}^{+}$for some $\epsilon_{n}>0$, such that

$$
g_{n}(0)=1, \quad g_{n}(\omega)\left(u_{n}-\omega\right) \in \mathcal{N}_{f}, \quad \forall \omega \in B_{e_{n}}(0) .
$$

We choose $0<\rho<\epsilon_{n}$. Let $u \in H^{2}\left(\mathbb{R}^{N}\right) \backslash\{0\}$ and $\omega_{\rho}=\frac{\rho u}{\|u\|}$. Since $g_{n}\left(\omega_{\rho}\right)\left(u_{n}-\omega_{\rho}\right) \in \mathcal{N}_{f}$, we deduce from (3.8) that

$$
\begin{align*}
& \frac{1}{n}\left[\left|g_{n}\left(\omega_{\rho}\right)-1\right|\left\|u_{n}\right\|+\rho g_{n}\left(\omega_{\rho}\right)\right] \\
& \geq \frac{1}{n}\left\|g_{n}\left(\omega_{\rho}\right)\left(u_{n}-\omega_{\rho}\right)-u_{n}\right\| \\
& \geq I_{f}\left(u_{n}\right)-I_{f}\left(g_{n}\left(\omega_{\rho}\right)\left(u_{n}-\omega_{\rho}\right)\right) \\
&= \frac{1}{2}\left\|u_{n}\right\|^{2}-\frac{1}{q} \int_{\mathbb{R}^{N}} f(x)\left|u_{n}\right|^{q} d x-\frac{1}{p} \int_{\mathbb{R}^{N}}\left|u_{n}\right|^{p} d x-\frac{1}{2}\left(g_{n}\left(\omega_{\rho}\right)\right)^{2}\left\|u_{n}-\omega_{\rho}\right\|^{2} \\
&+\frac{1}{q}\left(g_{n}\left(\omega_{\rho}\right)\right)^{q} \int_{\mathbb{R}^{N}} f(x)\left|u_{n}-\omega_{\rho}\right|^{q} d x+\frac{1}{p}\left(g_{n}\left(\omega_{\rho}\right)\right)^{p} \int_{\mathbb{R}^{N}}\left|u_{n}-\omega_{\rho}\right|^{p} d x  \tag{3.9}\\
&=-\frac{\left(g_{n}\left(\omega_{\rho}\right)\right)^{2}-1}{2}\left\|u_{n}-\omega_{\rho}\right\|^{2}-\frac{1}{2}\left(\left\|u_{n}-\omega_{\rho}\right\|^{2}-\left\|u_{n}\right\|^{2}\right) \\
&+\frac{\left(g_{n}\left(\omega_{\rho}\right)\right)^{q}-1}{q} \int_{\mathbb{R}^{N}} f(x)\left|u_{n}-\omega_{\rho}\right|^{q} d x \\
&+\frac{1}{q}\left(\int_{\mathbb{R}^{N}} f(x)\left|u_{n}-\omega_{\rho}\right|^{q} d x-\int_{\mathbb{R}^{N}} f(x)\left|u_{n}\right|^{q} d x\right) \\
&+\frac{\left(g_{n}\left(\omega_{\rho}\right)\right)^{p}-1}{p} \int_{\mathbb{R}^{N}}\left|u_{n}-\omega_{\rho}\right|^{p} d x+\frac{1}{p}\left(\int_{\mathbb{R}^{N}}\left|u_{n}-\omega_{\rho}\right|^{p} d x-\int_{\mathbb{R}^{N}}\left|u_{n}\right|^{p} d x\right) .
\end{align*}
$$

Note that

$$
\lim _{\rho \rightarrow 0^{+}} \frac{g_{n}\left(\omega_{\rho}\right)-1}{\rho}=\lim _{\rho \rightarrow 0^{+}} \frac{g_{n}\left(0+\rho \frac{u}{\|u\|}\right)-g_{n}(0)}{\rho}=\left\langle\left(g_{n}\right)^{\prime}(0), \frac{u}{\|u\|}\right\rangle .
$$

If we divide the ends of (3.9) by $\rho$ and let $\rho \rightarrow 0^{+}$, we have

$$
\begin{aligned}
\frac{1}{n}[\mid & \left.\left.\left\langle\left(g_{n}\right)^{\prime}(0), \frac{u}{\|u\|}\right\rangle \right\rvert\,\left\|u_{n}\right\|+1\right] \\
\geq & -\left\langle\left(g_{n}\right)^{\prime}(0), \frac{u}{\|u\|}\right\rangle\left\|u_{n}\right\|^{2}-\int_{\mathbb{R}^{N}} \Delta u_{n} \Delta\left(-\frac{u}{\|u\|}\right)+\nabla u_{n} \nabla\left(-\frac{u}{\|u\|}\right)+u_{n}\left(-\frac{u}{\|u\|}\right) d x \\
& +\left\langle\left(g_{n}\right)^{\prime}(0), \frac{u}{\|u\|}\right\rangle \int_{\mathbb{R}^{N}} f(x)\left|u_{n}\right|^{q} d x+\int_{\mathbb{R}^{N}} f(x)\left|u_{n}\right|^{q-2} u_{n}\left(-\frac{u}{\|u\|}\right) d x \\
& +\left\langle\left(g_{n}\right)^{\prime}(0), \frac{u}{\|u\|}\right\rangle \int_{\mathbb{R}^{N}}\left|u_{n}\right|^{p} d x+\int_{\mathbb{R}^{N}}\left|u_{n}\right|^{p-2} u_{n}\left(-\frac{u}{\|u\|}\right) d x \\
= & -\left\langle\left(g_{n}\right)^{\prime}(0), \frac{u}{\|u\|}\right\rangle\left(\left\|u_{n}\right\|^{2}-\int_{\mathbb{R}^{N}} f(x)\left|u_{n}\right|^{q} d x-\int_{\mathbb{R}^{N}}\left|u_{n}\right|^{p} d x\right)-\frac{1}{\|u\|} \int_{\mathbb{R}^{N}}\left|u_{n}\right|^{p-2} u_{n} u d x \\
& +\frac{1}{\|u\| \|} \int_{\mathbb{R}^{N}}\left(\Delta u_{n} \Delta u+\nabla u_{n} \nabla u+u_{n} u\right) d x-\frac{1}{\|u\|} \int_{\mathbb{R}^{N}} f(x)\left|u_{n}\right|^{q-2} u_{n} u d x \\
= & -\left\langle\left(g_{n}\right)^{\prime}(0), \frac{u}{\|u\|}\right\rangle\left\langle I_{f}^{\prime}\left(u_{n}\right), u_{n}\right\rangle+\frac{1}{\|u\|}\left\langle I_{f}^{\prime}\left(u_{n}\right), u\right\rangle \\
= & \frac{1}{\|u\|}\left\langle I_{f}^{\prime}\left(u_{n}\right), u\right\rangle,
\end{aligned}
$$

that is,

$$
\frac{1}{n}\left[\left|\left\langle\left(g_{n}\right)^{\prime}(0), u\right\rangle\right|\left\|u_{n}\right\|+\|u\|\right] \geq\left\langle I_{f}^{\prime}\left(u_{n}\right), u\right\rangle
$$

By the boundedness of $\left\|u_{n}\right\|$ and Lemma 3.2, there exists $\hat{C}>0$ such that

$$
\frac{\hat{C}}{n} \geq\left\langle I_{f}^{\prime}\left(u_{n}\right), \frac{u}{\|u\|}\right\rangle
$$

Therefore, we have

$$
\left\|I_{f}^{\prime}\left(u_{n}\right)\right\|_{H^{-2}\left(\mathbb{R}^{N}\right)}=\sup _{u \in H^{2}\left(\mathbb{R}^{N}\right) \backslash\{0\}} \frac{\left\langle I_{f}^{\prime}\left(u_{n}\right), u\right\rangle}{\|u\|} \leq \frac{\hat{C}}{n}
$$

that is, $I_{f}^{\prime}\left(u_{n}\right)=o_{n}(1)$ as $n \rightarrow+\infty$. This completes the proof of $(\mathrm{i})$.
(ii) Similarly, by using Lemma 3.2, we can prove (ii). We will omit the details here.

Now, we establish the existence of minimum for $I_{f}$ on $\mathcal{N}_{f}^{+}$.
Theorem 3.4. Assume that (F) holds. If $|f|_{r_{q}} \in(0, \sigma)$, then the functional $I_{f}$ has a minimizer $u^{+}$in $\mathcal{N}_{f}^{+}$and it satisfies
(i) $I_{f}\left(u^{+}\right)=\alpha_{f}=\alpha_{f}^{+}$;
(ii) $u^{+}$is a solution of equation (1.1).

Proof. From Lemma 3.3, let $\left\{u_{n}\right\}$ be a $(P S)_{\alpha_{f}}$ sequence for $I_{f}$ on $\mathcal{N}_{f}$, i.e.,

$$
\begin{equation*}
I_{f}\left(u_{n}\right)=\alpha_{f}+o_{n}(1), \quad I_{f}^{\prime}\left(u_{n}\right)=o_{n}(1) \quad \text { in } H^{-2}\left(\mathbb{R}^{N}\right) \tag{3.10}
\end{equation*}
$$

Then it follows from Lemma 2.3 that $\left\{u_{n}\right\}$ is bounded in $H^{2}\left(\mathbb{R}^{N}\right)$. Hence, up to a subsequence, there exists $u^{+} \in H^{2}\left(\mathbb{R}^{N}\right)$ such that

$$
\left\{\begin{array}{l}
u_{n} \rightharpoonup u^{+} \quad \text { in } H^{2}\left(\mathbb{R}^{N}\right) ;  \tag{3.11}\\
u_{n} \rightarrow u^{+} \quad \text { in } L_{l o c}^{s}\left(\mathbb{R}^{N}\right)\left(2 \leq s<2_{*}\right) ; \\
u_{n}(x) \rightarrow u^{+}(x) \quad \text { a.e. in } \mathbb{R}^{N} .
\end{array}\right.
$$

By (F), Hölder inequality and (3.11), we can infer that

$$
\begin{equation*}
\int_{\mathbb{R}^{N}} f(x)\left|u_{n}\right|^{q} d x=\int_{\mathbb{R}^{N}} f(x)\left|u^{+}\right|^{q} d x+o_{n}(1) \quad \text { as } n \rightarrow \infty . \tag{3.12}
\end{equation*}
$$

In fact, for any $\epsilon>0$, there exists $M$ sufficiently large such that

$$
\left(\int_{|x|>M}|f(x)|^{r_{q}} d x\right)^{\frac{1}{r_{q}}}<\epsilon
$$

And from $\left\{u_{n}\right\} \subset \mathcal{N}_{f}$ in $H^{2}\left(\mathbb{R}^{N}\right)$ is bounded, we obtain that $\left(\int_{\mathbb{R}^{N}}\left|u_{n}-u^{+}\right|^{r} d x\right)^{\frac{q}{r}}$ is bounded. Therefore, we have

$$
\begin{aligned}
\int_{\mathbb{R}^{N}}\left|f(x)\left(\left|u_{n}\right|^{q}-\left|u^{+}\right|{ }^{q}\right)\right| d x \leq & \int_{\mathbb{R}^{N}} f(x)\left|u_{n}-u^{+}\right|^{q} d x \\
= & \int_{|x| \leq M} f(x)\left|u_{n}-u^{+}\right|{ }^{q} d x+\int_{|x|>M} f(x)\left|u_{n}-u^{+}\right|^{q} d x \\
\leq & \left(\int_{|x| \leq M}|f(x)|^{r_{q}} d x\right)^{\frac{1}{r_{q}}}\left(\int_{|x| \leq M}\left|u_{n}-u^{+}\right|^{r} d x\right)^{\frac{q}{r}} \\
& +\left(\int_{|x|>M}|f(x)|^{r_{q}} d x\right)^{\frac{1}{r_{q}}}\left(\int_{|x|>M}\left|u_{n}-u^{+}\right|^{r} d x\right)^{\frac{q}{r}} \\
\rightarrow & 0 \quad \text { as } n \rightarrow \infty .
\end{aligned}
$$

First, we can claim that $u^{+}$is a nontrivial solution of (1.1). Indeed, by (3.10) and (3.11), it is easy to see that $u^{+}$is a solution of (1.1). Next we show that $u^{+}$is nontrivial. From $u_{n} \in \mathcal{N}_{f}$, we have that

$$
\begin{equation*}
I_{f}\left(u_{n}\right)=\left(\frac{1}{2}-\frac{1}{p}\right)\left\|u_{n}\right\|^{2}-\left(\frac{1}{q}-\frac{1}{p}\right) \int_{\mathbb{R}^{N}} f(x)\left|u_{n}\right|^{q} d x . \tag{3.13}
\end{equation*}
$$

Let $n \rightarrow \infty$ in (3.13), we can get

$$
\alpha_{f} \geq-\frac{p-q}{p q} \int_{\mathbb{R}^{N}} f(x)\left|u^{+}\right|^{q} d x
$$

In view of Lemma 2.6, we have $0>\alpha_{f}^{+} \geq \alpha_{f}$, which implies $\int_{\mathbb{R}^{N}} f(x)\left|u^{+}\right|^{q} d x>0$. Thus, $u^{+}$is a nontrivial solution of (1.1). Now we prove that $u_{n} \rightarrow u^{+}$strongly in $H^{2}\left(\mathbb{R}^{N}\right)$ and $I_{f}\left(u^{+}\right)=\alpha$. In fact, by $u_{n}, u \in \mathcal{N}_{f}$, (3.12) and weak lower semicontinuity of norm, we have

$$
\begin{aligned}
\alpha_{f} & \leq I_{f}\left(u^{+}\right)=\left(\frac{1}{2}-\frac{1}{p}\right)\left\|u^{+}\right\|^{2}-\left.\left(\frac{1}{q}-\frac{1}{p}\right) \int_{\mathbb{R}^{N}} f(x)\left|u^{+}\right|\right|^{q} d x \\
& \leq \lim _{n \rightarrow \infty}\left(\left(\frac{1}{2}-\frac{1}{p}\right)\left\|u_{n}\right\|^{2}-\left(\frac{1}{q}-\frac{1}{p}\right) \int_{\mathbb{R}^{N}} f(x)\left|u_{n}\right|^{q} d x\right) \\
& =\lim _{n \rightarrow \infty} I_{f}\left(u_{n}\right)=\alpha_{f},
\end{aligned}
$$

which implies that $I_{f}\left(u^{+}\right)=\alpha_{f}$ and $\lim _{n \rightarrow \infty}\left\|u_{n}\right\|^{2}=\left\|u^{+}\right\|^{2}$. Noting that $u_{n} \rightharpoonup u^{+}$in $H^{2}\left(\mathbb{R}^{N}\right)$, so $u_{n} \rightarrow u^{+}$strongly in $H^{2}\left(\mathbb{R}^{N}\right)$. Furthermore, we have $u^{+} \in \mathcal{N}_{f}^{+}$. On the contrary, if $u^{+} \in \mathcal{N}_{f}^{-}$, then by Lemma 2.5 (i), there are unique $t^{+}$and $t^{-}$such that $t^{+} u^{+} \in \mathcal{N}_{f}^{+}$and $t^{-} u^{+} \in \mathcal{N}_{f}^{-}$. In particular, we have $t^{+}<t^{-}=1$ and so $I_{f}\left(t^{+} u^{+}\right)<I_{f}\left(t^{-} u^{+}\right)=I_{f}\left(u^{+}\right)=\alpha_{f}$, which is a contradiction. By Lemma 2.4 we may assume that $u^{+}$is a solution of (1.1). This completes the proof.

In order to obtain the existence of the second local minimum, we consider the following minimization problem:

$$
S_{0}=\inf \left\{I_{0}(u) \mid u \in H^{2}\left(\mathbb{R}^{N}\right) \backslash\{0\}, I_{0}^{\prime}(u)=0\right\},
$$

where

$$
I_{0}(u)=\frac{1}{2}\|u\|^{2}-\frac{1}{p} \int_{\mathbb{R}^{N}}|u|^{p} d x .
$$

From $[17,21]$, we know that $S_{0}$ is achieved at $u_{0} \in H^{2}\left(\mathbb{R}^{N}\right)$. Moreover,

$$
S_{0}=I_{0}\left(u_{0}\right)=\sup _{t \geq 0} I_{0}\left(t u_{0}\right) .
$$

Then, we have the following lemma.
Lemma 3.5. If $|f|_{r_{g}} \in(0, \sigma)$, then $\alpha_{f}^{-}<\alpha_{f}+S_{0}$.
Proof. From Lemma 2.5 (iii), $\mathcal{N}_{f}^{-}$disconnects $H^{2}\left(\mathbb{R}^{N}\right) \backslash\{0\}$ in exactly two components:

$$
\begin{aligned}
& \Lambda_{1}=\left\{u \left\lvert\, \frac{1}{\|u\|} t^{-}\left(\frac{u}{\|u\|}\right)>1\right.\right\}, \\
& \Lambda_{2}=\left\{u \left\lvert\, \frac{1}{\|u\|} t^{-}\left(\frac{u}{\|u\|}\right)<1\right.\right\},
\end{aligned}
$$

and $\mathcal{N}_{f}^{+} \subset \Lambda_{1}$. Moreover, there exists $t_{1}$ such that $u^{+}+t_{1} u_{0} \in \Lambda_{2}$. Indeed, denote $t_{0}=$ $t^{-}\left(\left(u^{+}+t u_{0}\right) /\left\|u^{+}+t u_{0}\right\|\right)$. Since

$$
t^{-}\left(\frac{u^{+}+t u_{0}}{\left\|u^{+}+t u_{0}\right\|}\right)\left(\frac{u^{+}+t u_{0}}{\left\|u^{+}+t u_{0}\right\|}\right) \in \mathcal{N}_{f}^{-},
$$

we have

$$
0 \leq \frac{t_{0}^{q} \int_{R^{N}} f(x)\left|u^{+}+t u_{0}\right|^{q} d x}{\left\|u^{+}+t u_{0}\right\|^{q}}=t_{0}^{2}-\frac{t_{0}^{p} \int_{R^{N}}\left|u^{+}+t u_{0}\right|^{p} d x}{\left\|u^{+}+t u_{0}\right\|^{p}} .
$$

Thus

$$
t_{0} \leq\left[\frac{\left\|u^{+} / t+u_{0}\right\|}{\left(\int_{R^{N}}\left|u^{+} / t+u_{0}\right|^{p}\right)^{1 / p}}\right]^{p /(p-2)} \rightarrow\left\|u_{0}\right\| \quad \text { as } t \rightarrow \infty .
$$

Therefore, there exists $t_{2}>0$ such that $t_{0}<l\left\|u_{0}\right\|$, for some $l>1$ and $t \geq t_{2}$. Set $t_{1}>t_{2}+l$, then

$$
\begin{aligned}
\left(t^{-}\left(\frac{u^{+}+t_{1} u_{0}}{\left\|u^{+}+t_{1} u_{0}\right\|}\right)\right)^{2} & <l^{2}\left\|u_{0}\right\|^{2} \\
& \leq\left\|u^{+}\right\|^{2}+t_{1}^{2}\left\|u_{0}\right\|^{2}+2 t_{1} \int_{\mathbb{R}^{N}}\left(\Delta u^{+} \Delta u_{0}+\nabla u^{+} \nabla u_{0}+u^{+} u_{0}\right) d x \\
& =\left\|u^{+}+t_{1} u_{0}\right\|^{2}
\end{aligned}
$$

that is, $u^{+}+t_{1} u_{0} \in \Lambda_{2}$. So there exists $k \in(0,1)$ such that $u^{+}+k t_{1} u_{0} \in \mathcal{N}_{f}^{-}$. Furthermore, we have

$$
\begin{aligned}
\alpha_{f}^{-} & \leq I_{f}\left(u^{+}+k t_{1} u_{0}\right) \\
& =\frac{1}{2}\left\|u^{+}+k t_{1} u_{0}\right\|^{2}-\frac{1}{q} \int_{\mathbb{R}^{N}} f(x)\left|u^{+}+k t_{1} u_{0}\right|^{q} d x-\frac{1}{p} \int_{\mathbb{R}^{N}}\left|u^{+}+k t_{1} u_{0}\right|^{p} d x \\
& <I_{f}\left(u^{+}\right)+\frac{1}{2}\left\|k t_{1} u_{0}\right\|^{2}-\frac{1}{p} \int_{\mathbb{R}^{N}}\left|k t_{1} u_{0}\right|^{p} d x \\
& =I_{f}\left(u^{+}\right)+I_{0}\left(k t_{1} u_{0}\right) \\
& \leq \alpha_{f}+I_{0}\left(u_{0}\right) \\
& =\alpha_{f}+S_{0} .
\end{aligned}
$$

This completes the proof.
Next, we establish the existence of minimum for $I_{f}$ on $\mathcal{N}_{f}^{-}$.
Theorem 3.6. Assume that (F) holds. If $|f|_{r_{q}} \in(0, \sigma)$, then the functional $I_{f}$ has a minimizer $u^{-}$in $\mathcal{N}_{f}^{-}$and it satisfies
(i) $I_{f}\left(u^{-}\right)=\alpha_{f}^{-}$;
(ii) $u^{-}$is a solution of equation (1.1).

Proof. From Lemma 3.3, let $\left\{u_{n}\right\}$ be a $(P S)_{\alpha_{f}^{-}}$sequence for $I_{f}$ on $\mathcal{N}_{f}^{-}$, i.e.,

$$
\begin{equation*}
I_{f}\left(u_{n}\right)=\alpha_{f}^{-}+o_{n}(1), \quad I_{f}^{\prime}\left(u_{n}\right)=o_{n}(1) \quad \text { in } H^{-2}\left(\mathbb{R}^{N}\right) . \tag{3.14}
\end{equation*}
$$

From Lemma 2.3 we have $\left\{u_{n}\right\}$ is bounded in $H^{2}\left(\mathbb{R}^{N}\right)$. Hence, up to a subsequence, there exists $u^{-} \in H^{2}\left(\mathbb{R}^{N}\right)$ such that

$$
\left\{\begin{array}{l}
u_{n} \rightharpoonup u^{-} \quad \text { in } H^{2}\left(\mathbb{R}^{N}\right) ;  \tag{3.15}\\
u_{n} \rightarrow u^{-} \quad \text { in } L_{\text {loc }}^{s}\left(\mathbb{R}^{N}\right)\left(2 \leq s<2_{*}\right) ; \\
u_{n}(x) \rightarrow u^{-}(x) \quad \text { a.e. in } \mathbb{R}^{N} .
\end{array}\right.
$$

From (3.14) and (3.15), we have $\left\langle I_{f}^{\prime}\left(u^{-}\right), v\right\rangle=0, \forall v \in H^{2}\left(\mathbb{R}^{N}\right)$, that is, $u^{-}$is a weak solution of (1.1) and $u^{-} \in \mathcal{N}_{f}$. Let $v_{n}=u_{n}-u^{-}$. Then

$$
\left\{\begin{array}{l}
v_{n} \rightarrow 0 \quad \text { in } H^{2}\left(\mathbb{R}^{N}\right) ;  \tag{3.16}\\
v_{n} \rightarrow 0 \quad \text { in } L_{l o c}^{s}\left(\mathbb{R}^{N}\right)\left(2 \leq s<2_{*}\right) ; \\
v_{n}(x) \rightarrow 0 \quad \text { a.e. in } \mathbb{R}^{N} .
\end{array}\right.
$$

Now we prove that $u_{n} \rightarrow u^{-}$strongly in $H^{2}\left(\mathbb{R}^{N}\right)$, that is, $v_{n} \rightarrow 0$ strongly in $H^{2}\left(\mathbb{R}^{N}\right)$. Arguing by contradiction, we assume that there is $c>0$ such that $\left\|v_{n}\right\| \geq c>0$. By the Brézis-Lieb theorem [3],

$$
\begin{align*}
I_{f}\left(u_{n}\right) & =\frac{1}{2}\left\|u_{n}\right\|^{2}-\frac{1}{q} \int_{\mathbb{R}^{N}} f(x)\left|u_{n}\right|^{q} d x-\frac{1}{p} \int_{\mathbb{R}^{N}}\left|u_{n}\right|^{p} d x \\
& =I_{f}\left(u^{-}\right)+\frac{1}{2}\left\|v_{n}\right\|^{2}-\frac{1}{q} \int_{\mathbb{R}^{N}} f(x)\left|v_{n}\right|^{q} d x-\frac{1}{p} \int_{\mathbb{R}^{N}}\left|v_{n}\right|^{p} d x+o_{n}(1)  \tag{3.17}\\
& =I_{f}\left(u^{-}\right)+\frac{1}{2}\left\|v_{n}\right\|^{2}-\frac{1}{p} \int_{\mathbb{R}^{N}}\left|v_{n}\right|^{p} d x+o_{n}(1),
\end{align*}
$$

where $\int_{\mathbb{R}^{N}} f(x)\left|v_{n}\right|^{q} d x \rightarrow 0$ as $n \rightarrow \infty$. In fact, for any $\epsilon>0$, there exists $M$ sufficiently large such that

$$
\left(\int_{|x|>M}|f(x)|^{r_{q}} d x\right)^{\frac{1}{r_{q}}}<\epsilon .
$$

By ( F ), Hölder's inequality and (3.16), we have

$$
\begin{aligned}
\int_{\mathbb{R}^{N}} f(x)\left|v_{n}\right|^{q} d x= & \int_{|x| \leq M} f(x)\left|v_{n}\right|^{q} d x+\int_{|x|>M} f(x)\left|v_{n}\right|^{q} d x \\
\leq & \left(\int_{|x| \leq M}|f(x)|^{r_{q}} d x\right)^{\frac{1}{r_{q}}}\left(\int_{|x| \leq M}\left|v_{n}\right|^{r} d x\right)^{\frac{q}{r}} \\
& +\left(\int_{|x|>M}|f(x)|^{r} d x\right)^{\frac{q}{r_{q}}}\left(\int_{|x|>M}\left|v_{n}\right|^{r} d x\right)^{\frac{q}{r}} \\
\rightarrow & \text { as } n \rightarrow \infty .
\end{aligned}
$$

Moreover,

$$
\begin{align*}
o_{n}(1) & =\left\langle I_{f}^{\prime}\left(u_{n}\right), u_{n}\right\rangle=\left\|u_{n}\right\|^{2}-\int_{\mathbb{R}^{N}} f(x)\left|u_{n}\right|^{q} d x-\int_{\mathbb{R}^{N}}\left|u_{n}\right|^{p} d x \\
& =\left\langle I_{f}^{\prime}\left(u^{-}\right), u^{-}\right\rangle+\left\|v_{n}\right\|^{2}-\int_{\mathbb{R}^{N}} f(x)\left|v_{n}\right|^{q} d x-\int_{\mathbb{R}^{N}}\left|v_{n}\right|^{p} d x+o_{n}(1)  \tag{3.18}\\
& =\left\|v_{n}\right\|^{2}-\int_{\mathbb{R}^{N}}\left|v_{n}\right|^{p} d x+o_{n}(1) .
\end{align*}
$$

Combining (3.17) and (3.18), we obtain

$$
\left\|v_{n}\right\|^{2}-\int_{\mathbb{R}^{N}}\left|v_{n}\right|^{p} d x=o_{n}(1), \quad I_{f}\left(u_{n}\right) \geq \alpha_{f}+\frac{1}{2}\left\|v_{n}\right\|^{2}-\frac{1}{p} \int_{\mathbb{R}^{N}}\left|v_{n}\right|^{p} d x+o_{n}(1) .
$$

Since $\left\|v_{n}\right\| \geq c>0$, we can get a sequence $k_{n}, k_{n}>0, k_{n} \rightarrow 1$ as $n \rightarrow \infty$, such that $s_{n}=k_{n} v_{n}$ satisfying $\left\|s_{n}\right\|^{2}-\int_{\mathbb{R}^{N}}\left|s_{n}\right|^{p} d x=0$. Thus

$$
I_{f}\left(u_{n}\right) \geq \alpha_{f}+\frac{1}{2}\left\|s_{n}\right\|^{2}-\frac{1}{p} \int_{\mathbb{R}^{N}}\left|s_{n}\right|^{p} d x+o_{n}(1) \geq \alpha_{f}+S_{0}+o_{n}(1),
$$

that is, $\alpha_{f}^{-} \geq \alpha_{f}+S_{0}$, contradicting Lemma 3.5. Hence $u_{n} \rightarrow u^{-}$strongly in $H^{2}\left(\mathbb{R}^{N}\right)$. This implies

$$
I_{f}\left(u_{n}\right) \rightarrow I_{f}\left(u^{-}\right)=\alpha_{f}^{-} \quad \text { as } n \rightarrow \infty .
$$

Furthermore, from Lemma 2.2, $\mathcal{N}_{f}^{-}$is closed set and bounded away from 0 . We have $u^{-} \in \mathcal{N}_{f}^{-}$ and $u^{-}$is nontrivial. By Lemma 2.4 we may assume that $u^{-}$is a solution of (1.1). This completes the proof.

Proof of Theorem 1.1. By Theorems 3.4 and 3.6, for (1.1) there exist two solutions $u^{+}$and $u^{-}$ such that $u^{+} \in \mathcal{N}_{f}^{+}, u^{-} \in \mathcal{N}_{f}^{-}$. Since $\mathcal{N}_{f}^{+} \cap \mathcal{N}_{f}^{-}=\varnothing$, this implies that $u^{+}$and $u^{-}$are different. Moreover, $u^{+}$is the ground state solution. It completes the proof of Theorem 1.1.

## Acknowledgements

The authors thank the anonymous referees for their valuable suggestions and comments.

## References

[1] A. Ambrosetti, H. Brézis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, J. Funct. Anal. 122(1994), No. 2, 519-543. https: //doi.org/10.1006/jfan.1994.1078; MR1276168; Zbl 0805.35028
[2] Adimurthi, F. Pacella, S. L. Yadava, On the number of positive solutions of some semilinear Dirichlet problems in a ball, Differential Integral Equations 10(1997), No. 6, 11571170. MR1608057; Zbl 0940.35069
[3] H. Brézis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Am. Math. Soc. 88(1983), No. 3, 486-490. https://doi . org/10. 2307/2044999; MR0699419; Zbl 0526.46037
[4] K. J. Brown, Y. P. Zhang, The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function, J. Differential Equations 193(2003), No. 2, 481-499. https: //doi.org/10.1016/S0022-0396(03)00121-9; MR1998965; Zbl 1074.35032
[5] K. J. Chen, Combined effects of concave and convex nonlinearities in elliptic equation on $\mathbb{R}^{N}$, J. Math. Anal. Appl. 355(2009), No. 2, 767-777. https://doi.org/10.1016/j . jmaa. 2009.02.029; MR2521751; Zbl 1185.35091
[6] L. Damascelli, M. Grossi, F. Pacella, Qualitative properties of positive solutions of semilinear elliptic equations in symmetric domains via the maximum principle, Ann. Inst. Henri Poincaré Anal. Non Linéaire 16(1999), No. 5, 631-652. https://doi.org/10. 1016/S0294-1449(99) 80030-4; MR1712564; Zbl 0935.35049
[7] I. Ekeland, On the variational principle, J. Math. Anal. Appl. 47(1974), 324-353. https: //doi.org/10.1016/0022-247X (74)90025-0; MR0346619; Zbl 0286.49015
[8] J. Giacomoni, S. Prashanth, K. Sreenadh, A global multiplicity result for N-Laplacian with critical nonlinearity of concave-convex type, J. Differential Equations 232(2007), No. 2, 544-572. https://doi.org/10.1016/j.jde.2006.09.012; MR2286391; Zbl 1165.35022
[9] T. S. Hsu, H. L. Lin, Multiple positive solutions for semilinear elliptic equations in $\mathbb{R}^{N}$ involving concave-convex nonlinearities and sign-changing weight functions, Abstr. Appl. Anal. 2010(2010), Art. ID 658397, 21 pp. https://doi.org/10.1155/2010/658397; MR2669082; Zbl 1387.35309
[10] P. Korman, On uniqueness of positive solutions for a class of semilinear equations, Discrete Contin. Dyn. Syst. 8(2002), No. 4, 865-871. https://doi.org/10.3934/dcds.2002.8. 865; MR1920648; Zbl 1090.35082
[11] S. B. Liu, Z. H. Zhao, Solutions for fourth order elliptic equations on $\mathbb{R}^{N}$ involving $u \Delta\left(u^{2}\right)$ and sign-changing potentials, J. Differential Equations 267(2019), No. 3, 1581-1599. https://doi.org/10.1016/j.jde.2019.02.017; MR3945610; Zbl 1418.35128
[12] W. M. Ni, I. Takagi, On the shape of least-energy solutions to a semilinear Neumann problem, Commun. Pure Appl. Math. 44(1991), No. 7, 819-851. https://doi.org/10.1002/ сра.3160440705; MR1115095; Zbl 0754.35042
[13] T. C. Ouyang, J. P. Shi, Exact multiplicity of positive solutions for a class of semilinear problem. II, J. Differential Equations 158(1999), No. 1, 94-151. https://doi.org/10.1006/ jdeq.1999.3644; MR1721723; Zbl 0947.35067
[14] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, CBMS Regional Conference Series in Mathematics, Vol. 65, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986. MR0845785; Zbl 0609.58002
[15] K. Silva, A. Macedo, Local minimizers over the Nehari manifold for a class of concaveconvex problems with sign changing nonlinearity, J. Differential Equations 265(2018), No. 5, 1894-1921. https://doi.org/10.1016/j.jde.2018.04.018; MR3800105; Zbl 1392.35172
[16] G. Tarantello, On nonhomogeneous elliptic equations involving critical Sobolev exponent, Ann. Inst. Henri Poincaré Anal. Non Linéaire 9(1992), No. 3, 281-304. https: //doi.org/10.1016/S0294-1449 (16) 30238-4; MR1168304; Zbl 0785.35046
[17] M. Willem, Minimax theorems, Progress in Nonlinear Differential Equations and their Applications, Vol. 24, Birkhäuser, Boston, 1996. https://doi.org/10.1007/978-1-4612-4146-1; MR1400007; Zbl 0856.49001
[18] T. F. Wu, Multiple positive solutions for a class of concave-convex elliptic problems in $\mathbb{R}^{N}$ involving sign-changing weight, J. Funct. Anal. 258(2010), No. 1, 99-131. https:// doi.org/10.1016/j.jfa.2009.08.005; MR2557956; Zbl 1182.35119
[19] F. L. Wang, M. Avci, Y. K. An, Existence of solutions for fourth order elliptic equations of Kirchhoff type, J. Math. Anal. Appl. 409(2014), No. 1, 140-146. https://doi.org/10. 1016/j.jmaa.2013.07.003; MR3095024; Zbl 1311.35093
[20] W. H. Xie, H. B. Chen, Multiple positive solutions for the critical Kirchhoff type problems involving sign-changing weight functions, J. Math. Anal. Appl. 479(2019), No. 1, 135-161. https://doi.org/10.1016/j.jmaa.2019.06.020; MR3987029; Zbl 1425.35045
[21] W. Zou, M. Schechter, Critical point theory and its applications, Springer, New York, 2006. https://doi.org/10.1007/0-387-32968-4 MR2232879; Zbl 1125.58004

# Hopf bifurcation in a reaction-diffusive-advection two-species competition model with one delay 

Qiong Meng ${ }^{\boxtimes 1}$, Guirong Liu ${ }^{1}$ and Zhen Jin ${ }^{2}$<br>${ }^{1}$ School of Mathematical Sciences, Shanxi University, Taiyuan 030006, Shanxi, China<br>${ }^{2}$ Complex Systems Research Center, Shanxi University, Taiyuan 030006, Shanxi, China

Received 26 October 2020, appeared 19 September 2021
Communicated by Péter L. Simon


#### Abstract

In this paper, we investigate a reaction-diffusive-advection two-species competition model with one delay and Dirichlet boundary conditions. The existence and multiplicity of spatially non-homogeneous steady-state solutions are obtained. The stability of spatially nonhomogeneous steady-state solutions and the existence of Hopf bifurcation with the changes of the time delay are obtained by analyzing the distribution of eigenvalues of the infinitesimal generator associated with the linearized system. By the normal form theory and the center manifold reduction, the stability and bifurcation direction of Hopf bifurcating periodic orbits are derived. Finally, numerical simulations are given to illustrate the theoretical results.


Keywords: reaction-diffusive, advection, delay, Hopf bifurcation, spatial heterogeneity.
2020 Mathematics Subject Classification: 34K18, 35K57, 92D25.

## 1 Introduction

In this paper, we consider a two-species competition model in a reaction-diffusive-advection with one delay

$$
\left\{\begin{array}{l}
\frac{\partial u(x, t)}{\partial t}=\nabla \cdot\left[d_{1} \nabla u(x, t)-a_{1} u(x, t) \nabla m\right]+u(x, t)\left[m(x)-b_{1} u(x, t-r)-c_{1} v(x, t-r)\right],  \tag{1.1}\\
\frac{v v(x, t)}{\partial t}=\nabla \cdot\left[d_{2} \nabla v(x, t)-a_{2} v(x, t) \nabla m\right]+v(x, t)\left[m(x)-b_{2} u(x, t-r)-c_{2} v(x, t-r)\right],
\end{array}\right.
$$

where $u(x, t), v(x, t)$ represents the population density at location $x \in \Omega$ and time $t$, time delay $r>0$ represents the maturation time, and $\Omega$ is a bounded domain in $\mathbb{R}^{k}(1 \leq k \leq 3)$ in (1.1) with a smooth boundary $\partial \Omega$. $a_{i}, b_{i}, c_{i}, d_{i}>0(i=1,2)$.

In (1.1), we assume that both species have the same per-capita growth rates at place $x \in \Omega$, denoted by $m(x)$. This scenario can occur if the two species are competing for the same resources. To reflect the heterogeneity of environment, we assume that $m(x)$ is a nonconstant function. In some sense, $m(x)$ can reflect the quality and quantity of resources available at the location $x$, where the favorable region $\{x \in \Omega: m(x)>0\}$ acts as a source and the unfavorable part $\{x \in \Omega: m(x)<0\}$ is a sink region, see [26]. When $m(x) \equiv 1$, see [15,18].

[^34]Under our assumptions in (1.1), the dispersal of the two competitors can be described in terms of their fluxes

$$
J_{u}=-d_{1} \nabla u+a_{1} u \nabla m, \quad J_{v}=-d_{2} \nabla v+a_{2} v \nabla m,
$$

respectively, where $d_{1} \nabla u$ and $d_{2} \nabla v$ account for random diffusion, and $a_{1} u \nabla m$ and $a_{2} v \nabla m$ represent movement upward along the environmental gradient. The two non-negative constants $a_{1}$ and $a_{2}$ measure the tendency of the two populations to move up along the gradient of $m(x)$, and $d_{1}$ and $d_{2}$ represent the random diffusion rates of two species, respectively. See [1,2,4-8,10, 11, 13, 17, 20, 22-29].

When $b_{1}=b_{2}=c_{1}=c_{2}=1, r=0$ in (1.1), Chen, Hambrock and Lou [6] investigated the following model

$$
\left\{\begin{array}{l}
\frac{\partial u(x, t)}{\partial t}=\nabla \cdot\left[d_{1} \nabla u(x, t)-a_{1} u(x, t) \nabla m\right]+u(x, t)[m(x)-u(x, t)-v(x, t)]  \tag{1.2}\\
\frac{\partial v(x, t)}{\partial t}=\nabla \cdot\left[d_{2} \nabla v(x, t)-a_{2} v(x, t) \nabla m\right]+v(x, t)[m(x)-u(x, t)-v(x, t)]
\end{array}\right.
$$

They showed that at least two scenarios can occur: if only one species has a strong tendency to move upward the environmental gradients, the two species can coexist since one species mainly pursues resources at places of locally most favorable environments while the other relies on resources from other parts of the habitat; if both species have such strong biased movements, it can lead to overcrowding of the whole population at places of locally most favorable environments, which causes the extinction of the species with stronger biased movement. These results provided a new mechanism for the coexistence of competing species, and they also implied that selection is against excessive advection along environmental gradients, and an intermediate biased movement rate may evolve.

When $v=0$ in (1.1), Chen, Lou and Wei [8] investigated the following model,

$$
\left\{\begin{array}{l}
\frac{\partial u(x, t)}{\partial t}=\nabla \cdot\left[d \nabla u-a_{1} u \nabla m\right]+u(x, t)[m(x)-u(x, t-r)],  \tag{1.3}\\
u(x, t)=0 .
\end{array}\right.
$$

They investigated a reaction-diffusion-advection model with time delay effect. The stability and instability of the spatially nonhomogeneous positive steady state were investigated when the given parameter of the model is near the principle eigenvalue of an elliptic operator. Their results implied that time delay can make the spatially nonhomogeneous positive steady state unstable for a reaction-diffusion-advection model, and the model can exhibit oscillatory pattern through Hopf bifurcation. The effect of advection on Hopf bifurcation values was also considered, and their results suggested that Hopf bifurcation is more likely to occur when the advection rate increases. See $[3,9,12,14-16,18,19,21,30-34]$.

When $d_{1}=d_{2}=d, a_{2}=a_{1}$ in (1.1), we study the following model with homogeneous Dirichlet boundary and initial value conditions

$$
\left\{\begin{array}{l}
\frac{\partial u(x, t)}{\partial t}=\nabla \cdot\left[d \nabla u-a_{1} u \nabla m\right]+u(x, t)\left[m(x)-b_{1} u(x, t-r)-c_{1} v(x, t-r)\right],  \tag{1.4}\\
\frac{\partial v(x, t)}{\partial t}=\nabla \cdot\left[d \nabla v-a_{1} v \nabla m\right]+v(x, t)\left[m(x)-b_{2} u(x, t-r)-c_{2} v(x, t-r)\right], \\
x \in \Omega, \quad t>0, \\
u(x, t)=v(x, t)=0, \quad x \in \partial \Omega, \quad t \geq 0, \\
u(x, t)=\varphi_{1}(x, t) \geq 0, \quad v(x, t)=\varphi_{2}(x, t) \geq 0, \quad(x, t) \in \bar{\Omega} \times[-r, 0],
\end{array}\right.
$$

with the initial value functions

$$
\varphi_{i}(x, \cdot) \in C\left([-r, 0], \mathbb{R}_{0}^{+}\right) \quad(x \in \bar{\Omega}), \quad \varphi_{i}(\cdot, t) \in H_{0}^{1}(\bar{\Omega}) \quad(t \in[-r, 0]), \quad i=1,2 .
$$

In this paper, we mainly investigate whether time delay $r$ can induce Hopf bifurcation for reaction-diffusion-advection model (1.4).

As in [2,8], Let $\widetilde{u}=e^{\left(-a_{1} / d\right) m(x)} u, \widetilde{v}=e^{\left(-a_{1} / d\right) m(x)} v, \tilde{t}=t d$, dropping the tilde sign, and denoting $\lambda=1 / d, a=a_{1} / d, \tau=d r$, system (1.4) can be transformed as follows:

$$
\left\{\begin{array}{l}
\frac{\partial u}{\partial t}=e^{-a m(x)} \nabla \cdot\left[e^{a m(x)} \nabla u\right]+\lambda u\left[m(x)-b_{1} e^{a m(x)} u(x, t-\tau)-c_{1} e^{a m(x)} v(x, t-\tau)\right],  \tag{1.5}\\
\frac{\partial v}{\partial t}=e^{-a m(x)} \nabla \cdot\left[e^{a m(x)} \nabla v\right]+\lambda v\left[m(x)-b_{2} e^{a m(x)} u(x, t-\tau)-c_{2} e^{a m(x)} v(x, t-\tau)\right], \\
x \in \Omega, \quad t>0, \\
u(x, t)=v(x, t)=0, \quad x \in \partial \Omega, \quad t \geq 0, \\
u(x, t)=\varphi_{1}(x, t) \geq 0, \quad v(x, t)=\varphi_{2}(x, t) \geq 0, \quad(x, t) \in \bar{\Omega} \times[-\tau, 0] .
\end{array}\right.
$$

Throughout the paper, unless otherwise specified, $m(x)$ satisfies the following assumption
(H) $m \in C^{2}(\bar{\Omega})$, and $\max _{x \in \bar{\Omega}} m(x)>0$.

The following eigenvalue problem

$$
\left\{\begin{array}{l}
-e^{-a m(x)} \nabla \cdot\left[e^{a m(x)} \nabla u\right]=-\Delta u-a \nabla m \cdot \nabla u=\lambda m(x) u, \quad x \in \Omega,  \tag{1.6}\\
u(x)=0, \quad x \in \partial \Omega,
\end{array}\right.
$$

is crucial to derive our main results. It follows from $[2,8,26]$ that, under assumption (H), (1.6) has a unique positive principal eigenvalue $\lambda_{*}$ admitting a strictly positive eigenfunction $\varphi \in C_{0}^{1+\delta}(\bar{\Omega})$ for some $\delta \in(0,1)$ and $\int_{\Omega} \varphi^{2} d x=1$.

The rest of the paper is organized as follows. In Section 2, we study the existence of positive steady state solutions of (1.5). In Section 3, we focus on the eigenvalue problem of the linearized system of the steady-state solution of (1.5). In Section 4, we study the stability and Hopf bifurcation of the spatially nonhomogeneous positive steady state of (1.5). In Section 5, we derive an explicit formula, which can be used to determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic orbits. In Section 6, we give some numerical simulations are illustrated to support our analytical results.

Throughout the paper, we also denote the spaces $X=H^{2}(\Omega) \cap H_{0}^{1}(\Omega), Y=L^{2}(\Omega)$. Moreover, we denote the complexification of a linear space $Z$ to be $Z_{C}=Z \oplus i Z=\left\{x_{1}+\right.$ $\left.i x_{2} \mid x_{1}, x_{2} \in Z\right\}$, the domain of a linear operator $L$ by $\mathcal{D}(L)$, the kernel of $L$ by $\mathcal{N}(L)$, and the range of $L$ by $\mathcal{R}(L)$. For Hilbert space $Y_{\mathrm{C}}$, we use the standard inner product $\langle u, v\rangle=$ $\int_{\Omega} \bar{u}(x)^{T} v(x) d x, u, v \in Y_{\mathrm{C}}^{2}$.

## 2 Existence of positive steady state solutions

Denote

$$
L:=\nabla \cdot\left[e^{a m(x)} \nabla\right]+\lambda_{*} e^{a m(x)} m(x),
$$

where $\lambda_{*}$ is a unique positive principal eigenvalue of problem (1.6) admitting a strictly positive eigenfunction $\varphi \in C_{0}^{1+\delta}(\bar{\Omega})$ for some $\delta \in(0,1)$ and $\int_{\Omega} \varphi^{2} d x=1$.

Now, we have the following decompositions:

$$
\begin{gathered}
X=\mathcal{N}(L) \oplus X_{1}, \quad Y=\mathcal{N}(L) \oplus Y_{1}, \quad \mathcal{N}(L)=\operatorname{span}\{\varphi\}, \\
X_{1}=\left\{y \in X: \int_{\Omega} \varphi(x) y(x) d x=0\right\}, \quad Y_{1}=\mathcal{R}(L)=\left\{y \in Y: \int_{\Omega} \varphi(x) y(x) d x=0\right\} .
\end{gathered}
$$

Clearly, the operator $L: X \rightarrow Y$ is Fredholm with index zero. $\left.L\right|_{X_{1}}: X_{1} \rightarrow Y_{1}$ is invertible and has a bounded inverse.

In this section, we consider the existence of positive spatially nonhomogeneous steady states solutions of system (1.5), which satisfy

$$
\left\{\begin{array}{l}
\nabla \cdot\left[e^{a m(x)} \nabla u\right]+\lambda e^{a m(x)} u\left[m(x)-b_{1} e^{a m(x)} u(x, t)-c_{1} e^{a m(x)} v(x, t)\right]=0,  \tag{2.1}\\
\nabla \cdot\left[e^{a m(x)} \nabla v\right]+\lambda e^{a m(x)} v\left[m(x)-b_{2} e^{a m(x)} u(x, t)-c_{2} e^{a m(x)} v(x, t)\right]=0,
\end{array}\right.
$$

Suppose that the solution of (2.1) has the following expressions:

$$
\left\{\begin{array}{l}
u_{\lambda}=\alpha\left(\lambda-\lambda_{*}\right)\left[\varphi+\left(\lambda-\lambda_{*}\right) \xi(x)\right]  \tag{2.2}\\
v_{\lambda}=\beta\left(\lambda-\lambda_{*}\right)\left[\varphi+\left(\lambda-\lambda_{*}\right) \eta(x)\right]
\end{array}\right.
$$

where $\alpha, \beta \in \mathbb{R}, \xi, \eta \in X_{1}$. Substitute (2.2) into (2.1) we have

$$
\left\{\begin{align*}
L \xi+m(x) e^{a m(x)}\left[\varphi+\left(\lambda-\lambda_{*}\right) \xi\right]-\lambda \alpha b_{1} e^{2 a m(x)}\left[\varphi+\left(\lambda-\lambda_{*}\right) \xi\right]^{2}  \tag{2.3}\\
-\lambda \beta c_{1} e^{2 a m(x)}\left[\varphi+\left(\lambda-\lambda_{*}\right) \xi\right]\left[\varphi+\left(\lambda-\lambda_{*}\right) \eta\right]=0 \\
L \eta+m(x) e^{a m(x)}\left[\varphi+\left(\lambda-\lambda_{*}\right) \eta\right]-\lambda \alpha c_{2} e^{2 a m(x)}\left[\varphi+\left(\lambda-\lambda_{*}\right) \eta\right]^{2} \\
-\lambda \beta b_{2} e^{2 a m(x)}\left[\varphi+\left(\lambda-\lambda_{*}\right) \xi\right]\left[\varphi+\left(\lambda-\lambda_{*}\right) \eta\right]=0
\end{align*}\right.
$$

When $\lambda=\lambda_{*,}$ (2.3) becomes following equations

$$
\left\{\begin{array}{l}
L \xi+m(x) e^{a m(x)} \varphi-\lambda_{*} \alpha b_{1} e^{2 a m(x)} \varphi^{2}-\lambda_{*} \beta c_{1} e^{2 a m(x)} \varphi^{2}=0  \tag{2.4}\\
L \eta+m(x) e^{a m(x)} \varphi-\lambda_{*} \beta b_{2} e^{2 a m(x)} \varphi^{2}-\lambda_{*} \alpha c_{2} e^{2 a m(x)} \varphi^{2}=0
\end{array}\right.
$$

Multiplying both sides of each equation in (2.4) by $\varphi$ and integrating on $\Omega$, we have

$$
\alpha_{\lambda_{*}}=\frac{c_{2}-c_{1}}{b_{1} c_{2}-b_{2} c_{1}} d_{1}, \quad \beta_{\lambda_{*}}=\frac{b_{1}-b_{2}}{b_{1} c_{2}-b_{2} c_{1}} d_{1},
$$

where $d_{1}=\frac{\int_{\Omega} m(x) e^{a m(x)} \varphi^{2} d x}{\lambda_{*} \int_{\Omega} e^{2 m m(x)} \varphi^{3} d x}>0$, see [8]. And $\xi_{\lambda_{*}}, \eta_{\lambda_{*}} \in X_{1}$ is the unique solution of the following equations

$$
\left\{\begin{array}{l}
L \xi+m(x) e^{a m(x)} \varphi-\lambda_{*} \alpha_{\lambda_{*}} b_{1} e^{2 a m(x)} \varphi^{2}-\lambda_{*} \beta_{\lambda_{*}} c_{1} e^{2 a m(x)} \varphi^{2}=0  \tag{2.5}\\
L \eta+m(x) e^{a m(x)} \varphi-\lambda_{*} \beta_{\lambda_{*}} b_{2} e^{2 a m(x)} \varphi^{2}-\lambda_{*} \alpha_{\lambda_{*}} c_{2} e^{2 a m(x)} \varphi^{2}=0
\end{array}\right.
$$

To guarantee positive steady states solutions of system (2.1), we need following conditions:
(H1) $\left(\lambda-\lambda_{*}\right) \frac{c_{2}-c_{1}}{b_{1} c_{2}-b_{2} c_{1}}>0, \quad\left(\lambda-\lambda_{*}\right) \frac{b_{1}-b_{2}}{b_{1} c_{2}-b_{2} c_{1}}>0$.
Theorem 2.1. Assume that (H1) holds. Then there exist a constant $\delta>0$ and a continuously differentiable mapping which defined by $\lambda \rightarrow\left(\xi_{\lambda}, \eta_{\lambda}, \alpha_{\lambda}, \beta_{\lambda}\right)$, from $\left(\lambda_{*}-\delta, \lambda_{*}+\delta\right)$ to $X_{1}^{2} \times\left(\mathbb{R}^{+}\right)^{2}$ such that system (1.5) has a positive spatially nonhomogeneous steady-state solution:

$$
\left\{\begin{array}{l}
u_{\lambda}=\alpha_{\lambda}\left(\lambda-\lambda_{*}\right)\left[\varphi+\left(\lambda-\lambda_{*}\right) \xi_{\lambda}(x)\right]  \tag{2.6}\\
v_{\lambda}=\beta_{\lambda}\left(\lambda-\lambda_{*}\right)\left[\varphi+\left(\lambda-\lambda_{*}\right) \eta_{\lambda}(x)\right]
\end{array}\right.
$$

Proof. Let $F=\left(F_{1}, F_{2}, F_{3}, F_{4}\right)$ be defined as the following

$$
\left\{\begin{aligned}
F_{1}(\xi, \eta, \alpha, \beta, \lambda)= & L \xi+m(x) e^{a m(x)}\left[\varphi+\left(\lambda-\lambda_{*}\right) \xi\right]-\lambda \alpha b_{1} e^{2 a m(x)}\left[\varphi+\left(\lambda-\lambda_{*}\right) \xi\right]^{2} \\
& -\lambda \beta c_{1} e^{2 a m(x)}\left[\varphi+\left(\lambda-\lambda_{*}\right) \xi\right]\left[\varphi+\left(\lambda-\lambda_{*}\right) \eta\right]=0, \\
F_{2}(\xi, \eta, \alpha, \beta, \lambda)= & L \eta+m(x) e^{a m(x)}\left[\varphi+\left(\lambda-\lambda_{*}\right) \eta\right]-\lambda \alpha c_{2} e^{2 a m(x)}\left[\varphi+\left(\lambda-\lambda_{*}\right) \eta\right]^{2} \\
& -\lambda \beta b_{2} e^{2 a m(x)}\left[\varphi+\left(\lambda-\lambda_{*}\right) \xi\right]\left[\varphi+\left(\lambda-\lambda_{*}\right) \eta\right]=0, \\
F_{3}(\xi, \eta, \alpha, \beta, \lambda)= & \langle\varphi, \xi\rangle=0, \\
F_{4}(\xi, \eta, \alpha, \beta, \lambda)= & \langle\varphi, \eta\rangle=0 .
\end{aligned}\right.
$$

It is easy to obtain that from (2.5)

$$
F_{i}\left(\xi_{\lambda_{*}}, \eta_{\lambda_{*},} \alpha_{\lambda_{*}}, \beta_{\lambda_{*}}, \lambda_{*}\right)=0, \quad(i=1,2,3,4)
$$

The Fréchet derivative of $F$ at $\left(\xi_{\lambda_{*}}, \eta_{\lambda_{*}}, \alpha_{\lambda_{*},} \beta_{\lambda_{*},} \lambda_{*}\right)$ is

$$
\left.\frac{\partial F}{\partial(\xi, \eta, \alpha, \beta)}\right|_{\left(\xi_{\lambda_{*},} \eta_{\lambda_{*}}, \alpha_{\lambda_{*}}, \beta_{\lambda_{*}}, \lambda_{*}\right)}\left(\begin{array}{l}
\widehat{\xi} \\
\widehat{\eta} \\
\widehat{\alpha} \\
\widehat{\beta}
\end{array}\right)=\left(\begin{array}{l}
L \widehat{\xi}-\lambda_{*}\left(\widehat{\alpha} b_{1}+\widehat{\beta} c_{1}\right) e^{a m(x)} \varphi^{2} \\
L \widehat{\eta}-\lambda_{*}\left(\widehat{\alpha} c_{2}+\widehat{\beta} b_{2}\right) e^{a m(x)} \varphi^{2} \\
\langle\varphi, \widehat{\xi}\rangle \\
\langle\varphi, \widehat{\eta}\rangle
\end{array}\right) .
$$

It is clear that the derivative operator $\left.\frac{\partial F}{\partial(\xi, \eta, \alpha, \beta)}\right|_{\left(\xi_{\lambda_{*}}, \eta_{\lambda_{*}}, \alpha_{\lambda_{*}}, \beta_{\lambda_{*}}, \lambda_{*}\right)}$ is bijective. By using the implicit function theorem we know that there exist a constant $\delta>0$ and a continuously differentiable mapping which defined by $\lambda \rightarrow\left(\xi_{\lambda}, \eta_{\lambda}, \alpha_{\lambda}, \beta_{\lambda}\right)$ from $\left(\lambda_{*}-\delta, \lambda_{*}+\delta\right)$ to $X_{1}^{2} \times\left(\mathbb{R}^{+}\right)^{2}$ such that system (1.5) has a positive spatially nonhomogeneous steady-state solution (2.6).

## 3 Eigenvalue problems of the linearized system

For the convenience of discussion, we always suppose that $\Lambda=\left(\lambda_{*}-\delta, \lambda_{*}\right) \cup\left(\lambda_{*}, \lambda_{*}+\delta\right)$.
Let $\left(u_{\lambda}, v_{\lambda}\right)^{T}$ is a spatially nonhomogeneous steady-state solution of (1.5) which is determined by (2.6). Let

$$
\tilde{u}=u-u_{\lambda}, \quad \tilde{v}=v-v_{\lambda},
$$

dropping the tilde sign, system (1.5) can be transformed as follows:

$$
\left\{\begin{align*}
\frac{\partial u}{\partial t}= & e^{-a m(x)} \nabla \cdot\left[e^{a m(x)} \nabla u\right]+\lambda u(x, t)\left[m(x)-b_{1} e^{a m(x)} u_{\lambda}-c_{1} e^{a m(x)} v_{\lambda}\right]  \tag{3.1}\\
& -\lambda e^{a m(x)} u_{\lambda}\left[b_{1} u(x, t-\tau)+c_{1} v(x, t-\tau)\right] \\
\frac{\partial v}{\partial t}= & e^{-a m(x)} \nabla \cdot\left[e^{a m(x)} \nabla v\right]+\lambda v(x, t)\left[m(x)-b_{2} e^{a m(x)} u_{\lambda}-c_{2} e^{a m(x)} v_{\lambda}\right] \\
& -\lambda e^{a m(x)} v_{\lambda}\left[b_{2} u(x, t-\tau)+c_{2} v(x, t-\tau)\right] .
\end{align*}\right.
$$

Denote $A_{\lambda}, B_{\lambda}$ :

$$
A_{\lambda}=\left(\begin{array}{ll}
A_{1} & 0 \\
0 & A_{2}
\end{array}\right), \quad B_{\lambda} \psi=\binom{\lambda e^{a m(x)} u_{\lambda}\left[b_{1} \psi_{1}(-\tau)+c_{1} \psi_{2}(-\tau)\right]}{\lambda e^{a m(x)} v_{\lambda}\left[b_{2} \psi_{1}(-\tau)+c_{2} \psi_{2}(-\tau)\right]}
$$

where

$$
\begin{aligned}
& A_{1}=e^{-a m(x)} \nabla \cdot\left[e^{a m(x)} \nabla\right]+\lambda\left[m(x)-b_{1} e^{a m(x)} u_{\lambda}-c_{1} e^{a m(x)} v_{\lambda}\right], \\
& A_{2}=e^{-a m(x)} \nabla \cdot\left[e^{a m(x)} \nabla\right]+\lambda\left[m(x)-b_{2} e^{a m(x)} u_{\lambda}-c_{2} e^{a m(x)} v_{\lambda}\right]
\end{aligned}
$$

and $\psi=\left(\psi_{1}, \psi_{2}\right)^{T} \in X_{\mathbf{C}}^{2}$.
It follows from $[14,33]$ that the semigroup induced by the solutions of the linearized system (3.1) has the infinitesimal generator $T_{\tau, \lambda}$ satisfying

$$
\begin{equation*}
T_{\tau, \lambda} \psi=\dot{\psi} \tag{3.2}
\end{equation*}
$$

where

$$
\mathcal{D}\left(T_{\tau, \lambda}\right)=\left\{\psi \in C_{\mathbb{C}} \cap C_{\mathbb{C}}^{1} \mid \psi(0) \in X_{\mathbb{C}}, \dot{\psi}(0)=A_{\lambda} \psi(0)-B_{\lambda} \psi(-\tau)\right\},
$$

where

$$
C_{\mathrm{C}}=C\left([-\tau, 0], Y_{\mathrm{C}}^{2}\right), \quad C_{\mathrm{C}}^{1}=C^{1}\left([-\tau, 0], Y_{\mathrm{C}}^{2}\right) .
$$

Moreover, $\mu \in \mathbb{C}$ an eigenvalue of $T_{\tau, \lambda}$ if and only if there exists $\psi=\left(\psi_{1}, \psi_{2}\right)^{T} \in X_{C}^{2} \backslash\left\{(0,0)^{T}\right\}$ such that

$$
\begin{equation*}
\Delta(\lambda, \mu, \tau) \psi=A_{\lambda} \psi-B_{\lambda} \psi e^{-\mu \tau}-\mu \psi=0 . \tag{3.3}
\end{equation*}
$$

Lemma 3.1. 0 is not an eigenvalue of $T_{\tau, \lambda}$.
Proof. If 0 is an eigenvalue of $T_{\tau, \lambda}$, that is, there exists some $\psi=\left(\psi_{1}, \psi_{2}\right)^{T} \in X_{C}^{2} \backslash\left\{(0,0)^{T}\right\}$ such that

$$
\begin{equation*}
\Delta(\lambda, 0, \tau) \psi=0 . \tag{3.4}
\end{equation*}
$$

Note that $\Delta\left(\lambda_{*}, 0, \tau\right)=\left(\begin{array}{cc}L & 0 \\ 0 & L\end{array}\right)$ and $\mathcal{N}(L)=\operatorname{span}\{\varphi\}$. We let that $\psi$ takes the form

$$
\left\{\begin{array}{l}
\psi_{1}=p_{1} \varphi+\left(\lambda-\lambda_{*}\right) q_{1}(x),  \tag{3.5}\\
\psi_{2}=p_{2} \varphi+\left(\lambda-\lambda_{*}\right) q_{2}(x),
\end{array}\right.
$$

where $p_{1}, p_{2} \in \mathbb{R}, q_{1}(x), q_{2}(x) \in X_{1}$. Then substituting (3.5) into (3.4) and let $\lambda=\lambda_{*}$, by calculation, we have

$$
\left\{\begin{array}{l}
L q_{1}+\left[m(x) e^{a m(x)} \varphi-\lambda_{*} e^{2 a m(x)}\left(b_{1} \alpha_{\lambda_{*}}+c_{1} \beta_{\lambda_{*}}\right) \varphi^{2}\right] p_{1}-\lambda_{*} e^{2 a m(x)} \alpha_{\lambda_{*}} \varphi^{2}\left(b_{1} p_{1}+c_{1} p_{2}\right)=0,  \tag{3.6}\\
L q_{2}+\left[m(x) e^{a m(x)} \varphi-\lambda_{*} e^{2 a m(x)}\left(b_{2} \alpha_{\lambda_{*}}+c_{2} \beta_{\lambda_{*}}\right) \varphi^{2}\right] p_{2}-\lambda_{*} e^{2 a m(x)} \beta_{\lambda_{*}} \varphi^{2}\left(b_{2} p_{1}+c_{2} p_{2}\right)=0
\end{array}\right.
$$

By (2.5), (3.6) becomes

$$
\left\{\begin{array}{l}
L\left(q_{1}-\xi_{\lambda_{*}} p_{1}\right)-\lambda_{*} e^{2 a m(x)} \alpha_{\lambda_{*}} \varphi^{2}\left(b_{1} p_{1}+c_{1} p_{2}\right)=0,  \tag{3.7}\\
L\left(q_{2}-\eta_{\lambda_{*}} p_{2}\right)-\lambda_{*} e^{2 a m(x)} \beta_{\lambda_{*}} \varphi^{2}\left(b_{2} p_{1}+c_{2} p_{2}\right)=0 .
\end{array}\right.
$$

Multiplying both sides of each equation in (3.7) by $\varphi$ and integrating on $\Omega$, we have

$$
\left\{\begin{array}{l}
b_{1} p_{1}+c_{1} p_{2}=0  \tag{3.8}\\
b_{2} p_{1}+c_{2} p_{2}=0
\end{array}\right.
$$

By the condition (H1), we have $b_{1} c_{2}-b_{2} c_{1} \neq 0$. So we get $p_{1}=p_{2}=0$ from (3.8). By (3.6), we get $q_{1}=q_{2}=0$. Then $\psi_{1}=0, \psi_{2}=0$. The Lemma 3.1 is now proved.

We will show that the eigenvalues of $T_{\tau, \lambda}$ could pass through the imaginary axis when time delay $\tau$ increases. It is obvious that $T_{\tau, \lambda}$ has an imaginary eigenvalue $\mu=i \omega(\omega \neq 0)$ for some $\tau \geq 0$ if and only if

$$
\begin{equation*}
m(\lambda, \omega, \theta) \psi=\Delta(\lambda, \omega, \theta) \psi=A_{\lambda} \psi-B_{\lambda} \psi e^{-i \theta}-i \omega \psi=0 \tag{3.9}
\end{equation*}
$$

is solvable for some $\omega>0, \theta \in[0,2 \pi), \tau=\frac{\theta+2 n \pi}{\omega}, n \in \mathbb{N}_{0}=\{0,1,2, \ldots\}$ and $\psi=\left(\psi_{1}, \psi_{2}\right)^{T} \in$ $X_{\text {C }}^{2} \backslash\left\{(0,0)^{T}\right\}$.

Lemma 3.2. If $(\omega, \theta, \psi) \in(0, \infty) \times[0,2 \pi) \times\left(X_{C}^{2} \backslash\left\{(0,0)^{T}\right\}\right)$ solves (3.9), then $\frac{\omega}{\lambda-\lambda_{*}}$ is bounded for $\lambda \in \Lambda$.

Proof. Assume that $(\omega, \theta, \psi) \in(0, \infty) \times[0,2 \pi) \times\left(X_{\mathbf{C}}^{2} \backslash\left\{(0,0)^{T}\right\}\right)$ satisfy the following equation

$$
\begin{equation*}
\left\langle A_{\lambda} \psi-B_{\lambda} \psi e^{-i \theta}-i \omega \psi, \psi\right\rangle=0 . \tag{3.10}
\end{equation*}
$$

Separating the real and imaginary parts of system (3.10), we obtain

$$
\begin{gathered}
\omega\langle\psi, \psi\rangle=\sin \theta\left\langle B_{\lambda} \psi, \psi\right\rangle . \\
\frac{|\omega|}{\left|\lambda-\lambda_{*}\right|}=\lambda e^{a m(x)}|\sin \theta| \frac{\left|\left\langle\binom{\alpha_{\lambda}\left[\varphi+\left(\lambda-\lambda_{*}\right) \tilde{S}_{\lambda}(x)\right]\left(b_{1} \psi_{1}+c_{1} \psi_{2}\right)}{\beta_{\lambda}\left[\varphi+\left(\lambda-\lambda_{*}\right) \eta_{\lambda}(x)\right]\left(b_{2} \psi_{1}+c_{2} \psi_{2}\right)}, \psi\right\rangle\right|}{\langle\psi, \psi\rangle} \\
\leq\left(\lambda_{*}+\delta\right) e^{a \max _{x \in \Omega} m(x)} \max \{M, N\} \max \left\{\left|b_{1}\right|,\left|c_{1}\right|,\left|b_{2}\right|,\left|c_{2}\right|\right\} .
\end{gathered}
$$

where $M=\max _{\lambda \in \Lambda}\left\{\left|\alpha_{\lambda}\right|\left[\|\varphi\|_{\infty}+\left(\lambda+\lambda_{*}\right)\left\|\xi_{\lambda}(x)\right\|_{\infty}\right]\right\}$,
$N=\max _{\lambda \in \Lambda}\left\{\left|\beta_{\lambda}\right|\left[\|\varphi\|_{\infty}+\left(\lambda+\lambda_{*}\right)\left\|\eta_{\lambda}(x)\right\|_{\infty}\right]\right\}$. The boundedness of $\frac{\omega}{\lambda-\lambda_{*}}$ follows from the continuity of $\lambda \mapsto\left(\alpha_{\lambda}, \beta_{\lambda},\left\|\xi_{\lambda}(x)\right\|_{\infty},\left\|\eta_{\lambda}(x)\right\|_{\infty}\right)$. The Lemma 3.2 is now proved.

Note that $X=\mathcal{N}(L) \oplus X_{1}$. If $(\omega, \theta, \psi)$ satisfies (3.9), let $\psi=\left(\psi_{1}, \psi_{2}\right)^{T} \in X_{\mathbf{C}}^{2} \backslash\left\{(0,0)^{T}\right\}$ can be represented as

$$
\left\{\begin{array}{l}
\psi_{1}=p_{1} \varphi+\left(\lambda-\lambda_{*}\right) q_{1}(x),  \tag{3.11}\\
\psi_{2}=p_{2} \varphi+\left(\lambda-\lambda_{*}\right) q_{2}(x),
\end{array}\right.
$$

where $p_{1}, p_{2} \in \mathbb{R}, q_{1}(x), q_{2}(x) \in X_{1}$. Let

$$
\begin{equation*}
G\left(q_{1}, q_{2}, p_{1}, p_{2}, h, \theta, \lambda\right) \psi=\frac{m\left(\lambda,\left(\lambda-\lambda_{*}\right) h, \theta\right)}{\lambda-\lambda_{*}} \psi=0 \tag{3.12}
\end{equation*}
$$

where $m(\lambda, \omega, \theta)$ is defined as in (3.9).
Obviously, we have

$$
G\left(q_{1}, q_{2}, p_{1}, p_{2}, h, \theta, \lambda_{*}\right) \psi=0,
$$

that is

$$
\left\{\begin{array}{l}
L\left(q_{1}-\xi_{\lambda_{*}} p_{1}\right)-\lambda_{*} e^{2 a m(x)} \alpha_{\lambda_{*}} \varphi^{2}\left(b_{1} p_{1}+c_{1} p_{2}\right) e^{-i \theta}-i h \varphi e^{a m(x)} p_{1}=0  \tag{3.13}\\
L\left(q_{2}-\eta_{\lambda_{*}} p_{2}\right)-\lambda_{*} e^{2 a m(x)} \beta_{\lambda_{*}} \varphi^{2}\left(b_{2} p_{1}+c_{2} p_{2}\right) e^{-i \theta}-i h \varphi e^{a m(x)} p_{2}=0 .
\end{array}\right.
$$

Multiplying both sides of each equation in (3.13) by $\varphi$ and integrating on $\Omega$, we have

$$
\begin{equation*}
-\lambda_{*} d_{2} e^{-i \theta} M p=i h p, \tag{3.14}
\end{equation*}
$$


Separating the real and imaginary parts of (3.14), we get

$$
\left\{\begin{array}{l}
\lambda_{*} d_{2} \sin \theta M p=h p  \tag{3.15}\\
\lambda_{*} d_{2} \cos \theta M p=0
\end{array}\right.
$$

It is easy to obtain the following lemma.

## Lemma 3.3.

(1) When $\theta=\frac{\pi}{2}$ in (3.15), $\lambda_{*} d_{2} M$ has two real eigenvalues $h_{1}=\lambda_{*} d_{1} d_{2}, h_{2}=\lambda_{*} d_{1} d_{2} \frac{\left(c_{2}-c_{1}\right)\left(b_{1}-b_{2}\right)}{b_{1} c_{2}-b_{2} c_{1}}$, and $\left(c_{2}-c_{1}, b_{1}-b_{2}\right)^{T}$ and $\left(-c_{1}, b_{2}\right)^{T}$ are two eigenvectors associated with eigenvalues $h_{1}$ and $h_{2}$, respectively.
(2) When $\theta=\frac{3 \pi}{2}$ in (3.15), $-\lambda_{*} d_{2}$ M has two real eigenvalues $h_{1}=-\lambda_{*} d_{1} d_{2}, h_{2}=-\lambda_{*} d_{1} d_{2} \frac{\left(c_{2}-c_{1}\right)\left(b_{1}-b_{2}\right)}{b_{1} c_{2}-b_{2} c_{1}}$, and $\left(c_{2}-c_{1}, b_{1}-b_{2}\right)^{T}$ and $\left(-c_{1}, b_{2}\right)^{T}$ are two eigenvectors associated with eigenvalues $h_{1}$ and $h_{2}$, respectively.

For each $j=1,2$, set

$$
h_{\lambda *}^{j}= \begin{cases}\left|h_{j}\right|, & \text { if } \lambda>\lambda_{*}  \tag{3.16}\\ -\left|h_{j}\right|, & \text { if } \lambda<\lambda_{*}\end{cases}
$$

which satisfies $\omega_{\lambda *}^{j}=\left(\lambda-\lambda_{*}\right) h_{\lambda *}^{j}>0$, and their corresponding eigenvectors

$$
\begin{align*}
& \begin{cases}\left(p_{1 \lambda *}^{1} p_{2 \lambda *}^{1}\right)^{T}=\left(c_{2}-c_{1}, b_{1}-b_{2}\right)^{T}, & \text { if } h_{\lambda *}^{1}=\left|h_{1}\right|, \\
\left(p_{1 \lambda *}^{2} p_{2 \lambda *}^{2}\right)^{T}=\left(-c_{1}, b_{2}\right)^{T}, & \text { if } h_{\lambda *}^{2}=\left|h_{2}\right|,\end{cases}  \tag{3.17}\\
& \begin{cases}\left(p_{1 \lambda *}^{1} p_{2 \lambda *}^{1}\right)^{T}=\left(c_{2}-c_{1}, b_{1}-b_{2}\right)^{T}, & \text { if } h_{\lambda * *}^{1}=-\left|h_{1}\right|, \\
\left(p_{1 \lambda *}^{2} p_{2 \lambda *}^{2}\right)^{T}=\left(-c_{1}, b_{2}\right)^{T}, & \text { if } h_{\lambda *}^{2}=-\left|h_{2}\right| .\end{cases} \tag{3.18}
\end{align*}
$$

And set

$$
\theta_{\lambda *}^{j}= \begin{cases}\frac{\pi}{2}, & \text { if } \lambda>\lambda_{* \prime}  \tag{3.19}\\ \frac{3 \pi}{2}, & \text { if } \lambda>\lambda_{* \prime}\end{cases}
$$

which satisfies $-e^{-i \theta_{\lambda *}^{j} h_{\lambda *}^{j}}=i h_{\lambda *}^{j}$.
Thus, $q_{1 \lambda * *}^{j} q_{2 \lambda *}^{j} \in X_{1}$ is the unique solution of the following equations

$$
\left\{\begin{array}{l}
L\left(q_{1}^{j}-\xi_{\lambda_{*}} p_{1 \lambda *}^{j}\right)-\lambda_{*} e^{2 a m(x)} \alpha_{\lambda_{*}} \phi^{2}\left(b_{1} p_{1 \lambda_{*}}^{j}+c_{1} p_{2 \lambda *}^{j}\right) e^{-i \theta_{\lambda_{*}}^{j}-i h_{\lambda *}^{j} \phi e^{a m(x)} p_{1 \lambda_{*}}^{j}=0,}  \tag{3.20}\\
L\left(q_{2}^{j}-\eta_{\lambda_{*}} p_{2 \lambda *}^{j}\right)-\lambda_{*} e^{2 a m(x)} \beta_{\lambda_{*}} \phi^{2}\left(b_{2} p_{1 \lambda *}^{j}+c_{2} p_{2 \lambda *}^{j}\right) e^{-i \theta_{\lambda_{*}}^{j}}-i h_{\lambda *}^{j} \phi e^{a m(x)} p_{2 \lambda *}^{j}=0,
\end{array}\right.
$$

## Remark 3.4.

(1) When $v=0, b_{1}=1$ in (1.4), $h_{1}$ in Lemma 3.3 (1) is the same as $h_{\lambda_{*}}$ in (2.20) in [8].
(2) When $a_{1}=0$ in (1.4), $h_{1}, h_{2}$ in Lemma 3.3 (1) are the same as that in Lemma 3.4 (i) in [15].

Then we get the following lemma.
Lemma 3.5. Assume that (H1) holds. For $j=1,2$, the following equation

$$
\left\{\begin{array}{l}
G\left(q_{1}^{j}, q_{2}^{j}, p_{1}^{j}, p_{2}^{j}, \theta^{j}, h^{j}, \lambda_{*}\right)=0,  \tag{3.21}\\
q_{1}^{j}, q_{2}^{j} \in X_{1}, p_{1}^{j}, p_{2}^{j}, h^{j} \in \mathbb{R}, \theta^{j} \in[0,2 \pi]
\end{array}\right.
$$

has a unique solution $\left(q_{1 \lambda_{*},}^{j} q_{2 \lambda_{*}}^{j}, p_{1 \lambda_{*},}^{j} p_{2 \lambda_{*^{\prime}}}^{j} \theta_{\lambda_{*^{*}}}^{j}, h_{\lambda_{*}}^{j}\right)$, see (3.16)-(3.20).
Theorem 3.6. Assume that (H1) holds. Then for $j=1,2$, there exist a constant $\delta>0$ and a continuously differentiable mapping which defined by $\lambda \rightarrow\left(q_{1 \lambda}^{j}, q_{2 \lambda}^{j}, p_{1 \lambda}^{j}, p_{2 \lambda}^{j}, \theta_{\lambda}^{j}, h_{\lambda}^{j}\right)$ from $\Lambda$ to $X_{1}^{2} \times \mathbb{R}^{2} \times \mathbb{R}^{+} \times \mathbb{R}$ such that $G\left(q_{1 \lambda}^{j}, q_{2 \lambda}^{j}, p_{1 \lambda}^{j}, p_{2 \lambda}^{j}, \theta_{\lambda}^{j}, h_{\lambda}^{j}, \lambda\right)=0$.

Proof. Let $G=\left(g_{1}, g_{2}, g_{3}, g_{4}, g_{5}, g_{6}\right)$ be defined as the following:

$$
\left\{\begin{aligned}
g_{1}= & m\left(\lambda,\left(\lambda-\lambda_{*}\right) h_{\lambda}^{j}, \theta_{\lambda}^{j}\right) q_{1 \lambda}^{j}+m(x) e^{a m(x)} \varphi p_{1 \lambda}^{j} \\
& -\lambda e^{a m(x)} \varphi\left[\alpha_{\lambda} b_{1}\left(\varphi+\left(\lambda-\lambda_{*}\right) \xi\right)+\beta_{\lambda} c_{1}\left(\varphi+\left(\lambda-\lambda_{*}\right) \xi\right)\right] p_{1 \lambda}^{j} \\
& +\lambda \varphi \alpha_{\lambda} e^{a m(x)}\left(\varphi+\left(\lambda-\lambda_{*}\right) \xi\right)\left(b_{1} p_{1 \lambda}^{j}+c_{1} p_{2 \lambda}^{j}\right) e^{-i \theta_{\lambda}^{j}}-i h_{\lambda}^{j} \varphi p_{1 \lambda}^{j}=0, \\
g_{2}= & m\left(\lambda,\left(\lambda-\lambda_{*}\right) h_{\lambda}^{j}, \theta_{\lambda}^{j}\right) q_{2 \lambda}^{j}+m(x) e^{a m(x)} \varphi p_{2 \lambda}^{j} \\
& -\lambda e^{a m(x)} \varphi\left[\alpha_{\lambda} b_{2}\left(\varphi+\left(\lambda-\lambda_{*}\right) \xi\right)+\beta_{\lambda} c_{2}\left(\varphi+\left(\lambda-\lambda_{*}\right) \xi\right)\right] p_{2 \lambda}^{j} \\
& +\lambda \varphi \alpha_{\lambda} e^{a m(x)}\left(\varphi+\left(\lambda-\lambda_{*}\right) \xi\right)\left(b_{2} p_{1 \lambda}^{j}+c_{2} p_{2 \lambda}^{j}\right) e^{-i \theta_{\lambda}^{j}}-i h_{\lambda}^{j} \varphi p_{2 \lambda}^{j}=0, \\
g_{3}= & \operatorname{Re}\left\langle\varphi, q_{1 \lambda}^{j}\right\rangle=0, \quad g_{4}=\operatorname{Im}\left\langle\varphi, q_{1 \lambda}^{j}\right\rangle=0, \\
g_{5}= & \operatorname{Re}\left\langle\varphi, q_{2 \lambda}^{j}\right\rangle=0, \quad g_{6}=\operatorname{Im}\left\langle\varphi, q_{2 \lambda}^{j}\right\rangle=0 .
\end{aligned}\right.
$$

The Fréchet derivative of $G$ at $\left(q_{1 \lambda_{*}}^{j}, q_{2 \lambda_{*}}^{j} p_{1 \lambda_{*^{\prime}}}^{j} p_{2 \lambda_{*^{\prime}}}^{j} \theta_{\lambda_{*^{\prime}}}^{j}, h_{\lambda_{*^{\prime}}}^{j} \lambda_{*}\right)$ is

$$
\left.\frac{\partial G\left(q_{1 \lambda_{*}}^{j}, q_{2 \lambda^{\prime}}^{j}\right.}{j} p_{1 \lambda^{\prime}}^{j} p_{2 \lambda^{\prime}}^{j} \theta_{\lambda_{*^{\prime}}}^{j} h_{\lambda_{*^{\prime}}}^{j} \lambda_{*}\right)\left(\begin{array}{l}
\hat{q}_{1 \lambda}^{j} \\
\partial\left(q_{1 \lambda^{\prime}}^{j} q_{2 \lambda^{\prime}}^{j} p_{1 \lambda^{\prime}}^{j} p_{2 \lambda^{\prime}}^{j} \theta_{\lambda^{\prime}}^{j} h_{\lambda}^{j}\right)
\end{array}\left(\begin{array}{l}
e^{-a m(x)} L \hat{q}_{1 \lambda}^{j}+\widetilde{g}_{1} \hat{p}_{1 \lambda}^{j}+\widetilde{g}_{2} \hat{p}_{2 \lambda}^{j}+\widetilde{g}_{3} \hat{\theta}_{\lambda}^{j}+\widetilde{g}_{4} \hat{h}_{\lambda}^{j} \\
\hat{p}_{1 \lambda}^{j} \\
e^{-a m(x)} \hat{q}_{2 \lambda}^{j}+\widetilde{g}_{5} \hat{p}_{1 \lambda}^{j}+\widetilde{g}_{6} \hat{p}_{2 \lambda}^{j}+\widetilde{g}_{7} \hat{\theta}_{\lambda}^{j}+\widetilde{g}_{8} \hat{h}_{\lambda}^{j} \\
\operatorname{Re}\left\langle\varphi, \hat{q}_{1 \lambda}^{j}\right\rangle \\
\operatorname{im}\left\langle\varphi, \hat{q}_{2 \lambda}^{j}\right\rangle \\
\hat{\theta}_{\lambda \lambda}^{j} \\
\hat{h}_{\lambda}^{j}
\end{array}\right)=\left(\varphi, \hat{q}_{2 \lambda}^{j}\right\rangle,\right.
$$

where

It is clear that the derivative operator

$$
\frac{\partial G\left(q_{1 \lambda_{*}}^{j} q_{2 \lambda^{\prime}}{ }^{\prime} p_{1 \lambda_{*^{\prime}}}^{j} p_{2 \lambda^{\prime}}^{j}, \theta_{\lambda_{*^{\prime}}}^{j}, h_{\lambda_{*}}^{j}, \lambda_{*}\right)}{\partial\left(q_{1 \lambda^{\prime}}^{j} q_{2 \lambda}^{j}, p_{1 \lambda^{\prime}}^{j} p_{2 \lambda^{\prime}}^{j} \theta_{\lambda^{\prime}}^{j} h_{\lambda}^{j}\right)}
$$

is bijective. By using the implicit function theorem we know that there exist a constant $\delta>0$ and a continuously differentiable mapping which defined by $\lambda \rightarrow\left(q_{1 \lambda}^{j}, q_{2 \lambda}^{j}, p_{1 \lambda^{\prime}}^{j}, p_{2 \lambda^{\prime}}^{j} \theta_{\lambda^{\prime}}^{j} h_{\lambda}^{j}\right)$, from from $\Lambda$ to $X_{1}^{2} \times \mathbb{R}^{2} \times \mathbb{R}^{+} \times \mathbb{R}$ such that $G\left(q_{1 \lambda}^{j}, q_{2 \lambda}^{j}, p_{1 \lambda^{\prime}}^{j} p_{2 \lambda^{\prime}}^{j} \theta_{\lambda^{\prime}}^{j}, h_{\lambda}^{j}, \lambda\right)=0$. The proof of Theorem 3.6 is complete.

From Theorem 3.6, we derive the following result.
Theorem 3.7. Assume that (H1) holds. For $j=1,2, \lambda \in \Lambda$ and $n \in \mathbb{N}_{0}$, let

$$
\tau_{n}^{j}=\frac{\theta_{1 \lambda}^{j}+2 n \pi}{\omega_{\lambda}^{j}}, \quad \omega_{\lambda}^{j}=\left(\lambda-\lambda_{*}\right) h_{\lambda^{\prime}}^{j}
$$

and $\psi_{\lambda}^{j}=\left(\psi_{1 \lambda}^{j}, \psi_{2 \lambda}^{j}\right)^{T}$,

$$
\left\{\begin{array}{l}
\psi_{1 \lambda}^{j}=p_{1 \lambda}^{j} \varphi+\left(\lambda-\lambda_{*}\right) q_{1 \lambda}^{j}(x),  \tag{3.22}\\
\psi_{2 \lambda}^{j}=p_{2 \lambda}^{j} \varphi+\left(\lambda-\lambda_{*}\right) q_{2 \lambda}^{j}(x),
\end{array}\right.
$$

where $q_{1 \lambda}^{j}, q_{2 \lambda}^{j}, p_{1 \lambda}^{j}, p_{2 \lambda}^{j}, \theta_{\lambda}^{j}, h_{\lambda}^{j}$ are defined as in Theorem 3.6. Then
(1) $T_{\tau_{n}^{j}, \lambda}$ has a pair of purely imaginary eigenvalues $\pm i \omega_{\lambda}^{j}$;
(2) $T_{\tau_{n}^{j}, \lambda} e^{i \omega_{\lambda}^{j}} \psi^{j}=i \omega_{\lambda}^{j} e^{i \omega_{\lambda}^{j}} \psi_{\lambda}^{j}, T_{\tau_{n}^{j}, \lambda} e^{-i \omega_{\lambda}^{j}} \bar{\psi}^{j}=-i \omega_{\lambda}^{j} e^{-i \omega_{\lambda}^{j}} \bar{\psi}_{\lambda}^{j}$.

Now, we give some estimates to prove the simplicity of $i \omega_{\lambda}^{j}$. The proof of the following Lemmas 3.8-3.10 is similar to [14]. For the sake of the integrity of the article, we are going to prove them again.

Lemma 3.8. Assume that (H1) holds. For $j=1,2, \lambda \in \Lambda$ and $n \in \mathbb{N}_{0}$, we have

$$
\begin{equation*}
S_{n}^{j}=\left\langle\bar{\psi}_{\lambda}^{j}, \psi_{\lambda}^{j}-\tau_{n}^{j} e^{-i \theta_{\lambda}^{j}} B_{\lambda} \psi_{\lambda}^{j}\right\rangle \neq 0, \tag{3.23}
\end{equation*}
$$

where $\psi_{\lambda}^{j}, \tau_{n}^{j}$ and $\theta_{\lambda}^{j}$ are defined as in Theorem 3.7.
Proof. It is easy to obtain that

$$
\operatorname{Re}\left\{S_{n}^{j}\right\}=\left\langle\bar{\psi}_{\lambda},\left(\psi_{\lambda}-\tau_{n}^{j} e^{-i \theta_{\lambda}^{j}} B_{\lambda} \psi_{\lambda}\right)\right\rangle \rightarrow\left[\left(p_{1 \lambda_{*}}^{j}\right)^{2}+\left(p_{2 \lambda_{*}}^{j}\right)^{2}\right] \neq 0, \quad \text { as } \lambda \rightarrow \lambda_{*},
$$

where using $\int_{\Omega} \varphi^{2} d x=1$ in Section 1 .
Lemma 3.9. Assume that (H1) holds. For $j=1,2, \lambda \in \Lambda$ and $n \in \mathbb{N}_{0}, i \omega_{\lambda}^{j}$ is a simple eigenvalue of $T_{\tau_{n}^{j}, \lambda}$.

Proof. It follows from Theorem 3.7 that $\mathcal{N}\left[T_{\tau_{n, \lambda}^{j}}-i \omega_{\lambda}^{j}\right]=\operatorname{span}\left[\psi_{\lambda}^{j} e^{i \omega_{\lambda}^{j}(\cdot)}\right]$. If $\widetilde{\psi} \in \mathcal{N}\left[T_{\tau_{n}^{j}, \lambda}-i \omega_{\lambda}^{j}\right]^{2}$, that is

$$
\left(T_{\tau_{n}^{j}, \lambda}-i \omega_{\lambda}^{j}\right)^{2} \widetilde{\psi}=0,
$$

then

$$
\left(T_{\tau_{n, \lambda}^{j}}-i \omega_{\lambda}^{j}\right) \widetilde{\psi} \in \mathcal{N}\left[T_{\tau_{n}^{j}, \lambda}-i \omega_{\lambda}^{j}\right]=\operatorname{span}\left[\psi_{\lambda}^{j} e^{i \omega_{\lambda}^{j}(\cdot)}\right] .
$$

We assume that a constant $\rho$ satisfies

$$
\left(T_{\tau_{n}^{j}, \lambda}-i \omega_{\lambda}^{j}\right) \widetilde{\psi}=\rho \psi_{\lambda}^{j} e^{i \omega_{\lambda}^{j}(\cdot)},
$$

which leads to

$$
\left\{\begin{array}{l}
\widetilde{\psi}^{\prime}(s)=i \omega_{\lambda}^{j} \widetilde{\psi}(s)+\rho \psi_{\lambda^{j}}^{i} \lambda^{i \omega_{\lambda}^{j} s}, \quad s \in\left[-\tau_{n}^{j}, 0\right),  \tag{3.24}\\
\widetilde{\psi}^{\prime}(0)=A_{\lambda} \widetilde{\psi}(0)-B_{\lambda} \widetilde{\psi}\left(-\tau_{n}^{j}\right) .
\end{array}\right.
$$

From the first equation of (3.24), we have

$$
\left\{\begin{array}{l}
\widetilde{\psi}(s)=\widetilde{\psi}(0) e^{i \omega_{\lambda}^{j} s}+\rho s \psi_{\lambda}^{j} e^{i \omega_{\lambda}^{j} s}, \quad s \in\left[-\tau_{n}^{j}, 0\right),  \tag{3.25}\\
\widetilde{\psi}^{\prime}(0)=i \omega_{\lambda}^{j} \widetilde{\psi}(0)+\rho \psi_{\lambda}^{j} .
\end{array}\right.
$$

Eq. (3.24) and Eq. (3.25) imply that

$$
\left\{\begin{array}{l}
A_{\lambda} \widetilde{\psi}(0)-B_{\lambda} \widetilde{\psi}\left(-\tau_{n}^{j}\right)=i \omega_{\lambda}^{j} \widetilde{\psi}(0)+\rho \psi_{\lambda}^{j}  \tag{3.26}\\
\widetilde{\psi}\left(-\tau_{n}^{j}\right)=\widetilde{\psi}(0) e^{-i \omega_{\lambda}^{j} \tau_{n}^{j}}-\tau_{n}^{j} \rho \psi_{\lambda}^{j} e^{-i \omega_{\lambda}^{j} \tau_{n}^{j}}
\end{array}\right.
$$

Then we have

$$
\begin{aligned}
\Delta\left(\lambda, i \omega_{\lambda}^{j}, \tau_{n}^{j}\right) \widetilde{\psi}(0) & =\left(A_{\lambda}-i \omega_{\lambda}^{j}\right) \widetilde{\psi}(0)-B_{\lambda} \widetilde{\psi}(0) e^{-i \theta_{\lambda}^{j}} \\
& =\rho\left(\psi_{\lambda}^{j}-\tau_{n}^{j} e^{-i \theta_{\lambda}^{j}} B_{\lambda} \psi_{\lambda}^{j}\right)
\end{aligned}
$$

Since $\Delta\left(\lambda, i \omega_{\lambda}^{j}, \tau_{n}^{j}\right) \psi_{\lambda}^{j}=0$, so

$$
\begin{equation*}
\Delta\left(\lambda,-i \omega_{\lambda}^{j}, \tau_{n}^{j}\right) \bar{\psi}_{\lambda}^{j}=0 \tag{3.27}
\end{equation*}
$$

Then

$$
0=\left\langle\Delta\left(\lambda,-i \omega_{\lambda}^{j}, \tau_{n}^{j}\right) \bar{\psi}_{\lambda}^{j}, \widetilde{\psi}(0)\right\rangle=\left\langle\bar{\psi}_{\lambda}^{j}, \Delta\left(\lambda, i \omega_{\lambda}^{j}, \tau_{n}^{j}\right) \widetilde{\psi}(0)\right\rangle=\rho\left\langle\bar{\psi}_{\lambda^{\prime}}^{j}\left(\psi_{\lambda}^{j}-\tau_{n}^{j} e^{-i \theta_{\lambda}^{j}} B_{\lambda} \psi_{\lambda}^{j}\right)\right\rangle
$$

As a consequence of Lemma 3.8, we have $\rho=0$ and $\left(T_{\tau_{n}^{j}, \lambda}-i \omega_{\lambda}^{j}\right) \widetilde{\psi}=0$, that is $\widetilde{\psi} \in$ $\mathcal{N}\left[T_{\tau_{n}^{j}, \lambda}-i \omega_{\lambda}^{j}\right]$. By induction, we obtain

$$
\mathcal{N}\left(\left[T_{\tau_{n}^{j}, \lambda}-i \omega_{\lambda}^{j}\right]^{s}\right)=\mathcal{N}\left[T_{\tau_{n, \lambda}^{j}}-i \omega_{\lambda}^{j}\right]
$$

for all $s \in\{1,2,3, \ldots\}$. Hence, $i \omega_{\lambda}^{j}$ is a simple eigenvalue of $T_{\tau_{n}^{j}, \lambda}$.
Note that $\mu=i \omega_{\lambda}^{j}$ is a simple eigenvalue of $T_{\tau_{n}^{j}, \lambda}$. It follows from the implicit function theorem that there are a neighborhood On $O_{n} \times D_{n} \times H_{n} \subset \mathbb{R} \times \mathbb{C} \times X_{\mathbb{C}}$ of $\left(\tau_{n}^{j}, i \omega_{\lambda}^{j}, \psi_{\lambda}^{j}\right)$ and a continuously differential function $(\mu(\tau), \psi(\tau)): O_{n} \rightarrow D_{n} \times H_{n}$ such that for each $\tau \in O_{n}$, the only eigenvalue of $T_{\tau, \lambda}$ in $D_{n}$ is $\mu(\tau)$, and the following equality holds

$$
\begin{equation*}
\Delta(\lambda, \mu(\tau), \tau) \psi(\tau)=A_{\lambda} \psi(\tau)-e^{-\mu(\tau) \tau} B_{\lambda} \psi(\tau)-\mu(\tau) \psi(\tau)=0 \tag{3.28}
\end{equation*}
$$

Moreover, $\mu\left(\tau_{n}^{j}\right)=i \omega_{\lambda}^{j}$ and $\psi\left(\tau_{n}^{j}\right)=\psi_{\lambda}^{j}$. Then we have the following transversality condition, see [21].

Lemma 3.10. Assume that (H1) holds. For $j=1,2, \lambda \in \Lambda$ and $n \in \mathbb{N}_{0}$, assume that $\mu(\tau)$ is the eigenvalue of $T_{\tau, \lambda}$, then

$$
\left.\frac{d \operatorname{Re}\{\mu(\tau)\}}{d \tau}\right|_{\tau=\tau_{n}^{j}}>0
$$

Proof. Differentiating Eq. (3.28) with respect to $\tau$ at $\tau=\tau_{n}^{j}$, we have

$$
\begin{equation*}
\Delta\left(\lambda, i \omega_{\lambda}^{j}, \tau_{n}^{j}\right) \frac{d \psi_{\lambda}^{j}\left(\tau_{n}^{j}\right)}{d \tau}+\left[\tau_{n}^{j} e^{-i \theta_{\lambda}^{j}} B_{\lambda} \psi_{\lambda}^{j}-\psi_{\lambda}^{j}\right] \frac{d \mu\left(\tau_{n}^{j}\right)}{d \tau}+i \omega_{\lambda}^{j} e^{-i \theta_{\lambda}^{j}} B_{\lambda} \psi_{\lambda}^{j}=0 \tag{3.29}
\end{equation*}
$$

By (3.27), we get

$$
\begin{equation*}
\left\langle\bar{\psi}_{\lambda}^{j}, \Delta\left(\lambda, i \omega_{\lambda}^{j}, \tau_{n}^{j}\right) \frac{d \psi_{\lambda}^{j}\left(\tau_{n}^{j}\right)}{d \tau}\right\rangle=\left\langle\Delta\left(\lambda,-i \omega_{\lambda}^{j}, \tau_{n}^{j}\right) \bar{\psi}_{\lambda}^{j} \frac{d \psi_{\lambda}^{j}\left(\tau_{n}^{j}\right)}{d \tau}\right\rangle=0 \tag{3.30}
\end{equation*}
$$

Calculating the inner product with $\psi_{\lambda}^{j}$ in Eq. (3.29) and using Eq. (3.30), we have

$$
S_{n}^{j} \frac{d \mu\left(\tau_{n}^{j}\right)}{d \tau}=\left\langle\bar{\psi}_{\lambda}^{j}, i \omega_{\lambda}^{j} e^{-i \theta_{\lambda}^{j}} B_{\lambda} \psi_{\lambda}^{j}\right\rangle,
$$

where $S_{n}^{j}$ is defined as in Lemma 3.8. Then we have

$$
\frac{d \mu\left(\tau_{n}^{j}\right)}{d \tau}=\frac{I_{1}+I_{2}}{\left(S_{n}^{j}\right)^{2}},
$$

where

$$
I_{1}=\left\langle\bar{\psi}_{\lambda}^{j}, \psi_{\lambda}^{j}\right\rangle\left\langle\bar{\psi}_{\lambda}^{j}, i \omega_{\lambda}^{j} e^{-i \theta_{\lambda}^{j}} B_{\lambda} \psi_{\lambda}^{j}\right\rangle, \quad I_{2}=i \omega_{\lambda}^{j} \tau_{n}^{j}\left|\left\langle\bar{\psi}_{\lambda}^{j}, B_{\lambda} \psi_{\lambda}^{j}\right\rangle\right|^{2} .
$$

Hence, it is clear that

$$
\left.\frac{d \operatorname{Re}\{\mu(\tau)\}}{d \tau}\right|_{\tau=\tau_{n}^{j}}=\operatorname{Re} \frac{I_{1}}{\left|S_{n}^{j}\right|^{2}} .
$$

In fact,

$$
\left\langle\bar{\psi}_{\lambda}^{j}, \psi_{\lambda}^{j}\right\rangle \rightarrow\left[\left(p_{1 \lambda_{*}}^{j}\right)^{2}+\left(p_{2 \lambda_{*}}^{j}\right)^{2}\right], \quad \text { as } \lambda \rightarrow \lambda_{*},
$$

and

$$
\frac{1}{\left(\lambda-\lambda_{*}\right)^{2}}\left\langle\bar{\psi}_{\lambda}^{j}, i \omega_{\lambda}^{j} e^{-i \theta_{\lambda}^{j}} B_{\lambda} \psi_{\lambda}^{j}\right\rangle \rightarrow\left(h_{\lambda_{*}}^{j}\right)^{2}, \quad \text { as } \lambda \rightarrow \lambda_{*} .
$$

Therefore, for $\delta$ enough small, we have $\left.\frac{d \operatorname{Re}\{\mu(\tau)\}}{d \tau}\right|_{\tau=\tau_{n}^{j}}>0$.
From above analysis, we obtain that a pair of purely imaginary eigenvalues will occur as $\tau$ passes $\tau=\tau_{n}^{j}$. The proof of Lemma 3.10 is complete.

From Lemmas 3.8-3.10, we have the result on the distribution of eigenvalues of $T_{\tau, \lambda}$.
Theorem 3.11. Assume that (H1) holds. For $j=1,2, \lambda \in \Lambda$ and $n \in \mathbb{N}_{0}$, the infinitesimal generator $T_{\tau, \lambda}$ has exactly $2(n+1)$ eigenvalues with positive real parts when $\tau \in\left(\tau_{n}^{j}, \tau_{n+1}^{j}\right)$.

## 4 Stability analysis

In this section, we study the stability of the steady state solutions $\left(u_{\lambda}, v_{\lambda}\right)$ of (1.5) by regarding the delay $\tau$ as a parameter. We first investigate the stability of when $\tau=0$, and then discuss the stability and bifurcation when $\tau \neq 0$. We need the following condition (H2).
(H2) $b_{1} c_{2}-b_{2} c_{1}>0$.
Theorem 4.1. Assume (H1)-(H2) hold. For $\lambda \in\left(\lambda_{*}, \lambda_{*}+\delta\right)$ (respectively, $\lambda \in\left(\lambda_{*}-\delta, \lambda_{*}\right)$, then all eigenvalues of $T_{0, \lambda}$ have negative (respectively, positive) real parts, and hence the steady state solution $\left(u_{\lambda}, v_{\lambda}\right)^{T}$ of (1.5) with $\tau=0$ is locally asymptotically stable (respectively, unstable).

Proof. When $\tau=0$, the eigenvalue problem (3.3) reduces to

$$
\begin{equation*}
\Delta(\lambda, \mu, 0) \psi=A_{\lambda} \psi-B_{\lambda} \psi=\mu \psi . \tag{4.1}
\end{equation*}
$$

with $\psi=\left(\psi_{1}, \psi_{2}\right)^{T} \in X_{C}^{2} \backslash\left\{(0,0)^{T}\right\}$. We suppose that

$$
\left\{\begin{array}{l}
\psi_{1}=p_{1} \varphi+\left(\lambda-\lambda_{*}\right) q_{1}(x),  \tag{4.2}\\
\psi_{2}=p_{2} \varphi+\left(\lambda-\lambda_{*}\right) q_{2}(x),
\end{array}\right.
$$

where $p_{1}, p_{2} \in \mathbb{R}, q_{1}(x), q_{2}(x) \in X_{1}$. Then substituting (4.2) into (4.1) and let $\lambda \rightarrow \lambda_{*}$, by calculation, we have

$$
\left\{\begin{align*}
L q_{1}+\left[m(x) e^{a m(x)} \varphi\right. & \left.-\lambda_{*} e^{2 a m(x)}\left(b_{1} \alpha_{\lambda_{*}}+c_{1} \beta_{\lambda_{*}}\right) \varphi^{2}\right] p_{1}  \tag{4.3}\\
& -\lambda_{*} e^{2 a m(x)} \alpha_{\lambda_{*}} \varphi^{2}\left(b_{1} p_{1}+c_{1} p_{2}\right)=\tilde{\mu} e^{a m(x)} \varphi p_{1}, \\
L q_{2}+\left[m(x) e^{a m(x)} \varphi\right. & \left.-\lambda_{*} e^{2 a m(x)}\left(b_{2} \alpha_{\lambda_{*}}+c_{2} \beta_{\lambda_{*}}\right) \varphi^{2}\right] p_{2} \\
& -\lambda_{*} e^{2 a m(x)} \beta_{\lambda_{*}} \varphi^{2}\left(b_{2} p_{1}+c_{2} p_{2}\right)=\tilde{\mu} e^{a m(x)} \varphi p_{2}
\end{align*}\right.
$$

where $\tilde{\mu}=\lim _{\lambda \rightarrow \lambda_{*}} \frac{\mu}{\lambda-\lambda_{*}}$. By (2.5), (4.3) becomes

$$
\left\{\begin{array}{l}
L\left(q_{1}-\xi_{\lambda_{*}} p_{1}\right)-\lambda_{*} e^{2 a m(x)} \alpha_{\lambda_{*}} \varphi^{2}\left(b_{1} p_{1}+c_{1} p_{2}\right)=\tilde{\mu} \varphi p_{1}  \tag{4.4}\\
L\left(q_{2}-\eta_{\lambda_{*}} p_{2}\right)-\lambda_{*} e^{2 a m(x)} \beta_{\lambda_{*}} \varphi^{2}\left(b_{2} p_{1}+c_{2} p_{2}\right)=\tilde{\mu} \varphi p_{2}
\end{array}\right.
$$

Multiplying both sides of each equation in (4.4) by $\varphi$ and integrating on $\Omega$, we have

$$
\left\{\begin{array}{l}
\lambda_{*} d_{2} \alpha_{\lambda_{*}}\left(b_{1} p_{1}+c_{1} p_{2}\right)+\tilde{\mu} p_{1}=0  \tag{4.5}\\
\lambda_{*} d_{2} \beta_{\lambda_{*}}\left(b_{2} p_{1}+c_{2} p_{2}\right)+\tilde{\mu} p_{2}=0
\end{array}\right.
$$

Thus, we get that the eigenvalue equation of $\tilde{\mu}$

$$
\begin{equation*}
\tilde{\mu}^{2}+\lambda_{*} d_{2}\left(\alpha_{\lambda_{*}} b_{1}+\beta_{\lambda_{*}} c_{2}\right) \tilde{\mu}+\lambda_{*}^{2} d_{2}^{2} \alpha_{\lambda_{*}} \beta_{\lambda_{*}}\left(b_{1} c_{2}-b_{2} c_{1}\right)=0 . \tag{4.6}
\end{equation*}
$$

By (H2), we have that the eigenvalue of (4.6) $\tilde{\mu}_{1}, \tilde{\mu}_{2}<0$. Then the conclusion of Theorem 4.1 is obtained.

Theorem 4.2. Assume (H1)-(H2) hold. For $j=1,2, \lambda \in \Lambda, n \in \mathbb{N}_{0}$, then
(1) the steady state solution $\left(u_{\lambda}, v_{\lambda}\right)^{T}$ of (1.5) is locally asymptotically stable when $\tau \in\left[0, \tau_{0}\right)$, where $\tau_{0}=\min \left\{\tau_{0}^{1}, \tau_{0}^{2}\right\}$;
(2) the system (1.5) undergoes a Hopf bifurcation at the steady state solution $\left(u_{\lambda}, v_{\lambda}\right)^{T}$ when $\tau=\tau_{n}^{j}$, i.e., system (1.5) has a branch of periodic solutions bifurcating from the steady state solution $\left(u_{\lambda}, v_{\lambda}\right)^{T}$ near $\tau=\tau_{n}^{j}$.

## 5 Direction of Hopf bifurcation

From the analysis of section 4, we obtained conditions for Hopf bifurcation to occur when $\tau=\tau_{n}^{j}\left(j=1,2, n \in \mathbb{N}_{0}\right)$. In this section, we shall derive the explicit formulae determining the direction, stability, and period of these periodic solutions bifurcating from the equilibrium $\left(u_{\lambda}, v_{\lambda}\right)^{T}$ at $\tau=\tau_{n}^{j}\left(j=1,2, n \in \mathbb{N}_{0}\right)$, by using techniques from normal form and center manifold theory [9,12,14,19,33].

Let $\left(u_{\lambda}, v_{\lambda}\right)^{T}$ is a spatially nonhomogeneous steady-state solution of (1.5). Let

$$
\tilde{u}(t)=u(\cdot, \tau t)-u_{\lambda}, \tilde{v}(t)=v(\cdot, \tau t)-v_{\lambda} .
$$

For the simple, let $U(t)=(u(t), v(t))^{T}=(\tilde{u}(t), \tilde{v}(t))^{T}$, then system (1.5) can be written as follows:

$$
\begin{equation*}
\frac{d U(t)}{d t}=\tau L_{0}\left(U_{t}\right)-\tau L_{1}\left(U_{t}\right)+f\left(U_{t}, \tau\right) \tag{5.1}
\end{equation*}
$$

where $U_{t} \in \mathcal{C}=C^{1}\left([-1,0], Y^{2}\right)$, and

$$
\begin{gather*}
L_{0}(U(t))=\binom{e^{-a m(x)} \nabla \cdot\left[e^{a m(x)} \nabla u\right]+\lambda u(t)\left[m(x)-b_{1} e^{a m(x)} u_{\lambda}-c_{1} e^{a m(x)} v_{\lambda}\right]}{e^{-a m(x)} \nabla \cdot\left[e^{a m(x)} \nabla v\right]+\lambda v(t)\left[m(x)-b_{2} e^{a m(x)} u_{\lambda}-c_{2} e^{a m(x)} v_{\lambda}\right]},  \tag{5.2}\\
L_{1}\left(U_{t}\right)=\binom{\lambda e^{a m(x)} u_{\lambda}\left[b_{1} u(t-1)+c_{1} v(t-1)\right]}{\lambda e^{a m(x)} v_{\lambda}\left[b_{2} u(t-1)+c_{2} v(t-1)\right]},  \tag{5.3}\\
f\left(U_{t}, \tau\right)=\binom{-\tau \lambda e^{a m(x)}\left[b_{1} u(t) u(t-1)+c_{1} u(t) v(t-1)\right]}{-\tau \lambda e^{a m(x)}\left[b_{2} v(t) u(t-1)+c_{2} v(t) v(t-1)\right]} . \tag{5.4}
\end{gather*}
$$

Let $\tau=\tau_{n}^{j}+\varepsilon$, then (5.1) can be rewritten as

$$
\begin{equation*}
\frac{d U(t)}{d t}=\tau_{n}^{j} L_{0}(U(t))-\tau_{n}^{j} L_{1}\left(U_{t}\right)+F\left(U_{t}, \varepsilon\right), \tag{5.5}
\end{equation*}
$$

where

$$
\begin{equation*}
F\left(U_{t}, \varepsilon\right)=\varepsilon L_{0}(U(t))-\varepsilon L_{1}\left(U_{t}\right)+f\left(U_{t}, \tau_{n}^{j}+\varepsilon\right) . \tag{5.6}
\end{equation*}
$$

From the previous discussion, it is clear that when $\varepsilon=0$ (i.e., $\tau=\tau_{n}^{j}$ ) system (5.5) undergoes Hopf bifurcation at the equilibrium $(0,0)$.

It follows from $[14,33]$ that

$$
\begin{equation*}
T_{\tau_{n}^{\prime}} \psi=\dot{\psi}, \tag{5.7}
\end{equation*}
$$

and the domain

$$
\mathcal{D}\left(T_{\tau_{n}^{j}}\right)=\left\{\psi \in \mathcal{C}_{\mathbb{C}} \cap \mathcal{C}_{\mathbb{C}}^{1}: \psi(0) \in X_{\mathbb{C}}, \dot{\psi}(0)=\tau_{n}^{j} L_{0} \psi(0)-\tau_{n}^{j} L_{1} \psi(-1)\right\},
$$

where

$$
\mathcal{C}_{\mathrm{C}}=C\left([-1,0], Y_{\mathrm{C}}^{2}\right), \quad \mathcal{C}_{\mathrm{C}}^{1}=C^{1}\left([-1,0], Y_{\mathrm{C}}^{2}\right) .
$$

We can compute the formal adjoint operator $T_{\tau_{n}^{\prime}}^{*}$ of $T_{\tau_{n}^{\prime}}$ with respect to the formal duality,

$$
\begin{equation*}
T_{\tau_{n}^{\prime}}^{*} \phi=-\dot{\phi}, \tag{5.8}
\end{equation*}
$$

and the domain

$$
\mathcal{D}\left(T_{\tau_{n}^{\prime}}^{*}\right)=\left\{\phi \in \mathcal{C}_{\mathbb{C}}^{*} \cap\left(\mathcal{C}_{\mathbb{C}}^{*}\right)^{1}: \phi(0) \in X_{\mathbb{C}},-\dot{\phi}(0)=\tau_{n}^{j} L_{0} \phi(0)-\tau_{n}^{j} L_{1} \phi(1)\right\},
$$

where

$$
\mathcal{C}_{\mathbb{C}}^{*}=C\left([0,1], Y_{\mathrm{C}}^{2}\right), \quad\left(\mathcal{C}_{\mathrm{C}}^{*}\right)^{1}=C^{1}\left([0,1], Y_{\mathrm{C}}^{2}\right) .
$$

Following [30], we introduce the formal duality $\langle\langle\cdot, \cdot\rangle\rangle$ in $\mathcal{C}_{\mathrm{C}} \times \mathcal{C}_{\mathbb{C}}^{*}$ by

$$
\begin{equation*}
\langle\langle\phi, \psi\rangle\rangle=\langle\phi(0), \psi(0)\rangle_{1}-\tau_{n}^{j} \int_{-1}^{0}\left\langle\phi(s+1), L_{1} \psi(s)\right\rangle_{1} d s, \tag{5.9}
\end{equation*}
$$

for $\psi \in \mathcal{C}_{\mathrm{C}}$ and $\phi \in \mathcal{C}_{\mathrm{C}^{*}}^{*}$, where $\langle\psi, \phi\rangle_{1}=\int_{\Omega} e^{a m(x)} \bar{\psi}^{T} \phi d x$, see [8].
Lemma 5.1. $T_{\tau_{n}^{j}}$ and $T_{\tau_{n}^{\prime}}^{*}$ are adjoint operators, that is

$$
\left\langle\left\langle\phi, T_{\tau_{n}^{\prime}} \psi\right\rangle\right\rangle=\left\langle\left\langle T_{\tau_{n}^{\prime}}^{*} \phi, \psi\right\rangle\right\rangle,
$$

for $\psi \in \mathcal{C}_{\mathbb{C}}, \phi \in \mathcal{C}_{\mathbb{C}}^{*}$.

Proof. It follows from (5.9) and the definition of $T_{\tau_{n}^{j}}, T_{\tau_{n}^{j}}^{*}$ that,

$$
\begin{aligned}
\left\langle\left\langle\phi, T_{\tau_{n}^{i}} \psi\right\rangle\right\rangle= & \left\langle\phi(0), T_{\tau_{n}^{j}} \psi(0)\right\rangle_{1}-\tau_{n}^{j} \int_{-1}^{0}\left\langle\phi(s+1), L_{1} \dot{\psi}(s)\right\rangle_{1} d s \\
= & \left\langle\phi(0), \tau_{n}^{j} L_{0} \psi(0)-\tau_{n}^{j} L_{1} \psi(-1)\right\rangle_{1}-\tau_{n}^{j}\left[\left\langle\phi(s+1), L_{1} \psi(s)\right\rangle_{1}\right]_{-1}^{0} \\
& +\tau_{n}^{j} \int_{-1}^{0}\left\langle\dot{\phi}(s+1), L_{1} \psi(s)\right\rangle_{1} d s \\
= & \left\langle T_{\tau_{n}^{j}}^{*} \phi(0), \psi(0)\right\rangle_{1}+\tau_{n}^{j} \int_{-1}^{0}\left\langle-\dot{\phi}(s+1), L_{1} \psi(s)\right\rangle_{1} d s \\
= & \left\langle\left\langle T_{\tau_{n}^{*}}^{*} \phi, \psi\right\rangle\right\rangle .
\end{aligned}
$$

The proof of Lemma 5.1 is complete.
From Theorem 3.7 and Lemma 5.1, we have that $\pm i \omega_{\lambda}^{j} \tau_{n}^{j}$ are the eigenvalues of $T_{\tau_{n}^{\prime}}$, and they are the eigenvalues of $T_{\tau_{n}^{j}}^{*}$. The vectors $p(\theta)=e^{i \omega_{\lambda}^{j} \tau_{n}^{j} \theta} \psi_{\lambda}^{j}(\theta \in[-1,0])$ and $q(s)=$ $e^{i \omega_{\lambda}^{j} \tau_{n}^{j} s} \bar{\psi}_{\lambda}^{j}(s \in[0,1])$ satisfy

$$
T_{\tau_{n}^{\tau_{n}}} p=i \omega_{\lambda}^{j} \tau_{n}^{j} p, \quad \text { and } \quad T_{\tau_{n}^{j}}^{*} q=i \omega_{\lambda}^{j} \tau_{n}^{j} q,
$$

respectively. Let

$$
\begin{gathered}
\Phi=(p(\theta), \bar{p}(\theta))^{T}, \quad \Psi=\left(\frac{q(s)}{\bar{R}_{n}^{j}}, \frac{\bar{q}(s)}{R_{n}^{j}}\right)^{T}, \\
R_{n}^{j}=\left\langle\bar{\psi}_{\lambda}^{j}, \psi_{\lambda}^{j}-\tau_{n}^{j} e^{-i \theta_{\lambda}^{j}} B_{\lambda} \psi_{\lambda}^{j}\right\rangle_{1} .
\end{gathered}
$$

and $\operatorname{Re}\left\{R_{n}^{j}\right\}=\left\langle\bar{\psi}_{\lambda^{\prime}}\left(\psi_{\lambda}-\tau_{n}^{j} e^{-i \theta_{\lambda}^{j}} B_{\lambda} \psi_{\lambda}\right)\right\rangle_{1} \rightarrow\left[\left(p_{1 \lambda_{*}}^{j}\right)^{2}+\left(p_{2 \lambda_{*}}^{j}\right)^{2}\right] \int_{\Omega} e^{a m(x)} \varphi^{2} d x \neq 0$, as $\lambda \rightarrow \lambda_{*}$. One can easily check that $\langle\langle\Psi, \Phi\rangle\rangle=I$, where $I$ is the identity matrix in $\mathbb{R}^{2 \times 2}$. Moreover, can be decomposed as $\mathcal{C}_{\mathrm{C}}=P \oplus Q$, where

$$
\begin{gathered}
P=\operatorname{span}\{p(\theta), \bar{p}(\theta)\}, \quad P^{*}=\operatorname{span}\{q(s), \bar{q}(s)\}, \\
Q=\left\{\widetilde{\psi} \in \mathcal{C}_{\mathrm{C}}:\langle\langle\widetilde{\psi}, \psi\rangle\rangle=0, \quad \text { for all } \widetilde{\psi} \in P^{*}\right\} .
\end{gathered}
$$

By (5.7), system (5.5) can be transformed into the following

$$
\begin{equation*}
\frac{d U_{t}}{d t}=T_{\tau_{n}^{\prime}} U_{t}+X_{0} F\left(U_{t}, \varepsilon\right), \tag{5.10}
\end{equation*}
$$

where

$$
X_{0}(\theta)= \begin{cases}0, & \theta \in[-1,0)  \tag{5.11}\\ 1, & \theta=0\end{cases}
$$

Let $U_{t}$ be the solution of system (5.10) with $\varepsilon=0$ and set

$$
\begin{equation*}
z(t)=\frac{1}{R_{n}^{j}}\left\langle\left\langle q(s), U_{t}\right\rangle\right\rangle, \tag{5.12}
\end{equation*}
$$

then

$$
\begin{equation*}
\bar{z}(t)=\frac{1}{\bar{R}_{n}^{j}}\left\langle\left\langle\bar{q}(s), U_{t}\right\rangle\right\rangle . \tag{5.13}
\end{equation*}
$$

Let

$$
\begin{equation*}
W(z, \bar{z}, \theta)=W_{20}^{j}(\theta) \frac{z^{2}}{2}+W_{11}^{j}(\theta) z \bar{z}+W_{02}^{j}(\theta) \frac{\bar{z}^{2}}{2}+\cdots \tag{5.14}
\end{equation*}
$$

be the center manifold with the range in $Q$, and the flow of Eq. (5.10) on center manifold can be written as

$$
\begin{equation*}
U_{t}=W(z, \bar{z}, \theta)+p(\theta) z(t)+\bar{p}(\theta) \bar{z}(t) . \tag{5.15}
\end{equation*}
$$

From (5.12) and (5.10), we have that

$$
\begin{align*}
\dot{z}(t) & =\frac{1}{R_{n}^{j}} \frac{d}{d t}\left\langle\left\langle q(s), U_{t}\right\rangle\right\rangle \\
& =\frac{1}{R_{n}^{j}}\left\langle\left\langle q, T_{\tau_{n}^{j}} U_{t}\right\rangle+\frac{1}{R_{n}^{j}}\left\langle\left\langle q(s), X_{0} F\left(U_{t}, 0\right)\right\rangle\right\rangle\right. \\
& =\frac{1}{R_{n}^{j}}\left\langle\left\langle T_{\tau_{n}^{j}}^{*} q, U_{t}\right\rangle+\frac{1}{R_{n}^{j}}\left\langle q(0), F\left(U_{t}, 0\right)\right\rangle_{1}\right.  \tag{5.16}\\
& =i w_{\lambda}^{j} \tau_{n}^{j} z(t)+\frac{1}{R_{n}^{j}}\langle q(0), F(W(z, \bar{z}, \theta)+p(\theta) z(t)+\bar{p}(\theta) \bar{z}(t), 0)\rangle_{1} \\
& =i w_{\lambda}^{j} \tau_{n}^{j} z(t)+g(z, \bar{z}),
\end{align*}
$$

where

$$
g(z, \bar{z})=\frac{1}{R_{n}^{j}}\langle q(0), F(W(z, \bar{z}, \theta)+p(\theta) z(t)+\bar{p}(\theta) \bar{z}(t), 0)\rangle_{1} .
$$

From (5.15), we get

$$
\begin{aligned}
U_{t}(0) & =\psi_{\lambda}^{j} z+\bar{\psi}_{\lambda}^{j} \bar{z}+W_{20}^{j}(0) \frac{z^{2}}{2}+W_{11}^{j}(0) z \bar{z}+W_{02}^{j}(0) \frac{\bar{z}^{2}}{2}+\cdots, \\
U_{t}(-1) & =\psi_{\lambda}^{j} z e^{-i w_{\lambda}^{j} \tau_{n}^{j}}+\bar{\psi}_{\lambda}^{j} \bar{z} e^{i e^{i} \omega_{\lambda}^{j} \tau_{n}^{j}}+W_{20}^{j}(-1) \frac{z^{2}}{2}+W_{11}^{j}(-1) z \bar{z}+W_{02}^{j}(-1) \frac{\bar{z}^{2}}{2}+\cdots
\end{aligned}
$$

From the above three equalities, we get

$$
g(z, \bar{z})=-\frac{\lambda \tau_{n}^{j}}{R_{n}^{j}} \int_{\Omega} e^{2 a m(x)}\left(\psi_{\lambda}^{j}\right)^{T}(U(0) \times C U(-1)) d x=g_{20}^{j} \frac{z^{2}}{2}+g_{11}^{j} z \bar{z}+g_{02}^{j} \frac{\bar{z}^{2}}{2}+g_{21}^{j} \frac{z^{2} \bar{z}}{2}+\cdots,
$$

where

$$
C=\left(\begin{array}{ll}
b_{1} & c_{1} \\
b_{2} & c_{2}
\end{array}\right) \quad \text { and } \quad\binom{a}{b} \times\binom{ c}{d}=\binom{a c}{b d} .
$$

Thus we get

$$
\begin{aligned}
& g_{20}^{j}=-\frac{2 \lambda \tau_{n}^{j}}{R_{n}^{j}} e^{-i w_{\lambda}^{j}} \tau_{n}^{j} \\
& g_{\Omega}^{j} e^{2 a m(x)}\left(\psi_{\lambda}^{j}\right)^{T}\left(\psi_{\lambda}^{j} \times C \psi_{\lambda}^{j}\right) d x, \\
& g_{011}^{j}=\left.-\frac{\lambda \tau \tau_{n}^{j}}{R_{n}^{j}} e^{i w_{\lambda}^{j} \tau_{n}^{j}} \int_{\Omega} e^{2 a m(x)}\left(\psi_{\lambda}^{j}\right)^{T}\left(\psi_{\lambda}^{j} \times C \bar{\psi}_{\lambda}^{j}\right) d x+e^{-i w_{\lambda}^{j} \tau_{n}^{j}} \int_{\Omega} e^{2 a m(x)}\left(\psi_{\lambda}^{j}\right)^{T}\left(\bar{\psi}_{\lambda}^{j} \times C \psi_{\lambda}^{j}\right) d x\right] \\
& g_{02}^{j} \tau_{\lambda}^{j} \tau_{n}^{j} e_{\Omega}^{2 a m(x)}\left(\psi_{\lambda}^{j}\right)^{T}\left(\bar{\psi}_{\lambda}^{j} \times C \bar{\psi}_{\lambda}^{j}\right) d x, \\
& g_{21}^{j}=-\frac{2 \lambda \tau_{n}^{j}}{R_{n}^{j}} \int_{\Omega} e^{2 a m(x)}\left[e^{i w_{\lambda}^{j} \tau_{n}^{j}}\left(\psi_{\lambda}^{j}\right)^{T}\left(W_{20}^{j}(0) \times C \bar{\psi}_{\lambda}^{j}\right)+\left(\psi_{\lambda}^{j}\right)^{T}\left(\bar{\psi}_{\lambda}^{j} \times C W_{20}^{j}(-1)\right)\right] d x \\
&-\frac{2 \lambda \tau_{n}^{j}}{R_{n}^{j}} \int_{\Omega} e^{2 a m(x)}\left[e^{-i w_{\lambda}^{j} \tau_{n}^{j}}\left(\psi_{\lambda}^{j}\right)^{T}\left(W_{11}^{j}(0) \times C \psi_{\lambda}^{j}\right)+\left(\psi_{\lambda}^{j}\right)^{T}\left(\psi_{\lambda}^{j} \times C W_{11}^{j}(-1)\right)\right] d x .
\end{aligned}
$$

Similarly, by (5.16), we have

$$
\begin{equation*}
\dot{\bar{z}}(t)=-i w_{\lambda}^{j} \tau_{n}^{j} \bar{z}(t)+\bar{g}(z, \bar{z})=-i w_{\lambda}^{j} \tau_{n}^{j} \bar{z}(t)+\bar{g}_{20}^{j} \frac{z^{2}}{2}+\bar{g}_{11}^{j} z \bar{z}+\bar{g}_{02}^{j} \frac{\bar{z}^{2}}{2}+\bar{g}_{21}^{j} \frac{z^{2} \bar{z}}{2}+\cdots \tag{5.17}
\end{equation*}
$$

From (5.15), we have

$$
\begin{align*}
\dot{W}_{t}(z, \bar{z}, \theta) & =\frac{d U_{t}}{d t}-p(\theta) \dot{z}(t)-\bar{p}(\theta) \dot{\bar{z}}(t) \\
& =T_{\tau_{n}^{j}} U_{t}+X_{0} F\left(U_{t}, 0\right)-p(\theta) \dot{z}(t)-\bar{p}(\theta) \dot{z}(t) \\
& =T_{\tau_{n}^{j}} W+T_{\tau_{n}^{j}}(p(\theta) z(t)+\bar{p}(\theta) \bar{z}(t))+X_{0} F\left(U_{t}, 0\right)-p(\theta) \dot{z}(t)-\bar{p}(\theta) \dot{\bar{z}}(t)  \tag{5.18}\\
& =T_{\tau_{n}^{j}} W+X_{0} F\left(U_{t}, 0\right)-p(\theta) g(z, \bar{z})-\bar{p}(\theta) \bar{g}(z, \bar{z}) \\
& =T_{\tau_{n}^{j}} W+H(z, \bar{z}, \theta),
\end{align*}
$$

where

$$
\begin{align*}
H(z, \bar{z}, \theta) & =X_{0} F\left(U_{t}, 0\right)-p(\theta) g(z, \bar{z})-\bar{p}(\theta) \bar{g}(z, \bar{z}) \\
& =H_{20}^{j}(\theta) \frac{z^{2}}{2}+H_{11}^{j}(\theta) z \bar{z}+H_{02}^{j}(\theta) \frac{\bar{z}^{2}}{2}+\cdots \tag{5.1}
\end{align*}
$$

By using chain rule,

$$
\begin{equation*}
\dot{W}_{t}=\frac{\partial W(z, \bar{z}, \theta)}{\partial z} \dot{z}+\frac{\partial W(z, \bar{z}, \theta)}{\partial \bar{z}} \dot{\bar{z}} . \tag{5.20}
\end{equation*}
$$

It is from (5.18)-(5.20) and (5.15) that

$$
\left\{\begin{array}{l}
\left(2 i w_{\lambda}^{j} \tau_{n}^{j}-T_{\tau_{n}^{j}}\right) W_{20}^{j}(\theta)=H_{20}^{j}(\theta),  \tag{5.21}\\
-T_{\tau_{n}^{j}} W_{11}^{j}(\theta)=H_{11}^{j}(\theta) .
\end{array}\right.
$$

From (5.19), we get for $\theta \in[-1,0)$,

$$
\left\{\begin{array}{l}
H_{20}^{j}(\theta)=-g_{20}^{j} p(\theta)-\bar{g}_{20}^{j} \bar{p}(\theta),  \tag{5.22}\\
H_{11}^{j}(\theta)=-g_{11}^{j} p(\theta)-\bar{g}_{11}^{j} \bar{p}(\theta),
\end{array}\right.
$$

and for $\theta=0$,

$$
\begin{gathered}
H_{20}^{j}(0)=-g_{20}^{j} p(0)-\bar{g}_{20}^{j} \bar{p}(0)-2 \lambda \tau_{n}^{j} e^{-i w_{\lambda}^{j} \tau_{n}^{j}} e^{a m(x)}\left(\psi_{\lambda}^{j} \times C \psi_{\lambda}^{j}\right), \\
H_{11}^{j}(0)=-g_{11}^{j} p(0)-\bar{g}_{11}^{j} \bar{p}(0)-\lambda \tau_{n}^{j} e^{2 a m(x)}\left[e^{i w_{\lambda}^{j} \tau_{n}^{j}}\left(\psi_{\lambda}^{j} \times C \bar{\psi}_{\lambda}^{j}\right)+e^{-i w_{\lambda}^{j} \tau_{n}^{j}}\left(\bar{\psi}_{\lambda}^{j} \times C \psi_{\lambda}^{j}\right)\right] .
\end{gathered}
$$

It follows (5.21)-(5.22) and the definition of $T_{\tau_{n}^{j}}$ that

$$
\left(W_{20}^{j}\right)^{\prime}(\theta)=2 i w_{\lambda}^{j} \tau_{n}^{j} W_{20}^{j}(\theta)+g_{20}^{j} p(\theta)+\bar{g}_{20}^{j} \bar{p}(\theta) .
$$

Hence,

$$
\begin{equation*}
W_{20}^{j}(\theta)=\frac{i g_{20}^{j}}{w_{\lambda}^{j} \tau_{n}^{j}} p^{j}(0) e^{i w_{\lambda}^{j} \tau_{n}^{j} \theta}+\frac{i \bar{\phi}_{20}^{j}}{3 w_{\lambda}^{j} \tau_{n}^{j}} \bar{p}(0) e^{-i w_{\lambda}^{j} \tau_{n}^{j} \theta}+C_{1 \lambda}^{j} e^{2 i w_{\lambda}^{j} \tau_{n}^{j} \theta}, \tag{5.23}
\end{equation*}
$$

where $C_{1 \lambda}^{j} \in \mathbb{R}^{2}$ is a constant vector. From (5.22), we have that

$$
\begin{equation*}
T_{\tau_{n}^{\prime}} W_{20}^{j}(0)=2 i w_{\lambda}^{j} \tau_{n}^{j} W_{20}^{j}(0)-H_{20}^{j}(0) \tag{5.24}
\end{equation*}
$$

From (5.23)-(5.25) and the definition of $T_{\tau_{n}^{j}}$ in (5.7), we get that

$$
\left.\left(2 i w_{\lambda}^{j} \tau_{n}^{j}-T_{\tau_{n}^{j}}\right) C_{1} e^{2 i w_{\lambda}^{j} \tau_{n}^{j} \theta}\right|_{\theta=0}=-2 \lambda \tau_{n}^{j} e^{-i w_{\lambda}^{j} \tau_{n}^{j}} e^{a m(x)}\left(\psi_{\lambda}^{j} \times C \psi_{\lambda}^{j}\right),
$$

or equivalently,

$$
\begin{equation*}
\triangle\left(\lambda, 2 i w_{\lambda}^{j}, \tau_{n}^{j}\right) C_{1 \lambda}^{j}=2 \lambda e^{-i w_{\lambda}^{j} \tau_{n}^{j}} e^{a m(x)}\left(\psi_{\lambda}^{j} \times C \psi_{\lambda}^{j}\right) . \tag{5.25}
\end{equation*}
$$

Note that $2 i w_{\lambda}^{j}$ is not the eigenvalue of $T_{\tau_{n}^{j}}$ for $\lambda \in \Lambda$ and hence

$$
\begin{equation*}
C_{1 \lambda}^{j}=2 \lambda e^{-i w_{\lambda}^{j} \tau_{n}^{j}} \triangle\left(\lambda, 2 i w_{\lambda}^{j}, \tau_{n}^{j}\right)^{-1}\left(e^{a m(x)}\left(\psi_{\lambda}^{j} \times C \psi_{\lambda}^{j}\right)\right) . \tag{5.26}
\end{equation*}
$$

Similarly, from (5.21)-(5.22) and the definition of $T_{\tau_{n}^{j}}$, we get that

$$
\left(W_{11}^{j}\right)^{\prime}(\theta)=g_{11} p(\theta)+\bar{g}_{11} \bar{p}(\theta) .
$$

Hence,

$$
\begin{equation*}
W_{11}^{j}(\theta)=\frac{i g_{11}}{w_{\lambda}^{j} \tau_{n}^{j}} p(0) e^{i w_{\lambda}^{j} \tau_{n}^{j} \theta}+\frac{i \bar{g}_{11}}{3 w_{\lambda}^{j} \tau_{n}^{j}} \bar{p}(0) e^{-i w_{\lambda}^{j} \tau_{n}^{j} \theta}+C_{2 \lambda}^{j} \tag{5.27}
\end{equation*}
$$

where

$$
\begin{equation*}
C_{2 \lambda}^{j}=\lambda\left(e^{-i w_{\lambda}^{j} \tau_{n}^{j}}+e^{i w_{\lambda}^{j} \tau_{n}^{j}}\right) \triangle\left(\lambda, w_{\lambda}^{j}, \tau_{n}^{j}\right)^{-1}\left(e^{a m(x)}\left(\psi_{\lambda}^{j} \times C \psi_{\lambda}^{j}\right)\right) . \tag{5.28}
\end{equation*}
$$

Lemma 5.2. For $j=1,2, \lambda \in \Lambda$ and $n \in N_{0}, C_{1 \lambda}^{j}$ and $C_{2 \lambda}^{j}$ are defined in (5.26) and (5.27), then

$$
\begin{equation*}
C_{1 \lambda}^{j}=\frac{1}{\lambda-\lambda_{*}}\left(c_{\lambda}^{j} U_{\lambda}+\eta_{\lambda}^{j}\right), \quad C_{2 \lambda}^{j}=\frac{1}{\lambda-\lambda_{*}} \zeta_{\lambda^{\prime}}^{j} \tag{5.29}
\end{equation*}
$$

where $U_{\lambda}=\left(u_{\lambda}, v_{\lambda}\right)^{T}$,

$$
\begin{equation*}
\left\langle U_{\lambda}, \eta_{\lambda}^{j}\right\rangle=0, \quad \lim _{\lambda \rightarrow \lambda_{*}}\left\|\eta_{\lambda}^{j}\right\|_{Y_{\mathrm{C}}^{2}}=0, \quad \lim _{\lambda \rightarrow \lambda_{*}}\left\|\zeta_{\lambda}^{j}\right\|_{Y_{\mathrm{C}}^{2}}=0 \tag{5.30}
\end{equation*}
$$

Moreover,

$$
\lim _{\lambda \rightarrow \lambda_{*}}\left(\lambda-\lambda_{*}\right) c_{\lambda}^{j}= \begin{cases}\frac{2 i}{(2 i-1)} \frac{\left(b_{1} c_{2}-b_{2} c_{1}\right)^{2},}{d_{1}^{2}}, & j=1,  \tag{5.31}\\ \frac{2 i \alpha_{\lambda_{*}}+\lambda_{*}\left(c_{2}^{2}+b_{2}^{2}\right)\left(b_{1} c_{2}-b_{2} c_{1}\right)}{\bar{d}_{1}^{2}\left(\alpha_{\lambda_{*}}^{2}+\beta_{\lambda_{*}}^{2}\right)\left(2 i \alpha_{\lambda *} \beta_{\alpha_{*}}\left(b_{1} c_{2}-b_{2} c_{1}\right)-1\right]}, & j=2 .\end{cases}
$$

Proof. Since $e^{a m(x)} A_{\lambda} U_{\lambda}=0$. Substituting (5.29) into the equation $e^{a m(x)} \times(5.25)$, we obtain

$$
\begin{align*}
& e^{a m(x)} A_{\lambda} \eta_{\lambda}^{j}-e^{a m(x)} B_{\lambda} \eta_{\lambda}^{j} e^{-2 i w_{\lambda}^{j} \tau_{n}^{j}}-2 i \omega_{\lambda}^{j} e^{a m(x)} \eta_{\lambda}^{j}-c_{\lambda}^{j} B_{\lambda} U_{\lambda} e^{a m(x)} e^{-2 i w_{\lambda}^{j} \tau_{n}^{j}}-2 i w_{\lambda}^{j} c_{\lambda}^{j} e^{a m(x)} U_{\lambda} \\
&=2 \lambda\left(\lambda-\lambda_{*}\right) e^{-i w_{\lambda}^{j} \tau_{n}^{j}} e^{2 a m(x)}\left(\psi_{\lambda}^{j} \times C \psi_{\lambda}^{j}\right) . \tag{5.32}
\end{align*}
$$

Calculating the inner product of (5.32) with $U_{\lambda}$, we get

$$
\begin{align*}
&\left\langle U_{\lambda}, e^{a m(x)} B_{\lambda} \eta_{\lambda}^{j}\right\rangle e^{-2 i w_{\lambda}^{j} \tau_{n}^{j}}+c_{\lambda}^{j}\left\langle U_{\lambda}, e^{a m(x)} B_{\lambda} U_{\lambda}\right\rangle e^{-2 i w_{\lambda}^{j} \tau_{n}^{j}}+2 i w_{\lambda}^{j}{ }_{\lambda}^{j}\left\langle U_{\lambda}, e^{a m(x)} U_{\lambda}\right\rangle \\
&=-2 \lambda\left(\lambda-\lambda_{*}\right) e^{-i w_{\lambda}^{j} \tau_{n}^{\tau}}\left\langle U_{\lambda}, e^{2 a m(x)}\left(\psi_{\lambda}^{j} \times C \psi_{\lambda}^{j}\right)\right\rangle . \tag{5.33}
\end{align*}
$$

Then
$\left(\lambda-\lambda_{*}\right) c_{\lambda}^{j}=-\frac{2 \lambda\left(\lambda-\lambda_{*}\right)^{2} e^{-i w_{\lambda}^{j}} \tau_{n}^{j}\left\langle U_{\lambda}, e^{2 a m(x)}\left(\psi_{\lambda}^{j} \times C \psi_{\lambda}^{j}\right)\right\rangle+\left(\lambda-\lambda_{*}\right)\left\langle U_{\lambda}, e^{a m(x)} B_{\lambda} \eta_{\lambda}\right\rangle e^{-2 i w_{\lambda}^{j} \tau_{n}^{j}}}{\left\langle U_{\lambda}, e^{a m(x)} B_{\lambda} U_{\lambda}\right\rangle e^{-2 i w_{\lambda}^{j} \tau_{n}^{j}}+2 i w_{\lambda}^{j}\left\langle U_{\lambda}, e^{a m(x)} U_{\lambda}\right\rangle}$.

Since

$$
\begin{aligned}
& \lim _{\lambda \rightarrow \lambda_{*}}\left(\lambda-\lambda_{*}\right)^{-3}\left\langle U_{\lambda}, e^{a m(x)} B_{\lambda} U_{\lambda}\right\rangle=\lambda_{*} d_{1}\left(\alpha_{\lambda_{*}}^{2}+\beta_{\lambda_{*}}^{2}\right) \int_{\Omega} e^{2 a m(x)} \varphi^{3} d x, \\
& \lim _{\lambda \rightarrow \lambda_{*}}\left(\lambda-\lambda_{*}\right)^{-3} w_{\lambda}^{j}\left\langle U_{\lambda}, e^{a m(x)} U_{\lambda}\right\rangle= \begin{cases}\lambda_{*} d_{1}\left(\alpha_{\lambda_{*}}^{2}+\beta_{\lambda_{*}}^{2}\right) \int_{\Omega} e^{2 a m(x)} \varphi^{3} d x, & j=1, \\
\lambda_{*} d_{1}\left(\alpha_{\lambda_{*}}^{2}+\beta_{\lambda_{*}}^{2}\right) \frac{\left(c_{2}-c_{1}\right)\left(b_{1}-b_{2}\right)}{b_{1} c_{2}-b_{2} c_{1}} \int_{\Omega} e^{2 a m(x)} \varphi^{3} d x, & j=2,\end{cases} \\
& \lim _{\lambda \rightarrow \lambda_{*}}\left(\lambda-\lambda_{*}\right)^{-1} \lambda\left\langle U_{\lambda}, e^{2 a m(x)}\left(\psi_{\lambda}^{j} \times C \psi_{\lambda}^{j}\right)\right\rangle \\
& \quad= \begin{cases}\frac{1}{d_{1}} \lambda_{*}\left(\alpha_{\lambda_{*}}^{2}+\beta_{\lambda_{*}}^{2}\right)\left(b_{1} c_{2}-b_{2} c_{1}\right)^{2} \int_{\Omega} e^{2 a m(x)} \varphi^{3} d x, & j=1, \\
\frac{1}{d_{1}} \lambda_{*} \alpha_{\lambda_{*}} \beta_{\lambda_{*}}\left(c_{1}^{2}+b_{2}^{2}\right)\left(b_{1} c_{2}-b_{2} c_{1}\right) \int_{\Omega} e^{2 a m(x)} \varphi^{3} d x, & j=2 .\end{cases}
\end{aligned}
$$

Hence, there exist $\delta_{1}<\delta, M_{0}, M_{1}>0$ such that for any $\lambda \in\left(\lambda-\delta_{1}, \lambda+\delta_{1}\right)$,

$$
\begin{equation*}
\left(\lambda-\lambda_{*}\right) c_{\lambda}^{j} \leq M_{0}\left\|\eta_{\lambda}^{j}\right\|_{\gamma_{C}^{2}}+M_{1} . \tag{5.34}
\end{equation*}
$$

Calculating the inner product of (5.32) with $\eta_{\lambda}$, we obtain

$$
\begin{align*}
& \left\langle\eta_{\lambda}^{j}, e^{a m(x)} A_{\lambda} \eta_{\lambda}^{j}\right\rangle-\left\langle\eta_{\lambda}^{j}, e^{a m(x)} B_{\lambda} \eta_{\lambda}^{j}\right\rangle e^{-2 i w_{\lambda}^{j} \tau_{n}^{j}} \\
& \quad-2 i w_{\lambda}^{j}\left\langle\eta_{\lambda}^{j}, e^{a m(x)} \eta_{\lambda}^{j}\right\rangle-c_{\lambda}^{j}\left\langle\eta_{\lambda}^{j}, e^{a m(x)} B_{\lambda} U_{\lambda}\right\rangle e^{-2 i w_{\lambda}^{j} \tau_{n}^{j}}  \tag{5.35}\\
& =2 \lambda\left(\lambda-\lambda_{*}\right) e^{-i w_{\lambda}^{j} \tau_{n}^{j}}\left\langle\eta_{\lambda}^{j}, e^{2 a m(x)}\left(\psi_{\lambda}^{j} \times C \psi_{\lambda}^{j}\right)\right\rangle .
\end{align*}
$$

From (5.35), it follows that there exist constants $\delta_{2}<\delta_{1}, M_{2}, M_{3}>0$ such that for any $\lambda \in$ $\left(\lambda-\delta_{2}, \lambda+\delta_{2}\right)$,

$$
\begin{equation*}
\lambda_{2}(\lambda)\left\|\eta_{\lambda}^{j}\right\|_{Y_{\mathrm{C}}^{2}}^{2} \leq\left(\lambda-\lambda_{*}\right) M_{2}\left\|\eta_{\lambda}^{j}\right\|_{Y_{\mathrm{C}}}^{2}+\left(\lambda-\lambda_{*}\right) M_{3}\left\|\eta_{\lambda}^{j}\right\|_{Y_{\mathrm{C}}} . \tag{5.36}
\end{equation*}
$$

Similar to the proof of Lemma 2.3 of [3], we have $\left|\left\langle e^{a m(x)} A_{\lambda} U_{\lambda}, U_{\lambda}\right\rangle\right| \geq\left|\lambda_{2}(\lambda)\right|\left\|U_{\lambda}\right\|_{Y_{\mathrm{C}}^{2}}^{2}$ and $\lambda_{2}(\lambda)$ is the second eigenvalue of $e^{a m(x)} A_{\lambda}$. Then we have $\lim _{\lambda \rightarrow \lambda_{*}}\left\|\zeta_{\lambda}^{j}\right\|_{\gamma_{C}^{2}}=0$. From all, we can obtain (5.31). This completes the proof of Lemma 5.2.

## Remark 5.3.

(1) When $v=0, b_{1}=1$ in (1.4), (5.31) in Lemma 5.2 is as same as that in Lemma 3.2 in [8].
(2) When $a_{1}=0$ in (1.4), (5.31) in Lemma 5.2 is as same as that [18, p. 106].

Therefore, one can easily check

$$
\lim _{\lambda \rightarrow \lambda_{*}}\left(\lambda-\lambda_{*}\right) g_{11}^{j}=0, \quad \lim _{\lambda \rightarrow \lambda_{*}} \operatorname{Re}\left[\left(\lambda-\lambda_{*}\right)^{2} g_{21}^{j}\right]<0
$$

It is well-known that the real part of the following quantity determines the direction and stability of bifurcating periodic orbits (see [14, 19,33]):

$$
c_{1}^{j}(0)=\frac{i}{2 w_{\lambda}^{j} \tau_{n}^{j}}\left(g_{20}^{j} g_{11}^{j}-2\left|g_{11}^{j}\right|^{2}-\frac{1}{3}\left|g_{02}^{j}\right|^{2}\right)+\frac{1}{2} g_{21}^{j} .
$$

It follows from (5.31) that $\lim _{\lambda \rightarrow \lambda_{*}} \operatorname{Re}\left[\left(\lambda-\lambda_{*}\right)^{2} c_{1}^{j}(0)\right]<0$. Hence we have the following result.
Theorem 5.4. Assume (H1)-(H2) hold. Then for $j=1,2, \lambda \in \Lambda$ and $n \in N_{0}$, system (1.5) has a branch of bifurcating periodic solutions emerging from the steady state solution $\left(u_{\lambda}, v_{\lambda}\right)^{T}$ for $\tau$ near $\tau_{n}^{j}$. More precisely, the direction of the Hopf bifurcation at $\tau_{n}^{j}$ is forward and the bifurcating periodic solution from $\tau_{n}^{j}$ have the same stability as the steady state solution $\left(u_{\lambda}, v_{\lambda}\right)^{T}$.

## 6 Simulations

In this section, some numerical simulations for model (1.4) are given to illustrate the results of Theorem 5.4.

In (1.4), choose

$$
\begin{gathered}
\Omega=(0, \pi), \quad m(x)=\sin x \\
d=0.20, \quad b_{1}=0.04, \quad b_{2}=0.02, \quad c_{1}=0.03, \quad c_{2}=0.04
\end{gathered}
$$

and the initial value conditions:

$$
u(x, t)=v(x, t)=\sin x, \quad \text { for } t \in[-r, 0] .
$$

Example 6.1. Model (1.4) without advection, that is $a_{1}=0$.
(1) When $r=1$, solutions of model (1.4) without advection tend to a positive steady state. See Fig. 6.1.
(2) When $r=5$, solutions of model (1.4) without advection tend to periodically oscillatory orbit, that is, model (1.4) undergoes a Hopf bifurcation. See Fig. 6.2.



Figure 6.1: Solutions of model (1.4) without advection $\left(a_{1}=0\right)$ tend to a positive steady state when $r=1$.


Figure 6.2: Model (1.4) without advection $\left(a_{1}=0\right)$ undergoes a Hopf bifurcation when $r=4$.

Example 6.2. Model (1.4) with advection, that is $a_{1}=0.5$.
When $r=1$, solutions of model (1.4) with advection tend to a positive steady state. See Fig. 6.3.

When $r=5$, solutions of model (1.4) with advection tend to periodically oscillatory orbit, that is, model (1.4) undergoes a Hopf bifurcation. See Fig. 6.4.


Figure 6.3: Solutions of model (1.4) with advection $\left(a_{1}=0.5\right)$ tend to a positive steady state when $r=1$.


Figure 6.4: Model (1.4) with advection $\left(a_{1}=0.5\right)$ undergoes a Hopf bifurcation when $r=4$.

## Acknowledgements

We are grateful to the editor and the referee for their helpful comments and suggestions. This work was supported by the National Natural Science Foundation of China (No. 61873154, A011403) and Shanxi Natural Science Foundation (No. 201901D111009).

## References

[1] I. Averill, Y. Lou, D. Munther, On several conjectures from evolution of dispersal, J. Biol. Dyn. 6(2012), No. 2, 117-130. https://doi.org/10.1080/17513758.2010.529169; Zbl 1444.92132.
[2] F. Belgacem, C. Cosner, The effects of dispersal along environmental gradients on the dynamics of populations in heterogeneous environment, Canad. Appl. Math. Quart. 3(1995), 379-397. MR1372792; Zbl 0854.35053.
[3] S. Busenberg, W. Huang, Stability and Hopf bifurcation for a population delay model with diffusion effects, J. Differential Equations 124(1996), No. 1, 80-107. https://doi .org/ 10.1006/jdeq.1996.0003; MR1368062; Zbl 0854.35120.
[4] R. S. Cantrell, C. Cosner, Y. Lou, Movement towards better environments and the evolution of rapid diffusion, Math. Biosci. 240(2006), 199-214. https ://doi.org/10.1016/ j.mbs.2006.09.003; MR2290095; Zbl 1105.92036.
[5] R. S. Cantrell, C. Cosner, Y. Lou, Advection-mediated coexistence of competing species, Proc. Roy. Soc. Edinburgh Sect. A 137(2007), 497-518. https://doi.org/10.1017/ S0308210506000047; MR2332679; Zbl 1139.35048.
[6] X. Chen, R. Hambrock, Y. Lou, Evolution of conditional dispersal: a reaction-diffusionadvection model, J. Math. Biol. 57(2008), 361-386. https://doi.org/10.1007/s00285-008-0166-2; MR2411225; Zbl 1141.92040.
[7] X. Chen, K. Y. Lam, Y. Lou, Dynamics of a reaction-diffusion-advection model for two competing species, Discrete Contin. Dyn. Syst. 32(2012), 3841-3859. https://doi. org/10. 3934/dcds.2012.32.3841; MR2945810; Zbl 1258.35118.
[8] S. Chen, Y. Lou, J. Wei, Hopf bifurcation in a delayed reaction-diffusion-advection population model, J. Differential Equations 264(2018), 5333-5359. https://doi.org/10.1016/ j.jde.2018.01.008; MR3760176; Zbl 1383.35021.
[9] S. N. Chow, J. K. Hale, Methods of bifurcation theory, Springer, New York, 1982. https: //doi.org/10.1002/zamm.19840640613;MR0660633; Zbl 0487.47039.
[10] C. Cosner, Beyond diffusion: Conditional dispersal in ecological models, in: Infinite dimensional dynamical systems, Fields Inst. Commun., Vol. 64, Springer, New York, 2013, pp. 305-317. https://doi.org/10.1007/978-1-4614-4523-4_12; MR2986941.
[11] C. Cosner, Reaction-diffusion-advection models for the effects and evolution of dispersal, Discrete Contin. Dyn. Syst. 34(2014), 1701-1745. https://doi.org/10.3934/dcds. 2014.34.1701; MR3124710; Zbl 1277.35002.
[12] T. Faria, L. T. Magalhães, Normal form for retarded functional differential equations and applications to Bogdanov-Takens singularity, J. Differential Equations 122(1995), 201224. https://doi.org/10.1006/jdeq.1995.1145; MR1355889; Zbl 0836.34069.
[13] R. Gejji, Y. Lou, D. Munther, J. Peyton, Evolutionary convergence to ideal free dispersal strategies and coexistence, Bull. Math. Biol. 74(2012), 257-299. https://doi. org/10. 1007/s11538-011-9662-4; MR2881462; Zbl 1319.92035.
[14] S. Guo, J. Wu, Bifurcation of functional differential equations, Springer-Verlag, New York, 2013. https://doi.org/10.1007/978-1-4614-6992-6; MR3098815.
[15] S. Guo, S. Yan, Hopf bifurcation in a diffusive Lotka-Volterra type system with nonlocal delay effect, J. Differential Equations 260(2016), 781-817. https://doi.org/10.1016/j . jde.2015.09.031; MR3411690; Zbl 1330.35029.
[16] J. K. Hale, Theory of functional differential equation, Springer-Verlag, New York, 1977. https://doi.org/10.1007/978-1-4612-9892-2; MR0508721.
[17] R. Hambrock, Y. Lou, The evolution of conditional dispersal strategy in spatially heterogeneous habitats, Bull. Math. Biol. 71(2009), 1793-1817. https://doi.org/10.1007/ s11538-009-9425-7; MR2551690.
[18] R. Han, B. Dai, Hopf bifurcation in a reaction-diffusive two-species model with nonlocal delay effect and general functional response, Chaos Solitons Fractals 96(2017), 90-109. https://doi.org/10.1016/j.chaos.2016.12.022; MR3612685; Zbl 1372.35033.
[19] B. D. Hassard, N. D. Kazarinoff, Y. H. Wan, Theory and applications of Hopf bifurcation, Cambridge University Press, Cambridge, 1981. https://doi.org/10.1002/zamm. 19820621221; MR0603442; Zbl 0474.34002.
[20] R. Hu, Y. Yuan, Spatially nonhomogeneous equilibrium in a reaction-diffusion system with distributed delay, J. Differential Equations 250(2011), No. 6, 2779-2806. https://doi . org/10.1016/j.jde.2011.01.011; MR2771266; Zbl 1228.35034.
[21] Y. Kuang, Delay differential equations with applications in population dynamics, Mathematics in Science and Engineering, Vol. 191, Academic Press, Boston, 1993. MR1218880.
[22] K. Y. Lam, Y. Lou, Evolution of dispersal: ESS in spatial models, J. Math. Biol. 68(2014), 851-877. https://doi.org/10.1007/s00285-013-0650-1; MR3169066.
[23] K. Y. Lam, Y. Lou, Evolutionarily stable and convergent stable strategies in reactiondiffusion models for conditional dispersal, Bull. Math. Biol. 76(2014), 261-291. https: //doi.org/10.1007/s00285-013-0650-1-y; MR3165580; Zbl 1302.92348.
[24] Y. Lou, On the effects of migration and spatial heterogeneity on single and multiple species, J. Differential Equations 223(2006), 400-426. https://doi.org/10.1016/j.jde. 2005.05.010; MR2214941; Zbl 1097.35079.
[25] Y. Lou, Some challenging mathematical problems in evolution of dispersal and population dynamics, in: Tutorials in mathematical biosciences. IV, Lecture Notes in Mathematics, Vol. 1922, Springer, Berlin, 2008, pp. 171-205. https://doi.org/10.1007/978-3-540-74331-6_5; MR2392287; Zbl 1300.92083.
[26] Y. Lou, Some reaction diffusion models in spatial ecology, Sci. Sin. Math 45(2015), No. 10, 1619-1634. https://doi.org/10.1360/N012015-00233.
[27] Y. Lou, F. Lutscher, Evolution of dispersal in open advective environments, J. Math. Biol. 69(2014), 1319-1342. https://doi.org/10.1007/s00285-013-0730-2; MR3275198; Zbl 1307.35144.
[28] Y. Lou, P. Zhou, Evolution of dispersal in advective homogeneous environment: the effect of boundary conditions, J. Differential Equations 259(2015), 141-171. https://doi. org/10.1016/j.jde.2015.02.004; MR3335923; Zbl 1433.35171.
[29] W. M. Ni, The mathematics of diffusion, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 82, SIAM, Philadelphia, 2011. MR2866937; Zbl 1230.35003.
[30] Y. Su, J. Wei, J. Shi, Hopf bifurcation in a diffusive logistic equation with mixed delayed and instantaneous density dependence, J. Dynam. Differential Equations 24(2012), No. 4, 897-925. https://doi.org/10.1007/s10884-012-9268-z; MR3000609; Zbl 1263.35028.
[31] X. P. Yan, C. Zhang, Direction of Hopf bifurcation in a delayed Lotka-Volterra competition diffusion system, Nonlinear Anal. Real World Appl. 10(2009), 2758-2773. https: //doi.org/10.1016/j.nonrwa. 2008.08.004; MR2523239; Zbl 1162.92044.
[32] X. P. Yan, W. T. Li, Stability of bifurcating periodic solutions in a delayed reactiondiffusion population model, Nonlinearity 23(2010), No. 6, 1413-1431. https://doi.org/ 10.1088/0951-7715/23/6/008; MR2646072; Zbl 1198.37080.
[33] J. Wu, Theory and applications of partial functional differential equations, Springer-Verlag, New York, 1996. https://doi.org/10.1007/978-1-4612-4050-1; MR1415838.
[34] W. Zuo, J. Wei, Stability and Hopf bifurcation in a diffusive predator-prey system with delay effect, Nonlinear Anal. 12(2011), No. 4, 1998-2011. https://doi.org/10.1016/j. nonrwa.2010.12.016; MR2800995; Zbl 1221.35053.

# New results on the existence of ground state solutions for generalized quasilinear Schrödinger equations coupled with the Chern-Simons gauge theory 

Yingying Xiao ${ }^{1,2}$ and Chuanxi Zhu ${ }^{\boxtimes 1}$<br>${ }^{1}$ Department of Mathematics, Nanchang University, Nanchang 330031, Jiangxi, P. R. China<br>${ }^{2}$ Nanchang JiaoTong Institute, Nanchang, 330031, Jiangxi, P. R. China

Received 16 February 2021, appeared 22 September 2021
Communicated by Roberto Livrea

Abstract. In this paper, we study the following quasilinear Schrödinger equation

$$
\begin{aligned}
-\Delta u & +V(x) u-\kappa u \Delta\left(u^{2}\right)+\mu \frac{h^{2}(|x|)}{|x|^{2}}\left(1+\kappa u^{2}\right) u \\
& +\mu\left(\int_{|x|}^{+\infty} \frac{h(s)}{s}\left(2+\kappa u^{2}(s)\right) u^{2}(s) \mathrm{d} s\right) u=f(u) \quad \text { in } \mathbb{R}^{2}
\end{aligned}
$$

where $\kappa>0, \mu>0, V \in \mathcal{C}^{1}\left(\mathbb{R}^{2}, \mathbb{R}\right)$ and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$. By using a constraint minimization of Pohožaev-Nehari type and analytic techniques, we obtain the existence of ground state solutions.
Keywords: gauged Schrödinger equation, Pohožaev identity, ground state solutions.
2020 Mathematics Subject Classification: 35J60, 35J20.

## 1 Introduction

In this paper, we are interested in the existence of ground state solutions for the following nonlocal quasilinear Schrödinger equation

$$
\begin{align*}
-\Delta u & +V(x) u-\kappa u \Delta\left(u^{2}\right)+\mu \frac{h^{2}(|x|)}{|x|^{2}}\left(1+\kappa u^{2}\right) u  \tag{1.1}\\
& +\mu\left(\int_{|x|}^{+\infty} \frac{h(s)}{s}\left(2+\kappa u^{2}(s)\right) u^{2}(s) \mathrm{d} s\right) u=f(u) \quad \text { in } \mathbb{R}^{2},
\end{align*}
$$

where $u: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is a radially symmetric function, $\kappa, \mu$ are positive constants, $h(s)=$ $\int_{0}^{s} u^{2}(l) l \mathrm{~d} l(s \geq 0)$ and the nonlinearity $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following suitable assumptions:

[^35]$\left(f_{1}\right) \lim _{|s| \rightarrow 0} \frac{f(s)}{s}=0$ and there exist constants $C>0$ and $q \in(2,+\infty)$ such that
$$
|f(s)| \leq C\left(1+|s|^{q-1}\right), \quad \forall s \in \mathbb{R} ;
$$
$\left(f_{2}\right)$ there exists a constant $p \in(6,8)$ such that $\lim _{|s| \rightarrow+\infty} \frac{F(s)}{|s|^{p}}=+\infty$, where $F(s)=\int_{0}^{s} f(t) \mathrm{d} t$;
$\left(f_{3}\right) \frac{[f(s) s-(8-p) F(s)]}{\left.|s|\right|^{p-1}}$ is nondecreasing on both $(-\infty, 0)$ and $(0,+\infty)$.
Moreover, we assume that potential $V: \mathbb{R}^{2} \rightarrow \mathbb{R}$ verifies:
$\left(V_{1}\right) V \in \mathcal{C}^{1}\left(\mathbb{R}^{2}, \mathbb{R}\right)$ and $V_{\infty}:=\lim _{|y| \rightarrow+\infty} V(y)>V_{0}:=\min _{x \in \mathbb{R}^{2}} V(x)>0$ for all $x \in \mathbb{R}^{2}$;
$\left(V_{2}\right) t \rightarrow t^{6 \alpha-2}[(2 \alpha-2) V(t x)-\nabla V(t x) \cdot(t x)]$ is nondecreasing on $(0,+\infty)$ for any $x \in \mathbb{R}^{2}$, where $\alpha:=\frac{2}{8-p}>1$, which is inspired by [6] where Kirchhoff-type problems were studied.

If $\kappa=0,(1.1)$ turns into the following nonlocal elliptic problem

$$
\begin{equation*}
-\Delta u+V(x) u+\mu \frac{h^{2}(|x|)}{|x|^{2}} u+2 \mu\left(\int_{|x|}^{+\infty} \frac{h(s)}{s} u^{2}(s) \mathrm{d} s\right) u=f(u) \quad \text { in } \mathbb{R}^{2} . \tag{1.2}
\end{equation*}
$$

(1.2) appears in the study of the following Chern-Simons-Schrödinger system

$$
\left\{\begin{array}{l}
i D_{0} \phi+\left(D_{1} D_{1}+D_{2} D_{2}\right) \phi+f(\phi)=0,  \tag{1.3}\\
\partial_{0} A_{1}-\partial_{1} A_{0}=-\operatorname{Im}\left(\bar{\phi} D_{2} \phi\right), \\
\partial_{0} A_{2}-\partial_{2} A_{0}=-\operatorname{Im}\left(\bar{\phi} D_{1} \phi\right), \\
\partial_{1} A_{2}-\partial_{2} A_{1}=-\frac{1}{2}|\phi|^{2},
\end{array}\right.
$$

where $i$ denotes the imaginary unit, $\partial_{0}=\frac{\partial}{\partial t}, \partial_{1}=\frac{\partial}{\partial x_{1}}, \partial_{2}=\frac{\partial}{\partial x_{2}}$ for $\left(t, x_{1}, x_{2}\right) \in \mathbb{R}^{1+2}, \phi$ : $\mathbb{R}^{1+2} \rightarrow \mathbb{C}$ is the complex scalar field, $A_{\mu}: \mathbb{R}^{1+2} \rightarrow \mathbb{R}$ is the gauge field, $D_{\mu}=\partial_{\mu}+i A_{\mu}$ is the covariant derivative for $\mu=0,1,2$. Model (1.3) was first proposed and studied in [12,13], which described the non-relativistic thermodynamic behavior of large number of particles in an electromagnetic field. In [1], the authors considered the standing waves of system (1.3) with power type nonlinearity, that is, $f(u)=\lambda|u|^{p-1} u$, and established the existence and nonexistence of positeve solutions for (1.3) of type

$$
\begin{array}{rlrl}
\phi(t, x) & =u(|x|) e^{i w t}, & A_{0}(t, x) & =k(|x|) \\
A_{1}(t, x) & =\frac{x_{2}}{|x|^{2}} h(|x|), & A_{2}(t, x)=-\frac{x_{1}}{|x|^{2}} h(|x|), \tag{1.4}
\end{array}
$$

where $w>0$ is a given frequency, $\lambda>0$ and $p>1, u, k, h$ are real valued functions depending only on $|x|$. The ansatz (1.4) satisfies the Coulomb gauge condition $\partial_{1} A_{1}+\partial_{2} A_{2}=0$. Byeon et al. [1] got the following nonlocal semi-linear elliptic equation

$$
\begin{equation*}
-\Delta u+w u+\frac{h^{2}(|x|)}{|x|^{2}} u+\left(\int_{|x|}^{+\infty} \frac{h(s)}{s} u^{2}(s) \mathrm{d} s\right) u=\lambda|u|^{p-1} u \quad \text { in } \mathbb{R}^{2} . \tag{1.5}
\end{equation*}
$$

Later, based on the work of [1], the results for the case $p \in(1,3)$ have been extended by Pomponio and Ruiz in [20]. They investigated the geometry of the functional associated with (1.5) and obtained an explicit threshold value for $w$. The existence and properties of ground
state solutions of (1.5) have also been studied widely by many researchers, see, e.g., [2,7,10,11, $14,19,21,29,31,33,35]$ and references therein. If we replace $w>0$ with the radially symmetric potential $V$ and more general nonlinearity $f$, then (1.5) will turns into (1.2). Very recently, by using variational methods, Chen et al. in [4] studied the existence of sign-changing multibump solutions for (1.2) with deepening potential. In [25], when $f$ satisfied more general 6 -superlinear conditions, Tang et al. proved the existence and multiplicity results of (1.2). For more related work about the problem (1.2), we refer to $[9,15,28,35]$ and references therein.

If $\mu=0$, (1.1) reduces to the following quasilinear elliptic problem

$$
\begin{equation*}
-\Delta u+V(x) u-\kappa u \Delta\left(u^{2}\right)=f(u) \quad \text { in } \mathbb{R}^{2} . \tag{1.6}
\end{equation*}
$$

(1.6) is obtained from the quasilinear Schrödinger equation

$$
i \hat{\phi}_{t}+\Delta \hat{\phi}-W(x) \hat{\phi}+\kappa \hat{\phi} \Delta\left(|\hat{\phi}|^{2}\right)+\hat{h}\left(|\hat{\phi}|^{2}\right) \hat{\phi}=0 \quad \text { in } \mathbb{R}^{2},
$$

by setting $\hat{\phi}=e^{-i w t} u(x), V(x)=W(x)-w$, where $w \in \mathbb{R}, W$ is a given potential, $\hat{h}$ is a suitable function. The existence and properties of ground state solutions of (1.6) as well as the stability of standing wave solutions have also been studied widely in $[16,32]$ and references therein.

Motivated by [3,8], we try to establish the existence of positive ground state solutions for (1.1) involving radially symmetric variable potential $V$ and more general nonlinearity $f$ than [8]. Compared to [3], the equation (1.1) has appearance the Chern-Simons terms

$$
\left(\int_{|x|}^{+\infty} \frac{h(s)}{s} u^{2}(s) \mathrm{d} s+\frac{h^{2}(|x|)}{|x|^{2}}\right) u
$$

so that the equation (1.1) is no longer a pointwise identity. This nonlocal term causes some mathematical difficulties that make the study of it is rough and particularly interesting. To overcome these difficulties, we adopted a constraint minimization of the Pohožaev-Nehari type as in $[5,8]$ and establish some new inequalities.

In order to state our main theorem, let us define the metric space

$$
\chi=\left\{u \in H_{r}^{1}\left(\mathbb{R}^{2}\right): \int_{\mathbb{R}^{2}} u^{2}\left|\nabla u^{2}\right| \mathrm{d} x<+\infty\right\}=\left\{u \in H_{r}^{1}\left(\mathbb{R}^{2}\right): u^{2} \in H_{r}^{1}\left(\mathbb{R}^{2}\right)\right\},
$$

endowed with the distance

$$
d_{\chi}(u, v)=\|u-v\|+\left\|\nabla\left(u^{2}\right)-\nabla\left(v^{2}\right)\right\|_{L^{2}} .
$$

We will show that (1.1) can obtain the following energy functional: $I: \chi \rightarrow \mathbb{R}$,

$$
\begin{align*}
I(u)= & \frac{1}{2} \int_{\mathbb{R}^{2}}\left[\left(1+2 \kappa u^{2}\right)|\nabla u|^{2}+V(x) u^{2}\right] \mathrm{d} x+\frac{\mu}{2} \int_{\mathbb{R}^{2}} \frac{u^{2}(x)}{|x|^{2}}\left(\int_{0}^{|x|} s u^{2}(s) \mathrm{d} s\right)^{2} \mathrm{~d} x  \tag{1.7}\\
& +\frac{\mu}{4} \kappa \int_{\mathbb{R}^{2}} \frac{u^{4}(x)}{|x|^{2}}\left(\int_{0}^{|x|} s u^{2}(s) \mathrm{d} s\right)^{2} d x-\int_{\mathbb{R}^{2}} F(u) \mathrm{d} x, \quad \forall u \in \chi .
\end{align*}
$$

Similarly to $[1,8,16,22,29]$, any weak solution $u$ of (1.1) satisfies the Pohožaev identity, that is, $P(u)=0$. For the nice properties of the generalized Nehari manifold, we refer to previous works in $[17,18,34]$ and references therein. Inspired by this fact, we define the following Pohožaev-Nehari functional $\Gamma(u)=\alpha N(u)-P(u)$ and the Pohožaev-Nehari manifold of $I$

$$
\mathcal{M}:=\{u \in \chi \backslash\{0\}: \Gamma(u)=0\} .
$$

Although $\chi$ is not a vector space (it is not close with the respect to the sum), it is easy to check that $I$ is well-defined and continuous on $\chi$. For any $\varphi \in \mathcal{C}_{0, r}^{\infty}\left(\mathbb{R}^{2}\right), u \in \chi$ and $u+\varphi \in \chi$, we can compute the Gateaux derivative

$$
\begin{array}{r}
\left\langle I^{\prime}(u), \varphi\right\rangle=\int_{\mathbb{R}^{2}}\left\{\left(1+2 \kappa u^{2}\right) \nabla u \cdot \nabla \varphi+2 \kappa u|\nabla u|^{2} \varphi+V(x) u \varphi+\mu \frac{h^{2}(|x|)}{|x|^{2}}\left(1+\kappa u^{2}\right) u \varphi\right\} \mathrm{d} x \\
+\mu \int_{\mathbb{R}^{2}}\left(\int_{|x|}^{+\infty} \frac{h(s)}{s}\left(2+\kappa u^{2}(s)\right) u^{2}(s) \mathrm{d} s\right) u \varphi \mathrm{~d} x-\int_{\mathbb{R}^{2}} f(u) \varphi \mathrm{d} x . \tag{1.8}
\end{array}
$$

Then $u \in \chi$ is a weak solution of (1.1) if and only if the Gateaux derivative of $I$ along any direction $\varphi \in \mathcal{C}_{0, r}^{\infty}\left(\mathbb{R}^{2}\right)$ vanishes (see Proposition 2.2 below). A radial weak solution is called a radial ground state solution if it has the least energy among all nontrivial radial weak solutions.

Our main result is the following theorem.
Theorem 1.1. Assume that $\left(V_{1}\right)-\left(V_{2}\right)$ and $\left(f_{1}\right)-\left(f_{3}\right)$ are satisfied. Then (1.1) has a positive ground state solution $\bar{u} \in \chi \backslash\{0\} \cap \mathcal{C}^{2}\left(\mathbb{R}^{2}\right)$, such that $I(\bar{u})=\inf _{u \in \mathcal{M}} I(u)=\inf _{u \in \chi \backslash\{0\}} \max _{t>0} I\left(u_{t}\right)$ where $u_{t}=(u)_{t}:=t^{\alpha} u(t x)$.

Remark 1.2. Theorem 1.1 can be viewed as a partial extension to the counterpart of the result and method in [8]. The assumptions on $f$ in this paper are from the reference [5]. Furthermore, by [5, Remark 1.4],

$$
f(u)=\left(|u|^{p-2}-a|u|^{q-2}\right) u,
$$

satisfies $\left(f_{1}\right)-\left(f_{3}\right)$ when $a>0$ and $2<q<p \in(6,8]$.
To prove the Theorem 1.1, by using some new techniques and inequalities related to $I(u)$, $I\left(u_{t}\right)$ and $\Gamma(u)$, as performed in $[3,5,24]$, we prove that a minimizing sequence $\left\{u_{n}\right\} \subset \chi$ of $\inf _{u \in \mathcal{M}} I(u)$ weakly converges to some nontrivial $\bar{u}$ in $\chi$ (after a translation and extraction of a subsequence ) and $\bar{u} \in \mathcal{M}$ is a minimizer of $\inf _{u \in \mathcal{M}} I(u)$.

Notations. Throughout this paper, we make use of the following notations:

- $V_{\infty}$ is a positive constant;
- $C, C_{0}, C_{1}, C_{2}, \ldots$ denote positive constants, not necessarily the same one;
- $L^{r}\left(\mathbb{R}^{2}\right)$ denotes the Lebesgue space with norm $\|u\|_{L^{r}}=\left(\int_{\mathbb{R}^{2}}|u|^{r} \mathrm{~d} x\right)^{1 / r}$, where $1 \leq r<$ $+\infty$;
- $H^{1}\left(\mathbb{R}^{2}\right)$ denotes a Sobolev space with norm $\|u\|=\left(\int_{\mathbb{R}^{2}} u^{2}+|\nabla u|^{2} \mathrm{~d} x\right)^{1 / 2}$;
- $H_{r}^{1}\left(\mathbb{R}^{2}\right):=\left\{u \in H^{1}\left(\mathbb{R}^{2}\right): u\right.$ is radially symmetric $\}$;
- $\mathcal{C}_{0, r}^{\infty}\left(\mathbb{R}^{2}\right):=\left\{u \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right): u\right.$ is radially symmetric $\} ;$
- For any $x \in \mathbb{R}^{2}$ and $r>0, B_{r}(x)=\left\{y \in \mathbb{R}^{2}:|y-x|<r\right\}$;
- " $\rightarrow$ " and " $\rightarrow$ " denote weak and strong convergence, respectively.


## 2 Variational framework and preliminaries

In this section, we will give the variational framework of (1.1) and some preliminaries. Now we find that if $u \in \chi$ is a solution of (1.1), then it solves $Q(u)=0$, where

$$
Q(u)=\operatorname{div} A(u, \nabla u)+B(x, u, \nabla u),
$$

with

$$
\begin{align*}
A(u, \nabla u) & =\left(1+2 \kappa u^{2}\right) \nabla u, \\
B(x, u, \nabla u) & =-\left(2 \kappa|\nabla u|^{2}+V(x)+\mu K_{1}(x)\left(1+\kappa u^{2}\right)+\mu K_{2}(x)\right) u+f(u), \tag{2.1}
\end{align*}
$$

and

$$
K_{1}(x)=\left\{\begin{array}{ll}
\frac{h^{2}(|x|)}{|x|^{2}}, & x \neq 0, \\
0, & x=0,
\end{array} \quad K_{2}(x)=\int_{|x|}^{+\infty} \frac{h(s)}{s}\left(2+\kappa u^{2}(s)\right) u^{2}(s) \mathrm{d} s .\right.
$$

We observe from (2.1) that (1.1) is a quasilinear elliptic equation with principal part in divergence form and it satisfies all the structure conditions in [19] or [26].

In order to show that any weak solutions of (1.1) are classical ones, we introduce the following lemma.

Lemma 2.1 ([8]). Let us fix $u \in \chi$. We have:
(i) $K_{1}, K_{2}$ are nonnegative and bounded;
(ii) if we suppose further that $u \in \mathcal{C}\left(\mathbb{R}^{2}\right)$, then $K_{1}, K_{2} \in \mathcal{C}^{1}\left(\mathbb{R}^{2}\right)$.

Arguing as in $[1,8]$, standard computations show that
Proposition 2.2. The functional I in (1.7) is well-defined and continuous in $\chi$ and if the Gateaux derivative of I evaluated in $u \in \chi$ is zero in every direction $\varphi \in \mathcal{C}_{0, r}^{\infty}\left(\mathbb{R}^{2}\right)$, then $u$ is a weak solution of (1.1). Furthermore, the weak solution of (1.1) belongs to $\mathcal{C}^{2}\left(\mathbb{R}^{2}\right)$, so the weak solution $u$ is a classical solution of (1.1).

Lemma 2.3. Any weak solution $u$ of (1.1) satisfies the Nehari identity $N(u)=0$ and the Pohožaev identity $P(u)=0$, where

$$
\begin{align*}
N(u) & =\int_{\mathbb{R}^{2}}\left[\left(1+4 \kappa u^{2}\right)|\nabla u|^{2}+V(x) u^{2}+\mu \frac{h^{2}(|x|)}{|x|^{2}}\left(3+2 \kappa u^{2}\right) u^{2}\right] d x-\int_{\mathbb{R}^{2}} f(u) u d x,  \tag{2.2}\\
P(u) & =\int_{\mathbb{R}^{2}}\left[V(x) u^{2}+\frac{1}{2} \nabla V(x) \cdot x|u|^{2}+\mu \frac{h^{2}(|x|)}{|x|^{2}}\left(2+\kappa u^{2}\right) u^{2}\right] d x-2 \int_{\mathbb{R}^{2}} F(u) d x . \tag{2.3}
\end{align*}
$$

Proof. By a density argument, we can use $u \in \chi$ as a test function in (1.8), we have

$$
\begin{align*}
\int_{\mathbb{R}^{2}}[ & \left.\left(1+2 \kappa u^{2}\right)|\nabla u|^{2}+2 \kappa u^{2}|\nabla u|^{2}+V(x) u^{2}-f(u) u\right] \mathrm{d} x \\
& \quad+\mu \int_{\mathbb{R}^{2}} \frac{h^{2}(|x|)}{|x|^{2}}\left(1+\kappa u^{2}\right) u^{2}+\mu \int_{\mathbb{R}^{2}}\left(\int_{|x|}^{+\infty} \frac{h(s)}{s}\left(2+\kappa u^{2}(s)\right) u^{2}(s) \mathrm{d} s\right) u^{2} \mathrm{~d} x=0 . \tag{2.4}
\end{align*}
$$

We claim that: for $\beta=2$ or $\beta=4$, we have

$$
\int_{\mathbb{R}^{2}} \frac{h^{2}(|x|)}{|x|^{2}} u^{\beta} \mathrm{d} x=\int_{\mathbb{R}^{2}}\left(\int_{|x|}^{+\infty} \frac{u^{\beta}(s) h(s)}{s} \mathrm{~d} s\right) u^{2} \mathrm{~d} x .
$$

Now we using the integration by parts to prove the claim. A simple computation yields that

$$
\begin{aligned}
\int_{\mathbb{R}^{2}}\left[\frac{u^{\beta} h(|x|)}{|x|^{2}}\left(\int_{0}^{|x|} s u^{2}(s) \mathrm{d} s\right)\right] \mathrm{d} x & =\int_{0}^{2 \pi}\left[\int_{0}^{+\infty} \frac{u^{\beta} h(r)}{r^{2}}\left(\int_{0}^{r} s u^{2}(s) \mathrm{d} s\right) r \mathrm{~d} r\right] \mathrm{d} \theta \\
& =\int_{0}^{2 \pi} \int_{0}^{+\infty}\left(\int_{r}^{+\infty} \frac{u^{\beta}(s) h(s)}{s} \mathrm{~d} s\right) u^{2} r \mathrm{~d} r \mathrm{~d} \theta \\
& =\int_{\mathbb{R}^{2}}\left(\int_{|x|}^{+\infty} \frac{u^{\beta}(s) h(s)}{s} \mathrm{~d} s\right) u^{2} \mathrm{~d} x .
\end{aligned}
$$

Then, we conclude that the identity $N(u)=0$ holds.
Next, let $u \in \chi \cap \mathcal{C}^{2}\left(\mathbb{R}^{2}\right)$ be a solution of (1.1). Then multiplying by $\nabla u \cdot x$ and integrating by parts on $B_{R}$. Arguing as in $[1,8]$, we get the following identities:

$$
\begin{aligned}
\int_{B_{R}} \Delta u(\nabla u \cdot x) \mathrm{d} x & =\int_{\partial B_{R}} \frac{\partial u}{\partial \vec{n}}(\nabla u \cdot x) \mathrm{d} S_{x}-\int_{B_{R}} \nabla u \cdot \nabla(\nabla u \cdot x) \mathrm{d} x \\
& =R \int_{\partial B_{R}}\left(\frac{\partial u}{\partial \vec{n}}\right)^{2} \mathrm{~d} S_{x}-\frac{R}{2} \int_{\partial B_{R}}|\nabla u|^{2} \mathrm{~d} S_{x} \\
& =\frac{R}{2} \int_{\partial B_{R}}|\nabla u|^{2} \mathrm{~d} S_{x}=: \mathrm{I}, \\
\int_{B_{R}} u \Delta\left(u^{2}\right)(\nabla u \cdot x) \mathrm{d} x & =\int_{\partial B_{R}} \frac{\partial u^{2}}{\partial \vec{n}} u(\nabla u \cdot x) \mathrm{d} S_{x}-\int_{B_{R}} \nabla u^{2} \cdot \nabla(u(\nabla u \cdot x)) \mathrm{d} x \\
& =\frac{R}{2} \int_{\partial B_{R}}\left(\frac{\partial u^{2}}{\partial \vec{n}}\right)^{2} \mathrm{~d} S_{x}-\frac{1}{2} \int_{B_{R}} \nabla u^{2} \cdot \nabla\left(\nabla u^{2} \cdot x\right) \mathrm{d} x \\
& =\frac{R}{4} \int_{\partial B_{R}}\left|\nabla u^{2}\right|^{2} \mathrm{~d} S_{x}=: \mathrm{II}, \\
& =-\int_{B_{R}} V(x) u^{2} \mathrm{~d} x-\frac{1}{2} \int_{B_{R}}(\nabla V(x) \cdot x) u^{2} \mathrm{~d} x+\frac{R}{2} \int_{\partial B_{R}} V(x) u^{2} \mathrm{~d} S_{x} \\
& =-\int_{B_{R}} V(x) u^{2} \mathrm{~d} x-\frac{1}{2} \int_{B_{R}}(\nabla V(x) \cdot x) u^{2} \mathrm{~d} x+\mathrm{III}, \\
\int_{B_{R}} V(x) u(\nabla u \cdot x) \mathrm{d} x & =\int_{B_{R}} V(x)\left(\nabla\left(\frac{1}{2} u^{2}\right) \cdot x\right) \mathrm{d} x \\
\int_{B_{R}} f(u)(\nabla u \cdot x) \mathrm{d} x & =\int_{B_{R}} \nabla(F(u)) \cdot x \mathrm{~d} x \\
& =-2 \int_{B_{R}} F(u) \mathrm{d} x+R \int_{\partial B_{R}} F(u) \mathrm{d} S_{x} \\
& =:-2 \int_{B_{R}} F(u) \mathrm{d} x+\mathrm{IV} .
\end{aligned}
$$

We note that if $f(x) \geq 0$ is integrable on $\mathbb{R}^{2}$, then $\liminf _{R \rightarrow+\infty} R \int_{\partial B_{R}} f \mathrm{~d} S=0$. Since $u \in \chi$, then $u^{2} \in H^{1}\left(\mathbb{R}^{2}\right)$ and the integrands in the terms I, II, III and IV are all nonnegative and contained in $L^{1}\left(\mathbb{R}^{2}\right)$, one can take a sequence $\left\{R_{j}\right\}$ such that the terms I, II, III and IV with $R_{j}$
replacing $R$ converge to 0 as $j \rightarrow+\infty$. Moreover, for $\beta=2$ or $\beta=4$, we have

$$
\begin{aligned}
& \frac{4}{\beta} \int_{B_{R_{j}}}\left(\int_{|x|}^{+\infty} \frac{h(s)}{s} u^{\beta}(s) \mathrm{d} s\right) u(\nabla u \cdot x) \mathrm{d} x+\int_{B_{R_{j}}} \frac{h^{2}(|x|)}{|x|^{2}} u^{\beta-1}(\nabla u \cdot x) \mathrm{d} x \\
&= \int_{B_{R_{j}}} \frac{h^{2}(|x|)}{|x|^{2}} u^{\beta-1}(\nabla u \cdot x) \mathrm{d} x+\frac{4}{\beta} \int_{B_{R_{j}}} \frac{u^{\beta}(x)}{|x|^{2}}\left(\int_{0}^{|x|} s u^{2}(s) \mathrm{d} s\right)\left(\int_{0}^{|x|} s^{2} u(s) u^{\prime}(s) \mathrm{d} s\right) \mathrm{d} x \\
&-\frac{4}{\beta} \int_{B_{R_{j}}} \frac{u^{\beta}(x)}{|x|^{2}}\left(\int_{0}^{|x|} s u^{2}(s) \mathrm{d} s\right)\left(\int_{0}^{|x|} s^{2} u(s) u^{\prime}(s) \mathrm{d} s\right) \mathrm{d} x \\
&+\frac{4}{\beta} \int_{B_{R_{j}}}\left(\int_{|x|}^{+\infty} \frac{h(s)}{s} u^{\beta}(s) \mathrm{d} s\right) u(\nabla u \cdot x) \mathrm{d} x \\
&=\left.\frac{1}{\beta} \frac{d}{d t}\right|_{t=1} \int_{B_{R_{j}}} \frac{u^{\beta}(t x)}{|x|^{2}}\left(\int_{0}^{|x|} s u^{2}(t s) \mathrm{d} s\right)^{2} \mathrm{~d} x \\
&-\frac{4}{\beta} \int_{B_{R_{j}}} \frac{u^{\beta}(x)}{|x|^{2}}\left(\int_{0}^{|x|} s u^{2}(s) \mathrm{d} s\right)\left(\int_{0}^{|x|} s^{2} u(s) u^{\prime}(s) \mathrm{d} s\right) \mathrm{d} x \\
&+\frac{4}{\beta} \int_{B_{R_{j}}}\left(\int_{|x|}^{+\infty} \frac{h(s)}{s} u^{\beta}(s) \mathrm{d} s\right) u(\nabla u \cdot x) \mathrm{d} x \\
&=-\frac{4}{\beta} \int_{B_{R_{j}}} \frac{u^{\beta}(x)}{|x|^{2}}\left(\int_{0}^{|x|} s u^{2}(s) \mathrm{d} s\right)^{2} \mathrm{~d} x+\frac{R_{j}}{\beta} \int_{\partial B_{R_{j}}} \frac{u^{\beta}(x)}{|x|^{2}}\left(\int_{0}^{|x|} s u^{2}(s) \mathrm{d} s\right)^{2} \mathrm{~d} S_{x} \\
&+\frac{4}{\beta}\left(\int_{\left(\mathbb{R}^{2} \backslash B_{R_{j}}\right)} \frac{u^{\beta}(x) h(|x|)}{|x|^{2}} \mathrm{~d} x\right)^{2} \int_{0}^{R_{j}} s^{2} u(s) u^{\prime}(s) \mathrm{d} s \\
&=-\frac{4}{\beta} \int_{B_{R_{j}}} \frac{u^{\beta}(x)}{|x|^{2}}\left(\int_{0}^{|x|} s u^{2}(s) \mathrm{d} s\right)^{2} \mathrm{~d} x+o_{n}(1) .
\end{aligned}
$$

Then, from (1.1), we get

$$
\int_{B_{R_{j}}}\left[V(x) u^{2}+\frac{1}{2} \nabla V(x) \cdot x|u|^{2}+\mu \frac{h^{2}(|x|)}{|x|^{2}}\left(2+\kappa u^{2}\right) u^{2}\right] \mathrm{d} x-2 \int_{B_{R_{j}}} F(u) \mathrm{d} x+o_{n}(1)=0 .
$$

This implies that $P(u)=0$ holds. The proof is completed.
Remark 2.4. From (2.2) and (2.3), by Lemma 2.3, any weak solution of (1.1) belongs to $\mathcal{M}$.
For functionals $D(u), E(u)$ (see Section 3 below), we have the following compactness lemma:

Lemma 2.5 ([8]). Suppose that a sequence $\left\{u_{n}\right\}$ converges weakly to a function $u$ in $H_{r}^{1}\left(\mathbb{R}^{2}\right)$ as $n \rightarrow+\infty$. Then for each $\psi \in H_{r}^{1}\left(\mathbb{R}^{2}\right), D\left(u_{n}\right), D^{\prime}\left(u_{n}\right) \psi$ and $D^{\prime}\left(u_{n}\right) u_{n}, E\left(u_{n}\right), E^{\prime}\left(u_{n}\right) \psi$ and $E^{\prime}\left(u_{n}\right) u_{n}$ converges up to a subsequence to $D(u), D^{\prime}(u) \psi$ and $D^{\prime}(u) u, E(u), E^{\prime}(u) \psi$, and $E^{\prime}(u) u$, respectively, as $n \rightarrow+\infty$.

## 3 Existence of ground state solutions

Throughout this section, for any $u \in \chi$, we denote

$$
A(u)=\int_{\mathbb{R}^{2}}|\nabla u|^{2} \mathrm{~d} x, \quad B(u)=\int_{\mathbb{R}^{2}} V(x) u^{2} \mathrm{~d} x, \quad C(u)=\int_{\mathbb{R}^{2}} u^{2}|\nabla u|^{2} \mathrm{~d} x,
$$

$$
\begin{aligned}
& D(u)=\int_{\mathbb{R}^{2}} \frac{u^{2}(x)}{|x|^{2}}\left(\int_{0}^{|x|} s u^{2}(s) \mathrm{d} s\right)^{2} \mathrm{~d} x, \\
& E(u)=\int_{\mathbb{R}^{2}} \frac{u^{4}(x)}{|x|^{2}}\left(\int_{0}^{|x|} s u^{2}(s) \mathrm{d} s\right)^{2} \mathrm{~d} x .
\end{aligned}
$$

To complete the proof of Theorem 1.1, we prepare several lemmas.
Lemma 3.1. Assume that $\left(f_{1}\right)$ and $\left(f_{3}\right)$ hold. Then

$$
\begin{equation*}
g_{1}(t, \varrho):=t^{-2} F\left(t^{\alpha} \varrho\right)-F(\varrho)+\frac{1-t^{8 \alpha-4}}{4(2 \alpha-1)}[\alpha f(\varrho) \varrho-2 F(\varrho)] \geq 0, \quad \forall t>0, \varrho \in \mathbb{R}, \tag{3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
f(\varrho) \varrho-\frac{(8 \alpha-2)}{\alpha} F(\varrho) \geq 0, \quad \forall \varrho \in \mathbb{R} . \tag{3.2}
\end{equation*}
$$

Proof. It is easy to see that $g_{1}(t, 0) \geq 0$. For $\varrho \neq 0$, by $\left(f_{3}\right)$, we have

$$
\begin{aligned}
\frac{d}{d t} g_{1}(t, \varrho) & =t^{8 \alpha-5}|\varrho|^{\frac{8 \alpha-2}{\alpha}}\left[\frac{\alpha f\left(t^{\alpha} \varrho\right) t^{\alpha} \varrho-2 F\left(t^{\alpha} \varrho\right)}{\left.\left|t^{\alpha} \varrho\right|^{\frac{8 \alpha-2}{\alpha}}-\frac{\alpha f(\varrho) \varrho-2 F(\varrho)}{|\varrho|^{\frac{8 \alpha-2}{\alpha}}}\right]}\right. \\
& =\frac{2 t^{\frac{5 p-2 t}{8-p}}|\varrho|^{p}}{8-p}\left[\frac{f\left(t^{\frac{2}{8-p}} \varrho\right) t^{\frac{2}{8-p}} \varrho-(8-p) F\left(t^{\frac{2}{8-p}} \varrho\right)}{\left|t^{\frac{2}{8-p}} \varrho\right|^{p}}-\frac{f(\varrho) \varrho-(8-p) F(\varrho)}{|\varrho|^{p}}\right],
\end{aligned}
$$

and this expression is greater than or equal to zero for $t \geq 1$ and less than or equal to zero for $0<t<1$. Together with the continuity of $g_{1}(\cdot, \varrho)$, this implies that $g_{1}(t, \varrho) \geq g_{1}(1, \varrho)=0$ for all $t \geq 0$ and $\varrho \in \mathbb{R} \backslash\{0\}$. This shows that (3.1) holds. By ( $f_{1}$ ) and (3.1), we have

$$
\lim _{t \rightarrow 0} g_{1}(t, \varrho)=\frac{1}{4(2 \alpha-1)}[\alpha f(\varrho) \varrho-(8 \alpha-2) F(\varrho)] \geq 0, \quad \forall \varrho \in \mathbb{R},
$$

which implies that (3.2) holds.
Lemma 3.2. Assume that $\left(V_{1}\right)-\left(V_{2}\right)$ hold. Then

$$
\begin{align*}
g_{2}(t, x) & :=V(x)-t^{2 \alpha-2} V\left(t^{-1} x\right)-\frac{1-t^{8 \alpha-4}}{4(2 \alpha-1)}[(2 \alpha-2) V(x)-\nabla V(x) \cdot x]  \tag{3.3}\\
& \geq 0, \quad \forall t \geq 0, x \in \mathbb{R}^{2} \backslash\{0\},
\end{align*}
$$

and

$$
\begin{equation*}
(6 \alpha-2) V(x)+\nabla V(x) \cdot x \geq 0, \quad \forall x \in \mathbb{R}^{2} . \tag{3.4}
\end{equation*}
$$

Proof. For any $x \in \mathbb{R}^{2}$, by $\left(V_{1}\right)$ and $\left(V_{2}\right)$, we have

$$
\begin{aligned}
\frac{d}{d t} g_{2}(t, x)=t^{8 \alpha-5}\{(2 \alpha-2) V(x)-\nabla V & (x) \cdot x \\
& \left.-t^{-(6 \alpha-2)}\left[(2 \alpha-2) V\left(t^{-1} x\right)-\nabla V\left(t^{-1} x\right) \cdot\left(t^{-1} x\right)\right]\right\},
\end{aligned}
$$

and this expression is greater than or equal to zero for $t \geq 1$ and less than or equal to zero for $0<t<1$. Together with the continuity of $g_{2}(\cdot, x)$, this implies that $g_{2}(t, x) \geq g_{2}(1, x)$ for all $t \geq 0$ and $x \in \mathbb{R}^{2}$. This shows that (3.3) holds. By (3.3), one has

$$
\lim _{t \rightarrow 0} g_{2}(t, x)=\frac{(6 \alpha-2) V(x)+\nabla V(x) \cdot x}{4(2 \alpha-1)} \geq 0,
$$

which implies that (3.4) holds.

For $t \geq 0$, let

$$
\begin{align*}
& \tau_{1}(t)=\alpha t^{8 \alpha-4}-(4 \alpha-2) t^{2 \alpha}+3 \alpha-2  \tag{3.5}\\
& \tau_{2}(t)=\alpha t^{8 \alpha-4}-(2 \alpha-1) t^{4 \alpha}+\alpha-1  \tag{3.6}\\
& \tau_{3}(t)=(3 \alpha-2) t^{8 \alpha-4}-(4 \alpha-2) t^{6 \alpha-4}+\alpha . \tag{3.7}
\end{align*}
$$

Since $\alpha>1$, for all $t \in(0,1) \cup(1,+\infty)$,

$$
\begin{equation*}
\tau_{1}(t)>\tau_{1}(1)=0, \quad \tau_{2}(t)>\tau_{2}(1)=0, \quad \tau_{3}(t)>\tau_{3}(1)=0 \tag{3.8}
\end{equation*}
$$

Lemma 3.3. Assume that $\left(V_{1}\right)-\left(V_{2}\right),\left(f_{1}\right)$ and $\left(f_{3}\right)$ hold. Then for all $u \in H^{1}\left(\mathbb{R}^{2}\right)$ and $t>0$,

$$
\begin{equation*}
I(u) \geq I\left(u_{t}\right)+\frac{1-t^{8 \alpha-4}}{4(2 \alpha-1)} \Gamma(u)+\frac{\tau_{1}(t)}{4(2 \alpha-1)} A(u)+\frac{\tau_{2}(t)}{(2 \alpha-1)} C(u) . \tag{3.9}
\end{equation*}
$$

Proof. Note that

$$
\begin{align*}
I\left(u_{t}\right)= & \frac{t^{2 \alpha}}{2} A(u)+\frac{t^{2 \alpha-2}}{2} \int_{\mathbb{R}^{2}} V\left(t^{-1} x\right) u^{2} \mathrm{~d} x+t^{4 \alpha} \kappa C(u)  \tag{3.10}\\
& +\frac{t^{6 \alpha-4}}{2} \mu D(u)+\frac{t^{8 \alpha-4}}{4} \mu \kappa E(u)-\frac{1}{t^{2}} \int_{\mathbb{R}^{2}} F\left(t^{\alpha} u\right) \mathrm{d} x, \quad \forall u \in H^{1}\left(\mathbb{R}^{2}\right) .
\end{align*}
$$

Since $\Gamma(u)=\alpha N(u)-P(u)$ for $u \in \chi$, then (1.7) and (1.8) imply that

$$
\begin{align*}
\Gamma(u)= & \alpha A(u)+\frac{1}{2} \int_{\mathbb{R}^{2}}[(2 \alpha-2) V(x)-\nabla V(x) \cdot x] u^{2} \mathrm{~d} x  \tag{3.11}\\
& +4 \alpha \kappa C(u)+(3 \alpha-2) \mu D(u)+(2 \alpha-1) \mu \kappa E(u)+\int_{\mathbb{R}^{2}}[2 F(u)-\alpha f(u) u] \mathrm{d} x .
\end{align*}
$$

Then, it follows from (1.7), (3.1)-(3.7), (3.10)-(3.11) that

$$
\begin{aligned}
I(u)- & I\left(u_{t}\right) \\
= & \frac{1-t^{2 \alpha}}{2} A(u)+\frac{1}{2} \int_{\mathbb{R}^{2}}\left[V(x)-t^{2 \alpha-2} V\left(t^{-1} x\right)\right] u^{2} \mathrm{~d} x+\left(1-t^{4 \alpha}\right) \kappa C(u) \\
& +\left(\frac{1-t^{6 \alpha-4}}{2}\right) \mu D(u)+\left(\frac{1-t^{8 \alpha-4}}{4}\right) \mu \kappa E(u)+\int_{\mathbb{R}^{2}}\left[t^{-2} F\left(t^{\alpha} u\right)-F(u)\right] \mathrm{d} x \\
= & \frac{1-t^{8 \alpha-4}}{4(2 \alpha-1)}\left\{\alpha A(u)+\frac{1}{2} \int_{\mathbb{R}^{2}}[(2 \alpha-2) V(x)-\nabla V(x) \cdot x] u^{2} \mathrm{~d} x\right. \\
& \left.+4 \alpha \kappa C(u)+(3 \alpha-2) \mu D(u)+(2 \alpha-1) \mu \kappa E(u)+\int_{\mathbb{R}^{2}}[2 F(u)-\alpha f(u) u] \mathrm{d} x\right\} \\
& +\left[\frac{1-t^{2 \alpha}}{2}-\frac{\alpha\left(1-t^{8 \alpha-4}\right)}{4(2 \alpha-1)}\right] A(u)+\left[\left(\frac{1-t^{6 \alpha-4}}{2}\right)-\frac{\left(1-t^{8 \alpha-4}\right)(3 \alpha-2)}{4(2 \alpha-1)}\right] \mu D(u) \\
& +\frac{1}{2} \int_{\mathbb{R}^{2}}\left\{V(x)-t^{2 \alpha-2} V\left(t^{-1} x\right)-\frac{1-t^{8 \alpha-4}}{4(2 \alpha-1)}[(2 \alpha-2) V(x)-\nabla V(x) \cdot x]\right\} u^{2} \mathrm{~d} x \\
& +\left[1-t^{4 \alpha}-\frac{4 \alpha\left(1-t^{8 \alpha-4}\right)}{4(2 \alpha-1)}\right] \kappa C(u) \\
& +\int_{\mathbb{R}^{2}}\left\{t^{-2} F\left(t^{\alpha} u\right)-F(u)+\frac{1-t^{8 \alpha-4}}{4(2 \alpha-1)}[\alpha f(u) u-2 F(u)]\right\} \mathrm{d} x \\
\geq & \frac{1-t^{8 \alpha-4}}{4(2 \alpha-1)} \Gamma(u)+\frac{\tau_{1}(t)}{4(2 \alpha-1)} A(u)+\frac{\tau_{2}(t)}{(2 \alpha-1)} C(u),
\end{aligned}
$$

for all $u \in H^{1}\left(\mathbb{R}^{2}\right)$ and $t>0$. This implies that (3.9) holds.

From Lemma 3.3, we have the following corollary.
Corollary 3.4. Assume that $\left(V_{1}\right)-\left(V_{2}\right),\left(f_{1}\right)$ and $\left(f_{3}\right)$ hold. Then for all $u \in \mathcal{M}$,

$$
I(u)=\max _{t>0} I\left(u_{t}\right)
$$

Lemma 3.5. Assume that $\left(V_{1}\right)-\left(V_{2}\right),\left(f_{1}\right)-\left(f_{3}\right)$ hold. Then for any $\chi \backslash\{0\}$, there exists a unique $t_{u}>0$, such that $(u)_{t_{u}} \in \mathcal{M}$.

Proof. Inspired by [3,5], we let $u \in \chi \backslash\{0\}$ be fixed and define the function $\gamma(t):=I\left(u_{t}\right)$ on $(0,+\infty)$. Clearly by (3.10), (3.11), we have

$$
\begin{aligned}
\gamma^{\prime}(t)=0 \Longleftrightarrow & \alpha A(u) t^{2 \alpha-1}+\frac{t^{2 \alpha-3}}{2} \int_{\mathbb{R}^{2}}\left[2(\alpha-1) V\left(t^{-1} x\right)-\nabla V\left(t^{-1} x\right) \cdot\left(t^{-1} x\right)\right] u^{2} \mathrm{~d} x \\
& +4 \alpha \kappa C(u) t^{4 \alpha-1}+(3 \alpha-2) \mu D(u) t^{6 \alpha-5}+(2 \alpha-1) \mu \kappa E(u) t^{8 \alpha-5} \\
& +t^{-3} \int_{\mathbb{R}^{2}}\left[2 F\left(t^{\alpha} u\right)-\alpha f\left(t^{\alpha} u\right) t^{\alpha} u\right] \mathrm{d} x=0 \\
\Longleftrightarrow & \Gamma\left(u_{t}\right)=0 \Longleftrightarrow u_{t} \in \mathcal{M} .
\end{aligned}
$$

From $\left(V_{1}\right)$ and $\left(V_{2}\right),\left(f_{1}\right)$ and (3.10), it follows that $\lim _{t \rightarrow 0} \gamma(t)=0, \gamma(t)>0$ for $t>0$ small. Moreover, from $\left(f_{1}\right)$ and $\left(f_{2}\right)$, for every $\theta>0$, there exists $C_{\theta}>0$ such that

$$
\begin{equation*}
F(\varrho) \geq \theta|\varrho|^{p}-C_{\theta} \varrho^{2}, \quad \forall \varrho \in \mathbb{R} . \tag{3.12}
\end{equation*}
$$

We note from Lemma 2.1 and Hölder inequality that for some $C_{0}>0$,

$$
\begin{equation*}
h(s)=\int_{0}^{s} u^{2}(r) r \mathrm{~d} r=\int_{B_{s}} \frac{1}{2 \pi} u^{2}(y) \mathrm{d} y \leq C_{0} s\|u\|_{L^{4}}^{2} \tag{3.13}
\end{equation*}
$$

then

$$
\begin{gather*}
D(u)=\int_{\mathbb{R}^{2}} \frac{u^{2}(x)}{|x|^{2}}\left(\int_{0}^{|x|} s u^{2}(s) \mathrm{d} s\right)^{2} \mathrm{~d} x \leq C_{0}\|u\|_{L^{4}}^{4}\|u\|_{L^{2}}^{2}  \tag{3.14}\\
E(u)=\int_{\mathbb{R}^{2}} \frac{u^{4}(x)}{|x|^{2}}\left(\int_{0}^{|x|} s u^{2}(s) \mathrm{d} s\right)^{2} \mathrm{~d} x \leq C_{0}\|u\|_{L^{4}}^{8} . \tag{3.15}
\end{gather*}
$$

By $\left(V_{1}\right)$, we have $V_{\max }:=\max _{x \in \mathbb{R}^{2}} V(x)>0$ and by (3.10), (3.12) and (3.14), (3.15), we have

$$
\begin{align*}
I\left(u_{t}\right) \leq & \frac{t^{2 \alpha}}{2} A(u)+\frac{t^{2 \alpha-2}}{2} V_{\max }\|u\|^{2}+t^{4 \alpha} \kappa C(u) \\
& +\frac{t^{6 \alpha-4}}{2} \mu C_{0}\|u\|_{L^{4}}^{4}\|u\|_{L^{2}}^{2}+\frac{t^{8 \alpha-4}}{4} \mu \kappa\|u\|_{L^{4}}^{8}-\theta t^{8 \alpha-4}\|u\|_{L^{p}}^{p}  \tag{3.16}\\
& +t^{2 \alpha-2} C_{\theta}\|u\|_{L^{2}}^{2} .
\end{align*}
$$

Let $\theta$ be large enough in (3.16), then $\gamma(t)<0$ for $t$ large. Therefore, $\max _{t>0} \gamma(t)$ is achieved at some $t_{u}>0$, so that $\gamma^{\prime}\left(t_{u}\right)=0$ and $(u)_{t_{u}} \in \mathcal{M}$.

Next, we claim that $t_{u}>0$ is unique for any $u \in \chi \backslash\{0\}$. If there exist two positive constants $t_{1} \neq t_{2}$, such that both $u_{t_{1}}, u_{t_{2}} \in \mathcal{M}$, that is, $\Gamma\left(u_{t_{1}}\right)=\Gamma\left(u_{t_{2}}\right)=0$, then (3.5)-(3.7), (3.10) imply

$$
\begin{aligned}
I\left(u_{t_{1}}\right) & >I\left(u_{t_{2}}\right)+\frac{t_{1}^{6 \alpha-4}-t_{2}^{6 \alpha-4}}{4(2 \alpha-1) t_{1}^{6 \alpha-4}} \Gamma\left(u_{t_{1}}\right)=I\left(u_{t_{2}}\right) \\
& >I\left(u_{t_{1}}\right)+\frac{t_{2}^{6 \alpha-4}-t_{1}^{6 \alpha-4}}{4(2 \alpha-1) t_{2}^{6 \alpha-4}} \Gamma\left(u_{t_{2}}\right)=I\left(u_{t_{1}}\right)
\end{aligned}
$$

This contradiction shows that $t_{u}>0$ is unique for any $u \in \chi \backslash\{0\}$.

Arguing as in [5], standard computations show that
Lemma 3.6. Assume that $\left(V_{1}\right)-\left(V_{2}\right)$ hold. Then there exist constants $C_{1}, C_{2}>0$, such that

$$
\begin{equation*}
(2 \alpha-2) V(x)-\nabla V(x) \cdot x \geq C_{1}, \quad \forall x \in \mathbb{R}^{2} \tag{3.17}
\end{equation*}
$$

and

$$
\begin{equation*}
(6 \alpha-2) V(x)+\nabla V(x) \cdot x \geq C_{2}, \quad \forall x \in \mathbb{R}^{2} \tag{3.18}
\end{equation*}
$$

Lemma 3.7. Assume that $\left(V_{1}\right)$ and $\left(V_{2}\right),\left(f_{1}\right)-\left(f_{3}\right)$ hold. Then
(i) there exists $\rho_{0}>0$ such that $\|u\| \geq \rho_{0}, \quad \forall u \in \mathcal{M}$;
(ii) $m:=\inf _{u \in \mathcal{M}} I(u)=\inf _{u \in \chi \backslash\{0\}} \max I\left(u_{t}\right)>0$.

Proof. (i) Since $\Gamma(u)=0$ for $u \in \mathcal{M}$, it follows from $\left(f_{1}\right)$, (3.11), (3.17) and Sobolev embedding inequality, there exists a constant $C_{3}>0$, such that

$$
\begin{aligned}
\alpha A(u) & +4 \alpha \kappa C(u)+\frac{1}{2} C_{1}\|u\|_{L^{2}}^{2} \\
\leq & \alpha A(u)+4 \alpha \kappa C(u)+\frac{1}{2} \int_{\mathbb{R}^{2}}[(2 \alpha-2) V(x)-\nabla V(x) \cdot x] u^{2} \mathrm{~d} x \\
\leq & \int_{\mathbb{R}^{2}}[\alpha f(u) u-2 F(u)] \mathrm{d} x \\
\leq & \frac{1}{4} C_{1}\|u\|_{L^{2}}^{2}+C_{3}\|u\|^{p}
\end{aligned}
$$

for all $u \in \mathcal{M}$. This implies that there exists $\rho_{0}>0$ such that

$$
\begin{equation*}
\|u\| \geq \rho_{0}:=\left(\frac{\min \left\{4 \alpha, C_{1}\right\}}{4 C_{3}}\right)^{\frac{1}{p-2}}, \quad \forall u \in \mathcal{M} \tag{3.19}
\end{equation*}
$$

(ii) From Corollary 3.4 and Lemma 3.5, we have

$$
\mathcal{M} \neq \varnothing \quad \text { and } \quad m=\inf _{u \in \chi \backslash\{0\}} \max I\left(u_{t}\right)
$$

Next, we prove that $m>0$. Let

$$
\begin{align*}
\Psi(u):= & I(u)-\frac{1}{4(2 \alpha-1)} \Gamma(u) \\
= & \frac{3 \alpha-2}{4(2 \alpha-1)} A(u)+\frac{1}{8(2 \alpha-1)} \int_{\mathbb{R}^{2}}[(6 \alpha-2) V(x)+\nabla V(x) \cdot x] u^{2} \mathrm{~d} x  \tag{3.20}\\
& +\frac{\alpha-1}{(2 \alpha-1)} \kappa C(u)+\frac{\alpha}{4(2 \alpha-1)} \mu D(u) \\
& +\frac{1}{4(2 \alpha-1)} \int_{\mathbb{R}^{2}}[\alpha f(u) u-(8 \alpha-2) F(u)] \mathrm{d} x, \quad \forall u \in H^{1}\left(\mathbb{R}^{2}\right)
\end{align*}
$$

Since $\Gamma(u)=0$ for all $u \in \mathcal{M}$, then it follows from (3.2), (3.4), (3.18) and (3.19), (3.20) that

$$
\begin{aligned}
I(u) & \geq \frac{3 \alpha-2}{4(2 \alpha-1)} A(u)+\frac{1}{8(2 \alpha-1)} \int_{\mathbb{R}^{2}}[(6 \alpha-2) V(x)+\nabla V(x) \cdot x] u^{2} \mathrm{~d} x \\
& \geq \frac{\min \left\{2(3 \alpha-2), C_{2}\right\}}{8(2 \alpha-1)}\|u\|^{2} \geq \frac{\min \left\{2(3 \alpha-2), C_{2}\right\}}{8(2 \alpha-1)} \rho_{0}^{2}:=\rho_{1}>0, \quad \forall u \in \mathcal{M} .
\end{aligned}
$$

This shows that $m=\inf _{u \in \mathcal{M}} I(u) \geq \rho_{1}>0$.

Next, we establish the following lemma.
Lemma 3.8. Assume that $\left(V_{1}\right)-\left(V_{2}\right)$ and $\left(f_{1}\right)-\left(f_{3}\right)$ hold. If $u \in \mathcal{M}$ and $I(u)=m$, then $u$ is a radial ground state solution of (1.1). Moreover, it is positive (up to a change of sign).

Proof. We argue as in [8,22]. Suppose by contradiction that $u$ is not a weak solution of (1.2). Then, we can choose $\varphi \in C_{0, r}^{\infty}\left(\mathbb{R}^{2}\right)$ such that

$$
\left\langle I^{\prime}(u), \varphi\right\rangle<-1 .
$$

Hence, we fix $\varepsilon>0$ sufficiently small such that

$$
\begin{equation*}
\left\langle I^{\prime}\left(u_{t}+\vartheta \varphi\right), \varphi\right\rangle \leq-\frac{1}{2}, \quad \text { for }|t-1|,|\vartheta| \leq \varepsilon, \tag{3.21}
\end{equation*}
$$

and introduce $\zeta \in C_{0}^{\infty}(\mathbb{R})$ be a cut-off function $0 \leq \zeta \leq 1$ such that $\zeta(t)=1$ for $|t-1| \leq \frac{\varepsilon}{2}$ and $\zeta(t)=0$ for $|t-1| \geq \varepsilon$. For $t \geq 0$, we construct a path $\sigma: \mathbb{R}^{+} \rightarrow \chi$ defined by

$$
\sigma(t)= \begin{cases}u_{t}, & \text { if }|t-1| \geq \varepsilon \\ u_{t}+\varepsilon \zeta(t) \varphi, & \text { if }|t-1|<\varepsilon\end{cases}
$$

Note that $\eta$ is continuous on the metric space $\left(\chi, d_{\chi}\right)$ and eventually, choosing a smaller $\varepsilon$, if necessary, we obtain that $d_{\chi}(\sigma(t), 0)>0$ for $|t-1|<\varepsilon$.

We claim that

$$
\begin{equation*}
\sup _{t \geq 0} I(\sigma(t))<m . \tag{3.22}
\end{equation*}
$$

Indeed, if $|t-1| \geq \varepsilon$, from Corollary 3.4, we have $I(\sigma(t))=I\left(u_{t}\right)<I(u)=m$. If $|t-1|<\varepsilon$, by using the mean value theorem, we get

$$
\begin{aligned}
I(\sigma(t))=I\left(u_{t}+\varepsilon \zeta(t) \varphi\right) & =I\left(u_{t}\right)+\int_{0}^{\varepsilon}\left\langle I^{\prime}\left(u_{t}+\vartheta \zeta(t) \varphi\right), \zeta(t) \varphi\right\rangle \mathrm{d} \tau \\
& \leq I\left(u_{t}\right)-\frac{1}{2} \varepsilon \zeta(t)<m,
\end{aligned}
$$

where in the first inequality we have used (3.21).
To conclude that $\Gamma(\sigma(1+\varepsilon))<0$ and $\Gamma(\sigma(1-\varepsilon))>0$. By the continuity of the map $t \rightarrow$ $\Gamma(\sigma(t))$, there exists $t_{0} \in(1-\varepsilon, 1+\varepsilon)<0$ such that $\Gamma\left(\sigma\left(t_{0}\right)\right)=0$. This implies that $\sigma\left(t_{0}\right)=$ $u_{t_{0}}+\varepsilon \zeta\left(t_{0}\right) \varphi \in \mathcal{M}$ and $I\left(\sigma\left(t_{0}\right)\right)<m$. By Lemma 3.7, this gives the desired contradiction, hence $u$ is a weak solution of (1.2). By Remark 2.4, we conclude that $u$ is a radial ground state solution. Moreover, if $u \in \mathcal{M}$ is a minimizer of $\left.I\right|_{\mathcal{M}}$, then $|u|$ is also a minimizer and a solution. So we can assume that $u$ is nonnegative. By Proposition 2.2 , we know that $u \in \mathcal{C}^{2}\left(\mathbb{R}^{2}\right)$ and by the Harnack inequality [27], we know that $u>0$. This completes the proof.

Lemma 3.9. Assume that $\left(V_{1}\right)-\left(V_{2}\right)$ and $\left(f_{1}\right)-\left(f_{3}\right)$ hold. Then $m$ is achieved.
Proof. Let $\left\{u_{n}\right\} \subset \mathcal{M}$ be such that $I\left(u_{n}\right) \rightarrow m$, then by (3.20),

$$
m+o(1)=I\left(u_{n}\right) \geq \frac{3 \alpha-2}{4(2 \alpha-1)} A\left(u_{n}\right)+\frac{C_{2}}{8(2 \alpha-1)}\left\|u_{n}\right\|_{L^{2}}^{2}+\frac{\alpha-1}{(2 \alpha-1)} \kappa C\left(u_{n}\right)
$$

which implies that $\left\{u_{n}\right\}$ and $\left\{u_{n}^{2}\right\}$ are bounded in $H_{r}^{1}\left(\mathbb{R}^{2}\right)$. Therefore, by the compactness result due to [23], there exists $\bar{u} \in \chi$ such that, up to a subsequence,

$$
\begin{array}{ll}
u_{n} \rightharpoonup \bar{u} & \text { in } H_{r}^{1}\left(\mathbb{R}^{2}\right), \\
u_{n}^{2} \rightharpoonup \bar{u}^{2} & \text { in } H_{r}^{1}\left(\mathbb{R}^{2}\right), \\
u_{n} \rightarrow \bar{u} & \text { in } L^{q}\left(\mathbb{R}^{2}\right) \text { for any } \mathrm{q}>2, \\
u_{n} \rightarrow \bar{u} & \text { a.e. in } \mathbb{R}^{2} .
\end{array}
$$

There are two possible cases (i) $\bar{u}=0$ and (ii) $\bar{u} \neq 0$. Next, we prove that $\bar{u} \neq 0$.
Arguing by contradiction, suppose that $\bar{u}=0$, that is $u_{n} \rightharpoonup 0$ in $H_{r}^{1}\left(\mathbb{R}^{2}\right)$ and $u_{n}^{2} \rightharpoonup 0$ in $H_{r}^{1}\left(\mathbb{R}^{2}\right)$. Then $u_{n} \rightarrow 0$ in $L^{q}\left(\mathbb{R}^{2}\right)$ for $q>2$ and $u_{n} \rightarrow 0$ a.e. in $\mathbb{R}^{2}$. From $\Gamma\left(u_{n}\right)=0$, (3.17) and (3.19), one has

$$
\begin{align*}
\min \left\{\alpha, \frac{1}{2} C_{1}\right\} \rho_{0}{ }^{2} \leq & \min \left\{\alpha, \frac{1}{2} C_{1}\right\}\left\|u_{n}\right\|^{2} \\
\leq & \alpha A(u)+\frac{1}{2} C_{1}\left\|u_{n}\right\|_{L^{2}}^{2} \\
\leq & \alpha A\left(u_{n}\right)+\frac{1}{2} \int_{\mathbb{R}^{2}}[(2 \alpha-2) V(x)-\nabla V(x) \cdot x] u_{n}^{2} \mathrm{~d} x  \tag{3.23}\\
& +4 \alpha \kappa C\left(u_{n}\right)+(3 \alpha-2) \mu D\left(u_{n}\right)+(2 \alpha-1) \mu \kappa E\left(u_{n}\right) \\
= & \int_{\mathbb{R}^{2}}\left[\alpha f\left(u_{n}\right) u_{n}-2 F\left(u_{n}\right)\right] \mathrm{d} x+o(1) .
\end{align*}
$$

Using $\left(f_{1}\right),\left(f_{2}\right)$, clearly, (3.23) contradicts with $u_{n} \rightarrow 0$ in $L^{q}\left(\mathbb{R}^{2}\right)$ for $q>2$, therefore $\bar{u} \neq 0$.
Let $v_{n}=u_{n}-\bar{u}$. Then by Lemma 2.5 and the Brezis-Lieb Lemma (see [22,24,30]), yield

$$
\begin{equation*}
I\left(u_{n}\right)=I(\bar{u})+I\left(v_{n}\right)+o(1), \tag{3.24}
\end{equation*}
$$

and

$$
\begin{equation*}
\Gamma\left(u_{n}\right)=\Gamma(\bar{u})+\Gamma\left(v_{n}\right)+o(1) . \tag{3.25}
\end{equation*}
$$

Since $I\left(u_{n}\right) \rightarrow m, \Gamma\left(u_{n}\right)=0$, then it follows from (3.20), (3.24) and (3.25), we have

$$
\begin{align*}
\Psi\left(v_{n}\right) & :=I\left(v_{n}\right)-\frac{1}{4(2 \alpha-1)} \Gamma\left(v_{n}\right) \\
& =m-\Psi(\bar{u})+o(1)  \tag{3.26}\\
& =m-\left[I(\bar{u})-\frac{1}{4(2 \alpha-1)} \Gamma(\bar{u})\right]+o(1),
\end{align*}
$$

and

$$
\begin{equation*}
\Gamma\left(v_{n}\right)=-\Gamma(\bar{u})+o(1) . \tag{3.27}
\end{equation*}
$$

If there eixsts a subsequence $\left\{v_{n_{i}}\right\}$ of $\left\{v_{n}\right\}$ such that $v_{n_{i}}=0$, then

$$
\begin{equation*}
I(\bar{u})=m, \quad \Gamma(\bar{u})=0, \tag{3.28}
\end{equation*}
$$

which implies that the conclusion of Lemma 3.9 holds. Next, we assume that $v_{n} \neq 0$. In view of Lemma 3.5, there exists $t_{n}>0$ such that $\left(v_{n}\right)_{t_{n}} \in \mathcal{M}$ for large $n$, we claim that $\Gamma(\bar{u}) \leq 0$, otherwise, if $\Gamma(\bar{u})>0$, then (3.27) implies that $\Gamma\left(v_{n}\right)<0$ for large $n$. From (1.7), (3.9) and (3.26), we obtain

$$
\begin{aligned}
m-\Psi(\bar{u})+o(1) & =\Psi\left(v_{n}\right)=I\left(v_{n}\right)-\frac{1}{4(2 \alpha-1)} \Gamma\left(v_{n}\right) \\
& \geq I\left(\left(v_{n}\right)_{t_{n}}\right)-\frac{t_{n}^{8 \alpha-4}}{4(2 \alpha-1)} \Gamma\left(v_{n}\right)+\frac{\tau_{1}\left(t_{n}\right)}{4(2 \alpha-1)} A\left(v_{n}\right)+\frac{\tau_{2}\left(t_{n}\right)}{(2 \alpha-1)} C\left(v_{n}\right) \\
& \geq I\left(\left(v_{n}\right)_{t_{n}}\right)-\frac{t_{n}^{8 \alpha-4}}{4(2 \alpha-1)} \Gamma\left(v_{n}\right) \geq m \quad \text { for large } n \in \mathbb{N},
\end{aligned}
$$

which implies that $\Gamma(\bar{u}) \leq 0$ due to $\Psi(\bar{u})>0$. Applying Lemma 3.5, there exists $\bar{t}>0$ such that $\bar{u}_{\bar{t}} \in \mathcal{M}$. From (1.7), (3.5), (3.6) and (3.9), the weak semicontinuity of norm and Fatou's Lemma, one has

$$
\begin{aligned}
m & =\lim _{n \rightarrow \infty} \Psi\left(u_{n}\right) \\
& =\lim _{n \rightarrow \infty}\left[I\left(u_{n}\right)-\frac{1}{4(2 \alpha-1)} \Gamma\left(u_{n}\right)\right] \\
& \geq I(\bar{u})-\frac{1}{4(2 \alpha-1)} \Gamma(\bar{u}) \\
& \geq I\left(\bar{u}_{\bar{t}}\right)-\frac{\bar{t}^{8 \alpha-4}}{4(2 \alpha-1)} \Gamma(\bar{u})+\frac{\tau_{1}(\bar{t})}{4(2 \alpha-1)} A(\bar{u})+\frac{\tau_{2}(\bar{t})}{(2 \alpha-1)} C(\bar{u}) \\
& \geq m-\frac{\bar{t}^{8 \alpha-4}}{4(2 \alpha-1)} \Gamma(\bar{u}) \geq m,
\end{aligned}
$$

which implies that (3.28) holds.
Proof of Theorem 1.1. In view of Lemmas 3.7, 3.8, 3.9, there exists $\bar{u} \in \mathcal{M}$ such that $I^{\prime}(\bar{u})=0$, $I(\bar{u})=m=\inf _{u \in \chi \backslash\{0\}} \max I\left(u_{t}\right)$, we can conclude that, actually, $\bar{u}$ is a positive radial ground state solution of (1.1). This completes the proof.

## Acknowledgements

Authors are grateful to the referees for their very constructive comments and valuable suggestions. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11771198 ).

## References

[1] J. Byeon, H. Нuh, J. Seok, Standing waves of nonlinear Schrödinger equations with the gauge field, J. Funct. Anal. 263(2012), No. 6, 1575-1608. https://doi.org/10.1016/j. jfa. 2012.05.024; MR2948224; Zbl 1248.35193
[2] J. Byeon, Н. Нuh, J. Seok, On standing waves with a vortex point of order $N$ for the nonlinear Chern-Simons-Schrödinger equations, J. Differential Equations 261(2016), No. 2, 1285-1316. https://doi.org/10.1016/j.jde.2016.04.004; MR3494398; Zbl 1342.35321
[3] S. Chen, Z. Gao, An improved result on ground state solutions of quasilinear Schrödinger equations with super-linear nonlinearities, Bull. Aust. Math. Soc. 99(2019), No. 2, 231-241. https://doi.org/10.1017/S0004972718001235; MR3917237; Zbl 1412.35083
[4] Z. Chen, X. Tang, J. Zhang, Sign-changing multi-bump solutions for the Chern-SimonsSchrödinger equations in $\mathbb{R}^{2}$, Adv. Nonlinear Anal. 9(2020), No. 1, 1066-1091. https :// doi.org/10.1515/anona-2020-0041; MR4012183; Zbl 1429.35081
[5] S. Chen, B. Zhang, X. Tang, Existence and concentration of semiclassical ground state solutions for the generalized Chern-Simons-Schrödinger system in $H^{1}\left(\mathbb{R}^{2}\right)$, Nonlinear Anal. 185(2019), 68-96. https://doi.org/10.1016/j.na.2019.02.028; MR3924547; Zbl 1421.35146
[6] S. Chen, B. Zhang, X. Tang, Existence and non-existence results for Kirchhoff-type problem with convolution nonlinearity, Adv. Nonlinear Anal. 9(2020), No. 1, 148-167. https://doi.org/10.1515/anona-2018-0147; MR3935867; Zbl 1421.35100
[7] P. L. Cunha, P. d'Avenia, A. Pomponio, G. Siciliano, A multiplicity result for Chern-Simons-Schrödinger equation with a general nonlinearity, NoDEA Nonlinear Differential Equations Appl. 22(2015), No. 6, 1831-1850. https://doi.org/10.1007/s00030-015-0346-x; MR3415024; Zbl 1330.35397
[8] P. d'Avenia, A. Pomponio, T. Watanabe, Standing waves of modified Schrödinger equations coupled with the Chern-Simons gauge theory, Proc. Roy. Soc. Edinburgh Sect. A 150(2020), No. 4, 1915-1936. https://doi.org/10.1017/prm.2019.9; MR4122440; Zbl 1444.35073
[9] Y. Deng, S. Peng, W. Shuai, Nodal standing waves for a gauged nonlinear Schrödinger equation in $\mathbb{R}^{2}$, J. Differential Equations 264(2018), No. 6, 4006-4035. https://doi.org/10. 1016/j.jde.2017.12.003; MR3747436; Zbl 1383.35064
[10] H. Huh, Standing waves of the Schrödinger equation coupled with the Chern-Simons gauge field, J. Math. Phys. 53(2012), No. 6, 063702, 8 pp. https://doi.org/10.1063/1. 4726192; MR2977697; Zbl 1276.81053
[11] H. Huh, Energy Solution to the Chern-Simons-Schrödinger equations, Abstr. Appl. Anal. 2013, Art. ID 590653, 7 pp. https://doi.org/10.1155/2013/590653; MR3035224; Zbl 1276.35138
[12] R. Jackiw, S. Pi, Soliton solutions to the gauged nonlinear Schrödinger equation on the plane, Phys. Rev. Lett. 64(1990), No. 25, 2969-2972. https://doi.org/10.1103/ PhysRevLett.64.2969; MR1056846; Zbl 1050.81526
[13] R. Jackiw, S. Pi, Classical and quantal nonrelativistic Chern-Simons theory, Phys. Rev. D (3) 42(1990), No. 10, 3500-3513. https://doi.org/110.1103/PhysRevD.48.3929; MR1242533;
[14] Y. Jiang, A. Pomponio, D. Ruiz, Standing waves for a gauged nonlinear Schrödinger equation with a vortex point, Commun. Contemp. Math. 18(2016), No. 4, 1550074, 20 pp. https://doi.org/10.1142/S0219199715500741; MR3493222; Zbl 1341.35150
[15] G. Li, X. Luo, W. Shuai, Sign-changing solutions to a gauged nonlinear Schrödinger equation, J. Math. Anal. Appl. 455(2017), No. 2, 1559-1578. https://doi.org/10.1016/j . jmaa.2017.06.048; MR3671239; Zbl 1375.35199
[16] J. Liu, Y. Wang, Z. Wang, Solutions for quasilinear Schrödinger equations via the Nehari method, Comm. Partial Differential Equations 29(2004), No. 5-6, 879-901. https://doi. org/10.1081/PDE-120037335; MR2059151; Zbl 1140.35399
[17] A. Pankov, Homoclinics for strongly indefinite almost periodic second order Hamiltonian systems, Adv. Nonlinear Anal. 8(2019), No. 1, 372-385. https://doi.org/10.1515/ anona-2017-0041; MR3918382; Zbl 1430.37067
[18] N. S. Papageorgiou, V. D. Rădulescu, D. D. Repovš, Nonlinear analysis-theory and methods, Springer Monographs in Mathematics, Springer, Cham, 2019. https://doi.org/10. 1007/978-3-030-03430-6; MR3890060
[19] A. Pomponio, Some results on the Chern-Simons-Schrödinger equation, in: Recent advances in nonlinear PDEs theory, Lect. Notes Semin. Interdiscip. Mat., Vol. 13, Semin. Interdiscip. Mat. (S. I. M.), Potenza, 2016, pp. 67-93. MR3587788; Zbl 1375.35129
[20] A. Pomponio, D. Ruiz, A variational analysis of a gauged nonlinear Schrödinger equation, J. Eur. Math. Soc. (JEMS) 17(2015), No. 6, 1463-1486. https://doi. org/doi.org/10. 4171/JEMS/535; MR3353806; Zbl 1328.35218
[21] A. Pomponio, D. Ruiz, Boundary concentration of a gauged nonlinear Schrödinger equation, Calc. Var. Partial Differential Equations 53(2015), No. 1-2, 289-316. https://doi.org/ 10.1007/s00526-014-0749-2; MR3336321; Zbl 1331.35326
[22] D. Ruiz, G. Siciliano, Existence of ground states for a modified nonlinear Schrödinger equation, Nonlinearity 23(2010), No. 5, 1221-1233. https://doi.org/10. 1088/0951-7715/ 23/5/011; MR2630099; Zbl 1189.35316
[23] W. A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55(1977), No. 2, 149-162. https://doi.org/10.1007/BF01626517; MR0454365; Zbl 0356.35028
[24] X. Tang, S. Chen, Ground state solutions of Nehari-Pohožaev type for SchrödingerPoisson problems with general potentials, Discrete Contin. Dyn. Syst. 37(2017), No. 9, 4973-5002. https://doi.org/10.3934/dcds.2017214; MR3661829; Zbl 1371.35051
[25] X. Tang, J. Zhang, W. Zhang, Existence and concentration of solutions for the Chern-Simons-Schrödinger system with general nonlinearity, Results Math. 71(2017), No. 3-4, 643-655. https://doi.org/10.1007/s00025-016-0553-8; MR3648437; Zbl 1376.35026
[26] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, $J$. Differential Equations 51(1984), No. 1, 126-150. https: //doi. org/10.1016/0022-0396 (84) 90105-0; MR0727034; Zbl 0488.35017
[27] N. S. Trudinger, On Harnack type inequalities and their application to quasilinear elliptic equations, Comm. Pure Appl. Math. 20(1967), 721-747. https://doi.org/10.1002/cpa. 3160200406; MR0226198; Zbl 0153.42703
[28] Y. Wan, J. Tan, Standing waves for the Chern-Simons-Schrödinger systems without (AR) condition, J. Math. Anal. Appl. 415(2014), No. 1, 422-434. https://doi.org/10.1016/j. jmaa.2014.01.084; MR3173176; Zbl 1314.35174
[29] Y. Wan, J. Tan, The existence of nontrivial solutions to Chern-Simons-Schrödinger systems, Discrete Contin. Dyn. Syst. 37(2017), No. 5, 2765-2786. https://doi.org/10.3934/ dcds. 2017119; MR3619082; Zbl 1376.35039
[30] M. Willem, Minimax theorems, Birkhäuser Boston, Inc., Boston, MA, 1996. https://doi . org/10.1007/978-1-4612-4146-1; MR1400007; Zbl 0856.49001
[31] Y. Xiao, C. Zhu, J. Chen, Ground state solutions for modified quasilinear Schrödinger equations coupled with the Chern-Simons gauge theory, Appl. Anal., published online: 20 October 2020. https://doi.org/10.1080/00036811.2020.1836355
[32] M. Yang, Existence of solutions for a quasilinear Schrödinger equation with subcritical nonlinearities, Nonlinear Anal. 75(2012), No. 13, 5362-5373. https://doi.org/10.1016/ j.na.2012.04.054; MR2927594; Zbl 1258.35077
[33] J. Yuan, Multiple normalized solutions of Chern-Simons-Schrödinger system, NoDEA Nonlinear Differential Equations Appl. 22(2015), No. 6, 1801-1816. https://doi.org/10. 1007/s00030-015-0344-z; MR3415022; Zbl 1328.35224
[34] J. Zhang, W. Zhang, X. Tang, Ground state solutions for Hamiltonian elliptic system with inverse square potential, Discrete Contin. Dyn. Syst. 37(2017), 4565-4583. https:// doi.org/10.3934/dcds.2017195; MR3642277; Zbl 1370.35111
[35] J. Zhang, W. Zhang, X. Xie, Infinitely many solutions for a gauged nonlinear Schrödinger equation, Appl. Math. Lett. 88(2019), 21-27. https://doi.org/10.1016/j. aml.2018.08.007; MR3862708; Zbl 1411.35098

# Convective instability in a diffusive predator-prey system 

Hui Chen ${ }^{* 1}$ and Xuelian $\mathbf{X u}{ }^{\boxtimes 2}$<br>${ }^{1}$ School of Science, Heilongjiang University of Science and Technology, Harbin, 150022, China<br>${ }^{2}$ School of Mathematical Sciences, Harbin Normal University, Harbin, 150025, China

Received 29 January 2021, appeared 23 September 2021
Communicated by Péter L. Simon


#### Abstract

It is well known that biological pattern formation is the Turing mechanism, in which a homogeneous steady state is destabilized by the addition of diffusion, though it is stable in the kinetic ODEs. However, steady states that are unstable in the kinetic ODEs are rarely mentioned. This paper concerns a reaction diffusion advection system under Neumann boundary conditions, where steady states that are unstable in the kinetic ODEs. Our results provide a stabilization strategy for the same steady state, the combination of large advection rate and small diffusion rate can stabilize the homogeneous equilibrium. Moreover, we investigate the existence and stability of nonconstant positive steady states to the system through rigorous bifurcation analysis.


Keywords: diffusion, advection, predator-prey, instability.
2020 Mathematics Subject Classification: 92D40, 35K57, 35B40, 35 B35.

## 1 Introduction

The central player in mathematical biology models is the stability of steady states. It is well known that biological pattern formation is the Turing mechanism, in which a homogeneous steady state that is stable in the kinetic ODEs is destabilised by the addition of diffusion terms. However, steady states that are unstable in the kinetic ODEs are almost never mentioned. As a result, there is a widespread assumption that unstable steady states are not biologically significant as PDE solutions.

The objective of this paper is to explain how diffusion and advection can turn an unstable steady state of kinetic ODEs to a stable one, to illustrate their implications for PDE models of biological systems. For that purpose, we investigate the spatially extended Rosenzweig-

[^36]MacArthur model for predator-prey interaction in river, which was proposed in [3]:

$$
\begin{cases}P_{t}=d_{1}\left(P_{x x}-\alpha P_{x}\right)+P\left(1-P-\frac{m N}{a+P}\right), & (0, L) \times(0,+\infty)  \tag{1.1}\\ N_{t}=N_{x x}-\alpha N_{x}-d N+\frac{m P N}{a+P}, & (0, L) \times(0,+\infty) \\ \left.P_{x}(0, t)=P_{x}(L, t)=N_{x}(0, t)=N_{( } L, t\right)=0, & t>0 \\ P(x, 0)=P_{0}(x) \geq 0, N(x, 0)=N_{0}(x) \geq 0, & x \in(0, L)\end{cases}
$$

where $P(x, t)$ and $N(x, t)$ denote predator and prey densities, which depend on space $x$ and time $t$. Here, and throughout this paper, we restrict attention to one space dimension ( $0, L$ ), though our analysis carries over to multi-dimensions. Most predator-prey studies do not include advection terms, advection of this type arises naturally in river-based predator-prey systems [3] and $\alpha$ is the convective rate of unidirectional flow. The parameter $d_{1}$ is the random diffusion rate of the prey and the random diffusion rate of the predator is rescaled to 1 . The prey consumption rate per predator is an increasing saturating function of the prey density with Holling type II form: $m$ reflects how quickly the consumption rate saturates as prey density increases, $a$ is the density of prey necessary to achieve one half the rate. $d$ is the death rate of the predator, also see [9].

Here the zero Neumann boundary conditions correspond to a long river in which the downstream boundary has little influence, see e.g., [3,7]. For the same parameter values as used ODEs, the stability of constant steady states does not change in diffusive systems under zero Neumann boundary conditions, see e.g., [12]. We will find a distinguished result for the reaction-diffusion-advection system (1.1): the coexistence steady state of (1.1) becomes stable for large advection rates though it is unstable for the corresponding diffusive system.

Over the past few decades, reaction-diffusion systems have been widely applied and extensively studied to model the spatial-temporal predator-prey dynamics, which can greatly explain the invasion of a prey by predators (e.g., [8]). For the spatial model with advection, there are some recent related works to understand how the diffusion and advection jointly effect population persist over large temporal scales and resist washout in such environment [ $5,6,13]$. Our purpose is to investigate the stabilization effect of advection.

In Section 2, we perform linear stability of the unique equilibrium ( $P^{*}, N^{*}$ ) with respect to (1.1). Our results in Theorem 2.4 indicate that advection and diffusion stabilize the homogeneous equilibrium when the advection is large and diffusion is small, while it still destabilizes predator-prey interactions when the advection is small. This extends the work of [9]. Section 3 is devoted to the steady state bifurcation analysis of (1.1) which establishes the existence of its nonconstant steady states, with advection rate $\alpha$ being the bifurcation parameter, see Theorem 3.2.

## 2 Linearized stability driven by advection

The system (1.1) has three non-negative constant equilibrium solution $(0,0),(1,0),\left(P^{*}, N^{*}\right)$, where

$$
\left(P^{*}, N^{*}\right)=\left(\frac{a d}{m-d^{\prime}}, \frac{\left(a+P^{*}\right)\left(1-P^{*}\right)}{m}\right) .
$$

The coexistence equilibrium ( $P^{*}, N^{*}$ ) is in the first quadrant if and only if $0<\frac{a d}{m-d}<1$. First we recall some well known results on the ODE dynamics of (1.1), see for example [4,11]:

$$
\left\{\begin{array}{l}
P_{t}=P\left(1-P-\frac{m N}{a+P}\right)  \tag{2.1}\\
N_{t}=-d N+\frac{m P N}{a+P}
\end{array}\right.
$$

Lemma 2.1. The following statements hold for system (2.1):

1. when $P^{*} \geq 1,(1,0)$ is globally asymptotically stable, see [4];
2. when $1-a<P^{*}<1,\left(P^{*}, N^{*}\right)$ is globally asymptotically stable, see [4];
3. $P^{*}=\frac{1-a}{2}$ is the unique bifurcation point where a Hopf bifurcation occurs, and the Hopf bifurcation is supercritical and backward;
4. when $0<P^{*}<\frac{1-a}{2},\left(P^{*}, N^{*}\right)$ is unstable and there is a globally asymptotically stable periodic orbit, see [11];
5. when $\frac{1-a}{2}<P^{*}<1$, then (2.1) has no closed orbits in the first quadrant and the positive equilibrium $\left(P^{*}, N^{*}\right)$ is globally asymptotically stable in the first quadrant, see [11].
Based on this, we always assume that the constants satisfy $0<a<1, P^{*}>0$ and $N^{*}>0$ throughout the paper. Following the same process of Theorem 2.1 in [12], we have the existence of solution and a priori bound of the solution to the dynamical equation (1.1).

Lemma 2.2. The following statements hold:
(a) If $P_{0}(x) \geq 0(\not \equiv 0), N_{0}(x) \geq 0(\not \equiv 0)$, then (1.1) has a unique solution $(P(x, t), N(x, t))$ such that $P(x, t)>0, N(x, t)>0$ for $t \in(0, \infty)$ and $x \in[0, L]$;
(b) For any solution $(P(x, t), N(x, t))$ of (1.1),

$$
\limsup _{t \rightarrow \infty} P(x, t) \leq 1, \quad \int_{0}^{L} N(x, t) d x \leq\left(1+\frac{(a+1) L}{4 d}\right)
$$

Moreover, there exists $C>0$ such that

$$
\limsup _{t \rightarrow+\infty} N(x, t) \leq C,
$$

where $C$ is independent of $P_{0}, N_{0}, d_{1}, \alpha$. If $d_{1}=1$, then $N(x, t) \leq\left(1+\frac{(a+1) L}{4 d}\right)$ for all $t>0$, $x \in[0, L]$.

In the following, we investigate the effect of diffusion and advection on the stability of $\left(P^{*}, N^{*}\right)$. For the convenience, we denote

$$
\left\{\begin{array}{l}
f(P, N)=P\left(1-P-\frac{m N}{a+P}\right) \\
g(P, N)=-d N+\frac{m P N}{a+P}
\end{array}\right.
$$

Then the linearization of (1.1) at $\left(P^{*}, N^{*}\right)$ can be expressed by:

$$
\begin{equation*}
\binom{\phi_{t}}{\psi_{t}}=L(\alpha)\binom{\phi}{\psi}:=D\binom{\phi_{x x}-\alpha \phi_{x}}{\psi_{x x}-\alpha \psi_{x}}+J_{(P, N)}\binom{\phi}{\psi} \tag{2.2}
\end{equation*}
$$

with domain $X=\left\{(\phi, \psi) \in H^{2}((0, L)) \times H^{2}((0, L)): \phi_{x}=\psi_{x}=0, x=0, L\right\}$, where

$$
D=\left(\begin{array}{cc}
d_{1} & 0 \\
0 & 1
\end{array}\right), \quad J_{(P, N)}=\left(\begin{array}{cc}
f_{P} & f_{N} \\
g_{P} & g_{N}
\end{array}\right)
$$

and

$$
\begin{array}{ll}
f_{P}=\frac{P^{*}\left(1-a-2 P^{*}\right)}{\left(a+P^{*}\right)}, & f_{N}=-\frac{m P^{*}}{\left(a+P^{*}\right)^{\prime}}, \\
g_{P}=\frac{a\left(1-P^{*}\right)}{\left(a+P^{*}\right)}, & g_{N}=0 .
\end{array}
$$

From Theorem 5.1.1 of [2], it is known that if all the eigenvalues of the operator $L$ have negative real parts, then $\left(P^{*}, N^{*}\right)$ is asymptotically stable, otherwise, $\left(P^{*}, N^{*}\right)$ is unstable.

Thus $\lambda$ is an eigenvalue of $L$ if and only if $\lambda$ is an eigenvalue of the matrix $J_{k}=-\mu_{k} D+$ $J_{(P, N)}$ for some $k \geq 0$, where $\mu_{k}(k=0,1,2, \ldots)$ is the $k$ th eigenvalue of the following eigenvalue problem:

$$
\left\{\begin{array}{l}
\phi_{x x}-\alpha \phi_{x}=-\mu_{k} \phi, \quad x \in(0, L)  \tag{2.3}\\
\phi_{x}(0)=\phi_{x}(L)=0
\end{array}\right.
$$

Since $x \in(0, L)$, we can directly calculate the eigenvalue $\mu_{k}$ and eigenfunction $\phi_{k}(x)$ as following:

$$
\left\{\begin{array}{l}
\mu_{k}=\left(\frac{k \pi}{L}\right)^{2}+\frac{\alpha^{2}}{4}, k=0,1,2, \ldots  \tag{2.4}\\
\phi_{k}(x)=\alpha e^{\frac{\alpha x}{2}} \cos \left(\frac{k \pi x}{L}\right)+\frac{2 k \pi}{L} e^{\frac{\alpha x}{2}} \sin \left(\frac{k \pi x}{L}\right), k=0,1,2, \ldots
\end{array}\right.
$$

So the stability is reduced to consider the characteristic equation

$$
\begin{equation*}
\lambda^{2}-\operatorname{Trace}\left(J_{k}\right) \lambda+\operatorname{Det}\left(J_{k}\right)=0, \quad k=0,1,2, \ldots \tag{2.5}
\end{equation*}
$$

with

$$
\begin{align*}
& \operatorname{Trace}\left(J_{k}\right)=-\left(d_{1}+1\right) \mu_{k}+f_{P}+g_{N}:=-\left(d_{1}+1\right) \mu_{k}+\operatorname{Trace}(J), \\
& \operatorname{Det}\left(J_{k}\right)=d_{1} \mu_{k}^{2}-\left(d_{1} g_{N}+f_{P}\right) \mu_{k}+f_{P} g_{N}-f_{N} g_{P}:=d_{1} \mu_{k}^{2}-\left(d_{1} g_{N}+f_{P}\right) \mu_{k}+\operatorname{Det}(J) . \tag{2.6}
\end{align*}
$$

We take $\alpha$ as the main bifurcation parameter to observe its effect on the local stability $\left(P^{*}, N^{*}\right)$. First of all, we list four conditions for the sake of following discussion.
(A1) $f_{P}^{2}+4 d_{1} f_{N} g_{P}<0$,
(A2) $f_{P}^{2}+4 d_{1} f_{N} g_{P}>0$,
(A3) $d_{1} \mu_{0}^{2}-f_{P} \mu_{0}-f_{N} g_{P} \leq 0$,
(A4) $d_{1} \mu_{0}^{2}-f_{P} \mu_{0}-f_{N} g_{P}>0$.
Theorem 2.3. Suppose $P^{*} \geq \frac{1-a}{2}$. Then $\left(P^{*}, N^{*}\right)$ is always locally asymptotically stable for any advection rate $\alpha>0$.

Proof. It can find that $f_{P} \leq 0$ when $P^{*} \geq \frac{1-a}{2}$. Thus Trace $\left(J_{k}\right)<0$ and $\operatorname{Det}\left(J_{k}\right)>0$ for all $k=0,1,2, \ldots$, which implies the desired results.

Theorem 2.4. Suppose $P^{*}<\frac{1-a}{2}$.

1. If $-\left(d_{1}+1\right) \mu_{0}+f_{P}>0$, then $\left(P^{*}, N^{*}\right)$ is unstable.
2. If there is some $k \geq 0$ such that $-\left(d_{1}+1\right) \mu_{k}+f_{P}=0$, then system (1.1) generates a heterogeneous Hopf bifurcation at ( $P^{*}, N^{*}$ ) provided either (A1) holds or (A2), (A4) and $\mu_{0}>\frac{f_{p}}{2}$ holds.
3. If $-\left(d_{1}+1\right) \mu_{0}+f_{P}<0$, then ( $\left.P^{*}, N^{*}\right)$ is locally asymptotically stable provided either (A1) holds or (A2), (A4) and $\mu_{0}>\frac{f_{P}}{2}$ holds; and $\left(P^{*}, N^{*}\right)$ is unstable provided (A3) holds,

Proof. It just notices that $\operatorname{Det}\left(J_{k}\right)>0$ for all $k=0,1,2, \ldots$ if either (A1) or (A2) holds; and $\operatorname{Det}\left(J_{0}\right)<0$ if (A3) holds.

Remark 2.5. Theorem 2.3 and Theorem 2.4 imply that the advection rate $\alpha$ makes $\left(P^{*}, N^{*}\right)$ more stable compared with that for the corresponding ODE system in Lemma 2.1. The periodic solution bifurcating from ( $P^{*}, N^{*}$ ) will disappear when introducing the advection and diffusion; Moreover under the same condition that $\left(P^{*}, N^{*}\right)$ is unstable for (2.1), there is newborn homogeneous/heterogeneous Hopf bifurcation solutions at $\left(P^{*}, N^{*}\right)$ or $\left(P^{*}, N^{*}\right)$ even becomes stable for small diffusion rate $d_{1}$ or large advection rate $\alpha$.

## 3 Existence of non-constant positive steady state

In this section we show that when $\left(P^{*}, N^{*}\right)$ is unstable, there exist positive non-constant steady state solutions of (1.1). In order to show that we use bifurcation theory to prove the existence of positive non-constant steady state solutions. The bifurcations can be shown with parameter $\alpha_{k}\left(\right.$ or $\left.\mu_{k}\right)$ as shown in Theorem 2.4. From the relation given in (2.6), we define the potential bifurcation points:

$$
\begin{equation*}
\alpha_{k, \pm}^{2}=\frac{2 f_{P} \pm 2 \sqrt{f_{P}^{2}+4 d_{1} f_{N} g_{P}}}{d_{1}}-4\left(\frac{k \pi}{L}\right)^{2}, \quad k=0,1,2, \ldots \tag{3.1}
\end{equation*}
$$

We have the following properties of $\alpha_{k, \pm}$ :
Lemma 3.1. Assume that (A2) holds. Then

1. $\lim _{k \rightarrow \infty} \alpha_{k, \pm}=-\infty$;
2. Both $\alpha_{k,+}$ and $\alpha_{k,-}$ are monotonically decreasing with respect to $k$, there exists $m, n$ such that $\alpha_{0,+}>\alpha_{1,+}>\cdots>\alpha_{m,+} \geq 0$ and $\alpha_{0,-}>\alpha_{1,-}>\cdots>\alpha_{m,-} \geq 0$.

Theorem 3.2. Assume that (A2) holds. Let $\alpha_{k, \pm}$ be defined as in (3.1) such that $\alpha_{i,+} \neq \alpha_{j,-}$ for any $0 \leq i \leq m$ and $0 \leq j \leq n$. Then

1. Near $\left(\alpha_{i, \pm}, P^{*}, N^{*}\right)$, the set of positive non-constant steady state solutions of (1.1) is a smooth curve $\Sigma_{i}=\left\{\alpha_{i}(s), P_{i}(s), N_{i}(s): s \in(-\varepsilon, \varepsilon)\right\}$, where where $P_{i}(s)=P^{*}+s a_{i} \phi_{i}(x)+s^{2} \psi_{1, i}(s)+$ $O\left(s^{3}\right), N_{i}(s)=N^{*}+s b_{i} \phi_{i}(x)+s^{2} \psi_{2, i}(s)+O\left(s^{3}\right)$ for some smooth functions $\psi_{1, i}, \psi_{2, i}$ such that $\alpha_{i}(s)=\alpha_{i, \pm}+O(s)$ and $\psi_{1, i}(0)=\psi_{2, i}(0)=0$; Here $\left(a_{i}, b_{i}\right)$ satisfies

$$
L\left(\alpha_{i}\right)\left[\left(a_{i}, b_{i}\right)^{T} \phi_{i}(x)\right]=(0,0)^{T} .
$$

2. The smooth curve $\Sigma_{i}$ in part (1) is contained in a connected component $C_{i}$ of $\Gamma$, which is the closure of the set of positive non-constant steady state solutions of (1.1), and either $C_{i}$ is unbounded or $C_{i}$ contains another $\left(\alpha_{j, \pm}, P^{*}, N^{*}\right)$ with $\alpha_{i, \pm} \neq \alpha_{j, \pm}$.

Proof. The existence and uniqueness of $\alpha_{i, \pm}$ follows from discussions above. Then the local bifurcation result follows the bifurcation theorem in [1], and it is an application of a more general result Theorem 4.3 in [10].

Define a nonlinear mapping

$$
F(\alpha, P, N)=\binom{d_{1}\left(P_{x x}-\alpha P_{x}\right)+f(P, N)}{N_{x} x-\alpha N_{x}+g(P, N)}
$$

with domain $V=\left\{(\alpha, P, N): 0<\alpha<\alpha_{0,+},(P, N) \in X \times X\right\}$, where $X=\left\{\omega \in H^{2}((0, L))\right.$ : $\left.\omega^{\prime}(0)=\omega^{\prime}(L)=0\right\}$. Then $F(\alpha, P, N)=0$ is equivalent to the steady state system of (1.1):

$$
\begin{cases}d_{1}\left(P_{x x}-\alpha P_{x}\right)+P\left(1-P-\frac{m N}{a+P}\right)=0, & x \in(0, L),  \tag{3.2}\\ N_{x x}-\alpha N_{x}-d N+\frac{m P N}{a+P}=0, & x \in(0, L), \\ P_{x}(0)=P_{x}(L)=N_{x}(0)=N_{x}(L)=0 . & \end{cases}
$$

It is observed that $F(\alpha, P, N)=0$ for all $\alpha>0$. For any $\left(\alpha, P^{*}, N^{*}\right) \in V$, the The Fréchet derivative of $F$ is given by

$$
D_{(P, N)} F\left(\alpha, P^{*}, N^{*}\right)(P, N)=\binom{d_{1}\left(P_{x x}-\alpha P_{x}\right)+f_{P} P+f_{N} N}{N_{x x}-\alpha N_{x}+g_{P} P+g_{N} N} .
$$

Then $D_{(P, N)} F\left(\alpha_{i, \pm}, P^{*}, N^{*}\right)(P, N)$ is a Fredholm operator with index zero by Corollary 2.11 in [10].

We show that the conditions for Theorem 4.3 in [10] are satisfied in several steps.
Step 1. $\operatorname{dim} N\left(D_{(P, N)} F\left(\alpha_{i, \pm}, P^{*}, N^{*}\right)\right)=1$.
From the definition of $\alpha_{i, \pm}$, it is easy to verify that $\operatorname{Det}\left(J_{i}\right)=0$, hence zero is an eigenvalue of $J_{i}$ with an eigenvector $\left(a_{i}, b_{i}\right)=\left(g_{P}, d_{1} \mu_{i}\right)$. Then $V_{i}=\left(g_{P}, d_{1} \mu_{i}\right) \phi_{i}(x)$ is an eigenfunction of $L\left(\alpha_{i, \pm}\right)$ defined in (2.2) and evaluated at $\left(P^{*}, N^{*}\right)$ with eigenvalue zero. Since $\mu_{i}(i=0,1,2 \ldots)$ is a simple eigenvalue from (2.4), then the eigenvector is unique up to a constant multiple. Thus one has $N\left(D_{(P, N)} F\left(\alpha_{i, \pm}, P^{*}, N^{*}\right)\right)=\operatorname{span}\left\{V_{i}\right\}$ which is one-dimensional. Note that we also have that $\operatorname{codim} R\left(D_{(P, N)} F\left(\alpha_{i, \pm}, P^{*}, N^{*}\right)\right)=1$ as $D_{(P, N)} F\left(\alpha_{i, \pm}, P^{*}, N^{*}\right)$ is Fredholm with index zero.

Step 2. $D_{(P, N) \alpha} F\left(\alpha_{i, \pm}, P^{*}, N^{*}\right)\left(V_{i}\right) \notin R\left(D_{(P, N)} F\left(\alpha_{i, \pm}, P^{*}, N^{*}\right)\right)$.
It is easy to see that an eigenvector of $L^{*}\left(\alpha_{i, \pm}\right)$ corresponding to zero eigenvalue is $V_{i}^{*}=$ $\left(a_{i}^{*}, b_{i}^{*}\right)=\left(-\mu_{i}+f_{P}, g_{P}\right) \phi(x)$, here $L^{*}\left(\alpha_{i, \pm}\right)$ is the adjoint matrix of $-L\left(\alpha_{i, \pm}\right)$. If $\left(h_{1}, h_{2}\right) \in$ $R\left(D_{(P, N)} F\left(\alpha_{i, \pm}, P^{*}, N^{*}\right)\right)$, then there exists $\left(\varphi_{1}, \varphi_{2}\right)$ such that $D_{(P, N) \alpha} F\left(\alpha_{i, \pm}, P^{*}, N^{*}\right)\left(\varphi_{1}, \varphi_{2}\right)^{T}=$ $-L\left(\alpha_{i, \pm}\right)\left(\varphi_{1}, \varphi_{2}\right)^{T}=\left(h_{1}, h_{2}\right)^{T}$. Thus

$$
\int_{0}^{L}\left(a_{i}^{*} h_{1}+b_{i}^{*} h_{2}\right) \phi_{i}(x) d x=0
$$

It is noticed that $D_{(P, N) \alpha} F\left(\alpha_{i, \pm}, P^{*}, N^{*}\right)\left(V_{i}\right)=\left(0,-\mu_{i} b_{i} \phi_{i}(x)\right)^{T}$, and

$$
\int_{0}^{L} a_{i}^{*} \cdot 0+b_{i}^{*} \cdot\left(-\mu_{i} b_{i} \phi_{i}(x)\right) d x=\int_{0}^{L} d_{1} \mu_{i}^{2} g_{P} \phi_{i}^{2}(x) d x>0 .
$$

Thus $D_{(P, N) \alpha} F\left(\alpha_{i, \pm}, P^{*}, N^{*}\right)\left(V_{i}\right) \notin R\left(D_{(P, N)} F\left(\alpha_{i, \pm}, P^{*}, N^{*}\right)\right)$.
Step 3. It is noticed that $\left\{\left(\alpha, P^{*}, N^{*}\right): 0<\alpha<\alpha_{0,+}\right\}$ is a line of trivial solutions for $F=0$, thus Theorem 4.3 in [10] can be applied to each continuum $C_{i}$ bifurcated from ( $\alpha_{i, \pm}, P^{*}, N^{*}$ ). The solutions of (3.2) on $C_{i}$ near the bifurcation point are apparently positive. For each continuum $C_{i}$, either $\bar{C}_{i}$ contains another ( $\alpha_{j, \pm}, P^{*}, N^{*}$ ) or $C_{i}$ is not compact. (Here we do not make an extinction between the solutions of (3.2) and $F=0$ as they are essentially same, hence we use $C_{i}$ for solution continuum for both equations.) Therefore, either $C_{i}$ is unbounded or $C_{i}$ contains another $\left(\alpha_{j, \pm}, P^{*}, N^{*}\right)$ with $\alpha_{i, \pm} \neq \alpha_{j, \pm}$.

## 4 Conclusions and numerical simulations

It is a general result that a steady state that is unstable as a solution of the kinetic ODEs is also unstable as a PDE solution on a finite domain under zero Neumann conditions [12]. Our results in Theorem 2.4 indicate that the combination of advection and diffusion can stabilize the homogeneous equilibrium. For the constant steady state that are unstable in the kinetic ODEs, it becomes stable when the advection is large and diffusion is small, while it keeps instability when the advection is small. Moreover, we obtain non-constant steady states by bifurcation theory when the constant steady state is unstable. These results extend the work of [9]. Our analysis and methods are also suitable for higher dimensional systems, we can obtain the concrete bifurcation value in one dimensional interval. From a theoretical point of view, this paper introduces a new class of reaction-diffusion models with advection, which may be of independent interest.

Consider system (3.2) and fix $d=0.5, m=1, a=0.6$. Then $P^{*}>\frac{1-a}{2}$. Lemma 2.1 says that $\left(P^{*}, N^{*}\right)=(0.6,0.48)$ is locally asymptotically stable for any $d_{1}>0$ and $\alpha=0$, and Theorem 2.3 shows that $\left(P^{*}, N^{*}\right)=(0.6,0.48)$ keeps stable for $\alpha>0$, see Figure 4.1.


Figure 4.1: (Left): $d_{1}=1, \alpha=0$, and ( $P^{*}, N^{*}$ ) is locally asymptotically stable; (Right): $d_{1}=1, \alpha=10$, and $\left(P^{*}, N^{*}\right)$ is still locally asymptotically stable, the same initial value $\left(P_{0}, N_{0}\right)=(0.56,0.4)$.

Fix $d=0.5, m=1, a=0.33$. Then $P^{*}=\frac{1-a}{2}$. Lemma 2.1 says that (3.2) has a homogeneous Hopf bifurcation solution at $\left(P^{*}, N^{*}\right)=(0.33,0.44)$ for any $d_{1}>0$ and $\alpha=0$, while Theorem 2.3 shows that the homogeneous periodic solutions disappears for $\alpha>0$, see Figure 4.2.

Fix $d=0.5, m=1, a=0.32$. Then $P^{*}<\frac{1-a}{2}$. Lemma 2.1 says that $\left(P^{*}, N^{*}\right)=(0.32,0.44)$ is unstable for any $d_{1}>0$ and $\alpha=0$, while Theorem 2.3 shows that $\left(P^{*}, N^{*}\right)=(0.32,0.44)$ becomes stable for large $d_{1}>0$ and large $\alpha>0$, see Figure 4.3.


Figure 4.2: (Left): $d_{1}=1, \alpha=0$, and (3.2) has a homogeneous Hopf bifurcation solution at $\left(P^{*}, N^{*}\right)$; (Right): $d_{1}=0.3, \alpha=1$, and the periodic solution disappears, the same initial value $\left(P_{0}, N_{0}\right)=(0.33,0.4)$.


Figure 4.3: (Left): $d_{1}=1, \alpha=0$, and and $\left(P^{*}, N^{*}\right)$ is unstable; (Right): $d_{1}=1500$, $\alpha=1$, and $\left(P^{*}, N^{*}\right)$ becomes locally asymptotically stable, the same initial value $\left(P_{0}, N_{0}\right)=(0.3,0.4)$.

## Acknowledgements

The authors would like to thank the referee for valuable comments and suggestions. The work is supported by Natural Science Foundation of China (No. 11971135).

## References

[1] M. G. Crandall, P. H. Rabinowitz, Bifurcation perturbation of simple eigenvalues and linearized stability, Arch. Ration. Mech. Anal. 52(1973), 161-180. https://doi.org/10. 1007/BF00282325; MR1058648; Zbl 0275.47044
[2] D. Henry, Geometric theory of semilinear parabolic equations, Lecture Notes in Mathematics, Vol. 4, Springer-Verlag, Berlin-Heidelberg-New York, 1981. https://doi.org/10.1007/ BFb0089647; MR0610244; Zbl 0456.35001
[3] F. M. Hilker, M. A. Lewis, Predator-prey systems in streams and rivers, Theor. Ecol. 3(2010), No. 3, 175-193. https://doi.org/10.1007/s12080-009-0062-4;
[4] S. B. Hsu, On global stability of a predator-prey system, Math. Biosci. 39(1978), 1-10. https://doi.org/10.1016/0025-5564(78)90025-1; MR0472126; Zbl 0383.92014
[5] Q. H. Huang, Y. Jin, M. A. Lewis, $R_{0}$ analysis of a benthic-drift model for a stream population, SIAM J. Appl. Dyn. Syst. 15(2016), No. 1, 287-321. https ://doi. org/10.1137/ 15M1014486; MR3457691; Zbl 1364.92059
[6] K. Y. Lam, Y. Lou, F. Lutscher, The emergence of range limits in advective environments, SIAM J. Appl. Math. 76(2016), No. 2, 641-662. https://doi.org/10.1137/15M1027887; MR3477764; Zbl 1338.92146
[7] F. Lutscher, E. Расhepsky, M. A. Lewis, The effect of dispersal patterns on stream populations, SIAM Rev. 47(2005), No. 4, 749-772. https://doi.org/10.1137/050636152; MR2147329; Zbl 1076.92052
[8] M. R. Owen, M. A. Lewis, How predation can slow, stop or reverse a prey invasion, Bull. Math. Biol. 63(2001), No. 4, 655-684. https://doi.org/10.1137/050636152; MR3363426; Zbl 1323.92181
[9] J. A. Sherratt, A. S. Dagbovie, F. M. Hilker, A mathematical biologist's guide to absolute and convective instability, Bull. Math. Biol. 76(2014), No. 1, 1-26. https://doi.org/ /10.1007/s11538-013-9911-9; MR3150815; Zbl 1283.92007
[10] J. P. Shi, X. F. Wang, On global bifurcation for quasilinear elliptic systems on bounded domains, J. Differential Equations 246(2009), No. 7, 2788-2812. https://doi.org/10.1016/ j.jde.2008.09.009; MR2503022; Zbl 1165.35358
[11] J. F. Wang, J. P. Shi, J. J. Wei, Dynamics and pattern formation in a diffusive predatorprey system with strong Allee effect in prey, J. Differential Equations 251(2011), No. 4, 1276-1304. https://doi.org/10.1016/j.jde.2011.03.004; MR1222168; Zbl 1228.35037
[12] J. F. Wang, J. J. Wei, J. P. Shi, Global bifurcation analysis and pattern formation in homogeneous diffusive predator-prey system, J. Differential Equations 260(2016), No. 4, 34953523. https://doi.org/10.1016/j.jde.2015.10.036; MR2812590; Zbl 1332.35176
[13] X.-Q. Zhao, P. Zhou, On a Lotka-Volterra competition model: the effects of advection and spatial variation, Calc. Var. Partial Differential Equations 55(2016), No. 4. https://doi. org/10.1007/s00526-016-1021-8; MR3513211; Zbl 1366.35088

# A saddle point type solution for a system of operator equations 

Piotr M. Kowalski ${ }^{\boxtimes}$<br>Institute of Mathematics, Lodz University of Technology, Al. Politechniki 10, Lodz, 93-590, Poland

Received 6 April 2021, appeared 28 September 2021
Communicated by László Simon


#### Abstract

Let $\Omega \subset \mathbb{R}^{n}, n>1$ and let $p, q \geq 2$. We consider the system of nonlinear Dirichlet problems


$$
\left\{\begin{aligned}
(A u)(x) & =N_{u}^{\prime}(x, u(x), v(x)), & & x \in \Omega \\
-(B v)(x) & =N_{v}^{\prime}(x, u(x), v(x)), & & x \in \Omega \\
u(x) & =0, & & x \in \partial \Omega \\
v(x) & =0, & & x \in \partial \Omega
\end{aligned}\right.
$$

where $N: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is $\mathrm{C}^{1}$ and is partially convex-concave and $A: \mathrm{W}_{0}^{1, p}(\Omega) \rightarrow$ $\mathrm{W}^{-1, p^{\prime}}(\Omega), B: \mathrm{W}_{0}^{1, q}(\Omega) \rightarrow \mathrm{W}^{-1, q^{\prime}}(\Omega)$ are monotone and potential operators. The solvability of this system is reached via the Ky-Fan minimax theorem.
Keywords: Ky-Fan minimax theorem, Dirichlet problem, potential operators, monotone operators
2020 Mathematics Subject Classification: 35M12.

## 1 Introduction

Let $\Omega$ be any bounded domain in $\mathbb{R}^{n}$, where $n \in \mathbb{N}$ and let $p, q \geq 2, p, q \in \mathbb{R}$ be fixed. The aim of this work is to consider the system of two nonlinear Dirichlet boundary value problems whose solvability is reached via the Ky-Fan minimax theorem (consult [14] for details) which is a more general version of classical Sion's minimax theorem [10]. We also use some reasoning applied usually in the monotonicity approach. Namely we use direct method of Calculus of Variations, and the fact that monotone and potential operators are actually convex and l.s.c. To be precise we investigate the following problem. Let $N: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be an $L^{1}$-Carathéodory function, with some more requirement for its derivative with respect to second and third variables, and let $\mathcal{A}: \mathrm{W}_{0}^{1, p}(\Omega) \rightarrow \mathrm{W}^{-1, p^{\prime}}(\Omega), \mathcal{B}: \mathrm{W}_{0}^{1, q}(\Omega) \rightarrow \mathrm{W}^{-1, q^{\prime}}(\Omega)$ be some monotone and potential operators (pertaining to the classical negative $p$-Laplacian).

[^37]Problem 1 (Main problem). Find $(u, v) \in W_{0}^{1, p}(\Omega) \times W_{0}^{1, q}(\Omega)$ such that

$$
\begin{aligned}
\langle\mathcal{A}(u) ; \bar{u}\rangle & =\int_{\Omega} N_{u}^{\prime}(x, u(x), v(x)) \bar{u}(x) \mathrm{d} x \\
-\langle\mathcal{B}(v) ; \bar{v}\rangle & =\int_{\Omega} N_{v}^{\prime}(x, u(x), v(x)) \bar{u}(x) \mathrm{d} x .
\end{aligned}
$$

for all $\bar{u} \in \mathrm{~W}_{0}^{1, p}(\Omega), \bar{v} \in \mathrm{~W}_{0}^{1, q}(\Omega)$.
We see that the above is a system of mixed operator and integral type formulas, which under certain assumption appears to admit a solution of saddle point type. The existence of boundary value problems with the $p$-Laplacian is well covered in the literature, see for example $[4,7-9,15]$. Some results investigating the relation between the monotonicity and variational approaches are given in [5]. The case in which operators on LHS are both monotone is well studied, and existence result was proved by the critical point theory. In our situation one of the operators (namely $\mathcal{A}$ ) is monotone while the other (namely $-\mathcal{B}$ ) only becomes monotone in case it is multiplied by -1 . This observation forces us to adapt the approach known for elliptic systems, see for example $[6,11]$ to the case that could include also more non-linear equations. When compared with [11] we adapt their methods to the nonlinear setting and also simplify whenever possible their arguments by using direct links to the monotonicity theory. For an approach using the mixture of abstract formulation of the operator together with the explicitly written RHS we refer to [3] while underlying that these authors considered single equations.

## 2 Some preliminary results

The following properties are well known, but the full proofs are actually quite hard to be found. Some short proofs are indicated in [13], here we provide a full proof of a slightly modified result.

Lemma 2.1 (On properties of the pointwise maximum [13, Th. 3.3.3]). Assume $f: U \times V \rightarrow \mathbb{R}$, where $U, V$ are some vector spaces over $\mathbb{R}$ and let for any $u \in U$ there exists such $\hat{v} \in Y$ that $f(u, \hat{v})=\max _{v} f(u, v)$. If $u \mapsto f(u, v)$ is convex for any $v \in V$, then $u \mapsto \max _{v} f(u, v)$, is convex. If $u \mapsto f(u, v)$ is l.s.c. for any $v \in V$ then $u \mapsto \max _{v} f(u, v)$, is also lower semicontinuous.

Proof. Let $u, w \in U$ and let $\alpha \in(0,1)$. Lets denote $\hat{v}$ be such element of $V$ that

$$
f(\alpha u+(1-\alpha) w, \hat{v})=\max _{v} f(\alpha u+(1-\alpha) w, v) .
$$

Then

$$
\begin{aligned}
f(\alpha u+(1-\alpha) w, \hat{v}) & \leq \alpha f(u, \hat{v})+(1-\alpha) f(w, \hat{v}) \\
& \leq \alpha \max _{v} f(u, v)+(1-\alpha) f(w, \hat{v}) \\
& \leq \alpha \max _{v} f(u, v)+(1-\alpha) \max _{v} f(w, v) .
\end{aligned}
$$

Thus it follows that $u \mapsto \max _{v} f(u, v)$, is convex. For the second part we assume $u_{0} \in U$ and $\bar{v} \in V$ to be an arbitrary element. Then

$$
\liminf _{u \rightarrow u_{0}} \max _{v} f(u, v) \geq \liminf _{u \rightarrow u_{0}} f(u, \bar{v}) \geq f\left(u_{0}, \bar{v}\right) .
$$

As we apply maximum over $\bar{v}$ we gets

$$
\liminf _{u \rightarrow u_{0}} \max _{v} f(u, v) \geq \max _{v} f\left(u_{0}, v\right) .
$$

Since $u_{0} \in X$ was arbitrary, thus $u \mapsto \max _{v} f(u, v)$, is also lower semicontinuous.
Corollary 2.2 (On properties of the pointwise minimum). Assume $f: U \times V \rightarrow \mathbb{R}$, where $U, V$ are some vector spaces and let for any $u$ there exists such $\hat{v} \in Y$ that $f(u, \hat{v})=\min _{v} f(u, v)$. Let $u \mapsto f(u, v)$ be concave for any $v \in V$, then $u \mapsto \min _{v} f(u, v)$, is concave. If $u \mapsto f(u, v)$ be u.s.c. for any $v \in V$ then $u \mapsto \min _{v} f(u, v)$, is also upper semicontinuous.
$E$ will stand for a real and reflexive Banach space in this section. Since we shall use monotone operator approach lets recall its definition. We refer to [2] and [16] for some background.

Definition 2.3 (Properties of operators). Let $\mathcal{A}: E \rightarrow E^{*}$. Then

- $\mathcal{A}$ is called monotone iff

$$
\langle\mathcal{A}(u)-\mathcal{A}(v) ; u-v\rangle \geq 0,
$$

for all $u, v \in E$;

- $\mathcal{A}$ is called coercive iff

$$
\lim _{\|u\|_{E} \rightarrow \infty} \frac{\langle\mathcal{A}(u) ; u\rangle}{\|u\|_{E}}=+\infty .
$$

- $\mathcal{A}$ is called anticoercive iff operator $-\mathcal{A}$ is coercive.
- $\mathcal{A}$ is said to be demicontinuous iff $u_{n} \rightarrow u$ as $n \rightarrow \infty$ implies that

$$
\mathcal{A} u_{n} \rightharpoonup \mathcal{A} u,
$$

as $n \rightarrow \infty$.

- $\mathcal{A}$ is potential if there exists a functional $f: E \rightarrow \mathbb{R}$ differentiable in the sense of Gâteaux and such that

$$
f^{\prime}=\mathcal{A},
$$

Then $f$ is called potential of $\mathcal{A}$.
Lemma 2.4. Assume $\mathcal{A}: E \rightarrow E^{*}$ is potential and monotone. Then its potential is convex and weakly lower semicontinuous (w.l.s.c. for short). Also $\mathcal{A}$ is demicontinuous.

Lemma 2.5. Assume $\mathcal{A}: E \rightarrow E^{*}$ is potential and demicontinuous. Then

$$
v \mapsto \int_{0}^{1}\langle\mathcal{A}(t v) ; v\rangle \mathrm{d} t, v \in E,
$$

is a potential of $\mathcal{A}$.
Sufficient conditions for existence of solution may describe in terms of some constant provided by the following Sobolev embedding theorem.

Theorem 2.6 (Sobolev imbedding theorem [1, Th. 4.12]). Let $\Omega$ be a bounded domain in $\mathbb{R}^{n}$. Then

- if $p \geq n$ then

$$
\mathrm{W}_{0}^{1, p}(\Omega) \rightarrow \mathrm{L}^{q}(\Omega),
$$

for $1 \leq q \leq \infty$, and

- if $p<n$ then

$$
\mathrm{W}_{0}^{1, p}(\Omega) \rightarrow \mathrm{L}^{q}(\Omega),
$$

for $1 \leq q \leq \frac{n p}{n-p}$.
We shall require following two constants. Let $\lambda_{1, p}>0$ be such that for all $u \in W_{0}^{1, p}(\Omega)$ :

$$
\lambda_{1, p}\|u\|_{\mathrm{L}^{p}(\Omega)}^{p} \leq\|u\|_{\mathrm{W}_{0}^{1, p}(\Omega)}^{p} .
$$

Also let $\lambda_{1, q}>0$ satisfy similar condition for $q$ and $v \in \mathrm{~W}_{0}^{1, q}(\Omega)$.
Definition 2.7 (L's-Carathéodory function [5]). Assume $f: \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $s \geq 1$ holds. We shall say that $f$ is $\mathrm{L}^{s}$-Carathéodory, if

- for all $(u, v) \in \mathbb{R} \times \mathbb{R}$ function $x \mapsto f(x, u, v)$ is measurable;
- for a.e. $x \in \Omega$ function $(u, v) \mapsto f(x, u, v)$ is continuous;
- for each $d>0$ there exists a function $f_{d} \in \mathrm{~L}^{s}(\Omega)$ such that for a. e. $x \in \Omega$

$$
\max _{(u, v) \in[-d, d] \times[-d, d]}|f(x, u, v)| \leq f_{d}(x) ;
$$

## 3 Variational framework and the existence of a solution

(A) Operator $\mathcal{A}$ is potential and monotone.
(B) Operator $\mathcal{B}$ is potential and monotone.
(C) Operator $\mathcal{A}$ fulfils that there exists $\hat{\alpha_{1}}>0$,

$$
\langle\mathcal{A}(u) ; u\rangle \geq \hat{\alpha_{1}}\|u\|_{\mathrm{W}_{0}^{1, p}(\Omega)}^{p},
$$

for all $u \in W_{0}^{1, p}(\Omega)$.
(D) Operator $\mathcal{B}$ fulfils that there exists $\hat{\alpha_{2}}>0$,

$$
\langle\mathcal{B}(v) ; v\rangle \geq \hat{\alpha_{2}}\|v\|_{\mathrm{W}_{0}^{1, q}(\Omega)}^{q},
$$

for all $v \in \mathrm{~W}_{0}^{1, q}(\Omega)$.
(E) Function $N: \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is $L^{1}$-Carathéodory. Moreover, derivatives $N_{u}^{\prime}, N_{v}^{\prime}$ exists and $N_{u}^{\prime}: \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is $\mathrm{L}^{p^{\prime}}$-Carathéodory, and $N_{v}^{\prime}: \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is $\mathrm{L}^{q^{\prime}}$ Carathéodory.
(F) for each $v \in \mathrm{~W}_{0}^{1, q}(\Omega)$ there exists functions $\beta_{1} \in \mathrm{~L}^{2}(\Omega), \gamma_{1} \in \mathrm{~L}^{1}(\Omega)$ and $0<\alpha_{1}<\lambda_{1, p} \frac{\alpha_{1}}{p}$ that

$$
N(x, u, v(x)) \geq-\alpha_{1}|u|^{p}+\beta_{1}(x) \cdot u+\gamma_{1}(x),
$$

for almost every $x \in \Omega$ and all $u \in \mathbb{R}$.
(G) for each $u \in \mathrm{~W}_{0}^{1, p}(\Omega)$ there exists functions $\beta_{2} \in \mathrm{~L}^{2}(\Omega), \gamma_{2} \in \mathrm{~L}^{1}(\Omega)$ and $0<\alpha_{2}<\lambda_{1, q} \frac{\hat{\alpha}_{2}}{q}$ that

$$
N(x, u(x), v) \leq \alpha_{2}|v|^{q}+\beta_{2}(x) \cdot v+\gamma_{2}(x),
$$

for almost every $x \in \Omega$ and all $v \in \mathbb{R}$.
(H) For any fixed $v \in \mathrm{~W}_{0}^{1, q}(\Omega)$ functional

$$
u \mapsto \int_{\Omega} N(x, u(x), v(x)) \mathrm{d} x
$$

is convex.
(I) For any fixed $u \in \mathrm{~W}_{0}^{1, p}(\Omega)$ functional

$$
v \mapsto \int_{\Omega} N(x, u(x), v(x)) \mathrm{d} x
$$

is concave.
Let A be a potential to $\mathcal{A}$, and B to $\mathcal{B}$. Also by N be shall denote the Nemyckij's operator to $N$.

In order to obtain the existence result, we consider the following reformulation of Problem 1 to a critical point-type problem:

Problem 2 (Variational form of the main problem). Consider the following functional

$$
J: \mathrm{W}_{0}^{1, p}(\Omega) \times \mathrm{W}_{0}^{1, q}(\Omega) \rightarrow \mathbb{R}
$$

given by the formula

$$
\mathrm{J}(u, v)=\int_{0}^{1}\langle\mathcal{A}(t u) ; u\rangle \mathrm{d} t-\int_{0}^{1}\langle\mathcal{B}(t v) ; v\rangle \mathrm{d} t+\int_{\Omega} N(x, u(x), v(x)) \mathrm{d} x .
$$

Find such $\hat{u}, \hat{v}$ that

$$
\sup _{v \in \mathrm{~W}_{0}^{1, q}(\Omega)} \inf _{u \in \mathrm{~W}_{0}^{1, p}(\Omega)} \mathrm{J}(u, v)=\inf _{u \in \mathrm{~W}_{0}^{1, p}(\Omega)} \sup _{v \in \mathrm{~W}_{0}^{1, q}(\Omega)} \mathrm{J}(u, v)=\mathrm{J}(\hat{u}, \hat{v}) .
$$

We can easily observe that if conditions (A), (B), (E), (F), (G) holds then any solution to problem 2 is a solution to Problem 1.

Lemma 3.1 (Growth estimate on A and B). Under (C) for any $u \in W_{0}^{1, p}(\Omega)$ the following holds:

$$
\mathrm{A}(u)=\int_{0}^{1}\langle\mathcal{A}(t u) ; u\rangle \mathrm{d} t \geq \frac{\hat{\alpha_{1}}}{p}\|u\|_{\mathrm{W}_{0}^{1, p}(\Omega)}^{p} .
$$

Similarly under (D) for any $v \in \mathrm{~W}_{0}^{1, q}(\Omega)$ the following holds:

$$
\mathrm{B}(v)=\int_{0}^{1}\langle\mathcal{B}(t v) ; v\rangle \mathrm{d} t \geq \frac{\hat{\alpha_{2}}}{q}\|v\|_{\mathrm{W}_{0}^{1, q}(\Omega)}^{q} .
$$

Proof of Lemma 3.1. Let $u \in \mathrm{~W}_{0}^{1, p}(\Omega)$. Then

$$
\begin{aligned}
\int_{0}^{1}\langle\mathcal{A}(t u) ; u\rangle \mathrm{d} t & =\int_{0}^{1} \frac{1}{t}\langle\mathcal{A}(t u) ; t u\rangle \mathrm{d} t \\
& \geq \int_{0}^{1} \frac{1}{t}\|t u\|_{\mathrm{W}_{0}^{1, p}(\Omega)}^{p} \hat{\alpha_{1}} \mathrm{~d} t \\
& =\|u\|_{\mathrm{W}_{0}^{1, p}(\Omega)}^{p} \hat{\alpha_{1}} \int_{0}^{1} t^{p-1} \mathrm{~d} t=\|u\|_{\mathrm{W}_{0}^{1, p}(\Omega)}^{p} \frac{\hat{\alpha_{1}}}{p} .
\end{aligned}
$$

Similarly we prove the second part.
We also need a following auxiliary result used in order to prove the main theorem.
Lemma 3.2 (Properties of $\mathrm{F}_{\mathrm{v}}$ ). Assume ( E ), ( F$)$, ( $A$ ), (C), (H). Let $v \in \mathrm{~W}_{0}^{1, q}(\Omega)$ be fixed. The functional $\mathrm{F}_{\mathrm{v}}: \mathrm{W}_{0}^{1, p}(\Omega) \rightarrow \mathbb{R}$, given by formula

$$
\mathrm{F}_{\mathrm{v}}:=u \mapsto \mathrm{~A}(u)+\mathrm{N}(u, v),
$$

has a minimizer, is convex and w.l.s.c.
Proof. Let $v \in \mathrm{~W}_{0}^{1, q}(\Omega)$ be fixed. Potential A is convex and 1.s.c. Also N is convex and l.s.c. Thus functional $F_{v}$ is convex and weakly l.s.c. In order to show that $F_{v}$ has a minimizer it suffices to estimate it from below by some coercive functional.

Let $u \in \mathrm{~W}_{0}^{1, p}(\Omega), \hat{\beta}_{1}^{v}$ denotes $\left\|\beta_{1}^{v}\right\|_{L^{2}(\Omega)}$ multiplied by a constant from embedding of $\mathrm{W}_{0}^{1, p}(\Omega) \rightarrow \mathrm{L}^{2}(\Omega)$. By (F) we have

$$
\begin{aligned}
\mathrm{N}(u, v) & =\int_{\Omega} N(x, u(x), v(x)) \mathrm{d} x \\
& \geq \int_{\Omega}-\alpha_{1}|u(x)|^{p}+\beta_{1}^{v}(x) u(x)+\gamma_{1}^{v}(x) \mathrm{d} x \\
& \geq-\alpha_{1}\|u\|_{\mathrm{L}^{p}(\Omega)}^{p}-\left\|\beta_{1}^{v}\right\|_{\mathrm{L}^{2}(\Omega)}\|u\|_{\mathrm{L}^{2}(\Omega)}-\left\|\gamma_{1}\right\|_{\mathrm{L}^{1}(\Omega)} \\
& \geq-\frac{\alpha_{1}}{\lambda_{1, p}}\|u\|_{\mathrm{W}_{0}^{1, p}(\Omega)}^{p}-\hat{\beta}_{1}^{v}\|u\|_{\mathrm{W}_{0}^{1, p}(\Omega)}-\left\|\gamma_{1}\right\|_{\mathrm{L}^{1}(\Omega)} .
\end{aligned}
$$

By (C) and Lemma 3.1 we have

$$
\begin{aligned}
\mathrm{F}_{\mathrm{v}}(u) & =\mathrm{A}(u)+\mathrm{N}(u, v) \\
& \geq\left(\frac{\hat{\alpha}_{1}}{p}-\frac{\alpha_{1}}{\lambda_{1, p}}\right)\|u\|_{\mathrm{W}_{0}^{1, p}(\Omega)}^{p}-\hat{\beta}_{1}^{v}\|u\|_{\mathrm{W}_{0}^{1, p}(\Omega)}-\left\|\gamma_{1}\right\|_{\mathrm{L}^{1}(\Omega)} .
\end{aligned}
$$

Since $\left(\frac{\hat{\alpha}_{1}}{p}-\frac{\alpha_{1}}{\lambda_{1, p}}\right)>0$ we know that $\mathrm{F}_{\mathrm{v}}$ is bounded from below by a coercive functional. Thus since it is also w.l.s.c. functional, it must have a minimizer, however not necessarily unique.

Lemma 3.3 (Properties of $G_{u}$ ). Assume (E), (G), (B), (D), (I). Let $u \in W_{0}^{1, p}(\Omega)$ be fixed. The functional $\mathrm{G}_{\mathrm{u}}: \mathrm{W}_{0}^{1, q}(\Omega) \rightarrow \mathbb{R}$ given by formula

$$
\mathrm{G}_{\mathrm{u}}:=v \mapsto-\mathrm{B}(v)+\mathrm{N}(u, v) .
$$

has a maximizer (not necessarily unique), is concave and weakly upper semicontinuous (w.u.s.c. for short).

## 4 Main result - the existence of a saddle point

Theorem 4.1 (Existence of saddle point). Assume (A)-(I). There exists a solution to Problem 2.
Lets recall the main abstract result we use:
Theorem 4.2 (Ky-Fan minimax theorem [14, Th. 5.2.2.]). Let $X$ and $Y$ be Hausdorff topological vector spaces, $A \subset X$ and $B \subset Y$ be convex sets and $f: A \times B \rightarrow \mathbb{R}$ be a function which satisfies the following conditions
(i) for each $z_{2} \in B$ the function $z_{1} \mapsto f\left(z_{1}, z_{2}\right)$ is convex and lower semicontinuous on $A$;
(ii) for each $z_{1} \in A$ the function $z_{2} \mapsto f\left(z_{1}, z_{2}\right)$ is concave and upper semicontinuous on $B$;
(iii) for some $\hat{z_{1}} \in A$ and some

$$
\delta_{0}<\inf _{z_{1} \in A z_{2} \in B} f\left(z_{1}, z_{2}\right),
$$

the set $\left\{z_{2} \in B: f\left(\hat{z}_{1}, z_{2}\right) \geq \delta_{0}\right\}$ is compact. Then

$$
\sup _{z_{2}} \inf _{z_{1}} f\left(z_{1}, z_{2}\right)=\inf _{z_{1}} \sup _{z_{2}} f\left(z_{1}, z_{2}\right) .
$$

It is almost immediate to have (i) and (ii) fulfilled for our problem. But the hardest part is to obtain the last technical condition.

Proof of Theorem 4.1. First we start by proving (i) and (ii). Lets recall that for all $(u, v) \in$ $\mathrm{W}_{0}^{1, p}(\Omega) \times \mathrm{W}_{0}^{1, q}(\Omega)$ :

$$
\mathrm{J}(u, v)=\mathrm{A}(u)-\mathrm{B}(v)+\mathrm{N}(u, v) .
$$

Let us begin with (i). Let $v \in \mathrm{~W}_{0}^{1, q}(\Omega)$ be fixed. Since (A) holds by Lemma 2.4, A is convex and w.l.s.c. By (H) and since $N$ is $\mathrm{L}^{1}$-Carathéodory $u \mapsto \mathrm{~N}(u, v)$ is convex and w.l.s.c. $\mathrm{B}(v)$ is a constant - thus (i) holds. Similarly (ii) holds.

Actually we shall not use Ky -Fan theorem directly for J but for $\left.\mathrm{J}\right|_{A \times B}$ where $A, B$ are some closed balls respectively in $W_{0}^{1, p}(\Omega)$ and $W_{0}^{1, q}(\Omega)$. Since J fulfils (i) and (ii), those properties will remain unchanged for $\left.\mathrm{J}\right|_{A \times B}$.

We shall proceed as follows:

1. We shall define two more auxiliary functionals $\mathrm{J}^{+}, \mathrm{J}^{-}$, and bound each of them by yet another functional.
2. We prove that both

$$
\sup _{v} \inf _{u} \mathrm{~J}(u, v) \quad \text { and } \quad \inf _{u} \sup _{v} \mathrm{~J}(u, v)
$$

are attained.
3. We prove that each minimax argument must lie within balls of certain radius.
4. We deduce a suitable constant $\delta$ and show the compactness of the required set.

We consider the following functional $\mathrm{J}^{-}: \mathrm{W}_{0}^{1, q}(\Omega) \rightarrow \mathbb{R}$ given by the formula

$$
\mathrm{J}^{-}:=v \mapsto \min _{u \in \mathrm{~W}_{0}^{1, p}(\Omega)} \mathrm{J}(u, v) .
$$

We shall prove that that this functional is: well defined, concave and w.u.s.c. and anticoercive.
Let start with fixing $v \in \mathrm{~W}_{0}^{1, q}(\Omega)$. Then we see that $u \mapsto J(u, v)$ differs from $\mathrm{F}_{\mathrm{v}}$ by only a constant element -B $(v)$. Then by Lemma 3.2 a minimum must be attained. Since $v \in \mathrm{~W}_{0}^{1, q}(\Omega)$ was arbitrary thus $\mathrm{J}^{-}$is well defined.

Let fix $u \in \mathrm{~W}_{0}^{1, p}(\Omega)$. Then we see that $v \mapsto J(u, v)$ differs from $\mathrm{G}_{u}$ by only a constant element A $(u)$. Then by Lemma 3.3 each of such functionals must be u.s.c. and concave. Then by Corollaries 2.2 and 2.2 its is clear that $J^{-}$is u.s.c. and concave.

Let $v \in \mathrm{~W}_{0}^{1, q}(\Omega)$. By Assumption (G) and (D) we have

$$
\begin{aligned}
\mathrm{J}^{-}(v) & \leq \mathrm{J}(0, v) \\
& =-\int_{0}^{1}\langle\mathcal{B}(t v) ; v\rangle \mathrm{d} t+\int_{\Omega} N(x, 0, v(x)) \mathrm{d} x \\
& \leq\left(\frac{\alpha_{2}}{\lambda_{1, q}}-\frac{\hat{\alpha_{2}}}{q}\right)\|v\|_{\mathrm{W}_{0}^{1, q}(\Omega)}^{q}+\hat{\beta}_{2}^{0}\|v\|_{\mathrm{W}_{0}^{1, q}(\Omega)}+\left\|\gamma_{2}^{0}\right\|_{\mathrm{L}^{1}(\Omega)} .
\end{aligned}
$$

Since $\left(\frac{\alpha_{2}}{\lambda_{1, q}}-\frac{\hat{\alpha}_{2}}{q}\right)<0, \mathrm{~J}^{-}$, it follows that is anticoercive.
Since $\mathrm{J}^{-}$is concave, u.s.c. (weakly) and anticoercive it must attain a maximum. Thus there must exist a pair $(\hat{u}, \hat{v})$ such that

$$
\sup _{v} \inf _{u} \mathrm{~J}(u, v)=\mathrm{J}(\hat{u}, \hat{v}) .
$$

Lets use the previous estimate to define a functional $j^{-}: \mathrm{W}_{0}^{1, q}(\Omega) \rightarrow \mathbb{R}$ :

$$
\mathrm{J}^{-}(v) \leq\left(\frac{\alpha_{2}}{\lambda_{1, q}}-\frac{\hat{\alpha_{2}}}{q}\right)\|v\|_{\mathrm{W}_{0}^{1, q}(\Omega)}^{q}+\hat{\beta}_{2}^{0}\|v\|_{\mathrm{W}_{0}^{1, q}(\Omega)}+\left\|\gamma_{2}^{0}\right\|_{\mathrm{L}^{1}(\Omega)}=: j^{-}(v) .
$$

It is obviously a concave, continuous and anticoercive functional. Similarly we can define the following functional $J^{+}: \mathrm{W}_{0}^{1, p}(\Omega) \rightarrow \mathbb{R}$ given by the formula

$$
J^{+}(u)=\max _{v \in \mathrm{~W}_{0}^{1, q}(\Omega)} J(u, v) .
$$

By using the same argument we prove that that this functional is well defined, convex, w.l.s.c., coercive.

Let us fix $u \in \mathrm{~W}_{0}^{1, p}(\Omega)$. Then we see that $v \mapsto J(u, v)$ differs from $\mathrm{G}_{\mathrm{u}}$ by only a constant element $\mathrm{A}(u)$. Then by Lemma 3.3 a maximum must be attained. So $\mathrm{J}^{+}$is well defined since $u$ was set arbitrary.

Let set $v \in \mathrm{~W}_{0}^{1, q}(\Omega)$. Then we see that $u \mapsto J(u, v)$ differs from $\mathrm{F}_{\mathrm{v}}$ by only a constant element $-\mathrm{B}(v)$. Then by Lemma 3.2 each of such functionals must be w.l.s.c. and convex. Then by Lemmas 2.1 and 2.1 its is clear that $J^{+}$is w.l.s.c. and convex.

Let $u \in \mathrm{~W}_{0}^{1, p}(\Omega)$. By Assumption (F) and (C) we have

$$
\begin{aligned}
\mathrm{J}^{+}(u) & \geq \mathrm{J}(u, 0) \\
& =\mathrm{A}(u)+\mathrm{N}(u, 0) \\
& \geq\left(\frac{\hat{\alpha_{1}}}{p}-\frac{\alpha_{1}}{\lambda_{1, p}}\right)\|u\|_{\mathrm{W}_{0}^{1, p}(\Omega)}^{p}-\hat{\beta_{1}^{0}}\|u\|_{\mathrm{W}_{0}^{1, p}(\Omega)}+\hat{\gamma_{1}^{0}} .
\end{aligned}
$$

Where $\hat{\beta_{1}^{0}}$ and $\hat{\gamma_{1}^{0}}$ are some nonnegative constants. Since $\left(\frac{\hat{\alpha}_{1}}{p}-\frac{\alpha_{1}}{\lambda_{1, p}}\right)>0$, it follows $\mathrm{J}^{+}$is coercive.

Since $\mathrm{J}^{+}$is convex, l.s.c. (weakly) and coercive it must attain a minimum. Thus there must exist a pair $(\hat{u}, \hat{v})$ which satisfies that

$$
\inf _{u} \sup _{v} \mathrm{~J}(u, v)=\mathrm{J}(\hat{u}, \hat{v}) .
$$

Let us use the previous estimate to define a functional $j^{+}: \mathrm{W}_{0}^{1, p}(\Omega) \rightarrow \mathbb{R}$

$$
\mathrm{J}^{+}(u) \geq\left(\frac{\alpha_{1}}{\lambda_{1, p}}-\frac{\hat{\alpha_{1}}}{p}\right)\|u\|_{\mathrm{W}_{0}^{1, p}(\Omega)}^{q}+\hat{\beta}_{1}^{0}\|u\|_{\mathrm{W}_{0}^{1, p}(\Omega)}+\left\|\gamma_{1}^{0}\right\|_{\mathrm{L}^{1}(\Omega)}=: j^{+}(u) .
$$

It is a continuous, coercive and convex functional.
Now we shall focus on the balls which contain all the minimax points. Assume that $(\bar{u}, \bar{v}) \in \mathrm{W}_{0}^{1, p}(\Omega) \times \mathrm{W}_{0}^{1, q}(\Omega)$ is a pair such that

$$
\mathrm{J}(\bar{u}, \bar{v})=\max _{v} \min _{u} \mathrm{~J}(u, v) .
$$

Then

$$
\begin{aligned}
\mathrm{J}^{-}(\bar{v}) & \geq \mathrm{J}^{-}(0)=\min _{u} \mathrm{~J}(u, 0) \\
& \geq \min _{u} j^{+}(u) .
\end{aligned}
$$

Minimum of a coercive functional is in this case obviously a finite number which we shall denote as $\delta_{2}$.

Similarly assume $(\bar{u}, \bar{v}) \in \mathrm{W}_{0}^{1, p}(\Omega) \times \mathrm{W}_{0}^{1, q}(\Omega)$ be a point such that

$$
\mathrm{J}(\bar{u}, \bar{v})=\min _{u} \max _{v} \mathrm{~J}(u, v) .
$$

Then

$$
\begin{aligned}
\mathrm{J}^{+}(\bar{u}) & \leq \mathrm{J}^{+}(0)=\max _{v} \mathrm{~J}(0, v) \\
& \leq \max _{v} j^{-}(v) .
\end{aligned}
$$

Maximum of an anticoercive functional is in this case obviously a finite number which we shall denote as $\delta_{1}$.

Assume that $(\bar{u}, \bar{v}) \in \mathrm{W}_{0}^{1, p}(\Omega) \times \mathrm{W}_{0}^{1, q}(\Omega)$ is a pair such that

$$
\mathrm{J}(\bar{u}, \bar{v})=\max _{v} \min _{u} \mathrm{~J}(u, v)=\min _{u} \max _{v} \mathrm{~J}(u, v) .
$$

Then

$$
\begin{aligned}
\bar{v} & \in\left\{v \in \mathrm{~W}_{0}^{1, q}(\Omega): \mathrm{J}^{-}(v) \geq \delta_{2}\right\} \\
& \subset\left\{v \in \mathrm{~W}_{0}^{1, q}(\Omega): j^{-}(v) \geq \delta_{2}\right\} .
\end{aligned}
$$

The set $\left\{v \in W_{0}^{1, q}(\Omega): j^{-}(v) \geq \delta_{2}\right\}$, since $j^{-}$is anticoercive, must be a bounded one. Thus one could choose such a radius $r_{2}$ that zero-centred ball $B\left(r_{2}\right) \supset\left\{v \in \mathrm{~W}_{0}^{1, q}(\Omega): j^{-}(v) \geq \delta_{2}\right\} \ni \bar{v}$. Also

$$
\begin{aligned}
\bar{u} & \in\left\{u \in \mathrm{~W}_{0}^{1, p}(\Omega): \mathrm{J}^{+}(u) \leq \delta_{1}\right\} \\
& \subset\left\{u \in \mathrm{~W}_{0}^{1, p}(\Omega): j^{+}(u) \leq \delta_{1}\right\} .
\end{aligned}
$$

Then again, the set $\left\{u \in \mathrm{~W}_{0}^{1, p}(\Omega): j^{+}(u) \leq \delta_{1}\right\}$, since $j^{+}$is coercive, must be bounded. Thus one could choose such a radius $r_{1}$ that zero-centred ball

$$
B\left(r_{1}\right) \supset\left\{u \in \mathrm{~W}_{0}^{1, p}(\Omega): J^{+}(u) \leq \delta_{1}\right\} \ni \bar{u} .
$$

It follows that $(\bar{u}, \bar{v}) \in B\left(r_{1}\right) \times B\left(r_{2}\right)$. So if we restrict the domain of J to $B\left(r_{1}\right) \times B\left(r_{2}\right)$ we will not exclude any solution to Problem 2.

Finally we deduce (iii). Take $\hat{z_{1}}=0, A=B\left(r_{1}\right), B=B\left(r_{2}\right)$ and $\delta_{0}<\delta_{2}$. Then

- $\hat{z}_{1} \in A$ obviously holds.
- It follows that:

$$
\min _{u} \max _{v} \mathrm{~J}(u, v) \geq \min _{u} \mathrm{~J}(u, 0) \geq \min _{u} j^{+}(u)=\delta_{2}>\delta_{0} .
$$

- And finally

$$
\left\{v \in B: \mathrm{J}(0, v) \geq \delta_{0}\right\} \subset\left\{v \in B: j^{-}(v) \geq \delta_{0}\right\}
$$

is bounded since $j^{-}$is anticoercive and weakly closed (since $J(0, v)$ is concave and w.u.s.c). Thus by Banach-Alouglu theorem, and since a closed subset of compact set is compact - it is a weakly compact set. All the requirements of the Ky-Fan minimax theorem are fulfilled, so there exists $\hat{u} \in \mathrm{~W}_{0}^{1, p}(\Omega)$, and $\hat{v} \in \mathrm{~W}_{0}^{1, q}(\Omega)$ that

$$
\max _{v} \min _{u} \mathrm{~J}(u, v)=\min _{u} \max _{v} \mathrm{~J}(u, v)=\mathrm{J}(\hat{u}, \hat{v}) .
$$

This concludes the proof.
The following corollary follows instantly from the prove above.
Corollary 4.3. Assume that we replace convexity with strict convexity in $(H)$ and concavity with strict concavity in (I). If we also assume conditions (A)-(G) from Theorem 4.1 then Problem 2 has exactly 1 solution.

Assumptions ( F ), ( G ) can obviously have a stronger form, but without the upper bound requirement on constants $\hat{\alpha_{1}}, \hat{\alpha_{2}}$.
(F1) for each $v \in \mathrm{~W}_{0}^{1, q}(\Omega)$ there exists functions $\beta_{1} \in \mathrm{~L}^{2}(\Omega), \gamma_{1} \in \mathrm{~L}^{1}(\Omega), 1<\hat{p}<p$ and $\alpha_{1} \in \mathbb{R}^{+}$that

$$
N(x, u, v(x)) \geq-\alpha_{1}|u|^{\hat{p}}+\beta_{1}(x) \cdot u(x)+\gamma_{1}(x),
$$

for almost every $x \in \Omega$ and all $u \in \mathbb{R}$.
(G1) for each $u \in W_{0}^{1, p}(\Omega)$ there exists functions $\beta_{2} \in L^{2}(\Omega), \gamma_{2} \in L^{1}(\Omega) 1<\hat{q}<q$ and $\alpha_{2} \in \mathbb{R}^{+}$that

$$
N(x, u(x), v) \leq \alpha_{2}|v|^{\hat{q}}+\beta_{2}(x) \cdot v(x)+\gamma_{2}(x),
$$

for almost every $x \in \Omega$ and all $v \in \mathbb{R}$.
Corollary 4.4. Assume (A), (B), (C), (D), (E), (F1), (G1), (H), (I). Then Problem 2 has a solution.
It is easy to check that each step of proof to Theorem 4.1 can be used with the above setting.

## 5 Example

Lets consider a constants [12] $\lambda_{p}$ and $\lambda_{q}$ which are the first nonlinear eigenvalues of $-\Delta_{p}$ and $-\Delta_{q}$ respectively, namely

$$
\lambda_{p}=\min _{u \in \mathrm{~W}_{0}^{1, p}([0,1]), u \neq 0} \frac{\int_{0}^{1}\left|u^{\prime}(x)\right|^{p} \mathrm{~d} x}{\int_{0}^{1}|u(x)|^{p} \mathrm{~d} x}, \quad \lambda_{q}=\min _{u \in \mathrm{~W}_{0}^{1, q}([0,1]), u \neq 0} \frac{\int_{0}^{1}\left|u^{\prime}(x)\right|^{q} \mathrm{~d} x}{\int_{0}^{1}|u(x)|^{q} \mathrm{~d} x} .
$$

Example 5.1. Lets $p=6, q=4$, and $\Omega=[0,1]$. We consider the system of the following form

$$
\begin{align*}
\int_{0}^{1}\left|u^{\prime}(t)\right|^{p-2} u^{\prime}(t) \bar{u}^{\prime}(t) \mathrm{d} t & =-\frac{\lambda}{2 p} \int_{0}^{1}\left(|u(t)|^{p-2} u(t)+v(t)\right) \bar{u}(t) \mathrm{d} t, \\
\int_{0}^{1}\left|v^{\prime}(t)\right|^{q-2} v^{\prime}(t) \bar{v}^{\prime}(t) \mathrm{d} t & =\frac{\lambda}{2 p} \int_{0}^{1}\left(|v(t)|^{q-2} v(t)-u(t)\right) \bar{v}(t) \mathrm{d} t \tag{Ex1}
\end{align*}
$$

for all $\bar{u} \in \mathrm{~W}_{0}^{1, p}([0,1]), \bar{v} \in \mathrm{~W}_{0}^{1, q}([0,1])$. We consider a functional which critical points corresponds to solution Problem (Ex1). Such a functional has a form:

$$
\mathrm{J}(u, v)=\frac{1}{p} \int_{0}^{1}\left|u^{\prime}(t)\right|^{p} \mathrm{~d} t-\frac{1}{q} \int_{0}^{1}\left|v^{\prime}(t)\right|^{q} \mathrm{~d} t+\int_{0}^{1} \frac{\lambda}{2 p}\left(\frac{1}{p} u(t)^{p}-\frac{1}{q} v(t)^{q}+u(t) v(t)\right) \mathrm{d} t .
$$

We shall apply Theorem 4.1 to prove the existence of a critical point (saddle point) to this functional. Lets check all the required assumptions
(A), (B) Negative $p$-Laplace operator $\left(-\Delta_{p}\right)$ is know to be potential and monotone.
(C), (D) the conditions are fulfilled with $\hat{\alpha_{1}}=\frac{1}{p}$ and $\hat{\alpha_{2}}=\frac{1}{q}$.
(E) is obviously fulfilled.
(F), (G) If $v \in \mathrm{~W}_{0}^{1, q}([0,1])$ then it must be bounded a.e. as a continuous function by a positive constant. Lets check the condition on $\alpha_{1}$. It is easy to observe that

$$
\alpha_{1}:=\frac{\lambda}{p^{2}} \leq \lambda_{p} \frac{1}{p^{2}}=\lambda_{p} \frac{\hat{\alpha_{1}}}{p}=\lambda_{1, \frac{\hat{p}}{}}^{p} .
$$

Thus the condition holds. (G) follows in a similar manner.
(H) With $v$ fixed functional $u \mapsto \mathrm{~N}(u, v)$ has a plot similar to a function

$$
u \mapsto \frac{1}{2} u^{p}+c u+C .
$$

Since its second derivative is nonnegative $(p=6)-$ it is a convex function.
(I) With $u$ fixed functional $v \mapsto \mathrm{~N}(u, v)$ has a plot similar to a function

$$
v \mapsto-\frac{1}{2} v^{q}+c v+C .
$$

Since its second derivative is nonpositive $(q=4)$ - it is a concave function.
Thus from Theorem 4.1 it follows that Problem (Ex1) admits a solution.

## References

[1] R. A. Adams, J. J. Fournier, Sobolev spaces, Elsevier, 2003. Zbl 1098.46001
[2] J. H. Chabrowski, Variational methods for potential operator equations: with applications to nonlinear elliptic equations, Vol. 24, Walter de Gruyter, 2011. Zbl 1157.35338
[3] G. Dinca, P. Jebelean, J. Mawhin, Variational and topological methods for Dirichlet problems with $p$-Laplacian, Port. Math. (NS) 58(2001), No. 3, 339-378. Zbl 0991.35023
[4] P. Drábek, J. Мilota, Methods of nonlinear analysis: applications to differential equations, Birkhäuser Verlag, Basel, 2007. Zbl 1176.35002
[5] M. Galewski, On variational nonlinear equations with monotone operators, Adv. Nonlinear Anal. 10(2020), No. 1, 289-300. https://doi.org/10.1515/anona-2020-0102
[6] M. Galewski, L. Vilasi, Saddle-point solutions to Dirichlet problems on the Sierpiński gasket, Expo. Math. 37(2019), No. 4, 485-497. https://doi.org/10.1016/j .exmath. 2018. 04.001
[7] L. Gasinski, N. S. Papageorgiou, Nodal and multiple solutions for nonlinear elliptic equations involving a reaction with zeros, Dyn. Partial Differ. Equ. 12(2015), No. 1, 13-42. https://doi.org/10.4310/DPDE.2015.v12.n1.a2
[8] L. Gasiński, N. S. Papageorgiou, Multiple solutions for ( $p, 2$ )-equations with resonance and concave terms, Results Math. 74(2019), No. 2, Paper No. 79. https://doi.org/10. 1007/s00025-019-0996-9
[9] L. Gasinski, N. S. Papageorgiou, K. Winowski, Positive solutions for nonlinear robin problems with concave terms, J. Convex Anal. 26(2019), No. 4, 1145-1174. Zbl 1436.35113
[10] C. Ha. A noncompact minimax theorem, Pacific J. Math. 97(1981), No. 1, 115-117. MR638178
[11] M. Jakszto, A. Skowron, Existence of optimal controls via continuous dependence on parameters, Comput. Math. Appl. 46(2003), No. 10-11, 1657-1669. https://doi.org/10. 1016/S0898-1221(03)90200-8
[12] P. Lindqvist, On the equation $\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)+\lambda|u|^{p-2} u=0$, Proc. Amer. Math. Soc. 109(1990), No. 1, 157-164. https://doi.org/10.2307/2048375
[13] S. Łojasiewicz, An introduction to the theory of real functions, John Wiley \& Sons Inc, 1988. Zbl 0653.26001
[14] L. Nirenberg, Topics in nonlinear functional analysis, Vol. 6, American Mathematical Soc., 1974. Zbl 0992.47023
[15] V. D. Rădulescu, Qualitative analysis of nonlinear elliptic partial differential equations: monotonicity, analytic, and variational methods, Contemporary Mathematics and Its Applications, Vol. 6, Hindawi Publishing Corporation, 2008. Zbl 1190.35003
[16] E. Zeidler, Nonlinear functional analysis and its applications: II/B: nonlinear monotone operators, Springer Science \& Business Media, 2013. https://doi.org/10.1007/978-1-4612-0981-2

# Attractivity analysis on a neoclassical growth system incorporating patch structure and multiple pairs of time-varying delays 

Qian Cao ${ }^{\boxtimes}$<br>College of Mathematics and Physics, Hunan University of Arts and Science, Changde 415000, Hunan, P. R. China

Received 27 June 2021, appeared 5 October 2021
Communicated by Leonid Berezansky


#### Abstract

In this paper, we focus on the global dynamics of a neoclassical growth system incorporating patch structure and multiple pairs of time-varying delays. Firstly, we prove the global existence, positiveness and boundedness of solutions for the addressed system. Secondly, by employing some novel differential inequality analyses and the fluctuation lemma, both delay-independent and delay-dependent criteria are established to ensure that all solutions are convergent to the unique positive equilibrium point, which supplement and improve some existing results. Finally, some numerical examples are afforded to illustrate the effectiveness and feasibility of the theoretical findings.


Keywords: global attractivity, neoclassical growth system, patch structure, multiple pairs of time-varying delay.
2020 Mathematics Subject Classification: 34C25, 34D05, 34K13, 34K25.

## 1 Introduction

Under the assumptions that labor and capital are fully allocated and the output market is adjusted immediately, Day proposed a discrete-time neoclassical growth model in literature [5], which has unimodal feedback production function. As we all know, there is an inevitable time lag between the acquisition of information and the implementation of decisions, but the model proposed by Day ignores the influence of delays and cannot fully explain the actual economic situation. To revise this drawback and better characterize the long-term behavior of economics, Matsumoto and Szidarovszky [25] introduced the delayed neoclassical growth equation

$$
\begin{equation*}
x^{\prime}(t)=-\delta x(t)+P x^{\gamma}(t-\tau) e^{-\sigma x(t-\tau)} \tag{1.1}
\end{equation*}
$$

where $x(t)$ labels the capital per labor at time $t, \delta$ is the sum of labor growth rate and capital depreciation rate multiplied by average saving rate, $\tau$ designates the delay in the production

[^38]function, $\gamma$ denotes a proxy for measuring returns to scale of the production function, $\sigma$ is regarded as a strength of a 'negative influence' produced by adding concentration of capital and is settled via a damaging degree of energy resources or natural environment. If $\gamma=1$, the model (1.1) is the famous Nicholson's blowflies model, whose dynamic behavior has been extensively studied in recent years $[1,3,13,15-20,22,23,27,31,32,37]$. However, for the case of $\gamma \neq 1$, there are relatively few studies devoted to model (1.1) and its extended models [4,7,24,26,33,34].

Recently, regarding that the identical production function usually contains different delays, L. Berezansky and E. Braverman put forward a dynamic model of the form in [2],

$$
\begin{equation*}
x^{\prime}(t)=\sum_{j=1}^{m} F_{j}\left(t, x\left(t-\tau_{1}(t)\right), \ldots, x\left(t-\tau_{l}(t)\right)\right)-G(t, x(t)), \quad t \geq t_{0} \tag{1.2}
\end{equation*}
$$

where $l$ and $m$ are positive integers, $G$ describes the instantaneous mortality rate, and each $F_{j}(j \in I:=\{1,2, \cdots, m\})$ is the feedback control relying on the values of the stable variable with distinctive delays $\tau_{1}(t), \tau_{2}(t), \ldots, \tau_{l}(t)$. Manifestly, (1.2) contains the modified delayed differential neoclassical growth model

$$
\begin{equation*}
x^{\prime}(t)=\beta(t)\left[-\delta x(t)+\sum_{j=1}^{m} P_{j} x^{\gamma}\left(t-g_{j}(t)\right) e^{-\sigma x\left(t-h_{j}(t)\right)}\right], \quad \gamma \in(0,1), \tag{1.3}
\end{equation*}
$$

which in the case $h_{k} \equiv g_{k}$ agrees with the traditional model [33].
In general, when each nonlinear function of the model contains only a small enough time delay, it will inherit some features of non time delay systems. For example, all the nonoscillatory solutions with respect to the unique positive equilibrium point are convergent. Moreover, as long as the time delay is small enough, the global attractivity for the positive equilibrium point has been shown in $[2,30]$. And the existence, oscillation, persistence, periodicity and stability of positive solutions have been widely explored for the single time-delay system (1.3) and similar models with $g_{j}(t) \equiv h_{j}(t)[4,7,24,26,33,34]$. However, when the same nonlinear function of the model incorporates two or more time delays, chaotic oscillation of the system will occur, which will increase the difficulty in the study of the dynamics of such systems. Therefore, this issue has attracted the attention of many scholars. More recently, Huang et al. [21] studied the attractivity for the scalar equation (1.3). Meanwhile, since the financial environment of some capitals is fragmented, and the natural separation of the space area is separate, the above scalar neoclassical growth model can be naturally generalized to the patch structure system $[8,36]$, the scalar equation (1.3) can be normally extended to the following system incorporating patch structure and multiple pairs of time-varying delays:

$$
\begin{equation*}
x_{i}^{\prime}(t)=\beta(t)\left[-\bar{\delta}_{i} x_{i}(t)+\sum_{j=1, j \neq i}^{n} a_{i j} x_{j}(t)+\sum_{j=1}^{m} P_{i j} x_{i}^{\gamma}\left(t-g_{i j}(t)\right) e^{-\sigma_{i j} x_{i}\left(t-h_{i j}(t)\right)}\right], \quad \gamma \in(0,1), \tag{1.4}
\end{equation*}
$$

where $i \in Q:=\{1,2, \ldots, n\}, x_{i}$ stands for the amount of the capital per labor in the patch $i, a_{i j}$ designates the dispersal coefficient of the capital from patch $j$ to patch $i, m$ accounts for the number of population reproductive types, $P_{i j} x_{i}^{\gamma}\left(t-g_{i j}(t)\right) e^{-\sigma_{i j} x_{i}\left(t-h_{i j}(t)\right)}$ describes the time-dependent reproduction function which is related to the incubation delay $h_{i j}(t)$ and the maturation delay $g_{i j}(t)$, and $x_{i}^{\gamma} e^{-\sigma_{i j} x_{i}}$ acquires the maximum reproduce rate at $x_{i}(t)=\frac{\gamma}{\sigma_{i j}}$. For more detailed biological significance, one can directly refer to $[8,21,36]$ and their references quoted therein.

Hereafter, by changing the variables

$$
\bar{\delta}_{i}=\delta_{i}-a_{i i} \quad \text { with } a_{i i}<0
$$

(1.4) can be rewritten as

$$
\begin{equation*}
x_{i}^{\prime}(t)=\beta(t)\left[-\delta_{i} x_{i}(t)+\sum_{j=1}^{n} a_{i j} x_{j}(t)+\sum_{j=1}^{m} P_{i j} x_{i}^{\gamma}\left(t-g_{i j}(t)\right) e^{-\sigma_{i j} x_{i}\left(t-h_{i j}(t)\right)}\right], \quad \gamma \in(0,1), i \in Q . \tag{1.5}
\end{equation*}
$$

It should be pointed out that, the dynamic characteristics of neoclassical growth model incorporating patch structure and multiple pairs of time-varying delays have not been fully studied. To the best of our knowledge, we have only found that the author of [36] established the attractivity results of the system (1.5) when $g_{i j}(t) \equiv h_{i j}(t)(i \in Q, j \in I)$. However, there is no research on the dynamic behavior of the model (1.5) with $g_{i j}(t) \neq h_{i j}(t)(i \in Q, j \in I)$.

According to the above discussions, our goal is to establish the global attractivity conditions of the unique positive equilibrium point for the system (1.5) under $g_{i j}(t) \neq h_{i j}(t)(i \in$ $Q, j \in I)$. Briefly speaking, the contributions of this article can be summarized as below. 1) The boundedness and persistence on the solutions of system (1.5) are established by exploiting some novel differential inequality analyses; 2) Under certain assumptions, with the aid of the fluctuation lemma, some sufficient criteria ensuring the global attractivity of system (1.5) are obtained for the first time, which improve and generalize all recent works reported in $[21,36]$; 3) Numerical simulations involving comparison discussions are afforded to reveal the obtained theoretical results.

The remaining of this work is arranged as follows. In Section 2, some necessary lemmas and assumptions are listed. In Section 3, the global attractivity of the unique positive equilibrium point for the addressed system is demonstrated. To evidence our theoretical results, some numerical experiments are carried out in Section 4. Conclusions are given in Section 5.

## 2 Preliminary results

Throughout this manuscript, $\mathbb{N}^{+}$labels the set of all positive integers and $\mathbb{R}^{n}\left(\mathbb{R}^{1}=\mathbb{R}\right)$ designates the $n$-dimensional real vectors set. For a bounded real function $u$, let $u^{+}=$ $\sup _{\vartheta \in \mathbb{R}} u(\vartheta), u^{-}=\inf _{\vartheta \in \mathbb{R}} u(\vartheta)$.

With the biological applications in mind, we assume that $\delta_{i}>0, P_{i j}>0, \sigma_{i j}>0, \beta^{-}>0$ and

$$
r_{i}=\max \left\{\max _{1 \leq j \leq m} \sup _{t \in \mathbb{R}} g_{i j}(t), \max _{1 \leq j \leq m} \sup _{t \in \mathbb{R}} h_{i j}(t)\right\}, \quad r=\max _{1 \leq i \leq n}\left\{r_{i}\right\}
$$

Likewise, $g_{i j}, h_{i j}, \beta: \mathbb{R} \longrightarrow(0,+\infty)(i \in Q, j \in I)$ are bounded and continuous functions, $A=\left(a_{i j}\right)_{n \times n}$ is an irreducible and cooperative matrix with $a_{i j} \geq 0(i \neq j)$, and

$$
\begin{equation*}
\sum_{j=1, j \neq i}^{n} a_{i j}=-a_{i i}, \quad \text { for all } i \in Q \tag{2.1}
\end{equation*}
$$

In addition, suppose that there exists a positive constant $N^{*}$ such that

$$
\begin{equation*}
-\delta_{i}\left(N^{*}\right)^{1-\gamma}+\sum_{j=1}^{m} P_{i j} e^{-\sigma_{i j} N^{*}}=0, \quad \text { for all } i \in Q \tag{2.2}
\end{equation*}
$$

which implies that ( $\left.N^{*}, N^{*}, \ldots, N^{*}\right)$ is a positive equilibrium point of system (1.5).
Denote $C=\prod_{i=1}^{n} C\left(\left[-r_{i}, 0\right], \mathbb{R}\right)$ be a Banach space involving the supremum norm $\|\cdot\|$, and $C_{+}=\prod_{i=1}^{n} C\left(\left[-r_{i}, 0\right],[0,+\infty)\right)$. Also, we set $x_{t}\left(t_{0}, \varphi\right)\left(x\left(t ; t_{0}, \varphi\right)\right)$ for an admissible solution of (1.5) obeying the initial conditions:

$$
\begin{equation*}
x_{t_{0}}=\varphi, \quad \varphi \in C_{+} \quad \text { and } \quad \varphi_{i}(0)>0, \quad i \in Q, \tag{2.3}
\end{equation*}
$$

and $\left[t_{0}, \eta(\varphi)\right)$ be the maximal right-interval of existence.
Now, we present two lemmas to reveal the positiveness and boundedness of (1.5).
Lemma 2.1. $x(t)=x\left(t ; t_{0}, \varphi\right)$ has positiveness and boundedness on $\left[t_{0},+\infty\right)$.
Proof. By Theorem 5.2.1 in [28], we have that $x_{t}\left(t_{0}, \varphi\right) \in C_{+}$for all $t \in\left[t_{0}, \eta(\varphi)\right)$. This, together with (1.5) and (2.3), follows that

$$
\begin{align*}
x_{i}(t)= & \varphi_{i}(0) e^{-\int_{t_{0}}^{t}\left(\delta_{i}-a_{i i}\right) \beta(s) d s}+e^{-\int_{t_{0}}^{t}\left(\delta_{i}-a_{i i}\right) \beta(s) d s} \int_{t_{0}}^{t} \beta(s) \\
& \times\left[\sum_{j=1, j \neq i}^{n} a_{i j} x_{j}(s)+\sum_{j=1}^{m} P_{i j} x_{i}^{\gamma}\left(s-g_{i j}(s)\right) e^{-\sigma_{i j} x_{i}\left(s-h_{i j}(s)\right)}\right] e^{\int_{t_{0}}^{s}\left(\delta_{i}-a_{i i}\right) \beta(v) d v} d s \\
> & 0 \quad \text { for all } t \in\left[t_{0}, \eta(\varphi)\right) \text { and } i \in Q . \tag{2.4}
\end{align*}
$$

For $t>t_{0}$, let $i_{0} \in Q$ and $T_{i_{0}} \in\left[t_{0}-r_{i_{0}}, t\right]$ such that

$$
x_{i_{0}}\left(T_{i_{0}}\right)=\max _{t_{0}-r_{i_{0}} \leq s \leq t} x_{i_{0}}(s)=\max _{i \in Q}\left\{\max _{t_{0}-r_{i} \leq s \leq t} x_{i}(s)\right\} .
$$

When $T_{i_{0}} \in\left[t_{0}-r_{i_{0}}, t_{0}\right]$, it is easily seen that

$$
\begin{equation*}
\left\|x_{s}\left(t_{0}, \varphi\right)\right\| \leq x_{i_{0}}\left(T_{i_{0}}\right)=\|\varphi\| \quad \text { for all } s \in\left[t_{0}, t\right] . \tag{2.5}
\end{equation*}
$$

If $T_{i_{0}} \in\left(t_{0}, t\right],(1.5),(2.1)$ and (2.4) lead to

$$
\begin{aligned}
0 & \leq x_{i_{0}}^{\prime}\left(T_{i_{0}}\right) \\
& =\beta\left(T_{i_{0}}\right)\left[-\delta_{i_{0}} x_{i_{0}}\left(T_{i_{0}}\right)+\sum_{j=1}^{n} a_{i_{0} j} x_{j}\left(T_{i_{0}}\right)+\sum_{j=1}^{m} P_{i_{0} j} x_{i_{0}}^{\gamma}\left(T_{i_{0}}-g_{i_{0} j}\left(T_{i_{0}}\right)\right) e^{-\sigma_{i_{0} j} x_{i_{0}}\left(T_{i_{0}}-h_{i_{0} j}\left(T_{i_{0}}\right)\right)}\right] \\
& \leq \beta\left(T_{i_{0}}\right)\left[-\delta_{i_{0}} x_{i_{0}}\left(T_{i_{0}}\right)+\sum_{j=1}^{n} a_{i_{0} j} x_{i_{0}}\left(T_{i_{0}}\right)+\sum_{j=1}^{m} P_{i_{0} j} x_{i_{0}}^{\gamma}\left(T_{i_{0}}\right) e^{-\sigma_{i_{0}} x_{i_{0}}\left(T_{i_{0}}-h_{i_{0} j}\left(T_{i_{0}}\right)\right)}\right] \\
& \leq \beta\left(T_{i_{0}}\right) x_{i_{0}}^{\gamma}\left(T_{i_{0}}\right)\left[-\delta_{i_{0}} x_{i_{0}}^{1-\gamma}\left(T_{i_{0}}\right)+\sum_{j=1}^{m} P_{i_{0} j}\right],
\end{aligned}
$$

which yields

$$
\begin{equation*}
\left\|x_{s}\left(t_{0}, \varphi\right)\right\| \leq x_{i_{0}}\left(T_{i_{0}}\right) \leq \max _{i \in Q}\left(\frac{\sum_{j=1}^{m} P_{i j}}{\delta_{i}}\right)^{\frac{1}{1-\gamma}} \quad \text { for all } s \in\left(t_{0}, t\right] . \tag{2.6}
\end{equation*}
$$

From (2.5) and (2.6), we obtain that $x(t)$ has boundedness on $\left[t_{0}, \eta(\varphi)\right)$, and

$$
\begin{equation*}
\left\|x_{t}\left(t_{0}, \varphi\right)\right\| \leq x_{i_{0}}\left(T_{i_{0}}\right) \leq \max _{i \in Q}\left(\frac{\sum_{j=1}^{m} P_{i j}}{\delta_{i}}\right)^{\frac{1}{1-\gamma}}+\|\varphi\|=: X^{\varphi} \quad \text { for all } t \in\left[t_{0}, \eta(\varphi)\right) . \tag{2.7}
\end{equation*}
$$

This, together with Theorem 2.3.1 in [9], follows $\eta(\varphi)=+\infty$, and finishes the evidence of Lemma 2.1.

Lemma 2.2. $\liminf _{t \rightarrow+\infty} x_{i}(t)>0$ for all $i \in Q$.
Proof. To obtain a contradiction, we suppose that $l=\min _{i \in Q} \liminf _{t \rightarrow+\infty} x_{i}(t)=0$. Let

$$
m(t)=\max \left\{\xi: \xi \leq t \mid \text { there is } \hat{i} \in Q \text { satisfying } x_{\hat{i}}(\xi)=\min _{i \in Q}\left\{\min _{t_{0} \leq s \leq t} x_{i}(s)\right\}\right\}
$$

Then, $\lim _{t \rightarrow+\infty} m(t)=+\infty$. Likewise, for a strictly monotone increasing infinite sequence $\left\{t_{p}\right\}_{p \geq 1}$, there are $\hat{i} \in Q$ and a subsequence $\left\{t_{p_{k}}\right\}_{k \geq 1} \subseteq\left\{t_{p}\right\}_{p \geq 1}$ agreeing with

$$
\begin{equation*}
x_{\hat{i}}\left(m\left(t_{p_{k}}\right)\right)=\min _{t_{0} \leq s \leq t_{p_{k}}} x_{\hat{i}}(s)=\min _{i \in Q}\left\{\min _{t_{0} \leq s \leq t_{p_{k}}} x_{i}(s)\right\} \quad \text { and } \quad \lim _{k \rightarrow+\infty} x_{\hat{i}}\left(m\left(t_{p_{k}}\right)\right)=0 \tag{2.8}
\end{equation*}
$$

Owing to (1.5), (2.1), (2.7) and (2.8), we derive

$$
\begin{aligned}
0 \geq & x_{\hat{i}}^{\prime}\left(m\left(t_{p_{k}}\right)\right) \\
\geq & \beta\left(m\left(t_{p_{k}}\right)\right)\left[-\delta_{\hat{i}} x_{\hat{i}}\left(m\left(t_{p_{k}}\right)\right)+x_{\hat{i}}\left(m\left(t_{p_{k}}\right)\right) \sum_{j=1}^{n} a_{\hat{i} j}\right. \\
& \left.+\sum_{j=1}^{m} P_{\hat{i} j} x_{\hat{i}}^{\gamma}\left(m\left(t_{p_{k}}\right)-g_{\hat{i} j}\left(m\left(t_{p_{k}}\right)\right)\right) e^{-\sigma_{\hat{i} j} x_{\hat{i}}\left(m\left(t_{p_{k}}\right)-h_{\hat{i} j}\left(m\left(t_{p_{k}}\right)\right)\right)}\right] \\
\geq & \beta\left(m\left(t_{p_{k}}\right)\right)\left[-\delta_{\hat{i}} x_{\hat{i}}\left(m\left(t_{p_{k}}\right)\right)+\sum_{j=1}^{m} P_{\hat{i} j} x_{\hat{i}}^{\gamma}\left(m\left(t_{p_{k}}\right)\right) e^{-\sigma_{\hat{i} j} X^{\varphi}}\right] \quad \text { for all } m\left(t_{p_{k}}\right)>t_{0}
\end{aligned}
$$

and

$$
\begin{equation*}
\delta_{\hat{i}} \geq \sum_{j=1}^{m} P_{\hat{i} j} \frac{1}{x_{\hat{i}}^{1-\gamma}\left(m\left(t_{p_{k}}\right)\right)} e^{-\sigma_{\hat{i} \hat{j}} X^{\varphi}}, \quad \text { for all } m\left(t_{p_{k}}\right)>t_{0} \tag{2.9}
\end{equation*}
$$

By taking limits, (2.8) and (2.9) give us $\delta_{\hat{i}} \geq+\infty$, which yields a contradiction and finishes the proof.

Lemma 2.3. Lemma 2.2 indicates that $(0,0, \ldots, 0)$ is unstable.

## 3 Global attractivity analysis

First, we present a delay-independent criterion to assure the attractivity for nonoscillatory solutions of system (1.5).

Proposition 3.1. If

$$
\min _{i \in Q} \liminf _{t \rightarrow+\infty} x_{i}(t) \geq N^{*} \quad\left(\text { or } \max _{i \in Q} \limsup _{t \rightarrow+\infty} x_{i}(t) \leq N^{*}\right)
$$

then $\lim \sup _{t \rightarrow+\infty} x_{i}(t)=N^{*}\left(\right.$ or $\left.\liminf _{t \rightarrow+\infty} x_{i}(t)=N^{*}\right)$ for all $i \in Q$.
Proof. We just need to deal with the case that

$$
\min _{i \in Q} \liminf _{t \rightarrow+\infty} x_{i}(t) \geq N^{*}
$$

since the situation is entirely analogous for the case that $\max _{i \in Q} \lim \sup _{t \rightarrow+\infty} x_{i}(t) \leq N^{*}$.

Set $y_{i}(t)=x_{i}(t)-N^{*}(i \in Q)$, it is evident that

$$
\begin{equation*}
\limsup _{t \rightarrow+\infty} y_{i}(t) \geq 0 \quad \text { for all } i \in Q . \tag{3.1}
\end{equation*}
$$

Let $i^{*} \in Q$ be such an index as $\lim \sup _{t \rightarrow+\infty} y_{i^{*}}(t)=\max _{i \in Q} \lim \sup _{t \rightarrow+\infty} y_{i}(t)$. We state that

$$
\limsup _{t \rightarrow+\infty} y_{i^{*}}(t)=0 .
$$

Otherwise, $\lim \sup _{t \rightarrow+\infty} y_{i^{*}}(t)>0$. Owing to the fluctuation lemma [29, Lemma A.1.], it is an easy matter to find a sequence $\left\{t_{k}\right\}_{k \geq 1}$ obeying

$$
\begin{equation*}
\lim _{k \rightarrow+\infty} t_{k}=+\infty, \quad \lim _{k \rightarrow+\infty} y_{i^{*}}\left(t_{k}\right)=\limsup _{t \rightarrow+\infty} y_{i^{*}}(t), \quad \lim _{k \rightarrow+\infty} y_{i^{*}}^{\prime}\left(t_{k}\right)=0 . \tag{3.2}
\end{equation*}
$$

Due to (1.5) and (2.1), we gain

$$
\begin{equation*}
y_{i^{*}}^{\prime}\left(t_{k}\right)=\beta\left(t_{k}\right)\left[-\delta_{i^{*}} x_{i^{*}}\left(t_{k}\right)+\sum_{j=1}^{n} a_{i^{*} j} y_{j}\left(t_{k}\right)+\sum_{j=1}^{m} P_{i^{*} j} x_{i^{*}}^{\gamma}\left(t_{k}-g_{i^{*} j}\left(t_{k}\right)\right) e^{-\sigma_{i^{*} j} x_{*}^{*}\left(t_{k}-h_{i^{*} j}\left(t_{k}\right)\right)}\right] . \tag{3.3}
\end{equation*}
$$

Because $\beta(t), x_{i^{*}}\left(t-g_{i^{*} j}(t)\right)$ and $x_{i^{*}}\left(t-h_{i^{*} j}(t)\right)$ are bounded on $\left[t_{0},+\infty\right)$, we can select a subsequence of $\left\{t_{k}\right\}$ (for convenience of exposition, we still label by $\left\{t_{k}\right\}$ ) satisfying that $\lim _{k \rightarrow+\infty} \beta\left(t_{k}\right), \lim _{k \rightarrow+\infty} y_{l}\left(t_{k}\right), \lim _{k \rightarrow+\infty} x_{i^{*}}\left(t_{k}-g_{i^{*} j}\left(t_{k}\right)\right)$ and $\lim _{k \rightarrow+\infty} x_{i^{*}}\left(t_{k}-h_{i^{*} j}\left(t_{k}\right)\right)$ exist for all $l \in Q \backslash\left\{i^{*}\right\}$ and $j \in I$. Moreover, $0<\beta^{-} \leq \lim _{k \rightarrow+\infty} \beta\left(t_{k}\right)$, and

$$
\begin{equation*}
N^{*} \leq \lim _{k \rightarrow+\infty} x_{i^{*}}\left(t_{k}-h_{i^{*} j}\left(t_{k}\right)\right), \quad \lim _{k \rightarrow+\infty} x_{i^{*}}\left(t_{k}-g_{i^{*} j}\left(t_{k}\right)\right) \leq N^{*}+\lim _{k \rightarrow+\infty} y_{i^{*}}\left(t_{k}\right) . \tag{3.4}
\end{equation*}
$$

With the help of (3.4), we regard two cases as follow.
Case 1. If $\lim _{k \rightarrow+\infty} x_{i^{*}}\left(t_{k}-h_{i^{*} j}\left(t_{k}\right)\right)=N^{*}$ for all $j \in I$, by taking limits, (2.1), (2.2), (3.2), and (3.3) reveal that

$$
\begin{aligned}
0= & \lim _{k \rightarrow+\infty} y_{i^{*}}^{\prime}\left(t_{k}\right) \\
\leq & \lim _{k \rightarrow+\infty} \beta\left(t_{k}\right)\left[-\delta_{i^{*}}\left(\limsup _{t \rightarrow+\infty} y_{i^{*}}(t)+N^{*}\right)+\limsup _{t \rightarrow+\infty} y_{i^{*}}(t) \sum_{j=1}^{n} a_{i^{*} j}\right. \\
& \left.+\sum_{j=1}^{m} P_{i^{*} j}\left(\limsup _{t \rightarrow+\infty} y_{i^{*}}(t)+N^{*}\right)^{\gamma} e^{-\sigma_{i^{*}} N^{*}}\right] \\
\leq & \lim _{k \rightarrow+\infty} \beta\left(t_{k}\right)\left(\limsup _{t \rightarrow+\infty} y_{i^{*}}(t)+N^{*}\right)^{\gamma}\left[-\delta_{i^{*}}\left(\limsup _{t \rightarrow+\infty} y_{i^{*}}(t)+N^{*}\right)^{1-\gamma}+\sum_{j=1}^{m} P_{i^{*} j} e^{-\sigma_{i^{*} j} N^{*}}\right] \\
< & \lim _{k \rightarrow+\infty} \beta\left(t_{k}\right)\left(\limsup _{t \rightarrow+\infty} y_{i^{*}}(t)+N^{*}\right)^{\gamma}\left[-\delta_{i^{*}( }\left(N^{*}\right)^{1-\gamma}+\sum_{j=1}^{m} P_{i^{*} j} e^{-\sigma_{i^{*} j} N^{*}}\right] \\
= & 0,
\end{aligned}
$$

which leads to a contradiction, and suggests that $\lim \sup _{t \rightarrow+\infty} y_{i^{*}}(t)=0$.

Case 2. If for some $j \in I, N^{*}<\lim _{k \rightarrow+\infty} x_{i^{*}}\left(t_{k}-h_{i^{*} j}\left(t_{k}\right)\right)$, it follows from (2.1), (2.2), (3.2) and (3.3) that

$$
\begin{aligned}
0= & \lim _{k \rightarrow+\infty} y_{i^{*}}^{\prime}\left(t_{k}\right) \\
< & \lim _{k \rightarrow+\infty} \beta\left(t_{k}\right)\left[-\delta_{i^{*}} \lim _{k \rightarrow+\infty} x_{i^{*}}\left(t_{k}\right)+\sum_{j=1}^{n} a_{i^{*} j} \lim _{k \rightarrow+\infty} y_{j}\left(t_{k}\right)\right. \\
& \left.+\sum_{j=1}^{m} P_{i^{*} j}\left(\lim _{k \rightarrow+\infty} x_{i^{*}}^{\gamma}\left(t_{k}-g_{i^{*} j}\left(t_{k}\right)\right)\right) e^{-\sigma_{i^{*} j} N^{*}}\right] \\
< & \lim _{k \rightarrow+\infty} \beta\left(t_{k}\right)\left(\limsup _{k \rightarrow+\infty} y_{i^{*}}(t)+N^{*}\right)^{\gamma}\left[-\delta_{i^{*}}\left(N^{*}\right)^{1-\gamma}+\sum_{j=1}^{m} P_{i^{*} j} e^{-\sigma_{i^{*} j} N^{*}}\right] \\
= & 0
\end{aligned}
$$

which is also a contradiction and proves the above statement. This finishes the proof of Proposition 3.1.

Corollary 3.2. If for any $i \in Q, x_{i}(t)$ is eventually nonoscillatory about $N^{*}$, i.e., there is $T^{*}$ obeying that

$$
x_{i}(t) \geq N^{*}\left(\text { or } x_{i}(t) \leq N^{*}\right) \quad \text { for all } t \geq T^{*} \text { and } i \in Q
$$

Then $\lim _{t \rightarrow+\infty} x_{i}(t)=N^{*}$ for all $i \in Q$.
Remark 3.3. Corollary 3.2 shows that a delay-independent criterion has been established to guarantee that all non-oscillatory solutions of the system (1.5) are convergent to its unique positive equilibrium point.

Remark 3.4. It is obvious that all conclusions in Theorem 3.1, Theorem 3.2 of [21] and the results of Theorem 3.1 in [36] are special ones of Proposition 3.1.

Theorem 3.5. Let $\sigma=\max _{i \in Q} \max _{j \in I} \sigma_{i j}$, suppose that, for all $i \in Q$,

$$
\begin{equation*}
\frac{\delta_{i} \sigma N^{*}\left(e^{\left(\delta_{i}-a_{i i}\right) \beta^{+} r}-1\right)}{\delta_{i}-a_{i i}} \leq 1 \tag{3.5}
\end{equation*}
$$

and

$$
\begin{equation*}
0<\sigma N^{*} \delta_{i} \frac{1-e^{-r\left(\delta_{i}-a_{i i}\right) \beta^{+}}}{\delta_{i}\left[1-e\left(1-e^{-r\left(\delta_{i}-a_{i i}\right) \beta^{+}}\right)\right]-a_{i i} e^{-r\left(\delta_{i}-a_{i i}\right) \beta^{+}}} \leq 1 \tag{3.6}
\end{equation*}
$$

hold. Then $\lim _{t \rightarrow+\infty} x_{i}(t)=N^{*}$ for all $i \in Q$.
Proof. Let

$$
z_{i}(t)=\sigma\left(x_{i}(t)-N^{*}\right), \quad i \in Q
$$

we have from (1.5) that

$$
\begin{align*}
z_{i}^{\prime}(t) & +\sigma \delta_{i} \beta(t) N^{*}+\delta_{i} \beta(t) z_{i}(t) \\
& =\beta(t) \sum_{j=1}^{n} a_{i j} z_{j}(t)+\sigma \beta(t) \sum_{j=1}^{m} P_{i j}\left[\frac{z_{i}\left(t-g_{i j}(t)\right)}{\sigma}+N^{*}\right]^{\gamma} e^{-\frac{\sigma_{i j} z_{i}\left(t-h_{i j}(t)\right)}{\sigma}-\sigma_{i j} N^{*}} \tag{3.7}
\end{align*}
$$

and

$$
\begin{align*}
\left(z_{i}(t) e^{\int_{t_{0}}^{t}\left(\delta_{i}-a_{i i}\right) \beta(v) d v}\right)^{\prime} & =\left[\sum_{j=1, j \neq i}^{n} a_{i j} \beta(t) z_{j}(t)+\sigma \beta(t) \sum_{j=1}^{m} P_{i j}\left(\frac{z_{i}\left(t-g_{i j}(t)\right)}{\sigma}+N^{*}\right)^{\gamma}\right. \\
& \left.\times e^{-\frac{\sigma_{i j} z_{i}\left(t-k_{i j}(t)\right)}{\sigma}-\sigma_{i j} N^{*}}-\sigma \beta(t) \delta_{i} N^{*}\right] e^{\int_{t_{0}}^{t}\left(\delta_{i}-a_{i j}\right) \beta(v) d v}, \quad t \geq t_{0}, i \in Q . \tag{3.8}
\end{align*}
$$

To finish the verification, we shall reveal that

$$
\min _{i \in Q} \liminf _{t \rightarrow+\infty} z_{i}(t)=\max _{i \in Q} \limsup _{t \rightarrow+\infty} z_{i}(t)=0 .
$$

In view of Corollary 3.2, we only need to treat the case that for each $T^{*}>t_{0}$, there are $t^{*}, t^{* *} \in\left(T^{*},+\infty\right)$ such that

$$
\begin{equation*}
\min _{i \in Q} z_{i}\left(t^{*}\right)<0 \text { and } \max _{i \in Q} z_{i}\left(t^{* *}\right)>0 . \tag{3.9}
\end{equation*}
$$

Set

$$
\begin{equation*}
\mu=\limsup _{t \rightarrow+\infty} z_{i_{1}}(t)=\max _{i \in Q} \limsup _{t \rightarrow+\infty} z_{i}(t), \quad \lambda=\liminf _{t \rightarrow+\infty} z_{i_{2}}(t)=\min _{i \in Q} \liminf _{t \rightarrow+\infty} z_{i}(t) . \tag{3.10}
\end{equation*}
$$

Owing to (3.9), we gain

$$
\lambda \leq 0 \leq \mu .
$$

Now, it suffices to evidence that $\lambda=\mu=0$. Contrarily, either $\mu>0$ or $\lambda<0$ is valid.
We only deal with the case that $\mu>0$ occurs. ( $\lambda<0$ can be treated similarly.)
If $\lambda=0$, i.e., $\lambda=\min _{i \in Q} \liminf _{t \rightarrow+\infty} z_{i}(t)=0$. By Proposition 3.1, one can see that $\mu=\limsup t_{t \rightarrow+\infty} z_{i_{1}}(t)=0$.

When $\mu>0$ and $\lambda<0$, on account of the fluctuation lemma [29, Lemma A.1.], one can take two strictly monotone increasing infinite sequences $\left\{l_{q}\right\}_{q \geq 1},\left\{s_{q}\right\}_{q \geq 1}$ satisfying that

$$
\begin{equation*}
z_{i_{1}}\left(l_{q}\right)>0, \quad l_{q} \rightarrow+\infty, \quad z_{i_{1}}\left(l_{q}\right) \rightarrow \mu, \quad z_{i_{1}}^{\prime}\left(l_{q}\right) \rightarrow 0 \quad \text { as } q \rightarrow+\infty, \tag{3.11}
\end{equation*}
$$

and

$$
\begin{equation*}
z_{i_{2}}\left(s_{q}\right)<0, \quad s_{q} \rightarrow+\infty, \quad z_{i_{2}}\left(s_{q}\right) \rightarrow \lambda, \quad z_{i_{2}}^{\prime}\left(s_{q}\right) \rightarrow 0 \quad \text { as } q \rightarrow+\infty . \tag{3.12}
\end{equation*}
$$

Note that a bounded sequence has a convergent subsequence, we can presume that for all $j \in$ I,

$$
\begin{equation*}
\lim _{q \rightarrow+\infty} \beta\left(l_{q}\right)=\beta^{*}, \quad \lim _{q \rightarrow+\infty} z_{i_{1}}\left(l_{q}-g_{i_{1}} j\left(l_{q}\right)\right)=z_{i_{1}}^{j} \quad \lim _{q \rightarrow+\infty} z_{i}\left(l_{q}\right)=z_{i}^{l} \quad\left(i \in Q \backslash\left\{i_{1}\right\}\right), \tag{3.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{q \rightarrow+\infty} \beta\left(s_{q}\right)=\beta^{* *}, \quad \lim _{q \rightarrow+\infty} z_{i_{2}}\left(s_{q}-g_{i_{2} j}\left(s_{q}\right)\right)=z_{i_{2}}^{j} \quad \lim _{q \rightarrow+\infty} z_{i}\left(s_{q}\right)=z_{i}^{s} \quad\left(i \in Q \backslash\left\{i_{2}\right\}\right) . \tag{3.14}
\end{equation*}
$$

To obtain a contradiction, we divide our proof into three steps.
First, we assert that there exists $H_{1}>0$ obeying that, for any $q \geq H_{1}$, there is $L_{q} \in$ [ $l_{q}-r_{i_{1}}, l_{q}$ ) agreeing with

$$
\begin{equation*}
z_{i_{1}}\left(L_{q}\right)=0, \quad \text { and } \quad z_{i_{1}}(t)>0, \quad \text { for all } t \in\left(L_{q}, l_{q}\right) . \tag{3.15}
\end{equation*}
$$

If not, there exists a subsequence of $\left\{l_{q}\right\}$ (do not relabel) such that

$$
\begin{equation*}
z_{i_{1}}(t)>0, \quad \text { for all } t \in\left[l_{q}-r_{i_{1}}, l_{q}\right), q=1,2, \ldots \tag{3.16}
\end{equation*}
$$

Subsequently,

$$
\begin{equation*}
0 \leq \lim _{q \rightarrow+\infty} z_{i_{1}}\left(l_{q}-g_{i_{1} j}\left(l_{q}\right)\right) \leq \mu \quad \text { for all } j \in I \tag{3.17}
\end{equation*}
$$

and

$$
\begin{align*}
z_{i_{1}}^{\prime}\left(l_{q}\right)= & \beta\left(l_{q}\right) \sum_{j=1}^{n} a_{i_{1} j} z_{j}\left(l_{q}\right)+\sigma \beta\left(l_{q}\right) \sum_{j=1}^{m} P_{i_{1} j}\left[\frac{z_{i_{1}}\left(l_{q}-g_{i_{1} j}\left(l_{q}\right)\right)}{\sigma}+N^{*}\right]^{\gamma} e^{-\frac{\sigma_{i_{i} j} z_{i_{1}}\left(l_{q}-h_{i_{j} j}\left(l_{q}\right)\right)}{\sigma}}-\sigma_{i_{1 j} j} N^{*} \\
& -\sigma \delta_{i_{1}} \beta\left(l_{q}\right) N^{*}-\delta_{i_{1}} \beta\left(l_{q}\right) z_{i_{1}}\left(l_{q}\right) \\
< & \beta\left(l_{q}\right) \sum_{j=1}^{n} a_{i_{1} j} z_{j}\left(l_{q}\right)+\sigma \beta\left(l_{q}\right) \sum_{j=1}^{m} P_{i_{1} j}\left[\frac{z_{i_{1}}\left(l_{q}-g_{i_{1} j}\left(l_{q}\right)\right)}{\sigma}+N^{*}\right]^{\gamma} e^{-\sigma_{i_{j} j} N^{*}} \\
& -\sigma \delta_{i_{1}} \beta\left(l_{q}\right) N^{*}-\delta_{i_{1}} \beta\left(l_{q}\right) z_{i_{1}}\left(l_{q}\right) . \tag{3.18}
\end{align*}
$$

By taking limit, (3.11), (3.13), (3.17) and (3.18) lead to

$$
\begin{aligned}
0 \leq & a_{i_{1} i_{1}} \beta^{*} \lim _{q \rightarrow+\infty} z_{i_{1}}\left(l_{q}\right)+\beta^{*} \sum_{j=1, j \neq i_{1}}^{n} a_{i_{1} j} \lim _{q \rightarrow+\infty} z_{j}\left(l_{q}\right) \\
& +\sigma \beta^{*} \sum_{j=1}^{m} P_{i_{1} j}\left[\frac{\lim _{q \rightarrow+\infty} z_{i_{1}}\left(l_{q}-g_{i_{i j}}\left(l_{q}\right)\right)}{\sigma}+N^{*}\right]^{\gamma} e^{-\sigma_{i_{j}} N^{*}}-\sigma \delta_{i_{1}} \beta^{*} N^{*}-\delta_{i_{1}} \beta^{*} \lim _{q \rightarrow+\infty} z_{i_{1}}\left(l_{q}\right) \\
\leq & \sigma \beta^{*} \sum_{j=1}^{m} P_{i_{1} j}\left[\frac{\lim _{q \rightarrow+\infty} z_{i_{1}}\left(l_{q}-g_{i_{1} j}\left(l_{q}\right)\right)}{\sigma}+N^{*}\right]^{\gamma} e^{-\sigma_{i_{1} j} N^{*}}-\sigma \beta^{*} \delta_{i_{1}}\left(N^{*}+\frac{\mu}{\sigma}\right) \\
\leq & \sigma \beta^{*}\left(N^{*}+\frac{\mu}{\sigma}\right)^{\gamma}\left[\sum_{j=1}^{m} P_{i_{1 j} j} e^{-\sigma_{i_{1 j} j} N^{*}}-\delta_{i_{1}}\left(N^{*}+\frac{\mu}{\sigma}\right)^{1-\gamma}\right] \\
< & 0
\end{aligned}
$$

which is a contradiction and validates the above assertion.
Similarly, from (3.12) and (3.14), one can find $H_{1}^{*}>0$ such that for any $q \geq H_{1}^{*}$, there is $S_{q} \in\left[s_{q}-r_{i_{2}}, s_{q}\right)$ such that

$$
\begin{equation*}
z_{i_{2}}\left(S_{q}\right)=0, \quad \text { and } \quad z_{i_{2}}(t)<0, \quad \text { for all } t \in\left(S_{q}, s_{q}\right) \tag{3.19}
\end{equation*}
$$

Secondly, we show

$$
\begin{equation*}
e^{-\mu}-1 \leq \lambda \leq 0 \leq \mu \leq e^{-\lambda}-1 \tag{3.20}
\end{equation*}
$$

For any $0<\varepsilon<\sigma\left(N^{*}+\frac{\lambda}{\sigma}\right)=\sigma \liminf _{t \rightarrow+\infty} x_{i_{2}}(t)$, (3.10) suggests that one can select a positive integer $q^{*}>H_{1}+H_{1}^{*}$ satisfying

$$
\begin{equation*}
\lambda-\varepsilon<z_{i}(t)<\mu+\varepsilon \quad \text { for all } t>\min \left\{l_{q^{*}}, s_{q^{*}}\right\}-2 r \text { and } \quad i \in Q . \tag{3.21}
\end{equation*}
$$

With the aid of (2.1), (2.2), (3.8), (3.19), (3.21) and (3.23), we obtain

$$
\begin{aligned}
& z_{i_{2}}\left(s_{q}\right) e^{\int_{t_{0}}^{s q}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v} \\
& =-\sigma \delta_{i_{2}} N^{*} \frac{e^{\int_{t_{0}}^{s_{q}}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{s_{q}}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v}}{\delta_{i_{2}}-a_{i_{2} i_{2}}} \\
& +\sum_{j=1, j \neq i_{2}}^{n} a_{i_{2} j} \int_{S_{q}}^{s_{q}} z_{j}(t) \beta(t) e^{\int_{t_{0}}^{t}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v} d t+\sigma \sum_{j=1}^{m} P_{i_{2} j} \int_{S_{q}}^{s_{q}}\left[N^{*}+\frac{z_{i_{2}}\left(t-g_{i_{2} j}(t)\right)}{\sigma}\right]^{\gamma} \\
& \times e^{-\sigma_{i_{2}} N^{*}-\frac{\sigma_{i_{2} j}}{\sigma} z_{i_{2}}\left(t-h_{i_{2} j}(t)\right)} \beta(t) e^{\int_{t_{0}}^{t}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v} d t \\
& >-\sigma \delta_{i_{2}} N^{*} \frac{e^{\int_{t_{0}}^{s_{q}}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{S_{q}}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v}}{\delta_{i_{2}}-a_{i_{2} i_{2}}} \\
& +(\lambda-\varepsilon) \frac{e^{\int_{t_{0}}^{s_{q}}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{s_{q}}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v}}{\delta_{i_{2}}-a_{i_{2} i_{2} .}} \sum_{j=1, j \neq i_{2}}^{n} a_{i_{2} j} \\
& +\sigma \sum_{j=1}^{m} P_{i_{2} j} \int_{S_{q}}^{s_{q}}\left(N^{*}\right)^{\gamma}\left[\frac{N^{*}+\frac{\lambda-\varepsilon}{\sigma}}{N^{*}}\right]^{\gamma} e^{-\sigma_{i_{2} j} N^{*}-\frac{\sigma_{i_{2} j}}{\sigma}(\mu+\varepsilon)} \beta(t) e^{\int_{t_{0}}^{t}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v} d t \\
& >-\sigma \delta_{i_{2}} N^{*} \frac{e^{\int_{t_{0}}^{s_{q}}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{S_{q}}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v}}{\delta_{i_{2}}-a_{i_{2} i_{2}}} \\
& +(\lambda-\varepsilon) \frac{e^{\int_{t_{0}}^{s_{q}}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{s_{q}}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v}}{\delta_{i_{2}}-a_{i_{2} i_{2}}} \sum_{j=1, j \neq i_{2}}^{n} a_{i_{2} j} \\
& +\sigma \sum_{j=1}^{m} P_{i_{2} j} \int_{S_{q}}^{s_{q}}\left(N^{*}\right)^{\gamma-1}\left(N^{*}+\frac{\lambda-\varepsilon}{\sigma}\right) e^{-\sigma_{i_{2} j} N^{*}-\frac{\sigma_{i_{2} j}}{\sigma}(\mu+\varepsilon)} \beta(t) e^{\int_{t_{0}}^{t}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v} d t \\
& \geq \sigma \delta_{i_{2}} N^{*} \frac{e^{\int_{t_{0}}^{s_{q}}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{s_{q}}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v}}{\delta_{i_{2}}-a_{i_{2} i_{2}}}\left[e^{-(\mu+\varepsilon)}-1\right] \\
& +(\lambda-\varepsilon)\left(e^{\int_{t_{0}}^{s_{q}}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{S_{q}}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v}\right), \quad q>q^{*}
\end{aligned}
$$

and

$$
\begin{align*}
& z_{i_{2}}\left(s_{q}\right)+(\lambda-\varepsilon)\left(e^{-\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta^{+} r}-1\right) \\
& \quad \geq z_{i_{2}}\left(s_{q}\right)+(\lambda-\varepsilon)\left(e^{-\int_{S_{q}}^{s_{q}}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v}-1\right) \\
& \quad>\sigma N^{*}\left(1-e^{-\int_{S_{q}}^{s_{q}}\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta(v) d v}\right) \frac{\delta_{i_{2}}}{\delta_{i_{2}}-a_{i_{2} i_{2}}}\left[e^{-(\mu+\varepsilon)}-1\right] \\
& \quad \geq \sigma N^{*}\left(1-e^{-\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta^{+} r}\right) \frac{\delta_{i_{2}}}{\delta_{i_{2}}-a_{i_{2} i_{2}}}\left[e^{-(\mu+\varepsilon)}-1\right], \quad q>q^{*} \tag{3.22}
\end{align*}
$$

Letting $q \rightarrow \infty$ and $\varepsilon \rightarrow 0$, (3.5) and (3.22) give us

$$
\begin{equation*}
\lambda \geq \sigma N^{*}\left(e^{\left(\delta_{i_{2}}-a_{i_{2} i_{2}}\right) \beta^{+} r}-1\right) \frac{\delta_{i_{2}}}{\delta_{i_{2}}-a_{i_{2} i_{2}}}\left(e^{-\mu}-1\right) \geq\left(e^{-\mu}-1\right) \geq-1 \tag{3.23}
\end{equation*}
$$

In view of (2.1), (2.2), (3.8), (3.15) and (3.21), we acquire

$$
\begin{aligned}
& z_{i_{1}}\left(l_{q}\right) e^{\int_{t_{0}}^{l_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v} \\
& =-\sigma \delta_{i_{1}} N^{*} \frac{e^{\int_{t_{0}}^{l_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{L_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}}{\delta_{i_{1}}-a_{i_{1} i_{1}}} \\
& +\sum_{j=1, j \neq i_{1}}^{n} a_{i_{1} j} \int_{L_{q}}^{l_{q}} z_{j}(t) \beta(t) e^{\int_{t_{0}}^{t}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v} d t \\
& +\sigma \sum_{j=1}^{m} P_{i_{1} j} \int_{L_{q}}^{l_{q}}\left[N^{*}+\frac{z_{i_{1}}\left(t-g_{i_{1} j}(t)\right)}{\sigma}\right]^{\gamma} \\
& \times e^{-\sigma_{i_{1} j} N^{*}-\frac{\sigma_{i_{1} j}}{\sigma} z_{i_{1}}\left(t-h_{i_{1} j}(t)\right)} \beta(t) e^{\int_{t_{0}}^{t}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v} d t \\
& <-\sigma \delta_{i_{1}} N^{*} \frac{e^{\int_{t_{0}}^{l_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{L_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}}{\delta_{i_{1}}-a_{i_{1} i_{1}}} \\
& +(\mu+\varepsilon) \frac{e^{\int_{t_{0}}^{l_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{L_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}}{\delta_{i_{1}}-a_{i_{1} i_{1}}} \sum_{j=1, j \neq i_{1}}^{n} a_{i_{1} j} \\
& +\sigma \sum_{j=1}^{m} P_{i_{1} j} \int_{L_{q}}^{l_{q}}\left[N^{*}+\frac{\mu+\varepsilon}{\sigma}\right]^{\gamma} e^{-\sigma_{i_{1} j} N^{*}-\frac{\sigma_{i_{1} j}}{\sigma}(\lambda-\varepsilon)} \beta(t) e^{\int_{t_{0}}^{t}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v} d t \\
& =-\sigma \delta_{i_{1}} N^{*} \frac{e^{\int_{t_{0}}^{l_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{L q}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}}{\delta_{i_{1}}-a_{i_{1} i_{1}}} \\
& +(\mu+\varepsilon) \frac{e^{\int_{t_{0}}^{l_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{L q}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}}{\delta_{i_{1}}-a_{i_{1} i_{1}}} \sum_{j=1, j \neq i_{1}}^{n} a_{i_{1} j} \\
& +\sigma \sum_{j=1}^{m} P_{i_{1} j} \int_{L_{q}}^{l_{q}}\left(N^{*}\right)^{\gamma}\left[\frac{N^{*}+\frac{\mu+\varepsilon}{\sigma}}{N^{*}}\right]^{\gamma} e^{-\sigma_{i_{1} j} N^{*}-\frac{\sigma_{i_{1} j}}{\sigma}(\lambda-\varepsilon)} \beta(t) e^{\int_{t_{0}}^{t}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v} d t \\
& <-\sigma \delta_{i_{1}} N^{*} \frac{e^{\int_{t_{0}}^{l_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{L_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}}{\delta_{i_{1}}-a_{i_{1} i_{1}}} \\
& +(\mu+\varepsilon) \frac{e^{\int_{t_{0}}^{l_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{L_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}}{\delta_{i_{1}}-a_{i_{1} i_{1}}} \sum_{j=1, j \neq i_{1}}^{n} a_{i_{1} j} \\
& +\sigma \sum_{j=1}^{m} P_{i_{1} j} \int_{L_{q}}^{l_{q}}\left(N^{*}\right)^{\gamma-1}\left(N^{*}+\frac{\mu+\varepsilon}{\sigma}\right) e^{-\sigma_{i_{1} j} N^{*}-\frac{\sigma_{i_{1} j}}{\sigma}(\lambda-\varepsilon)} \beta(t) e^{\int_{t_{0}}^{t}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v} d t \\
& \leq-\sigma \delta_{i_{1}} N^{*} \frac{e^{\int_{t_{0}}^{l_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{L q}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}}{\delta_{i_{1}}-a_{i_{1} i_{1}}} \\
& +(\mu+\varepsilon) \frac{e^{\int_{t_{0}}^{l_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{L q}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}}{\delta_{i_{1}}-a_{i_{1} i_{1}}} \sum_{j=1, j \neq i_{1}}^{n} a_{i_{1} j} \\
& +\frac{e^{\int_{t_{0}}^{l_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{L q}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}}{\delta_{i_{1}}-a_{i_{1} i_{1}}} \\
& \times\left[\sigma \sum_{j=1}^{m} P_{i_{1} j}\left(N^{*}\right)^{\gamma} e^{-\sigma_{i_{1}} N^{*}} e^{-(\lambda-\varepsilon)}+(\mu+\varepsilon) e^{1+\varepsilon} \sum_{j=1}^{m} P_{i_{1} j}\left(N^{*}\right)^{\gamma-1} e^{-\sigma_{i_{1} j} N^{*}}\right]
\end{aligned}
$$

$$
\begin{aligned}
& =\sigma \delta_{i_{1}} N^{*} \frac{\int_{t_{t_{0}}^{l_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{L_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}}{\delta_{i_{1}}-a_{i_{1} i_{1}}}\left[e^{-(\lambda-\varepsilon)}-1\right] \\
& +(\mu+\varepsilon) \frac{e^{1+\varepsilon} \delta_{i_{1}}-a_{i_{1} i_{1}}}{\delta_{i_{1}}-a_{i_{1} i_{1}}}\left(e^{\int_{t_{0}}^{l_{\eta}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}-e^{\int_{t_{0}}^{L_{q}}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}\right), \quad q>q^{*},
\end{aligned}
$$

and

$$
\begin{align*}
z_{i_{1}}\left(l_{q}\right)< & \sigma N^{*} \delta_{i_{1}} \frac{1-e^{\int_{l_{q}}^{L q}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}}{\delta_{i_{1}}-a_{i_{1} i_{1}}}\left[e^{-(\lambda-\varepsilon)}-1\right] \\
& +(\mu+\varepsilon) \frac{e^{1+\varepsilon} \delta_{i_{1}}-a_{i_{1} i_{1}}}{\delta_{i_{1}}-a_{i_{1} i_{1}}}\left(1-e^{\int_{L_{q}}^{L q}\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta(v) d v}\right) \\
\leq & \sigma N^{*} \delta_{i_{1}} \frac{1-e^{-r\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta^{+}}}{\delta_{i_{1}}-a_{i_{1} i_{1}}}\left[e^{-(\lambda-\varepsilon)}-1\right] \\
& +(\mu+\varepsilon) \frac{e^{1+\varepsilon} \delta_{i_{1}}-a_{i_{1} i_{1}}}{\delta_{i_{1}}-a_{i_{1} i_{1}}}\left(1-e^{-r\left(\delta_{\delta_{1}}-a_{i_{1} i_{1}}\right) \beta^{+}}\right), q>q^{*} . \tag{3.24}
\end{align*}
$$

Letting $q \rightarrow \infty$ and $\varepsilon \rightarrow 0$, (3.6) and (3.24) entail that

$$
\begin{equation*}
\left.\left.\mu \leq \sigma N^{*} \delta_{i_{1}} \frac{1-e^{-r\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta^{+}}}{\delta_{i_{1}}\left[1-e\left(1-e^{-r\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right)} \beta^{+}\right.\right.}\right)\right]-a_{i_{1} i_{1}} e^{-r\left(\delta_{i_{1}}-a_{i_{1} i_{1}}\right) \beta^{+}}\left(e^{-\lambda}-1\right) \leq\left(e^{-\lambda}-1\right), \tag{3.25}
\end{equation*}
$$

which, together with (3.23), involves that (3.20) holds.
Finally, from the proof in Theorem 4.1 of [30], (3.20) implies that $\lambda=\mu=0$, which yields a clear contradiction of the fact that $\mu>0$. This finishes the proof.

Remark 3.6. Apparently, $\lim _{r \rightarrow 0^{+}} e^{\left(\delta_{i}-a_{i i}\right) \beta^{+} r}=1$, then the conditions (3.5) and (3.6) naturally hold, which means that sufficiently small pairs of timing-varying delays have little influence on the global attractivity of the positive equilibrium point for system (1.5). On the other hand, $\lim _{r \rightarrow+\infty} e^{\left(\delta_{i}-a_{i j}\right) \beta^{+} r}=+\infty$, then the assumptions (3.5) and (3.6) do not hold, which indicates that large enough pairs of time-varying delays will lead to chaotic oscillation of the system (1.5). We will verify this through some numerical simulations in the next section.

## 4 Numerical example

Example 4.1. Regard the following patch structure neoclassical growth model incorporating multiple pairs of time-varying delays:

$$
\left\{\begin{align*}
x_{1}^{\prime}(t)= & \left(3+\sin ^{2}(t)\right)\left[\left(-\frac{1}{20} x_{1}(t)+\frac{1}{20} x_{2}(t)\right)-\frac{1}{20} x_{1}(t)\right.  \tag{4.1}\\
& +\frac{11}{100} e^{\frac{8}{5}} x_{1}^{\frac{1}{3}}\left(t-g_{11}(t)\right) e^{-\frac{1}{5} x_{1}\left(t-h_{11}(t)\right)} \\
& \left.+\frac{9}{100} e^{\frac{16}{5}} x_{1}^{\frac{1}{3}}\left(t-g_{12}(t)\right) e^{-\frac{2}{5} x_{1}\left(t-h_{12}(t)\right)}\right], \\
x_{2}^{\prime}(t)= & \left(3+\sin ^{2}(t)\right)\left[\left(-\frac{1}{80} x_{2}(t)+\frac{1}{80} x_{1}(t)\right)-\frac{1}{80} x_{2}(t)\right. \\
& +\frac{3}{200} e^{\frac{8}{3}} x_{2}^{\frac{1}{3}}\left(t-g_{21}(t)\right) e^{-\frac{1}{3} x_{2}\left(t-h_{21}(t)\right)} \\
& \left.+\frac{7}{200} e^{2} x_{2}^{\frac{1}{3}}\left(t-g_{22}(t)\right) e^{-\frac{1}{4} x_{2}\left(t-h_{22}(t)\right)}\right],
\end{align*}\right.
$$

which possesses a unique positive equilibrium point $\left(N^{*}, N^{*}\right)=(8,8)$.
Now, one can easily check that

$$
\begin{equation*}
g_{i j}(t)=\frac{1}{20}|\cos (i+j) t|, h_{i j}(t)=\frac{1}{40}|\sin (i+j) t|, \quad i, j=1,2 \tag{4.2}
\end{equation*}
$$

satisfy (3.5) and (3.6). By Theorem 3.5, we obtain that the positive equilibrium point $(8,8)$ is a global attractor of (4.1) incorporating delays (4.2). The numeric simulations in Figure 4.1 support this theoretical assertions.


Figure 4.1: Numerical solutions of example (4.1) obeying (4.2) and the initial values: $(3,1),(10,7),(17,13)$.

Moreover, if we choose

$$
\begin{equation*}
g_{i j}(t)=40 j, \quad h_{i j}(t)=60 j, \quad i, j=1,2 \tag{4.3}
\end{equation*}
$$

it is an elementary computation to show that (3.5) and (3.6) do not hold for system (4.1) with delays (4.3). It can be seen from Figure 4.2 that $(8,8)$ maybe not the global attractor of (4.1) with delays (4.3). This confirms the conclusions reached in Remark 3.6.


Figure 4.2: Numerical solutions of example (4.1) satisfying (4.3) and the initial value $(35,19)$.

Remark 4.2. From the above simulations, we can make the following observations. First, small delays will make the positive equilibrium point be attractive. Second, big delays maybe yield complex dynamic behavior. In addition, the latest literature [ $8,21,36$ ] and $[6,10-12,14,35]$ have not touched the global attractivity of the positive equilibrium point for the patch structure neoclassical growth system with multiple pairs of time-varying delays. It can be found that all the conclusions in the above mentioned literature and the references cited therein cannot be used to reveal the global attractivity of (4.1). It should be pointed out that, in equations (20) and (21) on page 3861 of [36],

$$
\lim _{q \rightarrow+\infty} y_{i_{1}}^{\prime}\left(l_{q}\right) \geq 0 \quad \text { and } \quad \lim _{q \rightarrow+\infty} y_{i_{2}}^{\prime}\left(s_{q}\right) \leq 0
$$

maybe not hold. For a counterexample, consider $y_{i_{1}}(t)=1+\frac{1}{1+t^{2}}$ and $y_{i_{2}}(t)=-1-\frac{1}{1+t^{2}}$. In the proof of Theorem 3.5, we have successfully corrected these errors by adopting new proof strategies and ideas. This implies that our results generalize and improve all the ones in the above-mentioned references.

## 5 Conclusions

By introducing two time-varying delays in the same time-dependent reproduction function, this paper proposed a neoclassical growth system incorporating patch structure and multiple pairs of time-varying delays. Via some novel differential inequality analyses and the fluctuation lemma, the persistence on the positive solutions, as well as the global attractivity on the positive equilibrium point have firstly been established for the addressed model. The obtained results reveal that, by controlling labor growth rate, capital depreciation rate and the related parameters in the reproduction function, the attractivity of the positive equilibrium point can be guaranteed if the time-varying delays are sufficiently small in the development process. The adopted strategies could be taken into consideration in the area of dynamics problems on other patch structure population systems incorporating two or more distinctive delays in the same time-dependent reproduction function.

## Acknowledgements

The author would like to express the sincere appreciation to the associate editor and reviewers for their helpful comments in improving the presentation and quality of the paper. In particular, the author expresses the sincere gratitude to Prof. Gang Yang (Hunan University of Technology and Business, Changsha, China) for the helpful discussion when this revision work was being carried out. This work was supported by the Natural Science Foundation of Hunan Province (No. 2019JJ40142), and Hunan University of Arts and Science (STIT): Numerical calculation \& stochastic process with their applications.

## References

[1] L. Berezansky, E. Braverman, L. Idels, Nicholson's blowflies differential equations revisited: Main results and open problems, Appl. Math. Model. 34(2010), No. 6, 1405-1417. https://doi.org/10.1016/j.apm.2009.08.027; MR2592579; Zbl 1193.34149
[2] L. Berezansky, E. Braverman, A note on stability of Mackey-Glass equations with two delays, J. Math. Anal. Appl. 450(2017), No. 2, 1208-1228. https://doi.org/10.1016/j. jmaa. 2017.01.050; MR3639098; Zbl 1381.34093
[3] Q. Cao, G. Wang, H. Zhang, S. Gong, New results on global asymptotic stability for a nonlinear density-dependent mortality Nicholson's blowflies model with multiple pairs of time-varying delays, J. Inequal. Appl. 2020, Paper No. 7, 12 pp. https://doi.org/10. 1186/s13660-019-2277-2; MR4062072
[4] W. Chen, W. Wang, Global exponential stability for a delay differential neoclassical growth model, Adv. Difference Equ. 2014, Paper No. 325, 9 pp. https ://doi .org/10.1186/ 1687-1847-2014-325; MR3360574; Zbl 1417.37295
[5] R. Day, The emergence of chaos from classical economic growth, Q. J. Econ. 98(1983), No. 2, 201-213. https://doi.org/10.2307/1891124
[6] L. Duan, X. Fang, C. Huang, Global exponential convergence in a delayed almost periodic Nicholson's blowflies model with discontinuous harvesting, Math. Meth. Appl. Sci. 41(2018), No. 5, 1954-1965. https://doi.org/10.1002/mma.4722; MR3778099; Zbl 1446.65033
[7] L. Duan, C. Huang, Existence and global attractivity of almost periodic solutions for a delayed differential neoclassical growth model, Math. Methods Appl. Sci. 40(2017), No. 3, 814-822. https://doi.org/814-822.10.1002/mma.4019; MR3596571; Zbl 1359.34091
[8] T. Faria, Asymptotic behaviour for a class of delayed cooperative models with patch structure, Discrete Contin. Dyn. Syst. Ser. B. 18(2013), No. 6, 1567-1579. https://doi. org/ 10.3934/dcdsb.2013.18.1567 ; MR3038769 ; Zbl 1288.34061
[9] J. Hale, S. Verduyn Lunel, Introduction to functional differential equations, SpringerVerlag, New York, 1993. https://doi.org/10.1007/978-1-4612-4342-7; MR1243878; Zbl 0787.34002
[10] H. Hu, T. Yi, X. Zou, On spatial-temporal dynamics of a Fisher-KPP equation with a shifting environment, Proc. Amer. Math. Soc. 148(2020), No. 1, 213-221. https://doi. org/ 10.1090/proc/14659; MR4042844; Zbl 1430.35140
[11] H. Hu, X. Yuan, L. Huang, C. Huang, Global dynamics of an SIRS model with demographics and transfer from infectious to susceptible on heterogeneous networks, Math. Biosci. Eng. 16(2019), No. 5, 5729-5749. https://doi.org/10.3934/mbe.2019286; MR4032648
[12] H. Hu, X. Zou, Existence of an extinction wave in the Fisher equation with a shifting habitat, Proc. Amer. Math. Soc. 145(2017), No. 11, 4763-4771. https ://doi. org/10.1090/ proc/13687; MR3691993; Zbl 1372.34057
[13] C. Huang, L. Huang, J. Wu, Global population dynamics of a single species structured with distinctive time-varying maturation and self-limitation delays, Discrete Contin. Dyn. Syst. Ser. B., published online, 2021. https://doi. org/10.3934/dcdsb. 2021138
[14] C. Huang, B. Liu, C. Qian, J. Cao, Stability on positive pseudo almost periodic solutions of HPDCNNs incorporating D operator, Math. Comput. Simulation 190(2021), 1159-1163. https://doi.org/10.1016/j.matcom.2021.06.027
[15] C. Huang, X. Long, L. Huang, S. Fu, Stability of almost periodic Nicholson's blowflies model involving patch structure and mortality terms, Canad. Math. Bull. 63(2020), No. 2, 405-422. https://doi.org/10.4153/S0008439519000511; MR4092890; Zbl 1441.34088
[16] C. Huang, J. Wang, L. Huang, Asymptotically almost periodicity of delayed Nicholsontype system involving patch structure, Electron. J. Differential Equations 2020, No. 61, 1-17. MR4113459; Zbl 07244080
[17] C. Huang, X. Yang, J. Cao, Stability analysis of Nicholson's blowflies equation with two different delays, Math. Comput. Simulation 171(2020), No. 2, 201-206. https://doi.org/ 10.1016/j.matcom.2019.09.023; MR4066177; Zbl 07318015
[18] C. Huang, L. Yang, J. Cao, Asymptotic behavior for a class of population dynamics, AIMS Math. 5(2020), No. 4, 3378-3390. https://doi.org/10.3934/math.2020218; MR4146101
[19] C. Huang, Z. Yang, T. Yi, X. Zou, On the basins of attraction for a class of delay differential equations with non-monotone bistable nonlinearities, J. Differential Equations 256(2014), No. 7, 2101-2114. https://doi.org/10.1016/j.jde.2013.12.015; MR3160438; Zbl 1297.34084
[20] C. Huang, H. Zhang, L. Huang, Almost periodicity analysis for a delayed Nicholson's blowflies model with nonlinear density-dependent mortality term, Commun. Pure Appl. Anal. 18(2019), No. 6, 3337-3349. https://doi.org/10.3934/cpaa. 2019150
[21] C. Huang, X. Zhao, J. Cao, F. E. Alsaadi, Global dynamics of neoclassical growth model with multiple pairs of variable delays, Nonlinearity 33(2020), No. 12, 6819-6834. https: //doi.org/10.1088/1361-6544; MR4164693 ; Zbl 1456.34079
[22] B. Liv, Global exponential stability of positive periodic solutions for a delayed Nicholson's blowflies model, J. Anal. Math. Appl. 412(2014), No. 1, 212-221. https://doi.org/ 10.1016/j.jmaa.2013.10.049; MR3145795; Zbl 308.34096
[23] X. Long, S. Gong, New results on stability of Nicholson's blowflies equation with multiple pairs of time-varying delays, Appl. Math. Lett. 100(2020), Article ID 106027, 6 pp. https://doi.org/10.1016/j.aml.2019.106027; MR4008616; Zbl 1436.92011
[24] Z. Long, Y. TAN, Global attractivity for Lasota-Wazewska-type system with patch structure and multiple time-varying delays, Complexity 2020, Article ID 1947809, 7 pp. https://doi.org/10.1155/2020/1947809; Zbl 1435.34085
[25] A. Matsumoto, F. Szidarovszky, Delay differential neoclassical growth model, J. Econom. Behavior Organization 78(2011), No. 3, 272-289. https://doi.org/10.1016/j.jebo. 2011. 01.014
[26] Z. Ning, W. Wang, The existence of two positive periodic solutions for the delay differential neoclassical growth model, Adv. Difference Equ. 2016, Paper No. 266, 6 pp. https://doi.org/10.1186/s13662-016-0995-z; MR3563449; Zbl 1419.34188
[27] C. Qian, Y. Hu, Novel stability criteria on nonlinear density-dependent mortality Nicholson's blowflies systems in asymptotically almost periodic environments, J. Inequal. Appl. 13(2020), No. 13, 1-18. https://doi.org/10.1186/s13660-019-2275-4
[28] H. L. Smith, Monotone dynamical systems: An introduction to the theory of competitive and cooperative systems, Mathematical Surveys and Monographs, Vol. 41, American Mathematical Society, Providence, RI, 1955. https://doi.org/10.1090/surv/041; MR1319817; Zbl 0821.34003
[29] H. L. Smith, An introduction to delay differential equations with applications to the life sciences, Springer New York, 2011. https://doi.org/10.1007/978-1-4419-7646-8; MR2724792
[30] J. W. So, J. Yu, Global attractivity and uniform persistence in Nicholson's blowflies, Differential Equations Dynam. Systems 2(1994), No. 1, 11-18. MR1386035; Zbl 0869.34056
[31] Y. Tan, Dynamics analysis of Mackey-Glass model with two variable delays, Math. Biosci. Eng. 17(2020), No. 5, 4513-4526. https://doi.org/10.3934/mbe.2020249; MR4160190; Zbl 07378460
[32] Y. Tan, C. Huang, B. Sun, T. Wang, Dynamics of a class of delayed reaction-diffusion systems with Neumann boundary condition, J. Math. Anal. Appl. 458(2018), No. 2, 11151130. https://doi.org/10.1016/j.jmaa.2017.09.045; MR3724719; Zbl 1378.92077
[33] W. Wang, The exponential convergence for a delay differential neoclassical growth model with variable delay, Nonlinear Dyn. 86(2016), No. 3, 1875-1883. https://doi.org/10. 1007/s11071-016-3001-0; MR3562458; Zbl 1372.91067
[34] Y. $\mathrm{X}_{\mathrm{U}}$, New result on the global attractivity of a delay differential neoclassical growth model, Nonlinear Dyn. 89(2017), No. 1, 281-288. https ://doi.org/10.1007/s11071-017-3453-x; MR3663693; Zbl 1374.34140
[35] Y. Xu , Q. Cao , X. Guo, Stability on a patch structure Nicholson's blowflies system involving distinctive delays, Appl. Math. Lett. 105(2020), Article ID 106340, 7 pp. https://doi.org/ 10.1016/j .aml.2020.106340; MR4076842; Zbl 1372.91067
[36] G. Yang, Dynamical behaviors on a delay differential neoclassical growth model with patch structure, Math. Methods Appl. Sci. 41(2018), No. 10, 3856-3867. https://doi.org/ 10.1002/mma.4872; MR3820187
[37] H. Zhang, Q. Cao, H. Yang, Asymptotically almost periodic dynamics on delayed Nicholson-type system involving patch structure, J. Inequal. Appl. 2020, Paper No. 102, 27 pp. https://doi.org/10.1186/s13660-020-02366-0; MR4086030

Electronic Journal of Qualitative Theory of Differential Equations

# Higher order stroboscopic averaged functions: a general relationship with Melnikov functions 

Douglas D. Novaes ${ }^{\boxtimes}$<br>Departamento de Matemática - Instituto de Matemática, Estatística e Computação Científica (IMECC) - Universidade Estadual de Campinas (UNICAMP), Rua Sérgio Buarque de Holanda, 651, Cidade Universitária Zeferino Vaz, 13083-859, Campinas, SP, Brazil

Received 17 June 2021, appeared 9 October 2021
Communicated by Armengol Gasull


#### Abstract

In the research literature, one can find distinct notions for higher order averaged functions of regularly perturbed non-autonomous $T$-periodic differential equations of the kind $x^{\prime}=\varepsilon F(t, x, \varepsilon)$. By one hand, the classical (stroboscopic) averaging method provides asymptotic estimates for its solutions in terms of some uniquely defined functions $\mathbf{g}_{i}$ 's, called averaged functions, which are obtained through nearidentity stroboscopic transformations and by solving homological equations. On the other hand, a Melnikov procedure is employed to obtain bifurcation functions $\mathbf{f}_{i}$ 's which controls in some sense the existence of isolated $T$-periodic solutions of the differential equation above. In the research literature, the bifurcation functions $\mathbf{f}_{i}{ }^{\prime}$ s are sometimes likewise called averaged functions, nevertheless, they also receive the name of Poincaré-Pontryagin-Melnikov functions or just Melnikov functions. While it is known that $\mathbf{f}_{1}=T \mathbf{g}_{1}$, a general relationship between $\mathbf{g}_{i}$ and $\mathbf{f}_{i}$ is not known so far for $i \geq 2$. Here, such a general relationship between these two distinct notions of averaged functions is provided, which allows the computation of the stroboscopic averaged functions of any order avoiding the necessity of dealing with near-identity transformations and homological equations. In addition, an Appendix is provided with implemented Mathematica algorithms for computing both higher order averaging functions.


Keywords: averaging theory, Melnikov method, averaged functions, Melnikov functions, higher order analysis.
2020 Mathematics Subject Classification: 34C29, 34E10, 34C25.

## 1 Introduction

This paper is dedicated to investigate the link between two distinct notions of higher order averaged functions of regularly perturbed non-autonomous $T$-periodic differential equations of the kind $x^{\prime}=\varepsilon F(t, x, \varepsilon)$.

The first notion comes from the classical averaging method, which provides asymptotic estimates for the solutions of the differential equation $x^{\prime}=\varepsilon F(t, x, \varepsilon)$ in terms of some uniquely

[^39]defined functions, called averaged functions, which are obtained through near-identity stroboscopic transformations and by solving homological equations.

The second notion is provided by the Melnikov method, where the averaged functions are obtained by expanding the time- $T$ map of the differential equation $x^{\prime}=\varepsilon F(t, x, \varepsilon)$ around $\varepsilon=0$ and control, in some sense, the bifurcation of isolated $T$-periodic solutions.

In the sequel, these notions will be discussed in detail.

### 1.1 The averaging method

An important and celebrated tool for dealing with nonlinear oscillating systems in the presence of small perturbations is the averaging method, which has its foundations in the works of Clairaut, Laplace, and Lagrange, in the development of celestial mechanics, and was rigorous formalized by the works of Fatou, Krylov, Bogoliubov, and Mitropolsky [1,2,7,11] (for a brief historical review, see [15, Chapter 6] and [20, Appendix A]). It is mainly concerned in providing long-time asymptotic estimates for solutions of non-autonomous differential equations given in the following standard form

$$
\begin{equation*}
x^{\prime}=\sum_{i=1}^{k} \varepsilon F_{i}(t, x)+\varepsilon^{k+1} R(t, x, \varepsilon) \tag{1.1}
\end{equation*}
$$

Here, $F_{i}: \mathbb{R} \times D \rightarrow \mathbb{R}^{n}$, for $i=1, \ldots, k$, and $R: \mathbb{R} \times D \times\left[-\varepsilon_{0}, \varepsilon_{0}\right] \rightarrow \mathbb{R}^{n}$ are assumed to be smooth functions $T$-periodic in the variable $t$, with $D$ being an open subset of $\mathbb{R}^{n}$ and $\varepsilon_{0}>0$ small. Such asymptotic estimates are given in terms of solutions of an autonomous truncated averaged equation

$$
\begin{equation*}
\xi^{\prime}=\sum_{i=1}^{k} \varepsilon^{i} \mathbf{g}_{i}(\xi), \tag{1.2}
\end{equation*}
$$

where $\mathbf{g}_{i}: D \rightarrow \mathbb{R}^{n}$, for $i \in\{1, \ldots, k\}$, are obtained by the following result:
Theorem 1.1 ([20, Lemma 2.9.1]). There exists a smooth T-periodic near-identity transformation

$$
x=U(t, \xi, \varepsilon)=\xi+\sum_{i=1}^{k} \varepsilon^{i} \mathbf{u}_{i}(t, \xi)
$$

satisfying $U(0, \xi, \varepsilon)=\xi$, such that the differential equation (1.1) is transformed into

$$
\begin{equation*}
\xi^{\prime}=\sum_{i=1}^{k} \varepsilon^{i} \mathbf{g}_{i}(\xi)+\varepsilon^{k+1} r_{k}(t, \xi, \varepsilon) \tag{1.3}
\end{equation*}
$$

The averaging theory states that, for $|\varepsilon| \neq 0$ sufficiently small, the solutions of the original differential equation (1.1) and the truncated averaged equation (1.2), starting at the same initial condition, remains $\varepsilon^{k}$-close for a time interval of order $1 / \varepsilon$ (see [20, Theorem 2.9.2]).

The functions $\mathbf{g}_{i}$ and $\mathbf{u}_{i}$ can be algorithmically computed by solving homological equations. Section 3.2 of [20] is devoted to discuss how is the best way to work with such near-identity transformations based on Lie theory (see also [6,19]). One can see that, in general, $\mathbf{g}_{1}$ is the average of $F_{1}(t, \cdot)$, that is,

$$
\mathbf{g}_{1}(z)=\frac{1}{T} \int_{0}^{T} F_{1}(t, x) d t .
$$

It is worth mentioning that the so-called stroboscopic condition $U(\xi, 0, \varepsilon)=\xi$ does not have to be assumed in order to get (1.3). However, in that case, the functions $\mathbf{g}_{i}$, for $i \geq 2$, are not
uniquely determined. For the stroboscopic averaging, the uniqueness of each $\mathbf{g}_{i}$ is guaranteed and so it is natural to call it by averaged function of order $i$ (or ith-order averaged function) of the differential equation (1.1). Here, these functions are referred by stroboscopic averaged functions to indicate that the stroboscopic condition is being assumed.

The averaging method has been employed in the investigation of invariant manifolds of differential equations (see, for instance, [8]). In particular, it has been extensively used to study periodic solutions of differential equations. One can find results, in the classical research literature, that relate simple zeros of the first-order averaged function $\mathbf{g}_{1}$ with isolated $T$ periodic solutions of (1.1) (see, for instance, [9,20,21]).

### 1.2 The Melnikov method

The mentioned results relating simple zeros of the first-order averaged function $\mathbf{g}_{1}$ with isolated $T$-periodic solutions of (1.1) have been generalized in several directions (see, for instance, $[4,5,12-14,17,18])$. In particular, a recursively defined sequence of functions $\mathbf{f}_{i}: D \rightarrow \mathbb{R}^{n}$, $i \in\{1, \ldots, k\}$, was obtained in [13], for which the following result holds:

Theorem 1.2 ([13]). Denote $\mathbf{f}_{0}=0$. Let $\ell \in\{1, \ldots, k\}$ satisfying $\mathbf{f}_{0}=\cdots=\mathbf{f}_{\ell-1}=0$ and $\mathbf{f}_{\ell} \neq 0$. Assume that $z^{*} \in D$ is a simple zero of $\mathbf{f}_{\ell}$. Then, for $|\varepsilon| \neq 0$ sufficiently small, the differential equation (1.1) admits an isolated T-periodic solution $\varphi(t, \varepsilon)$ such that $\varphi(0, \varepsilon) \rightarrow z^{*}$ as $\varepsilon \rightarrow 0$.

The bifurcation functions $\mathbf{f}_{i}, i \in\{1, \ldots, k\}$, are obtained through a Melnikov procedure, which consists in expanding the time- $T$ map of the differential equation (1.1) around $\varepsilon=0$ by using the following result:

Lemma 1.3 ([13,16]). Let $x(t, z, \varepsilon)$ be the solution of (1.1) satisfying $x(0, z, \varepsilon)=z$. Then,

$$
x(t, z, \varepsilon)=z+\sum_{i=1}^{k} \varepsilon^{i} \frac{y_{i}(t, z)}{i!}+\mathcal{O}\left(\varepsilon^{k+1}\right),
$$

where

$$
\begin{align*}
& y_{1}(t, z)=\int_{0}^{t} F_{1}(s, z) d s \text { and } \\
& y_{i}(t, z)=\int_{0}^{t}\left(i!F_{i}(s, z)+\sum_{j=1}^{i-1} \sum_{m=1}^{j} \frac{i!}{j!} \partial_{x}^{m} F_{i-j}(s, z) B_{j, m}\left(y_{1}, \ldots, y_{j-m+1}\right)(s, z)\right) d s, \tag{1.4}
\end{align*}
$$

for $i \in\{2, \ldots, k\}$.
As usual, for $p$ and $q$ positive integers, $B_{p, q}$ denotes the partial Bell polynomials:

$$
B_{p, q}\left(x_{1}, \ldots, x_{p-q+1}\right)=\sum \frac{p!}{b_{1}!b_{2}!\cdots b_{p-q+1}!} \prod_{j=1}^{p-q+1}\left(\frac{x_{j}}{j!}\right)^{b_{j}} .
$$

The sum above is taken over all the tuples of nonnegative integers $\left(b_{1}, b_{2}, \ldots, b_{p-q+1}\right)$ satisfying $b_{1}+2 b_{2}+\cdots+(p-q+1) b_{p-q+1}=p$ and $b_{1}+b_{2}+\cdots+b_{p-q+1}=q$. Here, $\partial_{x}^{m} F_{i-j}(s, z)$ denotes the Frechet's derivative of $F_{i-j}$ with respect to the variable $x$ evaluated at $x=z$, which is a symmetric $m$-multilinear map that is applied to combinations of "products" of $m$ vectors in $\mathbb{R}^{n}$, in the present case $B_{j, m}\left(y_{1}, \ldots, y_{j-m+1}\right)$.

Accordingly, the functions $\mathbf{f}_{i}$, for $i \in\{1, \ldots, k\}$, are defined by

$$
\begin{equation*}
\mathbf{f}_{i}(z)=\frac{y_{i}(T, z)}{i!} . \tag{1.5}
\end{equation*}
$$

Notice that $\mathbf{f}_{1}$ is the average of $F_{1}(t, \cdot)$ multiplied by a factor $T$, that is, $\mathbf{f}_{1}=T \mathbf{g}_{1}$. Usually, $\mathbf{f}_{i}$ is likewise called by averaged function of order $i$ (or $i$ th-order averaged function) of the differential equation (1.1). It is worth mentioning that, in the research literature, the bifurcation functions $\mathbf{f}_{i}$ 's also receive the name of Poincaré-Pontryagin-Melnikov functions or just Melnikov functions. Such functions can be easily formally computed from (1.4) and (1.5), for instance

$$
\begin{align*}
\mathbf{f}_{2}(z) & =\int_{0}^{T}\left(F_{2}(t, z)+\partial_{x} F_{1}(t, z) y_{1}(t, z)\right) d t \\
& =\int_{0}^{T}\left(F_{2}(t, z)+\partial_{x} F_{1}(t, z) \int_{0}^{t} F_{1}(s, z) d s\right) d t . \tag{1.6}
\end{align*}
$$

### 1.3 Main goals

In a first view, the functions $\mathbf{f}_{i}$ and $\mathbf{g}_{i}$, for $i \geq 2$, do not hold a clear relationship. Thus, the present study is mainly concerned in establishing a link between them.

In Section 3, the main result of this paper, Theorem A, provides a general relationship between such distinct notions of higher order averaged functions, which allows the computation of the higher order stroboscopic averaged functions avoiding the necessity of dealing with near-identity transformations and homological equations. In addition, an Appendix is provided with implemented Mathematica algorithms for computing both higher order averaging functions.

In Section 3.2, some consequences of the main result are presented. First, Corollary A states that $\mathbf{f}_{i}=T \mathbf{g}_{i}$, for $i \in\{1, \ldots, \ell\}$, provided that either $\mathbf{f}_{1}=\cdots=\mathbf{f}_{\ell-1}=0$ or $\mathbf{g}_{1}=\cdots=$ $\mathbf{g}_{\ell-1}=0$. This gives a relatively simple and computable expression for the first non-vanishing stroboscopic averaged function (see Corollary B). This last result has been reported in [10] for differential equations coming from planar near-Hamiltonian systems.

## 2 Related results in research literature

In this section, some known results in research literature regarding the relationship between Melnikov functions and averaged functions are discussed.

In [10], the authors have investigated the relationship between averaged functions and Melnikov functions for planar near-Hamiltonian systems

$$
\dot{x}=H_{y}+\varepsilon f(x, y, \varepsilon), \quad \dot{y}=-H_{x}+\varepsilon g(x, y, \varepsilon), \quad(x, y) \in \mathbb{R}^{2},
$$

assuming that the unperturbed system $\dot{x}=H_{y}, \dot{y}=-H_{x}$ has a continuous family of periodic solutions $L_{h}, h \in J \subset \mathbb{R}$. It is worthy mentioning that this is the natural context where Melnikov theory is applied. After a change of variables $(x, y) \in \mathbb{R}^{2} \mapsto(\theta, h) \in[0,2 \pi) \times J$, the near-Hamiltonian system can be written in the standard form (1.1),

$$
\frac{d h}{d \theta}=\varepsilon F(\theta, h, \varepsilon), \quad(\theta, h) \in[0,2 \pi) \times J
$$

(see [10, Lemma 2.2]), for which the Melnikov functions $\mathbf{f}_{i}$ 's and the stroboscopic averaged functions $\mathbf{g}_{i}$ 's can be computed. Then, in [10, Theorem 3.1], they showed that $\mathbf{f}_{i}=T \mathbf{g}_{i}$, for $i \in\{1, \ldots, \ell\}$, provided that $\mathbf{f}_{1}=\cdots=\mathbf{f}_{\ell-1}=0$.

Although less related with the present study, another interesting paper to be mention is [3], where the author considered planar autonomous differential equations given by

$$
\begin{equation*}
\dot{x}=X_{0}(x)+\varepsilon X(x, \varepsilon), \tag{2.1}
\end{equation*}
$$

for which the unperturbed system $\dot{x}=X_{0}(x)$ has a continuous period annulus $\mathcal{P} \subset \mathbb{R}^{2}$ without equilibria. A polar-like change of variables is employed in order to write the planar system as the standard form (1.1),

$$
\begin{equation*}
\frac{d h}{d \theta}=\varepsilon F(\theta, h, \varepsilon), \quad(\theta, h) \in[0,2 \pi) \times J, \tag{2.2}
\end{equation*}
$$

(see [3, Propositions 4 and 5]). The averaging method for scalar periodic equations, described in [3, Section 1], corresponds to the Melnikov method described in Section 1.2 of this present paper, where the averaged functions $f_{i}^{\prime}$ s are given as the coefficients of the expansion of the time- $T$ map of the differential equation (2.2) around $\varepsilon=0$. The Melnikov function method for planar systems, also described in [3, Section 1], consider a Poincaré map $P^{\gamma}$ of the autonomous differential equation (2.1) defined on an analytic transversal section given by $\Sigma=\{\gamma(h): h \in I\}$. Accordingly, the Melnikov functions $M_{i}$ 's are given as the coefficients of the expansion of the Poincaré map around $\varepsilon=0$, which may depend on both the section $\Sigma$ and its parametrization $\gamma$. As the conclusion of [3], it was showed that both procedure correspond to the study of some Poincaré map and that $M_{\ell}=f_{\ell}$, where $\ell$ is the index of the first non-vanishing Melnikov function.

## 3 Main result

The main result of this paper establishes a general relationship between the distinct notions of higher order averaged functions provided by the stroboscopic averaging method in Theorem 1.1 and by the Melnikov procedure in Lemma 1.3.

Theorem A. For $i \in\{1, \ldots, k\}$, the following recursive relationship between $\mathbf{g}_{i}$ and $\mathbf{f}_{i}$ holds:

$$
\begin{align*}
& \mathbf{g}_{1}(z)=\frac{1}{T} \mathbf{f}_{1}(z), \\
& \mathbf{g}_{i}(z)=\frac{1}{T}\left(\mathbf{f}_{i}(z)-\sum_{j=1}^{i-1} \sum_{m=1}^{j} \frac{1}{j!} d^{m} \mathbf{g}_{i-j}(z) \int_{0}^{T} B_{j, m}\left(\tilde{y}_{1}, \ldots, \tilde{y}_{j-m+1}\right)(s, z) d s\right), \tag{3.1}
\end{align*}
$$

where $\tilde{y}_{i}(t, z)$, for $i \in\{1, \ldots, k\}$, are polynomial in the variable $t$ recursively defined as follows:

$$
\begin{align*}
& \tilde{y}_{1}(t, z)=t \mathbf{g}_{1}(z) \\
& \tilde{y}_{i}(t, z)=i!t \mathbf{g}_{i}(z)+\sum_{j=1}^{i-1} \sum_{m=1}^{j} \frac{i!}{j!} d^{m} \mathbf{g}_{i-j}(z) \int_{0}^{t} B_{j, m}\left(\tilde{y}_{1}, \ldots, \tilde{y}_{j-m+1}\right)(s, z) d s \tag{3.2}
\end{align*}
$$

Theorem A is proven in Section 3.1. An Appendix is provided with Mathematica algorithms implementing the recursive formulae (1.4), (3.1), and (3.2) for computing both higher order averaging functions.

Remark 3.1. By applying the formula above for $i=2$, one has

$$
\mathbf{g}_{2}(z)=\frac{1}{T}\left(\mathbf{f}_{2}(z)-\frac{1}{2} d \mathbf{f}_{1}(z) \mathbf{f}_{1}(z)\right)
$$

where $\mathbf{f}_{2}$ is explicitly given by (1.6). Thus,

$$
\mathbf{g}_{2}(z)=\frac{1}{T} \int_{0}^{T}\left(F_{2}(t, z)+\partial_{x} F_{1}(t, z) \int_{0}^{t}\left(F_{1}(s, z)-\frac{1}{2} \mathbf{g}_{1}(z)\right) d s\right) d t
$$

which coincides with the expression provided by [20, Section 2.9.1].

### 3.1 Proof of Theorem A

From Theorem 1.1, there exists a T-periodic near-identity transformation $x=U(t, \xi, \varepsilon)$, satisfying $U(0, \xi, \varepsilon)=\xi$, such that the differential equation (1.1) is transformed into (1.3). Let $x(t, z, \varepsilon)$ and $\xi(t, z, \varepsilon)$ be, respectively, the solutions of (1.1) and (1.3) satisfying $x(0, z, \varepsilon)=\xi(0, z, \varepsilon)=z$. From Lemma 1.3,

$$
\begin{equation*}
x(T, z, \varepsilon)=z+\sum_{i=1}^{k} \varepsilon^{i} \mathbf{f}_{i}(z)+\mathcal{O}\left(\varepsilon^{k+1}\right) \tag{3.3}
\end{equation*}
$$

where $\mathbf{f}_{i}$, for $i \in\{1, \ldots, k\}$, are given by (1.5), and

$$
\begin{equation*}
\xi(T, z, \varepsilon)=z+\sum_{i=1}^{k} \varepsilon^{i} \tilde{\mathbf{f}}_{i}(z)+\mathcal{O}\left(\varepsilon^{k+1}\right) \tag{3.4}
\end{equation*}
$$

where, for $i \in\{1, \ldots, k\}$,

$$
\begin{equation*}
\tilde{\mathbf{f}}_{i}(z)=\frac{\tilde{y}_{i}(T, z)}{i!} \tag{3.5}
\end{equation*}
$$

and the functions $\tilde{y}_{i}$ 's are obtained recursively from (1.4) as (3.2):

$$
\begin{aligned}
& \tilde{y}_{1}(t, z)=t \mathbf{g}_{1}(z) \\
& \tilde{y}_{i}(t, z)=i!t \mathbf{g}_{i}(z)+\sum_{j=1}^{i-1} \sum_{m=1}^{j} \frac{i!}{j!} d^{m} \mathbf{g}_{i-j}(z) \int_{0}^{t} B_{j, m}\left(\tilde{y}_{1}, \ldots, \tilde{y}_{j-m+1}\right)(s, z) d s
\end{aligned}
$$

Now, taking the transformation $x=U(t, \xi, \varepsilon)$ into account, given $z \in D$, there exists $\hat{z} \in D$ such that $x(t, z, \varepsilon)=U(t, \xi(t, \hat{z}, \varepsilon), \varepsilon)$. By the stroboscopic condition, $U(0, \xi, \varepsilon)=\xi$, it follows that

$$
z=x(0, z, \varepsilon)=U(0, \xi(0, \hat{z}, \varepsilon), \varepsilon)=\xi(0, \hat{z}, \varepsilon)=\hat{z}
$$

In addition, since $U$ is $T$ periodic in the variable $t$, one also has that

$$
x(T, z, \varepsilon)=U(T, \xi(T, z, \varepsilon), \varepsilon)=\xi(T, z, \varepsilon)
$$

Thus, from (3.3) and (3.4), one obtains the following relationship

$$
\begin{equation*}
\mathbf{f}_{i}=\tilde{\mathbf{f}}_{i} \quad \text { for every } \quad i \in\{1, \ldots, k\} \tag{3.6}
\end{equation*}
$$

The proof follows by substituting (3.5) into the equality (3.6) and, then, isolating $\mathbf{g}_{i}(z)$ in the resulting relation by taking (3.2) into account.

### 3.2 Some consequences

Two main consequences of Theorem A are given in the sequel. The first one states that $\mathbf{f}_{i}=T \mathbf{g}_{i}$, for $i \in\{1, \ldots, \ell\}$, provided that either $\mathbf{f}_{1}=\cdots=\mathbf{f}_{\ell-1}=0$ or $\mathbf{g}_{1}=\cdots=\mathbf{g}_{\ell-1}=0$. This generalizes the result from [10] discussed in Section 2.

Corollary A. Let $\ell \in\{2, \ldots, k\}$. If either $\mathbf{f}_{1}=\cdots=\mathbf{f}_{\ell-1}=0$ or $\mathbf{g}_{1}=\cdots=\mathbf{g}_{\ell-1}=0$, then $\mathbf{f}_{i}=T \mathbf{g}_{i}$, for $i \in\{1, \ldots, \ell\}$.

Proof. First, assume that $\mathbf{g}_{1}=\cdots=\mathbf{g}_{\ell-1}=0$. Then, from (3.2), $\tilde{y}_{i}=0$, for $i \in\{1, \ldots, \ell-1\}$, and $\tilde{y}_{\ell}(t, z)=\ell!t \mathbf{g}_{\ell}(z)$. Thus, from (3.5) and (3.6), it follows that $\mathbf{f}_{i}=\tilde{\mathbf{f}}_{i}=0$, for $i \in\{1, \ldots$, $\ell-1\}$, and $\mathbf{f}_{\ell}=\tilde{\mathbf{f}}_{\ell}=T \mathbf{g}_{\ell}$.

Finally, assume that $\mathbf{f}_{1}=\cdots=\mathbf{f}_{\ell-1}=0$. Then, from (3.5), (3.2), and (3.6), one concludes that $\mathbf{g}_{i}$, for $i \in\{1, \ldots, \ell-1\}$, satisfy the following system of equations

$$
\begin{aligned}
& 0=T \mathbf{g}_{1}(z), \\
& 0=T \mathbf{g}_{i}(z)+\sum_{j=1}^{i-1} \sum_{m=1}^{j} \frac{1}{j!} d^{m} \mathbf{g}_{i-j}(z) \int_{0}^{t} B_{j, m}\left(\tilde{y}_{1}, \ldots, \tilde{y}_{j-m+1}\right)(s, z) d s .
\end{aligned}
$$

Hence, $\mathbf{g}_{i}=0$ (and, then, $\mathbf{f}_{i}=T \mathbf{g}_{i}$ ) for $i \in\{1, \ldots, \ell-1\}$. Consequently, applying (3.2) for $i=\ell$ and taking (3.5) and (3.6) into account, if follows that

$$
\mathbf{f}_{\ell}(z)=\tilde{\mathbf{f}}_{\ell}(z)=\frac{\tilde{y}_{\ell}(T, z)}{\ell!}=T \mathbf{g}_{\ell}(z) .
$$

Now, as a direct consequence of Corollary A, the first non-vanishing stroboscopic averaged function can be computed in relatively simple way. In particular, the following result holds:

Corollary B. Denote $\mathbf{f}_{0}=0$ and let $\ell \in\{1, \ldots, k\}$ satisfy $\mathbf{f}_{1}=\cdots=\mathbf{f}_{\ell-1}=0$. Then, there exists a smooth $T$-periodic near-identity transformation $x=U(t, \xi, \varepsilon)$ satisfying $U(\xi, 0, \varepsilon)=\xi$, such that the differential equation (1.1) is transformed into

$$
\xi^{\prime}=\varepsilon^{\ell} \frac{1}{T} \mathbf{f}_{\ell}(\xi)+\varepsilon^{\ell+1} r_{\ell}(t, \xi, \varepsilon)
$$

## Appendix A: Algorithms

This appendix is devoted to provide implemented Mathematica algorithm, based on recursive formulae (1.4), (3.1), and (3.2), for computing the higher order Melnikov functions and the higher order stroboscopic averaged functions.

In what follows, $F_{i}(t, x)$ is denoted by $\mathrm{F}[i, \mathrm{t}, \mathrm{x}], y_{i}(t, x)$ is denoted by $\mathrm{y} 0[\mathrm{i}, \mathrm{t}], \tilde{y}_{i}(t, x)$ is denoted by $\mathrm{y} 1[i, t], \mathbf{f}_{i}(z)$ is denoted by $f[i, z]$, and $\mathbf{g}_{i}(z)$ is denoted by $\mathrm{g}[i, z]$. The order of perturbation k must be specified in order to run the code.

Listing 1: Mathematica's algorithm for computing $\mathbf{f}_{i}$

```
y0[1, t_] = Integrate[F[1, s, z], {s, 0, t }];
YO[1] = {y0[1, t]};
For[i=2,i}<=k,i++
y0[i, t_]:= Integrate[i! F[i, s, z] + Sum[Sum[i!/j! D[F[i - j, t, z], {z, m}] BelIY[j, m, Y0[j - m + 1]], {
    m, 1, j}], {j, 1, i - 1}], {s, 0, t}];
Y0[i] = Join[YO[i - 1], {y0[i, t]}];
f[i, z_] = y0[i, T]/i !];
```

Listing 2: Mathematica's algorithm for computing $\mathbf{g}_{i}$

```
\(\mathrm{g}\left[1, z_{-}\right]=\mathrm{f}[1, \mathrm{z}] / \mathrm{T} ;\)
\(\mathrm{y} 1[1, \mathrm{t}]=\mathrm{tg}[1, \mathrm{z}]\);
\(\mathrm{Y} 1[1, \mathrm{t}]=\{\mathrm{y} 1[1, \mathrm{t}]\}\)
For \([i=2, i<=k, i++\),
\(\mathrm{g}\left[\mathrm{i}, \mathrm{z}_{-}\right]=1 / \mathrm{T}(f[\mathrm{i}, \mathrm{z}]-\operatorname{Sum}[\operatorname{Sum}[1 / \mathrm{j}!\mathrm{D}[\mathrm{g}[\mathrm{i}-\mathrm{j}, \mathrm{z}],\{\mathrm{z}, \mathrm{m}\}]\) Integrate[BellY[j, m, Y1[j\(-\mathrm{m}+1, \mathrm{~s}]], \quad\{\mathrm{s}, 0\),
    \(T\}],\{m, 1, j\}],\{j, 1, i-1\}]) ;\)
\(y 1[i, t]]=i!t g[i, z]+\operatorname{Sum}[\operatorname{Sum}[i!/ j!\mathrm{D}[g[i-j, z], \quad\{z, m\}]\) Integrate[BellY[j, m, Y1[j \(-m+1, s]],\{s, 0\),
    \(T\}],\{m, 1, j\}],\{j, 1, i-1\}] ;\)
\(\mathrm{Y} 1[\mathrm{i}, \mathrm{t}]\) ] \(=\operatorname{Join}[\mathrm{Y} 1[\mathrm{i}-1, \mathrm{t}], \quad\{\mathrm{y} 1[\mathrm{i}, \mathrm{t}]\}]] ;\)
```


## Acknowledgements

The author thanks the referee for the constructive comments and suggestions which led to an improved version of the manuscript.

The author was partially supported by São Paulo Research Foundation (FAPESP) grants 2018/16430-8, 2018/ 13481-0, and 2019/10269-3, and by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) grants 306649/2018-7 and 438975/2018-9.

## References

[1] N. N. Bogoliubov, Y. A. Mitropolsky, Asymptotic methods in the theory of non-linear oscillations, Translated from the second revised Russian edition, International Monographs on Advanced Mathematics and Physics, Hindustan Publishing Corp., Delhi, Gordon and Breach Science Publishers, New York, 1961. MR0141845
[2] N. Bogolyubov, O nekotoryh statističeskih metodah v matematičeskol̆ fizike [On some statistical methods in mathematical physics], Akademiya Nauk Ukrainskoŭ SSR, Kiev, 1945. MR0016575
[3] A. Buică, On the equivalence of the Melnikov functions method and the averaging method, Qual. Theory Dyn. Syst. 16(2017), No. 3, 547-560. https://doi.org/10.1007/ s12346-016-0216-x; MR3703514
[4] A. Buică, J. Llibre, Averaging methods for finding periodic orbits via Brouwer degree, Bull. Sci. Math. 128(2004), No. 1, 7-22. https://doi.org/10.1016/j.bulsci. 2003.09. 002; MR2033097
[5] J. A. Cid, J. Mawhin, M. Zima, An abstract averaging method with applications to differential equations, J. Differential Equations 274(2021), 231-250. https://doi.org/10.1016/ j.jde.2020.11.051; MR4186657
[6] J. A. Ellison, A. W. Sáenz, H. S. Dumas, Improved Nth order averaging theory for periodic systems, J. Differential Equations 84(1990), No. 2, 383-403. https://doi. org/10. 1016/0022-0396(90)90083-2; MR1047574
[7] P. Fatou, Sur le mouvement d'un système soumis à des forces à courte période, Bull. Soc. Math. France 56(1928), 98-139. https://doi.org/10.24033/bsmf.1131; MR1504928
[8] J. K. Hale, Integral manifolds of perturbed differential systems, Ann. of Math. (2) 73(1961), 496-531. https://doi.org/10.2307/1970314; MR0123786
[9] J. K. Hale, Ordinary differential equations, Second edition, Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., 1980. MR0587488
[10] M. Han, V. G. Romanovski, X. Zhang, Equivalence of the Melnikov function method and the averaging method, Qual. Theory Dyn. Syst. 15(2015), No. 2, 471-479. https:// doi.org/10.1007/s12346-015-0179-3; MR3563430
[11] N. Krylov, N. Bogolyubov, Prilozhenie metodov nelineinoi mekhaniki $k$ teorii statsionarnykh kolebanii [The application of methods of nonlinear mechanics to the theory of stationary oscillations], Akademiya Nauk Ukrainsko1̆ SSR, Kiev, 1934.
[12] J. Llibre, D. D. Novaes, C. A. B. Rodrigues, Averaging theory at any order for computing limit cycles of discontinuous piecewise differential systems with many zones, Phys. D 353/354(2017), 1-10. https://doi.org/10.1016/j.physd.2017.05.003; MR3669353
[13] J. Llibre, D. D. Novaes, M. A. Teixeira, Higher order averaging theory for finding periodic solutions via Brouwer degree, Nonlinearity 27(2014), 563-583. https://doi. org/ 10.1088/0951-7715/27/3/563; MR3177572
[14] J. Llibre, D. D. Novaes, M. A. Teixeira, On the birth of limit cycles for non-smooth dynamical systems, Bull. Sci. Math. 139(2015), No. 3, 229-244. https: //doi . org/10.1016/ j.bulsci.2014.08.011; MR3336682
[15] Y. A. Mitropolskir, N. V. Dao, Applied asymptotic methods in nonlinear oscillations. Solid mechanics and its applications, Vol. 55, Kluwer Academic Publishers Group, Dordrecht, 1997. https://doi.org/10.1007/978-94-015-8847-8; MR1490834
[16] D. D. Novaes, An equivalent formulation of the averaged functions via Bell polynomials, in: Extended abstracts Spring 2016-nonsmooth dynamics, Vol. 8 of Trends Math. Res. Perspect. CRM Barc., Birkhäuser/Springer, Cham, 2017, pp. 141-145. https://doi.org/10.1007/ 978-3-319-55642-0_25; MR3741174
[17] D. D. Novaes, An averaging result for periodic solutions of Carathéodory differential equations, Proc. Amer. Math. Soc., published online, 2021. https://doi.org/10.1090/ proc/15810
[18] D. D. Novaes, F. B. Silva, Higher order analysis on the existence of periodic solutions in continuous differential equations via degree theory, SIAM J. Math. Anal. 53(2021), No. 2, 2476-2490. https://doi.org/10.1137/20M1346705; MR4249059
[19] L. M. Perko, Higher order averaging and related methods for perturbed periodic and quasi-periodic systems, SIAM J. Appl. Math. 17(1969), 698-724. https://doi.org/10. 1137/0117065; MR0257479
[20] J. Sanders, F. Verhulst, J. Murdock, Averaging methods in nonlinear dynamical systems, Second edition, Applied Mathematical Sciences, Vol. 59, Springer, New York, 2007. https://doi.org/10.1007/978-0-387-48918-6; MR2316999
[21] F. Verhulst, Nonlinear differential equations and dynamical systems, Second edition, Universitext, Springer-Verlag, Berlin, 1996. https://doi.org/10.1007/978-3-642-61453-8; MR1422255

Electronic Journal of Qualitative Theory of Differential Equations

# A converse of Sturm's separation theorem 

Leila Gholizadeh and Angelo B. Mingarelli ${ }^{\boxtimes}$

School of Mathematics and Statistics, Carleton University, Ottawa, Canada

Received 6 May 2021, appeared 11 October 2021
Communicated by Leonid Berezansky


#### Abstract

We show that Sturm's classical separation theorem on the interlacing of the zeros of linearly independent solutions of real second order two-term ordinary differential equations necessarily fails in the presence of a turning point in the principal part of the equation. Related results are discussed.


Keywords: Sturm separation theorem, recurrence relations, Sturm separation property, indefinite principal part, indefinite leading term.
2020 Mathematics Subject Classification: 34B24, 34C10, 47B50.

## 1 Introduction

In the sequel we will always assume, unless otherwise stated that

$$
\begin{equation*}
\frac{1}{p}, q \in L(I), \quad[a, b] \subset I \tag{1.1}
\end{equation*}
$$

where $I$ is a closed and bounded interval and the functions $p, q: I \rightarrow \mathbb{R}$. In this paper there are generally no sign restrictions on the principal part of (1.2), i.e., the values, $p(x)$, are generally unrestricted as to their sign and $p(x)$ may even be infinite on sets of positive measure. As usual the symbol $\|*\|_{1}$ will denote the $L(I)$ norm.

It is well known [5] that the conditions (1.1) imply the existence and uniqueness of Carathéodory solutions of initial value problems associated with (1.2),

$$
\begin{equation*}
-\left(p(x) y^{\prime}\right)^{\prime}+q(x) y=0, \quad x \in[a, b] \tag{1.2}
\end{equation*}
$$

that is, solutions $y$ such that both $y$ and $p y^{\prime}$ are absolutely continuous on $[a, b]$ and satisfy

$$
\begin{equation*}
y(a)=y_{a}, \quad p y^{\prime}(a)=y_{a^{\prime}}, \tag{1.3}
\end{equation*}
$$

for given $y_{a}, y_{a^{\prime}}$. The study of problems with an indefinite leading term (a.k.a. an indefinite principal part) are few and far between. For example, the failure of Sturm's oscillation theorem in such indefinite cases was observed in [2, p. 381] where, in the presence of an indefinite weight function, it may occur that the spectrum is, in fact, the whole complex plane and the

[^40]eigenfunctions behave in a totally non-Sturmian fashion. The example in question consists in choosing $p(x)=q(x)=\operatorname{sgn} x$ for $x \in[-1,1], y(-1)=y(1)=0$. Then the two solutions $y_{1}(x)=\sin P(x)$ and $y_{2}(x)=\cos P(x)$ where $P(x)=|x|-1$ have non interlacing zeros. Indeed, $y_{2}(x) \neq 0$ on $[-1,1]$ while $y_{1}(x)$ vanishes at both ends there. This special case is contained in Theorem 2.2 below.

Recall that, in its simplest most classical form, Sturm's Separation Theorem states that given any non-trivial solution $y$ of (1.2) having consecutive zeros at $a, b, a<b$, where $[a, b] \subset I$ then every other linearly independent solution of (1.2) must vanish only once in (a,b). An equation (1.2) is said to have the Sturm Separation Property (abbr. SSP) on $[a, b]$ if Sturm's Separation Theorem holds for the given equation on the given interval.

The framework described above normally assumes that the principal part, $p$, appearing in (1.2) is a.e. finite on $[a, b]$. However, still greater generality can be obtained by allowing $p(x)$ to be identically infinite on subintervals. In this case one needs to rewrite (1.2) as a vector system in two dimensions, e.g.,

$$
\begin{equation*}
u^{\prime}=\frac{v}{p^{\prime}} \quad v^{\prime}=q u . \tag{1.4}
\end{equation*}
$$

This now defines a problem of Atkinson-type (see [1, Chapter 8], [3, p. 558] for more details). The advantage of using this formulation is that it can be used to study three-term recurrence relations as well, see [1], [8]. We summarize this approach briefly: we divide $[a, b]$ into a finite union of subintervals

$$
\begin{equation*}
\left[a, b_{0}\right],\left[b_{0}, a_{1}\right],\left[a_{1}, b_{1}\right],\left[b_{1}, a_{2}\right],\left[a_{2}, b_{2}\right], \ldots,\left[b_{m-1}, a_{m}\right],\left[a_{m}, b\right] . \tag{1.5}
\end{equation*}
$$

on each of which alternately $p(x)=\infty$ or $q(x)=0$ (but $p(x)$ is not infinite when $q(x)=0$ ). Direct integration of (1.4) then shows that $y_{n}=u\left(a_{n}\right)$ satisfies the three-term recurrence relation

$$
\begin{equation*}
c_{n} y_{n+1}+c_{n-1} y_{n-1}-d_{n} y_{n}=0 \tag{1.6}
\end{equation*}
$$

where

$$
c_{n}^{-1}=\int_{b_{n}}^{a_{n+1}} \frac{d s}{p(s)}, \quad d_{n}=c_{n}+c_{n+1}+\int_{a_{n}}^{b_{n}} q(s) d s,
$$

or, equivalently, a second order difference equation

$$
\begin{equation*}
-\triangle\left(c_{n-1} \triangle y_{n-1}\right)+\left(\int_{a_{n}}^{b_{n}} q(s) d s\right) y_{n}=0 \tag{1.7}
\end{equation*}
$$

where, as usual, $\triangle$ represents the forward difference operator $\Delta y_{n}=y_{n+1}-y_{n}$.
Recall that by a zero of a solution of (1.6) is meant the zero of that absolutely continuous polygonal curve with vertices at $\left(n, y_{n}\right)$. (This interpretation arises directly by integrating (1.4).) Thus, zeros of solutions of (1.6) are said to interlace if the corresponding polygonal curves have interlacing zeros.

The failure of Sturm's Separation Theorem (or SSP) in the case of recurrence relations (or difference equations) is old but chronicled by both Bôcher [4] and Moulton, [9], and not independently of one another. (Moulton [9] actually cites Bôcher in reference to the question.) Bôcher [4] goes on to give, as an example, two independent solutions of the Fibonacci sequence recurrence relation,

$$
y_{n+1}=y_{n}+y_{n-1}, \quad y_{-1}=0, y_{0}=1 ; \quad y_{-1}=-10, y_{0}=6,
$$

with no interlacing features whereas Moulton [9] went on to show (at Bôcher's prodding) that (1.6) has the SSP provided $c_{n} c_{n-1}>0$ for all $n$. To the best of our knowledge, a converse of

Sturm's Separation Theorem has not been addressed. In [8, p. 209] we showed, by means of an example, that the SSP may fail in the case where Moulton's condition $c_{n} c_{n-1}>0$ fails.

This failure suggests that $p(x)$ must change its sign in the continuous case and that intervals in which violations to SSP occur must be neighborhoods of a "turning point" of $p$. The existence of such a point is necessitated by the fact that otherwise $p(x)$ would be (a.e.) of one sign in $[a, b]$ and so SSP must hold there by Sturmian arguments.

Below we present a converse to SSP as a consequence of more general results dealing with (1.4). Said result will then apply to both differential and difference equations.

Specifically, we will prove (Theorem 2.2) that whenever the leading term $p(x)$ has a turning point in $(a, b)$ then SSP must fail. This is equivalent to showing that if the SSP holds then $p(x)$ cannot have a turning point inside $(a, b)$ and thus $p(x)$ is a.e. of one sign. This is the actual converse of Sturm's Separation Theorem. We illustrate this result by means of explicit examples.

We also present (Theorem 2.10) an effective necessary condition for the existence of a solution vanishing at the end-points of an interval in the case of sign-indefinite $p$ and $q$. Examples are provided illustrating the various theorems. Other results of independent interest are also presented thus demonstrating the complexities of qualitative behavior of solutions in the case of indefinite leading terms.

We conclude by a conjecture which gives an upper and lower bound to the difference in the number of zeros in $[a, b]$ between two independent solutions in the case of an arbitrary but finite number of turning points in $p(x)$.

## 2 Main results

We recall that if $p$ is continuous or piecewise continuous on $[a, b]$ then a turning point is a point $c \in(a, b)$ around which $p(x)$ changes its sign. If $p$ is merely measurable then $c$ is defined by requiring that, in some interval containing $c$ in its interior, we have $(x-c) p(x)>0$ a.e. (or $(x-c) p(x)<0$ a.e.) This somewhat restrictive definition implies that the set of turning points of $p$ cannot be everywhere dense in $(a, b)$. Indeed, this definition implies that turning points must be separated from one another.

In the sequel we always assume that solutions of (1.4) or (2.1) below are deemed nontrivial. In addition, we take it that $1 / p(x)$ may vanish a.e. on sets of positive measure, but not vanish a.e. on $[a, b]$, and that $p(x)$ is unrestricted as to its sign there.

Lemma 2.1. For $i=1,2$, let $u_{1}, u_{2}$ be solutions of

$$
\begin{equation*}
u_{i}^{\prime}=\frac{v_{i}}{p}, \quad v_{i}^{\prime}=q u_{i}, \tag{2.1}
\end{equation*}
$$

where $p, q$ satisfy (1.1). Then

$$
\begin{equation*}
u_{2}(x) v_{1}(x)-u_{1}(x) v_{2}(x)=C, \tag{2.2}
\end{equation*}
$$

where $C$ is a constant.
We will assume that, without loss of generality, $C=1$. The main result shows that SSP fails whenever $p$ has a turning point and thus the a.e. positivity (or negativity) of $p(x)$ is a necessary condition for the validity of SSP as well as sufficient, as is well known.

Theorem 2.2. Let $p(x)$ have a unique turning point at $x=c, a<c<b$ and let $u_{i}, i=1,2$ be linearly independent solutions of (2.1) such that $u_{1}(a)=u_{1}(b)=0, u_{1}(x) \neq 0$ in $(a, b)$. Then either
$u_{2}(x) \neq 0$ on $[a, b]$, or $u_{2}(x)$ is of constant sign except only at $x=c$ where $u_{2}(c)=0$, or finally $u_{2}(x)$ has exactly two zeros in $(a, b)$. In every case it follows that SSP fails on $[a, b]$.

Remark 2.3. The previous theorem is independent of the sign of $q(x)$ and assumes only that a solution exists vanishing at two points around a given turning point. This is the general case as otherwise the existence of two consecutive zeros in a turning-point-free set would lead to SSP there by classical Sturm theory since $p(x)$ is a.e. of one sign.

The result also includes an analog for the difference equation (1.7) above. Basically, if the $c_{n}$ change sign once, then the solutions, viewed as polygonal curves, have the property stated in the theorem.

The next example illustrates the result in the continuous case.
Example 2.4. Let $I=[0, \pi]$, and consider the differential equation

$$
u^{\prime}=\cos (x) v, \quad v^{\prime}=-\cos (x) u
$$

with a unique turning point at $c=\pi / 2$. Then the general solution is

$$
y(x)=c_{1} \sin (\sin x)+c_{2} \cos (\sin (x))
$$

where $c_{1}, c_{2}$ are constants. First, note that solution $u_{1}(x)=\sin (\sin x)$ satisfies the conditions of Theorem 2.2. We now exhibit solutions of the type guaranteed by said theorem.

- The solution $u_{2}(x)=\cos (\sin x)$ has no zeros in $[0, \pi]$.
- The solution $u_{2}(x)=-\cos 1 \sin (\sin x)+\sin 1 \cos (\sin (x)) \geq 0$ on $[0, \pi]$ and it has exactly one zero at the turning point $x=\pi / 2$ bouncing positively there.
- The solution $u_{2}(x)=\cos (\sin x)-\sin (\sin x)$ has exactly two zeros, in conformity with said theorem.
- Every solution of this equation has at most two zeros.

The latter result is most readily proved by contradiction. Assuming three such zeros $x_{i}$, $i=1,2,3, x_{i} \in[0, \pi]$, we can easily deduce that the three quantities $\tan \left(\sin \left(x_{i}\right)\right)$ have a common value (i.e., independent of $i$ ) and this is impossible on $[0, \pi]$. As a result, SSP fails for this equation.

Next we consider the problem of finding necessary and sufficient conditions for the existence of two zeros of (1.4) on $[a, b]$, i.e., in particular, we are asking for conditions under which this equation not disconjugate. For the notion of disconjugacy we refer the reader to [3,7].

Theorem 2.5. The equation (1.4) with $q(x)=0$ a.e. on $[a, b]$ has a non-trivial solution satisfying $u(a)=u(b)=0$ if and only if

$$
\begin{equation*}
\int_{a}^{b} \frac{d s}{p(s)}=0 \tag{2.3}
\end{equation*}
$$

Corollary 2.6. Let $c_{n}$ satisfy

$$
\begin{equation*}
\sum_{n=0}^{m-1} c_{n}^{-1}=0 \tag{2.4}
\end{equation*}
$$

Then SSP fails for three-term recurrence relations of the form

$$
\begin{equation*}
c_{n} y_{n+1}+c_{n-1} y_{n-1}-\left(c_{n}+c_{n-1}\right) y_{n}=0 \tag{2.5}
\end{equation*}
$$

Example 2.7. Let $c_{n-1}=(-1)^{n}, n=0, \ldots, m$, where $m$ is even. Then (2.5) reduces to $y_{n+1}=$ $y_{n-1}$. This has two linearly independent solutions defined by the initial conditions, $y_{-1}=0$, $y_{0}=1$ and $y_{-1}=1, y_{0}=2$ the former of which has numerous zeros while the second has none. We can see that SSP fails both by direct computation and by Corollary 2.6.

On the other hand, the same initial conditions $y_{-1}=0, y_{0}=1$ and $y_{-1}=1, y_{0}=2$ for the slightly modified recurrence relation $y_{n+1}=-y_{n-1}$ gives two solutions satisfying SSP by Moulton's theorem, [9].

The separation property for the zeros of the quasi-derivatives of solutions, i.e., terms of the form $\left(p y^{\prime}\right)(x)$, is next. Although the result is simply proved we have been unable to find a reference to it and so present it here for the sake of completeness.
Proposition 2.8. For $p, q$ as in (1.1), let $p(x)$ be sign indefinite. In addition, let $q(x)$ be a.e. of one sign on $[a, b]$ and let $y$ be a non-trivial solution of (1.2) satisfying

$$
\begin{equation*}
\left(p y^{\prime}\right)(a)=0=\left(p y^{\prime}\right)(b) . \tag{2.6}
\end{equation*}
$$

Then for any linearly independent solution $y_{1}$ of (1.2) there is exactly one point $c \in(a, b)$ such that $\left(p y_{1}^{\prime}\right)(c)=0$.
Remark 2.9. This proposition seems to be the closest that one can get to a SSP-type result for positive $q$. In other words, as we have seen earlier, the SSP fails even if $q(x)>0$ on $[a, b]$, and $p(x)$ is sign indefinite (i.e., has a turning point in $(a, b)$ ).

Next, we give a necessary condition for the existence of a solution vanishing at the endpoints of a typical interval, $[a, b]$, and positive in its interior in the presence of an indefinite principal part or leading term, $p(x)$, in (1.4).
Theorem 2.10. Let $\|q\|_{1}>0$ and let (2.3) hold. Let $u$ be a solution of (1.4) such that $u(a)=u(b)=0$, and $u(x)>0$ for $x \in(a, b)$. Then, writing,

$$
\begin{equation*}
P(x):=\int_{a}^{x} \frac{d s}{p(s)}, \tag{2.7}
\end{equation*}
$$

either $P(x) q(x)=0 \quad$ a.e. on $(a, b)$ or there is a set of positive measure on which $P(x) q(x)>0$ a.e. in $(a, b)$ and a set of positive measure on which $P(x) q(x)<0$ a.e. in $(a, b)$ (i.e., Pq changes its "sign" on ( $a, b$ ).)

The next result is of independent interest, Example 2.4 being a special case.
Lemma 2.11. Let $I=[a, b], \lambda>0$. The general solution of either

$$
\left(p y^{\prime}\right)^{\prime}+\frac{\lambda}{p} y=0, \quad \text { or } \quad u^{\prime}=\frac{v}{p}, \quad v^{\prime}=-\frac{\lambda}{p} u .
$$

is given by

$$
y(x)=u(x)=c_{1} \cos (\sqrt{\lambda} P(x))+c_{2} \sin (\sqrt{\lambda} P(x))
$$

where

$$
v(x)=-c_{1} \sqrt{\lambda} \sin (\sqrt{\lambda} P(x))+c_{2} \sqrt{\lambda} \cos (\sqrt{\lambda} P(x))
$$

where $c_{1}, c_{2}$ are constants.
Remark 2.12. It is well known and easy to derive that in the case where the leading term $p(x)$ is a.e. positive (or negative) then the absolute value of the difference of the number of zeros of two independent solutions is equal to 1, due to the interlacing property of such zeros. In the case of an indefinite leading term we make the following conjecture.

## 3 Conjecture

Let $p(x)$ have at least one turning point in $(a, b)$ and let $y$ be a solution satisfying $y(a)=$ $y(b)=0$ having $n$ zeros in $[a, b]$. Then, given any integer $k, 0 \leq k \leq n$, there are examples for which the absolute value of the difference of the number of zeros of two independent solutions on $[a, b]$ is equal to $k$.

This totally non-Sturmian behavior appears to be typical in cases where the principal part changes sign.

## 4 Proofs

Proof of Lemma 2.1. The proof is by differentiation of the expression on the left of (2.2) making use of (2.1). Note that all $u_{i}, v_{i}$, and so their products, are absolutely continuous on the interval under consideration.

Proof of Theorem 2.2. There are only two logical possibilities. Either $u_{2}(x) \neq 0$ in $[a, b]$ or $u_{2}(x)=0$ at $x=x_{0}$ in $(a, b]$. Clearly $u_{2}(a) \neq 0$ as its negation would violate (2.2). For the sake of simplicity we may assume that $u_{2}(a)>0$ (or else we may replace $u_{2}$ by $-u_{2}$ in the ensuing discussion along with other minor changes).

In addition, we may assume, without loss of generality, that this first zero is at, say $x_{0} \in$ $(a, c)$, that is, to the left of the turning point. A similar argument applies in the event that this zero is in $(c, b]$. Thus, $u_{2}(x) \geq 0$ for $x \in\left[a, x_{0}\right)$.

Next, we show that, unless $x_{0}=c$ (see below), $u_{2}(x)$ cannot "bounce" off $x=x_{0}$ and remain positive for some $x>x_{0}$. To see this observe that (2.2) implies that $v_{2}\left(x_{0}\right)<0$. The continuity of $v_{2}$ now implies the existence of a $\delta>0$ and a neighborhood $J=\left(x_{0}-\delta, x_{0}+\delta\right) \in$ $(a, c)$ in which $v_{2}(x)<0$. It follows that, for $x \in\left(x_{0}, x_{0}+\delta\right)$,

$$
u_{2}(x)=\int_{x_{0}}^{x} \frac{v_{2}(s)}{p(s)} d s
$$

Since $p(x)>0$ a.e. in $J$ and $v_{2}(x)<0$ there as well, we see that $u_{2}(x)<0$ to the right of $x_{0}$ and thus $u_{2}$ must cross the axis whenever it is zero. Summarizing, we have shown that there exists a $\delta>0$ such that $u_{2}(x)>0$ on $\left[a, x_{0}\right)$ and $u_{2}(x)<0$ on $\left(x_{0}, x_{0}+\delta\right)$. Now, since $p(x)>0$ a.e. in $[a, c]$, by ordinary Sturm theory we get that it is impossible for $u_{2}(x)=0$ again in $\left(x_{0}+\delta, c\right]$. This is because SSP applies on intervals in which $p(x)$ is a.e. of one sign, and so $u_{2}(x)$ can have at most one zero there. It follows that $u_{2}(c)<0$.

As before we know that $(2.2)$ forces $u_{2}(b) \neq 0$. We show that $u_{2}(b)>0$. Assume the contrary, i.e., $u_{2}(b)<0$. Since $p(x)<0$ a.e. on $(c, b)$ we have from (2.2) that $u_{2}(b) v_{1}(b)=1$ and so that $v_{1}(b)<0$. A continuity argument again implies the existence of a $\eta>0$ such that $v_{1}(x)<0$ for $x \in(b-\eta, b)$. For such $x$,

$$
u_{1}(b)-u_{1}(x)=-u_{1}(x)=\int_{x}^{b} \frac{v_{1}(s)}{p(s)} d s
$$

However, $p(x)<0$ a.e. in $(b-\eta, b)$. Hence $u_{1}(x)<0$ in $(b-\eta, b)$ and this contradicts the fact that $u_{1}(x)>0$ on $(a, b)$. Hence $u_{2}(b) \geq 0$. As before, the case $u_{2}(b)=0$ being excluded by (2.2), we find that $u_{2}(b)>0$. Since $u_{2}$ is continuous and $u_{2}(c)<0$ there must exist another zero $x_{1} \in(c, b)$. This zero must be unique by Sturm theory since $p(x)$ is a.e. of one sign on $(c, b)$, i.e., SSP applies here.

Finally, let us consider the case where $x_{0}=c$, that is, the first zero of $u_{2}$ occurs at the turning point itself. This case may occur and a bounce is possible here. The reason for this is that previous argument fails on account that $p(x)$ a.e. changes its sign on every interval of the form $(c-\delta, c+\delta)$, by definition. Since $p(x)<0$ a.e. on $(c, c+\delta)$, and arguing as above, we get that for all $x \in(c, c+\delta)$ and $\delta$ sufficiently small,

$$
u_{2}(x)=\int_{c}^{x} \frac{v_{2}(s)}{p(s)} d s>0
$$

Thus, a bounce may occur there. Finally, $u_{2}(x)$ may not vanish again in $(c, b)$ since $p(x)$ is a.e. of one sign and so can only have at most one zero in $[c, b]$ by Sturm theory. This completes the proof.

Prof of Corollary 2.6. This follows from the discussion leading to the recurrence relations.
Proof of Proposition 2.8. We use the so-called reciprocal transformation [3]: let $z=p y^{\prime}$ where $y$ satisfies (1.2). Then $z$ satisfies the equation

$$
-\left(\frac{1}{q} z^{\prime}\right)^{\prime}+\frac{1}{p} z=0
$$

and

$$
z(a)=z(b)=0 .
$$

Since $q$ is a.e. of one sign, classical Sturmian results apply so that the previous equation has the SSP on said interval. Thus, for any other linearly independent solution $z_{1}(x)$ there is a unique $c \in(a, b)$ such that $z_{1}(c)=0$. In particular, if we define a solution $y_{1}$ via $z_{1}=p y_{1}^{\prime}$, then $z_{1}(c)=0$ for some $c$, and the result follows.

Proof of Theorem 2.10. Without loss of generality we can assume that $u(a)=0, v(a)=M$ where $M \neq 0$ is arbitrary but fixed. Then

$$
\begin{aligned}
u(x) & =M \int_{a}^{x} \frac{d s}{p(s)}+\int_{a}^{x} \frac{1}{p(s)} \int_{a}^{s} q(t) u(t) d t d s \\
& =M \int_{a}^{x} \frac{d s}{p(s)}+P(x) \int_{a}^{x} q(t) u(t) d t-\int_{a}^{x} P(t) q(t) u(t) d t
\end{aligned}
$$

Since $u(b)=0$ and $P(b)=0$, it follows that

$$
\int_{a}^{b} P(t) q(t) u(t) d t=0
$$

and, since $u(x)>0$ in $(a, b)$, the result follows.
Proof of Lemma 2.11. This is a direct calculation and so the proof is omitted.

Note added in proof: For an extension of some of the main results of this paper to the case of finitely many turning points, see [6].

## Acknowledgements

The authors should like to thank Mr. Seyifunmi Ayeni, of Carleton University, who provided the basis for some of the examples considered here during a summer research project at Carleton University under the supervision of the second author. We must also thank the referee for a careful reading of the manuscript.

## References

[1] F. V. Atkinson, Discrete and continuous boundary problems, Academic Press, New York, (1964), xiv, 570 pp. MR176141; Zbl 0117.05806
[2] F. V. Atkinson, A. B. Mingarelli, Asymptotics of the number of zeros and of the eigenvalues of general weighted Sturm-Liouville problems, J. Reine Angew. Math., 375/376(1987), 380-393. MR882305; Zbl 1222.34001
[3] J. H. Barrett, Disconjugacy of second order linear differential equations with nonnegative coefficients, Proc. Amer. Math. Soc. 10(1959), 552-561. https://doi.org/10. 2307/2033650; MR108613; Zbl 0092.08103
[4] M. Bôcher, Boundary problems and Green's functions for linear differential and linear difference equations, Ann. of Math. (2) 13(1911-12), 71-88. https://doi.org/10.2307/ 1968072; MR1502418; Zbl 42.0332.02
[5] W. N. Everitt, D. Race, On necessary and sufficient conditions for the existence of Carathéodory solutions of ordinary differential equations, Quaestiones Math 2(1978), 507512. MR477222; Zbl 0392.34002
[6] L. Gholizadeh, A. B. Mingarelli, The converse of Sturm's separation theorem, available on arXiv:2109.06953 [math.CA], 14 Sep 2021.
[7] P. Hartman, Ordinary differential equations, Wiley, NY, (1964). MR171038; Zbl 0125.32102
[8] A. B. Mingarelli, Volterra-Stieltjes integral equations and generalized ordinary differential expressions, Lecture Notes in Mathematics, Vol. 989, Springer, Berlin, 1983. https://doi. org/10.1007/BFb0070768; MR706255; Zbl 0516.45012
[9] E. J. Moulton, A theorem in difference equations on the alternation of nodes of linearly independent solutions, Ann. of Math. (2) 13(1911-12), 137-139. https://doi.org/ 10.2307/1968079; MR1502425; Zbl 43.0415.03

# Special cases of critical linear difference equations 

Jan Jekl ${ }^{\boxtimes}$<br>Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, CZ-611 37 Brno, Czech Republic

Received 15 April 2021, appeared 11 October 2021
Communicated by Stevo Stević


#### Abstract

In this paper, we investigate even-order linear difference equations and their criticality. However, we restrict our attention only to several special cases of the general Sturm-Liouville equation. We wish to investigate on such cases a possible converse of a known theorem. This theorem holds for second-order equations as an equivalence; however, only one implication is known for even-order equations. First, we show the converse in a sense for one term equations. Later, we show an upper bound on criticality for equations with nonnegative coefficients as well. Finally, we extend the criticality of the second-order linear self-adjoint equation for the class of equations with interlacing indices. In this way, we can obtain concrete examples aiding us with our investigation.


Keywords: critical equations, linear difference equations, equations with interlacing indices.

2020 Mathematics Subject Classification: 39A06, 39A21, 47B36.

## 1 Introduction

The concept of criticality for second-order equations was developed in [15] and for equations of general even-order in [8]. It is established for continuous case as well which the reader can find for example in $[14,16,27,32,36-38]$ and in other references. This work was intended as an attempt to investigate a converse of the main result obtained in [8] through observing subclasses of the Sturm-Liouville difference equation. We obtain several new properties of said subclasses and concrete examples whose behaviour motivates further research.

Section 2 contains a summary of necessary definitions and theorems together with some minor improvements. Nevertheless, it is worth pointing out that critical linear equations create a subclass of disconjugated equations. When we work with second-order equations we have only two options, that a disconjugated equation is either critical or subcritical. For higher-order equations of order $2 k$ we have to separate this approach into subsequent cases, that equations can be $p$-critical for $0 \leq p \leq k, p \in \mathbb{Z}$ and when it is 0 -critical we say that the equation is subcritical.

In Section 3 we work with the one term linear equation

$$
\begin{equation*}
(-\triangle)^{k}\left(r_{n} \triangle^{k} y_{n-k}\right)=0, \quad r_{n}>0, \quad n \in \mathbb{Z} \tag{1.1}
\end{equation*}
$$

[^41]Eq. (1.1) gives a subclass of the general Sturm-Liouville equation where only one of the coefficients is non-zero. Our main result of Section 3 considers a situation where we make one of the zero coefficients of Eq. (1.1) arbitrarily smaller. When this change leads to the situation where Eq. (1.1) loses disconjugacy, then the original Eq. (1.1) is at least $p$-critical where the assumptions give the number $p$. Later, we extend this approach for equations with more terms, where we use mainly equations with nonnegative coefficients. We will introduce an upper bound on the number $p$ in the $p$-criticality of such equations. Our approach also partially covers two term equations used in [7,39].

Section 4 focuses on the following class of linear difference equations with interlacing indices

$$
\begin{equation*}
a_{n} y_{n+2}+b_{n} y_{n}+a_{n-2} y_{n-2}=0, \quad n \in \mathbb{Z} \tag{1.2}
\end{equation*}
$$

The equations with interlacing indices from time to time appear in the literature (see, e.g., [19, 40-42]). They, among others, can be used in getting some counterexamples. Here we describe a space of recessive solutions of Eq. (1.2) at $\pm \infty$ and link the criticality of the secondorder self-adjoint equation to the criticality of Eq. (1.2). The important fact to note here is that for even-order equations, we cannot use several tools which are available for second-order equations. Hence, we work with equations with interlacing indices to apply these tools at least on a subclass of the Sturm-Liouville equation. By this, we obtain concrete examples where the possible behaviour of the converse shows clearly.

Overall, we develop a background for further research even though no attempt has been made to postulate the form of the possible converse. Additionally, our results show that there are still many uncharted territories in regard to the criticality of even-order linear equations. For other examples of the recent development in this field, we refer the reader to see, for example, $[13,17,22,25,28]$. The important point to note here is that the topic of critical equations is also close to the topic of oscillation. Hence, other closely related results about the critical case concerning non-oscillation are stated in [23,24], see also [9].

## 2 Preliminaries

The article [8] works with linear even-order Sturm-Liouville equation in the form

$$
\begin{equation*}
\sum_{i=0}^{k}(-\triangle)^{i}\left(r_{n}^{[i]} \triangle^{i} y_{n-i}\right)=0, \quad n \in \mathbb{Z} \tag{2.1}
\end{equation*}
$$

and its criticality is developed. To show this, we have to link solutions of Eq. (2.1) to the solutions of linear Hamiltonian difference system (see for example [1,3,8])

$$
\begin{equation*}
\triangle x_{n}=A x_{n+1}+B_{n} u_{n}, \quad \triangle u_{n}=C_{n} x_{n+1}-A^{T} u_{n} \tag{2.2}
\end{equation*}
$$

through the substitution

$$
x_{n}=\left(\begin{array}{c}
y_{n}  \tag{2.3}\\
\triangle y_{n-1} \\
\cdots \\
\triangle^{k-1} y_{n+1-k}
\end{array}\right), \quad u_{n}=\left(\begin{array}{c}
\sum_{i=1}^{k}(-\triangle)^{i-1}\left(r_{n+1}^{[i]} \triangle^{i} y_{n+1-i}\right) \\
\vdots \\
-\triangle\left(r_{n+1}^{[k]} \Delta^{k} y_{n+1-k}\right)+r_{n+1}^{[k-1]} \triangle^{k-1} y_{n+2-k} \\
r_{n+1}^{[k]} \triangle^{k} y_{n+1-k}
\end{array}\right) .
$$

Here $A, B_{n}, C_{n}$ are $k \times k$ matrices $B_{n}=\operatorname{diag}\left(0, \ldots, 0, \frac{1}{r_{n+1}^{k]}}\right), C_{n}=\operatorname{diag}\left(r_{n+1}^{[0]}, \ldots, r_{n+1}^{[k-1]}\right)$ and

$$
A=a_{i j}= \begin{cases}1, & i=j \\ -1, & i+1=j \\ 0, & \text { otherwise }\end{cases}
$$

A $2 k \times k$ matrix solution $\binom{X_{n}}{U_{n}}$ of Eq. (2.2) is said to be a conjoined basis when $X_{n}^{T} U_{n}$ is symmetric and rank $\binom{X_{n}}{u_{n}}=k$. A conjoined basis $\binom{X_{n}}{U_{n}}$ is said to be recessive solution at $\infty$ provided that for some $N$ sufficiently large holds $X_{n} X_{n+1}^{-1} A^{-1} B_{n} \geq 0$, for all $n \geq N$ and

$$
\lim _{h \rightarrow \infty}\left(\sum_{n=N}^{h} X_{n+1}^{-1} A^{-1} B_{n}\left(X_{n}^{T}\right)^{-1}\right)^{-1}=0
$$

If matrix solution $\binom{x_{n}}{U_{n}}$ is a recessive solution at $\infty$, then solutions $y_{n}^{1}, \ldots, y_{n}^{k}$ generating columns of $X_{n}$ form the system of recessive solutions of Eq. (2.1) at $\infty$. The system of recessive solutions at $-\infty$ is defined similarly. For analysis of recessive solutions of second-order equations, see for example $[4,5,35]$.

Here and subsequently, we denote the spaces of recessive solutions at $\pm \infty$ as $v^{ \pm}$, i.e.

$$
v^{ \pm}=\operatorname{Lin}\{\text { recessive solution of Eq. (2.1) at } \pm \infty\} .
$$

With this notation we shall call a disconjugate Eq. (2.1) as $p$-critical on $\mathbb{Z}$ when $\operatorname{dim} v^{+} \cap v^{-}=$ $p$. The main result of [8] reads as follows.

Theorem 2.1. Let disconjugate Eq. (2.1) be p-critical on $\mathbb{Z}$, and let $H \in \mathbb{Z}, \varepsilon>0$ be arbitrary. Furthermore, let arbitrary $J \subset\{0, \ldots, n-1\}$ satisfy $|J|=k-p+1$ and consider the sequences

$$
s_{H}^{[j]}= \begin{cases}r_{H}^{[j]}-\varepsilon, & \text { for } j \in J, \\ r_{H}^{[j]}, & \text { otherwise },\end{cases}
$$

and $s_{n}^{[i]}=r_{n}^{[i]}$, for all $i$ and $n \neq H$. Then the equation

$$
\sum_{i=0}^{k}(-\triangle)^{i}\left(s_{n}^{[i]} \triangle^{i} y_{n-i}\right)=0
$$

is not disconjugate.
This theorem has been later extended in $[26,44]$ and shows that critical equations create a borderline where appears a bifurcation with respect to disconjugacy. Nevertheless, Theorem 2.1 holds for second-order equations as an equivalence. One may ask whether this is still true if we consider a general even-order equation. Such question also serves as the primary motivation for our work.

Final conjecture of [8] is proved in [20] and they both focus on the one term equation

$$
\begin{equation*}
(-\triangle)^{k}\left(r_{n} \triangle^{k} y_{n-k}\right)=0, \quad r_{n}>0, \quad n \in \mathbb{Z}, k \in \mathbb{N} \tag{2.4}
\end{equation*}
$$

With a notation that $n^{[p]}=n \cdot(n-1) \cdot(n-2) \cdot \ldots \cdot(n-p+1), p \in \mathbb{N}$, the results state the following.

Theorem 2.2. Let $p \in\{1, \ldots, k\}$ and suppose that

$$
\sum_{j=-\infty}^{0} \frac{j^{2(k-p)}}{r_{j+k}}=\infty=\sum_{j=0}^{\infty} \frac{j^{2(k-p)}}{r_{j+k}} .
$$

Then $\operatorname{Lin}\left\{1, \ldots, n^{[p-1]}\right\} \subset v^{+} \cap v^{-}$and (2.4) is at least $p$-critical. Moreover, if either

$$
\sum_{j=-\infty}^{0} \frac{j^{2(k-1)}}{r_{j+k}}<\infty \quad \text { or } \quad \sum_{j=0}^{\infty} \frac{j^{2(k-1)}}{r_{j+k}}<\infty
$$

then $v^{+} \cap v^{-}=\varnothing$.
The converse of Theorem 2.2 can be found in [21]. Eq. (2.4) will be the main objective of the following section and so let us mention that when dealing with Eq. (2.4) it is useful to utilize the fact that if $\triangle^{j+1} y_{n-j-1}=z_{n}$ then

$$
\begin{equation*}
y_{n}=\frac{1}{j!} \sum_{i=-\infty}^{n-1}(n-i-1)^{[j]} z_{i+j+1} . \tag{2.5}
\end{equation*}
$$

Another useful result of [8] is the following lemma. However, first of all, let us mention that we follow the notation of $[8]$ and by $l_{0}^{2}(\mathbb{Z})$ we denote the set of sequences

$$
l_{0}^{2}(\mathbb{Z})=\left\{\left\{u_{n}\right\} \mid \text { only for finitely many } n \in \mathbb{Z} \text { is } u_{n} \neq 0\right\} .
$$

Lemma 2.3. Suppose that Eq. (2.1) is $p$-critical for some $p \in\{1, \ldots, k\}$. Then for every $\varepsilon>0$ there exists a sequence $u_{n} \in l_{0}^{2}(\mathbb{Z})$ such that

$$
F(u)=\sum_{n=-\infty}^{\infty} \sum_{i=0}^{k} r_{n}^{[i]}\left(\Delta^{i} u_{n-i}\right)<\varepsilon .
$$

Proof of Lemma 2.3 obtains for any $y_{n} \in v^{+} \cap v^{-}$such an $u_{n} \in l_{0}^{2}(\mathbb{Z})$ that $y_{n}=u_{n}$ on arbitrary compact $[A, B]$ and which satisfies that $F(u)<\varepsilon$, for arbitrary small $\varepsilon>0$. In light of this, we can reformulate ideas of the proof of Theorem 2.1 to obtain the following theorem.

Theorem 2.4. Let Eq. (2.1) be disconjugate and $p$-critical on $\mathbb{Z}$, and let $\varepsilon>0$ be arbitrary. For any $H \in \mathbb{Z}$ there is $J \subset\{0, \ldots, n-1\}$ with $|J| \geq p$ such that if for any $j \in J$ we replace

$$
s_{H}^{[i]}= \begin{cases}r_{H}^{[i]}-\varepsilon, & \text { for } i=j \\ r_{H}^{[i]}, & \text { for } i \neq j\end{cases}
$$

and $s_{n}^{[i]}=r_{n}^{[i]}$ for all $i, n \neq H$, then the equation

$$
\sum_{i=0}^{k}(-\triangle)^{i}\left(s_{n}^{[i]} \triangle^{i} y_{n-i}\right)=0
$$

is not disconjugate.
Proof. The proof of Theorem 2.1 (see also $[8,10]$ ) shows that there are $p$ solutions $y_{n}^{1}, \ldots, y_{n}^{p}$ of Eq. (2.1) with the following property. For any $H \in \mathbb{Z}$ there is $J \subset\{0, \ldots, n-1\}$ with $|J|=p$
such that there is a surjection from $J$ to $y_{n}^{j}$ where for any $j \in J$ holds $\triangle^{j} y_{H-j}^{j}=1$. Hence, for any $\varepsilon>0, H \in \mathbb{Z}$ and $j \in J$ we replace $r_{n}^{[i]}$ by $s_{n}^{[i]}$ to obtain

$$
\sum_{n=-\infty}^{\infty} \sum_{i=0}^{k} s_{n}^{[i]}\left(\triangle^{i} u_{n-i}\right)=-\varepsilon\left(\triangle^{j} u_{H-j}\right)+\sum_{n=-\infty}^{\infty} \sum_{i=0}^{k} r_{n}^{[i]}\left(\triangle^{i} u_{n-i}\right)=-\varepsilon\left(\triangle^{j} u_{H-j}\right)+F(u) .
$$

However, from the proof of Lemma 2.3 we have that we can choose such $u_{n}$ which satisfies $F(u)<\frac{\varepsilon}{2}$ and that $\triangle^{j} u_{H-j}=\Delta^{j} y_{H-j}^{j}=1$.

The principal difference between Theorems 2.1 and 2.4 is that in Theorem 2.1 we make $k-p+1$ coefficients arbitrarily smaller, and then we lose disconjugancy. On the other hand, in Theorem 2.4 it is enough to make only one of $p$ coefficients smaller to obtain the same. The problem in Theorem 2.4 is identifying the right coefficients. In contrast, because conditions of Theorem 2.4 are less restrictive, we assume that we could find a converse of Theorem 2.4 in the future.

We would like to also remind the reader about the following results concerning the selfadjoint second-order linear equation

$$
\begin{equation*}
a_{n-1} y_{n-1}+b_{n} y_{n}+a_{n} y_{n+1}=0 . \tag{2.6}
\end{equation*}
$$

In [15] it is shown, that Eq. (2.6) is disconjugate if and only if there are positive solutions $u_{n}^{ \pm}$, which are recessive at $\pm \infty$. Moreover, in [35] (see also [15]) appears the following theorem.

Theorem 2.5. If Eq. (2.6) is disconjugate, then

$$
\sum_{n}^{\infty} \frac{1}{\left(-a_{n}\right) u_{n}^{+} u_{n+1}^{+}}=\infty=\sum_{n=-\infty} \frac{1}{\left(-a_{n}\right) u_{n}^{-} u_{n+1}^{-}} .
$$

Additionally, Eq. (2.6) is critical if and only if $u_{n}^{+}=u_{n}^{-}$and Theorem 2.1 and 2.4 are for Eq. (2.6) the same. They hold as an equivalence for the second-order equations and therefore we have another way how to define criticality of Eq. (2.6). Other equivalent ways to define critical equations can be found in [15] or [29].

## 3 One term even-order linear equations

Following section deals with one term difference equation

$$
\begin{equation*}
(-\triangle)^{k}\left(r_{n} \triangle^{k} y_{n-k}\right)=0, \quad r_{n}>0, \quad n \in \mathbb{Z}, k \in \mathbb{N} . \tag{3.1}
\end{equation*}
$$

Such equation is investigated in [20] and according to [2] Eq. (3.1) is disconjugate if and only if

$$
\sum_{n=-\infty}^{\infty} r_{n}\left(\triangle^{k} u_{n-k}\right)^{2}>0, \quad \text { for all } u_{n} \in l_{0}^{2}(\mathbb{Z}), u_{n} \neq 0
$$

Of course, this sum can be rewritten in different shapes and forms, as we can see for example in [8]. Our main result is the following theorem. For simplicity of formulas, we denote in the proof $|0|^{k-p}=1$, because otherwise, we would have to define a new sequence

$$
\chi_{n}= \begin{cases}|n|^{k-p}, & n \neq 0, \\ 1, & n=0 .\end{cases}
$$

Theorem 3.1. Assume that for any $\varepsilon>0$ and $H \in \mathbb{Z}$ exists nontrivial $u_{n} \in l_{0}^{2}(\mathbb{Z})$ such that

$$
\begin{equation*}
\sum_{n=-\infty}^{\infty} r_{n}\left(\triangle^{k} u_{n-k}\right)^{2}<\varepsilon\left(\triangle^{p-1} u_{H-(p-1)}\right)^{2} \tag{3.2}
\end{equation*}
$$

Then Eq. (3.1) is at least $p$-critical and $\operatorname{Lin}\left\{1, \ldots, n^{[p-1]}\right\} \subset v^{+} \cap v^{-}$.
Proof. We first start by a series of substitution. Let us set $v_{n}=\triangle^{p-1} u_{n-p+1}$ and then (3.2) transforms as

$$
\sum_{n=-\infty}^{\infty} r_{n}\left(\triangle^{k-p+1} v_{n-(k-p+1)}\right)^{2}<\varepsilon\left(v_{H}\right)^{2}
$$

Because $u_{n} \in l_{0}^{2}(\mathbb{Z})$ and because differencing a zero sequence gives us only a zero sequence then also $v_{n} \in l_{0}^{2}(\mathbb{Z})$ and additionally $\triangle^{k-p+1} v_{n-(k-p+1)} \in l_{0}^{2}(\mathbb{Z})$. Bearing that in mind consider also $x_{n}=|n|^{k-p} \frac{1}{v_{H}} \triangle^{k-p+1} v_{n-(k-p+1)}$ to obtain that $x_{n} \in l_{0}^{2}(\mathbb{Z})$ as well and that

$$
\begin{equation*}
\sum_{n=-\infty}^{\infty} \frac{r_{n}}{n^{2(k-p)}} x_{n}^{2}<\varepsilon \tag{3.3}
\end{equation*}
$$

It is clear from the sum (3.3) that $\lim _{\varepsilon \rightarrow 0} x_{n}=0$ pointwise, for all $n \in \mathbb{Z}$. Through (2.5) we get via $x_{n}=|n|^{k-p} \frac{1}{v_{H}} \triangle^{k-p+1} v_{n-(k-p+1)}$ that

$$
v_{n}=\frac{v_{H}}{(k-p)!} \sum_{j=-\infty}^{n-1} \frac{(n-j-1)^{[k-p]}}{|j+(k-p+1)|^{k-p}} x_{j+(k-p+1)}
$$

Hence, for all $\varepsilon>0$ it has to hold that

$$
\begin{equation*}
1=\frac{1}{(k-p)!} \sum_{j=-\infty}^{H-1} \frac{(H-j-1)^{[k-p]}}{|j+(k-p+1)|^{k-p}} x_{j+(k-p+1)}=\frac{1}{(k-p)!} \sum_{i=-\infty}^{H+k-p} \frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}} x_{i} \tag{3.4}
\end{equation*}
$$

Next, we claim that we can obtain easily that

$$
\lim _{i \rightarrow-\infty} \frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}}=1
$$

Therefore, for some $\omega>0$ and some $i_{0}$ is eventually

$$
\begin{equation*}
1-\omega<\frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}}<1+\omega, \quad \text { for all } i \leq i_{0} \tag{3.5}
\end{equation*}
$$

Having disposed of the preliminary steps, we can now assume for contradiction that it holds $\sum_{n=-\infty} \frac{n^{2(k-p)}}{r_{n}}<\infty$. However, this would mean that

$$
\lim _{\substack{n \rightarrow-\infty \\ \varepsilon \rightarrow 0}} \frac{r_{n}}{n^{2(k-p)}} x_{n} \neq 0
$$

Otherwise, we get for arbitrarily small $\delta>0$ some $\varepsilon_{0}, n_{0}$ such that $\frac{r_{n}}{n^{2(k-p)}} x_{n}<\delta$, for any $n \leq n_{0}$ and $\varepsilon<\varepsilon_{0}$. It is a simple fact that because of

$$
\sum_{n=-\infty}^{n_{0}} x_{n}<\delta \sum_{n=-\infty}^{n_{0}} \frac{n^{2(k-p)}}{r_{n}}
$$

is the sum $\sum_{n=-\infty}^{n_{0}} x_{n}$ arbitrarily small. However, such situation cannot happen because by (3.4) and (3.5) we get that

$$
\begin{aligned}
1 & =\frac{1}{(k-p)!} \sum_{i=-\infty}^{H+k-p} \frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}} x_{i} \\
& <\frac{(1+\omega) \delta}{(k-p)!} \sum_{i=-\infty}^{\min \left\{n_{0}, i_{0}\right\}} \frac{i^{2(k-p)}}{r_{i}}+\frac{1}{(k-p)!} \sum_{i=\min \left\{n_{0}, i_{0}\right\}+1}^{H+k-p} \frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}} x_{i} \\
& \xrightarrow{\varepsilon \rightarrow 0} \frac{(1+\omega) \delta}{(k-p)!} \sum_{i=-\infty}^{\min \left\{n_{0}, i_{0}\right\}} \frac{i^{2(k-p)}}{r_{i}}<1, \quad \text { for } \delta \text { sufficiently small. }
\end{aligned}
$$

Therefore,

$$
\lim _{\substack{n \rightarrow-\infty \\ \varepsilon \rightarrow 0}} \frac{r_{n}}{n^{2(k-p)}} x_{n} \neq 0
$$

and by the definition of the limit we can find a positive constant $C$ for which there is a sequence $\varepsilon_{k} \rightarrow 0$ with the following property. For any given $\varepsilon_{k}$ there is a subsequence $n_{l} \rightarrow-\infty$ such that

$$
\frac{r_{n_{l}}}{n_{l}^{2(k-p)}}\left|x_{n_{l}}\left(\varepsilon_{k}\right)\right|>C .
$$

Before we proceed any further, let us consider, that for $\varepsilon_{k}$ there can also be a subsequence $n_{\hat{l}}$ for which is

$$
\frac{r_{n_{\hat{l}}}}{n_{\hat{l}}^{2(k-p)}}\left|x_{n_{\hat{l}}}\left(\varepsilon_{k}\right)\right|<\delta
$$

Altogether, we obtain the inequality

$$
\begin{aligned}
1 & =\frac{1}{(k-p)!} \sum_{i=-\infty}^{H+k-p} \frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}} x_{i} \\
& <\frac{(1+\omega) \delta}{(k-p)!} \sum_{i \in\left\{n_{\imath}\right\}} \frac{i^{2(k-p)}}{r_{i}}+\frac{1}{(k-p)!} \sum_{i \notin\left\{n_{\hat{l}}\right\}}^{H+k-p} \frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}} x_{i} \\
& \leq \frac{(1+\omega) \delta}{(k-p)!} \sum_{i \in\left\{n_{\imath}\right\}} \frac{i^{2(k-p)}}{r_{i}}+\frac{1+\omega}{(k-p)!} \sum_{i \notin\left\{n_{i}\right\}}^{i_{0}}\left|x_{i}\right|+\frac{1}{(k-p)!} \sum_{i=i_{0}+1}^{H+k-p} \frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}} x_{i} .
\end{aligned}
$$

We continue in this fashion by singling out

$$
\begin{aligned}
\sum_{i \notin\left\{n_{i}\right\}}^{i_{0}}\left|x_{i}\right| & >\frac{(k-p)!}{1+\omega}-\delta \sum_{i \in\left\{n_{i}\right\}} \frac{i^{2(k-p)}}{r_{i}}-\frac{1}{1+\omega} \sum_{i=i_{0}+1}^{H+k-p} \frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}} x_{i} \\
& \geq \frac{(k-p)!}{1+\omega}-\delta \sum_{i=-\infty}^{H+k-p} \frac{i^{2(k-p)}}{r_{i}}-\frac{1}{1+\omega} \sum_{i=i_{0}+1}^{H+k-p} \frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}} x_{i} .
\end{aligned}
$$

Because $x_{n}$ converges pointwise to the zero sequence, then the sum

$$
\frac{1}{1+\omega} \sum_{i=i_{0}+1}^{H+k-p} \frac{(H+k-p-i)^{[k-p]}}{|i|^{k-p}} x_{i}
$$

can be arbitrarily small if we make given $\varepsilon_{k}$ sufficiently small. Hence, by letting $\varepsilon_{k} \rightarrow 0$ we can find $\delta$ sufficiently small so that

$$
\sum_{i \notin\left\{n_{\hat{l}}\right\}}^{i_{0}}\left|x_{i}\right|>\delta \quad \text { and } \quad \frac{r_{n}}{n^{2(k-p)}}\left|x_{n}\left(\varepsilon_{k}\right)\right|>\delta, \text { for all } n \notin\left\{n_{\hat{l}}\right\}
$$

The result is that for a given $\varepsilon_{k}$ sufficiently small we have through (3.3) that

$$
\varepsilon_{k}>\sum_{j=-\infty}^{i_{0}} \frac{r_{j}}{j^{2(k-p)}} x_{j}^{2}>\sum_{i \notin\left\{n_{i}\right\}}^{i_{0}} \frac{r_{i}}{i^{2(k-p)}}\left|x_{i}\right| \cdot\left|x_{i}\right|>\delta \sum_{i \notin\left\{n_{\hat{l}}\right\}}^{i_{0}}\left|x_{i}\right|>\delta^{2}
$$

This contradicts our assumption as we have $\varepsilon_{k}$ arbitrarily small and $\delta$ is independent from $\varepsilon_{k}$.
Hence, it has to be $\sum_{n=-\infty} \frac{n^{2(k-p)}}{r_{n}}=\infty$. Divergence of the other sum $\sum^{\infty} \frac{n^{2(k-p)}}{r_{n}}=\infty$ is obtained analogously. Only this time we have to use that

$$
v_{n}=\frac{v_{H}}{(k-p)!} \sum_{j=n-1}^{\infty} \frac{(n-j-1)^{[k-p]}}{|j+(k-p+1)|} x_{j+(k-p+1)}
$$

The rest of the proof follows from Theorem 2.2.
As an example let us consider the case of $k=2$ with $r_{n}=\frac{1}{(n+1)^{2}}$. We know by Theorem 2.2 that such an equation is 2 -critical. Furthermore, from Eq. (3.3) we have that for any $\varepsilon>0$ there is $x_{n} \in l_{0}^{2}(\mathbb{Z})$ such that

$$
\sum_{n=-\infty}^{\infty} \frac{1}{(n+1)^{2}} x_{n}^{2}<\varepsilon
$$

It is verified easily that an example of such $x_{n}$ is the almost zero sequence where only $x_{p}=1$, for $p$ sufficiently large.

One question we can ask is whether Eq. (3.1) can be $p$-critical even when $\left\{1, \ldots, n^{[p-1]}\right\} \not \subset$ $v^{+} \cap v^{-}$. However, from Theorem 3.1 we get that this cannot happen.

Corollary 3.2. If Eq. (3.1) is p-critical, then $\operatorname{Lin}\left\{1, \ldots, n^{[p-1]}\right\} \subset v^{+} \cap v^{-}$.
Proof. Let $H \in \mathbb{Z}$ be arbitrary. Because of Theorem 2.4 there is a set $J \subset\{0, \ldots, k-1\},|J| \geq p$ such that for any $j \in J$ is

$$
\sum_{n=-\infty}^{\infty} r_{n}\left(\triangle^{k} u_{n-k}\right)^{2}<\varepsilon\left(\triangle^{j} u_{H-j}\right)^{2}
$$

However, because of Theorem 3.1 if $j \in J$, then $\operatorname{Lin}\left\{1, \ldots, n^{[j-1]}\right\} \subset v^{+} \cap v^{-}$. This can be satisfied only for $J=\{1, \ldots, p-1\}$.

We will formulate the following theorem to complete in a sense the equivalence with Theorem 3.1.

Theorem 3.3. Suppose Eq. (3.1) is p-critical and $\operatorname{Lin}\left\{1, \ldots, n^{[p-1]}\right\} \subset v^{+} \cap v^{-}$, then for any $\varepsilon>0$ and $H \in \mathbb{Z}$ exists $u_{n} \in l_{0}^{2}(\mathbb{Z})$ such that

$$
\sum_{n=-\infty}^{\infty} r_{n}\left(\triangle^{k} u_{n-k}\right)^{2}<\varepsilon\left(\triangle^{p-1} u_{H-p+1}\right)^{2}
$$

Proof. This is a direct result of Theorem 2.4.
We see one drawback of Theorem 3.1 in that we do not know whether Eq. (3.1) is $p$ critical or $q$-critical for some $q \geq p$. We could probably deal with this issue if we formulate Theorem 3.1 in a more precise way and with some workaround through Theorem 2.4. Note also that in Eq. (3.3) it holds for $s<p$ that

$$
\sum_{n=-\infty}^{\infty} \frac{r_{n}}{n^{2(k-s)}} x_{n}^{2}<\sum_{n=-\infty}^{\infty} \frac{r_{n}}{n^{2(k-p)}} x_{n}^{2}<\varepsilon
$$

### 3.1 Even-order equations with nonnegative coefficients

The following subsection works with Eq. (2.1) where

$$
\begin{equation*}
r_{n}^{[k]}>0 \text { and either } r_{n}^{[i]}>0 \text { for all } n \in \mathbb{Z}, \text { or } r_{n}^{[i]} \equiv 0, i \in\{0, \ldots, k-1\} \tag{3.6}
\end{equation*}
$$

Similar ideas as those in the proof of Theorem 3.1 lead us to the following result.
Theorem 3.4. Assume that Eq. (2.1) satisfies condition (3.6) and that for a given $i$ is $r_{n}^{[i]}$ a positive sequence. Then Eq. (2.1) is at most i-critical.

Proof. First consider the situation where $r_{n}^{[j]}>0$, for all $j>i$. Then replacing $r_{H}^{[j]}$ by $r_{H}^{[j]}-\varepsilon>0$ for $j \geq i$ does not lose disconjugacy. Hence, it means that Eq. (2.1) is at most $i$-critical by Theorem 2.4.

Next, for contradiction assume that Eq. (2.1) is at least $(i+1)$-critical. Therefore, for some $j>i$ and any $\varepsilon>0$ there is $H \in \mathbb{Z}$ such that $r_{H}^{[j]}=0$ and

$$
\sum_{n=-\infty}^{\infty} r_{n}^{[i]}\left(\triangle^{i} u_{n-i}\right)^{2}<\sum_{n=-\infty}^{\infty} \sum_{l=0}^{k} r_{n}^{[l]}\left(\triangle^{l} u_{n-l}\right)^{2}<\varepsilon\left(\triangle^{j} u_{H-j}\right)^{2}, \quad u_{n} \in l_{0}^{2}(\mathbb{Z})
$$

With convenient substitution $v_{n}=\triangle^{i} u_{n-i}$ we can rewrite this inequality as

$$
\sum_{n=-\infty}^{\infty} r_{n}^{[i]}\left(v_{n}\right)^{2}<\varepsilon\left(\triangle^{j-i} v_{H-j+i}\right)^{2}, \quad \text { for some } v_{n} \in l_{0}^{2}(\mathbb{Z})
$$

Another substitution

$$
\begin{equation*}
\left(\triangle^{j-i} v_{H-j+i}\right) x_{n}=v_{n} \tag{3.7}
\end{equation*}
$$

yields

$$
\sum_{n=-\infty}^{\infty} r_{n}^{[i]}\left(x_{n}\right)^{2}<\varepsilon, \quad \text { for some } x_{n} \in l_{0}^{2}(\mathbb{Z})
$$

It is clear that letting $\varepsilon \rightarrow 0$ gives that $x_{n} \rightarrow 0$ pointwise, for all $n \in \mathbb{Z}$. On the other side, by differentiating (3.7) with respect to $n$ for all $\varepsilon>0$ we obtain

$$
\left(\triangle^{j-i} v_{H-j+i}\right) \triangle^{j-i} x_{n}=\triangle^{j-i} v_{n}
$$

Note that $\triangle^{j-i} v_{H-j+i}$ is independent on $n$. And then by putting $n=H-j+i$ we obtain that $\triangle^{j-i} x_{H-j+i}=1$. However, we can rewrite (see for example [30]) the equality for all $\varepsilon>0$ as

$$
1=\triangle^{j-i} x_{H-j+i}=\sum_{q=0}^{j-i}(-1)^{q}\binom{j-i}{q} x_{H-q}
$$

Taking $\varepsilon \rightarrow 0$ together with the fact that we have a finite sum yields

$$
1=\lim _{\varepsilon \rightarrow 0} \sum_{q=0}^{j-i}(-1)^{q}\binom{j-i}{q} x_{H-q}=\sum_{q=0}^{j-i}(-1)^{q}\binom{j-i}{q} \lim _{\varepsilon \rightarrow 0} x_{H-q}=\sum_{q=0}^{j-i}(-1)^{q}\binom{j-i}{q} \cdot 0=0 .
$$

This contradicts our assumption.

As a simple example take the equation

$$
\begin{equation*}
-2 \triangle^{2} y_{n}+\triangle^{4} y_{n-1}=0, \tag{3.8}
\end{equation*}
$$

which can be by Theorem 3.4 at most 1 -critical. In fact, results of [29] show that such an equation is 1 -critical. However, [29] works only with equations of fourth-order and we do not have any results about equation

$$
\begin{equation*}
2 \triangle^{4} y_{n}-\triangle^{6} y_{n-1}=0 . \tag{3.9}
\end{equation*}
$$

As a result, we can only say that Eq. (3.9) is at most 2-critical and everything else we would have to work through its recessive solutions.

Corollary 3.5. Assume condition (3.6). If for a given $i$ is $r_{n}^{[i]}$ a positive sequence and $E q$. (2.1) is p-critical, then

$$
\sum^{\infty} \frac{n^{2(i-p)}}{r_{n}^{[i]}}=\infty=\sum_{-\infty} \frac{n^{2(i-p)}}{r_{n}^{[i]}} .
$$

Proof. First, because of Theorem 2.4 there is $j \geq p$ such that

$$
\sum_{n=-\infty}^{\infty} r_{n}^{[i]}\left(\triangle^{i} u_{n-i}\right)^{2}<\varepsilon\left(\triangle^{j-1} u_{H-j+1)}\right)^{2}, \quad \text { for some } u_{n} \in l_{0}^{2}(\mathbb{Z})
$$

Then in the same way as was done in Theorem 3.1 we see that

$$
\sum^{\infty} \frac{n^{2(i-j)}}{r_{n}^{[i]}}=\infty=\sum_{-\infty} \frac{n^{2(i-j)}}{r_{n}^{[i]}}
$$

However, it holds

$$
\begin{aligned}
& \infty=\sum^{\infty} \frac{n^{2(i-j)}}{r_{n}^{[i]}} \leq \sum^{\infty} \frac{n^{2(i-p)}}{r_{n}^{[i]}}, \\
& \infty=\sum_{-\infty}^{n^{2(i-j)}} \frac{r_{n}^{[i]}}{r_{n}^{2}} \leq \sum_{-\infty} \frac{n^{2(i-p)}}{r_{n}^{[i]}} .
\end{aligned}
$$

For introducing a nonhomogeneity into studied equations, we could use, for example, results obtained in $[33,34]$. Other possible ways forward may be hidden in extending the concept of criticality for half-linear difference equations. See for example [11,12] together with [44]. For symplectic systems, see also [43].

## 4 A class of linear equations with interlacing indices

To better understand critical equations of higher-order, we can consider other special cases. In the next part we utilize the second-order linear equation with interlacing indices

$$
\begin{equation*}
a_{n} y_{n+2}+b_{n} y_{n}+a_{n-2} y_{n-2}=0, \quad n \in \mathbb{Z}, \tag{4.1}
\end{equation*}
$$

where $b_{n}>0, a_{n}<0$, for all $n \in \mathbb{Z}$. Through the relations

$$
\begin{aligned}
& r_{n}^{[2]}=a_{n-2}, \\
& r_{n}^{[1]}=-2 a_{n-1}-2 a_{n-2}, \\
& r_{n}^{[0]}=b_{n}+a_{n}+a_{n-2},
\end{aligned}
$$

we directly link Eq. (2.1) and Eq. (4.1). For equations of general even-order we can find such formulas in [31]. On top of that, Eq. (4.1) has the functional

$$
F(u)=\sum_{n=-\infty}^{\infty} a_{n} u_{n+2} u_{n}+b_{n} u_{n}^{2}+a_{n-2} u_{n} u_{n-2}=\sum_{n=-\infty}^{\infty} b_{n} u_{n}^{2}+2 a_{n-2} u_{n} u_{n-2}, \quad \text { for } u_{n} \in l_{0}^{2}(\mathbb{Z})
$$

Eq. (4.1) consists of two equations of the second-order, where we separate Eq. (4.1) into two cases for even and odd $n$, i.e.

$$
\begin{array}{ll}
a_{n} y_{n+2}+b_{n} y_{n}+a_{n-2} y_{n-2}=0, & n=2 k+1, k \in \mathbb{Z}, \\
a_{n} y_{n+2}+b_{n} y_{n}+a_{n-2} y_{n-2}=0, & n=2 k, k \in \mathbb{Z} . \tag{4.3}
\end{array}
$$

This property is useful because there are more known results about second-order equations, and through them, we can extend some known results for higher-order equations. Moreover, we have corresponding functionals $F_{1}(u)$ for Eq. (4.2) and $F_{2}(u)$ for Eq. (4.3). It holds that

$$
\begin{aligned}
F(u) & =\sum_{k=-\infty}^{\infty} b_{2 k+1} u_{2 k+1}^{2}+2 a_{2 k-1} u_{2 k+1} u_{2 k-1}+\sum_{k=-\infty}^{\infty} b_{2 k} u_{2 k}^{2}+2 a_{2 k-2} u_{2 k} u_{2 k-2} \\
& =F_{1}\left(u^{1}\right)+F_{2}\left(u^{2}\right)
\end{aligned}
$$

where $u_{k}^{1}=u_{2 k+1}$ and $u_{k}^{2}=u_{2 k}$. It is clear that if $u_{2 k}=0$, for all $k \in \mathbb{Z}$ then $F(u)=F_{1}\left(u^{1}\right)$ and vice versa for $F_{2}\left(u^{2}\right)$. By these arguments, Eq. (4.1) is disconjugate if and only if Eq. (4.2) and (4.3) are both disconjugate. See also [2] and [30].

Theorem 4.1. Assume that Eq. (4.1) is disconjugate then Eq. (4.1) is $p$-critical, for $p \in\{1,2\}$ if and only if $p$ of the equations (4.2), (4.3) are critical. Additionally, disconjugated Eq. (4.1) is subcritical if and only if neither of the equations (4.2), (4.3) is critical.

Proof. Because of [15] Eq. (4.2) has a positive solutions $u_{n}^{ \pm}$, for $n=2 k+1, k \in \mathbb{Z}$ and Eq. (4.3) has a positive solutions $v_{n}^{ \pm}$, for $n=2 k, k \in \mathbb{Z}$. Both $u_{n}^{ \pm}, v_{n}^{ \pm}$are recessive at $\pm \infty$. Let us define two solutions of Eq. (4.1) as

$$
\alpha_{n}^{ \pm}=\left\{\begin{array}{ll}
u_{n}^{ \pm}, & n=2 k+1, \\
0, & n=2 k,
\end{array} \quad \text { and } \quad \beta_{n}^{ \pm}= \begin{cases}v_{n}^{ \pm}, & n=2 k, \\
0, & n=2 k+1 .\end{cases}\right.
$$

Through substitution (2.3) we obtain for $n=2 k+1$ odd a matrix solution

$$
\begin{aligned}
X_{n}^{ \pm} & =\left(\begin{array}{cc}
u_{n}^{ \pm} & 0 \\
u_{n}^{ \pm} & -v_{n-1}^{ \pm}
\end{array}\right), \\
U_{n}^{ \pm} & =\left(\begin{array}{cc}
-r_{n+1}^{[2]} u_{n+2}^{ \pm}-\left(r_{n+1}^{[2]}+2 r_{n}^{[2]}+r_{n}^{[1]}\right) u_{n}^{ \pm} & \left(2 r_{n+1}^{[2]}+r_{n}^{[2]}+r_{n}^{[1]}\right) v_{n+1}^{ \pm}+r_{n}^{[2]} v_{n-1}^{ \pm} \\
-2 r_{n}^{[2]} u_{n}^{ \pm} & r_{n}^{[2]}\left(v_{n+1}^{ \pm}+v_{n-1}^{ \pm}\right)
\end{array}\right) .
\end{aligned}
$$

For $n=2 k$ even, we get

$$
\begin{aligned}
X_{n}^{ \pm} & =\left(\begin{array}{cc}
0 & v_{n}^{ \pm} \\
-u_{n-1}^{ \pm} & v_{n}^{ \pm}
\end{array}\right), \\
U_{n}^{ \pm} & =\left(\begin{array}{cc}
\left(2 r_{n+1}^{[2]}+r_{n}^{[2]}+r_{n}^{[1]}\right) u_{n+1}^{ \pm}+r_{n}^{[2]} u_{n-1}^{ \pm} & -r_{n+1}^{[2]} v_{n+2}^{ \pm}-\left(r_{n+1}^{[2]}+2 r_{n}^{[2]}+r_{n}^{[1]}\right) v_{n}^{ \pm} \\
r_{n}^{[2]}\left(u_{n+1}^{ \pm}+u_{n-1}^{ \pm}\right) & -2 r_{n}^{[2]} v_{n}^{ \pm}
\end{array}\right) .
\end{aligned}
$$

Such matrix solution is a conjoined basis because $X_{n}^{ \pm}$will always have rank 2 and it holds for $n$ odd that

$$
\left(X_{n}^{ \pm}\right)^{T} U_{n}=\binom{\text { something } \underbrace{\left(2 r_{n+1}^{[2]}+2 r_{n}^{[2]}+r_{n}^{[1]}\right)}_{=0} v_{n+1}^{ \pm} u_{n}^{ \pm}+2 r_{n}^{[2]} u_{n}^{ \pm} v_{n-1}^{ \pm}}{2 r_{n}^{[2]} u_{n}^{ \pm} v_{n-1}^{ \pm}}
$$

is symmetrical. For $n$ even is the situation the same.
Subsequently, we will show that $\binom{X_{n}^{+}}{u_{n}^{+}}$is a recessive solution at $\infty$. If $n=2 k$ is even, then

$$
X_{n}^{+}\left(X_{n+1}^{+}\right)^{-1} A^{-1} B_{n}=\left(\begin{array}{cc}
0 & 0 \\
0 & \frac{-u_{n-1}^{+}}{u_{n+1}^{+} a_{n-1}}
\end{array}\right) \geq 0 .
$$

By properly multiplying matrices we conclude that it holds

$$
\left(X_{n+1}^{+}\right)^{-1} A^{-1} B_{n}\left(X_{n}^{+T}\right)^{-1}=\left(\begin{array}{cc}
\frac{1}{u_{n+1}^{+}} & 0 \\
\frac{1}{v_{n}^{+}} & \frac{-1}{v_{n}^{+}}
\end{array}\right)\left(\begin{array}{cc}
0 & \frac{1}{a_{n-1}} \\
0 & \frac{1}{a_{n-1}}
\end{array}\right)\left(\begin{array}{cc}
\frac{1}{u_{n-1}^{+}} & \frac{1}{v_{n}^{+}} \\
\frac{-1}{u_{n-1}^{+}} & 0
\end{array}\right)=\left(\begin{array}{cc}
\frac{-1}{a_{n-1} u_{n-1}^{+} u_{n+1}^{+}} & 0 \\
0 & 0
\end{array}\right) .
$$

Combining this with similar equality means for $n$ odd we obtain that

$$
\sum_{n=M}^{h}\left(X_{n+1}^{+}\right)^{-1} A^{-1} B_{n}\left(X_{n}^{+T}\right)^{-1}=\left(\begin{array}{cc}
\sum_{i=M, i}^{h} \text { even } \frac{-1}{a_{i-1} u_{i-1}^{+} u_{i+1}^{+}} & 0 \\
0 & \sum_{j=M, j \text { odd }}^{h} \frac{-1}{a_{j-1} v_{j-1}^{+} v_{j+1}^{+}}
\end{array}\right)
$$

Hence, because of Theorem 2.5 it holds that

$$
\lim _{h \rightarrow \infty}\left(\sum_{n=M}^{h}\left(X_{n+1}^{+}\right)^{-1} A^{-1} B_{n}\left(X_{n}^{+T}\right)^{-1}\right)^{-1}=0
$$

and $\binom{x_{n}^{+}}{u_{n}^{+}}$is indeed a recessive solution at $\infty$. Analogously we assert that $\binom{x_{n}^{-}}{u_{n}^{-}}$is a recessive solution at $-\infty$. The proof is complete by comparing definitions of criticality for Eq. (2.1) and both Eq. (4.2) and (4.3).

We assume that Theorem 4.1 can be extended for any tridiagonal equation of any evenorder in a similar fashion. Additionally, a system of recessive solutions of Eq. (4.1) is defined in [18] through the relation that, if there are solutions $u_{n}^{1}, \ldots, u_{n}^{4}$ of Eq. (4.1) where for any $C>0$ there is $K$ such that

$$
u_{n}^{k-1}<C u_{n}^{k}, \quad \text { for all } n \geq K,
$$

then $u_{n}^{1}, u_{n}^{2}$ create a system of recessive solutions. However, this does not work with $\alpha_{n}^{ \pm}, \beta_{n}^{ \pm}$, and we work around that through the recessive solutions of Hamiltonian systems.

### 4.1 Final remarks and examples

Consider the following example where we set in Eq. (4.1) sequences $a_{n}, b_{n}$ as

$$
a_{n}=\left\{\begin{array}{ll}
-1, & n \text { even }, \\
-3, & n \text { odd },
\end{array} \quad b_{n}= \begin{cases}2, & n \text { even }, \\
6, & n \text { odd } .\end{cases}\right.
$$

We know by the results of [29] and Theorem 4.1 that such an equation is 2 -critical. It is simple matter to verify that coefficients of (2.1) are $r_{n}^{[2]}=a_{n-2}, r_{n}^{[1]} \equiv 8$ and $r_{n}^{[1]} \equiv 0$. And therefore we have a concrete example of 4th order Sturm-Liouville equations which is 2-critical.

Another interesting situation appears provided that Eq. (4.1) is 1-critical. Through Theorem 4.1 we know that in such a case one of the equations (4.2) or (4.3) has to be critical. Without loss of generality let us say that it is Eq. (4.2). Because Theorem 2.1 holds for Eq. (2.6) as an equivalence, thus for any $\varepsilon>0$ and $H$ odd there is such $u_{n} \in l_{0}^{2}(\mathbb{Z})$ that

$$
F(u)=F_{1}\left(u^{1}\right)<\varepsilon\left(u_{H}^{1}\right)^{2}=\varepsilon\left(\triangle u_{H}\right)^{2}=\varepsilon\left(\triangle u_{H-1}\right)^{2} .
$$

Hence, we have seen two different behaviours of $F(u)$ in regard to 1-critical equations. We have seen, that 1-critical Eq. (3.1) satisfies $F(u)<\varepsilon u_{H}^{2}$ for any $H \in \mathbb{Z}$. On the other hand, 1-critical Eq. (4.1) satisfies that $F(u)<\varepsilon u_{H}^{2}$ and $F(u)<\varepsilon\left(\triangle u_{H-1}\right)^{2}$ for all $H$ either odd or even. We obtain simple example of 1 -critical Eq. (4.1) if we take $b_{n} \equiv 6$ and

$$
a_{n}= \begin{cases}-1, & n \text { even }, \\ -3, & n \text { odd } .\end{cases}
$$

Such an equation is again 1-critical by the results of [29]. Furthermore, we can compare this equation to Eq. (3.8) which is also 1 -critical.

Possible applications of Eq. (4.1) arise when we consider the second-order self-adjoint linear differential equation

$$
\begin{equation*}
\left(p(x) z^{\prime}(x)\right)^{\prime}+q(x) z(x)=0, \tag{4.4}
\end{equation*}
$$

where $p(x)>0$. We usually link Eq. (4.4) to the self-adjoint linear difference equation by approximating

$$
f^{\prime}(x) \approx \frac{f(x+h)-f(x)}{h}
$$

for some small $h$. See for example [30]. However, from numerical analysis we know (see for example [6]), that we can get better numerical results by approximating

$$
f^{\prime}(x) \approx \frac{f(x+h)-f(x-h)}{2 h} .
$$

Using this approximation with a bit of work in Eq. (4.4) we obtain approximation

$$
\begin{equation*}
p_{n} y_{n+2}-\left(p_{n}+p_{n-2}\right) y_{n}+p_{n-2} y_{n-2}+4 q_{n} y_{n} \approx 0 . \tag{4.5}
\end{equation*}
$$

Furthermore, fixing $\triangle_{s} y_{n}=\frac{y_{n+1}-y_{n-1}}{2}$ yields

$$
\triangle_{s}\left(p_{n-1} \triangle_{s} y_{n}\right)+q_{n} y_{n} \approx 0
$$

This way, we arrive to a second-order self-adjoint linear equation with a different definition of $\Delta$. It is a simple matter to link such an equation through (4.5) to Eq. (4.1) by $b_{n}=p_{n}+p_{n-2}-$ $4 q_{n}, a_{n}=-p_{n}$, for $p_{n}+p_{n-2}>4 q_{n}$.

## Competing interests

The author declares that he is also affiliated with the University of Defence, Brno, Czech Republic.

## Acknowledgements

The author wishes to express his thanks to Prof. W. Kratz for his support during his stay in Ulm during the spring of 2020. The author thanks the anonymous referees for their suggestions and references which improved the final version of the paper. This research is supported by the Czech Science Foundation under Grant GA20-11846S and by the Masaryk University under Grant MUNI/A/1160/2020.

## References

[1] C. D. Ahlbrandt, A. C. Peterson, Discrete Hamiltonian systems: Difference equations, continued fractions, and Riccati equations, Kluwer Academic Publishers, Boston, 1996. ISBN 0792342771. MR1423802; Zbl 0860.39001
[2] M. Bohner, Linear Hamiltonian difference systems: Disconjugacy and Jacobi-type conditions, J. Math. Anal. Appl. 199(1996), No. 3, 804-826. https://doi.org/10.1006/jmaa. 1996.0177; MR1386607; Zbl 0855.39018
[3] M. Bohner, O. Došlý, W. Kratz, A Sturmian theorem for recessive solutions of linear Hamiltonian difference systems, Appl. Math. Lett. 12(1999), No. 2, 101-106. https: //doi. org/10.1016/S0893-9659(98)00156-6; MR1749755 Zbl 0933.39034
[4] M. Bohner, S. Stević, Trench's perturbation theorem for dynamic equations, Discrete Dyn. Nat. Soc. 2007, Art. ID 75672, 11 pp. https://doi.org/10.1155/2007/75672; MR2375475; Zbl 1203.34151
[5] M. Bohner, S. Stević, Linear perturbations of a nonoscillatory second-order dynamic equation, J. Difference Equ. Appl. 15(2009), No. 11-12, 1211-1221. https://doi.org/10. 1080/10236190903022782; MR2569142; Zbl 1187.34127
[6] R. L. Burden, J. D. Faires, Numerical analysis, 9th ed., Boston: Brooks/Cole, 1997. ISBN 9780538733519. MR0519124; Zbl 0419.65001
[7] O. DošLý, Oscillation criteria for higher order Sturm-Liouville difference equations, J. Difference Equ. Appl. 4(1998), No. 5, 425-450. MR1665162; Zbl 0921.39005
[8] O. Došlý, P. Hasil, Critical higher order Sturm-Liouville difference operators, J. Difference Equ. Appl. 17(2011), No. 9, 1351-1363. https://doi.org/10.1080/ 10236190903527251; MR2825251; Zbl 1233.39002
[9] Z. Došlá, P. Hasil, S. Matucci, M. Veselý, Euler type linear and half-linear differential equations and their non-oscillation in the critical oscillation case, J. Inequal. Appl. 2019, Paper No. 189, 1-30. https://doi.org/10.1186/s13660-019-2137-0; MR3978958
[10] O. DošLý, J. Komenda, Conjugacy criteria and principal solutions of self-adjoint differential equations, Arch. Math. (Brno) 31(1995), No. 3, 217-238. MR1368260; Zbl 0841.34033
[11] O. Došıý, V. RůžIčka, Nonoscillation of higher order half-linear differential equations, Electron. J. Qual. Theory Differ. Equ. 2015, No. 19, 1-15. https://doi.org/10.14232/ ejqtde.2015.1.19; MR3325922; Zbl 1349.34109
[12] O. DošĽ́, V. RŮžǏčKa, Nonoscillation criteria and energy functional for even-order halflinear two-term differential equations, Electron. J. Differential Equations 2016, No. 95, 1-17. MR3489973; Zbl 1345.34042
[13] A. M. Encimas, M. J. Jimenez, Second order linear difference equations, J. Difference Equ. Appl. 24(2018), No. 3, 305-343. https://doi.org/10.1080/10236198.2017.1408608; MR3757171; Zbl 1444.39002
[14] F. Gesztesy, Z. Zhao, On critical and subcritical Sturm-Liouville operators, J. Funct. Anal. 98(1991), No. 2, 311-345. https://doi.org/10.1016/0022-1236(91)90081-F; MR1111572; Zbl 0726.35119
[15] F. Gesztesy, Z. Zhao, Critical and subcritical Jacobi operators defined as Friedrichs extensions, J. Differential Equations 103(1993), No. 1, 68-93. https://doi.org/10.1006/jdeq. 1993.1042; MR1218739; Zbl 0807.47004
[16] F. Gesztesy, Z. Zhao, On positive solutions of critical Schrödinger operators in two dimensions, J. Funct. Anal. 127(1995), No. 1, 235-256. https://doi.org/10.1006/jfan. 1995.1010; MR1308624; Zbl 0821.35035
[17] G. A. Grigorian, Oscillatory criteria for the second order linear ordinary differential euqations, Math. Slovaca 69(2019), No. 4, 857-870. https://doi.org/10.1515/ ms-2017-0274; MR3985023; Zbl 07289564
[18] P. Hartman, Difference equations: disconjugacy, principal solutions, Green's functions, complete monotonicity, Trans. Amer. Math. Soc. 246(1978), 1-30. https://doi.org/10. 2307/1997963; MR0515528; Zbl 0409.39001
[19] P. Hasil, On positivity of the three term $2 n$-order difference operators, Stud. Univ. Žilina Math. Ser. 23(2009), No. 1, 51-58. MR2741998; Zbl 1217.47064
[20] P. Hasil, Criterion of $p$-criticality for one term $2 n$-order difference operators, Arch. Math. (Brno) 47(2011), No. 2, 99-109. MR2813536; Zbl 1249.39001
[21] P. Hasil, Conjugacy of self-adjoint difference equations of even order, Abstr. Appl. Anal. 2011, Art. ID 814962, 16 pp. https://doi.org/10.1155/2011/814962; MR2817278; Zbl 1220.39007
[22] P. Hasil, J. Juránek, M. Veselý, Non-oscillation of half-linear difference equations with asymptotically periodic coefficients, Acta Math. Hungar. 159(2019), No. 1, 323-348. https: //doi.org/10.1007/s10474-019-00940-7; MR4003712
[23] P. Hasil, M. Veselý, Critical oscillation constant for difference equations with almost periodic coefficients, Abstr. Appl. Anal. 2012, Art. ID 471435, 19 pp. https: //doi. org/10. 1155/2012/471435; MR2984042; Zbl 1253.39006
[24] P. Hasil, M. Veselý, Non-oscillation of periodic half-linear equations in the critical case, Electron. J. Differential Equations 2016, No. 120, 1-12. MR3504436; Zbl 1345.34043
[25] P. Hasil, M. Veselý, Oscillation and non-oscillation criteria for linear and half-linear difference equations, J. Math. Anal. Appl. 452(2017), No. 1, 401-428. https://doi.org/ 10.1016/j.jmaa.2017.03.012; MR3628027; Zbl 1372.39015
[26] P. Hasil, P. Zemánek, Critical second order operators on time scales, Discrete Contin. Dyn. Syst. 2011, 653-659. https://doi.org/10.3934/proc.2011.2011.653; MR2987447; Zbl 1306.39004
[27] N. Ichihara, Criticality of viscous Hamilton-Jacobi equations and stochastic ergodic control, J. Math. Pures Appl. (9) 100(2013), No. 3, 368-390. https://doi.org/10.1016/j . matpur.2013.01.005; MR3095206; Zbl 1295.35092
[28] J. Jeкl, Linear even order homogenous difference equation with delay in coefficient, Electron. J. Qual. Theory Differ. Equ. 2020, No. 45, 1-19. https://doi.org/10.14232/ejqtde. 2020.1.45; MR4118160; Zbl 07307858
[29] J. Jekl, Properties of critical and subcritical second order self-adjoint linear equations, Math. Slovaca 71(2021), No. 5, 1149-1166. https://doi.org/10.1515/ms-2021-0045; MR4320180
[30] W. G. Kelley, A. C. Peterson, Difference equations: An introduction with applications, San Diego: Academic Press, 2001. ISBN 012403330X. MR1765695; Zbl 0970.39001
[31] W. Kratz, Banded matrices and difference equations, Linear Algebra Appl. 337(2001), No. 1, 1-20. https://doi.org/10.1016/S0024-3795(01)00328-7; MR1856849; Zbl 1002.39028
[32] M. Lucia, S. Prashanth, Criticality theory for Schrödinger operators with singular potential, J. Differential Equations 265(2018), No. 8, 3400-3440. https://doi.org/10.1016/j . jde.2018.05.006; MR3823973; Zbl 1396.35006
[33] J. Migda, Asymptotic properties of solutions to difference equations of Emden-Fowler type, Electron. J. Qual. Theory Differ. Equ. 2019, No. 77, 1-17. https://doi.org/10.14232/ ejqtde.2019.1.77; MR4028909; Zbl 1449.39008
[34] J. Migda, M. Nockowska-Rosiak, Asymptotic properties of solutions to difference equations of Sturm-Liouville type, Appl. Math. Comput. 2019, No. 340, 126-137. https: //doi.org/10.1016/j.amc.2018.08.001; MR3855172; Zbl 1428.39006
[35] W.T. Patula, Growth and oscillation properties of second order linear difference equations, SIAM J. Math. Anal. 10(1970), No. 1, 55-61. https://doi.org/10.1137/0510006; MR0516749; Zbl 0397.39001
[36] Y. Pinchover, Criticality and ground states for second-order elliptic equations, J. Differential Equations 80(1989), No. 2, 237-250. https://doi.org/10.1016/0022-0396(89) 90083-1; MR1011149; Zbl 0697.35036
[37] Y. Pinchover, On criticality and ground states of second order elliptic equations, II, J. Differential Equations 87(1990), No. 2, 353-364. https://doi.org/10.1016/0022-0396(90) 90007-C; MR1072906; Zbl 0714.35055
[38] Y. Pinchover, On positivity, criticality, and the spectral radius of the shuttle operator for elliptic operators, Duke Math. J. 85(1996), No. 2, 431-445. https://doi.org/10.1215/ S0012-7094-96-08518-X; MR1417623; Zbl 0901.35016
[39] Р. 尺̌ЕнА́к, Asymptotic formulae for solutions of linear second-order difference equations, J. Difference Equ. Appl. 22(2016), No. 1, 107-139. https://doi.org/10.1080/10236198. 2015.1077815; MR3473801; Zbl 1338.39025
[40] S. Stević, J. Diblík, B. IričAnin, Z. Šmarda, On some solvable difference equations and systems of difference equations, Abstr. Appl. Anal. 2012, Art. ID 541761, 11 pp. https: //doi.org/10.1155/2012/541761; MR2991014; Zbl 1253.39001
[41] S. Stević, A. M. A. El-Sayed, W. Kosmala, Z. Šmarda, On a class of difference equations with interlacing indices, Adv. Difference Equ. 2021, No. 297, 16 pp. https://doi.org/10. 1186/s13662-021-03452-3; MR4273249
[42] S. Stević, B. Iričanin, W. Kosmala, Z. Šmarda, Note on the bilinear difference equation with a delay, Math. Methods Appl. Sci. 41(2018), No. 18, 9349-9360. https://doi. org/10. 1002/mma.5293; MR3897790; Zbl 1404.39001
[43] P. Šepitka, R. Šimon Hilscher, Recessive solutions for nonoscillatory discrete symplectic systems, Linear Algebra Appl. 2015, No. 469, 243-275. https://doi.org/10.1016/j.laa. 2014.11.029; MR3299064; Zbl 1307.39007
[44] M. Veselý, P. Hasil, Criticality of one term $2 n$-order self-adjoint differential equations, Electron. J. Qual. Theory Differ. Equ. 2012, No. 18, 1-12. https://doi.org/10.14232/ ejqtde.2012.3.18; MR3338537; Zbl 1324.47139

# An extension to the planar Markus-Yamabe Jacobian conjecture 

Marco Sabatini ${ }^{\boxtimes}$<br>Università di Trento, Via Sommarive, 14, I-38123 Povo, Trento (TN), Italy

Received 30 June 2021, appeared 13 October 2021
Communicated by Armengol Gasull


#### Abstract

We extend the planar Markus-Yamabe Jacobian conjecture to differential systems having Jacobian matrix with eigenvalues with negative or zero real parts.


Keywords: Markus-Yamabe, Jacobian conjecture, global asymptotic stability, global center.
2020 Mathematics Subject Classification: 34D23, 34D45.

## 1 Introduction

Let

$$
\begin{equation*}
\dot{X}=F(X), \quad X \in \mathbb{R}^{n}, \quad F \in C^{1}\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right) \tag{1.1}
\end{equation*}
$$

be a first order differential system. Let us denote by $J_{F}(X)$ the Jacobian matrix of $F(X)$. If $O$ is a critical point of (1.1) and the eigenvalues of $J_{F}(O)$ have negative real parts, then $O$ is asymptotically stable [2]. In particular, all orbits starting close enough to $O$ tend asymptotically to $O$.

In [7] the question was raised, whether $J_{F}(X)$ having eigenvalues with negative real parts for every $X \in \mathbb{R}^{n}$ imply $O$ to be globally asymptotically stable, i. e. whether all orbits in $\mathbb{R}^{n}$ tend asymptotically to $O$. Such a problem was named Markus-Yamabe Jacobian conjecture and several results were obtained under various additional hypotheses. A key step was made in [8], where it was proved that under Markus-Yamabe hypotheses, for planar systems the global asymptotic stability of $O$ is equivalent to the injectivity of $F(X)$. Such a result led to study the problem applying methods previously used to study injectivity. The Markus-Yamabe Jacobian conjecture was solved in the positive in [4-6] for planar systems, and was proved to have negative answer in higher dimensions [1,3]. The three approaches proposed in in [4-6] first prove the injectivity of $F(X)$, then as a consequence get the global asymptotic stability. Actually, in all such papers injectivity is proved under much weaker hypotheses than that of negative real parts. In fact, it is sufficient to assume that the Jacobian matrix has nowhere real positive eigenvalues.

[^42]Such general results did not lead to similarly general results in the study of the systems dynamics. This is likely due to the fact that accepting the possibility of eigenvalues with different real parts (positive, zero or negative) at different points of the plane does not allow to apply the procedure developed in [8] to establish the equivalence of injectiviy and global asymptotic stability. On the other hand, eigenvalues with zero real parts are compatible with asymptotic stability, even if not sufficient to imply it.

In this paper we assume $J_{F}(X)$ to be non-singular and have eigenvalues with non-positive real parts for all $X \in \mathbb{R}^{2}$. Differently from the classical case, in this case a system does not necessarily have a globally asymptotically stable critical point. If a critical point exists, we prove that either such a system has a global center, or there exists a globally asymptotically stable compact set. We show by an example that such a global attractor is not necessarily a critical point. If the system is analytic the conclusion can be sharpened, proving that either there exists a global center, or a globally asymptotically stable critical point. Our results follow from Olech approach to global attractivity [8] and Fessler theorem about global injectivity [4].

## 2 Results

We consider maps $F \in C^{1}\left(\mathbb{R}^{2}, \mathbb{R}^{2}\right), F(x, y)=(P(x, y), Q(x, y))$. We denote partial derivatives by subscripts. Let

$$
J_{F}(x, y)=\left(\begin{array}{cc}
P_{x}(x, y) & P_{y}(x, y) \\
Q_{x}(x, y) & Q_{y}(x, y)
\end{array}\right) .
$$

be the Jacobian matrix of $F$ at $(x, y)$. We denote by $D(x, y)=\operatorname{det} J_{F}(x, y)=P_{x}(x, y) Q_{y}(x, y)-$ $P_{y}(x, y) Q_{x}(x, y)$ its determinant and by $T(x, y)=P_{x}(x, y)+Q_{y}(x, y)$ its trace. $T(x, y)$ is the divergence of the vector field $F(x, y)$.

In what follows we consider the differential system associated to $F$ :

$$
\left\{\begin{array}{l}
\dot{x}=P(x, y),  \tag{2.1}\\
\dot{y}=Q(x, y) .
\end{array}\right.
$$

We denote by $\phi(t, x, y)$ the local flow defined by (2.1). We say that a critical point $O$ of (2.1) is a center if it has a punctured neighbourhood filled with non-trivial cycles surrounding $O$. The largest connected set $N_{O}$ filled with such cycles is called period annulus of $O$. If $N_{O}=\mathbb{R}^{2} \backslash\{O\}$, then $O$ is said to be a global center. We say that a critical point $O$ of (2.1) is asymptotically stable if it is stable and attractive [2]. In this case we denote by $A_{O}$ its attraction region. If $A_{O}=\mathbb{R}^{2}$ then $O$ is said to be globally asymptotically stable.

In the proof of Theorem 2.2 we repeatedly use $F$ injectivity. We report here the theorem applied, proved in [4].

Theorem 2.1. Let $F \in C^{1}\left(\mathbb{R}^{2}, \mathbb{R}^{2}\right)$ be such that:

1) $D(x, y)>0$ for all $(x, y) \in \mathbb{R}^{2}$;
2) there is a compact set $K \subset \mathbb{R}^{2}$ such that $J_{F}(x, y)$ has no real positive eigenvalues for any $(x, y) \notin K$.

Then $F$ is injective.

For the sake of simplicity, without loss of generality from now on we assume $O=(0,0)$. The hypotheses we consider rely only on derivatives properties, hence they do not change after a translation. We set

$$
T_{-}=\{(x, y): T(x, y)<0\}
$$

and denote by $\overline{T_{-}}$its closure. We denote by $\mu$ the 2 -dimensional Lebesgue measure.
Theorem 2.2. Assume $D(x, y)>0$ and $T(x, y) \leq 0$ for all $(x, y) \in \mathbb{R}^{2}$. Let $O$ be a critical point of (2.1). Then:
i) $O$ is a center if and only if it has a neighbourhood $U_{O}$ such that $T(x, y)$ vanishes identically on $U_{O}$; in such a case (2.1) is Hamiltonian on all of $N_{O}$; if, additionally, $F$ is analytic, then the system is Hamiltonian and $O$ is a global center.
ii) $O$ is asymptotically stable if and only if it belongs to $\overline{T_{-}}$; in such a case $O$ is globally asymptotically stable.
iii) If $T(x, y)$ does not vanish identically, then there exists a globally asymptotically stable compact set $M$.

Proof. i.1) We claim that if $O$ is a center, then $T(x, y)$ vanishes identically on $N_{O}$. By absurd, assume $T\left(x^{*}, y^{*}\right)<0$ for some $\left(x^{*}, y^{*}\right) \in N_{O}$. By continuity there exists a neighbourhood $U^{*}$ of $\left(x^{*}, y^{*}\right)$ such that $T(x, y)<0$ for all $(x, y) \in U^{*}$. Let $\gamma^{*}$ be the cycle passing through $\left(x^{*}, y^{*}\right)$ and $\Delta^{*}$ the bounded planar region having $\gamma^{*}$ as boundary. $\Delta^{*}$ is invariant, hence $\mu\left(\Delta^{*}\right)=\mu\left(\phi\left(t, \Delta^{*}\right)\right)$ for all $t \in \mathbb{R}$. By Liouville's theorem one has

$$
0=\frac{d}{d t} \mu\left(\phi\left(t, \Delta^{*}\right)\right)=\int_{\phi\left(t, \Delta^{*}\right)} T(x, y) d x d y<0,
$$

because $T(x, y)<0$ on $\phi\left(t, \Delta^{*} \cap U^{*}\right)$, contradiction.
i.2) Vice-versa, assume $T(x, y)$ to vanish identically on a neighbourhood $U_{O}$ of $O$. Then the system is Hamiltonian on a simply connected neighbourhood $V_{O} \subset U_{O}$. Let $H(x, y)$ be its Hamiltonian function. One has

The Hessian matrix of $H(x, y)$ is

$$
J_{F}(x, y)=\left(\begin{array}{cc}
H_{x x} & H_{x y}  \tag{2.2}\\
H_{y x} & H_{y y}
\end{array}\right)=\left(\begin{array}{c}
Q_{x} \\
Q_{y} \\
-P_{x}-P_{y}
\end{array}\right) .
$$

The Hessian determinant is $H_{x x} H_{y y}-H_{x y} H_{y x}=P_{x} Q_{y}-P_{y} Q_{x}=D(x, y)>0$, hence $H(x, y)$ has a minimum at $O$. As a consequence, $O$ is a center.
i.3) If additionally $F$ is analytic, then also $T(x, y)$ is analytic. If it vanishes in a neighbourhood of $O$ then it vanishes on all of $\mathbb{R}^{2}$, hence the system is Hamiltonian on all of $\mathbb{R}^{2}$. We claim that $N_{O}$ is unbounded. In fact, let us assume by absurd $N_{O}$ is bounded, hence also $\partial N_{O}$ is bounded. By $F$ injectivity [4], $\partial N_{O}$ contains no critical points, hence by PoincaréBendixson theorem $\partial N_{O}$ is a non-trivial cycle. One can consider the Poincaré map defined on a section $\Sigma$ of $\partial N_{O}$. Such a map is analytic and coincides with the identity map on $\Sigma \cap N_{O}$, hence it coincides with the identity map on all of $\Sigma$. As a consequence every orbit meeting $\Sigma \cap \partial N_{O}$ is a cycle, hence $\partial N_{O}$ is contained in the period annulus, contradicting the fact that it is the boundary of $N_{O}$. Moreover, every connected components of $\partial N_{O}$ is unbounded. In fact, if a connected components of $\partial N_{O}$ was bounded, then by its invariance and by PoincaréBendixson theorem either it would be a cycle or it would contain a critical point. The former case has already been considered above, the latter one can be excluded by the injectivity of $F$.
i.4) In order to prove that $O$ is a global center we use again the injectivity of $F$. For $\varepsilon>0$ let $B_{\varepsilon}$ be the open disk of radius $\varepsilon>0$ centered at $O$. $F$ is a diffeomorphism, hence the antiimage $D_{\varepsilon}=F^{-1}\left(B_{\varepsilon}\right)$ is an open neighbourhood of $O$. By construction and by the injectivity of $F, D_{\varepsilon}$ contains all the points of $(x, y) \in \mathbb{R}^{2}$ such that $|F(x, y)|<\varepsilon$, hence for all $(x, y) \notin D_{\varepsilon}$ one has $|F(x, y)| \geq \varepsilon$. Let us choose $\varepsilon$ small enough such that $\partial N_{O} \cap D_{\varepsilon}=\varnothing$. Let $\partial N_{O}^{u}$ be an unbounded component of $\partial N_{O}$. Then working as in [8], since $T(x, y) \leq 0$ and $|F(x, y)| \geq \varepsilon$ outside $D_{\varepsilon}$, one proves that every orbit starting close enough to $\partial N_{O}^{u}$ is unbounded too, hence it is not a cycle, contradicting the fact that $\partial N_{O}^{u}$ is in the boundary of $N_{O}$. As a consequence $\partial N_{O}=\varnothing$ and $N_{O}=\mathbb{R}^{2} \backslash\{O\}$.
ii) Assume $O$ to be asymptotically stable and $A_{O}$ its region of attraction. By hypothesis, in every neighbourhood of $O$ there are points such that $T(x, y)<0$, and by continuity this occurs in an open subset of $A_{O}$. If by absurd $A_{O}$ is bounded, then by its invariance, for all $t$

$$
0=\frac{d}{d t} \mu\left(\phi\left(t, A_{O}\right)\right)=\int_{\phi\left(t, A_{O}\right)} T(x, y) d x d y<0
$$

contradiction. Hence $A_{O}$ is unbounded. Assume by absurd there exists a bounded connected component $\partial A_{O}^{b}$ of $\partial A_{O}$. As above, by Poincaré-Bendixson theorem either it is a cycle or contains a critical point. If it is a cycle, it cannot surround $O$, since in such a case $A_{O}$ would be bounded. Hence it surrounds another critical point, violating $F$ injectivity. The same violation would occur if $\partial A_{O}^{b}$ contained a critical point. Then the argument proceeds as in point $i .4$ ), showing that $\partial A_{O}=\varnothing$ and $A_{O}=\mathbb{R}^{2}$.

Vice-versa, assume $O \in \overline{T_{-}}$. Then $T(x, y)$ does not vanish identically on any neighbourhood $U_{O}$ of $O$, hence by point $i$ ) it is not a center. By the hypotheses on $D(0,0)$ and $T(0,0), O$ is a non degenerate elementary critical point of center-focus type, according to the real part of its eigenvalues. If such real parts are negative $O$ is a focus, hence asymptotically stable. If such real parts are zero, one proves, as at the beginning of point $i$ ), that $O$ cannot be accumulation point of cycles, hence it is asymptotically stable. Working as in point i.4) one proves that it is globally asymptotically stable.
iii) If $O \in \overline{T_{-}}$, then point $i i$ ) applies and one can take $M=\{O\}$.

If $O \notin \overline{T_{-}}$, it has a neighbourhood $U_{O}$ where $T(x, y)$ vanishes identically, hence it is a center. We claim that $N_{O}$ is bounded. In fact, if $N_{O}$ is unbounded one can proceed as in point $i .4$ ), in order to prove that every orbit starting close enough to $\partial N_{O}^{u}$ is unbounded, contradicting the fact that $\partial N_{O}^{u}$ is part of the boundary. The boundedness of $N_{O}$ implies the boundedness of $\partial N_{O}$, which is a cycle, by the absence of critical points on $\partial N_{O}^{u}$. Let us consider a section $\Sigma$ of $\partial N_{O}$ and its Poincaré map. Such a map is the identity on $\Sigma \cap N_{O}$, and has no fixed points on $\Sigma \backslash N_{O}$, otherwise there would be a cycle $\gamma$ containing $\partial N_{O}, T(x, y)$ would vanish identically inside $\gamma$ and every orbit inside $\gamma$ would be a cycle, contradicting the fact that $\partial N_{O}$ is the boundary of $N_{O}$. Hence the Poincaré map is strictly monotone, which implies either attractivity or repulsivity of $\partial N_{O}$. Repulsivity is not compatible with the sign of the divergence, hence $\partial N_{O}$ is attractive, and $\overline{N_{O}}$ is asymptotically stable. Its global attractivity can be proved as in $i .4$ ) and $i i$ ), proving that the boundary of its region of attraction is empty.

An example of globally asymptotically stable critical point belonging to $\overline{T_{-}}$is the origin in the following differential system,

$$
\left\{\begin{array}{l}
\dot{x}=y,  \tag{2.3}\\
\dot{y}=-x-y^{3},
\end{array}\right.
$$

for which one has

$$
J_{F}(x, y)=\left(\begin{array}{rc}
0 & 1 \\
-1 & -3 y^{2}
\end{array}\right) .
$$

One has $D(x, y)=1, T(x, y)=-3 y^{2} \leq 0$, hence $T_{-}$is $x$-axis.
If (2.1) is not analytic, then a center need not be global. We construct a system satisfying the hypotheses of Theorem 2.2, having a non-global center and a globally asymptotically stable compact set. Let $\alpha \in C^{\infty}(\mathbb{R}, \mathbb{R})$ be such that

$$
\left\{\begin{aligned}
\alpha(r)=0, & r \leq 1, \\
\alpha(r)>0, & r>1, \\
\alpha^{\prime}(r)>0, & r>1 .
\end{aligned}\right.
$$

Let us set $r=\sqrt{x^{2}+y^{2}}$. The vector field defined by the system

$$
\left\{\begin{array}{l}
\dot{x}=y-x \alpha(r),  \tag{2.4}\\
\dot{y}=-x-y \alpha(r) .
\end{array}\right.
$$

Setting $c r=x, s r=y$, the Jacobian matrix of the vector field is

$$
J_{F}(x, y)=\left(\begin{array}{rl}
-\alpha(r)-x c \alpha^{\prime}(r) & 1-x s \alpha^{\prime}(r) \\
-1-y c \alpha^{\prime}(r) & -\alpha(r)-y s \alpha^{\prime}(r)
\end{array}\right) .
$$

Its determinant is $1+\alpha^{2}(r)+r \alpha(r) \alpha^{\prime}(r)>0$ and its trace is $-2 \alpha(r)-2 r \alpha^{\prime}(r) \leq 0$. For $r \leq 1$ the trace is zero, for $r>1$ the trace is negative. The system (2.4) is Hamiltonian for $r \leq 1$, with a center at $O$ whose central region is the disk of radius 1 centered at $O$. Such a disk is a global attractor, since $\dot{r}<0$ for $r>1$.

## References

[1] J. Bernat, J. Llibre, Counterexample to Kalman and Markus-Yamabe conjectures in dimension larger than 3, Dynam. Contin. Discrete Impuls. Systems 2(1996), No. 3, 337-379. MR1411771; Zbl 0889.34047
[2] N. P. Bhatia, G. P. Szegö, Stability theory of dynamical systems, Die Grundlehren der mathematischen Wissenschaften, Band 161, Springer-Verlag, New York-Berlin, 1970. MR0289890; Zbl 0213.10904
[3] A. Cima, A. van den Essen, A. Gasull, E. Hubbers, F. Mañosas, A polynomial counterexample to the Markus-Yamabe conjecture, Adv. Math. 131(1997), No. 2, 453-457. https://doi.org/10.1006/aima.1997.1673; MR1483974; Zbl 0896.34042
[4] R. Fessler, A proof of the two-dimensional Markus-Yamabe stability conjecture and a generalization, Ann. Pol. Math. 62(1995), No. 1, 45-74. https://doi.org/10.4064/ ap-62-1-45-74; MR1348217; Zbl 0835.34052
[5] A. A. Glutsyuk, Complete solution of the Jacobian problem for planar vector fields (in Russian), Uspekhi Mat. Nauk 49(1994), No. 3, 179-180, translation in Russian Math. Surveys 49(1994), No. 3, 185-186. https://doi.org/10.1070/RM1994v049n03ABEH002262; MR1289394; Zbl 0828.34041
[6] C. Gutiérrez, A solution to the bidimensional global asymptotic stability conjecture, Ann. Inst. H. Poincaré Anal. Non Linéaire 12(1995), No. 6, 627-671. https://doi.org/10. 1016/S0294-1449(16) 30147-0; MR1360540; Zbl 0837.34057
[7] L. Markus, H. Yamabe, Global stability criteria for differential systems, Osaka Math. J. 12(1960), 305-317. MR0126019; Zbl 0096.28802
[8] C. Olech, On the global stability of an autonomous system on the plane, Contributions to Differential Equations 1(1963), 389-400. MR0147734

Electronic Journal of Qualitative Theory of Differential Equations

# Well-posedness for a fourth-order equation of Moore-Gibson-Thompson type 

Carlos Lizama ${ }^{\boxtimes 1}$ and Marina Murillo*2<br>${ }^{1}$ Universidad de Santiago de Chile, Facultad de Ciencias, Departamento de Matemática y Ciencia de la Computación, Las Sophoras 173, Estación Central, Santiago, Chile<br>${ }^{2}$ Universitat Politècnica de València, Instituto Universitario de Matemática Pura y Aplicada, 46022 València, Spain

Received 21 March 2021, appeared 14 October 2021
Communicated by Roberto Livrea


#### Abstract

In this paper, we completely characterize, only in terms of the data, the well-posedness of a fourth order abstract evolution equation arising from the Moore-Gibson-Thomson equation with memory. This characterization is obtained in the scales of vector-valued Lebesgue, Besov and Triebel-Lizorkin function spaces. Our characterization is flexible enough to admit as examples the Laplacian and the fractional Laplacian operators, among others. We also provide a practical and general criteria that allows $L^{p}-L^{q}$-well-posedness.


Keywords: well-posedness, Moore-Gibson-Thompson equation, operator-valued multipliers, $R$-boundedness.
2020 Mathematics Subject Classification: Primary: 45Q05, 49K40. Secondary: 35K65, 35L80.

## 1 Introduction

In recent years, there has been considerable interest in mathematical models that are close to practical situations of the real life. In the context of acoustics, and in order to gain a better understanding of the nonlinear model, a typical and standard reference is the linearized part of the Westervelt equation [25] i.e.

$$
\frac{\delta}{c_{0}^{4}} u^{\prime \prime \prime}(t)+\Delta u(t)-\frac{1}{c_{0}^{2}} u^{\prime \prime}(t)=0, \quad t \geq 0,
$$

where $u$ denotes the sound pressure, $c_{0}$ is the small signal sound speed, $\delta$ is the sound diffusivity and $\Delta$ denotes the Laplacian operator. An extension of the Westervelt equation that takes into account second sound effects and the associated thermal relaxation in viscous fluids is the Moore-Gibson-Thomson (MGT) equation

$$
\begin{equation*}
\tau u^{\prime \prime \prime}(t)+u^{\prime \prime}(t)-c^{2} \Delta u^{\prime}(t)-b_{0} \Delta u(t)=0, \quad t \geq 0, \tag{1.1}
\end{equation*}
$$

[^43]where $b_{0}=\delta+\tau c^{2}$, see $[16,27-29,35]$. The MGT equation with memory
\[

$$
\begin{equation*}
u^{\prime \prime \prime}(t)+a u^{\prime \prime}(t)-b \Delta u^{\prime}(t)-c \Delta u(t)+\int_{0}^{t} g(t-s) \Delta u(s) d s=0, \quad t \geq 0, \tag{1.2}
\end{equation*}
$$

\]

has been treated in $[13,20,32,33]$. When $g \neq 0$, the memory term introduces further dissipation. From the physical point of view, the most relevant case in connection with (1.2) is

$$
g(s)=d e^{-\ell s}, \quad d, \ell>0
$$

Motivated by the above kernel, the following model

$$
\begin{equation*}
u^{\prime \prime \prime \prime}(t)+\alpha u^{\prime \prime \prime}(t)+\beta u^{\prime \prime}(t)-\gamma \Delta u^{\prime \prime}(t)-\delta \Delta u^{\prime}(t)-\rho \Delta u(t)=0, \quad t \geq 0, \tag{1.3}
\end{equation*}
$$

has been recently proposed [21,34]. It can be obtained from (1.2) summing $\partial_{t}(1.2)+\ell(1.2)$.
It should be pointed out that third and fourth order derivatives in time are observed in various areas of research. In physics and engineering third and fourth order derivatives should always be considered when vibration occurs and particularly when this excitation induces multi-resonant modes of vibration [6]. They should also be considered at all times when a transition occurs such as: start up and shutdown; take-off and landing; and accelerating and decelerating [23]. Fourth order derivatives in time appear, for instance, in the study of chaotic hyperjerk systems [17], in the Taylor series expansion of the Hubble law [37] and in the kinematic performance of long-dwell mechanisms of linkage type, which are used in automatic machines to generate intermittent motions [24].

The model (1.3) was introduced and first studied by Dell'Oro and Pata [21] in their abstract version

$$
\begin{equation*}
u^{\prime \prime \prime \prime}(t)+\alpha u^{\prime \prime \prime}(t)+\beta u^{\prime \prime}(t)+\gamma A u^{\prime \prime}(t)+\delta A u^{\prime}(t)+\rho A u(t)=0, \tag{1.4}
\end{equation*}
$$

where $A$ is a strictly positive unbounded linear operator with domain $D(A)$ densely embedded in a separable real Hilbert space $H$ and $\alpha, \beta, \gamma, \delta, \rho \in \mathbb{R}$. In such abstract model, the equation (1.3) corresponds to the choice $H=L^{2}(\Omega)$ and $A=-\Delta$ with $D(A)=H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$. In [21] it was established the well-posedness for (1.4) by means of the existence of the solution semigroup, providing a detailed description of the spectrum of its infinitesimal generator and its relation with the growth bound. The stability properties of the related solution semigroup were then investigated and, in particular, a necessary and sufficient condition for exponential stability was established, in terms of the values of the stability numbers

$$
\chi=\gamma-\frac{\delta}{\alpha}, \quad \omega=\beta-\frac{\rho \alpha}{\delta}
$$

where $\alpha, \beta, \gamma, \delta$ and $\rho$ are strictly positive. Later, Liu et al. [34] discussed the well-posedness of the solution for (1.4) with an additional memory term like in (1.2) by using the Faedo-Galerkin method. Then, the authors in [34] proved general decay results for the case $\chi>0$ and $\omega>0$ based on the perturbed energy method and on some properties of convex functions.

However, we note that all above mentioned references studied (1.4) in the context of Hilbert spaces, and they do not include the important cases of the Lebesgue spaces $L^{q}(\Omega)$ except, of course, the case $q=2$. Furthermore, the class of operators $A$ studied so far does not allow the admissibility of more general types of differential operators like the Stokes operator, the fractional Laplacian operator or the biharmonic $\Delta^{2}$, equipped with suitable boundary conditions.

On the other hand, using the method of operator-valued Fourier multipliers due to Arendt and $\mathrm{Bu}[4,5]$, well-posedness of the solutions for the nonhomogeneous MGT equation (1.2) in the class of $\mathcal{H} \mathcal{T}$ (or UMD) spaces, that includes the scale of Lebesgue spaces $L^{q}(\Omega)$ among others, has been studied by Poblete and Pozo [36], Bu and Cai [7] and Conejero et al. [19]. This method allows the admissibility of very general linear operators $A$ but, depending on the regularity on the time variable, sometimes needs a more restrictive condition on the associated operator-valued symbols, namely: $R$-boundedness [4,10]. This restrictive condition can be replaced by uniform boundedness if we assume, for instance, that time-regularity is needed in the scales of Besov spaces (that includes the class of Hölder continuous functions) [5,11,12] or the scale of Triebel-Lizorkin spaces [8,9,14].

In this paper we will take this last approach as method. We succeed in obtaining a completely new characterization of strongly well-posedness for the nonhomogeneous equation (1.4) in the the scales of Lebesgue, Besov and Triebel-Lizorkin spaces. For that purpose, we take advantage of a recent result proved in [19, Theorem 1.1] in order to simplify complex computations on the operator-valued symbols associated to the corresponding nonhomogeneous model (1.4). In the case of the scale of Lebesgue spaces, our result reads as follows: Assume that $A$ is a closed linear operator with (not necessarily dense) domain $D(A)$ defined on a $U M D$ space $X$. The following assertions are equivalent:
(i) The equation

$$
\begin{equation*}
u^{\prime \prime \prime \prime}(t)+\alpha u^{\prime \prime \prime}(t)+\beta u^{\prime \prime}(t)+\gamma A u^{\prime \prime}(t)+\delta A u^{\prime}(t)+\rho A u(t)=f(t), \quad t \in \mathbb{T}:=[0,2 \pi], \tag{1.5}
\end{equation*}
$$

is strongly $L^{p}$-well-posed, i.e. for each $f \in L^{p}(\mathbb{T}, X)$, there exists a unique solution

$$
u \in W_{p e r}^{4, p}(\mathbb{T}, X) \cap W_{p e r}^{2, p}(\mathbb{T},[D(A)]) .
$$

(ii) $\mathbb{Z} \subset \rho_{s}(A)$ and the set $\left\{k^{4}\left[k^{4}-\alpha i k^{3}-\beta k^{2}-\gamma k^{2} A+\delta i k A+\rho A\right]^{-1}: k \in \mathbb{Z}\right\}$ is $R$-bounded.

Moreover, if (i) (or (ii)) holds, then the following maximal regularity estimate

$$
\begin{aligned}
\|u\|_{L^{p}(\mathbb{T}, X)}+ & \left\|u^{\prime \prime}\right\|_{W_{p e r}^{2, p}(\mathbb{T}, X)}+\left\|u^{\prime \prime \prime}\right\|_{W_{p e r}^{3, p}(\mathbb{T}, X)}+\left\|u^{\prime \prime \prime \prime}\right\|_{W_{p e r}^{4, p}(\mathbb{T}, X)} \\
& \quad+\|A u\|_{L^{p}(\mathbb{T},[D(A)])}+\left\|A u^{\prime}\right\|_{W_{p e r}^{1, p}(\mathbb{T},[D(A)])}+\left\|A u^{\prime \prime}\right\|_{W_{p e r}^{2, p}(\mathbb{T},[D(A)])} \leq C\|f\|_{L^{p}(\mathbb{T}, X)},
\end{aligned}
$$

holds. The last estimate has many important applications. It is the central tool in the study of the following problems: existence and uniqueness of solutions of nonautonomous evolution equations; existence and uniqueness of solutions of quasilinear and nonlinear partial differential equations; stability theory for evolution equations; maximal regularity of solutions of elliptic differential equations; existence and uniqueness of solutions of Volterra integral equations; and uniqueness of mild solutions of the NavierâĂŞStokes equations. In these applications, a maximal regularity estimate is frequently used to reduce, via a fixed-point argument, a nonautonomous (resp. nonlinear) problem to an autonomous (resp. linear) problem. In some cases, maximal regularity is needed to apply an implicit function theorem. According to the literature, there has been a substantial amount of work, as one can see, for example, in Amann [2], Denk, Hieber and Prüss [22], Clément, Londen and Simonett [18], the survey by Arendt [3], and the bibliography therein.

Our new characterization of strongly $L^{p}$-well-posedness shows to be flexible in certain combination of strictly positive parameters $\alpha, \beta, \gamma, \delta$ and $\rho$, and that is amenable enough to
allow fractional powers of operators. In fact, as a consequence of our results we deduce that if $A$ is an $R$-sectorial operator of angle $\pi / 2$ on $L^{q}(\Omega), \Omega \subset \mathbb{R}^{N}, 1<q<\infty$ and

$$
\rho+\beta \gamma<\alpha \delta
$$

then for any given $f \in L^{p}\left(\mathbb{T}, L^{q}(\Omega)\right), 1<p<\infty$, the initial value problem (1.5) admits a unique solution $u \in W_{p e r}^{4, p}\left(\mathbb{T}, L^{q}(\Omega)\right) \cap W_{p e r}^{2, p}(\mathbb{T},[D(A)])$. As a consequence, we obtain optimal results, that we illustrate with two examples: $A=\Delta$ the Laplacian, and $A=-(-\Delta)^{s}$ the fractional Laplacian of order $1 / 2<s<1$.

## 2 Preliminaries

We start this section introducing the notion of $L^{p}$-Fourier multiplier. We will denote the space of bounded linear operators from $X$ into $Y$ endowed with the uniform operator topology as $\mathcal{B}(X, Y)$. If $X=Y$ we simply abbreviate $\mathcal{B}(X)$.

Definition 2.1. Let $X$ and $Y$ be Banach spaces and $1 \leq p<\infty$. We say that $\left(M_{k}\right)_{k \in \mathbb{Z}} \subset \mathcal{B}(X, Y)$ is an $L^{p}$-Fourier multiplier if, for each $f \in L^{p}(\mathbb{T}, X)$, there exists $u \in L^{p}(\mathbb{T}, Y)$ such that $\hat{u}(k)=M_{k} \hat{f}(k)$ for all $k \in \mathbb{Z}$, where

$$
\hat{f}(k):=\frac{1}{2 \pi} \int_{0}^{2 \pi} e^{-i k t} f(t) d t
$$

denotes the $k$-th Fourier coefficient of $f$.
Our characterization will be provided in terms of the $R$-boundedness of certain sets of operators. For that purpose, we need to recall the notion of $R$-boundedness.

Definition 2.2. Let $X$ and $Y$ be Banach spaces. A set $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called $R$-bounded if there is a constant $c \geq 0$ such that

$$
\begin{equation*}
\left\|\left(T_{1} x_{1}, \ldots, T_{n} x_{n}\right)\right\|_{R} \leq c\left\|\left(x_{1}, \ldots, x_{n}\right)\right\|_{R} \tag{2.1}
\end{equation*}
$$

for all $T_{1}, \ldots, T_{n} \in \mathcal{T}, x_{1}, \ldots, x_{n} \in X, n \in \mathbb{N}$ where

$$
\left\|\left(x_{1}, \ldots, x_{n}\right)\right\|_{R}:=\frac{1}{2^{n}} \sum_{\epsilon_{j} \in\{-1,1\}^{n}}\left\|\sum_{j=1}^{n} \epsilon_{j} x_{j}\right\| .
$$

The least $c$ such that (2.1) is satisfied is called the $R$-bound of $\mathcal{T}$ and is denoted $R(\mathcal{T})$.
The property of $R$-boundedness is preserved under sum or product by a constant. Moreover, if $X$ and $Y$ are Hilbert spaces, $R$ - boundedness is equivalent to uniform boundedness. More information about these properties are summarized in [22].

The class of Banach spaces $X$ such that the Hilbert transform defined by

$$
(H f)(t)=\lim _{\epsilon, R \rightarrow \infty} \frac{1}{\pi} \int_{\epsilon \leq|s| \leq R} \frac{f(t-s)}{s} d s, \quad t \in \mathbb{R},
$$

is bounded in $L^{p}(\mathbb{R} ; X)$ for some $p \in(1, \infty)$ is denoted by $\mathcal{H} \mathcal{T}$. The basic reference for the class $\mathcal{H \mathcal { T }}$ is the survey article by Burkholder [15], where two other characterizations for the class $\mathcal{H} \mathcal{T}$ are also given, a probabilistic one, and a geometrical one. To describe the latter,
recall that a Banach space $X$ is termed $\xi$-convex, if there is a function $\xi: X \times X \rightarrow \mathbb{R}$ which is convex in each of its variables and such that $\xi(0,0)>0$ and

$$
\xi(x, y) \leq|x+y| \quad \text { for all } x, y \in X \text { with }|x|=|y|=1 .
$$

A Banach space $X$ belongs to the class $\mathcal{H} \mathcal{T}$ if and only if $X$ is $\xi$-convex if and only if $X$ has the unconditional martingale difference property (UMD) [15]. The UMD spaces include Hilbert spaces, Sobolev spaces $H_{p}^{s}(\Omega), 1<p<\infty$, Lebesgue spaces $L^{p}(\Omega, \mu), \ell_{p}, 1<p<\infty$, vectorvalued Lebesgue spaces $L^{p}(\Omega, \mu ; X)$ where $X$ is a UMD space, Hardy spaces, Lorentz and Orlicz spaces, any von Neumann algebra, and the Schatten-von Neumann classes $C_{p}(H) ; 1<$ $p<1$; of operators on Hilbert spaces. On the other hand, the space of continuous functions $C(K)$ does not have the $U M D$ property.

We need to recall the notion of $M$-bounded sequence ( $M R$-bounded sequence) of operators.

Definition 2.3 ([31]). We say that a sequence $\left\{T_{k}\right\}_{k \in \mathbb{Z}} \subset \mathcal{B}(X, Y)$ is $M$-bounded of order $n$ ( $n \in \mathbb{N} \cup\{0\}$ ), if

$$
\begin{equation*}
\sup _{0 \leq l \leq n} \sup _{k \in \mathbb{Z}}\left\|k^{l} \Delta^{l} T_{k}\right\|<\infty, \tag{2.2}
\end{equation*}
$$

where

$$
\Delta^{0} T_{k}:=T_{k}, \quad \Delta T_{k}:=\Delta^{1} T_{k}:=T_{k+1}-T_{k}
$$

and for $n \in \mathbb{N}$ with $n \geq 2$ we have

$$
\Delta^{n} T_{k}:=\Delta\left(\Delta^{n-1} T_{k}\right) .
$$

## Remark 2.4.

(i) Given $\left\{M_{k}\right\}_{k \in \mathbb{Z}}$ and $\left\{N_{k}\right\}_{k \in \mathbb{Z}}$ be such that they are both $M$-bounded of order $n$, then the sum is also $M$-bounded of the same order. Moreover, if $\left\{M_{k}\right\}_{k \in \mathbb{Z}}$ and $\left\{N_{k}\right\}_{k \in \mathbb{Z}}$ are sequences in $\mathcal{B}(Y, Z)$ and $\mathcal{B}(X, Y)$ that are $M$-bounded of order $n$, then $\left\{M_{k} N_{k}\right\}_{k \in \mathbb{Z}} \subset$ $\mathcal{B}(X, Z)$ is also $M$-bounded of the same order.
(ii) If we replace condition (2.2) in Definition 2.3 by the condition that the set

$$
\begin{equation*}
\left\{k^{l} \Delta^{l} M_{k}: k \in \mathbb{Z}\right\} \tag{2.3}
\end{equation*}
$$

is $R$-bounded for each $0 \leq l \leq n$, then we say that $\left\{M_{k}\right\}_{k \in \mathbb{Z}} \subset \mathcal{B}(X, Y)$ is $M R$-bounded of order $n$.

We also recall the definition of $n$-regular scalar sequences which was first considered in [31].

Definition 2.5. A sequence $\left\{c_{k}\right\}_{k \in \mathbb{Z}} \subset \mathbb{C}$ is called $n$-regular if the set $\left\{k^{p} \frac{\Delta c_{k}}{c_{k}}\right\}_{k \in \mathbb{Z}}$ is bounded for all $p=1, \ldots, n$.

We finally recall the following result recently shown in [19] which provides an important criterion for $M R$-boundedness in the context of maximal regularity for abstract evolution equations.

Theorem 2.6. Let $T: D(T) \subset X \rightarrow X$ be a closed linear operator defined in a Banach space $X$. For each $k \in \mathbb{Z}$ let $H_{k}: X \rightarrow D(T)$ be a sequence of bounded and linear operators such that $0 \in \rho\left(H_{k}\right)$ for all $k \in \mathbb{Z}$. Suppose that $\left(s_{k}\right)_{k \in \mathbb{Z}} \subset \mathbb{C}$ is a 1 -regular sequence and denote

$$
\begin{equation*}
M_{k}:=s_{k} T H_{k} \tag{2.4}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{k}:=\left(H_{k}^{-1}-H_{k+1}^{-1}\right) H_{k} \tag{2.5}
\end{equation*}
$$

If $\left\{M_{k}: k \in \mathbb{Z}\right\}$ and $\left\{k L_{k}: k \in \mathbb{Z}\right\}$ are $R$-bounded (uniformly bounded) sets, then $\left\{M_{k}: k \in \mathbb{Z}\right\}$ is $M R$-bounded ( $M$-bounded) of order 1. If, in addition, $\left(s_{k}\right)_{k \in \mathbb{Z}}$ is 2 -regular and the set $\left\{k^{2} \Delta L_{k}: k \in \mathbb{Z}\right\}$ is $R$-bounded (uniformly bounded), then $\left\{M_{k}: k \in \mathbb{Z}\right\}$ is MR-bounded (M-bounded) of order 2.

## 3 Well-posedness in $L^{p}$-spaces

Let $1 \leq p<\infty$ and $X$ be a Banach space. In this section, we want to give optimal conditions that can describe the well-posedness of the problem

$$
\begin{equation*}
u^{\prime \prime \prime \prime}(t)+\alpha u^{\prime \prime \prime}(t)+\beta u^{\prime \prime}(t)+\gamma A u^{\prime \prime}(t)+\delta A u^{\prime}(t)+\rho A u(t)=f(t), \quad t \in \mathbb{T}:=[0,2 \pi] \tag{3.1}
\end{equation*}
$$

in $2 \pi$-periodic vector valued $L^{p}$-spaces. In other words, we want to obtain a complete characterization on the existence, uniqueness and well-posedness of the problem only in terms of the data of the problem. Here $A$ is a closed linear operator with domain $D(A)$.

We now introduce the notion of the following set denoted as $\rho_{s}(A)$ as follows:

$$
\begin{align*}
\rho_{s}(A):=\left\{s \in \mathbb{R}: s^{4}-\alpha i s^{3}-\beta s^{2}-\gamma s^{2} A+\delta i s A+\rho A:\right. & {[D(A)] \rightarrow X } \\
& \text { is invertible and } \left.\left[s^{4}-\alpha i s^{3}-\beta s^{2}-\gamma s^{2} A+\delta i s A+\rho A\right]^{-1} \in \mathcal{B}(X)\right\}, \tag{3.2}
\end{align*}
$$

where $[D(A)]$ denotes a Banach space under the norm $\|x\|_{[D(A)]}:=\|x\|+\|A x\|$.
For any $n \in \mathbb{N}$ and $1 \leq p<\infty$ we define the vector-valued function spaces [7, Definition 2.4]:

$$
W_{p e r}^{n, p}(\mathbb{T}, X):=\left\{u \in L^{p}(\mathbb{T}, X): \text { there exists } v \in L^{p}(\mathbb{T}, X), \hat{v}(k)=(i k)^{n} \hat{u}(k) \text { for all } k \in \mathbb{Z}\right\} .
$$

Remark 3.1. It is important to point out that the following properties hold
(i) Given $n, m \in \mathbb{N}$, if $n \leq m$ then $W_{p e r}^{m, p}(\mathbb{T}, X) \subset W_{p e r}^{n, p}(\mathbb{T}, X)$.
(ii) If $u \in W_{p e r}^{n, p}(\mathbb{T}, X)$ then for all $0 \leq k \leq n-1$ it follows that $u^{(k)}(0)=u^{(k)}(2 \pi)$.

Note that [4]:

$$
\begin{aligned}
& W_{p e r}^{n, p}(\mathbb{T}, X)=\left\{u \in L^{p}(\mathbb{T}, X): u \text { is } n\right. \text {-times differentiable a.e., } \\
& \left.\qquad u^{(n)} \in L^{p}(\mathbb{T}, X) \text { and } u^{(k)}(0)=u^{(k)}(2 \pi), 0 \leq k \leq n-1\right\} .
\end{aligned}
$$

We refer to [4, Lemma 2.1] and [7] for more information about these spaces. In order to consider maximal regularity for our problem we need to define the following space:

$$
S_{p}(A):=W_{p e r}^{4, p}(\mathbb{T}, X) \cap W_{p e r}^{2, p}(\mathbb{T},[D(A)]) .
$$

The space $S_{p}(A)$ is a Banach space with the norm

$$
\|u\|_{S_{p}(A)}:=\|A u\|_{p}+\left\|A u^{\prime}\right\|_{p}+\left\|A u^{\prime \prime}\right\|_{p}+\|u\|_{p}+\left\|u^{\prime \prime}\right\|_{p}+\left\|u^{\prime \prime \prime}\right\|_{p}+\left\|u^{\prime \prime \prime \prime}\right\|_{p} .
$$

We now introduce the following definition.

Definition 3.2. Let $1 \leq p<\infty$ and $f \in L^{p}(\mathbb{T}, X)$ be given. We say that $u \in S_{p}(A)$ is a strong $L^{p}$-solution of equation (3.1) if it satisfies (3.1) for almost all $t \in \mathbb{T}$. We say that equation (3.1) is strongly $L^{p}$-well-posed if for each $f \in L^{p}(\mathbb{T}, X)$, there exists a unique strong $L^{p}$-solution of equation (3.1).

As a very important consequence, we obtain the following: There exists a constant $C>0$ such that for each $f \in L^{p}(\mathbb{T}, X)$, we have

$$
\|u\|_{S_{p}(A)} \leq C\|f\|_{L^{p}}
$$

Before we provide our main result, we need the following two theorems from [4] that establish the equivalence between $R$-boundedness and the fact of being an $L^{p}$-multiplier. They will be needed in order to characterize $L^{p}$-well-posedness for equation (3.1).

Theorem 3.3. Let $X, Y$ be UMD spaces. If a sequence $\left\{M_{k}\right\}_{k \in \mathbb{Z}} \subset \mathcal{B}(X, Y)$ is $M R$-bounded of order 1 , then $\left(M_{k}\right)_{k \in \mathbb{Z}}$ defines an $L^{p}$-Fourier multiplier whenever $1<p<\infty$.

Theorem 3.4. Let $X, Y$ be Banach spaces, $1 \leq p<\infty$ and let $\left(M_{k}\right)_{k \in \mathbb{Z}} \subset \mathcal{B}(X, Y)$ be an $L^{p}$-Fourier multiplier. Then the set $\left\{M_{k}: k \in \mathbb{Z}\right\}$ is $R$-bounded.

Let $A$ be a closed linear operator such that $\mathbb{Z} \subset \rho_{s}(A)$. We denote

$$
\begin{equation*}
N_{k}:=\left[a_{k}-\alpha b_{k}-\beta c_{k}-\gamma c_{k} A+\operatorname{sik} A+\rho A\right]^{-1}, \quad a_{k}=k^{4}, \quad b_{k}=i k^{3}, \quad c_{k}=k^{2}, \quad k \in \mathbb{Z} \tag{3.3}
\end{equation*}
$$

where $\alpha, \beta, \gamma, \delta, \rho \in \mathbb{R}$ are fixed constants.
The following proposition will be an important tool for proving the main result of this section.

Proposition 3.5. Let $A$ be a closed linear operator defined on a $U M D$ space $X$ and $\alpha, \beta, \gamma, \delta, \rho \in \mathbb{R}$. If $\mathbb{Z} \subset \rho_{s}(A)$ and $\left\{k^{4} N_{k}: k \in \mathbb{Z}\right\}$ and $\left\{k^{2} A N_{k}: k \in \mathbb{Z}\right\}$ are $R$-bounded sets, then $\left(k^{4} N_{k}\right)_{k \in \mathbb{Z}}$, $\left(i k^{3} N_{k}\right)_{k \in \mathbb{Z}^{\prime}}\left(k^{2} N_{k}\right)_{k \in \mathbb{Z}^{\prime}}\left(k^{2} A N_{k}\right)_{k \in \mathbb{Z}^{\prime}}\left(k A N_{k}\right)_{k \in \mathbb{Z}}$ and $\left(A N_{k}\right)_{k \in \mathbb{Z}}$ are $L^{p}$-Fourier multipliers.

Proof. We first point out that the $R$-boundedness of $\left\{k^{4} N_{k}: k \in \mathbb{Z}\right\}$ immediately implies the $R$-boundedness of the sets $\left\{k^{3} N_{k}: k \in \mathbb{Z}\right\}$ and $\left\{k^{2} N_{k}: k \in \mathbb{Z}\right\}$. Similarly, if by hypothesis $\left\{k^{2} A N_{k}: k \in \mathbb{Z}\right\}$ is $R$-bounded then the sets $\left\{k A N_{k}: k \in \mathbb{Z}\right\}$ and $\left\{A N_{k}: k \in \mathbb{Z}\right\}$ are so. Let $M_{k}:=k^{4} N_{k}$. In order to show that $M_{k}$ is an $L^{p}$-multiplier we only need to show that $\left\{k \Delta M_{k}: k \in \mathbb{Z}\right\}$ is $R$-bounded. We apply Theorem 2.6 with $s_{k}=k^{4}$, which is 1-regular, $H_{k}=N_{k}$ and $T=I$. By hypothesis $\left\{M_{k}: k \in \mathbb{Z}\right\}$ is $R$-bounded, then we only need to show that $\left\{k L_{k}: k \in \mathbb{Z}\right\}$ is $R$-bounded. Indeed, we have

$$
\begin{aligned}
k L_{k} & =k\left(N_{k}^{-1}-N_{k+1}^{-1}\right) N_{k} \\
& =k\left[-\Delta a_{k}+\alpha \Delta b_{k}+\beta \Delta c_{k}+\gamma \Delta c_{k} A-\delta i A\right] N_{k} \\
& =-\frac{k \Delta a_{k}}{a_{k}} M_{k}+\alpha \frac{k \Delta b_{k}}{b_{k}}\left(b_{k} N_{k}\right)+\beta \frac{k \Delta c_{k}}{c_{k}}\left(c_{k} N_{k}\right)+\gamma \frac{k \Delta c_{k}}{c_{k}}\left(c_{k} A N_{k}\right)-\delta i k A N_{k}
\end{aligned}
$$

By hypothesis then it follows that $\left\{k L_{k}: k \in \mathbb{Z}\right\}$ is $R$-bounded. The $R$-boundedness of $\left\{k \Delta\left(i k^{3} N_{k}\right)\right\}_{k \in \mathbb{Z}}$ and $\left\{k \Delta\left(k^{2} N_{k}\right)\right\}_{k \in \mathbb{Z}}$ follows similarly applying Theorem 2.6 with $s_{k}=i k^{3}$, $T=I$ and $H_{k}=N_{k}$ in the first case, $s_{k}=k^{2}, T=I$ and $H_{k}=N_{k}$ in the second case. As a consequence of Theorem 3.3 they are $L^{p}$-Fourier multipliers. On the other hand, the $R$-boundedness of $\left\{k \Delta\left(k^{2} A N_{k}\right)\right\}_{k \in \mathbb{Z}},\left\{k \Delta\left(k A N_{k}\right)\right\}_{k \in \mathbb{Z}}$ and $\left\{k \Delta\left(A N_{k}\right)\right\}_{k \in \mathbb{Z}}$ also follows from Theorem 2.6 with $s_{k}=k^{2}, T=A$ and $H_{k}=N_{k}$ in the first case, $s_{k}=k, T=A$ and $H_{k}=N_{k}$ in the second case and $s_{k}=1, T=A$ and $H_{k}=N_{k}$ in the last case.

We now show the main result of this section that provides a computable criterion to characterize the well-posedness of equation (3.1).

Theorem 3.6. Let $1<p<\infty$ and $\alpha, \beta, \gamma, \delta, \rho \in \mathbb{R}$ be given with $(\gamma, \delta, \rho) \neq(0,0,0)$. Assume that $A$ is a closed linear operator defined on a UMD space $X$. The following assertions are equivalent:
(i) Equation (3.1) is strongly L ${ }^{p}$-well-posed;
(ii) $\mathbb{Z} \subset \rho_{s}(A)$ and the set $\left\{k^{4} N_{k}: k \in \mathbb{Z}\right\}$ is $R$-bounded.

Proof. We first prove $(i) \Longrightarrow(i i)$. Given $k \in \mathbb{Z}$ and $y \in X$ we define the function $f \in L^{p}(\mathbb{T}, X)$ as $f(t)=e^{i k t} y$. It is not difficult to check that $\hat{f}(k)=y$ and 0 otherwise. By hypothesis, equation (3.1) is $L^{p}$-well-posed and then there exists a unique $u \in S_{p}(A)$ which solves equation (3.1). If we take

Fourier transform in both sides of (3.1) we get:

$$
\begin{equation*}
\left[a_{k}-\alpha b_{k}-\beta c_{k}-\gamma c_{k} A+\delta i k A+\rho A\right] \hat{u}(k)=y, \tag{3.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\left[a_{n}-\alpha b_{n}-\beta c_{n}-\gamma c_{n} A+\sin A+\rho A\right] \hat{u}(n)=0, \quad n \neq k . \tag{3.5}
\end{equation*}
$$

This shows that $\left[a_{k}-\alpha b_{k}-\beta c_{k}-\gamma c_{k} A+\delta i k A+\rho A\right]$ is surjective. On the other hand, let $x \in D(A)$ be such that

$$
\left[a_{k}-\alpha b_{k}-\beta c_{k}-\gamma c_{k} A+\delta i k A+\rho A\right] x=0 .
$$

We define $u \in S_{p}(A)$ as $u(t)=e^{i k t} x$ for $t \in \mathbb{T}$. It is not difficult to see that $u$ is a solution for equation (3.1) when $f=0$. By uniqueness, then it necessarily follows that $x=0$ and then $\left[a_{k}-\alpha b_{k}-\beta c_{k}-\gamma c_{k} A+\delta i k A+\rho A\right]$ is bijective from $D(A)$ onto $X$. Moreover, $\left[a_{k}-\alpha b_{k}-\beta c_{k}-\right.$ $\left.\gamma c_{k} A+\operatorname{\delta ik} A+\rho A\right]^{-1} \in \mathcal{B}(X)$. Indeed, given $y \in X$ and $k \in \mathbb{Z}$ let $f(t)=e^{i k t} y$ and let $u$ be the corresponding solution of (3.1) for $f$. Then $\hat{u}(k)=\left[a_{k}-\alpha b_{k}-\beta c_{k}-\gamma c_{k} A+\delta i k A+\rho A\right]^{-1} y$ and 0 otherwise.

This implies $u(t)=-e^{-i k t}\left[a_{k}-\alpha b_{k}-\beta c_{k}-\gamma c_{k} A+\delta i k A+\rho A\right]^{-1} y$ by uniqueness. As a consequence, there exists a positive constant $C>0$ independent of $y$ and $k$ such that

$$
\|u\|_{S_{p}(A)} \leq C\|f\|_{L^{p}},
$$

which implies

$$
\left\|\left[a_{k}-\alpha b_{k}-\beta c_{k}-\gamma c_{k} A+\delta i k A+\rho A\right]^{-1}\right\| \leq C
$$

for all $k \in \mathbb{Z}$. This proves the claim. We have shown that $\mathbb{Z} \subset \rho_{s}(A)$. Let $M_{k}=k^{4} N_{k}$ with $k \in \mathbb{Z}$, where $N_{k}$ is defined in (3.3). To finish this implication it only remains to show that $\left(M_{k}\right)_{k \in \mathbb{Z}}$ is $L^{p}$-Fourier multiplier. Given $f \in L^{p}(\mathbb{T}, X)$, there exists $u \in S_{p}(A)$ which is a solution of equation (3.1) by assumption. Taking Fourier transforms on both sides of (3.1), we get that $\hat{u}(k) \in D(A)$ and

$$
\left[a_{k}-\alpha b_{k}-\beta c_{k}-\gamma c_{k} A+\delta i k A+\rho A\right] \hat{u}(k)=\hat{f}(k), \quad k \in \mathbb{Z} .
$$

Due to the invertibility of $\left[a_{k}-\alpha b_{k}-\beta c_{k}-\gamma c_{k} A+\delta i k A+\rho A\right]$ we can assert that $\hat{u}(k)=$ $N_{k} \hat{f}(k), k \in \mathbb{Z}$. As $u \in S_{p}(A)$ we obtain that

$$
\widehat{\left[u^{\prime \prime \prime \prime}\right]}(k)=k^{4} \hat{u}(k)=k^{4} N_{k} \hat{f}(k)=M_{k} \hat{f}(k) .
$$

Finally, since $u^{\prime \prime \prime \prime \prime} \in L^{p}(\mathbb{T}, X)$ we get that $\left(M_{k}\right)_{k \in \mathbb{Z}}$ is $L^{p}$-Fourier multipliers and, by Theorem 3.4, we conclude that the set $\left\{M_{k}: k \in \mathbb{Z}\right\}$ is $R$-bounded, proving (ii).

Let now show $(i i) \Longrightarrow(i)$. We assume that $\mathbb{Z} \subset \rho_{s}(A)$ and the set $\left\{k^{4} N_{k}: k \in \mathbb{Z}\right\}$ is $R$-bounded. A simple calculation shows the following identity

$$
\begin{equation*}
k^{2} A N_{k}=\frac{k^{2}}{\gamma k^{2}-\rho-i \delta k}\left[1-\frac{\beta}{k^{2}}-\frac{i \alpha}{k}\right] k^{4} N_{k}-\frac{k^{2}}{\gamma k^{2}-\rho-i \delta k^{\prime}} \quad k \in \mathbb{Z} \backslash\{0\} \tag{3.6}
\end{equation*}
$$

proving that the set $\left\{k^{2} A N_{k}: k \in \mathbb{Z}\right\}$ is $R$-bounded, too. Let $M_{k}=k^{4} N_{k}$ and $S_{k}=k^{2} A N_{k}$. It follows from Proposition 3.5 that $\left(M_{k}\right)_{k \in \mathbb{Z}},\left(i k^{3} N_{k}\right)_{k \in \mathbb{Z}}$ and $\left(k^{2} N_{k}\right)_{k \in \mathbb{Z}}$ are $L^{p}$-Fourier multipliers.

Note that the $R$-boundedness of the set $\left\{k^{4} N_{k}\right\}_{k \in \mathbb{Z}}$ implies that $\left\{k N_{k}\right\}_{k \in \mathbb{Z}}$ is $R$-bounded and then the set $\left\{k\left(N_{k+1}-N_{k}\right)\right\}$ is also $R$-bounded. It follows from Theorem 3.3 that $\left\{N_{k}\right\}_{k \in \mathbb{Z}}$ is an $L^{p}$-Fourier multiplier. In particular, $N_{k} \in \mathcal{B}(X,[D(A)])$.

Then, for all $f \in L^{p}(\mathbb{T}, X)$ there exist $w, u_{1}, u_{2}, u_{3} \in L^{p}(\mathbb{T},[D(A)])$ satisfying:

$$
\hat{w}(k)=N_{k} \hat{f}(k), \hat{u}_{1}(k)=M_{k} \hat{f}(k), \hat{u}_{2}(k)=-i k^{3} N_{k} \hat{f}(k), \hat{u}_{3}(k)=-k^{2} N_{k} \hat{f}(k) .
$$

Consequently, $\hat{u}_{1}(k)=k^{4} \hat{w}(k)$ when $k \in \mathbb{Z}$. This implies that $w \in W_{p e r}^{4, p}(\mathbb{T} ;[D(A)])$ [4, Lemma 2.1] and $w^{\prime \prime \prime \prime}(t)=u_{1}(t)$ a.e. [4, Lemma 3.1]. In particular, $w^{\prime \prime \prime \prime} \in L^{p}(\mathbb{T},[D(A)])$. Similarly, we obtain:

$$
\hat{u}_{2}(k)=(i k)^{3} \hat{w}(k)=\widehat{w^{\prime \prime \prime}}(k), \quad \hat{u}_{3}(k)=(i k)^{2} \hat{w}(k)=\widehat{w^{\prime \prime}}(k)
$$

and then $w^{\prime \prime \prime}(t)=u_{2}(t)$ and $w^{\prime \prime}(t)=u_{3}(t)$. In particular, $w^{\prime \prime}, w^{\prime \prime \prime} \in L^{p}(\mathbb{T},[D(A)])$.
By hypothesis and Proposition 3.5, it follows that $\left\{S_{k}\right\}_{k \in Z},\left\{k A N_{k}\right\}_{k \in Z}$ and $\left\{A N_{k}\right\}_{k \in Z}$ are $L^{p}$-Fourier multipliers, and then we can ensure that there exist $u_{4}, u_{5}, u_{6} \in L^{p}(\mathbb{T}, X)$ such that

$$
\hat{u}_{4}(k)=-k^{2} A N_{k} \hat{f}(k)=A \widehat{w^{\prime \prime}}(k)=\widehat{A w^{\prime \prime}}(k)
$$

and

$$
\hat{u}_{5}(k)=i k A N_{k} \hat{f}(k)=A \widehat{w^{\prime}}(k)=\widehat{A w^{\prime}}(k)
$$

as well as

$$
\hat{u}_{6}(k)=A N_{k} \hat{f}(k)=A \widehat{w}(k)=\widehat{A w}(k)
$$

where we have used that $A$ is closed. It follows from [4, Lemma 3.1] that $w(t), w^{\prime}(t), w^{\prime \prime}(t) \in$ $D(A)$ and $A w^{\prime \prime}(t)=u_{4}(t), A w^{\prime}(t)=u_{5}(t)$ and $A w^{\prime}(t)=u_{6}(t)$. In addition, $A w, A w^{\prime}, A w^{\prime \prime} \in$ $L^{p}(\mathbb{T}, X)$. As a consequence, $w \in S_{p}(A)$. Moreover, the following identity holds:

$$
\begin{equation*}
I_{X}=k^{4} N_{k}-\alpha i k^{3} N_{k}-\beta k^{2} N_{k}-\gamma k^{2} A N_{k}+\delta i k A N_{k}+\rho A N_{k} \tag{3.7}
\end{equation*}
$$

and then we obtain

$$
\begin{aligned}
\hat{f}(k) & =\left[k^{4} N_{k}-\alpha i k^{3} N_{k}-\beta k^{2} N_{k}-\gamma k^{2} A N_{k}+\operatorname{\delta i} A N_{k}+\rho A N_{k}\right] \hat{f}(k) \\
& =\widehat{w^{\prime \prime \prime \prime}}(k)+\alpha \widehat{w^{\prime \prime \prime}}(k)+\beta \widehat{w^{\prime}}(k)+\gamma \widehat{A w^{\prime \prime}}(k)+\delta \widehat{A w^{\prime}}(k)+\rho \widehat{A w}(k)
\end{aligned}
$$

This implies that

$$
w^{\prime \prime \prime \prime}(t)+\alpha w^{\prime \prime \prime}(t)+\beta w^{\prime \prime}(t)+\gamma A w^{\prime \prime}(t)+\delta A w^{\prime}(t)+\rho A w(t)=f(t)
$$

by the uniqueness theorem (see [4, p. 314]). It only remains to prove that the solution is unique. Indeed, for a given $w \in S_{p}(A)$ that satisfies equation (3.1) for $f=0$, if we take Fourier transform we get that $\left[a_{k}-\alpha b_{k}-\beta c_{k}-\gamma c_{k} A+\delta i k A+\rho A\right] \hat{w}(k)=0$ for all $k \in \mathbb{Z}$. Hence $w=0$ since $\mathbb{Z} \subset \rho_{s}(A)$. Thus, equation (3.1) is strongly $L^{p}$-well-posed.

We point out that $L^{p}$-well-posedness does not depend on the parameter $p$, that is, if equation (3.1) is strongly $L^{p}$-well-posed for some $1<p<\infty$, then it is strongly $L^{p}$-well-posed for all $1<p<\infty$.

## 4 Well-posedness in Besov and Triebel-Lizorkin spaces

In this section, we now analyze the well-posedness of equation (3.1) in periodic Besov spaces $B_{p, q}^{s}(\mathbb{T}, X)$ and periodic Triebel-Lizorkin spaces $F_{p, q}^{s}(\mathbb{T}, X)$. The definition and properties of vector-valued periodic Besov spaces can be found in [5].

Given $1 \leq p, q \leq \infty$ and $s>0$, we define the maximal regularity space that describes the strongly $B_{p, q}^{s}-$ well-posedness of the equation (3.1) by

$$
S_{p, q, s}(A):=B_{p, q}^{s+4}(\mathbb{T}, X) \cap B_{p, q}^{s+2}(\mathbb{T},[D(A)]) .
$$

The vectorial space $S_{p, q, s}(A)$ is a Banach space with the norm

$$
\|u\|_{S_{p, q s}(A)}:=\left\|u^{\prime \prime}\right\|_{B_{p, q}^{s}}+\left\|u^{\prime \prime \prime}\right\|_{B_{p, q}^{s}}+\left\|u^{\prime \prime \prime \prime}\right\|_{B_{p, q}^{s}}+\|A u\|_{B_{p, q}^{s}}+\left\|A u^{\prime}\right\|_{B_{p, q}^{s}}+\left\|A u^{\prime \prime}\right\|_{B_{p, q}^{s}} .
$$

Analogously to the case $L^{p}$ we can define the strongly $B_{p, q}^{s}-$ well-posedness for equation (3.1) as follows.

Definition 4.1. Let $1 \leq p, q<\infty, s>0$ and $f \in B_{p, q}^{s}(\mathbb{T}, X)$ be given. We say that $u \in S_{p, q, s}(A)$ is a strong $B_{p, q}$-solution of (3.1) if it satisfies (3.1) for all $t \in \mathbb{T}$. We say that (3.1) is strongly $B_{p, q}^{s}$-well-posed if for each $f \in B_{p, q}^{s}(\mathbb{T}, X)$, there exists a unique strong $B_{p, q}^{s}$-solution of (3.1).

Note that if (3.1) is strongly $B_{p, q}^{s}$-well-posed, by the Closed Graph Theorem, there exists a constant $C>0$ such that for each $f \in B_{p, q}^{s}(\mathbb{T}, X)$, we have

$$
\|u\|_{S_{p, q, s}(A)} \leq C\|f\|_{B_{p, q}^{s}} .
$$

We now introduce the following notion that corresponds to $B_{p, q}^{s}$-Fourier multiplier (see [4]).
Definition 4.2. Let $X, Y$ be Banach spaces, $1 \leq p, q<\infty, s \in \mathbb{R}$ and $\left(M_{k}\right)_{k \in \mathbb{Z}} \subset \mathcal{B}(X, Y)$. We say that $\left(M_{k}\right)_{k \in \mathbb{Z}}$ is a $B_{p, q}^{s}$-Fourier multiplier if, for each $f \in B_{p, q}^{s}(\mathbb{T}, X)$ there exists $u \in B_{p, q}^{s}(\mathbb{T}, Y)$ such that

$$
\hat{u}(k)=M_{k} \hat{f}(k)
$$

for all $k \in \mathbb{Z}$.
The following theorem contained in [5] states that $M$-boundedness of order 2 is sufficient for an operator valued symbol to be a $B_{p, q^{-}}^{s}$ - Fourier multiplier.

Theorem 4.3. Let $X, Y$ be Banach spaces. If $\left(M_{k}\right)_{k \in \mathbb{Z}} \subset \mathcal{B}(X, Y)$ is $M$-bounded of order 2 , then for $1 \leq p, q \leq \infty, s \in \mathbb{R}$ the set $\left(M_{k}\right)_{k \in \mathbb{Z}}$ is a $B_{p, q}^{s}-$ Fourier multiplier.

The following result provides necessary conditions for certain sets which will be needed to characterize strongly $B_{p, q}^{s}$-well-posedness.

Proposition 4.4. Let $A$ be a closed linear operator defined on a $U M D$ space $X$ and $\alpha, \beta, \gamma, \delta, \rho \in$ $\mathbb{R}$. If $\mathbb{Z} \subset \rho_{s}(A)$ and the sets $\left\{k^{4} N_{k}: k \in \mathbb{Z}\right\}$ and $\left\{k^{2} A N_{k}: k \in \mathbb{Z}\right\}$ are uniformly bounded, then $\left(k^{4} N_{k}\right)_{k \in \mathbb{Z}^{\prime}}\left(i k^{3} N_{k}\right)_{k \in \mathbb{Z}}\left(k^{2} N_{k}\right)_{k \in \mathbb{Z}^{\prime}}\left(k^{2} A N_{k}\right)_{k \in \mathbb{Z}}\left(k A N_{k}\right)_{k \in \mathbb{Z}}$ and $\left(A N_{k}\right)_{k \in \mathbb{Z}}$ are $B_{p, q^{\prime}}^{s}$-Fourier multipliers.

Proof. Let $M_{k}=k^{4} N_{k}$. In order to show that $M_{k}$ is a $B_{p, q}^{s}$-Fourier multiplier and according to Theorem 4.3, we need to prove that $\sup _{k \in \mathbb{Z}}\left(\left\|M_{k}\right\|+\left\|k \Delta M_{k}\right\|\right)<\infty$ and $\sup _{k \in \mathbb{Z}}\left\|k^{2} \Delta^{2} M_{k}\right\|<\infty$. The first inequality holds as a consequence of the hypothesis and Proposition 3.5. Therefore, we only need to show the second one which will be done applying Theorem 2.6 to $s_{k}=k^{4}$, which is clearly a 2-regular sequence, $H_{k}=N_{k}$ and $T=I$. By hypothesis, $\sup _{k \in \mathbb{Z}}\left\|M_{k}\right\|<\infty$. Moreover, by Proposition 3.5 it follows that $\sup _{k \in \mathbb{Z}}\left\|k L_{k}\right\|<\infty$, then it only remains to show that $\sup _{k \in \mathbb{Z}}\left\|k^{2} \Delta L_{k}\right\|<\infty$. Indeed, we have

$$
L_{k}=\left(N_{k}^{-1}-N_{k+1}^{-1}\right) N_{k}=\left[-\Delta a_{k}+\alpha \Delta b_{k}+\beta \Delta c_{k}+\gamma \Delta c_{k} A-\delta i A\right] N_{k}
$$

Then,

$$
\begin{align*}
k^{2} \Delta L_{k}= & k^{2}\left[\left(a_{k+1}-a_{k+2}\right) N_{k+1}-\left(a_{k}-a_{k+1}\right) N_{k}\right] \\
& +\alpha k^{2}\left[\left(b_{k+2}-b_{k+1}\right) N_{k+1}-\left(b_{k+1}-b_{k}\right) N_{k}\right] \\
& +\beta k^{2}\left[\left(c_{k+2}-c_{k+1}\right) N_{k+1}-\left(c_{k+1}-c_{k}\right) N_{k}\right] \\
& +\gamma k^{2}\left[\left(c_{k+2}-c_{k+1}\right) A N_{k+1}-\left(c_{k+1}-c_{k}\right) A N_{k}\right] \\
& -\delta i\left(A N_{k+1}-A N_{k}\right), \tag{4.1}
\end{align*}
$$

where $a_{k}=k^{4}$ and $b_{k}=i k^{3}$ and $c_{k}=k^{2}$. We only need to prove that each term is bounded. First of all, a simple calculus shows that:

$$
\left(a_{k+1}-a_{k+2}\right) N_{k+1}-\left(a_{k}-a_{k+1}\right) N_{k}=-\left(\Delta^{2} a_{k}\right) N_{k+1}+\frac{\Delta a_{k}}{a_{k}}\left[\left(a_{k} N_{k}-a_{k+1} N_{k+1}\right)+N_{k+1}\left(\Delta a_{k}\right)\right]
$$

Therefore

$$
\begin{aligned}
& k^{2}\left[\left(a_{k+1}-a_{k+2}\right) N_{k+1}-\left(a_{k}-a_{k+1}\right) N_{k}\right]= \\
& \quad-k^{2} \frac{\left(\Delta^{2} a_{k}\right)}{a_{k}} \frac{a_{k}}{a_{k+1}}\left(a_{k+1} N_{k+1}\right)+k \frac{\Delta a_{k}}{a_{k}}\left[k\left(a_{k} N_{k}-a_{k+1} N_{k+1}\right)+a_{k+1} N_{k+1} \frac{a_{k}}{a_{k+1}}\left\{\frac{k\left(\Delta a_{k}\right)}{a_{k}}\right\}^{2}\right]
\end{aligned}
$$

Since the sequence $a_{k}$ is 2-regular, $M_{k}=a_{k} N_{k}$ and $k \Delta M_{k}$ are bounded, the above identity shows that

$$
\sup _{k \in \mathbb{Z}}\left\|k^{2}\left[\left(a_{k+1}-a_{k+2}\right) N_{k+1}-\left(a_{k}-a_{k+1}\right) N_{k}\right]\right\|<\infty
$$

Analogously and following the same procedure as above, using the fact that $b_{k}$ is also 2regular, $b_{k} N_{k}$ and $k \Delta\left(b_{k} N_{k}\right)$ are bounded, we obtain that

$$
\sup _{k \in \mathbb{Z}}\left\|k^{2}\left[\left(b_{k+2}-b_{k+1}\right) N_{k+1}-\left(b_{k+1}-b_{k}\right) N_{k}\right]\right\|<\infty
$$

Following the same idea we get that

$$
\sup _{k \in \mathbb{Z}}\left\|k^{2}\left[\left(c_{k+2}-c_{k+1}\right) N_{k+1}-\left(c_{k+1}-c_{k}\right) N_{k}\right]\right\|<\infty
$$

and

$$
\sup _{k \in \mathbb{Z}}\left\|k^{2}\left[\left(c_{k+2}-c_{k+1}\right) A N_{k+1}-\left(c_{k+1}-c_{k}\right) A N_{k}\right]\right\|<\infty
$$

since $c_{k}$ is 2-regular and $S_{k}=c_{k} N_{k}$ and $k \Delta S_{k}$ are bounded in the first case, meanwhile $R_{k}=$ $c_{k} A N_{k}$ and $k \Delta R_{k}$ are bounded for proving the second inequality. Finally, the fact that $k \Delta R_{k}$ is
bounded immediately implies the boundedness for the last summand $-\delta i\left(A N_{k+1}-A N_{k}\right)$ in (4.1). Consequently, $\left(k^{4} N_{k}\right)_{k \in \mathbb{Z}}$ is a $B_{p, q}^{s}$-Fourier multiplier.

We now consider $M_{k}=i k^{3} N_{k}$. In order to prove that it is a $B_{p, q}^{s}$-Fourier multiplier it only remains to show again that $\sup _{k \in \mathbb{Z}}\left\|k^{2} \Delta^{2} M_{k}\right\|<\infty$, which can be do using the second part of Theorem 2.6 with $s_{k}=i k^{3}, H_{k}=N_{k}$ and $T=I$. By hypothesis and Proposition 3.5 it follows that $\sup _{k \in \mathbb{Z}}\left\|M_{k}\right\|<\infty$ and $\sup _{k \in \mathbb{Z}}\left\|k L_{k}\right\|<\infty$, respectively. The inequality $\sup _{k \in \mathbb{Z}}\left\|k^{2} \Delta L_{k}\right\|<\infty$ has already been shown since $L_{k}$ is exactly the same that in the above computation. Therefore, $\left(i k^{3} N_{k}\right)_{k \in \mathbb{Z}}$ is a $B_{p, q}^{s}$-Fourier multiplier. Similarly, we obtain that $\left(k^{2} N_{k}\right)_{k \in \mathbb{Z}}$ is a $B_{p, q}^{s}$-Fourier multiplier.

Let now $M_{k}=k^{2} A N_{k}$. From Proposition 3.5 it follows that $\sup _{k \in \mathbb{Z}}\left(\left\|M_{k}\right\|+\left\|k \Delta M_{k}\right\|\right)<\infty$. To prove that $\sup _{k \in \mathbb{Z}}\left\|k^{2} \Delta^{2} M_{k}\right\|<\infty$ we apply Theorem 2.6 with $s_{k}=k^{2}, H_{k}=N_{k}$ and $T=A$. It remains to show that $\sup _{k \in \mathbb{Z}}\left\|k^{2} \Delta L_{k}\right\|<\infty$, where $L_{k}$ is the same that in the above calculus. Therefore, $\left(k^{2} A N_{k}\right)_{k \in \mathbb{Z}}$ is a $B_{p, q}^{s}-$-Fourier multiplier. The same procedure can be applied to $M_{k}=k A N_{k}$ with $s_{k}=k, H_{k}=N_{k}$ and $T=A$ and $M_{k}=A N_{k}$ with $s_{k}=1, H_{k}=N_{k}$ and $T=A$. The conclusion then holds and consequently $\left(k A N_{k}\right)_{k \in \mathbb{Z}}$ and $\left(A N_{k}\right)_{k \in \mathbb{Z}}$ are $B_{p, q^{-}}^{s}$ Fourier multipliers.

We now enunciate the main result of this section. The proof follows essentially the same steps than the one of Theorem 3.6. However, we include here the essential changes of the proof that differ from Theorem 3.6 in order to make it clear to the reader.
Theorem 4.5. Let $1 \leq p, q \leq \infty, s>0$ and $\alpha, \beta, \gamma, \delta, \rho \in \mathbb{R}$ be given with $(\gamma, \delta, \rho) \neq(0,0,0)$. Assume $A$ is a closed linear operator defined on a Banach space $X$. The following assertions are equivalent:
(i) The equation

$$
u^{\prime \prime \prime \prime}(t)+\alpha u^{\prime \prime \prime}(t)+\beta u^{\prime \prime}(t)+\gamma A u^{\prime \prime}(t)+\delta A u^{\prime}(t)+\rho A u(t)=f(t), \quad t \in[0,2 \pi]
$$

is strongly $B_{p, q}^{s}$-well-posed;
(ii) $\mathbb{Z} \subset \rho_{s}(A)$ and $\sup _{k \in \mathbb{Z}}\left\|k^{4} N_{k}\right\|<\infty$.

Proof. $(i) \Longrightarrow$ (ii) follows the same lines of Theorem 3.6 and therefore is omitted. We prove (ii) $\Longrightarrow(i)$. We assume that $\mathbb{Z} \subset \rho_{s}(A)$ and the set $\left\{k^{4} N_{k}: k \in \mathbb{Z}\right\}$ is uniformly bounded. The identity (3.6) shows that the set $\left\{k^{2} A N_{k}: k \in \mathbb{Z}\right\}$ is uniformly bounded.

Analogously, the identities $k N_{k}=\frac{1}{k^{3}}\left(k^{4} N_{k}\right)$ and $k^{2} N_{k}=\frac{1}{k^{2}}\left(k^{4} N_{k}\right)$ show that the sets $\left\{k N_{k}\right.$ : $k \in \mathbb{Z}\}$ and $\left\{k^{2} N_{k}: k \in \mathbb{Z}\right\}$ are also uniformly bounded. Therefore the sets $\left\{k\left(N_{k+1}-N_{k}\right)\right\}_{k \in \mathbb{Z}}$ and $\left\{k^{2}\left(N_{k+2}-2 N_{k+1}+N_{k}\right)\right\}_{k \in \mathbb{Z}}$ are uniformly bounded and hence, by Theorem 4.3, the set $\left\{N_{k}\right\}_{k \in \mathbb{Z}}$ is a $B_{p, q}^{s}-$-Fourier multiplier. Moreover, by hypothesis and Proposition 4.4 it follows that $\left(k^{4} N_{k}\right)_{k \in \mathbb{Z}^{\prime}}\left(i k^{3} N_{k}\right)_{k \in \mathbb{Z}^{\prime}}\left(k^{2} N_{k}\right)_{k \in \mathbb{Z}^{\prime}}\left(k^{2} A N_{k}\right)_{k \in \mathbb{Z}^{\prime}}\left(k A N_{k}\right)_{k \in \mathbb{Z}}$ and $\left(A N_{k}\right)_{k \in \mathbb{Z}}$ are $B_{p, q^{-}}^{s}$-Fourier multipliers.

Let $f \in B_{p, q}^{s}(\mathbb{T}, X)$ be given. Since $\left(k^{4} N_{k}\right)_{k \in \mathbb{Z},}\left(i k^{3} N_{k}\right)_{k \in \mathbb{Z},}\left(k^{2} N_{k}\right)_{k \in \mathbb{Z}}$ are $B_{p, q}^{s}$ multipliers, there exist $w, u_{1}, u_{2}, u_{3} \in B_{p, q}^{s}(\mathbb{T},[D(A)])$ satisfying:

$$
\begin{equation*}
\hat{w}(k)=N_{k} \hat{f}(k), \hat{u}_{1}(k)=k^{4} N_{k} \hat{f}(k), \hat{u}_{2}(k)=-i k^{3} N_{k} \hat{f}(k), \hat{u}_{3}(k)=-k^{2} N_{k} \hat{f}(k) . \tag{4.2}
\end{equation*}
$$

Consequently, $\hat{u}_{1}(k)=k^{4} \hat{w}(k)$ when $k \in \mathbb{Z}$. This implies that $w \in B_{p, q}^{s+4}(\mathbb{T} ;[D(A)])$ and $w^{\prime \prime \prime \prime}(t)=u_{1}(t)$. In particular, $w^{\prime \prime \prime \prime} \in B_{p, q}^{s}(\mathbb{T},[D(A)])$. Similarly, we obtain:

$$
\hat{u}_{2}(k)=(i k)^{3} \hat{w}(k)=\widehat{w^{\prime \prime \prime}}(k), \quad \hat{u_{3}}(k)=(i k)^{2} \hat{w}(k)=\widehat{w^{\prime \prime}}(k),
$$

and then $w^{\prime \prime \prime}(t)=u_{2}(t)$ and $w^{\prime \prime}(t)=u_{3}(t)$. In particular, $w^{\prime \prime}, w^{\prime \prime \prime} \in B_{p, q}^{s}(\mathbb{T},[D(A)])$.
By hypothesis and Proposition 4.4 the sets $\left\{k^{2} A N_{k}\right\}_{k \in Z},\left\{k A N_{k}\right\}_{k \in Z}$ and $\left\{A N_{k}\right\}_{k \in Z}$ are $B_{p, q}^{s}$-Fourier multipliers, and then we have that there exist $u_{4}, u_{5}, u_{6} \in B_{p, q}^{s}(\mathbb{T}, X)$ such that

$$
\begin{align*}
& \hat{u}_{4}(k)=-k^{2} A N_{k} \hat{f}(k)=A \widehat{w^{\prime \prime}}(k)=\widehat{A w^{\prime \prime}}(k), \\
& \hat{u}_{5}(k)=i k A N_{k} \hat{f}(k)=A \widehat{w^{\prime}}(k)=\widehat{A w^{\prime}}(k),  \tag{4.3}\\
& \hat{u}_{6}(k)=A N_{k} \hat{f}(k)=A \widehat{w}(k)=\widehat{A w}(k) .
\end{align*}
$$

where we have used that $A$ is closed. It follows from [4, Lemma 3.1] that $w(t), w^{\prime}(t), w^{\prime \prime}(t) \in$ $D(A)$ and $A w^{\prime \prime}(t)=u_{4}(t), A w^{\prime}(t)=u_{5}(t)$ and $A w^{\prime}(t)=u_{6}(t)$ a.e. In addition, $A w, A w^{\prime}$, $A w^{\prime \prime} \in B_{p, q}^{s}(\mathbb{T}, X)$. Replacing (4.2) - (4.3) in the following identity:

$$
\hat{f}(k)=k^{4} N_{k} \hat{f}(k)-\alpha i k^{3} N_{k} \hat{f}(k)-\beta k^{2} N_{k} \hat{f}(k)-\gamma k^{2} A N_{k} \hat{f}(k)+\delta i k A N_{k} \hat{f}(k)+\rho A N_{k} \hat{f}(k),
$$

we obtain by the uniqueness of the Fourier coefficients that $w$ solves equation (3.1). The uniqueness follows the same lines as in Theorem 3.6.

We point out that the second assertion in Theorem 4.5 does not depend on the parameters $p, q$ and $s$, and then strongly $B_{p, q}^{s}$-well-posedness for equation (3.1) holds for some $1 \leq p, q \leq$ $\infty, s>0$ if and only if it is strongly $B_{p, q}^{s}$-well-posed for all $1 \leq p, q \leq \infty, s>0$. To finish this section, we consider well-posedness in periodic Triebel-Lizorkin spaces $F_{p, q}^{s}$ with $1 \leq p<\infty$, $1 \leq q \leq \infty, s \in \mathbb{R}$. We do not include the formal definition of these spaces but we refer the reader to [14] for the details and properties of these spaces.

Using a similar argument as the one in the proof of Theorem 4.5, we obtain the following characterization of the strongly $F_{p, q}^{s}$-well-posedness of equation (3.1). In order to prove this result we use the operator-valued Fourier multiplier theorem proved in [14]. We omit the details.

Theorem 4.6. Let $1 \leq p<\infty, 1 \leq q \leq \infty, s>0$ and $\alpha, \beta, \gamma, \delta, \rho \in \mathbb{R}$ be given with $(\gamma, \delta, \rho) \neq$ $(0,0,0)$. Assume that $A$ is a closed linear operator defined on a Banach space X. The following assertions are equivalent:
(i) The equation

$$
u^{\prime \prime \prime \prime}(t)+\alpha u^{\prime \prime \prime}(t)+\beta u^{\prime \prime}(t)+\gamma A u^{\prime \prime}(t)+\delta A u^{\prime}(t)+\rho A u(t)=f(t), \quad t \in[0,2 \pi]
$$

is strongly $F_{p, q}^{s}$-well-posed;
(ii) $\mathbb{Z} \subset \rho_{s}(A)$ and $\sup _{k \in \mathbb{Z}}\left\|k^{4} N_{k}\right\|<\infty$.

As it was pointed out for $B_{p, q}^{s}-$-well-posedness, the problem (3.1) is strongly $F_{p, q}^{s}-$-well-posed for all $1 \leq p<\infty, 1 \leq q \leq \infty, s>0$ if it is so for some $1 \leq p<\infty, 1 \leq q \leq \infty, s>0$.

## 5 Sufficient conditions: $L^{p}-L^{q}$-well-posedness

Based on the previous abstract results, we give in this section a practical criteria to widely solve the following Cauchy problem in $L^{p}-L^{q}$ spaces with periodic boundary conditions:

$$
\left\{\begin{array}{l}
\partial_{t t t} u(x, t)+\alpha \partial_{t t t} u(x, t)+\beta \partial_{t t} u(x, t)+\gamma A_{x} \partial_{t t} u(x, t)+\delta A_{x} \partial_{t} u(x, t)+\rho A_{x} u(x, t)=f(x, t),  \tag{5.1}\\
u(x, 0)=u(x, 2 \pi), \partial_{t} u(x, 0)=\partial_{t} u(x, 2 \pi), \partial_{t t} u(x, 0)=\partial_{t t} u(x, 2 \pi), \partial_{t t t} u(x, 0)=\partial_{t t t} u(x, 2 \pi),
\end{array}\right.
$$

where $x \in \Omega \subset \mathbb{R}^{N}$ and $t \in(0,2 \pi)$. We begin with some preliminaries on $R$-sectorial operators. Given any $\theta \in(0, \pi)$, we denote $\Sigma_{\theta}:=\{z \in \mathbb{C}:|\arg (z)|<\theta, z \neq 0\}$. Recall that a closed operator $A: D(A) \subset X \rightarrow X$ with dense domain $D(A)$ is said to be $R$-sectorial of angle $\theta$ if the following conditions are satisfied:
(i) $\sigma(A) \subseteq \mathbb{C} \backslash \Sigma_{\theta}$;
(ii) The set $\left\{z(z-A)^{-1}: z \in \Sigma_{\theta}\right\}$ is $R$-bounded in $\mathcal{B}(X)$.

The permanence properties for $R$-sectorial operators are similar to those for sectorial operators. For instance, they behave well under perturbations. Sufficient conditions for $R$ sectoriality are studied in the monograph [22, Chapter 4]. As a consequence of our main theorem, we obtain the following remarkable result.

Theorem 5.1. Assume that $X$ is a UMD space, $1<p<\infty, \alpha, \beta, \gamma, \delta, \rho \in(0, \infty)$ and let $A$ be an $R$-sectorial operator on $X$ of angle $\pi / 2$. If $\rho+\beta \gamma<\alpha \delta$ then equation (5.1) is strongly $L^{p}$-well-posed.

Proof. Define $d_{k}=\frac{\left(k^{4}-\beta k^{2}\right)-i \alpha k^{3}}{\left(\gamma k^{2}-\rho\right)-i \delta k}$ and we note that

$$
\Re\left(d_{k}\right)=\frac{k^{2}\left[\gamma k^{4}-k^{2}(\rho+\beta \gamma-\alpha \delta)+\rho \beta\right]}{\left(\gamma k^{2}-\rho\right)^{2}+\delta^{2} k^{2}}>0,
$$

since $\rho+\beta \gamma<\alpha \delta$. Therefore $d_{k} \in \Sigma_{\pi / 2}$. The $R$-sectoriality of angle $\pi / 2$ of the operator $A$ ensures the invertibility of $d_{k} I-A$ and the set $\left\{d_{k}\left(d_{k}-A\right)^{-1}\right\}_{k \in \mathbb{Z}}$ is $R$-bounded. Finally, we note the following identity

$$
k^{4} N_{k}=\frac{k^{4}}{\left(k^{4}-\beta k^{2}\right)-i \alpha k^{3}} d_{k}\left(d_{k}-A\right)^{-1}, \quad k \in \mathbb{Z}
$$

which proves that the set $\left\{k^{4} N_{k}\right\}_{k \in \mathbb{Z}}$ is $R$-bounded. By Theorem 3.6 we conclude that the problem (3.1) is strongly $L^{p}$-well-posed.

Example 5.2. Let $1<p<\infty$ and $\alpha, \beta, \gamma, \delta, \rho$ be strictly positive real numbers satisfying $\rho+$ $\beta \gamma<\alpha \delta$. We consider the following equation in a bounded smooth domain $\Omega \subset \mathbb{R}^{N}$ :

$$
\left\{\begin{array}{l}
{\left[\partial_{t t t t} u+\alpha \partial_{t t t} u+\beta \partial_{t t} u+\gamma \Delta \partial_{t t} u+\delta \Delta \partial_{t} u+\rho \Delta u\right](x, t)=f(x, t), \text { for }(x, t) \in \Omega \times(0,2 \pi) ;}  \tag{5.2}\\
u(x, t)=0, \text { for }(x, t) \in \partial \Omega \times(0,2 \pi) ; \\
u(x, 0)=u(x, 2 \pi), \partial_{t} u(x, 0)=\partial_{t} u(x, 2 \pi), \partial_{t t} u(x, 0)=\partial_{t t} u(x, 2 \pi), \partial_{t t t} u(x, 0)=\partial_{t t t} u(x, 2 \pi),
\end{array}\right.
$$

where $\Delta$ denotes the Laplacian operator. By [26, Appendix] we have that the $L^{q}$ realization $\Delta_{q}$ in $X=L^{q}(\Omega)$ of $\Delta$ is an $R$-sectorial operator in $X$ with arbitrary angle $\theta \in(0, \pi)$, and that $\Delta_{q}$ coincides with $\Delta$ in the domain $D\left(\Delta_{q}\right)$ of $\Delta_{q}$. Therefore, we can denote $\left(\Delta_{q}, D\left(\Delta_{q}\right)\right)$ by $\left(\Delta, D_{q}(\Delta)\right)$. Thus, Theorem 5.1 implies that for any given $f \in L^{p}\left(\mathbb{T}, L^{q}(\Omega)\right)$ the solution $u$ of the problem (5.2) written in abstract form as:

$$
\left\{\begin{array}{l}
{\left[\partial_{t t t} u+\alpha \partial_{t t t} u+\beta \partial_{t t} u+\gamma \Delta \partial_{t t} u+\delta \Delta \partial_{t} u+\rho \Delta u\right](t)=f(t), \text { for } t \in(0,2 \pi) ;} \\
u(x, 0)=u(x, 2 \pi), \partial_{t} u(x, 0)=\partial_{t} u(x, 2 \pi), \partial_{t t} u(x, 0)=\partial_{t t} u(x, 2 \pi), \partial_{t t t} u(x, 0)=\partial_{t t t} u(x, 2 \pi),
\end{array}\right.
$$

exists, is unique and belongs to the space $W_{p e r}^{4, p}\left(\mathbb{T}, L^{q}(\Omega)\right) \cap W_{p e r}^{2, p}\left(\mathbb{T},\left[D\left(\Delta_{q}\right)\right]\right)$. Moreover, for any $1<p, q<\infty$ the estimate

$$
\begin{aligned}
& \|u\|_{L^{p}\left(\mathbb{T}, L^{q}(\Omega)\right)}+\left\|u^{\prime \prime}\right\|_{W_{p e r}^{2, p}\left(\mathbb{T}, L^{q}(\Omega)\right)}+\left\|u^{\prime \prime \prime}\right\|_{W_{p e r}^{3, p}\left(\mathbb{T}, L^{q}(\Omega)\right)}+\left\|u^{\prime \prime \prime \prime}\right\|_{W_{p e r}^{4, p}\left(\mathbb{T}, L^{q}(\Omega)\right)} \\
& \quad+\|A u\|_{L^{p}\left(\mathbb{T},\left[D\left(\Delta_{q}\right)\right]\right)}+\left\|A u^{\prime}\right\|_{W_{p e r}^{1, p}\left(\mathbb{T},\left[D\left(\Delta_{q}\right)\right]\right)}+\left\|A u^{\prime \prime}\right\|_{W_{p e r}^{2, p}\left(\mathbb{T},\left[D\left(\Delta_{q}\right)\right]\right)} \leq C \|_{L^{p}\left(\mathbb{T}, L^{q}(\Omega)\right)}
\end{aligned}
$$

holds.
We finish with the following example that considers the fractional Laplacian operator.
Example 5.3. Let $1<p<\infty, \frac{1}{2}<s<1$ and $\alpha, \beta, \gamma, \delta, \rho$ be strictly positive real numbers satisfying $\rho+\beta \gamma<\alpha \delta$. Consider the following nonlocal equation in a bounded smooth domain $\Omega \subset \mathbb{R}^{N}$ :

$$
\left\{\begin{array}{l}
{\left[\partial_{t t t t} u+\alpha \partial_{t t t} u+\beta \partial_{t t} u\right.}  \tag{5.3}\\
\left.-\gamma(-\Delta)^{s} \partial_{t t} u-\delta(-\Delta)^{s} \partial_{t} u-\rho(-\Delta)^{s} u\right](x, t)=f(x, t), \text { for }(x, t) \in \mathbb{R}^{N} \times(0,2 \pi) ; \\
u(x, t)=0, \text { for }(x, t) \in \partial \Omega \times(0,2 \pi) ; \\
u(x, 0)=u(x, 2 \pi), \partial_{t} u(x, 0)=\partial_{t} u(x, 2 \pi), \partial_{t t} u(x, 0)=\partial_{t t} u(x, 2 \pi), \partial_{t t t} u(x, 0)=\partial_{t t t} u(x, 2 \pi),
\end{array}\right.
$$

where the fractional Laplacian $-(-\Delta)^{s}$ is defined by

$$
(-\Delta)^{s} v:=\mathcal{F}_{\xi}^{-1}(|\xi|(\mathcal{F} v)(\xi)), \quad v \in H^{1, q}(\Omega)
$$

For $X=L^{q}(\Omega)$ and $D_{q}\left((-\Delta)^{s}\right):=H^{1, q}(\Omega), 1<q<\infty$, the fractional operator $-(-\Delta)^{s}$ : $H^{1, q}(\Omega) \rightarrow L^{q}(\Omega)$ is also $R$-sectorial of angle $\theta$ for an arbitrary $\theta \in(0, s \pi)$, see [1, Proposition 2.2]. Hence, by Theorem 5.1, for any $f \in L^{p}\left(\mathbb{T}, L^{q}(\Omega)\right)$ there exists a unique solution $u \in W_{\text {per }}^{4, p}\left(\mathbb{T}, L^{q}(\Omega)\right) \cap W_{\text {per }}^{2, p}\left(\mathbb{T}, H^{1, q}(\Omega)\right)$ of the problem (5.3) and satisfies the following maximal regularity estimate

$$
\begin{aligned}
& \|u\|_{L^{p}\left(\mathbb{T}, L^{q}(\Omega)\right)}+\left\|u^{\prime \prime}\right\|_{W_{p e r}^{2, p}\left(\mathbb{T}, L^{q}(\Omega)\right)}+\left\|u^{\prime \prime \prime}\right\|_{W_{p e r}^{3, p}\left(\mathbb{T}, L^{q}(\Omega)\right)}+\left\|u^{\prime \prime \prime \prime}\right\|_{W_{p e r}^{4, p}\left(\mathbb{T}, L^{q}(\Omega)\right)} \\
& \quad+\|A u\|_{L^{p}\left(\mathbb{T}, H^{1, q}(\Omega)\right)}+\left\|A u^{\prime}\right\|_{W_{p e r}^{1, p}\left(\mathbb{T}, H^{1, q}(\Omega)\right)}+\left\|A u^{\prime \prime}\right\|_{W_{p e r}^{2, p}\left(\mathbb{T}, H^{1, q}(\Omega)\right)} \leq C\|f\|_{L^{p}\left(\mathbb{T}, L^{q}(\Omega)\right)} .
\end{aligned}
$$

Analogous examples hold for the cases of the scales of Besov and Triebel-Lizorkin spaces, replacing $R$-sectorial operator by sectorial operator and $R$-boundedness by uniform boundedness. For instance, from Theorem 4.5 we obtain the following result.

Theorem 5.4. Let $X$ be a Banach space, $1<p<\infty, \alpha, \beta, \gamma, \delta, \rho \in(0, \infty)$ and let A be a sectorial operator on $X$ of angle $\pi / 2$. If $\rho+\beta \gamma<\alpha \delta$ then equation (5.1) is strongly $B_{p, q}^{s}$-well-posed.

## Acknowledgements

The first author is partially supported by FONDECYT grant number 1180041 and DICYT, Universidad de Santiago de Chile, USACH. The second author is supported by MEC, grants MTM2016-75963-P and PID2019-105011GB-I00.

## References

[1] G. Akrivis, B. Li, Maximum norm analysis of implicit-explicit backward difference formulas for nonlinear parabolic equations, IMA J. Numer. Anal. 38(2018), No. 1, 75-101. https://doi.org/10.1093/imanum/drx008; MR3800015
[2] H. Amann, Linear and quasilinear parabolic problems, Monographs in Mathematics, Vol. 89, Basel, Birkhäuser Verlag, 1995. https://doi.org/10.1007/978-3-030-11763-4; MR3930629
[3] W. Arendt, Semigroups and evolution equations: functional calculus, regularity and kernel estimates, in: Evolutionary equations, Vol. 1, Handbook of Differential Equations, North-Holland, Amsterdam, 2004, pp. 1-85. MR2103696
[4] W. Arendt, S. Bu, The operator-valued Marcinkiewicz multiplier theorem and maximal regularity, Math. Z. 240(2002), No. 2, 311-343. https ://doi.org/10.1007/s002090100384; MR1900314
[5] W. Arendt, S. Bu, Operator-valued Fourier multipliers on periodic Besov spaces and applications, Proc. Edinburgh Math. Soc. (2) 47(2004), 15-33. https://doi.org/10.1017/ S0013091502000378; MR2064734
[6] S. K. Bose, G. C. Gorain, Exact controllability and boundary stabilization of flexural vibrations of an internally damped flexible space structure, Appl. Math. Comput. 126(2002), No. 2-3, 341-360. https://doi.org/10.1016/S0096-3003(00)00112-0; MR1879167
[7] S. Bu, G. Cai, Periodic solutions of third-order degenerate differential equations in vectorvalued functional spaces, Israel J. Math. 212(2016), 163-188. https://doi.org/10.1007/ s11856-016-1282-0; MR3504322
[8] S. Bu, G. Cai, Well-posedness of second-order degenerate differential equations with finite delay in vector-valued function spaces, Pacific J. Math. 288(2017), No. 1, 27-46. https://doi.org/10.2140/pjm.2017.288.27; MR3667761
[9] S. Bu, G. Cai, Periodic solutions of third-order integro-differential equations in vectorvalued functional spaces, J. Evol. Equ. 17(2017), No. 2, 749-780. https://doi.org/10. 1007/s00028-016-0335-5; MR3665228
[10] S. Bu, G. CaI, Periodic solutions of fractional degenerate differential equations with delay in Banach spaces, Israel J. Math. 232(2019), No. 2, 695-717. https://doi.org/10.1007/ s11856-019-1884-4; MR3990956
[11] S. Bu, G. Cai, Well-posedness of fractional integro-differential equations in vector-valued functional spaces, Math. Nachr. 292(2019), No. 5, 969-982. https://doi.org/10.1002/ mana. 201800104; MR3953838
[12] S. Bu, G. Cai, Periodic solutions of second order degenerate differential equations with finite delay in Banach spaces, J. Fourier Anal. Appl. 25(2019), No. 1, 32-50. https://doi. org/10.1007/s00041-017-9560-8; MR3901916
[13] S. Bu, G. CaI, Solutions of third order degenerate equations with infinite delay in Banach spaces, Banach J. Math. Anal. 14(2020), No. 3, 1201-1221. https://doi. org/10.1007/ s43037-020-00058-x; MR4123328
[14] S. Bu, J. Kim, Operator-valued Fourier multipliers on periodic Triebel spaces, Acta Math. Sin. (Engl. Ser.) 21(2005), 1049-1056. https://doi.org/10.1007/s10114-004-04539; MR2176315
[15] D. L. Burkhölder, Martingales and Fourier analysis in Banach spaces, in: G. Letta, M. Pratelli (Eds.), Probability and analysis (Varenna, 1985), Lecture Notes in Mathematics, Vol. 1206, Berlin, Springer, 1986, pp. 61-108. https : //doi .org/10.1007/BFb0076300; MR0864712
[16] G. Cai, S. Bu, Periodic solutions of third-order degenerate differential equations in vectorvalued functional spaces, Israel J. Math. 212(2016), No. 1, 163-188. https://doi.org/10. 1007/s11856-016-1282-0; MR3504322
[17] K. E. Chlouverakis, J. C. Sprott, Chaotic hyperjerk systems, Chaos Solitons Fractals 28(2006), 739-746. https://doi.org/10.1016/j.chaos.2005.08.019; MR2204007
[18] Ph. Clément, S. O. Londen, G. Simonett, Quasilinear evolutionary equations and continuous interpolation spaces, J. Differential Equations 196(2004), No. 2, 418-447. https: //doi.org/10.1016/j.jde.2003.07.014; MR2028114
[19] J. A. Conejero, C. Lizama, M. Murillo-Arcila, J. B. Seoane-Sepúlveda, Wellposedness for degenerate third order equations with delay and applications to inverse problems, Israel J. Math. 229(2019), 219-254. https://doi.org/10.1007/s11856-018-1796-8; MR3905603
[20] F. Dell'Oro, I. Lasiecka, V. Pata, The Moore-Gibson-Thompson equation with memory in the critical case, J. Differential Equations 261(2016), 4188-4222. https://doi.org/10. 1016/j.jde.2016.06.025; MR3532069
[21] F. Dell'Oro, V. Pata, On a fourth-order equation of Moore-Gibson-Thompson type, Milan J. Math. 85(2017), 215-234. https://doi.org/10.1007/s00032-017-0270-0; MR3735562
[22] R. Denk, M. Hieber, J. Prüss, $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type, Mem. Amer. Math. Soc. 166(2003), No. 788. https://doi.org/ 10.1090/memo/0788; MR2006641
[23] D. Eager, A. M. Pendrill, N. Reistad, Beyond velocity and acceleration: jerk, snap and higher derivatives, Eur. J. Phys. 37(2016), 065008, 11 pp. https://doi.org/10.1088/01430807/37/6/065008
[24] G. Figliolini, C. Lanni, Jerk and jounce relevance for the kinematic performance of long-dwell mechanisms, Mech. Mach. Sci. 73(2019), 219-228. https://doi.org/10.1007/ 978-3-030-20131-9_22
[25] M. F. Hamilton, D. T. Blackstock, Nonlinear acoustics, Academic Press, New York, 1998.
[26] B. Jin, B. Li, Z. Zhou, Discrete maximal regularity of time-stepping schemes for fractional evolution equations. Numer. Math. 138(2018), No. 1, 101-131. https: //doi.org/10.1007/ s00211-017-0904-8; MR3745012
[27] P. M. Jordan, Nonlinear acoustic phenomena in viscous thermally relaxing fluids: Shock bifurcation and the emergence of diffusive solitons, J. Acoust. Soc. Am. 124(2008), No. 4, 2491-2491. https://doi.org/10.1121/1.4782790
[28] B. Kaltenbacher, I. Lasiecka, R. Marchand, Wellposedness and exponential decay rates for the Moore-Gibson-Thompson equation arising in high intensity ultrasound, Control Cybernet. 40(2011), No. 4, 971-988. MR2977496
[29] B. Kaltenbacher, I. Lasiecka, M. Pospieszalska, Well-posedness and exponential decay of the energy in the nonlinear Moore-Gibson-Thompson equation arising in high intensity ultrasound, Math. Models Meth. Appl. Sci. 22(2012), 1250035, 34 pp. https: //doi.org/10.1142/S0218202512500352; MR2974173
[30] N. Kalton, L. Weis, The $\mathcal{H}^{\infty}$ calculus and sums of closed operators, Math. Ann. 321(2001), 319-345. https://doi.org/10.1007/s002080100231; MR1866491
[31] V. Keyantuo, C. Lizama, Fourier multipliers and integro-differential equations in Banach spaces, J. London Math. Soc. (2) 69(2004), 737-750. https://doi.org/10.1112/ S0024610704005198; MR2050043
[32] I. Lasiecka, X. Wang, Moore-Gibson-Thompson equation with memory, part I: Exponential decay of energy, Z. Angew. Math. Phys. 67(2016), Art. No. 17, 23 pp. https: //doi.org/10.1007/s00033-015-0597-8; MR3480524
[33] I. Lasiecka, X. Wang, Moore-Gibson-Thompson equation with memory, part II: General decay of energy, J. Differential Equations 259(2015), 7610-7635. https ://doi. org/10.1016/ j.jde.2015.08.052; MR3401607
[34] W. Liv, Z. Chen, Z. Tu, New general decay result for a fourth-order Moore-GibsonThompson equation with memory, Electron. Res. Arch. 28(2020), No. 1, 433-457. https: //doi.org/10.3934/era. 2020025
[35] R. Marchand, T. Mcdevitt, R. Triggiani, An abstract semigroup approach to the thirdorder Moore-Gibson-Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability, Math. Meth. Appl. Sci. 35(2012), 1896-1929. https://doi.org/10.1002/mma.1576; MR2982472
[36] V. Poblete, J. C. Pozo, Periodic solutions of an abstract third-order differential equation, Studia Math. 215(2013), 195-219. https ://doi. org/10.4064/sm215-3-1; MR3080779
[37] M. Visser, Jerk, snap and the cosmological equation of state, Class. Quantum Gravity 21(2004), No. 11, 2603-2615. https://doi.org/10.1088/0264-9381/21/11/006

# Existence of solution for Kirchhoff model problems with singular nonlinearity 

Marcelo Montenegro ${ }^{\boxtimes}$<br>Universidade Estadual de Campinas, IMECC, Departamento de Matemática, Rua Sérgio Buarque de Holanda, 651, Campinas, SP, CEP 13083-859, Brasil

Received 12 June 2021, appeared 15 October 2021
Communicated by Patrizia Pucci


#### Abstract

We study the fourth order Kirchhoff equation $\Delta^{2} u-\left(a+b \int_{\Omega}|\nabla u|^{2}\right)^{\gamma} \Delta u=$ $f(u)$ in $\Omega$ with $-\Delta u>0$ and $u>0$ in $\Omega$, and $\Delta u=u=0$ on $\partial \Omega$, where $f(t)=$ $\alpha \frac{1}{t^{\theta}}+\lambda t^{q}+\mu t+g(t)$ for $t \geq 0, g$ has subcritical growth, $\alpha>0, \lambda>0, \mu \geq 0,0<\theta<1$, $0<q<1, \gamma \geq 0, a>0, b \geq 0$. We use the Galerkin projection method to show the existence of solution under some boundedness restriction on $\alpha, \lambda, \mu$. In some cases we study the behavior of the norm of the solution $u$ as $\lambda \rightarrow 0$ and as $\lambda \rightarrow \infty$. Similar issues are addressed for the equation $\left(a+b \int_{\Omega}|\nabla u|^{2}\right)^{\gamma} \Delta^{2} u-\varrho \Delta u=f(u), \varrho \geq 0$.


Keywords: existence of solution, Kirchhoff equation, singular nonlinearity, approximation scheme.
2020 Mathematics Subject Classification: Primary 35J40; Secondary 35J30, 37L65, 35J60, 65N30.

## 1 Introduction

Let $\Omega \subset \mathbb{R}^{N}, N \geq 1$, be a bounded domain with smooth boundary $\partial \Omega$. We solve the following problems.

$$
\begin{cases}\Delta^{2} u-\left(a+b \int_{\Omega}|\nabla u|^{2}\right)^{\gamma} \Delta u=f(u) & \text { in } \Omega  \tag{1.1}\\ -\Delta u>0, u>0 & \text { in } \Omega \\ \Delta u=u=0 & \text { on } \partial \Omega\end{cases}
$$

and

$$
\begin{cases}\left(a+b \int_{\Omega}|\nabla u|^{2}\right)^{\gamma} \Delta^{2} u-\varrho \Delta u=f(u) & \text { in } \Omega  \tag{1.2}\\ -\Delta u>0, u>0 & \text { in } \Omega \\ \Delta u=u=0 & \text { on } \partial \Omega\end{cases}
$$

[^44]Equation (1.1) is related to the study of Woinowsky-Krieger [31] in the analysis of buckling and vibrations dynamics of nonlinear beam models. The equation is given by

$$
u_{t t}+\tau u_{x x x x}-\left(a+b \int_{0}^{L}\left|u_{x}\right|^{2}\right) u_{x x}=f(x, u),
$$

where $\tau, a, b$ are physical quantities detailed in the sequel: $\tau=E I / \rho, a=H / \rho$ and $b=$ $E A / 2 \rho L$, where $L$ is the length of the beam in the initial position, $E$ is the modulus of elasticity in tension, $I$ is the cross-sectional moment of inertia, $\rho$ is the mass density, $H$ is the tension in the initial position, $A$ is the cross-sectional area. Here $u(t, x)$ is the deflection of the point $x$ of the beam at time $t$ subjected to a force $f$. More on wave equations in this field can be seen in $[6,10,14,21,32]$. In this respect, McKenna-Walter $[23,24]$ studied oscillations of a hanged bridge as it is conveyed by the equation

$$
u_{t t}+u_{x x x x}+\kappa u^{+}=f(x, u),
$$

where $\kappa>0$ belongs to a specific range.
Equation (1.1) is also associated to Berger's [5] plate model equation

$$
u_{t t}+\Delta^{2} u+\left(a+b \int_{\Omega}|\nabla u|^{2}\right) \Delta u=f\left(x, u, u_{t}\right)
$$

that describes the vertical wave vibration of a thin plate. It takes into account horizontal forces and material resistance represented by $a$ and $b$. Vertical loads $f$ forces the membrane up and down, and may depend on the displacement $u$ and speed $u_{t}$. Consult also Chueshov-Lasiecka [9] to appreciate the context of the continuum mechanics where such model is inserted.

Equation (1.2) is a fourth order generalization of the Kirchhoff's [16] wave equation

$$
u_{t t}-\left(a+b \int_{\Omega}|\nabla u|^{2}\right) \Delta u=f(x, u)
$$

that describes changes in length $u$ when a string is transversely fingered with force $f$, and where $a$ and $b$ stand for horizontal tensions magnitudes. This can be viewed as an extension of D'Alembert's wave equation for free vibration strings that gives a more accurate description of vibrations of an elastic string, see for instance [4]. Results dealing with variational methods applied to the stationary equation can be viewed in [11,18].

Recent works related to (1.1) and (1.2) dealing with variational methods are [2,7,8,12,17, $22,25,29,33]$. The list of papers in this subject is vast, we describe a fill of them below.

A similar equation to (1.1) was studied in [2], namely

$$
\begin{cases}\Delta^{2} u-\lambda_{0}\left(a+b \int_{\Omega}|\nabla u|^{2}\right) \Delta u=f(x, u) & \text { in } \Omega  \tag{1.3}\\ \Delta u=u=0 & \text { on } \partial \Omega\end{cases}
$$

where $\lambda_{0}>0$ is a parameter. Among other suitable hypotheses, $f$ is $o(|u|)$ at zero, has subcritical growth and satisfies the so-called Ambrosetti-Rabinowitz condition. By means of the mountain pass theorem, it was shown that there exists a $\bar{\lambda}>0$ such that the problem has a nontrivial solution for $0<\lambda_{0}<\bar{\lambda}$.

The Schrödinger-Kirchhoff equation

$$
\begin{equation*}
\Delta^{2} u-\left(a+b \int_{\mathbb{R}^{N}}|\nabla u|^{2}\right) \Delta u+V(x) u=f(x, u)+h(x) \text { in } \mathbb{R}^{N} \tag{1.4}
\end{equation*}
$$

was studied in [33]. When $h \geq 0$, by the mountain pass theorem, there is a nontrivial solution. For that matter the potential $V$ satisfies some suitable hypotheses and $f$ is $o(|u|)$ at zero, has subcritical growth and satisfies the so-called Ambrosetti-Rabinowitz condition. In case $h=0$ and $f$ has some symmetric properties, there are infinitely many high-energy solutions which are obtained by the symmetric mountain pass theorem. Moreover, there are infinitely many radial solutions.

The equation with critical growth

$$
\begin{equation*}
\Delta^{2} u-M\left(\int_{\mathbb{R}^{N}}|\nabla u|^{2}\right) \Delta u+u=\lambda_{0} f(u)+|u|^{\frac{8}{N-4}} u \quad \text { in } \mathbb{R}^{N} \tag{1.5}
\end{equation*}
$$

was studied in [7] for $N \geq 5$, where $M:[0, \infty) \rightarrow[0, \infty)$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions with $M(t) \geq m_{0}>0, f$ is $o(|u|)$ at zero, has subcritical growth, $f(t) / t$ is increasing and satisfies the so-called Ambrosetti-Rabinowitz condition. Using minimax critical point theorems, the authors show that there is a $\bar{\lambda}>0$ such that for $\lambda_{0}>\bar{\lambda}$ there is a nontrivial solution.

The critical problem with indefinite potentials was considered in [12], namely

$$
\begin{cases}\Delta^{2} u-\left(a+b \int_{\Omega}|\nabla u|^{2}\right) \Delta u=\lambda_{0} a_{0}(x)|u|^{q_{0}-2} u+b_{0}(x)|u|^{p_{0}-2} u & \text { in } \Omega  \tag{1.6}\\ \Delta u=u=0 & \text { on } \partial \Omega .\end{cases}
$$

Under suitable assumptions on the potentials $a_{0}$ and $b_{0}$, there is $\bar{\lambda}>0$ such that if $1<$ $q_{0}<2<p_{0} \leq 2 N /(N-4), N \geq 5$, then there exists a nontrivial nonnegative solution for $0<\lambda_{0}<\bar{\lambda}$. A second solution exists for $\lambda_{0}$ small if $1<q_{0}<2,4<p_{0} \leq 2 N /(N-4)$ and $N=5,6,7$. The first solution is obtained as the limit of a minimizing sequence by making use of Ekeland's variational principle and the second solution is found by means of the mountain pass theorem.

Using a similar strategy of the Galerkin method compared to the present paper, the following singular fourth order Kirchhoff equation with Hardy potential was studied in [3]. There $\Omega$ is a bounded domain with $0 \in \Omega, h$ and $k$ are positive continuous functions, $M:[0, \infty) \rightarrow[0, \infty)$ a continuous function such that $M(t) \geq m_{0}>0$ and $\bar{\mu}=\frac{(N(N-4))^{2}}{16}$ is the best constant of the Hardy inequality. The problem

$$
\begin{cases}\Delta^{2} u-\lambda_{0} M\left(\int_{\Omega}|\nabla u|^{2}\right) \Delta u=\mu_{0} \frac{1}{|x|^{4}} u+\frac{h(x)}{u^{\theta}}+k(x) u^{q} & \text { in } \Omega  \tag{1.7}\\ \Delta u=u=0 & \text { on } \partial \Omega\end{cases}
$$

has a positive solution for $\lambda_{0}>\mu_{0} / \bar{\mu} m_{0}$ and $0<\mu_{0}<\bar{\mu}$.
In contrast to some of the above papers, we prescribe mild assumptions on $f$, since we do not need the so-called Ambrosetti-Rabinowitz condition nor specific behavior of $f$ near zero. Instead, we adopt an approximation scheme inspired in [27,28].

Define

$$
\begin{equation*}
f(t)=\alpha \frac{1}{t^{\theta}}+\lambda t^{q}+\mu t+g(t) \quad \text { for } t \geq 0 \tag{1.8}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha>0, \quad \lambda>0, \quad \mu \geq 0, \quad 0<\theta<1, \quad 0<q<1 . \tag{1.9}
\end{equation*}
$$

The constants in the differential operators respect the following rules:

$$
\begin{equation*}
\gamma \geq 0, \quad a>0, \quad b \geq 0, \quad \varrho \geq 0 . \tag{1.10}
\end{equation*}
$$

The function

$$
\begin{equation*}
g: \mathbb{R} \rightarrow \mathbb{R} \text { is continuous } \tag{1.11}
\end{equation*}
$$

and satisfies

$$
\begin{equation*}
|g(t)| \leq c_{1}|t|^{p} \quad \text { for } t \in \mathbb{R} \text { and } 1 \leq p<2 N /(N-4) \quad(\text { or } 1 \leq p<\infty \text { if } N=1,2,3,4) \tag{1.12}
\end{equation*}
$$

where $c_{1}$ is a constant.
By a solution of (1.1) and (1.2) we mean a function $u \in H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$ such that

$$
\int_{\Omega} \Delta u \Delta \phi+\left(a+b \int_{\Omega}|\nabla u|^{2}\right)^{\gamma} \nabla u \nabla \phi-f(u) \phi=0, \quad \forall \phi \in H^{2}(\Omega) \cap H_{0}^{1}(\Omega)
$$

or

$$
\left(a+b \int_{\Omega}|\nabla u|^{2}\right)^{\gamma} \Delta u \Delta \phi+\varrho \nabla u \nabla \phi-f(u) \phi=0, \quad \forall \phi \in H^{2}(\Omega) \cap H_{0}^{1}(\Omega) .
$$

The underlying idea in the proof of the existence of solution, is to consider the function $f_{\varepsilon}(t)=\alpha \frac{1}{(t+\varepsilon)^{\theta}}+\lambda t^{q}+\mu t$ with $0<\varepsilon<1$, which is an approximation of $f(t)=\alpha \frac{1}{t^{\theta}}+\lambda t^{q}+\mu t$ that avoids the singular term at zero. We use the the spectral Galerkin projection method and transform the original equation into a family of finite dimensional nonlinear equations. In each of them we use Brouwer's theorem to get a solution. Due to the structure of the equations, we are able to obtain uniform estimates and to pass to the limit in the projected finite dimensional equations. We thus obtain a solution $u_{\varepsilon}$. And some extra reasoning is used to show that $u_{\varepsilon}$ converges to a nontrivial solution of the original equation as $\varepsilon \rightarrow 0$. Since we use the classical strong maximum principle, some arguments do not work if the boundary condition is $u=\frac{\partial u}{\partial v}=0$. A more general boundary condition related to the Kirchhoff-Love model for the vertical vibration of a thin elastic plate is presented in [13, pp. 5-7], motivated to earlier works [15,20], see also [26].

We state the main results.
Theorem 1.1. Assume (1.8)-(1.10) and $g \equiv 0$. There is $\mu^{*}>0$ such that for $0 \leq \mu<\mu^{*}$ and for every $\alpha, \lambda>0$, equation (1.1) has a solution.

Theorem 1.2. Assume (1.8)-(1.10) and $g \equiv 0$. There is $\mu^{*}>0$ such that for $0 \leq \mu<\mu^{*}$ and for every $\alpha, \lambda>0$, equation (1.2) has a solution.

Theorem 1.3. Assume (1.8)-(1.12). Then there exist $\alpha^{*}, \lambda^{*}, \mu^{*}>0$ such that for every $0<\alpha<\alpha^{*}$, $0<\lambda<\lambda^{*}$ and $0 \leq \mu<\mu^{*}$ equation (1.1) has a solution.

Theorem 1.4. Assume (1.8)-(1.12). Then there exist $\alpha^{*}, \lambda^{*}, \mu^{*}>0$ such that for every $0<\alpha<\alpha^{*}$, $0<\lambda<\lambda^{*}$ and $0 \leq \mu<\mu^{*}$ equation (1.2) has a solution.

Theorem 1.5. Let $f$ be such that $\alpha=\mu=0$ and $g(t)=t^{p}$ for $t \geq 0$ with $1<p<2 N /(N-4)$. And let $u_{\lambda}>0$ be the solution obtained in each Theorem 1.3 or 1.4. Then $\left\|u_{\lambda}\right\|_{H^{2} \cap H_{0}^{1}} \rightarrow 0$ as $\lambda \rightarrow 0$.

Theorem 1.6. Let $f(t)=\lambda\left(\frac{1}{t^{t}}+t^{q}+t\right)+t^{p}$ for $t \geq 0$ with $1<p<2 N /(N-4)$. And let $u_{\lambda}>0$ be the solution obtained in each Theorem 1.3 or 1.4. If $u_{\lambda}$ exists for every $\lambda$ large, then $\left\|u_{\lambda}\right\|_{H^{2} \cap H_{0}^{1}} \rightarrow \infty$ as $\lambda \rightarrow \infty$.

## 2 Preliminaries

The space $H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$ is Hilbert with

$$
\text { inner product }(u, v)=\int_{\Omega} \Delta u \Delta v \text { and norm }\|u\|_{H^{2} \cap H_{0}^{1}}=\left(\int_{\Omega}|\Delta u|^{2}\right)^{1 / 2} \text {. }
$$

The embedding $H^{2}(\Omega) \cap H_{0}^{1}(\Omega) \hookrightarrow L^{\sigma}(\Omega)$ is continuous if $1 \leq \sigma \leq 2 N /(N-4)$ and compact if $1 \leq \sigma<2 N /(N-4)$. The embedding is continuous if $N=1,2,3,4$ and $1 \leq \sigma<\infty$. Also, the embedding $H^{2}(\Omega) \cap H_{0}^{1}(\Omega) \hookrightarrow H_{0}^{1}(\Omega)$ is continuous and compact, see [1,12,30]. Moreover $\|u\|_{H_{0}^{1}}^{2} \leq\|u\|_{L^{2}}\|u\|_{H^{2} \cap H_{0}^{1}}$, since $\int_{\Omega}|\nabla u|^{2}=\int_{\Omega} u(-\Delta u)$. The Sobolev embedding constant $C_{\sigma}$ related to $\|u\|_{L^{\sigma}} \leq C_{\sigma}\|u\|_{H^{2} \cap H_{0}^{1}}$ will appear in some computations. The spectrum of $-\Delta$ in $H_{0}^{1}(\Omega)$ is given by the numbers $\lambda_{i}, i \in \mathbb{N}$, where $0<\lambda_{1}<\lambda_{2} \leq \lambda_{3} \leq \lambda_{4} \ldots$. The corresponding eigenfunctions are $w_{i} \in H_{0}^{1}(\Omega), i \in \mathbb{N}$. The first eigenfunction corresponding to $\lambda_{1}$ is $w_{1}>0$. For every $i \in \mathbb{N}$ one has

$$
\begin{cases}-\Delta w_{i}=\lambda_{i} w_{i} & \text { in } \Omega  \tag{2.1}\\ w_{i}=0 & \text { on } \partial \Omega\end{cases}
$$

By elliptic regularity $w_{i} \in C^{\infty}(\bar{\Omega}), i \in \mathbb{N}$. With respect to the biharmonic operator, for every $i \in \mathbb{N}$,

$$
\begin{cases}\Delta^{2} w_{i}=\lambda_{i}^{2} w_{i} & \text { in } \Omega  \tag{2.2}\\ \Delta w_{i}=w_{i}=0 & \text { on } \partial \Omega\end{cases}
$$

In other words, the spectrum of $\Delta^{2}$ in $H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$ is given by the numbers $\lambda_{i}^{2}, i \in \mathbb{N}$, where $0<\lambda_{1}^{2}<\lambda_{2}^{2} \leq \lambda_{3}^{2} \leq \lambda_{4}^{2} \ldots$ And the corresponding eigenfunctions are also $w_{i} \in$ $H^{2}(\Omega) \cap H_{0}^{1}(\Omega), i \in \mathbb{N}$. The following orthogonality relations take place

$$
\begin{equation*}
\int_{\Omega} \nabla w_{i} \nabla w_{j}=\int_{\Omega} w_{i}\left(-\Delta w_{j}\right)=\lambda_{j} \int_{\Omega} w_{i} w_{j}=0 \quad \text { if } i \neq j \tag{2.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{\Omega} \Delta w_{i} \Delta w_{j}=\int_{\Omega} w_{i}\left(\Delta^{2} w_{j}\right)=\lambda_{j}^{2} \int_{\Omega} w_{i} w_{j}=0 \quad \text { if } i \neq j . \tag{2.4}
\end{equation*}
$$

The set of eigenfunctions can be normalized either as $\left\|w_{i}\right\|_{H_{0}^{1}}=1$ or $\left\|w_{i}\right\|_{H^{2} \cap H_{0}^{1}}=1, i \in \mathbb{N}$. Hence $\mathbb{B}=\left\{w_{1}, w_{2}, \ldots, w_{m}, \ldots\right\}$ is an orthonormal basis of $H_{0}^{1}(\Omega)$ and of $H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$, according the inner product of each space.

An aside result that will be useful in the proofs is Brouwer's Theorem [19] that says: Let $F: \mathbb{R}^{m} \rightarrow \mathbb{R}^{m}$ be a continuous function such that $(F(\eta), \eta) \geq 0$ for every $\eta \in \mathbb{R}^{m}$ with $|\eta|=r$ for some $r>0$. Then, there exists $z_{0} \in \mathbb{R}^{m}$ with $\left|z_{0}\right| \leq r$ such that $F\left(z_{0}\right)=0$.

## 3 Proof of the theorems

We begin proving Theorem 1.1.
Proof. Define $f_{\varepsilon}(t)=\alpha \frac{1}{(t+\varepsilon)^{ब}}+\lambda t^{q}+\mu t$ with $0<\varepsilon<1$ and let $\mathbb{B}=\left\{w_{1}, w_{2}, \ldots, w_{m}, \ldots\right\}$ be an orthonormal basis of $H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$, see (2.3) and (2.4). (Here $w_{i}, i=1,2,3, \ldots$ need not to be eigenfuncitons, but we choose a such basis for convenience). Define

$$
\mathbb{W}_{m}=\left[w_{1}, w_{2}, \ldots, w_{m}\right],
$$

to be the space generated by $\left\{w_{1}, w_{2}, \ldots, w_{m}\right\}$. Define the function $F: \mathbb{R}^{m} \rightarrow \mathbb{R}^{m}$ such that

$$
F(\eta)=\left(F_{1}(\eta), F_{2}(\eta), \ldots, F_{m}(\eta)\right)
$$

where $\eta=\left(\eta_{1}, \eta_{2}, \ldots, \eta_{m}\right) \in \mathbb{R}^{m}$,

$$
F_{j}(\eta)=\int_{\Omega} \Delta u \Delta w_{j}+\left(a+b \int_{\Omega}|\nabla u|^{2}\right)^{\gamma} \int_{\Omega} \nabla u \nabla w_{j}-\int_{\Omega} f_{\varepsilon}(|u|) w_{j}, \quad j=1,2, \ldots, m
$$

and

$$
u=\sum_{i=1}^{m} \eta_{i} w_{i} \quad \in \mathbb{W}_{m} .
$$

Therefore

$$
\begin{align*}
(F(\eta), \eta) & =\int_{\Omega}|\Delta u|^{2}+\left(a+b \int_{\Omega}|\nabla u|^{2}\right)^{\gamma} \int_{\Omega}|\nabla u|^{2}-\int_{\Omega} f_{\varepsilon}(|u|) u \\
& \geq\|u\|_{H^{2} \cap H_{0}^{1}}^{2}-\alpha|\Omega|^{\theta} C_{1}^{1-\theta}\|u\|_{H^{2} \cap H_{0}^{1}}^{1-\theta}-\lambda C_{q+1}^{q+1}\|u\|_{H^{2} \cap H_{0}^{1}}^{q+1}-\mu C_{2}^{2}\|u\|_{H^{2} \cap H_{0}^{1}}^{2} . \tag{3.1}
\end{align*}
$$

The function $F$ is continuous because each $F_{j}$ is continuous by Sobolev embedding and dominated convergence theorem. Here $C_{1}, C_{q+1}$ and $C_{2}$ are Sobolev embedding constants appearing in $\|u\|_{L^{\sigma}} \leq C_{\sigma}\|u\|_{H^{2} \cap H_{0}^{1}}$, which are independent on $m$ and $\varepsilon$. Hence for $\mu<C_{2}^{-2}$, there is $R>0$ such that

$$
\begin{equation*}
(F(\eta), \eta)>0 \text { for }\|u\|_{H^{2} \cap H_{0}^{1}}=|\eta|=R . \tag{3.2}
\end{equation*}
$$

Brouwer's Theorem asserts that there exists $u_{m, \varepsilon} \in H^{2} \cap H_{0}^{1}$ with $\left\|u_{m, \varepsilon}\right\|_{H^{2} \cap H_{0}^{1}} \leq R$ satisfying

$$
\begin{equation*}
\int_{\Omega} \Delta u_{m, \varepsilon} \Delta w_{j}+\left(a+b \int_{\Omega}\left|\nabla u_{m, \varepsilon}\right|^{2}\right)^{\gamma} \int_{\Omega} \nabla u_{m, \varepsilon} \nabla w_{j}-\int_{\Omega} f_{\varepsilon}\left(\left|u_{m, \varepsilon}\right|\right) w_{j}=0, \quad j=1,2, \ldots, m . \tag{3.3}
\end{equation*}
$$

Hence

$$
\int_{\Omega} \Delta u_{m, \varepsilon} \Delta \zeta_{m}+\left(a+b \int_{\Omega}\left|\nabla u_{m, \varepsilon}\right|^{2}\right)^{\gamma} \int_{\Omega} \nabla u_{m, \varepsilon} \nabla \zeta_{m}-\int_{\Omega} f_{\varepsilon}\left(\left|u_{m, \varepsilon}\right|\right) \zeta_{m}=0, \quad \forall \zeta_{m} \in \mathbb{W}_{m} .
$$

Let $k \in \mathbb{N}$, then for every $m \geq k$ we obtain

$$
\begin{equation*}
\int_{\Omega} \Delta u_{m, \varepsilon} \Delta \zeta_{k}+\left(a+b \int_{\Omega}\left|\nabla u_{m, \varepsilon}\right|^{2}\right)^{\gamma} \int_{\Omega} \nabla u_{m, \varepsilon} \nabla \zeta_{k}-\int_{\Omega} f_{\varepsilon}\left(\left|u_{m, \varepsilon}\right|\right) \zeta_{k}=0, \quad \forall \zeta_{k} \in \mathbb{W}_{k} . \tag{3.4}
\end{equation*}
$$

Since $\left\|u_{m, \varepsilon}\right\|_{H^{2} \cap H_{0}^{1}} \leq R$ and $H^{2} \cap H_{0}^{1}$ is reflexive, there exists $u_{\varepsilon} \in H^{2} \cap H_{0}^{1}$ such that
$\left(i_{1}\right) u_{m, \varepsilon} \rightharpoonup u_{\varepsilon} \quad$ weakly in $H^{2} \cap H_{0}^{1}$ as $m \rightarrow \infty$
( $i_{2}$ ) $u_{m, \varepsilon} \rightarrow u_{\varepsilon}$ in $H_{0}^{1}$ as $m \rightarrow \infty$
(i3) $u_{m, \varepsilon} \rightarrow u_{\varepsilon}$ in $L^{\sigma}$ for $1 \leq \sigma<2 N /(N-4)$ (or $1 \leq \sigma<\infty$ if $N=1,2,3,4$ ) as $m \rightarrow \infty$
Letting $m \rightarrow \infty$, in the expression (3.4) we get

$$
\int_{\Omega} \Delta u_{\varepsilon} \Delta \zeta_{k}+\left(a+b \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right)^{\gamma} \int_{\Omega} \nabla u_{\varepsilon} \nabla \zeta_{k}-\int_{\Omega} f_{\varepsilon}\left(\left|u_{\varepsilon}\right|\right) \zeta_{k}=0, \quad \forall \zeta_{k} \in \mathbb{W}_{k} .
$$

Since the space of all subsapces $\left[\mathbb{W}_{m}\right]_{k \in \mathbb{N}}$ is dense in $H^{2} \cap H_{0}^{1}$, then

$$
\begin{equation*}
\int_{\Omega} \Delta u_{\varepsilon} \Delta \zeta+\left(a+b \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right)^{\gamma} \int_{\Omega} \nabla u_{\varepsilon} \nabla \zeta-\int_{\Omega} f_{\varepsilon}\left(\left|u_{\varepsilon}\right|\right) \zeta=0, \quad \forall \zeta \in H^{2} \cap H_{0}^{1} \tag{3.5}
\end{equation*}
$$

Hence $u_{\varepsilon}$ is a nontrivial weak solution of

$$
\begin{cases}\Delta^{2} u_{\varepsilon}-\left(a+b \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right)^{\gamma} \Delta u_{\varepsilon}=f_{\varepsilon}\left(\left|u_{\varepsilon}\right|\right) & \text { in } \Omega \\ \Delta u_{\varepsilon}=u_{\varepsilon}=0 & \text { on } \partial \Omega\end{cases}
$$

Notice that $-\Delta u_{\varepsilon}$ satisfy the equation with $f_{\varepsilon}\left(\left|u_{\varepsilon}\right|\right)>0$, hence the maximum principle applies. Consequently, $-\Delta u_{\varepsilon}>0$ and moreover $u_{\varepsilon}>0$ in $\Omega$. Thus $u_{\varepsilon}$ satisfies

$$
\begin{cases}\Delta^{2} u_{\varepsilon}-\left(a+b \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right)^{\gamma} \Delta u_{\varepsilon}=f_{\varepsilon}\left(u_{\varepsilon}\right) & \text { in } \Omega  \tag{3.6}\\ -\Delta u_{\varepsilon}>0, u_{\varepsilon}>0 & \text { in } \Omega \\ \Delta u_{\varepsilon}=u_{\varepsilon}=0 & \text { on } \partial \Omega .\end{cases}
$$

As we shall see $u_{\varepsilon} \geq \delta_{0} w_{1}$ in $\Omega$ for some $\delta_{0}>0$, see (2.1) and (2.2). For that matter denote $-\Delta u_{\varepsilon}=v$ and rewrite the equation (3.6) in the form

$$
\begin{equation*}
-\Delta v+\left(a+b \int_{\Omega}\left|\nabla u_{\mathcal{\varepsilon}}\right|^{2}\right)^{\gamma} v=f_{\varepsilon}\left(u_{\varepsilon}\right) \geq \vartheta \tag{3.7}
\end{equation*}
$$

where $\vartheta>0$ is a constant which does not depend on $\varepsilon$ such that

$$
f_{\varepsilon}(t)=\alpha \frac{1}{(t+\varepsilon)^{\theta}}+\lambda t^{q}+\mu t \geq \alpha \frac{1}{(t+1)^{\theta}}+\lambda t^{q} \geq \vartheta \quad \text { for } t \geq 0
$$

Let $V=\delta w_{1}$ with $\delta>0$ and notice that $\left\|u_{\varepsilon}\right\|_{H^{2} \cap H_{0}^{1}} \leq \liminf _{m \rightarrow \infty}\left\|u_{m, \varepsilon}\right\|_{H^{2} \cap H_{0}^{1}} \leq R$, then

$$
\begin{aligned}
-\Delta V+\left(a+b \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right)^{\gamma} V & =\delta w_{1}\left[\lambda_{1}+\left(a+b \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right)^{\gamma}\right] \\
& \leq \delta w_{1}\left[\lambda_{1}+\left(a+b \frac{1}{\lambda_{1}}\left\|u_{\varepsilon}\right\|_{H^{2} \cap H_{0}^{1}}^{2}\right)^{\gamma}\right] \\
& \leq \delta w_{1}\left[\lambda_{1}+\left(a+b \frac{R^{2}}{\lambda_{1}}\right)^{\gamma}\right] \leq \vartheta
\end{aligned}
$$

where the last inequality is valid by taking $\delta$ small enough, and it is independent on $\varepsilon$. Owing to (3.7) and remembering that $v=V=0$ on $\partial \Omega$, we obtain $-\Delta u_{\varepsilon}=v \geq \delta w_{1}$ in $\Omega$. By the maximum principle there is $\delta_{0}>0$ such that $u_{\varepsilon} \geq \delta_{0} w_{1}$ in $\Omega$.

Since $\left\|u_{\varepsilon}\right\|_{H^{2} \cap H_{0}^{1}} \leq R$. By Sobolev embedding and continuing to denote a subsequence $\varepsilon=\varepsilon_{n} \rightarrow 0$, then
$\left(j_{1}\right) u_{\varepsilon} \rightharpoonup u_{0} \quad$ weakly in $H^{2} \cap H_{0}^{1}$ as $\varepsilon \rightarrow 0$,
$\left(j_{2}\right) u_{\varepsilon} \rightarrow u_{0} \quad$ in $H_{0}^{1}$ as $\varepsilon \rightarrow 0$,
$\left(j_{3}\right) u_{\varepsilon} \rightarrow u_{0} \quad$ in $L^{\sigma}$ for $1 \leq \sigma<2 N /(N-4)$ (or $1 \leq \sigma<\infty$ if $\left.N=1,2,3,4\right)$ as $\varepsilon \rightarrow 0$,
$\left(j_{4}\right) u_{\varepsilon} \rightarrow u_{0} \quad$ a.e. in $\Omega$ as $\varepsilon \rightarrow 0$,
$\left(_{5}\right)\left|u_{\varepsilon}\right| \leq h(x) \quad$ a.e. in $\Omega$, for some $h$ in $L^{\sigma}, 1 \leq \sigma<2 N /(N-4)$ (or $1 \leq \sigma<\infty$ if $N=1,2,3,4)$.

We conclude that $u_{0} \geq \delta_{0} w_{1}$ in $\Omega$. We rewrite (3.5) below

$$
\begin{align*}
& \int_{\Omega} \Delta u_{\varepsilon} \Delta \zeta+\left(a+b \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right)^{\gamma} \int_{\Omega} \nabla u_{\varepsilon} \nabla \zeta \\
&-\int_{\Omega}\left(\alpha \frac{1}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta}}+\lambda u_{\varepsilon}^{q}+\mu u_{\varepsilon}\right) \zeta=0, \quad \forall \zeta \in H^{2} \cap H_{0}^{1} \tag{3.8}
\end{align*}
$$

Using $\left(j_{1}\right)-\left(j_{5}\right)$ and letting $\varepsilon \rightarrow 0$ in (3.8) we arrive at

$$
\begin{align*}
\int_{\Omega} \Delta u_{0} \Delta \zeta+\left(a+b \int_{\Omega}\left|\nabla u_{0}\right|^{2}\right)^{\gamma} \int_{\Omega} & \nabla u_{0} \nabla \zeta \\
& -\int_{\Omega}\left(\alpha \frac{1}{u_{0}^{\theta}} \zeta+\lambda u_{0}^{q}+\mu u_{0}\right) \zeta=0, \quad \forall \zeta \in H^{2} \cap H_{0}^{1} . \tag{3.9}
\end{align*}
$$

The first two integrals of (3.9) are consequences of $\left(j_{1}\right)$ and $\left(j_{2}\right)$. The integral involving $u_{0}^{q}$ follows from $\left(j_{4}\right),\left(j_{5}\right)$ and dominated convergence theorem. The integral with $\mu$ follows by $\left(j_{3}\right)$. It is useful to detail that

$$
\begin{equation*}
\int_{\Omega} \frac{1}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta}} \zeta \rightarrow \int_{\Omega} \frac{1}{u_{0}^{\theta}} \zeta, \quad \forall \zeta \in H^{2} \cap H_{0}^{1} . \tag{3.10}
\end{equation*}
$$

First notice that $\int_{\Omega} \frac{1}{u_{0}^{\epsilon}} \leq \frac{1}{\delta_{0}^{\theta}} \int_{\Omega} \frac{1}{w_{1}^{\epsilon}}<\infty$. By dominated convergence theorem we can write (3.10) with $\zeta \in C_{0}^{\infty}(\Omega)$, and by density we can take $\zeta \in H_{0}^{1}$, and finally (3.10) holds for every $\zeta \in H^{2} \cap H_{0}^{1}$.

We now prove Theorem 1.2.
Proof. We borrow $\mathbb{B}, \mathbb{W}_{m}$ and $F$ defined in the proof of Theorem 1.1. Define

$$
F_{j}(\eta)=\left(a+b \int_{\Omega}|\nabla u|^{2}\right)^{\gamma} \int_{\Omega} \Delta u \Delta w_{j}+\varrho \int_{\Omega} \nabla u \nabla w_{j}-\int_{\Omega} f_{\varepsilon}(|u|) w_{j}, \quad j=1,2, \ldots, m .
$$

Then

$$
\begin{align*}
(F(\eta), \eta) & =\left(a+b \int_{\Omega}|\nabla u|^{2}\right)^{\gamma} \int_{\Omega}|\Delta u|^{2}+\varrho \int_{\Omega}|\nabla u|^{2}-\int_{\Omega} f_{\varepsilon}(|u|) u \\
& \geq a^{\gamma}\|u\|_{H^{2} \cap H_{0}^{1}}^{2}-\alpha|\Omega|^{\theta} C_{1}^{1-\theta}\|u\|_{H^{2} \cap H_{0}^{1}}^{1-\theta}-\lambda C_{q+1}^{q+1}\|u\|_{H^{2} \cap H_{0}^{1}}^{q+1}-\mu C_{2}^{2}\|u\|_{H^{2} \cap H_{0}^{1}}^{2} . \tag{3.11}
\end{align*}
$$

For $\mu<a^{\gamma} C_{2}^{-2}$, there is $R>0$ verifying (3.2) and $u_{m, \varepsilon} \in H^{2} \cap H_{0}^{1}$ with $\left\|u_{m, \varepsilon}\right\|_{H^{2} \cap H_{0}^{1}} \leq R$ and satisfying

$$
\left(a+b \int_{\Omega}\left|\nabla u_{m, \varepsilon}\right|^{2}\right)^{\gamma} \int_{\Omega} \Delta u_{m, \varepsilon} \Delta w_{j}+\varrho \int_{\Omega} \nabla u_{m, \varepsilon} \nabla w_{j}-\int_{\Omega} f_{\varepsilon}\left(\left|u_{m, \varepsilon}\right|\right) w_{j}=0, \quad j=1,2, \ldots, m .
$$

After the same steps of the previous proof and using $\left(i_{1}\right)-\left(i_{3}\right)$ we reach

$$
\begin{equation*}
\left(a+b \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right)^{\gamma} \int_{\Omega} \Delta u_{\varepsilon} \Delta \zeta+\varrho \int_{\Omega} \nabla u_{\varepsilon} \nabla \zeta-\int_{\Omega} f_{\varepsilon}\left(\left|u_{\varepsilon}\right|\right) \zeta=0, \quad \forall \zeta \in H^{2} \cap H_{0}^{1} . \tag{3.12}
\end{equation*}
$$

We thus get a nontrivial weak solution $u_{\varepsilon}$ of

$$
\begin{cases}\left(a+b \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right)^{\gamma} \Delta^{2} u_{\varepsilon}-\varrho \Delta u_{\varepsilon}=f_{\varepsilon}\left(\left|u_{\varepsilon}\right|\right) & \text { in } \Omega \\ \Delta u_{\varepsilon}=u_{\varepsilon}=0 & \text { on } \partial \Omega\end{cases}
$$

We are in position to apply the maximum principle to the function $-\Delta u_{\varepsilon}$. Then $-\Delta u_{\varepsilon}>0$, thus $u_{\varepsilon}>0$ in $\Omega$ and

$$
\begin{cases}\left(a+b \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right)^{\gamma} \Delta^{2} u_{\varepsilon}-\varrho \Delta u_{\varepsilon}=f_{\varepsilon}\left(u_{\varepsilon}\right) & \text { in } \Omega  \tag{3.13}\\ -\Delta u_{\varepsilon}>0, u_{\varepsilon}>0 & \text { in } \Omega \\ \Delta u_{\varepsilon}=u_{\varepsilon}=0 & \text { on } \partial \Omega\end{cases}
$$

For $V=\delta w_{1}$ with $\delta>0$ and using $\left\|u_{\varepsilon}\right\|_{H^{2} \cap H_{0}^{1}} \leq R$, then for $\delta$ small enough

$$
-\left(a+b \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right)^{\gamma} \Delta V+\varrho V \leq \delta w_{1}\left[\left(a+b \frac{R^{2}}{\lambda_{1}}\right)^{\gamma} \lambda_{1}+\varrho\right] \leq \vartheta \leq f_{\varepsilon}\left(u_{\varepsilon}\right) .
$$

Comparing with (3.13) we obtain $-\Delta u_{\varepsilon} \geq \delta w_{1}$ and $u_{\varepsilon} \geq \delta_{0} w_{1}$ in $\Omega$ for $\delta_{0}>0$ small enough. The remaining steps are analogue to the proof of Theorem 1.1.

Next we describe the main steps of the proof of Theorem 1.3.
Proof. Define $f_{\varepsilon}(t)=\alpha \frac{1}{(t+\varepsilon)^{\theta}}+\lambda t^{q}+\mu t+g(t)$ with $0<\varepsilon<1$. As in the beginning of the proof of Theorem 1.1 we consider $\mathbb{B}, \mathbb{W}_{m}$ and $F$. Estimate (3.1) in this context turns out to be

$$
\begin{align*}
(F(\eta), \eta)= & \int_{\Omega}|\Delta u|^{2}+\left(a+b \int_{\Omega}|\nabla u|^{2}\right)^{\gamma} \int_{\Omega}|\nabla u|^{2}-\int_{\Omega} f_{\varepsilon}(|u|) u \\
\geq & \|u\|_{H^{2} \cap H_{0}^{1}}^{2}-\alpha|\Omega|^{\theta} C_{1}^{1-\theta}\|u\|_{H^{2} \cap H_{0}^{1}}^{1-\theta}-\lambda C_{q+1}^{q+1}\|u\|_{H^{2} \cap H_{0}^{1}}^{q+1} \\
& -c_{1} C_{p+1}^{p+1}\|u\|_{H^{2} \cap H_{0}^{1}}^{p+1}-\mu C_{2}^{2}\|u\|_{H^{2} \cap H_{0}^{1}}^{2} . \tag{3.14}
\end{align*}
$$

Hence, there is a constant $K>0$ such that

$$
\begin{equation*}
(F(\eta), \eta) \geq\|u\|_{H^{2} \cap H_{0}^{1}}^{2}-K\left(\alpha\|u\|_{H^{2} \cap H_{0}^{1}}^{1-\theta}+\lambda\|u\|_{H^{2} \cap H_{0}^{1}}^{q+1}+\|u\|_{H^{2} \cap H_{0}^{1}}^{p+1}+\mu\|u\|_{H^{2} \cap H_{0}^{1}}^{2}\right) . \tag{3.15}
\end{equation*}
$$

Next we will make the choice of $R, \alpha^{*}, \mu^{*}$ and $\lambda^{*}$. We need $\|u\|_{H^{2} \cap H_{0}^{1}}=R<(2 / 3 K)^{1 /(p-1)}$. Thus, let

$$
R=\min \left\{1,\left[(2 / 3 K)^{1 /(p-1)}\right] / 2\right\} .
$$

We require $\alpha<(1 / 2)^{1+\theta}(2 / 3 K)^{1+\theta /(p-1)}(2 / 3 K)$, then we select $\alpha^{*}$ with

$$
\alpha^{*}=\left[(1 / 2)^{1+\theta}(2 / 3 K)^{1+\theta /(p-1)}(2 / 3 K)\right] / 2 .
$$

We need $\mu<2 / 3 K$, thus we take $\mu^{*}=1 / 3 K$.
Once $R$ has been chosen, we want $\lambda^{*}$ such that $R^{2}-K \lambda R^{q+1}>0$, i.e., $\lambda<R^{1-q} / K$ for $\lambda<\lambda^{*}$. Hence we take

$$
\lambda^{*}=(1 / K) \min \left\{1,(1 / 2)^{2-q}(2 / 3 K)^{(1-q) /(p-1)}\right\} .
$$

With these these choices of $\alpha^{*}, \lambda^{*}, \mu^{*}$ announced in the statement of the theorem, we have the intervals where $\alpha, \lambda, \mu$ belong to, namely $0<\alpha<\alpha^{*}, 0<\lambda<\lambda^{*}$ and $0 \leq \mu<\mu^{*}$.

Thus, let $Y=R^{2}-K \lambda^{*} R^{q+1}>0$. Therefore,

$$
\begin{equation*}
(F(\eta), \eta)>\mathrm{Y} \quad \text { for }\|u\|_{H^{2} \cap H_{0}^{1}}=|\eta|=R \tag{3.16}
\end{equation*}
$$

Brouwer's Theorem asserts that there exists $u_{m, \varepsilon} \in H^{2} \cap H_{0}^{1}$ with $\left\|u_{m, \varepsilon}\right\|_{H^{2} \cap H_{0}^{1}} \leq R$ satisfying (3.3). Notice that there is a constant $\vartheta>0$, which does not depend on $\varepsilon$ such that

$$
f_{\varepsilon}(t)=\alpha \frac{1}{(t+\varepsilon)^{\theta}}+\lambda t^{q}+\mu t+g(t) \geq \alpha \frac{1}{(t+1)^{\theta}}+\lambda t^{q} \geq \vartheta \quad \text { for } t \geq 0
$$

The remaining parts of the proof run in the same manner as before, see all steps from (3.3) to (3.10).

The proof of Theorem 1.4 is similar.
Proof. The above proofs are well documented. It is a repetition of the arguments.
Next we prove Theorem 1.5.
Proof. The solution $u=u_{\lambda}$ satisfies

$$
\begin{aligned}
\|u\|_{H^{2} \cap H_{0}^{1}}^{2} & \leq \int_{\Omega}|\Delta u|^{2}+\left(a+b \int_{\Omega}|\nabla u|^{2}\right)^{\gamma} \int_{\Omega}|\nabla u|^{2}=\int_{\Omega} f(u) u \\
& =\int_{\Omega} \lambda u^{q+1}+u^{p+1} \leq \lambda C_{q+1}^{q+1}\|u\|_{H^{2} \cap H_{0}^{1}}^{q+1}+C_{p+1}^{p+1}\|u\|_{H^{2} \cap H_{0}^{1}}^{p+1} .
\end{aligned}
$$

Then

$$
\|u\|_{H^{2} \cap H_{0}^{1}}^{1-q} \leq \frac{\lambda C_{q+1}^{q+1}}{1-C_{p+1}^{p+1}\|u\|_{H^{2} \cap H_{0}^{1}}^{p-1}}
$$

By the choice of $R$ we get

$$
1-C_{p+1}^{p+1}\|u\|_{H^{2} \cap H_{0}^{1}}^{p-1} \geq 1 / 2
$$

Hence

$$
\|u\|_{H^{2} \cap H_{0}^{1}} \leq\left(2 \lambda C_{q+1}^{q+1}\right)^{1 /(1-q)} \rightarrow 0 \quad \text { as } \lambda \rightarrow 0
$$

The proof for (1.2) is similar.
We conclude the paper proving Theorem 1.6.
Proof. We denote the existing solution of Theorem 1.3 by $u=u_{\lambda}$ and assume that $\|u\|_{H^{2} \cap H_{0}^{1}} \leq$ $R$. Since $\|u\|_{H_{0}^{1}}^{2} \leq\|u\|_{L^{2}}\|u\|_{H^{2} \cap H_{0}^{1}}$, the term $a+b\|u\|_{H_{0}^{1}}^{2}$ is bounded. Multiply the equation (1.1) by $w_{1}$, integrate and use (2.1) and (2.2), hence

$$
\begin{equation*}
\int_{\Omega} f(u) w_{1}=\lambda_{1} \int_{\Omega} u w_{1}\left(1+\left(a+b\|u\|_{H_{0}^{1}}^{2}\right)^{\gamma}\right) \leq \lambda_{1} M \int_{\Omega} u w_{1} \tag{3.17}
\end{equation*}
$$

for a constant $M>0$ independent on $\lambda$. Notice that $f(t)=\lambda\left(\frac{1}{t^{\theta}}+t^{q}+t\right)+t^{p} \geq \lambda t^{q}+t^{p}$ for $t \geq 0$. Then $f(t) \geq \lambda^{(p-1)(p-q)} C_{p, q} t$ for $t \geq 0$, where $C_{p, q}>0$ is a constant depending only on $p$ and $q$. Hence (3.17) gives

$$
\lambda^{(p-1)(p-q)} C_{p, q} \int_{\Omega} u w_{1} \leq \lambda_{1} M \int_{\Omega} u w_{1}
$$

which makes $\lambda$ bounded, a contradiction. Again the reasoning for (1.2) is similar.

## Acknowledgements

The author has been supported by CNPq and FAPESP. He thanks the referee for the valuable suggestions and for pointing out references [3] and [33].

## References

[1] R. A. Adams, J. F. Fournier, Sobolev spaces, 2nd ed., Pure and Applied Mathematics, Vol. 140, Academic Press, New York, NY, 2003. MR2424078; Zbl 1098.46001
[2] Y. An, M. Avci, F. Wang, Existence of solutions for fourth order elliptic equations of Kirchhoff type, J. Math. Anal. Appl. 409(2014) 140-146. https://doi.org/10.1016/j. jmaa.2013.07.003; MR3095024; Zbl 1311.35093
[3] H. Ansari, M. Hesaaraki, S. M. Vaezpour, Existence of positive solution for nonlocal singular fourth order Kirchhoff equation with Hardy potential, Positivity 21(2017), 15451562. https://doi.org/10.1007/s11117-017-0484-y; MR3718554; Zbl 1375.35148
[4] A. Arosio, S. Panizzi, On the well-posedness of the Kirchhoff string, Trans. Amer. Math. Soc. 348(1996) 305-330. https : //doi .org/10.1090/S0002-9947-96-01532-2; MR1333386; Zbl 0858.35083
[5] M. Berger, A new approach to the large deflection of plate, J. Appl. Mech. 22(1955), 465472. MR0073407
[6] A. H. Bokhari, F. M. Mahomed, F. D. Zaman, Invariant boundary value problems for a fourth-order dynamic Euler-Bernoulli beam equation, J. Math. Phys. 53(2012), Paper No. 043703, 6 pp. https://doi.org/10.1063/1.4711131; MR2953160; Zbl 1276.74026
[7] A. Cabada, G. M. Figueiredo, A generalization of an extensible beam equation with critical growth in $\mathbb{R}^{N}$, Nonlinear Anal. Real World Appl. 20(2014), 134-142. https://doi. org/10.1016/j.nonrwa.2014.05.005; MR3233906; Zbl 1297.35094
[8] C. Chen, H. Song, Infinitely many solutions for Schrödinger-Kirchhoff-type fourth-order elliptic equations, Proc. Edinb. Math. Soc. (2) 60(2017), 1003-1020. https://doi.org/10. 1017/S001309151600047X; MR3715698; Zbl 1377.35083
[9] I. Chueshov, I. Lasiecka, Long-time behavior of second order evolution equations with nonlinear damping, Mem. Amer. Math. Soc. 195(2008), No. 912. https://doi.org/10. 1090/memo/0912; MR2438025; Zbl 1151.37059
[10] S. Dedoncker, W. Desmet, F. Maurin, F. Greco, Isogeometric analysis for nonlinear planar Kirchhoff rods: Weighted residual formulation and collocation of the strong form, Comput. Methods Appl. Mech. Engrg. 340(2018), 1023-1043. https://doi.org/10.1016/j. cma.2018.05.025; MR3845242; Zbl 1440.74186
[11] H. Fan, Positive solutions for a Kirchhoff-type problem involving multiple competitive potentials and critical Sobolev exponent, Nonlinear Anal. 198(2020), Paper No. 111869, 35 pp. https://doi.org/10.1016/j.na.2020.111869; MR4083146; Zbl 1442.35134
[12] G. M. Figueiredo, M. F. Furtado, J. P. da Silva, Two solutions for a fourth order nonlocal problem with indefinite potentials, Manuscripta Math. 160(2019), 199-215. https://doi. org/10.1007/s00229-018-1057-5; MR3983393; Zbl 1429.35083
[13] F. Gazzola, H.-C. Grunau, G. Sweers, Polyharmonic boundary value problems. Positivity preserving and nonlinear higher order elliptic equations in bounded domains, Lecture Notes in Mathematics, Vol. 1991, Springer-Verlag, Berlin, 2010. https://doi.org/10.1007/ 978-3-642-12245-3; MR2667016; Zbl 1239.35002
[14] J. R. Kang, Energy decay of solutions for an extensible beam equation with a weak nonlinear dissipation, Math. Methods Appl. Sci. 35(2012), 1587-1593. https://doi.org/ 10.1002/mma.2546; MR2957519; Zbl 1250.35032
[15] G. R. Кirchноғf, Über das Gleichgewicht und die Bewegung einer elastischen Scheibe (in German), J. Reine Angew. Math. 40(1850), 51-88. https://doi.org/10.1515/crll.1850. 40.51; MR1578677
[16] G. R. Kirchнoff, Vorlesungen über mathematische Physik (in German), Mechanik, Teubner, Leipzig, 1876.
[17] F. Li, D. L. Wu, Solutions for fourth-order Kirchhoff type elliptic equations involving concave-convex nonlinearities in $\mathbb{R}^{N}$, Comput. Math. Appl. 79(2020), 489-499. https:// doi.org/10.1016/j.camwa.2019.07.007; MR4046717; Zbl 1448.35252
[18] J. F. Liao, Y. Pu, X. F. Ke, C. L. Tang, Multiple positive solutions for Kirchhoff type problems involving concave-convex nonlinearities, Commun. Pure Appl. Anal. 16(2017), 2157-2175. https://doi.org/10.3934/cpaa. 2017107; MR3693877; Zbl 1376.35030
[19] J. L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires (in French), Dunod, Paris, 1969. MR0259693; Zbl 0189.40603
[20] A. E. H. Love, A treatise on the mathematical theory of elasticity, Fourth edition, Cambridge University Press, 1927. MR0010851; Zbl 1258.74003
[21] T. F. Ma, V. Narciso, Global attractor for a model of extensible beam with nonlinear damping and source terms, Nonlinear Anal. 73(2010), 3402-3412. https://doi.org/10. 1016/j.na. 2010.07.023; MR2680033; Zbl 1207.35071
[22] A. Mao, W. Wang, Nontrivial solutions of nonlocal fourth order elliptic equation of Kirchhoff type in $\mathbb{R}^{3}$, J. Math. Anal. Appl. 459(2018), 556-563. https://doi.org/10.1016/ j.jmaa.2017.10.020; MR3730455; Zbl 1382.35102
[23] P. J. McKenna, W. Walter, Nonlinear oscillations in a suspension bridge, Arch. Rational Mech. Anal. 98(1987), 167-177. https://doi.org/10.1007/BF00251232; MR0866720; Zbl 0676.35003
[24] P. J. McKenna, W. Walter, Travelling waves in a suspension bridge, SIAM J. Appl. Math. 50(1990), 703-715. https://doi.org/10.1137/0150041; MR1050908; Zbl 0699.73038
[25] D. Naimen, The critical problem of Kirchhoff type elliptic equations in dimension four, J. Differential Equations 257(2014), 1168-1193. https://doi.org/10.1016/j. jde. 2014.05. 002; MR3609014; Zbl 1301.35022
[26] J. Peradze, A Kirchhoff type equation in a nonlinear model of shell vibration, ZAMM Z. Angew. Math. Mech. 97(2017), 144-158. https://doi.org/10.1002/zamm.201600142; MR3609014
[27] Q. Ren, H. Tian, Legendre-Galerkin spectral approximation for a nonlocal elliptic Kirchhof-type problem, Int. J. Comput. Math. 94(2017), 1747-1758. https://doi.org/10. 1080/00207160.2016.1227799; MR3666463; Zbl 1394.65133
[28] J. Shen, Efficient spectral-Galerkin method I. Direct solvers for the second- and fourthorder equations using Legendre polynomials, SIAM J. Sci. Comput. 15(1994), 1489-1505. https://doi.org/10.1137/0915089; MR1298626; Zbl 0811.65097
[29] Y. Song, S. Shi, Multiplicity of solutions for fourth-order elliptic equations of Kirchhoff type with critical exponent, J. Dyn. Control Syst. 23(2017), 375-386. https://doi.org/10. 1007/s10883-016-9331-x; MR3625002; Zbl 1371.35061
[30] R. C. A. M. Van der Vorst, Best constant for the embedding of the space $H^{2} \cap H_{0}^{1}(\Omega)$ into $L^{2 N /(N-4)}(\Omega)$, Differential Integral Equations, 6(1993), 259-276. MR1195382; Zbl 0801.46033
[31] S. Woinowsky-Krieger, The effect of axial force on the vibration of hinged bars, J. Appl. Mech. 17(1950), 35-36. MR0034202; Zbl 0036.13302
[32] Z. Yang, On an extensible beam equation with nonlinear damping and source terms, J. Differential Equations 254(2013), 3903-3927. https://doi .org/10.1016/j . jde.2013.02. 008; MR3029139; Zbl 1329.35069
[33] J. Zuo, T. An, Y. Ru, D. Zhao, Existence and multiplicity of solutions for nonhomogeneous Schrödinger-Kirchhoff-type fourth-order elliptic equations in $\mathbb{R}^{N}$, Mediterr. J. Math. 16(2019), Paper No. 123, 14 pp. https://doi.org/10.1007/s00009-019-1402-2; MR3994869; Zbl 1427.35067

# Existence and multiplicity of nontrivial solutions to the modified Kirchhoff equation without the growth and Ambrosetti-Rabinowitz conditions 

Zhongxiang Wang ${ }^{\otimes 1}$ and Gao Jia ${ }^{2}$<br>${ }^{1}$ Business School, University of Shanghai for Science and Technology, Shanghai, 200093, China<br>${ }^{2}$ College of Science, University of Shanghai for Science and Technology, Shanghai, 200093, China

Received 7 June 2021, appeared 22 October 2021
Communicated by Roberto Livrea

Abstract. The paper focuses on the modified Kirchhoff equation

$$
-\left(a+b \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x\right) \Delta u-u \Delta\left(u^{2}\right)+V(x) u=\lambda f(u), \quad x \in \mathbb{R}^{N}
$$

where $a, b>0, V(x) \in C\left(\mathbb{R}^{N}, \mathbb{R}\right)$ and $\lambda<1$ is a positive parameter. We just assume that the nonlinearity $f(t)$ is continuous and superlinear in a neighborhood of $t=0$ and at infinity. By applying the perturbation method and using the cutoff function, we get existence and multiplicity of nontrivial solutions to the revised equation. Then we use the Moser iteration to obtain existence and multiplicity of nontrivial solutions to the above original Kirchhoff equation. Moreover, the nonlinearity $f(t)$ may be supercritical.
Keywords: modified Kirchhoff-type equation, cutoff function, perturbation approach.
2020 Mathematics Subject Classification: 35J20, 35J60.

## 1 Introduction

In this paper, we are devoted to studying the following modified Kirchhoff equation:

$$
\begin{equation*}
-\left(a+b \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x\right) \Delta u-u \Delta\left(u^{2}\right)+V(x) u=\lambda f(u), \quad x \in \mathbb{R}^{N} \tag{1.1}
\end{equation*}
$$

where $a, b>0, V(x) \in C\left(\mathbb{R}^{N}, \mathbb{R}\right), \lambda<1$ is a positive parameter and $f$ is continuous in $\mathbb{R}$. The equation (1.1) is the Euler-Lagrange equation of the energy functional
$I_{\lambda}(u)=\frac{a}{2} \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x+\frac{b}{4}\left(\int_{\mathbb{R}^{N}}|\nabla u|^{2} d x\right)^{2}+\frac{1}{2} \int_{\mathbb{R}^{N}}\left(V(x) u^{2}+2 u^{2}|\nabla u|^{2}\right) d x-\lambda \int_{\mathbb{R}^{N}} F(u) d x$,

[^45]where $F(t)=\int_{0}^{t} f(s) \mathrm{d} s$.
Kirchhoff's model is a general version of the equation
\[

$$
\begin{equation*}
\rho \frac{\partial^{2} u}{\partial t^{2}}-\left(\frac{P_{0}}{h}+\frac{E}{2 L} \int_{0}^{L}\left|\frac{\partial u}{\partial x}\right|^{2} d x\right) \frac{\partial^{2} u}{\partial x^{2}}=0, \tag{1.2}
\end{equation*}
$$

\]

which was first proposed by Kirchhoff in [6] for extending the classical D'Alembert's wave equations for free vibration of elastic strings. Kirchhoff's model takes into account the changes in string length produced by transverse vibration. In (1.2), $L$ is the length of the string, $h$ is the area of cross section, $E$ denotes the Young modulus of the material, $\rho$ is the mass density and $P_{0}$ denotes the initial tension. In addition, we have to point out that nonlocal problems also appear in other fields as biological systems, where $u$ describes a process which depends on the average of itself (for example, population density). Some early classical studies of Kirchhoff equations can be found in Bernstein [1] and Pohožaev [14]. Much attention was received after Lions [9] introducing an abstract functional framework to this problem. For more relevant mathematical and physical background, we refer readers to papers [8,13,21], and the references therein.

Especially, in recent paper [19], Wu studied the following problem:

$$
\begin{equation*}
-\left(a+b \int_{R^{N}}|\nabla u|^{2} d x\right) \Delta u+V(x) u=g(x, u), \quad x \in \mathbb{R}^{N} \tag{1.3}
\end{equation*}
$$

and obtained four new existence results of nontrivial solutions and a sequence of high energy solutions for equation (1.3).

When $a=1$ and $b=0,(1.3)$ is reduced to the well known quasilinear Schrödinger equation

$$
\begin{equation*}
-\Delta u+V(x) u-\Delta\left(u^{2}\right) u=g(x, u), \quad x \in \mathbb{R}^{N} . \tag{1.4}
\end{equation*}
$$

Several methods can be used to solve the equation (1.4), such as, the existence of a positive ground state solution has been studied in $[10,15]$ by using a constrained minimization argument; the problem is transformed to a semilinear one in $[2,11]$ by a change of variables (dual approach); Nehari method is used to get the existence results of ground state solutions in [12,17]. Especially, in [7], the existence of positive solutions, negative solutions and sequence of high energy solutions for the following problem

$$
-\Delta u+V(x) u-\Delta\left(|u|^{2 \alpha}\right)|u|^{2 \alpha-2} u=g(x, \psi), \quad x \in R^{N}
$$

was studied via a perturbation method, where $\alpha>\frac{3}{4}, V \in C\left(R^{N}, R\right)$ and $g \in C\left(R^{N} \times R, R\right)$.
Recently, Feng et al. [3] studied the following modified Kirchhoff type equation

$$
\begin{equation*}
-\left(a+b \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x\right) \Delta u-u \Delta\left(u^{2}\right)+V(x) u=h(x, u), \quad x \in \mathbb{R}^{N} \tag{1.5}
\end{equation*}
$$

where $a>0, b \geq 0, h \in C\left(\mathbb{R}^{N} \times \mathbb{R}, \mathbb{R}\right)$ and $V \in C\left(\mathbb{R}^{N}, \mathbb{R}\right)$. Under appropriate assumptions on $V(x)$ and $h(x, u)$, some existence results for positive solutions, negative solutions and sequence of high energy solutions were obtained via a perturbation method. Subsequently, in 2015, Wu [20] studied the existence of infinitely many small energy solutions for equation (1.5) by applying Clark's Theorem to a perturbation functional. And in the same year, He [4] proved the existence of infinitely many solutions for equation (1.5) by the dual method and the non-smooth critical point theory. Last year, Huang and Jia [5] obtained the existence of
infinitely many sign-changing solutions for equation (1.5) with $a=1$ and $h(x, u)=h(u)$ by genus theory.

In the present paper, we assume that $f \in C(\mathbb{R})$ and $V \in C\left(\mathbb{R}^{N}\right)$ satisfy the following conditions
$\left(f_{1}\right) \lim _{t \rightarrow 0} \frac{f(t)}{t}=0 ;$
$\left(f_{2}\right) \lim _{t \rightarrow+\infty} \frac{f(t)}{t}=+\infty ;$
$(V) V(x)$ satisfies $\inf _{x \in \mathbb{R}^{N}} V(x) \geq V_{0}>0$, and $\lim _{|x| \rightarrow \infty} V(x)=+\infty$.
Moreover, $f$ may be supercritical. But we do not assume the Ambrosetti-Rabinowitz condition or increasing condition.

Next, we give our main results.
Theorem 1.1. Assume that $(V),\left(f_{1}\right),\left(f_{2}\right)$ hold. Then equation (1.1) has a positive and a negative weak solutions for all $\lambda$ small enough.

Theorem 1.2. If $(V),\left(f_{1}\right),\left(f_{2}\right)$ hold and $f(t)$ is odd, then the equation (1.1) has a sequence $\left\{u_{n}\right\}$ of solutions such that $I_{\lambda}\left(u_{n}\right) \rightarrow+\infty$ for all $\lambda$ small enough.

This paper is organized as follows. In Section 2, we present the variational framework and some lemmas, which are bases of Section 3. In Section 3, we give the proof of Theorems 1.1 and 1.2.

In what follows, $C_{0}, C, c_{i}$ and $C_{i}(i=1,2, \ldots)$ denote positive generic constants.

## 2 Preliminaries and revised functional

In this section, we give work space, the revised functional and some lemmas.
Let $C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$ be the collection of smooth functions with compact supports. Let

$$
H^{1}\left(\mathbb{R}^{N}\right):=\left\{u \in L^{2}\left(\mathbb{R}^{N}\right): \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x<+\infty\right\}
$$

with the inner product

$$
\langle u, v\rangle_{H^{1}}=\int_{\mathbb{R}^{N}}(\nabla u \cdot \nabla v+u v) d x
$$

and the norm

$$
\|u\|_{H^{1}}=\langle u, u\rangle_{H^{1}}^{1 / 2} .
$$

Set

$$
H_{V}^{1}\left(\mathbb{R}^{N}\right):=\left\{u \in H^{1}\left(\mathbb{R}^{N}\right): \int_{\mathbb{R}^{N}} V(x) u^{2} d x<+\infty\right\}
$$

with the inner product

$$
\langle u, v\rangle_{H_{V}^{1}}=\int_{\mathbb{R}^{N}}[\nabla u \cdot \nabla v+V(x) u v] d x
$$

and the norm

$$
\|u\|_{H_{V}^{1}}=\langle u, u\rangle_{H_{V}^{1}}^{1 / 2} .
$$

Then both $H^{1}\left(\mathbb{R}^{N}\right)$ and $H_{V}^{1}\left(\mathbb{R}^{N}\right)$ are Hilbert spaces. Set $E=H_{V}^{1}\left(\mathbb{R}^{N}\right) \cap W^{1,4}\left(\mathbb{R}^{N}\right)$ with the norm $\|u\|_{E}=\|u\|_{H_{V}^{1}}+\|u\|_{W^{1,4}}$. Then $E$ is a reflexive Banach space.

Notice that there is no growth condition $|f(t)| \leqslant C|t|+C|t|^{q-1}$ and no AmbrosettiRabinowitz condition $t f(t)-4 F(t) \geqslant 0$. So we need the cutoff function.

By $\left(f_{2}\right)$, there exists $M>0$ large such that $f(M)>0$. And then given $M>0$, let

$$
h_{M}(t)= \begin{cases}f(t), & 0<t \leqslant M \\ C_{M} t^{p-1}, & t>M \\ 0, & t \leqslant 0\end{cases}
$$

where $C_{M}=f(M) / M^{p-1}$ and $4<p<22^{*}$. The continuity of $f$ implies the continuity of $h_{M}$. Moreover, by $\left(f_{1}\right)$ and $\left(f_{2}\right), h_{M}$ satisfies that
( $h_{1}$ ) There exists $4<p<22^{*}$ if $N \geq 3$ and $4<p<\infty$ if $N=1,2$ such that

$$
\left|h_{M}(t)\right| \leqslant C_{M}^{\prime}|t|+C_{M}|t|^{p-1} \leqslant C(M)\left(|t|+|t|^{p-1}\right), \quad \forall t \in \mathbb{R},
$$

where $C_{M}^{\prime}=\max _{t \in[0, M]}|f(t)| / t$ and $C(M)=\max \left\{C_{M}^{\prime}, C_{M}\right\}$;
( $h_{2}$ ) $\lim _{t \rightarrow 0} \frac{h_{M}(t)}{t}=0$;
$\left(h_{3}\right)$ There exists $\mu>4$ and $r>M$ such that

$$
\inf _{|t|=r} H_{M}(t)>0
$$

and

$$
\mu H_{M}(t) \leq h_{M}(t) t
$$

for $|t| \geq r$, where $H_{M}(t)=\int_{0}^{t} h_{M}(s) \mathrm{d} s$.
By [22, Lemma 3.4] and the condition $(V)$, we get that the embedding $H_{V}^{1}\left(\mathbb{R}^{N}\right) \hookrightarrow L^{s}\left(\mathbb{R}^{N}\right)$ is compact for each $2 \leq s<2^{*}$.

In what follows, we consider the revised problem

$$
\begin{equation*}
-\left(a+b \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x\right) \Delta u-u \Delta\left(u^{2}\right)+V(x) u=\lambda h_{M}(u), \quad x \in \mathbb{R}^{N} . \tag{2.1}
\end{equation*}
$$

Equation (2.1) is the Euler-Lagrange equation associated of the natural energy functional $J_{\lambda}(u): E \rightarrow \mathbb{R}$ given by

$$
\begin{aligned}
J_{\lambda}(u)= & \frac{a}{2} \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x+\frac{b}{4}\left(\int_{\mathbb{R}^{N}}|\nabla u|^{2} d x\right)^{2}+\frac{1}{2} \int_{\mathbb{R}^{N}}\left(V(x) u^{2}+2 u^{2}|\nabla u|^{2}\right) d x \\
& -\lambda \int_{\mathbb{R}^{N}} H_{M}(u) d x .
\end{aligned}
$$

For $\theta \in(0,1]$, let $J_{\theta, \lambda}(u)=\frac{1}{4} \theta \int_{\mathbb{R}^{N}}\left(|\nabla u|^{4}+u^{4}\right) d x+J_{\lambda}(u)$. Let $u^{+}=\max \{u, 0\}$ and $u^{-}=$ $\max \{-u, 0\}$. Set

$$
\begin{aligned}
J_{\lambda}^{ \pm}(u)= & \frac{a}{2} \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x+\frac{b}{4}\left(\int_{\mathbb{R}^{N}}|\nabla u|^{2} d x\right)^{2}+\frac{1}{2} \int_{\mathbb{R}^{N}}\left(V(x) u^{2}+2 u^{2}|\nabla u|^{2}\right) d x \\
& -\lambda \int_{\mathbb{R}^{N}} H_{M}\left(u^{ \pm}\right) d x
\end{aligned}
$$

and $J_{\theta, \lambda}^{ \pm}(u)=\frac{1}{4} \theta \int_{\mathbb{R}^{N}}\left(|\nabla u|^{4}+u^{4}\right) d x+J_{\lambda}^{ \pm}(u)$.
A sequence $\left\{u_{n}\right\} \subset E$ is called a P. S. sequence of $J_{\lambda}$ if $\left\{J_{\lambda}\left(u_{n}\right)\right\}$ is bounded and $J_{\lambda}^{\prime}\left(u_{n}\right) \rightarrow 0$ in $E^{*}$. We say that $J_{\lambda}$ satisfies the P. S. condition if every P. S. sequence possesses a convergent subsequence.

Our goal is to first prove that the critical point of $J_{\lambda}(u)$ can be obtained as limits of critical points of $J_{\theta, \lambda}(u)$. And then we need to prove that the nontrivial critical point $u$ of $J_{\lambda}(u)$ satisfying $\|u\|_{L^{\infty}} \leq M$ is a nontrivial solution of (1.1).

Lemma 2.1. Assume that $(V),\left(h_{1}\right)$ and $\left(h_{2}\right)$ hold. Then the functionals $J_{\lambda}$ and $J_{\theta, \lambda}^{ \pm}$are well defined in $E$ and $J_{\lambda}, J_{\theta, \lambda}^{ \pm} \in C^{1}(E, \mathbb{R})$.

Proof. The proof is similar to [3, Lemma 2.1], we omit it here.
Lemma 2.2. Assume that $(V),\left(h_{1}\right)$ and $\left(h_{2}\right)$ hold. Then every bounded P. S. sequence $\left\{u_{n}\right\} \subset E$ of $J_{\theta, \lambda}\left(\right.$ respectively, $\left.J_{\theta, \lambda}^{ \pm}\right)$possesses a convergent subsequence.

Proof. The proof is analogous to [3, Lemma 2.2], we omit it here.
Lemma 2.3. Assume that $(V)$ and $\left(h_{1}\right)-\left(h_{3}\right)$ hold. Let $\left\{\theta_{n}\right\} \subset(0,1]$ be such that $\theta_{n} \rightarrow 0$. Let $u_{n} \in E$ be a critical point of $J_{\theta_{n}, \lambda}$ with $J_{\theta_{n}, \lambda}\left(u_{n}\right) \leq c$ for some constant $c$ independent of $n$. Then, passing to a subsequence, we have $u_{n} \rightarrow u$ in $H_{V}^{1}\left(\mathbb{R}^{N}\right), u_{n} \nabla u_{n} \rightarrow u \nabla u$ in $L^{2}\left(\mathbb{R}^{N}\right), \theta_{n} \int_{\mathbb{R}^{N}}\left(\left|\nabla u_{n}\right|^{4}+\right.$ $\left.u_{n}^{4}\right) d x \rightarrow 0, J_{\theta_{n}, \lambda}\left(u_{n}\right) \rightarrow J_{\lambda}(u)$ and $u$ is a critical point of $J_{\lambda}$.

Proof. Step 1: We need to prove that the sequences $\left\{\int_{R^{N}} u_{n}^{2}\left|\nabla u_{n}\right|^{2} d x\right\},\left\{\theta_{n}\left\|u_{n}\right\|_{W^{1,4}}^{4}\right\}$ and $\left\{\left\|u_{n}\right\|_{H_{V}^{1}}^{2}\right\}$ are bounded.

By ( $h_{2}$ ), for $0<\varepsilon_{0}<\frac{1}{4}\left(\frac{1}{2}-\frac{1}{\mu}\right) V_{0}$, there exists $\delta>0$ such that

$$
\left|\frac{1}{\mu} t h_{M}(t)-H_{M}(t)\right| \leq \varepsilon_{0} t^{2}
$$

for all $|t| \leq \delta$. By $\left(h_{1}\right)$, for $\delta \leq|t| \leq r\left(r\right.$ is the constant appearing in the condition $\left.\left(h_{3}\right)\right)$, one obtains

$$
\left|\frac{1}{\mu} t h_{M}(t)-H_{M}(t)\right| \leq 2 C(M)\left(1+r^{p-2}\right) t^{2},
$$

where $C(M)$ is the constant appearing in the condition $\left(h_{1}\right)$. Thus, we get

$$
\left|\frac{1}{\mu} t h_{M}(t)-H_{M}(t)\right| \leq \varepsilon_{0} t^{2}+2 C(M)\left(1+r^{p-2}\right) t^{2}, \quad \forall t \in[-r, r] .
$$

Since $\lim _{|x| \rightarrow \infty} V(x)=+\infty$, there exists $\rho_{0}>0$ such that

$$
\frac{1}{4}\left(\frac{1}{2}-\frac{1}{\mu}\right) V(x)>2 \lambda C(M)\left(1+r^{p-2}\right)
$$

for all $|x| \geq \rho_{0}$. Thus,

$$
\begin{align*}
\left(\frac{1}{2}-\frac{1}{\mu}\right) \int_{\mathbb{R}^{N}} V(x) u_{n}^{2} d x+ & \lambda \int_{\left|u_{n}(x)\right| \leq r}\left[\frac{1}{\mu} u_{n} h_{M}\left(u_{n}\right)-H_{M}\left(u_{n}\right)\right] d x \\
& \geq\left(\frac{1}{4}-\frac{1}{2 \mu}\right) \int_{\mathbb{R}^{N}} V(x) u_{n}^{2} d x-2 \lambda C(M)\left(1+r^{p-2}\right) r^{2}\left|B_{\rho_{0}}\right| \tag{2.2}
\end{align*}
$$

where $B_{\rho_{0}}:=\left\{x \in R^{N}:|x|<\rho_{0}\right\},\left|B_{\rho_{0}}\right|:=$ meas $\left(B_{\rho_{0}}\right)$. Moreover, since $u_{n} \in E$ is a critical point of $J_{\theta_{n}, \lambda}$, for each $\phi \in E$, we have

$$
\begin{align*}
0= & \left\langle J_{\theta_{n}, \lambda}^{\prime}\left(u_{n}\right), \phi\right\rangle=\theta_{n} \int_{\mathbb{R}^{N}}\left[\left|\nabla u_{n}\right|^{2} \nabla u_{n} \nabla \phi+\left|u_{n}\right|^{2} u_{n} \phi\right] d x \\
& +\left(a+b \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x\right) \int_{\mathbb{R}^{N}} \nabla u_{n} \nabla \phi d x+2 \int_{\mathbb{R}^{N}}\left(u_{n}^{2} \nabla u_{n} \nabla \phi+\left|\nabla u_{n}\right|^{2} u_{n} \phi\right) d x  \tag{2.3}\\
& +\int_{\mathbb{R}^{N}} V(x) u_{n} \phi d x-\lambda \int_{\mathbb{R}^{N}} h_{M}\left(u_{n}\right) \phi d x .
\end{align*}
$$

Hence, it follows from ( $h_{3}$ ) and (2.2) that

$$
\begin{aligned}
c \geq & J_{\theta_{n}, \lambda}\left(u_{n}\right) \\
= & J_{\theta_{n}, \lambda}\left(u_{n}\right)-\frac{1}{\mu}\left\langle J_{\theta_{n}, \lambda}^{\prime}\left(u_{n}\right), u_{n}\right\rangle \\
= & \left(\frac{1}{4}-\frac{1}{\mu}\right) \theta_{n}\left\|u_{n}\right\|_{W^{1,4}}^{4}+\left(\frac{a}{2}-\frac{a}{\mu}\right) \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x+\left(\frac{b}{4}-\frac{b}{\mu}\right)\left(\int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x\right)^{2} \\
& +\left(1-\frac{4}{\mu}\right) \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} u_{n}^{2} d x+\left(\frac{1}{2}-\frac{1}{\mu}\right) \int_{\mathbb{R}^{N}} V(x) u_{n}^{2} d x \\
& +\lambda \int_{\mathbb{R}^{N}}\left[\frac{1}{\mu} u_{n} h_{M}\left(u_{n}\right)-H_{M}\left(u_{n}\right)\right] d x \\
\geq & \left(\frac{1}{4}-\frac{1}{\mu}\right) \theta_{n}\left\|u_{n}\right\|_{W^{1,4}}^{4}+\left(\frac{a}{2}-\frac{a}{\mu}\right) \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x+\left(\frac{b}{4}-\frac{b}{\mu}\right)\left(\int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x\right)^{2} \\
& +\left(1-\frac{4}{\mu}\right) \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} u_{n}^{2} d x+\left(\frac{1}{4}-\frac{1}{2 \mu}\right) \int_{R^{N}} V(x) u_{n}^{2} d x-2 \lambda C(M)\left(1+r^{p-2}\right) r^{2}\left|B_{\rho_{0}}\right| \\
\geq & \left(\frac{1}{4}-\frac{1}{\mu}\right) \theta_{n}\left\|u_{n}\right\|_{W^{1,4}}^{4}+c_{1}\left\|u_{n}\right\|_{H_{V}^{1}}^{2}+c_{2} \int_{\mathbb{R}^{N}} u_{n}^{2}\left|\nabla u_{n}\right|^{2} d x-\lambda C_{1}(M),
\end{aligned}
$$

where $C_{1}(M)=2 C(M)\left(1+r^{p-2}\right) r^{2}\left|B_{\rho_{0}}\right|$. Therefore, we get

$$
\begin{equation*}
\left(\frac{1}{4}-\frac{1}{\mu}\right) \theta_{n}\left\|u_{n}\right\|_{W^{1,4}}^{4}+c_{1}\left\|u_{n}\right\|_{H_{V}^{1}}^{2}+c_{2} \int_{\mathbb{R}^{N}} u_{n}^{2}\left|\nabla u_{n}\right|^{2} d x \leq C_{0}+\lambda C_{1}(M) \tag{2.4}
\end{equation*}
$$

By (2.4), going if necessary to a subsequence, we get $u_{n} \rightharpoonup u$ in $H_{V}^{1}\left(\mathbb{R}^{N}\right), u_{n} \nabla u_{n} \rightharpoonup u \nabla u$ in $L^{2}\left(\mathbb{R}^{N}\right), u_{n} \rightarrow u$ in $L^{s}\left(\mathbb{R}^{N}\right)$ for $s \in\left[2,22^{*}\right)$ and $u_{n}(x) \rightarrow u(x)$ a.e. $x \in \mathbb{R}^{N}$. This completes the proof of Step 1.

Step 2: We claim that $u_{n} \in L^{\infty}\left(\mathbb{R}^{N}\right),\left\|u_{n}\right\|_{L \infty} \leq M$ and $\|u\|_{L \infty} \leq M$, where the positive constant $M$ is independent of $n$.

Depending on (2.4), we infer

$$
\begin{equation*}
\left\|u_{n}\right\|_{L^{22^{*}}}^{4}=\left\|u_{n}^{2}\right\|_{L^{2^{*}}}^{2} \leq C\left\|\nabla u_{n}^{2}\right\|_{L^{2}}^{2} \leq C_{0}+\lambda C_{1}(M) . \tag{2.5}
\end{equation*}
$$

Set $T>2, r>0$ and $\tilde{u}_{n}^{T}=\gamma\left(u_{n}\right)$, where $\gamma: \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function satisfying $\gamma(t)=t$ for $|t| \leq T-1, \gamma(-t)=-\gamma(t) ; \gamma^{\prime}(t)=0$ for $t \geq T$ and $\gamma^{\prime}(t)$ is decreasing in $[T-1, T]$. This means that $\tilde{u}_{n}^{T}=u_{n}$ for $\left|u_{n}\right| \leq T-1 ;\left|\tilde{u}_{n}^{T}\right|=\left|\gamma\left(u_{n}\right)\right| \leq\left|u_{n}\right|$ for $T-1 \leq\left|u_{n}\right| \leq T ;\left|\tilde{u}_{n}^{T}\right|=C_{T}>0$ for $\left|u_{n}\right| \geq T$, where $T-1 \leq C_{T} \leq T$.

Setting $\phi=u_{n}\left|\tilde{u}_{n}^{T}\right|^{2 r}$, then we easily infer that $\phi \in E$. Therefore, it follows from (2.3) that

$$
\begin{align*}
& \lambda \int_{\mathbb{R}^{N}} h_{M}\left(u_{n}\right) \phi d x-\int_{\mathbb{R}^{N}} V(x) u_{n} \phi d x \\
&= \theta_{n} \int_{\mathbb{R}^{N}}\left[\left|\nabla u_{n}\right|^{2} \nabla u_{n} \nabla \phi+\left|u_{n}\right|^{2} u_{n} \phi\right] d x+\left(a+b \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x\right) \int_{\mathbb{R}^{N}} \nabla u_{n} \nabla \phi d x \\
&+2 \int_{\mathbb{R}^{N}}\left(u_{n}^{2} \nabla u_{n} \nabla \phi+\left|\nabla u_{n}\right|^{2} u_{n} \phi\right) d x \\
& \geq 2 \int_{\mathbb{R}^{N}} u_{n}^{2} \nabla u_{n} \nabla \phi d x \\
&= 2 \int_{\left|u_{n}\right| \geq T}\left|u_{n}\right|^{2}\left|\nabla u_{n}\right|^{2}\left|\tilde{u}_{n}^{T}\right|^{2 r} d x+2 \int_{\left|u_{n}\right| \leq T-1}(1+2 r)\left|u_{n}\right|^{2 r+2}\left|\nabla u_{n}\right|^{2} d x \\
&+2 \int_{T-1<\left|u_{n}\right|<T}\left[\left|\gamma\left(u_{n}\right)\right|^{2 r}+2 r u_{n} \gamma\left(u_{n}\right)\left|\gamma\left(u_{n}\right)\right|^{2 r-2} \gamma^{\prime}\left(u_{n}\right)\right]\left|u_{n}\right|^{2}\left|\nabla u_{n}\right|^{2} d x \\
& \geq \frac{1}{2} \int_{\left|u_{n}\right| \geq T}\left|\nabla\left[\left|u_{n}\right|^{2}\left(\tilde{u}_{n}^{T}\right)^{r}\right]\right|^{2} d x+\int_{\left|u_{n}\right| \leq T-1}\left|u_{n}\right|^{2 r+2}\left|\nabla u_{n}\right|^{2} d x \\
&+\frac{1}{2} \int_{T-1 \leq\left|u_{n}\right| \leq T}\left|\left(\tilde{u}_{n}^{T}\right)^{r} \nabla\left(\left|u_{n}\right|^{2}\right)\right|^{2} d x \\
&+2 r \int_{T-1 \leq\left|u_{n}\right| \leq T}\left|u_{n}\right|^{4}\left|\tilde{u}_{n}^{T}\right|^{2 r-2}\left(\gamma^{\prime}\left(u_{n}\right)\right)^{2}\left|\nabla u_{n}\right|^{2} d x  \tag{2.6}\\
&= \frac{1}{2} \int_{\left|u_{n}\right| \geq T}\left|\nabla\left[\left|u_{n}\right|^{2}\left(\tilde{u}_{n}^{T}\right)^{r}\right]\right|^{2} d x+\int_{\left|u_{n}\right| \leq T-1}\left|u_{n}\right|^{2 r+2}\left|\nabla u_{n}\right|^{2} d x \\
&+\frac{1}{2} \int_{T-1 \leq\left|u_{n}\right| \leq T}\left|\left(\tilde{u}_{n}^{T}\right)^{r} \nabla\left(\left|u_{n}\right|^{2}\right)\right|^{2} d x+\left.\left.\frac{2}{r} \int_{T-1 \leq\left|u_{n}\right| \leq T}| | u_{n}\right|^{2} \nabla\left(\tilde{u}_{n}^{T}\right)^{r}\right|^{2} d x \\
& \geq \frac{2}{(r+2)^{2}} \int_{\left|u_{n}\right| \geq T}\left|\nabla\left[\left|u_{n}\right|^{2}\left(\tilde{u}_{n}^{T}\right)^{r}\right]\right|^{2} d x+\frac{1}{(r+2)^{2}} \int_{\left|u_{n}\right| \leq T-1}\left|\nabla\left[\left|u_{n}\right|^{2}\left(\tilde{u}_{n}^{T}\right)^{r}\right]\right|^{2} d x \\
&+\frac{2}{(r+2)^{2}} \int_{T-1 \leq\left|u_{n}\right| \leq T}\left[\left|\left(\tilde{u}_{n}^{T}\right)^{r} \nabla\left(\left|u_{n}\right|^{2}\right)\right|^{2}+\left|\left|u_{n}\right|^{2} \nabla\left(\tilde{u}_{n}^{T}\right)^{r}\right|^{2}\right] d x \\
& \geq \frac{1}{(r+2)^{2}} \int_{\left|u_{n}\right| \geq T}\left|\nabla\left[\left|u_{n}\right|^{2}\left(\tilde{u}_{n}^{T}\right)^{r}\right]\right|^{2} d x+\frac{1}{(r+2)^{2}} \int_{\left|u_{n}\right| \leq T-1}\left|\nabla\left[\left|u_{n}\right|^{2}\left(\tilde{u}_{n}^{T}\right)^{r}\right]\right|^{2} d x \\
&+\frac{1}{(r+2)^{2}} \int_{T-1 \leq\left|u_{n}\right| \leq T}\left|\nabla\left[\left|u_{n}\right|^{2}\left(\tilde{u}_{n}^{T}\right)^{r}\right]\right|^{2} d x \\
&= 1 \\
&(r+2)^{2} r_{R^{N}}\left|\nabla\left[\left|u_{n}\right|^{2}\left(\tilde{u}_{n}^{T}\right)^{r}\right]\right|^{2} d x .
\end{align*}
$$

Choosing $0<\lambda \leq V_{0} / C_{M}^{\prime}$, then it follows from ( $h_{1}$ ) and (2.6) that

$$
\begin{equation*}
\frac{1}{(r+2)^{2}} \int_{\mathbb{R}^{N}}\left|\nabla\left[u_{n}^{2}\left(\tilde{u}_{n}^{T}\right)^{r}\right]\right|^{2} d x \leq \lambda C_{M} \int_{\mathbb{R}^{N}}\left|u_{n}\right|^{p}\left|\tilde{u}_{n}^{T}\right|^{2 r} d x . \tag{2.7}
\end{equation*}
$$

By (2.5) and Hölder inequality, we obtain

$$
\begin{aligned}
& \int_{\mathbb{R}^{N}}\left|u_{n}\right|^{p}\left|\tilde{u}_{n}^{T}\right|^{2 r} d x \\
& \quad=\int_{\mathbb{R}^{N}}\left|u_{n}\right|^{p-4}\left|\tilde{u}_{n}^{T}\right|^{2 r}\left|u_{n}\right|^{4} d x \\
& \quad \leq\left(\int_{\mathbb{R}^{N}}\left|u_{n}\right|^{(p-4) \frac{4 N}{(p-4)(N-2)}} d x\right)^{\frac{(p-4)(N-2)}{4 N}}\left(\int_{\mathbb{R}^{N}}\left(\left|\tilde{u}_{n}^{T}\right|^{2 r} u_{n}^{4}\right)^{\frac{4 N}{4 N-(p-4)(N-2)}} d x\right)^{\frac{4 N-(p-4)(N-2)}{4 N}}
\end{aligned}
$$

$$
\begin{align*}
& =\left(\int_{\mathbb{R}^{N}}\left|u_{n}\right|^{22^{*}} d x\right)^{\frac{(p-4)(N-2)}{4 N}}\left(\int_{\mathbb{R}^{N}}\left(\left|\tilde{u}_{n}^{T}\right|^{r} u_{n}^{2}\right)^{\frac{4 N}{4 N-(p-4)(N-2)}} d x\right)^{\frac{4 N-(p-4)(N-2)}{4 N}} \\
& \leq\left(C_{0}+\lambda C_{1}(M)\right)^{\frac{p-4}{4}}\left(\int_{\mathbb{R}^{N}}\left(\left|\tilde{u}_{n}^{T}\right|^{r} u_{n}^{2}\right)^{\frac{8 N}{4 N-(p-4)(N-2)}} d x\right)^{\frac{4 N-(p-4)(N-2)}{4 N}} \tag{2.8}
\end{align*}
$$

Since $u_{n}^{2}\left|\tilde{u}_{n}^{T}\right|^{r} \in D^{1,2}\left(\mathbb{R}^{N}\right)$, by the Sobolev embedding theorem, we infer

$$
\begin{equation*}
\left[\int_{\mathbb{R}^{N}}\left(u_{n}^{2}\left|\tilde{u}_{n}^{T}\right|^{r}\right)^{2^{*}} d x\right]^{\frac{2}{2^{*}}} \leq C \int_{\mathbb{R}^{N}}\left|\nabla\left[u_{n}^{2}\left(\tilde{u}_{n}^{T}\right)^{r}\right]\right|^{2} d x \tag{2.9}
\end{equation*}
$$

Then by (2.7), (2.8) and (2.9), one has

$$
\left[\int_{\mathbb{R}^{N}}\left(u_{n}^{2}\left|\tilde{u}_{n}^{T}\right|^{r}\right)^{2^{*}} d x\right]^{\frac{2}{2^{*}}} \leq \lambda C_{2}(M)(r+2)^{2}\left[\int_{\mathbb{R}^{N}}\left(\left|\tilde{u}_{n}^{T}\right|^{r} u_{n}^{2}\right)^{\frac{8 N}{4 N-(p-4)(N-2)}} d x\right]^{\frac{4 N-(p-4)(N-2)}{4 N}},
$$

where the constant $C_{2}(M)>0$ is dependent on $M$. Since $4<p<22^{*}, d:=2^{*} / q=\frac{2^{*}}{2}-\frac{p}{4}+$ $1>1$, where $q=\frac{8 N}{4 N-(p-4)(N-2)}$. Then

$$
\begin{equation*}
\left(\int_{\mathbb{R}^{N}}\left(u_{n}^{2}\left|\tilde{u}_{n}^{T}\right|^{r}\right)^{q d} d x\right)^{\frac{1}{q(r+2)}} \leq\left[\lambda C_{2}(M)(r+2)^{2}\right]^{\frac{1}{2(r+2)}}\left(\int_{\mathbb{R}^{N}}\left[u_{n}^{2}\left|\tilde{u}_{n}^{T}\right|^{r}\right]^{q} d x\right)^{\frac{1}{q(r+2)}} \tag{2.10}
\end{equation*}
$$

Take $r=r_{0}$ be such that $\left(2+r_{0}\right) q=22^{*}$. From $\left|\tilde{u}_{n}^{T}\right|=\left|\gamma\left(u_{n}\right)\right| \leq\left|u_{n}\right|$ and (2.5), one has

$$
\int_{\mathbb{R}^{N}}\left[\left|\tilde{u}_{n}^{T}\right|^{r_{0}} u_{n}^{2}\right]^{q} d x \leq \int_{\mathbb{R}^{N}}\left|u_{n}\right|^{\left(2+r_{0}\right) q} d x<C_{0}+\lambda C_{1}(M)
$$

Takeing the limit $T \rightarrow \infty$ in (2.10) with $r=r_{0}$, we obtain

$$
\left(\int_{\mathbb{R}^{N}}\left|u_{n}\right|^{\left(2+r_{0}\right) q d} d x\right)^{\frac{1}{q d\left(r_{0}+2\right)}} \leq\left[\lambda C_{2}(M)\left(r_{0}+2\right)^{2}\right]^{\frac{1}{2\left(r_{0}+2\right)}}\left(\int_{\mathbb{R}^{N}}\left|u_{n}\right|^{\left(2+r_{0}\right) q} d x\right)^{\frac{1}{q\left(r_{0}+2\right)}} .
$$

Further, setting $2+r_{1}=d\left(2+r_{0}\right)$, we get

$$
\left(\int_{\mathbb{R}^{N}}\left|u_{n}\right|^{\left(2+r_{1}\right) q} d x\right)^{\frac{1}{q\left(r_{1}+2\right)}} \leq\left[\lambda C_{2}(M)\left(r_{0}+2\right)^{2}\right]^{\frac{1}{2\left(r_{0}+2\right)}}\left(\int_{\mathbb{R}^{N}}\left|u_{n}\right|^{\left(2+r_{0}\right) q} d x\right)^{\frac{1}{q\left(r_{0}+2\right)}}
$$

Inductively, we have

$$
\begin{aligned}
\left(\int_{\mathbb{R}^{N}}\left|u_{n}\right|^{\left(2+r_{k+1}\right) q} d x\right)^{\frac{1}{q\left(r_{k+1}+2\right)}} & \leq\left[\lambda C_{2}(M)\left(r_{k}+2\right)^{2}\right]^{\frac{1}{2\left(r_{k}+2\right)}}\left(\int_{\mathbb{R}^{N}}\left|u_{n}\right|^{\left(2+r_{k}\right) q} d x\right)^{\frac{1}{q\left(r_{k}+2\right)}} \\
& \leq \prod_{i=0}^{k}\left[\lambda C_{2}(M)\left(r_{i}+2\right)^{2}\right]^{\frac{1}{2\left(r_{i}+2\right)}}\left(\int_{\mathbb{R}^{N}}\left|u_{n}\right|^{\left(2+r_{0}\right) q} d x\right)^{\frac{1}{q\left(r_{0}+2\right)}},
\end{aligned}
$$

where $\left(2+r_{i}\right)=d^{i}\left(2+r_{0}\right)(i=0,1, \ldots, k)$. Moreover,

$$
\begin{aligned}
\prod_{i=0}^{k}\left[\lambda C_{2}(M)\left(r_{i}+2\right)^{2}\right]^{\frac{1}{2\left(r_{i}+2\right)}} & =\exp \left\{\sum_{i=0}^{k} \frac{\ln \sqrt{\lambda C_{2}(M)} d^{i}\left(r_{0}+2\right)}{d^{i}\left(r_{0}+2\right)}\right\} \\
& =\exp \left\{\sum_{i=0}^{k}\left[\frac{\ln \sqrt{\lambda C_{2}(M)}\left(r_{0}+2\right)}{d^{i}\left(r_{0}+2\right)}+\frac{i \ln d}{d^{i}\left(r_{0}+2\right)}\right]\right\}
\end{aligned}
$$

is convergent as $k \rightarrow \infty$. Let $C_{k}=\prod_{i=0}^{k}\left[\lambda C_{2}(M)\left(r_{i}+2\right)^{2}\right]^{\frac{1}{2\left(r_{i}+2\right)}}$. For $C_{k}$, we can choose $0<\lambda_{0} \leq C_{0} / C_{1}(M)$ small enough and $\frac{1}{2} \lambda_{0}<\lambda<\lambda_{0}$ such that $C_{k} \rightarrow C_{\infty}>0$ as $k \rightarrow \infty$ and $C_{\infty} \leq M /\left(2 C_{0}^{\frac{1}{4}}\right)$. Then we get

$$
\left\|u_{n}\right\|_{\left.L^{\left(2+r_{0}\right.}\right) q d^{k+1}} \leq C_{k}\left\|u_{n}\right\|_{L^{22^{*}}}
$$

Let $k \rightarrow \infty$, for fixed constant $M$ and $\frac{1}{2} \lambda_{0}<\lambda<\lambda_{0}$, by (2.5) we have

$$
\begin{equation*}
\left\|u_{n}\right\|_{L^{\infty}} \leq C_{\infty}\left\|u_{n}\right\|_{L^{22^{*}}} \leq M, \quad\|u\|_{L^{\infty}} \leq M \tag{2.11}
\end{equation*}
$$

Step 3: We will show that $u$ is a critical point of $J_{\lambda}$.
For any $\psi \in C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$, there exists a bounded domain $\Omega \subset \mathbb{R}^{N}$ such that $\operatorname{supp}(\psi) \subset \Omega$. Thus, by (2.11), we know $\phi=\psi \exp \left(-K u_{n}\right) \in E$ for any $\psi \geq 0$ and $K>0$. Taking $\phi=$ $\psi \exp \left(-K u_{n}\right)$ as the test function in (2.3), we have

$$
\begin{align*}
0= & \theta_{n} \int_{\mathbb{R}^{N}} \exp \left(-K u_{n}\right)\left[\left|\nabla u_{n}\right|^{2} \nabla u_{n}\left(\nabla \psi-K \psi \nabla u_{n}\right)+\left|u_{n}\right|^{2} u_{n} \psi\right] d x \\
& +\left(a+b \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x\right) \int_{\mathbb{R}^{N}} \exp \left(-K u_{n}\right) \nabla u_{n}\left(\nabla \psi-K \psi \nabla u_{n}\right) d x \\
& +2 \int_{\mathbb{R}^{N}}\left[\exp \left(-K u_{n}\right) u_{n}^{2} \nabla u_{n}\left(\nabla \psi-K \psi \nabla u_{n}\right)+\exp \left(-K u_{n}\right) \psi\left|\nabla u_{n}\right|^{2} u_{n}\right] d x \\
& +\int_{\mathbb{R}^{N}} V(x) u_{n} \psi \exp \left(-K u_{n}\right) d x-\lambda \int_{\mathbb{R}^{N}} h_{M}\left(u_{n}\right) \psi \exp \left(-K u_{n}\right) d x \\
\leq & \theta_{n} \int_{\mathbb{R}^{N}} \exp \left(-K u_{n}\right)\left[\left|\nabla u_{n}\right|^{2} \nabla u_{n} \nabla \psi+\left|u_{n}\right|^{2} u_{n} \psi\right] d x  \tag{2.12}\\
& +\left(a+b \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x\right) \int_{\mathbb{R}^{N}} \exp \left(-K u_{n}\right) \nabla u_{n} \nabla \psi d x \\
& +2 \int_{\mathbb{R}^{N}} \exp \left(-K u_{n}\right) u_{n}^{2} \nabla u_{n} \nabla \psi d x \\
& -\int_{\mathbb{R}^{N}} \exp \left(-K u_{n}\right) \psi\left|\nabla u_{n}\right|^{2}\left[K\left(a+b \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x+2 u_{n}^{2}\right)-2 u_{n}\right] d x \\
& +\int_{\mathbb{R}^{N}} V(x) u_{n} \psi \exp \left(-K u_{n}\right) d x-\lambda \int_{\mathbb{R}^{N}} h_{M}\left(u_{n}\right) \psi \exp \left(-K u_{n}\right) d x .
\end{align*}
$$

Choose large $K>1$ be such that $K a>1$. Then, by

$$
\int_{\mathbb{R}^{N}} \exp \left(-K u_{n}\right) \psi\left|\nabla\left(u_{n}-u\right)\right|^{2}\left[K\left(a+b \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x+2 u_{n}^{2}\right)-2 u_{n}\right] d x \geq 0,
$$

one has

$$
\begin{aligned}
\int_{\mathbb{R}^{N}} & \exp \left(-K u_{n}\right) \psi\left|\nabla u_{n}\right|^{2}\left[K\left(a+b \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x+2 u_{n}^{2}\right)-2 u_{n}\right] d x \\
& \geq \int_{\mathbb{R}^{N}} \exp \left(-K u_{n}\right) \psi\left(2 \nabla u_{n} \nabla u-|\nabla u|^{2}\right)\left[K\left(a+b \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x+2 u_{n}^{2}\right)-2 u_{n}\right] d x \\
& \rightarrow \int_{\mathbb{R}^{N}} \exp (-K u) \psi|\nabla u|^{2}\left[K\left(a+b \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x+2 u^{2}\right)-2 u\right] d x .
\end{aligned}
$$

Because $\theta_{n} \rightarrow 0$ and $\left\|u_{n}\right\|_{\infty} \leq M$, (2.4) implies

$$
\theta_{n} \int_{\mathbb{R}^{N}} \exp \left(-K u_{n}\right)\left[\left|\nabla u_{n}\right|^{2} \nabla u_{n} \nabla \psi+\left|u_{n}\right|^{2} u_{n} \psi\right] d x \rightarrow 0
$$

as $n \rightarrow \infty$. By the weak convergence of $u_{n}$, the Hölder inequality and Lebesgue's dominated convergence theorem, we infer

$$
\begin{aligned}
\left(a+b \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x\right) \int_{\mathbb{R}^{N}} e^{\left(-K u_{n}\right)} \nabla u_{n} \nabla \psi d x & \rightarrow\left(a+b \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x\right) \int_{\mathbb{R}^{N}} e^{(-K u)} \nabla u \nabla \psi d x, \\
\int_{\mathbb{R}^{N}} \exp \left(-K u_{n}\right) u_{n}^{2} \nabla u_{n} \nabla \psi d x & \rightarrow \int_{\mathbb{R}^{N}} \exp (-K u) u^{2} \nabla u \nabla \psi d x, \\
\int_{\mathbb{R}^{N}} V(x) u_{n} \psi \exp \left(-K u_{n}\right) d x & \rightarrow \int_{\mathbb{R}^{N}} V(x) u \psi \exp (-K u) d x
\end{aligned}
$$

and

$$
\lambda \int_{\mathbb{R}^{N}} h_{M}\left(u_{n}\right) \psi \exp \left(-K u_{n}\right) d x \rightarrow \lambda \int_{\mathbb{R}^{N}} h_{M}(u) \psi \exp (-K u) d x .
$$

Hence, these together with (2.12) can deduce that

$$
\begin{align*}
0 \leq & \left(a+b \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x\right) \int_{\mathbb{R}^{N}} \exp (-K u) \nabla u \nabla \psi d x+2 \int_{\mathbb{R}^{N}} \exp (-K u) u^{2} \nabla u \nabla \psi d x \\
& -\int_{\mathbb{R}^{N}} \exp (-K u) \psi|\nabla u|^{2}\left[K\left(a+b \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x+2 u^{2}\right)-2 u\right] d x  \tag{2.13}\\
& +\int_{\mathbb{R}^{N}} V(x) u \psi \exp (-K u) d x-\lambda \int_{\mathbb{R}^{N}} h_{M}(u) \psi \exp (-K u) d x .
\end{align*}
$$

For any $\varphi \in E$ with $\varphi \geq 0$, by (2.11), we know $v:=\varphi \exp (K u) \in E$. By applying [18, Theorem 2.8], there exists a sequence $\left\{\psi_{n}\right\} \subset C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$ of functions such that $\psi_{n} \geq 0, \psi_{n} \rightarrow v$ in $H_{V}^{1}\left(\mathbb{R}^{N}\right)$ and $\psi_{n}(x) \rightarrow v(x)$ for a.e. $x \in \mathbb{R}^{N}$. Taking $\psi=\psi_{n}$ in (2.13) and letting $n \rightarrow \infty$, we have

$$
\begin{aligned}
0 \leq & \left(a+b \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x\right) \int_{\mathbb{R}^{N}} \nabla u \nabla \varphi d x+2 \int_{\mathbb{R}^{N}} u^{2} \nabla u \nabla \varphi d x \\
& +2 \int_{\mathbb{R}^{N}}|\nabla u|^{2} u \varphi d x+\int_{\mathbb{R}^{N}} V(x) u \varphi d x-\lambda \int_{\mathbb{R}^{N}} h_{M}(u) \varphi d x .
\end{aligned}
$$

The opposite inequality can be obtained in a similar way. Therefore,

$$
\begin{aligned}
&\left(a+b \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x\right) \int_{\mathbb{R}^{N}} \nabla u \nabla \varphi d x+2 \int_{\mathbb{R}^{N}}\left(u^{2} \nabla u \nabla \varphi+|\nabla u|^{2} u \varphi\right) d x \\
&+\int_{\mathbb{R}^{N}} V(x) u \varphi d x-\lambda \int_{\mathbb{R}^{N}} h_{M}(u) \varphi d x=0
\end{aligned}
$$

for all $\varphi \in E$. This shows that $u \in E$ is a critical point of $J_{\lambda}$ and

$$
\begin{align*}
&\left(a+b \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x\right) \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x+4 \int_{\mathbb{R}^{N}} u^{2}|\nabla u|^{2} d x \\
&+\int_{\mathbb{R}^{N}} V(x) u^{2} d x-\lambda \int_{\mathbb{R}^{N}} h_{M}(u) u d x=0 . \tag{2.14}
\end{align*}
$$

Finally, taking $\phi=u_{n}$ as the test function in (2.3), one has

$$
\begin{aligned}
0= & \theta_{n} \int_{\mathbb{R}^{N}}\left[\left|\nabla u_{n}\right|^{4}+\left|u_{n}\right|^{4}\right] d x+\left(a+b \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x\right) \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x \\
& +4 \int_{\mathbb{R}^{N}} u_{n}^{2}\left|\nabla u_{n}\right|^{2} d x+\int_{\mathbb{R}^{N}} V(x) u_{n}^{2} d x-\lambda \int_{\mathbb{R}^{N}} h_{M}\left(u_{n}\right) u_{n} d x .
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x & \geq 2 \int_{\mathbb{R}^{N}} \nabla u_{n} \nabla u d x-\int_{\mathbb{R}^{N}}|\nabla u|^{2} d x \longrightarrow \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x, \\
\int_{\mathbb{R}^{N}} u_{n}^{2}\left|\nabla u_{n}\right|^{2} d x & \geq 2 \int_{\mathbb{R}^{N}} u_{n}^{2} \nabla u_{n} \nabla u d x-\int_{\mathbb{R}^{N}} u_{n}^{2}|\nabla u|^{2} d x \longrightarrow \int_{\mathbb{R}^{N}} u^{2}|\nabla u|^{2} d x, \\
\lambda \int_{\mathbb{R}^{N}} h_{M}\left(u_{n}\right) u_{n} d x & \rightarrow \lambda \int_{\mathbb{R}^{N}} h_{M}(u) u d x
\end{aligned}
$$

and

$$
\liminf _{n \rightarrow \infty} \int_{\mathbb{R}^{N}} V(x) u_{n}^{2} d x \geq \int_{\mathbb{R}^{N}} V(x) u^{2} d x
$$

By (2.4) and (2.14), up to a subsequence, one has

$$
\theta_{n}\left\|u_{n}\right\|_{W^{1,4}}^{4} \rightarrow 0,\left\|u_{n}\right\|_{H_{V}^{1}} \rightarrow\|u\|_{H_{V}^{1}} \int_{\mathbb{R}^{N}} u_{n}^{2}\left|\nabla u_{n}\right|^{2} d x \rightarrow \int_{\mathbb{R}^{N}} u^{2}|\nabla u|^{2} d x
$$

Hence, $J_{\theta_{n}, \lambda}\left(u_{n}\right) \rightarrow J_{\lambda}(u)$ and $u_{n} \rightarrow u$ in $H_{V}^{1}\left(\mathbb{R}^{N}\right)$. This completes the proof.

## 3 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. First, we will show that for each $\theta \in(0,1], J_{\theta, \lambda}$ and $J_{\theta, \lambda}^{ \pm}$satisfy the $P$. S. condition. Indeed, by Lemma 2.2, it is sufficient to prove that any P. S. sequence of $J_{\theta, \lambda}$ is bounded.

Let $\left\{u_{n}\right\} \subset E$ be an arbitrary P. S. sequence for $J_{\theta, \lambda}$. If $\left\{u_{n}\right\}$ is unbounded in $E$, we can assume $\left\|u_{n}\right\|_{E} \rightarrow+\infty$. By (2.2) and ( $h_{3}$ ), we get

$$
\begin{align*}
J_{\theta, \lambda}\left(u_{n}\right) & -\frac{1}{\mu}\left\langle J_{\theta, \lambda}^{\prime}\left(u_{n}\right), u_{n}\right\rangle \\
= & \left(\frac{1}{4}-\frac{1}{\mu}\right) \theta\left\|u_{n}\right\|_{W^{1,4}}^{4}+\left(\frac{a}{2}-\frac{a}{\mu}\right) \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x+\left(\frac{b}{4}-\frac{b}{\mu}\right)\left(\int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x\right)^{2} \\
& +\left(1-\frac{4}{\mu}\right) \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} u_{n}^{2} d x+\left(\frac{1}{2}-\frac{1}{\mu}\right) \int_{\mathbb{R}^{N}} V(x) u_{n}^{2} d x \\
& +\lambda \int_{\mathbb{R}^{N}}\left[\frac{1}{\mu} u_{n} h_{M}\left(u_{n}\right)-H_{M}\left(u_{n}\right)\right] d x  \tag{3.1}\\
\geq & \left(\frac{a}{2}-\frac{a}{\mu}\right) \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x+\left(\frac{1}{2}-\frac{1}{\mu}\right) \int_{\mathbb{R}^{N}} V(x) u_{n}^{2} d x \\
& +\lambda \int_{\mathbb{R}^{N}}\left[\frac{1}{\mu} u_{n} h_{M}\left(u_{n}\right)-H_{M}\left(u_{n}\right)\right] d x \\
\geq & \left(\frac{a}{2}-\frac{a}{\mu}\right) \int_{\mathbb{R}^{N}}\left|\nabla u_{n}\right|^{2} d x+\left(\frac{1}{4}-\frac{1}{2 \mu}\right) \int_{\mathbb{R}^{N}} V(x) u_{n}^{2} d x-\lambda C_{1}(M) \\
\geq & \min \left\{\frac{a}{2}-\frac{a}{\mu}, \frac{1}{4}-\frac{1}{2 \mu}\right\}\left\|u_{n}\right\|_{H_{V}^{1}}^{2}-\lambda C_{1}(M) .
\end{align*}
$$

If $\left\{\left\|u_{n}\right\|_{W^{1,4}}\right\}$ is bounded, then $\frac{\left\|u_{n}\right\|_{H_{V}^{1}}}{\left\|u_{n}\right\|_{E}} \rightarrow 1$. Therefore, by (3.1), we infer

$$
\frac{J_{\theta, \lambda}\left(u_{n}\right)-\frac{1}{\mu}\left\langle J_{\theta, \lambda}^{\prime}\left(u_{n}\right), u_{n}\right\rangle}{\left\|u_{n}\right\|_{E}^{2}} \geq \min \left\{\frac{a}{2}-\frac{a}{\mu^{\prime}}, \frac{1}{4}-\frac{1}{2 \mu}\right\} \frac{\left\|u_{n}\right\|_{H_{V}^{1}}^{2}}{\left\|u_{n}\right\|_{E}^{2}}-\frac{\lambda C_{1}(M)}{\left\|u_{n}\right\|_{E}^{2}}
$$

which implies $0 \geq \min \left\{\frac{a}{2}-\frac{a}{\mu}, \frac{1}{4}-\frac{1}{2 \mu}\right\}>0$. That is to say, it is a contradiction. Hence, we can assume $\left\|u_{n}\right\|_{W^{1,4}} \rightarrow \infty$. For large $n$, it follows from (3.1) that

$$
\begin{aligned}
J_{\theta, \lambda} & \left(u_{n}\right)-\frac{1}{\mu}\left\langle J_{\theta, \lambda}^{\prime}\left(u_{n}\right), u_{n}\right\rangle \\
& \geq\left(\frac{1}{4}-\frac{1}{\mu}\right) \theta\left\|u_{n}\right\|_{W^{1,4}}^{4}+\min \left\{\frac{a}{2}-\frac{a}{\mu}, \frac{1}{4}-\frac{1}{2 \mu}\right\}\left\|u_{n}\right\|_{H_{V}^{1}}^{2}-\lambda C_{1}(M) \\
& \geq\left(\frac{1}{4}-\frac{1}{\mu}\right) \theta\left\|u_{n}\right\|_{W^{1,4}}^{2}+\min \left\{\frac{a}{2}-\frac{a}{\mu}, \frac{1}{4}-\frac{1}{2 \mu}\right\}\left\|u_{n}\right\|_{H_{V}^{1}}^{2}-\lambda C_{1}(M) \\
& \geq \frac{1}{2} \min \left\{\left(\frac{1}{4}-\frac{1}{\mu}\right) \theta, \frac{a}{2}-\frac{a}{\mu}, \frac{1}{4}-\frac{1}{2 \mu}\right\}\left\|u_{n}\right\|_{E}^{2}-\lambda C_{1}(M) .
\end{aligned}
$$

This together with $\left\|u_{n}\right\|_{W^{1,4}} \rightarrow \infty$ implies $0 \geq \frac{1}{2} \min \left\{\left(\frac{1}{4}-\frac{1}{\mu}\right) \theta, \frac{a}{2}-\frac{a}{\mu}, \frac{1}{4}-\frac{1}{2 \mu}\right\}>0$, a contradiction. This shows that $\left\{u_{n}\right\}$ is bounded in $E$.

Next, by $\left(h_{1}\right)$ and $\left(h_{2}\right)$, we get

$$
\begin{equation*}
\left|H_{M}(v)\right| \leq C_{M}^{\prime}|v|^{2}+C_{M}|v|^{22^{*}} \tag{3.2}
\end{equation*}
$$

for all $v \in \mathbb{R}$. For small $0<\rho \ll 1$, set

$$
S_{\rho}=\left\{v \in E:\|v\|_{E}=\rho\right\} .
$$

Then for $v \in S_{\rho}$ and $0<\lambda \leq V_{0} / 4 C_{M}^{\prime}$, by (3.2), we have

$$
\begin{aligned}
J_{\theta, \lambda}^{+}(v)= & \frac{1}{4} \theta \int_{\mathbb{R}^{N}}\left(|\nabla v|^{4}+v^{4}\right) d x+\frac{a}{2} \int_{\mathbb{R}^{N}}|\nabla v|^{2} d x+\frac{b}{4}\left(\int_{\mathbb{R}^{N}}|\nabla v|^{2} d x\right)^{2} \\
& +\frac{1}{2} \int_{\mathbb{R}^{N}}\left(V(x) v^{2}+2 v^{2}|\nabla v|^{2}\right) d x-\lambda \int_{\mathbb{R}^{N}} H_{M}\left(v^{+}\right) d x \\
\geq & \frac{1}{4} \theta\|v\|_{W^{1,4}}^{4}+\frac{1}{4} \min \{2 a, 1\}\|v\|_{H_{V}^{1}}^{2}+\int_{\mathbb{R}^{N}} v^{2}|\nabla v|^{2} d x-\lambda C_{M}\left(\int_{\mathbb{R}^{N}} v^{2}|\nabla v|^{2} d x\right)^{\frac{2^{*}}{2}} \\
\geq & \frac{1}{4} \theta\|v\|_{W^{1,4}}^{4}+\frac{1}{4} \min \{2 a, 1\}\|v\|_{H_{V}^{1}}^{2} \\
\geq & \frac{1}{4} \min \{\theta, 2 a, 1\}\left[\|v\|_{W^{1,4}}^{4}+\|v\|_{H_{V}^{1}}^{2}\right] \\
\geq & \frac{1}{64} \min \{\theta, 2 a, 1\} \rho^{4}:=\delta>0 .
\end{aligned}
$$

Moreover, for $|t| \geq r$, by $\left(h_{3}\right)$, we can infer $H_{M}(v) \geq C|v|^{\mu}$. Thus, by $\left(h_{1}\right)$ and $\left(h_{2}\right)$, there is a constant $C_{3}(M)>0$ that depends on $M$ such that

$$
\begin{equation*}
H_{M}(v) \geq C|v|^{\mu}-C_{3}(M) v^{2} \tag{3.3}
\end{equation*}
$$

for all $v \in E$. For any finite-dimensional subspace $\tilde{E} \subset E$, by the equivalency of all norms in the finite-dimensional space, there is a constant $\beta>0$ such that

$$
\begin{equation*}
\|v\|_{\mu} \geq \beta\|v\|_{E} \tag{3.4}
\end{equation*}
$$

for all $v \in \tilde{E}$. Hence, by (3.3) and (3.4), one has

$$
\begin{align*}
J_{\theta, \lambda}(v)= & \frac{1}{4} \theta \int_{\mathbb{R}^{N}}\left(|\nabla v|^{4}+v^{4}\right) d x+\frac{a}{2} \int_{\mathbb{R}^{N}}|\nabla v|^{2} d x+\frac{b}{4}\left(\int_{\mathbb{R}^{N}}|\nabla v|^{2} d x\right)^{2} \\
& +\frac{1}{2} \int_{\mathbb{R}^{N}}\left(V(x) v^{2}+2 v^{2}|\nabla v|^{2}\right) d x-\lambda \int_{\mathbb{R}^{N}} H_{M}(v) d x \\
\leq & \frac{1}{4} \theta\|v\|_{W^{1,4}}^{4}+\frac{1}{2} \max \{a, 1\}\|v\|_{H_{V}^{1}}^{2}+\int_{\mathbb{R}^{N}} v^{2}|\nabla v|^{2} d x \\
& +\frac{b}{4}\left(\int_{\mathbb{R}^{N}}|\nabla v|^{2} d x\right)^{2}-\lambda \int_{\mathbb{R}^{N}}\left[C|v|^{\mu}-C_{3}(M) v^{2}\right] d x  \tag{3.5}\\
\leq & \frac{3}{4}\|v\|_{W^{1,4}}^{4}+\frac{b}{4}\|v\|_{H_{V}^{1}}^{4}+\frac{1}{2} \max \{a, 1\}\|v\|_{H_{V}^{1}}^{2}-\lambda C\|v\|_{\mu}^{\mu}+\lambda C_{3}(M)\|v\|_{2}^{2} \\
\leq & \frac{1}{4} \max \{3, b\}\|v\|_{E}^{4}+\left(\lambda C_{3}(M)+\frac{1}{2} \max \{a, 1\}\right)\|v\|_{E}^{2}-\lambda C \beta^{\mu}\|v\|_{E}^{\mu}
\end{align*}
$$

for all $v \in \tilde{E}$ and $0<\theta \leq 1$. Thus, there is a large $R>0$ such that $J_{\theta, \lambda}<0$ on $\tilde{E} \backslash B_{R}$, where $B_{R}:=\left\{u \in E:\|u\|_{E}<R\right\}$. Set a fixed $e \in \tilde{E}$ with $e \geq 0$ and $\|e\|_{E}=1$. For any fixed constant $T>0$, define the path $h_{T}:[0,1] \rightarrow \tilde{E} \subset E$ by $h_{T}(t)=t T e$. Then for large $T>1$ and $\mu>4$, by (3.5), we get

$$
J_{\theta, \lambda}^{+}\left(h_{T}(1)\right) \leq \frac{1}{4} \max \{3, b\} T^{4}+\left(\lambda C_{3}(M)+\frac{1}{2} \max \{a, 1\}\right) T^{2}-\lambda C \beta^{\mu} T^{\mu}<0
$$

with $\left\|h_{T}(1)\right\|_{E}>\rho$, and

$$
\max _{t \in[0,1]} J_{\theta, \lambda}^{+}\left(h_{T}(t)\right) \leq C .
$$

Hence, by [16, Theorem 2.2], $J_{\theta, \lambda}^{+}$possesses a critical value

$$
c_{\theta}:=\inf _{\eta \in \Gamma} \max _{t \in[0,1]} J_{\theta, \lambda}^{+}(\eta(t)) \geq \delta>0
$$

and

$$
c_{\theta} \leq \max _{t \in[0,1]} J_{\theta, \lambda}^{+}\left(h_{T}(t)\right) \leq C,
$$

where

$$
\Gamma=\left\{\eta \in C([0,1], E): \eta(0)=0, \eta(1)=h_{T}(1)\right\} .
$$

Therefore, $J_{\theta, \lambda}^{+}$possesses the Mountain Pass geometry. Further, by Lemma 2.3 and Mountain Pass Theorem, we know that the equation (2.1) has a positive weak solution. This together with (2.11) implies that (1.1) has a positive weak solution. Moreover, by a similar argument, we infer that the equation (1.1) has a negative weak solution. This completes the proof.

Next, in order to prove Theorem 1.2, we need to revise the cutoff function. Let

$$
\hat{h}_{M}(t)= \begin{cases}f(t), & 0<t \leqslant M \\ C_{M} t^{p-1}, & t>M \\ -\hat{h}_{M}(-t), & t \leqslant 0\end{cases}
$$

Then for the odd function $f(t)$, it is easy to know that $\hat{h}_{M}(t)$ satisfies $\left(h_{1}\right)-\left(h_{3}\right)$ and the odd function property.

Hereinafter, we will concentrate on the following equation

$$
\begin{equation*}
-\left(a+b \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x\right) \Delta u-u \Delta\left(u^{2}\right)+V(x) u=\lambda \hat{h}_{M}(u), \quad x \in \mathbb{R}^{N} \tag{3.6}
\end{equation*}
$$

Here $\hat{J}_{\lambda}(u): E \rightarrow \mathbb{R}$ is the natural energy functional corresponding to (3.6)

$$
\begin{aligned}
\hat{J}_{\lambda}(u)= & \frac{a}{2} \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x+\frac{b}{4}\left(\int_{\mathbb{R}^{N}}|\nabla u|^{2} d x\right)^{2}+\frac{1}{2} \int_{\mathbb{R}^{N}}\left(V(x) u^{2}+2 u^{2}|\nabla u|^{2}\right) d x \\
& -\lambda \int_{\mathbb{R}^{N}} \hat{H}_{M}(u) d x
\end{aligned}
$$

where $\hat{H}_{M}(t)=\int_{0}^{t} \hat{h}_{M}(s) d s$. For $\theta \in(0,1]$, let $\hat{J}_{\theta, \lambda}(u)=\frac{1}{4} \theta \int_{\mathbb{R}^{N}}\left(|\nabla u|^{4}+u^{4}\right) d x+\hat{J}_{\lambda}(u)$.
Lemma 3.1. Assume that $(V),\left(f_{1}\right),\left(f_{2}\right)$ hold. If $f(t)$ is odd, then for all $\theta \in(0,1]$ fixed, $\hat{J}_{\theta, \lambda}$ has a sequence of critical points $u_{j}$ such that there exist $\alpha_{j}, \beta_{j}$ both of which are independent of $\theta$ to satisfy $\alpha_{j} \rightarrow \infty$ as $j \rightarrow \infty, \alpha_{j}<\beta_{j}$ and $c_{j}(\theta) \in\left[\alpha_{j}, \beta_{j}\right]$ for all $\theta>0$.

Proof. Consider the eigenvalue problem

$$
\begin{equation*}
\int_{\mathbb{R}^{N}}(\nabla u \cdot \nabla \varphi+V(x) u \varphi) d x=\xi \int_{\mathbb{R}^{N}} u \varphi d x, \quad \forall \varphi \in H_{V}^{1}\left(\mathbb{R}^{N}\right) \tag{3.7}
\end{equation*}
$$

For real number $\xi$, if there exists $u \in H_{V}^{1}\left(\mathbb{R}^{N}\right)(u \neq 0)$ to satisfy (3.7), then $\xi$ is called a eigenvalue of the operator $L=-\Delta+V$. Further, by the condition $(V)$ and the compactness of the embedding $H_{V}^{1}\left(\mathbb{R}^{N}\right) \hookrightarrow L^{2}\left(\mathbb{R}^{N}\right)$, we infer that the spectrum $\sigma(L)=\left\{\xi_{1}, \xi_{2}, \ldots, \xi_{n}, \ldots\right\}$ of $L$ satisfies

$$
0<\xi_{1}<\xi_{2}<\cdots<\xi_{n}<\cdots
$$

and $\xi_{n} \rightarrow+\infty$ as $n \rightarrow \infty$. Let $\phi_{n}$ be the eigenfunction corresponding to the eigenvalue $\xi_{n}$. By regularity argument, we know $\phi_{n} \in E$. Set $E_{n}=\operatorname{span}\left\{\phi_{1}, \phi_{2}, \ldots, \phi_{n}\right\}$. Then we decompose the space $E$ as a direct sum $E=E_{n} \oplus W_{n}$ for $n=1,2, \ldots$, where $W_{n}$ is orthogonal to $E_{n}$ in $H_{V}^{1}\left(\mathbb{R}^{N}\right)$. For $\rho>0$, set

$$
\mathcal{Z}_{\rho}=\left\{u \in E:\|u\|_{H_{V}^{1}}^{2}+\int_{\mathbb{R}^{N}} u^{2}|\nabla u|^{2} d x \leq \rho^{2}\right\}
$$

By (3.5), there exists $r_{n}>0$ independent of $\theta$ such that

$$
\begin{equation*}
\hat{J}_{\theta, \lambda}(u)<0, \quad \forall u \in \overline{E_{n} \backslash \mathcal{Z}_{r_{n}}} \tag{3.8}
\end{equation*}
$$

Set

$$
D_{n}=E_{n} \cap \mathcal{Z}_{r_{n}}, \quad G_{n}=\left\{\varphi \in C\left(D_{n}, E\right): \varphi \text { is odd and }\left.\varphi\right|_{\partial \mathcal{Z}_{r_{n}} \cap E_{n}}=i d\right\}
$$

and

$$
\Gamma_{j}=\left\{\varphi\left(\overline{D_{n} \backslash A}\right): \varphi \in G_{n}, n \geq j, A=-A \subset E_{n} \cap \mathcal{Z}_{r_{n}} \text { is closed and } \gamma(A) \leq n-j\right\}
$$

where $\gamma(\cdot)$ is the genus. Let

$$
c_{j}(\theta)=\inf _{B \in \Gamma_{j}} \sup _{u \in B} \hat{J}_{\theta, \lambda}(u), \quad j=1,2, \ldots
$$

We claim that $c_{j}(\theta)(j=1,2, \ldots)$ are critical values of $\hat{J}_{\theta, \lambda}$ and there exist $\beta_{j}>\alpha_{j}$ such that $c_{j}(\theta) \in\left[\alpha_{j}, \beta_{j}\right]$ and $\alpha_{j} \rightarrow \infty$ as $j \rightarrow \infty$.

Since $\hat{J}_{\theta, \lambda}$ is increasing with respect to $\theta$, we have $c_{j}(\theta) \leq c_{j}(1):=\beta_{j}(j=1,2, \ldots)$. And then we will estimate the lower bound for $c_{j}(\theta)$. Depending on the following Lemma 3.2, we have an intersection property: If $\rho<r_{n}$ for all $n \geq j$, then for $B \in \Gamma_{j}$, we have $B \cap \partial \mathcal{Z}_{\rho} \cap W_{j-1} \neq \varnothing$. Therefore,

$$
c_{j}(\theta) \geq \inf _{u \in \partial Z_{\rho} \cap W_{j-1}} \hat{J}_{\theta, \lambda}(u) \geq \inf _{u \in \partial \mathcal{Z}_{\rho} \cap W_{j-1}} \hat{J}_{\lambda}(u) .
$$

For small $\varepsilon>0$ and $u \in \partial \mathcal{Z}_{\rho} \cap W_{j-1}$, by $\left(h_{1}\right)$, for $0<\lambda \leq V_{0} / 4 C_{M}^{\prime}$ one has

$$
\begin{aligned}
\hat{J}_{\theta, \lambda}(u) \geq & \hat{J}_{\lambda}(u) \\
\geq & \frac{a}{2} \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x+\frac{b}{4}\left(\int_{\mathbb{R}^{N}}|\nabla u|^{2} d x\right)^{2}+\frac{1}{2} \int_{\mathbb{R}^{N}}\left(V(x) u^{2}+2 u^{2}|\nabla u|^{2}\right) d x \\
& -\lambda \int_{\mathbb{R}^{N}}\left(C_{M}^{\prime} u^{2}+C_{M}|u|^{p}\right) d x \\
\geq & \frac{1}{4} \min \{a, 1\}\|u\|_{H_{V}^{1}}^{2}+\int_{\mathbb{R}^{N}} u^{2}|\nabla u|^{2} d x-\lambda C_{M} \int_{\mathbb{R}^{N}}|u|^{p} d x \\
\geq & \frac{1}{4} \min \{a, 1\} \rho^{2}-\lambda C_{M}\|u\|_{2}^{(1-t) p}\|u\|_{22^{*}}^{t p} \\
\geq & \frac{1}{4} \min \{a, 1\} \rho^{2}-\lambda C_{M} \tilde{\zeta}_{j}^{-\frac{(1-t) p}{2}} \rho^{(1-t) p+\frac{t}{2}} \\
= & \rho^{2}\left(\frac{1}{4} \min \{a, 1\}-\lambda C_{M} \zeta_{j}^{Z_{j}^{\left(-\frac{(1-t p}{2}\right.}} \rho^{(1-t) p+\frac{p}{2}-2}\right),
\end{aligned}
$$

where $t \in(0,1)$ satisfies $\frac{1}{p}=\frac{t}{22^{*}}+\frac{1-t}{2}$. Take $\rho=\rho_{j}$ be such that $\rho_{j}^{(1-t) p+\frac{t p}{2}-2}=\frac{\min \{a, 1\}}{8 \lambda c_{M}} \xi_{j}^{(1-t) p}$. Then choosing $r_{n}>\rho_{n}$, we infer $\hat{J}_{\theta, \lambda}(u) \geq \frac{\min \{a, 1\}}{8} \rho_{j}^{2}:=\alpha_{j} \rightarrow+\infty$. Thus, $c_{j}(\theta) \in\left[\alpha_{j}, \beta_{j}\right]$ $\left(\alpha_{j} \rightarrow \infty\right.$ as $\left.j \rightarrow \infty\right)$.

Now we show that $c_{j}(\theta)(j=1,2, \ldots)$ are critical values of $\hat{J}_{\theta, \lambda}$. Indeed, if $c_{j}(\theta)$ is not a critical value of $\hat{J}_{\theta, \lambda}$, then by [16, Theorem A.4], we know that for given $0<\bar{\varepsilon}<$ $\min \left\{\alpha_{j}: j=1,2, \ldots\right\}$, there exist $\varepsilon \in(0, \bar{\varepsilon})$ and $\eta \in C([0,1] \times E, E)$ such that
(a) $\eta(t, u)=u$ for all $t \in[0,1]$ if $\hat{J}_{\theta, \lambda}(u) \notin\left[c_{j}(\theta)-\bar{\varepsilon}, c_{j}(\theta)+\bar{\varepsilon}\right]$.
(b) $\eta(t, \cdot): E \rightarrow E$ is a homeomorphism for each $t \in[0,1]$.
(c) $\eta\left(1, \hat{J}_{\theta, \lambda}^{c_{j}(\theta)+\varepsilon}\right) \subset \hat{J}_{\theta, \lambda}^{c_{j}^{( }(\theta)-\varepsilon}$, where $\hat{J}_{\theta, \lambda}^{\kappa}=\left\{u \in E: \hat{J}_{\theta, \lambda}(u) \leq \kappa\right\}$.
(d) $\eta(t, u)$ is odd in $u$.

Set $\psi=\eta(1, \cdot)$. Then, by (3.8), $\psi=i d$ on $\partial \mathcal{Z}_{r_{n}} \cap E_{n}$ for all $n$. By the definition of $c_{j}(\theta)$, there exists $B \in \Gamma_{j}$ such that

$$
\sup _{u \in B} \hat{J}_{\theta, \lambda}(u) \leq c_{j}(\theta)+\varepsilon .
$$

Notice that $A=\psi(B) \in \Gamma_{j}$. By (c), we know

$$
c_{j}(\theta) \leq \sup _{u \in A} \hat{J}_{\theta, \lambda}(u) \leq c_{j}(\theta)-\varepsilon,
$$

which is a contradiction. Hence, $c_{j}(\theta)(j=1,2, \ldots)$ are critical values of $\hat{J}_{\theta, \lambda}$. This completes the proof of Lemma 3.1.

Lemma 3.2. For $B \in \Gamma_{j}$, it follows that $B \cap \partial \mathcal{Z}_{\rho} \cap W_{j-1} \neq \varnothing$ provided $\rho<r_{n}$ for all $n \geq j$.
Proof. Set $B=\varphi\left(\overline{D_{n} \backslash A}\right)$ with $n \geq j$ and $\gamma(A) \leq n-j$. Let $\widetilde{\mathcal{X}}=\left\{u \in D_{n}: \varphi(u) \in \mathcal{Z}_{\rho}\right\}$. Then we can easily infer that 0 is an interior point of $\widetilde{\mathcal{X}}$. Let $\mathcal{X}$ be the connected component of $\widetilde{\mathcal{X}}$ containing 0 . Then $\mathcal{X}$ is a bounded symmetric neighborhood of 0 in $E_{n}$. Hence, by [16, Proposition 7.7], $\gamma(\partial \mathcal{X})=n$. Since $\left.\varphi\right|_{\partial \mathcal{Z}_{r_{n}} \cap E_{n}}=i d$, we obtain

$$
\begin{equation*}
\|\varphi(u)\|_{H_{V}^{1}}^{2}+\int_{\mathbb{R}^{N}} \varphi^{2}(u)|\nabla \varphi(u)|^{2} d x=r_{n}^{2}>\rho^{2}, \quad \forall u \in \partial \mathcal{Z}_{r_{n}} \cap E_{n} \tag{3.9}
\end{equation*}
$$

Then we get $\varphi(\partial \mathcal{X}) \subset \partial \mathcal{Z}_{\rho}$. In fact, for each $u \in \partial \mathcal{X}$, because $\varphi(u) \in \mathcal{Z}_{\rho}$, (3.9) implies that $u \in \operatorname{int}\left(\mathcal{Z}_{r_{n}}\right) \cap E_{n}$. Hence, if $\varphi(u) \in \operatorname{int}\left(\mathcal{Z}_{\rho}\right)$, then the continuity of $\varphi$ implies that there exists an open ball $B(u, r) \subset D_{n}$ centered at $u$ with radius $r$ such that $\varphi(B(u, r)) \subset \operatorname{int}\left(\mathcal{Z}_{\rho}\right)$. Since $B(u, r)$ is connected, $u \in \mathcal{X}$ and $B(u, r) \subset \mathcal{X}$, we know that $u$ is an interior point of $\mathcal{X}$. It contradicts that $u \in \partial \mathcal{X}$. Hence, $\varphi(u) \in \partial \mathcal{Z}_{\rho}$. Set $W=\left\{u \in D_{n}: \varphi(u) \in \partial \mathcal{Z}_{\rho}\right\}$. Then $\partial \mathcal{X} \subset W, \gamma(W)=n$ and $\gamma(W \backslash A) \geq n-(n-j)>j-1$. Hence [16, Proposition 7.5-2 ${ }^{0}$ ] implies $\gamma(\varphi(\overline{W \backslash A}))>j-1$. Notice that codim $\left(W_{j-1}\right)=j-1$. Consequently, $\varphi(\overline{W \backslash A}) \cap W_{j-1} \neq \varnothing$, that is to say, $B \cap \partial \mathcal{Z}_{\rho} \cap W_{j-1} \supset \varphi(\overline{W \backslash A}) \cap W_{j-1} \neq \varnothing$. The proof is finished.

Proof of Theorem 1.2. Depending on Lemma 2.3, Lemma 3.1 and Lemma 3.2, we get that the equation (2.1) has a sequence $\left\{u_{n}\right\}$ of solutions such that $\hat{J}_{\lambda}\left(u_{n}\right) \rightarrow+\infty$. Then for $\lambda$ small enough and fixed $M$, it follows from (2.11) that the equation (1.1) has a sequence $\left\{u_{n}\right\}$ of solutions such that $I_{\lambda}\left(u_{n}\right) \rightarrow+\infty$.

## References

[1] S. Bernstein, Sur une classe d'équations fonctionelles aux dérivées partielles, Bull. Acad. Sci. URSS. Sér. Math. (Izvestia Akad. Nauk SSSR) 4(1940), No. 1, 17-26. MR0002699; Zbl 0026.01901
[2] M. Colin, L. Jeanjean, Solutions for a quasilinear Schrödinger equation: a dual approach, Nonlinear Anal. 56(2004), No. 2, 213-226. https://doi.org/10.1016/j.na. 2003. 09.008; MR2029068; Zbl 1035.35038
[3] Z. Feng, X. Wu, H. Li, Multiple solutions for a modified Kirchhoff-type equation in $\mathbb{R}^{N}$, Math. Methods Appl. Sci. 38(2015), No. 4, 708-725. https://doi.org/10.1002/mma.3102; MR3310154; Zbl 1319.35040
[4] X. He, Multiplicity of solutions for a modified Schrödinger-Kirchhoff-type equation in $\mathbb{R}^{N}$, Discrete Dyn. Nat. Soc. 2015, Art. ID 179540, 9 pp. https://doi.org/10.1155/2015/ 179540; MR3407052; Zbl 1418.35151
[5] C. Huang, G. Jia, Infinitely many sign-changing solutions for modified Kirchhoff-type equations in $\mathbb{R}^{3}$, Complex Var. Elliptic Equ. 2020. https://doi.org/10.1080/17476933. 2020. 1807964
[6] G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883.
[7] Q. Li, W. Wang, K. Teng, X. Wu, Multiple solutions for a class of quasilinear Schrödinger equations, Math. Nachr. 292(2019), No. 7, 1530-1550. https://doi.org/10.1002/mana. 201700160; MR3982327; Zbl 1422.35024
[8] G. Li, C. Xiang, Nondegeneracy of positive solutions to a Kirchhoff problem with critical Sobolev growth, Appl. Math. Lett. 86(2018), 270-275. https://doi.org/10.1016/j.aml. 2018.07.010; MR3836833; Zbl 1412.35121
[9] J. L. Lions, On some questions in boundary value problems of mathematical physics, in: Contemporary Developments in Continuum Mechanics and Partial Differential Equations (Proceedings of International Symposium, Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977), North-Holland Math. Stud., Vol. 30, North-Holland, Amsterdam, 1978, pp. 284346. MR0519648; Zbl 0404.35002
[10] J. Liu, Z. Wang, Soliton solutions for quasilinear Schrödinger equations, I, Proc. Amer. Math. Soc. 131(2003), No. 2, 441-448. https://doi.org/10.1090/S0002-9939-02-067837 MR1933335; Zbl 1229.35269
[11] J. Liu, Y. Wang, Z. Wang, Soliton solutions for quasilinear Schrödinger equations, II, J. Differential Equations 187(2003), No. 2, 473-493. https://doi.org/10.1016/S00220396 (02) 00064-5; MR1949452; Zbl 1229.35268
[12] J. Liu, Y. Wang, Z. Wang, Solutions for quasilinear Schrödinger equations via the Nehari method, Comm. Partial Differential Equations 29(2004), No. 5-6, 879-901. https://doi. org/10.1081/PDE-120037335; MR2059151; Zbl 1140.35399
[13] S. Lu, An autonomous Kirchhoff-type equation with general nonlinearity in $\mathbb{R}^{N}$, Nonlinear Anal. Real World Appl. 34(2017), 233-249. https://doi.org/10.1016/j .nonrwa. 2016. 09.003; MR3567960; Zbl 1355.35058
[14] S. I. Роноžaev, A certain class of quasilinear hyperbolic equations, Mat. Sb. (N.S.) 96(1975), No. 138, 152-166. https://doi.org/10.1070/SM1975v025n01 ABEH002203; MR0369938; Zbl 0328.35060
[15] M. Poppenberg, K. Schmitt K, Z. Wang, On the existence of soliton solutions to quasilinear Schrödinger equations, Calc. Var. Partial Differential Equations 14(2002), No. 3, 329-344. https://doi.org/10.1007/s005260100105; MR1899450; Zbl 1052.35060
[16] P. H. Rabinowitz, Minimax methods in critical point theory with application to differential equations, CBMS Regional Conf. Ser. in Math., Vol. 65, American Mathematical Society, Providence, RI, 1986. https://doi.org/10.1090/cbms/065; MR0845785; Zbl 0609.58002
[17] D. Ruiz, G. Siciliano, Existence of ground states for a modified nonlinear Schrödinger equation, Nonlinearity 23(2010), No. 5, 1221-1233. https ://doi. org/10.1088/0951-7715/ 23/5/011; MR2630099; Zbl 1189.35316
[18] S. Wang, Introductions of Sobolev spaces and partial differential equations, Scientific Publishing House in China, 2009.
[19] X. Wu, Existence of nontrivial solutions and high energy solutions for Schrödinger-Kirchhoff-type equations in $\mathbb{R}^{N}$, Nonlinear Anal. Real World Appl. 12(2011), No. 2, 12781287. https://doi.org/10.1016/j.nonrwa.2010.09.023; MR2736309; Zbl 1208.35034
[20] K. Wu, X. Wu, Infinitely many small energy solutions for a modified Kirchhoff-type equation in $\mathbb{R}^{N}$, Comput. Math. Appl. 70(2015), No. 4, 592-602. https ://doi. org/10.1016/j . camwa.2015.05.014; MR3372045; Zbl 1443.35042
[21] Q. Xie, Singular perturbed Kirchhoff type problem with critical exponent, J. Math. Anal. Appl. 454(2017), No. 1, 144-180. https://doi.org/10.1016/j.jmaa.2017.04.048; MR3649848; Zbl 1367.35021
[22] W. Zou, M. Schechter, Critical point theory and its applications, Springer, New York, 2006. https://doi.org/10.1007/0-387-32968-4; MR2232879; Zbl 1125.58004

# Linear flows on compact, semisimple Lie groups: stability and periodic orbits 

Simão N. Stelmastchuk ${ }^{\boxtimes}$<br>Universidade Federal do Paraná, Jandaia do Sul, Brazil<br>Received 5 April 2021, appeared 24 October 2021<br>Communicated by Tibor Krisztin


#### Abstract

Our first purpose is to study the stability of linear flows on real, connected, compact, semisimple Lie groups. Our second purpose is to study periodic orbits of linear and invariant flows. As an application, we present periodic orbits of linear or invariant flows on $S O(3)$ and $S U(2)$ and we study periodic orbits of linear or invariant flows on $S O$ (4).


Keywords: stability, periodic orbits, linear flow, compact semisimple Lie group.
2020 Mathematics Subject Classification: 37C10, 37C75, 37C27, 22E46.

## 1 Introduction

Let $G$ be a real, connected Lie group. A vector field $\mathcal{X}$ on $G$ is called linear if its flow, which is denoted by $\varphi_{t}$, is a family of automorphisms of $G$. In this work, we assume that $G$ is a semisimple Lie group. Our wish is to study some aspects of stability of a linear flow $\varphi_{t}$ and periodic orbits of a linear or invariant flows.

Our first task is to study the stability in a fixed point of a linear flow $\varphi_{t}$. In a natural way, we follow the ideas presented in the classical literature of dynamical systems on a Euclidian space (see for instance [3], [6] and [7]). In [5], Da Silva, Santana, and Stelmastchuk show that a necessary and sufficient condition to the asymptotically and exponential stability of $\varphi_{t}$ at identity $e$ is that $\mathcal{X}$ is hyperbolic. However, if a linear vector field $\mathcal{X}$ on $G$ is hyperbolic, then $G$ is a nilpotent Lie group. Then, it is obstructed the use of hyperbolic property in the study of the stability of a linear flow on a semisimple Lie group. Thus, we choose to restrict our study to compact, semisimple Lie groups because their algebraic structure allows us to develop some results about stability.

Let $G$ be a real, connected, compact, semisimple Lie group. Consider a linear vector field $\mathcal{X}$ on $G$ and its linear flow $\varphi_{t}$. The first part of our work is about stability. We show that any fixed point of the linear flow $\varphi_{t}$ is stable (see Theorem 3.10). Furthermore, we proof that any periodic orbit of the linear flow $\varphi_{t}$ is stable (see Theorem 3.13). Also, we proof that the derivation $\mathcal{D}=-\operatorname{ad}(\mathcal{X})$ associated to $\mathcal{X}$ has only semisimple eigenvalues since the identity $e$ is stable. The last fact is the key to study periodic orbits of linear flow $\varphi_{t}$.

[^46]The second part of our work is about periodic orbits of the invariant and linear flows on compact, semisimple Lie groups. An important fact is that, in semisimple Lie groups, for any linear vector field $\mathcal{X}$ there is a right invariant vector field $X$ associated to it. Thus, our first step is to show that if the orbits of the invariant flow $\exp (t X)$ are periodic, then the orbits of the linear flow $\varphi_{t}$ are also periodic. Done this, we show that the orbits of the invariant flow $\exp (t X)$ are periodic if and only if the derivation $\mathcal{D}$ of $\mathcal{X}$ has as eigenvalues 0 or $\pm \alpha_{i}$ i with $\alpha_{i} \neq 0, i=1, \ldots, r$, and $\alpha_{i} / \alpha_{j}$ is a rational number for $i, j=1, \ldots, r$ (see Theorem 4.2). As a direct consequence, every orbit that is not a fixed point of an invariant flow $\exp (t X)$ on a 3-dimensional, compact, semisimple Lie group is periodic.

To end, we present the periodic orbits of a linear or invariant flows on $S O(3)$ and $S U(2)$, and we study the periodic orbits on $S O(4)$ (see Theorem 5.4).

This paper is organized as follows. Section 2 briefly reviews the notions of the linear vector fields. Section 3 works with stability on compact, semisimple Lie groups. Section 4 develops results in periodic orbits of a linear or invariant flows. Finally, section 5 applies the previous results on compact, semisimple Lie groups $S O(3), S U(2)$ and $S O(4)$.

## 2 Linear vector fields

Let $G$ be a connected Lie group and $\mathfrak{g}$ be its Lie algebra. We call a vector field $\mathcal{X}$ linear if its flow $\left(\varphi_{t}\right)_{t \in \mathbb{R}}$ is a family of automorphisms of the Lie group $G$. It is known that for any linear vector field $\mathcal{X}$ we can define a derivation $\mathcal{D}$ by

$$
\mathcal{D}(Y)=-[\mathcal{X}, Y], \quad Y \in \mathfrak{g} .
$$

Thus, the dynamical system

$$
\begin{equation*}
\dot{g}=\mathcal{X}(g), \quad g \in G, \tag{2.1}
\end{equation*}
$$

is associated to the derivation $\mathcal{D}$. In fact, the linearization of system above at the identity is

$$
\dot{X}=\mathcal{D}(X), \quad X \in \mathfrak{g}
$$

For the Euclidian case, if $A \in \mathbb{R}^{n \times n}$ and $b, x \in \mathbb{R}^{n}$, then $\mathcal{D}(b)(x)=[A x, b]=-A b$. Thus, we can view the dynamical system (2.1) as a generalization of dynamical system on $\mathbb{R}^{n}$ given by

$$
\dot{x}=A x .
$$

Da Silva, in [4], writes

$$
\mathfrak{g}^{+}=\bigoplus_{\alpha ; \operatorname{Re}(\alpha)>0} \mathfrak{g}_{\alpha}, \quad \mathfrak{g}^{0}=\bigoplus_{\alpha ; \operatorname{Re}(\alpha)=0} \mathfrak{g}_{\alpha}, \quad \text { and } \quad \mathfrak{g}^{-}=\bigoplus_{\alpha ; \operatorname{Re}(\alpha)<0} \mathfrak{g}_{\alpha},
$$

where $\alpha$ are eigenvalues of the derivation $\mathcal{D}$ such that

$$
\mathfrak{g}=\mathfrak{g}^{+} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{-} \quad \text { and } \quad\left[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}\right]=\mathfrak{g}_{\alpha+\beta}
$$

with $\alpha+\beta=0$ if the sum is not an eigenvalue. Let us denote by $G^{+}, G^{0}$ and $G^{-}$the Lie subgroups of the Lie algebras $\mathfrak{g}^{+}, \mathfrak{g}^{0}$ and $\mathfrak{g}^{-}$, respectively. It is simple to show that $G^{+}, G^{0}$ and $G^{+}$are $\varphi_{t}$-invariant. The Lie subgroups $G^{+}, G^{0}$ and $G^{-}$are called unstable, central and stable groups associated to $\varphi_{t}$, respectively.

For the convenience of the reader we resume some facts about a linear vector field $\mathcal{X}$ and its flow $\varphi_{t}$. The proof of these facts can be found in [2].

Proposition 2.1. Let $\mathcal{X}$ be a linear vector field, $\varphi_{t}$ be its flow, and $\mathcal{D}$ be the derivation associated to $\mathcal{X}$. The following assertions are true:
(i) $\varphi_{t}$ is an automorphism of Lie groups for each $t$;
(ii) $\mathcal{X}$ is linear iff $\mathcal{X}(g h)=R_{h *} \mathcal{X}(g)+L_{g *} \mathcal{X}(h)$ for $g, h \in G$;
(iii) $\left(d \varphi_{t}\right)_{e}=e^{t \mathcal{D}}$ for all $t \in \mathbb{R}$.

## 3 Stability of the linear flow

Let $G$ be a real, connected, semisimple Lie group and $\mathcal{X}$ be a linear vector field on $G$. In this section, our wish is to study the stability of the linear flow $\varphi_{t}$ that is the solution of the differential equation on $G$ given by

$$
\begin{equation*}
\dot{g}=\mathcal{X}(g), \quad g \in G \tag{3.1}
\end{equation*}
$$

Being $G$ semisimple, there is a right invariant vector field $X$ such that $\mathcal{X}=X+I_{*} X$, where $I_{*} X$ is the left invariant vector field associated to $X$ and $I_{*}$ is the differential of inverse map $\mathfrak{i}(g)=g^{-1}$ (more details is founded in [9]). It follows that the linear flow can be written as

$$
\varphi_{t}(g)=\exp (t X) \cdot g \cdot \exp (-t X), \quad \forall g \in G
$$

According to the above expression, we have that the identity $e$ is a fixed point for the linear flow $\varphi_{t}$. However, it may exist other fixed points.

Proposition 3.1. If a point $g$ belongs to center of the Lie group $G$, then $g$ is a fixed point of the linear flow $\varphi_{t}$.

Proof. Let $g$ be a point in the center of the Lie group $G$. Then, for all $t \in \mathbb{R}$,

$$
\varphi_{t}(g)=\exp (t X) \cdot g \cdot \exp (-t X)=\exp (t X) \cdot \exp (-t X) \cdot g=g
$$

which is the desired conclusion.
Our next step is to present the hyperbolic concept to the linear vector fields. We remember that the stability in Euclidian space is obtained if a dynamical system is hyperbolic (see for instance [7]). As one can see in [5], it is also true if $\mathcal{X}$ is hyperbolic on a Lie group $G$.
Definition 3.2. Let $\mathcal{X}$ be a linear vector field on a Lie group $G$. We call $\mathcal{X}$ hyperbolic if its associated derivation $\mathcal{D}$ is hyperbolic, that is, $\mathcal{D}$ has no eigenvalues with zero real part.

Let $\mathcal{X}$ be a hyperbolic linear vector field on a semisimple Lie group $G$. Then $\mathcal{D}$ has no eigenvalues with zero real part. Denoting by $\mathfrak{g}_{\alpha}$ the generalized eigenspace associated with an eigenvalue $\alpha$ of $\mathcal{D}$ we get

$$
\left[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}\right] \subset \mathfrak{g}_{\alpha+\beta},
$$

where $\alpha+\beta$ is an eigenvalue of $\mathcal{D}$ and zero otherwise (see for instance Proposition 3.1 in [9]). Since $\operatorname{dim} G<\infty$, it implies that the Lie algebra $\mathfrak{g}$ is nilpotent. In consequence, $G$ is nilpotent.

Proposition 3.3. There is not hyperbolic linear vector field on semisimple Lie groups.
We now begin studying the stability of linear flows on semisimple Lie groups. Firstly, we remember some concepts of stability.

Definition 3.4. Let $g \in G$ be a fixed point of the linear vector field $\mathcal{X}$. We call $g$

1) stable if for all $g$-neighborhood $U$ there is a $g$-neighborhood $V$ such that $\varphi_{t}(V) \subset U$ for all $t \geq 0$;
2) asymptotically stable if it is stable and there exists a $g$-neighborhood $W$ such that $\lim _{t \rightarrow \infty} \varphi_{t}(x)=g$ whenever $x \in W$;
3) exponentially stable if there exist $c, \mu$ and a $g$-neighborhood $W$ such that for all $x \in W$ it holds that

$$
\varrho\left(\varphi_{t}(x), g\right) \leq c \mathrm{e}^{-\mu t} \varrho(x, g), \quad \text { for all } \quad t \geq 0 ;
$$

4) unstable if it is not stable.

Since property 3) is local, it does not depend of the metric on $G$. Because of this reason, we will assume from now on that $\varrho$ is a left invariant Riemannian metric.

In order to study the stability, let us work with the Lyapunov exponents. We follow [5] in assuming that the Lyapunov exponent can be written as

$$
\lambda(e, v)=\underset{t \rightarrow \infty}{\limsup } \frac{1}{t} \log \left(\left\|\mathrm{e}^{\mathrm{t} \mathcal{D}}(\mathrm{v})\right\|\right)
$$

where $v$ is in $\mathfrak{g}$ and the norm $\|\cdot\|$ is given by the left invariant metric.
We will use $\lambda_{1}, \ldots, \lambda_{k}$ to denote $k$ distinct values of the real parts eigenvalues of the derivation $\mathcal{D}$. Then, the Lie algebra $\mathfrak{g}$ can be written as

$$
\mathfrak{g}=\bigoplus_{i=1}^{k} \mathfrak{g}_{\lambda_{i}} \quad \text { where } \mathfrak{g}_{\lambda_{i}}:=\bigoplus_{\alpha ; \operatorname{Re}(\alpha)=\lambda_{i}} \mathfrak{g}_{\alpha} .
$$

Furthermore, from Theorem 4.2 in [5] we see that

$$
\begin{equation*}
\lambda(e, v)=\lambda \quad \Leftrightarrow \quad v \in \mathfrak{g}_{\lambda}:=\bigoplus_{\alpha ; \operatorname{Re}(\alpha)=\lambda} \mathfrak{g}_{\alpha} . \tag{3.2}
\end{equation*}
$$

Using the Lyapunov exponent we show a first result about stability of linear flow $\varphi_{t}$.
Theorem 3.5. For any linear vector field $\mathcal{X}$ on a semisimple Lie group $G$, any fixed point is neither asymptotically nor exponentially stable to the linear flow $\varphi_{t}$.
Proof. We first observe that the Lyapunov exponents satisfy the following: $\lambda(g, v)=\lambda(e, v)$ for each $v \in \mathfrak{g}$. We need only consider the assertion at identity $e$. Suppose, contrary to our claim, that the identity $e$ is either asymptotically or exponentially stable. By Theorem 4.5 in [5], it follows that all Lyapunov exponents of $\mathcal{D}$ are negatives. From (3.2) it follows that any eigenvalue of $\mathcal{D}$ has the real part negative. It means that $\mathcal{X}$ is hyperbolic, and this contradicts Proposition 3.3.

Despite any fixed point is neither asymptotically nor exponentially stable, they are stable if $G$ is compact and semisimple as we will show. For this purpose, we begin by introducing an appropriate metric on $G$.

Let $G$ be a compact, semisimple Lie group. It implies that the Cartan-Killing form is negative defined. Thus we adopt the metric $\langle\cdot, \cdot\rangle$ given by negative of the Cartan-Killing form on $\mathfrak{g}$. Since $\langle\cdot, \cdot\rangle$ satisfies

$$
\langle\operatorname{Ad}(g) X, \operatorname{Ad}(g) Y\rangle=\langle X, Y\rangle, \quad \forall g \in G \text { and } X, Y \in \mathfrak{g},
$$

it follows that $\langle\cdot, \cdot\rangle$ is an invariant Riemannian metric on $G$ (see [1] for more details). From now on we make the assumption: every compact, semisimple Lie group is equipped with the Riemannian metric given by Cartan-Killing form.

Adopting these invariant metrics and using the Lyapunov exponents we obtain an algebraic characterization of linear vector fields on compact, semisimple Lie groups.

Proposition 3.6. Let $\mathcal{X}$ be a linear vector field on a compact, semisimple Lie group $G$. Then $G$ is the central group of linear flow $\varphi_{t}$.

Proof. We begin by writing $\mathcal{X}=X+I_{*}(X)$ with $X \in \mathfrak{g}$. It is clear that $\mathcal{D}=-\operatorname{Ad}(X)$. Then, for any $v \in \mathfrak{g}$ we have

$$
\left\|e^{t \mathcal{D}} v\right\|=\left\|e^{t(-\operatorname{Ad}(X))} v\right\|=\|\operatorname{Ad}(\exp (-t X)) v\|=\|v\|,
$$

where we used the Ad-invariance of metric at last equality. Thus, Lyapunov exponents can be written as

$$
\lambda(e, v)=\limsup _{t \rightarrow \infty} \frac{1}{t} \log \left(\left\|\mathrm{e}^{\mathfrak{D}} \mathrm{v}\right\|\right)=\underset{\mathrm{t} \rightarrow \infty}{\lim \sup } \frac{1}{\mathrm{t}} \log (\|\mathrm{v}\|)=0
$$

Therefore, $\lambda_{1}=\ldots=\lambda_{k}=0$. Using the relation (3.2) we conclude that $\mathfrak{g}=\mathfrak{g}_{0}$. Since $G$ is connected, $G=G_{0}$. It means that $G$ is the central group associated to the linear flow $\varphi_{t}$.

Despite Proposition above is presented in [4], we proved it because our proof is done by dynamical concepts instead of algebraic concepts.

Our next step is to show that the linear flow $\varphi_{t}$ satisfies some metric properties. Let $(M, g)$ be a Riemannian manifold, a Riemannian distance is $\rho$ associated to $g$ is defined by

$$
\rho(x, y)=\inf _{\sigma}\left\{\int_{0}^{1} g(\dot{\sigma}(s), \dot{\sigma}(s))^{1 / 2} d s\right\},
$$

where the infimum is taken over all smooth curves $\sigma$ such that $\sigma(0)=x$ and $\sigma(1)=y$.
Proposition 3.7. Let $\mathcal{X}$ be a linear vector field on a compact, semisimple Lie group $G$. Then $\varphi_{t}$ is an isometry for all $t$.

Proof. We begin writing $\mathcal{X}=X+I_{*} X$ where $X$ is a right invariant vector field. Thus for any $g, h \in G$ and $t \in \mathbb{R}$ we see that

$$
\rho\left(\varphi_{t}(g), \varphi_{t}(h)\right)=\rho\left(L_{\exp (t X)} \circ R_{\exp (-t X)}(g), L_{\exp (t X)} \circ R_{\exp (-t X)}(h)\right),
$$

where $L$ and $R$ stands for the left and right translation. Since left and right translations are isometries on $G$ to the invariant distance given by Cartan-Killing form, it follows that

$$
\rho\left(\varphi_{t}(g), \varphi_{t}(h)\right)=\rho(g, h),
$$

which shows that $\varphi_{t}$ is an isometry for any $t \in \mathbb{R}$.
Before our next result, we need to introduce some notations. For $r>0$ we will denote an sphere of radius $r$ with center $g$ by $\mathrm{S}_{r}(g)=\{x \in G: \rho(x, g)=r\}$ and an open ball of radius $r$ with center $g$ by $B_{r}(g)=\{x \in G ; \rho(x, g)<r\}$.

Proposition 3.8. If $G$ is a compact, semisimple Lie group, then for each $g \in G$ the linear flow $\varphi_{t}(g)$ is in a sphere.

Proof. We first choose an arbitrary point $g \in G$ and write $r=\rho(g, e)$. Then

$$
\rho\left(\varphi_{t}(g), e\right)=\rho\left(\varphi_{t}(g), \varphi_{t}(e)\right)=\rho(g, e)=r, \quad \forall t .
$$

It means that $\varphi_{t}(g) \in \mathrm{S}_{r}$ for all $t$, and the proof is complete.
A direct consequence of the proposition above is about $\omega$-limit and $\alpha$-limit sets.
Corollary 3.9. If $G$ is a compact, semisimple Lie group, then $\omega$-limit and $\alpha$-limit sets of $g$ are in spheres.

We can now to prove our main result of our section.
Theorem 3.10. Let $G$ be a compact, semisimple Lie Group. Then any fixed point of linear flow $\varphi_{t}$ is an stable point.

Proof. We begin by fixing an arbitrary fixed point $g$ of $G$. We also remember that a Riemannian distance induces the topology of Riemannian manifold. So it is sufficient to consider as neighborhoods of $g$ open balls $B_{r}(g)$ where $r>0$ is arbitrary. Choose $r_{0}>0$ such that $r_{0} \leq r$ and consider the ball $B_{r_{0}}(g)$. Taking any $y \in B_{r_{0}}(g)$ we see that

$$
\rho\left(\varphi_{t}(y), g\right)=\rho(y, g)<r_{0} \leq r,
$$

where we used Proposition 3.7 at first equality. It shows that $\varphi_{t}\left(B_{r_{0}}(g)\right) \subset B_{r}(g)$. Consequently, by definition, $g$ is a stable point to the linear flow $\varphi_{t}$.

Hereafter we give a characterization of derivations on compact, semisimple Lie groups. Before we need to introduce some concepts. Following [3], if for an eigenvalue $\mu$ all complex Jordan blocks are one-dimensional, i.e., a complete set of eigenvectors exists, it is called semisimple. Equivalently, the corresponding real Jordan blocks are one-dimensional if $\mu$ is real and two-dimensional if $\mu, \bar{\mu} \in \mathbb{C} \backslash \mathbb{R}$.

Theorem 3.11. On a compact, semisimple Lie group $G$, every derivation has only semisimple eigenvalues.

Proof. Let $\mathcal{D}$ be a derivation on $G$. From Theorem 3.10 we see that $e$ is a stable point of the linear flow $\varphi_{t}$ associated to $\mathcal{D}$. Since $\left(d \varphi_{t}\right)_{e}=e^{t \mathcal{D}}$, it follows that the linearization of $\dot{g}=\mathcal{X}(g)$ is $X=\mathcal{D}(X)$. Being exp a local diffeomorphism and $e$ stable, it follows that 0 is stable. From Proposition 3.6 we know that eigenvalues of $\mathcal{D}$ has real part null. Then Theorem 1.4.10 in [3] assures that every eigenvalue of $\mathcal{D}$ is semisimple, which gives the proof.

Theorem above is fundamental to study periodic orbits of linear flows.
To end this section, we study the stability of periodic orbits to the linear flows. A periodic orbit $\Gamma$ of a linear flow $\varphi_{t}$ is stable if for each open set $V$ that contains $\Gamma$, there is an open set $W \subset V$ such that every solution, starting at a point in $W$ at $t=0$, stays in $V$ for all $t \geq 0$.

Before presenting our next result, we need to introduce the following notation. Take $g \in G$ and consider the orbit $\varphi_{t}(g)$. Write for any $r>0$, $\operatorname{Tube}_{r}\left(\varphi_{t}(g)\right)=\left\{h \in G: \rho\left(h, \varphi_{t}(g)\right)<r\right.$ for some $t\}$.

Proposition 3.12. Let $\mathcal{X}$ be a linear vector field on a compact, semisimple Lie group $G$. If $h \in$ $\operatorname{Tube}_{r}\left(\varphi_{t}(g)\right)$, then $\varphi_{s}(h) \in \operatorname{Tube}_{r}\left(\varphi_{t}(g)\right)$ for any $s \in \mathbb{R}$.

Proof. Suppose that $h \in \operatorname{Tube}_{r}\left(\varphi_{t}(g)\right)$. Then for some $t$ we have $\rho\left(h, \varphi_{t}(g)\right)<r$. From Proposition 3.7 it follows for any $s \in \mathbb{R}$ that

$$
\rho\left(\varphi_{s}(h), \varphi_{t+s}(g)\right)=\rho\left(h, \varphi_{t}(g)\right)<r
$$

which implies that $\varphi_{s}(h) \in \operatorname{Tube}_{r}\left(\varphi_{t}(g)\right)$.
Theorem 3.13. Let $\mathcal{X}$ be a linear vector field on a compact, semisimple Lie group $G$. Then every periodic orbit is stable.

Proof. Let $g \in G$ such that $\varphi_{t}(g)$ is a periodic orbit of linear flow $\varphi_{t}$. We consider a open set $V$ such that $\varphi_{t}(g) \subset V$. Take $r_{0}=\inf \left\{r: B_{r}\left(\varphi_{t}(g)\right) \subset V, \forall t \geq 0\right\}$. Thus it is sufficient to take $U=$ Tube $_{r_{0}}\left(\varphi_{t}(g)\right)$ and to apply the proposition above.

## 4 Periodic orbits

In this section, we study periodic orbits of a linear flow in a compact, semisimple Lie group $G$. The key of our study is Theorem 3.11 because it describes all eigenvalues of any derivation on $G$.

We begin by recalling that a linear vector field $\mathcal{X}$ can be written as $\mathcal{X}=X+I_{*} X$, where $X$ is a right invariant vector field, $I_{*} X$ is the left invariant vector field associated to $X$, and $I_{*}$ is the differential of inverse map $\mathfrak{i}(g)=g^{-1}$. In this way, we can rewrite the differential equation (3.1) as

$$
\dot{g}=X(g)+\left(I_{*} X\right)(g) .
$$

It implies that there exists a relation between flows of the linear dynamical system $\dot{g}=\mathcal{X}(g)$ and of the invariant one $\dot{g}=X(g)$. In fact, direct accounts shows that, for all $g \in G, \varphi_{t}(g)$ is a solution of (3.1) if, and only if, $\varphi_{t}(g) \cdot \exp (t X)$ is a solution of $\dot{g}=X(g)$. It suggests us that there exists a relation between periodic orbits of the linear flow $\varphi_{t}$ and its associated invariant flow $\exp (t X)$. Therefore, our next step is to investigate this fact.

Proposition 4.1. Let $\mathcal{X}$ be a linear vector field on a compact, semisimple Lie group $G$. The following sentences are equivalent:
(i) for every $g \in G$ the invariant flow $\exp (t X) g$ is periodic;
(ii) the identity e is a periodic point of invariant flow $\exp (t X)$;
(iii) for each $g$ the point $\operatorname{Ad}(g)$ is periodic with respect to the flow $e^{t \mathcal{D}}$.

Furthermore, any assertion above implies that any point $g \in G$ is a periodic point of linear flow $\varphi_{t}$.
Proof. (i) $\Leftrightarrow$ (ii) If for every $g \in G$ the orbit $\exp (t X) g$ is periodic, then $e$ is a periodic point of the curve $\exp (t X)$. On contrary, suppose that $e$ is a periodic point of the flow $\exp (t X)$, that is, there is a $s>0$ such that $\exp ((t+s) X)=\exp (t X)$. Then for any $g \in G$

$$
\exp ((t+s) X) g=(\exp ((t+s) X) \cdot e) \cdot g=\exp (t X) g
$$

(i) $\Leftrightarrow$ (iii) Since $G$ is a semisimple Lie group, it follows

$$
\operatorname{Ad}(\exp (t X) \cdot g)=e^{t \operatorname{Ad}(X)} \operatorname{Ad}(g)=e^{t \mathcal{D}} \operatorname{Ad}(g)
$$

We thus get the equivalence.
Suppose now that $e$ is a periodic point of the flow $\exp (t X)$, then there is a $s>0$ such that $\exp (t X)=\exp ((t+s) X)$. Thus

$$
\varphi_{t+s}(g)=\exp ((t+s) X) \cdot g \cdot \exp (-(t+s) X)=\exp (t X) \cdot g \cdot \exp (-t X)=\varphi_{t}(g),
$$

which shows that $g$ is a periodic point of $\varphi_{t}$.
The interest of the proposition above is that periodic orbits of linear or invariant flows are equivalents on compact, semisimple Lie groups.

We are now in position to show our main result.
Theorem 4.2. Let $G$ be a compact, semisimple Lie group. Assume that $\mathcal{X}$ is a linear vector field on $G$, that $\mathcal{D}$ and $X$ are its associated derivation and invariant vector field, respectively. The following sentences are equivalent:
(i) there exists a periodic orbit for the right invariant flow $\exp (t X)$;
(ii) the eigenvalues of the derivation $\mathcal{D}=-\operatorname{Ad}(X)$ are the form 0 or $\pm \alpha_{1} \mathrm{i}, \ldots, \pm \alpha_{r} \mathrm{i}$ where $\alpha_{i} \neq 0$, $i=1, \ldots, r$, and $\alpha_{i} / \alpha_{j}$ is a rational for $i, j=1, \ldots, r$.

Furthermore, the sentences above implies that there exists a periodic orbit for the linear flow $\varphi_{t}$.
Proof. We first observe that (i) assures that $\varphi_{t}$ has a periodic orbit by Proposition 4.1. We are going to show that (i) is equivalent to (ii). For this, it is sufficient to consider $e$ as a periodic point to the flow $\exp (t X)$ with period $T>0$. Then for all $t \in \mathbb{R}$

$$
\exp ((t+T) X)=\exp (t X) \Leftrightarrow \exp (T X)=e \Leftrightarrow e^{T \mathcal{D}}=I d .
$$

Take the Jordan form $J$ of $\mathcal{D}$. A simple account shows that $e^{T J}=I d$. Since any eigenvalues of $\mathcal{D}$ is semisimple, its real Jordan Block has dimension 1 or 2 if it is real or complex, respectively. If 0 is eigenvalue of $\mathcal{D}$, then its real Jordan block is written as $J_{0}=[0]$. Therefore $e^{t)_{0}}$ is constant. It implies that in direction of 0 the $e^{t J}$ is constant. Consequently, solutions associated to 0 are trivially periodic. Suppose that there are non-null eigenvalues. From Proposition 3.6 these eigenvalues are of the form $\pm \alpha_{i} \mathrm{i}, i=1, \ldots, r$. By Theorem 3.11, its real Jordan blocks are

$$
\left(\begin{array}{cc}
\cos \left(t \alpha_{i}\right) & -\sin \left(t \alpha_{i}\right) \\
\sin \left(t \alpha_{i}\right) & \cos \left(t \alpha_{i}\right)
\end{array}\right), \quad i=1, \ldots r .
$$

As $e^{T J}=I d$ we have $\alpha_{i} \cdot T=p_{i} \cdot 2 \pi$ for some $p_{i} \in \mathbb{Z}, i=1, \ldots, r$. It entails for any $i, j=1, \ldots, r$ that $\alpha_{i} / \alpha_{j}=p_{i} / p_{j}$ is a rational number

Reciprocally, suppose that the eigenvalues of $\mathcal{D}$ are 0 or $\pm \alpha_{1} \mathrm{i}, \ldots, \pm \alpha_{r} \mathrm{i}$ where $\alpha_{i} \neq 0$, $i=1, \ldots, r$ and $\alpha_{i} / \alpha_{j}$ is rational for $i, j=1, \ldots n$. For the eigenvalue 0 we have that the solution is constant. We thus consider the eigenvalues $\pm \alpha_{i}$ i with $\alpha_{i} \neq 0$. Being $\pm \alpha_{i}$ i semisimple, every real Jordan block associated to it has dimension two and the solution applied at this block gives the following matrix

$$
\left(\begin{array}{cc}
\cos \left(t \alpha_{i}\right) & -\sin \left(t \alpha_{i}\right) \\
\sin \left(t \alpha_{i}\right) & \cos \left(t \alpha_{i}\right)
\end{array}\right) .
$$

By assumption, there exists $p_{i j}, q_{i j} \in \mathbb{Z}$ with $q_{i j}>0$ such that $\alpha_{i} / \alpha_{j}=p_{i j} / q_{i j}$ for $i, j=1, \ldots r$. In particular, we can written $\alpha_{i}=\left(p_{i 1} / q_{i 1}\right) \alpha_{1}$ for $i=2, \ldots, r$. Assuming that $\alpha_{1}>0$ it is sufficient to take $T=q_{21} q_{31} \ldots q_{r 1}\left(2 \pi / \alpha_{1}\right)$ to see that $J$ satisfies $e^{T J}=I d$. In other words, $I d$ is a periodic point of $e^{T J}$ with period $T>0$, which is equivalent $I d$ to be periodic point of $e^{T D}$. Consequently, by Proposition 4.1, the right invariant flow $\exp (t X)$ is periodic with period of $T>0$.

Remark 4.3. The theorem above fails if $\alpha_{i} / \alpha_{j}$ is irrational for some $i$ and $j$ in $\{1, \ldots r\}$ because the flow $e^{t J}$ in the proof of theorem above is a flow of a harmonic oscillator. The bidimensional case is treated in Section 6.2 of [6].

Corollary 4.4. Let $G$ be a compact, semisimple Lie group with dimension 3. If $X$ is a right invariant flow, then for every $g \in G$ the orbit $\exp (t X) \cdot g$ of the invariant flow is periodic. In consequence, the orbit $\varphi_{t}(g)$ of the linear flow is periodic for all $g \in G$.

Proof. It is sufficient to observe that the derivation $\mathcal{D}=-\operatorname{ad}(X)$ has only eigenvalues $0, \alpha \mathrm{i}$, and $-\alpha \mathrm{i}$ with $\alpha \in \mathbb{R}^{*}$.

## 5 Applications

In this section, our wish is to study the periodic orbits on compact, semisimple Lie groups of lower dimension. In fact, we are interested to describe the periodic orbits of linear flows on $S O(3)$ and $S U(2)$ and to study the periodic orbits of linear flows on $S O(4)$.

### 5.1 Linear flows on $S O$ (3) and $S U(2)$

Our first case is the orthogonal group

$$
S O(3)=\left\{g \in \mathbb{R}^{3 \times 3}: g g^{T}=1, \operatorname{det} g=1\right\} .
$$

It is well known that its Lie algebra is

$$
\mathfrak{s o}(3)=\left\{\left[\begin{array}{ccc}
0 & -z & y \\
z & 0 & -x \\
-y & x & 0
\end{array}\right]: x, y, z \in \mathbb{R}\right\} .
$$

Let $\mathcal{X}$ be a linear vector field on $S O(3)$. Then there exists a right invariant vector field $X$ such that $\mathcal{X}=X+I_{*} X$. A direct calculus shows that eigenvalues of $\mathcal{D}=-\operatorname{ad}(X)$ are

$$
\left\{0,-\sqrt{-x^{2}-y^{2}-z^{2}}, \sqrt{-x^{2}-y^{2}-z^{2}}\right\} .
$$

Write $\lambda_{1}=-\sqrt{-x^{2}-y^{2}-z^{2}}$ and $\lambda_{2}=\sqrt{-x^{2}-y^{2}-z^{2}}$. Using functional calculus we obtain

$$
\exp (t X)=\frac{\cosh \left(t \lambda_{1}\right)-1}{\lambda_{1}^{2}} X^{2}+\frac{\sinh \left(t \lambda_{1}\right)}{\lambda_{1}} X+I d
$$

Therefore it is possible to give the solution of linear flow $\varphi_{t}$ on $S O(3)$.
Proposition 5.1. Let $\mathcal{X}$ be a linear vector field on $S O(3)$. Then the solution of linear flow $\varphi_{t}(g)$ associated to $\mathcal{X}$ is

$$
\left(\frac{\cosh \left(t \lambda_{1}\right)-1}{\lambda_{1}^{2}} X^{2}+\frac{\sinh \left(t \lambda_{1}\right)}{\lambda_{1}} X+I d\right) \cdot g \cdot\left(\frac{\cosh \left(t \lambda_{2}\right)-1}{\lambda_{2}^{2}} X^{2}+\frac{\sinh \left(t \lambda_{2}\right)}{\lambda_{2}} X+I d\right),
$$

where X is the right invariant vector field associated to $\mathcal{X}$ and

$$
\lambda_{1}=-\sqrt{-x^{2}-y^{2}-z^{2}} \text { and } \lambda_{2}=\sqrt{-x^{2}-y^{2}-z^{2}}
$$

Corollary 4.4 now assures the characterization of periodic orbits of the linear flow $\varphi_{t}$.
Proposition 5.2. Under assumptions above,
(i) every orbit of the invariant flow $\exp (t X)$ is periodic;
(ii) every orbit of the linear flow $\varphi_{t}$ is periodic.

Our other case is the unitary group $\operatorname{SU}(2)$, which is a matrix group given by

$$
\mathrm{SU}(2)=\left\{g \in \mathbb{C}^{2 \times 2}: g g^{T}=1, \operatorname{det} g=1\right\} .
$$

The Lie algebra associated to $\operatorname{SU}(2)$ is described as

$$
\mathfrak{s u}(2)=\left\{\left[\begin{array}{cc}
\frac{i}{2} x & \frac{1}{2}(i z+y) \\
\frac{1}{2}(i z-y) & -\frac{1}{2} x
\end{array}\right]: x, y, z \in \mathbb{R}\right\} .
$$

Let $\mathcal{X}$ be a linear vector field on $S U(2)$ and $X$ the right invariant vector field associated to it. In analogous way to the case of $S O(3)$, it is easily to see that eigenvalues of a derivation $\mathcal{D}=-\operatorname{ad}(X)$ are

$$
\left\{0,-\sqrt{-x^{2}-y^{2}-z^{2}}, \sqrt{-x^{2}-y^{2}-z^{2}}\right\} .
$$

In consequence,
Proposition 5.3. Under assumptions above,
(i) every orbit of some invariant flow $\exp (t X)$ is periodic;
(ii) every orbit of some linear flow $\varphi_{t}$ is periodic.

### 5.2 Periodic orbits on $S O(4)$

In this subsection, our wish is to give a condition for the orbits of invariant or linear flow on $S O(4)$ be or not be periodic. Let $\mathfrak{s o}(4)$ be the Lie algebra of $S O(4)$ given by

$$
\left\{\left[\begin{array}{cccc}
0 & -x & -y & -z \\
x & 0 & -u & -v \\
y & u & 0 & -w \\
z & v & w & 0
\end{array}\right]: x, y, z, u, v, w \in \mathbb{R}\right\} .
$$

Consider the basis $\beta$ for $\mathfrak{s o}(4)$ that consists of $4 \times 4$ matrices $e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}$ that have 1 in the $(i, j)$ entry, -1 in the ( $j, i$ ) entry, and 0 elsewhere ( $1 \leq i<j \leq 4$ ). A computation of Lie brackets gives

$$
\begin{aligned}
& {\left[e_{12}, e_{13}\right]=e_{23}, \quad\left[e_{12}, e_{14}\right]=e_{24}, \quad\left[e_{12}, e_{23}\right]=-e_{13}, \quad\left[e_{12}, e_{24}\right]=-e_{14}, \quad\left[e_{12}, e_{34}\right]=0,} \\
& {\left[e_{13}, e_{14}\right]=e_{34}, \quad\left[e_{13}, e_{23}\right]=e_{12}, \quad\left[e_{13}, e_{24}\right]=0, \quad\left[e_{13}, e_{34}\right]=-e_{14}, \quad\left[e_{14}, e_{23}\right]=0,} \\
& {\left[e_{14}, e_{24}\right]=e_{12}, \quad\left[e_{14}, e_{34}\right]=e_{13}, \quad\left[e_{23}, e_{24}\right]=e_{34} \quad\left[e_{23}, e_{34}\right]=-e_{24}, \quad\left[e_{24}, e_{34}\right]=e_{23} .}
\end{aligned}
$$

Let $\mathcal{X}$ be a linear vector field on $S O(4)$. Let us denote by $\mathcal{D}=-\operatorname{Ad}(X)$ the associated derivation to $\mathcal{X}$ where $X$ is an right invariant vector field on $S O(4)$. Our next step is to describe the derivation $\mathcal{D}$. To do this, write

$$
X=a e_{12}+b e_{13}+c e_{14}+d e_{23}+e e_{24}+f e_{34}, \quad a, b, c, d, e, f \in \mathbb{R}
$$

By Lie brackets above, we compute

$$
\mathcal{D}=-\operatorname{ad}(X)=\left(\begin{array}{cccccc}
0 & -d & -e & b & c & 0 \\
d & 0 & -f & -a & 0 & c \\
e & f & 0 & 0 & -a & -b \\
-b & a & 0 & 0 & -f & e \\
-c & 0 & a & f & 0 & -d \\
0 & -c & b & -e & d & 0
\end{array}\right)
$$

Some calculus show that the eigenvalues of $\mathcal{D}=-\operatorname{ad}(X)$ are

$$
\left\{0,0, \pm \sqrt{-(a+f)^{2}-(b-e)^{2}-(c+d)^{2}}, \pm \sqrt{-(a+f)^{2}-(b+e)^{2}-(c-d)^{2}}\right\} .
$$

We observe that the eigenvalues are according to Theorem 3.11. We now are in a position to give a condition that characterizes periodic orbits of an invariant or linear flow.

Theorem 5.4. Let $\mathcal{X}$ be a linear vector field on $S O(4)$. Consider the derivation $\mathcal{D}=-\operatorname{ad}(X)$ of $\mathcal{X}$, where $X$ is a right invariant vector field such that

$$
X=a e_{12}+b e_{13}+c e_{14}+d e_{23}+e e_{24}+f e_{34}, \quad a, b, c, d, e, f \in \mathbb{R}
$$

A necessary and sufficient condition to every orbit that is not a fixed point of the invariant flow $\exp (t X)$ be periodic is that

$$
\begin{equation*}
\sqrt{\frac{(a+f)^{2}+(b-e)^{2}+(c+d)^{2}}{(a+f)^{2}+(b+e)^{2}+(c-d)^{2}}} \tag{5.1}
\end{equation*}
$$

is a rational number. The last condition is satisfies if be $=c d$.
Proof. It is a direct application of Theorem 4.2.
Corollary 5.5. Under conditions of Theorem above, if (5.1) is a rational number, then every orbit of the linear flow $\varphi_{t}$ associated to derivation $\mathcal{D}$ that is not a fixed point is periodic.

As a direct application of the theorem above, each right invariant vector field of the basis $\beta=\left\{e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\right\}$ yields periodic orbits for the linear or invariant flows.

## References

[1] A. Arvanitoyeorgos, An introduction to Lie groups and the geometry of homogeneous spaces, Student Mathematical Library, Vol. 22, American Mathematical Society, Providence, RI, 2003. https://doi.org/10.1090/stml/022; MR2011126; Zbl 1045.53001
[2] F. Cardetti, D. Mittenhuber, Local controllability for linear control systems on Lie groups, J. Dyn. Control Syst. 11(2005), No. 3., 353-373. https://doi.org/10.1007/ s10883-005-6584-1; MR2147190; Zbl 1085.93004
[3] F. Colonius, W. Kliemman, Dynamical systems and linear algebra, Graduate Studies in Mathematics, Vol. 158, American Mathematical Society, Providence, RI, 2014. https:// doi.org/10.1090/gsm/158; MR3242107; Zbl 1306.37001
[4] A. Da Silva, Controllability of linear systems on solvable Lie groups, SIAM J. Control Optim. 54(2016), No. 1, 372-390. https://doi.org/10.1137/140998342; MR3944253; Zbl 1440.93025
[5] A. Da Silva, A. J. Santana, S. N. Stelmastchuk, Topological conjugacy of linear systems on Lie groups, Discrete Contin. Dyn. Syst. 37(2017), No. 6, 3411-3421. https://doi.org/ 10.3934/dcds.2017144; MR3622087; Zbl 1362.37048
[6] M. W. Hirsch, S. Smale, R. L. Devaney, Differential equations, dynamical systems and an introduction to chaos, Academic Press, London, 2004. https://doi.org/10.1016/ C2009-0-61160-0; MR3293130; Zbl 1135.37002
[7] C. Robinson, Dynamical systems. Stability, symbolic dynamics, and chaos, 2nd Edition, CRC Press, London, 1999. MR1792240; Zbl 0914.58021
[8] Y. L. Sachкov, Control theory on Lie groups, J. Math. Sci. 156(2009), 381-439. https: //doi.org/10.1007/s10958-008-9275-0; MR2373391; Zbl 1211.93038
[9] L. B. A. San Martin, Álgebra de Lie (in Portuguese), Editora da Unicamp, Campinas, 2017.

# Ground state solution of a semilinear Schrödinger system with local super-quadratic conditions 

Jing Chen and Yiqing Li ${ }^{\boxtimes}$

College of Mathematics and Computing Science, Hunan University of Science and Technology, Xiangtan, Hunan 411201, P. R. China

Received 20 May 2021, appeared 28 October 2021
Communicated by Roberto Livrea


#### Abstract

This paper is dedicated to studying the following semilinear Schrödinger system $$
\begin{cases}-\Delta u+V_{1}(x) u=F_{u}(x, u, v) & \text { in } \mathbb{R}^{N}, \\ -\Delta v+V_{2}(x) v=F_{v}(x, u, v) & \text { in } \mathbb{R}^{N}, \\ u, v \in H^{1}\left(\mathbb{R}^{N}\right) & \end{cases}
$$ where the potential $V_{i}$ are periodic in $x, i=1,2$, the nonlinearity $F$ is assumed to be super-quadratic at some $x \in \mathbb{R}^{N}$ and asymptotically quadratic otherwise. Under a local super-quadratic condition of $F$, an approximation argument and variational method are used to prove the existence of Nehari-Pankov type ground state solutions and the least energy solutions.


Keywords: Schrödinger system, local super-quadratic condition, ground state solution.
2020 Mathematics Subject Classification: 35J20, 35 J61.

## 1 Introduction

We consider the following system of semilinear Schrödinger equations:

$$
\begin{cases}-\Delta u+V_{1}(x) u=F_{u}(x, u, v) & \text { in } \mathbb{R}^{N}  \tag{1.1}\\ -\Delta v+V_{2}(x) v=F_{v}(x, u, v) & \text { in } \mathbb{R}^{N} \\ u, v \in H^{1}\left(\mathbb{R}^{N}\right) & \end{cases}
$$

where $V_{1}, V_{2} \in \mathcal{C}\left(\mathbb{R}^{N}, \mathbb{R}\right), F: \mathbb{R}^{N} \times \mathbb{R}^{2} \rightarrow \mathbb{R}$ satisfy the following assumptions:
(V) $V_{1}, V_{2} \in \mathcal{C}\left(\mathbb{R}^{N}, \mathbb{R}\right)$ are 1-periodic in $x_{j}, j=1,2, \ldots, N$, and

$$
\sup \left[\sigma\left(-\Delta+V_{i}\right) \cap(-\infty, 0)\right]=: \underline{\Lambda}_{i}<0<\bar{\Lambda}_{i}:=\inf \left[\sigma\left(-\Delta+V_{i}\right) \cap(0, \infty)\right] ;
$$

[^47](F1) $F \in \mathcal{C}^{1}\left(\mathbb{R}^{N} \times \mathbb{R}^{2},[0, \infty)\right)$ and there exist constants $p \in\left(2,2^{*}\right), C_{1}>0$ such that
$$
\left|F_{z}(x, z)\right| \leq C_{1}\left(1+|z|^{p-1}\right), \quad \forall(x, z) \in \mathbb{R}^{N} \times \mathbb{R}^{2}
$$
where $F_{z}:=\left(F_{u}, F_{v}\right)=\nabla F, 2^{*}:=2 N /(N-2)$ if $N \geq 3$ and $2^{*}:=+\infty$ if $N=1$ or 2 ;
(F2) $\left|F_{z}(x, z)\right|=o(|z|)$ as $|z| \rightarrow 0$ uniformly in $x \in \mathbb{R}^{N}$.
From (V), (F1) and (F2), we can easily get that the critical points of functional $\Phi$ are the solutions of (1.1), here $\Phi$ is defined as:
\[

$$
\begin{equation*}
\Phi(z)=\frac{1}{2} \int_{\mathbb{R}^{N}}\left[|\nabla u|^{2}+V_{1}(x)|u|^{2}+|\nabla v|^{2}+V_{2}(x)|v|^{2}\right] \mathrm{d} x-\int_{\mathbb{R}^{N}} F(x, z) \mathrm{d} x, \quad z=(u, v) \in E, \tag{1.2}
\end{equation*}
$$

\]

where $E=H_{1} \times H_{2}$ is defined in Section 2 .
There is a scalar case of the Schrödinger system:

$$
\left\{\begin{array}{l}
-\Delta u+V(x) u=\nabla F(x, u), \quad x \in \mathbb{R}^{N}  \tag{1.3}\\
u \in H^{1}\left(\mathbb{R}^{N}\right)
\end{array}\right.
$$

we can easily obtain that case when $V_{1}=V_{2}$ and $u=v$. That equation has been widely studied in the literature, such as $[2,9,15,16,30,32]$.

Solution of (1.1) was related to the following system:

$$
\begin{cases}-i \frac{\partial \Psi}{\partial t}=\Delta \Psi-V_{1}(x) \Psi+F_{1}(x, \Psi), & x \in \mathbb{R}^{N}, t \geq 0 \\ -i \frac{\partial \Phi}{\partial t}=\Delta \Phi-V_{2}(x) \Phi+F_{2}(x, \Phi), & x \in \mathbb{R}^{N}, t \geq 0\end{cases}
$$

where $i$ denotes the imaginary unit, $V_{1}$ and $V_{2}$ are the relevant potentials, $\Phi$ and $\Psi$ represent the condensate wave functions. This type of Schrödinger systems arise in nonlinear optics, and have extensively been applied in many areas, such as the investigation of pulse propagation, Bose-Einstein condensates, Hartree-Fock theory for a double condensate, gap solitons in photonic crystals and so on, see as $[6,10,13,14,22,31]$. In recent years, many researchers were interested in such type of systems, we refer the readers to [1,3-7,17-20,24,25].

Manassés and João [29] investigated the existence of nontrivial solutions for the following strongly coupled system in $\mathbb{R}^{2}$ :

$$
\begin{cases}-\Delta u+V(x) u=g(x, v), & v>0 \text { in } \mathbb{R}^{2},  \tag{1.4}\\ -\Delta v+V(x) v=f(x, u), & u>0 \text { in } \mathbb{R}^{2}\end{cases}
$$

where $V: \mathbb{R}^{2} \rightarrow \mathbb{R}$ may change sign and vanish, $f, g$ are superlinear at infinity and satisfy critical or subcritical growth of Trudinger-Moser type. By using the linking geometry and a Trudinger-Moser type inequality, they obtained the boundedness of a Palais-Smale sequence, and proved there exists a subsequence that converges to a weak solution of (1.4). Finally, applying a Galerkin approximation procedure, they proved the existence of solutions in the subcritical case and critical case respectively.

Qin and Tang [23] established a nontrivial solution for the following elliptic system:

$$
\begin{cases}-\Delta u+U_{1}(x) u=F_{u}(x, u, v) & \text { in } \mathbb{R}^{N}, \\ -\Delta v+U_{2}(x) v=F_{v}(x, u, v) & \text { in } \mathbb{R}^{N}, \\ u, v \in H^{1}\left(\mathbb{R}^{N}\right), & \end{cases}
$$

where $U_{i}(x) \in \mathcal{C}\left(\mathbb{R}^{N}, \mathbb{R}\right), i=1,2, F \in \mathcal{C}^{1}\left(\mathbb{R}^{N} \times \mathbb{R}^{2}, \mathbb{R}\right)$ and $\nabla F=\left(F_{u}, F_{v}\right)$. In that paper, the authors distinguished two situations about $U_{i}$ and $F$ : periodic and asymptotically periodic case. For the periodic case, by using the diagonal method [32], the authors found a minimizing Cerami sequence outside the Nehari-Pankov manifold, then they proved the existence of the least energy solution and the ground state solution. For the latter case, by using a generalized linking theorem, they obtained a nontrivial solution. In that paper, $F$ satisfies the following super-quadratic assumption:
(SQ) $\lim _{|z| \rightarrow \infty} \frac{F(x, z)}{|z|^{2}}=\infty$ uniformly in $x$.
By using (SQ), one can prove the linking geometry, mountain pass geometry and verify the boundedness of Cerami or Palais-Smale sequence. Moreover, it is standard to show that $\mathcal{N}^{-} \neq \varnothing$, where

$$
\begin{equation*}
\mathcal{N}^{-}:=\left\{z \in E \backslash E^{-}:\left\langle\Phi^{\prime}(z), z\right\rangle=\left\langle\Phi^{\prime}(z), \zeta\right\rangle=0, \forall \zeta \in E^{-}\right\}, \tag{1.5}
\end{equation*}
$$

here $E^{-}$defined in (2.11). Introduced by Pankov [22], $\mathcal{N}^{-}$is a natural constraint and contains all nontrivial critical points of the energy functional $\Phi$, and every minimizer $u$ of $\Phi$ on the manifold $\mathcal{N}^{-}$is a solution which is called a ground state solution of Nehari-Pankov type. Also, the set $\mathcal{N}^{-}$plays a crucial role in proving the existence of the ground state solution.

Later, Tang et al. [33] investigated the existence of the ground state solutions about (1.3) under the assumptions (V), (F1), (F2) and the following assumptions:
(F3) There exists a domain $G \subset \mathbb{R}^{N}$ such that $\lim _{|z| \rightarrow \infty} \frac{F(x, z)}{|z|^{2}}=\infty$ a.e. $x \in G$.
(F4) $z \mapsto \frac{F_{z}(x, z)}{|z|}$ is non-decreasing on $|z| \neq 0$.
(F5) $\mathcal{F}(x, z):=\frac{1}{2} F_{z}(x, z) \cdot z-F(x, z) \geq 0$, and there exist some constants $C_{2}>0, R_{0}>0$ and $\sigma \in(0,1)$, such that

$$
\left(\frac{\left|F_{z}(x, z)\right|}{|z|^{\sigma}}\right)^{\kappa} \leq C_{2} \mathcal{F}(x, z), \quad \forall|z| \geq R_{0}
$$

holds with $\kappa=\frac{2 N}{2 N-(1+\sigma)(N-2)}$ if $N \geq 3$, or with $\kappa \in\left(1, \frac{2}{1-\sigma}\right)$ if $N=1,2$.
Since they relaxed condition (SQ) to the above local version (F3), it is difficult to demonstrate $\mathcal{N}^{-} \neq \varnothing$ and prove the boundedness of Cerami or Palais-Smale sequences for the energy functional $\Phi$. They use some new techniques to conquer the above difficulties. For the first one, by using linking geometry and verifying sup $\Phi(z)<\infty$ for $z \in E^{-} \oplus \mathbb{R}^{+} \bar{e}^{+}$, they illustrate that $\Phi$ is weakly upper semi-continuous, hence, they can prove that $\mathcal{N}^{-} \neq \varnothing$. For the second, they consider an approximation argument to find a minimizing sequence satisfying the PS condition for the corresponding functional. Finally, by using the uniqueness of the continuous spectrum about the operator $\mathcal{A}_{i}=-\Delta+V_{i}$, they make a contradiction to get the boundedness of the above sequence.

Recently, Qin et al. [26] proved the existence of nontrivial solutions for (1.1) by using generalized linking theorem and variational methods. More precisely, they found a Cerami sequence for the corresponding energy functional, and then proved the boundedness of the Cerami sequence. By applying linking geometry, they proved there exists a ground state solution of (1.1) with assumptions (V), (F1)-(F3). Besides, they used the following assumption to prove the boundedness of Cerami sequences:
(F6') $\mathcal{F}(x, z) \geq 0$, and there exist some constants $\tilde{C}_{1}>0, \delta_{0} \in\left(0, \Lambda_{0}\right)$ and $\sigma \in(0,1)$, such that

$$
\frac{\left|F_{z}(x, z)\right|}{|z|} \geq \frac{\sqrt{2}}{2} \tau \Longrightarrow\left(\frac{\left|F_{z}(x, z)\right|}{|z|^{\sigma}}\right)^{\kappa} \leq \tilde{C}_{1} \mathcal{F}(x, z), \quad \forall(x, z) \in \mathbb{R}^{N} \times \mathbb{R}^{2}
$$

holds with $\kappa=\frac{2 N}{2 N-(1+\sigma)(N-2)}$ if $N \geq 3$, or with $\kappa \in\left(1, \frac{2}{1-\sigma}\right)$ if $N=1,2$, where

$$
\begin{equation*}
\tau:=\Lambda_{0}-\delta_{0}, \quad \Lambda_{0}:=\min \left\{-\underline{\Lambda}_{1}, \bar{\Lambda}_{1},-\underline{\Lambda}_{2}, \bar{\Lambda}_{2}\right\} . \tag{1.6}
\end{equation*}
$$

To the best of our knowledge, there is few result about the ground state solution of system (1.1). Motivated by $[26,33]$, we aim to prove the existence of ground state solutions about system (1.1) by using approximation argument and variational method. We try to obtain the ground state solutions of Nehari-Pankov type and least energy solutions under assumptions (V), (F1)-(F5) and the following conditions:
(F6) $F(x, z) \geq 0, \mathcal{F}(x, z) \geq 0$, and there exist constants $C_{3}>0, \delta_{0} \in\left(0, \Lambda_{0}\right)$ and $\sigma \in(0,1)$, such that

$$
\frac{\left|F_{z}(x, z)\right|}{|z|} \geq \tau \Longrightarrow\left(\frac{\left|F_{z}(x, z)\right|}{|z|^{\sigma}}\right)^{\kappa} \leq C_{3} \mathcal{F}(x, z), \quad \forall(x, z) \in \mathbb{R}^{N} \times \mathbb{R}^{2}
$$

holds with $\kappa=\frac{2 N}{2 N-(1+\sigma)(N-2)}$ if $N \geq 3$, or with $\kappa \in\left(1, \frac{2}{1-\sigma}\right)$ if $N=1,2$, note that $\tau$ is the same with (1.6).

Now, we state our results of this paper.
Theorem 1.1. Let (V), (F1)-(F5) be satisfied. Then (1.1) has a Nehari-Pankov type ground state solution.

Theorem 1.2. Let (V), (F1)-(F3) and (F6) be satisfied. Then (1.1) has a least energy solution $\bar{z}$ in $K$, where $K:=\left\{z \in E \backslash\{0\}: \Phi^{\prime}(z)=0\right\}$.

There is an example to illustrate that the assumptions (F3)-(F6) can be satisfied.
Let $N \geq 3$ and $F(x, z)=\cos ^{2}\left(2 \pi x_{1}\right)|z|^{2} \ln \left(1+|z|^{2}\right)$, it is easy to verify that

$$
F_{z}(x, z)=2 z \cos ^{2}\left(2 \pi x_{1}\right)\left[\ln \left(1+|z|^{2}\right)+\frac{|z|^{2}}{1+|z|^{2}}\right]
$$

and

$$
\mathcal{F}(x, z)=\frac{\cos \left(2 \pi x_{1}\right)|z|^{4}}{1+|z|^{2}} \geq 0 .
$$

It is clear that $F$ satisfies (F1)-(F6) with $G=\left(-\frac{1}{8}, \frac{1}{8}\right) \times \mathbb{R}^{N-1}$, but does not satisfy (SQ).
Remark 1.3. Assume that (F1), (F2), (F4) and (F5) hold. Then (F6) holds also. See as [33, Lemma 3.8]. Moreover, (F6') implies (F6).

To prove the existence of ground state solutions about (1.1), at first, we show that $\mathcal{N}^{-} \neq \varnothing$. Inspired by Tang [33], we consider an approximation argument about the auxiliary functionals $I_{\epsilon}(z)=\Phi(z)-\epsilon \int_{\mathbb{R}^{N}}|z|^{p} \mathrm{~d} x$, which makes the corresponding problem superlinear in $\mathbb{R}^{N}$. Moreover, by demonstrating a key inequality (3.3) and using $\mathcal{N}^{-} \neq \varnothing$, we prove that $I_{\epsilon_{n}}\left(z_{\epsilon_{n}}\right)$ is bounded and $I_{\epsilon_{n}}^{\prime}\left(z_{\epsilon_{n}}\right)=0$, here $\epsilon_{n} \rightarrow 0$ as $n \rightarrow \infty$. Finally, by using Sobolev embedding theorem and Lion's concentration compactness principle, we prove the sequence $\left\{z_{\epsilon_{n}}\right\}$ is bounded, then we can get that $\left\{z_{\epsilon_{n}}\right\}$ is convergent to a solution of (1.1).

The reminder of this paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, we give the proof of Theorem 1.1 and Theorem 1.2. For convenience, let $C_{0}, \tilde{C}_{0}, C_{1}, \tilde{C}_{1}, \ldots$ denote different constants in different places.

## 2 Preliminaries

Let $\mathcal{A}_{i}=-\Delta+V_{i}$, here and in what follows $i=1,2$. Then $\mathcal{A}_{i}$ are self-adjoint in $L^{2}\left(\mathbb{R}^{N}\right)$ with domain $\mathfrak{D}\left(\mathcal{A}_{i}\right)=H^{2}\left(\mathbb{R}^{N}\right)$ (see [12, Theorem 4.26]). Let $\left\{\mathcal{E}_{i}(\lambda):-\infty \leq \lambda \leq+\infty\right\}$ and $\left|\mathcal{A}_{i}\right|$ be the spectral family and the absolute value of $\mathcal{A}_{i}$, respectively, and $\left|\mathcal{A}_{i}\right|^{1 / 2}$ be the square root of $\left|\mathcal{A}_{i}\right|$. Set $\mathcal{U}_{i}=i d-\mathcal{E}_{i}(0)-\mathcal{E}_{i}(0-)$. Then $\mathcal{U}_{i}$ commutes with $\mathcal{A}_{i},\left|\mathcal{A}_{i}\right|$ and $\left|\mathcal{A}_{i}\right|^{1 / 2}$. Furthermore, $\mathcal{A}_{i}=\mathcal{U}_{i}\left|\mathcal{A}_{i}\right|$ is the polar decomposition of $\mathcal{A}_{i}$ (see [11, Theorem IV 3.3]). Let

$$
H_{i}=\mathfrak{D}\left(\left|\mathcal{A}_{i}\right|^{1 / 2}\right), \quad H_{i}^{-}=\mathcal{E}_{i}(0-) H_{i}, \quad H_{i}^{+}=\left[i d-\mathcal{E}_{i}(0)\right] H_{i} .
$$

For any $u_{i} \in H_{i}$, fixing $i=1$ or $i=2$, it is easy to see that $u_{i}=u_{i}{ }^{-}+u_{i}{ }^{+}$with

$$
\begin{equation*}
u_{i}^{-}:=\mathcal{E}_{i}(0-) u_{i} \in H_{i}^{-}, \quad u_{i}^{+}:=\left[i d-\mathcal{E}_{i}(0)\right] u_{i} \in H_{i}^{+} \tag{2.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{A}_{i} u_{i}^{-}=-\left|\mathcal{A}_{i}\right| u_{i}^{-}, \quad \mathcal{A}_{i} u_{i}^{+}=\left|\mathcal{A}_{i}\right| u_{i}^{+}, \quad \forall u_{i}=u_{i}^{-}+u_{i}^{+} \in H_{i} \cap \mathfrak{D}\left(\mathcal{A}_{i}\right) . \tag{2.2}
\end{equation*}
$$

For fixed $i$ taking 1 or 2 , we define an inner product

$$
\begin{equation*}
(u, v)_{H_{i}}=\left(\left|\mathcal{A}_{i}\right|^{1 / 2} u,\left|\mathcal{A}_{i}\right|^{1 / 2} v\right)_{L^{2}}, \quad u, v \in H_{i} \tag{2.3}
\end{equation*}
$$

and the corresponding norm

$$
\|u\|_{H_{i}}=\left\|\left|\mathcal{A}_{i}\right|^{1 / 2} u\right\|_{L^{2}}, \quad u \in H_{i},
$$

where $(\cdot, \cdot)_{L^{2}}$ denotes the inner product of $L^{2}\left(\mathbb{R}^{N}\right),\|\cdot\|_{L^{s}}$ stands for the usual $L^{s}\left(\mathbb{R}^{N}\right)$ norm, $1 \leq s<\infty$. There are induced decompositions $H_{i}=H_{i}^{-} \oplus H_{i}^{+}$which are orthogonal with respect to both $(\cdot, \cdot)_{L^{2}}$ and $(\cdot, \cdot)_{H_{i}}$. Then

$$
\int_{\mathbb{R}^{N}}\left(\left|\nabla u_{i}\right|^{2}+V_{i}(x)\left|u_{i}\right|^{2}\right) \mathrm{d} x=\left\|u_{i}^{+}\right\|_{H_{i}}^{2}-\left\|u_{i}^{-}\right\|_{H_{i^{\prime}}}^{2} \quad \forall u_{i}=u_{i}^{-}+u_{i}^{+} \in H_{i}, \quad i=1,2 .
$$

Under condition (V), $H_{i}^{-} \oplus H_{i}^{+}=H_{i}=H^{1}\left(\mathbb{R}^{N}\right)$ with equivalent norms. Therefore, $H_{i}$ embeds continuously in $L^{s}\left(\mathbb{R}^{N}\right)$ for all $2 \leq s<2^{*}$. Then, there exists a constant $\gamma_{s}>0$ such that

$$
\begin{equation*}
\|z\|_{s} \leq \gamma_{s}\|z\|, \quad \forall z \in E, s \in\left[2,2^{*}\right] \tag{2.4}
\end{equation*}
$$

where $\|\cdot\|_{s}$ stands for the usual $L^{s}\left(\mathbb{R}^{N}, \mathbb{R}^{2}\right)$ norm.
Let

$$
\begin{equation*}
E=H_{1} \times H_{2} \tag{2.5}
\end{equation*}
$$

equipped with the inner product

$$
\begin{equation*}
\langle z, \xi\rangle=(u, \chi)_{H_{1}}+(v, \psi)_{H_{2}}, \quad z=(u, v), \xi=(\chi, \psi) \in E=H_{1} \times H_{2} \tag{2.6}
\end{equation*}
$$

and the corresponding norm

$$
\begin{equation*}
\|z\|=\left[\|u\|_{H_{1}}^{2}+\|v\|_{H_{2}}^{2}\right]^{1 / 2}, \quad z=(u, v) \in E . \tag{2.7}
\end{equation*}
$$

For any $\varepsilon>0$, (F1) and (F2) yield the existence of $C_{\varepsilon}>0$ such that

$$
\begin{equation*}
\left|F_{z}(x, z)\right| \leq \varepsilon|z|+C_{\varepsilon}|z|^{p-1}, \quad \forall(x, z) \in \mathbb{R}^{N} \times \mathbb{R}^{2} . \tag{2.8}
\end{equation*}
$$

Under (V), a standard argument (see [8,36]) shows that the solutions of problem (1.1) are critical points of the functional

$$
\begin{equation*}
\Phi(z)=\frac{1}{2} \int_{\mathbb{R}^{N}}\left[|\nabla u|^{2}+V_{1}(x)|u|^{2}+|\nabla v|^{2}+V_{2}(x)|v|^{2}\right] \mathrm{d} x-\int_{\mathbb{R}^{N}} F(x, z) \mathrm{d} x, \quad z=(u, v) \in E, \tag{2.9}
\end{equation*}
$$

$\Phi$ is of class $\mathcal{C}^{1}(E, \mathbb{R})$, and

$$
\begin{align*}
\left\langle\Phi^{\prime}(z), \xi\right\rangle= & \int_{\mathbb{R}^{N}}\left(\nabla u \nabla \chi+V_{1}(x) u \chi\right) \mathrm{d} x+\int_{\mathbb{R}^{N}}\left(\nabla v \nabla \psi+V_{2}(x) v \psi\right) \mathrm{d} x \\
& -\int_{\mathbb{R}^{N}}\left(F_{u}(x, z) \chi+F_{v}(x, z) \psi\right) \mathrm{d} x, \quad \forall z=(u, v), \xi=(\chi, \psi) \in E . \tag{2.10}
\end{align*}
$$

Let

$$
\begin{equation*}
E^{+}=H_{1}^{+} \times H_{2}^{+}, \quad E^{-}=H_{1}^{-} \times H_{2}^{-}, \tag{2.11}
\end{equation*}
$$

then for any $z=(u, v) \in E$, (2.1) yields $z=z^{+}+z^{-}$with the corresponding summands

$$
\begin{equation*}
z^{+}=\left(u^{+}, v^{+}\right) \in E^{+}, \quad z^{-}=\left(u^{-}, v^{-}\right) \in E^{-} . \tag{2.12}
\end{equation*}
$$

Moreover, $E^{+}$and $E^{-}$are orthogonal with respect to the inner products $\langle\cdot, \cdot\rangle$ and $(\cdot, \cdot)_{2}$, where $(\cdot, \cdot)_{2}$ is chosen by $((u, v),(\chi, \psi))_{2}=(u, \chi)_{L^{2}}+(v, \psi)_{L^{2}}$ for any $(u, v),(\chi, \psi) \in L^{2}\left(\mathbb{R}^{N}, \mathbb{R}^{2}\right)$. Hence

$$
E=E^{+} \oplus E^{-} .
$$

It follows from (2.2), (2.3), (2.6) and (2.12) that

$$
\begin{align*}
\int_{\mathbb{R}^{N}} & {\left[\nabla u \nabla \chi+V_{1}(x) u \chi+\nabla v \nabla \psi+V_{2}(x) v \psi\right] \mathrm{d} x } \\
& =\left(\mathcal{A}_{1} u, \chi\right)_{L^{2}}+\left(\mathcal{A}_{2} v, \psi\right)_{L^{2}} \\
& =\left(u_{1}^{+}, \chi_{1}^{+}\right)_{H_{1}}+\left(v_{2}^{+}, \psi_{2}^{+}\right)_{H_{2}}-\left(u_{1}^{-}, \chi_{1}^{-}\right)_{H_{1}}-\left(v_{2}^{-}, \psi_{2}^{-}\right)_{H_{2}} \\
& =\left\langle z^{+}, \xi^{+}\right\rangle-\left\langle z^{-}, \xi^{-}\right\rangle, \quad \forall z=(u, v), \xi=(\chi, \psi) \in E . \tag{2.13}
\end{align*}
$$

and

$$
\begin{equation*}
\int_{\mathbb{R}^{N}}\left[|\nabla u|^{2}+V_{1}(x)|u|^{2}+|\nabla v|^{2}+V_{2}(x)|v|^{2}\right] \mathrm{d} x=\left\|z^{+}\right\|^{2}-\left\|z^{-}\right\|^{2}, \quad \forall z=(u, v) \in E \tag{2.14}
\end{equation*}
$$

Lemma 2.1. Assume that (V), (F1), (F2) and (F4) hold. Then there exists $\rho>0$ such that

$$
\begin{equation*}
\inf \left\{\Phi(z): z \in E^{+},\|z\|=\rho\right\}>0 \tag{2.15}
\end{equation*}
$$

We omit the proof here since it is standard.
Suppose that $G \in \mathbb{R}^{N}$ is a bounded domain. We can choose $\bar{e}:=\left(\bar{e}_{u}, \bar{e}_{v}\right) \in \mathcal{C}_{0}^{\infty}\left(\mathbb{R}^{N}, \mathbb{R}^{+}\right) \cap$ $\mathcal{C}_{0}^{\infty}\left(G, \mathbb{R}^{+}\right)$satisfying

$$
\begin{aligned}
\left\|\bar{e}^{+}\right\|^{2}-\left\|\bar{e}^{-}\right\|^{2} & =\int_{\mathbb{R}^{N}}\left[\left|\nabla \bar{e}_{u}\right|^{2}+V_{1}(x)\left|\bar{e}_{u}\right|^{2}+\left|\nabla \bar{e}_{v}\right|^{2}+V_{2}(x)\left|\bar{e}_{v}\right|^{2}\right] \mathrm{d} x \\
& =\int_{G}\left[\left|\nabla \bar{e}_{u}\right|^{2}+V_{1}(x)\left|\bar{e}_{u}\right|^{2}+\left|\nabla \bar{e}_{v}\right|^{2}+V_{2}(x)\left|\bar{e}_{v}\right|^{2}\right] \mathrm{d} x \geq 1,
\end{aligned}
$$

then $\bar{e}^{+}=\left(\bar{e}_{u}^{+}, \bar{e}_{v}^{+}\right) \neq(0,0)$.
Owing to prove $\mathcal{N}^{-} \neq \varnothing$, we also need the following lemma.

Lemma 2.2. Assume that (V), (F1), (F2) and (F5) hold. Then $\sup \Phi\left(E^{-} \oplus \mathbb{R}^{+} \bar{e}^{+}\right)<\infty$ and there is $R_{\bar{e}}>0$ such that

$$
\begin{equation*}
\Phi(z) \leq 0, \quad \text { for } z \in E^{-} \oplus \mathbb{R}^{+} \bar{e}^{+} \text {with }\|z\| \geq R_{\bar{e}} . \tag{2.16}
\end{equation*}
$$

Proof. As the ideal of [34, Lemma 3.2 and Corollary 3.3], we can prove Lemma 2.2 by verifying that there is $r>\rho$ such that $\sup \Phi(\partial Q) \leq 0$, where $Q=\left\{w+s e^{+}: w \in E^{-}, s \geq 0\right.$, $\left.\left\|w+s e^{+}\right\| \leq r\right\}$.

Lemma 2.3. Assume that (V), (F1), (F2) and (F5) hold. Then $\mathcal{N}^{-} \neq \varnothing$.
Proof. From Lemma 2.1, $\Phi\left(t \bar{e}^{+}\right)>0$ for small $t>0$. Moreover, by Lemma 2.2, there exists $R_{\bar{e}}>0$ such that $\Phi(z) \leq 0$ for $z \in\left(E^{-} \oplus \mathbb{R}^{+} \bar{e}^{+}\right) \backslash B_{R_{\bar{e}}}(0)$. Since that, $0<\sup \Phi\left(E^{-} \oplus \mathbb{R}^{+} \bar{e}^{+}\right)<$ $\infty$. Hence, we can easily get that $\Phi$ is weakly upper semi-continuous on $E^{-} \oplus \mathbb{R}^{+} \bar{e}^{+}$. Then, there exists $z_{0} \in E^{-} \oplus \mathbb{R}^{+} \bar{e}^{+}$such that $\Phi\left(z_{0}\right)=\sup \Phi\left(E^{-} \oplus \mathbb{R}^{+} \bar{e}^{+}\right)$. It is obvious that $z_{0}$ is a critical point of $\Phi$, that is $\left\langle\Phi^{\prime}\left(z_{0}\right), z_{0}\right\rangle=\left\langle\Phi^{\prime}\left(z_{0}\right), \zeta\right\rangle=0$ for all $\zeta \in E^{-} \oplus \mathbb{R}^{+} \bar{e}^{+}$. Therefore, $z_{0} \in \mathcal{N}^{-} \cap\left(E^{-} \oplus \mathbb{R}^{+} \bar{e}^{+}\right)$.

## 3 The existence of ground state solutions

To prove Theorem 1.1 and Theorem 1.2, we define $I_{\epsilon}(z)$ for any $\epsilon \geq 0$ as follows:

$$
\begin{equation*}
I_{\epsilon}(z)=\Phi(z)-\epsilon \int_{\mathbb{R}^{N}}|z|^{p} \mathrm{~d} x . \tag{3.1}
\end{equation*}
$$

Let

$$
\begin{equation*}
\mathcal{N}_{\epsilon}^{-}=\left\{z \in E \backslash E^{-}:\left\langle I_{\epsilon}^{\prime}(z), z\right\rangle=\left\langle I_{\epsilon}^{\prime}(z), \zeta\right\rangle=0, \quad \forall \zeta \in E^{-}\right\} . \tag{3.2}
\end{equation*}
$$

Similar to Lemma 2.3, for $\epsilon \geq 0$, we have $\mathcal{N}_{\epsilon}^{-} \neq \varnothing$. Then we define $m_{\epsilon}:=\inf _{\mathcal{N}_{\epsilon}^{-}} I_{\epsilon}$.
Lemma 3.1. Assume that (V), (F1), (F2) and (F4) hold. Then

$$
\begin{equation*}
I_{\epsilon}(z) \geq I_{\epsilon}(t z+\zeta)+\frac{1}{2}\|\zeta\|^{2}+\frac{1-t^{2}}{2}\left\langle I_{\epsilon}^{\prime}(z), z\right\rangle-t\left\langle I_{\epsilon}^{\prime}(z), \zeta\right\rangle, \quad \forall t \geq 0, z \in E, \zeta \in E^{-} . \tag{3.3}
\end{equation*}
$$

Proof. From (2.9), (2.10) and (3.1), we have

$$
\begin{align*}
I_{\epsilon}(z)- & I_{\epsilon}(t z+\zeta) \\
= & \frac{1}{2}\left\|z^{+}\right\|^{2}-\frac{1}{2}\left\|z^{-}\right\|^{2}-\int_{\mathbb{R}^{N}} F(x, z) d x-\epsilon \int_{\mathbb{R}^{N}}|z|^{p} \mathrm{~d} x \\
& -\frac{t^{2}}{2}\left\|z^{+}\right\|^{2}+\frac{1}{2}\left\langle t z^{-}+\zeta, t z^{-}+\zeta\right\rangle+\int_{\mathbb{R}^{N}} F(x, t z+\zeta) d x-\epsilon \int_{\mathbb{R}^{N}}|t z+\zeta|^{p} \mathrm{~d} x \\
= & \frac{1}{2}\|\zeta\|^{2}+\frac{1-t^{2}}{2}\left\langle I_{\epsilon}^{\prime}(z), z\right\rangle-t\left\langle I_{\epsilon}^{\prime}(z), \zeta\right\rangle \\
& +\frac{1-t^{2}}{2} \int_{\mathbb{R}^{N}} F_{z}(x, z) \cdot z d x-t \int_{\mathbb{R}^{N}} F_{z}(x, z) \cdot \zeta d x+\int_{\mathbb{R}^{N}} F(x, t z+\zeta) d x-\int_{\mathbb{R}^{N}} F(x, z) d x \\
& +\frac{1-t^{2}}{2} p \epsilon \int_{\mathbb{R}^{N}}|z|^{p} \mathrm{~d} x-\epsilon \int_{\mathbb{R}^{N}}|z|^{p} \mathrm{~d} x+\epsilon \int_{\mathbb{R}^{N}}|t z+\zeta|^{p} \mathrm{~d} x-t p \epsilon \int_{\mathbb{R}^{N}}|z|^{p-2} z \cdot \zeta \mathrm{~d} x . \tag{3.4}
\end{align*}
$$

From [35, Lemma 4.3], one has

$$
\begin{equation*}
\frac{1-t^{2}}{2} F_{z}(x, z) z-t F_{z}(x, z) \zeta+F(x, t z+\zeta)-F(x, z) \geq 0, \quad \forall z \in E, \zeta \in E^{-}, t \geq 0 \tag{3.5}
\end{equation*}
$$

As in [28, Remark 6], we can get that

$$
\begin{equation*}
\frac{1-t^{2}}{2} p|z|^{p}-|z|^{p}+|t z+\zeta|^{p}-t p|z|^{p-2} z \cdot \zeta \geq 0, \quad \forall z \in E, \zeta \in E^{-}, t \geq 0 \tag{3.6}
\end{equation*}
$$

Then, from (3.4), (3.5) and (3.6), we have

$$
I_{\epsilon}(z)-I_{\epsilon}(t z+\zeta) \geq \frac{1}{2}\|\zeta\|^{2}+\frac{1-t^{2}}{2}\left\langle I_{\epsilon}^{\prime}(z), z\right\rangle-t\left\langle I_{\epsilon}^{\prime}(z), \zeta\right\rangle .
$$

The proof is completed.
From the above lemma, we can get the following two corollaries.
Corollary 3.2. Assume that (V), (F1), (F2) and (F4) hold. Then for $z \in \mathcal{N}_{\epsilon}^{-}$,

$$
\begin{equation*}
I_{\epsilon}(z) \geq I_{\epsilon}(t z+\zeta), \quad \forall t \geq 0, \zeta \in E^{-} . \tag{3.7}
\end{equation*}
$$

Corollary 3.3. Assume that (V), (F1), (F2) and (F4) hold. Then
$I_{\epsilon}(z) \geq \frac{t^{2}}{2}\|z\|^{2}-\int_{\mathbb{R}^{N}}\left[F\left(x, t z^{+}\right)+\epsilon\left|t z^{+}\right|^{p}\right] \mathrm{d} x+\frac{1-t^{2}}{2}\left\langle I_{\epsilon}^{\prime}(z), z\right\rangle+t^{2}\left\langle I_{\epsilon}^{\prime}(z), z^{-}\right\rangle, \quad \forall t \geq 0, z \in E$.

Lemma 3.4. Assume that (V), (F1), (F2) and (F4) hold. Then, for $\in \in[0,1]$,
(i) there exists $\hat{\kappa}>0$ which does not depend on $\epsilon \in[0,1]$ such that

$$
\begin{equation*}
I_{\epsilon}(z) \geq m_{\epsilon} \geq \hat{\kappa}, \quad \forall z \in \mathcal{N}_{\epsilon}^{-} ; \tag{3.9}
\end{equation*}
$$

(ii) $\left\|z^{+}\right\| \geq \max \left\{\left\|z^{-}\right\|, \sqrt{2 m_{\epsilon}}\right\}$ for all $z \in \mathcal{N}_{\epsilon}^{-}$.

Proof. (i) By (F1) and (F2), there exists a constant $C_{4}>0$ such that

$$
\begin{equation*}
F(x, z)+\epsilon|z|^{p} \leq \frac{1}{4 \gamma_{2}^{2}}|z|^{2}+C_{4}|z|^{p}, \quad \forall x \in \mathbb{R}^{N}, z \in \mathbb{R}^{2}, \epsilon \in[0,1] . \tag{3.10}
\end{equation*}
$$

In virtue of (2.4), (3.1), (3.7) and (3.10), one has

$$
\begin{align*}
I_{\epsilon}(z) & \geq I_{\epsilon}\left(t z^{+}\right)=\frac{t^{2}}{2}\left\|z^{+}\right\|^{2}-\int_{\mathbb{R}^{N}}\left[F\left(x, t z^{+}\right)+\epsilon\left|t z^{+}\right|^{p}\right] \mathrm{d} x \\
& \geq \frac{t^{2}}{4}\left\|z^{+}\right\|^{2}-t^{p} C_{4}\left\|z^{+}\right\|_{p}^{p} \\
& \geq \frac{t^{2}}{4}\left\|z^{+}\right\|^{2}-t^{p} C_{4} \gamma_{p}^{p}\left\|z^{+}\right\|^{p}, \quad \forall z \in \mathcal{N}_{\epsilon}^{-}, \epsilon \in[0,1], t \geq 0 . \tag{3.11}
\end{align*}
$$

Choose $t=t_{z}:=\frac{1}{\left[2 \mathrm{C}_{4} \gamma_{p}^{p}\right]^{\frac{1}{p-2}}\left\|z^{+}\right\|}$, then it follows from above inequality that

$$
\begin{align*}
I_{\epsilon}(z) & \geq \frac{t_{z}^{2}}{4}\left\|z^{+}\right\|^{2}-t_{z}^{p} C_{4} \gamma_{p}^{p}\left\|z^{+}\right\|^{p} \\
& =\frac{p-2}{4 p\left[2 C_{4} \gamma_{p}^{p} p\right]^{\frac{2}{p-2}}}=: \hat{\kappa}>0, \quad \forall \epsilon \in[0,1], z \in \mathcal{N}_{\epsilon}^{-} . \tag{3.12}
\end{align*}
$$

Hence, (3.9) holds.
(ii) (F4) shows that $F(x, z) \geq 0$. Then, it follows from (3.1), (3.2) and (3.9) that (ii) holds.

Lemma 3.5. Assume that (V), (F1), (F2) and (F4) hold. Then for any $\epsilon \in(0,1]$, there exists $z_{\epsilon} \in \mathcal{N}_{\epsilon}^{-}$ such that

$$
\begin{equation*}
I_{\epsilon}\left(z_{\epsilon}\right)=m_{\epsilon}, \quad I_{\epsilon}^{\prime}\left(z_{\epsilon}\right)=0 \tag{3.13}
\end{equation*}
$$

Proof. By virtue of [26, Lemma 4.2 and Lemma 4.3], we can get that there exists a bounded sequence $\left\{z_{\varepsilon_{n}}\right\} \in E$ such that

$$
\begin{equation*}
I_{\epsilon}\left(z_{\epsilon_{n}}\right) \rightarrow c, \quad\left\|I_{\epsilon}^{\prime}\left(z_{\epsilon_{n}}\right)\right\|\left(1+\left\|z_{\epsilon_{n}}\right\|\right) \rightarrow 0, \quad n \rightarrow \infty \tag{3.14}
\end{equation*}
$$

where $c \in\left[\hat{\kappa}, m_{\epsilon}\right]$. Hence, there exists a constant $\tilde{C}_{2}>0$ such that $\left\|z_{\epsilon_{n}}\right\|_{2} \leq \tilde{C}_{2}$. If

$$
\delta:=\limsup _{n \rightarrow \infty} \sup _{y \in \mathbb{R}^{N}} \int_{B_{1}(y)}\left|z_{\epsilon_{n}}\right|^{2} \mathrm{~d} x=0,
$$

applying Lion's concentration compactness principle [36, Lemma 1.21], $z_{\epsilon_{n}} \rightarrow 0$ in $L^{s}\left(\mathbb{R}^{N}\right)$ for $2<s<2^{*}$. By (F1) and (F2), for $\epsilon=\frac{c}{4 \tilde{C}_{2}^{2}}>0$, there exists $\tilde{C}_{\epsilon}>0$ such that

$$
\begin{aligned}
\left|F_{z}(x, z)\right| & \leq \epsilon|z|+\tilde{C}_{\epsilon}|z|^{p-1}, \\
|F(x, z)| & \leq \epsilon|z|^{2}+\tilde{C}_{\epsilon}|z|^{p}, \quad \forall(x, z) \in \mathbb{R}^{N} \times \mathbb{R}^{2} .
\end{aligned}
$$

Thus,

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \int_{\mathbb{R}^{N}}\left[\mathcal{F}\left(x, z_{\varepsilon_{n}}\right)+\frac{p-2}{2} \epsilon_{n}\left|z_{\epsilon_{n}}\right|^{p}\right] \mathrm{d} x \leq \frac{3}{2} \epsilon \tilde{C}_{2}^{2}+\left(\frac{3}{2} \tilde{C}_{\epsilon}+\tilde{C}_{3}\right) \lim _{n \rightarrow \infty}\left\|z_{\varepsilon_{n}}\right\|_{p}^{p}=\frac{3}{8} c . \tag{3.15}
\end{equation*}
$$

From (3.1), (3.14) and (3.15), one has

$$
\begin{aligned}
c & =I_{\epsilon_{n}}\left(z_{\epsilon_{n}}\right)-\frac{1}{2}\left\langle I_{\epsilon_{n}}^{\prime}\left(z_{\epsilon_{n}}\right), z_{\epsilon_{n}}\right\rangle+o(1) \\
& =\int_{\mathbb{R}^{N}}\left[\mathcal{F}\left(x, z_{\epsilon_{n}}\right)+\frac{p-2}{2} \epsilon_{n}\left|z_{\epsilon_{n}}\right|^{p}\right] \mathrm{d} x+o(1) \\
& \leq \frac{3}{8} c+o(1) .
\end{aligned}
$$

That is a contradiction, so we have $\delta>0$.
Going if necessary to a subsequence, we may assume there exists $k_{n} \in \mathbb{Z}^{N}$ such that

$$
\int_{B_{1+\sqrt{N}}\left(k_{n}\right)}\left|z_{n}\right|^{2} \mathrm{~d} x>\frac{\delta}{2} .
$$

Define $w_{n}(x):=z_{n}\left(x+k_{n}\right)$ such that

$$
\begin{equation*}
\int_{B_{1+\sqrt{N}}(0)}\left|w_{n}\right|^{2} \mathrm{~d} x>\frac{\delta}{2} . \tag{3.16}
\end{equation*}
$$

In view of $V_{i}(x)$ and $F_{z}(x, z)$ are periodic on $x, i=1,2$, we have $\left\|w_{n}\right\|=\left\|z_{n}\right\|$ and

$$
\begin{equation*}
I_{\epsilon_{n}}\left(w_{n}\right) \rightarrow c, \quad\left\|I_{\epsilon_{n}}^{\prime}\left(w_{n}\right)\right\|\left(1+\left\|w_{n}\right\|\right) \rightarrow 0 \tag{3.17}
\end{equation*}
$$

Going if necessary to a subsequence, we have $w_{n} \rightharpoonup \bar{w}$ in $E, w_{n} \rightarrow \bar{w}$ in $L_{l o c}^{s}\left(\mathbb{R}^{N}\right), 2<s<2^{*}$ and $w_{n} \rightarrow \bar{w}$ a.e. on $\mathbb{R}^{N}$. Obviously, (3.16) implies that $\bar{w} \neq 0$. By a standard argument, we
have $I_{\epsilon_{n}}^{\prime}(\bar{w})=0$. Then $\bar{w} \in \mathcal{N}^{-}$and $I_{\epsilon_{n}}\left(w_{n}\right) \geq m_{\epsilon}$. Moreover, from (3.17), (F4) and Fatou's Lemma, one has

$$
\begin{aligned}
m_{\epsilon} & \geq c=\lim _{n \rightarrow \infty}\left[I_{\epsilon_{n}}\left(w_{n}\right)-\frac{1}{2}\left\langle I_{\epsilon_{n}}^{\prime}\left(w_{n}\right), w_{n}\right\rangle\right] \\
& =\lim _{n \rightarrow \infty} \int_{\mathbb{R}^{N}}\left[\mathcal{F}\left(x, w_{n}\right)+\frac{p-2}{2} \epsilon_{n}\left|w_{n}\right|^{p}\right] \mathrm{d} x \\
& \geq \int_{\mathbb{R}^{N}} \lim _{n \rightarrow \infty}\left[\mathcal{F}\left(x, w_{n}\right)+\frac{p-2}{2} \epsilon_{n}\left|w_{n}\right|^{p}\right] \mathrm{d} x \\
& =\int_{\mathbb{R}^{N}}\left[\mathcal{F}(x, \bar{w})+\frac{p-2}{2} \epsilon_{n}|\bar{w}|^{p}\right] \mathrm{d} x \\
& =I_{\epsilon_{n}}(\bar{w})-\frac{1}{2}\left\langle I_{\epsilon_{n}}^{\prime}(\bar{w}), \bar{w}\right\rangle=I_{\epsilon_{n}}(\bar{w}) .
\end{aligned}
$$

This shows that $I_{\epsilon_{n}}(\bar{w}) \leq m_{\epsilon}$ and then $I_{\epsilon_{n}}(\bar{w})=m_{\epsilon}$.
Lemma 3.6. Assume that (V), (F1), (F2) and (F4) hold. Then for any $\epsilon \in(0,1]$ and $z \in E \backslash E^{-}$, there exist $t_{\epsilon}(z)>0$ and $\zeta_{\epsilon}(z) \in E^{-}$such that $t_{\epsilon}(z) z+\zeta_{\epsilon}(z) \in \mathcal{N}_{\epsilon}^{-}$.

We can easily prove this lemma in a similar way as Lemma 2.3 , so we omit it.
Proof of Theorem 1.1. Consider the case $N \geq 3$. By Lemma 3.5, there exists $z_{\epsilon} \in \mathcal{N}_{\epsilon}^{-}$such that (3.13) holds, where $\epsilon \in(0,1]$.

By Lemma 2.3, $\mathcal{N}^{-} \neq \varnothing$. Then, for $z_{0} \in \mathcal{N}^{-}$and $\zeta \in E^{-}, \Phi\left(z_{0}\right):=\bar{c} \geq 0$ and $\left\langle\Phi^{\prime}\left(z_{0}\right), z_{0}\right\rangle=$ $\left\langle\Phi^{\prime}\left(z_{0}\right), \zeta\right\rangle=0$ hold. In virtue of Lemma 3.6, there exist $t_{\epsilon}>0$ and $\zeta_{\epsilon} \in E^{-}$such that $t_{\epsilon} z_{0}+\zeta_{\epsilon} \in \mathcal{N}_{\epsilon}^{-}$. By Corollary 3.2 and Lemma 3.4, one has

$$
\begin{align*}
\bar{c} & =\Phi\left(z_{0}\right)=I_{0}\left(z_{0}\right) \geq I_{0}\left(t_{\epsilon} z_{0}+\zeta_{\epsilon}\right) \\
& \geq I_{\epsilon}\left(t_{\epsilon} z_{0}+\zeta_{\epsilon}\right) \geq m_{\epsilon} \geq \hat{\kappa}, \quad \forall \epsilon \in(0,1) . \tag{3.18}
\end{align*}
$$

Choose a sequence $\left\{\epsilon_{n}\right\} \subset(0,1]$ satisfy $\epsilon_{n} \rightarrow 0$ as $n \rightarrow \infty$, and

$$
\begin{equation*}
z_{\epsilon_{n}} \in \mathcal{N}_{\epsilon_{n_{n}}}^{-} \quad I_{\epsilon_{n}}\left(z_{\epsilon_{n}}\right)=m_{\epsilon_{n}} \rightarrow \bar{m} \in[\hat{\kappa}, \bar{c}], \quad I_{\epsilon_{n}}^{\prime}\left(z_{\epsilon_{n}}\right)=0 . \tag{3.19}
\end{equation*}
$$

There are three steps to prove Theorem 1.1.
Step 1: We prove that $\left\{z_{\epsilon_{n}}\right\}$ is bounded in $E$.
Arguing by contradiction, suppose that $\left\|z_{\varepsilon_{n}}\right\| \rightarrow \infty$. Set $w_{n}=\frac{z_{e_{n}}}{\left\|z_{e_{n}}\right\|}$, then $\left\|w_{n}\right\|=1$. By the Sobolev embedding theorem, going if necessary to a subsequence, we have

$$
\left\{\begin{array}{l}
w_{n} \rightharpoonup w, \quad \text { in } E ; \\
w_{n} \rightarrow w, \quad \text { in } L_{l o c}^{s}\left(\mathbb{R}^{N}\right), \forall s \in\left[2,2^{*}\right) ; \\
w_{n} \rightarrow w, \\
\text { a.e. on } \mathbb{R}^{N} .
\end{array}\right.
$$

From (3.19), we have

$$
\begin{equation*}
\bar{c} \geq I_{\epsilon_{n}}\left(z_{\epsilon_{n}}\right)-\frac{1}{2}\left\langle I_{\epsilon_{n}}^{\prime}\left(z_{\epsilon_{n}}\right), z_{\epsilon_{n}}\right\rangle=\int_{\mathbb{R}^{N}}\left[\mathcal{F}\left(x, z_{\epsilon_{n}}\right)+\frac{p-2}{2} \epsilon_{n}\left|z_{\epsilon_{n}}\right|^{p}\right] \mathrm{d} x . \tag{3.20}
\end{equation*}
$$

In view of Sobolev embedding theorem, there exists a constant $\tilde{C}_{4}>0$ such that $\left\|w_{n}\right\|_{2} \leq \tilde{C}_{4}$. If

$$
\begin{equation*}
\delta:=\limsup _{n \rightarrow \infty} \sup _{y \in \mathbb{R}^{N}} \int_{B_{1}(y)}\left|w_{n}^{+}\right|^{2} \mathrm{~d} x=0 \tag{3.21}
\end{equation*}
$$

by Lion's concentration compactness principle, $w_{n}^{+} \rightarrow 0$ in $L^{s}\left(\mathbb{R}^{N}\right)$ for $2<s<2^{*}$. Let $R>[2(1+\bar{c})]^{\frac{1}{2}}$. From (F1) and (F2), choose $\varepsilon=\frac{1}{4\left(R \tilde{C}_{4}\right)^{2}}>0$, there exists $\tilde{C}_{5}>0$ such that

$$
\begin{align*}
\limsup _{n \rightarrow \infty} \int_{\mathbb{R}^{N}}\left[F\left(x, R w_{n}^{+}\right)+\epsilon_{n}\left|R w_{n}^{+}\right|^{p}\right] \mathrm{d} x & \leq \limsup _{n \rightarrow \infty}\left[\varepsilon R^{2}\left\|w_{n}^{+}\right\|_{2}^{2}+\tilde{C}_{5} R^{p}\left\|w_{n}^{+}\right\|_{p}^{p}\right] \\
& \leq \varepsilon\left(R \tilde{C}_{4}\right)^{2}=\frac{1}{4} \tag{3.22}
\end{align*}
$$

Let $t_{n}=\frac{R}{\left\|z_{e_{n}}\right\|}$. From (3.19), (3.22) and Corollary 3.3, one has

$$
\begin{aligned}
\bar{c} & \geq m_{\epsilon_{n}}=I_{\epsilon_{n}}\left(z_{\epsilon_{n}}\right) \\
& \geq \frac{t_{n}^{2}}{2}\left\|z_{\epsilon_{n}}\right\|^{2}-\int_{\mathbb{R}^{N}}\left[F\left(x, t_{n} z_{\epsilon_{n}}^{+}\right)+\epsilon_{n}\left|t_{n} z_{\epsilon_{n}}^{+}\right|^{p}\right] \mathrm{d} x \\
& =\frac{R^{2}}{2}-\int_{\mathbb{R}^{N}}\left[F\left(x, R w_{n}^{+}\right)+\epsilon_{n}\left|R w_{n}^{+}\right|^{p}\right] \mathrm{d} x \\
& \geq \frac{R^{2}}{2}-\frac{1}{4}+o(1) \\
& >\bar{c}+\frac{3}{4}+o(1)
\end{aligned}
$$

which is a contradiction, then $\delta>0$.
Passing to a subsequence, we may assume there exists $k_{n} \in \mathbb{Z}^{N}$ such that

$$
\int_{B_{1+\sqrt{n}}\left(k_{n}\right)}\left|w_{n}^{+}\right|^{2} \mathrm{~d} x>\frac{\delta}{2}
$$

Let $\tilde{w}_{n}=w_{n}\left(x+k_{n}\right)$. Since $V_{1}(x)$ and $V_{2}(x)$ are 1-periodic in each of $x_{1}, x_{2}, \ldots, x_{N}$, then $\mathcal{A}_{i}=-\Delta+V_{i}, E^{+}$and $E^{-}$are $\mathbb{Z}^{N}$-translation invariance. Thereby, $\left\|\tilde{w}_{n}\right\|=\left\|w_{n}\right\|=1$, and

$$
\begin{equation*}
\int_{B_{1+\sqrt{n}}(0)}\left|\tilde{w}_{n}^{+}\right|^{2} \mathrm{~d} x>\frac{\delta}{2} . \tag{3.23}
\end{equation*}
$$

Going if necessary to a subsequence, we have

$$
\begin{cases}\tilde{w}_{n} \rightharpoonup \tilde{w}, & \text { in } E ; \\ \tilde{w}_{n} \rightarrow \tilde{w}, & \text { in } L_{l o c}^{s}\left(\mathbb{R}^{N}\right), \forall s \in\left[2,2^{*}\right) ; \\ \tilde{w}_{n} \rightarrow \tilde{w}, & \text { a.e. on } \mathbb{R}^{N}\end{cases}
$$

Then (3.23) shows that $\tilde{w} \neq 0$.
Define $\tilde{z}_{n}=\left(\tilde{u}_{n}, \tilde{v}_{n}\right)=z_{\epsilon_{n}}\left(x+k_{n}\right)$, note that $z_{\epsilon_{n}}=\left(u_{\epsilon_{n}}, v_{\epsilon_{n}}\right)$. Hence, $\frac{z_{n}}{\left\|z_{e_{n}}\right\|}=\tilde{w}_{n} \rightarrow \tilde{w}$ a.e. on $\mathbb{R}^{N}$ and $\tilde{w} \neq 0$, here $\tilde{w}_{n}=\left(\tilde{\eta}_{n}, \tilde{\theta}_{n}\right)$. For any $\varphi=(\mu, v) \in \mathcal{C}_{0}^{\infty}\left(\mathbb{R}^{N}\right)$, let $\phi_{n}=\left(\mu_{n}, v_{n}\right)=$ $\varphi\left(x-k_{n}\right)$. From (3.1) and (3.19), we have

$$
\begin{aligned}
0= & \left\langle I_{\epsilon_{n}}^{\prime}\left(z_{\epsilon_{n}}\right),\left\|z_{\epsilon_{n}}\right\| \phi_{n}\right\rangle \\
= & \left\|z_{\epsilon_{n}}\right\| \int_{\mathbb{R}^{N}}\left(\nabla u_{\epsilon_{n}} \cdot \nabla \mu_{n}+V_{1}(x) u_{\epsilon_{n}} \cdot \mu_{n}+\nabla v_{\epsilon_{n}} \cdot \nabla v_{n}+V_{2}(x) v_{\epsilon_{n}} \cdot v_{n}\right) \mathrm{d} x \\
& -\left\|z_{\epsilon_{n}}\right\| \int_{\mathbb{R}^{N}}\left[F_{z}\left(x, z_{\epsilon_{n}}\right)+p \epsilon_{n}\left|z_{\epsilon_{n}}\right|^{p-2} z_{\epsilon_{n}}\right] \varphi_{n} \mathrm{~d} x \\
= & \left\|z_{\epsilon_{n}}\right\| \int_{\mathbb{R}^{N}}\left(\nabla \tilde{u}_{n} \cdot \nabla \mu+V_{1}(x) \tilde{u}_{n} \cdot \mu+\nabla \tilde{v}_{n} \cdot \nabla v+V_{2}(x) \tilde{v}_{n} \cdot v\right) \mathrm{d} x \\
& -\left\|z_{\epsilon_{n}}\right\| \int_{\mathbb{R}^{N}}\left[F_{z}\left(x, \tilde{z}_{n}\right)+p \epsilon_{n}\left|\tilde{z}_{n}\right|^{p-2} \tilde{z}_{n}\right] \varphi \mathrm{d} x
\end{aligned}
$$

$$
\begin{align*}
= & \left\|z_{\epsilon_{n}}\right\|^{2} \int_{\mathbb{R}^{N}}\left(\nabla \tilde{\eta}_{n} \cdot \nabla \mu+V_{1}(x) \tilde{\eta}_{n} \cdot \mu+\nabla \tilde{\theta}_{n} \cdot \nabla v+V_{2}(x) \tilde{\theta}_{n} \cdot v\right) \mathrm{d} x \\
& -\left\|z_{\epsilon_{n}}\right\| \int_{\mathbb{R}^{N}}\left[F_{z}\left(x, \tilde{z}_{n}\right)+p \epsilon_{n}\left|\tilde{z}_{n}\right|^{p-2} \tilde{z}_{n}\right] \varphi \mathrm{d} x, \tag{3.24}
\end{align*}
$$

which implies

$$
\begin{align*}
\int_{\mathbb{R}^{N}} & \left(\nabla \tilde{\eta}_{n} \cdot \nabla \mu+V_{1}(x) \tilde{\eta}_{n} \cdot \mu+\nabla \tilde{\theta}_{n} \cdot \nabla v+V_{2}(x) \tilde{\theta}_{n} \cdot v\right) \mathrm{d} x \\
& =\frac{1}{\left\|z_{\epsilon_{n}}\right\|} \int_{\mathbb{R}^{N}}\left[F_{z}\left(x, \tilde{z}_{n}\right)+p \epsilon_{n}\left|\tilde{z}_{n}\right|^{p-2} \tilde{z}_{n}\right] \varphi \mathrm{d} x . \tag{3.25}
\end{align*}
$$

By virtue of (F1), (F2), (F6), (3.20) and the Hölder inequality, one can get that

$$
\begin{align*}
& \frac{1}{\left\|z_{\epsilon_{n}}\right\|} \int_{\mathbb{R}^{N}}\left|\left[F_{z}\left(x, \tilde{z}_{n}\right)+p \epsilon_{n}\left|\tilde{z}_{n}\right|^{p-2} \tilde{z}_{n}\right] \varphi\right| \mathrm{d} x \\
& \leq \frac{1}{\left\|z_{e_{n}}\right\|^{1-\sigma}} \int_{\tilde{z}_{n} \neq 0}\left(\frac{\left|F_{z}\left(x, \tilde{z}_{n}\right)\right|}{\left|\tilde{z}_{n}\right|^{\sigma}}+p \epsilon_{n}\left|\tilde{z}_{n}\right|^{p-1-\sigma}\right)\left|\tilde{w}_{n}\right|^{\sigma}|\varphi| \mathrm{d} x \\
& =\frac{1}{\left\|z_{e_{n}}\right\|^{1-\sigma}}\left[\int_{0<\left|\tilde{z}_{n}\right|<R_{0}}\left(\frac{\left|F_{z}\left(x, \tilde{z}_{n}\right)\right|}{\left|\tilde{z}_{n}\right|^{\sigma}}+p \epsilon_{n}\left|\tilde{z}_{n}\right|^{p-1-\sigma}\right)\left|\tilde{w}_{n}\right|^{\sigma}|\varphi| \mathrm{d} x\right. \\
& \left.+\frac{1}{\|\left. z_{\mathcal{E}_{n}}\right|^{1-\sigma}} \int_{\left|\tilde{z}_{n}\right| \geq R_{0}}\left(\frac{\left|F_{z}\left(x, \tilde{z}_{n}\right)\right|}{\left|\tilde{z}_{n}\right|^{\sigma}}+p \epsilon_{n}\left|\tilde{z}_{n}\right|^{p-1-\sigma}\right)\left|\tilde{w}_{n}\right|^{\sigma}|\varphi| \mathrm{d} x\right] \\
& \leq \frac{\left\|\tilde{w}_{n}\right\|_{2^{*}}^{\sigma}\|\varphi\|_{2^{*}}}{\left\|z_{e_{n}}\right\|^{1-\sigma}}\left[\int_{\left|\tilde{z}_{n}\right| \geq R_{0}}\left(\frac{\left|F_{z}\left(x, \tilde{z}_{n}\right)\right|}{\left|\tilde{z}_{n}\right|^{\sigma}}+p \epsilon_{n}\left|\tilde{z}_{n}\right|^{p-1-\sigma}\right)^{\frac{2^{*}}{2^{*}-1-\sigma}} \mathrm{d} x\right]^{\frac{2^{*}-1-\sigma}{2^{*}}} \\
& +\frac{C_{5}\left\|\tilde{w}_{n}\right\|_{2}^{\sigma}\|\varphi\|_{\frac{2}{2-\sigma}}}{\left\|z_{\varepsilon_{n}}\right\|^{1-\sigma}} \\
& \leq \frac{C_{6}}{\left.\left\|z_{\epsilon_{n}}\right\|\right|^{1-\sigma}}\left\{\|\varphi\|_{\frac{2}{2-\sigma}}+\|\varphi\|_{2^{*}}\left[\int_{\left|\tilde{z}_{n}\right| \geq R_{0}}\left(\mathcal{F}\left(x, \tilde{z}_{n}\right)+\frac{p-2}{2} \epsilon_{n}\left|\tilde{z}_{n}\right|^{p}\right) \mathrm{d} x\right]^{\frac{2^{*}-1-\sigma}{2^{*}}}\right\} \\
& \leq \frac{C_{6}}{\left\|z_{\mathcal{E}_{n}}\right\|^{1-\sigma}}\left\{\|\varphi\|_{\frac{2}{2-\sigma}}+\|\varphi\|_{2^{*}}\left[\int_{\mathbb{R}^{N}}\left(\mathcal{F}\left(x, \tilde{z}_{n}\right)+\frac{p-2}{2} \epsilon_{n}\left|\tilde{z}_{n}\right|^{p}\right) \mathrm{d} x\right]^{\frac{2^{*}-1-\sigma}{2^{*}}}\right\} \\
& \leq \frac{\tilde{C}_{6}}{\left\|z_{e_{n}}\right\|^{1-\sigma}}\left[\|\varphi\|_{2-\sigma}+\|\varphi\|_{2^{*}}\right]=o(1) . \tag{3.26}
\end{align*}
$$

It follows from (3.25) and (3.26) that

$$
\begin{equation*}
\int_{\mathbb{R}^{N}}\left(\nabla \tilde{\eta}_{n} \cdot \nabla \mu+V_{1}(x) \tilde{\eta}_{n} \cdot \mu+\nabla \tilde{\theta}_{n} \cdot \nabla v+V_{2}(x) \tilde{\theta}_{n} \cdot v\right) \mathrm{d} x=o(1), \quad \forall(\mu, v) \in \mathcal{C}_{0}^{\infty}\left(\mathbb{R}^{N}\right) . \tag{3.27}
\end{equation*}
$$

In view of $\tilde{w}_{n} \rightharpoonup \tilde{w}$, one has

$$
\begin{equation*}
\int_{\mathbb{R}^{N}}\left(\nabla \tilde{\eta} \cdot \nabla \mu+V_{1}(x) \tilde{\eta} \cdot \mu+\nabla \tilde{\theta} \cdot \nabla v+V_{2}(x) \tilde{\theta} \cdot v\right) \mathrm{d} x=0, \quad \forall(\mu, v) \in \mathcal{C}_{0}^{\infty}\left(\mathbb{R}^{N}\right) \tag{3.28}
\end{equation*}
$$

This implies that $\mathcal{A}_{i} \tilde{w}=-\Delta \tilde{w}+V_{i}(x) \tilde{w}=0$. Then $\tilde{w}$ is an eigenfunction of the operator $\mathcal{A}_{i}$, where $i=1,2$. Note that $\mathcal{A}_{i}$ has only a continuous spectrum. That is a contradiction. Hence, $\left\{\left\|z_{\epsilon_{n}}\right\|\right\}$ is bounded.
Step 2: We prove that there exists $\bar{z} \in E$ such that $\Phi^{\prime}(\bar{z})=0$ and $\Phi(\bar{z}) \geq m_{0}:=\inf _{\mathcal{N}_{0}} I_{0}=$ $\inf _{\mathcal{N}^{-}} \Phi$.

Applying Lion's concentration principle like in Step 1, we can deduce that there exist a constant $\delta_{1}>0$, a sequence $y_{n} \in \mathbb{Z}^{N}$ and a subsequence of $\left\{z_{\varepsilon_{n}}\right\}$, which is still denoted by $\left\{z_{e_{n}}\right\}$, such that

$$
\begin{equation*}
\int_{B_{1}\left(y_{n}\right)}\left|z_{\epsilon_{n}}\right|^{2} \mathrm{~d} x>\delta_{1} . \tag{3.29}
\end{equation*}
$$

Define $\hat{z}_{n}=z_{\epsilon_{n}}\left(x+y_{n}\right)$. By $E^{+}$and $E^{-}$are $\mathbb{Z}^{N}$-translation invariance, we have $\left\|\hat{z}_{n}\right\|=\left\|z_{\varepsilon_{n}}\right\|$ and

$$
\begin{equation*}
\hat{z}_{n} \in \mathcal{N}_{\epsilon_{n}}^{-}, \quad I_{\epsilon_{n}}\left(\hat{z}_{n}\right)=m_{\epsilon_{n}} \rightarrow \bar{m} \in[\hat{\kappa}, \bar{c}] \quad, I_{\epsilon_{n}}^{\prime}\left(\hat{z}_{n}\right)=0 \tag{3.30}
\end{equation*}
$$

Hence, there exists $\bar{z} \in H^{1}\left(\mathbb{R}^{N}\right) \backslash\{0\}$ such that, going if necessary to a subsequence,

$$
\begin{cases}\hat{z}_{n} \rightharpoonup \bar{z}, & \text { in } H^{1}\left(\mathbb{R}^{N}\right) ;  \tag{3.31}\\ \hat{z}_{n} \rightarrow \bar{z}, & \text { in } L_{l o c}^{s}\left(\mathbb{R}^{N}\right), \forall s \in\left[1,2^{*}\right) ; \\ \hat{z}_{n} \rightarrow \bar{z}, & \text { a.e. on } \mathbb{R}^{N} .\end{cases}
$$

Noting that $\hat{z}_{n}=\left(\hat{u}_{n}, \hat{v}_{n}\right), \varphi=(\mu, v)$. By virtue of (2.10), (3.1) and (3.31), we have

$$
\begin{aligned}
\left\langle\Phi^{\prime}(\bar{z}), \varphi\right\rangle= & \int_{\mathbb{R}^{N}}\left(\nabla \hat{u}_{n} \nabla \mu+V_{1}(x) \hat{u}_{n} \mu+\nabla \hat{v}_{n} \nabla v+V_{2}(x) \hat{v}_{n} v\right) \mathrm{d} x-\int_{\mathbb{R}^{N}} F_{z}(x, \bar{z}) \varphi \mathrm{d} x \\
= & \lim _{n \rightarrow \infty}\left\{\int_{\mathbb{R}^{N}}\left(\nabla \hat{u}_{n} \nabla \mu+V_{1}(x) \hat{u}_{n} \mu+\nabla \hat{v}_{n} \nabla v+V_{2}(x) \hat{v}_{n} v\right) \mathrm{d} x\right. \\
& \left.-\int_{\mathbb{R}^{N}}\left[F_{z}\left(x, \hat{z}_{n}\right)+\epsilon_{n} p\left|\hat{z}_{n}\right|^{p-2} \hat{z}_{n}\right] \varphi \mathrm{d} x\right\} \\
= & \lim _{n \rightarrow \infty}\left\langle I_{\epsilon_{n}}^{\prime}\left(\hat{z}_{n}\right), \varphi\right\rangle=0, \quad \forall \varphi \in \mathcal{C}_{0}^{\infty}(\Omega) .
\end{aligned}
$$

This implies that $\Phi^{\prime}(\bar{z})=0$. Then, $\bar{z} \in \mathcal{N}^{-}, \Phi(\bar{z}) \geq m_{0}$.
Step 3: We prove that $\Phi(\bar{z})=m_{0}$.
In view of (2.9), (2.10), (3.1), (3.30), (3.31) and Fatou's Lemma, we have

$$
\begin{align*}
\bar{m} & =\lim _{n \rightarrow \infty} m_{\epsilon_{n}} \\
& =\lim _{n \rightarrow \infty}\left[I_{\epsilon_{n}}\left(\hat{z}_{n}\right)-\frac{1}{2}\left\langle I_{\epsilon_{n}}^{\prime}\left(\hat{z}_{n}\right), \hat{z}_{n}\right\rangle\right] \\
& =\lim _{n \rightarrow \infty} \int_{\mathbb{R}^{N}}\left[\mathcal{F}\left(x, \hat{z}_{n}\right)+\frac{p-2}{2} \epsilon_{n}\left|\hat{z}_{n}\right|^{p}\right] \mathrm{d} x \\
& \geq \int_{\mathbb{R}^{N}} \mathcal{F}(x, \bar{z}) \mathrm{d} x=\Phi(\bar{z})-\frac{1}{2}\left\langle\Phi^{\prime}(\bar{z}), \bar{z}\right\rangle \geq m_{0} . \tag{3.32}
\end{align*}
$$

Let $\varepsilon>0$. Then there exists $w_{\varepsilon} \in \mathcal{N}^{-}$such that $\Phi\left(w_{\varepsilon}\right)<m_{0}+\varepsilon$. By Lemma 3.6, there exist $t_{n}>0$ and $\zeta_{n} \in E^{-}$such that $t_{n} w_{\varepsilon}+\zeta_{n} \in \mathcal{N}_{\epsilon_{n}}^{-}$. From (3.1) and Corollary 3.2, one has

$$
\begin{equation*}
m_{0}+\varepsilon>\Phi\left(w_{\varepsilon}\right)=I_{0}\left(w_{\varepsilon}\right) \leq I_{0}\left(t_{n} w_{\varepsilon}+\zeta_{n}\right) \geq I_{\epsilon_{n}}\left(t_{n} w_{\varepsilon}+\zeta_{n}\right) \geq m_{\epsilon_{n}} . \tag{3.33}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\bar{m}=\lim _{n \rightarrow \infty} m_{\epsilon_{n}} \leq m_{0}+\varepsilon . \tag{3.34}
\end{equation*}
$$

Since $\varepsilon$ can be any positive number, we have $\bar{m} \leq m_{0}$. In view of (3.32), we can get that $\bar{m}=m_{0}=\Phi(\bar{z})$.

Since the case $N=1,2$ can be dealt with similarly, we omit it. The proof is completed.

Lemma 3.7. Assume that (V), (F1)-(F3) and (F6) hold. Then
(i) $\vartheta:=\inf \{\|z\|: z \in K\}>0$;
(ii) $\varrho:=\inf \{\Phi(z): z \in K\}>0$.

Proof. We only consider the case where $N \geq 3$, since $N=1,2$ can be dealt with similarity.
(i) Similar to [26, Theorem 1.1], we have $K \neq \varnothing$. Let $\left\{z_{n}\right\} \subset K$ such that $\left\|z_{n}\right\| \rightarrow \vartheta$. From (2.10), we have

$$
\begin{equation*}
\left\|z_{n}\right\|^{2}=\int_{\mathbb{R}^{N}} F_{z}\left(x, z_{n}\right)\left(z_{n}^{+}-z_{n}^{-}\right) \mathrm{d} x . \tag{3.35}
\end{equation*}
$$

In view of $F(x, z) \geq 0$ and $\mathcal{F}(x, z) \geq 0$, then $F_{z}(x, z) z \geq 0$. From (F1), (F2), (2.4) and (3.35), one has

$$
\begin{aligned}
\left\|z_{n}\right\|^{2} & =\int_{z_{n} \neq 0} \frac{F_{z}\left(x, z_{n}\right)}{z_{n}}\left(\left|z_{n}^{+}\right|^{2}-\left|z_{n}^{-}\right|^{2}\right) \mathrm{d} x \\
& \leq \frac{1}{2 \gamma_{2}^{2}}\left\|z_{n}^{+}\right\|_{2}^{2}+C_{7}\left\|z_{n}\right\|_{p}^{p-2}\left\|z_{n}^{+}\right\|_{p}^{2} \\
& \leq \frac{1}{2}\left\|z_{n}\right\|_{2}^{2}+C_{8}\left\|z_{n}\right\|^{p},
\end{aligned}
$$

then,

$$
\begin{equation*}
\vartheta+o(1)=\left\|z_{n}\right\| \geq\left(2 C_{8}\right)^{-\frac{1}{p-2}}>0 . \tag{3.36}
\end{equation*}
$$

This implies that (i) holds.
(ii) Let $\left\{z_{n}\right\} \subset K$ such that $\Phi\left(z_{n}\right) \rightarrow \varrho$. Then $\left\langle\Phi^{\prime}\left(z_{n}\right), \bar{z}\right\rangle=0$ for any $\bar{z} \in E$. From (2.9) and (2.10), we have

$$
\begin{equation*}
\varrho+o(1)=\Phi\left(z_{n}\right)-\frac{1}{2}\left\langle\Phi^{\prime}\left(z_{n}\right), z_{n}\right\rangle=\int_{\mathbb{R}^{N}} \mathcal{F}\left(x, z_{n}\right) \mathrm{d} x . \tag{3.37}
\end{equation*}
$$

Let $w_{n}=\frac{z_{n}}{\left\|z_{n}\right\|}$. Then $\left\|z_{n}\right\|^{2}=1$. Set

$$
\begin{equation*}
\Omega_{n}:=\left\{x \in \mathbb{R}^{N}: \frac{\left|F_{z}\left(x, z_{n}\right)\right|}{|z|} \leq \tau\right\} . \tag{3.38}
\end{equation*}
$$

Since $\Lambda_{0}\left\|w_{n}^{+}\right\|_{2}^{2} \leq\left\|w_{n}^{+}\right\|^{2}$, we have

$$
\begin{align*}
& \int_{\Omega_{n}} \frac{F_{z}\left(x, z_{n}\right)}{z_{n}}\left|w_{n}\right|\left(\left|w_{n}^{+}\right|+\left|w_{n}^{-}\right|\right) \mathrm{d} x \\
& \quad \leq \tau\left\|w_{n}\right\|_{2}\left[\int_{\mathbb{R}^{N}}\left(\left|w_{n}^{+}\right|+\left|w_{n}^{-}\right|\right)^{2} \mathrm{~d} x\right]^{\frac{1}{2}} \\
& \quad \leq \tau\left\|w_{n}\right\|_{2}\left(\left\|w_{n}^{+}\right\|_{2}^{2}+\left\|w_{n}^{-}\right\|_{2}^{2}\right)^{\frac{1}{2}} \leq 1-\frac{\delta_{0}}{\Lambda_{0}} . \tag{3.39}
\end{align*}
$$

From (F6), (3.36), (3.37) and the Hölder inequality, we have

$$
\begin{aligned}
& \frac{1}{\left\|z_{n}\right\|^{1-\delta}} \int_{\mathbb{R}^{N} \backslash \Omega_{n}} \frac{\left|F_{z}\left(x, z_{n}\right)\right|}{\left|z_{n}\right|^{\sigma}}\left|w_{n}\right|^{\sigma}\left|w_{n}^{+}-w_{n}^{-}\right| \mathrm{d} x \\
& \quad \leq \frac{1}{\left\|z_{n}\right\|^{1-\delta}}\left[\int_{\mathbb{R}^{N} \backslash \Omega_{n}}\left(\frac{\left|F_{z}\left(x, z_{n}\right)\right|}{\left|z_{n}\right|^{\sigma}}\right)^{\frac{2^{*}}{2^{*}-1-\sigma}} \mathrm{d} x\right]^{\frac{2^{*}-1-\sigma}{2^{*}}}\left\|w_{n}\right\|_{2^{*}}^{\sigma}\left\|w_{n}^{+}-w_{n}^{-}\right\|_{2^{*}}
\end{aligned}
$$

$$
\begin{equation*}
\leq \frac{C_{9}}{\left\|w_{n}\right\|^{1-\sigma}}\left[\int_{\mathbb{R}^{N} \backslash \Omega_{n}} \mathcal{F}\left(x, z_{n}\right) \mathrm{d} x\right]^{\frac{2^{*}-1-\sigma}{2^{*}}} \leq C_{10}[\varrho+o(1)]^{\frac{2^{*}-1-\sigma}{2^{*}}} . \tag{3.40}
\end{equation*}
$$

By virtue of (3.39), (3.40) and (2.10), one has

$$
\begin{aligned}
1 & =\frac{\left\|z_{n}\right\|^{2}-\left\langle\Phi^{\prime}\left(z_{n}\right), z_{n}^{+}-z_{n}^{-}\right\rangle}{\left\|z_{n}\right\|^{2}} \\
& =\frac{1}{\left\|z_{n}\right\|} \int_{\mathbb{R}^{N}} F_{z}\left(x, z_{n}\right)\left(z_{n}^{+}-z_{n}^{-}\right) \mathrm{d} x \\
& =\int_{\Omega_{n}} \frac{F_{z}\left(x, z_{n}\right)}{z_{n}}\left[\left(w_{n}^{+}\right)^{2}-\left(w_{n}^{-}\right)^{2}\right] \mathrm{d} x+\frac{1}{\left\|z_{n}\right\|^{1-\sigma}} \int_{\mathbb{R}^{N} \backslash \Omega_{n}} \frac{F_{z}\left(x, z_{n}\right)}{\left|z_{n}\right|^{\sigma}}\left|w_{n}\right|^{\sigma}\left(w_{n}^{+}-w_{n}^{-}\right) \mathrm{d} x \\
& \leq \int_{\Omega_{n}} \frac{F_{z}\left(x, z_{n}\right)}{z_{n}}\left(w_{n}^{+}\right)^{2} \mathrm{~d} x+\frac{1}{\left\|z_{n}\right\|^{1-\sigma}} \int_{\mathbb{R}^{N} \backslash \Omega_{n}} \frac{F_{z}\left(x, z_{n}\right)}{\left.\left|z_{n}\right|\right|^{\sigma}}\left|w_{n}\right|^{\sigma}\left(w_{n}^{+}-w_{n}^{-}\right) \mathrm{d} x \\
& \leq 1-\frac{\delta_{0}}{\Lambda_{0}}+C_{10}[\varrho+o(1)]^{2^{*}-1^{2-\sigma}}
\end{aligned}
$$

Then we can get that $\varrho>0$.
Proof of Theorem 1.2. Let $z_{n} \in K$ such that $\Phi\left(z_{n}\right) \rightarrow \varrho$. As [26, Lemma 4.3], we can easily prove the boundedness of $\left\{z_{n}\right\}$ in $E$, so we omit it. Then, similar to the proof of Theorem 1.1, we can get that there exists $\bar{z} \in E \backslash\{0\}$ such that $\Phi^{\prime}(\bar{z})=0$ and $\Phi(\bar{z})=\varrho>0$.

## Acknowledgements

The authors would like to thank the referees for their useful suggestions. The authors are supported financially by National Natural Science Foundation of China (No:11501190), Hunan provincial Natural Science Foundation (No:2019JJ50146) and Scientific Research Fund of Hunan Provincial Education Department (No:20B243).

## References

[1] A. Ambrosetti, G. Cerami, D. Ruiz, Solitons of linearly coupled systems of semilinear non-autonomous equations on $\mathbb{R}^{N}$, J. Funct. Anal. 254(2008), No. 11, 2816-2845. https: //doi.org/10.1016/j.jfa.2007.11.013; MR2414222; Zbl 1148.35080
[2] T. Bartsch, A. Pankov, Z.-Q. Wang, Nonlinear Schrödinger equations with steep potential well, Commun. Contemp. Math. 3(2001), No. 4, 549-569. https://doi.org/10.1142/ S0219199701000494; MR1869104; Zbl 1076.35037
[3] G. W. Chen, S. W. Ma, Asymptotically or super linear cooperative elliptic systems in the whole space, Sci. China Math. 56(2013), No. 6, 1181-1194. https://doi.org/10.1007/ s11425-013-4567-3; MR3063964; Zbl 1279.35037
[4] G. W. Chen, S. W. Ma, Infinitely many solutions for resonant cooperative elliptic systems with sublinear or superlinear terms, Calc. Var. 49(2014), No. 1-2, 271-286. https: //doi. org/10.1007/s00526-012-0581-5; MR3148116; Zbl 1288.35234
[5] G. W. Chen, S. W. Ma, Nonexistence and multiplicity of solutions for nonlinear elliptic systems of $\mathbb{R}^{N}$, Nonlinear Anal. Real World Appl. 36(2017), 233-248. https://doi. org/10. 1016/j.nonrwa.2017.01.012; MR3621240; Zbl 1368.35109
[6] R. Cipolatti, W. Zumpichiatti, On the existence and regularity of ground states for a nonlinear system of coupled Schrödinger equations in $\mathbb{R}^{N}$, Comput. Appl. Math. 18(1999), No. 1, 15-29. MR1935085; Zbl 0927.35104
[7] D. G. Costa, C. A. Magalhães, A variational approach to subquadratic perturbations of elliptic systems, J. Differential Equation 111(1994), No. 1, 103-122. https://doi.org/10. 1006/jdeq.1994.1077; MR1280617; Zbl 0803.35052
[8] Y. H. Ding, Varitional methods for strongly indefinite problems, World Scientific, Singapore, 2008. https://doi.org/10.3724/SP.J.1160.2011.00209; Zbl 1463.35020
[9] Y. H. Ding, C. Lee, Multiple solutions of Schrödinger equations with indefinite linear part and super or asymptotically linear terms, J. Differential Equations 222(2006), No. 1, 137-163. https://doi.org/10.1016/j.jde.2005.03.011; MR2200749; Zbl 1090.35077
[10] B. D. Esry, C. H. Greene, J. P. Burke, Jr., J. L. Bohn, Hartree-Fock theory for double condensates, Phys. Rev. Lett. 78(1997), No. 19, 3594-3597. https://doi.org/10.1103/ PhysRevLett.78.3594
[11] D. E. Edmunds, W. D. Evans, Spectral theory and differential operators, Clarendon Press, Oxford, 1987. https://doi.org/10.1017/CB09780511623721
[12] Y. Egorov, V. Kondratiev, On spectral theory of elliptic operators, Birkhäuser, Basel, 1996. https://doi.org/10.1007/978-3-0348-9029-8; MR1409364
[13] A. Hasegawa, Y. Kodama, Solitons in optical communications, Oxford University Press, Oxford, 1995.
[14] M. N. Islam, Ultrafast fiber switching devices and systems, Cambridge University Press, New York, 1992.
[15] W. Kryszewski, A. Szulkin, Generalized linking theorem with an application to a semilinear Schrödinger equations, Adv. Differential Equations 3(1998), No. 3, 441-472. MR1751952; Zbl 0947.35061
[16] G. Li, A. Szulkin, An asymptotically periodic Schrödinger equation with indefinite linear part, Commun. Contemp. Math. 4(2002), No. 4, 763-776. https://doi.org/10.1142/ S0219199702000853; MR1938493; Zbl 1056.35065
[17] L. Li, C-L. Tang, Infinitely many solutions for resonance elliptic systems, C. R. Acad. Sci. Paris, Ser. I 353(2015), No. 1, 35-40. https://doi.org/10.1016/j.crma.2014.10.010; MR3285144; Zbl 1310.35096
[18] L. A. Maia, E. Montefusco, B. Pellacci, Positive solutions for a weakly coupled nonlinear Schrödinger system, J. Differential Equations 229(2006), No. 2, 743-767. https: //doi.org/10.1016/j.jde.2006.07.002; MR2263573; Zbl 1104.35053
[19] S. W. Ma, Nontrivial solutions for resonant cooperative elliptic systems via computations of the critical groups, Nonlinear Anal. 73(2010), No. 12, 3856-3872. https ://doi.org/10. 1016/j.na.2010.08.013; MR2728560; Zbl 1202.35086
[20] L. Ma, L. Zhao, Uniqueness of ground states of some coupled nonlinear Schrödinger systems and their application, J. Differential Equations 245(2008), No. 9, 2551-2565. https : //doi.org/10.1016/j.jde.2008.04.008; MR2455776; Zbl 1154.35083
[21] J. M. do Ó, J. C. de Albuquerque, Positive ground state of coupled systems of Schrödinger equations in $\mathbb{R}^{2}$ involving critical exponential growth, Math. Meth. Appl. Sci. 40(2017), No. 18, 6864-6879. https://doi.org/10.1002/mma.4498; MR3742101; Zbl 1387.35182
[22] A. Pankov, Periodic nonlinear Schrödinger equation with application to photonic crystals, Milan J. Math. 73(2005), No. 1, 259-287. https://doi.org/10.1007/s00032-005-0047-8; MR2175045; Zbl 1225.35222
[23] D. D. Qin, X. H. Tang, Solutions on asymptotically periodic elliptic system with new conditions, Results. Math. 70(2016), No. 3, 539-565. https://doi.org/10.1007/s00025-015-0491-x; MR3544877; Zbl 1358.35032
[24] D. D. Qin, Y. B. He, X. H. Tang, Ground and bound states for non-linear Schrödinger systems with indefinite linear terms, Complex Var. Elliptic Equ. 62(2017), No. 12, 17581781. https://doi.org/10.1080/17476933.2017.1281256; MR3698470; Zbl 1377.35076
[25] D. D. Qin, J. Chen, X. H. Tang, Existence and non-existence of nontrivial solutions for Schrödinger systems via Nehari-Pohozaev manifold, Comput. Math. Appl. 74(2017), No. 12, 3141-3160. https://doi.org/10.1016/j.camwa.2017.08.010; MR3725943; Zbl 1400.35095
[26] D. D. Qin, X. H. Tang, Q. F. Wu, Ground state of nonlinear Schrödinger systems with periodic or non-periodic potentials, Commun. Pure Appl. Anal. 18(2019), No. 3, 1261-1280. https://doi.org/10.3934/cpaa.2019061; MR3917706; Zbl 1415.35115
[27] D. D. Qin, X. H. Tang, Q. F. Wu, Existence and concentration properties of ground state solutions for elliptic systems, Complex Var. Elliptic Equ. 65(2020), No. 8, 1257-1286. https://doi.org/10.1080/17476933.2019.1579210; MR4118686; Zbl 1454.35065
[28] M. Schechter, Superlinear Schrödinger operators, J. Funct. Anal. 262(2012), No. 6, 26772694. https://doi.org/10.1016/j.jfa.2011.12.023; MR2885962; Zbl 1243.35049
[29] M. de Souza, J. M. do Ó, Hamiltonian elliptic systems in $\mathbb{R}^{2}$ with subcritical and critical exponential growth, Ann. Mat. Pura. Appl. 195(2016), No. 3, 935-956. https://doi.org/ 10.1007/s10231-015-0498-7; MR3500314; Zbl 1341.35046
[30] A. Szulkin, T. Weth, Ground state solutions for some indefinite variational problems, J. Funct. Anal. 257(2009), No. 12, 3802-3822. https://doi.org/10.1016/j.jfa.2009.09. 013; MR2557725; Zbl 1178.35352
[31] E. Timmermans, Phase seperation of Bose-Einstein condensates, Phys. Rev. Lett. 81(1998), No. 26, 5718-5721. https://doi.org/10.1103/PhysRevLett. 81.5718
[32] X. H. Tang, Non-Nehari manifold method for asymptotically periodic Schrödinger equation, Sci. China Math. 58(2015), No. 4, 715-728. https://doi.org/10.1007/s11425-014-4957-1; MR3319307; Zbl 1321.35055
[33] X. H. Tang, S. T. Chen, X. Y. Lin, J. S. Yu, Ground state solutions of Nehari-Pankov type for Schrödinger equations with local super-quadratic conditions, J. Differential Equations 268(2020), No. 8, 4663-4690. https://doi.org/10.1016/j.jde.2019.10.041; MR4066032; Zbl 1437.35224
[34] X. H. Tang, X. Y. Lin, J. S. Yu, Nontrivial solutions for Schrödinger equation with local super-quadratic conditions, J. Dyn. Differential Equations 31(2019), No. 1, 369-383. https: //doi.org/10.1007/s10884-018-9662-2; MR3935147; Zbl 1414.35062
[35] X. H. Tang, D. D. Qin, Ground state solutions for semilinear time-harmonic Maxwell equations, J. Math. Phys. 57(2016), No. 4, 041505. https://doi.org/10.1063/1.4947179; MR3490057; Zbl 1339.35307
[36] M. Willem, Minimax theorems, Birkhäuser, Boston, 1996. https://doi.org/10.1007/978-1-4612-4146-1; MR1400007; Zbl 0856.49001

# Sign-changing solutions for a Schrödinger-Kirchhoff-Poisson system with 4-sublinear growth nonlinearity 

Shubin Yu ${ }^{1}$, Ziheng Zhang ${ }^{\boxtimes 1}$ and Rong Yuan ${ }^{2}$<br>${ }^{1}$ School of Mathematical Sciences, TianGong University, Tianjin 300387, People's Republic of China<br>${ }^{2}$ Laboratory of Mathematics and Complex Systems (Ministry of Education), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People's Republic of China

Received 22 February 2021, appeared 16 November 2021
Communicated by Dimitri Mugnai

Abstract. In this paper we consider the following Schrödinger-Kirchhoff-Poisson-type system

$$
\begin{cases}-\left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \Delta u+\lambda \phi u=Q(x)|u|^{p-2} u & \text { in } \Omega \\ -\Delta \phi=u^{2} & \text { in } \Omega \\ u=\phi=0 & \text { on } \partial \Omega\end{cases}
$$

where $\Omega$ is a bounded smooth domain of $\mathbb{R}^{3}, a>0, b \geq 0$ are constants and $\lambda$ is a positive parameter. Under suitable conditions on $Q(x)$ and combining the method of invariant sets of descending flow, we establish the existence and multiplicity of signchanging solutions to this problem for the case that $2<p<4$ as $\lambda$ sufficiently small. Furthermore, for $\lambda=1$ and the above assumptions on $Q(x)$, we obtain the same conclusions with $2<p<\frac{12}{5}$.
Keywords: Schrödinger-Kirchhoff-Poisson type system, invariant sets of descending flow, sign-changing solutions.
2020 Mathematics Subject Classification: 35A15, 35J20, 35J50.

## 1 Introduction

In this paper we are concerned with the existence of sign-changing solutions to the following Schrödinger-Kirchhoff-Poisson-type system

$$
\begin{cases}-\left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \Delta u+\lambda \phi u=Q(x)|u|^{p-2} u & \text { in } \Omega  \tag{1.1}\\ -\Delta \phi=u^{2} & \text { in } \Omega \\ u=\phi=0 & \text { on } \partial \Omega\end{cases}
$$

[^48]where $\Omega$ is a bounded smooth domain of $\mathbb{R}^{3}, a>0, b \geq 0$ are constants and $\lambda$ is a positive parameter.

When $a=1$ and $b=0$, problem (1.1) reduces to the classical Schrödinger-Poisson system on bounded domain. We rewrite it in the following more general form

$$
\begin{cases}-\Delta u+\lambda \phi u=f(x, u) & \text { in } \Omega,  \tag{1.2}\\ -\Delta \phi=u^{2} & \text { in } \Omega, \\ u=\phi=0 & \text { on } \partial \Omega\end{cases}
$$

It is well known that system (1.2) has a great importance in the study of stationary solution $\psi(x, t)=e^{-i t} u(x)$ of time-dependent Schrödinger-Poisson equations, which describes quantum (nonrelativistic) particles interacting with the electromagnetic field generated by the motion. For more details about the physical background of system (1.2), we refer to $[3,4,31]$. Compared with the researches about system (1.2) on the whole space $\mathbb{R}^{3}$, there are few works concerning the Schrödinger-Poisson system on bounded domain, see for instance [2, $8,12,33,37]$. In [33], the authors considered the existence, nonexistence and multiplicity results by using variational methods when $f(x, u)=|u|^{p-1} u$ with $p \in(1,5)$. Siciliano [37] studied the same nonlinearity as in [33], and, by means of Lusternik-Schnirelmann theory,
 ponent 6 , where $\operatorname{cat}(\cdot)$ denotes the Lusternik-Schnirelmann category. Alves and Souto [2] studied system (1.2) when $f$ has a subcritical growth and obtained the existence of least energy sign-changing solution by means of variational methods. Using a new sign-changing version of the symmetric mountain pass theorem, Batkam [12] proved the existence of infinitely many sign changing solutions for system (1.2) with critical growth. Bai and He [8] considered system (1.2) with a general 4 -superlinear nonlinearity $f$ and proved the existence of ground state solution by the aid of the Nehari manifold; moreover, they also obtained the existence of infinitely many solutions.

On the other hand, if setting $\phi=0$ and considering the first equation of problem (1.1), we get the Kirchhoff-Dirichlet problem

$$
\begin{cases}-\left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \Delta u=f(x, u) & \text { in } \Omega  \tag{1.3}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

When $b \neq 0$, problem (1.3) is nonlocal due to the emergence of $b \int_{\Omega}|\nabla u|^{2} d x \Delta u$ and is related to the stationary analogue of the following problem

$$
\begin{cases}u_{t t}-\left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \Delta u=f(x, u) & \text { in } \Omega \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

which was introduced by Kirchhoff [22] as a generalization of the classical d'Alembert wave equation

$$
\rho \frac{\partial^{2} u}{\partial t^{2}}-\left(\frac{P_{0}}{h}+\frac{E}{2 L} \int_{0}^{L}\left|\frac{\partial u}{\partial x}\right|^{2} d x\right) \frac{\partial^{2} u}{\partial x^{2}}=0
$$

for free vibration of elastic string, where $L$ is the length of the string, $h$ is the area of crosssection, $E$ is the Young modulus of the material and $P_{0}$ is the initial tension. The Kirchhoff's
model takes into account the length variation of the string produced by the transverse vibration, which gives rise to the appearance of nonlocal term. For more mathematical and physical relevance on problem (1.3), we refer the reader to $[6,16]$ and the references therein. Recently, different methods and techniques are used to deal with the existence of sign-changing solutions to problem (1.3) or similar Kirchhoff-type equations, and indeed, some interesting results were obtained. For example, the method of invariant sets of descent flow was used in $[21,30,44]$ to obtain the existence of a sign-changing solution for problem (1.3). The authors in $[20,36,40]$ considered problem (1.3) or more general Kirchhoff-type equations respectively, combining constraint variational methods and quantitative deformation lemma. Later, under some more weak assumptions on $f$ (especially, Nehari-type monotonicity condition has been removed), with the aid of some new analytical skills and non-Nehari manifold method, Tang and Cheng [39] improved and generalized some results obtained in [36].

Now we turn our attention to problem (1.1). As far as we know, for the first time Batkam and Santos Júnior [13] introduced this type problem with bi-nonlocal terms and proved that problem (1.1) with $\lambda=1$ has at least three solutions: one positive, one negative, and one changing its sign by imposing the conditions on the nonlinear term $f$ (more general form than $\left.Q(x)|u|^{p-2} u\right)$ as follows
$\left(f_{1}\right) f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ and there exists a constant $c>0$ such that

$$
|f(x, t)| \leq c\left(1+|t|^{p-1}\right), \quad \text { where } 4<p<6 \text {; }
$$

$\left(f_{2}\right) f(x, t)=o(|t|)$ uniformly in $x \in \bar{\Omega}$ as $t \rightarrow 0$;
$\left(f_{3}\right)$ there exists $\mu>4$ such that $0<\mu F(x, t) \leq t f(x, t)$ for all $t \neq 0, x \in \bar{\Omega}$, where $F(x, t)=$ $\int_{0}^{t} f(x, s) d s$.

Furthermore, in such case, if $f$ is odd with respect to $t$, the authors obtained an unbounded sequence of sign-changing solutions. After this pioneer work, several interesting results have been obtained about the existence of positive solutions, multiple solutions, ground state solutions and sign-changing solutions, we refer the reader to [ $5,14,25,27,28,34,35,41,43,45,46,48]$ and their references. Here, we must point out that, to obtain their results in the above references, various 4 -superlinear growth conditions or asymptotical 4 -linear assumptions or the Nehari-type monotonicity condition on $f$ are needed, especially for the discussion of signchanging solutions. So, a natural question is that, for the case that $f(x, u)$ is 4 -sublinear, here special form $f(x, u)=Q(x)|u|^{p-2} u$ being considered, does problem (1.1) admit the existence of sign-changing solutions? Meanwhile, due to the oddness of $Q(x)|u|^{p-2} u$ on $u$, does there exist infinitely many sign-changing solutions as usual?

Motivated by the above discussion, the purpose of this paper is to deal with the existence and multiplicity of sign-changing solutions to problem (1.1) for the case that $2<p<4$. For this case, to our best knowledge, during the existing literatures there is no result concerned with sign-changing solutions for problem (1.1). To state our main results, $Q(x)$ is supposed to be satisfied the following condition
(Q) $Q(x)>0$ and $Q \in L^{\infty}(\Omega)$.

Now we are in the position to state our first result.
Theorem 1.1. If $2<p<4$ and ( $Q$ ) holds true, there exists $\lambda^{*}>0$ such that for all $0<\lambda \leq \lambda^{*}$, problem (1.1) admits one sign-changing solution. Moreover, problem (1.1) has infinitely many signchanging solutions.

In what follows, we list some difficulties during the process of dealing with sign-changing solutions of nonlocal elliptic problems as usual. Problem (1.1) is a bi-nonlocal problem as the appearance of the two terms $b \int_{\Omega}|\nabla u|^{2} d x \Delta u$ and $\phi_{u} u$ implies that problem (1.1) is not a pointwise identity, where $\phi_{u}$ is defined in Lemma 2.1. This causes some mathematical difficulties in finding sign-changing solutions. In fact, since the nonlocal terms $\frac{b}{4}\left(\int_{\Omega}|\nabla u|^{2} d x\right)^{2}$ and $\frac{1}{4} \int_{\Omega} \phi_{u} u^{2} d x$ in the associated variational functional are homogeneous of order 4 , it seems difficult to get the boundedness and compactness for any (PS) sequence or Cerami sequence. Inspired by [21], we overcome this difficulty by adding a reasonable potential $Q(x)$. On the other hand, we observe that

$$
\int_{\Omega}|\nabla u|^{2} d x=\int_{\Omega}\left|\nabla u^{+}\right|^{2} d x+\int_{\Omega}\left|\nabla u^{-}\right|^{2} d x
$$

but the following decomposition relationship

$$
\int_{\Omega} \phi_{u} u^{2} d x=\int_{\Omega} \phi_{u^{+}} u^{+2} d x+\int_{\Omega} \phi_{u^{-}} u^{-2} d x
$$

does not hold in $H_{0}^{1}(\Omega)$. In order to overcome these difficulties, we adopt the idea from [21, 24,26 ] to introduce an auxiliary operator $A$, which will be used to construct a pseudo-gradient vector field to ensure existence of the desired invariant sets of the flow. However, since $A$ is merely continuous (see Lemma 3.1 below), it may not be used to define the descending flow. Fortunately, one can construct a suitable locally Lipschitz continuous operator $B$ inheriting the properties of $A$ in a similar way as [11] to define the flow. Finally, by restricting the parameter $\lambda$ small enough during the minimax arguments in the presence of invariant sets, we complete the proof of Theorem 1.1.

Remark 1.2. As we discussed above, the necessary restriction must be added to the parameter $\lambda$ to obtain the existence and multiplicity of sign-changing solutions. Indeed, similar requirements have emerged in the literatures to discuss the nonexistence of nontrivial solutions or the existence of positive solutions of Schrödinger-Poisson systems. Explicitly, we observe that, in [31], system

$$
\begin{cases}-\Delta u+V(x) u+\lambda \phi u=|u|^{p-2} u & \text { in } \mathbb{R}^{3}  \tag{1.4}\\ -\Delta \phi=u^{2} & \text { in } \mathbb{R}^{3}\end{cases}
$$

admits only one trivial solution with $p \in(2,3)$ if $\lambda \geq \frac{1}{4}$. Moreover, the authors in [4] considered the bounded states of system (1.4), and showed that for any $n \in \mathbb{N}$ there exists $\lambda_{n} \in\left(0, \frac{1}{4}\right)$ such that for all $\lambda \in\left(0, \lambda_{n}\right)$ system (1.4) with $p \in(2,3)$ has at least $n$ pairs of radially symmetric solutions with positive energies.

However, it will be noted that, different from the whole space $\mathbb{R}^{3}$ discussed in $[4,31]$, we can also obtain the existence and multiplicity of sign-changing solutions for our problem (1.1) considered on bounded domain without any restriction on the parameter $\lambda$. To discuss this case simply, we set $\lambda=1$. Nevertheless, the range of $p$ will be limited to a small range as follows.

Theorem 1.3. If $2<p<\frac{12}{5}$ and ( $Q$ ) is satisfied, then for $\lambda=1$, the results of Theorem 1.1 still hold true.

Remark 1.4. Here, two recent papers $[21,38]$ must be mentioned. In fact, as particular cases, the existence of sign-changing solutions for problems (1.2) and (1.3) are considered, when
the nonlinearity is the form of $Q(x)|u|^{p-2} u$ or general form covering the pure power type $Q(x)|u|^{p-2} u$ with $2<p<4$ for the case $\Omega=\mathbb{R}^{3}$. Certainly, different hypotheses on $Q(x)$ are presented to obtain their conclusions. To this point, it should be pointed out that $Q(x)$ can be equal to constants in our Theorems 1.1 and 1.3, which is different from the assumptions on $Q(x)$ in $[21,38]$. In addition, compared with the situations investigated in [21,38], it is worth pointing out that the technique of constructing nonempty nodal set used in [38] is invalid for our problem. To finish the proof of our Theorems 1.1 and 1.3, as in [21] we apply the method of invariant sets of descending flow. In fact, we make use of an abstract critical point theory developed in [24] that is very useful to deal with elliptic equations, see for instance [ $9-11,21,26$ ] and the references therein. Meanwhile, it is must be mentioned that, although similar conditions on $Q(x)$ have been given, we could not make the estimation for the energy functional as in [21], and some new difficulties need to be addressed due to the combination of two nonlocal terms $b \int_{\Omega}|\nabla u|^{2} d x \Delta u$ and $\phi_{u} u$, which is also the reason that we restrict the parameter $\lambda$ small enough in Theorem 1.1. The difference between the proof of Theorem 1.1 and Theorem 1.3 is just that the functions satisfying particular properties in Properties 3.7 and 3.11 are needed to make necessary changes.

Remark 1.5. Note that, our Theorem 1.3 is valid for the case that $b=0$, this observation and Remark 1.2 indicate that Schrödinger-Poisson system has differently dynamical behavior on bounded domain $\Omega$ and the whole space $\mathbb{R}^{3}$. Meanwhile, the results of signchanging solutions in [38] are also based on the fundamental assumption that $\lambda$ is sufficiently small. Compared this fact with our Theorem 1.3, it is natural to ask whether one can show that Schrödinger-Poisson system defined on the whole space $\mathbb{R}^{3}$ possesses nontrivial signchanging solutions such as the case dealt with in [38] without any parameter. In addition, Theorem 1.3 is actually an extension of Theorem 1.1 in the sense that there is not any restriction on the parameter $\lambda$ in Theorem 1.3. Here, we point out that, using the invariant sets of descending flow, $2<p<\frac{12}{5}$ is the optimal range (in fact, it is the optimal range to guarantee that (4.2) holds). For $\frac{12}{5} \leq p<4$, it remains an open question about the existence and multiplicity of sign-changing solutions when $\lambda=1$ in problem (1.1).

This paper is organized as follows. In Section 2, we present some useful preliminary results. Theorems 1.1 and 1.3 are proved in Sections 3 and 4, respectively.

## 2 Preliminaries

In this section, we first introduce the variational framework associated with problem (1.1). Before that, we define $E$ to be the usual Sobolev space $H_{0}^{1}(\Omega)$ with the inner product

$$
(u, v)=\int_{\Omega} \nabla u \cdot \nabla v d x
$$

and endowed with the norm $\|u\|=(u, u)^{\frac{1}{2}}$ for $u, v \in E$. From [42, Theorem 1.9], the embed$\operatorname{ding} E \hookrightarrow L^{q}(\Omega)$ is compact for any $1 \leq q<6$. The usual norm in the Lebesgue space $L^{q}(\Omega)$ is denoted by $\|u\|_{q}$ and $C$ is the positive constant whose precise value can change from line to line. The following result is well known and is a collection of results in [17] and [31].

Lemma 2.1. For each $u \in H_{0}^{1}(\Omega)$, there exists a unique element $\phi_{u} \in H_{0}^{1}(\Omega)$ such that $-\Delta \phi_{u}=u^{2}$. Moreover,

$$
\phi_{u}=\int_{\Omega} \frac{u^{2}(y)}{4 \pi|x-y|} d y
$$

has following properties:
(1) $\phi_{u} \geq 0$ and $\phi_{t u}=t^{2} \phi_{u}, \forall t>0$;
(2) there exists $C>0$ independent of $u$ such that $\left\|\phi_{u}\right\| \leq C\|u\|^{2}$ and

$$
\int_{\Omega} \phi_{u} u^{2} d x \leq C\|u\|^{4}
$$

(3) if $u_{n} \rightharpoonup u$ in $H_{0}^{1}(\Omega)$, then $\phi_{u_{n}} \rightharpoonup \phi_{u}$ in $H_{0}^{1}(\Omega)$ and

$$
\lim _{n \rightarrow \infty} \int_{\Omega} \phi_{u_{n}} u_{n}^{2} d x=\int_{\Omega} \phi_{u} u^{2} d x
$$

In view of Lemma 2.1, we can substitute $\phi=\phi_{u}$ into problem (1.1) and rewrite it as a single equation

$$
\begin{equation*}
-\left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \Delta u+\lambda \phi_{u} u=Q(x)|u|^{p-2} u, \quad u \in E . \tag{2.1}
\end{equation*}
$$

For the equivalent problem (2.1), the corresponding functional $I: E \mapsto \mathbb{R}$

$$
I(u)=\frac{a}{2} \int_{\Omega}|\nabla u|^{2} d x+\frac{b}{4}\left(\int_{\Omega}|\nabla u|^{2} d x\right)^{2}+\frac{\lambda}{4} \int_{\Omega} \phi_{u} u^{2} d x-\frac{1}{p} \int_{\Omega} Q(x)|u|^{p} d x
$$

is well defined. In addition, standard discussion shows that $I \in C^{1}(E, \mathbb{R})$ and

$$
\begin{equation*}
\left\langle I^{\prime}(u), \varphi\right\rangle=\left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \int_{\Omega} \nabla u \cdot \nabla \varphi d x+\lambda \int_{\Omega} \phi_{u} u \varphi d x-\int_{\Omega} Q(x)|u|^{p-2} u \varphi d x \tag{2.2}
\end{equation*}
$$

for any $u, \varphi \in E$. Clearly, critical points of $I$ are weak solutions of problem (2.1).
To estimate the second nonlocal term in (2.2) conveniently, we define

$$
D(g, h)=\int_{\Omega} \int_{\Omega} \frac{g(x) h(y)}{4 \pi|x-y|} d x d y
$$

Obviously, for each $u \in H_{0}^{1}(\Omega), D\left(u^{2}, u^{2}\right)=\int_{\Omega} \phi_{u} u^{2}$. Moreover, the following properties can be reached. For the proof, we refer to [32] and [23, p. 250].

## Lemma 2.2.

(1) $D(g, h)^{2} \leq D(g, g) D(h, h)$ for any $g, h \in L^{\frac{6}{5}}(\Omega)$;
(2) $D(u v, u v)^{2} \leq D\left(u^{2}, u^{2}\right) D\left(v^{2}, v^{2}\right)$ for any $u, v \in L^{\frac{12}{5}}(\Omega)$.

Next we prove a compactness condition for the functional $I$ which will be used later.
Lemma 2.3. Assume that ( $Q$ ) holds, then the functional I satisfies the Cerami condition.
Proof. Let $\left\{u_{n}\right\} \subset E$ be a Cerami sequence of $I$, that is, $\left|I\left(u_{n}\right)\right| \leq C$ and $\left(1+\left\|u_{n}\right\|\right)\left\|I^{\prime}\left(u_{n}\right)\right\|_{E^{-1}} \rightarrow$ 0 in the dual space $E^{-1}$. Firstly, we show that $\left\{u_{n}\right\}$ is bounded in $E$. In fact, for $\mu>4$, we have

$$
\begin{aligned}
C+o_{n}(1) \geq & I\left(u_{n}\right)-\frac{1}{\mu}\left\langle I^{\prime}\left(u_{n}\right), u_{n}\right\rangle \\
= & \left(\frac{1}{2}-\frac{1}{\mu}\right) a \int_{\Omega}\left|\nabla u_{n}\right|^{2} d x+\left(\frac{1}{4}-\frac{1}{\mu}\right) b\left(\int_{\Omega}\left|\nabla u_{n}\right|^{2} d x\right)^{2} \\
& +\left(\frac{1}{4}-\frac{1}{\mu}\right) \lambda \int_{\Omega} \phi_{u_{n}} u_{n}^{2} d x+\left(\frac{1}{\mu}-\frac{1}{p}\right) \int_{\Omega} Q(x)\left|u_{n}\right|^{p} d x .
\end{aligned}
$$

Due to the fact that $L^{\infty}(\Omega) \subset L^{\frac{6}{6-p}}(\Omega)$, by means of Hölder's inequality and the Sobolev embedding theorem, we obtain that

$$
\begin{equation*}
\left(\frac{1}{\mu}-\frac{1}{p}\right) \int_{\Omega} Q(x)\left|u_{n}\right|^{p} d x \geq\left(\frac{1}{\mu}-\frac{1}{p}\right)\|Q\|_{\frac{6}{6-p}}\left\|u_{n}\right\|_{6}^{p} \geq\left(\frac{1}{\mu}-\frac{1}{p}\right) C\left(\int_{\Omega}\left|\nabla u_{n}\right|^{2} d x\right)^{\frac{p}{2}} \tag{2.3}
\end{equation*}
$$

which yields that

$$
\begin{aligned}
\left(\frac{1}{2}-\frac{1}{\mu}\right) a \int_{\Omega}\left|\nabla u_{n}\right|^{2} d x+\left(\frac{1}{4}-\frac{1}{\mu}\right) b\left(\int_{\Omega}\left|\nabla u_{n}\right|^{2} d x\right)^{2} & +\left(\frac{1}{4}-\frac{1}{\mu}\right) \lambda \int_{\Omega} \phi_{u_{n}} u_{n}^{2} d x \\
& +\left(\frac{1}{\mu}-\frac{1}{p}\right) C\left(\int_{\Omega}\left|\nabla u_{n}\right|^{2} d x\right)^{\frac{p}{2}} \leq C+1 .
\end{aligned}
$$

Since $2<p<4<\mu$, the above inequality indicates that $\left\{u_{n}\right\}$ is bounded in $E$. Then, there exists a subsequence of $\left\{u_{n}\right\}$ (still denoted by $\left\{u_{n}\right\}$ ) such that

$$
\begin{equation*}
u_{n} \rightharpoonup u \quad \text { in } E ; \quad u_{n} \rightarrow u \quad \text { in } L^{q}(\Omega), \quad 1 \leq q<6 ; \quad u_{n} \rightarrow u \quad \text { a.e. in } \Omega . \tag{2.4}
\end{equation*}
$$

Since $u_{n} \rightarrow u$ in $L^{\frac{12}{5}}(\Omega)\left(q=\frac{12}{5}\right.$ in (2.4)), using Hölder's inequality, we have

$$
\begin{aligned}
& \left|\int_{\Omega}\left(\phi_{u_{n}} u_{n}-\phi_{u} u\right)\left(u_{n}-u\right) d x\right| \\
& \quad \leq\left\|\phi_{u_{n}}\right\|_{6}\left\|u_{n}-u\right\|_{\frac{12}{5}}^{2}+\left\|\phi_{u_{n}}-\phi_{u}\right\|_{6}\|u\|_{\frac{12}{5}}\left\|u_{n}-u\right\|_{\frac{12}{5}} \\
& \quad=o_{n}(1)
\end{aligned}
$$

From $(Q)$ and (2.4), using Hölder's inequality again gives that

$$
\begin{aligned}
& \left|\int_{\Omega} Q(x)\left(\left|u_{n}\right|^{p-2} u_{n}-|u|^{p-2} u\right)\left(u_{n}-u\right) d x\right| \\
& \quad \leq C\left(\left\|u_{n}\right\|_{p}^{p-1}\left\|u_{n}-u\right\|_{p}+\|u\|_{p}^{p-1}\left\|u_{n}-u\right\|_{p}\right) \\
& \quad=o_{n}(1) .
\end{aligned}
$$

Hence, the above two facts and the weak convergence of $u_{n} \rightharpoonup u$ in $E$ bring that

$$
\begin{aligned}
o_{n}(1)= & \left\langle I^{\prime}\left(u_{n}\right)-I^{\prime}(u), u_{n}-u\right\rangle \\
= & \left(a+b \int_{\Omega}\left|\nabla u_{n}\right|^{2} d x\right) \int_{\Omega}\left|\nabla\left(u_{n}-u\right)\right|^{2} d x \\
& +b\left(\int_{\Omega}\left|\nabla u_{n}\right|^{2} d x-\int_{\Omega}|\nabla u|^{2} d x\right) \int_{\Omega} \nabla u \cdot \nabla\left(u_{n}-u\right) d x \\
& +\lambda \int_{\Omega}\left(\phi_{u_{n}} u_{n}-\phi_{u} u\right)\left(u_{n}-u\right) d x \\
& -\int_{\Omega} Q(x)\left(\left|u_{n}\right|^{p-2} u_{n}-|u|^{p-2} u\right)\left(u_{n}-u\right) d x \\
\geq & a\left\|u_{n}-u\right\|^{2}+o_{n}(1),
\end{aligned}
$$

which implies that $u_{n} \rightarrow u$ in $E$.

## 3 Proof of Theorem 1.1

In this section, we use the method of invariant sets of descending flow to study the existence of sign-changing solutions for problem (1.1). To do this, we introduce an auxiliary operator $A: E \rightarrow E$. Explicitly, for any $u \in E$, we define $v=A u$ to be the unique solution for the equation

$$
\begin{equation*}
-\left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \Delta v+\lambda \phi_{u} v=Q(x)|u|^{p-2} u, \quad v \in E \tag{3.1}
\end{equation*}
$$

Clearly, $u$ is a fixed point of $A$ if and only if $u$ is a solution of (2.1).
Lemma 3.1. The operator $A$ is well defined, maps bounded sets to bounded sets and is continuous.
Proof. For any $u \in E$, define

$$
\begin{equation*}
J(v)=\frac{1}{2}\left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \int_{\Omega}|\nabla v|^{2} d x+\frac{\lambda}{2} \int_{\Omega} \phi_{u} v^{2} d x-\int_{\Omega} Q(x)|u|^{p-2} u v d x \tag{3.2}
\end{equation*}
$$

Then, $J \in C^{1}(E, \mathbb{R})$ and

$$
\begin{equation*}
\left\langle J^{\prime}(v), \omega\right\rangle=\left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \int_{\Omega} \nabla v \cdot \nabla \omega d x+\lambda \int_{\Omega} \phi_{u} v \omega d x-\int_{\Omega} Q(x)|u|^{p-2} u \omega d x \tag{3.3}
\end{equation*}
$$

for any $\omega \in E$. From $(Q)$ and the Sobolev embedding theorem, it is easy to verify that $J$ is coercive, bounded below and weakly lower semicontinuous. Thus, $J$ admits a unique minimizer $v=A u \in E$, which is the unique solution to (3.1), that is to say, $A$ is well defined.

Taking $v=\omega=A u$ in (3.3) leads to

$$
\left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \int_{\Omega}|\nabla A u|^{2} d x+\lambda \int_{\Omega} \phi_{u}(A u)^{2} d x=\int_{\Omega} Q(x) A u|u|^{p-2} u d x
$$

which implies, using ( $Q$ ) and the Sobolev embedding theorem, that

$$
a\|A u\| \leq C\|u\|^{p-1}
$$

Therefore, $A u$ is bounded whenever $u$ is bounded.
In the following, we prove that $A$ is continuous. Assuming $\left\{u_{n}\right\} \subset E$ with $u_{n} \rightarrow u$ in $E$ and taking $v=A u, v_{n}=A u_{n}$, we need to prove that $\left\|v_{n}-v\right\| \rightarrow 0$ in $E$. Based on the observation that $\left\langle J^{\prime}\left(v_{n}\right)-J^{\prime}(v), v_{n}-v\right\rangle=0$, that is,

$$
\begin{align*}
(a+b & \left.\int_{\Omega}\left|\nabla u_{n}\right|^{2} d x\right) \int_{\Omega}\left|\nabla\left(v_{n}-v\right)\right|^{2} d x \\
= & b\left(\int_{\Omega}|\nabla u|^{2} d x-\int_{\Omega}\left|\nabla u_{n}\right|^{2} d x\right) \int_{\Omega} \nabla v \cdot \nabla\left(v_{n}-v\right) d x  \tag{3.4}\\
& +\lambda \int_{\Omega}\left(\phi_{u_{n}} v_{n}-\phi_{u} v\right)\left(v-v_{n}\right) d x \\
& +\int_{\Omega} Q(x)\left(\left|u_{n}\right|^{p-2} u_{n}-|u|^{p-2} u\right)\left(v_{n}-v\right) d x
\end{align*}
$$

and $u_{n} \rightarrow u$ in $E$, it is sufficient to estimate the second and third terms in the right side of (3.4). Indeed, using Hölder's inequality, one has

$$
\begin{align*}
\int_{\Omega}\left(\phi_{u_{n}} v_{n}-\phi_{u} v\right)\left(v-v_{n}\right) d x & \leq \int_{\Omega}\left(\phi_{u_{n}} v-\phi_{u} v\right)\left(v-v_{n}\right) d x \\
& \leq\left\|\phi_{u_{n}}-\phi_{u}\right\|_{3}\|v\|_{3}\left\|v-v_{n}\right\|_{3}  \tag{3.5}\\
& \leq C\left\|\phi_{u_{n}}-\phi_{u}\right\|_{3}\|v\|\left\|v-v_{n}\right\|
\end{align*}
$$

where $\left\|\phi_{u_{n}}-\phi_{u}\right\|_{3} \rightarrow 0$ due to Lemma 2.1. In addition, according to $(Q)$ and Theorem A. 2 in [42], we have

$$
Q(x)\left(\left|u_{n}\right|^{p-2} u_{n}-|u|^{p-2} u\right) \rightarrow 0 \quad \text { in } \quad L^{\frac{p}{p-1}}(\Omega)
$$

which, combining with Hölder's inequality, states that

$$
\begin{equation*}
\left|\int_{\Omega} Q(x)\left(\left|u_{n}\right|^{p-2} u_{n}-|u|^{p-2} u\right)\left(v_{n}-v\right) d x\right| \leq C\left\|Q(x)\left(\left|u_{n}\right|^{p-2} u_{n}-|u|^{p-2} u\right)\right\|_{\frac{p}{p-1}}\left\|v_{n}-v\right\| . \tag{3.6}
\end{equation*}
$$

Thus, (3.4), (3.5) and (3.6) imply that

$$
\begin{aligned}
a\left\|v_{n}-v\right\|^{2} \leq & \left.b\left|\int_{\Omega}\right| \nabla u\right|^{2} d x-\int_{\Omega}\left|\nabla u_{n}\right|^{2} d x \mid\|v\|\| \| v_{n}-v \| \\
& +C \lambda\left\|\phi_{u_{n}}-\phi_{u}\right\|_{3}\|v\|\left\|v-v_{n}\right\| \\
& +C\left\|Q(x)\left(\left|u_{n}\right|^{p-2} u_{n}-|u|^{p-2} u\right)\right\|_{\frac{p}{p-1}}\left\|v_{n}-v\right\| \\
= & o_{n}(1)\left\|v_{n}-v\right\|,
\end{aligned}
$$

which means that $A$ is continuous.

## Lemma 3.2.

(1) $\left\langle I^{\prime}(u), u-A u\right\rangle \geq a\|u-A u\|^{2}$ for any $u \in E$;
(2) $\left\|I^{\prime}(u)\right\|_{E^{-1}} \leq\left[a+(C \lambda+b)\|u\|^{2}\right]\|u-A u\|$ for some $C>0$ and all $u \in E$.

Proof. Since $A u$ is the solution of (3.1), we see that

$$
\begin{aligned}
\left\langle I^{\prime}(u), u-A u\right\rangle= & \left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \int_{\Omega} \nabla u \cdot \nabla(u-A u) d x \\
& +\lambda \int_{\Omega} \phi_{u} u(u-A u) d x-\int_{\Omega} Q(x)|u|^{p-2} u(u-A u) d x \\
= & \left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \int_{\Omega}|\nabla(u-A u)|^{2} d x+\lambda \int_{\Omega} \phi_{u}(u-A u)^{2} d x
\end{aligned}
$$

which implies that $\left\langle I^{\prime}(u), u-A u\right\rangle \geq a\|u-A u\|^{2}$ for any $u \in E$.
By Hölder's inequality, Lemmas 2.1 and 2.2, for any $\varphi \in E$, we have

$$
\begin{aligned}
\left\langle I^{\prime}(u), \varphi\right\rangle & =\left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \int_{\Omega} \nabla(u-A u) \cdot \nabla \varphi d x+\lambda \int_{\Omega} \phi_{u}(u-A u) \varphi d x \\
& =\left(a+b\|u\|^{2}\right)(u-A u, \varphi)+\lambda D\left(u^{2},(u-A u) \varphi\right) \\
& \leq\left(a+b\|u\|^{2}\right)\|u-A u\|\|\varphi\|+C \lambda\|u\|^{2}\|u-A u\|\|\varphi\| \\
& \leq\left[a+(C \lambda+b)\|u\|^{2}\right]\|u-A u\|\|\varphi\| .
\end{aligned}
$$

Thus, $\left\|I^{\prime}(u)\right\|_{E^{-1}} \leq\left[a+(C \lambda+b)\|u\|^{2}\right]\|u-A u\|$ for any $u \in E$.
Lemma 3.3. Let $\delta_{1}<\delta_{2}$ and $\alpha>0$, there exists $\beta>0$ such that $\|u-A u\| \geq \beta$ if $u \in E$, $I(u) \in\left[\delta_{1}, \delta_{2}\right]$ and $\left\|I^{\prime}(u)\right\|_{E^{-1}} \geq \alpha$.

Proof. Let $\mu>4$, for $u \in E$, using $\left\langle J^{\prime}(A u), u\right\rangle=0$, we have

$$
\begin{aligned}
I(u)- & \frac{1}{\mu}\left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \int_{\Omega} \nabla u \cdot \nabla(u-A u) d x-\frac{\lambda}{\mu} \int_{\Omega} \phi_{u} u(u-A u) d x \\
= & \left(\frac{1}{2}-\frac{1}{\mu}\right) a \int_{\Omega}|\nabla u|^{2} d x+\left(\frac{1}{4}-\frac{1}{\mu}\right) b\left(\int_{\Omega}|\nabla u|^{2} d x\right)^{2} \\
& +\left(\frac{1}{4}-\frac{1}{\mu}\right) \lambda \int_{\Omega} \phi_{u} u^{2} d x+\left(\frac{1}{\mu}-\frac{1}{p}\right) \int_{\Omega} Q(x)|u|^{p} d x .
\end{aligned}
$$

Using (2.3), Hölder's inequality, Lemmas 2.1 and 2.2, we obtain

$$
\begin{aligned}
\left(\frac{1}{2}\right. & \left.-\frac{1}{\mu}\right) a\|u\|^{2}+\left(\frac{1}{4}-\frac{1}{\mu}\right) b\|u\|^{4}+\left(\frac{1}{4}-\frac{1}{\mu}\right) \lambda \int_{\Omega} \phi_{u} u^{2} d x+\left(\frac{1}{\mu}-\frac{1}{p}\right) C\|u\|^{p} \\
& \leq|I(u)|+\frac{1}{\mu}\left(a+b \int_{\Omega}|\nabla u|^{2} d x\right)\left|\int_{\Omega} \nabla u \cdot \nabla(u-A u) d x\right|+\left|\frac{\lambda}{\mu} \int_{\Omega} \phi_{u} u(u-A u) d x\right| \\
& \leq|I(u)|+\frac{1}{\mu}\left(a+b\|u\|^{2}\right)\|u\|\|u-A u\|+\frac{\lambda}{\mu}\left(\int_{\Omega} \phi_{u}(u-A u)^{2} d x\right)^{\frac{1}{2}}\left(\int_{\Omega} \phi_{u} u^{2} d x\right)^{\frac{1}{2}} \\
& \leq|I(u)|+\frac{1}{\mu}\left(a+b\|u\|^{2}\right)\|u\|\|u-A u\|+C\|u\|\|u-A u\|\left(\int_{\Omega} \phi_{u} u^{2} d x\right)^{\frac{1}{2}} \\
& \leq|I(u)|+\frac{1}{\mu}\left(a+b\|u\|^{2}\right)\|u\|\|u-A u\|+\frac{C}{2}\left(\|u\|^{2}+\int_{\Omega} \phi_{u} u^{2} d x\right)\|u-A u\| .
\end{aligned}
$$

If there exists $\left\{u_{n}\right\} \subset E$ with $I\left(u_{n}\right) \in\left[\delta_{1}, \delta_{2}\right]$ and $\left\|I^{\prime}\left(u_{n}\right)\right\|_{E^{-1}} \geq \alpha$ such that $\left\|u_{n}-A u_{n}\right\| \rightarrow 0$ as $n \rightarrow \infty$, since $2<p<4$, from the above inequality, we deduce that $\left\{u_{n}\right\}$ is bounded. Then, from Lemma 3.2-(2), we see that $\left\|I^{\prime}\left(u_{n}\right)\right\|_{E^{-1}} \rightarrow 0$ as $n \rightarrow \infty$, which is a contradiction. Thus, the proof is completed.

In the sequel, we introduce the positive and negative cones in $E$ defined as follows

$$
P^{+}:=\{u \in E: u \geq 0\} \quad \text { and } \quad P^{-}:=\{u \in E: u \leq 0\} .
$$

For given $\varepsilon>0$, two open convex subsets of $E$ are chosen in the following forms

$$
P_{\varepsilon}^{+}:=\left\{u \in E: \operatorname{dist}\left(u, P^{+}\right)<\varepsilon\right\} \quad \text { and } \quad P_{\varepsilon}^{-}:=\left\{u \in E: \operatorname{dist}\left(u, P^{-}\right)<\varepsilon\right\},
$$

where $\operatorname{dist}\left(u, P^{ \pm}\right)=\inf _{v \in P^{ \pm}}\|u-v\|$. Obviously, $P_{\varepsilon}^{-}=-P_{\varepsilon}^{+}$. Let $W=P_{\varepsilon}^{+} \cup P_{\varepsilon}^{-}$, then $W$ is an open and symmetric subset of $E$, and $E \backslash W$ contains only sign-changing functions. To find solutions for problem (1.1) in $E \backslash W$, we establish the following result which provides invariance properties for the convex sets $P_{\varepsilon}^{ \pm}$.

Lemma 3.4. There exists $\varepsilon_{0}>0$ such that for all $0<\varepsilon \leq \varepsilon_{0}$, there hold

$$
A\left(\partial P_{\varepsilon}^{+}\right) \subset P_{\varepsilon}^{+} \quad \text { and } \quad A\left(\partial P_{\varepsilon}^{-}\right) \subset P_{\varepsilon}^{-} .
$$

Proof. Let $u \in E$ and $v=A u$ satisfying (3.1). Notice that for any $2 \leq q \leq 6$, there exists $C_{q}>0$ such that

$$
\begin{equation*}
\left\|u^{+}\right\|_{q}=\inf _{v \in P^{-}}\|u-v\|_{q} \leq C_{q} \inf _{v \in P^{-}}\|u-v\|=C_{q} \operatorname{dist}\left(u, P^{-}\right) . \tag{3.7}
\end{equation*}
$$

Then, due to the fact that $\operatorname{dist}\left(v, P^{-}\right) \leq\left\|v^{+}\right\|$, we have

$$
\begin{aligned}
a \operatorname{dist}\left(v, P^{-}\right)\left\|v^{+}\right\| & \leq a\left\|v^{+}\right\|^{2} \\
& \leq\left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \int_{\Omega}\left|\nabla v^{+}\right|^{2} d x+\lambda \int_{\Omega} \phi_{u} v v^{+} d x \\
& =\int_{\Omega} Q(x)|u|^{p-2} u v^{+} d x \leq \int_{\Omega} Q(x)\left|u^{+}\right|^{p-2} u^{+} v^{+} d x \\
& =\int_{\Omega} Q(x)\left|u^{+}\right|^{p-1} v^{+} d x \leq\|Q\|_{\frac{6}{6-p}}\left\|u^{+}\right\|_{6}^{p-1}\left\|v^{+}\right\|_{6} \\
& \leq C\left(\operatorname{dist}\left(u, P^{-}\right)\right)^{p-1}\left\|v^{+}\right\|,
\end{aligned}
$$

which means that

$$
\operatorname{dist}\left(v, P^{-}\right) \leq \frac{C}{a}\left(\operatorname{dist}\left(u, P^{-}\right)\right)^{p-1} .
$$

Therefore, for $\varepsilon_{0} \in\left(0,\left(\frac{a}{2 C}\right)^{\frac{1}{p-2}}\right)$ with $0<\varepsilon \leq \varepsilon_{0}$, it holds that

$$
\operatorname{dist}\left(v, P^{-}\right) \leq \frac{1}{2} \varepsilon<\varepsilon \quad \text { for any } \quad u \in P_{\varepsilon}^{-} .
$$

That is to say, $A\left(\partial P_{\varepsilon}^{-}\right) \subset P_{\varepsilon}^{-}$. In a similar way, one also has $A\left(\partial P_{\varepsilon}^{+}\right) \subset P_{\varepsilon}^{+}$.
Let $K=\left\{u \in E: I^{\prime}(u)=0\right\}$. Since $A$ is merely continuous, it may by itself not be the right operator to construct a descending flow for the functional $I$, and we need an improved operator $B: E \backslash K \rightarrow E$ which is locally Lipschitz continuous and inherits the main properties of $A$.

Lemma 3.5. For $0<\varepsilon \leq \varepsilon_{0}$, there exists a locally Lipschitz continuous odd operator $B: E \backslash K \rightarrow E$ such that
(1) $\frac{1}{2}\|u-A u\| \leq\|u-B u\| \leq 2\|u-A u\| ;$
(2) $\left\langle I^{\prime}(u), u-B u\right\rangle \geq \frac{1}{2} a\|u-A u\|^{2}$;
(3) $B\left(\partial P_{\varepsilon}^{+}\right) \subset P_{\varepsilon}^{+}, B\left(\partial P_{\varepsilon}^{-}\right) \subset P_{\varepsilon}^{-}$.

Proof. The proof is similar to that of [9, Lemma 4.1] and [11, Lemma 2.1], so we omit the details.

By means of the invariant set of descending flow, we are intended to establish the existence of sign-changing solutions for problem (1.1). Here, we use the known abstract critical theorem given by [24, Theorem 2.4], and include its statement for the sake of completeness in the form of a proposition.

Let $X$ be a complete metric space with the metric $d, h \in C^{1}(X, \mathbb{R}), P_{1}, P_{2} \subset X$ be open subsets, $M=P_{1} \cap P_{2}, \Sigma=\partial P_{1} \cap \partial P_{2}$ and $W=P_{1} \cup P_{2}$. For $c \in \mathbb{R}, h^{c}=\{x \in X: h(x) \leq c\}$ and $K_{c}=\left\{x \in X: h(x)=c, h^{\prime}(x)=0\right\}$.

Definition 3.6 ([24]). $\left\{P_{1}, P_{2}\right\}$ is called an admissible family of invariant sets with respect to $h$ at level $c$, provided that the following deformation property holds: if $K_{c} \backslash W=\varnothing$ there exists $\varepsilon_{0}>0$ such that for $0<\varepsilon<\varepsilon_{0}$, there exists a continuous map $\eta: X \rightarrow X$ satisfying
(1) $\eta\left(\overline{P_{i}}\right) \subset \overline{P_{i}}, i=1,2 ;$
(2) $\left.\eta\right|_{h^{c-2 e}}=I d$;
(3) $\eta\left(h^{c+\varepsilon} \backslash W\right) \subset h^{c-\varepsilon}$.

Proposition 3.7 ([24]). Assume $\left\{P_{1}, P_{2}\right\}$ is an admissible family of invariant sets with respect to $h$ at level c for $c \geq c_{*}:=\inf _{u \in \Sigma} h(u)$ and there exists a map $\psi_{0}: \triangle \rightarrow X$ satisfying
(1) $\psi_{0}\left(\partial_{i} \triangle\right) \subset P_{i}, i=1,2 ;$
(2) $\psi_{0}\left(\partial_{0} \triangle\right) \cap M=\varnothing$;
(3) $c_{0}=\sup _{u \in \psi_{0}\left(\partial_{0} \Delta\right)} h(u)<c_{*}$,
where $\triangle=\left\{\left(t_{1}, t_{2}\right) \in \mathbb{R}^{2}: t_{1}, t_{2} \geq 0, t_{1}+t_{2} \leq 1\right\}, \partial_{0} \triangle=\triangle \cap\left\{t_{1}+t_{2}=1\right\}$ and $\partial_{i} \Delta=$ $\Delta \cap\left\{t_{i}=0\right\}, i=1,2$. Define

$$
c=\inf _{\psi \in \Gamma} \sup _{u \in \psi(\Delta) \backslash W} h(u),
$$

where $\Gamma:=\left\{\psi \in C(\triangle, X): \psi\left(\partial_{i} \triangle\right) \subset P_{i}, i=1,2,\left.\psi\right|_{\partial_{0} \Delta}=\psi_{0}\right\}$. Then $c$ is a critical value of $h$ and $K_{c} \backslash W \neq \varnothing$.

Now we use Proposition 3.7 to obtain the existence of one sign-changing solution for problem (1.1). Here, we choose $X=E, h=I, P_{1}=P_{\varepsilon}^{+}$and $P_{2}=P_{\varepsilon}^{-}$, then, $M=P_{\varepsilon}^{+} \cap P_{\varepsilon}^{-}, \Sigma=$ $\partial P_{\varepsilon}^{+} \cap \partial P_{\varepsilon}^{-}$and $W=P_{\varepsilon}^{+} \cup P_{\varepsilon}^{-}$. The following lemma implies that $\left\{P_{\varepsilon}^{+}, P_{\varepsilon}^{-}\right\}$is an admissible family of invariant sets for the functional $I$ at any level $c \in \mathbb{R}$.

Lemma 3.8. Assume ( $Q$ ) holds. If $K_{c} \backslash W=\varnothing$, then there exists $\varepsilon_{0}>0$ such that for $0<\varepsilon<\varepsilon^{\prime}<\varepsilon_{0}$, there exists a continuous map $\sigma:[0,1] \times E \rightarrow E$ satisfying
(1) $\sigma(0, u)=u$ for all $u \in E$;
(2) $\sigma(t, u)=u$ for $t \in[0,1], u \notin I^{-1}\left(\left[c-\varepsilon^{\prime}, c+\varepsilon^{\prime}\right]\right)$;
(3) $\sigma\left(1, I^{c+\varepsilon} \backslash W\right) \subset I^{c-\varepsilon}$;
(4) $\sigma\left(t, \overline{P_{\varepsilon}^{+}}\right) \subset \overline{P_{\varepsilon}^{+}}, \sigma\left(t, \overline{P_{\varepsilon}^{-}}\right) \subset \overline{P_{\varepsilon}^{-}}, t \in[0,1]$.

Proof. The proof is similar to that of many existing literatures (see [21,26]). We include its proof for the sake of completeness. If $K_{c} \backslash W=\varnothing$, then $K_{c} \subset W$. Thus, $2 \delta:=\operatorname{dist}\left(K_{c}, \partial W\right)>0$ on account of $K_{c}$ is compact by Lemma 2.3. For this $\delta$, we have $N_{\delta}\left(K_{c}\right):=\left\{u \in E: \operatorname{dist}\left(u, K_{c}\right)<\delta\right\} \subset$ $W$. Since I satisfies the Cerami condition, there exist $\varepsilon_{0}, \alpha>0$ such that

$$
\left\|I^{\prime}(u)\right\|_{E^{-1}} \geq \alpha \quad \text { for } u \in I^{-1}\left(\left[c-\varepsilon_{0}, c+\varepsilon_{0}\right]\right) \backslash N_{\frac{\delta}{2}}\left(K_{c}\right) .
$$

By Lemmas 3.3 and 3.5-(1), there exists $\beta>0$ such that

$$
\|u-B u\| \geq \frac{\beta}{2} \quad \text { for } u \in I^{-1}\left(\left[c-\varepsilon_{0}, c+\varepsilon_{0}\right]\right) \backslash N_{\frac{\delta}{2}}\left(K_{c}\right) .
$$

Furthermore, owing to Lemma 3.5-(2), we obtain

$$
\begin{equation*}
\left\langle I^{\prime}(u), \frac{u-B u}{\|u-B u\|}\right\rangle \geq \frac{1}{8} a\|u-B u\| \geq \theta:=\frac{a \beta}{16} . \tag{3.8}
\end{equation*}
$$

Decreasing $\varepsilon_{0}$ if necessary, we assume $\varepsilon_{0} \leq \frac{\theta \delta}{4}$. Take two even Lipschitz continuous functions $p, q: E \rightarrow[0,1]$ such that

$$
p(u)=\left\{\begin{array}{ll}
0, & u \in N_{\frac{\delta}{4}}\left(K_{c}\right), \\
1, & u \notin N_{\frac{\delta}{2}}\left(K_{c}\right),
\end{array} \quad \text { and } \quad q(u)= \begin{cases}0, & u \notin I^{-1}\left(\left[c-\varepsilon^{\prime}, c+\varepsilon^{\prime}\right]\right), \\
1, & u \in I^{-1}([c-\varepsilon, c+\varepsilon]),\end{cases}\right.
$$

and consider the following initial value problem

$$
\left\{\begin{array}{l}
\frac{d \tau(t, u)}{d t}=-\Phi(\tau(t, u))  \tag{3.9}\\
\tau(0, u)=u
\end{array}\right.
$$

where $\Phi(u)=p(u) q(u) \frac{u-B u}{\|u-B u\|}$. Obviously, $\Phi(u)$ is locally Lipschitz continuous, so the existence and uniqueness theory of ODE in Banach space implies that (3.9) admits a unique solution $\tau(\cdot, u) \in C(\mathbb{R}, E)$. Define $\sigma$ on $[0,1] \times E$ by $\sigma(t, u):=\tau\left(\frac{2 \varepsilon}{\theta} t, u\right)$, it is sufficient to check (3), because (1)-(2) are obvious, and (4) is a consequence of Lemma 3.5-(3).

To do this, choose $u \in I^{c+\varepsilon} \backslash W$. By (3.9), it easy to see $\frac{d I(\tau(t, u))}{d t} \leq 0$, namely, $I(\tau(t, u))$ is nonincreasing for $t \geq 0$. Then, if there exists $t_{0} \in\left[0, \frac{2 \varepsilon}{\theta}\right]$ such that $I\left(\tau\left(t_{0}, u\right)\right)<c-\varepsilon$, we have

$$
I(\sigma(1, u))=I\left(\tau\left(\frac{2 \varepsilon}{\theta}, u\right)\right)<c-\varepsilon
$$

Otherwise, for any $t \in\left[0, \frac{2 \varepsilon}{\theta}\right], I(\tau(t, u)) \geq c-\varepsilon$, then $\tau(t, u) \in I^{-1}([c-\varepsilon, c+\varepsilon])$. We claim that for any $t \in\left[0, \frac{2 \varepsilon}{\theta}\right], \tau(t, u) \notin N_{\frac{\delta}{2}}\left(K_{c}\right)$. If not, there exists $t_{0} \in\left[0, \frac{2 \varepsilon}{\theta}\right]$ such that $\tau\left(t_{0}, u\right) \in N_{\frac{\delta}{2}}\left(K_{c}\right)$, then, since $N_{\delta}\left(K_{c}\right) \subset W$, we obtain

$$
\frac{\delta}{2} \leq\left\|\tau\left(t_{0}, u\right)-u\right\| \leq \int_{0}^{t_{0}}\left\|\tau^{\prime}(s, u)\right\| d s \leq t_{0}<\frac{2 \varepsilon_{0}}{\theta} \leq \frac{\delta}{2}
$$

which is a contradiction. Therefore, $p(\tau(t, u)) q(\tau(t, u)) \equiv 1$ for $t \in\left[0, \frac{2 \varepsilon}{\theta}\right]$. Hence, by (3.8) and (3.9),

$$
\begin{aligned}
I(\sigma(1, u)) & =I\left(\tau\left(\frac{2 \varepsilon}{\theta}, u\right)\right) \\
& =I(u)-\int_{0}^{\frac{2 \varepsilon}{\theta}}\left\langle I^{\prime}(\tau(s, u)), \Phi(\tau(s, u))\right\rangle d s \\
& \leq c+\varepsilon-\frac{2 \varepsilon}{\theta} \theta \\
& =c-\varepsilon .
\end{aligned}
$$

Thus, the proof is completed.
Lemma 3.9. If $\varepsilon>0$ small enough, then $I(u) \geq a \varepsilon^{2}$ for any $u \in \Sigma=\partial P_{\varepsilon}^{+} \cap \partial P_{\varepsilon}^{-}$.
Proof. For any $u \in \Sigma$, it has $\left\|u^{ \pm}\right\|=\left\|u-u^{\mp}\right\| \geq \operatorname{dist}\left(u, P^{\mp}\right)=\varepsilon$. Then, using ( $Q$ ) and (3.7), we have

$$
\begin{aligned}
I(u) & =\frac{a}{2} \int_{\Omega}|\nabla u|^{2} d x+\frac{b}{4}\left(\int_{\Omega}|\nabla u|^{2} d x\right)^{2}+\frac{\lambda}{4} \int_{\Omega} \phi_{u} u^{2} d x-\frac{1}{p} \int_{\Omega} Q(x)|u|^{p} d x \\
& \geq \frac{a}{2}\|u\|^{2}-\frac{C}{p}\|u\|_{p}^{p} \geq 2 a \varepsilon^{2}-\frac{C}{p} \varepsilon^{p} \geq a \varepsilon^{2}
\end{aligned}
$$

for $\varepsilon>0$ small enough.

Proof of Theorem 1.1 (Existence part). We construct a suitable map $\psi_{0}$ satisfying the properties in Proposition 3.7. Choose $v_{1} \in P_{\varepsilon}^{-}, v_{2} \in P_{\varepsilon}^{+}$such that $\operatorname{supp}\left(v_{1}\right) \cap \operatorname{supp}\left(v_{2}\right)=\varnothing$ and $\inf _{\text {supp }\left(v_{1}\right) \cup \operatorname{supp}\left(v_{2}\right)} Q(x)>0$. For

$$
\rho=\left(\rho_{1}, \rho_{2}\right) \in \Delta=\left\{t \in \mathbb{R}^{2}: t=\left(t_{1}, t_{2}\right), t_{1}, t_{2} \geq 0, t_{1}+t_{2} \leq 1\right\}
$$

define

$$
\psi_{0}(\rho)(x)=R\left(\rho_{1} v_{1}\left(R^{-2} x\right)+\rho_{2} v_{2}\left(R^{-2} x\right)\right),
$$

where $R$ is a positive constant to be determined later. It is obvious that, for any $\rho=\left(0, \rho_{2}\right) \in$ $\partial_{1} \Delta$ and $\rho=\left(\rho_{1}, 0\right) \in \partial_{2} \Delta$, we have

$$
\psi_{0}(\rho)(x)=R\left(\rho_{2} v_{2}\left(R^{-2} x\right)\right) \in P_{\varepsilon}^{+} \quad \text { and } \quad \psi_{0}(\rho)(x)=R\left(\rho_{1} v_{1}\left(R^{-2} x\right)\right) \in P_{\varepsilon}^{-}
$$

respectively. Thus, $\psi_{0}\left(\partial_{1} \Delta\right) \subset P_{\varepsilon}^{+}$and $\psi_{0}\left(\partial_{2} \Delta\right) \subset P_{\varepsilon}^{-}$.
From Lemma 3.9, we have $c_{\lambda}^{*}:=\inf _{u \in \Sigma} I(u) \geq a \varepsilon^{2}$. Next, we verify that

$$
c_{0}=\sup _{u \in \psi_{0}\left(\partial_{0} \Delta\right)} I(u)<c_{\lambda}^{*} .
$$

Set $u_{s}=\psi_{0}(s, 1-s)$ for $s \in[0,1]$, a direct computation shows that

$$
\begin{aligned}
\int_{\Omega}\left|\nabla u_{s}\right|^{2} d x & =R^{4} \int_{\Omega}\left(s^{2}\left|\nabla v_{1}\right|^{2}+(1-s)^{2}\left|\nabla v_{2}\right|^{2}\right) d x \\
\int_{\Omega} \phi_{u_{s}} u_{s}^{2} d x & =R^{14} \int_{\Omega} \phi_{\hat{u}_{s}} \hat{u}_{s}^{2} d x, \quad \text { where } \hat{u}_{s}=s v_{1}+(1-s) v_{2}, \\
\int_{\Omega}\left|u_{s}\right|^{p} d x & =R^{p+6} \int_{\Omega}\left(s^{p}\left|v_{1}\right|^{p}+(1-s)^{p}\left|v_{2}\right|^{p}\right) d x .
\end{aligned}
$$

Based on the equalities above, we have

$$
\begin{align*}
I\left(u_{s}\right)= & \frac{a}{2} \int_{\Omega}\left|\nabla u_{s}\right|^{2} d x+\frac{b}{4}\left(\int_{\Omega}\left|\nabla u_{s}\right|^{2} d x\right)^{2}+\frac{\lambda}{4} \int_{\Omega} \phi_{u_{s}} u_{s}^{2} d x \\
& -\frac{1}{p} \int_{\operatorname{supp}\left(v_{1}\right) \cup \operatorname{supp}\left(v_{2}\right)} Q(x)\left|u_{s}\right|^{p} d x \\
\leq & \frac{a}{2} \int_{\Omega}\left|\nabla u_{s}\right|^{2} d x+\frac{b}{4}\left(\int_{\Omega}\left|\nabla u_{s}\right|^{2} d x\right)^{2}+\frac{\lambda}{4} \int_{\Omega} \phi_{u_{s}} u_{s}^{2} d x \\
& -\min _{\operatorname{supp}\left(v_{1}\right) \cup \operatorname{supp}\left(v_{2}\right)} Q(x) \int_{\Omega}\left|u_{s}\right|^{p} d x  \tag{3.10}\\
= & \frac{a}{2} R^{4} \int_{\Omega}\left(s^{2}\left|\nabla v_{1}\right|^{2}+(1-s)^{2}\left|\nabla v_{2}\right|^{2}\right) d x \\
& +\frac{b}{4} R^{8}\left(\int_{\Omega}\left(s^{2}\left|\nabla v_{1}\right|^{2}+(1-s)^{2}\left|\nabla v_{2}\right|^{2}\right) d x\right)^{2} \\
& +\frac{\lambda}{4} R^{14} \int_{\Omega} \phi_{\hat{u}_{s}} u_{s}^{2} d x-C R^{p+6} \int_{\Omega}\left(s^{p}\left|v_{1}\right|^{p}+(1-s)^{p}\left|v_{2}\right|^{p}\right) d x .
\end{align*}
$$

Then, taking $0<\lambda \leq \lambda_{R}=R^{-6}=: \lambda^{*}$ in (3.10) (which means that $\lambda$ is sufficiently small for $R$ large enough), we obtain

$$
\begin{align*}
I\left(u_{s}\right) \leq & \frac{a}{2} R^{4} \int_{\Omega}\left(s^{2}\left|\nabla v_{1}\right|^{2}+(1-s)^{2}\left|\nabla v_{2}\right|^{2}\right) d x \\
& +\frac{b}{4} R^{8}\left(\int_{\Omega}\left(s^{2}\left|\nabla v_{1}\right|^{2}+(1-s)^{2}\left|\nabla v_{2}\right|^{2}\right) d x\right)^{2}  \tag{3.11}\\
& +\frac{1}{4} R^{8} \int_{\Omega} \phi_{u_{s}} \hat{u}_{s}^{2} d x-C R^{p+6} \int_{\Omega}\left(s^{p}\left|v_{1}\right|^{p}+(1-s)^{p}\left|v_{2}\right|^{p}\right) d x .
\end{align*}
$$

Since $2<p<4$, by (3.11), it is evident that $I\left(u_{s}\right) \rightarrow-\infty$ as $R \rightarrow \infty$ uniformly for $s \in[0,1]$. Consequently, choosing $R$ large enough and independent of $s$, we have

$$
\begin{equation*}
c_{0}=\sup _{u \in \psi_{0}\left(\partial_{0} \Delta\right)} I(u)<c_{\lambda}^{*}:=\inf _{u \in \Sigma} I(u) . \tag{3.12}
\end{equation*}
$$

Moreover, we observe that

$$
\begin{equation*}
\int_{\Omega}\left|u_{s}\right|^{2} d x=R^{4} \int_{\Omega}\left(s^{2}\left|v_{1}\right|^{2}+(1-s)^{2}\left|v_{2}\right|^{2}\right) d x \rightarrow \infty \quad \text { as } R \rightarrow \infty \tag{3.13}
\end{equation*}
$$

uniformly with respect to $s \in[0,1]$, which, combining with (3.7), indicates that $\psi_{0}\left(\partial_{0} \Delta\right) \cap M=$ $\varnothing$. Define

$$
c=\inf _{\psi \in \Gamma} \sup _{u \in \psi(\Delta) \backslash W} I(u),
$$

where $\Gamma:=\left\{\psi \in C(\triangle, E): \psi\left(\partial_{1} \triangle\right) \subset P_{\varepsilon}^{+}, \psi\left(\partial_{2} \triangle\right) \subset P_{\varepsilon}^{-},\left.\psi\right|_{\partial_{0} \triangle}=\psi_{0}\right\}$, and apply Proposition 3.7, there is a critical point $u \in K_{c} \backslash W$ which is a sign-changing solution of problem (1.1).

Next we turn to the existence of infinitely many sign-changing solutions for problem (1.1). To do this, we make use of Theorem 2.5 in [24] recalled below. Explicitly, let $X$ be a complete metric space with the metric $d$ and $h \in C^{1}(X, \mathbb{R})$, then we say $G: X \rightarrow X$ is an isometric involution if $G$ satisfies $G^{2}=I d$ and $d(G x, G y)=d(x, y)$ for $x, y \in X$. A subset $O \subset X$ is said to be symmetric if $G x \in O$ for any $x \in O$. The genus of a closed symmetric subset $O$ of $X \backslash\{0\}$ is denoted by $\gamma(O)$.
Definition 3.10 ([24]). Assume $G$ is an isometric involution of $X$ and $h$ is a $G$-invariant continuous functional on $X$ that is $h(G x)=h(x)$ for any $x \in X$. We say $P$ is a $G$-admissible invariant set with respect to $h$ at level $c$ if the following deformation property holds: there exist a symmetric open neighbourhood $N$ of $K_{c} \backslash(P \cup Q)$ with $\gamma(\bar{N})<\infty$ and $\varepsilon_{0}$ such that for $0<\varepsilon<\varepsilon_{0}$ there exists a continuous map $\eta: X \rightarrow X$ satisfying
(1) $\eta(\bar{P}) \subset \bar{P}, \eta(\bar{Q}) \subset \bar{Q}$, here $Q=G P$;
(2) $\eta \circ G=G \circ \eta$;
(3) $\left.\eta\right|_{h^{c-2 e}}=I d$;
(4) $\eta\left(h^{c+\varepsilon} \backslash(N \cup(P \cup Q))\right) \subset h^{c-\varepsilon}$.

Proposition 3.11 ([24]). Assume that $P$ is a G-admissible invariant set with respect to $h$ at level $c$ for $c \geq c_{*}:=\inf _{u \in \partial P \cap \partial Q} h(u)$ and for any $n \in \mathbb{N}$ there exists a continuous map $\psi_{n}: B_{n} \rightarrow X$ satisfying
(1) $\psi_{n}(0) \in P \cap Q$;
(2) $\psi_{n}\left(\partial B_{n}\right) \cap(P \cap Q)=\varnothing$;
(3) $\sup _{u \in \operatorname{Fix}_{G} \cup \psi_{n}\left(\partial B_{n}\right)} h(u)<c_{*}$,
where $B_{n}:=\left\{x \in \mathbb{R}^{n}:|x| \leq 1\right\}$ and Fix $_{G}:=\{u \in X: G u=u\}$. Define

$$
c_{j}=\inf _{B \in \Gamma_{j}} \sup _{u \in B \backslash(P \cup Q)} h(u),
$$

where $\Gamma_{j}:=\left\{B: B=\psi\left(B_{n} \backslash Y\right), \psi \in G_{n}, n \geq j\right.$, and open subset $\left.Y=-Y \subset B_{n}, \gamma(\bar{Y}) \leq n-j\right\}$ and $G_{n}:=\left\{\psi: \psi \in C\left(B_{n}, X\right), \psi(-t)=G \psi(t), t \in B_{n}, \psi(0) \in P \cap Q\right.$ and $\left.\left.\psi\right|_{\partial B_{n}}=\psi_{n}\right\}$. Then $c_{j}, j \geq 2$, are critical values of $h$ with $c_{j} \rightarrow \infty$ and $K_{c_{j}} \backslash(P \cup Q) \neq \varnothing$.

To apply Proposition 3.11, we set $X=E, h=I, G=-I d, P=P_{\varepsilon}^{+}$. In addition, thanks to the nonlinearity in problem (1.1) is odd, as a sequence, $G$ is an isometric involution on $E, Q=-P_{\varepsilon}^{+}=P_{\varepsilon}^{-}$, and the functional $I$ is $G$-invariant continuous functional. Since $K_{c}$ is compact, there exists a symmetric open neighborhood $N$ of $K_{c} \backslash\left(P_{\varepsilon}^{+} \cup P_{\varepsilon}^{-}\right)$with $\gamma(\bar{N})<\infty$.

Lemma 3.12. Assume $(Q)$ holds true, then there exists $\varepsilon_{0}>0$ such that for $0<\varepsilon<\varepsilon^{\prime}<\varepsilon_{0}$, there exists a continuous map $\sigma:[0,1] \times E \rightarrow E$ satisfying
(1) $\sigma(0, u)=u$ for $u \in E$;
(2)
$\sigma(t, u)=u$ for $t \in[0,1], u \notin I^{-1}\left(\left[c-\varepsilon^{\prime}, c+\varepsilon^{\prime}\right]\right) ;$
(3) $\sigma(t,-u)=-\sigma(t, u)$ for $(t, u) \in[0,1] \times E$;
(4) $\sigma\left(1, I^{c+\varepsilon} \backslash\left(N \cup\left(P_{\varepsilon}^{+} \cup P_{\varepsilon}^{-}\right)\right)\right) \subset I^{c-\varepsilon}$;
(5) $\sigma\left(t, \overline{P_{\varepsilon}^{+}}\right) \subset \overline{P_{\varepsilon}^{+}}, \sigma\left(t, \overline{P_{\varepsilon}^{-}}\right) \subset \overline{P_{\varepsilon}^{-}}, t \in[0,1]$.

Proof. The proof is similar to Lemma 3.8. Since $I$ is even, thus $\sigma$ is odd in $u$. Here, we omit the details.

Combining Definition 3.10 with Lemma 3.12, we conclude that $P_{\varepsilon}^{+}$is a $G$-admissible set for the function $I$ at any level $c \in \mathbb{R}$.

Proof of Theorem 1.1 (Multiplicity part). According to the above discussion, we need to construct an appropriate continuous map $\psi_{n}: B_{n} \rightarrow E$ to apply Proposition 3.11. In order to achieve this point, for any $n \in \mathbb{N}$, we choose $\left\{v_{i}\right\}_{1}^{n} \in E$ with disjoint supports and $\inf _{\text {supp }\left(v_{i}\right)} Q(x)>0$, and define

$$
\psi_{n}(t)(x)=R_{n}\left(t_{1} v_{1}\left(R_{n}^{-2} x\right)+\cdots+t_{n} v_{n}\left(R_{n}^{-2} x\right)\right),
$$

where $t=\left(t_{1}, t_{2}, \ldots, t_{n}\right) \in B_{n}, R_{n}$ is a large number such that $\psi_{n}\left(\partial B_{n}\right) \cap\left(P_{\varepsilon}^{+} \cap P_{\varepsilon}^{-}\right)=\varnothing$ and

$$
\sup _{u \in \psi_{n}\left(\partial B_{n}\right)} I(u)<0<\inf _{u \in \partial P_{e}^{+} \cap \partial P_{\varepsilon}^{-}} I(u)
$$

as in (3.12) and (3.13). Obviously, $\psi_{n}(0)=0 \in P_{\varepsilon}^{+} \cap P_{\varepsilon}^{-}$and $\psi_{n}(-t)=-\psi_{n}(t)$ for $t \in B_{n}$. Define

$$
c_{j}=\inf _{B \in \Gamma_{j}} \sup _{u \in B \backslash\left(P_{\varepsilon}^{+} \cup P_{e}^{-}\right)} I(u),
$$

where $\Gamma_{j}$ is given in Proposition 3.11, then it follows that $c_{j}(j \geq 2)$ are critical values of $I$ with $c_{j} \rightarrow \infty$ as $j \rightarrow \infty$, and the corresponding critical points $u_{j} \in K_{c_{j}} \backslash\left(P_{\varepsilon}^{+} \cup P_{\varepsilon}^{-}\right)$are sign-changing solutions of problem (1.1).

## 4 Proof of Theorem 1.3

Under the assumptions of Theorem 1.3, we establish the existence and multiplicity of signchanging solutions for problem (1.1) in this section. Before proceeding, we point out that the energy functional is still denoted by $I$, and obviously the conclusions of Lemma 2.3 and Lemmas 3.1-3.9 are effective for $\lambda=1$. However, due to the fact that $p \in\left(2, \frac{12}{5}\right)$, we need to construct $\bar{\psi}_{0}$ different from $\psi_{0}$ in Theorem 1.1 to establish the proof of Theorem 1.3.

Proof of Theorem 1.3. Define

$$
\bar{\psi}_{0}=\bar{\psi}_{0}(\rho)(x)=R^{-1}\left(\rho_{1} v_{1}\left(R^{-m} x\right)+\rho_{2} v_{2}\left(R^{-m} x\right)\right),
$$

where $v_{1}, v_{2}, \rho=\left(\rho_{1}, \rho_{2}\right)$ are the same as in the proof of Theorem 1.1 and $m \in\left(\frac{p}{3}, \frac{4-p}{2}\right)$ is a constant dependent on $p$. Next, we check that $\bar{\psi}_{0}$ satisfies the properties in Proposition 3.11. Similar to the proof of Theorem 1.1, we obtain $\bar{\psi}_{0}\left(\partial_{1} \Delta\right) \subset P_{\varepsilon}^{+}$and $\bar{\psi}_{0}\left(\partial_{2} \Delta\right) \subset P_{\varepsilon}^{-}$. Therefore, it suffices to verify (2) and (3) of Proposition 3.11. Indeed, set $\bar{u}_{s}=\bar{\psi}_{0}(s, 1-s)$ for $s \in[0,1]$, the direct computations show that

$$
\begin{aligned}
\int_{\Omega}\left|\nabla \bar{u}_{s}\right|^{2} d x & =R^{-2+m} \int_{\Omega}\left(s^{2}\left|\nabla v_{1}\right|^{2}+(1-s)^{2}\left|\nabla v_{2}\right|^{2}\right) d x, \\
\int_{\Omega} \phi_{\bar{u}_{s}} \bar{u}_{s}^{2} d x & =R^{-4+5 m} \int_{\Omega} \phi_{\hat{u}_{s}} \hat{u}_{s}^{2} d x, \text { where } \hat{u}_{s}=s v_{1}+(1-s) v_{2}, \\
\int_{\Omega}\left|\bar{u}_{s}\right|^{p} d x & =R^{-p+3 m} \int_{\Omega}\left(s^{p}\left|v_{1}\right|^{p}+(1-s)^{p}\left|v_{2}\right|^{p}\right) d x,
\end{aligned}
$$

which signify that

$$
\begin{align*}
I\left(\bar{u}_{s}\right) \leq & \frac{a}{2} R^{-2+m} \int_{\Omega}\left(s^{2}\left|\nabla v_{1}\right|^{2}+(1-s)^{2}\left|\nabla v_{2}\right|^{2}\right) d x \\
& +\frac{b}{4} R^{-4+2 m}\left(\int_{\Omega}\left(s^{2}\left|\nabla v_{1}\right|^{2}+(1-s)^{2}\left|\nabla v_{2}\right|^{2}\right) d x\right)^{2}  \tag{4.1}\\
& +\frac{1}{4} R^{-4+5 m} \int_{\Omega} \phi_{\hat{u}_{s}} \hat{u}_{s}^{2} d x \\
& -C R^{-p+3 m} \int_{\Omega}\left(s^{p}\left|v_{1}\right|^{p}+(1-s)^{2}\left|v_{2}\right|^{p}\right) d x .
\end{align*}
$$

Since $2<p<\frac{12}{5}$ and $m \in\left(\frac{p}{3}, \frac{4-p}{2}\right)$, we get

$$
\begin{equation*}
\max \{-2+m,-4+2 m,-4+5 m,-p+3 m\}=-p+3 m>0 . \tag{4.2}
\end{equation*}
$$

Considering the above relationship in (4.1), we are led to $I\left(\bar{u}_{s}\right) \rightarrow-\infty$ as $R \rightarrow \infty$ uniformly for $s \in[0,1]$. In addition, from Lemma 3.9, we have known that $c_{1}^{*}:=\inf _{u \in \Sigma} I(u) \geq a \varepsilon^{2}$. Therefore, choosing $R$ large enough and independent of $s$ can guarantee that

$$
\bar{c}_{0}=\sup _{u \in \bar{\psi}_{0}\left(\partial_{0} \Delta\right)} I(u)<c_{1}^{*} .
$$

Meanwhile, it is obvious that

$$
\int_{\Omega}\left|\bar{u}_{s}\right|^{2} d x=R^{-2+3 m} \int_{\Omega}\left(s^{2}\left|v_{1}\right|^{2}+(1-s)^{2}\left|v_{2}\right|^{2}\right) d x \rightarrow \infty \quad \text { as } R \rightarrow \infty
$$

uniformly for $s \in[0,1]$, which, combining with (3.7), indicates that that $\bar{\psi}_{0}\left(\partial_{0} \Delta\right) \cap M=\varnothing$. Based on the above facts, define

$$
\bar{c}=\inf _{\psi \in \bar{\Gamma} u \in \psi(\Delta) \backslash W} \sup _{u} I(u),
$$

where $\bar{\Gamma}:=\left\{\psi \in C(\triangle, E): \psi\left(\partial_{1} \triangle\right) \subset P_{\varepsilon}^{+}, \psi\left(\partial_{2} \triangle\right) \subset P_{\varepsilon}^{-},\left.\psi\right|_{\partial_{0} \Delta}=\bar{\psi}_{0}\right\}$ and apply Proposition 3.7, we obtain the existence of sign-changing solution. The rest of proof with respect to multiplicity is very similar to that of Theorem 1.1. Actually, it is just necessary to use

$$
\bar{\psi}_{n}(t)(x)=R_{n}^{-1}\left(t_{1} v_{1}\left(R_{n}^{-m} x\right)+\cdots+t_{n} v_{n}\left(R_{n}^{-m} x\right)\right)
$$

instead of $\psi_{n}(t)(x)$ in the process of the proof of Theorem 1.1. Once $\bar{\psi}_{n}(t)(x)$ is determined as above, the remainder is just to repeat the proof of Theorem 1.1, so we omit the details.

## Acknowledgements

The authors would like to express their appreciation to the referee for valuable suggestions. This work is supported by National Natural Science Foundation of China (No. 11771044 and No. 12171039).

## References

[1] J. Albuquerque, R. Clemente, D. Ferraz, Existence of infinitely many small solutions for sublinear fractional Kirchhoff-Schrödinger-Poisson systems, Electron. J. Differential Equations 2019, No. 13, 1-16. MR3919653
[2] C. Alves, M. Souto, Existence of least energy nodal solution for a Schrödinger-Poisson system in bounded domains, Z. Angew. Math. Phys. 65(2014), 1153-1166. https://doi. org/10.1007/s00033-013-0376-3
[3] A. Ambrosetti, On Schrödinger-Poisson systems, Milan J. Math. 76(2008), 257-274. https://doi.org/10.1007/s00032-008-0094-z
[4] A. Ambrosetti, D. Ruiz, Multiple bound states for the Schrödinger-Poisson problem, Commun. Contemp. Math. 10(2008), 391-404. https://doi.org/10.1142/ S021919970800282X
[5] C. An, J. Yao, W. Han, The existence of the sign-changing solutions for the Kirchhoff-Schrödinger-Poisson system in bounded domains, Adv. Math. Phys. 2020, Article ID 8254898, 10 pp. https://doi.org/10.1155/2020/8254898
[6] A. Arosio, S. Panizzi, On the well-posedness of the Kirchhoff string, Trans. Amer. Math. Soc. 348(1996), 305-331. MR1333386
[7] A. Azzollini, A. Pomponio, Ground state solutions for the nonlinear SchrödingerMaxwell equations, J. Math. Anal. Appl. 345(2008), 90-108. https://doi.org/10.1016/ j.jmaa.2008.03.057
[8] Z. BaI, X. He, Solutions for a class of Schrödinger-Poisson system in bounded domains, J. Appl. Math. Comput. 51(2016), 287-297. MR3490972
[9] T. Bartsch, Z. Liu, On a superlinear elliptic $p$-Laplacian equation, J. Differential Equations 198(2004), 149-175. https://doi.org/10.1016/j.jde.2003.08.001
[10] T. Bartsch, Z. Liu, T. Weth, Sign changing solutions of superlinear Schrödinger equations, Comm. Partial Differential Equations 29(2004), 25-42. https://doi.org/10.1081/ PDE-120028842
[11] T. Bartsch, Z. Liu, T. Weth, Nodal solutions of a $p$-Laplacian equation, Proc. London Math. Soc. 91(2005), 129-152. MR2149532
[12] C. J. BatKam, High energy sign-changing solutions to Schrödinger-Poisson type systems, available on arXiv:1501.05942v1 [math.AP].
[13] C. J. Batkam, J. R. Santos Júnior, Schrödinger-Kirchhoff-Poisson type systems, Commun. Pure Appl. Anal. 15(2016), 429-444. https://doi.org/10.3934/cpaa.2016.15.429
[14] G. Che, H. Chen, Existence and multiplicity of positive solutions for Kirchhoff-Schrödinger-Poisson system with critical growth, RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 114(2020), Paper No. 78, 27 pp. https://doi.org/10.1007/s13398-020-00809-3
[15] J. Chen, X. Tang, Z. Gao, Existence of multiple solutions for modified Schrödinger-Kirchhoff-Poisson type systems via perturbation method with sign-changing potential, Comput. Math. Appl. 3(2017), 505-519. https://doi.org/10.1016/j.camwa.2016.12.006
[16] P. D'Ancona, S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, Invent. Math. 108(1992), 247-262. https://doi.org/10.1007/ BF02100605; MR1161092
[17] T. D'Aprile, D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations, Proc. Roy. Soc. Edinburgh Sect. A. 134(2004), 893-906. https://doi.org/10.1017/S030821050000353X
[18] J. Deng, J. Yang, Nodal solutions for Schrödinger-Poisson type equations in $\mathbb{R}^{3}$, Electron. J. Differential Equations 2016, No. 277, 1-21. MR3578298
[19] G. M. Figueiredo, N. Ікоma, J. R. Santos Júnior, Existence and concentration result for the Kirchhoff type equations with general nonlinearities, Arch. Ration. Mech. Anal. 231(2014), 931-979. https://doi.org/10.1007/s00205-014-0747-8
[20] G. M. Figueiredo, R. G. Nascimento, Existence of a nodal solution with minimal energy for a Kirchhoff equation, Math. Nachr. 288(2015), 48-60. https://doi.org/10.1002/mana. 201300195
[21] W. Huang, L. Wang, Infinitely many sign-changing solutions for Kirchhoff type equations, Complex Var. Elliptic Equ. 65(2020), 920-935. https://doi.org/10.1080/17476933. 2019.1636790
[22] G. Kirchioff, Mechanik, Teubner, Leipzig, 1883.
[23] E. Lieb, M. Loss, Analysis, Graduate Studies in Mathematics, Vol 14. AMS, Providence, 1997. MR1415616
[24] J. Liu, X. Liu, Z. Wang, Multiple mixed states of nodal solutions for nonlinear Schrödinger systems, Calc. Var. Partial Differential Equations 52(2015), 565-586. https: //doi.org/10.1007/s00526-014-0724-y
[25] F. Liu, S. Wang, Positive solutions of Schrödinger-Kirchhoff-Poisson system without compact condition, Bound. Value Probl. 2017, Art. No. 156. https://doi.org/10.1186/ s13661-017-0884-8
[26] Z. Liu, Z. Wang, J. Zhang, Infinitely many sign-changing solutions for the nonlinear Schrödinger-Poisson system, Ann. Mat. Pura Appl. 195(2016), 775-794. https ://doi. org/ 10.1007/s10231-015-0489-8
[27] C. Liv, H. Zhang, Ground state and nodal solutions for critical Kirchhoff-SchrödingerPoisson systems with an asymptotically 3-linear growth nonlinearity, Bound. Value Probl. 2020, Art. No. 133. https://doi.org/10.1186/s13661-020-01421-5
[28] D. Lü, Positive solutions for Kirchhoff-Schrödinger-Poisson systems with general nonlinearity, Comm. Pure Appl. Anal. 17(2018), 605-626. https://doi.org/10.3934/cpaa. 2018033
[29] A. Mao, W. Wang, Signed and sign-changing solutions of bi-nonlocal fourth order elliptic problem, J. Math. Phys. 60(2019), 1-13. https://doi.org/10.1063/1.5093461
[30] A. Mao, Z. Zhang, Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition, Nonlinear Anal. 70(2009), 1275-1287. https://doi.org/10. 1016/j.na.2008.02.011
[31] D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinear local term, J. Funct. Anal. 237(2006), 655-674. https://doi.org/10.1016/j . jfa.2006.04.005
[32] D. Ruiz, On the Schrödinger-Poisson-Slater system: behavior of minimizers, radial and nonradial cases, Arch. Ration. Mech. Anal. 198(2010), 349-368. https://doi .org/10.1007/ s00205-010-0299-5
[33] D. Ruiz, G. Siciliano, A note on the Schrödinger-Poisson-Salter equation on bounded domain, Adv. Nonlinear Stud. 8(2008), 179-190. https://doi.org/10.1515/ans-20080106
[34] L. Shao, B. Chen, Existence of solutions for the Schrödinger-Kirchhoff-Poisson systems with a critical nonlinearity, Bound. Value Probl. 2016, Art. No. 210. https://doi. org/10. 1186/s13661-016-0718-0
[35] H. Shi, Positive solutions for a class of nonhomogeneous Kirchhoff-Schrödinger-Poisson systems, Bound. Value Probl. 2019, Art. No. 139. https ://doi. org/10.1186/s13661-019-1252-7
[36] W. Shual, Sign-changing solutions for a class of Kirchhoff-type problem in bounded domains, J. Differential Equations 259(2015), 1256-1274. https://doi. org/10.1016/j. jde. 2015.02.040
[37] G. Siciliano, Multiple positive solutions for a Schrödinger-Poisson-Slater system, J. Math. Anal. Appl. 365(2010), 288-299. https://doi.org/10.1016/j.jmaa.2009.10.061
[38] J. Sun, Z. Wu, Bound state nodal solutions for the non-autonomous Schrödinger-Poisson system in $\mathbb{R}^{3}$, J. Differential Equations 268(2020), 7121-7163. https://doi.org/10.1016/j . jde.2019.11.070
[39] X. Tang, B. Cheng, Ground state sign-changing solutions for Kirchhoff type problems in bounded domains, J. Differential Equations 261(2016), 2384-2402. https://doi. org/10. 1016/j.jde.2016.04.032
[40] D. WANG, Least energy sign-changing solutions of Kirchhoff-type equation with critical growth, J. Math. Phys. 61(2020), 011501. https://doi.org/10.1063/1.5074163
[41] D. Wang, T. Li, X. Hao, Least-energy sign-changing solutions for Kirchhoff-SchrödingerPoisson systems in $\mathbb{R}^{3}$, Bound. Value Probl. 2019, Art. No. 75. https://doi.org/10.1186/ s13661-019-1183-3
[42] M. Willem, Minimax theorem, Birkhäuser Boston, 1996. https://doi.org/10.1007/ 978-1-4612-4146-1; MR1400007
[43] D. Yang, C. BaI, Multiplicity results for a class of Kirchhoff-Schrödinger-Poisson system involving sign-changing weight functions, J. Funct. Spaces 2019, Art. ID 6059459, 11 pp. MR3963604
[44] Z. Zhang, K. Perera, Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow, J. Math. Anal. Appl. 317(2006), 456-463. https://doi.org/10.1016/ j.jmaa.2005.06.102
[45] J. Zhang, D. Wang, Existence of least energy nodal solution for Kirchhoff-SchrödingerPoisson system with potential vanishing, Bound. Value Probl. 2020, Art. No. 111. https: //doi.org/10.1186/s13661-020-01408-2
[46] J. Zhang, D. Wang, Existence of least energy nodal solution for Kirchhoff-type system with Hartree-type nonlinearity, AIMS Math 5(2020), 4494-4511. https://doi.org/ 10.3934/math. 2020289
[47] F. Zhao, L. Zhao, Positive solutions for Schrödinger-Poisson equations with a critical exponent, Nonlinear Anal. 70(2009), 2150-2164. https://doi.org/10.1016/j.na.2008.02. 116
[48] G. Zhao, X. Zhu, Y. Li, Existence of infinitely many solutions to a class of Kirchhoff-Schrödinger-Poisson system, Appl. Math. Comput. 256(2015), 572-581. https://doi. org/ 10.1016/j.amc.2015.01.038

# Positive solutions for a class of generalized quasilinear Schrödinger equations involving concave and convex nonlinearities in Orlicz space 

Yan Meng, Xianjiu Huang ${ }^{\boxtimes}$ and Jianhua Chen<br>Department of Mathematics, Nanchang University, Nanchang 330031, Jiangxi, P. R. China

Received 4 January 2021, appeared 12 December 2021
Communicated by Dimitri Mugnai


#### Abstract

In this paper, we study the following generalized quasilinear Schrödinger equation $$
-\operatorname{div}\left(g^{2}(u) \nabla u\right)+g(u) g^{\prime}(u)|\nabla u|^{2}+V(x) u=\lambda f(x, u)+h(x, u), \quad x \in \mathbb{R}^{N},
$$ where $\lambda>0, N \geq 3, g \in \mathcal{C}^{1}\left(\mathbb{R}, \mathbb{R}^{+}\right)$. By using a change of variable, we obtain the existence of positive solutions for this problem with concave and convex nonlinearities via the Mountain Pass Theorem. Our results generalize some existing results. Keywords: generalized quasilinear Schrödinger equation, positive solutions, concave and convex nonlinearities.


2020 Mathematics Subject Classification: 35J60, 35J20.

## 1 Introduction

In this paper, we are concerned with a class of generalized quasilinear Schrödinger equation

$$
\begin{equation*}
-\operatorname{div}\left(g^{2}(u) \nabla u\right)+g(u) g^{\prime}(u)|\nabla u|^{2}+V(x) u=\lambda f(x, u)+h(x, u), \tag{1.1}
\end{equation*}
$$

where $\lambda>0, N \geq 3,2^{*}=\frac{2 N}{N-2}$, and $g$ satisfies:
$(g) g \in \mathcal{C}^{1}(\mathbb{R},(0,+\infty))$ is even with $g^{\prime}(t) \geq 0$, for all $t \in[0,+\infty), g(0)=1$ and satisfies

$$
\begin{equation*}
g_{\infty}:=\lim _{t \rightarrow \infty} \frac{g(t)}{t} \in(0, \infty) \tag{1.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\beta:=\sup _{t \in \mathbb{R}} \frac{\operatorname{tg}^{\prime}(t)}{g(t)} \leq 1 . \tag{1.3}
\end{equation*}
$$

[^49]Mathematically, it is also a hot issue in nonlinear analysis to study the existence of solitary wave solutions for the following quasi-linear Schrödinger equation

$$
\begin{equation*}
i \partial_{t} z=-\Delta z+W(x) z-k(x,|z|)-\Delta l\left(|z|^{2}\right) l^{\prime}\left(|z|^{2}\right) z \tag{1.4}
\end{equation*}
$$

where $z: \mathbb{R} \times \mathbb{R}^{N} \rightarrow \mathbb{C}, W: \mathbb{R}^{N} \rightarrow \mathbb{R}$ is a given potential, $l: \mathbb{R} \rightarrow \mathbb{R}$ and $k: \mathbb{R}^{N} \times \mathbb{R} \rightarrow \mathbb{R}$ are suitable functions.

The quasilinear equation of the form (1.4) appears naturally in mathematical physics and has been derived as models of several physical phenomena corresponding to various types of nonlinear terms $l$. Kurihara [23] considered the case where $l(s)=s$ in (1.4), and this kind of equation was used for the superfluid film [23,24] equation in fluid mechanics. [29-31] studied the equation which corresponds to the case $l(t)=t^{\alpha}$ for some $\alpha \geq 1$. For more details see [ $2-4,21,25,32,34,38$ ] and references therein. Moreover, many conclusions about the equation (1.4) with $l(s)=1$ have been studied, see [35-37] and the references therein.

Cuccagna [11] was interested in the existence of standing wave solutions, that is, solutions of type $z(t, x)=\exp (-i E t) u(x)$, where $E \in \mathbb{R}$ and $u$ is a real function. It is well known that $z$ satisfies (1.4) if and only if the function $u(x)$ solves the following equation of elliptic type with the formal variational structure

$$
\begin{equation*}
-\Delta u+V(x) u-\Delta l\left(u^{2}\right) l^{\prime}\left(u^{2}\right) u=a(x, u), \quad x \in \mathbb{R}^{N} . \tag{1.5}
\end{equation*}
$$

If we take

$$
g^{2}(u)=1+\frac{\left[\left(l^{2}(u)\right)^{\prime}\right]^{2}}{2},
$$

then (1.5) turns into quasilinear elliptic equations (see [39])

$$
\begin{equation*}
-\operatorname{div}\left(g^{2}(u) \nabla u\right)+g(u) g^{\prime}(u)|\nabla u|^{2}+V(x) u=a(x, u), \quad x \in \mathbb{R}^{N} . \tag{1.6}
\end{equation*}
$$

The existence of solutions for (1.6) have been extensively investigated in the literature over the past several decades (see $[5,9,12,13,26,27,39-42]$ ). For example, Shen et al. studied the existence of positive solutions for two types of quasilinear elliptic equations with degenerate coerciveness and slightly superlinear growth in [40]. By introducing a new variable replacement, Cheng et al. proved the existence of positive and soliton solutions to a class of relativistic nonlinear Schrödinger equations in [12,13]. In [9], Chen et al. proved existence and asymptotic behavior of standing wave solutions for a class of generalized quasilinear Schrödinger equations with critical Sobolev exponents. In [42], Shi et al. proved the positive solutions for generalized quasilinear Schrödinger equations with potential vanishing at infinity. Besides, Li et al. in [26] considered a class of generalized quasilinear Schrödinger equations with a Kirchhoff-type perturbation. Under the assumption that the potential may be vanishing at infinity, the existence of both the ground state and the ground state sign-changing solutions is established. Furthermore, the behavior of these solutions is studied when the perturbation vanishes. More concretely, Deng et al. [17,18] proved the existence of positive solutions with critical exponents, where critical exponents are $2^{*}$ and $\alpha 2^{*}$, respectively. Moreover, the existence of nodal solutions have been proved by Deng et al. in [15,16]. Very recently, in [27], Li et al. via Nehari manifold method proved the existence of ground state solutions and geometrically distinct solutions. In [28], the authors by using symmetric mountain theorem, considered the existence of a positive solution, a negative solution and infinitely many solutions. For generalized quasilinear Schrödinger equation of Kirchhoff type and generalized quasilinear Schrödinger-Maxwell system, we can refer to $[6,7,26,44]$ and references therein.

If we set

$$
g^{2}(u)=1+2 u^{2},
$$

then (1.6) reduces to the following well-known quasilinear Schrödinger equation

$$
\begin{equation*}
-\Delta u+V(x) u-u \Delta\left(u^{2}\right)=h(x, u) . \tag{1.7}
\end{equation*}
$$

For (1.7), Liu-Wang-Wang [30] and Colin-Jeanjean [14] made the change of variable by $v=$ $f^{-1}(u)$, where $f$ is defined by

$$
\begin{equation*}
f^{\prime}(t)=\frac{1}{\left(1+2 f^{2}(t)\right)^{\frac{1}{2}}} \quad \text { on }[0, \infty) \quad \text { and } \quad f(t)=-f(-t) \text { on }(-\infty, 0] \tag{1.8}
\end{equation*}
$$

and then equation (1.7) in form can be transformed into a semilinear equation. Afterwards, many recent studies has focused on the above quasilinear equation via the variable $f$, see for example $[8,14,33]$ and references therein. Especially, in [33], the authors considered the existence of positive solutions for (1.7) with concave and convex nonlinearities.

To our knowledge, there are few papers studying the existence of positive solutions for (1.6) with concave and convex nonlinearities. Motivated by the previously mentioned papers, especially [33], we study the existence of positive solutions with concave and convex nonlinearities. Next, we give the following conditions on $V$ :
$\left(V_{1}\right) V \in \mathcal{C}\left(\mathbb{R}^{N}, \mathbb{R}\right)$ and $0<V_{0} \leq \inf _{x \in \mathbb{R}^{N}} V(x) ;$
$\left(V_{2}\right)[V(x)]^{-1} \in L^{1}\left(\mathbb{R}^{N}\right)$.
Moreover, the nonlinearities term $f$ and $h$ should satisfy the following assumptions:
(FH) $f, h \in \mathcal{C}\left(\mathbb{R}^{N} \times \mathbb{R}, \mathbb{R}\right)$ and $f(x, s)=0, h(x, s)=0$ for all $s \leq 0$ and $x \in \mathbb{R}^{N}$;
$\left(F H_{1}\right)$ there exist constant $c_{1}>0$ and $q \in(1,2)$ such that

$$
0 \leq f(x, s) \leq c_{1}|s|^{q-1}, \quad \text { for all }(x, s) \in \mathbb{R}^{N} \times \mathbb{R}
$$

$\left(F H_{2}\right) \lim _{s \rightarrow 0} \frac{h(x, s)}{s}=0$ uniformly in $x \in \mathbb{R}^{N}$;
$\left(F H_{3}\right)$ there exist $c_{2}>0$ and $4<p<2 \cdot 2^{*}$ such that

$$
h(x, s) \leq c_{2}\left(1+|s|^{p-1}\right), \quad \text { for all }(x, s) \in \mathbb{R}^{N} \times \mathbb{R} ;
$$

( $F H_{4}$ ) there exists $\mu \geq 2$ such that

$$
0<2 \mu g(s) H(x, s) \leq G(s) h(x, s),
$$

where $H(x, s)=\int_{0}^{s} h(x, t) d t$ and $G(s)=\int_{0}^{s} g(t) d t ;$
$\left(F H_{5}\right)$ there exist $c_{3}>0$ and $q_{1} \in(1,2)$ such that

$$
h(x, s) \geq c_{3} g(s)|G(s)|^{q_{1}-1} \quad \text { for all }(x, s) \in \mathbb{R}^{N} \times \mathbb{R} .
$$

Next, we recall some basic notions. Let

$$
H^{1}\left(\mathbb{R}^{N}\right)=\left\{u \in L^{2}\left(\mathbb{R}^{N}\right): \nabla u \in L^{2}\left(\mathbb{R}^{N}\right)\right\},
$$

endowed with the norm

$$
\|u\|_{H^{1}}=\left(\int_{\mathbb{R}^{N}}\left(|\nabla u|^{2}+u^{2}\right)\right)^{\frac{1}{2}}
$$

In the study of the elliptic equations, it is well known that the potential function $V$ plays an important role in choosing of a right working space and some suitable compactness methods. Generally speaking, many papers study (1.1) under the following working space:

$$
X=\left\{u \in H^{1}\left(\mathbb{R}^{N}\right): \int_{\mathbb{R}^{3}} V(x) u^{2}<\infty\right\},
$$

endowed with the norm

$$
\|u\|_{X}=\left(\int_{\mathbb{R}^{N}}\left(|\nabla u|^{2}+V(x) u^{2}\right)\right)^{\frac{1}{2}}, \quad u \in X .
$$

But in this paper, we define the following working space:

$$
E=\left\{v \in H^{1}\left(\mathbb{R}^{N}\right): \int_{\mathbb{R}^{N}} V(x)\left|G^{-1}(v)\right|^{2}<\infty\right\},
$$

which is called Orlicz-Sobolev space. Then $E$ is a Banach space endowed with the following norm

$$
\begin{equation*}
\|v\|:=\|\nabla v\|_{2}+\inf _{\xi>0} \frac{1}{\tilde{\xi}}\left\{1+\int_{\mathbb{R}^{N}} V(x)\left|G^{-1}(\tilde{\xi} v)\right|^{2}\right\} . \tag{1.9}
\end{equation*}
$$

To resolve the equation (1.1), due to the appearance of the nonlocal term $\int_{\mathbb{R}^{N}} g^{2}(u)|\nabla u|^{2}$, the right working space seems to be

$$
E_{0}=\left\{u \in E: \int_{\mathbb{R}^{N}} g^{2}(u)|\nabla u|^{2}<\infty\right\} .
$$

But under the assumption of $(g)$, it is easy to see that $E_{0}$ is not a linear space. To overcome this difficulty, we follow the idea developed by Shen and Wang in [39], that is, we make the change of variable substitution

$$
u=G^{-1}(v) \quad \text { and } \quad G(u)=\int_{0}^{u} g(t) \mathrm{d} t, \quad v \in E,
$$

then

$$
\int_{\mathbb{R}^{N}} g^{2}(u)|\nabla u|^{2}=\int_{\mathbb{R}^{N}} g^{2}\left(G^{-1}(v)\right)\left|\nabla G^{-1}(v)\right|^{2}:=|\nabla v|_{2}^{2}<+\infty, \quad v \in E .
$$

In such a case, we obtain the following Euler-Lagrange functional associated with the equation (1.1)

$$
J_{\lambda}(u)=\frac{1}{2} \int_{\mathbb{R}^{N}}\left[g^{2}(u)|\nabla u|^{2}+V(x) u^{2}\right]-\lambda \int_{\mathbb{R}^{N}} F(x, u)-\int_{\mathbb{R}^{N}} H(x, u) .
$$

Therefore, after this change of variable, $E$ can be used as the working space and the equation (1.1) in form can be transformed into the following functional

$$
\begin{equation*}
\mathcal{J}_{\lambda}(v)=\frac{1}{2} \int_{\mathbb{R}^{N}}\left(|\nabla v|^{2}+V(x)\left|G^{-1}(v)\right|^{2}\right)-\Psi_{\lambda}(v), \tag{1.10}
\end{equation*}
$$

where

$$
\begin{equation*}
\Psi_{\lambda}(v)=\lambda \int_{\mathbb{R}^{N}} F\left(x, G^{-1}(v)\right)+\int_{\mathbb{R}^{N}} H\left(x, G^{-1}(v)\right) . \tag{1.11}
\end{equation*}
$$

Because $g$ is a nondecreasing positive function, we get $\left|G^{-1}(v)\right| \leq \frac{1}{g(0)}|v|$. From this and our hypotheses, it is clear that $\mathcal{J}$ is well defined in $E$ and $\mathcal{J} \in \mathcal{C}^{1}$.

Moreover, we can easily derive that if $v \in \mathcal{C}^{2}\left(\mathbb{R}^{N}\right)$ is a critical point of (1.10), then $u=$ $G^{-1}(v) \in \mathcal{C}^{2}\left(\mathbb{R}^{N}\right)$ is a classical solution to the equation (1.1). In order to obtain a critical point of (1.10), we only need to find the weak solution to the following equation

$$
\begin{equation*}
-\Delta v+V(x) \frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}=\lambda \frac{f\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)}+\frac{h\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)}, \quad x \in \mathbb{R}^{N} . \tag{1.12}
\end{equation*}
$$

Here, we call that $v \in E$ is a weak solution to the equation (1.12) if it holds that

$$
\left\langle\mathcal{J}_{\lambda}^{\prime}(v), \varphi\right\rangle=\int_{\mathbb{R}^{N}} \nabla v \cdot \nabla \varphi+\int_{\mathbb{R}^{N}} V(x) \frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)} \varphi-\left\langle\Psi_{\lambda}(v), \varphi\right\rangle,
$$

where

$$
\left\langle\Psi_{\lambda}(v), \varphi\right\rangle=\lambda \int_{\mathbb{R}^{N}} \frac{f\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)} \varphi+\int_{\mathbb{R}^{N}} \frac{h\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)} \varphi, \quad \varphi \in E .
$$

Then it is standard to obtain that $v \in E$ is a weak solution to the equation (1.12) if and only if $v$ is a critical point of the functional $\mathcal{J}$ in $E$. All in all, if we find a critical point of the functional $\mathcal{J}$ in $E$, then we will get a classical solution to the equation (1.1).

Now, we state the results by the following theorems.
Theorem 1.1. Suppose that $(g),\left(V_{1}\right),\left(V_{2}\right)$ and $(F H)-\left(F H_{4}\right)$ are satisfied. Then there exist $\lambda_{0}, C_{0}>$ 0 such that for all $\lambda \in\left[0, \lambda_{0}\right]$, problem (1.1) has one positive solution $u_{\lambda, 1} \in H^{1}\left(\mathbb{R}^{N}\right)$ such that $\left\|u_{\lambda, 1}\right\|_{H^{1}} \leq C_{0}$. Moreover, if $\lambda=0$, then there exist constants $M, \zeta>0$ such that

$$
u_{0,1} \leq M \exp (-\zeta|x|), \quad \text { for all } x \in \mathbb{R}^{N} .
$$

Theorem 1.2. Suppose that $(g),\left(V_{1}\right),\left(V_{2}\right)$ and $(F H)-\left(F H_{5}\right)$ are satisfied. Then for all $\lambda>0$, (1.1) possesses a positive solution $v_{\lambda, 2} \in H^{1}\left(\mathbb{R}^{N}\right)$, which is different of $v_{\lambda, 1}$ when $\lambda \in\left(0, \lambda_{0}\right]$.

Remark 1.3. Condition (g) originates from [19]. In fact, as [10], there are many functions satisfying $(g)$. For example:

$$
g(s)= \begin{cases}\sqrt{1+s^{2}}, & \text { if } 0 \leq s \leq 1 \\ \frac{\sqrt{2}}{2}(s+1), & \text { if } s>1, \\ g(-s), & \text { if } s<0,\end{cases}
$$

and

$$
g(s)=\sqrt{1+2 s^{2}} .
$$

Note that if we choose $g(s)=\sqrt{1+2 s^{2}}$ in (1.1), then (1.1) will become the classical quasilinear Schrödinger equation

$$
-\Delta u+V(x) u-\Delta\left(u^{2}\right) u=\lambda f(x, u)+h(x, u) .
$$

Remark 1.4. By Remark 1.3, our results extend [33].

Remark 1.5. Is easy to check that the following function satisfies (V1) and (V2):

$$
V(x)= \begin{cases}2, & \text { if }|x|<1 \\ |x|^{N-1}\left(1+|x|^{2}\right), & \text { if }|x| \geq 1,\end{cases}
$$

where $N \geq 3$.
For the above problem, there are many difficulties in treating this class of generalized quasilinear Schrödinger equations in $\mathbb{R}^{N}$. The first difficulty is the possible lack of compactness besides the concave term. The second difficulty is lack of natural functions space for the associated energy functional to be well defined. The function space $H^{1}\left(\mathbb{R}^{N}\right)$ cannot be applied directly to handle with this class of generalized quasilinear Schrödinger equations. To overcome these difficulties, we refer [33] and establish a different approach based on an appropriate Orlicz space. It was crucial in our argument the fact that this function space considered in our approach can be embedded into the usual Lebesgue spaces $L^{r}\left(\mathbb{R}^{N}\right)$ for all $1 \leq r<2^{*}$.

Motivated by the argument used in [39], we use a change of variable to turn the problem into a semilinear one so that it has the associated functional well defined and Gateaux differentiable in a suitable Orlicz space. We prove that the energy functional satisfies the geometric hypotheses of the mountain-pass theorem and the Palais-Smale condition. In this paper, the first result is proved by using a version of the mountain-pass theorem and the second solution is obtained as a consequence of a minimization argument based on the Ekeland variational principle.

The paper is organized as follows. In Section 2, we give some preliminary lemmas. In Section 3, we give the proof of Theorem 1.1 and Theorem 1.2, respectively.

Notations. Throughout this paper, we make use of the following notations:

- $\int_{\mathbb{R}^{N}} \boldsymbol{d}$ denotes $\int_{\mathbb{R}^{N}} \boldsymbol{d} \mathrm{~d} x$.
- $C$ will denote a positive constant, not necessarily the same one.
- $L^{r}\left(\mathbb{R}^{N}\right)$ denotes the Lebesgue space with norm

$$
\|u\|_{r}=\left(\int_{\mathbb{R}^{N}}|u|^{r}\right)^{1 / r},
$$

where $1 \leq r<\infty$.

- For any $z \in \mathbb{R}^{N}$ and $R>0, B_{R}(z):=\left\{x \in \mathbb{R}^{N}:|x-z|<R\right\}$.
- The weak convergence in $H^{1}\left(\mathbb{R}^{N}\right)$ is denoted by $\rightarrow$, and the strong convergence by $\rightarrow$.


## 2 Some preliminary lemmas

In this section, we present some useful lemmas and corollaries. Now, let us recall the following lemma which has been proved in [19].

Lemma 2.1 ([19]). For the function $g, G$, and $G^{-1}$, the following properties hold:
$\left(g_{1}\right)$ the functions $G(\cdot)$ and $G^{-1}(\cdot)$ are strictly increasing and odd;
$\left(g_{2}\right) 0<\frac{d}{d t}\left(G^{-1}(t)\right)=\frac{1}{g\left(G^{-1}(t)\right)} \leq \frac{1}{g(0)}$ for all $t \in \mathbb{R}$;
$\left(g_{3}\right)\left|G^{-1}(t)\right| \leq \frac{1}{g(0)}|t|$ for all $t \in \mathbb{R}$;
( $\left.g_{4}\right) \lim _{||t| \rightarrow 0} \frac{G^{-1}(t)}{t}=\frac{1}{g(0)}$;
(g5) $\lim _{|t| \rightarrow+\infty} \frac{G^{-1}(t)}{g\left(G^{-1}(t)\right)}= \pm \frac{1}{g \infty}$;
$\left(g_{6}\right) 1 \leq \frac{\operatorname{tg}(t)}{G(t)} \leq 2$ and $1 \leq \frac{\mathrm{G}^{-1}(t) g\left(G^{-1}(t)\right)}{t} \leq 2$ for all $t \neq 0$;
$\left(g_{7}\right) \frac{G^{-1}(t)}{\sqrt{t}}$ is non-decreasing in $(0,+\infty)$ and $\left|G^{-1}(t)\right| \leq\left(2 / g_{\infty}\right)^{1 / 2} \sqrt{|t|}$ for all $t \in \mathbb{R}$;
$\left(g_{8}\right)$

$$
\left|G^{-1}(t)\right| \geq \begin{cases}G^{-1}(1)|t|, & \text { for all }|t| \leq 1 \\ G^{-1}(1) \sqrt{|t|}, & \text { for all }|t| \geq 1\end{cases}
$$

(g9) $\frac{t}{g(t)}$ is increasing and $\left|\frac{t}{g(t)}\right| \leq \frac{1}{\delta_{\infty}}$ for all $t \in \mathbb{R}$;
$\left(g_{10}\right)$ the function $\left[G^{-1}(t)\right]^{2}$ is convex. In particular, $\left[G^{-1}(\theta t)\right]^{2} \leq \theta\left[G^{-1}(t)\right]^{2}$ for all $t \in \mathbb{R}, \theta \in$ [0,1];
( $\left.g_{11}\right)\left[G^{-1}(\theta t)\right]^{2} \leq \theta^{2}\left[G^{-1}(t)\right]^{2}$ for all $t \in \mathbb{R}, \theta \geq 1$;
(g12) $\left[G^{-1}\left(t_{1}-t_{2}\right)\right]^{2} \leq 4\left(\left[G^{-1}\left(t_{1}\right)\right]^{2}+\left[G^{-1}\left(t_{2}\right)\right]^{2}\right)$ for all $t_{1}, t_{2} \in \mathbb{R} ;$
( $g_{13}$ ) the function $G^{-1}(t)$ is concave. In particular, $G^{-1}(\theta t) \leq \theta G^{-1}(t)$ for all $t \in \mathbb{R}, \theta \in[1,+\infty)$;
$\left(g_{14}\right) G^{-1}(\theta t) \geq \theta G^{-1}(t)$ for all $t \in \mathbb{R}, \theta \in[0,1]$.
Proposition 2.2 ([19]). Assume that $V$ satisfies $\left(V_{1}\right)-\left(V_{2}\right)$. Then the space $E$ has the following properties:
(1) if $\left\{v_{n}\right\} \subset E$ is such that $v_{n}(x) \rightarrow v(x)$ a.e. in $\mathbb{R}^{N}$ and

$$
\lim _{n \rightarrow+\infty} \int_{\mathbb{R}^{N}} V(x)\left[G^{-1}\left(v_{n}\right)\right]^{2}=\int_{\mathbb{R}^{N}} V(x)\left[G^{-1}(v)\right]^{2}
$$

then

$$
\inf _{\xi>0} \frac{1}{\xi}\left\{1+\int_{\mathbb{R}^{N}} V(x)\left|G^{-1}\left(\xi\left(v_{n}-v\right)\right)\right|^{2}\right\} \rightarrow 0 ;
$$

(2) the embedding $E \hookrightarrow D^{1,2}\left(\mathbb{R}^{N}\right), E \hookrightarrow H^{1}\left(\mathbb{R}^{N}\right)$ and $X \hookrightarrow E$ are continuous;
(3) the map $v \mapsto G^{-1}(v)$ from $E$ to $L^{r}\left(\mathbb{R}^{N}\right)$ is continuous for each $r \in\left[2,2 \cdot 2^{*}\right]$;
(4) if $v \in E$ and $u=G^{-1}(v)$, then

$$
\|u g(u)\| \leq 4\|v\| ;
$$

(5) if $v_{n} \rightharpoonup v$ in $D^{1,2}\left(\mathbb{R}^{N}\right)$ and $\left\{\int_{\mathbb{R}^{N}} V(x)\left[G^{-1}\left(v_{n}\right)\right]^{2} d x\right\}$ is bounded then, up to a subsequence, $G^{-1}\left(v_{n}\right) \rightarrow 0$ strongly in $L^{r}\left(\mathbb{R}^{N}\right)$ for any $2 \leq r<2 \cdot 2^{*}$.

Proposition 2.3 ([10]). Assume that $V$ satisfies $\left(V_{1}\right)-\left(V_{2}\right)$. Then the space $E$ has the following properties:
(i) $E$ is a normed linear space with respect to the norm given in (1.9);
(ii) there exists a positive constant $C>0$ such that for all $v \in E$,

$$
\begin{equation*}
\frac{\int_{\mathbb{R}^{\mathbb{N}}} V(x)\left|G^{-1}(v)\right|^{2}}{\left[1+\left(\int_{\mathbb{R}^{N}} V(x)\left|G^{-1}(v)\right|^{2}\right)^{1 / 2}\right]} \leq C\|v\| ; \tag{2.1}
\end{equation*}
$$

(iii) if $v_{n} \rightarrow v$ in $E$, then

$$
\left.\int_{\mathbb{R}^{N}} V(x)| | G^{-1}\left(v_{n}\right)\right|^{2}-\left|G^{-1}(v)\right|^{2} \mid \rightarrow 0
$$

and

$$
\int_{\mathbb{R}^{N}} V(x)\left|G^{-1}\left(v_{n}\right)-G^{-1}(v)\right|^{2} \rightarrow 0
$$

Lemma 2.4. Suppose that $\left(V_{1}\right)-\left(V_{2}\right)$ holds. Then the embedding

$$
X \hookrightarrow L^{r}\left(\mathbb{R}^{N}\right)
$$

is continuous for $1 \leq r \leq 2^{*}$ and compact for $1 \leq r<2^{*}$.
Proof. Similar to the proof of [33], by $\left(V_{1}\right)$, the embedding $X \hookrightarrow H^{1}\left(\mathbb{R}^{N}\right)$ is continuous. Thus $X \hookrightarrow L^{r}\left(\mathbb{R}^{N}\right)$ is continuous for $2 \leq r \leq 2^{*}$. Moreover, if $u \in X$ we get

$$
\int_{\mathbb{R}^{N}}|u| \leq\left(\int_{\mathbb{R}^{N}} V(x)^{-1}\right)^{1 / 2}\|u\|_{X} .
$$

Therefore, by interpolation the first part the lemma is proved. Next, let $\left\{u_{n}\right\} \subset X$ be a bounded sequence. Hence up to a subsequence, $u_{n} \rightharpoonup u_{0}$ in $X$. Given $\varepsilon>0$, for enough large $R>0$, we have

$$
\int_{|x|>R} V(x)^{-1} \leq\left[\frac{\varepsilon}{2\left(C+\left\|u_{0}\right\|_{X}\right)}\right]^{2},
$$

which shows that

$$
\int_{|x|>R}\left|u_{n}-u_{0}\right| \leq\left(\int_{|x|>R} V(x)^{-1}\right)^{1 / 2}\left\|u_{n}-u_{0}\right\|_{X} \leq \frac{\varepsilon}{2}
$$

and since $X \hookrightarrow L^{1}\left(B_{R}\right)$ is compact, it follows that there exists $\mathbf{N}$ such that for all $n \geq \mathbf{N}$

$$
\int_{B_{R}}\left|u_{n}-u_{0}\right| \leq \frac{\varepsilon}{2} .
$$

Thus $u_{n} \rightarrow u_{0}$ in $L^{1}\left(\mathbb{R}^{N}\right)$. Now, if $r \in\left[1,2^{*}\right)$, by interpolation inequality, for some $0<\sigma \leq 1$, we have

$$
\left\|u_{n}-u_{0}\right\|_{r} \leq\left\|u_{n}-u_{0}\right\|_{1}^{\sigma}\left\|u_{n}-u_{0}\right\|_{2^{*}}^{1-\sigma} \leq C\left\|u_{n}-u_{0}\right\|_{1}^{\sigma} \rightarrow 0,
$$

and this completes the proof.
Lemma 2.5. The map $v \mapsto G^{-1}(v)$ from $E$ to $L^{r}\left(\mathbb{R}^{N}\right)$ is continuous for each $r \in\left[1,2 \cdot 2^{*}\right]$. Moreover, under assumption $\left(V_{2}\right)$, the above map is compact for $r \in\left[1,2 \cdot 2^{*}\right)$.

Proof. Let $v \in E$. By definition, we know that $G^{-1}(v) \in X$, which together with Lemma 2.4 and $\left(g_{2}\right)$ in Lemma 2.1 implies that

$$
\begin{equation*}
\left\|G^{-1}(v)\right\|_{r} \leq C\left\|G^{-1}(v)\right\|_{X} \leq C\left[\int_{\mathbb{R}^{N}}\left(|\nabla v|^{2}+V(x)\left|G^{-1}(v)\right|^{2}\right)\right]^{1 / 2} \tag{2.2}
\end{equation*}
$$

for all $1 \leq r \leq 2^{*}$. Moreover, by the Gagliardo-Nirenberg inequality and $\left(g_{9}\right)$ in Lemma 2.1, we have

$$
\begin{align*}
\left\|G^{-1}(v)\right\|_{2 \cdot 2^{*}} & =\left\|G^{-1}(v)^{2}\right\|_{2^{*}}^{1 / 2} \\
& \leq C\left\|\nabla\left(G^{-1}(v)^{2}\right)\right\|_{2}^{1 / 2} \\
& =C\left[\int_{\mathbb{R}^{N}}\left|\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}\right|^{2}|\nabla v|^{2}\right]^{1 / 2}  \tag{2.3}\\
& \leq \frac{C}{g_{\infty}}\|v\|
\end{align*}
$$

Thus for all $v \in E$, we know $G^{-1}(v) \in L^{2 \cdot 2^{*}}\left(\mathbb{R}^{N}\right)$.
Let $\left\{v_{n}\right\}$ be a sequence in $E$ such that $v_{n} \rightarrow v$ in $E$. Thus

$$
\frac{\partial v_{n}}{\partial x_{i}} \rightarrow \frac{\partial v}{\partial x_{i}} \quad \text { in } L^{2}\left(\mathbb{R}^{N}\right)
$$

for $i=1,2, \ldots, N$. By (iii) in Proposition 2.3, we have

$$
\begin{equation*}
\int_{\mathbb{R}^{N}} V(x)| | G^{-1}\left(v_{n}\right)\left|-\left|G^{-1}(v)\right|\right|^{2} d x \rightarrow 0 \tag{2.4}
\end{equation*}
$$

Therefore, by Lemma A. 1 in [43], up to a subsequence, there exists $U_{i} \in L^{2}\left(\mathbb{R}^{N}\right)$ for $i=$ $1,2, \ldots, N$ such that

$$
\left|\frac{\partial v_{n}}{\partial x_{i}}\right| \leq U_{i}(x), \quad \text { a.e. } x \in \mathbb{R}^{N}
$$

Hence

$$
\left|\frac{\partial G^{-1}\left(v_{n}\right)}{\partial x_{i}}\right|=\left|\frac{1}{g\left(G^{-1}\left(v_{n}\right)\right)} \frac{\partial v_{n}}{\partial x_{i}}\right| \leq \frac{1}{g(0)} U_{i}(x)
$$

and

$$
\frac{\partial G^{-1}\left(v_{n}\right)}{\partial x_{i}}=\frac{1}{g\left(G^{-1}\left(v_{n}\right)\right)} \frac{\partial v_{n}}{\partial x_{i}} \rightarrow \frac{1}{g\left(G^{-1}(v)\right)} \frac{\partial v}{\partial x_{i}}=\frac{\partial G^{-1}(v)}{\partial x_{i}} \quad \text { a.e. } x \in \mathbb{R}^{N}
$$

for $i=1,2, \ldots, N$. So by the Lebesgue Dominated Converge Theorem, we have

$$
G^{-1}\left(v_{n}\right) \rightarrow G^{-1}(v) \quad \text { in } D^{1,2}\left(\mathbb{R}^{N}\right)
$$

which together with (2.4), we have

$$
G^{-1}\left(v_{n}\right) \rightarrow G^{-1}(v) \quad \text { in } X
$$

Moreover, by Lemma 2.4, one has

$$
G^{-1}\left(v_{n}\right) \rightarrow G^{-1}(v) \quad \text { in } L^{r}\left(\mathbb{R}^{N}\right) \quad \text { for } 1 \leq r \leq 2^{*}
$$

By (2.3), we have

$$
\left|G^{-1}\left(v_{n}-v\right)\right|^{2} \rightarrow 0 \quad \text { in } L^{2^{*}}\left(\mathbb{R}^{N}\right)
$$

Again by Lemma A. 1 in [43], there exists $W \in L^{2^{*}}\left(\mathbb{R}^{N}\right)$ such that

$$
\left|G^{-1}\left(v_{n}-v\right)\right|^{2} \leq W(x) \quad \text { a.e. } x \in \mathbb{R}^{N} .
$$

By the convexity of $G^{-1}(v)^{2}$, we get

$$
\begin{aligned}
\left|G^{-1}\left(v_{n}\right)^{2 \cdot 2^{*}}\right| & \leq\left|\frac{1}{2} G^{-1}\left(2\left(v_{n}-v\right)\right)^{2}+\frac{1}{2} G^{-1}(2 v)^{2}\right|^{2^{*}} \\
& \leq\left|\frac{C_{0}}{2} G^{-1}\left(v_{n}-v\right)^{2}+\frac{C_{0}}{2} G^{-1}(v)^{2}\right|^{2^{*}} \\
& \leq \frac{C_{0} 2^{2^{*}-1}}{2}\left(\left|G^{-1}\left(v_{n}-v\right)^{2}\right|^{2^{*}}+\left|G^{-1}(v)^{2}\right|^{2^{*}}\right) \\
& \leq \frac{C_{0} 2^{2^{*}-1}}{2}\left(W(x)^{2^{*}}+\left|G^{-1}(v)^{2}\right|^{2^{*}}\right) \in L^{1}\left(\mathbb{R}^{N}\right) .
\end{aligned}
$$

Hence by the Lebesgue dominated converge theorem, one has

$$
G^{-1}\left(v_{n}\right) \rightarrow G^{-1}(v) \quad \text { in } L^{2 \cdot 2^{*}}\left(\mathbb{R}^{N}\right)
$$

Therefore, this completes the proof of continuity.
Next, we will prove the compactness. Let $\left\{v_{n}\right\} \subset E$ be a bounded sequence. Then $\left\{v_{n}\right\}$ is bounded in $D^{1,2}\left(\mathbb{R}^{N}\right)$ and by (2.1), we conclude that there exists a constant $C>0$ such that

$$
\int_{\mathbb{R}^{N}} V(x) G^{-1}\left(v_{n}\right)^{2} \leq C .
$$

Form (2.2) and (2.3), we can know that $\left\{G^{-1}\left(v_{n}\right)\right\}$ is bounded in $X$ and in $L^{2 \cdot 2 \cdot 2^{*}}\left(\mathbb{R}^{N}\right)$. The compact embedding $X \hookrightarrow L^{1}\left(\mathbb{R}^{N}\right)$ implies that, up to a subsequence, there is $w \in L^{1}\left(\mathbb{R}^{N}\right)$ such that $G^{-1}\left(v_{n}\right) \rightarrow w$ in $L^{1}\left(\mathbb{R}^{N}\right)$ and almost everywhere in $\mathbb{R}^{N}$. Thus, by the Brezis-Lieb Lemma we conclude that $w \in L^{2 \cdot 2^{*}}\left(\mathbb{R}^{N}\right)$ and according to interpolation inequality, given any $1 \leq q<2 \cdot 2^{*}$, there exists $0<\varsigma \leq 1$ such that

$$
\left\|G^{-1}\left(v_{n}\right)-w\right\|_{q} \leq\left\|G^{-1}\left(v_{n}\right)-w\right\|_{1}^{\varsigma}\left\|G^{-1}\left(v_{n}\right)-w\right\|_{2 \cdot 2^{*}}^{1-\varsigma} \leq C\left\|G^{-1}\left(v_{n}\right)-w\right\|_{1}^{\zeta},
$$

which shows that $G^{-1}\left(v_{n}\right) \rightarrow w$ in $L^{r}\left(\mathbb{R}^{N}\right)$ for $1 \leq r<2 \cdot 2^{*}$. This completes the proof.
Lemma 2.6. The embedding $E \hookrightarrow L^{r}\left(\mathbb{R}^{N}\right)$ is continuous for $1 \leq r \leq 2^{*}$.
Proof. Firstly, by $\left(g_{8}\right)$ in Lemma 2.1, we can get

$$
\begin{equation*}
|t| \leq \frac{1}{G^{-1}(0)}\left|G^{-1}(t)\right|+\frac{1}{G^{-1}(0)^{2}}\left|G^{-1}(t)\right|^{2} . \tag{2.5}
\end{equation*}
$$

Moreover, by Lemma 2.5, if $v \in E$, then $v \in L^{1}\left(\mathbb{R}^{N}\right)$. That is to say that if $v_{n} \rightarrow 0$ in $E$, then we have $G^{-1}\left(v_{n}\right) \rightarrow 0$ in $L^{1}\left(\mathbb{R}^{N}\right)$ and $L^{2}\left(\mathbb{R}^{N}\right)$. Thus by (2.5), we know $v_{n} \rightarrow 0$ in $L^{1}\left(\mathbb{R}^{N}\right)$. Thus $E \hookrightarrow L^{1}\left(\mathbb{R}^{N}\right)$ is continuous. Using one more time (2.5), we have

$$
|t|^{2^{*}} \leq \frac{1}{G^{-1}(0)}\left|G^{-1}(t)\right|^{2^{*}}+\frac{1}{G^{-1}(0)^{2}}\left|G^{-1}(t)\right|^{2 \cdot 2^{*}}
$$

It follows from Lemma 2.5 that $v_{n} \rightarrow 0$ in $L^{2^{*}}\left(\mathbb{R}^{N}\right)$. Finally, by interpolation the results obviously holds.

Proposition 2.7 ([10]). E is a Banach space. Moreover, $C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$ is dense in $E$.
Proposition 2.8. The functional $\mathcal{J}_{\lambda}$ is well defined, continuous and Gateaux-differentiable in E with

$$
\left\langle\mathcal{J}_{\lambda}^{\prime}(v), \varphi\right\rangle=\int_{\mathbb{R}^{N}} \nabla v \nabla \varphi+\int_{\mathbb{R}^{N}} V(x) \frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)} \varphi-\left\langle\Psi_{\lambda}(v), \varphi\right\rangle,
$$

where

$$
\left\langle\Psi_{\lambda}(v), \varphi\right\rangle=\lambda \int_{\mathbb{R}^{N}} \frac{f\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)} \varphi+\int_{\mathbb{R}^{N}} \frac{h\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)} \varphi, \quad v, \varphi \in E .
$$

Moreover, for $v \in E$ we know that $\mathcal{J}_{\lambda}^{\prime}(v) \in E^{*}$ and if $v_{n} \rightarrow v$ in $E$ then

$$
\left\langle\mathcal{J}_{\lambda}^{\prime}\left(v_{n}\right), \varphi\right\rangle \rightarrow\left\langle\mathcal{J}_{\lambda}^{\prime}(v), \varphi\right\rangle,
$$

for each $\varphi \in E$, that is, $\mathcal{J}_{\lambda}^{\prime}\left(v_{n}\right) \rightarrow \mathcal{J}_{\lambda}^{\prime}(v)$ in the weak $*$ topology of $E^{*}$.
Proof. By $\left(F H_{1}\right)-\left(F H_{3}\right)$, for each $v \in E$, we have

$$
\begin{equation*}
\int_{\mathbb{R}^{N}} F\left(x, G^{-1}(v)\right) \leq \frac{c_{1}}{q} \int_{\mathbb{R}^{N}}\left|G^{-1}(v)\right|^{q}, \tag{2.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{\mathbb{R}^{N}} H\left(x, G^{-1}(v)\right) \leq C \int_{\mathbb{R}^{N}}\left(\left|G^{-1}(v)\right|^{2}+\left|G^{-1}(v)\right|^{p}\right) . \tag{2.7}
\end{equation*}
$$

Hence, by Lemma 2.6, $\Psi_{\lambda}(v)$ is well defined.
Let $v_{n} \rightarrow v$ in $E$, then by the continuous embedding $E \hookrightarrow D^{1,2}\left(\mathbb{R}^{N}\right) \hookrightarrow L^{2^{*}}\left(\mathbb{R}^{N}\right)$, we have $v_{n} \rightarrow v$ in $D^{1,2}\left(\mathbb{R}^{N}\right), v_{n} \rightarrow v$ in $L^{r}\left(\mathbb{R}^{N}\right)$ for $1 \leq r \leq 2^{*}$ and

$$
\int_{\mathbb{R}^{N}} V(x) G^{-1}\left(v_{n}\right)^{2} \rightarrow \int_{\mathbb{R}^{N}} V(x) G^{-1}(v)^{2} .
$$

It follows from (2.6), (2.7) and Lebesgue's Dominated Converge Theorem implies that

$$
\begin{aligned}
& \int_{\mathbb{R}^{N}} F\left(x, G^{-1}\left(v_{n}\right)\right) \rightarrow \int_{\mathbb{R}^{N}} F\left(x, G^{-1}(v)\right), \\
& \int_{\mathbb{R}^{N}} H\left(x, G^{-1}\left(v_{n}\right)\right) \rightarrow \int_{\mathbb{R}^{N}} H\left(x, G^{-1}(v)\right) .
\end{aligned}
$$

Thus $\mathcal{J}_{\lambda}$ is continuous.
Next, we prove that $\mathcal{J}_{\lambda}$ is Gateaux-differentiable in $E$. Note that for any fixed $v, \varphi \in E$, by the mean value theorem, there exists $0<\theta<1$ such that

$$
\frac{1}{2} \int_{\mathbb{R}^{N}} \frac{V(x)\left(\left|G^{-1}(v+t \varphi)\right|^{2}-\left|G^{-1}(v)\right|^{2}\right)}{t} d x=\int_{\mathbb{R}^{N}} V(x) \frac{G^{-1}(v+\theta t \varphi)}{g\left(G^{-1}(v+\theta t \varphi)\right)} \varphi d x .
$$

For any $|t| \leq 1$, we have

$$
\begin{aligned}
\left|V(x) \frac{G^{-1}(v+\theta t \varphi)}{g\left(G^{-1}(v+\theta t \varphi)\right)} \varphi\right| & \leq C V(x)|(v+\theta t \varphi) \varphi| \\
& \leq C V(x)\left|v \varphi+\varphi^{2}\right| \\
& \leq C V(x)\left(|v \varphi|+|\varphi|^{2}\right) \in L^{1}\left(\mathbb{R}^{N}\right) .
\end{aligned}
$$

Since

$$
\frac{G^{-1}(v+\theta t \varphi)}{g\left(G^{-1}(v+\theta t \varphi)\right)} \varphi \rightarrow \frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)} \varphi, \quad \text { a.e. on } \mathbb{R}^{N}, \quad \text { as } t \rightarrow 0,
$$

then by the Lebesgue Dominated Convergence Theorem, we have

$$
\frac{1}{2} \int_{\mathbb{R}^{N}} \frac{V(x)\left(\left|G^{-1}(v+t \varphi)\right|^{2}-\left|G^{-1}(v)\right|^{2}\right)}{t} \rightarrow \int_{\mathbb{R}^{N}} V(x) \frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)} \varphi, \quad \text { as } t \rightarrow 0,
$$

Similar to the proof and using $(F H)-\left(F H_{1}\right)$, we can conclude that

$$
\int_{\mathbb{R}^{N}} \frac{F\left(x, G^{-1}(v+t \varphi)\right)-F\left(x, G^{-1}(v)\right)}{t} \rightarrow \int_{\mathbb{R}^{N}} \frac{f\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)} \varphi, \quad \text { as } t \rightarrow 0,
$$

and

$$
\int_{\mathbb{R}^{N}} \frac{H\left(x, G^{-1}(v+t \varphi)\right)-F\left(x, G^{-1}(v)\right)}{t} \rightarrow \int_{\mathbb{R}^{N}} \frac{h\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)} \varphi, \quad \text { as } t \rightarrow 0,
$$

Based on the above discussion, we have that $\mathcal{J} \in \mathcal{C}^{1}(E, \mathbb{R})$.
To prove $\mathcal{J}^{\prime} \in E^{*}$ for $v \in E$, we only need to check the term $\int_{\mathbb{R}^{N}} V(x) \frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)} \varphi$. In fact, let $\omega_{n} \rightarrow 0$ in $E$. By Proposition 2.3-(iii), we have

$$
\int_{\mathbb{R}^{N}} V(x)\left|G^{-1}\left(\omega_{n}\right)\right|^{2} \rightarrow 0, \quad \text { as } n \rightarrow \infty
$$

Moreover, it follows from $\left(g_{2}\right),\left(g_{9}\right)$ in Lemma 2.1 and (2.5) that

$$
\begin{aligned}
\left|\int_{\mathbb{R}^{N}} V(x) \frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)} \omega_{n}\right| \leq & \int_{\mathbb{R}^{N}} V(x)\left|\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}\right|\left|\omega_{n}\right| \\
\leq & \frac{1}{G^{-1}(0)} \int_{\mathbb{R}^{N}} V(x)\left|\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}\right|\left|G^{-1}\left(\omega_{n}\right)\right| \\
& \quad+\frac{1}{G^{-1}(0)^{2}} \int_{\mathbb{R}^{N}} V(x)\left|\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}\right|\left|G^{-1}\left(\omega_{n}\right)\right|^{2} \\
\leq & \frac{1}{G^{-1}(0)} \int_{\mathbb{R}^{N}} V(x)\left|G^{-1}(v)\right|\left|G^{-1}\left(\omega_{n}\right)\right| \\
& \quad+\frac{1}{g_{\infty} G^{-1}(0)^{2}} \int_{\mathbb{R}^{N}} V(x)\left|G^{-1}\left(\omega_{n}\right)\right|^{2} \\
\leq & \frac{1}{G^{-1}(0)}\left[\int_{\mathbb{R}^{N}} V(x)\left|G^{-1}(v)\right|^{2}\right]^{1 / 2}\left[\int_{\mathbb{R}^{N}} V(x)\left|G^{-1}\left(\omega_{n}\right)\right|^{2}\right]^{1 / 2} \\
& \quad+\frac{1}{g_{\infty} G^{-1}(0)^{2}} \int_{\mathbb{R}^{N}} V(x)\left|G^{-1}\left(\omega_{n}\right)\right|^{2},
\end{aligned}
$$

which implies that

$$
\int_{\mathbb{R}^{N}} V(x) \frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)} \omega_{n} \rightarrow 0 .
$$

Thus $\mathcal{J}^{\prime} \in E^{*}$ for any $v \in E$.
Similar to the proof of the first part in this proposition, we can prove that if $v_{n} \rightarrow v$ in $E$, then

$$
\left\langle\mathcal{J}_{\lambda}^{\prime}\left(v_{n}\right), \varphi\right\rangle \rightarrow\left\langle\mathcal{J}_{\lambda}^{\prime}(v), \varphi\right\rangle,
$$

for each $\varphi \in E$, that is, $\mathcal{J}_{\lambda}^{\prime}\left(v_{n}\right) \rightarrow \mathcal{J}_{\lambda}^{\prime}(v)$ in the weak $*$ topology of $E^{*}$.

## 3 Proofs of Theorem 1.1 and Theorem 1.2

This section is devoted to prove Theorem 1.1 and Theorem 1.2. To this end, we will present two lemmas to show that the functional $\mathcal{J}_{\lambda}$ verifies the mountain pass geometry. Before proving the two lemmas, we need to the following version mountain pass theorem, which is a consequence of the Ekeland variational principle as developed in [1].

Theorem 3.1 ([1]). Let $E$ be a Banach space and $\Phi \in \mathcal{C}(E, R)$, Gateaux-differentiable for all $v \in E$, with $G$-derivative $\Phi^{\prime}(v) \in E^{*}$ continuous from the norm topology of $E$ to the weak $*$ topology of $E^{*}$, $\Phi$ satisfies (PS) condition and $\Phi(0)=0$. Let $\mathcal{S}$ be a closed subset of $E$ which disconnects (archwise) $E$. Let $v_{0}=0$ and $v_{1} \in E$ be points belonging to distinct connected components of $E \backslash S$. Suppose that

$$
\inf _{\mathcal{S}} \Phi \geq \eta>0 \quad \text { and } \quad \Phi\left(v_{1}\right) \leq 0 .
$$

Then

$$
c=\inf _{\gamma \in \Gamma} \max _{t \in[0,1]} \Phi(\gamma(t)) \geq \alpha,
$$

and there exists a $(P S)_{c}$ sequence for $\Phi$. c is critical value of $\Phi$.
Next, we prove that there exists $\lambda_{0}>0$ such that for all $\lambda \in\left[0, \lambda_{0}\right], \mathcal{J}_{\lambda}$ satisfies all the conditions of Theorem 3.1. To this end, for $\rho>0$, let us define the following set

$$
\mathcal{S}(\rho)=\left\{x \in \mathbb{R}^{N}: \mathcal{P}(v)=\rho^{2}\right\}
$$

where $\mathcal{P}: E \rightarrow \mathbb{R}$ is given by

$$
\mathcal{P}(v)=\int_{\mathbb{R}^{N}}\left(|\nabla v|^{2}+V(x)\left|G^{-1}(v)\right|^{2}\right) .
$$

Since $\mathcal{P}(v)$ is continuous then $\mathcal{S}(\rho)$ is a closed subset and disconnects the space $E$ for $\rho>0$.
Lemma 3.2. Suppose that $\left(V_{1}\right)-\left(V_{2}\right)$ and $(F H)-\left(F H_{4}\right)$ are satisfied. Then there exist $\lambda_{0}, \eta, \rho>0$ such that for all $\lambda \in\left[0, \lambda_{0}\right], \mathcal{J}_{\lambda}(v) \geq \eta$ for all $v \in \mathcal{S}(\rho)$.

Proof. By $\left(F H_{2}\right)$ and $\left(F H_{3}\right)$, for any $\varepsilon>0$ there exists $C_{\varepsilon}>0$ such that

$$
\left|H\left(x, G^{-1}(s)\right)\right| \leq \varepsilon\left|G^{-1}(s)\right|^{2}+C_{\varepsilon}\left|G^{-1}(s)\right|^{p} \quad \text { for all }(x, s) \in \mathbb{R}^{N} \times \mathbb{R}
$$

Thus for $v \in \mathcal{S}_{\rho}$, by $\left(V_{1}\right)$ and Hölder's inequality, we get

$$
\begin{aligned}
\int_{\mathbb{R}^{N}} H\left(x, G^{-1}(v)\right) & \leq \varepsilon \int_{\mathbb{R}^{N}}\left|G^{-1}(v)\right|^{2}+C_{\varepsilon} \int_{\mathbb{R}^{N}}\left|G^{-1}(v)\right|^{p} \\
& \leq \frac{\varepsilon}{V_{0}} \int_{\mathbb{R}^{N}} V(x)\left|G^{-1}(v)\right|^{2}+C_{\varepsilon} \int_{\mathbb{R}^{N}}\left|G^{-1}(v)^{2}\right|^{p / 2} \\
& \leq \frac{\varepsilon}{V_{0}} \rho^{2}+C_{\varepsilon}\left[\int_{\mathbb{R}^{N}}\left|G^{-1}(v)\right|^{2}\right]^{\frac{p \kappa}{2}}\left[\int_{\mathbb{R}^{N}}\left|G^{-1}(v)^{2}\right|^{2^{*}}\right]^{1-\frac{p \kappa}{2}} \\
& \leq \frac{\varepsilon}{V_{0}} \rho^{2}+C_{\varepsilon} \rho^{p \kappa}\left[\int_{\mathbb{R}^{N}}\left|\nabla\left(G^{-1}(v)^{2}\right)\right|^{2}\right]^{\frac{2^{*}}{2}\left(1-\frac{p \kappa}{2}\right)},
\end{aligned}
$$

where

$$
\kappa=\frac{2 \cdot 2^{*}-p}{p\left(2^{*}-1\right)} .
$$

Moreover, since $v \in \mathcal{S}_{\rho}$, then

$$
\int_{\mathbb{R}^{N}}\left|\nabla\left(G^{-1}(v)^{2}\right)\right|^{2}=4 \int_{\mathbb{R}^{N}}\left|\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}\right|^{2}|\nabla v|^{2} \leq \frac{4}{g_{\infty}^{2}} \int_{\mathbb{R}^{N}}|\nabla v|^{2} \leq \frac{4}{g_{\infty}^{2}} \rho^{2} .
$$

Therefore, one has

$$
\begin{align*}
\int_{\mathbb{R}^{N}} H\left(x, G^{-1}(v)\right) & \leq \frac{\varepsilon}{V_{0}} \rho^{2}+C_{\varepsilon} \rho^{p \kappa}\left[\frac{4}{g_{\infty}^{2}} \rho^{2}\right]^{\frac{2^{*}}{2}\left(1-\frac{p \kappa}{2}\right)}  \tag{3.1}\\
& \leq \frac{\varepsilon}{V_{0}} \rho^{2}+C \rho^{\frac{2(p+N)}{N+2}}
\end{align*}
$$

Next, by $\left(F H_{1}\right)$ and (2.2), we conclude that

$$
\begin{aligned}
\int_{\mathbb{R}^{N}} F\left(x, G^{-1}(v)\right) & \leq c_{1} \int_{\mathbb{R}^{N}}\left|G^{-1}(v)\right|^{q} \\
& \leq\left\|G^{-1}(v)\right\|_{X}^{q} \\
& \leq C\left[\int_{\mathbb{R}^{N}}\left(|\nabla v|^{2}+V(x)\left|G^{-1}(v)\right|^{2}\right)\right]^{q / 2},
\end{aligned}
$$

and so $v \in \mathcal{S}_{\rho}$, we get

$$
\begin{equation*}
\int_{\mathbb{R}^{N}} F\left(x, G^{-1}(v)\right) \leq C \rho^{q} . \tag{3.2}
\end{equation*}
$$

It follows from (3.1) and (3.2) that for $v \in \mathcal{S}_{\rho}$,

$$
\begin{aligned}
\mathcal{J}_{\lambda}(v) & \geq\left(\frac{1}{2}-\frac{\varepsilon}{V_{0}}\right) \rho^{2}-C \lambda \rho^{q}-C \rho^{\frac{2(p+N)}{N+2}} \\
& =\rho^{2}\left(\frac{1}{2}-\frac{\varepsilon}{V_{0}}-C \rho^{\frac{2(p-2)}{N+2}}\right)-C \lambda \rho^{q} .
\end{aligned}
$$

Next, we choose $0<2 \varepsilon<V_{0}$ and $\rho_{0}>0$ such that

$$
\alpha_{0}:=\frac{1}{2}-\frac{\varepsilon}{V_{0}}-C \rho_{0}^{\frac{2(p-2)}{N+2}}>0
$$

where implies that

$$
\mathcal{J}_{\lambda}(v) \geq \rho_{0}^{q}\left(\alpha_{0} \rho_{0}^{2-q}-\lambda C\right)
$$

In the above inequality, choosing $\lambda_{0}=\frac{\alpha_{0} \rho_{0}^{2-\eta}}{4 C}$ and $\eta:=\frac{3 \alpha_{0} \rho_{0}^{2}}{4}>0$ such that for all $\lambda \in\left[0, \lambda_{0}\right]$,

$$
\mathcal{J}_{\lambda}(v) \geq \eta>0
$$

This completes the proof.
Lemma 3.3. Suppose that $\left(V_{1}\right)-\left(V_{2}\right)$ and $(F H)-\left(F H_{4}\right)$ are satisfied. Then for $\lambda \in\left[0, \lambda_{0}\right]$, there exists $v \in E$ such that $\mathcal{P}(v)>\rho_{0}$ and $\mathcal{J}_{\lambda}(v)<0$.

Proof. To this end, we prove that for fixed $\psi \in E \backslash\{0\}$,

$$
\mathcal{J}_{\lambda}(t \psi) \rightarrow-\infty \quad \text { as } t \rightarrow+\infty .
$$

By $\left(F H_{4}\right)$, there exist $C_{1}, C_{2}>0$ such that $H\left(x, G^{-1}(s)\right) \geq C_{1} s^{2 \mu}-C_{2}$ for all $(x, s) \in \mathbb{R}^{N} \times \mathbb{R}$. Choosing $\psi \in\left(\mathcal{C}_{0}^{\infty}\left(\mathbb{R}^{N}\right),[0,1]\right)$ such that $\operatorname{supp} \psi=\bar{\Omega}$, we have

$$
\begin{aligned}
\mathcal{J}_{\lambda}(t \psi) & \leq \frac{1}{2} \int_{\mathbb{R}^{N}}\left(|t \nabla \psi|^{2}+V(x)\left|G^{-1}(t \psi)\right|^{2}\right)-\int_{\mathbb{R}^{N}} H\left(x, G^{-1}(t \psi)\right) \\
& \leq t^{2} \int_{\bar{\Omega}}\left(|\nabla \psi|^{2}+V(x) \psi^{2}\right)-C_{1} \int_{\bar{\Omega}}\left|G^{-1}(t \psi)\right|^{2 \mu}+C_{2}|\bar{\Omega}| \\
& \leq t^{2}\left[\int_{\bar{\Omega}}\left(|\nabla \psi|^{2}+V(x) \psi^{2}\right)-C_{1} \int_{\bar{\Omega}} \frac{\left|G^{-1}(t \psi)\right|^{2 \mu}}{t^{2}}+C_{2} \frac{|\bar{\Omega}|}{t^{2}}\right],
\end{aligned}
$$

where $|\bar{\Omega}|$ denotes the Lebesgue measure of $\bar{\Omega}$. Moreover, by $\left(g_{7}\right)$ in Lemma 2.1, we have

$$
\int_{\bar{\Omega}} \frac{\left|G^{-1}(t \psi)\right|^{2 \mu}}{t^{2}}=\int_{\bar{\Omega}}\left(\frac{G^{-1}(t \psi)}{\sqrt{t|\psi|}}\right)^{4}\left|G^{-1}(t \psi)\right|^{2 \mu-4} \psi^{2} \rightarrow+\infty \quad \text { as } t \rightarrow+\infty
$$

Hence, we take $v=t \psi$ with $t$ large enough. This completes the proof.
A sequence $\left\{v_{n}\right\} \subset E$ is said to be a $(P S)_{c}$-sequence if $\mathcal{J}_{\lambda}\left(v_{n}\right) \rightarrow c$ and $\mathcal{J}_{\lambda}^{\prime}\left(v_{n}\right) \rightarrow 0 . \mathcal{J}_{\lambda}$ is said to satisfy the $(P S)_{c}$-condition if any $(P S)_{c}$-sequence has a convergent subsequence. Now, we will prove that $\mathcal{J}_{\lambda}$ satisfies $(P S)_{c}$-condition.

Lemma 3.4. Any $(P S)_{c}$ sequence for $\mathcal{J}_{\lambda}$ is bounded in $E$.
Proof. Suppose that $\left\{v_{n}\right\}$ is a $(P S)_{c}$ for $\mathcal{J}_{\lambda}$, that is, $\mathcal{J}_{\lambda}\left(v_{n}\right) \rightarrow c$ and $\mathcal{J}_{\lambda}^{\prime}\left(v_{n}\right) \rightarrow 0$. Using $(g)$

$$
\begin{align*}
\mathcal{J}_{\lambda}\left(v_{n}\right) & -\frac{1}{2 \mu}\left\langle\mathcal{J}_{\lambda}^{\prime}\left(v_{n}\right), v_{n}\right\rangle \\
= & \left(\frac{1}{2}-\frac{1}{2 \mu}\right) \int_{\mathbb{R}^{N}}\left|\nabla v_{n}\right|^{2}+\int_{\mathbb{R}^{N}} V(x)\left(\frac{1}{2}\left|G^{-1}\left(v_{n}\right)\right|^{2}-\frac{1}{2 \mu} \frac{G^{-1}\left(v_{n}\right)}{g\left(G^{-1}\left(v_{n}\right)\right)} v_{n}\right) \\
& -\frac{1}{2 \mu} \int_{\mathbb{R}^{N}}\left[2 \mu H\left(x, G^{-1}\left(v_{n}\right)\right)-\frac{h\left(x, G^{-1}\left(v_{n}\right)\right)}{g\left(G^{-1}\left(v_{n}\right)\right)} v_{n}\right] \\
& -\lambda \int_{\mathbb{R}^{N}} F\left(x, G^{-1}\left(v_{n}\right)\right)+\frac{\lambda}{2 \mu} \int_{\mathbb{R}^{N}} \frac{f\left(x, G^{-1}\left(v_{n}\right)\right)}{g\left(G^{-1}\left(v_{n}\right)\right)} v_{n}  \tag{3.3}\\
\geq & \left(\frac{1}{2}-\frac{1}{2 \mu}\right) \int_{\mathbb{R}^{N}}\left|\nabla v_{n}\right|^{2}+\int_{\mathbb{R}^{N}} V(x)\left(\frac{1}{2}\left|G^{-1}\left(v_{n}\right)\right|^{2}-\frac{1}{2 \mu} \frac{G^{-1}\left(v_{n}\right)}{g\left(G^{-1}\left(v_{n}\right)\right)} v_{n}\right) \\
& -\frac{1}{2 \mu} \int_{\mathbb{R}^{N}}\left[2 \mu H\left(x, G^{-1}\left(v_{n}\right)\right)-\frac{h\left(x, G^{-1}\left(v_{n}\right)\right)}{g\left(G^{-1}\left(v_{n}\right)\right)} v_{n}\right]-\lambda \int_{\mathbb{R}^{N}} F\left(x, G^{-1}\left(v_{n}\right)\right) .
\end{align*}
$$

Using the definition of $G$, we can get $G(t) \leq g(t) t$ for all $t \geq 0$. In fact, by $g^{\prime}(t) \geq 0$ for all $t \geq 0$

$$
\begin{equation*}
G(t)=\int_{0}^{t} g(s) d s=\left.s g(s)\right|_{0} ^{t}-\int_{0}^{t} s g^{\prime}(s) d s \leq g(t) t \tag{3.4}
\end{equation*}
$$

By (3.4) and ( $g_{6}$ ) in Lemma 2.1, it is easy to check that

$$
\begin{equation*}
\frac{1}{2}\left|G^{-1}(s)\right|^{2} \leq \frac{G^{-1}(s) s}{g\left(G^{-1}(s)\right)} \leq\left|G^{-1}(s)\right|^{2}, \quad \text { for all } s \in \mathbb{R} \tag{3.5}
\end{equation*}
$$

By (3.3) and (3.5), we have

$$
\begin{equation*}
c+o_{n}(1)\left\|v_{n}\right\| \geq\left(\frac{1}{2}-\frac{1}{2 \mu}\right) \int_{\mathbb{R}^{N}}\left(\left|\nabla v_{n}\right|^{2}+V(x)\left|G^{-1}\left(v_{n}\right)\right|^{2}\right)-\lambda C\left\|G^{-1}\left(v_{n}\right)\right\|_{q}^{q} . \tag{3.6}
\end{equation*}
$$

It follows from (2.2) and (3.6) that

$$
\begin{gathered}
c+o_{n}(1)\left\|v_{n}\right\|+\lambda C\left[\int_{\mathbb{R}^{N}}\left(\left|\nabla v_{n}\right|^{2}+V(x)\left|G^{-1}\left(v_{n}\right)\right|^{2}\right)\right]^{q / 2} \\
\geq\left(\frac{1}{2}-\frac{1}{2 \mu}\right) \int_{\mathbb{R}^{N}}\left(\left|\nabla v_{n}\right|^{2}+V(x)\left|G^{-1}\left(v_{n}\right)\right|^{2}\right),
\end{gathered}
$$

which implies that there exists $C>0$ such that

$$
\begin{equation*}
\int_{\mathbb{R}^{N}}\left(\left|\nabla v_{n}\right|^{2}+V(x)\left|G^{-1}\left(v_{n}\right)\right|^{2}\right) \leq C, \tag{3.7}
\end{equation*}
$$

due to $q \in(1,2)$. Since $s^{1 / 2} \leq 1+s$ for all $s \geq 0$, then we have the following estimate

$$
\begin{align*}
\left\|v_{n}\right\| & \leq\left(\int_{\mathbb{R}^{N}}\left|\nabla v_{n}\right|^{2}\right)^{1 / 2}+1+\int_{\mathbb{R}^{N}} V(x)\left|G^{-1}\left(v_{n}\right)\right|^{2}  \tag{3.8}\\
& \leq 2+\int_{\mathbb{R}^{N}}\left(\left|\nabla v_{n}\right|^{2}+V(x)\left|G^{-1}\left(v_{n}\right)\right|^{2}\right) .
\end{align*}
$$

It follows from (3.7) and (3.8) that $\left\{v_{n}\right\}$ is bounded in $E$.
Lemma 3.5. Any $(P S)_{c}$ sequence for $\mathcal{J}_{\lambda}$ has a converge subsequence.
Proof. Let $\left\{v_{n}\right\}$ be a $(P S)_{c}$ for $\mathcal{J}_{\lambda}$. By Lemma 3.4, we know that $\left\{v_{n}\right\}$ is bounded in $E$. Since $E \hookrightarrow H^{1}\left(\mathbb{R}^{N}\right),\left\{v_{n}\right\}$ is bounded in $H^{1}\left(\mathbb{R}^{N}\right)$. Hence up to a subsequence, there exists $v \in H^{1}\left(\mathbb{R}^{N}\right)$ such that

$$
v_{n} \rightharpoonup v \text { in } H^{1}\left(\mathbb{R}^{N}\right), \quad v_{n} \rightharpoonup v \quad \text { in } L^{r}\left(\mathbb{R}^{N}\right) \quad \text { for all } 1 \leq r \leq 2^{*}, \quad v_{n} \rightarrow v \quad \text { a.e. in } \mathbb{R}^{N} .
$$

Using (2.1) and Fatou's Lemma, we have

$$
\int_{\mathbb{R}^{N}} V(x)\left|G^{-1}(v)\right|^{2} \leq \liminf _{n \rightarrow \infty} \int_{\mathbb{R}^{N}} V(x)\left|G^{-1}\left(v_{n}\right)\right|^{2} \leq C,
$$

which shows that $v \in E$. Moreover, by Lemma 2.5 , one has

$$
\begin{equation*}
G^{-1}\left(v_{n}\right) \rightarrow G^{-1}(v) \quad \text { in } L^{r}\left(\mathbb{R}^{N}\right) \quad \text { for all } 1 \leq r<2 \cdot 2^{*} . \tag{3.9}
\end{equation*}
$$

Since $\left[G^{-1}(s)\right]^{2}$ is convex, then $\mathcal{P}(s)$ is also convex function. Therefore by Lemma 15.3 in [22], we have

$$
\begin{aligned}
\frac{1}{2} \mathcal{P}(v)-\frac{1}{2} \mathcal{P}\left(v_{n}\right) & \geq \frac{1}{2}\left\langle\mathcal{P}^{\prime}\left(v_{n}\right), v-v_{n}\right\rangle \\
& =\int_{\mathbb{R}^{N}} \nabla v_{n} \nabla\left(v-v_{n}\right)+\int_{\mathbb{R}^{N}} V(x) \frac{G^{-1}\left(v_{n}\right)}{g\left(G^{-1}\left(v_{n}\right)\right)}\left(v-v_{n}\right) .
\end{aligned}
$$

Moreover,

$$
\begin{align*}
& \frac{1}{2} \int_{\mathbb{R}^{N}}\left[|\nabla v|+V(x)\left|G^{-1}(v)\right|^{2}\right]-\frac{1}{2} \int_{\mathbb{R}^{N}}\left[\left|\nabla v_{n}\right|+V(x)\left|G^{-1}\left(v_{n}\right)\right|^{2}\right] \\
& \geq  \tag{3.10}\\
& \quad \lambda \int_{\mathbb{R}^{N}} \frac{f\left(x, G^{-1}\left(v_{n}\right)\right)}{g\left(G^{-1}\left(v_{n}\right)\right)}\left(v-v_{n}\right)+\int_{\mathbb{R}^{N}} \frac{h\left(x, G^{-1}\left(v_{n}\right)\right)}{g\left(G^{-1}\left(v_{n}\right)\right)}\left(v-v_{n}\right) \\
& \quad+\left\langle\mathcal{J}_{\lambda}^{\prime}\left(v_{n}\right), v-v_{n}\right\rangle .
\end{align*}
$$

Writing

$$
\begin{aligned}
\int_{\mathbb{R}^{N}} \frac{h\left(x, G^{-1}\left(v_{n}\right)\right)}{g\left(G^{-1}\left(v_{n}\right)\right)}\left(v-v_{n}\right)= & \int_{\mathbb{R}^{N}}\left[\frac{h\left(x, G^{-1}\left(v_{n}\right)\right)}{g\left(G^{-1}\left(v_{n}\right)\right)}-\frac{h\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)}\right]\left(v-v_{n}\right) \\
& +\int_{\mathbb{R}^{N}} \frac{h\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)}\left(v-v_{n}\right) .
\end{aligned}
$$

Due to $\frac{h\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)} \in L^{2 N /(N+2)}\left(\mathbb{R}^{N}\right)$ and $v_{n} \rightharpoonup v$ in $L^{2^{*}}\left(\mathbb{R}^{N}\right)$, we have

$$
\int_{\mathbb{R}^{N}} \frac{h\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)}\left(v-v_{n}\right) \rightarrow 0 \quad \text { as } n \rightarrow \infty .
$$

By $\left(F H_{2}\right)-\left(F H_{3}\right),(3.9)$ and the Lebesgue Dominated Converge Theorem, we have

$$
\frac{h\left(x, G^{-1}\left(v_{n}\right)\right)}{g\left(G^{-1}\left(v_{n}\right)\right)} \rightarrow \frac{h\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)} \quad \text { in } L^{2 N /(N+2)}\left(\mathbb{R}^{N}\right) .
$$

By Hölder inequality and the boundedness of $\left\{v_{n}\right\}$ in $L^{2^{*}}\left(\mathbb{R}^{N}\right)$, one has

$$
\int_{\mathbb{R}^{N}}\left[\frac{h\left(x, G^{-1}\left(v_{n}\right)\right)}{g\left(G^{-1}\left(v_{n}\right)\right)}-\frac{h\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)}\right]\left(v-v_{n}\right) \rightarrow 0 \quad \text { as } n \rightarrow \infty .
$$

Thus

$$
\int_{\mathbb{R}^{N}} \frac{h\left(x, G^{-1}\left(v_{n}\right)\right)}{g\left(G^{-1}\left(v_{n}\right)\right)}\left(v-v_{n}\right) \rightarrow 0 \quad \text { as } n \rightarrow \infty .
$$

Similarly, we can prove the following

$$
\int_{\mathbb{R}^{N}} \frac{f\left(x, G^{-1}\left(v_{n}\right)\right)}{g\left(G^{-1}\left(v_{n}\right)\right)}\left(v-v_{n}\right) \rightarrow 0 \quad \text { as } n \rightarrow \infty,
$$

which dues to

$$
v_{n} \rightharpoonup v \text { in } L^{q}\left(\mathbb{R}^{N}\right) \quad \text { and } \quad \frac{f\left(x, G^{-1}\left(v_{n}\right)\right)}{g\left(G^{-1}\left(v_{n}\right)\right)} \rightarrow \frac{f\left(x, G^{-1}(v)\right)}{g\left(G^{-1}(v)\right)} \quad \text { in } L^{q /(q-1)}\left(\mathbb{R}^{N}\right) .
$$

By virtue of $\left\langle\mathcal{J}_{\lambda}^{\prime}\left(v_{n}\right), v-v_{n}\right\rangle=o_{n}(1)$, by (3.10), we get

$$
\liminf _{n \rightarrow \infty} \int_{\mathbb{R}^{N}}\left[\left|\nabla v_{n}\right|+V(x)\left|G^{-1}\left(v_{n}\right)\right|^{2}\right] \leq \int_{\mathbb{R}^{N}}\left[|\nabla v|+V(x)\left|G^{-1}(v)\right|^{2}\right] .
$$

In addition, by the semicontinuity of norm and Fatou's Lemma, we have

$$
\begin{aligned}
\liminf _{n \rightarrow \infty} \int_{\mathbb{R}^{N}}\left|\nabla v_{n}\right|^{2} & \geq \int_{\mathbb{R}^{N}}|\nabla v|^{2} \\
\liminf _{n \rightarrow \infty} \int_{\mathbb{R}^{N}} V(x)\left|G^{-1}\left(v_{n}\right)\right|^{2} & \geq \int_{\mathbb{R}^{N}} V(x)\left|G^{-1}(v)\right|^{2} .
\end{aligned}
$$

Therefore we have

$$
\liminf _{n \rightarrow \infty} \int_{\mathbb{R}^{N}}\left|\nabla v_{n}\right|^{2}=\int_{\mathbb{R}^{N}}|\nabla v|^{2},
$$

and

$$
\liminf _{n \rightarrow \infty} \int_{\mathbb{R}^{N}} V(x)\left|G^{-1}\left(v_{n}\right)\right|^{2}=\int_{\mathbb{R}^{N}} V(x)\left|G^{-1}(v)\right|^{2}
$$

By (1) in Proposition 2.2, we get

$$
\inf _{\xi>0} \frac{1}{\xi}\left\{1+\int_{\mathbb{R}^{N}} V(x)\left|G^{-1}\left(\xi\left(v_{n}-v\right)\right)\right|^{2}\right\} \rightarrow 0,
$$

which together with $\nabla v_{n} \rightarrow \nabla v$ in $L^{2}\left(\mathbb{R}^{N}\right)$, implies that $v_{n} \rightarrow v$ in $E$.

Proof of Theorem 1.1. By Lemmas 3.2-3.5, all conditions of Theorem 3.1 are satisfied. Thus there exists a critical point $c_{\lambda}$ for $\mathcal{J}_{\lambda}$ at mountain pass level

$$
c_{\lambda}=\inf _{\gamma \in \Gamma_{\lambda}} \max _{t \in[0,1]} \mathcal{J}_{\lambda}(\gamma(t))>0,
$$

where

$$
\Gamma_{\lambda}=\left\{\gamma \in \mathcal{C}([0,1], E): \gamma(0)=0 \text { and } \mathcal{J}_{\lambda}(\gamma(1))<0\right\}
$$

Therefore for all $\phi \in E$, we get

$$
\begin{equation*}
\int_{\mathbb{R}^{N}} \nabla v_{\lambda} \nabla \phi+V(x) \frac{G^{-1}\left(v_{\lambda}\right)}{g\left(G^{-1}\left(v_{\lambda}\right)\right)} \phi=\lambda \int_{\mathbb{R}^{N}} \frac{f\left(x, G^{-1}\left(v_{\lambda}\right)\right)}{g\left(G^{-1}\left(v_{\lambda}\right)\right)} \phi+\int_{\mathbb{R}^{N}} \frac{h\left(x, G^{-1}\left(v_{\lambda}\right)\right)}{g\left(G^{-1}\left(v_{\lambda}\right)\right)} \phi . \tag{3.11}
\end{equation*}
$$

Choosing $\phi=-v_{\lambda}^{-}$, where $v_{\lambda}^{-}=\max \left\{-v_{\lambda}, 0\right\}$, we have

$$
\int_{\mathbb{R}^{N}}\left[\left|\nabla v_{\lambda}^{-}\right|^{2}+V(x) \frac{G^{-1}\left(v_{\lambda}\right)}{g\left(G^{-1}\left(v_{\lambda}\right)\right)}\left(-v_{\lambda}^{-}\right)\right]=0 .
$$

Since $G^{-1}\left(v_{\lambda}\right)\left(-v_{\lambda}^{-}\right) \geq 0$, we conclude that

$$
\int_{\mathbb{R}^{N}}\left|\nabla v_{\lambda}^{-}\right|^{2}=0 \quad \text { and } \quad \int_{\mathbb{R}^{N}} V(x) \frac{G^{-1}\left(v_{\lambda}\right)}{g\left(G^{-1}\left(v_{\lambda}\right)\right)}\left(-v_{\lambda}^{-}\right)=0 .
$$

Thus $v_{\lambda}^{-}=0$ a.e. in $\mathbb{R}^{N}$ and we have $v_{\lambda} \geq 0$. It follows from the strong maximum principle that $v_{\lambda}>0$ in $\mathbb{R}^{N}$, therefore $u_{\lambda, 1}=G^{-1}\left(v_{\lambda}\right)$ is a positive solution for (1.1).

Next, we shall prove that there exists $C>0$ such that $\left\|u_{\lambda, 1}\right\| \leq C$ for all $\lambda \in\left[0, \lambda_{0}\right]$. In (3.11), taking $\phi=v_{\lambda}$ and using (3.5), we get

$$
\begin{align*}
2 \int_{\mathbb{R}^{N}}\left|\nabla v_{\lambda}\right|^{2}+2 \int_{\mathbb{R}^{N}} V(x)\left|G^{-1}\left(v_{\lambda}\right)\right|^{2} & \geq 2 \int_{\mathbb{R}^{N}}\left|\nabla v_{\lambda}\right|^{2}+2 \int_{\mathbb{R}^{N}} V(x) \frac{G^{-1}\left(v_{\lambda}\right) v_{\lambda}}{g\left(G^{-1}\left(v_{\lambda}\right)\right)}  \tag{3.12}\\
& \geq \int_{\mathbb{R}^{N}} \frac{h\left(x, G^{-1}\left(v_{\lambda}\right)\right) v_{\lambda}}{g\left(G^{-1}\left(v_{\lambda}\right)\right)} .
\end{align*}
$$

Since $\mathcal{J}_{\lambda}\left(v_{\lambda}\right)=c_{\lambda}$, we get

$$
\begin{align*}
2 \mu c_{\lambda}= & \mu \int_{\mathbb{R}^{N}}\left(\left|\nabla v_{\lambda}\right|^{2}+V(x)\left|G^{-1}\left(v_{\lambda}\right)\right|^{2}\right) d x-2 \mu \lambda \int_{\mathbb{R}^{N}} F\left(x, G^{-1}\left(v_{\lambda}\right)\right) d x  \tag{3.13}\\
& -2 \mu \int_{\mathbb{R}^{N}} H\left(x, G^{-1}\left(v_{\lambda}\right)\right) d x,
\end{align*}
$$

and $c_{\lambda} \leq c_{0}$, where

$$
c_{0}=\inf _{\gamma \in \Gamma_{0}} \max _{t \in[0,1]} \mathcal{J}_{0}(\gamma(t))>0,
$$

with $\mathcal{J}_{0}$ is given by

$$
\mathcal{J}_{0}(v)=\frac{1}{2} \int_{\mathbb{R}^{N}}\left(\left|\nabla v_{\lambda}\right|^{2}+V(x)\left|G^{-1}\left(v_{\lambda}\right)\right|^{2}\right) d x-\int_{\mathbb{R}^{N}} H\left(x, G^{-1}\left(v_{\lambda}\right)\right) d x,
$$

and

$$
\Gamma_{0}=\left\{\gamma \in \mathcal{C}([0,1], E): \gamma(0)=0 \text { and } \mathcal{J}_{0}(\gamma(1))<0\right\} .
$$

Thus, by (3.12), (3.13), $\left(F H_{4}\right)$ and (2.2), one has

$$
\begin{aligned}
(\mu-2) & \int_{\mathbb{R}^{N}}\left(\left|\nabla v_{\lambda}\right|^{2}+V(x)\left|G^{-1}\left(v_{\lambda}\right)\right|^{2}\right) \\
= & -2 \int_{\mathbb{R}^{N}}\left(\left|\nabla v_{\lambda}\right|^{2}+V(x)\left|G^{-1}\left(v_{\lambda}\right)\right|^{2}\right)+2 \mu \lambda \int_{\mathbb{R}^{N}} F\left(x, G^{-1}\left(v_{\lambda}\right)\right) \\
& +2 \mu \int_{\mathbb{R}^{N}} H\left(x, G^{-1}\left(v_{\lambda}\right)\right) d x+2 \mu c_{\lambda} \\
\leq & \int_{\mathbb{R}^{N}}\left[2 \mu H\left(x, G^{-1}\left(v_{\lambda}\right)\right)-\frac{h\left(x, G^{-1}\left(v_{\lambda}\right)\right)}{g\left(G^{-1}\left(v_{\lambda}\right)\right)} v_{\lambda}\right]+2 \mu c_{1} \int_{\mathbb{R}^{N}}\left|G^{-1}\left(v_{\lambda}\right)\right|^{q}+2 \mu c_{0} \\
\leq & 2 \mu c_{1}\left[\int_{\mathbb{R}^{N}}\left(\left|\nabla v_{\lambda}\right|^{2}+V(x)\left|G^{-1}\left(v_{\lambda}\right)\right|^{2}\right)\right]^{q / 2}+2 \mu c_{0},
\end{aligned}
$$

which implies that $\int_{\mathbb{R}^{N}}\left(\left|\nabla v_{\lambda}\right|^{2}+V(x)\left|G^{-1}\left(v_{\lambda}\right)\right|^{2}\right)$ is bounded in $\lambda$. Thus

$$
\left\|u_{\lambda, 1}\right\|_{H^{1}}=\left\|G^{-1}\left(v_{\lambda}\right)\right\|_{H^{1}} \leq C\left[\int_{\mathbb{R}^{N}}\left(\left|\nabla v_{\lambda}\right|^{2}+V(x)\left|G^{-1}\left(v_{\lambda}\right)\right|^{2}\right)\right]^{1 / 2} \leq C
$$

Next, we study the exponential decay property for solutions of (1.1) when $\lambda=0$. Let $v_{0}$ be a solution of (1.12) for $\lambda=0$. Now, we first prove that $v_{0} \in L^{\infty}\left(\mathbb{R}^{N}\right)$. Thus we conclude that

$$
\begin{equation*}
\int_{\mathbb{R}^{N}}\left(\nabla v_{0} \nabla \phi+V(x) \frac{G^{-1}\left(v_{0}\right)}{g\left(G^{-1}\left(v_{0}\right)\right)} \phi\right)=\int_{\mathbb{R}^{N}} \frac{h\left(x, G^{-1}\left(v_{0}\right)\right)}{g\left(G^{-1}\left(v_{0}\right)\right)} \phi, \quad \phi \in E . \tag{3.14}
\end{equation*}
$$

For each $k>0$, let

$$
v_{k}= \begin{cases}v_{0}, & \text { if } v_{0} \leq k \\ 0, & \text { if } v_{0} \geq k\end{cases}
$$

and

$$
\varrho_{k}=v_{k}^{2(\beta-1)} v_{0} \quad \text { and } \quad w_{k}=v_{k}^{\beta-1} v_{0},
$$

where $\beta>1$. By $\left(V_{1}\right),\left(F H_{2}\right)$ and $\left(F H_{3}\right)$, we have

$$
h\left(x, G^{-1}\left(v_{0}\right)\right) \leq \frac{V_{0}}{2}\left|G^{-1}\left(v_{0}\right)\right|+C_{V_{0}}\left|G^{-1}\left(v_{0}\right)\right|^{p-1} .
$$

Thus choosing $\varrho_{k}$ as a test function in (3.14), we know

$$
\begin{aligned}
\int_{\mathbb{R}^{N}} v_{k}^{2(\beta-1)}\left|\nabla v_{0}\right|^{2} & \leq \int_{\mathbb{R}^{N}} v_{k}^{2(\beta-1)}\left|\nabla v_{0}\right|^{2}+2(\beta-1) \int_{\mathbb{R}^{N}} v_{k}^{2(\beta-1)-1} v_{0} \nabla v_{k} \cdot \nabla v_{0} \\
& \leq C_{V_{0}} \int_{\mathbb{R}^{N}} \frac{\left|G^{-1}\left(v_{0}\right)\right|^{p-1}}{g\left(G^{-1}\left(v_{0}\right)\right)} v_{k}^{2(\beta-1)} v_{0} .
\end{aligned}
$$

By (3.5), we have

$$
\frac{v_{0}}{g\left(G^{-1}\left(v_{0}\right)\right)} \leq G^{-1}\left(v_{0}\right)
$$

It follows from $\left(g_{7}\right)$ in Lemma 2.1 and the above inequality that

$$
\begin{equation*}
\int_{\mathbb{R}^{v}} v_{k}^{2(\beta-1)}\left|\nabla v_{0}\right|^{2} \leq C \int_{\mathbb{R}^{N}} v_{k}^{2(\beta-1)} v_{0}^{\frac{p}{2}}=C \int_{\mathbb{R}^{N}} v_{0}^{\frac{p}{2}-2} w_{k}^{2} . \tag{3.15}
\end{equation*}
$$

Moreover, using the Gagliardo-Nirenberg-Sobolev inequality and (3.15), one has

$$
\begin{aligned}
\left(\int_{\mathbb{R}^{N}} w_{k}^{2^{*}}\right)^{2 / 2^{*}} & \leq C \int_{\mathbb{R}^{N}}\left|\nabla w_{k}\right|^{2} \\
& \leq C \int_{\mathbb{R}^{N}} v_{k}^{2(\beta-1)}\left|\nabla v_{0}\right|^{2}+C(\beta-1)^{2} \int_{\mathbb{R}^{N}} v_{0}^{2} v_{k}^{2(\beta-2)}\left|\nabla v_{k}\right|^{2} \\
& \leq C \beta^{2} \int_{\mathbb{R}^{N}} v_{k}^{2(\beta-1)}\left|\nabla v_{0}\right|^{2} \\
& \leq C \beta^{2} \int_{\mathbb{R}^{N}} v_{0}^{\frac{p}{2}-2} w_{k}^{2},
\end{aligned}
$$

which dues to $v_{k} \leq v_{0}, 1 \leq \beta^{2}$ and $(\beta-1)^{2} \leq \beta^{2}$. By Hölder's inequality,

$$
\left(\int_{\mathbb{R}^{N}} w_{k}^{2^{*}}\right)^{2 / 2^{*}} \leq C \beta^{2}\left(\int_{\mathbb{R}^{N}} v_{0}^{2^{*}}\right)^{\frac{\left(\frac{p}{2}-2\right)}{2^{*}}}\left(\int_{\mathbb{R}^{N}} w_{k}^{\frac{22^{*}-\frac{p}{2}}{2^{*}}+2}\right)^{\frac{2^{*}-\frac{p}{2}+2}{2^{*}}} .
$$

Since $w_{k} \leq v_{0}^{\beta}$, using the continuity of the embedding $E \hookrightarrow L^{2^{*}}\left(\mathbb{R}^{N}\right)$, we have

$$
\left(\int_{\mathbb{R}^{N}}\left[v_{0} v_{k}^{\beta-1}\right]^{2^{*}}\right)^{2 / 2^{*}} \leq C \beta^{2}\left\|v_{0}\right\|^{\left(\frac{p}{2}-2\right)}\left(\int_{\mathbb{R}^{N}} v_{0}^{\frac{2 \cdot 2^{*} \beta-\frac{p}{2}+2}{2^{*}}}\right)^{\frac{2^{*}-\frac{p}{2}+2}{2^{*}}} .
$$

Taking $\beta=1+\frac{2^{*}-\frac{p}{2}}{2}$, we get $\frac{2 \cdot 2^{*} \beta}{2^{*}-\frac{p}{2}+2}=2^{*}$. Let $\delta:=\frac{2 \cdot 2^{*}}{2^{*}-\frac{p}{2}+2}$. Thus

$$
\left(\int_{\mathbb{R}^{N}}\left|v_{0} v_{k}^{\beta-1}\right|^{2^{*}}\right)^{2 / 2^{*}} \leq C \beta^{2}\left\|v_{0}\right\|^{\left(\frac{p}{2}-2\right)}\left\|v_{0}\right\|_{\beta \delta}^{2 \beta}
$$

Using Fatou's Lemma in $k$, we have

$$
\begin{equation*}
\left\|v_{0}\right\|_{2^{*} \beta} \leq\left(C \beta^{2}\left\|v_{0}\right\|^{\left(\frac{p}{2}-2\right)}\right)^{\frac{1}{2 \beta}}\left\|v_{0}\right\|_{\beta \delta} . \tag{3.16}
\end{equation*}
$$

For $m=0,1,2, \ldots$, let $2^{*} \beta_{m}=\delta \beta_{m+1}$ with $\beta_{0}=\beta$. Hence, similar to (3.16), for $\beta_{1}$, we know that

$$
\begin{aligned}
\left\|v_{0}\right\|_{2^{*} \beta_{1}} & \leq\left(C \beta_{1}^{2}\left\|v_{0}\right\|^{\left(\frac{p}{2}-2\right)}\right)^{\frac{1}{2 \beta_{1}}}\left\|v_{0}\right\|_{\beta_{1} \delta} \\
& \leq\left(C \beta_{1}^{2}\left\|v_{0}\right\|^{\left(\frac{p}{2}-2\right)}\right)^{\frac{1}{2 \beta_{1}}}\left(C \beta^{2}\left\|v_{0}\right\|^{\left(\frac{p}{2}-2\right)}\right)^{\frac{1}{2 \beta}}\left\|v_{0}\right\|_{\beta \delta} \\
& \leq\left(C\left\|v_{0}\right\|^{\left(\frac{p}{2}-2\right)}\right)^{\frac{1}{2 \beta_{1}}+\frac{1}{2 \beta}} \beta^{\frac{1}{\beta}} \beta_{1}^{\frac{1}{\beta_{1}}}\left\|v_{0}\right\|_{2^{*}} .
\end{aligned}
$$

Since $\beta_{m+1}=\beta_{m} \cdot \beta$, we know $\beta_{m}=\beta^{m} \cdot \beta$. Thus by iteration, we have

$$
\left\|v_{0}\right\|_{2^{*} \beta_{m}} \leq\left(C\left\|v_{0}\right\|^{\left(\frac{p}{2}-2\right)}\right)^{\frac{1}{2 \beta} \sum_{i=0}^{m} \beta^{-i}} \beta^{\frac{1}{\beta} \sum_{i=0}^{m} \beta^{-i}} \beta^{\frac{1}{\beta}} \sum_{i=0}^{m} i \beta^{-i}\left\|v_{0}\right\|_{2^{*}} .
$$

Using $\beta>1$, we conclude that $\sum_{i=0}^{m} \beta^{-i}$ and $\sum_{i=0}^{m} i \beta^{-i}$. Thus letting $m \rightarrow \infty$, we have $v_{0} \in$ $L^{\infty}\left(\mathbb{R}^{N}\right)$ and

$$
\left\|v_{0}\right\|_{\infty} \leq C\left\|v_{0}\right\|^{\frac{2^{*}-2}{2^{*}-\frac{p}{2}}} .
$$

By $\left(V_{1}\right),\left(g_{9}\right)$ in Lemma 2.1 and (3.14), for all $\varphi \in \mathcal{C}_{0}^{\infty}\left(\mathbb{R}^{N}\right)$, we get

$$
\int_{\mathbb{R}^{N}} \nabla v_{0} \nabla \varphi \leq C \int_{\mathbb{R}^{N}} v_{0} \varphi .
$$

Thus using an elliptic estimate in [20], for $\iota>\frac{N}{2}$ and any ball $B_{R}(x)$ centered at any $x \in \mathbb{R}^{N}$, we have

$$
\sup _{y \in B_{R}(x)} v_{0}(y) \leq C\left[\left\|v_{0}\right\|_{L^{2}\left(B_{2 R}(x)\right)}+\left\|v_{0}\right\|_{L^{\prime}\left(B_{2 R}(x)\right)}\right] .
$$

Obviously,

$$
v_{0}(x) \leq C\left[\left\|v_{0}\right\|_{L^{2}\left(B_{2 R}(x)\right)}+\left\|v_{0}\right\|_{L^{\prime}\left(B_{2 R}(x)\right)}\right] .
$$

Since

$$
\left\|v_{0}\right\|_{L^{2}\left(B_{2 R}(x)\right)}+\left\|v_{0}\right\|_{L^{\prime}\left(B_{2 R}(x)\right)} \rightarrow 0 \quad \text { as }|x| \rightarrow \infty,
$$

it follows that

$$
v_{0}(x) \rightarrow 0 \quad \text { as }|x| \rightarrow \infty .
$$

At last, we give a proof of the exponential decay for $v_{0}$. By $\left(F H_{2}\right)$ and since

$$
\lim _{s \rightarrow 0} \frac{G^{-1}(s)}{s g\left(G^{-1}(s)\right)}=1
$$

we can choose $R_{0}>0$ such that for all $|x| \geq R_{0}$,

$$
\begin{equation*}
\frac{G^{-1}\left(v_{0}(x)\right)}{g\left(G^{-1}\left(v_{0}(x)\right)\right)} \geq \frac{3}{4} v_{0}(x) \tag{3.17}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{h\left(x, G^{-1}\left(v_{0}(x)\right)\right)}{g\left(G^{-1}\left(v_{0}(x)\right)\right)} \leq \frac{V_{0}}{2} v_{0}(x) . \tag{3.18}
\end{equation*}
$$

Now, we define

$$
\chi(x)=M \exp (-\zeta|x|)
$$

where $\zeta$ and $M$ are such that $4 \zeta^{2}<V_{0}$ and for all $|x|=R_{0}$,

$$
M \exp \left(-\zeta R_{0}\right) \geq v_{0}(x)
$$

It is easy to check that for all $x \neq 0$,

$$
\begin{equation*}
\Delta x \leq \zeta^{2} \chi \tag{3.19}
\end{equation*}
$$

Let $\vartheta=\chi-v_{0}$. Then it follows from (3.17)-(3.19) and

$$
-\Delta v_{0}+V(x) \frac{G^{-1}\left(v_{0}\right)}{g\left(G^{-1}\left(v_{0}\right)\right)}=\frac{h\left(x, G^{-1}\left(v_{0}\right)\right)}{g\left(G^{-1}\left(v_{0}\right)\right)}, \quad x \in \mathbb{R}^{N}
$$

that

$$
\begin{aligned}
&-\Delta \vartheta+\frac{V_{0}}{4} \vartheta \geq 0 \text { in }|x| \geq R_{0} \\
& \vartheta \geq 0 \quad \text { in }|x|=R_{0} \\
& \lim _{|x| \rightarrow \infty} \vartheta(x)=0
\end{aligned}
$$

By the maximum principle, we know that $\vartheta(x) \geq 0$ for all $|x| \geq R_{0}$. Hence

$$
\vartheta(x) \leq M \exp (-\zeta|x|) \quad \text { for all }|x| \geq R_{0}
$$

which implies that

$$
u_{0}=G^{-1}\left(v_{0}\right) \leq v_{0}(x) \leq M \exp (-\zeta|x|) \quad \text { for all } x \in \mathbb{R}^{N} .
$$

This completes the proof.

Next, we shall use the Ekeland variational principle in [43] to prove Theorem 1.2. To this end, we prove the following lemma.

Lemma 3.6. Suppose that $\left(V_{1}\right)-\left(V_{2}\right)$ and $(F H)-\left(F H_{5}\right)$ are satisfied. Then there exists $\psi \in E$ such that $\mathcal{J}_{\lambda}(t \psi)<0$ for $t$ enough small.

Proof. To this end, by $\left(g_{14}\right)$ in Lemma 2.1 and choosing $\psi \in\left(\mathcal{C}_{0}^{\infty}\left(\mathbb{R}^{N}\right),[0,1]\right) \backslash\{0\}$ such that $\operatorname{supp} \psi=\bar{\Omega}$, we have

$$
\begin{aligned}
\mathcal{J}_{\lambda}(t \psi) & \leq \frac{1}{2} \int_{\mathbb{R}^{N}}\left(|t \nabla \psi|^{2}+V(x)\left|G^{-1}(t \psi)\right|^{2}\right)-\int_{\mathbb{R}^{N}} H\left(x, G^{-1}(t \psi)\right) \\
& \leq t^{2} \int_{\bar{\Omega}}\left(|\nabla \psi|^{2}+V(x) \psi^{2}\right)-C_{1} \int_{\bar{\Omega}}\left|G^{-1}(t \psi)\right|^{q_{1}} \\
& \leq t^{2}\left[\int_{\bar{\Omega}}\left(|\nabla \psi|^{2}+V(x) \psi^{2}\right)-C_{1} t^{q_{1}-2} \int_{\bar{\Omega}}\left|G^{-1}(\psi)\right|^{q_{1}}\right]<0
\end{aligned}
$$

where $|\bar{\Omega}|$ denotes the Lebesgue measure of $\bar{\Omega}$ and $t$ enough small. This completes the proof.

Proof of Theorem 1.2. By the previous proof, we know that $\mathcal{J}_{\lambda}$ is bounded in $B_{R}$ for $R>0$. By Lemma 3.6, we have

$$
-\infty<b_{\lambda}:=\inf _{B_{R}} \mathcal{J}_{\lambda}<0
$$

Since $\mathcal{J}_{\lambda}$ satisfies (PS)-condition. By the Ekeland variational principle (see [43]) for $\mathcal{J}_{\lambda}$ in $\bar{B}_{R}$, there exists $\omega_{\lambda} \in E$ such that for all $\lambda>0$

$$
\mathcal{J}_{\lambda}\left(\omega_{\lambda}\right)=b_{\lambda} \quad \text { and } \quad \mathcal{J}_{\lambda}^{\prime}\left(\omega_{\lambda}\right)=0
$$

Therefore $u_{\lambda, 2}=G^{-1}\left(\omega_{\lambda}\right)$ is a solution of (1.1).
Moreover, for $\lambda \in\left(0, \lambda_{0}\right]$, we have $\mathcal{J}_{\lambda}\left(\omega_{\lambda}\right)<0<\eta \leq \mathcal{J}_{\lambda}\left(v_{\lambda}\right)$, which shows that $u_{\lambda, 1}$ is different from $u_{\lambda, 2}$, where $\left(0, \lambda_{0}\right.$ ].

## Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grant Nos. 11661053, 11771198, 11901276 and 11961045), and supported by Jiangxi Provincial Natural Science Foundation (Grant Nos. 20202BAB201001 and 20202BAB211004).

## References

[1] J. P. Aubin, I. Ekeland, Applied nonlinear analysis, Pure and Applied Mathematics, A Wiley-Interscience Publication, John Wiley \& Sons, Inc., New York, 1984. MR0749753; Zbl 0641.47066
[2] F. G. Bass, N. N. Nasanov, Nonlinear electromagnetic-spin waves, Phys. Rep. 189(1990), No. 4, 165-223. https://doi.org/10.1016/0370-1573(90) 90093-H
[3] A. D. Bouard, N. Hayashi, J. Saut, Global existence of small solutions to a relativistic nonlinear Schrödinger equation, Comm. Math. Phys. 189(1997), No. 1, 73-105. https :// doi.org/10.1007/s002200050191; MR1478531; Zbl 0948.81025
[4] X. L. Chen, R. N. Sudan, Necessary and sufficient conditions for self-focusing of short ultraintense laser pulse in underdense plasma, Phys. Rev. Lett. 70(1993), No. 14, 20822085. https://doi.org/10.1103/PhysRevLett. 70.2082
[5] J. H. Chen, X. H. Tang, B. T. Cheng, Non-Nehari manifold method for a class of generalized quasilinear Schrödinger equations, Appl. Math. Lett. 74(2017), 20-26. https: //doi.org/10.1016/j.aml.2017.04.032; MR3677837; Zbl 1379.35073
[6] J. H. Chen, X. H. Tang, B. T. Cheng, Ground state sign-changing solutions for a class of generalized quasilinear Schrödinger equations with a Kirchhoff-type perturbation, J. Fixed Point Theory Appl. 19(2017), No. 4, 3127-3149. https://doi.org/10.1007/s11784-017-0475-4; MR3720499; Zbl 1379.35111
[7] J. H. Chen, X. H. Tang, B. T. Cheng, Existence and nonexistence of positive solutions for a class of generalized quasilinear Schrödinger equations involving a Kirchhoff-type perturbation with critical Sobolev exponent, J. Math. Phys. 59(2018), No. 2, 021505. https: //doi.org/10.1063/1.5024898; MR3765736; Zbl 1420.35346
[8] J. H. Chen, X. H. Tang, B. T. Cheng, Existence of ground state solutions for a class of quasilinear Schrödinger equations with general critical nonlinearity, Comun. Pure Appl. Anal. 18(2019), No. 1, 493-517. https://doi.org/10.3934/cpaa.2019025; MR3845576; Zbl 1401.35047
[9] J. H. Chen, X. J. Huang, D. D. Qin, B. T. Cheng, Existence and asymptotic behavior of standing wave solutions for a class of generalized quasilinear Schrödinger equations with critical Sobolev exponents, Asymptotic Anal. 120(2020), No. 3, 199-248. https://doi. org/ 10.3233/ASY-191586; MR4169206; Zbl 07367935
[10] J. H. Chen, X. J. Huang, B. T. Cheng, X. H. Tang, Existence and concentration behavior of ground state solutions for a class of generalized quasilinear Schrödinger equations in $\mathbb{R}^{N}$, Acta Math. Sci. 40(2020), No. 5, 1495-1524. https://doi.org/10.1007/s10473-020-0519-5; MR4143605
[11] S. Cuccagna, On instability of excited states of the nonlinear Schödinger equation, Phys. D 238(2009), No. 1, 38-54. https://doi.org/10.1016/j.physd.2008.08.010; MR2571965; Zbl 1161.35500
[12] Y. K. Cheng, J. Yang, Positive solution to a class of relativistic nonlinear Schrödinger equation, J. Math. Anal. Appl. 411(2014), No. 2, 665-674. https://doi.org/10.1016/j. jmaa. 2013.10.006; MR3128421; Zbl 1333.35033
[13] Y. K. Cheng, Y. X. Yao, Soliton solutions to a class of relativistic nonlinear Schrödinger equations, Appl. Math. Comput. 260(2015), 342-350. https://doi.org/10.1016/j.amc. 2015.03.055; MR3343274; Zbl 1410.35035
[14] M. Colin, L. Jeanjean, Solutions for a quasilinear Schrödinger equation: a dual approach, Nonlinear Anal. 56(2004), No. 2, 213-226. https://doi.org/10.1016/j.na. 2003. 09.008; MR2029068; Zbl 1035.35038
[15] Y. B. Deng, S. J. Peng, J. X. Wang, Nodal soliton solutions for quasilinear Schrödinger equations with critical exponent, J. Math. Phys. 54(2013), No. 1, 011504. https://doi. org/10.1063/1.4774153; MR3059863; Zbl 1293.35292
[16] Y. B. Deng, S. J. Peng, J. X. Wang, Nodal soliton solutions for generalized quasilinear Schrödinger equations, J. Math. Phys. 55(2014), No. 5, 051501. https://doi .org/10.1063/ 1.4874108; MR3390611; Zbl 1292.81037
[17] Y. B. Deng, S. J. Peng, S. S. Yan, Positive solition solutions for generalized quasilinear Schrödinger equations with critical growth, J. Differential Equations 258(2015), No. 1, 115147. https://doi.org/10.1016/j.jde.2014.09.006; MR3271299; Zbl 1302.35345
[18] Y. B. Deng, S. J. Peng, S. S. Yan, Critical exponents and solitary wave solutions for generalized quasilinear Schrödinger equations, J. Differential Equations 260(2016), No. 2, 1228-1262. https://doi.org/10.1016/j.jde.2015.09.021; MR3419726; Zbl 1330.35099
[19] M. F. Furtado, E. D. Silva, M. L. Silva, Existence of solutions for a generalized elliptic problem, J. Math. Phys. 58(2017), No. 3, 031503. https://doi.org/10.1063/1.4977480; MR3620675; Zbl 1362.35138
[20] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, Second edition, Springer-Verlag, Berlin, 1983. https ://doi.org/10.1007/978-642-61798-0; MR0737190; Zbl 0562.35001
[21] R. W. Hasse, A general method for the solution of nonlinear soliton and kink Schrödinger equations, Z. Phys. B 37(1980), No. 1, 83-87. https://doi.org/10.1007/BF01325508; MR0563644
[22] O. Kavian, Introduction à la théorie des points critiques et applications aux problèmes elliptiques (in French) [Introduction to critical point theory and applications to elliptic problems], Mathématiques \& Applications, Vol. 13, Springer-Verlag, Paris, 1993. MR1276944; Zbl 0797.58005
[23] S. Kurihara, Large-amplitude quasi-solitons in superfluid films, J. Phys. Soc. Japan 50(1981), No. 10, 3262-3267. https://doi.org/10.1143/JPSJ.50. 3262
[24] E. W. Laedke, K. H. Spatschek, L. Stenflo, Evolution theorem for a class of perturbed envelope soliton solutions, J. Math. Phys. 24(1983), 2764-2769. https://doi. org/ 10.1063/1.525675; MR0727767; Zbl 0548.35101
[25] H. Lange, M. Poppenberg, H. Teismann, Nash-Moser methods for the solution of quasilinear Schrödinger equations, Comm. Partial Differential Equations 24(1999), No. 7-8, 13991418. https://doi.org/10.1080/03605309908821469; MR1697492; Zbl 0935.35153
[26] F. Y. Li, X. L. Zhu, Z. P. Liang, Multiple solutions to a class of generalized quasilinear Schrödinger equations with a Kirchhoff-type perturbation, J. Math. Anal. Appl. 443(2016), No. 1, 11-38. https://doi.org/10.1016/j.jmaa.2016.05.005; MR3508477; Zbl 1341.35069
[27] Q. Q. Li, K. M. Teng, X. Wu, Ground state solutions and geometrically distinct solutions for generalized quasilinear Schrödinger equation, Math. Methods Appl. Sci. 40(2017), No. 6, 2165-2176. https://doi.org/10.1002/mma.4131; MR3624089; Zbl 1368.35133
[28] Q. Q. Li, X. Wu, Multiple solutions for generalized quasilinear Schrödinger equations, Math. Methods Appl. Sci. 40(2017), No. 5, 1359-1366. https ://doi.org/10.1002/mma.4050; MR3622401; Zbl 1368.35134
[29] J. Q. Liu, Z. Q. Wang, Soliton solutions for quasilinear Schrödinger equations. I, Proc. Amer. Math. Soc. 131(2003), No. 2, 441-448. https://doi.org/10.1090/S0002-9939-02-06783-7; MR1933335; Zbl 1229.35269
[30] J. Q. Liu, Y. Q. Wang, Z. Q. Wang, Soliton solutions for quasilinear Schrödinger equations. II, J. Differential Equations 187(2003), No. 2, 473-493. https://doi.org/10.1016/ S0022-0396(02)00064-5; MR1949452; Zbl 1229.35268
[31] J. Q. Liu, Y. Q. Wang, Z. Q. Wang, Solutions for quasilinear Schrödinger equations via the Nehari method, Comm. Partial Differential Equations 29(2004), No. 5-6, 879-901. https : //doi.org/10.1081/PDE-120037335; MR2059151; Zbl 1140.35399
[32] V. G. Makhankov, V. K. Fedyanin, Nonlinear effects in quasi-one-dimensional models and condensed matter theory, Phys. Rep. 104(1984), No. 1, 1-86. https://doi.org/10. 1016/0370-1573(84) 90106-6; MR0740342
[33] J. M. do Ó, U. Severo, Quasilinear Schrödinger equations with concave and convex nonlinearities, Comm. Pure Appl. Anal. 8(2009), No. 2, 621-644. https://doi.org/10.3934/ сраа. 2009.8.621; MR2461565; Zbl 1171.35118
[34] M. Poppenberg, K. Schmitt, Z. Q. Wang, On the existence of soliton solutions to quasilinear Schrödinger equations, Calc. Var. Partial Differential Equations 14(2002), No. 3, 329344. https://doi.org/10.1007/s005260100105; MR1899450; Zbl 1052.35060
[35] D. D. Qin, V. D. Rǎdulescu, X. H. Tang, Ground states and geometrically distinct solutions for periodic Choquard-Pekar equations, J. Differential Equations 275(2021), 652-683. https://doi.org/10.1016/j.jde.2020.11.021; MR4191377; Zbl 1456.35187
[36] D. D. Qin, X. H. Tang, On the planar Choquard equation with indefinite potential and critical exponential growth, J. Differential Equations 285(2021), 40-98. https://doi.org/ 10.1016/j.jde.2021.03.011; MR4228403; Zbl 1465.35249
[37] D. D. Qin, L. Z. Lai, S. Yuan, Q. F. Wu, Ground states and multiple solutions for Choquard-Pekar equations with indefinite potential and general nonlinearity, J. Math. Anal. Appl. 500(2021), No. 2, 125143. https://doi.org/10.1016/j.jmaa.2021.125143; MR4232680; Zbl 1465.35248
[38] B. Ritchie, Relativistic self-focusing and channel formation in laser-plasma interaction, Phys. Rev. E 50(1994), No. 2, 687-689. https://doi.org/10.1103/PhysRevE.50.R687
[39] Y. T. Shen, Y. J. Wang, Soliton solutions for generalized quasilinear Schrödinger equations, Nonlinear Anal. 80(2013), 194-201. https://doi.org/10.1016/j.na.2012.10.005; MR3010765; Zbl 1278.35233
[40] Y. T. Shen, Y. J. Wang, Two types of quasilinear elliptic equations with degenerate coerciveness and slightly superlinear growth, Appl. Math. Lett. 47(2015), 21-25. https: //doi.org/10.1016/j.aml.2015.02.009; MR3339633; Zbl 1322.35037
[41] Y. T. Shen, Y. J. Wang, Standing waves for a class of quasilinear Schrödinger equations, Complex Var. Elliptic Equ. 61(2016), No. 6, 817-84. https://doi.org/10.1080/17476933. 2015.1119818; MR3508254; Zbl 1347.35101
[42] H. X. Shi, H. B. Chen, Positive solutions for generalized quasilinear Schrödinger equations with potential vanishing at infinity, Appl. Math. Lett. 61(2016), 137-142. https: //doi.org/10.1016/j.aml.2016.06.004; MR3518460; Zbl 1347.35116
[43] M. Willem, Minimax theorems, Progress in Nonlinear Differential Equations and Their Applications, Vol. 24, Birkhäuser, Boston, MA, 1996. https://doi.org/10.1007/978-1-4612-4146-1; MR1400007; Zbl 0856.49001
[44] X. L. Zhu, F. Y. Li, Z. P. Liang, Existence of ground state solutions to a generalized quasilinear Schrödinger-Maxwell system, J. Math. Phys. 57(2016), No. 10, 101505. https: //doi.org/10.1063/1.4965442; MR3564317; Zbl 1353.35128

# Asymptotic behavior of solutions to difference equations in Banach spaces 

Janusz Migda ${ }^{\boxtimes}$<br>Faculty of Mathematics and Computer Science, A. Mickiewicz University, ul. Uniwersytetu<br>Poznańskiego 4, 61-614 Poznań, Poland

Received 1 October 2021, appeared 14 December 2021
Communicated by Stevo Stević


#### Abstract

We investigate the asymptotic properties of solutions to higher order nonlinear difference equations in Banach spaces. We introduce a new technique based on a vector version of discrete L'Hospital's rule, remainder operator, and the regional topology on the space of all sequences on a given Banach space. We establish sufficient conditions for the existence of solutions with prescribed asymptotic behavior. Moreover, we are dealing with the problem of approximation of solutions. Our technique allows us to control the degree of approximation of solutions.


Keywords: difference equation in Banach space, prescribed asymptotic behavior, degree of approximation, remainder operator, regional topology.

2020 Mathematics Subject Classification: 39A10.

## 1 Introduction

Let $\mathbb{N}, \mathbb{R}$ denote the set of positive integers and the set of real numbers respectively. In this paper we assume that $m \in \mathbb{N}$ is fixed and $X$ is a real Banach space. We consider the equation

$$
\begin{gather*}
\Delta^{m} x_{n}=a_{n} f\left(n, x_{\sigma(n)}\right)+b_{n}  \tag{E}\\
n \in \mathbb{N}, \quad a_{n} \in \mathbb{R}, \quad b_{n} \in X, \quad f: \mathbb{N} \times X \rightarrow X, \quad \sigma: \mathbb{N} \rightarrow \mathbb{N}, \quad \lim \sigma(n)=\infty
\end{gather*}
$$

By a solution of (E) we mean a sequence $x: \mathbb{N} \rightarrow X$ satisfying (E) for all large $n$.
Nonlinear difference equations often appear in mathematical models used, for example, in technology, biology, physics, economics or medicine. Hence the study of behavior of solutions to difference equations is of great importance. Therefore, many papers are devoted to this topic, see for example $[3,4,6,12,14,15,17-22]$. In some papers the difference equations in Banach spaces are also investigated, see for example [1,2,5,7-9,16].

In this paper we deal with the problem of the existence of solutions to the equation (E), with prescribed asymptotic behavior and the problem of approximation of solutions to equation (E). More precisely, in Section 4 we establish conditions under which for a given sequence

[^50]$y: \mathbb{N} \rightarrow X$ such that $\Delta^{m} x_{n}=b_{n}$ and a given number $s \in(-\infty, 0]$ there exists a solution $x$ of (E) such that $x_{n}=y_{n}+\mathrm{o}\left(n^{s}\right)$ (then $x$ is called a solution with prescribed asymptotic behavior, and $y$ is called an approximative solution of (E)). Next, in Section 5, we establish conditions under which for a given solution $x$ of ( E ) and a given number $s \in(-\infty, 0]$ there exists a sequence $y: \mathbb{N} \rightarrow X$ such that $\Delta^{m} y_{n}=b_{n}$ and $x_{n}=y_{n}+\mathbf{o}\left(n^{s}\right)$. By selecting the number $s$, we can control the degree of approximation of solution.

The paper is organized as follows. In Section 2, we introduce notation and terminology. In Section 3, we present our technical tools, i.e. vector version of discrete L'Hospital's rule, the regional topology on the space of all sequences on a given Banach space, and remainder operator which are needed to get the main results. The next two sections contain our main results. In Section 4 we establish sufficient conditions for the existence of solutions with prescribed asymptotic behavior. Section 5 is devoted to approximation of solutions.

## 2 Notation and terminology

Let $\mathbb{Z}$, denote the set of all integers. If $p, k \in \mathbb{Z}, p \leq k$, then $\mathbb{N}(p), \mathbb{N}(p, k)$ denote the sets defined by

$$
\mathbb{N}(p)=\{p, p+1, \ldots\}, \quad \mathbb{N}(p, k)=\{p, p+1, \ldots, k\} .
$$

We use the symbol $|t|$ to denote the norm of a vector $t \in X$. The space of all sequences $x: \mathbb{N} \rightarrow \mathbb{R}$ we denote by $\mathbb{R}^{\mathbb{N}}$. Moreover, we use the symbol $X^{\mathbb{N}}$ to denote the space of all sequences $x: \mathbb{N} \rightarrow X$. If $a \in \mathbb{R}^{\mathbb{N}}$ and $x \in X^{\mathbb{N}}$, then $a x$ denotes the sequence defined by pointwise multiplication

$$
a x(n)=a_{n} x_{n} .
$$

Moreover, $|x|$ denotes the sequence defined by $|x|(n)=\left|x_{n}\right|$ for every $n$. Let

$$
\begin{gathered}
\operatorname{Fin}(X)=\bigcup_{p=1}^{\infty}\left\{x \in X^{\mathbb{N}}: x_{n}=0 \text { for } n \geq p\right\}, \\
\mathrm{o}_{X}(1)=\left\{x \in X^{\mathbb{N}}: \lim _{n \rightarrow \infty} x_{n}=0\right\}, \quad \mathrm{O}_{X}(1)=\left\{x \in X^{\mathbb{N}}: x \text { is bounded }\right\}
\end{gathered}
$$

and for $a \in \mathbb{R}^{\mathbb{N}}$ let

$$
\begin{aligned}
\mathrm{o}_{X}(a) & =\left\{a x: x \in \mathrm{o}_{X}(1)\right\}+\operatorname{Fin}(X), \\
\mathrm{O}_{X}(a) & =\left\{a x: x \in \mathrm{O}_{X}(1)\right\}+\operatorname{Fin}(X) .
\end{aligned}
$$

For a sequence $a \in \mathbb{R}^{\mathbb{N}}$ and $x \in X^{\mathbb{N}}$ we write $x_{n}=\mathrm{o}\left(a_{n}\right)$ to denote the relation

$$
x \in \mathrm{o}_{X}(a) .
$$

Analogously $x_{n}=\mathrm{O}\left(a_{n}\right)$ denotes the relation $x \in \mathrm{O}_{X}(a)$.
We use the symbol $\Delta$ to denote the difference operator defined by

$$
\Delta: X^{\mathbb{N}} \rightarrow X^{\mathbb{N}}, \quad(\Delta x)(n)=x_{n+1}-x_{n} .
$$

As usual we use $\Delta x_{n}$ to denote the value $(\Delta x)(n)$. For $k \in \mathbb{N}$ we denote by $\Delta^{k}$ the $k$-th iteration of the operator $\Delta$. Moreover, $\Delta^{0}$ denotes the identity operator. For $k \in \mathbb{N}(0)$ we define

$$
\operatorname{Pol}_{X}(k-1)=\operatorname{Ker}\left(\Delta^{k}\right)=\left\{x \in X^{\mathbb{N}}: \Delta^{k} x=0\right\} .
$$

Then $\operatorname{Pol}_{X}(k-1)$ is the space of all polynomial sequences of degree less than $k$. Note that

$$
\operatorname{Pol}_{X}(-1)=\operatorname{Ker}\left(\Delta^{0}\right)=0
$$

is the zero space. It is easy to see that $\varphi \in \operatorname{Pol}_{X}(k-1)$ if and only if there exist vectors $x_{0}, x_{1}, \ldots, x_{k-1} \in X$ such that

$$
\varphi(n)=x_{k-1} n^{k-1}+x_{k-2} n^{k-2}+\cdots+x_{1} n+x_{0}
$$

for any $n \in \mathbb{N}$. For $b \in X^{\mathbb{N}}$ we use the symbol $\Delta^{-k} b$ to denote the set

$$
\Delta^{-k} b=\left\{x \in X^{\mathbb{N}}: \Delta^{k} x=b\right\}
$$

Remark 2.1. If $y$ is an arbitrary element of $\Delta^{-k} b$, then

$$
\Delta^{-k} b=y+\operatorname{Pol}_{X}(k-1)
$$

Let $H$ be a metric space. For a subset $A$ of $H$ and $\varepsilon>0$, we define an $\varepsilon$-ball about $A$ by

$$
\mathrm{B}(A, \varepsilon)=\bigcup_{a \in A} \mathrm{~B}(a, \varepsilon)
$$

where $\mathrm{B}(a, \varepsilon)$ denotes an open ball of radius $\varepsilon$ centered at $a$. We say that a subset $U$ of $H$ is a uniform neighborhood of $A$ if there exists a positive $\varepsilon$ such that

$$
\mathrm{B}(A, \varepsilon) \subset U
$$

A subset $A$ of $H$ is called an $\varepsilon$-net for a subset $Z$ of $H$ if $Z \subset B(A, \varepsilon)$. A subset $Z$ of $H$ is said to be totally bounded if for any $\varepsilon>0$ there exist a finite $\varepsilon$-net for $Z$.

## 3 Preliminaries

In this section, we introduce the technical tools that form the basis of our technique for studying the asymptotic properties of solutions to difference equations.

### 3.1 Discrete L'Hospital's rule

Lemma 3.1. Assume $a, b, r$ are positive real numbers, $c \in X, a_{1}, a_{2}, \ldots, a_{n}$ are real numbers with the same nonzero sign. Then

$$
\begin{equation*}
a \mathrm{~B}(c, r)=\mathrm{B}(a c, a r), \quad a \mathrm{~B}(c, r)+b \mathrm{~B}(c, r)=(a+b) \mathrm{B}(c, r), \tag{3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
a_{1} \mathrm{~B}(c, r)+a_{2} \mathrm{~B}(c, r)+\cdots+a_{n} \mathrm{~B}(c, r)=\left(a_{1}+\cdots+a_{n}\right) \mathrm{B}(c, r) . \tag{3.2}
\end{equation*}
$$

Proof. The assertion (3.1) is an easy exercise, (3.2) is a consequence of (3.1).
Lemma 3.2. Assume $x \in X^{\mathbb{N}}, p \in \mathbb{N}, r, L \in \mathbb{R}$,

$$
c \in X, \quad r>0, \quad L \geq|c|+r
$$

$\left(y_{n}\right)$ is a sequence of real numbers, strictly monotonic for $n \geq p$. Moreover,

$$
\begin{equation*}
y_{n} \neq 0 \quad \text { and } \quad \frac{\Delta x_{n}}{\Delta y_{n}} \in \mathrm{~B}(c, r) \tag{3.3}
\end{equation*}
$$

for $n \geq p$. Then

$$
\begin{equation*}
\left|\frac{x_{n}}{y_{n}}-c\right|<r+L\left|\frac{y_{k}}{y_{n}}\right|+\left|\frac{x_{k}}{y_{n}}\right| \tag{3.4}
\end{equation*}
$$

for $n, k \geq p$.
Proof. Assume the sequence $\left(y_{n}\right)$ is increasing for $n \geq p$. Choose $n, k \geq p$. For $i \geq p$ we have $\Delta x_{i} \in\left(\Delta y_{i}\right) \mathrm{B}(c, r)$. Hence, using Lemma 3.1, we obtain

$$
\begin{aligned}
x_{n}-x_{k} & =\Delta x_{k}+\cdots+\Delta x_{n-1} \in\left(\Delta y_{k}\right) \mathrm{B}(c, r)+\cdots+\left(\Delta y_{n-1}\right) \mathrm{B}(c, r) \\
& =\left(\Delta y_{k}+\cdots+\Delta y_{n-1}\right) \mathrm{B}(c, r)=\left(y_{n}-y_{k}\right) \mathrm{B}(c, r) .
\end{aligned}
$$

for $n \geq k$. Similarly, for $k \geq n$, we have $x_{k}-x_{n} \in\left(y_{k}-y_{n}\right) \mathrm{B}(c, r)$. Hence

$$
x_{n}-x_{k} \in\left(y_{n}-y_{k}\right) \mathrm{B}(c, r) \quad \text { and } \quad \frac{x_{n}}{y_{n}}-\frac{x_{k}}{y_{n}} \in\left(1-\frac{y_{k}}{y_{n}}\right) \mathrm{B}(c, r) .
$$

Therefore, there exists a vector $b \in B(c, r)$ such that

$$
\frac{x_{n}}{y_{n}}-\frac{x_{k}}{y_{n}}=\left(1-\frac{y_{k}}{y_{n}}\right) b=b-\left(\frac{y_{k}}{y_{n}}\right) b .
$$

Hence

$$
\frac{x_{n}}{y_{n}}-c=b-c-\left(\frac{y_{k}}{y_{n}}\right) b+\frac{x_{k}}{y_{n}}
$$

Since $|b-c|<r$ and $|b| \leq|c|+r \leq L$, we have

$$
\left|\frac{x_{n}}{y_{n}}-c\right|<r+L\left|\frac{y_{k}}{y_{n}}\right|+\left|\frac{x_{k}}{y_{n}}\right|
$$

The case when $\left(y_{n}\right)$ is decreasing for $n \geq p$ is analogous.
Theorem 3.3 (Discrete L'Hospital's rule). Assume $\left(x_{n}\right) \in X^{\mathbb{N}},\left(y_{n}\right)$ is a sequence of real numbers which is strictly monotonic for large $n$. Moreover, we assume that the sequence $\left(\Delta x_{n} / \Delta y_{n}\right)$ is convergent and one of the following conditions is satisfied:
(a) $\lim _{n \rightarrow \infty} x_{n}=0$ and $\lim _{n \rightarrow \infty} y_{n}=0$,
(b) the sequence $\left(y_{n}\right)$ is unbounded.

Then the sequence $\left(x_{n} / y_{n}\right)$ is convergent and

$$
\lim _{n \rightarrow \infty} \frac{x_{n}}{y_{n}}=\lim _{n \rightarrow \infty} \frac{\Delta x_{n}}{\Delta y_{n}}
$$

Proof. Let $\varepsilon>0$. There exists an index $p$ such that

$$
\left|\frac{\Delta x_{n}}{\Delta y_{n}}-\frac{\Delta x_{k}}{\Delta y_{k}}\right|<\varepsilon
$$

for $n, k \geq p$. Let $c=\Delta x_{p} / \Delta y_{p}$. Then $\Delta x_{n} / \Delta y_{n} \in \mathrm{~B}(c, \varepsilon)$ for $n \geq p$. If condition $(a)$ is satisfied and $n \geq p$, then taking sufficiently large $k$ and using Lemma 3.2 we obtain $\left|x_{n} / y_{n}-c\right|<2 \varepsilon$. Similarly, if condition (b) is satisfied, then using Lemma 3.2 we obtain an index $q \geq p$ such that $\left|x_{n} / y_{n}-c\right|<2 \varepsilon$ for $n \geq q$. Then

$$
\left|\frac{x_{n}}{y_{n}}-\frac{\Delta x_{n}}{\Delta y_{n}}\right| \leq\left|\frac{x_{n}}{y_{n}}-c\right|+\left|c-\frac{\Delta x_{n}}{\Delta y_{n}}\right|<2 \varepsilon+\varepsilon .
$$

Lemma 3.4. If $x \in X^{\mathbb{N}}, m \in \mathbb{N}, s \in(-1, \infty)$, and $\Delta^{m} x_{n}=\mathrm{o}\left(n^{s}\right)$, then

$$
x_{n}=\mathrm{o}\left(n^{s+m}\right) .
$$

Proof. Induction on $m$. Let $m=1$. Using L'Hospital's rule we obtain

$$
\begin{aligned}
\lim _{t \rightarrow \infty} \frac{(t+1)^{s+1}-t^{s+1}}{t^{s}} & =\lim _{t \rightarrow \infty} \frac{(t+1)^{s+1}-t^{s+1}}{t^{-1} t^{s+1}}=\lim _{t \rightarrow \infty} \frac{\left(1+t^{-1}\right)^{s+1}-1}{t^{-1}} \\
& =\lim _{t \rightarrow \infty} \frac{(s+1)\left(1+t^{-1}\right)^{s}\left(-t^{-2}\right)}{-t^{-2}}=s+1 .
\end{aligned}
$$

Hence

$$
\lim _{n \rightarrow \infty} \frac{\Delta n^{s+1}}{n^{s}}=s+1 .
$$

So by assumption $\Delta x=\mathrm{o}\left(n^{s}\right)$ we obtain

$$
\lim \frac{\Delta x_{n}}{\Delta n^{s+1}}=\lim \frac{\Delta x_{n}}{n^{s}} \frac{n^{s}}{\Delta n^{s+1}}=\lim \frac{\Delta x_{n}}{n^{s}} \lim \frac{n^{s}}{\Delta n^{s+1}}=\frac{0}{s+1}=0 .
$$

Since $s>-1$, the sequence $\left(n^{s+1}\right)$ is increasing to infinity. By Theorem 3.3, we obtain $x_{n}=$ $\mathrm{o}\left(n^{s+1}\right)$. Hence the assertion is true for $m=1$. Assume it is true for certain $m \geq 1$ and let $\Delta^{m+1} x_{n}=\mathrm{o}\left(n^{s}\right)$. Then $\Delta^{m} \Delta x_{n}=\mathrm{o}\left(n^{s}\right)$ and by inductive hypothesis we get $\Delta x_{n}=\mathrm{o}\left(n^{s+m}\right)$. Hence by the first part of the proof we obtain $x_{n}=\mathrm{o}\left(n^{s+m+1}\right)$.

### 3.2 Regional topology

Let $Y$ be a real vector space. We say that a function $\|\cdot\|: Y \rightarrow[0, \infty]$ is regional norm if the condition $\|x\|=0$ is equivalent to $x=0$ and for any $x, y \in Y$ and $\alpha \in \mathbb{R}$ we have

$$
\|\alpha x\|=|\alpha|\|x\|, \quad\|x+y\| \leq\|x\|+\|y\| .
$$

Hence, the notion of regional norm generalizes the notion of usual norm. If a regional norm on $Y$ is given, then we say that $Y$ is a regional normed space. If there exists a vector $x \in Y$ such that $\|x\|=\infty$, then we say that $Y$ is extraordinary.

Assume $Y$ is a regional normed space. We say that a subset $Z$ of $Y$ is ordinary if $\|x-y\|<$ $\infty$ for any $x, y \in Z$. We regard every ordinary subset $Z$ of $Y$ as a metric space with metric defined by

$$
d(x, y)=\|x-y\| .
$$

Let $U \subset Y$. We say that $U$ is regionally open if $U \cap Z$ is open in $Z$ for any ordinary subset $Z$ of $Y$. The family of all regionally open subsets is a topology on $Y$ which we call the regional topology. We regard any subset of $Y$ as a topological space with topology induced by the regional topology. The subset

$$
Y_{0}=\{y \in Y:\|y\|<\infty\}
$$

is a linear subspace of $Y$ and regional norm induces an usual norm on $Y_{0}$. We say that $Y$ is a regional Banach space if $Y_{0}$ is a Banach space.

An important special case of a regional Banach space we obtain as follows. Let $D$ be an arbitrary nonempty set and let $\mathrm{F}(D, X)$ denote the space of all functions $f: D \rightarrow X$. Then the formula

$$
\|f\|=\sup \{|f(p)|: p \in D\}
$$

defines a regional norm on $F(D, X)$. This space is extraordinary if $D$ is infinite. In particular, we obtain the regional topology on the space

$$
X^{\mathbb{N}}=\mathrm{F}(\mathbb{N}, X)
$$

The regional topology in $F(D, X)$ is, simply, the topology of uniform convergence. In extraordinary case this topology is not linear but almost linear. For more details and for the proof of the following theorem we refer to [13].

Theorem 3.5 (Generalized Schauder theorem). Assume $Q$ is a closed and convex subset of a regional Banach space $Y$, a map $A: Q \rightarrow Q$ is continuous and the set $A(Q)$ is ordinary and totally bounded. Then there exists a point $x \in Q$ such that $A(x)=x$.

We say that a family $T \subset X^{\mathbb{N}}$ is pointwise totally bounded if for any $n$ the set $T(n)=\left\{t_{n}\right.$ : $t \in T\}$ is totally bounded. We say that $T$ is stable at infinity if for any $\varepsilon>0$ there exists an index $p$ such that $\left|x_{n}-y_{n}\right|<\varepsilon$ for any $n>p$ and any $x, y \in T$.

Lemma 3.6. If a family $T \subset X^{\mathbb{N}}$ is pointwise totally bounded and stable at infinity, then $T$ is totally bounded with respect to regional norm.

Proof. Let $t \in T$ and $\varepsilon>0$. Choose an index $p$ such that

$$
\left|x_{n}-y_{n}\right|<\varepsilon
$$

for any $x, y \in T$ and any $n>p$. For any $i=1, \ldots, p$ choose a finite $\varepsilon$-net $G_{i}$ for the set

$$
T(i)=\left\{x_{i}: x \in T\right\} .
$$

Let

$$
G=\left\{z \in X^{\mathbb{N}}: z_{n} \in G_{n} \text { for } n \leq p \text { and } z_{n}=t_{n} \text { for } n>p\right\}
$$

Fix an $x \in T$. For any $i \in \mathbb{N}(1, p)$ choose $g_{i} \in G_{i}$ such that $\left|x_{i}-g_{i}\right|<\varepsilon$. Let $h \in X^{\mathbb{N}}$ be defined by

$$
h_{n}=g_{n} \text { for } n \leq p, \quad h_{n}=t_{n} \text { for } n>p
$$

Then $h \in G$ and $|x-h|<\varepsilon$. Hence $G$ is a finite $\varepsilon$-net for $T$.

### 3.3 Remainder operator

In this section we define the iterated remainder operator. This operator will be used in the proofs of our main results. In Lemmas 3.7 and 3.8 we establish some basic properties this operator. Next in Lemma 3.10 we show that if $x \in X^{\mathbb{N}}$ and $\Delta^{m} x$ is asymptotically zero, then $x$ is asymptotically polynomial. In Lemmas 3.11 and 3.12 we present some useful consequences of Lemma 3.10.

Now we define the spaces $S_{X}(m)$ of $m$-times summable sequences and the remainder operator. Let

$$
\mathrm{S}_{X}(0)=\mathrm{o}_{X}(1), \quad \mathrm{S}_{X}(1)=\left\{x \in X^{\mathbb{N}}: \text { the series } \sum_{n=1}^{\infty} x_{n} \text { is convergent }\right\}
$$

For $x \in S_{X}(1)$, we define the sequence $r(x)$ by the formula

$$
r(x)(n)=\sum_{j=n}^{\infty} x_{j} .
$$

Then $r(x) \in \mathrm{S}_{X}(0)$ and we obtain the remainder operator

$$
r: \mathrm{S}_{X}(1) \rightarrow \mathrm{S}_{X}(0)
$$

For $m \in \mathbb{N}$, by induction, we define the linear space $S_{X}(m+1)$ and the linear operator

$$
r^{m+1}: \mathrm{S}_{X}(m+1) \rightarrow \mathrm{S}_{X}(0)
$$

by

$$
\mathrm{S}_{X}(m+1)=\left\{x \in \mathrm{~S}_{X}(m): r^{m}(x) \in \mathrm{S}_{X}(1)\right\}, \quad r^{m+1}(x)=r\left(r^{m}(x)\right) .
$$

Note that

$$
r^{m}(x)(n)=\sum_{i_{1}=n}^{\infty} \sum_{i_{2}=i_{1}}^{\infty} \cdots \sum_{i_{m}=i_{m-1}}^{\infty} x_{i_{m}}
$$

for any $x \in S_{X}(m)$ and any $n \in \mathbb{N}$.
In the proof of the next lemma we use the fact that in Banach space absolute convergence implies convergence of a series.
Lemma 3.7. Assume $x \in X^{\mathbb{N}}, m \in \mathbb{N}, p \in \mathbb{N}$, and $s \in(-\infty, 0]$. Then
(a) if $|x| \in \mathrm{S}_{\mathbb{R}}(m)$, then $x \in \mathrm{~S}_{X}(m)$ and $\left|r^{m}(x)\right| \leq r^{m}(|x|)$,
(b) $|x| \in \mathrm{S}_{\mathbb{R}}(m)$ if and only if $\sum_{n=1}^{\infty} n^{m-1}\left|x_{n}\right|<\infty$,
(c) if $|x| \in \mathrm{S}_{\mathbb{R}}(m)$, then $r^{m}(|x|)(p) \leq \sum_{n=p}^{\infty} n^{m-1}\left|x_{n}\right|$,
(d) if $x \in \mathrm{~S}_{X}(m)$, then $\Delta^{m}\left(r^{m}(x)\right)=(-1)^{m} x$,
(e) if $x \in \mathrm{o}_{X}(1)$, then $\Delta^{m} x \in \mathrm{~S}_{X}(m)$ and $r^{m}\left(\Delta^{m}(x)\right)=(-1)^{m} x$,
( $f$ ) if $\sum_{n=1}^{\infty} n^{m-1-s}\left|x_{n}\right|<\infty$, then $x \in \mathrm{~S}_{X}(m)$ and $r^{m}(x)(n)=\mathrm{o}\left(n^{s}\right)$.
Proof. Using our notation, the assertion (a) may be proved by repeating the proof of [10, Lemma 1]. Analogously, repeating the proof of [10, Lemma 2] we obtain (b). Similarly, we can obtain (c), (d), and (e) from [10, Lemma 2], [10, Lemma 5] and [10, Lemma 6] respectively. The assertion (f) we can obtain from [12, Lemma 4.2].

Lemma 3.8. If $x \in X^{\mathbb{N}}$ and $|x| \in \mathrm{S}_{\mathbb{R}}(m)$, then

$$
\begin{aligned}
r^{m}(x)(n) & =\sum_{i_{1}=n}^{\infty} \sum_{i_{2}=i_{1}}^{\infty} \cdots \sum_{i_{m}=i_{m-1}}^{\infty} x_{i_{m}}=\sum_{k=0}^{\infty}\binom{m+k-1}{m-1} x_{n+k} \\
& =\sum_{k=0}^{\infty} \frac{(k+1)(k+2) \cdots(k+m-1)}{(m-1)!} x_{n+k}=\sum_{j=n}^{\infty} \frac{(j-n+1) \cdots(j-n+m-1)}{(m-1)!} x_{j} .
\end{aligned}
$$

Proof. See [11, Lemma 4].

Lemma 3.9. If $a, b \in \mathrm{~S}_{\mathbb{R}}(m)$ and $a \leq b$, then $r^{m}(a) \leq r^{m}(b)$.
Proof. See [12, Lemma 4.1 (h)].
Lemma 3.10. Assume $a \in \mathbb{R}^{\mathbb{N}}, x \in X^{\mathbb{N}}, m \in \mathbb{N}, s \in(-\infty, m-1]$,

$$
\sum_{n=1}^{\infty} n^{m-1-s}\left|a_{n}\right|<\infty, \quad \text { and } \quad \Delta^{m} x_{n}=\mathrm{O}\left(a_{n}\right)
$$

Then

$$
x \in \operatorname{Pol}_{X}(m-1)+\mathrm{o}_{X}\left(n^{s}\right)
$$

Proof. Let $s \leq 0$. The condition $\Delta^{m} x_{n}=\mathrm{O}\left(a_{n}\right)$ implies

$$
\sum_{n=1}^{\infty} n^{m-1-s}\left|\Delta^{m} x_{n}\right|<\infty
$$

Let $u=\Delta^{m}(x)$. By Lemma 3.7 (f), $u \in \mathrm{~S}_{X}(m)$ and $r^{m}(u)(n)=\mathrm{o}\left(n^{s}\right)$. Let $w=(-1)^{m} r^{m}(u)$. Then $w_{n}=\mathrm{o}\left(n^{s}\right)$ and, by Lemma $3.7(\mathrm{~d}), \Delta^{m}(w)=u=\Delta^{m}(x)$. Hence

$$
x-w \in \operatorname{Ker}\left(\Delta^{m}\right)=\operatorname{Pol}_{X}(m-1)
$$

and we obtain

$$
x=x-w+w \in \operatorname{Pol}_{X}(m-1)+\mathrm{o}_{X}\left(n^{s}\right)
$$

Let $s \in(0, m-1]$. Choose $k \in \mathbb{N}(1, m-1)$ such that $k-1<s \leq k$. Then

$$
\sum_{n=1}^{\infty} n^{(m-k)-1-(s-k)}\left|u_{n}\right|<\infty
$$

and, by Lemma 3.7 (f), $u \in \mathrm{~S}(m-k)$ and $r^{m-k}(u)(n)=\mathrm{o}\left(n^{s-k}\right)$. Let $w=(-1)^{m-k} r^{m-k}(u)$. Then $w_{n}=\mathrm{o}\left(n^{s-k}\right)$ and, by Lemma $3.7(\mathrm{~d}), \Delta^{m-k} w=u$. Choose $z \in X^{\mathbb{N}}$ such that $\Delta^{k} z_{n}=$ $w_{n}=\mathrm{o}\left(n^{s-k}\right)$. Since $s-k>-1$, so by Lemma 3.4 we have $z_{n}=\mathrm{o}\left(n^{s}\right)$. Moreover

$$
\Delta^{m} z=\Delta^{m-k} \Delta^{k} z=\Delta^{m-k} w=u=\Delta^{m} x
$$

and

$$
x=x-z+z \in \operatorname{Pol}_{X}(m-1)+\mathrm{o}_{X}\left(n^{s}\right) .
$$

Lemma 3.11. Assume $a \in \mathbb{R}^{\mathbb{N}}, b, x \in X^{\mathbb{N}}, m \in \mathbb{N}, s \in(-\infty, m-1]$,

$$
\sum_{n=1}^{\infty} n^{m-1-s}\left|a_{n}\right|<\infty, \quad \text { and } \quad \Delta^{m} x \in \mathrm{O}_{X}(a)+b
$$

Then

$$
x \in \Delta^{-m} b+\mathrm{o}_{X}\left(n^{s}\right)
$$

Proof. Choose $u \in \Delta^{-m} b$. Then

$$
\Delta^{m}(x-u)=\Delta^{m} x-\Delta^{m} u=\Delta^{m} x-b \in \mathrm{O}_{X}(a) .
$$

Hence, by the previous lemma,

$$
x-u \in \operatorname{Pol}_{X}(m-1)+o_{X}\left(n^{s}\right) .
$$

On the other hand,

$$
u+\operatorname{Pol}_{X}(m-1)=\Delta^{-m} b .
$$

Hence

$$
x \in u+\operatorname{Pol}_{X}(m-1)+\mathrm{o}_{X}\left(n^{s}\right)=\Delta^{-m} b+\mathrm{o}_{X}\left(n^{s}\right) .
$$

Lemma 3.12. Assume $b \in X^{\mathbb{N}}, m \in \mathbb{N}, s \in(-\infty, m-1]$, and

$$
\sum_{n=1}^{\infty} n^{m-1-s}\left|b_{n}\right|<\infty .
$$

Then

$$
\Delta^{-m} b+\mathrm{o}_{X}\left(n^{s}\right)=\operatorname{Pol}_{X}(m-1)+\mathrm{o}_{X}\left(n^{s}\right) .
$$

Proof. Let $x \in \Delta^{-m} b$ and $z \in \mathrm{o}_{X}\left(n^{s}\right)$. By Lemma 3.10,

$$
x \in \operatorname{Pol}_{X}(m-1)+\mathrm{o}_{X}\left(n^{s}\right)
$$

Hence $x+z \in \operatorname{Pol}_{X}(m-1)+\mathrm{o}_{X}\left(n^{s}\right)$ and we have

$$
\Delta^{-m} b+\mathrm{o}_{X}\left(n^{s}\right) \subset \operatorname{Pol}_{X}(m-1)+\mathrm{o}_{X}\left(n^{s}\right) .
$$

By Lemma $3.7(\mathrm{f}), b \in \mathrm{~S}_{X}(m)$ and $r^{m}(b)(n)=\mathrm{o}\left(n^{s}\right)$. Let

$$
u=(-1)^{m} r^{m}(b) \quad \text { and } \quad \varphi \in \operatorname{Pol}_{X}(m-1) .
$$

Then $u=\mathrm{o}\left(n^{s}\right)$ and using Lemma 3.7 (d), we have

$$
\Delta^{m}(\varphi+u)=\Delta^{m} u=b .
$$

Hence

$$
\varphi+u \in \Delta^{-m} b \quad \text { and } \quad \varphi \in \Delta^{-m} b+\mathbf{o}\left(n^{s}\right) .
$$

Therefore

$$
\operatorname{Pol}_{X}(m-1)+\mathrm{o}_{X}\left(n^{s}\right) \subset \Delta^{-m} b+\mathrm{o}_{X}\left(n^{s}\right) .
$$

## 4 Solutions with prescribed asymptotic behavior

We say that a map $f: Y \rightarrow Z$ from a metric space $Y$ to a metric space $Z$ is a Heine map if it is completely continuous and is uniformly continuous on any bounded subset of $Y$. We define a metric $d$ on $\mathbb{N} \times X$ by

$$
d((k, s),(n, t))=\max (|n-k|,|t-s|) .
$$

Note that if the dimension of the space $X$ is finite then any continuous map $f: \mathbb{N} \times X \rightarrow X$ is a Heine map.

Theorem 4.1. Assume $f$ is a Heine map, $s \in(-\infty, 0]$,

$$
\begin{equation*}
\sum_{n=1}^{\infty} n^{m-s-1}\left|a_{n}\right|<\infty, \tag{4.1}
\end{equation*}
$$

$w \in \mathbb{R}^{\mathbb{N}}$ is positive and bounded, $g:[0, \infty) \rightarrow[0, \infty)$ is locally bounded,

$$
\begin{equation*}
|f(n, t)| \leq g\left(w_{n}|t|\right) \tag{4.2}
\end{equation*}
$$

for $(n, t) \in \mathbb{N} \times X, y \in X^{\mathbb{N}}, \Delta^{m} y=b$ and

$$
\begin{equation*}
w_{n} y_{\sigma(n)}=\mathrm{O}(1) . \tag{4.3}
\end{equation*}
$$

Then there exists a solution $x$ of (E) such that $x=y+\mathrm{o}\left(n^{s}\right)$.
Proof. For $x \in X^{\mathbb{N}}$ let $\bar{x} \in X^{\mathbb{N}}$ be defined by

$$
\bar{x}_{n}=f\left(n, x_{\sigma(n)}\right) .
$$

Choose a positive constant $c$. Let

$$
T=\left\{x \in X^{\mathbb{N}}:|x-y| \leq c\right\} .
$$

By boundedness of $w$ and (4.3), there exists a constant $K$ such that if $x \in T$ and $n \in \mathbb{N}$, then

$$
\begin{aligned}
\left|w_{n} x_{\sigma(n)}\right| & =\left|w_{n} x_{\sigma(n)}-w_{n} y_{\sigma(n)}+w_{n} y_{\sigma(n)}\right| \\
& \leq\left|w_{n}\right|\left|x_{\sigma(n)}-y_{\sigma(n)}\right|+\left|w_{n} y_{\sigma(n)}\right| \leq K .
\end{aligned}
$$

Since $g$ is locally bounded, there exists $M>0$ such that $g([0, K]) \subset[0, M]$. Therefore, we have

$$
\begin{equation*}
g\left(\left|w_{n} x_{\sigma(n)}\right|\right) \leq M \quad \text { and } \quad\left|\bar{x}_{n}\right| \leq g\left(\left|x_{\sigma(n)} w_{n}\right|\right) \leq M \tag{4.4}
\end{equation*}
$$

for $x \in T$ and $n \in \mathbb{N}$. Since $r^{m}(|a|)(n)=\mathrm{o}(1)$, there exists an index $p \geq 1$ such that

$$
\begin{equation*}
M r^{m}(|a|)(n) \leq c \quad \text { for } \quad n \geq p . \tag{4.5}
\end{equation*}
$$

Let $\mu, \rho \in \mathbb{R}^{\mathbb{N}}$,

$$
\mu_{n}=\left\{\begin{array}{ll}
0 & \text { for } n<p,  \tag{4.6}\\
1 & \text { for } n \geq p,
\end{array} \quad \rho=\mu M r^{m}(|a|)\right.
$$

Now, we define a subset $S$ of $X^{\mathbb{N}}$ and a map $A: S \rightarrow X^{\mathbb{N}}$ by

$$
S=\left\{x \in X^{\mathbb{N}}:|x-y| \leq \rho\right\}, \quad A(x)=y+(-1)^{m} \mu r^{m}(a \bar{x}) .
$$

Then $S \subset T$. Obviously, $S$ is convex, closed and ordinary subset of $X^{\mathbb{N}}$. If $x \in S$, then, using Lemma 3.7 (a), Lemma 3.9, (4.4) and (4.6) we get

$$
|A x-y|=\left|\mu r^{m}(a \bar{x})\right| \leq \mu r^{m}(|a \bar{x}|) \leq \rho .
$$

Hence $A(S) \subset S$. Choose $\varepsilon>0$. There exists $q \geq p$ and $\alpha>0$ such that

$$
\begin{equation*}
2 M \sum_{n=q}^{\infty} n^{m-1}\left|a_{n}\right|<\varepsilon \quad \text { and } \quad \alpha q^{m-1} \sum_{n=1}^{q}\left|a_{n}\right|<\varepsilon . \tag{4.7}
\end{equation*}
$$

Let

$$
L=\max \left\{\left|y_{\sigma(n)}-y_{n}\right|: n \in \mathbb{N}(1, q)\right\},
$$

and

$$
W=\left\{(n, t) \in \mathbb{N} \times X: n \in \mathbb{N}(1, q),\left|t-y_{n}\right| \leq L+c\right\} .
$$

The function $f$ is uniformly continuous on $W$. Hence, there exists a $\delta>0$ such that

$$
\begin{equation*}
\text { if } \quad(n, s),(n, t) \in W \quad \text { and } \quad|s-t|<\delta, \quad \text { then } \quad|f(n, s)-f(n, t)|<\alpha \tag{4.8}
\end{equation*}
$$

Assume $x, z \in S,|x-z|<\delta$. Let $u=\bar{x}-\bar{z}$. Then

$$
|A x-A z|=\left|\mu r^{m}(a u)\right|
$$

Using Lemma 3.7 we get

$$
\begin{aligned}
d(A x, A z) & =\sup _{n \in \mathbb{N}}\left|A x_{n}-A z_{n}\right|=\sup _{n \in \mathbb{N}}\left|r^{m}(a u)(n)\right| \\
& \leq \sup _{n \in \mathbb{N}} r^{m}(|a u|)(n) \leq \sum_{n=1}^{\infty} n^{m-1}\left|a_{n} u_{n}\right|
\end{aligned}
$$

Hence

$$
\begin{equation*}
d(A x, A z) \leq \sum_{n=1}^{q} n^{m-1}\left|a_{n} u_{n}\right|+\sum_{n=q}^{\infty} n^{m-1}\left|a_{n} u_{n}\right| \tag{4.9}
\end{equation*}
$$

By (4.4), $|u| \leq 2 M$. If $n \in \mathbb{N}(1, q)$, then

$$
\left|x_{\sigma(n)}-y_{n}\right| \leq\left|x_{\sigma(n)}-y_{\sigma(n)}\right|+\left|y_{\sigma(n)}-y_{n}\right| \leq \rho(n)+L \leq L+c .
$$

Hence $\left(n, x_{\sigma(n)}\right) \in W$. Analogously $\left(n, z_{\sigma(n)}\right) \in W$. Therefore, by (4.8), $\left|u_{n}\right| \leq \alpha$ for $n \leq q$. By (4.7) and (4.9) we get

$$
d(A x, A z) \leq \alpha q^{m-1} \sum_{n=1}^{q}\left|a_{n}\right|+2 M \sum_{n=q}^{\infty} n^{m-1}\left|a_{n}\right|<\varepsilon+\varepsilon
$$

Thus the map $A$ is continuous. Now, we will show that the family $A(S)$ is pointwise totally bounded. Fix an $n \in \mathbb{N}$. Then

$$
A(S)(n)=\left\{y_{n}+(-1)^{m} \mu_{n} r^{m}(a \bar{x})(n): x \in S\right\}
$$

and, by Lemma 3.8,

$$
r^{m}(a \bar{x})(n)=\sum_{k=0}^{\infty}\binom{m+k-1}{m-1} a_{n+k} f\left(n+k, x_{\sigma(n+k)}\right)
$$

Let

$$
Q_{n}=\left\{r^{m}(a \bar{x})(n): x \in S\right\} .
$$

For $k \in \mathbb{N}(0)$ let

$$
\begin{gathered}
\lambda_{k}=\binom{m+k-1}{m-1}, \\
V_{k}=\left\{\left(n+k, x_{\sigma(n+k)}\right): x \in S\right\}=\{n+k\} \times S(\sigma(n+k))
\end{gathered}
$$

and

$$
U_{k}=\left\{\lambda_{k} a_{n+k} \bar{x}_{n+k}: x \in S\right\}
$$

Then $V_{k}$ is a bounded subset of $\mathbb{N} \times X$ and, since $f$ is completely continuous, the set $f\left(V_{k}\right)$ is totally bounded. Hence

$$
U_{k}=\left\{\lambda_{k} a_{n+k} f\left(n+k, x_{\sigma(n+k)}\right): x \in S\right\}=\lambda_{k} a_{n+k} f\left(V_{k}\right)
$$

is also totally bounded. Let $\varepsilon>0$. By (4.1) and Lemma 3.7 (b), $|a| \in \mathrm{S}_{\mathbb{R}}(m)$. By Lemma 3.8 there exists an index $n_{1}$ such that

$$
M \sum_{k=n_{1}}^{\infty} \lambda_{k}\left|a_{n+k}\right|<\varepsilon .
$$

Let

$$
D=\left\{\sum_{k=n_{1}}^{\infty} \lambda_{k} a_{n+k} \bar{x}_{n+k}: x \in S\right\} \quad \text { and } \quad U=U_{0}+U_{1}+\cdots+U_{n_{1}} .
$$

Then

$$
Q_{n}=\left\{\sum_{k=0}^{\infty} \lambda_{k} a_{n+k} \bar{x}_{n+k}: x \in S\right\} \subset U+D
$$

By (4.4), $\left|\bar{x}_{n+k}\right| \leq M$ for any $k$. Hence $|z|<\varepsilon$ for any $z \in D$. Moreover, $U$ is totally bounded and there exists a finite $\varepsilon$-net $H$ for $U$. If $u \in U$, then there exists $h \in H$ such that $|u-h| \leq \varepsilon$. Moreover, if $z \in D$, then

$$
|u+z-h| \leq|u-h|+|z| \leq 2 \varepsilon .
$$

Hence $H$ is a finite $2 \varepsilon$-net for $U+D$ and for $Q_{n} \subset U+D$. Therefore $Q_{n}$ is totally bounded. Thus

$$
A(S)(n)=y_{n}+(-1)^{m} \mu_{n} Q_{n}
$$

is also totally bounded. Obviously the family $A(S)$ is stable at infinity. Hence, by Lemma 3.6, $A(S)$ is totally bounded. Therefore, by Theorem 3.5, there exists a sequence $x \in S$ such that $A(x)=x$. Then

$$
x_{n}=y_{n}+(-1)^{m} r^{m}(a \bar{x})(n)
$$

for $n \geq p$. This means that there exists a sequence $u \in X^{\mathbb{N}}$ such that $u_{n}=0$ for $n \geq p$ and

$$
\begin{equation*}
x=y+(-1)^{m} r^{m}(a \bar{x})+u . \tag{4.10}
\end{equation*}
$$

Hence, by Lemma 3.7 (d),

$$
\Delta^{m} x=\Delta^{m} y+a \bar{x}+\Delta^{m} u=a \bar{x}+b+\Delta^{m} u .
$$

It is easy to see that $\Delta^{m} u_{n}=0$ for $n \geq p$ and we obtain

$$
\Delta^{m} x_{n}=a_{n} f\left(n, x_{\sigma(n)}\right)+b_{n}
$$

for $n \geq p$. Moreover, using (4.10) and Lemma 3.7 (f), we get $x_{n}=y_{n}+\mathrm{o}\left(n^{s}\right)$.
Theorem 4.2. Assume $s \in(-\infty, 0], y \in X^{\mathbb{N}}, \Delta^{m} y=b$,

$$
\sum_{n=1}^{\infty} n^{m-s-1}\left|a_{n}\right|<\infty,
$$

$U \subset X$ is a uniform neighborhood of the set $y(\mathbb{N})$, and the map $f \mid \mathbb{N} \times U$ is Heine and bounded. Then there exists a solution $x$ of ( E ) such that $x_{n}=y_{n}+\mathrm{o}\left(n^{s}\right)$.

Proof. For $x \in X^{\mathbb{N}}$ let $\bar{x} \in X^{\mathbb{N}}$ be defined by

$$
\bar{x}_{n}=f\left(n, x_{\sigma(n)}\right) .
$$

Choose a positive constant $c$ such that $\mathrm{B}(y(\mathbb{N}), c) \subset U$. Let

$$
T=\left\{x \in X^{\mathbb{N}}:|x-y| \leq c\right\} \quad \text { and } \quad M=\sup \{|f(n, t)|:(n, t) \in \mathbb{N} \times U\} .
$$

If $x \in T$ and $n \in \mathbb{N}$, then $x_{n} \in \mathrm{~B}(y(\mathbb{N}), c) \subset U$. Hence

$$
\left|\bar{x}_{n}\right| \leq M
$$

for any $x \in T$ and $n \in \mathbb{N}$. There exists an index $p \geq 1$ such that

$$
M r^{m}(|a|)(n) \leq c \quad \text { for } \quad n \geq p
$$

The rest of the proof is analogous to the second part of the proof of Theorem 4.1.
Corollary 4.3. Assume the map $f$ is Heine, $s \in(-\infty, 0]$, and

$$
\sum_{n=1}^{\infty} n^{m-s-1}\left|a_{n}\right|<\infty .
$$

Moreover, for any bounded subset $Z$ of $X, f$ is bounded on $\mathbb{N} \times Z$. Then for any bounded solution $y$ of the equation $\Delta^{m} y=b$ there exists a solution $x$ of ( E ) such that $x_{n}=y_{n}+\mathrm{o}\left(n^{s}\right)$.

Proof. The assertion is an easy consequence of Theorem 4.2.

## 5 Approximations of solutions

Theorem 5.1. Assume $x$ is a solution of $(E), s \in(-\infty, m-1], p \in \mathbb{N}, U \subset X$,

$$
\sum_{n=1}^{\infty} n^{m-1-s}\left|a_{n}\right|<\infty, \quad g:[0, \infty) \rightarrow[0, \infty), \quad w \in \mathbb{R}^{\mathbb{N}},
$$

and one of the following conditions is satisfied:
(1) the sequence $\bar{x}_{n}=f\left(n, x_{\sigma(n)}\right)$ is bounded,
(2) $f$ is bounded on $\mathbb{N}(p) \times U$ and $x_{\sigma(n)} \in U$ for large $n$,
(3) $f$ is bounded on $\mathbb{N}(p) \times U$ and $x_{n} \in U$ for large $n$,
(4) $f$ is bounded,
(5) $g$ is locally bounded, $x_{\sigma(n)}=\mathrm{O}\left(w_{n}^{-1}\right)$ and $|f(n, t)| \leq g\left(\left|w_{n} t\right|\right)$ on $\mathbb{N} \times X$.

Then $x \in \Delta^{-m} b+\mathrm{o}_{X}\left(n^{s}\right)$. If, moreover,

$$
\sum_{n=1}^{\infty} n^{m-1-s}\left|b_{n}\right|<\infty,
$$

then $x \in \operatorname{Pol}_{X}(m-1)+\mathrm{o}\left(n^{s}\right)$.

Proof. Obviously $(4) \Rightarrow(3) \Rightarrow(2) \Rightarrow(1)$. Assume (5). Then the sequence

$$
z_{n}=g\left(\left|w_{n} x_{\sigma(n)}\right|\right)
$$

is bounded and $\left|f\left(n, x_{\sigma(n)}\right)\right| \leq g\left(\left|w_{n} x_{\sigma(n)}\right|\right)=z_{n}$. Hence (5) $\Rightarrow$ (1). If the sequence $\bar{x}$ is bounded, then by the equality

$$
\Delta^{m} x_{n}=a_{n} \bar{x}_{n}+b_{n}
$$

for large $n$ we obtain $\Delta^{m} x=\mathrm{O}(a)+b$. Hence the assertion follows from Lemma 3.11.
Corollary 5.2. Assume $f$ is bounded on $\mathbb{N} \times Z$ for any bounded subset $Z$ of $X, s \leq 0$,

$$
\sum_{n=1}^{\infty} n^{m-s-1}\left|a_{n}\right|<\infty, \quad \text { and } \quad \sum_{n=1}^{\infty} n^{m-s-1}\left|b_{n}\right|<\infty .
$$

Then any bounded solution $x$ of (E) is convergent. More precisely, there exists a vector $c \in X$ such that $x=c+\mathrm{o}\left(n^{s}\right)$.

Proof. Let $x$ be a bounded solution of $(\mathrm{E})$ and let $Z=x(\mathbb{N})$. Then $f$ is bounded on $\mathbb{N} \times Z$, and, by Theorem 5.1, $x \in \operatorname{Pol}_{X}(m-1)+\mathrm{o}\left(n^{s}\right)$. Using the boundedness of $x$ and assumption $s \leq 0$ we see that there exists a vector $c \in X$ such that $x=c+\mathrm{o}\left(n^{s}\right)$.

Corollary 5.3. Assume that for any bounded subset $Z$ of $X, f$ is bounded on $\mathbb{N} \times Z, s \leq 0, q \in \mathbb{N}, y$ is a $q$-periodic solution of the equation $\Delta^{m} y=b$ and

$$
\sum_{n=1}^{\infty} n^{m-s-1}\left|a_{n}\right|<\infty .
$$

Then any bounded solution $x$ of (E) is asymptotically $q$-periodic. More precisely, there exists a vector $c \in X$ such that $x=c+y+\mathrm{o}\left(n^{s}\right)$.

Proof. If $x$ is a bounded solution of (E), then, by Theorem 5.1,

$$
x \in \Delta^{-m} b+\mathbf{o}\left(n^{s}\right)=y+\operatorname{Pol}_{X}(m-1)+\mathbf{o}_{X}\left(n^{s}\right) .
$$

Using boundedness of $x$ and $y$ and assumption $s \leq 0$ we see that there exists a vector $c \in X$ such that $x=c+y+\mathrm{o}\left(n^{s}\right)$.

Lemma 5.4. Assume $a, u$ are nonnegative sequences, $p \in \mathbb{N}, \lambda, \mu>0$, and $b \geq 0$. Let $g:[0, \infty) \rightarrow$ $[0, \infty)$ be nondecreasing, $g(b)>0$,

$$
\sum_{k=0}^{\infty} a_{k}<\infty, \quad \int_{b}^{\infty} \frac{d t}{g(t)}=\infty, \quad \text { and } \quad u_{n} \leq b+\lambda \sum_{k=p}^{n-1} a_{k} g\left(\mu u_{k}\right)
$$

for $n \geq p$. Then the sequence $u$ is bounded.
Proof. See [12, Lemma 7.2].
Lemma 5.5. If $x \in X^{\mathbb{N}}, m \in \mathbb{N}$ and $p \in \mathbb{N}(m)$ then there exists a positive constant $L$ such that

$$
\left|x_{n}\right| \leq n^{(m-1)}\left(L+\sum_{i=p}^{n-1}\left|\Delta^{m} x_{i}\right|\right) \quad \text { for } \quad n \geq p
$$

Proof. The proof is analogous to the proof of [12, Lemma 7.3].

Theorem 5.6. Assume $\sigma(n) \leq n$ for large $n, s \in(-\infty, m-1]$,

$$
g:[0, \infty) \rightarrow[0, \infty), \quad w \in \mathbb{R}^{\mathbb{N}}, \quad w=\mathrm{O}\left(n^{1-m}\right)
$$

$|f(n, t)| \leq g\left(\left|w_{n} t\right|\right)$ on $\mathbb{N} \times X, g$ is nondecreasing, $g(t)>0$ for $t>1$,

$$
\sum_{n=0}^{\infty} n^{m-1-s}\left|a_{n}\right|<\infty, \quad \sum_{n=0}^{\infty} n^{m-1-s}\left|b_{n}\right|<\infty, \quad \int_{1}^{\infty} \frac{d t}{g(t)}=\infty
$$

and $x$ is a solution of $(\mathrm{E})$. Then $x \in \operatorname{Pol}_{X}(m-1)+\mathrm{o}_{X}\left(n^{s}\right)$.
Proof. Choose $M>0$ such that $\left|w_{n}\right| n^{m-1} \leq M$. Then $\left|w_{n}\right| n^{(m-1)} \leq M$. By assumption

$$
\begin{aligned}
\left|\Delta^{m} x_{n}\right| & =\left|a_{n} f\left(n, x_{\sigma(n)}\right)+b_{n}\right| \leq\left|a_{n}\right|\left|f\left(n, x_{\sigma(n)}\right)\right|+\left|b_{n}\right| \\
& \leq\left|a_{n}\right|\left|g\left(\left|w_{n} x_{\sigma(n)}\right|\right)\right|+\left|b_{n}\right| .
\end{aligned}
$$

By Lemma 5.5, there exists a positive constant $L$ such that

$$
\left|x_{\sigma(n)}\right| \leq \sigma(n)^{(m-1)}\left(L+\sum_{i=p}^{\sigma(n)-1}\left|\Delta^{m} x_{i}\right|\right) \leq n^{(m-1)}\left(L+\sum_{i=p}^{n-1}\left|\Delta^{m} x_{i}\right|\right)
$$

Hence

$$
\left|w_{n} x_{\sigma(n)}\right| \leq M L+M \sum_{j=1}^{n-1}\left|\Delta^{m} x_{j}\right|
$$

Then

$$
\begin{aligned}
\left|w_{n} x_{\sigma(n)}\right| & \leq M L+M \sum_{j=1}^{n-1}\left|a_{j}\right| g\left(\left|w_{j} x_{\sigma(j)}\right|\right)+M \sum_{j=1}^{n-1}\left|b_{j}\right| \\
& \leq K+M \sum_{j=1}^{n-1}\left|a_{j}\right| g\left(\left|w_{j} x_{\sigma(j)}\right|\right),
\end{aligned}
$$

where

$$
K=M L+M \sum_{j=1}^{n-1}\left|b_{j}\right|
$$

Obviously $\int_{K}^{\infty} g(t)^{-1} d t=\infty$. By Lemma 5.4 , the sequence $\left(w_{n} x_{\sigma(n)}\right)$ is bounded. Choose $Q>0$ such that $\left|w_{n} x_{\sigma(n)}\right| \leq Q$ for every $n$. Choose $P \geq 1$ such that $g(Q) \leq P$. Then $g\left(\left|w_{n} x_{\sigma(n)}\right|\right) \leq P$ for every $n$. Hence

$$
\left|\Delta^{m} x_{n}\right| \leq\left|a_{n}\right| g\left(\left|w_{n} x_{\sigma(n)}\right|\right)+\left|b_{n}\right| \leq P\left|a_{n}\right|+\left|b_{n}\right| \leq P\left(\left|a_{n}\right|+\left|b_{n}\right|\right)
$$

Therefore $\Delta^{m} x_{n}=\mathrm{O}\left(\left|a_{n}\right|+\left|b_{n}\right|\right)$. Now the conclusion follows from Lemma 3.10.

## References

[1] R. P. Agarwal, H. B. Thompson, C. C. Tisdell, Difference equations in Banach spaces, Comput. Math. Appl. 45(2003), No. 6-9, 1437-1444. https://doi.org/10.1016/S08981221 (03)00100-7 MR2000608; Zbl 1057.39007
[2] A. R. Baias, D. Popa, On Ulam stability of a linear difference equation in Banach spaces, Bull. Malays. Math. Sci. Soc. 43(2020), 1357-1371. https://doi.org/10.1007/s40840-019-00744-6; MR4061428; Zbl 1436.39002
[3] M. Bohner, S. Stević, Linear perturbations of a nonoscillatory second-order dynamic equation, J. Difference Equ. Appl. 15(2009), 11-12, 1211-1221. https://doi.org/10.1080/ 10236190903022782 ; MR2569142; Zbl 1187.34127
[4] S. S. Cheng, W. T. Patula, An existence theorem for a nonlinear difference equation, Nonlinear Anal. 20(1993), No. 3, 193-203. MR1202198; Zbl 0774.39001
[5] C. Cuevas, C. Lizama, Semilinear evolution equations of second order via maximal regularity, Adv. Difference Equ. 2008, Art. ID 316207, 20 pp. https://doi. org/10.1155/2008/ 316207; MR2393473; Zbl 1155.47056
[6] A. Drozdowicz, J. Popenda, Asymptotic behavior of the solutions of the second order difference equations, Proc. Amer. Math. Soc. 99(1987), No. 1, 135-140. https://doi.org/ 10.2307/2046284; MR866443; Zbl 0612.39003
[7] A. Kisiolek, I. Kubiaczyк, Asymptotic behaviour of solutions of nonlinear delay difference equations in Banach spaces, Int. J. Math. Math. Sci. 2005, No. 17, 2769-2774. https://doi.org/10.1155/IJMMS .2005.2769; MR2193832; Zbl 1103.39010
[8] R. Medina, Existence and boundedness of solutions for nonlinear Volterra difference equations in Banach spaces, Abstr. Appl. Anal. 2016, Article ID 1319049, 6 pp. https: //doi.org/10.1155/2016/1319049; MR3589503; Zbl 1470.39031
[9] R. Medina, M. I. Gil, Delay difference equations in Banach spaces, J. Difference Equ. Appl. 11(2005), No. 10, 889-895, https://doi.org/10.1080/10236190512331333860; MR2173719; Zbl 1081.39003
[10] J. Migda, Asymptotic properties of solutions of nonautonomous difference equations, Arch. Math. (Brno) 46(2010), 1-11. MR2644450; Zbl 1240.39009
[11] J. Migda, Asymptotic properties of solutions of higher order difference equations, Math. Bohem. 135(2010), No 1, 29-39. MR2643353; Zbl 1224.39021
[12] J. Migda, Approximative solutions of difference equations, Electron. J. Qual. Theory Differ. Equ. 2014, No. 13, 1-26. https://doi.org/10.14232/ejqtde.2014.1.13; MR3183611; Zbl 1417.39009
[13] J. Migda, Regional topology and approximative solutions of difference and differential equations, Tatra Mt. Math. Publ. 63(2015), 183-203. https://doi.org/10.1515/tmmp-2015-0031; MR3411445; Zbl 6545447
[14] M. Migda, J. Migda, Oscillatory and asymptotic properties of solutions of even order neutral difference equations, J. Difference Equ. Appl. 15(2009), No. 11-12, 1077-1084. https://doi.org/10.1080/10236190903032708; MR2569136; Zbl 1194.39009
[15] J. Popenda, Asymptotic properties of solutions of difference equations, Proc. Indian Acad. Sci. Math. Sci. 95(1986), No. 2, 141-153. https://doi.org/10.1007/bf02881078; MR913886; Zbl 0628.39003
[16] S. Stević, Bounded solutions of a class of difference equations in Banach spaces producing controlled chaos, Chaos Solitons Fractals 35(2008), No. 2, 238-245. https://doi .org/ 10.1016/j.chaos.2007.07.037; MR2357000; Zbl 1142.39013
[17] S. Stević, Existence of a unique bounded solution to a linear second-order difference equation and the linear first-order difference equation, Adv. Difference Equ. 2017, Paper No. 169, 13 pp. https://doi.org/10.1186/s13662-017-1227-x; MR3663764; Zbl 1413.39014
[18] S. Stević, General solution to a higher-order linear difference equation and existence of bounded solutions, Adv. Difference Equ. 2017, Article No. 377, 12 pp. https://doi.org/ 10.1186/s13662-017-1432-7; MR3736631; Zbl 1444.39003
[19] S. Stević, B. Iričanin, W. Kosmala, Z. Šmarda, Existence and global attractivity of periodic solutions to some classes of difference equations, Filomat 33(2019), No. 10, 31873201. https://doi.org/10.1186/s13662-019-1959-x; MR4038981; Zbl 1458.39003
[20] E. Thandapani, R. Arul, J. R. Graef, P. W. Spikes, Asymptotic behavior of solutions of second order difference equations with summable coefficients, Bull. Inst. Math. Acad. Sinica 27(1999), 1-22. MR1681601; Zbl 0920.39001
[21] W. F. Trench, Asymptotic behavior of solutions of Emden-Fowler difference equations with oscillating coefficients, J. Math. Anal. Appl. 179(1993), 135-153. https://doi.org/ 10.1006/jmaa.1993.1340; MR1244954; Zbl 0796.39001
[22] A. Zafer, Oscillatory and asymptotic behavior of higher order difference equations, Math. Comput. Modelling 21(1995), No. 4, 43-50. https://doi.org/10.1016/08957177 (95)00005-m; MR1317929; Zbl 0820.39001

# Regularity properties and blow-up of the solutions for improved Boussinesq equations 

Veli B. Shakhmurov ${ }^{\boxtimes 1,2}$ and Rishad Shahmurov ${ }^{2}$<br>${ }^{1}$ Antalya Bilim University, Department of Industrial Engineering, Dosemealti, 07190 Antalya, Turkey<br>${ }^{2}$ Azerbaijan State Economic University, Linking of Research Centers<br>194 M. Mukhtarov AZ1001 Baku, Azerbaijan

Received 10 March 2021, appeared 17 December 2021
Communicated by Maria Alessandra Ragusa


#### Abstract

In this paper, we study the Cauchy problem for linear and nonlinear Boussinesq type equations that include the general differential operators. First, by virtue of the Fourier multipliers, embedding theorems in Sobolev and Besov spaces, the existence, uniqueness, and regularity properties of the solution of the Cauchy problem for the corresponding linear equation are established. Here, $L^{p}$-estimates for a solution with respect to space variables are obtained uniformly in time depending on the given data functions. Then, the estimates for the solution of linearized equation and perturbation of operators can be used to obtain the existence, uniqueness, regularity properties, and blow-up of solution at the finite time of the Cauchy for nonlinear for same classes of Boussinesq equations. Here, the existence, uniqueness, $L^{p}$-regularity, and blow-up properties of the solution of the Cauchy problem for Boussinesq equations with differential operators coefficients are handled associated with the growth nature of symbols of these differential operators and their interrelationships. We can obtain the existence, uniqueness, and qualitative properties of different classes of improved Boussinesq equations by choosing the given differential operators, which occur in a wide variety of physical systems.


Keywords: Boussinesq equations, hyperbolic equations, differential operators, blowup, Fourier multipliers.
2020 Mathematics Subject Classification: 35Bxx, 35A01.

## 1 Introduction

The aim of this paper is to investigate the existence, uniqueness, and quality properties of the solution of the Cauchy problem for the following improved Boussinesq equation

$$
\begin{gather*}
u_{t t}+L_{0} u_{t t}+L_{1} u=L_{2} f(u), \quad x \in \mathbb{R}^{n}, \quad t \in(0, T),  \tag{1.1}\\
u(x, 0)=\varphi(x), \quad u_{t}(x, 0)=\psi(x), \tag{1.2}
\end{gather*}
$$

[^51]where $u(x, t)$ is the complex-valued unknown function, $f(u)=f(x, t, u)$ is the given nonlinear function, $L_{i}$ are differential operators with constant coefficients, $\varphi(x)$ and $\psi(x)$ are the given initial value functions.

Here, we find the sufficient conditions depending on the qualifications and mutual relevance of the elliptic operators included in the equation to ensure that there exists a unique solution of the problem, being $L^{p}$-regular and blow up infinite time. By choosing the operators $L_{i}$ we obtain different classes of Boussinesq type equations which occur in a wide variety of physical systems, such as in the propagation of longitudinal deformation waves in an elastic rod, a hydro-dynamical process in plasma, in materials science which describe spinodal decomposition and in the absence of mechanical stresses (see [2,6,9,18,21,30-32]). We think this article is useful in the context of $L^{p}$-regularity theory of improved Boussinesg equations. For the first time here, the existence, uniqueness, $L^{p}$-regularity, and blow-up properties of solution (at the finite time) of the Cauchy problem for these type Boussinesq equations are established depending on the symbol of the differential operators and their orders, contained in the equation. We can obtain different classes of Boussinesq equations, by choosing these differential operators, which occur in a wide variety of physical systems. Moreover, in this paper, the method of proofs naturally differs from those used in previous works. Indeed, since the problem includes a general differential operator in the leading part, we need some extra mathematics tools for deriving considered conclusions.

For example, if we choose $L_{0}=L_{1}=L_{2}=-\Delta$, where $\Delta$ is $n$-dimensional Laplace, we obtain the Cauchy problem for the Boussinesq equation

$$
\begin{gather*}
u_{t t}-\Delta u_{t t}-\Delta u=\Delta f(u), \quad x \in \mathbb{R}^{n}, \quad t \in(0, T),  \tag{1.3}\\
u(x, 0)=\varphi(x), \quad u_{t}(x, 0)=\psi(x) . \tag{1.4}
\end{gather*}
$$

Let

$$
L_{0}=L_{1}=L_{2}=A_{1}=\sum_{|\alpha|=2} a_{\alpha} D^{\alpha},
$$

where $a_{\alpha}$ are real numbers. Then the problem (1.1)-(1.2) is reduced to the Cauchy problem for the following Boussinesq equation

$$
\begin{align*}
u_{t t}+A_{1} u_{t t}+A_{1} u & =A_{1} f(x, t, u), \quad x \in \mathbb{R}^{2}, \quad t \in(0, T),  \tag{1.5}\\
u(x, 0) & =\varphi(x), \quad u_{t}(x, 0)=\psi(x),
\end{align*}
$$

here

$$
\varphi, \psi \in W_{p}^{s}\left(\mathbb{R}^{2}\right), \quad s>\frac{2}{p}, \quad p \in(1, \infty) .
$$

Now let

$$
L_{0}=L_{1}=L_{2}=A_{2}=\sum_{|\alpha|=4} a_{\alpha} D^{\alpha},
$$

where $a_{\alpha}$ are real numbers, $\alpha=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right), \alpha_{k}$ are natural numbers and $|\alpha|=\sum_{k=1}^{3} \alpha_{k}$.
Then we get the following Boussinesq equation

$$
\begin{align*}
u_{t t}+A_{2} u_{t t}+A_{2} u & =A_{2} f(x, t, u), \quad x \in \mathbb{R}^{3}, \quad t \in(0, T),  \tag{1.6}\\
u(x, 0) & =\varphi(x), \quad u_{t}(x, 0)=\psi(x) .
\end{align*}
$$

where

$$
\varphi, \psi \in W_{p}^{s}\left(\mathbb{R}^{3}\right), \quad s>\frac{3}{p^{\prime}} \quad p \in(1, \infty) .
$$

Finally, let

$$
L_{0}=\sum_{|\alpha|=4} a_{0 \alpha} D^{\alpha}, \quad L_{1}=\sum_{|\alpha|=2} a_{1 \alpha} D^{\alpha}, \quad L_{2}=\sum_{|\alpha|=4} a_{2 \alpha} D^{\alpha},
$$

where $a_{\alpha i}$ are real numbers, $\alpha=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right), \alpha_{k}$ are natural numbers and $|\alpha|=\sum_{k=1}^{3} \alpha_{k}$.
The problem (1.1)-(1.2) reduced to Cauchy problem for the following Boussinesq equation

$$
\begin{align*}
u_{t t}+L_{0} u_{t t}+L_{1} u & =L_{2} f(x, t, u), \quad x \in \mathbb{R}^{3}, \quad t \in(0, T),  \tag{1.7}\\
u(x, 0) & =\varphi(x), \quad u_{t}(x, 0)=\psi(x),
\end{align*}
$$

where

$$
\varphi, \psi \in W^{s, p}\left(\mathbb{R}^{3}\right), \quad s>\frac{3}{p^{\prime}} \quad p \in(1, \infty) .
$$

By using the general result for (1.1)-(1.2), we obtain the existence, uniqueness, $L^{p}$-regularity, and blow-up properties of the solutions of the problems (1.5), (1.6) and (1.7).

The equation (1.3) arises in different situations (see [18,30]). For example, for $n=1$ it describes a limit of a one-dimensional nonlinear lattice [32], shallow-water waves [12,31] and the propagation of longitudinal deformation waves in an elastic rod [4]. Rosenau [23] derived the equations governing dynamics of one, two and three-dimensional lattices. One of those equations is (1.3). Note that, the existence of solutions and regularity properties for different wave type equations are considered e.g. in $[1,7,8,14,15,17,20,22,24,29,33]$. In this respect we can show new results e.g. [1,7,8,14,15,22,29,33]. In [27] and [28] the existence of the global classical solutions and the blow-up of the solutions of the initial value problem (1.3)(1.4) are studied. In this paper, we obtain the existence, uniqueness of solution and regularity properties of the problem (1.1)-(1.2). The strategy is to express the Boussinesq equation as an integral equation. To treat the nonlinearity as a small perturbation of the linear part of the equation, the contraction mapping theorem is used. Also, a priori estimates on $L^{p}$ norm of solutions of the linearized version are utilized. The key step is the derivation of the uniform estimate of the solutions of the linearized Boussinesq equation. The methods of harmonic analysis, operator theory, interpolation of Banach spaces and embedding theorems in Sobolev spaces are the main tools implemented to carry out the analysis.

## 2 Definitions and background

In order to state our results precisely, we introduce some notations and some function spaces. Let $E$ be a Banach space. $L_{p}(\Omega ; E)$ denotes the space of strongly measurable $E$-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^{n}$ with the norm

$$
\begin{aligned}
\|f\|_{L_{p}} & =\|f\|_{L_{p}(\Omega ; E)}=\left(\int_{\Omega}\|f(x)\|_{E}^{p} d x\right)^{\frac{1}{p}}, \quad 1 \leq p<\infty, \\
\|f\|_{L_{\infty}(\Omega: E)} & =\underset{x \in \Omega}{\operatorname{ess} \sup }\|f(x)\|_{E} .
\end{aligned}
$$

Let $\mathbb{R}, \mathbb{C}$ denote the sets of all real and complex numbers, respectively. For $E=\mathbb{C}$ the $L_{p}(\Omega ; E)$ denotes by $L_{p}(\Omega)$. Let $m$ be a positive integer. $W_{p}^{m}(\Omega)$ denotes the Sobolev space, i.e. space of all functions $u \in L_{p}(\Omega)$ that have the generalized derivatives $\frac{\partial^{m} u}{\partial x_{k}^{m}} \in L_{p}(\Omega), 1 \leq p \leq \infty$ with the norm

$$
\|u\|_{W_{p}^{m}(\Omega)}=\|u\|_{L_{p}(\Omega)}+\sum_{k=1}^{n}\left\|\frac{\partial^{m} u}{\partial x_{k}^{m}}\right\|_{L_{p}(\Omega)}<\infty .
$$

Let $F$ denotes the Fourier transform defined by

$$
\hat{u}(\xi)=F u=(2 \pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n}} e^{-i x \xi} u(x) d x \quad \text { for } u \in S\left(\mathbb{R}^{n} ; E\right) \text { and } x, \xi \in \mathbb{R}^{n} .
$$

Let $S\left(\mathbb{R}^{n}\right)$ denote the Schwartz class, i.e., the space of rapidly decreasing smooth functions on $\mathbb{R}^{n}$, equipped with its usual topology generated by seminorms. Let $S^{\prime}\left(\mathbb{R}^{n}\right)$ denote the space of all continuous linear operators $L: S\left(\mathbb{R}^{n}\right) \rightarrow \mathbb{C}$, equipped with the bounded convergence topology. Recall $S\left(\mathbb{R}^{n}\right)$ is norm dense in $L_{p}\left(\mathbb{R}^{n}\right)$ when $1 \leq p<\infty$. Let $1 \leq p \leq q<\infty$. A function $\Psi \in L_{\infty}\left(\mathbb{R}^{n}\right)$ is called a Fourier multiplier from $L_{p}\left(\mathbb{R}^{n}\right)$ to $L_{q}\left(\mathbb{R}^{n}\right)$ if the map $u \rightarrow F^{-1} \Psi(\xi) F u$ for $u \in S\left(\mathbb{R}^{n}\right)$ is well defined and extends to a bounded linear operator

$$
T: L_{p}\left(\mathbb{R}^{n}\right) \rightarrow L_{q}\left(\mathbb{R}^{n}\right) .
$$

Let $L_{p}^{s}\left(\mathbb{R}^{n}\right),-\infty<s<\infty$ denotes Liouville-Sobolev space of order $s$ which is defined as:

$$
L_{p}^{s}=L_{p}^{s}\left(\mathbb{R}^{n}\right)=(I-\Delta)^{-\frac{s}{2}} L_{p}\left(\mathbb{R}^{n}\right)
$$

with the norm

$$
\|u\|_{L_{p}^{s}}=\left\|(I-\Delta)^{\frac{s}{2}} u\right\|_{L_{p}\left(\mathbb{R}^{n}\right)}=\left\|F^{-1}\left(1+|\xi|^{2}\right)^{\frac{s}{2}} \hat{u}\right\|_{L_{p}\left(\mathbb{R}^{n}\right)}<\infty .
$$

It clear that $L_{p}^{0}\left(\mathbb{R}^{n}\right)=L_{p}\left(\mathbb{R}^{n}\right)$. It is known that $L_{p}^{m}\left(\mathbb{R}^{n}\right)=W_{p}^{m}\left(\mathbb{R}^{n}\right)$ for the positive integer $m$ (see e.g. [26, § 15].

Let $L_{q}^{*}(E)$ denote the space of all $E$-valued function space such that

$$
\|u\|_{L_{q}^{*}(E)}=\left(\int_{0}^{\infty}\|u(t)\|_{E}^{q} \frac{d t}{t}\right)^{\frac{1}{q}}<\infty, 1 \leq q<\infty,\|u\|_{L_{\infty}^{*}(E)}=\sup _{t \in(0, \infty)}\|u(t)\|_{E} .
$$

Here, $F$ denotes the Fourier transform. Fourier-analytic representation of Besov space on $\mathbb{R}^{n}$ are defined as:

$$
\begin{aligned}
B_{p, q}^{s}\left(\mathbb{R}^{n}\right)=\left\{u \in S^{\prime}\left(\mathbb{R}^{n}\right):\|u\|_{B_{p, q}^{s}\left(\mathbb{R}^{n}\right)}=\left\|F^{-1} t^{\varkappa-s}\left(1+|\xi|^{\frac{\varkappa}{2}}\right) e^{-t|\xi|^{2}} F u\right\|_{L_{q}^{\star}\left(L_{p}\left(\mathbb{R}^{n}\right)\right)^{\prime}}\right. \\
\left.|\xi|^{2}=\sum_{k=1}^{n} \xi_{k}^{2}, \xi=\left(\xi_{1}, \xi_{2}, \ldots, \xi_{n}\right), p \in(1, \infty), q \in[1, \infty], \varkappa>s\right\} .
\end{aligned}
$$

Here,

$$
\begin{gathered}
X_{p}=L^{p}\left(\mathbb{R}^{n}\right), \quad 1 \leq p \leq \infty, \quad Y^{s, p}=L^{s, p}\left(\mathbb{R}^{n}\right), \\
Y_{1}^{s, p}=L_{p}^{s}\left(\mathbb{R}^{n}\right) \cap L_{1}\left(\mathbb{R}^{n}\right), \quad Y_{\infty}^{s, p}=L^{s, p}\left(\mathbb{R}^{n}\right) \cap L_{\infty}\left(\mathbb{R}^{n}\right),
\end{gathered}
$$

It should be note that, the norm of Besov space does not depends on $\varkappa$ (see e.g. [25, § 2.3]. For $p=q$ the space $B_{p, q}^{s}\left(\mathbb{R}^{n}\right)$ will be denoted by $B_{p}^{s}\left(\mathbb{R}^{n}\right)$.
Definition 2.1. For any $T>0$ the function $u \in C^{2}\left([0, T] ; Y_{\infty}^{2, s, p}\right)$ satisfies the equation (1.1)(1.2) a.e. in $\mathbb{R}_{T}^{n}=\mathbb{R}^{n} \times(0, T)$ is called the continuous solution or the strong solution of the problem (1.1)-(1.2). If $T<\infty$, then $u(x, t)$ is called the local strong solution of the problem (1.1)-(1.2). If $T=\infty$, then $u(x, t)$ is called the global strong solution of (1.1)-(1.2).

Sometimes we use one and the same symbol $C$ without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say $\alpha$, we write $C_{\alpha}$.

The paper is organized as follows: In Section 1, some definitions and background are given. In Section 2, we obtain the existence of a unique solution and priory estimates for the solution of the linearized problem (1.1)-(1.2). In Section 3, we show the existence and uniqueness of the local strong solution of the problem (1.1)-(1.2). Section 4 is devoted to the existence of the global solution. In Section 5 the blow-up properties of the solution are derived. In Section 6 we show some applications of the problem (1.1)-(1.2).

Sometimes we use one and the same symbol $C$ without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say $h$, we write $C_{h}$.

## 3 Estimates for linearized equation

In this section, we make the necessary estimates for solutions of the Cauchy problem for the following linear Boussinesq equation

$$
\begin{gather*}
u_{t t}+L_{0} u_{t t}+L_{1} u=L_{2} g(x, t), \quad x \in \mathbb{R}^{n}, \quad t \in(0, T),  \tag{3.1}\\
u(x, 0)=\varphi(x), \quad u_{t}(x, 0)=\psi(x),
\end{gather*}
$$

where

$$
L_{i} u=\sum_{|\alpha|=2 m_{i}} a_{i \alpha} D^{\alpha} u, a_{i \alpha} \in \mathbb{R}, \quad i=0,1,2,
$$

$\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right), \alpha_{k}$ are natural numbers, $|\alpha|=\sum_{k=1}^{n} \alpha_{k}$ and $m_{i}$ are positive integers. Let

$$
\begin{gather*}
L_{i}(\xi)=\sum_{|\alpha|=2 m_{i}} a_{i \alpha}\left(i \xi_{1}\right)^{\alpha_{1}}\left(i \xi \xi^{\alpha_{2}} \ldots(i \xi)^{\alpha_{n}}, \quad i=0,1,2,\right.  \tag{3.2}\\
Q=Q(\xi)=L_{1}(\xi)\left[1+L_{0}(\xi)\right]^{-1}, \quad L(\xi)=L_{2}(\xi)\left[1+L_{0}(\xi)\right]^{-1} .
\end{gather*}
$$

Condition 3.1. Assume that $L_{1}(\xi) \neq 0, L_{0}(\xi) \neq-1$ and there exist positive constants $M_{1}$ and $M_{2}$ depend only on $a_{i \alpha}$ such that

$$
\begin{equation*}
\left|Q^{\frac{1}{2}}(\xi)\right| \leq M_{1}\left(1+|\xi|^{2}\right)^{\frac{v}{2}}, \quad\left|L(\xi) Q^{\frac{1}{2}}(\xi)\right| \leq M_{2}\left(1+|\xi|^{2}\right)^{\frac{v}{2}} \tag{3.3}
\end{equation*}
$$

for all $\xi \in \mathbb{R}^{n}$ and a real number $v$.
Remark 3.2. It is not hard to see that if $v \geq m_{1}-m_{0}$, then the first inequality verified. Moreover, if $v \geq m_{1}+2 m_{2}-\left(2 m_{0}\right)^{\frac{3}{2}}$, then the second inequality holds.

First we need the following lemmas.
Lemma 3.3. Suppose that $Q(\xi) \neq 0$ for each $\xi \in \mathbb{R}^{n}$. Then problem (3.1) has a strong solution.
Proof. Since $L_{0}, L_{1}$ and $L_{2}$ are differential operators with constant coefficients, by using of Fourier transform and in view of (3.2), we get from (3.1):

$$
\begin{align*}
\hat{u}_{t t}(\xi, t)+Q(\xi) \hat{u}(\xi, t) & =L(\xi) \hat{g}(\xi, t), \\
\hat{u}(\xi, 0) & =\hat{\varphi}(\xi), \quad \hat{u}_{t}(\xi, 0)=\hat{\psi}(\xi), \quad \xi \in \mathbb{R}^{n}, \quad t \in(0, T), \tag{3.4}
\end{align*}
$$

where $\hat{u}(\xi, t)$ is a Fourier transform of $u(x, t)$ with respect to $x$. By using the variation of constants we get that there exists a solution of the problem (3.4) that can be written as the following

$$
\begin{equation*}
\hat{u}(\xi, t)=C(\xi, t) \hat{\varphi}(\xi)+S(\xi, t) \hat{\psi}(\xi)+O g(\xi) \tag{3.5}
\end{equation*}
$$

here,

$$
\begin{gathered}
C(\xi, t)=\cos \left(Q^{\frac{1}{2}} t\right), \quad S(\xi, t)=Q^{-\frac{1}{2}} \sin \left(Q^{\frac{1}{2}} t\right) \\
\hat{\Phi}(\xi, t)=L(\xi) Q^{-\frac{1}{2}}(\xi) \sin \left(Q^{\frac{1}{2}} t\right), \quad O g=O g(\xi)=\int_{0}^{t} \hat{\Phi}(\xi, t-\tau) \hat{g}(\xi, \tau) d \tau
\end{gathered}
$$

From (3.5) we get that the solution of the problem (3.1) can be expressed as

$$
\begin{equation*}
u(x, t)=S_{1}(t) \varphi(x)+S_{2}(t) \psi(x)+\int_{0}^{t} F^{-1} O g(\xi) d \xi, \quad t \in(0, T) \tag{3.6}
\end{equation*}
$$

where $F^{-1}$ denotes the inverse Fourier transformation, $S_{1}(t)$ and $S_{2}(t)$ are linear operators defined by

$$
\begin{aligned}
& S_{1}(t) \varphi=(2 \pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n}} e^{i x \xi} C(\xi, t) \hat{\varphi}(\xi) d \xi \\
& S_{2}(t) \psi=(2 \pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n}} e^{i x \xi} S(\xi, t) \hat{\psi}(\xi) d \xi
\end{aligned}
$$

Theorem 3.4. Assume that the Condition 3.1 holds and

$$
\begin{equation*}
s>n\left(\frac{2}{q}+\frac{1}{p}\right)+v \quad \text { if } v \geq 0, \quad s>n\left(\frac{2}{q}+\frac{1}{p}\right) \quad \text { if } v<0 \tag{3.7}
\end{equation*}
$$

for $p \in[1, \infty]$ and for a $q \in[1,2]$. Then for $\varphi, \psi, g(\cdot, t) \in Y_{1}^{s, p}$ for $t \in(0, T)$ and $g(x, \cdot) \in$ $L^{1}\left(0, T ; Y_{1}^{s, p}\right)$ for $x \in \mathbb{R}^{n}$ problem (3.1) has a unique solution $u(x, t)$ satisfies the following estimate

$$
\begin{align*}
& \|u\|_{X_{\infty}}+\left\|u_{t}\right\|_{X_{\infty}} \\
& \quad \leq C\left[\|\varphi\|_{Y^{s, p}}+\|\varphi\|_{X_{1}}+\|\psi\|_{Y^{s, p}}+\|\psi\|_{X_{1}}+\int_{0}^{t}\left(\|g(\cdot, \tau)\|_{Y^{s, p}}+\|g(\cdot, \tau)\|_{X_{1}}\right) d \tau\right] \tag{3.8}
\end{align*}
$$

uniformly with respect to $t \in[0, T]$.
Proof. Let $N \in \mathbb{N}$ and

$$
\Pi_{N}=\left\{\xi: \xi \in \mathbb{R}^{n},|\xi| \leq N\right\}, \quad \Pi_{N}^{\prime}=\left\{\xi: \xi \in \mathbb{R}^{n},|\xi| \geq N\right\}
$$

It is clear to see that

$$
\begin{align*}
\|u\|_{X_{\infty}} \leq & \left\|F^{-1} C(\xi, t) \hat{\varphi}(\xi)\right\|_{X_{\infty}}+\left\|F^{-1} S(\xi) \hat{\psi}(\xi, t)\right\|_{X_{\infty}} \\
\leq & \left\|\int_{\mathbb{R}^{n}} e^{i x \xi} C(\xi, t) \varphi(x) d x\right\|_{L_{\infty}\left(\Pi_{N}\right)}+\left\|\int_{\mathbb{R}^{n}} e^{i x \xi} S(\xi, t) \psi(x) d x\right\|_{L_{\infty}\left(\Pi_{N}\right)} \\
& +\left\|F^{-1} C(\xi, t) \hat{\varphi}(\xi)\right\|_{L_{\infty}\left(\Pi_{N}^{\prime}\right)}+\left\|F^{-1} S(\xi, t) \hat{\psi}(\xi)\right\|_{L \infty\left(\Pi_{N}^{\prime}\right)} \\
& +\left\|F^{-1} C(\xi, t) O g(\xi)\right\|_{L_{\infty}\left(\Pi_{N}^{\prime}\right)}+\left\|F^{-1} O g(\xi)\right\|_{L \infty\left(\Pi_{N}^{\prime}\right)} \tag{3.9}
\end{align*}
$$

Using Minkowski's inequality for integrals and uniformly boundedness of $C(\xi, t), S(\xi, t)$ on $\Pi_{N}$ we have

$$
\begin{equation*}
\left\|\int_{\mathbb{R}^{n}} e^{i x \xi} C(\xi, t) \varphi(x) d x\right\|_{L_{\infty}\left(\Pi_{N}\right)}+\left\|\int_{\mathbb{R}^{n}} e^{i x \xi} S(\xi, t) \psi(x) d x\right\|_{L_{\infty}\left(\Pi_{N}\right)} \leq C\left[\|\varphi\|_{X_{1}}+\|\psi\|_{X_{1}}\right] \tag{3.10}
\end{equation*}
$$

It is clear to see that

$$
\begin{align*}
& \left\|F^{-1} C(\xi, t) \hat{\varphi}(\xi)\right\|_{L^{\infty}\left(\Pi_{N}^{\prime}\right)}+\left\|F^{-1} S(\xi, t) \hat{\psi}(\xi)\right\|_{L^{\infty}\left(\Pi_{N}^{\prime}\right)} \\
& =\left\|F^{-1}\left(1+|\xi|^{2}\right)^{-\frac{s}{2}} C(\xi, t)\left(1+|\xi|^{2}\right)^{\frac{s}{2}} \hat{\varphi}(\xi)\right\|_{L^{\infty}\left(\Pi_{N}^{\prime}\right)} \\
& \quad+\left\|F^{-1}\left(1+|\xi|^{2}\right)^{-s} S(\xi, t)(1+|\xi|)^{\frac{s}{2}} \hat{\psi}(\xi)\right\|_{L^{\infty}\left(\Pi_{N}^{\prime}\right)} . \tag{3.11}
\end{align*}
$$

By using (3.5) and (3.3) we get the estimates

$$
\begin{align*}
& \left.\left.\sup _{\xi \in \mathbb{R}^{n}, t \in[0, T]}|\xi|\right|^{|\alpha|+\frac{n}{p}} D^{\alpha}\left[\left(1+|\xi|^{2}\right)^{-\frac{s}{2}} C(\xi, t)\right] \right\rvert\, \leq C_{2}  \tag{3.12}\\
& \left.\left.\sup _{\xi \in \mathbb{R}^{n}, t \in[0, T]}|\xi|\right|^{|\alpha|+\frac{n}{p}} D^{\alpha}\left[\left(1+|\xi|^{2}\right)^{-\frac{s}{2}} S(\xi, t)\right] \right\rvert\, \leq C_{2}
\end{align*}
$$

uniformly in $t \in[0, T]$ for $s>\frac{n}{p}, \alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right), \alpha_{k} \in\{0,1\}, \xi \in \mathbb{R}^{n}$ and $\xi \neq 0$.
Let we show that $G(\cdot, t), V(\cdot, t) \in B_{q, 1}^{\frac{n}{q}+\frac{1}{p}}\left(\mathbb{R}^{n} ; E\right)$ for some $q \in(1,2)$ and for all $t \in[0, T]$, where

$$
G(\xi, t)=\left(1+|\xi|^{2}\right)^{-\frac{s}{2}} Q^{\frac{1}{2}}(\xi) C(\xi, t), V(\cdot, t)=\left(1+|\xi|^{2}\right)^{-\frac{s}{2}} S(\xi, t)
$$

By embedding properties of Sobolev and Besov spaces it sufficient to derive that $G, V \in$ $W_{q}^{n\left(\frac{1}{q}+\frac{1}{p}\right)+\varepsilon}\left(\mathbb{R}^{n}\right)$ for some $\varepsilon>0$. Indeed by construction of solution, by Condition 3.1 and by (3.3) we get $G \in L_{q}\left(\mathbb{R}^{n}\right)$. Let $\sigma>n\left(\frac{1}{q}+\frac{1}{p}\right)$. For deriving the embedding relating $G \in$ $W_{q}^{\sigma+\varepsilon}\left(\mathbb{R}^{n}\right)$, it sufficient to show

$$
\left(1+|\xi|^{2}\right)^{\frac{\sigma}{2}} G(\cdot, t) \in L_{\sigma}\left(\mathbb{R}^{n}\right) \quad \text { for all } t \in[0, T]
$$

Indeed, in view of (3.3), (3.12) the function $\left(1+|\xi|^{2}\right)^{\frac{\sigma}{2}} G(\xi, t)$ is uniformly bounded for $\xi \in \mathbb{R}^{n}$ and $s>\sigma$. By virtue of (3.3), (3.12) and by assumption (3.7) we have

$$
\begin{aligned}
\int_{\mathbb{R}^{n}}\left(1+|\xi|^{2}\right)^{\frac{\sigma}{2} q}|G(\xi, t)|^{q} d \xi & \lesssim \int_{\mathbb{R}^{n}}\left(1+|\xi|^{2}\right)^{-\frac{(s-\sigma)}{2} q}|C(\xi, t)|^{q} d \xi \\
& \lesssim \int_{\mathbb{R}^{n}}\left(1+|\xi|^{2}\right)^{-\left(\frac{s-\sigma}{2}\right) q} d \xi<\infty
\end{aligned}
$$

In a similar way we obtain the following

$$
\int_{\mathbb{R}^{n}}\left(1+|\xi|^{2}\right)^{\frac{\sigma}{2} q}|V(\xi, t)|^{q} d \xi \lesssim \int_{\mathbb{R}^{n}}\left(1+|\xi|^{2}\right)^{-\left(\frac{s-\sigma}{2}\right) q}|S(\xi, t)|^{q} d \xi<\infty
$$

By the Fourier multiplier theorem [10, Theorem 4.3], from (3.12) we get that the functions $\left(1+|\xi|^{2}\right)^{-\frac{s}{2}} Q^{\frac{1}{2}}(\xi) C(\xi, t),\left(1+|\xi|^{2}\right)^{-\frac{s}{2}} L(\xi) Q^{\frac{1}{2}}(\xi) S(\xi, t)$ are $L_{p}\left(\mathbb{R}^{n}\right) \rightarrow L_{\infty}\left(\mathbb{R}^{n}\right)$ Fourier multipliers. Then by Minkowski's inequality for integrals from (3.10) and (3.11) we obtain

$$
\begin{equation*}
\left\|F^{-1} C(\xi, t) \hat{\varphi}(\xi)\right\|_{L_{\infty}\left(\Pi_{N}^{\prime}\right)}+\left\|F^{-1} S(\xi, t) \hat{\psi}(\xi)\right\|_{L_{\infty}\left(\Pi_{N}^{\prime}\right)} \leq C\left[\|\varphi\|_{Y^{s, p}}+\|\psi\|_{Y^{s, p}}\right] \tag{3.13}
\end{equation*}
$$

Moreover, by using the representation of $\hat{\Phi}(\xi, t)$ in (3.5) and the estimate (3.3) we get the uniform estimate

$$
\begin{equation*}
\left.\left.\sup _{\xi \in \mathbb{R}^{n}, t \in[0, T]}|\xi|\right|^{|\alpha|+\frac{n}{p}} D^{\alpha}\left[\left(1+|\xi|^{2}\right)^{-\frac{s}{2}} \hat{\Phi}(\xi, t)\right] \right\rvert\, \leq C_{3} . \tag{3.14}
\end{equation*}
$$

By reasoning as the above and in view of (3.3) we get that the function $\left(1+|\xi|^{2}\right)^{-\frac{5}{2}} O g(\xi)$ is a $L_{p}\left(\mathbb{R}^{n}\right) \rightarrow L_{\infty}\left(\mathbb{R}^{n}\right)$ Fourier multiplier, i.e. we have the following uniform estimate

$$
\left\|F^{-1} \int_{0}^{t} \hat{\Phi}(\xi, t-\tau) \hat{g}(\xi, \tau) d \tau\right\|_{X_{\infty}} \leq C \int_{0}^{t}\left(\|g(\cdot, \tau)\|_{Y^{s}}+\|g(\cdot, \tau)\|_{X_{1}}\right) d \tau
$$

Hence, from (3.9)-(3.11), we deduced the following

$$
\begin{equation*}
\|u\|_{X_{\infty}} \leq C\left[\|\varphi\|_{Y^{s, p}}+\|\varphi\|_{X_{1}}+\|\psi\|_{Y^{s, p}}+\|\psi\|_{X_{1}}+\int_{0}^{t}\left(\|g(\cdot, \tau)\|_{Y^{s, p}}+\|g(\cdot, \tau)\|_{X_{1}}\right) d \tau\right] \tag{3.15}
\end{equation*}
$$

By differentiating from (3.5) we get

$$
\begin{align*}
\hat{u}_{t}(\xi, t)= & -Q^{\frac{1}{2}}(\xi) \sin \left(Q^{\frac{1}{2}} t\right) \hat{\varphi}(\xi)+\cos \left(Q^{\frac{1}{2}} t\right) \hat{\psi}(\xi) \\
& +\int_{0}^{t} Q^{\frac{1}{2}}(\xi) L(\xi) \sin \left(Q^{\frac{1}{2}}(\xi, t-\tau)\right) \hat{g}(\xi, \tau) d \tau, \quad t \in(0, T) \tag{3.16}
\end{align*}
$$

By using (3.3) and (3.16) in a similar way, we get

$$
\begin{equation*}
\left\|u_{t}\right\|_{X_{\infty}} \leq C\left[\|\varphi\|_{Y^{s, p}}+\|\varphi\|_{X_{1}}+\|\psi\|_{Y^{s, p}}+\|\psi\|_{X_{1}}+\int_{0}^{t}\left(\|g(\cdot, \tau)\|_{Y^{s, p}}+\|g(\cdot, \tau)\|_{X_{1}}\right) d \tau\right] \tag{3.17}
\end{equation*}
$$

Then from (3.15) and (3.17), we obtain the estimate (3.8). Let us now show that problem (3.1) has a unique solution $u \in C^{(1)}\left([0, T] ; Y^{s, p}\right)$. Let us admit it is the opposite. So let us assume that the problem (3.1) has two solutions $u_{1}, u_{2} \in C^{(1)}\left([0, T] ; Y^{s, p}\right)$. Then by linearity of (3.1), we get that $v=u_{1}-u_{2}$ is also a solution of the corresponding homogenous equation

$$
v_{t t}+L_{0} v_{t t}+L_{1} v=0, \quad v(x, 0)=0, \quad v_{t}(x, 0)=0, \quad x \in \mathbb{R}^{n}, \quad t \in(0, T)
$$

Moreover, by (3.8) we have the following estimate

$$
\|v\|_{X_{\infty}} \leq 0
$$

The above estimate implies that $v=0$.
Remark 3.5. In view of Remark 3.2 we see that the assumption (3.7) is satisfied if $v \geq 0$ and

$$
s>n\left(\frac{2}{q}+\frac{1}{p}\right)+\max \left\{m_{1}-m_{0}, m_{1}+2 m_{2}-\left(2 m_{0}\right)^{\frac{3}{2}}\right\} .
$$

By reasoning as in Theorem 3.4 we obtain

Theorem 3.6. Let the Condition 3.1 hold. Then for $\varphi, \psi, g(\cdot, t) \in Y^{s, p}$ for $t \in(0, T), g(x, \cdot) \in$ $L^{1}\left(0, T ; Y_{1}^{s, p}\right)$ for $x \in \mathbb{R}^{n}$ problem (3.1) has a unique solution $u(x, t)$ and the following uniform estimate holds

$$
\begin{equation*}
\|u\|_{Y^{s, p}}+\left\|u_{t}\right\|_{Y^{s, p}} \leq C\left[\|\varphi\|_{Y^{s, p}}+\|\psi\|_{Y^{s, p}}+\int_{0}^{t}\|g(., \tau)\|_{Y^{s, p}} d \tau\right] . \tag{3.18}
\end{equation*}
$$

Proof. From (3.5) we have the following uniform estimate

$$
\begin{align*}
&\left\|F^{-1}\left(1+|\xi|^{2}\right)^{\frac{s}{2}} \hat{u}\right\|_{X_{p}}+\left\|F^{-1}\left(1+|\xi|^{2}\right)^{\frac{s}{2}} \hat{u}_{t}\right\|_{X_{p}} \\
& \leq C\left\{\left\|F^{-1}(1+|\xi|)^{\frac{s}{2}} C(\xi, t) \hat{\varphi}\right\|_{X_{p}}+\left\|F^{-1}(1+|\xi|)^{\frac{s}{2}} S(\xi, t) \hat{\psi}\right\|_{X_{p}}\right. \\
&\left.+\int_{0}^{t}\left\|(1+|\xi|)^{\frac{s}{2}} \hat{\Phi}(\xi, t-\tau) \hat{\delta}(\cdot, \tau)\right\|_{X_{p}} d \tau\right\} . \tag{3.19}
\end{align*}
$$

By Condition 3.1 and by virtue of Fourier multiplier theorems (see e.g. [10, Theorem 4.3], we get that $C(\xi, t), S(\xi, t)$ and $\hat{\Phi}(\xi, t)$ are Fourier multipliers in $L_{p}\left(\mathbb{R}^{n}\right)$ uniformly with respect to $t \in[0, T]$. So, the estimate (3.19) by using Minkowski's inequality for integrals implies (3.18).

The uniqueness of (3.3) is obtained by reasoning as in Theorem 3.4.

## 4 Initial value problem for nonlinear equation

In this section, we will show the local existence and uniqueness of solution for the Cauchy problem (1.1)-(1.2).

For the study of the nonlinear problem (1.1)-(1.2) we need the following lemmas
Lemma 4.1 (Nirenberg's inequality [19]). Assume that $u \in L_{p}(\Omega), D^{m} u \in L_{q}(\Omega), p, q \in(1, \infty)$. Then for $i$ with $0 \leq i \leq m, m>\frac{n}{q}$ we have

$$
\begin{equation*}
\left\|D^{i} u\right\|_{r} \leq C\|u\|_{p}^{1-\mu} \sum_{k=1}^{n}\left\|D_{k}^{m} u\right\|_{q^{\prime}}^{\mu} \tag{4.1}
\end{equation*}
$$

where

$$
\frac{1}{r}=\frac{i}{m}+\mu\left(\frac{1}{q}-\frac{m}{n}\right)+(1-\mu) \frac{1}{p^{\prime}}, \quad \frac{i}{m} \leq \mu \leq 1 .
$$

Lemma 4.2 ([19]). Assume that $u \in W_{p}^{m}(\Omega) \cap L_{\infty}(\Omega)$ and $f(u)$ possesses continuous derivatives up to order $m \geq 1$. Then $f(u)-f(0) \in W_{p}^{m}(\Omega)$ and

$$
\begin{gather*}
\|f(u)-f(0)\|_{p} \leq\left\|f^{(1)}(u)\right\|_{\infty}\|u\|_{p^{\prime}} \\
\left\|D^{k} f(u)\right\|_{p} \leq C_{0} \sum_{j=1}^{k}\left\|f^{(j)}(u)\right\|_{\infty}\|u\|_{\infty}^{j-1}\left\|D^{k} u\right\|_{p^{\prime}} \quad 1 \leq k \leq m \tag{4.2}
\end{gather*}
$$

where $C_{0} \geq 1$ is a constant.
Let

$$
E_{0}=\left(Y^{s, p}, X_{p}\right)_{\frac{1}{2 p}, p}=B_{p}^{s\left(1-\frac{1}{2 p}\right)}\left(\mathbb{R}^{n}\right)
$$

Remark 4.3. By using a result by J. Lions and I. Petree (see e.g. [25, § 1.8]) we obtain that the map $u \rightarrow u\left(t_{0}\right), t_{0} \in[0, T]$ is continuous and surjective from $W_{p}^{2}\left(0, T ; Y^{s, p}, X_{p}\right)$ onto $E_{0}$ and there is a constant $C_{1}$ such that

$$
\left\|u\left(t_{0}\right)\right\|_{E_{0}} \leq C_{1}\|u\|_{W_{p}^{2}\left(0, T ; \gamma^{s p}, X_{p}\right)^{\prime}} \quad 1 \leq p \leq \infty .
$$

Let

$$
C^{(m)}(p)=C^{(m)}\left([0, T] ; Y_{\infty}^{s, p}\right) .
$$

First all of, we define the space $Y(T)=C\left([0, T] ; Y_{\infty}^{s, p}\right)$ equipped with the norm defined by

$$
\|u\|_{Y(T)}=\max _{t \in[0, T]}\|u\|_{Y_{s, p}}+\max _{t \in[0, T]}\|u\|_{X_{\infty^{\prime}}} u \in Y(T) .
$$

It is easy to see that $Y(T)$ is a Banach space. For $\varphi, \psi \in Y^{s, p}$, let

$$
M=\|\varphi\|_{Y^{s, p}}+\|\varphi\|_{X_{\infty}}+\|\psi\|_{Y^{s, p}}+\|\psi\|_{X_{\infty}} .
$$

Condition 4.4. Assume:
(1) The Condition 3.1 holds, $\varphi, \psi \in Y_{1}^{s, p}$ and $s>n\left(\frac{2}{q}+\frac{1}{p}\right)+v$ if $v \geq 0, s>n\left(\frac{2}{q}+\frac{1}{p}\right)$ if $v<0$ for $p \in[1, \infty]$ and for a $q \in[1,2]$;
(2) (2) the function $u \rightarrow \hat{f}(\xi, t, u): \mathbb{R}^{n} \times[0, T] \times E_{0} \rightarrow \mathbb{C}$ is a measurable in $(\xi, t) \in \mathbb{R}^{n} \times[0, T]$ for $u \in E_{0}$; moreover, $\hat{f}(\xi, t, u)$ is continuous in $u \in E_{0}$ and $\hat{f}(\xi, t, u) \in C^{([s]+1)}\left(E_{0} ; \mathbb{C}\right)$ uniformly for $\xi \in \mathbb{R}^{n}$ and $t \in[0, T]$.

The main aim of this section is to prove the following result.
Theorem 4.5. Let the Condition 4.4 hold. Then problem (1.1)-(1.2) has a unique strong solution $u \in C^{(2)}(p)$, where $T_{0}$ is a maximal time that is appropriately small relative to $M$. Moreover, if

$$
\begin{equation*}
\sup _{t \in\left[0, T_{0}\right)}\left(\|u\|_{Y^{s, p}}+\|u\|_{X_{\infty}}+\left\|u_{t}\right\|_{Y^{s, p}}+\left\|u_{t}\right\|_{X_{\infty}}\right)<\infty \tag{4.3}
\end{equation*}
$$

then $T_{0}=\infty$.
Proof. First, we are going to prove the existence and the uniqueness of the local strong solution of (1.1)-(1.2) by contraction mapping principle. Consider a map $G$ on $Y(T)$ such that $G(u)$ is the operator defined by

$$
\begin{equation*}
G(u)=G(u)(x, t)=S_{1}(t) \varphi(x)+S_{2}(t) \psi(x)+O(u), \tag{4.4}
\end{equation*}
$$

where

$$
\begin{equation*}
O(u)=\int_{0}^{t} F^{-1}[S(\xi, t-\tau) L(\xi) \hat{f}(u)(\xi, \tau)] d \tau, \quad t \in(0, T) \tag{4.5}
\end{equation*}
$$

From Lemma 4.2 we know that $f(u) \in L_{p}\left(0, T ; Y_{\infty}^{s, p}\right)$ for any $T>0$. From Lemma 4.2 it is easy to see that the map $G$ is well defined for $f \in C^{(2)}\left(X_{0} ; \mathbb{C}\right)$. We put

$$
Q(M ; T)=\left\{u \mid u \in Y(T),\|u\|_{Y(T)} \leq M+1\right\} .
$$

First, by reasoning as in [12] let us prove that the map $G$ has a unique fixed point in $Q(M ; T)$. From Lemma 4.2 it is easy to see that the map $G$ is well defined for $f \in C^{(2)}\left(X_{0} ; \mathbb{C}\right)$. Let

$$
W(u)=F^{-1}[S(\xi, t-\tau) L(\xi) f(u)](x, \tau) .
$$

By assumption (2) of Condition 4.4 and by virtue [10, Theorem 4.3], the function $U(\xi, t-\tau) L(\xi)$ is a Fourier multiplier theorem in $X_{p}$, i.e. if $f(u) \in X_{p}$, then $W(u) \in X_{p}$.

First, by reasoning as in [12] let us prove that the map $G$ has a unique fixed point in $Q(M ; T)$. For this aim, it is sufficient to show that the operator $G$ maps $Q(M ; T)$ into $Q(M ; T)$ and $G: Q(M ; T) \rightarrow Q(M ; T)$ is strictly contractive if $T$ is appropriately small relative to $M$. Consider the function $W(\xi):[0, \infty) \rightarrow[0, \infty)$ defined by

$$
\bar{W}(\sigma)=\max _{|\xi| \leq \sigma}\left\{\left|\bar{W}^{(1)}(\xi)\right|,\left|\bar{W}^{(2)}(\xi)\right|, \ldots,\left|\bar{W}^{([\xi])}(\xi)\right|\right\}, \quad \sigma \geq 0 .
$$

It is clear to see that the function $\bar{W}(\sigma)$ is continuous and nondecreasing on $[0, \infty)$. From Lemma 4.2 we have

$$
\begin{align*}
\|W(u)\|_{Y^{2, p}} \leq & \left\|W^{(1)}(u)\right\|_{X_{\infty}}\|u\|_{X_{p}}+\left\|W^{(1)}(u)\right\|_{X_{\infty}}\|D u\|_{X_{p}} \\
& +C_{0}\left[\left\|W^{(1)}(u)\right\|_{X_{\infty}}\|u\|_{X_{p}}+\cdots+\left\|W^{([s])}(u)\right\|_{X_{\infty}}\|u\|_{X_{\infty}}\left\|D^{[s]} u\right\|_{X_{p}}\right] \\
\leq & 2 C_{0} \bar{W}(M+1)(M+1)\|u\|_{Y^{s, p}} . \tag{4.6}
\end{align*}
$$

By using Theorem 3.4 we obtain from (4.5):

$$
\begin{align*}
\|G(u)\|_{X_{\infty}} & \leq\|\varphi\|_{X_{\infty}}+\|\psi\|_{X_{\infty}}+\int_{0}^{t}\|W(x, \tau, u(\tau))\|_{X_{\infty}}  \tag{4.7}\\
\|G(u)\|_{Y^{2 s p}} & \leq\|\varphi\|_{Y^{s, p}}+\|\psi\|_{Y^{s, p}}+\int_{0}^{t}\|W(x, \tau, u(\tau))\|_{Y^{2, p}} d \tau . \tag{4.8}
\end{align*}
$$

Thus, from (4.6)-(4.8) and Lemma 4.2 we get

$$
\|G(u)\|_{Y(T)} \leq M+T(M+1)\left[1+2 C_{0}(M+1) \bar{f}(M+1)\right] .
$$

If $T$ satisfies

$$
\begin{equation*}
T \leq\left\{(M+1)\left[1+2 C_{0}(M+1) \bar{f}(M+1)\right]\right\}^{-1}, \tag{4.9}
\end{equation*}
$$

then

$$
\|G u\|_{Y(T)} \leq M+1 .
$$

Therefore, if (4.9) holds, then $G$ maps $Q(M ; T)$ into $Q(M ; T)$. Now, we are going to prove that the map $G$ is strictly contractive. Assume $T>0$ and $u_{1}, u_{2} \in Q(M ; T)$ given. We get

$$
G\left(u_{1}\right)-G\left(u_{2}\right)=\int_{0}^{t}\left[W\left(u_{1}\right)(x, \tau)-W\left(u_{2}\right)(x, \tau)\right] d \tau, \quad t \in(0, T) .
$$

By using the assumption (3) and the mean value theorem, we obtain

$$
\begin{aligned}
W\left(u_{1}\right)-W\left(u_{2}\right)= & W^{(1)}\left(u_{2}+\eta_{1}\left(u_{1}-u_{2}\right)\right)\left(u_{1}-u_{2}\right), \\
D\left[W\left(u_{1}\right)-W\left(u_{2}\right)\right]= & W^{(2)}\left(u_{2}+\eta_{2}\left(u_{1}-u_{2}\right)\right)\left(u_{1}-u_{2}\right) D_{\tilde{\xi}} u_{1}+W^{(1)}\left(u_{2}\right)\left(D u_{1}-D_{\tilde{\xi}} u_{2}\right), \\
D^{2}\left[\hat{f}\left(u_{1}\right)-\hat{f}\left(u_{2}\right)\right]= & W^{(3)}\left(u_{2}+\eta_{3}\left(u_{1}-u_{2}\right)\right)\left(u_{1}-u_{2}\right)\left(D u_{1}\right)^{2} \\
& +W^{(2)}\left(u_{2}\right)\left(D u_{1}-D u_{2}\right)\left(D u_{1}+D u_{2}\right) \\
& +W^{(2)}\left(u_{2}+\eta_{4}\left(u_{1}-u_{2}\right)\right)\left(u_{1}-u_{2}\right) D^{2} u_{1}+W^{(1)}\left(u_{2}\right)\left(D^{2} u_{1}-D^{2} u_{2}\right),
\end{aligned}
$$

where $0<\eta_{i}<1$. Thus, using Hormander's and Nirenberg's inequality, we have

$$
\begin{align*}
\left\|W\left(u_{1}\right)-W\left(u_{2}\right)\right\|_{X_{\infty}} \leq & \bar{W}(M+1)\left\|u_{1}-u_{2}\right\|_{X_{\infty}}  \tag{4.10}\\
\left\|\left(u_{1}\right)-W\left(u_{2}\right)\right\|_{X_{p}} \leq & \bar{W}(M+1)\left\|u_{1}-u_{2}\right\|_{X_{p^{\prime}}} \\
\left\|D\left[W\left(u_{1}\right)-W\left(u_{2}\right)\right]\right\|_{X_{p}} \leq & (M+1) \bar{W}(M+1)\left\|u_{1}-u_{2}\right\|_{X_{\infty}} \\
& +\bar{W}(M+1)\left\|W\left(u_{1}\right)-W\left(u_{2}\right)\right\|_{X_{p}} \tag{4.11}
\end{align*}
$$

$$
\begin{align*}
&\left\|D^{2}\left[W\left(u_{1}\right)-W\left(u_{2}\right)\right]\right\|_{X_{p}} \\
& \leq(M+1) \bar{W}(M+1)\left\|u_{1}-u_{2}\right\|_{X_{\infty}}\left\|D^{2} u_{1}\right\|_{Y^{2, p}}^{2} \\
&+\bar{W}(M+1)\left\|D\left(u_{1}-u_{2}\right)\right\|_{Y^{2}, p}\left\|D\left(u_{1}+u_{2}\right)\right\|_{Y^{2, p}} \\
&+\bar{W}(M+1)\left\|u_{1}-u_{2}\right\|_{X_{\infty}}\left\|D^{2} u_{1}\right\|_{X_{p}}+\bar{W}(M+1)\left\|D\left(u_{1}-u_{2}\right)\right\|_{X_{p}} \\
& \leq C^{2} \bar{W}(M+1)\left\|u_{1}-u_{2}\right\|_{X_{\infty}}\left\|u_{1}\right\|_{X_{\infty}}\left\|D^{2} u_{1}\right\|_{X_{p}} \\
&+C^{2} \bar{W}(M+1)\left\|u_{1}-u_{2}\right\|_{X_{\infty}}\left\|D^{2}\left(u_{1}-u_{2}\right)\right\|_{X_{p}}\left\|u_{1}+u_{2}\right\|_{X_{\infty}}\left\|D^{2}\left(u_{1}+u_{2}\right)\right\|_{X_{p}} \\
&+(M+1) \bar{W}(M+1)\left\|u_{1}-u_{2}\right\|_{X_{\infty}}+\bar{W}(M+1)\left\|D^{2}\left(u_{1}-u_{2}\right)\right\|_{X_{p}} \\
& \leq 3 C^{2}(M+1)^{2} \bar{W}(M+1)\left\|u_{1}-u_{2}\right\|_{X_{\infty}}+2 C^{2}(M+1) \bar{W}(M+1)\left\|D^{2}\left(u_{1}-u_{2}\right)\right\|_{X_{p}} . \tag{4.12}
\end{align*}
$$

In a similar way, we have

$$
\begin{equation*}
\left\|D^{[s]}\left[W\left(u_{1}\right)-W\left(u_{2}\right)\right]\right\|_{X_{p}} \leq C_{1}\left\|u_{1}-u_{2}\right\|_{X_{\infty}}+C_{2}\left\|D^{[s]}\left(u_{1}-u_{2}\right)\right\|_{X_{p}} . \tag{4.13}
\end{equation*}
$$

From (4.10)-(4.13), using Minkowski's inequality for integrals, Fourier multiplier theorem in $X_{p}$ spaces and Young's inequality, we obtain

$$
\begin{aligned}
\left\|G\left(u_{1}\right)-G\left(u_{2}\right)\right\|_{Y(T)} \leq & \int_{0}^{t}\left\|u_{1}-u_{2}\right\|_{X_{\infty}} d \tau+\int_{0}^{t}\left\|u_{1}-u_{2}\right\|_{Y^{s}, p} d \tau \\
& +\int_{0}^{t}\left\|W\left(u_{1}\right)-W\left(u_{2}\right)\right\|_{X_{\infty}} d \tau+\int_{0}^{t}\left\|W\left(u_{1}\right)-W\left(u_{2}\right)\right\|_{Y^{s, p}} d \tau \\
\leq & T\left[1+C_{1}(M+1)^{2} \bar{W}(M+1)\right]\left\|u_{1}-u_{2}\right\|_{Y(T)}
\end{aligned}
$$

where $C_{1}$ is a constant. If $T$ satisfies (4.9) and the following inequality

$$
\begin{equation*}
T \leq \frac{1}{2}\left[1+C_{1}(M+1)^{2} \bar{W}(M+1)\right]^{-1}, \tag{4.14}
\end{equation*}
$$

then

$$
\left\|G u_{1}-G u_{2}\right\|_{Y(T)} \leq \frac{1}{2}\left\|u_{1}-u_{2}\right\|_{Y(T)} .
$$

That is, $G$ is a contractive map. By contraction mapping principle we know that $G(u)$ has a fixed point $u(x, t) \in Q(M ; T)$ that is a solution of (1.1)-(1.2). From (3.6) we get that $u$ is a solution of the following integral equation

$$
u(t, x)=S_{1}(t) \varphi(x)+S_{2}(t) \psi(x)++\int_{0}^{t} W(u)(x, \tau) d \tau, \quad t \in(0, T)
$$

Let us show that this solution is a unique in $Y(T)$. Let $u_{1}, u_{2} \in Y(T)$ be two solutions of the problem (1.1)-(1.2). Then

$$
\begin{equation*}
u_{1}-u_{2}=\int_{0}^{t}\left[W\left(u_{1}\right)(x, \tau)-W\left(u_{2}\right)(x, \tau)\right] d \tau \tag{4.15}
\end{equation*}
$$

By the definition of the space $Y(T)$, we can assume that

$$
\left\|u_{1}\right\|_{X_{\infty}} \leq C_{1}(T), \quad\left\|u_{1}\right\|_{X_{\infty}} \leq C_{1}(T)
$$

Hence, by Minkowski's inequality for integrals and Theorem 3.6 we obtain from (4.15)

$$
\begin{equation*}
\left\|u_{1}-u_{2}\right\|_{Y^{s, p}} \leq C_{2}(T) \int_{0}^{t}\left\|u_{1}-u_{2}\right\|_{Y^{2, p}} d \tau \tag{4.16}
\end{equation*}
$$

From (4.16) and Gronwall's inequality, we have $\left\|u_{1}-u_{2}\right\|_{Y^{s, p}}=0$, i.e. problem (1.1)-(1.2) has a unique solution which belongs to $Y(T)$. That is, we obtain the first part of the assertion. Now, let $\left[0, T_{0}\right)$ be the maximal time interval of existence for $u \in Y\left(T_{0}\right)$. It remains only to show that if (4.3) is satisfied, then $T_{0}=\infty$. Assume contrary that, (4.3) holds and $T_{0}<\infty$. For $T \in\left[0, T_{0}\right)$, we consider the following integral equation

$$
\begin{equation*}
v(x, t)=S_{1}(t) u(x, T)+S_{2}(t) u_{t}(x, T)+\int_{0}^{t} W(v)(x, \tau) d \tau, \quad t \in(0, T) \tag{4.17}
\end{equation*}
$$

By virtue of (4.3), for $T^{\prime}>T$ we have

$$
\sup _{t \in[0, T)}\left(\|u\|_{Y^{s, p}}+\|u\|_{X_{\infty}}+\left\|u_{t}\right\|_{Y^{s, p}}+\left\|u_{t}\right\|_{X_{\infty}}\right)<\infty
$$

By reasoning as in the first part of the theorem and by the contraction mapping principle, there is a $T^{*} \in\left(0, T_{0}\right)$ such that for each $T \in\left[0, T_{0}\right)$ the equation (4.17) has a unique solution $v \in Y\left(T^{*}\right)$. The estimates (4.9) and (4.14) imply that $T^{*}$ can be selected independently of $T \in\left[0, T_{0}\right)$. Set $T=T_{0}-\frac{T^{*}}{2}$ and define

$$
\tilde{u}(x, t)=\left\{\begin{array}{l}
u(x, t), t \in[0, T] \\
v(x, t-T), t \in\left[T, T_{0}+\frac{T^{*}}{2}\right]
\end{array}\right.
$$

By construction $\tilde{u}(x, t)$ is a solution of the problem (1.1)-(1.2) on $\left[T, T_{0}+\frac{T^{*}}{2}\right]$ and in view of local uniqueness, $\tilde{u}(x, t)$ extends $u$. This is against to the maximality of $\left[0, T_{0}\right)$, i.e. we obtain $T_{0}=\infty$.

From [27], we have
Lemma 4.6. Let $s \geq 0, f \in C^{[s]+1}(\mathbb{R})$ with $f(0)=0$. Then for any $u \in Y^{s, p} \cap L^{\infty}$, we have $f(u) \in$ $Y^{s, p} \cap X_{\infty}$. Moreover, there is some constant $A(M)$ depending on $M$ such that for all $u \in Y^{s, p} \cap L^{\infty}$ with $\|u\|_{X_{\infty}} \leq M$,

$$
\left.\|f(u)\|_{Y^{s, p}} \leq C(M) \| u\right) \|_{Y^{s, p}} .
$$

By using Lemma 4.1 and properties of convolution operators we obtain
Corollary 4.7. Let $s \geq 0, f \in C^{[s]+1}(\mathbb{R})$ with $f(0)=0$. Moreover, assume $\Phi \in L^{\infty}\left(\mathbb{R}^{n}\right)$. Then for any $u \in Y^{s, p} \cap L^{\infty}$, we have $f(u) \in Y^{s, p} \cap X_{\infty}$. Moreover, there is some constant $A(M)$ depending on $M$ such that for all $u \in Y^{s, p} \cap L^{\infty}$ with $\|u\|_{X_{\infty}} \leq M$,

$$
\left.\|\Phi * f(u)\|_{Y^{s, p}} \leq C(M) \| u\right) \|_{Y^{s, p}} .
$$

Lemma 4.8. Let $s \geq 0, f \in C^{[s]+1}(\mathbb{R})$. Then for for any $M$ there is some constant $K(M)$ depending on $M$ such that for all $u, v \in Y^{s, p} \cap X_{\infty}$ with $\|u\|_{X_{\infty}} \leq M,\|v\|_{X_{\infty}} \leq M,\|u\|_{Y^{s, p}} \leq M,\|v\|_{Y^{s, p}} \leq M$,

$$
\| f(u)-f\left(v\left\|_{Y^{s, p}} \leq K(M)\right\| u-v\left\|_{Y^{s, p}},\right\| f(u)-f\left(v\left\|_{X_{\infty}} \leq K(M)\right\| u-v \|_{X_{\infty}} .\right.\right.
$$

By reasoning as in [27, Lemma 3.4] and [5, Lemma X 4] we have, respectively
Corollary 4.9. Let $s>\frac{n}{p}, f \in C^{[s]+1}(\mathbb{R})$. Then for any $M$ there is a constant $K(M)$ depending on $M$ such that for all $u, v \in Y^{s, p}$ with $\|u\|_{Y^{s, p}} \leq M,\|v\|_{Y^{s, p}} \leq M$,

$$
\| f(u)-f\left(v\left\|_{Y^{s, p}} \leq K(M)\right\| u-v \|_{Y^{s, p}} .\right.
$$

Lemma 4.10. If $s>0$, then $Y_{\infty}^{s, p}$ is an algebra. Moreover, for $f, g \in Y_{\infty}^{s, p}$,

$$
\|f g\|_{Y^{s, p}} \leq C\left[\|f\|_{X_{\infty}}+\|g\|_{Y^{s, p}}+\|f\|_{Y^{s, p}}+\|g\|_{X_{\infty}}\right] .
$$

By using Corollary 4.7 and Lemma 4.10 we obtain
Lemma 4.11. Let $s \geq 0, f \in C^{[s]+1}(\mathbb{R})$ and $f(u)=O\left(|u|^{\gamma+1}\right)$ for $u \rightarrow 0, \gamma \geq 1$ be a positive integer. If $u \in Y_{\infty}^{s, p}$ and $\|u\|_{X_{\infty}} \leq M$, then

$$
\begin{aligned}
\|f(u)\|_{Y^{s, p}} & \leq C(M)\left[\|u\|_{Y^{s, p}}\|u\|_{X_{\infty}}^{\gamma}\right] \\
\|f(u)\|_{X_{1}} & \leq C(M)\|u\|_{X_{p}}^{p}\|u\|_{X_{\infty}}^{\gamma-1} .
\end{aligned}
$$

The solution in Theorems $4.2-4.4$ can be extended to a maximal interval $\left[0, T_{\max }\right)$, where finite $T_{\text {max }}$ is characterized by the blow-up condition

$$
\limsup _{T \rightarrow T_{\max }}\|u\|_{Y^{s, p}(A ; E)}=\infty .
$$

Lemma 4.12. Let the Condition 4.4 hold and $u$ be a solution of (1.1)-(1.2). Then there is a global solution if for any $T<\infty$, we have

$$
\begin{equation*}
\sup _{t \in[0, T]}\left(\|u\|_{Y_{\infty}^{s, p}}+\left\|u_{t}\right\|_{Y_{\infty}^{s, p}}\right)<\infty \tag{4.18}
\end{equation*}
$$

Proof. Indeed, by reasoning as in the second part of the proof of Theorem 4.5, by using a continuation of local solution of (1.1)-(1.2) and assuming contrary that, (4.18) holds and $T_{0}<$ $\infty$, then we obtain contradiction, i.e. we get $T_{0}=T_{\max }=\infty$.

## 5 Conservation of energy and global existence.

Consider the problem for $p=2$. Let us denote $Y^{s, 2}$ by $W^{s}$. We prove the following results.
Condition 5.1. Assume the Condition 4.4 holds for $p=2$. Let $L_{0}=L_{1}=L_{2}=-L$ and $L$ be a negative symmetric operator in $L_{2}\left(\mathbb{R}^{n}\right)$. Suppose $(I-L)^{-1}, A=L(I-L)^{-1}$ are bounded in $L_{2}\left(\mathbb{R}^{n}\right)$ and assume

$$
\psi \in L_{2}\left(\mathbb{R}^{n}\right), \quad(A u, u) \in L^{2}\left(\mathbb{R}^{n}\right), \quad \Phi(\cdot) \in L^{1}\left(\mathbb{R}^{n}\right),
$$

where $(u, v)$ denotes the inner product in $L_{2}\left(\mathbb{R}^{n}\right)$.

Let

$$
F(u)=A[f(u)-u], \Phi(\eta)=\int_{0}^{\eta} F(\sigma) d \sigma
$$

Remark 5.2. Note that if $-L$ is self-adjoint positive operator in $L_{2}\left(\mathbb{R}^{n}\right)$, then the operators $(I-L)^{-1}, A$ are bounded in $L_{2}\left(\mathbb{R}^{n}\right)$.

Lemma 5.3. Let the Condition 4.4 hold and let $u \in C^{(2)}\left([0, T] ; W^{s}\right)$ be solution of (1.1)-(1.2) for any $t \in[0, T)$. Then the energy

$$
\begin{equation*}
E(t)=\left\|u_{t}\right\|^{2}+2 \int_{\mathbb{R}^{n}} \Phi(u) d x \tag{5.1}
\end{equation*}
$$

is constant.

Proof. By use of (1.1) and in view of Condition 4.4, it follows from straightforward calculation that

$$
\frac{d}{d t} E(t)=2\left(u_{t t}, u_{t}\right)+2 \int_{\mathbb{R}^{n}} \Phi_{u}(u) u_{t} d x=2\left(u_{t t}+A u-A f(u), u_{t}\right)=0
$$

Hence, we obtain the assertion.

By using the above lemmas we obtain the following results.
Theorem 5.4. Assume the Condition 5.1 is satisfied and $\varphi, \psi \in Y_{\infty}^{s, 2}$. Moreover, there is some $k>0$ so that

$$
\begin{equation*}
\Phi(s) \geq-k|s|^{2}, \quad \text { for all } s \in \mathbb{R} \tag{5.2}
\end{equation*}
$$

Then there is some $T>0$ such that problem (1.1)-(1.2) has a global solution

$$
u \in C^{(2)}\left([0, T] ; W^{s}\right)
$$

Proof. Since $r>2+\frac{n}{2}$, by Theorem 4.5 we get local existence in $u \in C^{(2)}\left([0, T] ; W^{s}\right)$ for some $T>0$. Assume that $u$ exists on $[0, T)$. By assumption (5.2), we obtain

$$
\begin{equation*}
E(t)=\left\|u_{t}\right\|^{2}+2 \int_{\mathbb{R}^{n}} \Phi(u) d x \leq E(0)+2 k\|u(\cdot, t)\|^{2} \tag{5.3}
\end{equation*}
$$

for all $t \in[0, T)$. By properties of norms in Hilbert spaces and by the Cauchy-Schwarz inequality, from (5.3) we get

$$
\begin{aligned}
\frac{d}{d t}\|u(\cdot, t)\|_{W^{s}}^{2} & \leq 2\left\|u_{t}(\cdot, t)\right\|_{W^{s}}\|u(\cdot, t)\|_{W^{s}} \\
& \leq\left\|u_{t}(\cdot, t)\right\|_{W^{s}}^{2}+\|u(\cdot, t)\|_{W^{s}}^{2} \leq C E(0)+(2 C k+1)\|u(t)\|_{W^{s}}^{2}
\end{aligned}
$$

Gronwall's lemma implies that $\|u(\cdot, t)\|_{W^{s}}$ is bounded in $[0, T)$. But, since $s>\frac{n}{2}$, we conclude that $\|u(t)\|_{L^{\infty}}$ also is bounded in $[0, T)$. By Lemma 4.12 this implies a global solution.

## 6 Blow up in finite time

We will use the following lemma to prove blow up in finite time.

Lemma 6.1 ([11]). Suppose $H(t), t \geq 0$ is a positive, twice differentiable function satisfying

$$
H^{(2)} H-(1+v)\left(H^{(1)}\right)^{2} \geq 0 \quad \text { for } v>0
$$

If $H(0)>0$ and $H^{(1)}(0)>0$, then $H(t) \rightarrow \infty$ when $t \rightarrow t_{1}$ for some

$$
t_{1} \leq H(0)\left[v H^{(1)}(0)\right]^{-1}
$$

We rewrite the energy identity as

$$
E(t)=\left\|u_{t}\right\|^{2}+2 \int_{\mathbb{R}^{n}} \Phi(u) d x=E(0)
$$

where

$$
\begin{equation*}
\Phi(\eta)=\int_{0}^{\eta} F(\sigma) d \sigma, \quad A=L(I-L)^{-1} . \tag{6.1}
\end{equation*}
$$

We prove here the following result.
Theorem 6.2. Assume the Condition 5.1 is satisfied and $\varphi, \psi \in Y_{\infty}^{s, 2}$. Let $u \in C^{(2)}\left([0, T] ; W^{s}\right)$ be solution of (1.1)-(1.2) for any $t \in[0, T)$. Suppose there are some positive numbers $v, t_{0}$ and $b$ such that

$$
\begin{equation*}
\sigma F(\sigma) \leq 2(1+2 v) \Phi(\sigma) \quad \text { for all } \sigma \in \mathbb{R} \tag{6.2}
\end{equation*}
$$

and

$$
\begin{equation*}
E(0)=\left\|u_{t}\right\|^{2}+2 \int_{\mathbb{R}^{n}} \Phi(u) d x<0 \tag{6.3}
\end{equation*}
$$

Then the solution $u$ of the problem (1.1)-(1.2) blows up in finite time.
Proof. Assume that there is a global solution. Let

$$
H(t)=\|u\|^{2}+b\left(t+t_{0}\right)^{2} .
$$

for some positive $b$ and $t_{0}$ that will be determined later. We have

$$
\begin{align*}
& H^{(1)}(t)=2\left(u, u_{t}\right)+2 b\left(t+t_{0}\right), \\
& H^{(2)}(t)=2\left\|u_{t}\right\|^{2}+2\left(u, u_{t t}\right)+2 b . \tag{6.4}
\end{align*}
$$

Hence, from (1.1)we get

$$
\begin{equation*}
\left(u, u_{t t}\right)=-(u, A F(u))=-\int_{\mathbb{R}^{n}} u A F(u) d x . \tag{6.5}
\end{equation*}
$$

From (6.2)-(6.3) and (6.5) we deduced

$$
\begin{equation*}
\left(u, u_{t t}\right) \geq-2(1+v) \int_{\mathbb{R}^{n}} \Phi(u) d x=2(1+v)\left[\left\|u_{t}\right\|^{2}-E(0)\right] . \tag{6.6}
\end{equation*}
$$

From (6.4) and (6.6), we obtain

$$
\begin{equation*}
H^{(2)}(t) \geq 2\left\|u_{t}\right\|^{2}+2(1+v)\left[E(0)-\left\|u_{t}\right\|^{2}\right]+2 b . \tag{6.7}
\end{equation*}
$$

On the other hand, in view of the Cauchy-Schwarz inequality, we have

$$
\begin{align*}
\left(H^{(1)}(t)\right)^{2} & =\left[2\left(u, u_{t}\right)+2 b\left(t+t_{0}\right)\right]^{2} \\
& \leq 4\left[\|u\|^{2}\left\|u_{t}\right\|^{2}+b^{2}\left(t+t_{0}\right)^{2}\left(\|u\|^{2}+\left\|u_{t}\right\|^{2}\right)\right]+4 b^{2}\left(t+t_{0}\right)^{2} \tag{6.8}
\end{align*}
$$

Hence, combining (6.4), (6.7) and (6.8) we obtain

$$
\begin{aligned}
& H^{(2)} H-(1+v)\left(H^{(1)}\right)^{2} \\
& \geq {\left[2\left\|u_{t}\right\|^{2}+4(1+v)\left\|u_{t}\right\|^{2}-2(1+2 v) E(0)+2 b\right]\left[\|u\|^{2}+b\left(t+t_{0}\right)^{2}\right] } \\
&-4(1+v)\left[\|u\|^{2}\left\|u_{t}\right\|^{2}+b^{2}\left(t+t_{0}\right)^{2}\left(\|u\|^{2}+\left\|u_{t}\right\|^{2}\right)\right]+4 b^{2}\left(t+t_{0}\right)^{2} \\
&=-2(1+2 v)[b+E(0)] H(t) .
\end{aligned}
$$

Hence, if we choose $b \leq-E(0)$, this gives

$$
H^{(2)} H-(1+v)\left(H^{(1)}\right)^{2} \geq 0
$$

Moreover,

$$
H^{(1)}(0)=2(\varphi, \psi)+2 b\left(t_{0}\right) \geq 0
$$

for sufficiently large $t_{0}$. According to Lemma 6.1, this implies that $H(t)$, and thus $\|u(t)\|^{2}$ blows up in finite time contradicting the assumption that the global solution exists.

## 7 Applications

In this section we give some application of Theorem 4.5.

1. Let

$$
L_{0}=L_{1}=L_{2}=A_{1}=\sum_{|\alpha|=2} a_{\alpha} D^{\alpha}
$$

where $a_{\alpha}$ are real numbers.
Then the problem (1.1)-(1.2) is reduced to the Cauchy problem for the following Boussinesq equation

$$
\begin{align*}
u_{t t}+A_{1} u_{t t}+A_{1} u & =A_{1} f(x, t, u), \quad x \in \mathbb{R}^{2}, \quad t \in(0, T), \\
u(x, 0) & =\varphi(x), \quad u_{t}(x, 0)=\psi(x), \tag{7.1}
\end{align*}
$$

Let

$$
X_{p}=L_{p}\left(\mathbb{R}^{2}\right), \quad 1 \leq p \leq \infty, \quad Y^{s, p}=L_{p}^{s}\left(\mathbb{R}^{2}\right)
$$

Assumption 7.1. Assume that $A_{2}(\xi) \neq 0, A_{2}(\xi) \neq-1$ for all $\xi=\left(\xi_{1}, \xi_{2}\right) \in \mathbb{R}^{2}$. Let $\varphi$, $\psi \in Y^{s, p} \cap X_{1}$ and

$$
M=\|\varphi\|_{Y^{s, p}}+\|\varphi\|_{X_{1}}+\|\psi\|_{Y^{s, p}}+\|\psi\|_{X_{1}}
$$

It is not hard to see that Assumtion 7.1 implies Condition 3.1. Hence, from Theorem 4.5 we obtain:

Theorem 7.2. Suppose that the Assumption 7.1 holds. Let $s>2\left(\frac{2}{q}+\frac{1}{p}\right)$ for $p \in[1, \infty]$ and for a $q \in[1,2]$. Assume that the function $u \rightarrow f(x, t, u): \mathbb{R}^{2} \times[0, T] \times B_{p}^{s\left(1-\frac{1}{2 p}\right)}\left(\mathbb{R}^{2}\right) \rightarrow L_{p}\left(\mathbb{R}^{2}\right)$ is measurable in $(x, t) \in \mathbb{R}^{2} \times[0, T]$ for $u \in B_{p}^{s\left(1-\frac{1}{2 p}\right)}\left(\mathbb{R}^{2}\right)$. Moreover, $f(x, t, u)$ is continuous in $u \in B_{p}^{s\left(1-\frac{1}{2 p}\right)}\left(\mathbb{R}^{2}\right)$ and

$$
f(x, t, u) \in C^{(3)}\left(B_{p}^{s\left(1-\frac{1}{2 p}\right)}\left(\mathbb{R}^{2}\right)\right)
$$

uniformly with respect to $(x, t) \in \mathbb{R}^{2} \times[0, T]$. Then for $\varphi, \psi \in Y^{s, p} \cap X_{1}$ problem (7.1) has a unique local strong solution $u \in C^{(2)}\left(\left[0, T_{0}\right) ; Y_{\infty}^{s, p}\right)$, where $T_{0}$ is a maximal time interval that is appropriately small relative to $M$. Moreover, if

$$
\sup _{t \in\left[0, T_{0}\right)}\left(\|u\|_{Y_{s, p}}+\|u\|_{X_{\infty}}+\left\|u_{t}\right\|_{Y^{s, p}}+\left\|u_{t}\right\|_{X_{\infty}}\right)<\infty
$$

then $T_{0}=\infty$.
2. Let

$$
L_{0}=L_{1}=L_{2}=A_{2}=\sum_{|\alpha|=4} a_{\alpha} D^{\alpha},
$$

where $a_{\alpha}$ are real numbers, $\alpha=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right), \alpha_{k}$ are natural numbers and

$$
|\alpha|=\sum_{k=1}^{3} \alpha_{k} .
$$

Then the problem (1.1)-(1.2) is reduced to the Cauchy problem for the following Boussinesq equation

$$
\begin{align*}
u_{t t}+A_{2} u_{t t}+A_{2} u & =A_{2} f(x, t, u), \quad x \in \mathbb{R}^{3}, \quad t \in(0, T),  \tag{7.2}\\
u(x, 0) & =\varphi(x), \quad u_{t}(x, 0)=\psi(x) .
\end{align*}
$$

where

$$
\varphi, \quad \psi \in L_{p}^{s}\left(\mathbb{R}^{3}\right), \quad s>\frac{3}{p}, \quad p \in[1, \infty] .
$$

Assumption 7.3. Assume that $A_{2}(\xi) \neq 0, A_{2}(\xi) \neq-1$ for all $\xi=\left(\xi_{1}, \xi_{2}, \xi_{3}\right) \in \mathbb{R}^{3}$. Let $\varphi$, $\psi \in Y^{s, p} \cap X_{1}$ and

$$
M=\|\varphi\|_{Y^{s, p}}+\|\varphi\|_{X_{1}}+\|\psi\|_{Y^{s, p}}+\|\psi\|_{X_{1}} .
$$

It is clear to see that if Assumption 7.1 holds, then Condition 3.1 is satisfied.
Let

$$
X_{p}=L_{p}\left(\mathbb{R}^{3}\right), 1 \leq p \leq \infty, \quad Y^{s, p}=L_{p}^{s}\left(\mathbb{R}^{3}\right) .
$$

Hence, from Theorem 4.5 we obtain:
Theorem 7.4. Suppose that the Assumption 7.3 holds. Let $s>3\left(\frac{2}{q}+\frac{1}{p}\right)$ for $p \in[1, \infty]$ and for a $q \in[1,2]$. Suppose that the function $u \rightarrow f(x, t, u): \mathbb{R}^{3} \times[0, T] \times B_{p}^{s\left(1-\frac{1}{2 p}\right)}\left(\mathbb{R}^{3}\right) \rightarrow L_{p}\left(\mathbb{R}^{3}\right)$ is measurable in $(x, t) \in \mathbb{R}^{3} \times[0, T]$ for $u \in B_{p}^{s\left(1-\frac{1}{2 p}\right)}\left(\mathbb{R}^{3}\right)$. Moreover, $f(x, t, u)$ is continuous in $u \in B_{p}^{s\left(1-\frac{1}{2 p}\right)}\left(\mathbb{R}^{3}\right)$ and

$$
f(x, t, u) \in C^{(3)}\left(B_{p}^{s\left(1-\frac{1}{2 p}\right)}\left(\mathbb{R}^{3}\right)\right)
$$

uniformly with respect to $(x, t) \in \mathbb{R}^{3} \times[0, T]$. Then problem (7.2) has a unique local strong solution

$$
u \in C^{(2)}\left(\left[0, T_{0}\right) ; Y_{\infty}^{s, p}\right),
$$

where $T_{0}$ is a maximal time interval that is appropriately small relative to $M$. Moreover, if

$$
\sup _{t \in\left[0, T_{0}\right)}\left(\|u\|_{Y^{s}, p}+\|u\|_{X_{\infty}}+\left\|u_{t}\right\|_{Y^{s, p}}+\left\|u_{t}\right\|_{X_{\infty}}\right)<\infty
$$

then $T_{0}=\infty$.
3. Let

$$
L_{0}=\sum_{|\alpha|=4} a_{0 \alpha} D^{\alpha}, \quad L_{1}=\sum_{|\alpha|=2} a_{1 \alpha} D^{\alpha}, \quad L_{2}=\sum_{|\alpha|=4} a_{2 \alpha} D^{\alpha}
$$

where $a_{\alpha i}$ are real numbers, $\alpha=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right), \alpha_{k}$ are natural numbers and

$$
|\alpha|=\sum_{k=1}^{3} \alpha_{k}
$$

Then the problem (1.1)-(1.2) is reduced to Cauchy problem for the following Boussinesq equation

$$
\begin{align*}
u_{t t}+L_{0} u_{t t}+L_{1} u & =L_{2} f(x, t, u), \quad x \in \mathbb{R}^{3}, \quad t \in(0, T),  \tag{7.3}\\
u(x, 0) & =\varphi(x), \quad u_{t}(x, 0)=\psi(x)
\end{align*}
$$

where

$$
\varphi, \psi \in L^{s, p}\left(\mathbb{R}^{3}\right), \quad p \in[1, \infty]
$$

Hence, from Theorem 4.5 we obtain:
Theorem 7.5. Assume that the Condition 3.1 is satisfied. Let $\varphi, \psi \in Y^{s, p} \cap X_{1}$ and

$$
M=\|\varphi\|_{Y^{s, p}}+\|\varphi\|_{X_{1}}+\|\psi\|_{Y^{s, p}}+\|\psi\|_{X_{1}}
$$

for $s>3\left(\frac{2}{q}+\frac{1}{p}\right)+v, p \in[1, \infty]$ and for a $q \in[1,2]$. Suppose that the function $u \rightarrow f(x, t, u)$ : $\mathbb{R}^{3} \times[0, T] \times B_{p}^{s\left(1-\frac{1}{2 p}\right)}\left(\mathbb{R}^{3}\right) \rightarrow L_{p}\left(\mathbb{R}^{3}\right)$ is measurable in $(x, t) \in \mathbb{R}^{3} \times[0, T]$ for $u \in B_{p}^{s\left(1-\frac{1}{2 p}\right)}\left(\mathbb{R}^{3}\right)$. Moreover, $f(x, t, u)$ is continuous in $u \in B_{p}^{s\left(1-\frac{1}{2 p}\right)}\left(\mathbb{R}^{3}\right)$ and

$$
f(x, t, u) \in C^{(3)}\left(B_{p}^{s\left(1-\frac{1}{2 p}\right)}\left(\mathbb{R}^{3}\right)\right)
$$

uniformly with respect to $(x, t) \in \mathbb{R}^{3} \times[0, T]$. Then problem (7.3) has a unique strong solution

$$
u \in C^{(2)}\left(\left[0, T_{0}\right) ; Y_{\infty}^{s, p}\right)
$$

where $T_{0}$ is a maximal time interval that is appropriately small relative to $M$. Moreover, if

$$
\sup _{t \in\left[0, T_{0}\right)}\left(\|u\|_{Y^{s, p}}+\|u\|_{X_{\infty}}+\left\|u_{t}\right\|_{Y^{s, p}}+\left\|u_{t}\right\|_{X_{\infty}}\right)<\infty
$$

then $T_{0}=\infty$.

## References

[1] R. Agarwal, S. Gala, M. A. Ragusa, Regularity criterion in weak spaces to Boussinesq equations, Mathematics 8(2020), 920-932. https://doi.org/10.1002/mma. 1367
[2] S. Benbernou, S. Gala, M. A. Ragusa, On the regularity criteria for the 3D magnetohydrodynamic equations via two components in terms of BMO space, Math. Meth. Appl. Sci. 37(2014), No. 15, 2320-2325. https://doi.org/10.1002/mma. 2981
[3] J. L. Bona, R. L. Sachs, Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation, Comm. Math. Phys. 118(1988), 15-29. MR954673
[4] A. Clarkson, R. J. LeVeque, R. Saxton, Solitary-wave interactions in elastic rods, Stud. Appl. Math. 75(1986), 95-122. https://doi.org/10.1002/sapm198675295
[5] R. Coifman, Y. Meyer, Wavelets. Calderón-Zygmund and multilinear operators, Cambridge University Press, 1997. MR1456993
[6] A. Constantin, L. Molinet, The initial value problem for a generalized Boussinesq equation, Differential Integral Equations 15(2002), 1061-1072. MR1919762; Zbl 1161.35445
[7] X. Cur, The regularity criterion for weak solutions to the $n$-dimensional Boussinesq system, Bound. Value Probl. 2017, Paper No. 44, 12 pp. https://doi.org/10.1186/s13661-017-0778-9
[8] Z. Dai, X. Wang, L. Zhang, W. Hou, Blow-up criterion of weak solutions for the 3D Boussinesq equations, J. Funct. Spaces 6(2015), Article ID 303025. https://doi.org/10. 1155/2015/303025
[9] S. Gala, E. Galakhov, M. Ragusa, O. Salieva, Beale-Kato-Majda regularity criterion of smooth solutions for the Hall-MHD equations with zero viscosity, Bull. Braz. Math. Soc. (N.S.), published online, 2021. https://doi.org/10.1007/s00574-021-00256-7
[10] M. Girardi, L. Weis, Operator-valued Fourier multiplier theorems on Besov spaces, Math. Nachr. 251(2003), 34-51. https://doi.org//10.1002/mana. 200310029
[11] V. K. Kalantarov, O. A. Ladyzhenskaya, The occurrence of collapse for quasilinear equation of parabolic and hyperbolic types, Journal of Soviet Mathematics 10(1978), 53-70. https://doi.org/10.1007/BF01109723
[12] T. Kano, T. Nishida, A mathematical justification for Korteweg-de Vries equation and Boussinesq equation of water surface waves, Osaka J. Math. 23(1986), 389-413. https: //doi.org/ojm/1200779331
[13] S. Klainerman, Global existence for nonlinear wave equations, Comm. Pure Appl. Math. 33(1980), 43-101. https://doi.org//10.1002/cpa.3160330104
[14] N. Kutev, N. Kolкovsкa, M. Dimova, Global existence of Cauchy problem for Boussinesq paradigm equation, Comput. Math. Appl. 65(2013), 500-511. https://doi.org/10. 1016/j.camwa.2012.05.024; MR3008555
[15] J. Li, H. Shang, J. Wu, X. Xu, Z. Ye, Regularity criteria for the 2D Boussinesq equations with supercritical dissipation, Commun. Math. Sci. International Press 14(2016), 1999-2022. https://doi.org/10.4310/CMS. 2016.v14.n7.a10
[16] F. Linares, Global existence of small solutions for a generalized Boussinesq equation, J. Differential Equations 106(1993), 257-293. https://doi.org/10.1006/jdeq.1993.1108; MR1251854
[17] Y. Liu, Instability and blow-up of solutions to a generalized Boussinesq equation, SIAM J. Math. Anal. 26(1995), 1527-1546. https://doi.org/10.1137/S0036141093258094; MR1356458
[18] V. Maкнankov, Dynamics of classical solitons (in non-integrable systems), Phys. Rep. 35(1978), 1-128. https://doi.org/10.1016/0370-1573(78) 90074-1
[19] L. Nirenberg, On elliptic partial differential equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 13(1959), 115-162. MR109940
[20] S. Piskarev, S.-Y. Shaw, Multiplicative perturbations of semigroups and applications to step responses and cumulative outputs, J. Funct. Anal. 128(1995), 315-340. https://doi. org/10.1006/jfan.1995.1034; MR1319959
[21] A. Razani, Subsonic detonation waves in porous media, Phys. Scr. 94(2019), No. 085209, 6 pp. https://doi.org/0.1088/1402-4896/ab029b
[22] Y. Ren, F. Xu, A logarithmically improved blow-up criterion of the 3D incompressible Boussinesq equations, Nonlin. Anal. Differ. Equ. 8(2020), 17-24. https://doi.org//10. 12988/nade. 2020.91018
[23] P. Rosenau, Dynamics of nonlinear mass-spring chains near continuum limit, Phys. Lett. 118A(1986), 1061-1072, 222-227. https ://doi.org/10.1016/0375-9601 (86) 90170-2
[24] C. D. Sogge, Lectures on nonlinear wave equations, 2nd edition, International Press, Cambridge, MA, 2008. MR2455195
[25] H. Triebel, Interpolation theory, function spaces, differential operators, North-Holland, Amsterdam, 1978.
[26] H. Triebel, Fractals and spectra, Birkhauser Verlag, Related to Fourier analysis and function spaces, Basel, 1997.
[27] S. Wang, G. Chen, Small amplitude solutions of the generalized IMBq equation, J. Math. Anal. Appl. 274(2002), 846-866. https://doi.org/10.1016/S0022-247X (02) 004018; MR1936734
[28] S. Wang, G. Chen, The Cauchy problem for the generalized $\operatorname{IMBq}$ equation in $W^{s, p}\left(\mathbb{R}^{n}\right)$, J. Math. Anal. Appl. 266(2002), 38-54. https://doi.org/10.1006/jmaa. 2001. 7670; MR1876769
[29] Y. Wanga, H. Gaob, Regularity criterion for weak solutions to the 3D Boussinesq equations, Science Asia 38(2012), 196-200. https://doi.org/10.2306/scienceasia15131874.2012.38.196
[30] G. B. Whitham, Linear and nonlinear waves, Wiley-Interscience, New York, 1975. https: //doi.org/10.1002/9781118032954
[31] Z. Yang, X. Wang, Blowup of solutions for improved Boussinesq type equation, J. Math. Anal. Appl. 278(2003), No. 2, 335-353. https://doi.org/10.1016/S0022-247X (02) 005164
[32] N. J. Zabusky, Nonlinear partial differential equations, Academic Press, New York, 1967. https://doi.org/10.1016/B978-1-4831-9647-3.50019-4
[33] Y. Zhuan, A Logarithmical improved regularity criterion of smooth solutions for the 3D Boussinesq equations, Osaka J. Math. 3(2016), 417-423. MR3492806

# Multiple positive solutions for a logarithmic Schrödinger-Poisson system with singular nonlinearity 

Linyan Peng, Hongmin Suo ${ }^{\boxtimes}$, Deke Wu, Hongxi Feng and Chunyu Lei<br>School of Sciences, Guizhou Minzu University, Guiyang, Guizhou, 550025, P. R. China

Received 5 February 2021, appeared 20 December 2021
Communicated by Dimitri Mugnai

Abstract. In this article, we devote ourselves to investigate the following logarithmic Schrödinger-Poisson systems with singular nonlinearity

$$
\begin{cases}-\Delta u+\phi u=|u|^{p-2} u \log |u|+\frac{\lambda}{u^{\gamma},} & \text { in } \Omega \\ -\Delta \phi=u^{2}, & \text { in } \Omega \\ u=\phi=0, & \text { on } \partial \Omega\end{cases}
$$

where $\Omega \subset \mathbb{R}^{3}$ is a smooth bounded domain with boundary $\partial \Omega, 0<\gamma<1, p \in$ $(4,6)$ and $\lambda>0$ is a real parameter. By using the critical point theory for nonsmooth functional and variational method, the existence and multiplicity of positive solutions are established.
Keywords: logarithmic Schrödinger-Poisson system, multiplicity, singularity, positive solutions.
2020 Mathematics Subject Classification: 35A15, 35A20, 35J10.

## 1 Introduction and main result

In this paper, we consider the following logarithmic Schrödinger-Poisson system with singular term

$$
\begin{cases}-\Delta u+\phi u=|u|^{p-2} u \log |u|+\frac{\lambda}{u^{\gamma},} & \text { in } \Omega,  \tag{1.1}\\ -\Delta \phi=u^{2}, & \text { in } \Omega, \\ u=\phi=0, & \text { on } \partial \Omega,\end{cases}
$$

where $\Omega \subset \mathbb{R}^{3}$ is a smooth bounded domain with boundary $\partial \Omega, 0<\gamma<1, p \in(4,6)$ and $\lambda>0$ is a real parameter.

[^52]Due to the wide applications in physics and other applied sciences, partial differential equations with logarithmic nonlinearity have attracted much attention in recent years, the logarithmic Schrödinger equation given by

$$
\begin{equation*}
-i \frac{\partial \Psi}{\partial t}=-\Delta \Psi+(W(x)+W) \Psi-|\Psi|^{p-1} \log |\Psi|, \quad \Psi:[0, \infty) \times \mathbb{R}^{N} \rightarrow \mathbb{C}, \quad N \geq 1 \tag{1.2}
\end{equation*}
$$

has also received a special attention. This class of equation has some important physics applications, such as quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum system, effective quantum gravity and Bose-Einstein condensation, for more details see [28] and the references therein. For the elliptic equations with logarithmic nonlinearity, we can refer to $[6,10-12,17,19,23,25]$ and the references therein. The authors in [10] considered the following logarithmic elliptic equations of the type

$$
\left\{\begin{array}{l}
-\Delta u+u=u \log u^{2}, \quad \text { in } \mathbb{R}^{\mathrm{N}}, \\
u \in H^{1}\left(\mathbb{R}^{N}\right) .
\end{array}\right.
$$

The authors obtained solutions for this equation by applying the non-smooth critical point theory. In addition, Chao et al. in [11] considered the following Schrödinger equation with logarithmic nonlinearity

$$
-\Delta u+V(x) u=u \log u^{2}, \quad x \in \mathbb{R}^{N},
$$

where the potential $V$ is continuous and satisfies the condition $\lim _{|x| \rightarrow \infty} V(x)=V_{\infty}$. When the potential is coercive, the author obtained infinitely many solutions by adapting some arguments of the Fountain theorem, and in the case of bounded potential obtained a ground state solution.

Returning to the singular Schrödinger-Poisson over bounded or unbounded domains, many papers have studied the following problem

$$
\begin{cases}-\Delta u+u+q \phi f(u)=g(x, u), & \text { in } \mathbb{R}^{3},  \tag{1.3}\\ -\Delta \phi=2 F(u), & \text { in } \mathbb{R}^{3} .\end{cases}
$$

Under various assumptions of nonlocal term $f$ and nonlinear term $g$, the existence, uniqueness and multiplicity of solutions to system (1.3) has been studied by using the modern variational methods, see [1,8,13-15,20-22,24,26,27].

There are also many references which investigated Schrödinger-Poisson system in bounded domain, see $[2,3,9]$. It is worth mentioning that the author in [27] considered the following singular Schrödinger-Poisson system

$$
\begin{cases}-\Delta u+\eta \phi u=\mu u^{-\gamma}, & \text { in } \Omega, \\ -\Delta \phi=u^{2}, & \text { in } \Omega, \\ u>0, & \text { in } \Omega, \\ u=\phi=0, & \text { on } \partial \Omega,\end{cases}
$$

where $\Omega \subset \mathbb{R}^{3}$ is a smooth bounded domain with boundary $\partial \Omega, \eta= \pm 1, \gamma \in(0,1)$ is a constant, $\mu>0$ is a parameter and he proved the existence and uniqueness result for $\eta=1$ and multiplicity of solutions for $\eta=-1$ and $\mu>0$ small enough by using Nehari manifold.

In [16] Liu et al. has considered the following singular $p$-Laplacian equation in $\mathbb{R}^{N}$

$$
\left\{\begin{array}{l}
\Delta_{p} u+f(x) u^{-\alpha}+\lambda g(x) u^{\beta}=0 \\
u \geq 0, \quad x \in \mathbb{R}^{N}
\end{array}\right.
$$

where $\Delta_{p} u=\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$ is the $p$-Laplacian operator, $N \geq 3,1<p<N, \lambda>0,0<$ $\alpha<1$, $\max (p, 2)<\beta+1<p^{*}=\frac{N p}{N-p}$. The existence and multiplicity of positive solutions for this equation are considered under some suitable condition by the critical point theory for non-smooth functional and supper-and sub-solutions method.

On the one hand, we find that most of Schrödinger-Poisson system contain only power terms and not the logarithmic terms $|t|^{p-2} t \log |t|$. This arouses the research interest of the Schrödinger-Poisson systems with logarithmic nonlinear term. On the other hand, it is noted that the logarithmic nonlinear term does not satisfy the monotonicity condition and (AR) condition, which makes system (1.1) more complex and challenging than the case without the logarithmic nonlinear term. Remarkably, the singular term leads to the non-differentiability of the energy functional corresponding to the system (1.1) on $H_{0}^{1}(\Omega)$, which make the study of system (1.1) particularly interesting. To our knowledge, the logarithmic Schrödinger-Poisson system with singular term has not been studied. Motivated by the above references, in this paper, we consider logarithmic Schrödinger-Poisson system (1.1) with singular term.

Now our main result is as follows:
Theorem 1.1. Assume that $0<\gamma<1, p \in(4,6)$, then there exists $\Lambda_{0}>0$ such that for any $\lambda \in\left(0, \Lambda_{0}\right)$, system (1.1) has at least two pair of different positive solutions.

## 2 Preliminaries

Throughout this paper, we denote the norm of $L^{p}(\Omega)$ by $|\cdot|_{p}=\left(\int_{\Omega}|u|^{p} d x\right)^{\frac{1}{p}}$, where $p \in$ $[1,+\infty)$. Let $H_{0}^{1}(\Omega)$ be the usual Sobolev space with the inner product and the norm $(u, v)=$ $\int_{\Omega}(\nabla u, \nabla v) d x,\|u\|^{2}=\int_{\Omega}|\nabla u|^{2} d x$. We denote by $B_{r}$ (respectively, $\partial B_{r}$ ) the closed ball (respectively, the sphere) of center zero and radius $r . u_{n}^{+}(x)=\max \left\{u_{n}(x), 0\right\}, u_{n}^{-}(x)=\max \left\{-u_{n}(x), 0\right\}$. $C, C_{1}, C_{2}, \ldots$ denote various positive constants, which may vary from line to line. Let $S$ be the best Sobolev constant, namely

$$
S:=\inf _{u \in H_{0}^{1}(\Omega) \backslash\{0\}} \frac{\int_{\Omega}|\nabla u|^{2} d x}{\left(\int_{\Omega}|u|^{6} d x\right)^{1 / 3}} .
$$

With the help of the Lax-Milgram theorem, for any given $u \in H_{0}^{1}(\Omega)$, the Dirichlet boundary problem $-\Delta \phi=u^{2}$ in $\Omega$ has a unique solution $\phi_{u} \in H_{0}^{1}$. Substituting $\phi_{u}$ to the first equation of system (1.1), system (1.1) is transformed into the following equation

$$
\begin{cases}-\Delta u+\phi_{u} u=|u|^{p-2} u \log |u|+\frac{\lambda}{u^{\gamma},} & \text { in } \Omega,  \tag{2.1}\\ u=0, & \text { on } \partial \Omega .\end{cases}
$$

The energy functional corresponding to the equation (2.1) is the following
$J(u)=\frac{1}{2} \int_{\Omega}|\nabla u|^{2} d x+\frac{1}{4} \int_{\Omega} \phi_{u} u^{2} d x+\frac{1}{p^{2}} \int_{\Omega}|u|^{p} d x-\frac{1}{p} \int_{\Omega}|u|^{p} \log |u| d x-\frac{\lambda}{1-\gamma} \int_{\Omega}|u|^{1-\gamma} d x$.

From (1.3) and (1.4) in [25], we have

$$
\begin{equation*}
\lim _{t \rightarrow 0} \frac{t^{p-1} \log |t|}{t}=0 \quad \text { and } \quad \lim _{t \rightarrow \infty} \frac{t^{p-1} \log |t|}{t^{q-1}}=0 \tag{2.2}
\end{equation*}
$$

where $q \in(p, 6)$, and for any $\epsilon>0$, there exists $C_{\epsilon}>0$ such that

$$
\begin{equation*}
|t|^{p-1} \log |t| \leq \epsilon|t|+C_{\epsilon}|t|^{q-1}, \quad \forall t \in \mathbb{R} \backslash\{0\} . \tag{2.3}
\end{equation*}
$$

If a function $u \in H_{0}^{1}(\Omega)$ satisfies

$$
\int_{\Omega}(\nabla u, \nabla \varphi) d x+\int_{\Omega} \phi_{u} u \varphi d x-\int_{\Omega}|u|^{p-1} \varphi \log |u| d x-\lambda \int_{\Omega} \frac{\varphi}{u^{\gamma}} d x=0
$$

for $\varphi \in H_{0}^{1}(\Omega)$, then we say $u$ is a weak solution of (2.1) and $\left(u, \phi_{u}\right)$ is a pair solution of system (1.1).

Before proving Theorem 1.1, we give the following important lemma.
Lemma 2.1 (See $[3,7,18,27])$. For every $u \in H_{0}^{1}(\Omega)$, there exists a unique solution $\phi_{u} \in H_{0}^{1}(\Omega)$ of

$$
\begin{cases}-\Delta \phi=u^{2}, & \text { in } \Omega \\ \phi=0, & \text { on } \partial \Omega\end{cases}
$$

and
(1) $\left\|\phi_{u}\right\|^{2}=\int_{\Omega} \phi_{u} u^{2} d x$;
(2) $\phi_{u} \geq 0$. Moreover, $\phi_{u}>0$ when $u \neq 0$;
(3) For $t \neq 0, \phi_{t u}=t^{2} \phi_{u}$;
(4) Assume that $u_{n} \rightharpoonup u$ in $H_{0}^{1}(\Omega)$, then $\phi_{u_{n}} \rightarrow \phi_{u}$ in $H_{0}^{1}(\Omega)$ and

$$
\int_{\Omega} \phi_{u_{n}} u_{n} v d x \rightarrow \int_{\Omega} \phi_{u} u v d x, \quad \forall v \in H_{0}^{1}(\Omega) ;
$$

(5) $\int_{\Omega} \phi_{u} u^{2} d x=\int_{\Omega}\left|\nabla \phi_{u}\right|^{2} d x \leq C\|u\|^{4}$;
(6) Set $\mathcal{F}(u)=\int_{\Omega} \phi_{u} u^{2} d x$, then $\mathcal{F}(u): H_{0}^{1}(\Omega) \rightarrow H_{0}^{1}(\Omega)$ is $C^{1}$ and

$$
\left\langle\mathcal{F}^{\prime}(u), v\right\rangle=4 \int_{\Omega} \phi_{u} u v d x, \quad \forall v \in H_{0}^{1}(\Omega) ;
$$

(7) For $u, v \in H_{0}^{1}(\Omega), \int_{\Omega}\left(\phi_{u} u-\phi_{v} v\right)(u-v) d x \geq \frac{1}{2}\left\|\phi_{u}-\phi_{v}\right\|^{2}$.

Lemma 2.2 (See [4]). For all $p, a, s>0$, we have the following results:

$$
\begin{equation*}
s^{p} \log (s) \leq \frac{1}{e a} s^{p+a}, \tag{2.4}
\end{equation*}
$$

and by simple calculation, we have

$$
s^{p} \log (s) \geq-\frac{1}{e p}
$$

Proof. We can repeat the proof of [4, Lemma 2], so we omit the detailed proof of (2.4). Next, we will prove that another inequality holds.

Let $h(t)=t^{p} \log t$ for all $t>0$. Clearly, one can obtain that $t_{*}=e^{-\frac{1}{p}}$ is the unique minimum point of function $h$. Thus, $h(t) \geq h\left(t_{*}\right)=-\frac{1}{e p}$ for all $t>0$.

In the following, we first recall some concepts and known results of the critical points theory for continuous functional. Let $(X, d)$ be a complete metric space with metric $d$ and $f: X \rightarrow R$ be a continuous functional in $X$. Denote by $|D f|(u)$ the supremum of $\delta$ in $[0, \infty)$ such that there exist $r>0$, and a continuous map $\sigma: U \times[0, r] \rightarrow X$ satisfying

$$
\begin{cases}f(\sigma(v, t)) \leq f(v)-\delta t, & (v, t) \in U \times[0, r]  \tag{2.5}\\ d(\sigma(v, t), v) \leq t, & (v, t) \in U \times[0, r] .\end{cases}
$$

The extended real number $|D f|(u)$ is called the weak slope of $f$ at $u$, we say that $u \in X$ is a critical point of $f$ if $|D f|(u)=0$, we say that $c \in R$ is a critical value of $f$ if there exists a critical point $u \in X$ of $f$ with $f(u)=c$.

Because of looking for positive solutions of system (1.1), we consider the functional $J$ defined on the closed positive cone $P$ of $H_{0}^{1}(\Omega)$, that is,

$$
P=\left\{u \mid u \in H_{0}^{1}(\Omega), u(x) \geq 0, \text { a.e. } x \in \Omega\right\} .
$$

Lemma 2.3. Assume $|D J|(u)<+\infty$, then for any $v \in P$ there holds

$$
\begin{align*}
\lambda \int_{\Omega} \frac{v-u}{u^{\gamma}} d x \leq & \int_{\Omega} \nabla u \nabla(v-u) d x+\int_{\Omega} \phi_{u} u(v-u) d x-\int_{\Omega}|u|^{p-1}(v-u) \log |u| d x  \tag{2.6}\\
& +|D J|(u)\|v-u\| .
\end{align*}
$$

Proof. We take a similar approach to [16, Lemma 3.1]. Let $|D J|(u)<c, \delta<\frac{1}{2}\|v-u\|, v \in P$ and $v \neq u$. Define the mapping $\sigma: U \times[0, \delta] \rightarrow P$ by

$$
\sigma(z, t)=z+t \frac{v-z}{\|v-z\|},
$$

where $U$ is a neighborhood of $u$. Then $\|\sigma(z, t)-z\|=t$, by (2.5), there exists a pair $(z, t) \in U \times$ $[0, \delta]$ such that $J(\sigma(z, t))>J(z)-c t$. Consequently, we assume that there exists a sequences $\left\{u_{n}\right\} \subset P$ and $\left\{t_{n}\right\} \subset[0, \infty)$, such that $u_{n} \rightarrow u, t_{n} \rightarrow 0^{+}$, and

$$
J\left(u_{n}+t_{n} \frac{v-u_{n}}{\left\|v-u_{n}\right\|}\right) \geq J\left(u_{n}\right)-c t_{n},
$$

that is,

$$
\begin{equation*}
J\left(u_{n}+s_{n}\left(v-u_{n}\right)\right) \geq J\left(u_{n}\right)-c s_{n}\left\|v-u_{n}\right\|, \tag{2.7}
\end{equation*}
$$

where $s_{n}=\frac{t_{n}}{\left\|v-u_{n}\right\|} \rightarrow 0^{+}$as $n \rightarrow \infty$. Let us divide (2.7) by $s_{n}$ and rewrite it as

$$
\begin{aligned}
& \frac{\lambda}{1-\gamma} \int_{\Omega} \frac{\left[u_{n}+s_{n}\left(v-u_{n}\right)\right]^{1-\gamma}-u_{n}^{1-\gamma}}{s_{n}} d x \\
& \leq \\
& \frac{1}{2} \int_{\Omega} \frac{\left|\nabla\left(u_{n}+s_{n}\left(v-u_{n}\right)\right)\right|^{2}-\left|\nabla u_{n}\right|^{2}}{s_{n}} d x+\frac{1}{4} \int_{\Omega} \frac{\phi_{u_{n}+s_{n}\left(v-u_{n}\right)}\left(u_{n}+s_{n}\left(v-u_{n}\right)\right)^{2}-\phi_{u_{n}} u_{n}^{2}}{s_{n}} d x \\
& \quad+\int_{\Omega} \frac{H\left(u_{n}+s_{n}\left(v-u_{n}\right)\right)-H\left(u_{n}\right)}{s_{n}}+c\left\|v-u_{n}\right\|,
\end{aligned}
$$

where

$$
H(u)=\frac{1}{p^{2}} \int_{\Omega}|u|^{p} d x-\frac{1}{p} \int_{\Omega}|u|^{p} \log |u| d x .
$$

Letting $n \rightarrow \infty$, we claim that we get

$$
\begin{align*}
\lim _{n \rightarrow \infty} \int_{\Omega} & \frac{H\left(u_{n}+s_{n}\left(v-u_{n}\right)\right)-H\left(u_{n}\right)}{s_{n}} d x \\
= & \lim _{n \rightarrow \infty} \int_{\Omega} \frac{\left[u_{n}+s_{n}\left(v-u_{n}\right)\right]^{p}-u_{n}^{p}}{p^{2} s_{n}} d x \\
& -\lim _{n \rightarrow \infty} \int_{\Omega} \frac{\left[u_{n}+s_{n}\left(v-u_{n}\right)\right]^{p} \log \left|u_{n}+s_{n}\left(v-u_{n}\right)\right|-u_{n}^{p} \log \left|u_{n}\right|}{p s_{n}} d x  \tag{2.8}\\
= & \frac{1}{p} \int_{\Omega}|u|^{p-1}(v-u) d x-\int_{\Omega}|u|^{p-1}(v-u) \log |u| d x-\frac{1}{p} \int_{\Omega}|u|^{p-1}(v-u) d x \\
= & -\int_{\Omega}|u|^{p-1}(v-u) \log |u| d x .
\end{align*}
$$

Indeed, we have only to justify the limit

$$
\begin{equation*}
\int_{\Omega}\left|u_{n}\right|^{p} \log \left|u_{n}\right| d x \rightarrow \int_{\Omega}|u|^{p} \log |u| d x . \tag{2.9}
\end{equation*}
$$

Since $u_{n}(x) \rightarrow u(x)$ a.e. in $\Omega$ and $u \rightarrow u^{p} \log (u)$ is continuous, then we get

$$
u_{n}^{p} \log u_{n} \rightarrow u^{p} \log u \quad \text { a.e. in } \Omega .
$$

Furthermore,

$$
u^{p} \log u \leq \frac{1}{e a} u^{p+a},
$$

where $a$ is a positive number small enough to ensure the compact embedding $H_{0}^{1}(\Omega) \hookrightarrow$ $L^{p+a}(\Omega)$. By Lemma 2.2, for $n$ large enough, we have

$$
-\frac{1}{e p} \leq u_{n}^{p} \log u_{n} \leq \frac{1}{e a} u^{p+a}+1 \in L^{1}(\Omega) .
$$

By using dominating convergence theorem, we justify (2.9). Thus, (2.8) holds.
Notice that

$$
\begin{aligned}
\int_{\Omega} \frac{\left[u_{n}+s_{n}\left(v-u_{n}\right)\right]^{1-\gamma}-u_{n}^{1-\gamma}}{s_{n}(1-\gamma)} d x= & \int_{\Omega} \frac{\left[u_{n}+s_{n}\left(v-u_{n}\right)\right]^{1-\gamma}-\left[\left(1-s_{n}\right) u_{n}\right]^{1-\gamma}}{s_{n}(1-\gamma)} d x \\
& +\int_{\Omega} \frac{\left[\left(1-s_{n}\right) u_{n}\right]^{1-\gamma}-u_{n}^{1-\gamma}}{s_{n}(1-\gamma)} d x \\
= & \int_{\Omega} \frac{\left[u_{n}+s_{n}\left(v-u_{n}\right)\right]^{1-\gamma}-\left[\left(1-s_{n}\right) u_{n}\right]^{1-\gamma}}{s_{n}(1-\gamma)} d x \\
& +\frac{\left(1-s_{n}\right)^{1-\gamma}-1}{s_{n}(1-\gamma)} \int_{\Omega} u_{n}^{1-\gamma} d x \\
= & J_{1, n}+J_{2, n} .
\end{aligned}
$$

Clearly, $J_{1, n}=\int_{\Omega} \frac{\xi^{-r} s_{n} v}{s_{n}} d x=\int_{\Omega} \frac{v}{\overline{\xi_{n}^{\gamma}}} d x$, where $\xi_{n} \in\left(u_{n}-u_{n} s_{n}, u_{n}+s_{n}\left(v-u_{n}\right)\right)$, which implies that $\xi_{n} \rightarrow u\left(u_{n} \rightarrow u\right)$ as $s_{n} \rightarrow 0^{+}$. Since $J_{1, n} \geq 0$ for all $n$, applying Fatou's Lemma to $J_{1, n}$, we obtain

$$
\liminf _{n \rightarrow \infty} J_{1, n} \geq \int_{\Omega} \frac{v}{u^{\gamma}} d x,
$$

for any $v \in P$. For $J_{2, n}$, by the dominated convergence theorem, we get

$$
\lim _{n \rightarrow \infty} J_{2, n}=-\int_{\Omega} u^{1-\gamma} d x
$$

From the above information, for every $v \in P$, it follows

$$
\begin{aligned}
\lambda \int_{\Omega} \frac{v-u}{u^{\gamma}} d x \leq & \int_{\Omega} \nabla u \nabla(v-u) d x+\int_{\Omega} \phi_{u} u(v-u) d x-\int_{\Omega}|u|^{p-1}(v-u) \log |u| d x \\
& +c\|v-u\|
\end{aligned}
$$

Since $|D J|(u)<c$ is arbitrary, this leads us to the proof of Lemma 2.3.
Lemma 2.4. J satisfies the (PS) condition.
Proof. Let $\left\{u_{n}\right\} \subset P$ be $(P S)$ sequence of $J$, that is

$$
|D J|\left(u_{n}\right) \rightarrow 0, \quad J\left(u_{n}\right) \rightarrow c \quad \text { as } n \rightarrow \infty
$$

By Lemma 2.3, for any $v \in P$, we have

$$
\begin{align*}
\lambda \int_{\Omega} \frac{v-u_{n}}{u_{n}^{\gamma}} d x \leq & \int_{\Omega} \nabla u_{n} \nabla\left(v-u_{n}\right) d x+\int_{\Omega} \phi_{u_{n}} u_{n}\left(v-u_{n}\right) d x  \tag{2.10}\\
& -\int_{\Omega} u_{n}^{p-1}\left(v-u_{n}\right) \log \left|u_{n}\right| d x+o(1)\left\|v-u_{n}\right\|
\end{align*}
$$

taking $v=2 u_{n} \in P$ in (2.10), we get

$$
\begin{equation*}
\lambda \int_{\Omega} u_{n}^{1-\gamma} d x \leq \int_{\Omega}\left|\nabla u_{n}\right|^{2} d x+\int_{\Omega} \phi_{u_{n}} u_{n}^{2} d x-\int_{\Omega} u_{n}^{p} \log \left|u_{n}\right| d x+o(1)\left\|u_{n}\right\| \tag{2.11}
\end{equation*}
$$

Since $J\left(u_{n}\right) \rightarrow c$,

$$
\begin{align*}
\frac{1}{2} \int_{\Omega}\left|\nabla u_{n}\right|^{2} d x+\frac{1}{4} \int_{\Omega} \phi_{u_{n}} u_{n}^{2} d x+\frac{1}{p^{2}} \int_{\Omega}\left|u_{n}\right|^{p} d x-\frac{1}{p} & \int_{\Omega}\left|u_{n}\right|^{p} \log \left|u_{n}\right| d x \\
& -\frac{\lambda}{1-\gamma} \int_{\Omega}\left|u_{n}\right|^{1-\gamma} d x=c+o(1) \tag{2.12}
\end{align*}
$$

It follows from (2.11) and (2.12) that

$$
\begin{aligned}
& \frac{p-2}{2 p} \int_{\Omega}\left|\nabla u_{n}\right|^{2}+\frac{p-4}{4 p} \int_{\Omega} \phi{u_{n}}_{n}^{2} d x+\frac{1}{p^{2}} \int_{\Omega}\left|u_{n}\right|^{p} d x \\
& \quad \leq \lambda\left(\frac{1}{1-\gamma}-\frac{1}{p}\right) \int_{\Omega} u_{n}^{1-\gamma} d x+c+o(1)+o(1)\left\|u_{n}\right\| \\
& \quad \leq C \lambda\left\|u_{n}\right\|^{1-\gamma}+C+o(1)\left\|u_{n}\right\|
\end{aligned}
$$

Which implies that $\left\{u_{n}\right\}$ is bounded in $H_{0}^{1}(\Omega)$. Thus, there exists a subsequence, still denoted by itself, and a function $u \in H_{0}^{1}(\Omega)$, such that $u_{n} \rightharpoonup u$ in $H_{0}^{1}(\Omega), u_{n}(x) \rightarrow u(x)$ a.e. in $\Omega$ as $n \rightarrow \infty$. Choosing $v=u_{m}$ as the test function in (2.10), we have

$$
\begin{aligned}
\lambda \int_{\Omega} \frac{u_{m}-u_{n}}{u_{n}^{\gamma}} d x \leq & \int_{\Omega} \nabla u_{n} \nabla\left(u_{m}-u_{n}\right) d x+\int_{\Omega} \phi_{u_{n}} u_{n}\left(u_{m}-u_{n}\right) d x \\
& -\int_{\Omega} u_{n}^{p-1}\left(u_{m}-u_{n}\right) \log \left|u_{n}\right| d x+o(1)\left\|u_{m}-u_{n}\right\|
\end{aligned}
$$

By changing the role of $u_{m}$ and $u_{n}$, we have a similar inequality, by adding the two inequalities, there holds

$$
\begin{aligned}
\left\|u_{n}-u_{m}\right\|^{2} \leq & \lambda \int_{\Omega}\left(u_{n}-u_{m}\right)\left(\frac{1}{u_{n}^{\gamma}}-\frac{1}{u_{m}^{\gamma}}\right) d x+\int_{\Omega}\left(\phi_{u_{m}} u_{m}-\phi_{u_{n}} u_{n}\right)\left(u_{n}-u_{m}\right) d x \\
& +\int_{\Omega}\left(u_{n}^{p-1} \log \left|u_{n}\right|-u_{m}^{p-1} \log \left|u_{m}\right|\right)\left(u_{n}-u_{m}\right) d x+o(1)\left\|u_{m}-u_{n}\right\| \\
\leq & \int_{\Omega}\left(\phi_{u_{m}} u_{m}-\phi_{u_{n}} u_{n}\right)\left(u_{n}-u_{m}\right) d x \\
& +\int_{\Omega}\left(u_{n}^{p-1} \log \left|u_{n}\right|-u_{m}^{p-1} \log \left|u_{m}\right|\right)\left(u_{n}-u_{m}\right) d x+o(1)\left\|u_{m}-u_{n}\right\| \\
\leq & -\frac{1}{2}\left\|\phi_{u_{m}}-\phi_{u_{n}}\right\|^{2}+\int_{\Omega} u_{n}^{p}\left(u_{n}-u_{m}\right) d x+\int_{\Omega} u_{m}^{p}\left(u_{n}-u_{m}\right) d x+o(1)\left\|u_{m}-u_{n}\right\|
\end{aligned}
$$

Note that

$$
\left\|\phi_{u_{m}}-\phi_{u_{n}}\right\| \rightarrow 0, \quad \int_{\Omega} u_{n}^{p}\left(u_{n}-u_{m}\right) d x \rightarrow 0, \quad \int_{\Omega} u_{m}^{p}\left(u_{n}-u_{m}\right) d x \quad \text { as } n \rightarrow \infty
$$

We have $\lim _{n \rightarrow \infty}\left\|u_{n}-u_{m}\right\|=0$. Therefore, $u_{n} \rightarrow u$ in $H_{0}^{1}(\Omega)$ as $n \rightarrow \infty$. The proof is complete.

Lemma 2.5. Assume that $|D J|(u)=0$, then $u$ is a weak solution of problem (2.1). Namely $u^{-\gamma} \varphi \in$ $L^{1}(\Omega)$ for all $\varphi \in H_{0}^{1}(\Omega)$, it holds that

$$
\begin{equation*}
\int_{\Omega} \nabla u \nabla \varphi d x+\int_{\Omega} \phi_{u} u \varphi d x=\int_{\Omega}|u|^{p-1} \varphi \log |u| d x+\lambda \int_{\Omega} \frac{\varphi}{u^{\gamma}} d x \tag{2.13}
\end{equation*}
$$

Proof. By Lemma 2.3, we have

$$
\lambda \int_{\Omega} \frac{v-u}{u^{\gamma}} d x \leq \int_{\Omega} \nabla u \nabla(v-u) d x+\int_{\Omega} \phi_{u} u(v-u) d x-\int_{\Omega}|u|^{p-1}(v-u) \log |u| d x
$$

for every $v \in P$. Letting $s \in \mathbb{R}, \varphi \in H_{0}^{1}(\Omega)$, taking $(u+s \varphi)^{+} \in P$ as a test function in (2.6), one has

$$
\begin{aligned}
0 \leq & \int_{\Omega} \nabla u \nabla\left((u+s \varphi)^{+}-u\right) d x+\int_{\Omega} \phi_{u} u\left((u+s \varphi)^{+}-u\right) d x \\
& -\int_{\Omega}|u|^{p-1}\left((u+s \varphi)^{+}-u\right) \log |u| d x-\lambda \int_{\Omega} \frac{(u+s \varphi)^{+}-u}{u^{\gamma}} d x \\
= & s\left[\int_{\Omega} \nabla u \nabla \varphi d x+\int_{\Omega} \phi_{u} u \varphi d x-\int_{\Omega}|u|^{p-1} \varphi \log |u| d x-\lambda \int_{\Omega} \frac{\varphi}{u^{\gamma}} d x\right] \\
& -\int_{u+s \varphi<0} \nabla u \nabla(u+s \varphi) d x-\int_{u+s \varphi<0} \phi_{u} u(u+s \varphi) d x+\int_{u+s \varphi<0}|u|^{p-1}(u+s \varphi) \log |u| d x \\
& +\lambda \int_{u+s \varphi<0} \frac{u+s \varphi}{u^{\gamma}} d x \\
\leq & s\left[\int_{\Omega} \nabla u \nabla \varphi d x+\int_{\Omega} \phi_{u} u \varphi d x-\int_{\Omega}|u|^{p-1} \varphi \log |u| d x-\lambda \int_{\Omega} \frac{\varphi}{u^{\gamma}} d x\right] \\
& -s \int_{u+s \varphi<0}\left[\nabla u \nabla \varphi+\phi_{u} u \varphi\right] d x+\int_{u+s \varphi<0}|u|^{p-1}(u+s \varphi) \log |u| d x .
\end{aligned}
$$

Since $\nabla u(x)=0$ for a.e. $x \in \Omega$ with $u(x)=0$ and $\operatorname{meas}\{x \in \Omega \mid u(x)+s \varphi(x)<0$, $u(x)>0\} \rightarrow 0$ as $s \rightarrow 0$, we have

$$
\int_{u+s \varphi<0}\left[\nabla u \nabla \varphi+\phi_{u} u \varphi\right] d x=\int_{\substack{u+s \varphi<0 \\ u>0}}\left[\nabla u \nabla \varphi+\phi_{u} u \varphi\right] d x \rightarrow 0
$$

and

$$
\int_{u+s \varphi<0}|u|^{p-1}(u+s \varphi) \log |u| d x=\int_{\substack{u+s \varphi<0, u>0}}|u|^{p-1}(u+s \varphi) \log |u| d x \rightarrow 0 \quad \text { as } s \rightarrow 0 .
$$

Therefore

$$
0 \leq s\left(\int_{\Omega} \nabla u \nabla \varphi d x+\int_{\Omega} \phi_{u} u \varphi d x-\int_{\Omega}|u|^{p-1} \varphi \log |u| d x-\lambda \int_{\Omega} \frac{\varphi}{u^{\gamma}} d x\right)+o(s)
$$

as $s \rightarrow 0$. we obtain that

$$
\int_{\Omega} \nabla u \nabla \varphi d x+\int_{\Omega} \phi_{u} u \varphi d x-\int_{\Omega}|u|^{p-1} \varphi \log |u| d x-\lambda \int_{\Omega} \frac{\varphi}{u^{\gamma}} d x \geq 0 .
$$

By the arbitrariness of $\varphi$, this inequality also holds for $-\varphi$,

$$
\int_{\Omega} \nabla u \nabla \varphi d x+\int_{\Omega} \phi_{u} u \varphi d x-\int_{\Omega}|u|^{p-1} \varphi \log |u| d x-\lambda \int_{\Omega} \frac{\varphi}{u^{\gamma}} d x=0 .
$$

Hence, we can deduce that (2.13) holds. The proof of Lemma 2.5 is complete.

## 3 Proof of Theorem 1.1

In this section, we firstly prove that the problem (2.1) has a negative energy solution.
Lemma 3.1. Given $0<\gamma<1$, there exist constants $r, \rho, \Lambda_{0}>0$ such that the functional J satisfies the following conditions for $0<\lambda<\Lambda_{0}$ :
(i) $\left.J(u)\right|_{u \in S_{\rho}} \geq r, \inf _{u \in B_{\rho}} J(u)<0$;
(ii) There exists $e \in H_{0}^{1}(\Omega)$ with $\|e\|>\rho$ such that $J(e)<0$.

Proof. (i) By (2.12) in [25], we have

$$
\begin{equation*}
\int_{\Omega}|u|^{p} \log |u| d x \leq \frac{1}{2}\|u\|^{2}+C_{1}\|u\|^{q} . \tag{3.1}
\end{equation*}
$$

Therefore, one has

$$
\begin{aligned}
J(u) & =\frac{1}{2}\|u\|^{2} d x+\frac{1}{4} \int_{\Omega} \phi_{u} u^{2} d x+\frac{1}{p^{2}} \int_{\Omega}|u|^{p} d x-\frac{1}{p} \int_{\Omega}|u|^{p} \log |u| d x-\frac{\lambda}{1-\gamma} \int_{\Omega}|u|^{1-\gamma} d x \\
& \geq \frac{p-1}{2 p}\|u\|^{2}+\frac{1}{4} \int_{\Omega} \phi_{u} u^{2} d x-C_{1}\|u\|^{q}-\frac{\lambda}{1-\gamma} \int_{\Omega}|u|^{1-\gamma} d x \\
& \geq \frac{p-1}{2 p}\|u\|^{2}-C_{1}\|u\|^{q}-C_{2} \lambda\|u\|^{1-\gamma} .
\end{aligned}
$$

Where $q \in(p, 6)$. Which implies that there exist constants $r, \rho, \Lambda_{0}>0$, such that $\left.J(u)\right|_{u \in S_{\rho}} \geq r$ for every $\lambda \in\left(0, \Lambda_{0}\right)$. Moreover, for $u \in H_{0}^{1}(\Omega) \backslash\{0\}$, it holds that

$$
\lim _{t \rightarrow 0^{+}} \frac{J(t u)}{t^{1-\gamma}}=-\frac{\lambda}{1-\gamma} \int_{\Omega}|u|^{1-\gamma} d x<0
$$

So we obtain that $J(t u)<0$ for all $u \neq 0$ and $t$ small enough. Therefore, for $\|u\|$ small enough, one has

$$
\begin{equation*}
m_{1}=\inf _{u \in B_{\rho}} J(u)<0 . \tag{3.2}
\end{equation*}
$$

(ii) For every $u^{+} \in H_{0}^{1}(\Omega), u^{+} \neq 0$ and $t>0$, we have

$$
\begin{aligned}
J(t u)= & \frac{t^{2}}{2}\|u\|^{2}+\frac{t^{4}}{4} \int_{\Omega} \phi_{u} u^{2} d x+\frac{t^{p}}{p^{2}} \int_{\Omega}|u|^{p} d x \\
& -\frac{t^{p}}{p} \int_{\Omega} u^{p} \log |t u| d x-\frac{\lambda t^{1-\gamma}}{1-\gamma} \int_{\Omega}|u|^{1-\gamma} d x \\
\rightarrow & -\infty
\end{aligned}
$$

as $t \rightarrow+\infty$. Therefore we can certainly find $e \in H_{0}^{1}(\Omega)$ such that $\|e\|>\rho$ and $J(e)<0$. The proof is complete.

Theorem 3.2. Suppose $0<\lambda<\Lambda_{0}$, then system (1.1) has a positive function pair solution ( $u_{*}, \phi_{u_{*}}$ ) $\in$ $H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega)$, satisfying $J\left(u_{*}\right)<0$.

Proof. First, we claim that there exists $u_{*} \in B_{\rho}$, such that $J\left(u_{*}\right)=m_{1}<0$.
By the definition of $m_{1}$, we know that there exists a minimizing sequence $\left\{u_{n}\right\} \subset B_{\rho} \subset P$ such that $\lim _{n \rightarrow \infty} J\left(u_{n}\right)=m_{1}<0$. Since $J\left(\left|u_{n}\right|\right)=J\left(u_{n}\right)$, we may assume that $u_{n}(x)>0$ for almost every $x$ in $\Omega$. Clearly, this minimizing sequence is of course bounded in $B_{\rho}$, up to a subsequence, there exists $u_{*}>0$ such that

$$
\left\{\begin{array}{l}
u_{n} \rightharpoonup u_{*}, \quad \text { weakly in } H_{0}^{1}(\Omega),  \tag{3.3}\\
u_{n} \rightarrow u_{*}, \quad \text { strongly in } L^{q}(\Omega), 1 \leq q<2^{*}, \\
u_{n}(x) \rightarrow u_{*}(x), \quad \text { a.e. in } \Omega
\end{array}\right.
$$

as $n \rightarrow \infty$. Set $\omega_{n}=u_{n}-u_{*}$, by the Brézis-Lieb Lemma, one has

$$
\begin{equation*}
\left\|u_{n}\right\|^{2}=\left\|\omega_{n}\right\|^{2}+\left\|u_{*}\right\|^{2}+o(1) . \tag{3.4}
\end{equation*}
$$

Hence, by Lemma 2.4, we have that

$$
m_{1}=\lim _{n \rightarrow \infty} J\left(u_{n}\right)=J\left(u_{*}\right)+\frac{1}{2} \lim _{n \rightarrow \infty}\left\|\omega_{n}\right\|^{2} \geq J\left(u_{*}\right)
$$

from $u_{*} \in B_{\rho}$ and by definition of $m_{1}$ equality holds. Hence, we obtain $J\left(u_{*}\right)=m_{1}<0$ and $u_{*} \not \equiv 0$. From the above arguments we know that $u_{*}$ is a local minimizer of $J$.

Now, we prove that $u_{*}$ is a critical point of $J$. Note that $u_{*} \geq 0$ and $u_{*} \not \equiv 0$. Then for any $\psi \in P \subset H_{0}^{1}(\Omega)$, let $t>0$ such that $u_{*}+t \psi \in H_{0}^{1}(\Omega)$ and one has

$$
\begin{align*}
0 \leq & J\left(u_{*}+t \psi\right)-J\left(u_{*}\right) \\
= & \frac{1}{2}\left\|u_{*}+t \psi\right\|^{2}+\frac{1}{4} \int_{\Omega} \phi_{u_{*}+t \psi}\left(u_{*}+t \psi\right)^{2} d x+\frac{1}{p^{2}} \int_{\Omega}\left|u_{*}+t \psi\right|^{p} d x \\
& -\frac{1}{p} \int_{\Omega}\left|u_{*}+t \psi\right|^{p} \log \left|u_{*}+t \psi\right| d x-\frac{\lambda}{1-\gamma} \int_{\Omega}\left|u_{*}+t \psi\right|^{1-\gamma} d x  \tag{3.5}\\
& -\frac{1}{2} \|\left. u_{*}\right|^{2}-\frac{1}{4} \int_{\Omega} \phi_{u_{*}} u_{*}^{2} d x-\frac{1}{p^{2}} \int_{\Omega}\left|u_{*}\right|^{p} d x \\
& +\frac{1}{p} \int_{\Omega}\left|u_{*}\right|^{p} \log \left|u_{*}\right| d x+\frac{\lambda}{1-\gamma} \int_{\Omega}\left|u_{*}\right|^{1-\gamma} d x .
\end{align*}
$$

Actually, from (3.5), we also get

$$
\begin{aligned}
& \frac{\lambda}{1-\gamma} \int_{\Omega}\left[\left(u_{*}+t \psi\right)^{1-\gamma}-\left(u_{*}\right)^{1-\gamma}\right] d x \\
& \leq \\
& \quad \frac{1}{2}\left(\left\|u_{*}+t \psi\right\|^{2}-\left\|u_{*}\right\|^{2}\right) d x+\frac{1}{4} \int_{\Omega}\left[\phi_{u_{*}+t \psi}\left(u_{*}+t \psi\right)^{2}-\phi_{u_{*}} u_{*}^{2}\right] d x \\
& \quad+\frac{1}{p^{2}} \int_{\Omega}\left[\left(u_{*}+t \psi\right)^{p}-u_{*}^{p}\right] d x-\frac{1}{p} \int_{\Omega}\left[\left(u_{*}+t \psi\right)^{p} \log \left|u_{*}+t \psi\right|-u_{*}^{p} \log \left|u_{*}\right|\right] d x .
\end{aligned}
$$

Dividing by $t>0$ and passing to the limit as $t \rightarrow 0^{+}$, it gives

$$
\begin{align*}
\frac{\lambda}{1-\gamma} \liminf _{t \rightarrow 0^{+}} \int_{\Omega} \frac{\left(u_{*}+t \psi\right)^{1-\gamma}-\left(u_{*}\right)^{1-\gamma}}{t} d x \leq & \int_{\Omega} \nabla u_{*} \nabla \psi d x+\int_{\Omega} \phi_{u_{*}} u_{*} \psi d x  \tag{3.6}\\
& -\int_{\Omega}\left|u_{*}\right|^{p-1} \psi \log \left|u_{*}\right| d x
\end{align*}
$$

Notice that

$$
\frac{\lambda}{1-\gamma} \int_{\Omega} \frac{\left(u_{*}+t \psi\right)^{1-\gamma}-\left(u_{*}\right)^{1-\gamma}}{t} d x=\lambda \int_{\Omega}\left(u_{*}+\xi t \psi\right)^{-\gamma} \psi d x
$$

Where $\xi \rightarrow 0^{+}$and $\left(u_{*}+\xi t \psi\right)^{-\gamma} \psi \rightarrow\left(u_{*}\right)^{-\gamma} \psi$ a.e. $x \in \Omega$ as $t \rightarrow 0^{+}$, since $\left(u_{*}+\xi t \psi\right)^{-\gamma} \psi \geq 0$. Thus by using Fatou's Lemma, we have

$$
\lambda \int_{\Omega}\left(u_{*}\right)^{-\gamma} \psi d x \leq \frac{\lambda}{1-\gamma} \liminf _{t \rightarrow 0^{+}} \int_{\Omega} \frac{\left(u_{*}+t \psi\right)^{1-\gamma}-\left(u_{*}\right)^{1-\gamma}}{t} d x
$$

Therefore, we deduce from (3.6) and the above estimate that

$$
\begin{equation*}
\int_{\Omega}\left(\nabla u_{*}, \nabla \psi\right) d x+\int_{\Omega} \phi_{u_{*}} u^{*} \psi d x-\int_{\Omega}\left|u_{*}\right|^{p-1} \psi \log \left|u_{*}\right| d x-\lambda \int_{\Omega}\left(u_{*}\right)^{-\gamma} \psi d x \geq 0, \quad \psi \geq 0 \tag{3.7}
\end{equation*}
$$

Since $J\left(u_{*}\right)<0$, this together with Lemma 3.1, imply that $u_{*} \notin S_{\rho}$, therefore we obtain $\left\|u_{*}\right\|<\rho$. For $u_{*}$ there is $\delta_{1} \in(0,1)$ such that $(1+t) u_{*} \in B_{\rho}$ for $|t| \leq \delta_{1}$. Define $k:\left[-\delta_{1}, \delta_{1}\right]$ by $k(t)=J\left((1+t) u_{*}\right)$. Clearly, $k(t)$ achieves its minimum at $t=0$, namely

$$
\begin{equation*}
\left.k^{\prime}(t)\right|_{t=0}=\left\|u_{*}\right\|^{2}+\int_{\Omega} \phi_{u_{*}}\left(u_{*}\right)^{2} d x-\int_{\Omega}\left|u_{*}\right|^{p} \log \left|u_{*}\right| d x-\lambda \int_{\Omega}\left(u_{*}\right)^{1-\gamma} d x=0 \tag{3.8}
\end{equation*}
$$

Suppose for any $v \in H_{0}^{1}(\Omega)$, and $\varepsilon>0$. Define $\Psi \in P$ by

$$
\Psi=\left(u_{*}+\varepsilon v\right)^{+} .
$$

By (3.7) and (3.8), we have

$$
\begin{align*}
& 0 \leq \int_{\Omega}\left[\left(\nabla u_{*}, \nabla \Psi\right)+\phi_{u_{*}} u_{*} \Psi-\left|u_{*}\right|^{p-1} \Psi \log \left|u_{*}\right|-\lambda\left(u_{*}\right)^{-\gamma} \Psi\right] d x \\
&= \int_{\left\{u_{*}+\varepsilon v>0\right\}}\left(\nabla u_{*} \nabla\left(u_{*}+\varepsilon v\right)\right) d x \\
&+\int_{\left\{u_{*}+\varepsilon v>0\right\}}\left[\phi_{u_{*}} u_{*}\left(u_{*}+\varepsilon v\right)-\left|u_{*}\right|^{p-1}\left(u_{*}+\varepsilon v\right) \log \left|u_{*}\right|-\lambda\left(u_{*}\right)^{-\gamma}\left(u_{*}+\varepsilon v\right)\right] d x \\
&=\left(\int_{\Omega}-\int_{\left\{u_{*}+\varepsilon v \leq 0\right\}}\right)\left[\left(\nabla u_{*}, \nabla\left(u_{*}+\varepsilon v\right)\right)\right. \\
&+\phi_{\left.u_{*} u_{*}\left(u_{*}+\varepsilon v\right)-\left|u_{*}\right|^{p-1}\left(u_{*}+\varepsilon v\right) \log \left|u_{*}\right|-\lambda\left(u_{*}\right)^{-\gamma}\left(u_{*}+\varepsilon v\right)\right] d x}^{\leq} \\
&\left|\| u_{*}\right|^{2}+\int_{\Omega} \phi_{u_{*}} u_{*}^{2} d x-\int_{\Omega}\left|u_{*}\right|^{p} \log \left|u_{*}\right| d x-\lambda \int_{\Omega}\left(u_{*}\right)^{1-\gamma} d x  \tag{3.9}\\
&+\varepsilon \int_{\Omega}\left[\left(\nabla u_{*}, \nabla v\right)+\phi_{u_{*}} u_{*} v-\left|u_{*}\right|^{p-1} v \log \left|u_{*}\right|-\lambda\left(u_{*}\right)^{-\gamma} v\right] d x \\
&-\int_{\left\{u_{*}+\varepsilon v \leq 0\right\}}\left[\left(\nabla u_{*}, \nabla\left(u_{*}+\varepsilon v\right)\right)+\phi_{u_{*}} u_{*}\left(u_{*}+\varepsilon v\right)\right] d x \\
&+\int_{\left\{u_{*}+\varepsilon v \leq 0\right\}}\left[\left|u_{*}\right|^{p-1}\left(u_{*}+\varepsilon v\right) \log \left|u_{*}\right|+\lambda\left(u_{*}\right)^{-\gamma}\left(u_{*}+\varepsilon v\right)\right] d x \\
& \leq \varepsilon \int_{\Omega}\left[\left(\nabla u_{*}, \nabla v\right)+\phi_{u_{*}} u_{*} v-\left|u_{*}\right|^{p-1} v \log \left|u_{*}\right|-\lambda\left(u_{*}\right)^{-\gamma} v\right] d x \\
&-\varepsilon \int_{\left\{u_{*}+\varepsilon v \leq 0\right\}}\left(\nabla u_{*} \nabla v+\phi_{u_{*}} u_{*} v\right) d x .
\end{align*}
$$

Since the measure of the domain of integration $\left\{u_{*}+\varepsilon v \leq 0\right\} \rightarrow 0$ as $\varepsilon \rightarrow 0$, it follows that

$$
\lim _{\varepsilon \rightarrow 0} \int_{\left\{u_{*}+\varepsilon v \leq 0\right\}}\left(\nabla u_{*} \nabla v+\phi_{u_{*}} u_{*} v\right) d x=0 .
$$

Therefore, dividing by $\varepsilon$ and setting $\varepsilon \rightarrow 0$ in (3.9), one has

$$
\begin{equation*}
\int_{\Omega}\left(\nabla u_{*}, \nabla v\right) d x+\int_{\Omega} \phi_{u_{*}} u_{*} v d x-\int_{\Omega}\left|u_{*}\right|^{p-1} v \log \left|u_{*}\right| d x-\lambda \int_{\Omega}\left(u_{*}\right)^{-\gamma} v d x \geq 0 . \tag{3.10}
\end{equation*}
$$

By the arbitrariness of $v$, the inequality also holds for $-v$,

$$
\begin{equation*}
\int_{\Omega}\left(\nabla u_{*}, \nabla v\right) d x+\int_{\Omega} \phi_{u_{*}} u_{*} v d x-\int_{\Omega}\left|u_{*}\right|^{p-1} v \log \left|u_{*}\right| d x-\lambda \int_{\Omega}\left(u_{*}\right)^{-\gamma} v d x=0 . \tag{3.11}
\end{equation*}
$$

Since $u_{*} \not \equiv 0$. From (3.10), there holds

$$
-\Delta u_{*}+\phi_{u_{*}} u_{*} \geq 0
$$

Note that $\phi_{u_{*}}>0$, then, by the strong maximum principle, it suggests that $u_{*}>0$ in $\Omega$. From the above arguments, we obtain that $\left(u_{*}, \phi_{u_{*}}\right)$ is a positive solution of system (1.1) with $J\left(u_{*}\right)=m_{1}<0$. This proof is complete.

Now, we only need prove that system (1.1) has another positive solution.
Theorem 3.3. Suppose $0<\lambda<\Lambda_{0}$, then system (1.1) has a positive function pair solution $\left(v_{*}, \phi_{v_{*}}\right) \in$ $H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega)$, such that $J\left(v_{*}\right)>0$.

Proof. By Lemma 3.1, J satisfies the geometric structure of mountain pass Lemma. Applying the Mountain pass Lemma [5] and Lemma 2.4, there exists a sequence $\left\{v_{n}\right\}$ such that

$$
|D J|\left(v_{n}\right) \rightarrow 0, \quad J\left(v_{n}\right) \rightarrow c \quad \text { as } n \rightarrow \infty .
$$

According to Lemma 2.4, we know that $\left\{v_{n}\right\} \subset H_{0}^{1}(\Omega)$ has a convergent subsequence, still denoted by $\left\{v_{n}\right\}$, we may assume that $v_{n} \rightarrow v_{*}$ in $H_{0}^{1}(\Omega)$, and

$$
J\left(v_{*}\right)=\lim _{n \rightarrow \infty} J\left(v_{n}\right)=c, \quad|D J|\left(v_{n}\right) \rightarrow 0 .
$$

Similar to Theorem 3.2, $v_{*}$ satisfies equation (2.1) with $J\left(v_{*}\right)=c>0$. Thus $\left(v_{*}, \phi_{v_{*}}\right)$ is a positive solution of system (1.1). Thereby, we obtain that the function pairs ( $u_{*}, \phi_{u_{*}}$ ) and $\left(v_{*}, \phi_{v_{*}}\right)$ are different positive solutions. This completes the proof of Theorem 1.1.

## Acknowledgements

The authors thanks an anonymous referees for careful reading and some helpful comments, which greatly improve the manuscript. This work was supported the National Natural Science Foundation of China (No. 11661021; No. 11861021); Science Fund Grants of Guizhou Minzu University (No. KY[2018]5773-YB03).

## References

[1] A. Ambrosetti, D. Ruiz, Multiple bound states for the Schrödinger-Poisson problem, Commun. Contemp. Math. 10(2008), No. 3, 391-404. https://doi.org/10.1142/ S021919970800282X; MR2417922; Zbl 1188.35171
[2] A. Azzollini, P. D'Avenia, On a system involving a critically growing nonlinearity, J. Math. Anal. Appl. 387(2012), No. 1, 433-438. https://doi.org/10.1016/j.jmaa.2011.09. 012; MR2845762; Zbl 1229.35060
[3] V. Benci, D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations, Topol. Methods. Nonlinear. Anal. 11(1998), No. 2, 283-293. https://doi.org/10.12775/ TMNA. 1998.019; MR1659454; Zbl 0926.35125
[4] Y. Bouizem, S. Boulaaras, B. Djebbar, Some existence results for an elliptic equation of Kirchhoff-type with changing sign data and a logarithmic nonlinearity, Math. Meth Appl. Sci. 42(2019), No. 7, 2465-2474. https://doi.org/10.1002/mma.5523; MR3936413; Zbl 1417.35031
[5] A. Canino, M. Degiovanni, Nonsmooth critical point theory and quasilinear elliptic equations, in: Topological methods in differential equations and inclusions (Montreal, PQ, 1994), NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci, Vol. 472, Kluwer, Dordrecht, 1995. https: //doi.org/10.1007/978-94-011-0339-8_1; MR1368669; Zbl 0851.35038
[6] S. Chen, X. Tang, Ground state sign-changing solutions for elliptic equations with logarithmic nonlinearity, Acta. Math. Hungar. 157(2019), No. 1, 27-38. https://doi. org/10. 1007/s10474-018-0891-y; MR3911157; Zbl 1438.35192
[7] T. D'Aprile, D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations, Proc. Roy. Soc. Edinburgh Sect. A 134(2004), No. 5, 893906. https://doi.org/10.1142/S021919970800282X; MR2099569; Zbl 1064.35182
[8] T. D'Aprile, D. Mugnai, Non-existence results for the coupled Klein-Gordon-Maxwell equations, Adv. Nonlinear. Stud. 4(2004), No. 3, 307-322. https://doi.org/10.1515/ ans-2004-0305; MR2079817; Zbl 1142.35406
[9] P. d’Avenia, A. Azzollini, V. Luisi, Generalized Schrödinger-Poisson type systems, J. Coттип. Pure. Appl. Anal. 12(2013), No. 2, 867-879. https://doi.org/10.3934/cpaa. 2013.12.867; MR2982795; Zbl 1270.35227
[10] P. d’Avenia, E. Montefusco, M. Squassina, On the logarithmic Schrödinger equation, Commun. Contemp. Math. 16(2014), No. 2, 706-729. https://doi.org/10.1142/ S0219199713500326; MR3195154; Zbl 1292.35259
[11] C. Ji, A. Szulkin, A logarithmic Schrödinger equation with asymptotic conditions on the potential, J. Math. Anal. Appl. 437(2016), No. 3, 241-254. https://doi.org/10.1016/j. jmaa.2015.11.071; MR3451965; Zbl 1333.35010
[12] M. Jing, Z. D. Yang, Existence of solutions to $p$-Laplace equations with logarithmic nonlinearity, Electron. J. Differential Equations 2009, No. 87, 1-10. MR2519912; Zbl 1175.35067
[13] A. C. Lazer, P. J. McKenna, On a singular nonlinear elliptic boundary-value problem, Proc. Amer. Math. Soc. 111(1991), No. 3, 721-730. https://doi.org/10.2307/2048410; MR1037213; Zbl 0727.35057
[14] C. Y. Lei, G. S. Liu, H. M. Suo, Positive solutions for a Schrödinger-Poisson system with singularity and critical exponent, J. Math. Anal. Appl. 483(2019), No. 2, 123647, 21 pp. https://doi.org/10.1016/j.jmaa.2019.123647; MR4037579; Zbl 1433.35073
[15] C. Y. Lei, H. M. Suo, Positive solutions for a Schrödinger-Poisson system involving concave-convex nonlinearities, Comput. Math. Appl. 74(2017), No. 6, 1516-1524. https: //doi.org/10.1007/s00526-017-1229-2; MR3693350; Zbl 1394.35172
[16] X. Q. Liu, Y. X. Guo, J. Q. Liu, Solutions for singular $p$-Laplacian equation in $\mathbb{R}^{N}$, J. Syst. Sci. Complex. 22(2009), No. 4, 597-613. https://doi.org/10.1007/s11424-009-9190-6; MR2565258; Zbl 1300.35039
[17] H. L. Liu, Z. S. Liu, Q. Z. Xiao, Ground state solution for a fourth-order nonlinear elliptic problem with logarithmic nonlinearity, Appl. Math. Lett. 79(2018), No. 1, 176-181. https://doi.org/10.1016/j.aml.2017.12.015; MR3748628; Zbl 1459.35123
[18] D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinear local term, J. Funct. Anal. 237(2006), No. 2, 655-674. https://doi.org/10.1016/j.jfa.2006.04.005; MR2230354; Zbl 1136.35037
[19] M. Squassina, A. Szulkin, Multiple solutions to logarithmic Schrödinger equations with periodic potential, Calc. Var. Partial Differ. Equ. 54(2015), No. 1, 585-597. https://doi. org/10.1007/s00526-014-0796-8; MR3385171; Zbl 1326.35358
[20] Y. J. Sun, S. J. Li, Some remarks on a superlinear-singular problem: Estimates of $\lambda^{*}$, Nonlinear. Anal. 69(2008), No. 8, 2636-2650. https://doi.org/10.1016/j.na.2007.08. 037; MR2446359; Zbl 1237.35076
[21] Y. J. Sun, X. P. Wu, An exact estimate result for a class of singular equations with critical exponents, J. Funct. Anal. 260(2011), No. 5, 1257-1284. https://doi.org/10.1016/j.jfa. 2010.11.018; MR2749428; Zbl 1237.35077
[22] Y. J. Sun, X. P. Wu, Y. M. Long, Combined effects of singular and superlinear nonlinearities in some singular boundary value problems, J. Differential. Equations 176(2001), No. 2, 511-531. https://doi.org/10.1006/jdeq.2000.3973; MR1866285; Zbl 1109.35344
[23] S. Tian, Multiple solutions for the semilinear elliptic equations with the sign-changing logarithmic nonlinearity, J. Math. Anal. Appl. 454(2017), No. 2, 816-828. https ://doi. org/ 10.1016/j.jmaa.2017.05.015; MR3658801; Zbl 1379.35140
[24] F. Y. Wang, J. L. Wu, Compactness of Schrödinger semigroups with unbounded below potentials, Bull. Sci. Math. 132(2008), No. 8, 679-689. https://doi.org/10.1016/j . bulsci.2008.06.004; MR2474487; Zbl 1156.47043
[25] L. Wen, X. H. Tang, S. T. Chen, Ground state sign-changing solutions for Kirchhoff equations with logarithmic nonlinearity, Electron. J. Qual. Theory Differ. Equ. 2019, No. 47, 1-13. https://doi.org/10.14232/ejqtde.2019.1.47; MR3991096; Zbl 1438.35159
[26] H. T. Yang, Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem, J. Differential Equations 189(2003), No. 2, 487-512. https :// doi.org/10.1016/S0022-0396(02) 00098-0; MR1964476; Zbl 1034.35038
[27] Q. Zhang, Existence, uniqueness and multiplicity of positive solutions for SchrödingerPoisson system with singularity, J. Math. Anal. Appl. 437(2016), No. 1, 160-180. https: //doi.org/10.1016/j.jmaa.2015.12.061; MR3451961; Zbl 1334.35048
[28] K. G. Zloshchastiev, Logarithmic nonlinearity in theories of quantum gravity: origin of time and observational consequences, Gravit. Cosmol. 16(2010), No. 4, 288-297. https : //doi.org/10.1134/S0202289310040067; MR2740900; Zbl 1232.83044

Electronic Journal of Qualitative Theory of Differential Equations

# A double phase equation with convection 

Zhenhai Liu ${ }^{\boxtimes 1,2}$ and Nikolaos S. Papageorgiou ${ }^{3}$<br>${ }^{1}$ Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, P.R. China.<br>${ }^{2}$ Guangxi Key Laboratory of Hybrid Computation and IC Design Analysis, Guangxi University for Nationalities, Nanning, Guangxi, 530006, P.R. China<br>${ }^{3}$ Department of Mathematics, National Technical University, Zografou Campus, 15780 Athens, Greece

Received 11 October 2021, appeared 20 December 2021
Communicated by Dimitri Mugnai


#### Abstract

We consider a double phase problem with a gradient dependent reaction (convection). Using the theory of nonlinear operators of monotone type, we show the existence of a nontrivial, positive, bounded solution.


Keywords: Gradient dependent reaction, Musielak-Orlicz spaces, pseudomonotone operator, strongly coercive operator, modular function.

2020 Mathematics Subject Classification: 35B02, 35B40, 35 J15.

## 1 Introduction

Let $\Omega \subseteq \mathbb{R}^{N}$ be a bounded domain with a Lipschitz boundary $\partial \Omega$. In this paper we study the following double phase Dirichlet problem with gradient dependent reaction (convection)

$$
\left\{\begin{array}{l}
-\Delta_{p}^{a} u(z)-\Delta_{q} u(z)=f(z, u(z))+E(z)|D u(z)|^{q-1} \quad \text { in } \Omega  \tag{1.1}\\
\left.u\right|_{\partial \Omega}=0, u>0,1<q<p
\end{array}\right.
$$

Here $\Delta_{p}^{a}$ denotes the weighted $p$-Laplace differential operator defined by

$$
\Delta_{p}^{a} u=\operatorname{div}\left(a(z)|D u|^{p-2} D u\right)
$$

Problem (1.1) has two interesting features. The first is that in the weighted operator, the weight $a \in L^{\infty}(\Omega)$ is not bounded away from zero. This means that the integrand

$$
\theta(z, x)=a(z) x^{p}+x^{q} \quad \forall z \in \Omega, \forall x \geq 0
$$

which is associated with the energy functional of the differential operator exhibits unbalanced growth, that is,

$$
x^{q} \leq \theta(z, x) \leq c_{1}\left[x^{p}+x^{q}\right] \quad \text { for all } z \in \Omega, \text { all } x \geq 0, \text { some } c_{1}>0
$$

[^53]Such functionals were first examined by Marcellini [11] and Zhikov [19] in the context of problems of the calculus of variations and of nonlinear elasticity theory. More recently Marcellini and co-workers and Mingione and co-workers, produced important local regularity results for such problems. We refer to the papers of Marcellini [12] and Baroni-ColomboMingione [1] and the references therein. We also mention the recent informative survey paper of Mingione-Rădulescu [13]. A global regularity theory (that is, regularity up to the boundary), remains so far elusive and this makes double phase problems more difficult to deal with. The second distinguishing feature of problem (1.1), is that the reaction (right hand side) of (1.1) is gradient dependent. This makes the problem nonvariational and this eliminates the use of minimax theorems from the critical point theory. For this reason, our approach is based on the theory of nonlinear operators of monotone type. Variational double phase problems have been studied recently using a variety of methods. We mention the works of Colasuonno-Squassina [2], Gasiński-Winkert [5], Ge-Lv-Lu [7], Liu-Dai [9], Liu-Papageorgiou [10], Papageorgiou-Rădulescu-Repovš [15], Papageorgiou-Vetro-Vetro [16]. On the other hand the study of double phase problems with convection, is lagging behind. There are only the works of GasińskiWinkert [6] and Zeng-Bai-Gasiński-Winkert [18].

Finally we should mention the very recent work of Repovš-Vetro [17], who studied parametric, variational (that is, no convection term is presented) Dirichlet problems, driven by a weighted ( $p, q$ )-Laplacian. However the weights in [17] are bounded away from zero and so the differential operator in [17] exhibits balanced growth. This facilitates the analysis since for such problems there is a global regularity theory available.

## 2 Mathematical background-hypotheses

The unbalanced growth of the integrand corresponding to the differential operator, leads to a functional framework based on Musielak-Orlicz spaces. We introduce the following conditions on the weight $a(\cdot)$, the coefficient $E(\cdot)$ and the exponents $p, q, r$. In what follows by $C^{0,1}(\bar{\Omega})$ we denote the space of locally Lipschitz functions from $\bar{\Omega}$ into $\mathbb{R}$.

$$
\begin{aligned}
H_{0}: & a \in C^{0,1}(\bar{\Omega}), a \neq 0, a(z) \geq 0 \text { for all } z \in \bar{\Omega}, E \in L^{\infty}(\Omega), E(z) \neq 0, E(z) \geq 0 \text { for a.a. } z \in \Omega, \\
& 1<q<p<N, \frac{p}{q}<1+\frac{1}{N} .
\end{aligned}
$$

Remark 2.1. The relation $\frac{p}{q}<1+\frac{1}{N}$ is standard in Dirichlet double phase problems and implies $p<q *=\frac{N q}{N-q}$. So the relation $p \leq r<q *$ makes sense and we have useful embeddings of the relevant Musielak-Orlicz-Sobolev spaces.

Let $\theta: \Omega \times \mathbb{R}_{+} \rightarrow \mathbb{R}_{+} \quad\left(\mathbb{R}_{+}=[0,+\infty)\right)$ be the integrand $\theta(z, x)=a(z) x^{p}+x^{q}$. Evidently $\theta(\cdot, \cdot)$ is continuous and uniformly convex in $x \in \mathbb{R}_{+}$. Let $M(\Omega)=\{u: \Omega \rightarrow$ $\mathbb{R}$ measurable function $\}$. As usual we identify two such functions which differ only on a Lebesgue-null set. The Musielak-Orlicz space $L^{\theta}(\Omega)$ is defined by

$$
L^{\theta}(\Omega)=\left\{u \in M(\Omega): \rho_{\theta}(u)<\infty\right\},
$$

with $\rho_{\theta}(\cdot)$ being the modular function defined by

$$
\rho_{\theta}(u)=\int_{\Omega}\left[a(z)|u|^{p}+|u|^{q}\right] d z .
$$

We equip $L^{\theta}(\Omega)$ with the so called "Luxemburg norm" defined by

$$
\|u\|_{\theta}=\inf \left[\lambda>0: \rho_{\theta}\left(\frac{u}{\lambda}\right) \leq 1\right] .
$$

Then $L^{\theta}(\Omega)$ becomes a Banach space which is also separable and reflexive (in fact uniformly convex). Using $L^{\theta}(\Omega)$ we can define the corresponding Musielak-Orlicz-Sobolev space $W^{1, \theta}(\Omega)$ by

$$
W^{1, \theta}(\Omega)=\left\{u \in L^{\theta}(\Omega):|D u| \in L^{\theta}(\Omega)\right\} .
$$

Here $D u$ denotes the weak gradient of $u(\cdot)$. We equip $W^{1, \theta}(\Omega)$ with the following norm

$$
\|u\|_{1, \theta}=\|u\|_{\theta}+\|D u\|_{\theta} \quad \text { for all } u \in W^{1, \theta}(\Omega)
$$

Here $\|D u\|_{\theta}=\||D u|\|_{\theta}$. Also, we define

$$
W_{0}^{1, \theta}(\bar{\Omega})={\overline{C_{0}^{\infty}(\Omega)}}^{\|\cdot\|_{1, \theta}} .
$$

For this space the Poincaré inequality holds and so on $W_{0}^{1, \theta}(\Omega)$ we consider the equivalent norm

$$
\|u\|=\|D u\|_{\theta} \quad \text { for all } u \in W_{0}^{1, \theta}(\Omega) .
$$

Both spaces are separable and reflexive (in fact uniformly convex).
Given $u \in W_{0}^{1, \theta}(\Omega)$, we define

$$
u^{+}=\max \{u, 0\}, \quad u^{-}=\max \{-u, 0\} .
$$

We know that $u^{+} \in W_{0}^{1, \theta}(\Omega), u=u^{+}-u^{-},|u|=u^{+}+u^{-}$.
We have the following useful embeddings.
Proposition 2.2. If hypotheses $H_{0}$ hold, then
(a) $L^{\theta}(\Omega) \hookrightarrow L^{r}(\Omega)$ and $W_{0}^{1, \theta}(\Omega) \hookrightarrow W_{0}^{1, r}(\Omega)$ continuously and densely for all $1 \leq r \leq q$;
(b) $W_{0}^{1, \theta}(\Omega) \hookrightarrow L^{r}(\Omega)$ continuously (resp. compactly) and densely for all $1 \leq r \leq q^{*}$ (resp. $\left.1 \leq r<q^{*}\right) ;$
(c) $L^{p}(\Omega) \hookrightarrow L^{\theta}(\Omega)$ continuously and densely.

Also there is a close relation between the norm $\|\cdot\|_{\theta}$ and the modular function $\rho_{\theta}(\cdot)$.
Proposition 2.3. If hypotheses $H_{0}$ hold, then
(a) $\|u\|_{\theta}=\lambda \Leftrightarrow \rho_{\theta}\left(\frac{u}{\lambda}\right)=1$;
(b) $\|u\|_{\theta}<1($ resp. $=1,>1) \Leftrightarrow \rho_{\theta}(u)<1($ resp. $=1,>1)$;
(c) $\|u\|_{\theta}<1 \Rightarrow\|u\|_{\theta}^{p} \leq \rho_{\theta}(u) \leq\|u\|_{\theta}^{q} ;$
(d) $\|u\|_{\theta}>1 \Rightarrow\|u\|_{\theta}^{q} \leq \rho_{\theta}(u) \leq\|u\|_{\theta}^{p}$;
(e) $\|u\|_{\theta} \rightarrow 0($ resp. $\rightarrow+\infty) \Leftrightarrow \rho_{\theta}(u) \rightarrow 0($ resp. $\rightarrow+\infty)$.

We consider the nonlinear operators $A_{p}^{a}, A_{q}: W_{0}^{1, \theta}(\Omega) \rightarrow W_{0}^{1, \theta}(\Omega)^{*}$ defined by

$$
\begin{aligned}
\left\langle A_{p}^{a}(u), h\right\rangle & =\int_{\Omega} a(z)|D u|^{p-2}(D u, D h)_{\mathbb{R}^{N}} d z \\
\left\langle A_{q}(u), h\right\rangle & =\int_{\Omega}|D u|^{q-2}(D u, D h)_{\mathbb{R}^{N}} d z \quad \text { for all } u, h \in W_{0}^{1, \theta}(\Omega)
\end{aligned}
$$

We set $V=A_{p}^{a}+A_{q}: W_{0}^{1, \theta}(\Omega) \rightarrow W_{0}^{1, \theta}(\Omega)^{*}$. This operator has the following properties.

Proposition 2.4. If hypotheses $H_{0}$ hold, then $V(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type $(S)_{+}$that is, " $u_{n} \xrightarrow{w} u$ in $W_{0}^{1, \theta}(\Omega)$ and $\lim \sup _{n \rightarrow \infty}\left\langle V\left(u_{n}\right), u_{n}-u\right\rangle \leq 0$ imply that $u_{n} \rightarrow u$ in $W_{0}^{1, \theta}(\Omega) .^{\prime \prime}$

For details on Musielak-Orlicz spaces, we refer to the book of Harjulehto-Hästo [8].
By $\widehat{\lambda}_{1}(q)$ we denote the principal eigenvalue of $\left(-\Delta_{q}, W_{0}^{1, q}(\Omega)\right)$. We know that $\hat{\lambda}_{1}(q)>0$, is simple, isolated and has the following variational characterization

$$
\begin{equation*}
\widehat{\lambda}_{1}(q)=\inf \left[\frac{\|D u\|_{q}^{q}}{\|u\|_{q}^{q}}: u \in W_{0}^{1, q}(\Omega), u \neq 0\right] \tag{2.1}
\end{equation*}
$$

The infimum in (2.1) is realized on the corresponding one-dimensional eigenspace. So, we see that the elements of this eigenspace have fixed sign. By $\widehat{u}_{1}(q)$ we denote the $L^{q}$-normalized (that is, $\|\widehat{u}(q)\|_{q}=1$ ), positive eigenfunction corresponding to $\widehat{\lambda}_{1}(q)$. We know that $\widehat{u}_{1}(q) \in$ $C^{1}(\Omega)$ and $\widehat{u}_{1}(q)(z)>0$ for all $z \in \Omega$. For details we refer to Gasinski-Papageorgiou [4].

The hypotheses on the perturbation $f(z, x)$ are the following:
$\left(H_{1}\right): f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function (that is, for all $x \in \mathbb{R}, z \rightarrow f(z, x)$ is measurable and for a.a. $z \in \Omega, x \rightarrow f(z, x)$ is continuous) such that $f(z, 0)=0$ for a.a. $z \in \Omega$ and
(i) $|f(z, x)| \leq \widehat{a}(z)\left[1+x^{r-1}\right]$ for a.a. $z \in \Omega$, all $x \geq 0$, with $\widehat{a} \in L^{\infty}(\Omega), p \leq r<q^{*}$;
(ii) there exists $M>1$ such that $f(z, x) \leq 0$ for a.a. $z \in \Omega$, all $x \geq M$;
(iii) there exists a function $\eta \in L^{\infty}(\Omega)$ such that

$$
\begin{aligned}
\hat{\lambda}_{1}(q) & \leq \eta(z) \quad \text { for a.a. } z \in \Omega, \quad \eta \not \equiv \hat{\lambda}_{1}(q) \\
\eta(z) & \leq \liminf _{x \rightarrow 0^{+}} \frac{f(z, x)}{x^{q-1}} \quad \text { uniformly for a.a. } z \in \Omega
\end{aligned}
$$

Remark 2.5. Since we look for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_{+}=[0,+\infty)$, without any loss of generality, we may assume that

$$
f(z, x)=0 \quad \text { for a.a. } z \in \Omega, \text { all } x \geq 0
$$

On account of hypotheses $H_{1}\left(\right.$ i), (iii), given $\varepsilon>0$, we can find $c_{\varepsilon}>0$ such that

$$
\begin{equation*}
f(z, x) \geq[\eta(z)-\varepsilon] x^{q-1}-c_{\varepsilon} x^{r-1} \quad \text { for a.a. } z \in \Omega \text { all } x \geq 0 \tag{2.2}
\end{equation*}
$$

## 3 An auxiliary problem

The unilateral growth restriction (2.2) on $f(z, \cdot)$, leads to the following auxiliary double phase problem:

$$
\left\{\begin{array}{l}
-\Delta_{p}^{a} u(z)-\Delta_{q} u(z)=[\eta(z)-\varepsilon] u(z)^{q-1}-c_{\varepsilon} u(z)^{r-1} \quad \text { in } \Omega,  \tag{3.1}\\
\left.u\right|_{\partial \Omega}=0, \quad u>0 .
\end{array}\right.
$$

Proposition 3.1. If hypotheses $H_{0}$ hold, then for all $\varepsilon>0$ small problem (2.2) has a unique solution $\underline{u} \in W_{0}^{1, \theta}(\Omega) \cap L^{\infty}(\Omega), \underline{u}(z)>0$ for a.a. $z \in \Omega$.
Proof. Consider the $C^{1}$-functional $\varphi_{\varepsilon}: W_{0}^{1, \theta}(\Omega) \rightarrow \mathbb{R}$ defined by

$$
\varphi_{\varepsilon}(u)=\frac{1}{p} \rho_{a}(D u)+\frac{1}{q}\|D u\|_{q}^{q}+\frac{c_{\varepsilon}}{r}\left\|u^{+}\right\|_{r}^{r}-\frac{1}{q} \int_{\Omega}[\eta(z)-\varepsilon]\left(u^{+}\right)^{q} d z .
$$

Here $\rho_{a}(D u)=\int_{\Omega} a(z)|D u|^{p} d z$. Since $q<p \leq r$, we see that $\varphi_{\varepsilon}(\cdot)$ is coercive. Also since $\rho_{a}(\cdot)$ is continuous, convex, exploiting the compact embedding of $W_{0}^{1, \theta}(\Omega)$ into $L^{r}(\Omega)$ (see Proposition 2.2), we infer that $\varphi_{\varepsilon}(\cdot)$ is sequentially weakly lower semi-continuous. So, by the Weierstrass-Tonelli theorem, we can find $\underline{u} \in W_{0}^{1, \theta}(\Omega)$ such that

$$
\begin{equation*}
\varphi_{\varepsilon}(\underline{u})=\inf \left[\varphi_{\varepsilon}(u): u \in W_{0}^{1, \theta}(\Omega)\right] . \tag{3.2}
\end{equation*}
$$

Let $\widehat{\lambda}_{1}=\widehat{\lambda}_{1}(q), \widehat{u}_{1}=\widehat{u}_{1}(q)$, and $t \in(0,1)$. We have

$$
\varphi_{\varepsilon}\left(t \widehat{u}_{1}\right)=\frac{t^{p}}{p} \rho_{a}\left(D \widehat{u}_{1}\right)+\frac{t^{q}}{q}\left[\int_{\Omega}\left(\widehat{\lambda}_{1}-\eta(z)\right) \widehat{u}_{1}^{q} d z+\varepsilon\right]+\frac{t^{r} c_{\varepsilon}}{r}\left\|\widehat{u}_{1}\right\|_{r}^{r}
$$

Since $\widehat{u}_{1}(z)>0$ for all $z \in \Omega$, hypotheses $H_{1}$ (iii) implies that

$$
\mu_{0}=\int_{\Omega}\left[\eta(z)-\widehat{\lambda}_{1}\right] \widehat{u}_{1}^{q} d z>0
$$

So, choosing $\varepsilon \in\left(0, \mu_{0}\right)$ and since $p \leq r$ and $t \in(0,1)$, we have

$$
\varphi_{\varepsilon}\left(t \widehat{u}_{1}\right) \leq c_{1} t^{p}-c_{2} t^{q} \text { for some } c_{1}, c_{2}>0
$$

Recall that $q<p$. So, choosing $t \in(0,1)$ even smaller if necessary, we see that

$$
\begin{aligned}
& \varphi_{\varepsilon}\left(t \widehat{u}_{1}\right)<0 \\
\Rightarrow & \varphi_{\varepsilon}(\underline{u})<0=\varphi_{\varepsilon}(0) \quad(\operatorname{see}(3.2)), \\
\Rightarrow & \underline{u} \neq 0
\end{aligned}
$$

From (3.2) we have $\varphi_{\varepsilon}^{\prime}(\underline{u})=0$,

$$
\begin{equation*}
\Rightarrow\langle V(u), h\rangle=\int_{\Omega}\left[(\eta(z)-\varepsilon) \underline{u}^{q-1}-c_{\varepsilon} \underline{u}^{r-1}\right] h d z \quad \text { for all } h \in W_{0}^{1, \theta}(\Omega) \tag{3.3}
\end{equation*}
$$

Choosing $h=-\underline{u}^{-} \in W_{0}^{1, \theta}(\Omega)$ in (3.3), we obtain

$$
\rho_{\theta}\left(D \underline{u}^{-}\right)=0 \quad \Rightarrow \quad \underline{u} \geq 0, \underline{u} \neq 0 .
$$

Therefore $\underline{u}$ is a weak solution of (3.1). From Theorem 3.1 of Gasiński-Winkert [5], we have that

$$
\underline{u} \in W_{0}^{1, \theta}(\Omega) \cap L^{\infty}(\Omega)
$$

Moreover, Proposition 2.4 of Papageorgiou-Vetro-Vetro [16] implies that

$$
\underline{u}(z)>0 \quad \text { for a.a. } z \in \Omega .
$$

Next we show that this positive solution of (3.1) is unique. So, suppose that $\underline{v} \in W_{0}^{1, \theta}(\Omega)$ is another positive solution of (3.1). Again we show that

$$
\underline{v} \in W_{0}^{1, \theta}(\Omega) \cap L^{\infty}(\Omega), \quad \underline{v}(z)>0 \quad \text { for a.a. } z \in \Omega .
$$

Let $\underline{u}_{\delta}=\underline{u}+\delta, \underline{v}_{\delta}=\underline{v}+\delta, \delta>0$. If $L^{\infty}(\Omega)_{+}=\left\{u \in L^{\infty}(\Omega): u(z) \geq 0 \quad\right.$ for a.a. $\left.z \in \Omega\right\}$ (the positive (order) cone of the ordered Banach space $L^{\infty}(\Omega)$ ), then $\underline{u}_{\delta}, \underline{v}_{\delta} \in \operatorname{int} L^{\infty}(\Omega)_{+}$. Hence using Proposition 4.1.22, p. 274, of Papageorgiou-Rădulescu-Repovš [14], we have

$$
\begin{array}{ll}
\underline{\underline{u}_{\delta}}  \tag{3.4}\\
\underline{v}_{\delta}
\end{array} L^{\infty}(\Omega), \quad \frac{\underline{v}_{\delta}}{\underline{u}_{\delta}} \in L^{\infty}(\Omega) .
$$

We consider the integral functional $j: L^{1}(\Omega) \rightarrow \overline{\mathbb{R}}=\mathbb{R} \cup\{+\infty\}$ defined by

$$
j(u)= \begin{cases}\frac{1}{p} \rho_{a}\left(D u^{\frac{1}{q}}\right)+\frac{1}{q}\left\|D u^{\frac{1}{q}}\right\|_{q}^{q} & \text { if } u \geq 0, u^{\frac{1}{q}} \in W^{1, \theta}(\Omega), \\ +\infty & \text { otherwise. }\end{cases}
$$

The convexity of $\rho_{a}(\cdot)$ implies that $j(\cdot)$ is convex (see Diaz-Saá [3]). On account of (3.4), if $h=\underline{u}_{\delta}^{q}-\underline{v}_{\delta}^{q} \in W^{1, \theta}(\Omega)$ and $|t|<1$ is small, we have

$$
\underline{u}_{\delta}^{q}+t h \in \operatorname{dom} j, \quad \underline{v}_{\delta}^{q}+\text { th } \in \operatorname{dom} j
$$

where $\operatorname{dom} j=\left\{u \in L^{1}(\Omega): j(u)<\infty\right\}$ (the effective domain of $j(\cdot)$ ). Then using the convexity of $j(\cdot)$, we see that $j(\cdot)$ is Gateaux differentiable at $\underline{u}_{\delta}^{q}$ and at $\underline{v}_{\delta}^{q}$ in the direction $h$. Moreover, using the chain rule and the nonlinear Green's identity (see [14, p. 34]), we have

$$
\begin{aligned}
j^{\prime}\left(\underline{u}_{\delta}^{q}\right)(h) & =\frac{1}{q} \int_{\Omega} \frac{-\Delta_{p}^{a} \underline{u}_{\delta}-\Delta_{q} \underline{u}_{\delta}}{\underline{u}_{\delta}^{q-1}} h d z \\
& =\frac{1}{q} \int_{\Omega} \frac{[\eta(z)-\varepsilon] \underline{u}_{\delta}^{q-1}-c_{\varepsilon} \underline{u}_{\delta}^{r-1}}{\underline{u}_{\delta}^{q-1}} h d z, \quad(\text { see (3.1)). }
\end{aligned}
$$

and

$$
\begin{aligned}
j^{\prime}\left(\underline{v}_{\delta}^{q}\right)(h) & =\frac{1}{q} \int_{\Omega} \frac{-\Delta_{p}^{a} \underline{v}_{\delta}-\Delta_{q} \underline{v}_{\delta}}{\underline{v}_{\delta}^{q-1}} h d z \\
& =\frac{1}{q} \int_{\Omega} \frac{[\eta(z)-\varepsilon] \underline{v}_{\delta}^{q-1}-c_{\varepsilon} v_{\delta}^{r-1}}{\underline{v}_{\delta}^{q-1}} h d z, \quad(\text { see (3.1)). }
\end{aligned}
$$

The convexity of $j(\cdot)$ implies the monotonicity of $j^{\prime}(\cdot)$, Hence

$$
\begin{equation*}
0 \leq \int_{\Omega}[\eta(z)-\varepsilon]\left[\frac{\underline{u}^{q-1}}{\underline{u}_{\delta}^{q-1}}-\frac{\underline{v}^{q-1}}{\underline{v}_{\delta}^{q-1}}\right]\left(\underline{u}_{\delta}^{q}-\underline{v}_{\delta}^{q}\right) d z+\int_{\Omega} c_{\varepsilon}\left[\frac{\underline{v}^{r-1}}{\underline{v}_{\delta}^{q-1}}-\frac{\underline{u}^{r-1}}{\underline{u}_{\delta}^{q-1}}\right]\left(\underline{u}_{\delta}^{q}-\underline{v}_{\delta}^{q}\right) d z . \tag{3.5}
\end{equation*}
$$

Note that for $\delta \in(0,1]$, we have

$$
\left|\frac{\underline{u}^{q-1}}{\underline{u}_{\delta}^{q-1}}-\frac{\underline{v}^{q-1}}{\underline{v}_{\delta}^{q-1}}\right|\left|\underline{u}_{\delta}^{q}-\underline{v}_{\delta}^{q}\right| \leq 2^{q}\left[\|u\|_{\infty}^{q}+\|v\|_{\infty}^{q}+2\right],
$$

$$
\left[\frac{\underline{u}^{q-1}}{\underline{u}_{\delta}^{q-1}}-\frac{\underline{v}^{q-1}}{\underline{v}_{\delta}^{q-1}}\right]\left(\underline{u}_{\delta}^{q}-\underline{v}_{\delta}^{q}\right) \rightarrow 0 \quad \text { for a.a. } z \in \Omega \text {, as } \delta \rightarrow 0^{+} .
$$

So, invoking the dominated convergence theorem, we obtain

Also, for $\delta \in(0,1]$ we have

$$
\begin{gathered}
\left\lvert\, \frac{\underline{v}^{r-1}}{\left.\frac{\underline{v}_{\delta}^{q-1}}{q-1}-\frac{\underline{u}^{r-1}}{\underline{u}_{\delta}^{q-1}}| | \underline{u}_{\delta}^{q}-\underline{v}_{\delta}^{q-1} \right\rvert\, \leq 2^{q-1}\left[\|\underline{v}\|_{\infty}^{r-q}+\|\underline{u}\|_{\infty}^{r-q}\right]\left[\|\underline{u}\|_{\infty}^{q}+\|\underline{v}\|_{\infty}^{q}+2\right],}\right. \\
\left|\frac{\underline{v}^{r-1}}{\underline{v}_{\delta}^{q-1}}-\frac{\underline{u}^{r-1}}{\underline{u}_{\delta}^{q-1}}\right|\left(\underline{u}_{\delta}^{q}-\underline{v}_{\delta}^{q}\right) \rightarrow\left(\underline{v}^{r-q}+\underline{u}^{r-q}\right)\left(\underline{u}^{q}-\underline{v}^{q}\right) \quad \text { for a.a. } z \in \Omega, \text { as } \delta \rightarrow 0^{+} .
\end{gathered}
$$

Then once again the dominated convergence theorem gives

$$
\begin{equation*}
\int_{\Omega} c_{\varepsilon}\left[\frac{\underline{v}^{r-1}}{\underline{v}_{\delta}^{q-1}}-\frac{\underline{u}^{r-1}}{\underline{u}_{\delta}^{q-1}}\right]\left(\underline{u}_{\delta}^{q}-\underline{v}_{\delta}^{q}\right) d z \rightarrow \int_{\Omega} c_{\varepsilon}\left[\underline{v}^{r-q}-\underline{u}^{r-q}\right]\left(\underline{u}^{q}-\underline{v}^{q}\right) d z \quad \text { as } \delta \rightarrow 0^{+} . \tag{3.7}
\end{equation*}
$$

We return to (3.5), pass to the limit as $\delta \rightarrow 0^{+}$and use (3.6) and (3.7). We obtain

$$
\begin{aligned}
& 0 \leq \int_{\Omega} c_{\varepsilon}\left[\underline{v}^{r-q}-\underline{u}^{r-q}\right]\left(\underline{u}^{q}-\underline{v}^{q}\right) d z \leq 0, \\
& \Rightarrow \underline{u}=\underline{v} .
\end{aligned}
$$

This proves the uniqueness of the positive solution of (3.1).
In the next section, we will use this solution $\underline{u} \in W_{0}^{1, \theta}(\Omega) \cap L^{\infty}(\Omega)$ of (3.1), to produce a nontrivial positive solution for problem (1.1).

## 4 Positive solution

Let $M>1$ be as in hypothesis $H_{1}$ (ii). Choose $\bar{u} \geq M>1$ big so that $\|\underline{u}\|_{\infty}<\bar{u}$. We have $\underline{u}<\bar{u}$. Then on account of hypothesis $H_{1}(\mathrm{iii})$, we have

$$
f(z, \bar{u}) \leq 0 \quad \text { a.a. } z \in \Omega .
$$

We introduce the truncation map $\tau: L^{q}(\Omega) \rightarrow L^{q}(\Omega)$ defined by

$$
\tau(u)(z)= \begin{cases}\underline{u}(z) & \text { if } u(z)<\underline{u}(z)  \tag{4.1}\\ u(z) & \text { if } \underline{u}(z) \leq u(z) \leq \bar{u} \\ \bar{u} & \text { if } \bar{u}<u(z)\end{cases}
$$

Evidently $\tau(\cdot)$ is continuous and $\tau(u) \in W_{0}^{1, \theta}(\Omega)$ if $u \in W_{0}^{1, \theta}(\Omega)$.
Let $N_{f}(\tau(u))(\cdot)=f(\cdot, \tau(u)(\cdot))$ (the Nemitsky map corresponding to $f$ ). We define

$$
N_{\tau}(u)(\cdot)=N_{f}(\tau(u))(\cdot)+E(\cdot)|D \tau(u)|^{q-1} \quad \text { for all } u \in W_{0}^{1, \theta}(\Omega) .
$$

We consider the map $K: W_{0}^{1, \theta}(\Omega) \rightarrow W_{0}^{1, \theta}(\Omega)^{*}$ defined by

$$
K(u)=V(u)-N_{\tau}(u) \quad \text { for all } u \in W_{0}^{1, \theta}(\Omega) .
$$

Proposition 4.1. If hypotheses $H_{0}, H_{1}$ hold, then $K(\cdot)$ is pseudomonotone.
Proof. Consider a sequence $\left\{u_{n}\right\}_{n \in \mathbb{N}} \subseteq W_{0}^{1, \theta}(\Omega)$ such that

$$
\left\{\begin{array}{l}
u_{n} \xrightarrow{w} u \text { in } W_{0}^{1, \theta}(\Omega), K\left(u_{n}\right) \xrightarrow{w} u^{*} \text { in } W_{0}^{1, \theta}(\Omega)^{*},  \tag{4.2}\\
\lim \sup _{n \rightarrow \infty}\left\langle K\left(u_{n}\right), u_{n}-u\right\rangle \leq 0 .
\end{array}\right\}
$$

From (4.2) and since $W_{0}^{1, \theta}(\Omega) \hookrightarrow L^{q}(\Omega)$ compactly (see Proposition 2.2), we have $u_{n} \rightarrow u$ in $L^{q}(\Omega)$. This implies that $\tau\left(u_{n}\right) \rightarrow \tau(u)$ in $L^{q}(\Omega)$. Then by Krasnoselskii's theorem (see Gasiński-Papageorgiou [4], p. 407), we have

$$
\begin{equation*}
N_{f}\left(\tau\left(u_{n}\right)\right) \rightarrow N_{f}(\tau(u)) \quad \text { in } L^{q^{\prime}}(\Omega) \quad\left(\frac{1}{q}+\frac{1}{q^{\prime}}=1\right) \tag{4.3}
\end{equation*}
$$

Moreover, we have

$$
\left\{D \tau\left(u_{n}\right)\right\}_{n \in \mathbb{N}} \subseteq L^{\theta}\left(\Omega, \mathbb{R}^{N}\right) \hookrightarrow L^{q}\left(\Omega, \mathbb{R}^{N}\right) \quad \text { is bounded (see Proposition 2.2). }
$$

Therefore

$$
\begin{equation*}
\left.\left.\langle E(\cdot)| D \tau\left(u_{n}\right)\right|^{q-1}, u_{n}-u\right\rangle=\int_{\Omega} E(z)\left|D \tau\left(u_{n}\right)\right|^{q-1}\left(u_{n}-u\right) d z \rightarrow 0 \tag{4.4}
\end{equation*}
$$

From (4.2), (4.3) and (4.4), it follows that

$$
\limsup _{n \rightarrow \infty}\left\langle V\left(u_{n}\right), u_{n}-u\right\rangle \leq 0 \Rightarrow u_{n} \rightarrow u \text { in } W_{0}^{1, \theta}(\Omega) \text { (see Proposition 2.4). }
$$

Then we have

$$
V\left(u_{n}\right) \rightarrow V(u) \text { in } W_{0}^{1, \theta}(\Omega)^{*},
$$

$$
\begin{aligned}
N_{f}\left(\tau\left(u_{n}\right)\right) \rightarrow & N_{f}(\tau(u)) \text { in } L^{q^{\prime}}(\Omega) \hookrightarrow W_{0}^{1, \theta}(\Omega)^{*} \quad \text { (see Gasiński-Papageorgiou [4], p. 141), } \\
& E(\cdot)\left|D \tau\left(u_{n}\right)\right|^{q-1} \rightarrow E(\cdot)|D \tau(u)|^{q-1} \quad \text { in } L^{q^{\prime}}(\Omega) \hookrightarrow W_{0}^{1, \theta}(\Omega)^{*}
\end{aligned}
$$

So, finally we have

$$
\begin{gathered}
u^{*}=V(u)-N_{\tau}(u)=K(u) \quad(\text { see }(4.2)) \\
\left\langle K\left(u_{n}\right), u_{n}\right\rangle \rightarrow\langle K(u), u\rangle
\end{gathered}
$$

This means that $K(\cdot)$ is generalized pseudomonotone and by Proposition 3.2.49, p. 333, of Gasiński-Papageorgiou [4], we conclude that $K(\cdot)$ is pseudomonotone.

Proposition 4.2. If hypotheses $H_{0}, H_{1}$ hold, then the map $K(\cdot)$ is strongly coercive (see [14], p. 130).
Proof. For every $u \in W_{0}^{1, \theta}(\Omega)$ with $\|u\|>1$, we have

$$
\begin{aligned}
\langle K(u), u\rangle & =\rho_{\theta}(D u)-\int_{\Omega} f(z, \tau(u)) u d z-\int_{\Omega} E(z)|D \tau(u)|^{q-1} u d z \\
& \geq c_{3}\|u\|^{q}-c_{4}\|u\|^{q-1} \text { for some } c_{3}, c_{4}>0 \quad \text { (see Proposition } 2.3 \text { and (4.1)) } \\
& \Rightarrow K(\cdot) \text { is strongly coercive. }
\end{aligned}
$$

Now we are ready for the existence theorem.
Theorem 4.3. If hypotheses $H_{0}$ and $H_{1}$ hold, then problem (1.1) has a positive solution $u_{0} \in$ $W_{0}^{1, \theta}(\Omega) \bigcap L^{\infty}(\Omega)$ with $u_{0}(z)>0$ for a.a. $z \in \Omega$.

Proof. Propositions 4.1 and 4.2 together with Theorem 3.2.52, p. 336, of Gasiński-Papageorgiou [4], imply that $K(\cdot)$ is surjective. So we can find $u_{0} \in W_{0}^{1, \theta}(\Omega)$ such that

$$
K\left(u_{0}\right)=0 .
$$

Then we have

$$
\begin{aligned}
\left\langle V\left(u_{0}\right),\left(\underline{u}-u_{0}\right)^{+}\right\rangle & \left.\geq \int_{\Omega} f(z, \underline{u})\left(\underline{u}-u_{0}\right)^{+} d z \text { (see (4.1) and recall } E \geq 0\right) \\
& \geq \int_{\Omega}\left([\eta(z)-\varepsilon] \underline{u}^{q-1}-c_{\varepsilon} \underline{u}^{r-1}\right)\left(\underline{u}-u_{0}\right)^{+} d z \quad \text { (see (2.2)) } \\
& =\left\langle V(\underline{u}),\left(\underline{u}-u_{0}\right)^{+}\right\rangle \quad \text { (see Proposition 4) } \\
& \Rightarrow \underline{u} \leq u_{0} \quad \text { (see Proposition 3). }
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\left\langle V\left(u_{0}\right),\left(u_{0}-\bar{u}\right)^{+}\right\rangle & \left.=\int_{\Omega} f(z, \bar{u})\left(u_{0}-\bar{u}\right)^{+} d z \quad \text { (see (4.1) and note } D \bar{u}=0\right) \\
& \leq 0=\left\langle V(\bar{u}),\left(u_{0}-\bar{u}\right)^{+}\right\rangle \quad\left(\text { see } H_{1}(i i)\right) \\
& \Rightarrow u_{0} \leq \bar{u} \quad(\text { see Proposition 3). }
\end{aligned}
$$

So we have proved

$$
\begin{aligned}
& u_{0} \in[\underline{u}, \bar{u}]=\left\{u \in W_{0}^{1, \theta}(\Omega): \underline{u}(z) \leq u(z) \leq \bar{u} \quad \text { for a.a. } z \in \Omega\right\}, \\
\Rightarrow & u_{0} \in W_{0}^{1, \theta}(\Omega) \cap L^{\infty}(\Omega) \quad \text { is a positive solution of (1.1). }
\end{aligned}
$$

Moreover, we have

$$
0<\underline{u}(z) \leq u_{0}(z) \quad \text { for a.a. } z \in \Omega .
$$

## Acknowledgements

The work was supported by NNSF of China Grant Nos. 12071413, 12111530282, NSF of Guangxi Grant No. 2018GXNSFDA138002. The authors wish to thank an anonymous referee for his/her corrections and remarks.

## References

[1] P. Baroni, M. Colombo, G. Mingione, Regularity for general functionals with double phase. Calc. Var. Partial Differential Equations 57(2019), Art. No. 62, 48pp. https://doi. org/10.1007/s00526-018-1332-z; MR3775180
[2] F. Colasuonno, M. Squassina, Eigenvalues for double phase variational integrals. Ann. Mat. Pura Appl. (4) 195(2016), No. 6, 1917-1959. https ://doi.org/10.1007/s10231-015-0542-7; MR3558314
[3] J. I. Diaz, J. E. Sấ, Existence et unicité de solutions positives pour certaines equations elliptique quasilineaires (in French) [Existence and uniqueness of positive solutions of some quasilinear elliptic equations], C. R. Acad. Sci. Paris Sér. I Math. 305(1987), 521-524. MR0916325
[4] L. Gasiński, N. S. Papageorgiou, Nonlinear analysis, Chapman \& Hall/CRC, Boca Raton, FL, 2006. MR2168068
[5] L. Gasiński, P. Winkert, Constant sign solutions for double phase problems with superlinear nonlinearity, Nonlinear Anal. 195(2020), 111739. https://doi.org/10.1016/j.na. 2019.111739; MR4050785.
[6] L. Gasí́ski, P. Winkert, Existence and uniqueness results for double phase problems with convection, J. Differential Equations 208(2020), 4183-4193. https://doi.org/ 10.1016/j.jde.2019.10.022; MR4066014
[7] B. Ge, D. Lv, J. Lu, Multiple solutions for a class of double phase problem without the Ambrosetti-Rabinowitz condition, Nonliear Anal. 188(2019), 294-315. https://doi.org/ 10.1016/j.na.2019.06.007; MR3964193
[8] P. Harjulehto, P. Hästö, Orlicz spaces and generalized Orlicz spaces, Lecture Notes in Mathematics, Vol. 2236, Springer, Cham, 2019. https://doi.org/10.1007/978-3-030-15100-3; MR3931352
[9] W. Liu, G. Dai, Existence and multiplicity results for double phase problems, J. Differential Equations 265(2018), 4311-4334. https://doi.org/10.1016/j.jde.2018.06.006; MR3843302
[10] Z. H. Liu, N. S. Papageorgiou, Solutions for parametric double phase Robin problems, Asymptot. Anal. 124(2021), 291-302. https://doi.org/10.3233/ASY-201645; MR4198477
[11] P. Marcellini, Regularity and existence of solutions of elliptic equations with $p, q-$ growth conditions, J. Differential Equations 90(1991), 1-30. https://doi.org/10.1016/ 0022-0396(91) 90158-6; MR1094446
[12] P. Marcellini, Growth conditions and regalarity for weak solutions to nonlinear elliptic pdes, J. Math. Anal. Appl. 501(2021), No. 1, 124408. https://doi.org/10.1016/j.jmaa. 2020.124408; MR4258802
[13] G. Mingione, V. D. Rădulescu, Recent developments in problems with nonstandard growth and nonuniform ellipticity, J. Math. Anal. Appl. 501(2021), 125197. https://doi. org/10.1016/j.jmaa.2021.125197; MR4258810
[14] N. S. Papageorgiou, V. D. Rădulescu, D. Repovš, Nonlinear analysis-theory and methods, Springer Monographs in Mathematics, Springer, Cham, 2019. https://doi. org/10. 1007/978-3-030-03430-6; MR3890060
[15] N. S. Papageorgiou, V. D. Rădulescu, D. Repovš, Existence and multiplicity of solutions for double-phase Robin problems, Bull. London Math. Soc. 52(2020), 546-560. https:// doi.org/10.1112/blms.12347; MR4171387
[16] N. S. Papageorgiou, C. Vetro, F. Vetro, Multiple solutions for parametric double phase Dirichlet problems, Comm. Contemp. Math. 23(2021), 2050006, 18 pp. https://doi.org/ 10.1142/S0219199720500066; MR4237564
[17] D. Repovš, C. Vetro, The behavior of solutions of a parametric weighted ( $p, q$ )-Laplacian equation, AIMS Math. 7(2022), No. 1, 499-517. https://doi. org/10.3934/math. 2022032; MR4332388
[18] S. D. Zeng, Y. R. Bai, L. Gasiński, P. Winkert, Existence results for double phase implicit obstacle problems involving multivalued operators, Calc. Var. Partial Differential Equations 59(2020), No. 5, Article No. 176, 18 pp. https://doi.org/10.1007/s00526-020-01841-2; MR4153902
[19] V. V. Zhiкоv, Averaging of functionals of the calculus of variations and elasticity (English; Russian original), Math. USSR-Izv. 29(1987), 33-66; translation from Izv. Akad. Nauk SSSR, Ser. Mat. 50(1986), No. 4, 675-710, 877. MR0864171; Zbl 599.49031

# Minimizing of the quadratic functional on Hopfield networks 

Oleksandr Boichuk ${ }^{\boxtimes}$, Oleksandr Pokutnyi, Viktor Feruk and Dmytro Bihun

Institute of Mathematics of the National Academy of Sciences, 3 Tereschenkivska Street, Kyiv, 01024, Ukraine

Received 12 May 2021, appeared 29 December 2021
Communicated by Michal Fečkan


#### Abstract

In this paper, we consider the continuous Hopfield model with a weak interaction of network neurons. This model is described by a system of differential equations with linear boundary conditions. Also, we consider the questions of finding necessary and sufficient conditions of solvability and constructive construction of solutions of the given problem, which turn into solutions of the linear generating problem, as the parameter $\varepsilon$ tends to zero. An iterative algorithm for finding solutions has been constructed. The problem of finding the extremum of the target functions on the given problem solution is considered. To minimize a functional, an accelerated method of conjugate gradients is used. Results are illustrated with examples for the case of three neurons.


Keywords: boundary-value problem, Moore-Penrose pseudo-inverse matrix, differential equations, Hopfield networks, quadratic functional.
2020 Mathematics Subject Classification: 34B05, 49K15.

## 1 Introduction

The study of various natural and social phenomena is carried out today by building and investigating their mathematical models. Practical applications contributed to the birth and development of many mathematical disciplines. Among them, there is a theory of dynamic neural networks, which are used to solve various optimization problems, control theory and mathematical modelling. The variety of tasks to be solved led to the existence of several models of such networks. An important place among them takes Hopfield model (see [22,25,43]), a single layer neural network with general non-linear and additional internal linear connections among neurons. Hopfield nets have a large number of publications. Both models with discrete and continuous time are considered. In particular, such questions as stability (see [47]), absolute stability of neural nets (see [15]), modelling of closed control systems, asymptotics and

[^54]stability of relaxation self-oscillations in Hopfield nets with delay (see [19,20]) are considered. The vital phenomenon is flow invariance for such systems (see [34]). Ill-posed problems with fractional derivative (see [46]), optimization problems (see [17,26,45]), deep neural networks (see [35]), relativistic Hopfield model (see [1]), quantum generalization of Hopfield model (see [40]), as well as its discrete analogue (see [2]) are studied. Chaos is explored in the corresponding models (see [14]). The Hopfield model is considered as a model of memory (see [23]). The impulsive Hopfield model with boundary conditions is studied in [38]. In this work, the case of weak interaction of network neurons is considered, to the study of which for other models, for example, the papers are devoted $[27,29]$. Using the theory of pseudo-inverse matrices (see [3-8,41]), an approach allows establishing necessary and sufficient conditions for the solvability of boundary-value problem for a system of differential equations that describe Hopfield network for $n$ neurons with weak interaction. We use hyperbolic tangent as the increasing activation function and symmetric matrix of weights as in [22, p. 690]). The application of the accelerated method of conjugate gradients (see [31,32]) for solving the problem of finding the extremum (minimum) of the loss function is explored on the solutions of the given problem in the form of a quadratic functional of synaptic communication scales.

## 2 Formulation of the problem

We consider a continuous Hopfield model with a weak interaction of the network neurons, the evolution in time of which is described by a system of $n$ non-linear differential equations (see [22, p. 690], [43, p. 140])

$$
\begin{equation*}
x_{j}^{\prime}(t)=-\frac{x_{j}(t)}{R_{j}}+\varepsilon\left(\hat{I}_{j}(t)+\sum_{i=1}^{n} w_{i j} \tanh \left(\frac{a_{i} x_{i}(t)}{2}\right)\right)+I_{j}(t), \quad j=\overline{1, n}, \tag{2.1}
\end{equation*}
$$

where $x_{j}(t) \in \mathbf{C}^{1}[0, T]$ is the potential of the $j$ th neuron; the real parameters $a_{j}$ are gain coefficients of the $j$ th neuron, and $w_{i j}$ are the elements of a symmetric matrix $W$ :

$$
W=\left(\begin{array}{cccc}
0 & w_{12} & \ldots & w_{1 n} \\
w_{12} & 0 & \ldots & w_{2 n} \\
\vdots & \vdots & \ddots & \vdots \\
w_{1 n} & w_{2 n} & \ldots & 0
\end{array}\right)
$$

which consists of synaptic weights of the connection of the $i$ th neuron with the $j$ th neuron, $R_{j}$ are the leakage resistances, $\hat{I}_{j}(t) \in \mathbf{C}^{1}[0, T], I_{j}(t) \in \mathbf{C}^{1}[0, T]$ are external signals, $\varepsilon \ll 1$ is a small parameter characterizing the strength of the interaction of network neurons.

As is known (see [22, p. 693], [43, p. 144]), in practice the property of a monotonic increasing of the activation function (in our case this is hyperbolic tangent) of the considered matrix $W\left(w_{i j}=w_{j i}, w_{i i}=0\right)$ and the asynchronous mode of network operation are often used. It provides the global asymptotic stability of the Hopfield network. These features persist in the case of a weak interaction of network neurons described by equation (2.1) and provide both practical and theoretical interest in Hopfield networks. For the convenience of further reasoning, we rewrite the equation (2.1) in the following form

$$
\begin{equation*}
x^{\prime}(t, w, \varepsilon)=A x(t, w, \varepsilon)+\varepsilon(\hat{I}(t)+W Z(x(t, w, \varepsilon)))+I(t), \tag{2.2}
\end{equation*}
$$

where

$$
x(t, w, \varepsilon)=\operatorname{col}\left(x_{1}(t, w, \varepsilon), \quad x_{2}(t, w, \varepsilon), \quad \ldots, \quad x_{n}(t, w, \varepsilon)\right),
$$

$$
\begin{gathered}
A=-\operatorname{diag}\left\{\frac{1}{R_{1}}, \frac{1}{R_{2}}, \ldots, \frac{1}{R_{n}}\right\}, \\
Z(x(t, w, \varepsilon))=\operatorname{col}\left(\tanh \left(\frac{a_{1} x_{1}(t, w, \varepsilon)}{2}\right), \quad \tanh \left(\frac{a_{2} x_{2}(t, w, \varepsilon)}{2}\right), \ldots, \tanh \left(\frac{a_{n} x_{n}(t, w, \varepsilon)}{2}\right)\right), \\
\hat{I}(t)=\operatorname{col}\left(\hat{I}_{1}(t), \quad \hat{I}_{2}(t), \ldots, \quad \hat{I}_{n}(t)\right), \quad I(t)=\operatorname{col}\left(I_{1}(t), \quad I_{2}(t), \ldots, \quad I_{n}(t)\right),
\end{gathered}
$$

$w$ is a vector of dimension $M=n(n-1) / 2$ formed from the elements of the matrix $W$ in the following way:

$$
w=\operatorname{col}\left(w_{12}, \quad w_{22}, \ldots, \quad w_{1 n}, \quad w_{23}, \quad w_{24}, \ldots, \quad w_{2 n}, \ldots, \quad w_{(n-1) n}\right)
$$

In some cases, the solutions of systems of equations describing the functioning of neural nets satisfy additional conditions due to particular properties of the modelled process. Various types of boundary-value problems are explored for such systems (see [12,38,44]). In our paper, we investigate the questions of finding conditions for the existence and effective construction of equation (2.2) solutions with $m$ boundary conditions

$$
\begin{equation*}
l x(\cdot, w, \varepsilon)=\alpha \tag{2.3}
\end{equation*}
$$

$l=\operatorname{col}\left(l_{1}, \quad l_{2}, \ldots, l_{m}\right): \mathbf{C}^{1}[0, T] \rightarrow \mathbb{R}^{m}$ is bounded linear vector functional, $l_{v}: \mathbf{C}^{1}[0, T] \rightarrow$ $\mathbb{R}, v=\overline{1, m}, \alpha=\operatorname{col}\left(\alpha_{1}, \alpha_{2}, \ldots, \quad \alpha_{m}\right) \in \mathbb{R}^{m}$, which for $\varepsilon=0$ turns into the solution of the generating problem

$$
\begin{align*}
& x^{\prime}(t)=A x(t)+I(t)  \tag{2.4}\\
& l x(\cdot)=\alpha . \tag{2.5}
\end{align*}
$$

These solutions will be called generating solutions of the boundary-value problem (2.2), (2.3). Note that the boundary-value problem (2.2), (2.3) includes both underdetermined ( $m<n$ ) and overdetermined $(m>n)$ boundary-value problems, the study of which for Hopfield models is not given enough attention, in our opinion.

## 3 Necessary condition for the solvability of the problem (2.2), (2.3)

Let us first consider the question of solution existence to the problem (2.2), (2.3). For this purpose, we use the general scheme for the exploration of boundary-value problems studied in detail in [7], which allows finding effective coefficients which are necessary and sufficient for the solvability of problem (2.2), (2.3). In particular, for the generating problem (2.4), (2.5), the following criterion holds (see [7]).

Theorem 3.1. The homogeneous problem (2.4), (2.5) $(I(t)=0, \alpha=0)$ has an $r$-parametric ( $r \leq n$ ) family of solutions $x\left(t, c_{r}\right) \in \mathbf{C}^{1}[0, T]$

$$
x\left(t, c_{r}\right)=U(t) P_{Q_{r}} c_{r} \quad \forall c_{r} \in \mathbb{R}^{r} .
$$

The inhomogeneous problem (2.4), (2.5) is solvable if and only if $g$ satisfies $d(d \leq m)$ linearly independent conditions:

$$
\begin{equation*}
P_{Q_{d}^{*}} g=0 \tag{3.1}
\end{equation*}
$$

In this case, the inhomogeneous problem (2.4), (2.5) has an r-parameter family of solutions $x\left(t, c_{r}\right) \in$ $\mathbf{C}^{1}[0, T]$ of the following form:

$$
\begin{equation*}
x\left(t, c_{r}\right)=U(t) P_{Q_{r}} c_{r}+(G[I, \alpha])(t) \quad \forall c_{r} \in \mathbb{R}^{r} \tag{3.2}
\end{equation*}
$$

where

$$
(G[I, \alpha])(t):=U(t)\left(Q^{+} g+\int_{0}^{t} U^{-1}(\tau) I(\tau) d \tau\right)
$$

is the generalized Green operator.
Here

$$
U(t)=\operatorname{diag}\left\{e^{-\frac{t}{R_{1}}}, e^{-\frac{t}{R_{2}}}, \ldots, \quad e^{-\frac{t}{R_{n}}}\right\}
$$

is a fundamental decision matrix of the linear homogeneous system (2.4),

$$
g=\alpha-l \int_{0}^{\cdot} U(\cdot) U^{-1}(\tau) I(\tau) d \tau
$$

$Q=l U(\cdot)$ is a matrix of dimension $(m \times n), P_{Q_{r}}\left(P_{Q_{d}^{*}}\right)$ is the matrix which consists of the complete system $r(d)$ of linearly independent columns (rows) of the projector matrix $P_{Q}$ $\left(P_{Q^{*}}\right)$, where $P_{Q}\left(P_{Q^{*}}\right)$ is projector onto the kernel (cokernel) of the matrix $Q, Q^{+}$is the Moore-Penrose pseudo-inverse (see [37]) to the $Q$ matrix.

Let us find necessary conditions for the existence of a solution $x(t, w, \varepsilon)$ to the problem (2.2), (2.3), which for $\varepsilon=0$ turns into one of the solutions $x\left(t, c_{r}\right)$ of the generating problem (2.4), (2.5). According to Theorem 3.1, boundary-value problem (2.2), (2.3) is solvable if and only if $d$ linearly independent conditions are satisfied

$$
\begin{equation*}
P_{Q_{d}^{*}}\left(g-\varepsilon l \int_{0} U(\cdot) U^{-1}(\tau)(\hat{I}(\tau)+W Z(x(\tau, w, \varepsilon))) d \tau\right)=0 . \tag{3.3}
\end{equation*}
$$

Taking into account (3.1), we obtain that condition (3.3) is equivalent to the following

$$
\begin{equation*}
P_{Q_{d}^{*}} l \int_{0}^{r} U(\cdot) U^{-1}(\tau)(\hat{I}(\tau)+W Z(x(\tau, w, \varepsilon))) d \tau=0 \tag{3.4}
\end{equation*}
$$

Considering the limit for (3.4) as $\varepsilon \rightarrow 0$ and also taking into account that $x(t, w, \varepsilon) \rightarrow x\left(t, c_{r}\right)$ in this case, we obtain the following solvability condition

$$
\begin{equation*}
F\left(c_{r}\right):=P_{Q_{d}^{*}} l \int_{0}^{r} U(\cdot) U^{-1}(\tau)\left(\hat{I}(\tau)+W Z\left(x\left(\tau, c_{r}\right)\right)\right) d \tau=0 . \tag{3.5}
\end{equation*}
$$

Note that in the case of the periodic boundary-value problem (2.2), (2.3) $(l x(\cdot, w, \varepsilon)=$ $x(0, w, \varepsilon)-x(T, w, \varepsilon)=\alpha=0)$ equation (3.5) corresponds to that known in the theory of nonlinear oscillations of the equation for the generating amplitudes (see [21,33]). Therefore, we will call the equation (3.5) the equation for the generating vectors of boundary-value problem (2.2), (2.3). If equation (3.5) has a solution $c_{r}=c_{r}^{0} \in \mathbb{R}^{r}$, then $c_{r}^{0}$ defines the solution

$$
x\left(t, c_{r}^{0}\right)=\operatorname{col}\left(x_{1}\left(t, c_{r}^{0}\right), \quad x_{2}\left(t, c_{r}^{0}\right), \ldots, \quad x_{n}\left(t, c_{r}^{0}\right)\right)
$$

of the generating problem (2.4), (2.5), which may correspond to the solution $x(t, w, \varepsilon)$ of the problem (2.2), (2.3). If the equation (3.5) has no solutions, then problem (2.2), (2.3) also does not have the desired solution. Note that since we are considering the original problem in real form, we are only talking about real solutions of equation (3.5).

Thus, the following statement is true.
Theorem 3.2 (Necessary condition). If the boundary-value problem (2.2), (2.3) has a solution, which for $\varepsilon=0$ turns into one of the solutions $x\left(t, c_{r}^{0}\right)$ generating boundary-value problem (2.4), (2.5), then the vector $c_{r}^{0} \in \mathbb{R}^{r}$ must be a real solution to the equation for the generating vectors (3.5).

## 4 Optimization of the objective function

One of the important research questions of exploring neural networks is finding the extremum (minimum) or the objective function by solving the considered model. In particular, the such problems, arising in medicine, neurobiology, machine learning, were studied in [11-13, 36, 42, 44]. In this paper, we consider the problem of finding the minimum of the objective function $L(x(t, w, \varepsilon), w)$ :

$$
\begin{gathered}
L(x(t, w, \varepsilon), w) \rightarrow \min _{w \in \mathbb{R}^{M}} \\
L(\cdot, w) \in \mathbf{C}\left[\left\|x-x_{0}\right\| \leq q\right], \quad L(x, \cdot) \in \mathbf{C}\left(\mathbb{R}^{M}\right)
\end{gathered}
$$

on the solutions of boundary-value problem (2.2), (2.3), which at $\varepsilon=0$ turn into generating solution of (2.4), (2.5). Here, $x_{0}$ is the generating solution and $q$ is a small parameter.

Suppose that when $\varepsilon$ tends to 0 function $L(x(t, w, \varepsilon), w)$ takes the form of quadratic functional by vector of parameters $w \in \mathbb{R}^{M}$, that is

$$
\begin{equation*}
L\left(x\left(t, c_{r}^{0}\right), w\right)=\Phi(w)=(S w, w)-2(f, w) \rightarrow \min _{w \in \mathbb{R}^{M}} \tag{4.1}
\end{equation*}
$$

where $x\left(t, c_{r}^{0}\right)$ is a solution of the generating problem (2.4), (2.5) and $S: \mathbb{R}^{M} \rightarrow \mathbb{R}^{M}$ is the linear self-adjoint bounded positive operator (positive definite quadratic form), that is

$$
\begin{equation*}
\gamma_{1}\|u\|^{2} \leq(S u, u) \leq \gamma_{2}\|u\|^{2}, \quad \gamma_{2}>\gamma_{1}>0, \quad \forall u \in \mathbb{R}^{M} \tag{4.2}
\end{equation*}
$$

$f \in \mathbb{R}^{M}$. Restriction (4.1), as known from [30], is equivalent to finding solutions $w$ of the following equation

$$
\begin{equation*}
S w=f \tag{4.3}
\end{equation*}
$$

To minimize functional (4.1) we use the accelerated method of conjugate gradients, which, as known from [31,32], improves the convergence of the method of steepest descent and the conjugate gradient method, expands their scope and is more robust to rounding errors. Since $S$ satisfies condition (4.2), the functional (4.1) has a unique minimum $w^{*}$ (equation (4.3) has a unique solution for any $f$ ) (see [30]).

Let us take a closer look at the accelerated method of conjugate gradients. Its essence for the minimization of functional (4.1) is, that based on some initial value approximation $w=w^{0}$, the following approximate solutions are determined according to the formulas

$$
\begin{gather*}
w^{k+1}=w^{k}+\alpha_{k} r_{k}+\beta_{k} \delta_{k}+\sigma_{k}  \tag{4.4}\\
r_{k}=f-S w^{k}, \quad \delta_{k}=w^{k}-w^{k-1}, \quad \sigma_{k}=\sum_{i=1}^{n_{0}} a_{i}^{k} \varphi_{i}, \tag{4.5}
\end{gather*}
$$

where $\varphi_{i}, i=\overline{1, n_{0}}, n_{0} \leq M$ is a system of linearly independent elements and the unknown parameters $\alpha_{k}, \beta_{k}$ and $a_{i}^{k}$ we will determine from the system of linear algebraic equations

$$
\begin{equation*}
\frac{\partial \Phi\left(w^{k+1}\right)}{\partial \alpha_{k}}=0, \quad \frac{\partial \Phi\left(w^{k+1}\right)}{\partial \beta_{k}}=0, \quad \frac{\partial \Phi\left(w^{k+1}\right)}{\partial a_{i}^{k}}=0 . \tag{4.6}
\end{equation*}
$$

Note that in [31], using the form of functional (4.1) and the rule of differentiation of scalar product, a convenient for practical application computational scheme of the method (4.4)-(4.6) is given.

Remark 4.1. As known (see [31]), the use of the method (4.4)-(4.6) in the space $\mathbb{R}^{M}$ allows to obtain an exact solution to equation (4.3) in $k \leq M$ iterations.

Remark 4.2. For $\sigma_{k}=0$, the accelerated method of conjugate gradients (4.4)-(4.6) transforms into the conjugate gradient method, and for $\delta_{k}=0, \sigma_{k}=0$ - into the method of steepest descent (see [31,32]).

Let us formulate, using the results of [31], an estimate for the rate of convergence of the corresponding conjugate gradient method (4.4)-(4.6) for our optimization problem. Let $H_{n_{0}}$, $n_{0} \leq M$ be the subspace spanned on a system of linearly independent elements $\left\{\varphi_{i}\right\}_{i=1}^{n_{0}}$. We introduce into consideration a self-adjoint mapping in the space $V_{n_{0}}$,

$$
\mathbb{R}^{M}=H_{n_{0}} \oplus V_{n_{0}},
$$

operator $K=S Z$ which satisfies the condition

$$
\eta_{1}\|v\|^{2} \leq(K v, v) \leq \eta_{2}\|v\|^{2}, \quad \gamma_{1} \leq \eta_{1} \leq \eta_{2} \leq \gamma_{2}, \quad \forall v \in V_{n_{0}} .
$$

Here the operator $Z$ is linear and is defined by the formula

$$
Z g=g+h
$$

where $g \in \mathbb{R}^{M}$ is an arbitrary element and $h \in H_{n_{0}}$ is a solution of equation

$$
P S(g+h)=0
$$

where $P$ is the operator of orthogonal projection $\mathbb{R}^{M}$ onto $H_{n_{0}}$.
The following statement is true (see [31]).
Theorem 4.3. Let the operator $S$ satisfy condition (4.2). Then, the accelerated method of conjugate gradients (4.4)-(4.6) converges and the rate of its convergence is characterized by estimate

$$
\left\|w^{*}-w^{k}\right\| \leq \frac{q_{k}}{\sqrt{\gamma_{1} \eta_{1}}}\left\|f-S w^{0}\right\|,
$$

where

$$
q_{k}=\frac{2 \rho^{k}}{1+\rho^{2 k}}, \quad \rho=\frac{\sqrt{\eta_{2}}-\sqrt{\eta_{1}}}{\sqrt{\eta_{2}}+\sqrt{\eta_{1}}} .
$$

## 5 A sufficient condition for the solvability of problem (2.2), (2.3)

For the further investigation of the problem (2.2), (2.3), let us fix the value of the vector of parameters $w=w^{*}$, which is found using the accelerated method of conjugate gradients (4.4)(4.6). To obtain a sufficient condition for the existence of a solution, we make the following change in variables in the boundary-value problem (2.2), (2.3):

$$
\begin{equation*}
x\left(t, w^{*}, \varepsilon\right)=x\left(t, c_{r}^{0}\right)+y\left(t, w^{*}, \varepsilon\right) \tag{5.1}
\end{equation*}
$$

where $x\left(t, c_{r}^{0}\right)$ is a solution of the generating boundary-value problem (2.4), (2.5),

$$
y(t, w, \varepsilon)=\operatorname{col}\left(y_{1}(t, w, \varepsilon), \quad y_{2}(t, w, \varepsilon), \quad \ldots, \quad y_{n}(t, w, \varepsilon)\right)
$$

and $c_{r}^{0} \in \mathbb{R}^{r}$ is a solution to the equation for the generating vectors (3.5). By replacing the variables in (5.1), the study of the existence of a solution to problem (2.2), (2.3) is reduced to the corresponding question for the boundary-value problem

$$
\begin{align*}
& y^{\prime}\left(t, w^{*}, \varepsilon\right)=A y\left(t, w^{*}, \varepsilon\right)+\varepsilon\left(\hat{I}(t)+W^{*} Z\left(x\left(t, c_{r}^{0}\right)+y\left(t, w^{*}, \varepsilon\right)\right)\right),  \tag{5.2}\\
& l y\left(\cdot, w^{*}, \varepsilon\right)=0 . \tag{5.3}
\end{align*}
$$

As follows from the vector-function $Z\left(x\left(t, w^{*}, \varepsilon\right)\right)$, it is differentiable in the neighbourhood of the generating solution $x\left(t, c_{r}^{0}\right)$, therefore, the following representation holds:

$$
Z\left(x\left(t, c_{r}^{0}\right)+y\left(t, w^{*}, \varepsilon\right)\right)=Z\left(x\left(t, c_{r}^{0}\right)\right)+A_{1}(t) y\left(t, w^{*}, \varepsilon\right)+\mathcal{R}\left(y\left(t, w^{*}, \varepsilon\right)\right),
$$

where

$$
Z\left(x\left(t, c_{r}^{0}\right)\right)=\operatorname{col}\left(\tanh \left(\frac{a_{1} x_{1}\left(t, c_{r}^{0}\right)}{2}\right), \tanh \left(\frac{a_{2} x_{2}\left(t, c_{r}^{0}\right)}{2}\right), \ldots, \tanh \left(\frac{a_{n} x_{n}\left(t, c_{r}^{0}\right)}{2}\right)\right)
$$

is a limit to which the function $Z\left(x\left(t, w^{*}, \varepsilon\right)\right)$ tends under $\varepsilon$ tends towards 0 and $c=c_{r}^{0}$,

$$
\begin{aligned}
A_{1}(t) & =\left.Z_{x}^{\prime}(v)\right|_{v=x\left(t, c_{r}^{0}\right)} \\
& =\frac{1}{2} \operatorname{diag}\left\{\frac{a_{1}}{\cosh ^{2}\left(\frac{a_{1} x_{1}\left(t, c_{r}^{0}\right)}{2}\right)}, \frac{a_{2}}{\cosh ^{2}\left(\frac{a_{2} x_{2}\left(t, c_{r}^{0}\right)}{2}\right)}, \ldots, \frac{a_{n}}{\cosh ^{2}\left(\frac{a_{n} x_{n}\left(t, c_{r}^{0}\right)}{2}\right)}\right\}
\end{aligned}
$$

is derivative in the sense of Fréchet, and $\mathcal{R}\left(y\left(t, w^{*}, \varepsilon\right)\right)$ are higher-order members

$$
\begin{aligned}
\mathcal{R}\left(y\left(t, w^{*}, \varepsilon\right)\right)= & Z\left(x\left(t, c_{r}^{0}\right)+y\left(t, w^{*}, \varepsilon\right)\right)-Z\left(x\left(t, c_{r}^{0}\right)\right)-A_{1}(t) y\left(t, w^{*}, \varepsilon\right) \\
= & \left(\begin{array}{c}
\tanh \left(\frac{a_{1}\left(x_{1}\left(t, c_{r}^{0}\right)+y_{1}\left(t, w^{*}, \varepsilon\right)\right)}{2}\right)-\tanh \left(\frac{a_{1} x_{1}\left(t, c_{r}^{0}\right)}{2}\right)-\frac{a_{1} y_{1}\left(t, w^{*}, \varepsilon\right)}{2 \cosh ^{2}\left(\frac{a_{1} x_{1}\left(t, c_{r}^{0}\right)}{2}\right)} \\
\tanh \left(\frac{a_{2}\left(x_{2}\left(t, c_{r}^{0}\right)+y_{2}\left(t, w^{*}, \varepsilon\right)\right)}{2}\right)-\tanh \left(\frac{a_{2} x_{2}\left(t, c_{r}^{0}\right)}{2}\right)-\frac{a_{2} y_{2}\left(t, w^{*}, \varepsilon\right)}{2 \cosh ^{2}\left(\frac{a_{2} x_{2}\left(t, c_{r}^{0}\right)}{2}\right)} \\
\vdots \\
\tanh \left(\frac{a_{n}\left(x_{n}\left(t, c_{r}^{0}\right)+y_{n}\left(t, w^{*}, \varepsilon\right)\right)}{2}\right)-\tanh \left(\frac{a_{n} x_{n}\left(t, c_{r}^{0}\right)}{2}\right)-\frac{a_{n} y_{n}\left(t, w^{*}, \varepsilon\right)}{2 \cosh ^{2}\left(\frac{a_{n} x_{n}\left(t, c_{r}^{0}\right)}{2}\right)}
\end{array}\right) .
\end{aligned}
$$

Thus, the boundary-value problem (5.2), (5.3) takes the following form

$$
\begin{align*}
y^{\prime}\left(t, w^{*}, \varepsilon\right)= & A y\left(t, w^{*}, \varepsilon\right) \\
& +\varepsilon\left(\hat{I}(t)+W^{*}\left(Z\left(x\left(t, c_{r}^{0}\right)\right)+A_{1}(t) y\left(t, w^{*}, \varepsilon\right)+\mathcal{R}\left(y\left(t, w^{*}, \varepsilon\right)\right)\right)\right)  \tag{5.4}\\
l y\left(\cdot, w^{*}, \varepsilon\right)= & 0 . \tag{5.5}
\end{align*}
$$

According to the Theorem 3.1, under $d$ conditions

$$
\begin{align*}
& P_{Q_{d}^{*}} l \int_{0} U(\cdot) U^{-1}(\tau) W^{*}\left(Z\left(x\left(\tau, c_{r}^{0}\right)\right)+A_{1}(\tau) y\left(\tau, w^{*}, \varepsilon\right)+\mathcal{R}\left(y\left(\tau, w^{*}, \varepsilon\right)\right)\right) d \tau \\
& \quad=-P_{Q_{d}^{*}} l \int_{0} U(\cdot) U^{-1}(\tau) \hat{I}(\tau) d \tau \tag{5.6}
\end{align*}
$$

boundary-value problem (5.4), (5.5) has an $r$-parametric family of solutions of the following form

$$
\begin{align*}
& y\left(t, w^{*}, \varepsilon\right)=U(t) P_{Q_{r}} c_{r}+\bar{y}\left(t, w^{*}, \varepsilon\right) \quad \forall c_{r} \in \mathbb{R}^{r}  \tag{5.7}\\
& \bar{y}\left(t, w^{*}, \varepsilon\right)=\varepsilon\left(G\left[\hat{I}(t)+W^{*} Z\left(x\left(t, c_{r}^{0}\right)+y\left(t, w^{*}, \varepsilon\right)\right), 0\right]\right)(t, \varepsilon)
\end{align*}
$$

Using condition (3.5), relation (5.6) can be rewritten as

$$
\begin{equation*}
P_{Q_{d}^{*}} l \int_{0} U(\cdot) U^{-1}(\tau) W^{*}\left(A_{1}(\tau) y\left(\tau, w^{*}, \varepsilon\right)+\mathcal{R}\left(y\left(\tau, w^{*}, \varepsilon\right)\right)\right) d \tau=0 \tag{5.8}
\end{equation*}
$$

Substituting (5.7) into (5.8), we obtain the following equation for $c_{r}$ :

$$
\begin{align*}
B c_{r} & =-P_{Q_{d}^{*}} l \int_{0} U(\cdot) U^{-1}(\tau) W^{*} H\left(\tau, y\left(\tau, w^{*}, \varepsilon\right), \bar{y}\left(\tau, w^{*}, \varepsilon\right)\right) d \tau  \tag{5.9}\\
H\left(t, y\left(t, w^{*}, \varepsilon\right), \bar{y}\left(t, w^{*}, \varepsilon\right)\right) & =A_{1}(t) \bar{y}\left(t, w^{*}, \varepsilon\right)+\mathcal{R}\left(y\left(t, w^{*}, \varepsilon\right)\right)
\end{align*}
$$

where matrix $B(d \times r)$ has the following form

$$
\begin{equation*}
B=P_{Q_{d}^{*}} l \int_{0} U(\cdot) U^{-1}(\tau) W^{*} A_{1}(\tau) U(\tau) P_{Q_{r}} d \tau \tag{5.10}
\end{equation*}
$$

The algebraic system (5.9) is solvable if and only if $d_{1}$ conditions hold

$$
\begin{equation*}
P_{B_{d_{1}}^{*}} P_{Q_{d}^{*}} l \int_{0}^{*} U(\cdot) U^{-1}(\tau) W^{*} H\left(\tau, y\left(\tau, w^{*}, \varepsilon\right), \bar{y}\left(\tau, w^{*}, \varepsilon\right)\right) d \tau=0 \tag{5.11}
\end{equation*}
$$

If, for example,

$$
\begin{equation*}
P_{B_{d_{1}}^{*}} P_{Q_{d}^{*}}=0 \tag{5.12}
\end{equation*}
$$

then the condition (5.11) is always valid, and system (5.9) has $r_{1}$-parametric solution

$$
\begin{aligned}
& c_{r}=P_{B_{r_{1}}} \hat{c}_{r_{1}}+\bar{c}_{r} \quad \forall \hat{c}_{r_{1}} \in \mathbb{R}^{r_{1}} \\
& \bar{c}_{r}=-B^{+} P_{Q_{d}^{*}} l \int_{0}^{\cdot} U(\cdot) U^{-1}(\tau) W^{*} H\left(\tau, y\left(\tau, w^{*}, \varepsilon\right), \bar{y}\left(\tau, w^{*}, \varepsilon\right)\right) d \tau
\end{aligned}
$$

Here $P_{B_{r_{1}}}\left(P_{B_{d_{1}}^{*}}\right)$ is a matrix that consists of the complete system $r_{1}\left(d_{1}\right)$ of linearly independent columns (rows) of the projector matrix $P_{B}\left(P_{B^{*}}\right)$, where $P_{B}\left(P_{B^{*}}\right)$ is the projector on kernel (cokernel) of the matrix $B, B^{+}$is the Moore-Penrose pseudo-inverse to the matrix $B$.

From now on we will restrict ourselves to the particular solution $c_{r}=\bar{c}$ of the system (5.9). So, for defining the solution of the problem (5.2), (5.3) we come to the system of equations

$$
\begin{aligned}
y\left(t, w^{*}, \varepsilon\right) & =U(t) P_{Q_{r}} c_{r}+\bar{y}\left(t, w^{*}, \varepsilon\right) \\
c_{r} & =-B^{+} P_{Q_{d}^{*}} l \int_{0} U(\cdot) U^{-1}(\tau) W^{*} H\left(\tau, y\left(\tau, w^{*}, \varepsilon\right), \bar{y}\left(\tau, w^{*}, \varepsilon\right)\right) d \tau \\
\bar{y}\left(t, w^{*}, \varepsilon\right) & =\varepsilon\left(G\left[\hat{I}(t)+W^{*} Z\left(x\left(t, c_{r}^{0}\right)+y\left(t, w^{*}, \varepsilon\right)\right), 0\right]\right)(t, \varepsilon),
\end{aligned}
$$

which can be solved using a convergent iterative process explained in detail in [7]. The following statement is true.

Theorem 5.1 (Sufficient condition). Let the generating problem for (2.2), (2.3) problem (2.4), (2.5), subject to the conditions of $d$ linearly independent conditions (3.1), have an r-parametric family of solutions $x\left(t, c_{r}^{0}\right)$ (3.2) and the operator $S$ satisfy condition (4.2). Then for every real value of the
vector $c_{r}^{0} \in \mathbb{R}^{r}$, which satisfies the equation for the generating vectors (3.5), for the value of the parameter vector $w^{*} \in \mathbb{R}^{M}$ minimizing the quadratic functional (4.1), and when conditions (5.12) hold, the boundary-value problem (2.2), (2.3) has a solution that can be found using the following iterative process

$$
\begin{gathered}
c_{r}^{k}=-B^{+} P_{Q_{d}^{*}} l \int_{0} U(\cdot) U^{-1}(\tau) W^{*} H\left(\tau, y^{k}\left(\tau, w^{*}, \varepsilon\right), \bar{y}^{k}\left(\tau, w^{*}, \varepsilon\right)\right) d \tau \\
\bar{y}^{k+1}\left(t, w^{*}, \varepsilon\right)=\varepsilon\left(G \left[\hat{I}(t)+W^{*}\left(Z\left(x\left(t, c_{r}^{0}\right)\right)+A_{1}(t) U(t) P_{Q_{r}} c_{r}^{k}\right.\right.\right. \\
\left.\left.\left.+H\left(t, y^{k}\left(t, w^{*}, \varepsilon\right), \bar{y}^{k}\left(t, w^{*}, \varepsilon\right)\right)\right), 0\right]\right)(t, \varepsilon), \\
y^{k+1}\left(t, w^{*}, \varepsilon\right)=U(t) P_{Q_{r}} c_{r}^{k}+\bar{y}^{k+1}\left(t, w^{*}, \varepsilon\right) \\
x^{k}\left(t, w^{*}, \varepsilon\right)=y^{k}\left(t, w^{*}, \varepsilon\right)+x\left(t, c_{r}^{0}\right), \quad x\left(t, w^{*}, \varepsilon\right)=\lim _{k \rightarrow \infty} x^{k}\left(t, w^{*}, \varepsilon\right) \\
y^{0}\left(t, w^{*}, \varepsilon\right)=\bar{y}^{0}\left(t, w^{*}, \varepsilon\right)=0
\end{gathered}
$$

Corollary 5.2. Let $r=d$ and nonlinearity $F\left(c_{r}\right)$ has the inverse to $F^{\prime}\left(c_{r}^{0}\right)$ for the vector $c_{r}^{0}$, that satisfies the equation (3.5). Then $F^{\prime}\left(c_{r}^{0}\right)=B$, and for such each $c_{r}^{0}$, the boundary-value problem (2.2), (2.3) has a unique solution.

Proof. Consider the difference

$$
\begin{aligned}
F\left(c_{r}+h\right)-F\left(c_{r}\right)= & P_{Q_{d}^{*}} l \int_{0} U(\cdot) U^{-1}(\tau) W^{*} Z\left(x\left(\tau, c_{r}+h\right)\right) d \tau \\
& -P_{Q_{d}^{*}} l \int_{0} U(\cdot) U^{-1}(\tau) W^{*} Z\left(x\left(\tau, c_{r}\right)\right) d \tau
\end{aligned}
$$

Based on the representation (3.2), that is $x\left(\tau, c_{r}\right)=U(\tau) P_{Q_{r}} c_{r}+(G[I, \alpha])(\tau)$, we obtain that

$$
Z\left(x\left(\tau, c_{r}+h\right)\right)=Z\left(x\left(\tau, c_{r}\right)+U(t) P_{Q_{r}} h\right)=Z\left(x\left(\tau, c_{r}\right)\right)+A_{1}(\tau) U(\tau) P_{Q_{r}} h+\mathcal{R}\left(U(\tau) P_{Q_{r}} h\right)
$$

where $\mathcal{R}\left(U(\tau) P_{Q_{r}} h\right)$ contains terms higher than the first order in $h$. Substituting the received equality in the difference $F\left(c_{r}+h\right)-F\left(c_{r}\right)$, we get the following:

$$
\begin{aligned}
F\left(c_{r}+h\right)-F\left(c_{r}\right)= & P_{Q_{d}^{*}} l \int_{0} U(\cdot) U^{-1}(\tau) W^{*}\left(Z\left(x\left(\tau, c_{r}\right)\right)+A_{1}(\tau) U(\tau) P_{Q_{r}} h+\mathcal{R}\left(U(\tau) P_{Q_{r}} h\right)\right) d \tau \\
& -P_{Q_{d}^{*}} l \int_{0} U(\cdot) U^{-1}(\tau) W^{*} Z\left(x\left(\tau, c_{r}\right)\right) d \tau \\
= & B h+P_{Q_{d}^{*}} l \int_{0} U(\cdot) U^{-1}(\tau) W^{*} \mathcal{R}\left(U(t) P_{Q_{r}} h\right) d \tau
\end{aligned}
$$

From the equation above we obtain that $F^{\prime}\left(c_{r}^{0}\right)=B$. Thus, reversibility of $F^{\prime}\left(c_{r}^{0}\right)$ implies the invertibility of the matrix $B$. From this follows, that $r_{1}=d_{1}=0, P_{B_{r_{1}}}=P_{B_{d_{1}}^{*}}=0$, condition (5.12) is satisfied, and non-linear boundary-value problem (2.2), (2.3) has a unique solution for each such $c_{r}^{0}$.

Remark 5.3. To calculate the projectors and the Moore-Penrose pseudo-inverse matrices, one can use the well-known formulas (see [7, p. 48], [28, p. 454]).

## 6 Examples

Example 6.1. Consider the underdetermined boundary-value problem (2.2), (2.3) for three equations in case when boundary condition (2.3) is $T$-periodic in part of the coordinates and has the form

$$
l\left(\begin{array}{l}
x_{1}(\cdot, w, \varepsilon) \\
x_{2}(\cdot, w, \varepsilon) \\
x_{3}(\cdot, w, \varepsilon)
\end{array}\right)=\binom{x_{1}(T, w, \varepsilon)-x_{1}(0, w, \varepsilon)}{x_{2}(T, w, \varepsilon)-x_{2}(0, w, \varepsilon)}=\binom{\alpha_{1}}{\alpha_{2}}=\binom{0}{0} .
$$

In this case, the matrix $Q$ is defined by the equality

$$
Q=\left(\begin{array}{ccc}
e^{-\frac{T}{R_{1}}}-1 & 0 & 0 \\
0 & e^{-\frac{T}{R_{2}}}-1 & 0
\end{array}\right) .
$$

The Moore-Penrose pseudo-inverse matrix $Q^{+}$and vector $g$ have the following form:

$$
Q^{+}=\left(\begin{array}{cc}
\frac{e^{\frac{T}{R_{1}}}}{1-e^{\frac{T}{R_{1}}}} & 0 \\
0 & \frac{e^{\frac{T}{R_{2}}}}{1-e^{\frac{T}{R_{2}}}} \\
0 & 0
\end{array}\right), \quad g=-\int_{0}^{T}\binom{e^{\frac{\tau-T}{R_{1}}} I_{1}(\tau)}{e^{\frac{\frac{-T}{R_{2}}}{R_{2}}} I_{2}(\tau)} d \tau
$$

and orthoprojectors $P_{Q}, P_{Q^{*}}$ look as follows

$$
P_{Q}=E_{3}-Q^{+} Q=\operatorname{diag}\{0,0,1\}, \quad P_{Q^{*}}=E_{2}-Q Q^{+}=O_{2},
$$

where $E_{2}, E_{3}$ identity matrices of dimensions 2 and 3 , respectively, $O_{2}$ is zero matrix of dimension 2. That is, $r=1, d=0$ and the condition of solvability (3.1) holds, and the linear boundary-value problem (2.4), (2.5) has a one-parameter set of solutions of the following form:

$$
x\left(t, c_{r}\right)=\left(\begin{array}{c}
\frac{-1}{1-e^{\frac{T}{R_{1}}}} \int_{0}^{T} e^{\frac{\tau-t}{R_{1}}} I_{1}(\tau) d \tau+\int_{0}^{t} e^{\frac{\tau-t}{R_{1}}} I_{1}(\tau) d \tau \\
\frac{-1}{1-e^{\frac{T}{R_{2}}}} \int_{0}^{T} e^{\frac{\tau-t}{R_{2}}} I_{2}(\tau) d \tau+\int_{0}^{t} e^{\frac{\tau-t}{R_{2}}} I_{2}(\tau) d \tau \\
e^{-\frac{t}{R_{3}}} c_{3}+\int_{0}^{t} e^{\frac{\tau-t}{R_{3}}} I_{3}(\tau) d \tau
\end{array}\right) .
$$

The equation for the generating vectors (3.5) takes the form of the identity $F\left(c_{r}\right) \equiv 0$ for any vector $c_{r}$.

Example 6.2. Consider the original problem for three equations with the boundary condition (2.3) of this form

$$
l\left(\begin{array}{l}
x_{1}(\cdot, w, \varepsilon)  \tag{6.1}\\
x_{2}(\cdot, w, \varepsilon) \\
x_{3}(\cdot, w, \varepsilon)
\end{array}\right)=\left(\begin{array}{l}
x_{1}(T, w, \varepsilon)-x_{1}(0, w, \varepsilon) \\
x_{2}(T, w, \varepsilon)-x_{2}(0, w, \varepsilon) \\
x_{3}(T, w, \varepsilon)-x_{3}(0, w, \varepsilon)
\end{array}\right)=\left(\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\alpha_{3}
\end{array}\right) .
$$

In this case

$$
Q=\operatorname{diag}\left\{e^{-\frac{T}{R_{1}}}-1, \quad e^{-\frac{T}{R_{2}}}-1, \quad e^{-\frac{T}{R_{3}}}-1\right\}
$$

and the Moore-Penrose pseudo-inverse matrix $Q^{+}$coincides with matrix $Q^{-1}$

$$
Q^{+}=Q^{-1}=\operatorname{diag}\left\{\frac{e^{\frac{T}{R_{1}}}}{1-e^{\frac{T}{R_{1}}}}, \frac{e^{\frac{T}{R_{2}}}}{1-e^{\frac{T}{R_{2}}}}, \frac{e^{\frac{T}{R_{3}}}}{1-e^{\frac{T}{R_{3}}}}\right\}
$$

the vector $g$ has the form

$$
g=\left(\begin{array}{l}
\alpha_{1} \\
\alpha_{2} \\
\alpha_{3}
\end{array}\right)-\int_{0}^{T}\left(\begin{array}{c}
\frac{\tau-T}{R_{1}} I_{1}(\tau) \\
e^{\frac{\tau-T}{R_{2}}} I_{2}(\tau) \\
e^{\frac{\tau-T}{R_{3}}} I_{2}(\tau)
\end{array}\right) d \tau
$$

The orthoprojectors $P_{Q}, P_{Q^{*}}$ are given by the relations

$$
P_{Q}=E_{3}-Q^{+} Q=P_{Q^{*}}=E_{3}-Q Q^{+}=O_{3}
$$

where $O_{3}$ is the zero matrix of dimension 3 . The condition of solvability (3.1) is fulfilled automatically, and the linear boundary-value problem (2.4), (2.5) has only one solution of the following form:

Any vector $c_{r}$ satisfies the equation for generating vectors (3.5) since $P_{Q^{*}}=O_{3}$. From (5.10) follows that matrix $B=O_{3}$. Thus, the boundary-value problem (2.2), (6.1) has a solution which, according to Theorem 5.1, can be found using the iterative process

$$
\begin{aligned}
x^{k+1}(t, w, \varepsilon) & =y^{k+1}(t, w, \varepsilon)+x(t) \\
y^{k+1}(t, w, \varepsilon) & =\bar{y}^{k+1}(t, w, \varepsilon) \\
& =\varepsilon\left(\hat{I}(t)+G\left[W\left(Z(x(t))+A_{1}(t) y^{k}(t, w, \varepsilon)+\mathcal{R}\left(y^{k}(t, w, \varepsilon)\right)\right), 0\right]\right)(t, \varepsilon)
\end{aligned}
$$

If we rewrite this componentwise, then we obtain the following iterative procedure for finding the solutions of the boundary-value problem (2.2), (6.1):

$$
\begin{aligned}
y_{1}^{k+1}(t, w, \varepsilon)= & \varepsilon \int_{0}^{t} e^{\frac{\tau-t}{R_{1}}}\left(\hat{I}_{1}(\tau)+w_{12} \tanh \left(\frac{a_{2}\left(x_{2}(\tau)+y_{2}^{k}(\tau, w, \varepsilon)\right)}{2}\right)\right) d \tau \\
& +\varepsilon w_{13} \int_{0}^{t} e^{\frac{\tau-t}{R_{1}}} \tanh \left(\frac{a_{3}\left(x_{1}(\tau)+y_{3}^{k}(\tau, w, \varepsilon)\right)}{2}\right) d \tau \\
& -\varepsilon \int_{0}^{T} \frac{e^{\frac{\tau-t}{R_{1}}}}{1-e^{\frac{T}{R_{1}}}}\left(\hat{I}_{1}(\tau)+w_{12} \tanh \left(\frac{a_{2}\left(x_{2}(\tau)+y_{2}^{k}(\tau, w, \varepsilon)\right)}{2}\right)\right) d \tau \\
& -\varepsilon w_{13} \int_{0}^{T} \frac{e^{\frac{\tau-t}{R_{1}}}}{1-e^{\frac{T}{R_{1}}}} \tanh \left(\frac{a_{3}\left(x_{1}(\tau)+y_{3}^{k}(\tau, w, \varepsilon)\right)}{2}\right) d \tau
\end{aligned}
$$

$$
\begin{aligned}
y_{2}^{k+1}(t, w, \varepsilon)= & \varepsilon \int_{0}^{t} e^{\frac{\tau-t}{R_{2}}}\left(\hat{I}_{2}(\tau)+w_{12} \tanh \left(\frac{a_{1}\left(x_{1}(\tau)+y_{1}^{k}(\tau, w, \varepsilon)\right)}{2}\right)\right) d \tau \\
& +\varepsilon w_{23} \int_{0}^{t} e^{\frac{\tau-t}{R_{2}}} \tanh \left(\frac{a_{3}\left(x_{3}(\tau)+y_{3}^{k}(\tau, w, \varepsilon)\right)}{2}\right) d \tau \\
& -\varepsilon \int_{0}^{T} \frac{e^{\frac{\tau-t}{R_{2}}}}{1-e^{\frac{T}{R_{2}}}}\left(\hat{I}_{2}(\tau)+w_{12} \tanh \left(\frac{a_{1}\left(x_{1}(\tau)+y_{1}^{k}(\tau, w, \varepsilon)\right)}{2}\right)\right) d \tau \\
& -\varepsilon w_{23} \int_{0}^{T} \frac{e^{\frac{\tau-t}{R_{2}}}}{1-e^{\frac{T}{R_{2}}}} \tanh \left(\frac{a_{3}\left(x_{3}(\tau)+y_{3}^{k}(\tau, w, \varepsilon)\right)}{2}\right) d \tau, \\
y_{3}^{k+1}(t, w, \varepsilon)= & \varepsilon \int_{0}^{t} e^{\frac{\tau-t}{R_{3}}}\left(\hat{I}_{3}(\tau)+w_{13} \tanh \left(\frac{a_{1}\left(x_{1}(\tau)+y_{1}^{k}(\tau, w, \varepsilon)\right)}{2}\right)\right) d \tau \\
& +\varepsilon w_{23} \int_{0}^{t} e^{\frac{\tau-t}{R_{3}}} \tanh \left(\frac{a_{2}\left(x_{2}(\tau)+y_{2}^{k}(\tau, w, \varepsilon)\right)}{2}\right) d \tau \\
& -\varepsilon \int_{0}^{T} \frac{e^{\frac{\tau-t}{R_{3}}}}{1-e^{\frac{T}{R_{3}}}}\left(\hat{I}_{3}(\tau)+w_{13} \tanh \left(\frac{a_{1}\left(x_{1}(\tau)+y_{1}^{k}(\tau, w, \varepsilon)\right)}{2}\right)\right) d \tau \\
& -\varepsilon w_{23} \int_{0}^{T} \frac{e^{\frac{\tau-t}{R_{3}}}}{1-e^{\frac{T}{R_{3}}}} \tanh \left(\frac{a_{2}\left(x_{2}(\tau)+y_{2}^{k}(\tau, w, \varepsilon)\right)}{2}\right) d \tau .
\end{aligned}
$$

Example 6.3. Let us consider the Hopfield model for three neurons described by the bounda-ry-value problem (2.2), (2.3) of the form

$$
\begin{gather*}
x_{1}^{\prime}(t, w, \varepsilon)=\varepsilon\left(w_{12} \tanh \left(x_{2}(t, w, \varepsilon)\right)+w_{13} \tanh \left(x_{3}(t, w, \varepsilon)\right)\right), \\
x_{2}^{\prime}(t, w, \varepsilon)=\varepsilon\left(w_{12} \tanh \left(x_{1}(t, w, \varepsilon)\right)+w_{23} \tanh \left(x_{3}(t, w, \varepsilon)\right)\right),  \tag{6.2}\\
x_{3}^{\prime}(t, w, \varepsilon)=\varepsilon\left(2+w_{13} \tanh \left(x_{1}(t, w, \varepsilon)\right)+w_{23} \tanh \left(x_{2}(t, w, \varepsilon)\right)\right), \\
x_{1}(1, w, \varepsilon)-x_{1}(0, w, \varepsilon)=0, \\
\int_{0}^{1} x_{1}(t, w, \varepsilon) d t=1 \tag{6.3}
\end{gather*}
$$

and the generating problem for it

$$
\begin{gather*}
x_{j}^{\prime}(t)=0, \quad j=1,2,3,  \tag{6.4}\\
x_{1}(1)-x_{1}(0)=0, \\
\int_{0}^{1} x_{1}(t) d t=1 . \tag{6.5}
\end{gather*}
$$

That is, in our case $R_{1}=R_{2}=R_{3}=\infty, a_{1}=a_{2}=a_{3}=2, I_{1}(t)=I_{2}(t)=I_{3}(t)=0$, $\hat{I}_{1}(t)=\hat{I}_{2}(t)=0, \hat{I}_{3}(t)=2, l=\operatorname{col}\left(l_{1}, l_{2}\right), \alpha=\operatorname{col}(0,1)$.

Let us investigate the problem of finding the extremum (minimum) of the loss function

$$
\begin{align*}
L(x(t, w, \varepsilon), w)= & 2\left(2 w_{12}^{2}+w_{13}^{2}+2 w_{23}^{2}-2 w_{12} w_{13}+w_{13} w_{23}\right) x_{1}(0, w, \varepsilon) \\
& -2 \operatorname{coth}(1)\left(6 w_{12}-4 w_{13}-w_{23}\right) x_{1}(1, w, \varepsilon) \rightarrow \min _{w \in \mathbb{R}^{3}} \tag{6.6}
\end{align*}
$$

on the solutions of the boundary-value problem (6.2), (6.3), which at $\varepsilon=0$ turn into solutions of the generating problem (6.4), (6.5), by the vector of parameters $w=\operatorname{col}\left(w_{12}, w_{13}, \quad w_{23}\right) \in$ $\mathbb{R}^{3}$.

Using the well-known formulas (see [7, p. 48], [28, p. 501]), we get $r=2, d=1$,

$$
Q=\left(\begin{array}{lll}
0 & 0 & 0  \tag{6.7}\\
1 & 0 & 0
\end{array}\right), \quad Q^{+}=\left(\begin{array}{ll}
0 & 1 \\
0 & 0 \\
0 & 0
\end{array}\right), \quad P_{Q_{2}}=\left(\begin{array}{ll}
0 & 0 \\
1 & 0 \\
0 & 1
\end{array}\right), \quad P_{Q_{1}^{*}}=\left(\begin{array}{ll}
1 & 0
\end{array}\right)
$$

and the vector $g$ has the following form

$$
g=\binom{0}{1}
$$

The solvability condition (3.1), in our case, is satisfied and due to Theorem 3.1, the solution of the generating problem (6.4), (6.5) takes the form

$$
x\left(t, c_{r}\right)=\left(\begin{array}{c}
1  \tag{6.8}\\
c_{2} \\
c_{3}
\end{array}\right)
$$

The necessary condition for the existence of a solution $x(t, w, \varepsilon)$ of the problem (6.2), (6.3), which by $\varepsilon=0$ turns into one of the solutions $x\left(t, c_{r}\right)$ (6.8) of the generating problem (6.4), (6.5), in our case has the following representation:

$$
\begin{equation*}
F\left(c_{r}\right)=\int_{0}^{1}\left(w_{12} \tanh \left(c_{2}\right)+w_{13} \tanh \left(c_{3}\right)\right) d t=0 \tag{6.9}
\end{equation*}
$$

or

$$
c_{2}=-\tanh ^{-1}\left(\frac{w_{13}}{w_{12}} \tanh \left(c_{3}\right)\right)
$$

The values of the parameters $c_{2}=c_{3}=0$, which are the solution of the system of equations (6.9), determine the generating solution

$$
x\left(t, c_{r}^{0}\right)=\left(\begin{array}{l}
1  \tag{6.10}\\
0 \\
0
\end{array}\right)
$$

to which corresponds the solution $x(t, w, \varepsilon)$ of the problem (6.2), (6.3).
Let us return to the problem of finding the minimum of functional (6.6). When $\varepsilon$ tends to 0 , taking into consideration $x(t, w, \varepsilon) \rightarrow x\left(t, c_{r}^{0}\right)$, where $x\left(t, c_{r}^{0}\right)$ taking into consideration (6.10), we obtain the quadratic functional for the vector of parameters $w$

$$
\begin{align*}
\Phi(w)= & 4 w_{12}^{2}+2 w_{13}^{2}+4 w_{23}^{2}-4 w_{12} w_{13}+2 w_{13} w_{23} \\
& -\operatorname{coth}(1)\left(12 w_{12}-8 w_{13}-2 w_{23}\right) \rightarrow \min _{w \in \mathbb{R}^{3}} . \tag{6.11}
\end{align*}
$$

The problem of finding the minimum of the quadratic functional (6.11) is equivalent to the solution of the following equation

$$
S w=\left(\begin{array}{ccc}
4 & -2 & 0  \tag{6.12}\\
-2 & 2 & 1 \\
0 & 1 & 4
\end{array}\right)\left(\begin{array}{c}
w_{12} \\
w_{13} \\
w_{23}
\end{array}\right)=\operatorname{coth}(1)\left(\begin{array}{c}
6 \\
-4 \\
-1
\end{array}\right)=f .
$$

To find the solution of the equation (6.12), or, which is the same, find the minimum of the quadratic functional (6.11), we use the accelerated method of conjugate gradients (4.4)-(4.6)
and compare it with the method of steepest descent and the conjugate gradient method. Now let us consider the case where

$$
w^{0}=\frac{3}{2} \operatorname{coth}(1)\left(\begin{array}{l}
1 \\
0 \\
0
\end{array}\right), \quad \sigma_{k}=a_{k} \varphi, \quad \varphi=\left(\begin{array}{l}
1 \\
0 \\
0
\end{array}\right) .
$$

If the system $\varphi_{i}, i=\overline{1, n}, n \leq 3$ consists of more than one linearly independent element, then the rate of convergence of the accelerated method of conjugate gradients increases. Without cluttering the example above with calculations that can be made following the computational scheme from [31], we present successive approximations to the minimum of functional (6.11) obtained by the accelerated method of conjugate gradients

$$
w^{1}=\frac{\operatorname{coth}(1)}{14}\left(\begin{array}{c}
19 \\
-4 \\
-4
\end{array}\right), \quad w^{2}=\operatorname{coth}(1)\left(\begin{array}{c}
1 \\
-1 \\
0
\end{array}\right)
$$

and the conjugate gradient method

$$
w^{1}=\frac{\operatorname{coth}(1)}{4}\left(\begin{array}{c}
6 \\
-1 \\
-1
\end{array}\right), \quad w^{2}=\frac{\operatorname{coth}(1)}{10}\left(\begin{array}{c}
11 \\
-6 \\
-2
\end{array}\right), \quad w^{3}=\operatorname{coth}(1)\left(\begin{array}{c}
1 \\
-1 \\
0
\end{array}\right)
$$

Therefore, as can be proved by substituting the obtained approximations into the equation (6.12), the minimum of functional (6.11) is achieved by the accelerated method of conjugate gradients in the second approximation $w^{2}$, and by the conjugate gradient method in the third approximation $w^{3}$, and is equal to

$$
\begin{equation*}
w_{12}^{*}=\operatorname{coth}(1), \quad w_{13}^{*}=-\operatorname{coth}(1), \quad w_{23}^{*}=0 . \tag{6.13}
\end{equation*}
$$

Note that the rate of convergence of the method of steepest descent, in our case, is considerably slower and even the approximation $w^{5}$ is far away from the value (6.13)

$$
w^{1}=\frac{\operatorname{coth}(1)}{4}\left(\begin{array}{c}
6 \\
-1 \\
-1
\end{array}\right), \quad w^{3}=\frac{\operatorname{coth}(1)}{32}\left(\begin{array}{c}
42 \\
-17 \\
-5
\end{array}\right), \quad w^{5} \approx \operatorname{coth}(1)\left(\begin{array}{c}
1,206881 \\
-0,683004 \\
-0,103441
\end{array}\right) .
$$

Let us fix the value of the vector of parameters $w=w^{*}$ (6.13). Let us now find the sufficient condition for the existence of solutions of the problem (6.2), (6.3). For this we make the substitution

$$
x\left(t, w^{*}, \varepsilon\right)=x\left(t, c_{r}^{0}\right)+y\left(t, w^{*}, \varepsilon\right)
$$

where $x\left(t, c_{r}^{0}\right)$ has the form (6.10). After such a substitution we obtain the following boundaryvalue problem

$$
\begin{aligned}
y_{1}^{\prime}\left(t, w^{*}, \varepsilon\right) & =\varepsilon \operatorname{coth}(1)\left(\tanh \left(y_{2}\left(t, w^{*}, \varepsilon\right)\right)-\tanh \left(y_{3}\left(t, w^{*}, \varepsilon\right)\right)\right), \\
y_{2}^{\prime}\left(t, w^{*}, \varepsilon\right) & =\varepsilon \operatorname{coth}(1) \tanh \left(1+y_{1}\left(t, w^{*}, \varepsilon\right)\right), \\
y_{3}^{\prime}\left(t, w^{*}, \varepsilon\right) & =\varepsilon\left(2-\operatorname{coth}(1) \tanh \left(1+y_{1}\left(t, w^{*}, \varepsilon\right)\right)\right), \\
y_{1}\left(1, w^{*}, \varepsilon\right)-y_{1}\left(0, w^{*}, \varepsilon\right) & =0 \\
\int_{0}^{1} y_{1}\left(t, w^{*}, \varepsilon\right) d t & =0 .
\end{aligned}
$$

For the vector-function $Z\left(x\left(t, c_{r}^{0}\right)+y\left(t, w^{*}, \varepsilon\right)\right)$, in the neighbourhood of the generating solution $x\left(t, c_{r}^{0}\right)$ (6.10), the following representation holds

$$
Z\left(x\left(t, c_{r}^{0}\right)+y\left(t, w^{*}, \varepsilon\right)\right)=Z\left(x\left(t, c_{r}^{0}\right)\right)+A_{1}(t) y\left(t, w^{*}, \varepsilon\right)+\mathcal{R}\left(y\left(t, w^{*}, \varepsilon\right)\right)
$$

where

$$
\begin{aligned}
Z\left(x\left(t, c_{r}^{0}\right)\right) & =\operatorname{col}(\tanh (1), 0,0) \\
A_{1}(t) & =\operatorname{diag}\left\{\cosh ^{-2}(1), \quad 1,1\right\} \\
\mathcal{R}\left(y\left(t, w^{*}, \varepsilon\right)\right) & =\left(\begin{array}{c}
\mathcal{R}_{1}\left(y\left(t, w^{*}, \varepsilon\right)\right) \\
\mathcal{R}_{2}\left(y\left(t, w^{*}, \varepsilon\right)\right) \\
\mathcal{R}_{3}\left(y\left(t, w^{*}, \varepsilon\right)\right)
\end{array}\right) \\
& =\left(\begin{array}{c}
\tanh \left(1+y_{1}\left(t, w^{*}, \varepsilon\right)\right)-\tanh (1)-\frac{y_{1}\left(t, w^{*}, \varepsilon\right)}{\cosh ^{2}(1)} \\
\tanh \left(y_{2}\left(t, w^{*}, \varepsilon\right)\right)-y_{2}\left(t, w^{*}, \varepsilon\right) \\
\tanh \left(y_{3}\left(t, w^{*}, \varepsilon\right)\right)-y_{3}\left(t, w^{*}, \varepsilon\right)
\end{array}\right)
\end{aligned}
$$

The function $H\left(t, y\left(t, w^{*}, \varepsilon\right), \bar{y}\left(t, w^{*}, \varepsilon\right)\right)$ has the form:

$$
\begin{aligned}
H\left(t, y\left(t, w^{*}, \varepsilon\right), \bar{y}\left(t, w^{*}, \varepsilon\right)\right) & =\left(\begin{array}{c}
H_{1}\left(t, y\left(t, w^{*}, \varepsilon\right), \bar{y}\left(t, w^{*}, \varepsilon\right)\right) \\
H_{2}\left(t, y\left(t, w^{*}, \varepsilon\right), \bar{y}\left(t, w^{*}, \varepsilon\right)\right) \\
H_{3}\left(t, y\left(t, w^{*}, \varepsilon\right), \bar{y}\left(t, w^{*}, \varepsilon\right)\right)
\end{array}\right) \\
& =\left(\begin{array}{c}
\frac{\bar{y}_{1}\left(t, w^{*}, \varepsilon\right)}{\cosh ^{2}(1)}+\mathcal{R}_{1}\left(y\left(t, w^{*}, \varepsilon\right)\right) \\
\bar{y}_{2}\left(t, w^{*}, \varepsilon\right)+\mathcal{R}_{2}\left(y\left(t, w^{*}, \varepsilon\right)\right) \\
\bar{y}_{3}\left(t, w^{*}, \varepsilon\right)+\mathcal{R}_{3}\left(y\left(t, w^{*}, \varepsilon\right)\right)
\end{array}\right)
\end{aligned}
$$

Matrices $B, B^{+}, P_{B_{r_{1}}}, P_{B_{d_{1}}^{*}}$ in our case takes the following view

$$
B=\operatorname{coth}(1)\left(\begin{array}{ll}
1 & -1 \tag{6.14}
\end{array}\right), \quad B^{+}=\frac{\tanh (1)}{2}\binom{1}{-1}, \quad P_{B_{1}}=\frac{1}{2}\binom{1}{1}, \quad P_{B^{*}}=0
$$

Using (6.7), (6.14), we verify the validity of condition (5.12). Following Theorem 5.1, we can find an approximate solution of (6.2), (6.3), which under $\varepsilon=0$ turns into $x\left(t, c^{0}\right)$ (6.10) of the generating problem (6.4), (6.5), following this algorithm

$$
\begin{align*}
c_{3}^{k}\left(w^{*}, \varepsilon\right)=-c_{2}^{k}\left(w^{*}, \varepsilon\right)= & \frac{1}{2} \int_{0}^{1} H_{2}\left(\tau, y^{k}\left(\tau, w^{*}, \varepsilon\right), \bar{y}^{k}\left(\tau, w^{*}, \varepsilon\right)\right) d \tau \\
& -\frac{1}{2} \int_{0}^{1} H_{3}\left(\tau, y^{k}\left(\tau, w^{*}, \varepsilon\right), \bar{y}^{k}\left(\tau, w^{*}, \varepsilon\right)\right) d \tau  \tag{6.15}\\
\bar{y}_{1}^{k+1}\left(t, w^{*}, \varepsilon\right)= & \varepsilon \operatorname{coth}(1) \int_{0}^{t}\left(2 c_{2}^{k}\left(w^{*}, \varepsilon\right)+H_{2}\left(\tau, y^{k}\left(\tau, w^{*}, \varepsilon\right), \bar{y}^{k}\left(\tau, w^{*}, \varepsilon\right)\right)\right) d \tau \\
& -\varepsilon \operatorname{coth}(1) \int_{0}^{t} H_{3}\left(\tau, y^{k}\left(\tau, w^{*}, \varepsilon\right), \bar{y}^{k}\left(\tau, w^{*}, \varepsilon\right)\right) d \tau \\
- & \varepsilon \operatorname{coth}(1) \int_{0}^{1} \int_{0}^{t}\left(2 c_{2}^{k}\left(w^{*}, \varepsilon\right)+H_{2}\left(\tau, y^{k}\left(\tau, w^{*}, \varepsilon\right), \bar{y}^{k}\left(\tau, w^{*}, \varepsilon\right)\right)\right) d \tau d t  \tag{6.16}\\
& +\varepsilon \operatorname{coth}(1) \int_{0}^{1} \int_{0}^{t} H_{3}\left(\tau, y^{k}\left(\tau, w^{*}, \varepsilon\right), \bar{y}^{k}\left(\tau, w^{*}, \varepsilon\right)\right) d \tau d t
\end{align*}
$$

$$
\begin{gather*}
\bar{y}_{2}^{k+1}\left(t, w^{*}, \varepsilon\right)=\varepsilon \operatorname{coth}(1) \int_{0}^{t} \tanh \left(1+y_{1}^{k}\left(\tau, w^{*}, \varepsilon\right)\right) d \tau \\
+\frac{2 \varepsilon}{\sinh (2)} \int_{0}^{t}\left(\bar{y}_{1}^{k}\left(\tau, w^{*}, \varepsilon\right)-y_{1}^{k}\left(\tau, w^{*}, \varepsilon\right)\right) d \tau  \tag{6.17}\\
\bar{y}_{3}^{k+1}\left(t, w^{*}, \varepsilon\right)=2 \varepsilon t-\bar{y}_{2}^{k+1}\left(t, w^{*}, \varepsilon\right),  \tag{6.18}\\
y_{1}^{k+1}\left(t, w^{*}, \varepsilon\right)=\bar{y}_{1}^{k+1}\left(t, w^{*}, \varepsilon\right), \quad y_{2}^{k+1}\left(t, w^{*}, \varepsilon\right)=c_{2}^{k}\left(w^{*}, \varepsilon\right)+\bar{y}_{2}^{k+1}\left(t, w^{*}, \varepsilon\right),  \tag{6.19}\\
y_{3}^{k+1}\left(t, w^{*}, \varepsilon\right)=c_{3}^{k}\left(w^{*}, \varepsilon\right)+\bar{y}_{3}^{k+1}\left(t, w^{*}, \varepsilon\right)  \tag{6.20}\\
x_{1}^{k}\left(t, w^{*}, \varepsilon\right)=1+y_{1}^{k}\left(t, w^{*}, \varepsilon\right), \quad x_{2}^{k}\left(t, w^{*}, \varepsilon\right)=y_{2}^{k}\left(t, w^{*}, \varepsilon\right), \quad x_{3}^{k}\left(t, w^{*}, \varepsilon\right)=y_{3}^{k}\left(t, w^{*}, \varepsilon\right)  \tag{6.21}\\
x_{i}\left(t, w^{*}, \varepsilon\right)=\lim _{k \rightarrow \infty} x_{i}^{k}\left(t, w^{*}, \varepsilon\right), \quad y_{i}^{0}\left(t, w^{*}, \varepsilon\right)=\bar{y}_{i}^{0}\left(t, w^{*}, \varepsilon\right)=0, \quad i=\overline{1,3} \tag{6.22}
\end{gather*}
$$

Let us construct the first approximation $x^{1}\left(t, w^{*}, \varepsilon\right)$. Since $y^{0}\left(t, w^{*}, \varepsilon\right)=\bar{y}^{0}\left(t, w^{*}, \varepsilon\right)=0$, then the constants $c_{2}^{0}\left(w^{*}, \varepsilon\right), c_{3}^{0}\left(w^{*}, \varepsilon\right)$ defined by formula (6.15) take the form $c_{2}^{0}\left(w^{*}, \varepsilon\right)=$ $c_{3}^{0}\left(w^{*}, \varepsilon\right)=0$. From (6.16)-(6.20) we obtain

$$
y^{1}\left(t, w^{*}, \varepsilon\right)=\bar{y}^{1}\left(t, w^{*}, \varepsilon\right)=\varepsilon\left(\begin{array}{l}
0 \\
t \\
t
\end{array}\right)
$$

and, following (6.21), we get

$$
x^{1}\left(t, w^{*}, \varepsilon\right)=\left(\begin{array}{c}
1  \tag{6.23}\\
\varepsilon t \\
\varepsilon t
\end{array}\right)
$$

Continuing calculations according to (6.15)-(6.22), we see that

$$
c_{2}^{k}\left(w^{*}, \varepsilon\right)=c_{3}^{k}\left(w^{*}, \varepsilon\right)=0, \quad \forall k \geq 1
$$

and all subsequent approximations $x^{k}\left(t, w^{*}, \varepsilon\right), k \geq 2$ are equal to the first approximation, that is vector-function (6.23), as can be seen by a simple substitution, is the solution of (6.2), (6.3), which at $\varepsilon=0$ turns into the generating solution (6.10) for the values of parameters (6.13), minimizing functional (6.11).

Note that one of the important concepts in the study of the problem for finding the extremum of a function on solutions of an equation, including problem (2.2), (2.3), (4.1), is the concept of solution sensitivity with respect to the parameters

$$
s(t, \varepsilon)=\frac{\partial x(t, w, \varepsilon)}{\partial w} .
$$

In the literature $[9,10,12,16,18,24,42]$ there are two approaches to find $s(t, \varepsilon)$ : the direct method, which uses the chain rule to find the complete derivative, and adjoint sensitivity method from Pontryagin papers [39]. The use of the conjugate sensitivity method reduces the computational costs when finding the gradient by parameters when the number of parameters is much greater than the dimension of the set of required functions. When the number of parameters is much less than the number of desired functions, the advantages of this method are lost due to the complexity of solving the auxiliary coupled system. In Example 6.3 we consider a boundary-value problem for systems of three differential equations with three parameters. Therefore, the use of the direct method and adjoint sensitivity method for finding the gradient by parameters $w_{12}, w_{13}, w_{23}$ are equivalent. However, when investigating the
problem of optimization of function on solutions of the Hopfield network for $n(n \gg 3)$ neurons, in which the number of weights by far exceeds the number of potentials ( $M \gg n$ ), the adjoint sensitivity method has advantages over the direct method. The study of relationships of the direct method, adjoint sensitivity method and the accelerated method of conjugate gradients for solving the presented paper tasks will be devoted to our future research.

## 7 Conclusions

Necessary and sufficient conditions for the solvability were established, as well as a constructive algorithm for finding solutions to a boundary-value problem for a system of weakly non-linear differential equations describing Hopfield network for $n$ neurons is presented. The problem of minimizing a functional on the solutions of the given problem was investigated and the application of the accelerated method of conjugate gradients to its solutions was explored. The results are demonstrated by examples of problems for the case of three neurons.

## Acknowledgements

The work was supported by the National Research Foundation of Ukraine (Project number 2020.20/0089).

## References

[1] E. Agliari, A. Barra, M. Notarnicola, The relativistic Hopfield network: rigorous results, J. Math. Phys. 60(2019), No. 3, Paper No. 033302, 11 pp. https://doi.org/10. 1063/1.5077060; MR3927839; Zbl 1421.82027
[2] M. Акнmet, E. M. Alejaily, Domain-structured chaos in a Hopfield neural network, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 29(2019), No. 14, Paper No. 1950205, 7 pp. https: //doi.org/10.1142/S0218127419502055; MR4045912; Zbl 1429.37021
[3] A. Boichuk, J. Diblík, D. Khusainov, M. Růžičková, Boundary-value problems for weakly nonlinear delay differential systems, Abstr. Appl. Anal. 2011, Paper No. 631412, 19 pp. https://doi.org/10.1155/2011/631412; MR2802836; Zbl 1222.34075
[4] A. A. Boichuk, I. A. Korostil, M. Fečkan, Bifurcation conditions for a solution of an abstract wave equation, Differ. Equ. 43(2007), No. 4, 495-502. https ://doi. org/10.1134/ S0012266107040076; MR2358690; Zbl 1146.35010
[5] A. A. Boichuk, M. Medved, V. P. Zhuravliov, Fredholm boundary-value problems for linear delay systems defined by pairwise permutable matrices, Electron. J. Qual. Theory Differ. Equ. 2015, No. 23, 1-9. https://doi.org/10.14232/ejqtde.2015.1.23; MR3346934; Zbl 1349.34254
[6] А. А. Воісник, А. А. Рокитnyi, Perturbation theory of operator equations in the Fréchet and Hilbert spaces, Ukrainian Math. J. 67(2016), No. 9, 1327-1335. https://doi .org/10. 1007/s11253-016-1156-y; MR3473723; Zbl 06817053
[7] A. A. Boichuk, A. M. Samoilenko, Generalized inverse operators and Fredholm boundaryvalue problems, 2nd ed., Inverse and Ill-Posed Problems Series, Vol. 59, De Gruyter, Berlin, 2016. https://doi.org/10.1515/9783110378443; MR3585692; Zbl 1394.47001
[8] A. A. Boichuk, V. F. Zhuravlev, Solvability criterion for integro-differential equations with degenerate kernel in Banach spaces, Nonlinear Dyn. Syst. Theory 18(2018), No. 4, 331-341. MR3891950; Zbl 1411.34105
[9] J. Calver, W. Enright, Numerical methods for computing sensitivities for ODEs and DDEs, Numer. Algorithms 74(2017), No. 4, 1101-1117. https://doi.org/10.1007/s11075-016-0188-6; MRMR3626330; Zbl 1362.65070
[10] Y. Cao, S. Li, L. Petzold, R. Serban, Adjoint sensitivity analysis for differential-algebraic equations: the adjoint DAE system and its numerical solution, SIAM J. Sci. Comput. 24(2003), No. 3, 1076-1089. https://doi.org/10.1137/S1064827501380630; MR1950525; Zbl 1034.65066
[11] Z. Che, S. Purushotham, K. Сho, D. Sontag, Y. Liu, Recurrent neural networks for multivariate time series with missing values, Sci. Rep. 8(2018), Paper No. 6085. https: //doi.org/10.1038/s41598-018-24271-9
[12] R. T. Q. Chen, Yu. Rubanova, J. Bettencourt, D. Duvenaud, Neural ordinary differential equations, in: NeurIPS 2018 (Proc. of the International Conference on Neural Information Processing Systems, 2018), Curran Associates Inc., New York, 2018, pp. 6571-6583.
[13] E. Choi, M. T. Bahadori, A. Schuetz, W. F. Stewart, J. Sun, Doctor AI: predicting clinical events via recurrent neural networks, in: Machine Learning for Healthcare Conference (Proceedings of the 1st Machine Learning for Healthcare Conference, Vol. 56 of Proceedings of Machine Learning Research), PMLR, 2016, pp. 301-318.
[14] M.-F. Danca, N. Kuznetsov, Hidden chaotic sets in a Hopfield neural system, Chaos Solitons Fractals 103(2017), 144-150. https://doi.org/10.1016/j.chaos.2017.06.002; MR3689806; Zbl 1380.65423
[15] E. E. Dudnikov, M. V. Rybashov, On the absolute stability of a class of neural networks with feedback, Automat. Remote Control 60(1999), No. 12, part 1, 1700-1706. MR1813073; Zbl 1088.34524
[16] R. M. Errico, What is an adjoint model?, Bull. Amer. Meteorol. Soc. 78(1997), No. 11, 2577-2591. https://doi.org/10.1175/1520-0477(1997)078<2577:WIAAM>2.0.C0;2
[17] R. Fuentes-García, R. H. Mena, S. G. Walker, Modal posterior clustering motivated by Hopfield's network, Comput. Statist. Data Anal. 137(2019), 92-100. https://doi.org/10. 1016/j.csda.2019.02.008; MR3921062; Zbl 07058808
[18] M. B. Giles, N. A. Pierce, An introduction to the adjoint approach to design, Flow Turbul. Combust. 65(2000), No. 3-4, 393-415. https://doi.org/10.1023/A:1011430410075; Zbl 0996.76023
[19] S. D. Glyzin, A. Yu. Kolesov, N. Кh. Rozov, Relaxation self-oscillations in Hopfield networks with delay, Izv. Math. 77(2013), No. 2, 271-312. https://doi.org/10.1070/ IM2013v077n02ABEH002636; MR3097567; Zbl 1271.34074
[20] S. D. Glyzin, A. Yu. Kolesov, N. Kh. Rozov, Self-excited relaxation oscillations in networks of impulse neurons, Russian Math. Surveys 70(2015), No. 3, 383-452. https: //doi.org/10.1070/RM2015v070n03ABEH004951; MR3400564; Zbl 1361.34093
[21] E. A. Grebenikov, Yu. A. Ryabov, Constructive methods in the analysis of nonlinear systems, Mir Publishers, Moscow, 1983. MR0733787; Zbl 0536.65058
[22] S. Haykin, Neural networks and learning machines, 3rd ed., Pearson Education, New York, 2009.
[23] C. J. Hillar, N. M. Tran, Robust exponential memory in Hopfield networks, J. Math. Neurosci. 8(2018), Paper No. 1, 20 pp. https://doi.org/10.1186/s13408-017-0056-2; MR3747011; Zbl 1395.92010
[24] A. C. Hindmarsh, R. Serban, User documentation for CVODES, an ODE solver with sensitivity analysis capabilities, Lawrence Livermore National Laboratory, Livermore, 2002.
[25] J. J. Hopfield, Neural networks and physical systems with emergent collective computational abilities, Proc. Nat. Acad. Sci. U.S.A. 79(1982), No. 8, 2554-2558. https: //doi.org/10.1073/pnas.79.8.2554; MR0652033; Zbl 1369.92007
[26] J. J. Hopfield, D. W. Tank, Neural computation of decisions in optimization problems, Biol. Cybernet. 52(1985), No. 3, 141-152. MR0824597; Zbl 0572.68041
[27] F. C. Hoppensteadt, E. M. Izhikevich, Weakly connected neural networks, Applied Mathematical Sciences, Vol. 126, Springer, New York, 1997. https://doi.org/10.1007/978-1-4612-1828-9; MR1458890; Zbl 0887.92003
[28] R. A. Horn, C. R. Johnson, Matrix analysis, 2nd ed., Cambridge University Press, Cambridge, 2013. MR2978290; Zbl 1267.15001
[29] E. M. Izhiкevich, Neural excitability, spiking and bursting, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 10(2000), No. 6, 1171-1266. https://doi.org/10.1142/S0218127400000840; MR1779667; Zbl 1090.92505
[30] L. V. Kantorovič, Functional analysis and applied mathematics (in Russian), Uspehi Matem. Nauk 3(1948), No. 6(28), 89-185. MR0027947; Zbl 0034.21203
[31] A. Yu. Luchka, O. E. Noshchenko, I. V. Sergiyenko, N. I. Tukalevskaya, Accelerated method of conjugate gradients, U.S.S.R. Comput. Math. and Math. Phys. 27(1987), No. 3, 6-12. https://doi.org/10.1016/0041-5553(87)90073-5; Zbl 0655.65075
[32] A. Yu. Luchka, O. E. Noshchenko, N. I. Tukalevskaya, The variational gradient method, U.S.S.R. Comput. Math. Math. Phys. 24(1984), No. 4, 1-6. https://doi.org/10. 1016/0041-5553(84)90222-2; Zbl 0601.47012
[33] I. G. Malkin, Some problems of the theory of nonlinear oscillations (in Russian), Gos. Izdt. Tekh.-Teor. Lit., Moscow, 1956. MR0081402; Zbl 0070.08703
[34] M.-H. Matcovschi, O. Pastravanu, Flow-invariance and stability analysis for a class of nonlinear systems with slope conditions, Eur. J. Control 10(2004), No. 4, 352-364. https: //doi.org/10.3166/ejc.10.352-364; MR2120426; Zbl 1293.93391
[35] M. Mézard, Mean-field message-passing equations in the Hopfield model and its generalizations, Phys. Rev. E 95(2017), No. 2, Paper No. 022117, 15 pp. https://doi.org/10. 1103/PhysRevE.95.022117; MR3778464
[36] B. A. Pearlmutter, Learning state space trajectories in recurrent neural networks, Neural Comput. 1(1989), No. 2, 263-269. https://doi.org/10.1162/neco.1989.1.2.263
[37] R. Penrose, A generalized inverse for matrices, Proc. Cambridge Philos. Soc. 51(1955), 406413. MR0069793 ; Zbl 0065.24603
[38] M. Pinto, D. Serulveda, R. Torres, Exponential periodic attractor of impulsive Hopfield-type neural network system with piecewise constant argument, Electron. J. Qual. Theory Differ. Equ. 2018, No. 34, 1-28. https://doi.org/10.14232/ejqtde.2018.1.34; MR3811497; Zbl 1413.34224
[39] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko, The mathematical theory of optimal processes, A Pergamon Press Book, The Macmillan Co., New York, 1964. MR0186436; Zbl 0117.31702
[40] P. Rotondo, M. Marcuzzi, J. P. Garrahan, I. Lesanovsky, M. Müller, Open quantum generalisation of Hopfield neural networks, J. Phys. A 51(2018), No. 11, Paper No. 115301, 11 pp. https://doi.org/10.1088/1751-8121/aaabcb; MR3766232; Zbl 1394.82014
[41] A. M. Samoilenko, A. A. Boichuk, S. M. Chuiko, Hybrid difference-differential boundary-value problem, Miskolc Math. Notes 18(2017), No. 2, 1015-1031. https://doi. org/10.18514/MMN. 2017.2280; MR3768335; Zbl 1399.34051
[42] B. Sengupta, K. J. Friston, W. D. Penny, Efficient gradient computation for dynamical models, NeuroImage 98(2014), 521-527. https://doi.org/10.1016/j.neuroimage. 2014. 04.040
[43] I. N. Da Silva, D. H. Spatti, R. A. Flauzino, L. H. B. Liboni, S. F. R. Alves, Artificial neural networks, Springer, Switzerland, 2017. https://doi.org/10.1007/978-3-319-43162-8; MR3526246
[44] P. Stapor, F. Fröhlich, J. Hasenauer, Optimization and profile calculation of ODE models using second order adjoint sensitivity analysis, Bioinformatics 34(2018), No. 13, 151159. https://doi.org/10.1093/bioinformatics/bty230
[45] A.-M. Stoica, I. Yaesh, Hopfield networks with multiplicative noise in an anisotropic norm setup, Ann. Acad. Rom. Sci. Ser. Math. Appl. 11(2019), No. 1, 79-97. MR3973488; Zbl 1438.93218
[46] C. A. Tavares, T. M. R. Santos, N. H. T. Lemes, J. P. C. Dos Santos, J. C. Ferreira, J. P. Braga, Solving ill-posed problems faster using fractional-order Hopfield neural network, J. Comput. Appl. Math. 381(2021), Paper No. 112984, 14 pp. https: //doi.org/10.1016/j.cam.2020.112984; MR4105673; Zbl 07241269
[47] Z. Zeng, W. X. Zheng, Multistability of two kinds of recurrent neural networks with activation functions symmetrical about the origin on the phase plane, IEEE Trans. Neural Netw. Learn. Syst. 24(2013), No. 11, 1749-1762. https://doi.org/10.1109/TnNLS. 2013. 2262638

# Pullback attractor for a nonlocal discrete nonlinear Schrödinger equation with delays 

Jardel Morais Pereira ${ }^{\boxtimes}$<br>Department of Mathematics, Federal University of Santa Catarina, Florianópolis SC, 88040-900, Brazil

Received 15 February 2021, appeared 30 December 2021
Communicated by Christian Pötzsche


#### Abstract

We consider a nonlocal discrete nonlinear Schrödinger equation with delays. We prove that the process associated with the non-autonomous model possesses a pullback attractor. As a consequence of our discussion, the existence of a global attractor for the autonomous model is derived.


Keywords: pullback attractor, discrete nonlinear Schrödinger equation, delay terms, global attractor differential equations, difference equations.
2020 Mathematics Subject Classification: 35Q55, 37L60, 37B55, 37L30.

## 1 Introduction

Discrete Schrödinger equations are widely used as models in Physics and other branches of science (see, e.g., $[3,6,11,12,14,19]$ and the references therein). These discrete equations belong to a large class of lattice dynamical systems which has been the object of extensive research (see, for example, $[4,5,7,9,12,13,19,22]$ and the references therein). Various properties related to the dynamics of such systems have been studied. Among them, the existence of global attractors is a theme which attracts a great deal of attention. However, most of the contributions in this line of research addressed to discrete Schrödinger models are concerned the discrete nonlinear Schrödinger equation (DNLS). In this paper, our main aim is to prove the existence of a pullback attractor for a nonlocal discrete nonlinear Schrödinger equation when delay terms are considered. The model is written as follows

$$
\begin{align*}
& i \dot{u}_{n}(t)+\sum_{m=-\infty}^{+\infty} J(n-m) u_{m}(t)+g_{n}\left(t, u_{n t}\right)+i \gamma u_{n}(t)=f_{n}(t), \quad t>\tau, n \in \mathbb{Z},  \tag{1.1}\\
& u_{n}(s)=\psi_{n}(s-\tau), \quad \forall s \in[\tau-h, \tau],
\end{align*}
$$

where $\tau, h$, and $\gamma$ are real numbers with $h>0$ and $\gamma>0$. In (1.1), $u_{n}(t), f_{n}(t)$, and $\psi_{n}$ are complex functions and $u_{n t}$ denotes the translation of $u_{n}$ at time $t$, defined by $u_{n t}(s)=$ $u_{n}(t+s), \forall s \in[-h, 0]$. The dispersive coupling parameters $J(m)$ are assumed to be real numbers, symmetric (i.e., $J(-m)=J(m)$, for all positive integer $m$ ) and $\sum_{m=1}^{+\infty}|J(m)|<+\infty$.

[^55]This includes important special cases as $J(m)=J_{0} e^{-\beta|m|}$ and $J(m)=J_{0}|m|^{-s}$, where $J_{0}, \beta$, and $s$ are positive real constants suitably chosen [8].

We assume that the nonlinear term $g_{n}\left(t, u_{n t}\right)$ in (1.1) includes delay terms as follows

$$
\begin{equation*}
g_{n}\left(t, u_{n t}\right)=g_{0, n}\left(u_{n}(t)\right)+g_{1, n}\left(u_{n}(t-\rho(t))+\int_{-h}^{0} b_{n}\left(s, u_{n}(t+s)\right) d s .\right. \tag{1.2}
\end{equation*}
$$

Appropriate hypotheses on the functions $\rho: \mathbb{R} \rightarrow[0, h], g_{i, n}: \mathbb{C} \rightarrow \mathbb{C}, i=0,1, b_{n}:$ $[0, h] \times \mathbb{C} \rightarrow \mathbb{C}$, and $f_{n}(t)$ are stated in Section 2.

Specific deterministic cases of equation (1.1) have been used in the study of physical phenomena in which long-range dispersive interactions cannot be disregarded (see the physical discussions in [8]). An example is the model proposed in [17] for the description of the nonlinear dynamics of the DNA molecule.

A class of discrete Schrödinger equations of great importance is

$$
\begin{equation*}
i u_{n}(t)+\Delta_{d}^{p} u_{n}(t)+g_{n}\left(t, u_{n t}\right)+i \gamma u_{n}(t)=f_{n}(t), \tag{1.3}
\end{equation*}
$$

where $\Delta_{d}^{p}=\Delta_{d} \circ \cdots \circ \Delta_{d}, p$ times, and $\Delta_{d}$ is the one-dimensional discrete Laplace operator defined by $\Delta_{d} u_{n}=u_{n+1}+u_{n-1}-2 u_{n}$. Equation (1.3) can be derived from (1.1) by choosing the coupling parameters $J(m)$ as

$$
J(m)=\sum_{j=0}^{2 p}\binom{2 p}{j}(-1)^{j} \delta_{m, j-p}
$$

where $p$ is any positive integer and $\delta_{m, k}$ is the Kronecker delta.
Many contributions on existence and properties of solutions of the DNLS equation (i.e, (1.3) with $p=1, g_{1, n}=b_{n}=0$ ) and $f_{n}$ independent of time can be found in the literature (see, e.g., $[3,4,11,19]$ and references therein). For example, the existence and approximation of attractors for the DNLS equation were investigated in [11] while the existence of attractors for the DNLS with retarded terms was studied in [4]. Concerning equation (1.1), in [19], the authors studied the existence of localized solutions for the homogeneous case without delays. Later, also for the autonomous deterministic model, the existence of a global attractor in weighted spaces was established in [20]. For the existence of attractors for some nonautonomous lattice dynamical systems with retarded terms of the type (1.2) and references about related works we refer the reader to the article [2]. Still concerning lattice models with nonlocal terms, we would like to mention the papers [1,10,15,18,21].

In this paper, under suitable conditions on the functions $\rho, g_{i, n}, i=0,1, b_{n}$, and $f_{n}$, we prove the existence of a pullback attractor for the process associated with problem (1.1). As a consequence of our discussion, the existence of a global attractor for the autonomous model is derived.

The paper is organized as follows. In Section 2, we prove that the initial value problem (1.1) is globally well posed. In Section 3, we establish the existence of a pullback attractor for the process associated with problem (1.1) using the results in [16]. Finally, in Section 4, we briefly show how the same ideas of the previous sections can be adapted to prove the existence of a global attractor for the autonomous model.

## 2 Existence of solutions

In this section, we discuss the existence of solutions for the problem (1.1). We denote by $\ell^{p}$ the usual space of complex sequences $u=\left(u_{n}\right)_{n \in \mathbb{Z}}$ such that $\|u\|_{\ell^{p}}<\infty$, where

$$
\|u\|_{\ell^{p}}=\left(\sum_{n=-\infty}^{+\infty}\left|u_{n}\right|^{p}\right)^{\frac{1}{p}}, \quad \text { if } 1 \leq p<\infty \text { and }\|u\|_{\ell^{\infty}}=\sup _{n \in \mathbb{Z}}\left|u_{n}\right| \text {, if } p=\infty \text {. }
$$

When $p=2, \ell^{2}$ is a Hilbert space with the inner product given by

$$
(u, v)_{\ell^{2}}=\sum_{n=-\infty}^{+\infty} u_{n} \bar{v}_{n}, \quad u, v \in \ell^{2},
$$

and, in this case, we denote by $\|\cdot\|$ the corresponding norm.
For $1 \leq p<\infty, L^{p}(-h, 0)$ denotes the usual Banach space of (class of ) real functions $f$ defined on $[-h, 0]$ such that $|f|^{p}$ is integrable in sense of Lebesgue and we recall that for the $\ell^{p}$ spaces the following embedding relation holds:

$$
\ell^{q} \subset \ell^{p}, \quad\|u\|_{\ell^{p}} \leq\|u\|_{\ell^{q}}, \quad 1 \leq q \leq p \leq \infty .
$$

Regarding the functions $g_{i, n}: \mathbb{C} \rightarrow \mathbb{C}, i=0,1, b_{n}:[-h, 0] \times \mathbb{C} \rightarrow \mathbb{C}, f=\left(f_{n}(t)\right)_{n \in \mathbb{Z}}$, and $\rho(t)$ in (1.1) and (1.2) we assume that
(A1) $\bar{z} g_{0, n}(z)$ is real for all $z \in \mathbb{C}$ and $n \in \mathbb{Z}$.
(A2) There exist a function $\kappa \in L^{2}(-h, 0)$ and functions $b_{0, n}: \mathbb{C} \rightarrow \mathbb{C}$ such that

$$
\left|b_{n}\left(s, z_{1}\right)-b_{n}\left(s, z_{2}\right)\right| \leq \kappa(s)\left|b_{0, n}\left(z_{1}\right)-b_{0, n}\left(z_{2}\right)\right|,
$$

$\forall s \in[-h, 0]$ and $\forall z_{1}, z_{2} \in \mathbb{C}$. We set $\kappa_{0}^{2}:=\int_{-h}^{0}|\kappa(s)|^{2} d s$.
(A3) For every $R>0$ there exist positive constants $L_{j}(R), j=1,2$, such that

$$
\begin{aligned}
& \left|g_{i, n}\left(z_{1}\right)-g_{i, n}\left(z_{2}\right)\right| \leq L_{1}(R)\left|z_{1}-z_{2}\right|, \quad i=0,1, \\
& \left|b_{0, n}\left(z_{1}\right)-b_{0, n}\left(z_{2}\right)\right| \leq L_{2}(R)\left|z_{1}-z_{2}\right|,
\end{aligned}
$$

for any $n \in \mathbb{Z}$ and any $z_{1}, z_{2} \in \mathbb{C}$ such that $\left|z_{j}\right| \leq R, j=1,2$. Moreover, $\left(g_{0, n}(0)\right)_{n \in \mathbb{Z}} \in \ell^{2}$.
(A4) There exist sequences of real numbers $k_{1}=\left(k_{1, n}\right)_{n \in \mathbb{Z}} \in \ell^{\infty}, k_{2}=\left(k_{2, n}\right)_{n \in \mathbb{Z}} \in \ell^{2}$ and non-negative real functions $\beta_{1, n}(\cdot) \in L^{2}(-h, 0)$ and $\beta_{2, n}(\cdot) \in L^{1}(-h, 0)$ such that

$$
\left|g_{1, n}(z)\right| \leq k_{1, n}|z|+k_{2, n} \quad \text { and } \quad\left|b_{n}(s, z)\right| \leq \beta_{1, n}(s)|z|+\beta_{2, n}(s),
$$

for all $n \in \mathbb{Z}, s \in[-h, 0]$, and $z \in \mathbb{C}$. We set $K_{1}=\left\|k_{1}\right\|_{\ell \infty}, K_{2}=\left\|k_{2}\right\|$, and

$$
B_{1}=\sup _{n \in \mathbb{Z}}\left(\int_{-h}^{0} \beta_{1, n}^{2}(s) d s\right)^{1 / 2}<\infty, B_{2}=\left[\sum_{n=-\infty}^{+\infty}\left(\int_{-h}^{0} b_{2, n}(s) d s\right)^{2}\right]^{1 / 2}<\infty .
$$

(A5) $f \in C\left(\mathbb{R} ; \ell^{2}\right)$.
(A6) $\rho \in C(\mathbb{R} ;[0, h])$.
(A7) $\int_{-\infty}^{t}\|f(s)\|^{2} d s<\infty, \forall t \in \mathbb{R}$.

Example 2.1. Let $0 \neq \chi=\left(\chi_{n}\right)_{n \in \mathbb{Z}} \in \ell^{p}$, for some $1 \leq p \leq \infty$, and $\varphi_{1}: \mathbb{R} \rightarrow \mathbb{R}$ defined by $\varphi_{1}(t)=\frac{t^{2}}{a+b t^{2}}$, where $a$ and $b$ are positive real constants. Also define the functions $g_{1, n}: \mathbb{C} \rightarrow \mathbb{C}$, $b_{0, n}: \mathbb{C} \xrightarrow{\mathbb{C}}$ and $b_{n}:[-h, 0] \times \mathbb{C} \rightarrow \mathbb{C}$ by

$$
\begin{gathered}
g_{1, n}(z)=b_{0, n}(z)=\chi_{n} \varphi_{1}(|z|) z, \\
b_{n}(s, z)=\chi_{n} \varphi_{1}(|z|) z \frac{1}{h}(s+h), \quad \forall n \in \mathbb{Z}, s \in[-h, 0] \text { and } z \in \mathbb{C} .
\end{gathered}
$$

Then, the hypotheses (A2)-(A4) are satisfied with

$$
\begin{gathered}
L_{1}(R)=L_{2}(R)=\left(\frac{1}{b}+\frac{R}{\sqrt{a b}}\right)\|\chi\|_{\ell^{p},} \\
\kappa(s)=\frac{1}{h}(s+h), \quad k_{1, n}=\frac{1}{b}\left|\chi_{n}\right|, \quad k_{2, n}=0, \quad \beta_{1, n}(s)=\frac{1}{b h}\left|\chi_{n}\right|(s+h), \quad \text { and } \quad \beta_{2, n}=0 .
\end{gathered}
$$

Conditions (A1) and (A3) concerning $g_{0, n}$ are satisfied, for example, if $g_{0, n}(z)=\chi_{n} \varphi_{2}(|z|) z$, with $\chi_{n}$ as before and any $\varphi_{2} \in C^{1}\left(\mathbb{R}^{+} ; \mathbb{R}\right)$, such that $\varphi_{2}(0)=0$.

Now let us write (1.1) as an evolution equation with a retarded term in $\ell^{2}$. For any $u=\left(u_{n}\right)_{n \in \mathbb{Z}}$ we define $(A u)_{n}=\sum_{m=-\infty}^{+\infty} J(n-m) u_{m}, \forall n \in \mathbb{Z}$.

Lemma 2.2. $A: \ell^{2} \rightarrow \ell^{2}$ is a bounded operator and $\|A u\| \leq 4\|J\|_{\ell^{1}}\|u\|, \forall u \in \ell^{2}$.
Proof. See Lemma 2.1 in [20].
We consider the space $E_{h}=C\left([-h, 0] ; \ell^{2}\right)$ with the usual norm given by $\|u\|_{E_{h}}=$ $\max _{s \in[-h, 0]}\|u(s)\|$ and define the map $g: \mathbb{R} \times E_{h} \rightarrow \ell^{2}$ by $(g(t, v))_{n \in \mathbb{Z}}=g_{n}\left(t, v_{n}\right)$, where $v(s)=\left(v_{n}(s)\right)_{n \in \mathbb{Z}}$, for any $s \in[-h, 0]$, and

$$
g_{n}\left(t, v_{n}\right)=g_{0, n}\left(v_{n}(0)\right)+g_{1, n}\left(v_{n}(-\rho(t))\right)+\int_{-h}^{0} b_{n}\left(s, v_{n}(s)\right) d s
$$

If we set $u_{t}=\left(u_{n t}\right)_{n \in \mathbb{Z}}$ for any $t \geq \tau$, then we can write the initial value problem (1.1) in $\ell^{2}$ as

$$
\begin{align*}
& i \dot{u}(t)+A u(t)+g\left(t, u_{t}\right)+i \gamma u(t)=f(t), \quad t>\tau,  \tag{2.1}\\
& u(s)=\psi(s-\tau), \quad \forall s \in[\tau-h, \tau],
\end{align*}
$$

where $\psi(s)=\left(\psi_{n}(s)\right)_{n \in \mathbb{Z}}$, for any $s \in[-h, 0]$.
We now define the map $\mathcal{B}: \mathbb{R} \times E_{h} \rightarrow \ell^{2}$ by

$$
\mathcal{B}(t, v)=-i[\operatorname{Av}(0)+g(t, v)+i \gamma v(0)-f(t)] .
$$

Then, problem (2.1) can be rewritten as the following functional equation in $\ell^{2}$

$$
\begin{align*}
& \frac{d u}{d t}+\mathcal{B}\left(t, u_{t}\right)=0, \quad t>\tau  \tag{2.2}\\
& u_{\tau}=\psi
\end{align*}
$$

The following two lemmas are sufficient to ensure the existence of a local solution for (2.1).

Lemma 2.3. Assume that (A2)-(A6) hold. Then the map $\mathcal{B}$ is continuous and satisfies the local Lipschitz condition: For any $v, w \in E_{h}$, with $\|v\|_{E_{h}} \leq R$ and $\|w\|_{E_{h}} \leq R$, there exists a positive constant $L=L(R)$ such that

$$
\|\mathcal{B}(t, v)-\mathcal{B}(t, w)\| \leq L\|v-w\|_{E_{h^{\prime}}} \quad \forall t \in \mathbb{R} .
$$

Proof. Using (A2)-(A6) we see that $\mathcal{B}$ is well defined. Fix $(t, v) \in \mathbb{R} \times E_{h}$ and consider $t^{m} \rightarrow t$ in $\mathbb{R}$ and $v^{m} \rightarrow v$ in $E_{h}$. We have that

$$
\begin{align*}
\left\|\mathcal{B}\left(t^{m}, v^{m}\right)-\mathcal{B}(t, v)\right\| \leq & \left\|A\left(v^{m}(0)-v(0)\right)\right\|+\left\|g\left(t^{m}, v^{m}\right)-g(t, v)\right\|  \tag{2.3}\\
& +\gamma\left\|v^{m}(0)-v(0)\right\|+\left\|f\left(t^{m}\right)-f(t)\right\| .
\end{align*}
$$

Since the sequence $\left(v^{m}\right)_{m \in \mathbf{N}}$ is bounded in $E_{h}$, then using the assumptions (A2), (A3), and (A6) we can find a positive constant $L$ depending only on $\|v\|_{E_{h}}$ such that

$$
\begin{align*}
\left\|g\left(t^{m}, v^{m}\right)-g(t, v)\right\|^{2} \leq & 4 \sum_{n=-\infty}^{+\infty}\left|g_{0, n}\left(v_{n}^{m}(0)\right)-g_{0, n}\left(v_{n}(0)\right)\right|^{2} \\
& +4 \sum_{n=-\infty}^{+\infty}\left|g_{1, n}\left(v_{n}^{m}\left(-\rho\left(t^{m}\right)\right)\right)-g_{1, n}\left(v_{n}(-\rho(t))\right)\right|^{2} \\
& +4 \sum_{n=-\infty}^{+\infty}\left(\int_{-h}^{0}\left|b_{n}\left(s, v_{n}^{m}(s)\right)-b_{n}\left(s, v_{n}(s)\right)\right| d s\right)^{2}  \tag{2.4}\\
\leq & 8 L^{2}\left\|v^{m}-v\right\|_{E_{h}}^{2}+4 L^{2} \sum_{n=-\infty}^{+\infty}\left(\int_{-h}^{0}|\kappa(s)|\left|v_{n}^{m}(s)-v_{n}(s)\right| d s\right)^{2} .
\end{align*}
$$

Using the Cauchy-Schwarz inequality and the fact that $\left\|v^{m}-v\right\|_{E_{h}}<\infty$ we can estimate the last term in (2.4) as follows

$$
\begin{align*}
& \sum_{n=-\infty}^{+\infty}\left(\int_{-h}^{0}|\kappa(s)|\left|v_{n}^{m}(s)-v_{n}(s)\right| d s\right)^{2} d s \leq \kappa_{0}^{2} \sum_{n=-\infty}^{+\infty} \int_{-h}^{0}\left|v_{n}^{m}(s)-v(s)\right|^{2} d s \\
& \quad \leq \kappa_{0}^{2} \int_{-h}^{0} \sum_{n=-\infty}^{+\infty}\left|v_{n}^{m}(s)-v(s)\right|^{2} d s \leq \kappa_{0}^{2}\left\|v^{m}-v\right\|_{E_{h}}^{2} h . \tag{2.5}
\end{align*}
$$

From (2.3), (2.4), (2.5), (A5), and Lemma 2.2 we deduce the continuity of $\mathcal{B}$. In a similar manner we prove the Lipschitz condition.

Lemma 2.4. Assume that (A2)-(A6) hold. Then the map $\mathcal{B}$ is bounded, i.e., it takes bounded subsets of $\mathbb{R} \times E_{h}$ onto bounded subsets of $\ell^{2}$.

Proof. Let $\mathcal{O}$ be a bounded subset of $\mathbb{R} \times E_{h}$. Then, there exists a positive constant $R$ such that $|t|^{2}+\|v\|_{E_{h}}^{2} \leq R^{2}, \forall(t, v) \in \mathcal{O}$. Using Lemma 2.3 we find a positive constant $L=L(R)$ such that

$$
\begin{aligned}
\|\mathcal{B}(t, v)\| & \leq\|\mathcal{B}(t, v)-\mathcal{B}(t, 0)\|+\|\mathcal{B}(t, 0)\| \\
& \leq L R+\max _{|t| \leq R}\|\mathcal{B}(t, 0)\|<\infty, \quad \forall(t, v) \in \mathcal{O} .
\end{aligned}
$$

Using Lemmas 2.3, 2.4 and applying the Theory of Functional Equations to problem (2.2) we deduce the following result of existence of local solution for (2.1).

Theorem 2.5. Assume that (A2)-(A6) hold. Then, for each $\psi \in E_{h}$, the initial value problem (2.1) has a unique solution $u=u(t)$ defined in $[\tau-h, T)$ such that $u \in C\left([\tau-h, T) ; \ell^{2}\right) \cap C^{1}\left([\tau, T) ; \ell^{2}\right)$. Moreover, if $T<\infty$ then $\lim _{t \rightarrow T^{-}}\|u(t)\|=\infty$.

Next let us show that the local solution obtained in Theorem 2.5 can be extended globally.
Lemma 2.6. Assume that (A1)-(A6) hold. Then the solution $u$ of (2.1) with $u_{\tau}=\psi \in E_{h}$ satisfies

$$
\begin{align*}
\frac{1}{2} \frac{d}{d t}\|u(t)\|^{2}+\frac{\gamma}{2}\|u(t)\|^{2} \leq & \frac{1}{2 \gamma}\|f(t)\|^{2}+\left(K_{1}\|u(t-\rho(t))\|+K_{2}\right)\|u(t)\| \\
& +\left[B_{1}\left(\int_{-h}^{0}\|u(t+s)\|^{2} d s\right)^{1 / 2}+B_{2}\right]\|u(t)\|, \quad \tau \leq t<T \tag{2.6}
\end{align*}
$$

Proof. Taking the imaginary part of the inner product of equation (2.1) with $u$ in $\ell^{2}$, we obtain

$$
\frac{1}{2} \frac{d}{d t}\|u(t)\|^{2}+\operatorname{Im}(A u(t), u(t))_{\ell^{2}}+\gamma\|u(t)\|^{2}+\operatorname{Im}\left(g\left(t, u_{t}\right), u(t)\right)_{\ell^{2}}=\operatorname{Im}(f(t), u(t))_{\ell^{2}},
$$

for all $\tau \leq t<T$. Since

$$
\begin{gathered}
\operatorname{Im}(f(t), u(t))_{\ell^{2}} \leq \frac{1}{2 \gamma}\|f(t)\|^{2}+\frac{\gamma}{2}\|u(t)\|^{2} \\
(A u(t), u(t))_{\ell^{2}}=J(0)\|u(t)\|^{2}+2 \sum_{m=1}^{+\infty} \sum_{n=-\infty}^{+\infty} J(m) \operatorname{Re}\left(\overline{u_{n+m}(t)} u_{n}(t)\right),
\end{gathered}
$$

then, using (A1), we get the inequality

$$
\begin{align*}
\frac{1}{2} \frac{d}{d t}\|u(t)\|^{2}+\frac{\gamma}{2}\|u(t)\|^{2} \leq & \frac{1}{2 \gamma}\|f(t)\|^{2}-\operatorname{Im} \sum_{n=-\infty}^{+\infty} g_{1, n}\left(u_{n}(t-\rho(t))\right) \bar{u}_{n} \\
& -\operatorname{Im} \sum_{n=-\infty}^{+\infty} \int_{-h}^{0} b_{n}\left(s, u_{n}(t+s)\right) d s \bar{u}_{n}, \quad \tau \leq t<T . \tag{2.7}
\end{align*}
$$

Let us estimate the last two terms in (2.7) using the assumption (A4) and the fact that $\left\|u_{t}\right\|_{E_{h}}<\infty, \forall \tau \leq t<T$. We have that

$$
\begin{align*}
&-\operatorname{Im} \sum_{n=-\infty}^{+\infty} g_{1, n}\left(u_{n}(t-\rho(t))\right) \bar{u}_{n} \leq \sum_{n=-\infty}^{+\infty}\left[k_{1, n}\left|u_{n}(t-\rho(t))\right|+k_{2, n}\right]\left|u_{n}\right|  \tag{2.8}\\
& \leq\left(K_{1}\|u(t-\rho(t))\|+K_{2}\right)\|u\|, \\
&-\operatorname{Im} \sum_{n=-\infty}^{+\infty} \int_{-h}^{0} b_{n}\left(s, u_{n}(t+s)\right) d s \bar{u}_{n} \leq \sum_{n=-\infty}^{+\infty} \int_{-h}^{0}\left[\beta_{1, n}(s)\left|u_{n}(t+s)\right|+\beta_{2, n}(s)\right] d s\left|u_{n}\right| \\
& \leq\left[\sum_{n=-\infty}^{+\infty}\left(\int_{-h}^{0} \beta_{1, n}^{2}(s) d s\right)\left(\int_{-h}^{0}\left|u_{n}(t+s)\right|^{2} d s\right)\right]^{1 / 2}\|u\|+B_{2}\|u\|  \tag{2.9}\\
& \leq\left[B_{1}\left(\int_{-h}^{0}\|u(t+s)\|^{2} d s\right)^{1 / 2}+B_{2}\right]\|u\| .
\end{align*}
$$

From (2.7)-(2.9) we obtain (2.6).

We now make the following assumptions on the constants $B_{1}, K_{1}, \gamma, h$, and a suitable positive parameter $\mu$, which will be used in Section 3 to define the universe where the pullback attractor will lie in.
(A8) We assume that there exists a positive real number $\mu$ such that
(i) If $K_{1}>0$ and $B_{1} \geq 0$ then

$$
\begin{equation*}
4 B_{1}^{2} h<e^{-\mu h} \gamma\left(\frac{\gamma}{2}-\mu\right) \tag{2.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\mu>2 K_{1} e^{\mu h} . \tag{2.11}
\end{equation*}
$$

(ii) If $K_{1}=0$ and $B_{1}>0$ then

$$
\begin{equation*}
\mu<\frac{\gamma}{2} \quad \text { and } \quad \mu>\frac{4}{\gamma} B_{1}^{2} e^{2 \mu h} h . \tag{2.12}
\end{equation*}
$$

(iii) If $K_{1}=B_{1}=0$ then $\mu=\frac{\gamma}{2}$ and $h$ is arbitrary.

Remark 2.7. Conditions in (A8) will be used in the next theorem to prove an estimate for the solution of (2.1) that allows us to extend it globally and that will be used in the proofs of Lemmas 3.1, 3.2 and 3.3 in Section 3. It is clear from (2.10) that $\mu<\frac{\gamma}{2}$. We also observe that (2.11) holds if and only if $0<2 K_{1}<\frac{1}{h e}$, where $\frac{1}{h e}$ is the maximum value of the real function $\phi(s)=s e^{-h s}, s \geq 0$. From this we see that $2 K_{1} e h<1$ and $\mu \in\left(\mu_{1}, \mu_{2}\right)$, where $\mu_{j}, j=1,2$, are the two positive solutions of the equation $\mu e^{-\mu h}=2 K_{1}$.

Theorem 2.8. Assume that (A1)-(A8) hold. Then, the solution $u=u(t)$ of (2.1) with $u_{\tau}=\psi \in E_{h}$ exists globally. Moreover, for each $\tau<T<\infty$, the map $\mathfrak{I}: E_{h} \rightarrow C\left([\tau, T] ; E_{h}\right)$, defined by $\mathfrak{I}(\psi)(t)=$ $u_{t}, \forall \tau \leq t \leq T$, is continuous.

Proof. Assume that (A8)(i) holds. Multiplying (2.6) by $e^{\mu t}$ and integrating the resulting inequality over $[\tau, t]$ we have, for any positive real constants $\varepsilon$ and $\varepsilon^{\prime}$,

$$
\begin{align*}
e^{\mu t}\|u(t)\|^{2} \leq & e^{\mu \tau}\|\psi\|_{E_{h}}^{2}+\left(\mu-\gamma+\varepsilon+\varepsilon^{\prime}\right) \int_{\tau}^{t} e^{\mu s}\|u(s)\|^{2} d s+\frac{1}{\gamma} \int_{\tau}^{t} e^{\mu s}\|f(s)\|^{2} d s \\
& +\left(\frac{2 B_{2}^{2}}{\varepsilon}+\frac{K_{2}^{2}}{\varepsilon^{\prime}}\right) \frac{e^{\mu t}}{\mu}+2 K_{1} \int_{\tau}^{t} e^{\mu s}\left\|u_{s}\right\|_{E_{h}}^{2} d s  \tag{2.13}\\
& +\frac{2 B_{1}^{2}}{\varepsilon} \int_{\tau}^{t} \int_{-h}^{0} e^{u t^{\prime}}\left\|u\left(t^{\prime}+s\right)\right\|^{2} d s d t^{\prime} .
\end{align*}
$$

Let us estimate the last term in (2.13) using the initial condition in (2.1). We have

$$
\begin{align*}
\int_{\tau}^{t} \int_{-h}^{0} e^{\mu t^{\prime}}\left\|u\left(t^{\prime}+s\right)\right\|^{2} d s d t^{\prime} & =\int_{-h}^{0} \int_{\tau}^{t} e^{-\mu s} e^{\mu\left(t^{\prime}+s\right)}\left\|u\left(t^{\prime}+s\right)\right\|^{2} d t^{\prime} d s \\
& \leq e^{\mu h} \int_{-h}^{0} \int_{\tau-h}^{t} e^{\mu \sigma}\|u(\sigma)\|^{2} d \sigma d s \\
& =e^{\mu h} h\left[\int_{\tau-h}^{\tau} e^{\mu \sigma}\|u(\sigma)\|^{2} d \sigma+\int_{\tau}^{t} e^{\mu \sigma}\|u(\sigma)\|^{2} d \sigma\right]  \tag{2.14}\\
& \leq \frac{e^{\mu(\tau+h)} h}{\mu}\|\psi\|_{E_{h}}^{2}+e^{\mu h} h \int_{\tau}^{t} e^{\mu \sigma}\|u(\sigma)\|^{2} d \sigma .
\end{align*}
$$

Substituting (2.14) into (2.13) we get

$$
\begin{align*}
e^{\mu t}\|u(t)\|^{2} \leq & e^{\mu \tau}\|\psi\|_{E_{h}}^{2}+\left(\mu-\gamma+\varepsilon+\varepsilon^{\prime}+\frac{2 B_{1}^{2} e^{\mu h} h}{\varepsilon}\right) \int_{\tau}^{t} e^{\mu s}\|u(s)\|^{2} d s \\
& +\frac{2 B_{1}^{2} e^{\mu h} h}{\mu \varepsilon} e^{\mu \tau}\|\psi\|_{E_{h}}^{2}+\left(\frac{2 B_{2}^{2}}{\varepsilon}+\frac{K_{2}^{2}}{\varepsilon^{\prime}}\right) \frac{e^{\mu t}}{\mu}  \tag{2.15}\\
& +\frac{1}{\gamma} \int_{\tau}^{t} e^{\mu s}\|f(s)\|^{2} d s+2 K_{1} \int_{\tau}^{t} e^{\mu s}\left\|u_{s}\right\|_{E_{h}}^{2} d s .
\end{align*}
$$

Using (2.10) we can choose $\varepsilon=\frac{\gamma}{2}$ and

$$
\begin{equation*}
\varepsilon^{\prime}=\frac{\gamma}{2}-\mu-\frac{4 B_{1}^{2} e^{\mu h} h}{\gamma} \tag{2.16}
\end{equation*}
$$

in (2.15) to obtain

$$
\begin{align*}
e^{\mu t}\|u(t)\|^{2} \leq & e^{\mu \tau}\left(1+\frac{4 B_{1}^{2} e^{\mu h} h}{\mu \gamma}\right)\|\psi\|_{E_{h}}^{2}+\left(\frac{4 B_{2}^{2}}{\gamma}+\frac{K_{2}^{2}}{\varepsilon^{\prime}}\right) \frac{e^{\mu t}}{\mu}  \tag{2.17}\\
& +\frac{1}{\gamma} \int_{\tau}^{t} e^{\mu s}\|f(s)\|^{2} d s+2 K_{1} \int_{\tau}^{t} e^{\mu s}\left\|u_{s}\right\|_{E_{h}}^{2} d s .
\end{align*}
$$

Since $\|u(s)\| \leq\|\psi\|_{E_{h}}, \forall s \in[\tau-h, \tau]$, then we can replace $t$ in (2.17) by $t+\sigma$, with $\sigma \in[-h, 0]$, to deduce that

$$
e^{\mu t}\left\|u_{t}\right\|_{E_{h}}^{2} \leq M(t)+L \int_{\tau}^{t} e^{\mu s}\left\|u_{s}\right\|_{E_{h}}^{2} d s
$$

where $L=2 K_{1} e^{\mu h}$ and

$$
M(t)=e^{\mu(\tau+h)}\left(1+\frac{4 B_{1}^{2} e^{\mu h} h}{\mu \gamma}\right)\|\psi\|_{E_{h}}^{2}+\left(\frac{4 B_{2}^{2}}{\gamma}+\frac{K_{2}^{2}}{\varepsilon^{\prime}}\right) \frac{e^{\mu(t+h)}}{\mu}+\frac{e^{\mu h}}{\gamma} \int_{\tau}^{t} e^{\mu s}\|f(s)\|^{2} d s
$$

The above inequality implies that

$$
\begin{equation*}
e^{\mu t}\left\|u_{t}\right\|_{E_{h}}^{2} \leq e^{L(t-\tau)} M(\tau)+e^{L t} \int_{\tau}^{t} e^{-L s} M^{\prime}(s) d s . \tag{2.18}
\end{equation*}
$$

Performing the calculations in (2.18) using $M(t)$ above and the fact that $\mu>L$ by (2.11), we find the following estimate for the solution of (2.1)

$$
\begin{equation*}
\left\|u_{t}\right\|_{E_{h}}^{2} \leq c_{1}\|\psi\|_{E_{h}}^{2} e^{(L-\mu) t} e^{(\mu-L) \tau}+\frac{2 \mu-L}{\mu-L} c_{2}+\frac{e^{\mu h}}{\gamma} \int_{-\infty}^{t}\|f(s)\|^{2} d s, \tag{2.19}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{1}=e^{\mu h}\left(1+\frac{4 B_{1}^{2} e^{\mu h} h}{\mu \gamma}\right) \quad \text { and } \quad c_{2}=\left(\frac{4 B_{2}^{2}}{\gamma}+\frac{K_{2}^{2}}{\varepsilon^{\prime}}\right) \frac{e^{\mu h}}{\mu} . \tag{2.20}
\end{equation*}
$$

Now, assume that (A8)(ii) holds. For this case we replace (2.14) by

$$
\begin{align*}
e^{\mu t}\|u(t)\|^{2} \leq & e^{\mu \tau}\|\psi\|_{E_{h}}^{2}+\left(\mu-\gamma+\varepsilon+\varepsilon^{\prime}\right) \int_{\tau}^{t} e^{\mu s}\|u(s)\|^{2} d s \\
& +\frac{2 B_{1}^{2} e^{\mu h} h}{\mu \varepsilon} e^{\mu \tau}\|\psi\|_{E_{h}}^{2}+\left(\frac{2 B_{2}^{2}}{\varepsilon}+\frac{K_{2}^{2}}{\varepsilon^{\prime}}\right) \frac{e^{\mu t}}{\mu}  \tag{2.21}\\
& +\frac{1}{\gamma} \int_{\tau}^{t} e^{\mu s}\|f(s)\|^{2} d s+\frac{2 B_{1}^{2} e^{\mu h} h}{\varepsilon} \int_{\tau}^{t} e^{\mu s}\left\|u_{s}\right\|_{E_{h}}^{2} d s .
\end{align*}
$$

Since that $\mu<\frac{\gamma}{2}$, then we can choose $\varepsilon=\frac{\gamma}{2}$ and $\varepsilon^{\prime}=\frac{\gamma}{2}-\mu$ in (2.21) and proceed as before to obtain

$$
\begin{align*}
e^{\mu t}\|u(t)\|^{2} \leq & e^{\mu(\tau+h)}\left(1+\frac{4 B_{1}^{2} e^{\mu h} h}{\mu \gamma}\right)\|\psi\|_{E_{h}}^{2}+\left(\frac{4 B_{2}^{2}}{\gamma}+\frac{K_{2}^{2}}{\varepsilon^{\prime}}\right) \frac{e^{\mu(t+h)}}{\mu}  \tag{2.22}\\
& +\frac{e^{\mu h}}{\gamma} \int_{\tau}^{t} e^{\mu s}\|f(s)\|^{2} d s+L \int_{\tau}^{t} e^{\mu s}\left\|u_{s}\right\|_{E_{h}}^{2} d s
\end{align*}
$$

where $L=\frac{4}{\gamma} B_{1}^{2} e^{2 \mu h} h$. By (2.12) we see that $\mu>L$. Therefore, we can deduce the estimate (2.19) with $c_{1}$ and $c_{2}$ as in (2.20), with $\varepsilon^{\prime}=\frac{\gamma}{2}-\mu$. Similarly, we can treat the case (A8)(iii) to obtain the estimate

$$
\begin{equation*}
\left\|u_{t}\right\|_{E_{h}}^{2} \leq c_{1}^{\prime}\|\psi\|_{E_{h}}^{2} e^{-\mu t} e^{\mu \tau}+2 c_{2}^{\prime}+\frac{e^{\mu h}}{\gamma} \int_{-\infty}^{t}\|f(s)\|^{2} d s \tag{2.23}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{1}^{\prime}=2 e^{\mu h} \text { and } c_{2}^{\prime}=\frac{e^{\mu h}}{\mu^{2}}\left(B_{2}^{2}+K_{2}^{2}\right) . \tag{2.24}
\end{equation*}
$$

From (2.19) or (2.23) and Theorem 2.5 we conclude that the solution of (2.1) exists globally. Next, let us prove that the map $\mathfrak{I}$ is continuous. Fix $\tau<T<\infty, \psi \in E_{h}$ and consider $\psi_{1} \in E_{h}$ such that $\left\|\psi-\psi_{1}\right\|_{E_{h}}<1$. Let us denote by $v=v(t)$ the solution of (2.1) with initial condition $v(s)=\psi_{1}(s-\tau), \forall s \in[\tau-h, \tau]$. Using the estimate (2.19) or (2.23) we can find a positive constant $K_{0}$ depending on $\|\psi\|_{E_{h}}$ and $T$ such that $\left\|u_{t}\right\|_{E_{h}} \leq K_{0}$ and $\left\|v_{t}\right\|_{E_{h}} \leq K_{0}$, for all $\tau \leq t \leq T$. Then, using the integral representations of $u$ and $v$ and Lemma 2.3 it follows that

$$
\begin{align*}
\|u(t)-v(t)\| & \leq\left\|\psi(0)-\psi_{1}(0)\right\|+\int_{\tau}^{t}\left\|\mathcal{B}\left(s, u_{s}\right)-\mathcal{B}\left(s, v_{s}\right)\right\| d s \\
& \leq\left\|\psi-\psi_{1}\right\|_{E_{h}}+L\left(K_{0}\right) \int_{\tau}^{t}\left\|u_{s}-v_{s}\right\|_{E_{h}} d s . \tag{2.25}
\end{align*}
$$

Replacing $t$ in (2.25) by $t+\sigma$, with $\sigma \in[-h, 0]$, taking into account that $\| u(t+\sigma)-$ $v(t+\sigma)\left\|_{E_{h}} \leq\right\| \psi-\psi_{1} \|_{E_{h}}$ if $t+\sigma \leq \tau$, we obtain

$$
\left\|u_{t}-v_{t}\right\|_{E_{h}} \leq\left\|\psi-\psi_{1}\right\|_{E_{h}}+L\left(K_{0}\right) \int_{\tau}^{t}\left\|u_{s}-v_{s}\right\|_{E_{h}} d s, \quad \forall \tau \leq t \leq T
$$

Then, by Gronwall's inequality, we conclude that $\left\|u_{t}-v_{t}\right\|_{E_{h}} \leq e^{L\left(K_{0}\right)(T-\tau)}\left\|\psi-\psi_{1}\right\|_{E_{h}}$, which implies the continuity of $\mathfrak{I}$.

## 3 Existence of a pullback attractor

By Theorem 2.8 we can associate to the initial value problem (2.1) a process $\{U(t, \tau)\}_{t \geq \tau}$ of continuous maps $U(t, \tau)$ in $E_{h}$ defined by $U(t, \tau) \psi=u_{t}$, where $\tau \leq t$ and $u=u(t)$ is the global solution of (2.1). In this section, we establish the existence of a pullback attractor for the process $\{U(t, \tau)\}_{t \geq \tau}$ using the results obtained in [16]. We are interested in the existence of a pullback attractor for a family of sets depending on time (see [16, Section 3]). Motivated by the estimate (2.19) we consider the set $\mathcal{R}_{\mu}$ of all functions $r: \mathbb{R} \rightarrow(0, \infty)$ such that

$$
\begin{equation*}
\lim _{t \rightarrow-\infty} e^{(\mu-L) t} r^{2}(t)=0 \tag{3.1}
\end{equation*}
$$

Let us denote by $\mathcal{D}_{\mu}$ the class of all families $\hat{D}=\{D(t) ; t \in \mathbb{R}\}$ of nonempty subsets of $E_{h}$ such that $D(t) \subset B_{E_{h}}\left[0 ; r_{\hat{D}}(t)\right]:=\left\{\psi \in E_{h} ;\|\psi\|_{E_{h}} \leq r_{\hat{D}(t)}\right\}$, for some radius $r_{\hat{D}} \in \mathcal{R}_{\mu}$. For the case (A8)(iii) we consider in (3.1) $L=0$. In what follows, we will assume that (A8)(i) or (A8)(ii) holds. Suitable modifications will be indicated for the case (A8)(iii). We will also consider $L$ as in the proof of Theorem 2.8 and the constants $c_{1}, c_{2}, c_{1}^{\prime}$ and $c_{2}^{\prime}$ given by (2.20) and (2.24).
Lemma 3.1. Assume that (A1)-(A8) hold. Then, the family $\hat{B}_{\mu}$ of closed balls $B_{\mu}(t)=B_{E_{h}}\left[0 ; R_{\mu}(t)\right]$, where for each $t \in \mathbb{R}$, the radius $R_{\mu}(t)$ is defined by

$$
\begin{equation*}
R_{\mu}^{2}(t)=\frac{2 \mu-L}{\mu-L} c_{2}+\frac{e^{\mu h}}{\gamma} \int_{-\infty}^{t}\|f(s)\|^{2} d s+1, \tag{3.2}
\end{equation*}
$$

is pullback $\mathcal{D}_{\mu}$-absorbing for the process $\{U(t, \tau)\}_{t \geq \tau}$.
Proof. Since $\mu>L$, then using (A7), we have

$$
\lim _{t \rightarrow-\infty} e^{(\mu-L) t} R_{\mu}^{2}(t)=\lim _{t \rightarrow-\infty} e^{(\mu-L) t}\left(\frac{2 \mu-L}{\mu-L} c_{2}+\frac{e^{\mu h}}{\gamma} \int_{-\infty}^{t}\|f(s)\|^{2} d s+1\right)=0
$$

which shows that $\hat{B}_{\mu} \in \mathcal{D}_{\mu}$. Now, fixed $t \in \mathbb{R}$ and $\hat{D} \in \mathcal{D}_{\mu}$, there exists a $\tau_{0}=\tau_{0}(t, \hat{D}) \leq t$ such that

$$
e^{(\mu-L) \tau} r_{\hat{D}}^{2}(\tau)<c_{1}^{-1} e^{(\mu-L) t}
$$

for any $\tau \leq \tau_{0}$. Then, for any $\psi \in D(\tau)$, using (2.19) we obtain

$$
\begin{aligned}
\|U(t, \tau) \psi\|_{E_{h}}^{2} & \leq c_{1} r_{\hat{D}}^{2}(\tau) e^{(\mu-L) \tau} e^{(L-\mu) t}+\frac{2 \mu-L}{\mu-L} c_{2}+\frac{e^{\mu h}}{\gamma} \int_{-\infty}^{t}\|f(s)\|^{2} d s \\
& \leq R_{\mu}^{2}(t) .
\end{aligned}
$$

Therefore, $U(t, \tau) D(\tau) \subset B_{\mu}(t)$, for all $\tau \leq \tau_{0}$, which proves that the family $\hat{B}_{\mu}$ is pullback $\mathcal{D}_{\mu}$-absorbing for the process $\{U(t, \tau)\}_{t \geq \tau}$.

In Lemma 3.1, in the case (A8)(iii), we take $L=0$ and replace $c_{2}$ by $c_{2}^{\prime}$ in (3.2). Next, let us prove an estimate for the tails of the solutions $u=u(t)$ of (2.1) when the initial conditions $u_{\tau}=\psi$ belong to $B_{\mu}(\tau)$.

Lemma 3.2. Assume that (A1)-(A8) hold. Let $\hat{\mathrm{B}}_{\mu}$ be the pullback $\mathcal{D}_{\mu}$-absorbing family defined in Lemma 3.1. Then, for any $\varepsilon>0$ and any $t^{\prime}<T$, there exist $\tau_{0}=\tau_{0}\left(\varepsilon, t^{\prime}, T, \hat{B}_{\mu}\right)$ and a positive integer $k=k\left(\varepsilon, T, \hat{B}_{\mu}\right)$, such that

$$
\max _{s \in[-h, 0]} \sum_{|n|>2 k}\left|u_{n}(t+s)\right|^{2}<\varepsilon, \forall \tau \leq \tau_{0}, t \in\left[t^{\prime}, T\right],
$$

for any solution $u=u(t)$ of (2.1) with initial condition $u_{\tau} \in B_{\mu}(\tau)$.
Proof. Assume that (A8)(i) holds. Similarly, we treat the case (A8)(ii). Let $u_{\tau}=\psi \in B_{\mu}(\tau)$ and consider the corresponding solution $u=u(t)$ of (2.1) defined in $[\tau, \infty)$. Let $\theta \in C^{1}\left(\mathbb{R}^{+} ; \mathbb{R}\right)$ be a function such that $\theta \equiv 0$ on $[0,1], \theta \equiv 1$ on $[2, \infty), \quad 0 \leq \theta \leq 1$, and $\left|\theta^{\prime}(t)\right| \leq 2, \forall t \geq 0$. Let $v=\left(v_{n}(t)\right)_{n \in \mathbb{Z}}$, where $v_{n}(t)=\theta\left(\frac{|n|}{k}\right) u_{n}(t)$, with $k>0$ fixed in $\mathbb{Z}$. In order to simplify notation, we will write $\theta_{n}=\theta\left(\frac{|n|}{k}\right),\|w\|_{\theta}=\sum_{n=-\infty}^{+\infty} \theta_{n}\left|w_{n}\right|^{2}$ and $\left\|u_{t}\right\|_{E_{h, \theta}}^{2}=\max _{s \in[-h, 0]}\left\|u_{t}(s)\right\|_{\theta}^{2}$. Taking the imaginary part of the inner product of equation (2.1) with $v$ in $\ell^{2}$ we find

$$
\begin{equation*}
\frac{1}{2} \frac{d}{d t}(u, v)_{\ell^{2}}+\gamma(u, v)_{\ell^{2}}=\operatorname{Im}(f, v)_{\ell^{2}}-\operatorname{Im}(A u, v)_{\ell^{2}}-\operatorname{Im}\left(g\left(t, u_{t}\right), v\right)_{\ell^{2}}, \quad \forall t \geq \tau \tag{3.3}
\end{equation*}
$$

Let us estimate the terms on the right-hand side of (3.3). Since $\psi \in B_{\mu}(\tau)$ then, using (2.19), we see that

$$
\|u(t)\| \leq r_{0}, \quad \forall t \in[\tau, T]
$$

with $r_{0}=\left(c_{1}+1\right) R_{\mu}(T)$. Moreover, by the definition of $\theta$, we have that $\left|\theta_{n+m}-\theta_{n}\right| \leq \frac{2}{k} m$ and $\left|\theta_{n+m}-\theta_{n}\right| \leq 2$. Then,

$$
\begin{aligned}
-\operatorname{Im}(A u(t), v(t))_{\ell^{2}} & =-\operatorname{Im}\left\{J(0)\|u(t)\|_{\theta}^{2}+\sum_{n=-\infty}^{+\infty} \sum_{m=1}^{+\infty} J(m)\left(\theta_{n+m}-\theta_{n}\right) \overline{u_{n+m}(t)} u_{n}(t)\right\} \\
& \leq \sum_{n=-\infty}^{+\infty} \sum_{m=1}^{+\infty}|J(m)|\left|\theta_{n+m}-\theta_{n}\right|\left|u_{n+m}(t)\right|\left|u_{n}(t)\right| \leq v(T, k, l),
\end{aligned}
$$

where $v(T, k, l)=\left(\frac{2}{k} \sum_{m=1}^{l} m|J(m)|+2 \sum_{m=l+1}^{+\infty}|J(m)|\right) r_{0}^{2}, l \geq 1$.
Using the hypotheses (A1) and (A4) and proceeding as in the proof of Lemma 2.6 we obtain the estimate

$$
\begin{aligned}
-\operatorname{Im}\left(g\left(t, u_{t}\right), v(t)\right)_{\ell^{2}} \leq & \sum_{n=-\infty}^{+\infty} \theta_{n}\left|g_{1, n}\left(t, u_{n}(t-\rho(t))\right)\right|\left|u_{n}(t)\right| \\
& +\sum_{n=-\infty}^{+\infty} \theta_{n} \int_{-h}^{0}\left|b_{n}\left(s, u_{n}(t+s)\right)\right| d s\left|u_{n}(t)\right| \\
\leq & \left(K_{1}\|u(t-\rho(t))\|_{\theta}+K_{2, \theta}\right)\|u\|_{\theta} \\
& +\left[B_{1}\left(\int_{-h}^{0}\|u(t+s)\|_{\theta}^{2} d s\right)^{1 / 2}+B_{2, \theta}\right]\|u\|_{\theta}
\end{aligned}
$$

where $B_{2, \theta}=\left[\sum_{n=-\infty}^{+\infty} \theta_{n}\left(\int_{-h}^{0} \beta_{2, n}(s) d s\right)^{2}\right]^{1 / 2}$ and $K_{2, \theta}=\left(\sum_{n=-\infty}^{+\infty} \theta_{n} k_{2, n}^{2}\right)^{1 / 2}$.
In addition, we know that

$$
-\operatorname{Im}(f(t), v(t))_{\ell^{2}} \leq \frac{1}{2 \gamma}\|f(t)\|_{\theta}^{2}+\frac{\gamma}{2}\|u(t)\|_{\theta}^{2} .
$$

Therefore,

$$
\begin{align*}
\frac{d}{d t}\|u(t)\|_{\theta}^{2}+\gamma\|u(t)\|_{\theta}^{2} \leq & \frac{1}{\gamma}\|f\|_{\theta}^{2}+2\left(K_{1}\|u(t-\rho(t))\|_{\theta}+K_{2, \theta}\right)\|u(t)\|_{\theta} \\
& +2\left[B_{1}\left(\int_{-h}^{0}\|u(t+s)\|_{\theta}^{2} d s\right)^{1 / 2}+B_{2, \theta}\right]\|u(t)\|_{\theta}+2 v(T, k, l), \tag{3.4}
\end{align*}
$$

for all $\tau \leq t \leq T$.
Now, we multiply (3.4) by $e^{\mu t}$ and use the inequalities

$$
\begin{gathered}
2\left[B_{1}\left(\int_{-h}^{0}\|u(t+s)\|_{\theta}^{2} d s\right)^{1 / 2}+B_{2, \theta}\right]\|u\|_{\theta} \leq \frac{4 B_{1}^{2}}{\gamma} \int_{-h}^{0}\|u(t+s)\|_{\theta}^{2} d s+\frac{4 B_{2, \theta}^{2}}{\gamma}+\frac{\gamma}{2}\|u\|_{\theta}^{2} \\
2\left(K_{1}\|u(t-\rho(t))\|_{\theta}+K_{2, \theta}\right)\|u(t)\|_{\theta} \leq 2 K_{1}\left\|u_{t}\right\|_{E_{h, \theta}}^{2}+\frac{K_{2, \theta}^{2}}{\varepsilon^{\prime}}+\varepsilon^{\prime}\|u\|_{\theta}^{2}
\end{gathered}
$$

where $\varepsilon^{\prime}>0$, to find

$$
\begin{align*}
\frac{d}{d t}\left(e^{\mu t}\|u(t)\|_{\theta}^{2}\right) \leq & \left(\mu-\frac{\gamma}{2}+\varepsilon^{\prime}\right) e^{\mu t}\|u(t)\|_{\theta}^{2}+\frac{1}{\gamma} e^{\mu t}\|f(t)\|_{\theta}^{2}+2 K_{1} e^{\mu t}\left\|u_{t}\right\|_{E_{h, \theta}}^{2} \\
& +\left(\frac{4 B_{2, \theta}^{2}}{\gamma}+\frac{K_{2, \theta}^{2}}{\varepsilon^{\prime}}\right) e^{\mu t}+2 v(T, k, l) e^{\mu t}  \tag{3.5}\\
& +\frac{4 B_{1}^{2} e^{\mu t}}{\gamma} \int_{-h}^{0}\|u(t+s)\|_{\theta}^{2} d s, \quad \forall \tau \leq t \leq T .
\end{align*}
$$

Integrating (3.5) over $[\tau, t]$ and using the following estimate analogous to (2.14)

$$
\int_{\tau}^{t} \int_{-h}^{0} e^{\mu t^{\prime}}\|u(t+s)\|_{\theta}^{2} d s d t^{\prime} \leq \frac{e^{\mu(\tau+h)} h}{\mu}\|\psi\|_{E_{h}}^{2}+e^{\mu h} h \int_{\tau}^{t} e^{\mu s}\|u(s)\|_{\theta}^{2} d s,
$$

we obtain

$$
\begin{aligned}
e^{\mu t}\|u(t)\|_{\theta}^{2} \leq & e^{\mu \tau}\left(1+\frac{4 B_{1}^{2} e^{\mu h} h}{\mu \gamma}\right)\|\psi\|_{E_{h}}^{2}+\left(\mu-\frac{\gamma}{2}+\varepsilon^{\prime}+\frac{4 B_{1}^{2} e^{\mu h} h}{\gamma}\right) \int_{\tau}^{t} e^{\mu s}\|u(s)\|_{\theta}^{2} d s \\
& +\left(\frac{4 B_{2, \theta}^{2}}{\gamma}+\frac{K_{2, \theta}^{2}}{\varepsilon^{\prime}}+2 v(T, k, l)\right) \frac{e^{\mu t}}{\mu}+2 K_{1} \int_{\tau}^{t} e^{\mu s}\left\|u_{s}\right\|_{E_{h, \theta}}^{2} d s \\
& +\frac{1}{\gamma} \int_{\tau}^{t} e^{\mu s}\|f(s)\|_{\theta}^{2} d s .
\end{aligned}
$$

By condition (2.10) we can choose $\varepsilon^{\prime}$ as in (2.16) in the above inequality to obtain

$$
\begin{align*}
e^{\mu t}\|u(t)\|_{\theta}^{2} \leq & e^{\mu \tau}\left(1+\frac{4 B_{1}^{2} e^{\mu h} h}{\mu \gamma}\right)\|\psi\|_{E_{h}}^{2}+\left(\frac{4 B_{2, \theta}^{2}}{\gamma}+\frac{K_{2, \theta}^{2}}{\varepsilon^{\prime}}+2 \nu(T, k, l)\right) \frac{e^{\mu t}}{\mu}  \tag{3.6}\\
& +2 K_{1} \int_{\tau}^{t} e^{\mu s}\left\|u_{s}\right\|_{E_{h, \theta}}^{2} d s+\frac{1}{\gamma} \int_{\tau}^{t} e^{\mu s}\|f(s)\|_{\theta}^{2} d s .
\end{align*}
$$

Replacing $t$ by $t+\sigma$, with $\sigma \in[-h, 0]$ in (3.6) and using the inequality $\|u(t+\sigma)\|=$ $\|\psi(t+\sigma)\| \leq\|\psi\|_{E_{h}}$, valid for $t+\sigma<\tau$, we deduce that

$$
\begin{equation*}
e^{\mu t}\left\|u_{t}\right\|_{E_{h, \theta}}^{2} \leq M_{\theta}(t)+L \int_{\tau}^{t} e^{\mu s}\left\|u_{s}\right\|_{E_{h, \theta}}^{2} d s \tag{3.7}
\end{equation*}
$$

where $L=2 K_{1} e^{\mu h}$ and

$$
\begin{aligned}
M_{\theta}(t)= & e^{\mu(\tau+h)}\left(1+\frac{4 B_{1}^{2} e^{\mu h} h}{\mu \gamma}\right)\|\psi\|_{E_{h}}^{2}+\left(\frac{4 B_{2, \theta}^{2}}{\gamma}+\frac{K_{2, \theta}^{2}}{\varepsilon^{\prime}}+2 v(T, k, l)\right) \frac{e^{\mu(t+h)}}{\mu} \\
& +\frac{e^{\mu h}}{\gamma} \int_{\tau}^{t} e^{\mu s}\|f(s)\|_{\theta}^{2} d s .
\end{aligned}
$$

We know that $\mu>L$. Then, from (3.7) and $\psi \in B_{\mu}(\tau)$, we obtain

$$
\begin{align*}
\left\|u_{t}\right\|_{E_{h, \theta}}^{2} \leq & c_{1} R_{\mu}^{2}(\tau) e^{(L-\mu) t} e^{(\mu-L) \tau}+\frac{2 \mu-L}{\mu-L} c_{2, \theta}+\frac{2(2 \mu-L)}{\mu(\mu-L)} e^{\mu h} v(T, k, l) \\
& +\frac{e^{\mu h}}{\gamma} \int_{-\infty}^{t}\|f(s)\|_{\theta}^{2} d s, \quad \forall t \geq \tau \tag{3.8}
\end{align*}
$$

where $c_{2, \theta}=\left(\frac{4 B_{2, \theta}^{2}}{\gamma}+\frac{K_{2, \theta}^{2}}{\varepsilon^{\prime}}\right) \frac{e^{\mu h}}{\mu}$. Similarly, if (A8)(ii) holds, we obtain (3.8) with $L=\frac{4}{\gamma} B_{1}^{2} e^{2 \mu h} h$.
To conclude the proof, let $\varepsilon>0$ be given. Since $\hat{B}_{\mu} \in \mathcal{D}_{\mu}$ and $\sum_{m=1}^{\infty}|J(m)|<\infty$, then there exist $\tau_{0}=\tau_{0}\left(t^{\prime}, T, \varepsilon, \hat{B}_{\mu}\right)<t^{\prime}$ and a positive integer $l(\varepsilon)$ such that

$$
c_{1} R_{\mu}^{2}(\tau) e^{(L-\mu) t} e^{(\mu-L) \tau}<\frac{\varepsilon}{4}, \quad \forall \tau \leq \tau_{0}, t \in\left[t^{\prime}, T\right],
$$

and

$$
\frac{4(2 \mu-L) e^{\mu h}}{\mu(\mu-L)} r_{0}^{2} \sum_{m=l(\varepsilon)+1}^{+\infty}|J(m)|<\frac{\varepsilon}{4} .
$$

Then, from (3.8) we have

$$
\left\|u_{t}\right\|_{E_{h, \theta}}^{2}<\frac{\varepsilon}{2}+\frac{2 \mu-L}{\mu-L} c_{2, \theta}+\frac{4(2 \mu-L) e^{\mu h}}{\mu(\mu-L)} \frac{r_{0}^{2}}{k} \sum_{m=1}^{l(\varepsilon)} m|J(m)|+\frac{e^{\mu h}}{\gamma} \int_{-\infty}^{T}\|f(s)\|_{\theta}^{2} d s,
$$

for all $\tau \leq \tau_{0}$ and $t^{\prime} \leq t \leq T$. Observe that the hypothesis (A7) and the Lebesgue Dominated Convergence Theorem imply that

$$
\lim _{k \rightarrow+\infty} \int_{-\infty}^{T} \sum_{|n|>k}\left|f_{n}(s)\right|^{2} d s=0 .
$$

Using this fact and also $\sum_{n=-\infty}^{\infty}\left(\int_{-h}^{0} \beta_{2, n}(s) d s\right)^{2}<\infty$ and $\sum_{-\infty}^{\infty} k_{2, n}^{2}<\infty$ we can find a positive integer $k=k\left(\varepsilon, T, \hat{B}_{\mu}\right)$ such that

$$
\frac{2 \mu-L}{\mu-L} c_{2, \theta}+\frac{4(2 \mu-L) e^{\mu h}}{\mu(\mu-L)} \frac{r_{0}^{2}}{k} \sum_{m=1}^{l(\varepsilon)} m|J(m)|+\frac{e^{\mu h}}{\gamma} \int_{-\infty}^{T}\|f(s)\|_{\theta}^{2} d s<\frac{\varepsilon}{2} .
$$

Therefore,

$$
\max _{s \in[-h, 0]} \sum_{|n|>2 k}\left|u_{n}(t+s)\right|^{2} \leq\left\|u_{t}\right\|_{E_{h, \theta}}^{2}<\varepsilon, \quad \text { if } \tau \leq \tau_{0}, t^{\prime} \leq t \leq T .
$$

In the case $\mathrm{A}(8)$ (iii), in (3.8), we take $L=0$, replace $c_{1}$ by $c_{1}^{\prime}$ and $c_{2, \theta}$ and $R_{\mu}^{2}(\tau)$ by

$$
c_{2, \theta}^{\prime}=\frac{e^{\mu h}}{\mu^{2}}\left(B_{2, \theta}^{2}+K_{2, \theta}^{2}\right) \quad \text { and } \quad R_{\mu}^{2}(\tau)=c_{2}^{\prime}+\frac{e^{\mu h}}{2 \mu} \int_{-\infty}^{t} \|\left. f(s)\right|^{2} d s
$$

Lemma 3.3. Under the assumptions (A1)-(A8), the process $\{U(t, \tau)\}_{t \geq \tau}$ is pullback $\mathcal{D}$-asymptotically compact.

Proof. Fixed $t \in \mathbb{R}$ and $\hat{D}_{\mu} \in \mathcal{D}_{\mu}$, consider the sequences $\left(\tau_{m}\right)_{m \in \mathbb{N}}$ and $\left(u_{t}^{m}\right)_{m \in \mathbb{N}}$, such that $\tau_{m} \rightarrow-\infty$ and $u_{t}^{m}=U\left(t, \tau_{m}\right) \psi^{m}$, with $\psi^{m} \in D\left(\tau_{m}\right)$. We want to prove that $\left(u_{t}^{m}\right)_{m \in \mathbb{N}}$ has a subsequence which is relatively compact in $E_{h}$. Given $\varepsilon>0$, by Lemma 3.2, there exist $\tau=\tau\left(\varepsilon, t, \hat{B}_{\mu}\right)<t-h$ and a positive integer $n_{1}=n_{1}\left(\varepsilon, t, \hat{B}_{\mu}\right)$ such that

$$
\begin{equation*}
\max _{s \in[-h, 0]} \sum_{|n|>n_{1}}\left|u_{n}(t+s)\right|^{2}<\frac{\varepsilon^{2}}{8}, \tag{3.9}
\end{equation*}
$$

where $u=u(t)=\left(u_{n}(t)\right)$ is any solution of the initial value problem (2.1) with $u_{\tau} \in B_{\mu}(\tau)$.

Since $\hat{B}_{\mu}$ is pullback $\mathcal{D}_{\mu}$-absorbing and $\tau_{m} \rightarrow-\infty$, without loss of generality, we can assume that

$$
\begin{equation*}
U\left(\tau, \tau_{m}\right) \psi^{m} \in B_{\mu}(\tau), \quad \forall m \geq 1 \tag{3.10}
\end{equation*}
$$

Also, by the definition of a process, we know that

$$
\begin{equation*}
U\left(t^{\prime}, \tau\right) U\left(\tau, \tau_{m}\right) \psi^{m}=U\left(t^{\prime}, \tau_{m}\right) \psi^{m}, \quad \forall \tau \leq t^{\prime} \leq t \tag{3.11}
\end{equation*}
$$

Using (3.10), (3.11), and the estimate (2.19) we see that

$$
\begin{equation*}
\left\|U\left(t^{\prime}, \tau_{m}\right) \psi^{m}\right\|_{E_{h}} \leq K, \quad \forall \tau \leq t^{\prime} \leq t \tag{3.12}
\end{equation*}
$$

where $K=K(t)=\left(c_{1}+1\right) R_{\mu}^{2}(t)$. In particular, the sequence $\left(u_{t}^{m}(s)\right)_{m \in \mathbb{N}}$ is bounded in $\ell^{2}$, for any $s \in[-h, 0]$. Therefore, for any fixed $s \in[-h, 0]$, there exists a subsequence, which we will still denote by $\left(u_{t}^{m}(s)\right)_{m \in \mathbb{N}}$ and $\zeta(s) \in \ell^{2}$, such that

$$
\begin{equation*}
u^{m}(t+s) \rightharpoonup \zeta(s) \quad \text { weakly in } \ell^{2} . \tag{3.13}
\end{equation*}
$$

Let us show that the convergence in (3.13) is strong in $\ell^{2}$. Since $\zeta(s) \in \ell^{2}$, then there exists a positive integer $n_{2}$ such that

$$
\begin{equation*}
\sum_{|n|>n_{2}}\left|\zeta_{n}(s)\right|^{2}<\frac{\varepsilon^{2}}{8} . \tag{3.14}
\end{equation*}
$$

Moreover, using the weak convergence (3.13), we can find a positive integer $m_{1}=m_{1}\left(\varepsilon, t, \hat{B}_{\mu}\right)$ such that

$$
\begin{equation*}
\sum_{|n| \leq n_{0}}\left|u_{n}^{m}(t+s)-\zeta_{n}(s)\right|^{2}<\frac{\varepsilon^{2}}{2}, \quad \forall m \geq m_{1} \tag{3.15}
\end{equation*}
$$

where $n_{0}=\max \left\{n_{1}, n_{2}\right\}$. From (3.14) and (3.15), for any $m \geq m_{1}$, we have that

$$
\begin{align*}
\left\|u^{m}(t+s)-\zeta(s)\right\|^{2} \leq & \sum_{|n| \leq n_{0}}\left|u_{n}^{m}(t+s)-\zeta_{n}(s)\right|^{2}+2 \sum_{|n|>n_{0}}\left|u_{n}^{m}(t+s)\right|^{2} \\
& +2 \sum_{|n|>n_{0}}\left|\zeta_{n}(s)\right|^{2}<\frac{3 \varepsilon^{2}}{4}+2 \sum_{|n|>n_{0}}\left|u_{n}^{m}(t+s)\right|^{2} . \tag{3.16}
\end{align*}
$$

Using the estimate (3.9) with $u_{\tau}=U\left(\tau, \tau_{m}\right) \psi^{m}, m \geq m_{1}$, from (3.16) we conclude that

$$
\left\|u^{m}(t+s)-\zeta(s)\right\|^{2}<\varepsilon^{2} .
$$

Therefore, $\left(u_{t}^{m}(s)\right)_{m \in \mathbb{N}}$ is relatively compact in $\ell^{2}$ for each $s \in[-h, 0]$.
Next, let us show that $\left(u_{t}^{m}\right)_{m \in \mathbb{N}}$ is equicontinuous in $[-h, 0]$. Using the integral representation of the solution of (2.1) we obtain

$$
\begin{equation*}
\left\|u^{m}\left(t+s_{1}\right)-u^{m}\left(t+s_{2}\right)\right\| \leq \int_{t+s_{1}}^{t+s_{2}}\left\|\mathcal{B}\left(r, u_{r}^{m}\right)\right\| d r \tag{3.17}
\end{equation*}
$$

for any $-h \leq s_{1} \leq s_{2} \leq 0$. Using (3.12) in (3.17) and Lemma 2.4, we deduce the existence of a positive constant $L(K)$ such that $\left\|u^{m}\left(t+s_{1}\right)-u^{m}\left(t+s_{2}\right)\right\| \leq L(K)\left(s_{2}-s_{1}\right), \forall m \in \mathbb{N}$, which implies the equicontinuity. By the Ascoli-Arzelà Theorem, we conclude that $\left(u_{t}^{m}\right)_{m \in \mathbb{N}}$ is relatively compact in $E_{h}$. This completes the proof of Lemma 3.3.

As consequence of Lemmas 3.1, 3.3 and of Theorem 18 in [16] we obtain the main result of this section.

Theorem 3.4. Assume that (A1)-(A8) hold. Then, the process $\{U(t, \tau)\}_{t \geq \tau}$ possesses a unique pullback $\mathcal{D}_{\mu}$-attractor $\hat{A}$ in $\mathcal{D}_{\mu}$.

Proof. By Lemmas 3.1, 3.3 and Theorem 18 in [16] the process $\{U(t, \tau\})_{t \geq \tau}$ possesses a pullback $\mathcal{D}_{\mu}$-attractor $\hat{A}$. Since the family $\mathcal{D}_{\mu}$ is inclusion-closed and each member $B_{\mu}(t)$ of the pullback family $\hat{B}_{\mu}$ is a closed subset of $E_{h}$, then $\hat{A} \in \mathcal{D}_{\mu}$ and it is the unique pullback $\mathcal{D}_{\mu}$-attractor belonging to the class $\mathcal{D}_{\mu}$.

## 4 The autonomous model

In this section, we consider the autonomous model

$$
\begin{align*}
& i \dot{u}_{n}(t)+\sum_{m=-\infty}^{+\infty} J(n-m) u_{m}(t)+g_{n}\left(u_{n t}\right)+i \gamma u_{n}(t)=f_{n}, \quad t>0, n \in \mathbb{Z},  \tag{4.1}\\
& u_{n}(s)=\psi_{n}(s), \quad \forall s \in[-h, 0],
\end{align*}
$$

where $f=\left(f_{n}\right)_{n \in \mathbb{Z}}$ and

$$
g_{n}\left(u_{n t}\right)=g_{0, n}\left(u_{n}(t)\right)+g_{1, n}\left(u_{n}(t-\rho)\right)+\int_{-h}^{0} b_{n}\left(s, u_{n}(t+s)\right) d s,
$$

with $0<\rho \leq h$. We assume that $f \in \ell^{2}$ and the functions $g_{0, n}, g_{1, n}$, and $b_{n}$ satisfy the assumptions (A1)-(A4) stated in Section 2.

Defining the map $g: E_{h} \rightarrow \ell^{2}$ by $(g(v))_{n \in \mathbb{Z}}=g_{n}\left(v_{n}\right)$, where

$$
g_{n}\left(v_{n}\right)=g_{0, n}\left(v_{n}(0)\right)+g_{1, n}\left(v_{n}(-\rho)\right)+\int_{-h}^{0} b_{n}\left(s, v_{n}(s)\right) d s
$$

we can write (4.1) in $\ell^{2}$ as

$$
\begin{align*}
& i \dot{u}(t)+A u(t)+g\left(u_{t}\right)+i \gamma u(t)=f, \quad t>0  \tag{4.2}\\
& u(s)=\psi(s), \quad \forall s \in[-h, 0],
\end{align*}
$$

where, as before, $u(t)=\left(u_{n}(t)\right)_{n \in \mathbb{Z}}$ and $\psi(s)=\left(\psi_{n}(s)\right)_{n \in \mathbb{Z}}$.
Using the assumptions (A1)-(A4) and the Theory of Functional Equations we obtain a local solution for the problem (4.2) with $\psi \in E_{h}$.

In what follows, we will use the same notations of Sections 2 and 3 and, as before, we will assume that (A8)(i) or (ii) holds. Similarly, we can prove the results for (A8)(iii). Proceeding as in the proof of Theorem 2.8 we can prove the following lemma.

Lemma 4.1. Assume that (A1)-(A4) and (A8) hold. Then, the solution $u=u(t)$ of (4.2) with initial condition $u_{0}=\psi \in E_{h}$, defined in the maximal interval of existence $[0, T)$, satisfies

$$
\begin{equation*}
\left\|u_{t}\right\|_{E_{h}}^{2} \leq c_{1}\|\psi\|_{E_{h}}^{2} e^{-(\mu-L) t}+\frac{2 \mu-L}{\mu-L}\left(c_{2}+\frac{e^{\mu h}}{\mu \gamma}\|f\|^{2}\right) . \tag{4.3}
\end{equation*}
$$

As a consequence of (4.3) we conclude that the solution $u=u(t)$ of (4.2) exists on $[0, \infty)$ and we can define a semigroup $\{S(t)\}_{t \geq 0}$ on $E_{h}$ associated with (4.2) as follows

$$
S(t) \psi=u_{t}, \quad \forall t \geq 0 .
$$

Moreover, from (4.3) we deduce that the closed ball $\mathcal{O}_{0}=B_{E_{h}}\left[0 ; r_{0}\right]$ in $E_{h}$, where

$$
\begin{equation*}
r_{0}=\left[\frac{2 \mu-L}{\mu-L}\left(c_{2}+\frac{e^{\mu h}}{\mu \gamma}\|f\|^{2}\right)+1\right]^{1 / 2}, \tag{4.4}
\end{equation*}
$$

is an absorbing set for $\{S(t)\}_{t \geq 0}$ in $E_{h}$.
Next, let us modify the proof of Lemma 3.2 to show that $\{S(t)\}_{t \geq 0}$ is asymptotically compact in $E_{h}$.
Lemma 4.2. Assume that (A1)-(A4) and (A8) hold. Also, assume that $\psi \in \mathcal{O}_{0}$. Then, for any $\epsilon>0$, there exist $T(\epsilon) \geq 0$ and a positive integer $k(\epsilon)$, such that the solution $u=u(t)$ of (4.2) satisfies

$$
\max _{s \in[-h, 0]} \sum_{|n|>k(\varepsilon)}\left|u_{n}(t+s)\right|^{2}<\epsilon, \quad \forall t \geq T(\epsilon) .
$$

Proof. Since $\psi \in \mathcal{O}_{0}$, then by (4.3) and (4.4), we have

$$
\begin{equation*}
\left\|u_{t}\right\|_{E_{h}} \leq r_{1}, \quad \forall t \geq 0 \tag{4.5}
\end{equation*}
$$

where $r_{1}=\left(c_{1}+1\right)^{1 / 2} r_{0}$.
Using (4.5) and proceeding as in the proof of Lemma 3.2 we can prove that

$$
\begin{equation*}
e^{\mu t}\left\|u_{t}\right\|_{E_{h, \theta}}^{2} \leq M_{\theta}(t)+L \int_{0}^{t} e^{\mu s}\left\|u_{s}\right\|_{E_{h, \theta}}^{2} d s \tag{4.6}
\end{equation*}
$$

with

$$
M_{\theta}(t)=e^{\mu h}\left(1+\frac{4 B_{1}^{2}}{\mu \gamma} e^{\mu h}\right)\|\psi\|_{E_{h}}^{2}+\left(\frac{4 B_{2, \theta}^{2}}{\gamma}+\frac{K_{2, \theta}^{2}}{\varepsilon^{\prime}}+2 \nu(k, l)+\frac{1}{\gamma}\|f\|_{\theta}^{2}\right) \frac{e^{\mu(t+h)}}{\mu},
$$

where

$$
v(k, l)=\left(\frac{2}{k} \sum_{m=1}^{l} m|J(m)|+2 \sum_{m=l+1}^{+\infty}|J(m)|\right) r_{1}^{2} .
$$

From (4.6) we obtain

$$
\begin{equation*}
\left\|u_{t}\right\|_{E_{h, \theta}}^{2} \leq c_{1} r_{1}^{2} e^{-(\mu-L) t}+\frac{2 \mu-L}{\mu-L} c_{2, \theta}+\frac{2(2 \mu-L)}{\mu(\mu-L)} e^{u h} v(k, l), \tag{4.7}
\end{equation*}
$$

where

$$
c_{2, \theta}=\left(\frac{4 B_{2, \theta}^{2}}{\gamma}+\frac{K_{2, \theta}^{2}}{\varepsilon^{\prime}}+\frac{1}{\gamma}\|f\|_{\theta}^{2}\right) \frac{e^{\mu h}}{\mu} .
$$

Finally, from (4.7) we can conclude the proof of Lemma 4.2.
Under the hypotheses of Lemma 4.1, using Lemma 4.2 and proceeding as in the proof of Lemma 3.3, we show that the semigroup $\{S(t)\}_{t \geq 0}$ is asymptotically compact in $E_{h}$. Thus, we can derive the desired result in this section.

Theorem 4.3. Under the same hypotheses of Lemma 4.1, the semigroup $\{S(t)\}_{t \geq 0}$ possesses a unique global attractor $\mathcal{A}$ in $E_{h}$.

Remark 4.4. When $\rho(t) \equiv \rho$ and $f(t) \equiv f$ in problem (2.1), then the constant family $\hat{A}=$ $\{A(t)=\mathcal{A} ; t \in \mathbb{R}\}$ is the pullback $\mathcal{D}$-attractor from Theorem 4.3.

## Acknowledgements

The content of this paper is part of a thesis submitted by the author to the Center of Physical and Mathematical Sciences of Federal University of Santa Catarina as part of the requirements to get the title of Full Professor. He would like to thank professors Pedro Alberto Barbetta (INE-UFSC), Paulo Ricardo de Ávila Zingano (DMPA-UFRGS), Julio Cesar Ruiz Claeyssen (IM-UFRGS) and Rolci de Almeida Cipolatti (IM-UFRJ) for their assessments and valuable comments on the subject of this paper. The author also wishes to thank the anonymous referee for his/her comments and suggestions which allowed him to improve the presentation of this work.

## References

[1] P. W. Bates, C. Zhang, Traveling pulses for the Klein-Gordon equation on a lattice or continuum with long-range interaction, Discrete Contin. Dyn. Syst. 16(2006), No. 1, 253277. https://doi.org/10.3934/dcds.2006.16.235
[2] T. Caraballo, F. Morillas, J. Valero, Attractors for non-autonomous retarded lattice dynamical systems, Nonauton. Dyn. Syst. 2(2015), No. 1, 31-51. https://doi.org/10. 1515/msds-2015-0003
[3] J. P. Chebab, S. Dumont, O. Goubet, H. Moatassime, M. Abounouh, Discrete Schrödinger equation and dissipative dynamical systems, Commun. Pure Appl. Anal. 7(2006), No. 2, 211-227. https://doi.org/10.3934/cpaa.2008.7.211
[4] T. Chen, S. Zhou, C. Zhao, Attractors for discrete nonlinear Schrödinger equation with delay, Acta Math. Appl. Sin. Engl. Ser. 26(2010), No. 4, 633-642. https://doi.org/10. 1007/s10255-007-7101-y
[5] N. Chow, J. Mallet-Paret, Traveling waves in lattice dynamical systems, J. Differential Equations 149(1988), 248-291. https://doi.org/10.1006/jdeq.1998.3478
[6] J. C. Eilbeck, M. Johansson, The discrete nonlinear Schrödinger equation-20 years on, in: L. Vásquez, R. S. MacKay, M. P. Zorzano (Eds.), Proc. of the 3rd Conf. Localization and Energy Transfer in Nonlinear Systems, World Scientific, 2003, pp. 44-67. https://doi. org/ 10.1142/9789812704627_0003
[7] T. Erneux, G. Nicolis, Propagating waves in discrete bistable reaction diffusion systems, Phys. D 67(1993), 237-244. https://doi.org/10.1016/0167-2789 (93) 90208-I
[8] Y. Gaididei, P. Christiansen, S. Mingaleev, Scale competition in nonlinear models, in: P. L. Christiansen, M. P. Sørensen , A. C. Scott (Eds.), Nonlinear science at the dawn of the 21st century, Lecture Notes in Physics, Vol. 542, Springer, Berlin, Heidelberg, 2000, pp. 307-321. https://doi.org/10.1007/3-540-46629-0_15
[9] Y. Hong, C. Yang, Uniform Strichartz estimates on the lattice, Discrete Contin. Dyn. Syst. 39(2019), No. 6, 3239-3264. https://doi.org/10.3934/dcds. 2019134
[10] I. L. Ignat, J. D. Rossi, Asymptotic behaviour for a nonlocal diffusion equation on a lattice, Z. Angew. Math. Phys. 59(2008), No. 5, 918-925. https://doi.org/10.1007/s00033-007-7011-0
[11] N. I. Karachalios, A. N. Yannacopoulos, Global existence and global attractors for the discrete nonlinear Schrödinger equation, J. Differential Equations 217(2005), No. 1, 88-123. https://doi.org/10.1016/j.jde.2005.06.002
[12] K. Kirkpatrick, E. Lenzman, G. Staffilani, On the continuum limit for discrete NLS with long range lattice interactions, Commun. Pure Appl. Anal. 317(2013), 563-591. https : //doi.org/10.1007/s00220-012-1621-x
[13] M. Kollman, H. Capel, T. Bountis, Breathers and multibreathers in a periodically driven damped discrete nonlinear Schrödinger equation, Phys. Rev. E 60(1999), 11951211. https://doi.org/10.1103/PhysRevE.60.1195
[14] P. G. Krevekides, K. Ø. Rasmussen, A. R. Bishop, The discrete nonlinear Schrödinger equation: a survey of recent results, Internat. J. Modern Phys. B 15(2001), 2833-2900. https: //doi.org/10.1142/S0217979201007105
[15] N. Laskin, G. Zaslavsky, Nonlinear fractional dynamics on a lattice with long range interactions, Phys. A 386(2006), No. 1, 38-54. https://doi.org/10.1016/j.physa. 2006. 02.027
[16] P. Marín-Rubio, J. Real, On the relation between two different concepts of pullback attractors for non-autonomous dynamical systems, Nonlinear Anal. 71(2009), No. 9, 39563963. https://doi.org/:10.1016/j.na.2009.02.065
[17] S. F. Mingaleev, P. L. Christiansen, Y. B. Gaididei, M. Johannson, K. Ø. Rasmussen, Models for energy and charge transport and storage in biomolecules, J. Biol. Phys. 25(1999), 41-63. https://doi.org/10.1023/A:1005152704984
[18] F. Morillas, J. Valero, On a nonlocal discrete diffusion system modeling life tables, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 108(2014), No. 2, 935-955. https: //doi.org/10.1007/s13398-013-0153-3
[19] P. Pacciani, V. V. Konotop, G. Perla Menzala, On localized solutions of discrete nonlinear Schrödinger equation. An exact result, Phys. D 204(2005), No. 1-2, 122-133. https://doi.org/10.1016/j.physd.2005.04.009
[20] J. M. Pereira, Global attractor for a generalized discrete nonlinear Schrödinger equation, Acta Appl. Math. 134(2014), No. 1, 173-183. https://doi.org/doi:10.1007/s10440-014-9877-0
[21] V. M. Schouten-Straatman, H. J. Hupkes, Nonlinear stability of pulses solutions for the discrete FitzHugh-Nagumo equation with infinite-range interactions, Discrete Contin. Dyn. Syst. 39(2019), No. 9, 5017-5088. https://doi.org/10.3934/dcds. 2019205
[22] S. Zhou, Attractors and approximations for lattice dynamical systems, J. Differential Equations 200(2000), No. 2, 342-368. https://doi.org/10.1016/j.jde.2004.02.005

# Solitary waves for a fractional Klein-Gordon-Maxwell equation 

Xin Zhang ${ }^{\boxtimes}$<br>Liaoning Normal University, 1 Malan Street, Dalian, China

Received 28 June 2021, appeared 31 December 2021
Communicated by Gabriele Bonanno


#### Abstract

We investigate existence of solutions for a fractional Klein-Gordon coupled with Maxwell's equation. On the basis of overcoming the lack of compactness, we obtain that there is a radially symmetric solution for the critical system by means of variational methods.


Keywords: variational methods, Klein-Gordon-Maxwell equations, solitary waves.
2020 Mathematics Subject Classification: 35A15.

## 1 Introduction and preliminaries

Recently, a great attention has been focused on the study of non-linear problems involving the fractional Laplacian, in view of concrete real-world applications. For instance, this type of operators arises in the thin obstacle problem, optimization, finance, phase transitions, stratified materials, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, materials science and water waves, see [13]. Moreover fractional Laplace equations can be applied to many subjects, such as anomalous diffusion, elliptic problems with measure data, gradient potential theory, minimal surfaces, non-uniformly elliptic problems, optimization, phase transitions, quasigeostrophic flows, singular set of minima of variational functionals and water waves (see $[2,5-7,13,16-21]$ and the references therein). In present paper, we consider the following fractional system

$$
\left\{\begin{array}{l}
(-\Delta)^{s} u+\left[m^{2}-(\omega+\phi)^{2}\right] u=\mu|u|^{q-2} u+|u|^{2_{s}^{*}-2} u, \quad x \in \mathbb{R}^{3}  \tag{1.1}\\
(\Delta)^{s} \phi=(\omega+\phi) u^{2}, \quad x \in \mathbb{R}^{3}
\end{array}\right.
$$

where $\frac{3}{4}<s<1, \mu>0$ and $4 \leq q<2_{s}^{*}=\frac{2 n}{n-2 s}=\frac{6}{3-2 s}, m$ and $\omega$ are real constants, $u \in H^{s}\left(\mathbb{R}^{3}\right), \phi \in D^{s, 2}\left(\mathbb{R}^{3}\right),(-\Delta)^{s}$ stands for the fractional Laplacian, $2_{s}^{*}$ is the fractional Sobolev critical exponent.

The Klein-Gordon-Maxwell equations have been introduced in [3] as a model describing solitary waves for the non-linear stationary Klein-Gordon equation coupled with Maxwell

[^56]equation in the three dimensional space interacting with the eletrostatic field. In recent years, some existence and nonexistence results for the Klein-Gordon-Maxwell equations have been proved. In $[3,4,12]$, the authors investigated the existence of infinitely many radially symmetric solutions $(u, \phi)$ in $H^{1}\left(\mathbb{R}^{3}\right) \times D^{1,2}\left(\mathbb{R}^{3}\right)$. In [1] the existence of a ground state solution $(u, \phi)$ in $H^{1}\left(\mathbb{R}^{3}\right) \times D^{1,2}\left(\mathbb{R}^{3}\right)$ was established; In [11], the nonexistence results for system related to Klein-Gordon-Maxwell system were obtained.

Cassani in [8] investigated the following system when $n=3$ and $s=1$

$$
\left\{\begin{array}{l}
-\Delta u+\left[m_{0}^{2}-(\omega+\phi)^{2}\right] u=\mu|u|^{q-2} u+|u|^{2^{*}-2} u, \quad x \in \mathbb{R}^{3} \\
\Delta \phi=(\omega+\phi) u^{2}, \quad x \in \mathbb{R}^{3}
\end{array}\right.
$$

where $\mu>0$ and $4 \leq q<6=2^{*}$. Cassani proved that the system has at least a radially symmetric (nontrivial) solution.

In [20], Servadei and Valdinoci showed the non-local fractional counterpart of the Laplace equation involving critical non-linearities studied in the famous paper of Brezis and Nirenberg (1983) by the following system

$$
\begin{cases}(-\Delta)^{s} u-\lambda u=|u|^{2_{s}^{*}-2} u & \text { in } \Omega  \tag{1.2}\\ u=0 & \text { in } \mathbb{R}^{n} \backslash \Omega\end{cases}
$$

and the authors firstly studyed the problem in a general framework

$$
\begin{cases}\mathcal{L}_{K} u+\lambda u+|u|^{2_{s}^{*}-2}+f(x, u)=0 & \text { in } \Omega,  \tag{1.3}\\ u=0 & \text { in } \mathbb{R}^{n} \backslash \Omega,\end{cases}
$$

where $\mathcal{L}_{K}$ is a general non-local integrodifferential operator of order $s, f$ is a lower order perturbation of the critical power $|u|^{2_{s}^{*}-2}$. In this setting they proved an existence result through variational techniques. Then, as a concrete example, they derived a Brezis-Nirenberg type result for the problem.

The authors in [15] explored the problem

$$
\begin{cases}(-\Delta)^{s} u+V(x) u-(2 \omega+\phi) \phi u=K(x) f(u), & \text { in } \mathbb{R}^{3}, \\ (\Delta)^{s} \phi=(\omega+\phi) u^{2}, & \text { in } \mathbb{R}^{3}\end{cases}
$$

where $K: \mathbb{R}^{3} \rightarrow \mathbb{R}$ is a function satisfying some decay condition, $V: \mathbb{R}^{3} \rightarrow \mathbb{R}$ is a positive continuous function, $\phi, u: \mathbb{R}^{3} \rightarrow \mathbb{R}$ are functions. Furthermore, they showed the existence and positivity of the ground state solution with zero mass potential for the problem, that is, when the potential $V(x) \rightarrow 0$, as $|x| \rightarrow \infty$ and they also studied the case when $V$ is bounded and considered carefully the weight $K(x)$. In addition, they treated the problem using the fractional Laplace operator instead of classical Laplace operator.

Next there are two ways to define fractional Sobolev space. One is via Gagliardo seminorm

$$
H^{s}\left(\mathbb{R}^{3}\right):=\left\{u \in L^{2}\left(\mathbb{R}^{3}\right): \frac{|u(x)-u(y)|}{|x-y|^{\frac{3}{2}+s}} \in L^{2}\left(\mathbb{R}^{3} \times \mathbb{R}^{3}\right)\right\}
$$

the other is via Fourier transformation

$$
\widehat{H}^{s}\left(\mathbb{R}^{3}\right):=\left\{u \in L^{2}\left(\mathbb{R}^{3}\right): \int_{\mathbb{R}^{3}}\left(1+|\xi|^{2 s}\right)|\mathcal{F} u(\xi)|^{2} \mathrm{~d} \xi<+\infty\right\}
$$

and $H^{s}\left(\mathbb{R}^{3}\right)=\widehat{H}^{s}\left(\mathbb{R}^{3}\right)$. In the present paper, as the norm of fractional Sobolev space, we define

$$
\|u\|_{H^{s}}^{2}:=\int_{\mathbb{R}^{3}}\left(m^{2}-\omega^{2}\right) u^{2} \mathrm{~d} x+\frac{C_{3, s}}{2} \iint_{\mathbb{R}^{3} \times \mathbb{R}^{3}} \frac{|u(x)-u(y)|^{2}}{|x-y|^{3+2 s}} \mathrm{~d} x \mathrm{~d} y .
$$

The fractional Laplacian is defined by

$$
\begin{aligned}
(-\Delta)^{s} u(x) & =C_{3, s} \text { P.V. } \int_{\mathbb{R}^{3}} \frac{u(x)-u(y)}{|x-y|^{3+2 s}} \mathrm{~d} y \\
& =C_{3, s} \lim _{\varepsilon \rightarrow 0^{+}} \int_{B_{\varepsilon}^{c}(x)} \frac{u(x)-u(y)}{|x-y|^{3+2 s}} \mathrm{~d} y \\
& =-\frac{1}{2} C_{3, s} \int_{\mathbb{R}^{3}} \frac{u(x+y)+u(x-y)-2 u(x)}{|y|^{3+2 s}} \mathrm{~d} y \\
& =\mathcal{F}^{-1}\left(|\xi|^{2 s} \mathcal{F} u(\xi)\right),
\end{aligned}
$$

where

$$
C_{3, s}=\left(\int_{\mathbb{R}^{3}} \frac{1-\cos \left(\zeta_{1}\right)}{|\zeta|^{3+2 s}} \mathrm{~d} \zeta\right)^{-1}
$$

and P.V. is the principle value defined by the latter formula.
Consider the Sobolev space

$$
D^{s, 2}\left(\mathbb{R}^{3}\right):=\left\{u \in L^{2_{s}^{*}}\left(\mathbb{R}^{3}\right): \frac{|u(x)-u(y)|}{|x-y|^{\frac{3}{2}+s}} \in L^{2}\left(\mathbb{R}^{3} \times \mathbb{R}^{3}\right)\right\},
$$

which is the completion of $C_{0}^{\infty}\left(\mathbb{R}^{3}\right)$ under the norm

$$
\|u\|_{D^{s, 2}}^{2}:=\frac{C_{3, s}}{2} \iint_{\mathbb{R}^{3} \times \mathbb{R}^{3}} \frac{|u(x)-u(y)|^{2}}{|x-y|^{3+2 s}} \mathrm{~d} x \mathrm{~d} y .
$$

Theorem 1.1. If $|m|>|\omega|$ and $4<q<2_{s}^{*}$, then the problem (1.1) has a radially symmetric solution $(u, \phi) \in H^{s}\left(\mathbb{R}^{3}\right) \times D^{s, 2}\left(\mathbb{R}^{3}\right)$ for each $\mu>0$.

Theorem 1.2. If $|m|>|\omega|$ and $q=4$, system (1.1) still possesses a radially symmetric solution provided that $\mu$ is sufficiently large.

According to system (1.1), one obtains the functional

$$
\begin{equation*}
F(u, \phi)=\frac{1}{2}\|u\|_{H^{s}}^{2}-\frac{1}{2}\|\phi\|_{D^{s, 2}}^{2}-\frac{1}{2} \int_{\mathbb{R}^{3}}\left(2 \omega \phi+\phi^{2}\right) u^{2} \mathrm{~d} x-\frac{\mu}{q} \int_{\mathbb{R}^{3}}|u|^{q} \mathrm{~d} x-\frac{1}{2_{s}^{*}} \int_{\mathbb{R}^{3}}|u|^{2_{s}^{*}} \mathrm{~d} x . \tag{1.4}
\end{equation*}
$$

It's easy to know that $F(u, \phi)$ exhibits a strong indefiniteness, namely it is unbounded from below and from above on infinite dimensional subspaces. This indefiniteness can be removed by using the reduction methods. For $u$ and $\phi$ defined above, we have the following lemmas.
Lemma 1.3. Let $u \in H^{s}\left(\mathbb{R}^{3}\right)$, then there exists a unique solution $\Phi(u)$ of the second equation for problem (1.1) such that $\phi=\Phi(u) \in D^{s, 2}\left(\mathbb{R}^{3}\right)$.

Proof. The proof is similar to the proof of in Reference [15, Lemma 2.1], so we omit its proof.

Remark 1.4. Define the map $\Phi: H^{s}\left(\mathbb{R}^{3}\right) \rightarrow D^{s, 2}\left(\mathbb{R}^{3}\right)$. We can get that for each $u \in H^{s}\left(\mathbb{R}^{3}\right)$, the map $\Phi$ gives the unique solution $\Phi(u)=\phi$, i.e., $\Phi(u)=\left((\Delta)^{s}-u^{2}\right)^{-1} \omega u^{2}$.

Next we state some properties of problem (1.1) as follows.
Lemma 1.5. For any $u \in H^{s}\left(\mathbb{R}^{3}\right)$, it results in $\Phi(u) \leq 0$. Moreover, $\Phi(u)(x) \geq-\omega$ if $u(x) \neq 0$ and $\omega>0$.

Proof. Multiplying the second equation of problem (1.1) by $\Phi^{+}(u)=\max \{\Phi(u), 0\}$, we get

$$
-\left\|\Phi^{+}(u)\right\|_{D^{s, 2}}^{2}=\omega \int_{\mathbb{R}^{3}} \Phi^{+}(u) u^{2} \mathrm{~d} x+\int_{\mathbb{R}^{3}} u^{2}\left(\Phi^{+}(u)\right)^{2} \mathrm{~d} x \geq 0
$$

so that $\Phi^{+}(u) \equiv 0$.
If we multiply the second equation of problem (1.1) by $(\omega+\Phi(u))^{-}$, one has

$$
\int_{\{x: \Phi(u)<-\omega\}}\left|(-\Delta)^{\frac{s}{2}} \Phi(u)\right|^{2} \mathrm{~d} x=-\int_{\{x: \Phi(u)<-\omega\}}(\omega+\Phi(u))^{2} u^{2} \mathrm{~d} x,
$$

so that $(\omega+\Phi(u))^{-}=0$ where $u(x) \neq 0$.
Lemma 1.6. The map $\Phi$ is $C^{1}$ and

$$
G_{\phi}=\left\{(u, \phi) \in H^{s}\left(\mathbb{R}^{3}\right) \times D^{s, 2}\left(\mathbb{R}^{3}\right) \mid F_{\phi}^{\prime}(u, \phi)=0\right\} .
$$

Proof. Noticing that $\Phi(u)$ is a solution of the second equation in problem (1.1), we have

$$
\begin{equation*}
-\|\Phi(u)\|_{D^{s, 2}}^{2}=\int_{\mathbb{R}^{3}}(\omega+\Phi(u)) \Phi(u) u^{2} \mathrm{~d} x=\int_{\mathbb{R}^{3}} \omega \Phi(u) u^{2} \mathrm{~d} x+\int_{\mathbb{R}^{3}} \Phi^{2}(u) u^{2} \mathrm{~d} x . \tag{1.5}
\end{equation*}
$$

In addition,

$$
\begin{aligned}
F(u, \Phi(u))= & \frac{1}{2}\|u\|_{H^{s}}^{2}-\frac{1}{2}\|\Phi(u)\|_{D^{s, 2}}^{2}-\frac{1}{2} \int_{\mathbb{R}^{3}}\left(2 \omega \Phi(u)+\Phi^{2}(u)\right) u^{2} \mathrm{~d} x \\
& -\frac{\mu}{q} \int_{\mathbb{R}^{3}}|u|^{q} \mathrm{~d} x-\frac{1}{2_{s}^{*}} \int_{\mathbb{R}^{3}}|u|^{2_{s}^{*}} \mathrm{~d} x
\end{aligned}
$$

and

$$
F_{\phi}^{\prime}(u, \Phi(u))=-\|\Phi(u)\|_{D^{s, 2}}^{2}-\int_{\mathbb{R}^{3}} \omega \Phi(u) u^{2} \mathrm{~d} x-\int_{\mathbb{R}^{3}} \Phi^{2}(u) u^{2} \mathrm{~d} x,
$$

according to (1.5), one gets that $F_{\phi}^{\prime}(u, \Phi(u))=0$ for any $(u, \phi) \in H^{s}\left(\mathbb{R}^{3}\right) \times D^{s, 2}\left(\mathbb{R}^{3}\right)$. Thus

$$
F^{\prime}(u, \Phi(u))=F_{u}^{\prime}(u, \Phi(u))+F_{\phi}^{\prime}(u, \Phi(u)) \Phi^{\prime}(u)=F_{u}^{\prime}(u, \Phi(u)) .
$$

Define $I(u):=F(u, \Phi(u))$ and if $u, v \in H^{s}\left(\mathbb{R}^{3}\right)$, one gets that

$$
\begin{equation*}
I^{\prime}(u) v=\langle u, v\rangle_{H^{s}}+\int_{\mathbb{R}^{3}}\left[\left(m^{2}-(\omega+\Phi(u))^{2}\right) u v-\mu|u|^{q-2} u v-|u|^{2_{s}^{*}-2} u v\right] \mathrm{d} x . \tag{1.6}
\end{equation*}
$$

Lemma 1.7. The following statements are equivalent:
(i) $(u, \phi) \in H^{s}\left(\mathbb{R}^{3}\right) \times D^{s, 2}\left(\mathbb{R}^{3}\right)$ is a solution of problem (1.1).
(ii) $u$ is a critical point of $I$ and $\phi=\Phi(u)$.

Proof. (ii) $\Longrightarrow$ (i) Obviously.
(i) $\Longrightarrow(i i)$ Let $F_{u}^{\prime}(u, \phi)$ and $F_{\phi}^{\prime}(u, \phi)$ denote the partial derivatives of $F$ at $(u, \phi) \in H^{s}\left(\mathbb{R}^{3}\right) \times$ $D^{s, 2}\left(\mathbb{R}^{3}\right)$. Then for every $v \in H^{s}\left(\mathbb{R}^{3}\right)$ and $\psi \in D^{s, 2}\left(\mathbb{R}^{3}\right)$, one obtains that

$$
\begin{gather*}
F_{u}^{\prime}(u, \phi)[v]=\langle u, v\rangle_{H^{s}}+\int_{\mathbb{R}^{3}}\left[\left(m^{2}-(\omega+\phi)^{2}\right) u v-\mu|u|^{q-2} u v-|u|^{2_{s}^{*}-2} u v\right] \mathrm{d} x,  \tag{1.7}\\
F_{\phi}^{\prime}(u, \phi)[\psi]=-\langle\phi, \psi\rangle_{D^{s, 2}}-\int_{\mathbb{R}^{3}}\left(\omega \psi u^{2}+\phi \psi u^{2}\right) \mathrm{d} x . \tag{1.8}
\end{gather*}
$$

By standard computations, we can prove that $F_{u}^{\prime}(u, \phi)$ and $F_{\phi}^{\prime}(u, \phi)$ are continuous. From (1.7) and (1.8), it is easy to obtain that its critical points are solutions of problem (1.1), moreover, by Lemma 1.3, one has $\phi=\Phi(u)$.

## 2 Proof of Theorem 1.1

Lemma 2.1. For $u \in H_{r}^{s}\left(\mathbb{R}^{3}\right)$, if $|m|>|\omega|$, then there exist some constants $\rho, \alpha>0$ such that $\left.I(u)\right|_{\|u\|_{H^{s}}=\rho} \geq \alpha>0$.
Proof. From (1.4) and (1.5), $I(u)$ can be written in the following form

$$
\begin{align*}
I(u) & =\frac{1}{2}\|u\|_{H^{s}}^{2}-\frac{1}{2}\|\phi\|_{D^{s, 2}}^{2}-\frac{1}{2} \int_{\mathbb{R}^{3}}\left(2 \omega \phi+\phi^{2}\right) u^{2} \mathrm{~d} x-\frac{\mu}{q} \int_{\mathbb{R}^{3}}|u|^{q} \mathrm{~d} x-\frac{1}{2_{s}^{*}} \int_{\mathbb{R}^{3}}|u|^{2_{s}^{*}} \mathrm{~d} x \\
& =\frac{1}{2}\|u\|_{H^{s}}^{2}+\frac{1}{2}\|\phi\|_{D^{s, 2}}^{2}+\frac{1}{2} \int_{\mathbb{R}^{3}} \phi^{2} u^{2} \mathrm{~d} x-\frac{\mu}{q} \int_{\mathbb{R}^{3}}|u|^{q} \mathrm{~d} x-\frac{1}{2_{s}^{*}} \int_{\mathbb{R}^{3}}|u|^{2_{s}^{*}} \mathrm{~d} x . \tag{2.1}
\end{align*}
$$

Then by the Sobolev inequality, we have

$$
I(u) \geq \frac{1}{2}\|u\|_{H^{s}}^{2}-C_{1}\|u\|_{H^{s}}^{q}-C_{2}\|u\|_{H^{s}}^{2_{s}^{*}} \geq \alpha>0, \text { for } u \in H^{s}\left(\mathbb{R}^{3}\right),\|u\|_{H^{s}}=\rho
$$

Thus

$$
\left.I(u)\right|_{\|u\|_{H^{s}}=\rho} \geq \alpha>0
$$

and the proof is completed.
Lemma 2.2. Under the assumptions of Theorem 1.1, there exists a function $e \in H^{s}\left(\mathbb{R}^{3}\right)$ with $\|e\|_{H^{s}}>$ $\rho$ such that $I(e)<0$.
Proof. For any $u \in H^{s}\left(\mathbb{R}^{3}\right) \backslash\{0\}$, in view of (1.4), it is easy to obtain that

$$
\begin{aligned}
\lim _{t \rightarrow+\infty} I(t u)= & \frac{t^{2}}{2}\|u\|_{H^{s}}^{2}-\frac{1}{2}\|\Phi(t u)\|_{D^{s, 2}}^{2}-\frac{t^{2}}{2} \int_{\mathbb{R}^{3}}\left(2 \omega \Phi(t u)+\Phi^{2}(t u)\right) u^{2} \mathrm{~d} x \\
& -\frac{t^{q} \mu}{q} \int_{\mathbb{R}^{3}}|u|^{q} \mathrm{~d} x-\frac{t^{2 *}}{2_{s}^{*}} \int_{\mathbb{R}^{3}}|u|^{2_{s}^{*}} \mathrm{~d} x \\
\leq & \frac{t^{2}}{2}\left(\|u\|_{H^{s}}^{2}+\int_{\mathbb{R}^{3}} 2 \omega^{2} u^{2} \mathrm{~d} x\right)-\frac{t^{q} u}{q} \int_{\mathbb{R}^{3}}|u|^{q} \mathrm{~d} x-\frac{t^{2 *}}{2_{s}^{*}} \int_{\mathbb{R}^{3}}|u|^{2_{s}^{*}} \mathrm{~d} x \\
\rightarrow & -\infty,
\end{aligned}
$$

which implies that $I(u) \rightarrow-\infty$, as $\|u\|_{H^{s}} \rightarrow \infty$.
The lemma is proved by taking $e=t u$ with $t>0$ large enough and $u \neq 0$. Therefore we know that there exists $e \in H^{s}\left(\mathbb{R}^{3}\right),\|e\|_{H^{s}}>\rho$ such that $I(e)<0$.

Define

$$
\begin{equation*}
c=\inf _{\gamma \in \Gamma} \max _{0 \leq t \leq 1} I(\gamma(t)), \tag{2.2}
\end{equation*}
$$

where $\Gamma=\left\{\gamma \in C\left([0,1], H^{s}\left(\mathbb{R}^{3}\right)\right) \mid \gamma(0)=0, \gamma(1)=e\right\}$ is the MP level. Obviously, $c \geq \alpha>0$. There exists a $(P S)_{c}$ sequence $\left\{u_{k}\right\} \subset E$ such that

$$
\begin{align*}
I\left(u_{k}\right) & \rightarrow c, \\
I^{\prime}\left(u_{k}\right) & \rightarrow 0, \quad k \rightarrow \infty . \tag{2.3}
\end{align*}
$$

Lemma 2.3. The $(P S)_{c}$ sequence $\left\{u_{k}\right\} \subset E$ given in (2.3) is bounded.
Proof. There is a positive constant $M$ such that

$$
\begin{align*}
M+o(1)\left\|u_{k}\right\| \geq & I\left(u_{k}\right)-\frac{1}{q}\left(I^{\prime}\left(u_{k}\right), u_{k}\right) \\
= & \left(\frac{1}{2}-\frac{1}{q}\right)\left\|u_{k}\right\|_{H^{s}}^{2}+\frac{1}{2}\left\|\Phi\left(u_{k}\right)\right\|_{D^{s, 2}}^{2}+\left(\frac{1}{2}+\frac{1}{q}\right) \int_{\mathbb{R}^{3}} \Phi^{2}\left(u_{k}\right) u_{k}^{2} \mathrm{~d} x  \tag{2.4}\\
& +\frac{2}{q} \int_{\mathbb{R}^{3}} \omega \Phi\left(u_{k}\right) u_{k}^{2} \mathrm{~d} x+\left(\frac{1}{q}-\frac{1}{2_{s}^{*}}\right) \int_{\mathbb{R}^{3}}\left|u_{k}\right|^{2_{s}^{*}} \mathrm{~d} x .
\end{align*}
$$

Substituting (1.5) into (2.4), we get that

$$
\begin{aligned}
M+o(1)\left\|u_{k}\right\| \geq & I\left(u_{k}\right)-\frac{1}{q}\left(I^{\prime}\left(u_{k}\right), u_{k}\right) \\
= & \left(\frac{1}{2}-\frac{1}{q}\right)\left\|u_{k}\right\|_{H^{s}}^{2}+\left(\frac{1}{2}-\frac{2}{q}\right)\left\|\Phi\left(u_{k}\right)\right\|_{D^{s, 2}}^{2} \\
& +\left(\frac{1}{2}-\frac{1}{q}\right) \int_{\mathbb{R}^{3}} \Phi^{2}\left(u_{k}\right) u_{k}^{2} \mathrm{~d} x+\left(\frac{1}{q}-\frac{1}{2_{s}^{*}}\right) \int_{\mathbb{R}^{3}}\left|u_{k}\right|^{2_{s}^{*}} \mathrm{~d} x \\
\geq & C_{4}\left\|u_{k}\right\|_{H^{s}}^{2} .
\end{aligned}
$$

Since $4<q<2_{s}^{*}$, as a consequence of the above inequality, $\left\{u_{k}\right\}$ is bounded in $H^{s}\left(\mathbb{R}^{3}\right)$.
Furthermore, according to (1.5), one gets that

$$
\begin{equation*}
\|\Phi(u)\|_{D^{s, 2}}^{2}=-\int_{\mathbb{R}^{3}}(\omega+\Phi(u)) \Phi(u) u^{2} \mathrm{~d} x=-\int_{\mathbb{R}^{3}} \omega \Phi(u) u^{2} \mathrm{~d} x-\int_{\mathbb{R}^{3}} \Phi^{2}(u) u^{2} \mathrm{~d} x . \tag{2.5}
\end{equation*}
$$

Then by Hölder inequality and Sobolev inequality, one obtains that

$$
\begin{aligned}
\left\|\Phi\left(u_{k}\right)\right\|_{D^{s, 2}}^{2} & \leq-\int_{\mathbb{R}^{3}} \omega \Phi\left(u_{k}\right) u_{k}^{2} \mathrm{~d} x \\
& \leq|\omega|\left(\int_{\mathbb{R}^{3}}\left|\Phi\left(u_{k}\right)\right|^{22_{s}^{*}} \mathrm{~d} x\right)^{\frac{1}{2_{s}^{s}}}\left(\int_{\mathbb{R}^{3}}\left|u_{k}\right|^{\frac{2 \cdot 22_{s}^{*}}{2_{s}^{s-1}}} \mathrm{~d} x\right)^{\frac{2_{5}^{*}-1}{22_{s}^{s}}} \\
& =|\omega|\left(\int_{\mathbb{R}^{3}}\left|\Phi\left(u_{k}\right)\right|^{\frac{6}{3-2 s}} \mathrm{~d} x\right)^{\frac{3-2 s}{6}}\left(\int_{\mathbb{R}^{3}}\left|u_{k}\right| \frac{12}{3+2 s} \mathrm{~d} x\right)^{\frac{3+2 s}{6}} \\
& \leq C_{5}\left\|\Phi\left(u_{k}\right)\right\|_{D^{s, 2}}\left\|u_{k}\right\|_{H^{s}}^{2} .
\end{aligned}
$$

Thus $\left\{\Phi\left(u_{k}\right)\right\}$ is bounded (even uniformly).
Due to the presence of the unbounded domain, the embedding $H^{s}\left(\mathbb{R}^{3}\right) \hookrightarrow L^{q}\left(\mathbb{R}^{3}\right)$
$\left(2 \leq q \leq \frac{2 n}{n-2 s}=\frac{6}{3-2 s}\right)$ is not compact. In order to overcome this kind, we restrict $I$ to radial
functions, namely, $H_{r}^{s}\left(\mathbb{R}^{3}\right)=\left\{u \in H^{s}\left(\mathbb{R}^{3}\right) \mid u(x)=u(|x|)\right\}$ is compactly embedded in $L_{r}^{q}\left(\mathbb{R}^{3}\right)$ for $2<q<\frac{2 n}{n-2 s}=\frac{6}{3-2 s}$ (see [9] and [14]). By the standard arguments we know that a critical point $u \in H_{r}^{s}\left(\mathbb{R}^{3}\right)$ for the functional $\left.I\right|_{H_{r}^{s}\left(\mathbb{R}^{3}\right)}$ is also a critical point for $I$.

Up to a subsequence, we may assume that there exist $u \in H_{r}^{s}\left(\mathbb{R}^{3}\right)$ and $\varphi \in D_{r}^{s, 2}$ such that

$$
\begin{align*}
u_{k} \rightharpoonup u & \text { in } H_{r}^{s}\left(\mathbb{R}^{3}\right),  \tag{2.6}\\
u_{k} \rightarrow u & \text { in } L_{r}^{q}\left(\mathbb{R}^{3}\right) \text { for } 2<q<2_{s}^{*},  \tag{2.7}\\
\Phi\left(u_{k}\right) \rightharpoonup \varphi & \text { in } D_{r}^{s, 2}\left(\mathbb{R}^{3}\right) . \tag{2.8}
\end{align*}
$$

Lemma 2.4. $\varphi=\Phi(u)$ and $\Phi\left(u_{k}\right) \rightarrow \Phi(u)$ in $D_{r}^{s, 2}\left(\mathbb{R}^{3}\right)$.
Proof. First we prove the uniqueness. For every fixed $u \in H_{r}^{s}\left(\mathbb{R}^{3}\right)$, we consider the following minimizing problem

$$
\inf _{\phi \in D_{r}^{s, 2}} E_{u}(\phi),
$$

where $E_{u}: D_{r}^{s, 2} \rightarrow \mathbb{R}$ defined as energy functional of the second equation in system (1.1).

$$
E_{u}(\phi)=\frac{1}{2}\|\phi\|_{D_{r}^{s, 2}}^{2}+\int_{\mathbb{R}^{3}} \omega \phi u^{2} \mathrm{~d} x+\frac{1}{2} \int_{\mathbb{R}^{3}} \phi^{2} u^{2} \mathrm{~d} x .
$$

In fact, by the proof of [22, Lemma 2.1], we know that

$$
\Phi\left(u_{k}\right) \rightarrow \varphi \text {, locally uniformly in } \mathbb{R}^{3},
$$

so we obtain that

$$
\int_{\mathbb{R}^{3}} \Phi\left(u_{k}\right) u_{k}^{2} \mathrm{~d} x \rightarrow \int_{\mathbb{R}^{3}} \varphi u^{2} \mathrm{~d} x, \quad \int_{\mathbb{R}^{3}} \Phi^{2}\left(u_{k}\right) u_{k}^{2} \mathrm{~d} x \rightarrow \int_{\mathbb{R}^{3}} \varphi^{2} u^{2} \mathrm{~d} x .
$$

From the weak lower semicontinuity of the norm in $D_{r}^{s, 2}\left(\mathbb{R}^{3}\right)$ and the convergence above, one has

$$
E_{u}(\varphi) \leq \liminf _{k \rightarrow \infty} E_{u_{k}}\left(\Phi\left(u_{k}\right)\right) \leq \liminf _{k \rightarrow \infty} E_{u_{k}}(\Phi(u))=E_{u}(\Phi(u)),
$$

then by Lemma 1.3, $\varphi=\Phi(u)$.
Next we prove that $\left\{\Phi\left(u_{k}\right)\right\}$ converges strongly in $D_{r}^{s, 2}\left(\mathbb{R}^{3}\right)$. Since $\Phi\left(u_{k}\right)$ and $\Phi(u)$ satisfy the second equation in problem (1.1).

$$
\left\{\begin{array}{l}
\left\langle\Phi\left(u_{k}\right), \psi\right\rangle_{D_{r}^{s, 2}}=-\int_{\mathbb{R}^{3}}\left[\omega u_{k}^{2} \psi+\Phi\left(u_{k}\right) u_{k}^{2} \psi\right] \mathrm{d} x, \\
\langle\Phi(u), \psi\rangle_{D_{r}^{s, 2}}=-\int_{\mathbb{R}^{3}}\left[\omega u^{2} \psi+\Phi(u) u^{2} \psi\right] \mathrm{d} x,
\end{array}\right.
$$

then we take the difference for $\Phi$, one obtains that

$$
\begin{equation*}
\left\langle\Phi\left(u_{k}\right)-\Phi(u), \psi\right\rangle_{D_{r}^{s, 2}}=-\int_{\mathbb{R}^{3}}\left[\omega\left(u_{k}^{2}-u^{2}\right) \psi+\left(\Phi\left(u_{k}\right) u_{k}^{2}-\Phi(u) u^{2}\right) \psi\right] \mathrm{d} x, \quad \psi \in D_{r}^{s, 2}\left(\mathbb{R}^{3}\right) . \tag{2.9}
\end{equation*}
$$

Thus

$$
\begin{align*}
& \left\langle\Phi\left(u_{k}\right)-\Phi(u), \psi\right\rangle_{D_{r}^{s, 2}}+\int_{\mathbb{R}^{3}}\left[u_{k}^{2}\left(\Phi\left(u_{k}\right)-\Phi(u)\right) \psi\right] \mathrm{d} x+\int_{\mathbb{R}^{3}}\left(u_{k}^{2}-u^{2}\right) \Phi(u) \psi \mathrm{d} x  \tag{2.10}\\
& \quad=-\omega \int_{\mathbb{R}^{3}}\left(u_{k}^{2}-u^{2}\right) \psi \mathrm{d} x, \quad \psi \in D_{r}^{s, 2}\left(\mathbb{R}^{3}\right) .
\end{align*}
$$

By the Hölder inequality and the Sobolev inequality, testing with $\psi=\left(\Phi\left(u_{k}\right)-\Phi(u)\right)$, the following holds:

$$
\begin{aligned}
\| \Phi\left(u_{k}\right) & -\Phi(u) \|_{D_{r}^{s, 2}}^{2} \\
= & -\omega \int_{\mathbb{R}^{3}}\left(u_{k}^{2}-u^{2}\right)\left(\Phi\left(u_{k}\right)-\Phi(u)\right) \mathrm{d} x \\
& -\int_{\mathbb{R}^{3}} u_{k}^{2}\left(\Phi\left(u_{k}\right)-\Phi(u)\right)^{2} \mathrm{~d} x-\int_{\mathbb{R}^{3}}\left(u_{k}^{2}-u^{2}\right) \Phi(u)\left(\Phi\left(u_{k}\right)-\Phi(u)\right) \mathrm{d} x \\
\leq & |\omega| \int_{\mathbb{R}^{3}}\left|u_{k}^{2}-u^{2}\right|\left|\Phi\left(u_{k}\right)-\Phi(u)\right| \mathrm{d} x+\int_{\mathbb{R}^{3}}\left|u_{k}^{2}-u^{2}\right||\Phi(u)|\left|\Phi\left(u_{k}\right)-\Phi(u)\right| \mathrm{d} x \\
\leq & |\omega|\left|\Phi\left(u_{k}\right)-\Phi(u)\right|_{\frac{6}{3-2 s}}\left|u_{k}^{2}-u^{2}\right|_{\frac{6}{3+2 s}}+\left|u_{k}^{2}-u^{2}\right|_{\frac{3}{2 s}}|\Phi(u)|_{\frac{6}{3-2 s}}\left|\Phi\left(u_{k}\right)-\Phi(u)\right|_{\frac{6}{3-2 s}} \\
\leq & C_{6}\left|u_{k}-u\right|_{\frac{12}{3+2 s}}+C_{7}\left|u_{k}-u\right|_{\frac{3}{5}} .
\end{aligned}
$$

Since $u_{k} \rightharpoonup u$ in $H_{r}^{s}\left(\mathbb{R}^{3}\right), u_{k} \rightarrow u$ in $L_{r}^{q}\left(\mathbb{R}^{3}\right)\left(2<q<2_{s}^{*}\right)$, one has $\Phi\left(u_{k}\right) \rightarrow \Phi(u)$ strongly in $D_{r}^{s, 2}\left(\mathbb{R}^{3}\right)$.

Lemma 2.5. The weak limit $(u, \Phi(u))$ solves problem (1.1).
Proof. From (1.6), we know that

$$
\begin{align*}
\left(I^{\prime}\left(u_{k}\right), v\right)= & \left\langle u_{k}, v\right\rangle_{H^{s}}+\int_{\mathbb{R}^{3}}\left[\left(m^{2}-\left(\omega+\Phi\left(u_{k}\right)\right)^{2}\right) u_{k} v\right] \mathrm{d} x  \tag{2.11}\\
& -\int_{\mathbb{R}^{3}}\left[\mu\left|u_{k}\right|^{q-2} u_{k} v+\left|u_{k}\right|^{2_{s}^{*}-2} u_{k} v\right] \mathrm{d} x, \quad v \in H_{r}^{s}\left(\mathbb{R}^{3}\right) .
\end{align*}
$$

All convergences in the sequel must be understood passing to a subsequence if necessary. Since $\left\{u_{k}\right\}$ is bounded in $L_{r}^{2_{s}^{*}}\left(\mathbb{R}^{3}\right)$,

$$
\left|u_{k}\right|^{2_{s}^{*}-2} u_{k} \rightharpoonup|u|^{2_{s}^{*}-2} u, \quad \text { in }\left(L_{r}^{2_{s}^{*}}\left(\mathbb{R}^{3}\right)\right)^{*} .
$$

Moreover by Lemma 2.4, for any $v \in H_{r}^{s}\left(\mathbb{R}^{3}\right)$, one gets that

$$
\int_{\mathbb{R}^{3}} u_{k} \Phi^{2}\left(u_{k}\right) v \mathrm{~d} x+2 \omega \int_{\mathbb{R}^{3}} \Phi\left(u_{k}\right) u_{k} v \mathrm{~d} x \rightarrow \int_{\mathbb{R}^{3}} u \Phi^{2}(u) v \mathrm{~d} x+2 \omega \int_{\mathbb{R}^{3}} \Phi(u) u v \mathrm{~d} x .
$$

In fact one obtains that

$$
\begin{align*}
& \int_{\mathbb{R}^{3}}\left|\Phi(u) u-\Phi\left(u_{k}\right) u_{k}\right||v| \mathrm{d} x  \tag{2.1.1}\\
& \quad \leq\left|\Phi(u)-\Phi\left(u_{k}\right)\right|_{\frac{6}{3-2 s}}|u|_{\frac{3}{2 s}}|v|_{\frac{6}{3-2 s}}+\left|\Phi\left(u_{k}\right)\right|_{\frac{6}{3-2 s}}|v|_{\frac{6}{3-2 s}}\left|u_{k}-u\right|_{\frac{3}{2 s}}
\end{align*}
$$

and

$$
\begin{align*}
& \int_{\mathbb{R}^{3}}\left|u_{k} \Phi^{2}\left(u_{k}\right)-u \Phi^{2}(u)\right||v| \mathrm{d} x  \tag{2.13}\\
& \quad \leq\left|u_{k}-u\right|_{\frac{3}{2 s}}\left|\Phi\left(u_{k}\right)\right|_{\frac{12}{3-2 s}}^{2}|v|_{\frac{6}{3-2 s}}+\left|\Phi\left(u_{k}\right)-\Phi(u)\right|_{\frac{6}{3-2 s}}\left|\Phi\left(u_{k}\right)+\Phi(u)\right|_{\frac{6}{3-2 s}}|u|_{\frac{3}{5}}|v|_{\frac{3}{5}} .
\end{align*}
$$

The compactness of the embedding $H_{r}^{s}\left(\mathbb{R}^{3}\right) \hookrightarrow L_{r}^{q}\left(\mathbb{R}^{3}\right)$ the lemma follows.
In the following we will prove $u \neq 0$, so we assume that $c$ denotes the $M P$ level.

Claim 2.6. $c<\frac{s}{3} S_{s}^{\frac{3}{2 s}}$, where $S_{s}$ corresponds to the best constant for the fractional Sobolev embedding $D^{s, 2}\left(\mathbb{R}^{3}\right) \hookrightarrow L^{2_{s}^{*}}\left(\mathbb{R}^{3}\right)$, precisely,

$$
\begin{equation*}
S_{s}:=\inf _{u \in D^{s, 2}\left(\mathbb{R}^{3}\right) \backslash\{0\}} \frac{\|u\|_{D^{s, 2}}^{2}}{|u|_{2_{s}^{*}}^{2}} . \tag{2.14}
\end{equation*}
$$

Proof. By [10], $S_{s}$ is attained by

$$
\widetilde{u}(x)=\kappa\left(\varepsilon^{2}+\left|x-x_{0}\right|\right)^{-\frac{3-2 s}{2}},
$$

i.e., $S_{s}=\frac{\|\widetilde{u}\|_{s, 2}^{2}}{|\widetilde{u}|_{2_{s}^{*}}^{2}}$, normalizing $\widetilde{u}$ by $|\widetilde{u}|_{2_{s}^{*}}$, one obtains that $\bar{u}=\frac{\widetilde{u}}{|\tilde{u}|_{2_{s}^{*}}^{*}}$. Thus

$$
S_{s}=\inf _{u \in D^{s, 2}\left(\mathbb{R}^{3}\right), \mid u_{2_{s}^{*}}=1}\|u\|_{D^{s, 2}}^{2}=\|\bar{u}\|_{D^{s, 2}}^{2} .
$$

Moreover $u_{1}=S_{s}^{\frac{1}{2_{s}^{-2}}} \bar{u}$ is a positive ground state solution of $(-\Delta)^{s}=|u|^{2_{s}^{*}-2}$ in $\mathbb{R}^{3}$ and

$$
\left\|u_{1}\right\|_{D^{s, 2}}^{2}=\left|u_{1}\right|_{2_{s}^{*}}^{2_{s}^{*}}=S_{s}^{\frac{3}{s}} .
$$

Now according to Reference [20], given $\varepsilon>0$, we consider the function

$$
\begin{equation*}
U_{\varepsilon}(x)=\varepsilon^{-\frac{3-2 s}{2}} u_{1}\left(\frac{x}{\varepsilon}\right), \quad U_{\varepsilon} \in D^{s, 2}\left(\mathbb{R}^{3}\right) . \tag{2.15}
\end{equation*}
$$

Let $\varphi \in C_{0}^{\infty}\left(\mathbb{R}^{3}\right)$ such that $0 \leq \varphi \leq 1$ in $\mathbb{R}^{3}, \varphi \equiv 1$ in $B_{\delta}(\delta>0)$ and $\varphi \equiv 0$ in $C B_{2 \delta}$, where $B_{\delta}=B(0, \delta)$ and $C B_{\delta}=\mathbb{R}^{3} \backslash B_{\delta}$. For every $\varepsilon>0$ we denote $u_{\varepsilon}$ by the following function: $u_{\varepsilon}=\varphi(x) U_{\varepsilon}(x), x \in \mathbb{R}^{3}$ and

$$
v_{\varepsilon}(x)=\frac{u_{\varepsilon}(x)}{\left|u_{\varepsilon}(x)\right|_{2_{s}^{*}}}
$$

Let $e>0$ and $\mu>0$, if $x \in C B_{e}$, then

$$
\left|\nabla u_{\varepsilon}(x)\right| \leq C \varepsilon^{\frac{3-2 s}{2}} \quad \text { for any } \varepsilon>0
$$

and for some positive constant $C$, possibly depending on $\mu, e$ and $s$. Suppose $s \in\left(\frac{3}{4}, 1\right)$. Then according to [20], the following estimate holds true:

$$
\begin{equation*}
X_{\varepsilon}:=\frac{C_{3, s}}{2} \iint_{\mathbb{R}^{3} \times \mathbb{R}^{3}} \frac{\left|v_{\varepsilon}(x)-v_{\varepsilon}(y)\right|^{2}}{|x-y|^{3+2 s}} \mathrm{~d} x \mathrm{~d} y \leq S_{s}^{\frac{3}{s}}+O\left(\varepsilon^{3-2 s}\right), \quad \text { as } \varepsilon \rightarrow 0 . \tag{2.16}
\end{equation*}
$$

Since as $t \rightarrow+\infty, I\left(t v_{\varepsilon}\right) \rightarrow-\infty$, we may assume that

$$
\sup _{t \geq 0} I\left(t v_{\varepsilon}\right)=I\left(t_{\varepsilon} v_{\varepsilon}\right)
$$

and without loss of generality that $t_{\varepsilon} \geq C_{0}>0$, for all $\varepsilon>0$. Otherwise, there exists a sequence $\varepsilon_{n}$ such that

$$
\lim _{n \rightarrow \infty} t_{\varepsilon_{n}}=0
$$

and then

$$
0<c \leq \lim _{n \rightarrow \infty} I\left(t_{\varepsilon_{n}} v_{\varepsilon_{n}}\right)=0 .
$$

Next we will prove the above bound of $t_{\varepsilon}$, that is, for any $\varepsilon>0$ small enough

$$
\begin{equation*}
t_{\varepsilon} \leq\left(X_{\varepsilon}+\int_{\mathbb{R}^{3}} m^{2} v_{\varepsilon}^{2} \mathrm{~d} x\right)^{\frac{1}{2_{s}^{2}-2}}=T \tag{2.17}
\end{equation*}
$$

Set $f(t)=I\left(t v_{\varepsilon}\right)$ and compute

$$
\begin{aligned}
f^{\prime}(t) & =\left(I^{\prime}\left(t v_{\varepsilon}\right), v_{\varepsilon}\right) \\
& =t T^{2_{s}^{*}-2}-t^{2_{s}^{*}-1}-t \int_{\mathbb{R}^{3}}\left(\omega+\Phi\left(t v_{\varepsilon}\right)\right)^{2} v_{\varepsilon}^{2} \mathrm{~d} x-\mu t^{q-1} \int_{\mathbb{R}^{3}}\left|v_{\varepsilon}\right|^{q} \mathrm{~d} x \leq 0, \quad t \geq T .
\end{aligned}
$$

Hence, $f^{\prime}(t) \leq 0$ if $t \geq T$ and (2.17) holds.
Since the function $t \mapsto \frac{1}{2} t^{2} T^{2_{s}^{*}-2}-\frac{1}{2_{s}^{*}}{ }^{2 *}$ is increasing in the internal [0,T), by (2.16), one obtains that

$$
\begin{aligned}
I\left(t_{\varepsilon} v_{\varepsilon}\right)= & \frac{t_{\varepsilon}^{2}}{2}\left(\frac{C_{3, s}}{2} \iint_{\mathbb{R}^{3} \times \mathbb{R}^{3}} \frac{\left|v_{\varepsilon}(x)-v_{\varepsilon}(y)\right|^{2}}{|x-y|^{3+2 s}} \mathrm{~d} x \mathrm{~d} y+\int_{\mathbb{R}^{3}} m^{2} v_{\varepsilon}^{2} \mathrm{~d} x\right)-\frac{t_{\varepsilon}^{2}}{2} \int_{\mathbb{R}^{3}}\left(\omega+\Phi\left(t_{\varepsilon} v_{\varepsilon}\right)\right)^{2} v_{\varepsilon}^{2} \mathrm{~d} x \\
& -\frac{1}{2}\left\|\Phi\left(t_{\varepsilon} v_{\varepsilon}\right)\right\|_{D_{r}^{s, 2}}^{2}-\frac{\mu t_{\varepsilon}^{q}}{q} \int_{\mathbb{R}^{3}}\left|v_{\varepsilon}\right|^{q} \mathrm{~d} x-\int_{\mathbb{R}^{3}} \frac{t_{\varepsilon}^{2 *}}{2_{s}^{*}}\left|v_{\varepsilon}\right|^{22_{s}^{*}} \mathrm{~d} x \\
\leq & \frac{s}{3}\left(S_{s}^{\frac{3}{2 s}}+O\left(\varepsilon^{3-2 s}\right)+\int_{\mathbb{R}^{3}} m^{2} v_{\varepsilon}^{2} \mathrm{~d} x\right)^{\frac{3}{2 s}}+\int_{\mathbb{R}^{3}} \frac{t_{\varepsilon}^{2}}{2} \omega^{2} v_{\varepsilon}^{2} \mathrm{~d} x-\frac{\mu t_{\varepsilon}^{q}}{q} \int_{\mathbb{R}^{3}}\left|v_{\varepsilon}\right|^{q} \mathrm{~d} x .
\end{aligned}
$$

Then using the inequality $(a+b)^{\sigma} \leq a^{\sigma}+\sigma(a+b)^{\sigma-1} b$, for all $\sigma \geq 1, a, b \geq 0$, we get that

$$
I\left(t_{\varepsilon} v_{\varepsilon}\right) \leq \frac{s}{3} S_{S_{s}^{42^{2}}}^{\frac{9}{2}}+O\left(\varepsilon^{3-2 s}\right)+C_{1}(\varepsilon) \int_{\mathbb{R}^{3}} v_{\varepsilon}^{2} \mathrm{~d} x-C_{2}(\varepsilon) \int_{\mathbb{R}^{3}}\left|v_{\varepsilon}\right|^{q} \mathrm{~d} x
$$

with constants $C_{i}(\varepsilon)>0(i=1,2)$. On the other hand, we may get the conclusion that

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{2 s}} \int_{\mathbb{R}^{3}}\left(v_{\varepsilon}^{2}-\mu\left|v_{\varepsilon}\right|^{q}\right) \mathrm{d} x=-\infty \quad \text { for } \varepsilon \text { small enough. } \tag{2.18}
\end{equation*}
$$

In fact, by the definition of $u_{\varepsilon}$, since for $\varepsilon \rightarrow 0$, as in [20],

$$
\begin{equation*}
\int_{\mathbb{R}^{3}}\left|u_{\varepsilon}\right|^{2_{s}^{*}} \mathrm{~d} x=S_{s}^{\frac{3}{s}}+O\left(\varepsilon^{3}\right), \tag{2.19}
\end{equation*}
$$

it suffices to evaluate (2.18) with $u_{\varepsilon}$ in place of $v_{\varepsilon}$. For $p \geq 1$, one has

$$
\begin{align*}
\int_{\mathbb{R}^{3}}\left|u_{\varepsilon}(x)\right|^{p} \mathrm{~d} x & =\int_{B_{\delta}}\left|U_{\varepsilon}(x)\right|^{p} \mathrm{~d} x+\int_{B_{2 \delta} \backslash B_{\delta}}\left|\varphi(x) U_{\varepsilon}(x)\right|^{p} \mathrm{~d} x \\
& =C_{8} \varepsilon^{\frac{p(3-2 s)}{2}} \int_{B_{\delta}}\left|u_{1}\left(\frac{x}{\varepsilon}\right)\right|^{p} \mathrm{~d} x \\
& =C_{8} \varepsilon^{\frac{6-3 p+2 p s}{2}} \int_{R}^{\frac{\delta}{\varepsilon}}\left|u_{1}(r)\right|^{p} r^{2} \mathrm{~d} r  \tag{2.20}\\
& =C_{8} \varepsilon^{\frac{6-3 p+2 p s}{2}} \int_{R}^{\frac{\delta}{\varepsilon}} r^{-3 p+2 p s+2} \mathrm{~d} r
\end{align*}
$$

for any $0<R<\frac{\delta}{\varepsilon}$ and therefore, one has for $4<q<2_{s}^{*}$, as $\varepsilon \rightarrow 0$,

$$
\begin{equation*}
\int_{\mathbb{R}^{3}} u_{\varepsilon}^{2} \mathrm{~d} x-\mu \int_{\mathbb{R}^{3}} u_{\varepsilon}^{q} \mathrm{~d} x \leq C_{9} \varepsilon^{2 s}-C_{10} \mu \varepsilon^{\frac{6-3 q+2 q s}{2}}, \tag{2.21}
\end{equation*}
$$

where $C_{i}>0(i=9,10)$ are independent from $\varepsilon$. According to (2.19) and (2.21), we complete the proof of (2.18).

Claim 2.7. The solution $u$ is nontrivial.

Proof. By contradiction, suppose that $u \equiv 0$. It follows that $\Phi(u)=0$ and as $k \rightarrow \infty$,

$$
\begin{aligned}
\left(I^{\prime}\left(u_{k}\right), u_{k}\right)= & \frac{C_{3, s}}{2} \iint_{\mathbb{R}^{3} \times \mathbb{R}^{3}} \frac{\left|u_{k}(x)-u_{k}(y)\right|^{2}}{|x-y|^{3+2 s}} \mathrm{~d} x \mathrm{~d} y+\int_{\mathbb{R}^{3}}\left(m^{2}-\omega^{2}\right) u_{k}^{2} \mathrm{~d} x \\
& -\int_{\mathbb{R}^{3}}\left(2 \omega \Phi\left(u_{k}\right)+\Phi^{2}\left(u_{k}\right)\right) u_{k}^{2} \mathrm{~d} x-\mu \int_{\mathbb{R}^{3}}\left|u_{k}\right|^{q} \mathrm{~d} x-\int_{\mathbb{R}^{3}}\left|u_{k}\right|^{2_{s}^{*}} \mathrm{~d} x \\
\rightarrow & 0
\end{aligned}
$$

and

$$
u_{k} \rightarrow 0 \text { in } L_{r}^{q}\left(\mathbb{R}^{3}\right)
$$

Thus one obtains that

$$
\int_{\mathbb{R}^{3}}\left|u_{k}\right|^{q} \mathrm{~d} x \rightarrow 0
$$

and

$$
\int_{\mathbb{R}^{3}}\left(2 \omega \Phi\left(u_{k}\right)+\Phi^{2}\left(u_{k}\right)\right) u_{k}^{2} \mathrm{~d} x \rightarrow 0 .
$$

Hence, up to a subsequence, if necessary, we can assume that

$$
\begin{equation*}
\frac{C_{3, s}}{2} \iint_{\mathbb{R}^{3} \times \mathbb{R}^{3}} \frac{\left|u_{k}(x)-u_{k}(y)\right|^{2}}{|x-y|^{3+2 s}} \mathrm{~d} x \mathrm{~d} y+\int_{\mathbb{R}^{3}}\left(m^{2}-\omega^{2}\right) u_{k}^{2} \mathrm{~d} x \rightarrow L \tag{2.22}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{\mathbb{R}^{3}}\left|u_{k}(x)\right|^{2_{s}^{*}} \mathrm{~d} x \rightarrow L, \quad L \geq 0 \tag{2.23}
\end{equation*}
$$

Furthermore, $I\left(u_{k}\right) \rightarrow c$, it follows that

$$
\begin{equation*}
c=\left(\frac{1}{2}-\frac{1}{2_{s}^{*}}\right) L=\frac{s}{3} L \tag{2.24}
\end{equation*}
$$

Since $c \geq \alpha>0$, it is easily seen that $L>0$. In addition,

$$
\frac{C_{3, s}}{2} \iint_{\mathbb{R}^{3} \times \mathbb{R}^{3}} \frac{\left|u_{k}(x)-u_{k}(y)\right|^{2}}{|x-y|^{3+2 s}} \mathrm{~d} x \mathrm{~d} y \geq S_{s}\left|u_{k}\right|_{2_{s}^{*},}^{2}
$$

so that taking into account (2.22) and (2.23), we get $L \geq S_{s} L^{\frac{2}{2_{s}^{*}}}$, which combined with (2.24) gives

$$
c \geq \frac{s}{3} S^{\frac{2_{s}^{*}}{2_{s}^{*}-2}}=\frac{s}{3} S^{\frac{3}{2 s}}
$$

this contradicts Claim 2.6. Hence $u$ is nontrivial.

## 3 Proof of Theorem 1.2

We can observe if $q=4$, in (2.21) one can stress the parameter choosing $\mu=\varepsilon^{-\sigma}, \sigma>0$, then to get (2.18), the rest proof of Theorem 1.2 is similar to proof of Theorem 1.1.

## References

[1] A. Azzollini, A. Pomponio, Ground state solutions for the nonlinear Klein-GordonMaxwell equations, Topol. Methods Nonlinear Anal. 35(2010), No. 1, 33-42. MR2677428; Zbl 1203.35274
[2] B. Barrios, E. Colorado, A. de Pablo, U. Sánchez, On some critical problems for the fractional Laplacian operator, J. Differential Equations 252(2012), No. 11, 6133-6162. https://doi.org/10.1016/j.jde.2012.02.023; MR2911424; Zbl 1245.35034
[3] V. Benci, D. Fortunato, Solitary waves of the nonlinear Klein-Gordon equation coupled with the Maxwell equations, Rev. Math. Phys. 14(2002), No. 4, 409-420. https://doi .org/ 10.1142/S0129055X02001168; MR1901222; Zbl 1037.35075
[4] V. Benci, D. Fortunato, The nonlinear Klein-Gordon equation coupled with the Maxwell equations, in: Proceedings of the Third World Congress of Nonlinear Analysts, Part 9 (Catania, 2000), Nonlinear Anal. 47(2011), No. 9, 6065-6072. https://doi.org/10.1016/ S0362-546X (01) 00688-5; MR1970778; Zbl 1042.78500
[5] K. Bogdan, The boundary Harnack principle for the fractional Laplacian, Studia Math. 123(1997), No. 1, 43-80. https://doi.org/10.4064/sm-123-1-43-80; MR1438304; Zbl 0870.31009
[6] L. A. Caffarelli, J.-M. Roquejoffre, Y. Sire, Variational problems with free boundaries for the fractional Laplacian, J. Eur. Math. Soc. (JEMS) 12(2010), No. 5, 1151-1179. https : //doi.org/10.4171/JEMS/226; MR2677613; Zbl 1221.35453
[7] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32(2007), No. 7-9, 1245-1260. https://doi.org/10. 1080/03605300600987306; MR2354493; Zbl 1143.26002
[8] D. Cassani, Existence and non-existence of solitary waves for the critical Klein-Gordon equation coupled with Maxwell's equations, Nonlinear Anal. 58(2004), No. 7-8, 733-747. https://doi.org/10.1016/j.na.2003.05.001; MR2085333; Zbl 1057.35041
[9] W. Chen, C. Li, B. Ou, Classification of solutions for an integral equation, Comm. Pure Appl. Math. 59(2006), No. 3, 330-343. https://doi.org/10.1002/cpa.20116; MR2200258; Zbl 1093.45001
[10] A. Cotsiolis, N. K. Tavoularis, Best constants for Sobolev inequalities for higher order fractional derivatives, J. Math. Anal. Appl. 295(2004), No. 1, 225-236. https://doi .org/ 10.1016/j.jmaa.2004.03.034; MR2064421; Zbl 1084.26009
[11] T. D'Aprile, D. Mugnai, Non-existence results for the coupled Klein-Gordon-Maxwell equations, Adv. Nonlinear Stud. 4(2004), No. 3, 307-322. https ://doi .org/10.1515/ans-2004-0305; MR2079817; Zbl 1142.35406
[12] T. D'Aprile, D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations, Proc. Roy. Soc. Edinburgh Sect. A 134(2004), No. 5, 893906. https://doi.org/10.1017/S030821050000353X; MR2099569; Zbl 1064.35182
[13] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136(2012), No. 5, 521-573. https://doi.org/10.1016/j .bulsci. 2011.12.004; MR2944369; Zbl 1252.46023
[14] R. L. Frank, E. Lenzmann, L. Silvestre, Uniqueness of radial solutions for the fractional Laplacian, Comm. Pure Appl. Math. 69(2016), No. 9, 1671-1726. https://doi.org/ 10.1002/cpa.21591; MR3530361; Zbl 1365.35206
[15] O. H. Miyagaki, E. L. de Moura, R. Ruviaro, Positive ground state solutions for quasicritical the fractional Klein-Gordon-Maxwell system with potential vanishing at infinity, Complex Var. Elliptic Equ. 64(2019), No. 2, 315-329. https://doi.org/10.1080/17476933. 2018.1434625; MR3895852; Zbl 1405.35119
[16] X. Ros-Oton, J. Serra, The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, J. Math. Pures Appl. (9) 101(2014), No. 3, 275-302. https://doi .org/10. 1016/j.matpur. 2013.06.003; MR3168912; Zbl 1285.35020
[17] X. Ros-Oton, J. Serra, The Pohozaev identity for the fractional Laplacian, Arch. Ration. Mech. Anal. 213(2014), No. 2, 587-628. https://doi.org/10.1007/s00205-014-0740-2; MR3211861; Zbl 1361.35199
[18] R. Servadei, E. Valdinoci, Mountain pass solutions for non-local elliptic operators, J. Math. Anal. Appl. 389(2012), No. 2, 887-898. https://doi.org/10.1016/j.jmaa.2011.12. 032; MR2879266; Zbl 1234.35291
[19] R. Servadei, E. Valdinoci, Weak and viscosity solutions of the fractional Laplace equation, Publ. Mat. 58(2014), No. 1, 133-154. https://doi .org/euclid.pm/1387570393, MR3161511; Zbl 1292.35315
[20] R. Servadei, E. Valdinoci, The Brezis-Nirenberg result for the fractional Laplacian, Trans. Amer. Math. Soc. 367(2015), No. 1, 67-102. https ://doi.org/10.1090/S0002-9947-2014-05884-4; MR3271254; Zbl 1323.35202
[21] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, Comm. Pure Appl. Math. 60(2007), No. 1, 67-112. https://doi.org/10.1002/ cpa.20153; MR2270163; Zbl 1141.49035
[22] Y. Yu, Solitary waves for nonlinear Klein-Gordon equations coupled with Born-Infeld theory, Ann. Inst. H. Poincaré Anal. Non Linéaire 27(2010), No. 1, 351-376. https://doi. org/10.1016/j.anihpc.2009.11.001; MR2580514; Zbl 1184.35286


[^0]:    ${ }^{\boxtimes}$ Corresponding author.
    E-mails: andreigrecu.cv@gmail.com (A. Grecu), denisa.stancu@yahoo.com (D. Stancu-Dumitru).

[^1]:    ${ }^{\boxtimes}$ Email: jan-christoph.schlage-puchta@uni-rostock.de

[^2]:    ${ }^{\boxtimes}$ Corresponding author. Email: m.reza.h56@gmail.com

[^3]:    ${ }^{\boxtimes}$ Corresponding author. Email: syhuang@mail.ntue.edu.tw

[^4]:    ${ }^{\boxtimes}$ Email: ignacio.marquez@usc.es

[^5]:    ${ }^{\boxtimes}$ Email: roberto.capistranofilho@ufpe.br
    *This work is dedicated to my daughter Helena.

[^6]:    *The interested readers are referred to $[18,30]$ for history and origins of the Korteweg-de Vries equation.

[^7]:    ${ }^{* *}$ Remark that the uncoupling is not possible in (4.4) unless $r=0$.

[^8]:    ${ }^{\boxtimes}$ Corresponding author. Email: peloe1@udayton.edu

[^9]:    ${ }^{\boxtimes}$ Corresponding author: regilene@icmc.usp.br

[^10]:    ${ }^{\boxtimes}$ Corresponding author. Email: John-Graef@utc.edu

[^11]:    ${ }^{\boxtimes}$ Corresponding author. Email: zhhliu@hotmail.com

[^12]:    ${ }^{\boxtimes}$ Email: maria_dem@mail.ru

[^13]:    ${ }^{\boxtimes}$ Corresponding author. Email: zegela1@yahoo.com

[^14]:    ${ }^{\boxtimes}$ Corresponding author. Email: Ifnie@163.com

[^15]:    ${ }^{\boxtimes}$ Corresponding author. Email: volok@math.nsc.ru

[^16]:    ${ }^{\boxtimes}$ Corresponding author. Email: hhattori@wvu.edu

[^17]:    ${ }^{\boxtimes}$ Corresponding author. Email: dosla@math.muni.cz

[^18]:    ${ }^{\boxtimes}$ Corresponding author. Email: z_aliyev@mail.ru

[^19]:    ${ }^{\boxtimes}$ Email: qianxiaotao1984@163.com

[^20]:    ${ }^{\boxtimes}$ Corresponding author. Email: wangpy@xynu.edu.cn

[^21]:    ${ }^{\boxtimes}$ Corresponding author. Email: fengxj@sxu.edu.cn

[^22]:    ${ }^{\boxtimes}$ Corresponding author. Email: min.wang@kennesaw.edu

[^23]:    ${ }^{\boxtimes}$ Corresponding author. Email: pdrabek@kma.zcu.cz

[^24]:    ${ }^{\boxtimes}$ Email: sremr@fme.vutbr.cz

[^25]:    ${ }^{\boxtimes}$ Corresponding author. Email: jan.tomecek@upol.cz

[^26]:    ${ }^{\boxtimes}$ Corresponding author. Email: gmweixy@163.com

[^27]:    ${ }^{\boxtimes}$ Corresponding author. Email: dang@math.msstate.edu

[^28]:    ${ }^{\boxtimes}$ Corresponding author. Email: tiziana.cardinali@unipg.it

[^29]:    ${ }^{\boxtimes}$ Corresponding author. Email: gpapas@env.duth.gr

[^30]:    ${ }^{\boxtimes}$ Corresponding author. Email: mlsong2004@163.com

[^31]:    ${ }^{\boxtimes}$ Corresponding author. Email:jjohanajimenez@gmail.com

[^32]:    ${ }^{\boxtimes}$ Corresponding author. Email: huiguo_math@163.com

[^33]:    ${ }^{\boxtimes}$ Corresponding author. Email: math_chb@163.com

[^34]:    ${ }^{\boxtimes}$ Corresponding author. Email: mengqiong@qq.com

[^35]:    ${ }^{\boxtimes}$ Corresponding author. Email: chuanxizhu@126.com

[^36]:    *Co-corresponding author. Email: chenhuialena@163.com
    ${ }^{\boxtimes}$ Corresponding author. Email: xuelian632@163.com

[^37]:    ${ }^{\boxtimes}$ Email: piotr.kowalski.1@p.lodz.pl

[^38]:    ${ }^{\boxtimes}$ Email: caoqianj2019@126.com

[^39]:    ${ }^{\boxtimes}$ Email: ddnovaes@unicamp.br

[^40]:    ${ }^{\boxtimes}$ Corresponding author. Email: angelo@math.carleton.ca

[^41]:    ${ }^{\boxtimes}$ Email: jekl@mail.muni.cz

[^42]:    ${ }^{\boxtimes}$ Email: marco.sabatini@unitn.it

[^43]:    ${ }^{\boxtimes}$ Corresponding author. Email: carlos.lizama@usach.cl
    *Email: mamuar1@upv.es

[^44]:    ${ }^{\boxtimes}$ Email: msm@ime.unicamp.br

[^45]:    ${ }^{\boxtimes}$ Corresponding author. Email: wzhx5016674@126.com

[^46]:    ${ }^{\boxtimes}$ Email: simnaos@gmail.com

[^47]:    ${ }^{\boxtimes}$ Corresponding author. Email: 19010701008@mail.hnust.edu.cn

[^48]:    ${ }^{\boxtimes}$ Corresponding author. Email: zhzh@mail.bnu.edu.cn

[^49]:    ${ }^{\boxtimes}$ Corresponding author. Email: xjhuangxwen@163.com

[^50]:    ${ }^{\boxtimes}$ Email: migda@amu.edu.pl

[^51]:    ${ }^{\boxtimes}$ Corresponding author. Email: veli.sahmurov@antalya.edu.tr

[^52]:    ${ }^{\boxtimes}$ Corresponding author. Email:11394861@qq.com

[^53]:    ${ }^{\boxtimes}$ Corresponding author. Email: zhhliu@hotmail.com

[^54]:    ${ }^{\boxtimes}$ Corresponding author. Email: boichuk.aa@gmail.com

[^55]:    ${ }^{\boxtimes}$ Email: jardel.m.pereira@ufsc.br

[^56]:    ${ }^{\boxtimes}$ Email: Zhang_Xinbb@163.com

