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Strongly formal Weierstrass non-integrability for polynomial differential systems in $\mathbb{C}^2$

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Abstract. Recently a criterion has been given for determining the weakly formal Weierstrass non-integrability of polynomial differential systems in $\mathbb{C}^2$. Here we extend this criterion for determining the strongly formal Weierstrass non-integrability which includes the weakly formal Weierstrass non-integrability of polynomial differential systems in $\mathbb{C}^2$. The criterion is based on the solutions of the form $y = f(x)$ with $f(x) \in \mathbb{C}[[x]]$ of the differential system whose integrability we are studying. The results are applied to a differential system that contains the famous force-free Duffing and the Duffing–Van der Pol oscillators.

Keywords: Liouville integrability, Weierstrass integrability, polynomial differential systems.

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1 Introduction and statement of the main result

One of the main problems in the qualitative theory of differential systems is the integrability problem. For differential systems in $\mathbb{C}^2$ this problem consists in to determine if the system has or not an explicit first integral. When this first integral can be expressed as quadratures of elementary functions we have the so-called Liouville integrability, which is the most studied, see for instance [16, 30, 31] and references therein. The Liouville integrability is based on the cofactors of the invariant algebraic curves and the exponential factors (see definitions below). Some generalizations of the Liouville integrability theory defining the generalized cofactors have been obtained, see [7, 8, 10, 11, 19, 20, 30, 31].

Some differential systems have an explicit first integral that cannot be expressed as quadratures of elementary functions. Hence these systems are not Liouville integrable. Sometimes these first integrals can be expressed in terms of special functions, as for instance functions that are solutions of second order linear differential equations (in [11, 19, 29] several examples are given). To determine when a differential system is not Liouville integrable is an open problem, see [25]. A partial answer to this question has been recently given in [23].

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In this work we present a criterion to detect the strongly formal Weierstrass non-integrability which is a generalization of the criterion for detecting weakly formal Weierstrass non-integrability given in [23]. Finally we apply this new criterion to some differential systems. Puiseux Weierstrass integrability is a generalization of formal Weierstrass integrability which includes the Liouville integrability and is based on the Puiseux Weierstrass polynomials, see again [23] and below.

First we provide some preliminary definitions and results.

In this paper we consider polynomial differential systems in the plane $\mathbb{C}^2$ that are given by

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1.1}$$

where the functions P and Q are polynomials in the complex variables x and y. We define by $m = \max\{\deg P, \deg Q\}$ the *degree* of system (1.1) with $P(0,0) = Q(0,0) = 0$. Along the paper we also consider the associated differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{Q(x,y)}{P(x,y)}, \tag{1.2}$$

and the associated *vector field* $\mathcal{X} = P(x,y)\partial/\partial x + Q(x,y)\partial/\partial y$.

An *invariant algebraic curve* of system (1.1) is an invariant curve $f = 0$ with $f \in \mathbb{C}[x,y]$, such that the orbital derivative $\dot{f} = \mathcal{X}f = P\partial f/\partial x + Q\partial f/\partial y$ vanishes on $f = 0$. This condition implies that there exists a polynomial $K(x,y) \in \mathbb{C}[x,y]$ of degree less than or equal to $m-1$ such that

$$\mathcal{X}f = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{1.3}$$

This polynomial K is called the *cofactor* of the curve $f(x,y) = 0$.

A function of the form $e^{f/g}$ with f and g polynomials is called an *exponential factor* if there is a polynomial L of degree at most $m-1$ such that

$$\mathcal{X}(e^{f/g}) = P\,\frac{\partial e^{f/g}}{\partial x} + Q\,\frac{\partial e^{f/g}}{\partial y} = L\,e^{f/g}.$$

The polynomial L is called the *cofactor* of the exponential factor $e^{f/g}$.

A non-locally constant function $H : U \subset \mathbb{C}^2 \to \mathbb{C}$ is a *first integral* of system (1.1) in the open set U if this function is constant on each solution $(x(t), y(t))$ of system (1.1) contained in U. In fact if $H \in C^1(U)$ is a first integral of system (1.1) on U if and only if $\mathcal{X}H = P\partial H/\partial x + Q\partial H/\partial y \equiv 0$ on U. A non-constant function $M : U \subset \mathbb{C}^2 \to \mathbb{C}$ is an *integrating factor* in U if

$$P\frac{\partial M}{\partial x} + Q\frac{\partial M}{\partial y} = -\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) M = -\mathrm{div}(\mathcal{X})M. \tag{1.4}$$

This integrating factor M is associated to a first integral H when $MP = -\partial H/\partial y$ and $MQ = \partial H/\partial x$. Moreover $V = 1/M$ is an *inverse integrating factor* in $U \setminus \{M = 0\}$.

A polynomial differential system (1.1) has a *Liouville first integral* H if its associated integrating factor is of the form

$$M = \exp\left(\frac{D}{E}\right) \prod_i C_i^{\alpha_i}, \tag{1.5}$$

where D, E and the C_i are polynomials in $\mathbb{C}[x,y]$ and $\alpha_i \in \mathbb{C}$, see [3, 17, 30, 31]. The curves $C_i = 0$ and $E = 0$ are invariant algebraic curves of the differential system (1.1), and the exponential $\exp(D/E)$ is a product of some exponential factors associated to the multiple invariant algebraic curves of system (1.1) or to the invariant straight line at infinity, see for instance [2,4,5,15] or Chapter 8 of [10].

The Liouville integrability is based on the existence of algebraic cofactors for the invariant algebraic curves and for the exponential factors. The first generalization of this theory is to consider non-algebraic invariant curves but still with algebraic cofactors, see [11]. In [12] a method for detecting non-algebraic invariant curves for polynomial differential systems was given. However there exist non-algebraic invariant curves without an algebraic cofactor, see [20].

Now we are recall the definition of Puiseux Weierstrass integrability introduced in [23].

Let $\mathbb{C}((x))$ be the set of series in fractionary powers in the variable x with coefficients in $\mathbb{C}$ (these series are called *Puiseux series*), and $\mathbb{C}[y]$ the set of the polynomials in the variable y with coefficients in the ring $\mathbb{C}$. A function of the form

$$\sum_{i=0}^{\ell} a_i(x)y^i \in \mathbb{C}((x))[y] \tag{1.6}$$

is a *Puiseux Weierstrass polynomial* in y of degree ℓ, i.e. a polynomial in the variable y with coefficients in $\mathbb{C}((x))$. Here we have privileged the variable y but of course we can privileged the variable x instead of the y.

In the next result we provide the expression of the cofactor of an invariant curve $y - g(x) = 0$ with $g(x)$ being a Puiseux series, for a proof see [23], see also [13].

Proposition 1.1. *Let $g(x) \in \mathbb{C}((x))$. An invariant curve of the form $y - g(x) = 0$ of a polynomial differential system* (1.1) *of degree m has a Puiseux Weierstrass polynomial cofactor of the form*

$$K(x,y) = k_{m-1}(x)y^{m-1} + \cdots + k_1(x)y + k_0(x). \tag{1.7}$$

A planar autonomous differential system is *Puiseux Weierstrass integrable* if admits an integrating factor of the form (1.5) where D, E and the C_i's are Puiseux Weierstrass polynomials. This definition is a generalization of the Weierstrass integrability given in [19] and studied in [21,22,24,28]. We remark that by definition that all the Liouvillian integrable systems are particular cases of the Puiseux Weierstrass integrable systems.

Let $\mathbb{C}[[x,y]]$ be the set of all formal power series in the variables x and y with coefficients in $\mathbb{C}$.

Theorem 1.2. *If $f \in \mathbb{C}[[x,y]]$ then it has a unique decomposition of the form*

$$f = ux^s \prod_{j=1}^{\ell}(y - g_j(x)), \tag{1.8}$$

where $g_j(x)$ are Puiseux series and $s \in \mathbb{Z}$, $s \geq 0$ and $u \in \mathbb{C}[[x,y]]$ is invertible inside the ring $\mathbb{C}[[x,y]]$.

For a proof of Theorem 1.2 see Corollary 1.5.6 of [1].

We note that a Darboux integrating factor (1.5) is analytic function where it is defined consequently by Theorem 1.2 it can be written into the form (1.8).

The first aim of this work was to give a necessary condition for detecting the Puiseux Weierstrass integrability but when $g_j(x) \in \mathbb{C}[[x]]$ of a polynomial differential system (1.1).

However this has been impossible using only the formal solutions of the form $y = f(x)$ of the associated differential equation for the reasons that we will see later on.

We say that a polynomial differential system (1.1) is *strongly formal Weierstrass integrable* if it has an integrating factor of the form

$$M(x,y) = \alpha(x) \prod_{k=1}^{\ell} (y - g_k(x))^{\alpha_k}, \tag{1.9}$$

where the functions $\alpha(x), g_k(x) \in \mathbb{C}[[x]]$ for $i = 1, \ldots, k$. Note that the definition of strongly formal Weierstrass integrability is a generalization of the definition of weakly formal Weierstrass integrability given in [23], where that the functions $\alpha(x)$ is constant equal to one.

In this work we give a criterion for detecting when a polynomial differential system (1.1) is not strongly formal Weierstrass integrable with $\alpha(x), g_k(x) \in \mathbb{C}[[x]]$. This criterion is based on the following result which provides a necessary condition in order that a polynomial differential system (1.1) be strongly formal Weierstrass integrable with $\alpha(x), g_k(x) \in \mathbb{C}[[x]]$.

Our main result is the following one.

Theorem 1.3. *Assume that a polynomial differential system (1.1) is strongly formal Weierstrass integrable with $\alpha(x), g_k(x) \in \mathbb{C}[[x]]$, and let $H(x,y)$ be a first integral provided by the strongly formal Weierstrass integrability.*

(a) Let $h(x) \in \mathbb{C}[[x]]$ and $y = h(x)$ be an invariant curve of the system such that $H(x,y)$ is defined on the curve $y = h(x)$. Then there exists an integrating factor $M(x,y)$ of the form (1.9) such that $M(x, h(x)) = 0$.

(b) Assume that the origin of system (1.1) is a singular point, and the first integral $H(x,y)$ and $M(x,y)$ of statement (a) are well-defined at the origin. Then a linear combination of the formal Weierstrass polynomial cofactors up to order r of the solutions of the form $y = f(x)$ satisfying $Eq := \dot{x}dy/dx - \dot{y} = 0$ must be equal to minus the divergence of system (1.1) up to order r.

Theorem 1.3 is proved in Section 2.

Now we apply Theorem 1.3 to a differential system that contains the force-free Duffing and Duffing–Van der Pol oscillators. Hence we consider the differential system

$$\dot{x} = y, \quad \dot{y} = -(\zeta x^2 + \alpha)y - (\varepsilon x^3 + \sigma x). \tag{1.10}$$

This system contains the famous force-free Duffing ($\zeta = 0$, $\varepsilon \neq 0$) and the Duffing–Van der Pol ($\zeta \neq 0$, $\varepsilon \neq 0$) oscillators that appear in several fields of mathematics, physics, biology, see [18] and references therein. The Liouville integrability of system (1.10) was studied in [9] where the following results were established.

Theorem 1.4. *System (1.10) with $\zeta = 0$ and $\varepsilon \neq 0$ is Liouvillian integrable if and only if either $\alpha = 0$, or $\sigma = 2\alpha^2/9$.*

In the case $\zeta \neq 0$ by a suitable rescaling of the variables for the Duffing–Van der Pol system we can take $\zeta = 3$ without loss of generality.

Theorem 1.5. *System (1.10) with $\zeta = 3$ and $\varepsilon \neq 0$ is Liouvillian integrable if and only if $\alpha = 4\varepsilon/3$ and $\sigma = \varepsilon^2/3$.*

Applying Theorem 1.3 to system (1.10) we obtain the following result.

Theorem 1.6. *System (1.10) can be strongly formal Weierstrass integrable with $\alpha(x), g_k(x) \in \mathbb{C}[[x]]$ if, and only if, one of the following cases holds:*

(a) $\sigma = 2\alpha^2/9$,

(b) $\sigma \neq 2\alpha^2/9$, $\sigma \neq 0$ *and* $3\alpha\varepsilon - 4\zeta\sigma = 0$,

(c) $\sigma \neq 2\alpha^2/9$, $\sigma \neq 0$ *and* $-21\alpha\varepsilon^2 + 6\alpha^2\varepsilon\zeta + 24\varepsilon\zeta\sigma - 7\alpha\zeta^2\sigma = 0$,

(d) $\sigma \neq 2\alpha^2/9$, $\sigma = 0$ *and* $-6\varepsilon(7\varepsilon - 2\alpha\zeta) = 0$.

We can see that all the Liouvillian integrable cases given in Theorems 1.4 and 1.5 are included in Theorem 1.6. In particular the case $\zeta = 3$ with $\alpha = 4\varepsilon/3$ and $\sigma = \varepsilon^2/3$ vanish the condition $-21\alpha\varepsilon^2 + 6\alpha^2\varepsilon\zeta + 24\varepsilon\zeta\sigma - 7\alpha\zeta^2\sigma = 0$.

Theorem 1.6 is proved in Section 3.

The following proposition shows that if a polynomial differential system has a Puiseux Weierstrass first integral of the form (1.5) then it has an integrating factor of the same form.

Proposition 1.7. *If system (1.1) has a Puiseux Weierstrass first integral of the form (1.5), then it has a Puiseux Weierstrass integrating factor of the form (1.5).*

The proof is straightforward because $M = (\partial H/\partial y)/P(x, y)$ which has the form (1.5). This proposition was generalized in [17] for non-Liouville integrable systems.

2 Proof of Theorem 1.3

Proof of statement (a) of Theorem 1.3. By assumptions the first integral $H(x, y)$ is defined on the invariant curve $y = h(x)$. So $H(x, h(x)) = c \in \mathbb{C}$, and the first integral $\bar{H}(x, y) = H(x, y) - c$ satisfies $\bar{H}(x, h(x)) = 0$. Now we consider the integrating factor $M(x, y)$ associated to the first integral $\bar{H}$. Perhaps this inverse integrating factor does not vanish at $y = h(x)$, but we consider the function $\bar{M} = MF(\bar{H})$ being F an arbitrary function of $\bar{H}$ such that $F(0) = 0$. This function $\bar{M}$ is also an inverse integrating factor of system (1.1) because

$$\begin{aligned}\mathcal{X}(\bar{M}) &= \mathcal{X}(MF(\bar{H})) = \mathcal{X}(M)F(\bar{H}) + M\mathcal{X}(F(\bar{H})) = F(\bar{H})\mathcal{X}(M)\\ &= -F(\bar{H})\mathrm{div}(\mathcal{X})\, M = -\mathrm{div}(\mathcal{X})\, MF(\bar{H}) = -\mathrm{div}(\mathcal{X})\, \bar{M}.\end{aligned}$$

Hence we obtain that $\bar{M}(x, h(x)) = 0$ because $F(0) = 0$. □

We can repeat this process to obtain an integrating factor that vanish in a finite number of the solutions of the form $y = h(x)$ such $H(x, h(x)) = c \in \mathbb{C}$.

In the proof of statement (b) of Theorem 1.3 we shall need the following result, for a proof see for instance Proposition 8.4 of [10].

Proposition 2.1. *Assume that $f \in \mathbb{C}[x, y]$ and let $f = f_1^{n_1} \dots f_r^{n_r}$ be its factorization into irreducible factors over $\mathbb{C}[x, y]$. Then for a polynomial system (1.1), $f = 0$ is an invariant algebraic curve with cofactor K_f if and only if $f_i = 0$ is an invariant algebraic curve for each $i = 1, \dots, r$ with cofactor K_{f_i}. Moreover $K_f = n_1 K_{f_1} + \dots + n_r K_{f_r}$.*

Proof of statement (b) of Theorem 1.3. We assume that the system is strongly formal Weierstrass integrable with $\alpha(x), g_k(x) \in C[[x]]$ this means by definition that the system has an integrating factor of the form (1.9). Hence we know that a first integral H and an integrating factor

M of the form given in statement (a) can be found. We compute the solutions $y = f_i(x)$ where $f_i(x) = \sum_{j=0}^{\infty} a_j x^j$ with a_i arbitrary coefficients that must satisfy the equation $Eq := \dot{x}dy/dx - \dot{y} = 0$ up to certain order r. Note that these solutions satisfy that

$$\text{either} \quad M(x, f_i(x)) = \mathcal{O}(x^r), \quad \text{or} \quad M(x, f_i(x)) = c_2 + \mathcal{O}(x^r),$$

with $c_2 \neq 0$, this case appears when the integrating factor (1.8) has $s = 0$. The first ones correspond to the $f_i(x)$ that approximate the invariant curves $y = g_k(x)$ that appear in the integrating factor (1.9). For such $f_i(x)$ we compute the cofactor K_i up to certain order r though the equation

$$\mathcal{X}(y - f_i(x)) = \bar{K}_i(y - f_i(x)) + \mathcal{O}(x^r). \tag{2.1}$$

Hence these cofactors $\bar{K}_i$ of the solutions $y - f_i(x)$ are the approximations up to order r of the cofactors K_k of the invariant curves $y - g_k(x)$ of the integrating factor (1.9).

The second ones satisfy

$$M(x, f_i(x)) = \alpha(x) \prod_{k=1}^{\ell} (f_i(x) - g_k(x))^{\alpha_k} = c_2 + \mathcal{O}(x^r). \tag{2.2}$$

Hence, since $c_2 \neq 0$, $M(x, f_i(x)) = c_2 + \mathcal{O}(x^r)$, and from (1.9) we have that $\alpha(0) \neq 0$. Then up to order r we have

$$\prod_{k=1}^{\ell} (f_i(x) - g_k(x))^{\alpha_k} = \left[\frac{c_2}{\alpha(x)} \right]_r + \mathcal{O}(x^r), \tag{2.3}$$

where here $[\cdot]_r$ means up to order r. Consequently $y = f_i(x)$ is an approximation up to order r of the equation

$$\prod_{k=1}^{\ell} (y - g_k(x))^{\alpha_k} = \frac{c_2}{\alpha(x)}. \tag{2.4}$$

We apply the vector field operator to (2.4) and we obtain

$$\mathcal{X}\left(\prod_{k=1}^{\ell} (y - g_k(x))^{\alpha_k} \right) = \mathcal{X}\left(\frac{c_2}{\alpha(x)} \right) = -\frac{c_2 \alpha'(x)}{\alpha(x)^2} P = -K_\alpha \frac{c_2}{\alpha(x)}, \tag{2.5}$$

because $\mathcal{X}(\alpha(x)) = K_\alpha(x, y)\alpha(x)$ where K_α is a formal Weierstrass polynomial cofactor. This happens because $\alpha(x) = 0$ is an invariant algebraic curve of the vector field $\mathcal{X}$. Indeed, $\alpha(x)$ is a factor of the integrating factor $M(x, y)$ given in (1.9), and $M(x, y) = 0$ is an invariant curve because it satisfies (1.4), and the factors of an invariant curve are also invariant curves. Moreover we have taken into account that $\mathcal{X}(\alpha(x)) = \alpha'(x)\dot{x} = \alpha'(x)P(x, y)$ and then $K_\alpha = \alpha'(x)P(x, y)/\alpha(x)$.

In summary from equations (2.4) and (2.5) we have

$$\mathcal{X}\left(\prod_{k=1}^{\ell} (y - g_k(x))^{\alpha_k} \right) = -K_\alpha \prod_{k=1}^{\ell} (y - g_k(x))^{\alpha_k}. \tag{2.6}$$

Now we apply the vector field operator to (2.3) and we obtain

$$\mathcal{X}\left(\prod_{k=1}^{\ell} (f_i(x) - g_k(x))^{\alpha_k} \right) = \mathcal{X}\left(\left[\frac{c_2}{\alpha(x)} \right]_r \right) + \mathcal{O}(x^r), \tag{2.7}$$

where $\mathcal{X}\left(\mathcal{O}(x^r)\right) = \mathcal{O}(x^{r-1})\,P(x, f_i(x)) = \mathcal{O}(x^r)$. Taking into account equation (2.5) we define the new cofactor $\tilde{K}_\alpha$ through the equation

$$\mathcal{X}\left(\left[\frac{c_2}{\alpha(x)}\right]_r\right) = -\tilde{K}_\alpha\left(\left[\frac{c_2}{\alpha(x)}\right]_r\right) \tag{2.8}$$

which is equation (2.5) taking the lower terms up to r and where $\tilde{K}_\alpha$ is an approximation up to r of the cofactor K_α. Therefore from (2.3), (2.7) and (2.8) we obtain an approximation of the cofactor of $\alpha(x)$ up to order r computing

$$\frac{\mathcal{X}\left(\prod_{k=1}^{\ell}(f_i(x) - g_k(x))^{\alpha_k}\right)}{\prod_{k=1}^{\ell}(f_i(x) - g_k(x))^{\alpha_k}} = -\tilde{K}_\alpha + \mathcal{O}(x^r). \tag{2.9}$$

By the definition of integrating factor (1.9) and from the extension of the Darboux theory to Weierstrass functions, see for instance Theorem 3 of [23], we have that

$$\mathcal{X}(M) = -\mathrm{div}(\mathcal{X})M. \tag{2.10}$$

In short the cofactors $\bar{K}_i$ of the solutions $y - f_i(x)$ passing through the origin are the approximations up to order r of the cofactors K_i of the solution $y = g_i(x)$. By Proposition 2.1 the other solutions $y - f_i(x)$ not passing through the origin with cofactor $\tilde{K}_i$ give by equation (2.9) an approximation up to order r of the cofactor $\tilde{K}_\alpha$ of $\alpha(x)$, i.e.

$$\sum_{i=1}^{s}\mu_i\tilde{K}_i = -\tilde{K}_\alpha. \tag{2.11}$$

Therefore, from (2.2), (2.10) and (2.11) we obtain that

$$\sum_{i=1}^{\ell}\lambda_i\bar{K}_i + \sum_{i=1}^{s}\mu_i\tilde{K}_i = -\mathrm{div}_r(\mathcal{X}) + \mathcal{O}(x^{r+1}). \tag{2.12}$$

This proves statement (b) of the theorem. □

In summary, if condition (2.12) is not satisfied then system (1.1) does not admit an integrating factor of the form (1.9) and consequently is not strongly formal Weierstrass integrable. Hence we have a necessary condition to have strongly formal Weierstrass integrability. Note that if we have that $\sum_{i=1}^{\ell}\lambda_i\bar{K}_i + \sum_{i=1}^{s}\mu_i\tilde{K}_i = \mathcal{O}(x^{r+1})$ system (1.1) satisfies a necessary condition to have a first integral of the form (1.9), see for more details statement (i) of Theorem 8.7 of [10].

3 Proof of Theorem 1.6

We apply the criterion provided by statement (b) of Theorem 1.3 to detect if system (1.10) can be strongly formal Weierstrass integrable, that is, if it can has an inverse integrating factor of the form (1.9). We propose a solution curve of the form

$$y = f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

Substituting this solution in the first ordinary differential equation $Eq := \dot{x}dy/dx - \dot{y} = 0$ we get an infinite system of equations. First we have studied the case when $a_0 \neq 0$, and in this case it is easy to see that we find two solutions not passing through the origin but we do not find any possible integrable case. So we consider the case $a_0 = 0$. In order to determine the first coefficients we fix up to certain order the developments of $f(x)$ and Eq in power series of the variable x. If we compute the solutions up to order 6 we obtain the following finite system of equations

$$\begin{aligned} a_2(3a_1+\alpha) &= 0, & a_1^2 + a_1\alpha + \sigma &= 0, \\ 2a_2^2 + 4a_1a_3 + a_3\alpha + \varepsilon + a_1\zeta &= 0, & 7a_3a_4 + 7a_2a_5 + a_4\zeta &= 0, \\ 5a_2a_3 + 5a_1a_4 + a_4\alpha + a_2\zeta &= 0, & 3a_3^2 + 6a_2a_4 + 6a_1a_5 + a_5\alpha + a_3\zeta &= 0. \end{aligned}$$

From the first equation we have two possibilities $a_2 = 0$ or $a_1 = -\alpha/3$. First we take $a_2 = 0$. The obtained system is compatible and we get two solutions. We denote them y_1 and y_2 but we do not write them here due to their long extensions. Now we study the case $a_1 = -\alpha/3$ with $a_2 \neq 0$. In this case the equation $a_1^2 + a_1\alpha + \sigma = 0$ takes the form $\sigma - 2\alpha^2/9 = 0$. Hence we must impose $\sigma = 2\alpha^2/9$ in order that the finite system of equations be compatible. Under this condition we find four more solutions that we denote by y_3, y_4, y_5 and y_6, but again we do not write them here due to their big extensions. We recall that all these solutions pass through the origin, i.e., $y_i(0) = 0$ for $i = 1, \ldots, 6$. Now we compute their cofactors using equation (2.1), that we denote by $\bar{K}_i$. Finally we verify if the equation

$$\sum_{i=1}^{6} \lambda_i \bar{K}_i = -\mathrm{div}_6 \mathcal{X} + \mathcal{O}(x^7),$$

has any solution, and since it has a solution statement (a) of the theorem follows.

Now we consider the case $\sigma \neq 2\alpha^2/9$. In this case the solutions y_i for $i = 3, \ldots, 6$ do not exist and we only have the solutions y_1 and y_2. We compute their cofactors from equation (2.1), that we denote by $\bar{K}_1$ and $\bar{K}_2$ and we verify if the equation

$$\lambda_1 \bar{K}_1 + \lambda_2 \bar{K}_2 = -\mathrm{div}_6 \mathcal{X} + \mathcal{O}(x^7),$$

is satisfied. This equation gives a system of three equations. The first one is

$$\alpha(2 + \lambda_1 + \lambda_2) - (\lambda_1 - \lambda_2)\sqrt{\alpha^2 - 4\sigma} = 0. \tag{3.1}$$

From this condition we can isolate λ_1 if $\sigma \neq 0$ (we will consider $\sigma = 0$ below) and we have

$$\lambda_1 = \frac{\alpha(2 + \lambda_2) + \lambda_2\sqrt{\alpha^2 - 4\sigma}}{-\alpha + \sqrt{\alpha^2 - 4\sigma}}.$$

From the second equation we obtain

$$\left(-\alpha + 2\sqrt{\alpha^2 - 4\sigma} + \lambda_2\sqrt{\alpha^2 - 4\sigma}\right)(3\alpha\varepsilon - 4\zeta\sigma) = 0.$$

Hence we have two possibilities: If $3\alpha\varepsilon - 4\zeta\sigma = 0$ the third equation can vanish choosing the value of λ_2 and this proves statement (b) of the theorem. If $-\alpha + 2\sqrt{\alpha^2 - 4\sigma} + \lambda_2\sqrt{\alpha^2 - 4\sigma} = 0$ we isolate the value of λ_2, i.e.

$$\lambda_2 = \frac{\alpha - 2\sqrt{\alpha^2 - 4\sigma}}{\sqrt{\alpha^2 - 4\sigma}},$$

and the third equation provides the condition $-21\alpha\varepsilon^2 + 6\alpha^2\varepsilon\zeta + 24\varepsilon\zeta\sigma - 7\alpha\zeta^2\sigma = 0$, which shows statement (c) of the theorem.

Now we study the case $\sigma = 0$. In this case condition (3.1) becomes $\alpha(1+\lambda_2) = 0$. Taking into account that we are in the case $\sigma \neq 2\alpha^2/9$, we must take $\lambda_2 = -1$. The second condition is $\varepsilon(3+\lambda_1) = 0$. The case $\varepsilon = 0$ gives a trivial integrable case. Hence we must consider $\lambda_1 = -3$. In this case the third condition gives $-6\varepsilon(7\varepsilon - 2\alpha\zeta) = 0$ which proves statement (d) of the theorem. Hence this completes the proof of theorem.

4 Examples

Example 4.1. Consider the differential system

$$\dot{x} = y + xy + x^2, \qquad \dot{y} = 2y(y+x). \tag{4.1}$$

This system was studied in [14] where an algorithmic method to determine integrability was given. Using the method developed in [14] it was shown that system (4.1) has an integrating factor of the form $M(x,y) = e^{x^2/(2y)}y^{-5/2}$ and the a Liouville first integral

$$H(x,y) = \frac{e^{\frac{x^2}{2y}}}{\sqrt{y}} + \sqrt{2}\int_0^{x/\sqrt{2y}} e^{t^2}dt.$$

Now we are going to apply the criterion provided by statement (b) of Theorem 1.3 for detecting if system (4.1) can have an inverse integrating factor of the form (1.9). We propose a solution curve of the form

$$y = f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

Substituting this solution in the first ordinary differential equation $Eq := \dot{x}dy/dx - \dot{y} = 0$ we get an infinite system of equations. In order to determine the first coefficients we fix up to certain order the developments of $f(x)$ and Eq in power series in variable x. If we do that up to order 6 and we solve the finite system of equations we obtain the following solutions.

1) $a_6 = a_5 = a_4 = a_3 = a_2 = a_1 = 0$,

2) $a_6 = \frac{-1368989-4007\sqrt{150829}}{607500}$, $a_5 = \frac{781+3\sqrt{150829}}{750}$, $a_4 = \frac{173-\sqrt{150829}}{900}$,
$a_3 = -\frac{2}{3}$, $a_2 = \frac{497-\sqrt{150829}}{70}$, $a_1 = \frac{427-\sqrt{150829}}{35}$, $a_0 = \frac{427-\sqrt{150829}}{70}$,

3) $a_6 = \frac{-1368989+4007\sqrt{150829})}{607500}$, $a_5 = \frac{781-3\sqrt{150829})}{750}$, $a_4 = \frac{173+\sqrt{150829})}{900}$,
$a_3 = -\frac{2}{3}$, $a_2 = \frac{497+\sqrt{150829}}{70}$, $a_1 = \frac{427+\sqrt{150829}}{35}$, $a_0 = \frac{427+\sqrt{150829}}{70}$.

The solutions correspond to the solution curves

1) $y_1 = 0 + \mathcal{O}(x^7)$,

2) $y_2 = f_2(x) = \frac{427-\sqrt{150829}}{70} + \frac{427-\sqrt{150829}}{35}x + \frac{497-\sqrt{150829}}{70}x^2 - \frac{3}{2}x^3 + \frac{173-\sqrt{150829})}{900}x^4 + \frac{781+3\sqrt{150829}}{750}x^5 - \frac{1368989+4007\sqrt{150829}}{607500}x^6 + \mathcal{O}(x^7)$,

3) $y_3 = f_3(x) = \frac{427+\sqrt{150829}}{70} + \frac{427+\sqrt{150829}}{35}x + \frac{497+\sqrt{150829}}{70}x^2 - \frac{3}{2}x^3 + \frac{173+\sqrt{150829})}{900}x^4 + \frac{781-3\sqrt{150829}}{750}x^5 - \frac{1368989-4007\sqrt{150829}}{607500}x^6 + \mathcal{O}(x^7)$,

respectively. The first one corresponds to the invariant algebraic curve $y = 0$ whose cofactor is $2x + 2y$. However, in general, we can have an approximation of a solution of the form $y = g_k(x)$ and an approximation of its cofactor. To compute the approximation of the Weierstrass polynomial cofactor of the solution curve $y = 0$, since the system is of degree 2, it must be of the form $\bar{K}_1 = k_0(x) + k_1(x)y$. Hence we have the equation

$$\frac{\partial(y-y_1)}{\partial x}\dot{x} + \frac{\partial(y-y_1)}{\partial y}\dot{y} = (k_0(x) + k_1(x)y)(y - y_1) + \mathcal{O}(x^7),$$

From here we obtain $k_0 = 2x$ and $k_1 = 2$.

For determining the cofactors of the other two solutions $y = f_i(x)$ for $i = 2, 3$ we use equation (2.7) that in this case are

$$\mathcal{X}\,(y_2(x) - y_1(x)) = \tilde{K}_2(x)(y_2(x) - y_1(x)) + \mathcal{O}(x^7),$$

$$\mathcal{X}\,(y_3(x) - y_1(x)) = \tilde{K}_3(x)(y_3(x) - y_1(x)) + \mathcal{O}(x^7),$$

We do not write here the expressions of $\tilde{K}_2$ and $\tilde{K}_3$ due to their extension but the reader can compute them straightforward. Now we study if the cofactors $\bar{K}_1$, $\bar{K}_2$ and $\bar{K}_3$ satisfy (2.12), i.e.

$$\lambda_1 \bar{K}_1 + \mu_1 \tilde{K}_2 + \mu_2 \tilde{K}_3 = -\text{div}_6 \mathcal{X} + \mathcal{O}(x^7),$$

and this equation has not solution. Hence system (4.1) has not an integrating factor of the form (1.9), this implies that system (4.1) is not strongly formal Weierstrass integrable.

If we try to see if there is a linear combination that gives zero, then the system has the unique solution $\lambda_1 = \mu_2 = \mu_3 = 0$. Therefore the system has not a first integral of the form (1.9).

The conclusion is that system (4.1) is not strongly formal Weierstrass integrable in the original coordinates (x, y). However we can ask if system (4.1) is strongly formal Weierstrass integrable after a change of variable. The answer to this question is positive as we will see below.

System (4.1) after doing the change of variables

$$z = \frac{x}{\sqrt{2y}}, \qquad u = \sqrt{y},$$

takes the form

$$\dot{u} = \sqrt{2}u^2 + 2uz, \qquad \dot{z} = 1.$$

First we rename the new variables of the form $u := x$ and $z := y$. So the equation associated to this differential system is the Bernoulli equation $dx/dy = \sqrt{2}x^2 + 2xy$, and then its integrability is straightforward. In fact an integrating factor is given by $M(x, y) = e^{-y^2}x^2$ and a first integral is

$$H(x, y) = \frac{e^{y^2}}{x} + \sqrt{2}\int_0^y e^{t^2}dt.$$

Anyway we are going to apply the necessary condition of strongly formal Weierstrass integrability to this system. Attending to the form of the integrating factor in this case the answer must be positive.

Hence consider the system of the form

$$\dot{x} = \sqrt{2}x^2 + 2xy, \qquad \dot{y} = 1. \tag{4.2}$$

Now we study if system (4.2) is strongly formal Weierstrass integrable. We propose a solution curve of the form

$$y = f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

Substituting this solution in the ordinary differential equation $Eq := \dot{x} dy/dx - \dot{y} = 0$ we get an infinite system of equations without any solution. Therefore privileging the variable y the system has no solutions curves. Next we propose a solution curve of the form

$$x = f(y) = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5 + \cdots$$

Substituting this solution in the first ordinary differential equation $Eq := \dot{x} dx/dy - \dot{x} = 0$ we get an infinite system of equations. We determine the first parameters fixing certain order in the developments of $f(y)$ and Eq in power series of the variable y. If we do that up to order 4 and we solve the finite system of equations we obtain the following solutions.

1) $x_1 = \mathcal{O}(x^5)$,

2) $x_2 = -\frac{i}{15}\sqrt{\frac{225-15\sqrt{15}}{2}} + \frac{-15+\sqrt{15}}{15\sqrt{2}} y - \frac{i}{15}\sqrt{\frac{15-\sqrt{15}}{2}} y^2 - \frac{\sqrt{2}}{45}(6+\sqrt{15}) y^3$
$+\frac{i}{450}\sqrt{\frac{15-\sqrt{15}}{2}}(10+3\sqrt{15}) y^4 + \mathcal{O}(x^5)$,

3) $x_3 = \frac{i}{15}\sqrt{\frac{225-15\sqrt{15}}{2}} + \frac{-15+\sqrt{15}}{15\sqrt{2}} y + \frac{i}{15}\sqrt{\frac{15-\sqrt{15}}{2}} y^2 - \frac{\sqrt{2}}{45}(6+\sqrt{15}) y^3$
$-\frac{i}{450}\sqrt{\frac{15-\sqrt{15}}{2}}(10+3\sqrt{15}) y^4 + \mathcal{O}(x^5)$,

4) $x_4 = -i\sqrt{\frac{15+\sqrt{15}}{30}} - \frac{15+\sqrt{15}}{15\sqrt{2}} y + \frac{i}{15}\sqrt{\frac{15+\sqrt{15}}{2}} y^2 - \frac{\sqrt{2}}{45}(6-\sqrt{15}) y^3$
$-\frac{i}{450}\sqrt{\frac{15+\sqrt{15}}{2}}(10-3\sqrt{15}) y^4 + \mathcal{O}(x^5)$,

5) $x_5 = i\sqrt{\frac{15+\sqrt{15}}{30}} - \frac{15+\sqrt{15}}{15\sqrt{2}} y - \frac{i}{15}\sqrt{\frac{15+\sqrt{15}}{2}} y^2 - \frac{\sqrt{2}}{45}(6-\sqrt{15}) y^3$
$+\frac{i}{450}\sqrt{\frac{15+\sqrt{15}}{2}}(10-3\sqrt{15}) y^4 + \mathcal{O}(x^5)$.

Next we compute their Weierstrass polynomial cofactor for the solution curve y_1 through the equation

$$\frac{\partial(x-x_1)}{\partial x}\dot{x} + \frac{\partial(x-x_1)}{\partial y}\dot{y} = (k_0(y) + k_1(y)x)(x-x_1) + \mathcal{O}(x^5),$$

which is $\bar{K}_1 = \sqrt{2}x + 2y$, and the cofactors of the other solutions through the equations

$$\mathcal{X}\,(x_i(x) - x_1(x)) = \tilde{K}_i(x)(x_i(x) - x_1(x)) + \mathcal{O}(x^5),$$

for $i = 2, 3, 4, 5$. We do not write here the expressions of these cofactors due to their extension. Finally we try to see if there is a linear combination of these cofactors equals to minus the divergence, that is,

$$\lambda_1 \bar{K}_1 + \mu_2 \tilde{K}_2 + \mu_3 \tilde{K}_3 + \mu_4 \tilde{K}_4 + \mu_5 \tilde{K}_5 = -\mathrm{div}_4 \mathcal{X} + \mathcal{O}(x^5),$$

and this system has the solution $\lambda_1 = -2$, $\mu_2 = 5/6 - \sqrt{5/3}$, $\mu_3 = 5/6 - \sqrt{5/3}$, $\mu_4 = 5/6 + \sqrt{5/3}$ and $\mu_5 = 5/6 + \sqrt{5/3}$. Hence system (4.2) satisfies the strongly formal Weierstrass integrability condition and it can have an integrating factor of the form (1.9) as indeed it has. Moreover we can also study if the system has a first integral of the form (1.9) using the equation

$$\lambda_1 \bar{K}_1 + \mu_2 \tilde{K}_2 + \mu_3 \tilde{K}_3 + \mu_4 \tilde{K}_4 + \mu_5 \tilde{K}_5 = \mathcal{O}(x^5),$$

and this system has the only solution $\lambda_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0$. Consequently system (4.2) has not a first integral of the form (1.9).

Example 4.2. In 1944 Kukles [27] studied the following system

$$\dot{x} = y, \quad \dot{y} = -x + Q(x,y), \tag{4.3}$$

where $Q(x,y) = a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3$, giving the conditions in order that the origin of (4.3) be a center. However some decades later in [6, 26] was proved that the conditions were uncompleted showing that the origin of the following system has also a center. Consider the system

$$\dot{x} = y, \quad \dot{y} = -x + x^2 - \frac{x^3}{3} - \frac{x^2y}{\sqrt{2}} - 2y^2 + \frac{y^3}{3\sqrt{2}}. \tag{4.4}$$

System (4.4) has an inverse integrating factor of the form

$$V(x,y) = e^{-x(1-\frac{x}{2})}(3\sqrt{2}(1-x) + x(\sqrt{2}\,x + y))^3,$$

and the following first integral

$$H(x,y) = \frac{y^2(x+1) + 2\sqrt{2}xy(x-2) + 6(3x-2) + 2x^3 - 10x^2}{(x(y+\sqrt{2}x) + 3\sqrt{2}(1-x))^2} e^{x(1-\frac{x}{2})} + \int e^{x(1-\frac{x}{2})} dx\,.$$

The analyticity of this first integral around the origin implies that the origin is a center.

Now we are going to apply the criterion to detect if system (4.1) can have a strongly formal Weierstrass first integral. We propose a solution curve of the form $y = f(x) = \sum_{i=0} a_i x^i$ and substitute this solution into the differential equation $Eq := \dot{x}dy/dx - \dot{y} = 0$ and we get an infinite system of equations. If we develop up to order 3 and we solve the finite system of equations we obtain the solutions curves.

1) $y_1 = ix - ix^2 + \mathcal{O}(x^3)$,

2) $y_2 = -ix + ix^2 + \mathcal{O}(x^3)$,

3) $y_{3,4} = 3\sqrt{2} \pm \sqrt{3(4-\sqrt{6})} - (\sqrt{2}+\sqrt{3})x + 1/12\Big(6\sqrt{2} - 6\sqrt{3} \pm \sqrt{3}(4-\sqrt{6})^{3/2}$
$\mp 10\sqrt{3(4-\sqrt{6})}\,\Big)x^2 + \mathcal{O}(x^3)$,

4) $y_{5,6} = 3\sqrt{2} \pm \sqrt{3(4 \mp \sqrt{6})} + (-\sqrt{2}+\sqrt{3})x + 1/12\Big(6\sqrt{2} + 6\sqrt{3} \pm \sqrt{3}(4+\sqrt{6})^{3/2}$
$\mp 10\sqrt{3(4+\sqrt{6})}\,\Big)x^2 + \mathcal{O}(x^3)$.

Now we compute the Weierstrass polynomial cofactor of the first two solutions curves. These cofactors, as the system is of degree 3 must be of the form $K = k_0(x) + k_1(x)y + k_2(x)y^2$. Applying equation (2.1) to the solution curves $y - f(x) = 0$ we obtain the Weierstrass polynomial cofactors up to order 3 in the variable x

1) $K_1 = -1/6(6i + 4(-3i+\sqrt{2})x^2) - 1/6(12 - \sqrt{2}ix + \sqrt{2}ix^2)y + 1/(3\sqrt{2})y^2 + \mathcal{O}(x^3)$,

2) $K_2 = 1/6(6i - 4(3i + \sqrt{2})x^2) - 1/6(12 + \sqrt{2}ix - \sqrt{2}ix^2))y + 1/(3\sqrt{2})y^2 + \mathcal{O}(x^3)$,

The cofactors of the other solutions must be computed through the equation (2.9) that in this case are

$$\mathcal{X}\Big((y_i(x) - y_1(x))(y_i(x) - y_2(x))\Big) = \tilde{K}_i(x)\Big((y_i(x) - y_1(x))(y_i(x) - y_2(x))\Big) + \mathcal{O}(x^3),$$

for $i = 3, 4, 5, 6$. We do not write here the expressions of these cofactors due to their extension.

From statement (b) of Theorem 1.3 we study if system (4.4) has a strongly formal Weierstrass first integral using the equation

$$\lambda_1 K_1 + \lambda_2 K_2 + \mu_3 K_3 + \mu_4 K_4 + \mu_5 K_5 + \mu_6 K_6 = \mathcal{O}(x^3),$$

and this system has the solution $\lambda_1 = \lambda_2 = 0$ and

$$\begin{aligned}
\mu_6 &= \frac{\mu_5\Big(6\sqrt{10} + 5\sqrt{15} - 6\sqrt{24 - 6\sqrt{6}} - 15\sqrt{4 - \sqrt{6}}\Big)}{6\sqrt{10} + 5\sqrt{15} + 6\sqrt{24 - 6\sqrt{6}} + 15\sqrt{4 - \sqrt{6}}},\\
\mu_3 &= \frac{\mu_5}{D_1}\Bigg[24\sqrt{10} + 20\sqrt{15} + 4\sqrt{60 - 15\sqrt{6}} + 5\sqrt{40 - 10\sqrt{6}}\\
&\qquad + 129\sqrt{24 - 6\sqrt{6}} + 316\sqrt{4 - \sqrt{6}} + 8\sqrt{4 + \sqrt{6}}\\
&\qquad + 3\sqrt{6(4 + \sqrt{6})} - 49\sqrt{10(4 + \sqrt{6})} - 40\sqrt{15(4 + \sqrt{6})}\Bigg],\\
\mu_4 &= \frac{\mu_5}{D_2}\Bigg[-6\sqrt{10} - 4\sqrt{15} + \sqrt{60 - 15\sqrt{6}} + \sqrt{40 - 10\sqrt{6}}\\
&\qquad - 29\sqrt{24 - 6\sqrt{6}} - 71\sqrt{4 - \sqrt{6}} + \sqrt{4 + \sqrt{6}} + \sqrt{6(4 + \sqrt{6})}\\
&\qquad + 11\sqrt{10(4 + \sqrt{6})} + 9\sqrt{15(4 + \sqrt{6})}\Bigg],
\end{aligned}$$

where we have $D_1 = 372 + 152\sqrt{6} + 80\sqrt{60 - 15\sqrt{6}} + 98\sqrt{40 - 10\sqrt{6}}$ and $D_2 = 84 + 34\sqrt{6} + 18\sqrt{60 - 15\sqrt{6}} + 22\sqrt{40 - 10\sqrt{6}}$. Consequently system (4.4) can have a strong formal Weierstrass first integral as indeed it has as we have seen before.

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Conflict of interest

The authors declare that they have no conflict of interest.

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Existence of limit cycles for some generalisation of the Liénard equations: the relativistic and the prescribed curvature cases

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Abstract. We study the problem of existence of periodic solutions for some generalisations of the relativistic Liénard equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + \hat{f}(x,\dot{x})\dot{x} + g(x) = 0\,,$$

and the prescribed curvature Liénard equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + \hat{f}(x,\dot{x})\dot{x} + g(x) = 0\,,$$

where the damping function depends both on the position and the velocity. In the associated phase-plane this corresponds to a term of the form $f(x,y)$ instead of the standard dependence on x alone. By controlling the continuability of the solutions, we are able to prove the existence of at least a limit cycle in the associated phase-plane for both cases, moreover we provide results with a prefixed arbitrary number of limit cycles. Some examples are given to show the applicability of these results.

Keywords: periodic orbits, limit cycles, Liénard relativistic equation, Liénard curvature equation.

2010 Mathematics Subject Classification: 34C25, 34C07.

1 Introduction

The problem of the existence of periodic solutions for the Liénard differential equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0\,, \tag{1.1}$$

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has been widely investigated in the literature and there is an enormous number of papers originated from the pioneering work by Liénard [7] in 1928. The interested reader can consult for instance [5,15,16] and the references quoted therein.

Recently, particular attention has been given to the study of generalisations of Eq. (1.1) of the form

$$\frac{d}{dt}\varphi(\dot{x}) + f(x)\dot{x} + g(x) = 0\,, \tag{1.2}$$

where a nonlinear function φ is involved. Once we replace the Newton acceleration $\ddot{x}$ in Eq. (1.2) by the relativistic one, we get the so called *relativistic Liénard* equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + f(x)\dot{x} + g(x) = 0\,, \tag{1.3}$$

originally introduced and studied in [9,11]. Similarly, one can study [10] the *prescribed curvature* equation of Liénard type

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + f(x)\dot{x} + g(x) = 0\,. \tag{1.4}$$

The former models have been studied using a phase-plane analysis. Indeed Eq. (1.3) is equivalent to the system

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1+y^2}} \\ \dot{y} = -f(x)\frac{y}{\sqrt{1+y^2}} - g(x)\,, \end{cases} \tag{1.5}$$

while Eq. (1.4) can be rewritten as

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1-y^2}} \\ \dot{y} = -f(x)\frac{y}{\sqrt{1-y^2}} - g(x)\,. \end{cases} \tag{1.6}$$

Let us observe that, at a first glance, Eqs. (1.3) and (1.4) may look similar each other, however once we pass to the phase-plane it is clear that system (1.6) is defined only in the strip $|y| < 1$, while the former in the whole plane. This geometric feature makes the analysis completely different; indeed in order to obtain a winding trajectory and then apply the Poincaré–Bendixson Theorem, it is crucial to control the possible blow-up of trajectories when $|y| \to 1$. To tackle this issue, Eq. (1.4) was studied in [10] in the particular case

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + \lambda f(x)\dot{x} + g(x) = 0\,, \tag{1.7}$$

where the real positive parameter λ should be taken sufficiently small, exactly to control the trajectories.

The aim of this paper is to consider the more general case where the term $f(x)$ in Eqs. (1.3) and (1.4) has been replaced by $f(x,\dot{x})$. Let us observe that a particular case involving a term $f(x,\dot{x})$, polynomial in $\dot{x}$, has been already investigated in [4] in the classical Liénard problem. The interested reader can found other cases in [3,6,12,17]. The goal of the present paper is thus to deal with the following equations

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + \hat{f}(x,\dot{x})\dot{x} + g(x) = 0\,, \tag{1.8}$$

and

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + \hat{f}(x,\dot{x})\dot{x} + g(x) = 0\,. \tag{1.9}$$

Observe that the first equation can be rewritten in the phase-plane as follows

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1+y^2}} \\ \dot{y} = -f(x,y)\frac{y}{\sqrt{1+y^2}} - g(x)\,, \end{cases} \tag{1.10}$$

with $f(x,y) = \hat{f}(x, y/\sqrt{1+y^2})$. In the following we will impose some conditions on $f(x,y)$ to guaranteed the existence of a winding orbit (e.g. Lemma 3.5), let us however notice that the above relation between $f(x,y)$ and $\hat{f}(x,\dot{x})$ allows us to conclude that the assumptions on $\hat{f}$ can be very mild, because system (1.10), covers more general cases, as well as the following system (1.11). The second Eq. (1.9) becomes

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1-y^2}} \\ \dot{y} = -f(x,y)\frac{y}{\sqrt{1-y^2}} - g(x)\,, \end{cases} \tag{1.11}$$

using $f(x,y) = \hat{f}(x, y/\sqrt{1-y^2})$. In particular, the phase portrait of these two systems will be studied, and the problem of existence, uniqueness and number of limit cycles will be investigated.

The structure of the paper is the following. In Section 2 some basic properties of the above systems will be presented, in particular we will develop further the method presented in [9,10] to deal with a case where the number of limit cycles can be completely determined studying the curves where $f(x,y) = 0$. The main results will be presented and proved in Section 3, while in Section 4 some examples will be presented and discussed.

2 Definitions and basic facts

Throughout the paper, we assume the functions $f(x,y)$ and $g(x)$ to be regular enough to ensure the existence and uniqueness of the associated Cauchy problem. Moreover we assume $xg(x) > 0$ for $x \neq 0$. Therefore $(0,0)$ is the unique equilibrium point for both systems (1.10) and (1.11).

The slope of the trajectories of latter systems is given by the following expressions, where y' denotes the derivative of y with respect to x:

$$y'(x) = -f(x,y) - g(x)\frac{\sqrt{1+y^2}}{y}\,, \tag{2.1}$$

and

$$y'(x) = -f(x,y) - g(x)\frac{\sqrt{1-y^2}}{y}\,. \tag{2.2}$$

Let us consider the phase-plane system equivalent to the classical Liénard system, Eq (1.1)

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x)\,, \end{cases} \tag{2.3}$$

whose slope is given by

$$y' = -f(x) - \frac{g(x)}{y}. \tag{2.4}$$

Then one can remark that the main difference between the classical case and the former ones (1.10) and (1.11), lies on the fact that trajectories of system (2.3) cannot cannot escape to infinity (i.e. no vertical asymptote is allowed) in any finite interval $\alpha < x < \beta$; this is no longer true for systems (1.10) and (1.11) due to the presence of the function $f(x, y)$. Another difference, as already mentioned, is that trajectories of system (1.11) are constrained in the horizontal strip $|y| < 1$.

Therefore, in the study of Eq. (1.10) in order to avoid the possibility of the existence of vertical asymptotes some growth restriction on $f(x, y)$ is required, as for instance that in any finite interval $\alpha < x < \beta$ there exist two positive constants L and D such that $|f(x, y)| < L|y|$ for every x in $\alpha < x < \beta$ and $|y| > D$. This growth restriction will be easily improved in the Lemma 3.5. On the other hand when dealing with Eq. (1.11) this lemma is not necessary due to the fact that solutions are constrained in the horizontal strip $|y| < 1$.

Continuing our analysis of the phase-plane we can compare the slope (2.1) with the one of system (2.4), this will allow us to use the former ones to drive the trajectories of the second one. We eventually conclude that if $f(x, y) > f(x)$ in $xy > 0$, then trajectories of system (1.10) "enter" trajectories of system (1.5), while in $xy < 0$ we have the opposite situation. More precisely we mean that once a trajectory of system (1.10) transversally crosses a given orbit of system (1.5), then the former will never intersect the latter again and so it will remain constrained in a region bounded by the second orbit. This property can be used in order to prove the intersection of the trajectories with the x-axis for $|x|$ large enough, in fact orbits of the system (1.10) will be guided by the ones of system (1.5), and this will be used for finding suitable conditions in order to prove that trajectories of system (1.10) turn clockwise.

When $f(x, y)$ is identically to zero, the system (1.10) becomes the *relativistic Duffing* system, namely

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1+y^2}} \\ \dot{y} = -g(x), \end{cases} \tag{2.5}$$

while system (1.11) becomes the *prescribed curvature Duffing* system, namely

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1-y^2}} \\ \dot{y} = -g(x). \end{cases} \tag{2.6}$$

The phase-portraits of systems (2.5) and (2.6) has been previously studied in [9, 10], however we decided to hereby present a short analysis because this will determine a crucial step in order to study the phase-portrait of systems (1.10) and (1.11).

The first observation is that both systems (2.5) and (2.6) have a Hamiltonian structure for a suitable function H

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y}(x, y) \\ \dot{y} = -\frac{\partial H}{\partial x}(x, y); \end{cases} \tag{2.7}$$

for system (2.5) we should use

$$H_r(x, y) = \sqrt{1 + y^2} - 1 + G(x), \tag{2.8}$$

while for system (2.6)

$$H_c(x,y) = -\sqrt{1-y^2} + 1 + G(x)\,, \tag{2.9}$$

where in both cases $G(x)$ is the integral of $g(x)$, $G(x) = \int_0^x g(s)\,ds$.

Notice that in the original case of the Liénard system we would have $H(x,y) = \frac{1}{2}y^2 + G(x)$; despite the difference, all these Hamiltonian functions will play the same role.

Moreover, it is possible to show that the origin is a global centre for both systems (2.5) and (2.6) if and only if $G(x)$ diverges to infinity for $|x| \to \infty$, exactly as in the classical case, while if $G(x)$ is bounded the origin is a local centre. We observe that $H_r(x,y)$ and $H_c(x,y)$ may be viewed as an energy first integral for both the modified Duffing systems (see [5] for a discussion of the energy function in the classical case).

Taking $H_r(x,y)$, respectively $H_c(x,y)$, as a Lyapunov function for systems (1.10), respectively (1.11), we obtain, for its time-derivative along the trajectories the relations

$$\frac{dH_r}{dt}(x,y) = \frac{y}{\sqrt{1+y^2}}\left(-f(x,y)\frac{y}{\sqrt{1+y^2}} - g(x)\right) + g(x)\frac{y}{\sqrt{1+y^2}} = -f(x,y)\frac{y^2}{1+y^2}\,, \tag{2.10}$$

for the system (1.10), while

$$\frac{dH_c}{dt} = \frac{y}{\sqrt{1-y^2}}\left(-f(x,y)\frac{y}{\sqrt{1-y^2}} - g(x)\right) + g(x)\frac{y}{\sqrt{1-y^2}} = -f(x,y)\frac{y^2}{1-y^2}\,, \tag{2.11}$$

in the case of system (1.11). Therefore, when $f(x,y)$ is positive the trajectories of systems (1.10) and (1.11) enter trajectories of systems (2.5) and (2.6) respectively, while when $f(x,y)$ is negative trajectories of system of systems (1.10) and (1.11) exit trajectories of systems (2.5) and (2.6). This behaviour will be crucial in the following and as a first result we get that if $f(0,0) < 0$ then the origin is a source for both systems (1.10) and (1.11). Let us observe that the latter result holds true even if $f(0,0) = 0$ provided that $f(x,y) < 0$ sufficiently close to the origin.

3 Main results

At this point we are able to present a first result, holding for both systems (1.10) and (1.11), which provides examples of a system with a prescribed number of periodic solutions in the light of the classical Poincaré example (the interested reader can consult the § 3.3 of the survey [8]), moreover it shows the flexibility given by the fact that $f(x,y)$ depends on two variables.

Theorem 3.1. *Consider system* (1.10)

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1+y^2}} \\ \dot{y} = -f(x,y)\frac{y}{\sqrt{1+y^2}} - g(x)\,, \end{cases}$$

let us assume $G(x) \to \infty$ for $|x| \to \infty$, and let for any fixed integer n and positive increasing sequence c_k

$$f(x,y) = \prod_{k=1}^{n} (H_r(x,y) - c_k)\,. \tag{3.1}$$

Then the system (1.10) exhibits exactly n limit cycles. Moreover their sizes become arbitrarily large as n increases.

On the other hand if G is bounded, namely $G(x) \leq K$, where $K < \min\{G(-\infty), G(+\infty)\}$, let

$$f(x,y) = \prod_{k=1}^{n} \left(H_r(x,y) - Kd_k\right), \tag{3.2}$$

for any positive decreasing sequence d_k, $d_1 < 1$, then the system (1.10) admits again exactly n limit cycles.

Proof. We observe that for unbounded G, once $H_r(x,y) = c_k$ (or Kd_k in the case of bounded G), the system (1.10) reduces to the Duffing one (2.4) and it clearly exhibits closed trajectories. The rest of the proof follows straightforwardly using the signs discussion previously done. □

Remark 3.2. A similar result holds for system (1.11). Namely if G is unbounded or bounded by some $K' > 1$ and one chooses

$$f(x,y) = \prod_{k=1}^{n} \left(H_c(x,y) - e_k\right), \tag{3.3}$$

for any positive decreasing sequence e_k, $e_1 < 1$ then the system has n limit cycles. Conversely if $G \leq K' \leq 1$, with $K' < \min\{G(-\infty), G(+\infty)\}$, one can take

$$f(x,y) = \prod_{k=1}^{n} \left(H_c(x,y) - K'e_k\right), \tag{3.4}$$

and again the system exhibits n limit cycles.

The latter result looks more interesting, in fact it allows to prove the existence of limit cycles without the need for a small parameter as done in Eq. (1.7) to force the solutions to remain into the horizontal strip.

At this point in order to get the existence of a limit cycle, it is necessary to create a winding trajectory, being the origin a source because of the assumptions made above. Therefore we consider again systems (1.10) and (1.11) and present our main results.

3.1 The relativistic case

Theorem 3.3. *Consider system (1.10) and let us assume the regularities conditions on $f(x,y)$ and $g(x)$ given in Section 2. If $f(0,0) < 0$, $G(x) \to \infty$ for $|x| \to \infty$, and there exist smooth functions $\phi(x) > 0$ and $\psi(x) < 0$ such that:*

$$f(x,\phi(x))\phi(x) > -\phi'(x)\phi(x) - g(x)\sqrt{1 + [\phi(x)]^2} \qquad \forall x \leq \alpha < 0, \tag{3.5}$$

and

$$f(x,\psi(x))\psi(x) < -\psi'(x)\psi(x) - g(x)\sqrt{1 + [\psi(x)]^2} \qquad \forall x \geq \beta > 0. \tag{3.6}$$

There exists a positive function $T(x)$ such that

$$|f(x,y)| \leq LT(x)|y| \quad \forall x \in [\alpha, \beta] \text{ and } |y| \geq D > 0. \tag{3.7}$$

Assume then $f(x,y) > 0$ for $x > \beta$ and $y > 0$ and $x < \alpha$ and $y < 0$.

Then the system (1.10) exhibits at least one stable limit cycle.

Let us present two preliminary lemmas, upon which the proof of our theorem will be based. We start with the following lemma which is based on previous results [1,2,13,14].

Lemma 3.4. *Let us assume there exist* $\alpha < 0$ *and* $\beta > 0$ *and two smooth functions* $\phi(x)$ *and* $\psi(x)$ *such that*

$$\phi(x) > 0 \quad \forall x \leq \alpha \text{ and } \psi(x) < 0 \quad \forall x \geq \beta\,.$$

Assume moreover

$$f(x,\phi(x))\phi(x) > -\phi'(x)\phi(x) - g(x)\sqrt{1+[\phi(x)]^2} \qquad \forall x \leq \alpha\,, \tag{3.8}$$

and

$$f(x,\psi(x))\psi(x) < -\psi'(x)\psi(x) - g(x)\sqrt{1+[\psi(x)]^2} \qquad \forall x \geq \beta\,. \tag{3.9}$$

Then the orbits enter the regions bounded by $y = \phi(x)$ *for* $x \leq \alpha$ *and* $y = \psi(x)$ *for* $x \geq \beta$ *(see Fig. 3.1).*

Proof. Let us consider the case involving $\phi(x)$ being the one for $\psi(x)$ similar. The slope of system (1.10) is given by

$$\frac{dy}{dx}(x) = -f(x,y) - g(x)\frac{\sqrt{1+y^2}}{y}\,, \tag{3.10}$$

and thus evaluated on the graph of the function $\phi(x)$ gives:

$$\begin{aligned} \frac{dy}{dx}(x)|_{y=\phi(x)} &= -f(x,\phi(x)) - g(x)\frac{\sqrt{1+[\phi(x)]^2}}{\phi(x)} \\ &= \frac{1}{\phi(x)}\left[-f(x,\phi(x))\phi(x) - g(x)\sqrt{1+[\phi(x)]^2}\right], \end{aligned} \tag{3.11}$$

where in the rightmost equality we factorised $\phi(x)$.

Recalling the assumption (3.8) and the positiveness of $\phi(x)$ for $x \leq \alpha$ we get:

$$\frac{dy}{dx}(x)|_{y=\phi(x)} < \phi'(x) \qquad \forall x \leq \alpha\,,$$

that is orbits starting on $y = \phi(x)$ will enter the region bounded by such curve if $x \leq \alpha$.

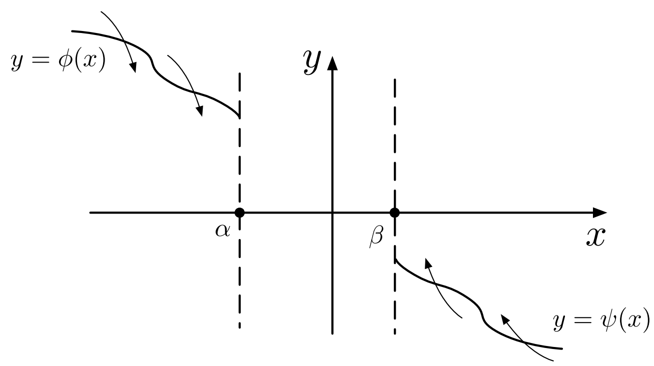

Figure 3.1: The curves $y = \phi(x)$ and $y = \psi(x)$ and their relation with the trajectories of the system (1.10).

□

Let us now emphasise some relevant cases obtained by specialising the functions ϕ and ψ:

- $\phi(x) = \psi(x) = -g(x)$, then the condition of the lemma becomes

$$f(x, -g(x)) > g'(x) + \sqrt{1 + [g(x)]^2} \qquad \forall x \leq \alpha < 0\,,$$

  and similarly for $x \geq \beta > 0$.

- $\phi(x) = \psi(x) = -x$, hence

$$f(x, -x) > 1 + \sqrt{1 + x^2} \qquad \forall x \leq \alpha\,,$$

  and similarly for $x \geq \beta > 0$.

- $\phi(x) = k > 0$, $\psi(x) = -k$, then

$$f(x, k) > -g(x)\sqrt{1 + k^2}/k \qquad \forall x \leq \alpha < 0\,,$$

  and

$$f(x, -k) > g(x)\sqrt{1 + k^2}/k \qquad \forall x \geq \beta > 0\,.$$

We are now presenting a lemma allowing us to avoid the presence of vertical asymptotes for the orbits.

Lemma 3.5. *Let us assume there exists a positive continuous function $T(x)$ and two positive constants L and D such that*

$$|f(x, y)| \leq LT(x)|y| \qquad \forall x \in [-M, M] \text{ and } |y| \geq D\,. \tag{3.12}$$

Then any orbit starting on the lines $x = \pm M$ cannot escape to infinity (i.e. no vertical asymptote is allowed).

Proof. Let us notice that using the bound (3.12) we can show that the orbit can be continued in the future starting from any $x_0 = -M$ and $y_0 \geq D$ up to $x = M$. □

We observe that this lemma is necessary once dealing with general $f(x, y)$ as in Eq. (1.10), on the contrary when such function is actually obtained by $\hat{f}(x, y/\sqrt{1 + y^2})$ the lemma is no longer necessary because the latter function is bounded for $|y| \to \infty$.

The last ingredient needed to show that orbits turn clockwise is the following lemmas.

Lemma 3.6. *Let us assume $G(x)$ is unbounded and $f(x, y) > 0$ for $x > b$, for some $b > 0$, then any trajectory starting on the vertical line (b, y_b), $y_b > 0$, will intersect the positive x-axis.*

Proof. The proof is straightforward; it is based on Eq. (2.10) and the properties of the Duffing system (2.5) which exhibits a global centre because of the assumption on $g(x)$. See Fig. 3.2 for a geometrical representation of such result. □

Let us note that assuming $f(x, y) > 0$ for $x < a$, for some $a < 0$. A similar result can be used to prove that any trajectory starting from (a, y_a), $y_a < 0$, will intersect the negative x-axis. We can now prove the Theorem 3.3

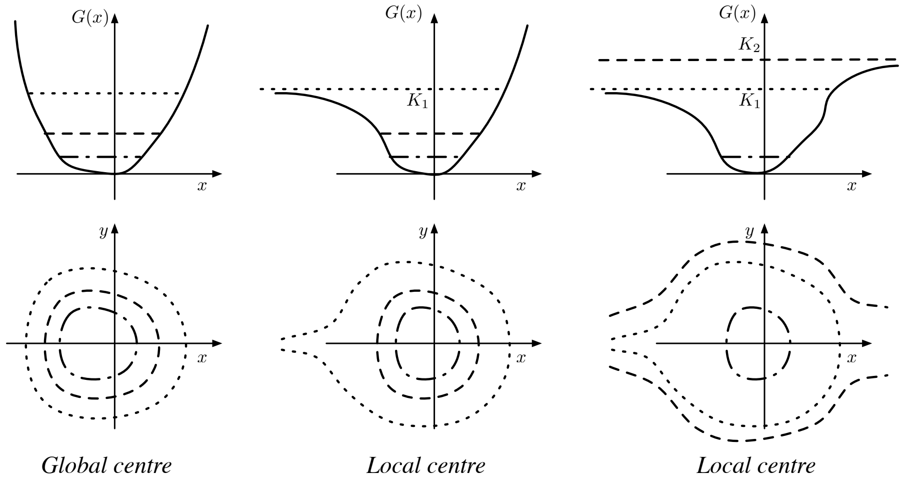

Figure 3.2: The global and local centre cases according to the behaviour of $G(x)$ for $|x| \to \infty$.

Proof. The reader can consult Fig. 3.3 to follow the progress of the proof. Let us consider a point $P(x_0, y_0)$, $x_0 < \alpha$ and $y_0 = \phi(x_0) > 0$, in virtue of Lemma 3.4 the trajectory originating from this point it is bounded away from the x-axis for negative t and it enters the graph of the function $\phi(x)$ and therefore it will intercept the line $x = \alpha$. From this point on, the trajectory can intersect the x-axis at some point in between $(0, \beta)$; on the contrary by virtue of Lemma 3.5 the trajectory will reach the line $x = \beta$ at some $y_1 > 0$. By Lemma 3.6 the trajectory will be guided by the trajectories of the Duffing system (2.5) and thus will reach the x-axis at some point $x > \beta$.

Using condition (3.9), again from Lemma 3.4, the trajectory will intersect the line $x = \beta$ at the point $\psi(\beta) < y_2 < 0$. As before, such trajectory can intersect either the negative x-axis in $(\alpha, 0)$, or it reaches twice the line $x = \alpha$ once for some negative y_3 and eventually for $0 < y_4 < \phi(\alpha)$. In both cases we produced a winding trajectory, which completes the proof by recalling that the origin is a source and using the Poincaré–Bendixson Theorem. □

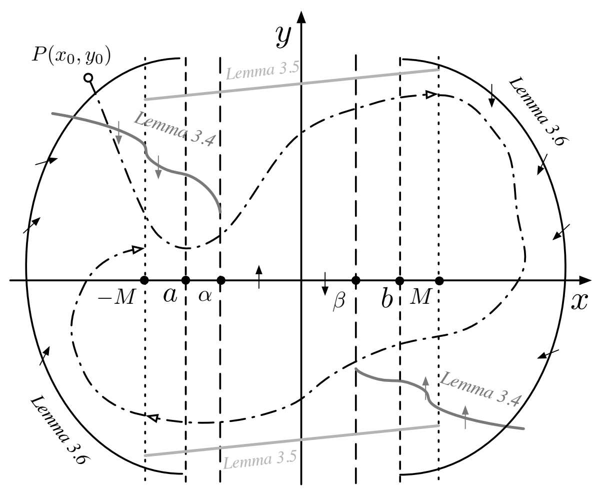

Figure 3.3: Scheme of the proof of the Theorem 3.3.

Let us observe that assuming both Eq. (3.5) and (3.6) is perhaps too restrictive. Indeed one can prove a similar result relaxing one of the above as the following result shows:

Theorem 3.7. *Consider system (1.10) and let us assume the regularities conditions on $f(x,y)$ and $g(x)$ given in Section 2. If $f(0,0) < 0$, $G(x) \to \infty$ for $|x| \to \infty$, and there exists a smooth function $\phi(x)$ such that:*

$$f(x,\phi(x))\phi(x) > -\phi'(x)\phi(x) - g(x)\sqrt{1+[\phi(x)]^2} \qquad \forall x \le \alpha\,, \tag{3.13}$$

moreover there exists a positive function $T(x)$ such that

$$|f(x,y)| \le LT(x)|y| \qquad \forall x \in [\alpha,\beta] \text{ and } |y| \ge D > 0\,. \tag{3.14}$$

Assume then $f(x,y) > 0$ for $x > \beta$ and every y. Then the system (1.10) exhibits a least 1 stable limit cycle

The proof is similar to the previous one and therefore it will be omitted. A similar result clearly holds if one assume Eq. (3.6) and $f(x,y) > 0$ for $x < \alpha$ and every y.

3.2 The prescribed curvature case

If we consider system (1.11) a lemma similar to 3.5 can be proved with the additional assumption that the orbit should be constrained in the strip $|y| < 1$; for the same reason, as previously mentioned, a result similar to Lemma 3.4 will not be interesting in this contest. However a result similar to the one provided by Theorem 3.3 holds true.

Theorem 3.8. *Consider system (1.11) and let us assume the regularities conditions on $f(x,y)$ and $g(x)$ given in Section 2. If $f(0,0) < 0$, $G(x) \to G_\infty \ge 1$ for $|x| \to \infty$ and there exists a smooth function $\phi(x)$ such that:*

$$f(x,\phi(x))\phi(x) > -\phi'(x)\phi(x) - g(x)\sqrt{1+[\phi(x)]^2} \quad \forall x \le \alpha\,, \tag{3.15}$$

and $\phi(\alpha) \le 1$. $f(x,y) > 0$ for $x < \alpha$ and all y. Then the system (1.11) exhibits at least one stable limit cycle provided $f(x,y)$ is sufficiently small for $x > \alpha$.

The proof is similar to the one of Theorem 3.7 and thus it will be omitted. A similar result can be stated using the function $\psi(x)$ with $\psi(\beta) \ge -1$. A slight different version of this result together with its dual version, will require a growth assumption on $f(x,y)$ only in the strip $(\alpha,\beta) \times \mathbb{R}$ and requiring $f(x,y) > 0$ also for $x > \beta$. However this result, even if supported by numerical evidence similar to the one of Fig. 7 [10], is less appealing for the heavy assumption on the growth of $f(x,y)$. Moreover, some restrictions on the growth of $G(x)$ seem necessary, as for instance $G(x) < 1$ for all x. A detailed discussion on this assumption may be found in [10].

4 Some examples

The aim of this section is to present some examples of application of the theory developed so far. The numerical results, in particular the 0-isocline and the orbits, have been realised using the *MATLAB* software [18], while the phase-space portraits using the open source *Field-Play* [19].

4.1 Using the 0-isocline for the functions $\phi(x)$and $\psi(x)$

Let us consider $f(x,y) = |y| \cos^2 y(x^2-1)$ and $g(x) = x$, then the system (1.10) rewrites

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1+y^2}} \\ \dot{y} = -\frac{y|y|(\cos y)^2(x^2-1)}{\sqrt{1+y^2}} - x\,, \end{cases} \tag{4.1}$$

whose slope is thus (see Eq. (2.1)) given by

$$y'(x) = -|y|(\cos y)^2(x^2-1) - x\frac{\sqrt{1+y^2}}{y}\,.$$

Finally the 0-isocline, $y_0(x)$, is implicitly given by:

$$(\cos y_0(x))^2 \frac{|y_0(x)|y_0(x)}{\sqrt{1+[y_0(x)]^2}} = -\frac{x}{x^2-1}\,. \tag{4.2}$$

This function is invariant with respect to the transformation $(x,y) \to (-x,-y)$ and thus it would be enough to study it for $y \geq 0$. For a sake of clarity let us define the functions

$$A(y) = (\cos y_0(x))^2 \frac{|y_0(x)|y_0(x)}{\sqrt{1+[y_0(x)]^2}} \quad \text{and} \quad B(x) = -\frac{x}{x^2-1}\,,$$

In the limit $x \to 0^+$, we have $B(x) \to 0^+$, hence to satisfy the equation $A(y) = B(x)$ we have $y_0 \to 0^+$ and $(\cos y_0)^2 \to 0$, that is $y_0 \to \pi/2 + k\pi$, for $k \in \mathbb{N} \cup \{0\}$. On the other hand once $x \to 0^-$, we get $B(x) \to 0^-$ and thus the only root of $A(y)$ is given by $y_0 \to 0^-$.

We then consider the case of large x. For $x \to +\infty$, $B(x) \to 0^-$ and thus the unique zero of $A(y)$ is $y_0 \to 0^-$. Finally, if $x \to -\infty$, $B(x) \to 0^+$, hence beside the zero $y_0 \to 0^+$ we have also the positive zeros of $(\cos y_0)^2$, that is $y_0 \to \pi/2 + k\pi$, for $k \in \mathbb{N} \cup \{0\}$.

We observe that $f(0,0) = 0$ but $f(x,y) < 0$ for $0 < x^2 < 1$ and thus one can prove that the origin is an unstable equilibrium. The function $G(x) = x^2/2$ and thus it is unbounded for $|x| \to \infty$. Taking $L = 1$ and $T(x) = 1 - x^2$ one can obtain $|f(x,y)| \leq LT(x)|y|$ for all $x \in [-1,1]$ and $y > 0$. Finally $f(x,y)$ is positive for $|x| > 1$. Observe that $\dot{x} > 0$ for $y > 0$ and thus the 0-isocline is traversed from left to right by the orbits, the latter can thus play the role of the function $\phi(x)$ because it prevents the orbits to grow.

The hypotheses of Theorem 3.3 are met and thus the system will exhibit at least one stable limit cycle. A numerical integration of (4.1) shows that indeed the limit cycle is unique (see Fig. 4.1). However we are not able to prove the uniqueness of the limit cycle, and at this stage this remains a conjecture.

4.2 Using a constant function for $\phi(x)$and $\psi(x)$

Let us consider $f(x,y) = (k^2 + x^2/2)(\sin y - 1/2) + x^2$, for some $k \in \mathbb{R}$, and $g(x) = x$, then the system (1.10) rewrites

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1+y^2}} \\ \dot{y} = -\frac{y}{\sqrt{1+y^2}} \left[\left(k^2 + \frac{x^2}{2}\right) \left(\sin y - \frac{1}{2}\right) + x^2 \right] - x\,. \end{cases} \tag{4.3}$$

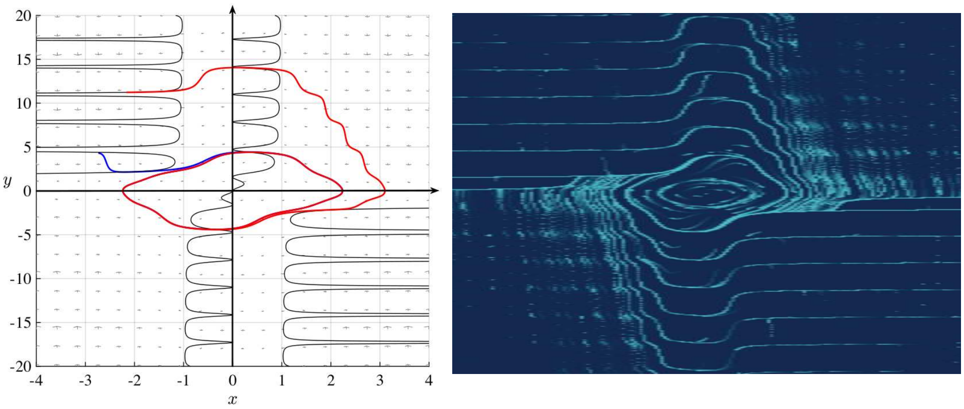

Figure 4.1: The phase-portrait of the system (1.10) with $f(x,y) = |y|\cos^2 y(x^2 - 1)$ and $g(x) = x$. On the left panel [18]: the black curves denote the different branches of the 0-isocline while the blue and red curves are two generic orbits of the system which accumulate on the (unique) stable limit cycle as provided by Theorem 3.3. On the right panel [19]: several orbits are traced to better appreciate a larger view of the phase-portrait; let us observe that the scales of the two figures are slightly different, this is the reason why the two limit cycles look different.

One has $f(0,0) = -k^2/2 < 0$, $|f(x,y)| \leq 3k^2/2 + 7x^2/4$ and thus Lemma 3.5 holds true, and finally $f(x,y) > 0$ for $|x|$ large enough, say $x^2 > 6k^2$. Once again $G(x) \to \infty$ for $|x| \to \infty$.

Let us finally assume $\phi(x) = a$ and $\psi(x) = -a$, where $a = \pi/2$, then conditions (3.5) and (3.6) are satisfied if

$$f(x,a) > -x\frac{\sqrt{1+a^2}}{a} \quad \text{and} \quad f(x,-a) > x\frac{\sqrt{1+a^2}}{a}\,.$$

A direct computation shows that if $k^2 > \frac{4}{5\pi}\sqrt{1+\frac{\pi^2}{4}}$, then the previous conditions are satisfied, indeed

$$f(x,a) = \frac{k^2}{2} + \frac{5}{4}x^2\,,$$

and with the chosen bound on k^2 we are sure that such parabola never intersects the line $y = -x\frac{\sqrt{1+a^2}}{a}$.

4.3 The curvature case

The last example concerns the prescribed curvature case. We will use that same function $f(x,y)$ of the first example, that is $f(x,y) = |y|(\cos 3y)^2(x^2-1)$. On the other hand for the $g(x)$ function we will use $g(x) = \mu x e^{-|x|}$, where $\mu < 1$, in such a way $\lim_{|x|\to\infty} G(x) = \mu < 1$ (see section 6 of [10]). Finally to constraint the orbits into the strip $|y| < 1$ we multiply $f(x,y)$ by a sufficiently small parameter λ, then the system (1.11) rewrites

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1-y^2}} \\ \dot{y} = -\lambda\frac{y|y|(\cos 3y)^2(x^2-1)}{\sqrt{1-y^2}} - \mu x e^{-|x|}\,, \end{cases} \tag{4.4}$$

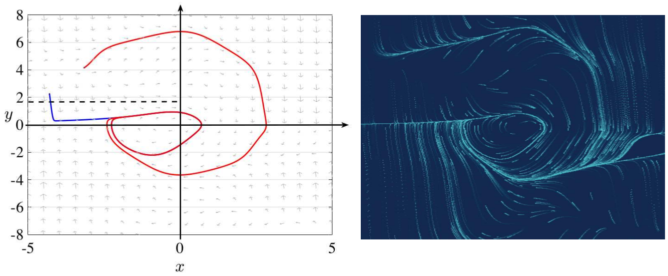

Figure 4.2: The phase-portrait of the system (1.10) with $f(x,y) = (1 + x^2/2)(\sin y - 1/2) + x^2$ and $g(x) = x$. On the left panel [18]: the dashed black line denotes the function $\phi(x) = \pi/2$, while the blue and red curves are two generic orbits of the system which accumulate on the stable limit cycle as provided by Theorem 3.3. On the right panel [19]: several orbits are traced to better appreciate a larger view of the phase-portrait.

whose slope is (see Eq. (2.2)):

$$y'(x) = -\lambda|y|(\cos 3y)^2(x^2-1) - \mu x e^{-|x|}\frac{\sqrt{1-y^2}}{y}\,. \tag{4.5}$$

Finally the 0-isocline, $y_0(x)$, is implicitly given by:

$$\lambda(\cos 3y_0(x))^2\frac{|y_0(x)|y_0(x)}{\sqrt{1-[y_0(x)]^2}} = -\frac{\mu x e^{-|x|}}{x^2-1}\,. \tag{4.6}$$

The study of the 0-isocline is very similar to the one done in Section 4.1 the main differences being now that not all the zeros of $(\cos 3y)^2$ should be taken into account because, only those falling inside the strip $|y| < 1$ do matter; secondly the positive-y branch stops at $(x,y) = (-1,1)$ and the negative one at $(x,y) = (1,-1)$ (see Fig. 4.3 for a numerical example). One can use again the 0-isocline as function $\phi(x)$ and $\psi(x)$ to apply the Theorem 3.8.

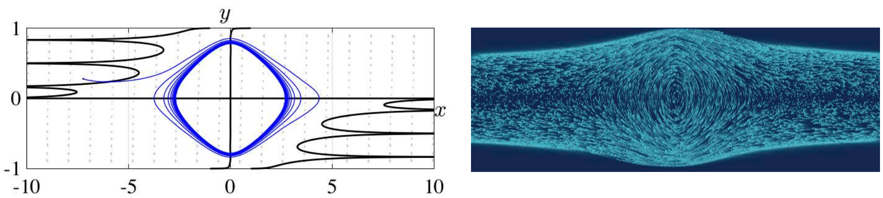

Figure 4.3: The phase-portrait of the system (4.4) with $f(x,y) = |y|(\cos(3y))^2(x^2-1)$, $g(x) = \mu x e^{-|x|}$, $\lambda = 0.01$ and $\mu = 1/2$. On the left panel [18]: the solid black lines denote the branches of the 0-isocline while the blue curve a generic orbit of the system which accumulate on the (unique) stable limit cycle as provided by Theorem 3.8. On the right panel [19]: several orbits are traced to better appreciate a larger view of the phase-portrait; let us observe that because of the small parameter λ the system is very close to the Duffing one which exhibits a global centre (in the strip $|y| < 1$), therefore it is hard to visualise the limit cycle for the system we are considering.

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Optimal control problem for 3D micropolar fluid equations

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Abstract. In this paper we study an optimal control problem related to strong solutions of 3D micropolar fluid equations. We deduce the existence of a global optimal solution with distributed control and, using a Lagrange multipliers theorem, we derive first-order optimality conditions for local optimal solutions.

Keywords: micropolar fluid equations, optimal control, optimality conditions.

2010 Mathematics Subject Classification: 49J20, 76D05, 76D55.

1 Introduction

The Navier–Stokes equations are a widely accepted model for the behavior of viscous incompressible fluids in the presence of convection. However, the classical Navier–Stokes theory is incapable of describing some physical phenomena for a class of fluids which exhibit certain microscopic effects arising from the local structure and micro-motions of the fluid elements. A subclass of these fluids is the micropolar fluids, which exhibit micro-rotational effects and micro-rotational inertia. Animal blood, liquid crystals, and certain polymeric fluids are a few examples of fluids which may be represented by the mathematical model of micropolar fluids, so that it is interesting to study the behavior of such fluids. The mathematical model that describes the movement of these fluids has been introduced by Eringen in [7] (see, also [6]). In this work we consider an optimal control problem restricted by the 3D micropolar fluid equations in which a distributed control acts on linear momentum as external source on the domain. Specifically, we consider $\Omega \subset \mathbb{R}^3$ be an open bounded domain with smooth boundary $\partial\Omega$ and $(0,T)$ a time interval, with $T > 0$. Then we study an optimal control problem related to the following system in the space-time domain $Q := \Omega \times (0,T)$

$$\begin{cases} \partial_t \mathbf{u} - (\nu + \nu_r)\Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 2\nu_r \operatorname{curl} \mathbf{w} + \mathbf{f}, \\ \partial_t \mathbf{w} - (c_a + c_d)\Delta \mathbf{w} + (\mathbf{u} \cdot \nabla)\mathbf{w} - (c_0 + c_d - c_a)\nabla \operatorname{div} \mathbf{w} + 4\nu_r \mathbf{w} = 2\nu_r \operatorname{curl} \mathbf{u} + \mathbf{g}, \\ \operatorname{div} \mathbf{u} = 0, \end{cases} \tag{1.1}$$

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where the unknowns are the linear velocity $\mathbf{u} = \mathbf{u}(x,t) \in \mathbb{R}^3$, the velocity of rotation of the particles $\mathbf{w} = \mathbf{w}(x,t) \in \mathbb{R}^3$ and the pressure $p = p(x,t) \in \mathbb{R}$. The functions $\mathbf{f}$ and $\mathbf{g}$ are given, and represent external sources of linear and angular momentum of particles, respectively. The positive real constant ν, ν_r, c_0, c_a and c_d characterize isotropic properties of the fluid; in particular, ν is the usual kinematic viscosity and ν_r, c_0, c_a and c_d are new viscosities related to the asymmetry of the stress tensor. These constants satisfy $c_0 + c_d > c_a$. For simplicity we denote $\nu_1 = \nu + \nu_r$, $\nu_2 = c_a + c_d$ and $\nu_3 = c_0 + c_d - c_a$. Without loss generality we can assume that density of the fluid is equal to one. The symbols Δ, ∇, curl and div denote the Laplacian, gradient, rotational and divergence operators, respectively; $\partial_t \mathbf{u}$ and $\partial_t \mathbf{w}$ stand for the time derivatives of $\mathbf{u}$ and $\mathbf{w}$, respectively. The i-th components of $(\mathbf{u} \cdot \nabla)\mathbf{u}$ and $(\mathbf{u} \cdot \nabla)\mathbf{w}$ are respectively given by

$$[(\mathbf{u} \cdot \nabla)\mathbf{u}]_i = \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} \quad \text{and} \quad [(\mathbf{u} \cdot \nabla)\mathbf{w}]_i = \sum_{j=1}^{3} u_j \frac{\partial w_i}{\partial x_j}.$$

When the microrotation viscous effects are not considered, that is, $\nu_r = 0$, or $\mathbf{w} = \mathbf{0}$, model (1.1) reduces to the well known incompressible Navier–Stokes system, which have been greatly studied (see, for instance, the classical text books [17], [18] and [31]).

We complete system (1.1) with initial conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \mathbf{w}(x,0) = \mathbf{w}_0(x) \quad \text{in } \Omega \tag{1.2}$$

and boundary conditions

$$\mathbf{u} = \mathbf{0}, \quad \mathbf{w} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,T). \tag{1.3}$$

From the mathematical point of view, the initial-value problem (1.1)–(1.3) has been studied by several authors, and important results on existence of weak solutions and local strong solutions, large time asymptotic behavior, regularity of solutions, and general qualitative analysis, have been obtained (see [1,8–11,20,26,27,33], for instance).

There is an extensive literature devoted to the study of optimal control problems related with the classical Navier–Stokes equations (see, for instance, [3–5,14–16,25,32] and references therein). As far as known, the literature related to optimal control problems for micropolar fluids is scarce. In [29], an optimal control problem associated with themotion of a micropolar fluid, with applications in the control of the blood pressure, was studied. In [30], in a two-dimensional domain, the relation between the microrotation and vorticity of the fluid was analyzed. Also, a boundary control problem for the stationary case with mixed boundary conditions, including a Navier slip condition on a part of the boundary for the velocity field, was studied in [22,23]. In [22], for three-dimensional flows with constant density is considered, while in [23], the 2D case with variable density is studied.

For two-dimensional flows, an existence and uniqueness theorem for a weak solution of (1.1)–(1.3) has been known for a long time (see [20]). The study for 3D domains is more complicated. Here we can distinguish two types of solutions: weak and *strong* solutions. Under minimal assumptions in the initial data and external forces $\mathbf{f}$ and $\mathbf{g}$ the existence of weak solutions for (1.1)–(1.3) can be proved; however, the uniqueness is an open question (this is similar to what happens with the 3D Navier–Stokes equations). The existence of weak solutions is not sufficient to carry out the study of the optimal control problem, due to the lack of regularity of weak solutions. Indeed, we cannot obtain first-order necessary optimality conditions. To overcome this, following the ideas of Casas [3] and Casas et al. [4], we consider a convenient cost functional. Instead of setting the L^2-norm of $\mathbf{u} - \mathbf{u}_d$ in the objective functional as usual, we consider the functional

$$J(\mathbf{u}, \mathbf{w}, \mathbf{f}) := \frac{\alpha}{6} \int_0^T \|\mathbf{u}(t) - \mathbf{u}_d(t)\|_{L^6}^6 dt + \frac{\beta}{2} \int_0^T \|\mathbf{w}(t) - \mathbf{w}_d(t)\|^2 dt + \frac{\gamma}{2} \int_0^T \|\mathbf{f}(t)\|^2 dt, \qquad (1.4)$$

where $\alpha > 0$, $\beta, \gamma \geq 0$, and the functions $\mathbf{u}_d$ and $\mathbf{w}_d$ to be fixed more precisely later. The objective is to minimize $J(\mathbf{u}, \mathbf{w}, \mathbf{f})$ in a certain set, with $(\mathbf{u}, \mathbf{w}, \mathbf{f})$ satisfying system (1.1)–(1.3). From Loayza and Rojas-Medar [19] we deduce that, if $(\mathbf{u}, \mathbf{w})$ is a weak solution of (1.1)–(1.3) such that $J(\mathbf{u}, \mathbf{w}, \mathbf{f}) < +\infty$, then the pair $(\mathbf{u}, \mathbf{w})$ is a strong solution. With this formulation we can prove the existence of an optimal solution and obtain first-order optimality conditions.

The paper is organized as follow: in Section 2 we fix the notation, introduce the functional spaces to be used and give the definition of weak and strong solutions for system (1.1)–(1.3). In Section 3 we establish the optimal control problem, proving the existence of a global optimal solution and we derive the first-order optimality conditions using a Lagrange multipliers theorem in Banach spaces. Finally, we improve the regularity of Lagrange multipliers.

2 Preliminaries

Through this paper, we will use the Lebesgue space $L^p(\Omega)$, $1 \leq p \leq +\infty$, with norm denoted by $\|\cdot\|_{L^p}$. In particular, the L^2-norm and its inner product will be denoted by $\|\cdot\|$ and $(\cdot, \cdot)$, respectively. We consider the standard Sobolev spaces $W^{m,p}(\Omega) = \{u \in L^p(\Omega) : \|\partial^\alpha u\|_{L^p} < +\infty, \forall |\alpha| \leq m\}$, with norm denoted by $\|\cdot\|_{W^{m,p}}$. When $p = 2$, we write $H^m(\Omega) := W^{m,2}(\Omega)$ and we denote the respective norm by $\|\cdot\|_{H^m}$. Corresponding functional spaces of vector-valued functions will be denoted by bold letter; for instance $\mathbf{H}^1(\Omega)$, $\mathbf{L}^2(\Omega)$, and so on. We will use the Hilbert space $\mathbf{H}_0^1(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega\}$, which is a Hilbert spaces with inner-product $(\mathbf{u}, \mathbf{v})_{\mathbf{H}_0^1} := (\nabla \mathbf{u}, \nabla \mathbf{v})$. Also, as usual we define $\mathcal{V} := \{\mathbf{u} \in \mathcal{C}_0^\infty(\Omega) : \operatorname{div} \mathbf{u} = 0\}$ and the spaces

$$\mathbf{H} := \text{ The closure of } \mathcal{V} \text{ in } \mathbf{L}^2(\Omega), \qquad \mathbf{V} := \text{ The closure of } \mathcal{V} \text{ in } \mathbf{H}^1(\Omega).$$

The spaces $\mathbf{H}$ and $\mathbf{V}$ are characterized by (see [31]):

$$\mathbf{H} = \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \operatorname{div} \mathbf{u} = 0 \text{ and } \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$
$$\mathbf{V} = \{\mathbf{u} \in \mathbf{H}_0^1(\Omega) : \operatorname{div} \mathbf{u} = 0\},$$

where $\mathbf{n}$ denotes the outward unit normal vector to $\partial\Omega$. If X is a Banach space, we denote by $L^p(0, T; X)$ the space of valued functions in X defined on the interval $[0, T]$ that are integrable in the Bochner sense, and its norm will denoted by $\|\cdot\|_{L^p(X)}$. For simplicity, we will denotes $L^p(Q) := L^p(0, T; L^p(\Omega))$ for $p \neq \infty$ and its norm by $\|\cdot\|_{L^p(Q)}$. In the case $p = +\infty$, $L^\infty(Q) := L^\infty(\Omega \times (0, T))$ and its respective norm will denoted by $\|\cdot\|_{L^\infty(Q)}$. Also, we denote by $C([0; T]; X)$ the space of continuous functions from $[0, T]$ into a Banach space X, and its norm by $\|\cdot\|_{C(X)}$. The topological dual space of a Banach space X will be denoted by X', and the duality for a pair X and X' by $\langle \cdot, \cdot \rangle_{X'}$ or simply by $\langle \cdot, \cdot \rangle$ unless this leads to ambiguity. In particular $\mathbf{V}'$ is the dual space of $\mathbf{V}$ and the space $\mathbf{H}^{-1}(\Omega)$ denotes the dual of $\mathbf{H}_0^1(\Omega)$. Moreover, the letters C, K, C_1, $K_1, \ldots$, are positive constants, independent of state $(\mathbf{u}, \mathbf{w})$ and control $\mathbf{f}$, but its value may change from line to line.

Now, we give the concept of weak solutions of system (1.1)–(1.3).

Definition 2.1 (Weak solutions)**.** Let $(\mathbf{f},\mathbf{g}) \in L^2(Q) \times L^2(Q)$ and $(\mathbf{u}_0,\mathbf{w}_0) \in \mathbf{H} \times \mathbf{L}^2(\Omega)$. A weak solution of (1.1)-(1.3) is a pair $(\mathbf{u},\mathbf{w})$ such that

$$\mathbf{u} \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V}), \quad \partial_t \mathbf{u} \in L^2(0,T;\mathbf{V}'), \tag{2.1}$$

$$\mathbf{w} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}_0^1(\Omega)), \quad \partial_t \mathbf{w} \in L^2(0,T;\mathbf{H}^{-1}(\Omega)), \tag{2.2}$$

and satisfies the following weak formulation

$$\begin{aligned}
&\int_0^T \langle \partial_t \mathbf{u}, \mathbf{v}\rangle + \nu_1 \int_0^T (\nabla \mathbf{u}, \nabla \mathbf{v}) + \int_0^T ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}) \\
&\quad = 2\nu_r \int_0^T (\operatorname{curl}\mathbf{w}, \mathbf{v}) + \int_0^T (\mathbf{f},\mathbf{v}) \qquad \forall \mathbf{v} \in L^2(0,T;\mathbf{V}), \qquad (2.3)\\
&\int_0^T \langle \partial_t \mathbf{w}, \mathbf{z}\rangle + \nu_2 \int_0^T (\nabla \mathbf{w}, \nabla \mathbf{z}) + \int_0^T ((\mathbf{u}\cdot\nabla)\mathbf{w}, \mathbf{z}) + \nu_3 \int_0^T (\operatorname{div}\mathbf{w}, \operatorname{div}\mathbf{z}) + 4\nu_r \int_0^T (\mathbf{w},\mathbf{z}) \\
&\quad = 2\nu_r \int_0^T (\operatorname{curl}\mathbf{u}, \mathbf{z}) + \int_0^T (\mathbf{g},\mathbf{z}) \qquad \forall \mathbf{z} \in L^2(0,T;\mathbf{H}_0^1(\Omega)), \qquad (2.4)\\
&\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{w}(0) = \mathbf{w}_0 \quad \text{in } \Omega, \qquad (2.5)\\
&\quad \mathbf{u} = \mathbf{w} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,T). \qquad (2.6)
\end{aligned}$$

Remark 2.2. We consider the usual Stokes operator $A := -P\Delta$ with domain $D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{V}$, where $P : \mathbf{L}^2(\Omega) \to \mathbf{H}$ is the Leray projector, and the strongly elliptic operator $L := -\nu_2\Delta - \nu_3\nabla\text{div}$ with domain $D(L) = \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$, then system (1.1)–(1.3) can be rewritten as follows

$$\begin{cases}
\partial_t \mathbf{u} + \nu A\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = 2\nu_r \operatorname{curl}\mathbf{w} + P\mathbf{f} \text{ in } Q,\\
\partial_t \mathbf{w} + L\mathbf{w} + (\mathbf{u}\cdot\nabla)\mathbf{w} + 4\nu_r\mathbf{w} = 2\nu_r \operatorname{curl}\mathbf{u} + \mathbf{g} \text{ in } Q,\\
\operatorname{div}\mathbf{u} = 0 \text{ in } Q,\\
\mathbf{u}(x,0) = \mathbf{u}_0(x), \ \mathbf{w}(x,0) = \mathbf{w}_0(x) \text{ in } \Omega,\\
\mathbf{u} = \mathbf{0}, \ \mathbf{w} = \mathbf{0} \text{ on } \partial\Omega \times (0,T).
\end{cases} \tag{2.7}$$

Thus, we have the following equivalent formulation of weak solutions of system (1.1)–(1.3).

Definition 2.3. Let $(\mathbf{f},\mathbf{g}) \in L^2(Q) \times L^2(Q)$ and $(\mathbf{u}_0,\mathbf{w}_0) \in \mathbf{H} \times \mathbf{L}^2(\Omega)$. Find a pair $(\mathbf{u},\mathbf{w})$ such that

$$\mathbf{u} \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V}), \quad \partial_t \mathbf{u} \in L^2(0,T;\mathbf{V}'), \tag{2.8}$$

$$\mathbf{w} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}_0^1(\Omega)), \quad \partial_t \mathbf{w} \in L^2(0,T;\mathbf{H}^{-1}(\Omega)), \tag{2.9}$$

and satisfies the system

$$\begin{cases}
\partial_t \mathbf{u} + \nu A\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = 2\nu_r \operatorname{curl}\mathbf{w} + P\mathbf{f} \text{ in } D(A)',\\
\partial_t \mathbf{w} + L\mathbf{w} + (\mathbf{u}\cdot\nabla)\mathbf{w} + 4\nu_r\mathbf{w} = 2\nu_r \operatorname{curl}\mathbf{u} + \mathbf{g} \text{ in } D(L)',\\
\mathbf{u}(x,0) = \mathbf{u}_0(x) \in \mathbf{H},\\
\mathbf{w}(x,0) = \mathbf{w}_0(x) \text{ in } \mathbf{L}^2(\Omega),\\
\mathbf{u} = \mathbf{0}, \ \mathbf{w} = \mathbf{0} \text{ on } \partial\Omega \times (0,T).
\end{cases} \tag{2.10}$$

We are interested in studying an optimal control problem related the strong solutions of system (1.1)–(1.3), the following definition is given in this sense.

Definition 2.4 (Strong solutions)**.** Let $(\mathbf{f}, \mathbf{g}) \in L^2(Q) \times L^2(Q)$ and $(\mathbf{u}_0, \mathbf{w}_0) \in \mathbf{V} \times \mathbf{H}_0^1(\Omega)$. We say that $(\mathbf{u}, \mathbf{w})$ is a strong solution of system (1.1)–(1.3) in $(0, T)$ if

$$\mathbf{u} \in \mathbf{X}_{\mathbf{u}} := \{\mathbf{u} \in L^\infty(0,T;\mathbf{V}) \cap L^2(0,T;\mathbf{H}^2(\Omega)) \,:\, \partial_t \mathbf{u} \in L^2(Q)\}, \tag{2.11}$$

$$\mathbf{w} \in \mathbf{X}_{\mathbf{w}} := \{\mathbf{w} \in L^\infty(0,T;\mathbf{H}_0^1(\Omega)) \cap L^2(0,T;\mathbf{H}^2(\Omega)) \,:\, \partial_t \mathbf{w} \in L^2(Q)\}, \tag{2.12}$$

and satisfies

$$\begin{cases} \partial_t \mathbf{u} + \nu A\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = 2\nu_r \mathrm{curl}\, \mathbf{w} + \mathbf{f} \text{ in } L^2(Q), \\ \partial_t \mathbf{w} + L\mathbf{w} + (\mathbf{u}\cdot\nabla)\mathbf{w} + 4\nu_r \mathbf{w} = 2\nu_r \mathrm{curl}\, \mathbf{u} + \mathbf{g} \text{ in } L^2(Q), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \in \mathbf{V}, \\ \mathbf{w}(x,0) = \mathbf{w}_0(x) \text{ in } \mathbf{H}_0^1(\Omega), \\ \mathbf{u} = \mathbf{0},\ \mathbf{w} = \mathbf{0} \text{ on } \partial\Omega \times (0,T). \end{cases} \tag{2.13}$$

The following result is a criterion of regularity that allows us to obtain a strong solution of system (1.1)–(1.3), the proof can be consulted in [19].

Theorem 2.5. *Let $(\mathbf{u}, \mathbf{w})$ be a weak solution of (1.1)–(1.3). If, in addition, the initial data $(\mathbf{u}_0, \mathbf{w}_0)$ belongs to $\mathbf{V} \times \mathbf{H}_0^1(\Omega)$ and*

$$\mathbf{u} \in L^4(0,T;\mathbf{L}^6(\Omega)), \tag{2.14}$$

then $(\mathbf{u}, \mathbf{w})$ is a strong solution of (1.1)–(1.3).

Moreover, there exists a positive constant $K := K(\|\mathbf{u}_0\|_{\mathbf{V}}, \|\mathbf{w}_0\|_{\mathbf{H}_0^1}, \|\mathbf{f}\|_{L^2(Q)}, \|\mathbf{g}\|_{L^2(Q)})$ such that

$$\|(\mathbf{u}, \mathbf{w})\|_{\mathbf{X}_{\mathbf{u}} \times \mathbf{X}_{\mathbf{w}}} \leq K. \tag{2.15}$$

3 The optimal control problem

In this section we establish the statement of control problem. We formulate the control problem un such way a that any admissible state is a strong solution of (1.1)–(1.3). Due to the is no existence result of strong solutions of (1.1)–(1.3), we have to choose a suitable objective functional.

We suppose that $\mathcal{U} \subset L^2(Q)$ is a nonempty, closed and convex set and we consider the initial data $\mathbf{u}_0 \in \mathbf{V}$, $\mathbf{w}_0 \in \mathbf{H}_0^1(\Omega)$, and the function $\mathbf{f} \in \mathcal{U}$ describing the distributed control on the linear momentum equation.

Now, we define the following constrained extremal problem related to PDE system (1.1)–(1.3):

$$\begin{cases} \text{Find } (\mathbf{u}, \mathbf{w}, \mathbf{f}) \in \mathbf{X}_{\mathbf{u}} \times \mathbf{X}_{\mathbf{w}} \times \mathcal{U} \text{ such that the functional} \\ J(\mathbf{u}, \mathbf{w}, \mathbf{f}) := \dfrac{\alpha}{6}\displaystyle\int_0^T \|\mathbf{u}(t) - \mathbf{u}_d(t)\|_{L^6}^6 dt + \dfrac{\beta}{2}\int_0^T \|\mathbf{w}(t) - \mathbf{w}_d(t)\|^2 dt + \dfrac{\gamma}{2}\int_0^T \|\mathbf{f}(t)\|^2 dt \\ \text{is minimized, subject to } (\mathbf{u}, \mathbf{w}, \mathbf{f}) \text{ be a strong solution of (1.1)–(1.3).} \end{cases} \tag{3.1}$$

Here $(\mathbf{u}_d, \mathbf{w}_d) \in L^{10}(Q) \times L^2(Q)$ represent the desires states (in the proof of Theorem 3.14 below is justified the fact that $\mathbf{u}_d \in L^{10}(Q)$) and the real numbers α, β and γ measure the cost of the states and control, respectively. These constants satisfy

$$\alpha > 0 \quad \text{and} \quad \beta, \gamma \geq 0.$$

The admissible set for the control problem (3.1) is defined by

$$\mathcal{S}_{ad} = \{\mathbf{s} = (\mathbf{u}, \mathbf{w}, \mathbf{f}) \in \mathbf{X}_{\mathbf{u}} \times \mathbf{X}_{\mathbf{w}} \times \mathcal{U} \,:\, \mathbf{s} \text{ is a strong solution of (1.1)–(1.3) in } (0,T)\}.$$

The functional J defined in (3.1) describes the deviation of the velocity of the fluid $\mathbf{u}$ and the microrotational velocity $\mathbf{w}$ from a desired velocity $\mathbf{u}_d$ and microrotational velocity $\mathbf{w}_d$ respectively, plus the control of the control measured in the L^2-norm.

Thus, we have the following definition.

Definition 3.1 (Optimal solution)**.** An element $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathcal{S}_{ad}$ will be called global optimal solution of problem (3.1) if

$$J(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) = \min_{(\mathbf{u},\mathbf{w},\mathbf{f})\in\mathcal{S}_{ad}} J(\mathbf{u}, \mathbf{w}, \mathbf{f}). \tag{3.2}$$

Remark 3.2. Notice that if $(\mathbf{u}, \mathbf{w})$ is a weak solution of (1.1)–(1.3) in $(0,T)$ such that $J(\mathbf{u}, \mathbf{w}, \mathbf{f}) < +\infty$, then, in particular $\mathbf{u} \in L^6(0,T;\mathbf{L}^4(\Omega))$; thus by Theorem 2.5 the pair $(\mathbf{u}, \mathbf{w})$ is a strong solution of (1.1)–(1.3) in $(0,T)$ (in sense of Definition 2.4). Due to there is no existence result of strong solutions, in what follows, we will assume that

$$\mathcal{S}_{ad} \neq \emptyset. \tag{3.3}$$

3.1 Existence of global optimal solution

In this subsection we will prove the existence of a global optimal solution of problem (3.1) in sense of Definition 3.1. Concretely, we will prove the following result.

Theorem 3.3. *Let $(\mathbf{u}_0, \mathbf{w}_0) \in \mathbf{V} \times \mathbf{H}_0^1(\Omega)$. We assume that either $\gamma > 0$ or $\mathcal{U}$ is bounded in $L^2(Q)$ and hypothesis (3.3), then the optimal control problem (3.1) has at least one global optimal solution $(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathcal{S}_{ad}$.*

Proof. From (3.3) the admissible set $\mathcal{S}_{ad} \neq \emptyset$. Since functional J is nonnegative, then is bounded below. Hence there exists the infimum over all the admissible elements $\mathbf{s} := (\mathbf{u}, \mathbf{w}, \mathbf{f})$ belongs to $\mathcal{S}_{ad}$; that is,

$$0 \leq \inf_{\mathbf{s}\in\mathcal{S}_{ad}} J(\mathbf{s}) < +\infty.$$

Then, by definition of the infimum, there exists a minimizing sequence

$$\{\mathbf{s}_m\}_{m\geq 1} := \{(\mathbf{u}_m, \mathbf{w}_m, \mathbf{f}_m)\}_{m\geq 1}$$

such that

$$\lim_{m\to+\infty} J(\mathbf{s}_m) = \inf_{\mathbf{s}\in\mathcal{S}_{ad}} J(\mathbf{s}).$$

From definition of $\mathcal{S}_{ad}$, for each $m \in \mathbb{N}$, $\mathbf{s}_m$ is a strong solution of (1.1)–(1.3), then by definition of J and the assumption $\gamma > 0$ or $\mathcal{U}$ is bounded in $L^2(Q)$ we deduce that

$$\{(\mathbf{u}_m, \mathbf{f}_m)\}_{m\geq 1} \text{ is bounded in } L^6(Q) \times L^2(Q). \tag{3.4}$$

Also, from estimate (2.15) (given in Theorem 2.5) there exists a positive constant, independent of m such that

$$\|(\mathbf{u}_m, \mathbf{w}_m)\|_{\mathbf{X}_{\mathbf{u}}\times\mathbf{X}_{\mathbf{w}}} \leq K. \tag{3.5}$$

Thus, from (3.4), (3.5), and using the fact that $\mathcal{U} \subset L^2(Q)$ is a closed and convex (then is weakly closed in $L^2(Q)$), we conclude that there exists an element $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathbf{X}_{\mathbf{u}} \times \mathbf{X}_{\mathbf{w}} \times \mathcal{U}$ such that, for some subsequence of $\{\mathbf{s}_n\}_{m\geq 1}$; which, for simplicity, still will denoted by $\{\mathbf{s}_m\}_{m\geq 1}$, the following convergences hold (as $m \to +\infty$):

$$\mathbf{u}_m \to \tilde{\mathbf{u}} \quad \text{weak in } L^2(0,T;\mathbf{H}^2(\Omega)) \text{ and weak* in } L^\infty(0,T;\mathbf{V}), \tag{3.6}$$

$$\mathbf{w}_m \to \tilde{\mathbf{w}} \quad \text{weak in } L^2(0,T;\mathbf{H}^2(\Omega)) \text{ and weak* in } L^\infty(0,T;\mathbf{H}_0^1(\Omega)), \tag{3.7}$$

$$\partial_t \mathbf{u}_m \to \partial_t \tilde{\mathbf{u}} \quad \text{weak in } L^2(Q), \tag{3.8}$$

$$\partial_t \mathbf{w}_m \to \partial_t \tilde{\mathbf{w}} \quad \text{weak in } L^2(Q), \tag{3.9}$$

$$\mathbf{f}_m \to \tilde{\mathbf{f}} \quad \text{weak in } L^2(Q). \tag{3.10}$$

Furthermore, from (3.6)–(3.9), the Aubin–Lions lemma (see [18, Théorème 5.1, p. 58]) and [28, Corollary 4], we deduce the strong convergences

$$\mathbf{u}_m \to \tilde{\mathbf{u}} \quad \text{in } L^2(0,T;\mathbf{H}^1(\Omega)) \cap C([0,T];\mathbf{L}^2(\Omega)), \tag{3.11}$$

$$\mathbf{w}_m \to \tilde{\mathbf{w}} \quad \text{in } L^2(0,T;\mathbf{H}^1(\Omega)) \cap C([0,T];\mathbf{L}^2(\Omega)). \tag{3.12}$$

From (3.11) and (3.12) we have that the pair $(\mathbf{u}_m(0), \mathbf{w}_m(0))$ converges to $(\tilde{\mathbf{u}}, \tilde{\mathbf{w}})$ in $\mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$, and since $\mathbf{u}_m(0) = \mathbf{u}_0$ and $\mathbf{w}_m(0) = \mathbf{w}_0$ we conclude that $(\tilde{\mathbf{u}}(0), \tilde{\mathbf{w}}(0)) = (\mathbf{u}_0, \mathbf{w}_0)$. Thus, the limit element $\tilde{\mathbf{s}}$ satisfies the initial conditions given in (1.2). The convergences (3.6)–(3.12), and a standard argument allow us to pass to the limit in system (2.3)–(2.6) written by $(\mathbf{u}_m, \mathbf{w}_m, \mathbf{f}_m)$, as m goes to $+\infty$; consequently we have that $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}})$ is a strong solution of (1.1)–(1.3), that is, $\tilde{\mathbf{s}}$ belongs to admissible set $\mathcal{S}_{ad}$. Therefore

$$\lim_{m\to+\infty} J(\mathbf{s}_m) = \inf_{\mathbf{s}\in\mathcal{S}_{ad}} J(\mathbf{s}) \leq J(\tilde{\mathbf{s}}). \tag{3.13}$$

Finally, taking into account that the functional J is weakly lower semicontinuous on $\mathcal{S}_{ad}$, we have

$$J(\tilde{\mathbf{s}}) \leq \liminf_{m\to+\infty} J(\mathbf{s}_m). \tag{3.14}$$

Therefore, from (3.13) and (3.14) we deduce (3.2), which implies that optimal control problem (3.1) has at least global optimal solution. □

3.2 Optimality system

In this subsection we will derive the first-order necessary optimality conditions for a local optimal solution $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}})$ of problem (3.1), using a Lagrange multiplier theorem in Banach spaces. We will base on a generic result given by Zowe et al. [34] (see, also [32, Chapter 6]). This method has been used by Guillén-González et al. [12,13] in the context of chemo-repulsion systems and in [21] for other models. In order to introduce the concepts and results given in [34] we consider the following extremal problem:

$$\min_{\mathbf{x}\in\mathcal{M}} J(\mathbf{x}) \text{ subject to } R(\mathbf{x}) = \mathbf{0}, \tag{3.15}$$

where $J : \mathbf{X} \to \mathbb{R}$ is a functional, $R : \mathbf{X} \to \mathbf{Y}$ is an operator, $\mathbf{X}$ and $\mathbf{Y}$ are Banach spaces, and $\mathcal{M} \subset \mathbf{X}$ is a nonempty, closed and convex set. The admissible set for problem (3.15) is given by

$$\mathcal{S} = \{\mathbf{x} \in \mathcal{M} \,:\, R(\mathbf{x}) = \mathbf{0}\}.$$

The so-called *Lagrangian functional* $\mathcal{L} : \mathbf{X} \times \mathbf{Y}' \to \mathbb{R}$ related to problem (3.15) is given by

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) := J(\mathbf{x}) - \langle \boldsymbol{\lambda}, R(\mathbf{x}) \rangle_{\mathbf{Y}'}. \tag{3.16}$$

Definition 3.4 (Lagrange multiplier)**.** Let $\tilde{\mathbf{x}} \in \mathcal{S}$ be a local optimal solution of (3.15). Suppose that J and R are Fréchet differentiable in $\tilde{\mathbf{x}}$, with derivatives denoted by $J'(\tilde{\mathbf{x}})$ and $R'(\tilde{\mathbf{x}})$, respectively. Then, $\boldsymbol{\lambda} \in \mathbf{Y}'$ is called Lagrange multiplier for problem (3.15) at the point $\tilde{\mathbf{x}}$ if

$$\begin{cases} \langle \boldsymbol{\lambda}, R(\tilde{\mathbf{x}}) \rangle_{\mathbf{Y}'} = 0, \\ \mathcal{L}'(\tilde{\mathbf{x}}, \boldsymbol{\lambda})[\mathbf{s}] := J'(\tilde{\mathbf{x}})[\mathbf{s}] - \langle \boldsymbol{\lambda}, R'(\tilde{\mathbf{x}})[\mathbf{s}] \rangle_{\mathbf{Y}'} \geq 0 \quad \forall \mathbf{s} \in \mathcal{C}(\tilde{\mathbf{x}}), \end{cases} \tag{3.17}$$

where $\mathcal{C}(\tilde{\mathbf{x}})$ is the conical hull of $\tilde{\mathbf{x}}$ in $\mathcal{M}$, that is, $\mathcal{C}(\tilde{\mathbf{x}}) = \{\theta(\mathbf{x} - \tilde{\mathbf{x}}) \, : \, \mathbf{x} \in \mathcal{M}, \ \theta \geq 0\}$.

Definition 3.5. Let $\tilde{\mathbf{x}} \in \mathcal{S}$ be a local optimal solution of problem (3.15). We say that $\tilde{\mathbf{x}}$ is a regular point if

$$R'(\tilde{\mathbf{x}})[\mathcal{C}(\tilde{\mathbf{x}})] = \mathbf{Y}. \tag{3.18}$$

The following result guarantees the existence of Lagrange multiplier for problem (3.15); the proof can be found in [34, Theorem 3.1] and [32, Theorem 6.3, p. 330].

Theorem 3.6. *Let $\tilde{\mathbf{x}} \in \mathcal{S}$ be a local optimal solution of problem* (3.15)*. Suppose that J is Fréchet differentiable in $\tilde{\mathbf{x}}$ and R is continuously Fréchet differentiable in $\tilde{\mathbf{x}}$. If $\tilde{\mathbf{x}}$ is a regular point, then the set of Lagrange multipliers for* (3.15) *at $\tilde{\mathbf{x}}$ is nonempty.*

Now, we will reformulate the optimal control problem (3.1) in the abstract setting (3.15). We consider the Banach spaces

$$\mathbf{X} := \widehat{\mathbf{X}}_{\mathbf{u}} \times \widehat{\mathbf{X}}_{\mathbf{w}} \times L^2(Q), \qquad \mathbf{Y} := L^2(Q) \times L^2(Q) \times \mathbf{V} \times \mathbf{H}_0^1(\Omega),$$

where

$$\widehat{\mathbf{X}}_{\mathbf{u}} := \{\mathbf{u} \in \mathbf{X}_{\mathbf{u}} \, : \, \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega \times (0,T)\}, \tag{3.19}$$
$$\widehat{\mathbf{X}}_{\mathbf{w}} := \{\mathbf{u} \in \mathbf{X}_{\mathbf{w}} \, : \, \mathbf{w} = \mathbf{0} \text{ on } \partial\Omega \times (0,T)\}, \tag{3.20}$$

and the operator $R = (R_1, R_2, R_3, R_4) : \mathbf{X} \to \mathbf{Y}$, where

$$R_1 : \mathbf{X} \to L^2(Q), \quad R_2(\mathbf{X}) \to L^2(Q), \quad R_3 : \mathbf{X} \to \mathbf{V}, \ R_4 : \mathbf{X} \to \mathbf{H}_0^1(\Omega)$$

are defined at each point $\mathbf{s} = (\mathbf{u}, \mathbf{w}, \mathbf{f}) \in \mathbf{X}$ by

$$\begin{cases} R_1(\mathbf{s}) = \partial_t \mathbf{u} + \nu A \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - 2\nu_r \operatorname{curl} \mathbf{w} - P\mathbf{f}, \\ R_2(\mathbf{s}) = \partial_t \mathbf{w} + L\mathbf{w} + (\mathbf{u} \cdot \nabla)\mathbf{w} + 4\nu_r \mathbf{w} - 2\nu_r \operatorname{curl} \mathbf{u} - \mathbf{g}, \\ R_3(\mathbf{s}) = \mathbf{u}(0) - \mathbf{u}_0, \\ R_4(\mathbf{s}) = \mathbf{w}(0) - \mathbf{w}_0. \end{cases} \tag{3.21}$$

Hence, the control problem (3.1) is reformulated as follows

$$\min_{\mathbf{s} \in \mathbf{M}} J(\mathbf{s}) \text{ subject to } R(\mathbf{s}) = \mathbf{0}. \tag{3.22}$$

Notice that $\mathbf{M} := \widehat{\mathbf{X}}_{\mathbf{u}} \times \widehat{\mathbf{X}}_{\mathbf{w}} \times \mathcal{U}$ is a closed convex subset of $\mathbf{X}$ and the admissible set is rewritten as follows

$$\mathcal{S}_{ad} = \{\mathbf{s} = (\mathbf{u}, \mathbf{w}, \mathbf{f}) \in \mathbf{M} \, : \, R(\mathbf{s}) = \mathbf{0}\}. \tag{3.23}$$

Concerning to differentiability of the functional J and constraint operator R we have the following lemmas.

Lemma 3.7. *The functional J is Fréchet differentiable and the Fréchet derivative of J in $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathbf{X}$ in the direction $\mathbf{t} = (\mathbf{U}, \mathbf{W}, \mathbf{F}) \in \mathbf{X}$ is given by*

$$J'(\tilde{\mathbf{s}})[\mathbf{t}] = \alpha \int_0^T \int_\Omega |\tilde{\mathbf{u}} - \mathbf{u}_d|^4 (\tilde{\mathbf{u}} - \mathbf{u}_d) \cdot \mathbf{U} + \beta \int_0^T \int_\Omega (\tilde{\mathbf{w}} - \mathbf{w}_d) \cdot \mathbf{W} + \gamma \int_0^T \int_\Omega \tilde{\mathbf{f}} \cdot \mathbf{F}. \tag{3.24}$$

Lemma 3.8. *The operator R is continuously-Fréchet differentiable and the Fréchet derivative of R in $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathbf{X}$, in the direction $\mathbf{t} = (\mathbf{U}, \mathbf{W}, \mathbf{F}) \in \mathbf{X}$, is the linear and bounded operator $R'(\tilde{\mathbf{s}})[\mathbf{t}] = (R'_1(\tilde{\mathbf{s}})[\mathbf{t}], R'_2(\tilde{\mathbf{s}})[\mathbf{t}], R'_3(\tilde{\mathbf{s}})[\mathbf{t}], R'_4(\tilde{\mathbf{s}})[\mathbf{t}])$ defined by*

$$\begin{cases} R'_1(\tilde{\mathbf{s}})[\mathbf{t}] = \partial_t \mathbf{U} + \nu A \mathbf{U} + (\mathbf{U} \cdot \nabla)\tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla)\mathbf{U} - 2\nu_r \operatorname{curl} \mathbf{W} - P\mathbf{F}, \\ R_2(\tilde{\mathbf{s}})[\mathbf{t}] = \partial_t \mathbf{W} + L\mathbf{W} + (\tilde{\mathbf{u}} \cdot \nabla)\mathbf{W} + (\mathbf{U} \cdot \nabla)\tilde{\mathbf{w}} + 4\nu_r \mathbf{W} - 2\nu_r \operatorname{curl} \mathbf{U}, \\ R'_3(\tilde{\mathbf{s}})[\mathbf{t}] = \mathbf{U}(0), \\ R'_4(\tilde{\mathbf{s}})[\mathbf{t}] = \mathbf{W}(0). \end{cases} \tag{3.25}$$

Remark 3.9. From Definition 3.5 we conclude that $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathcal{S}_{ad}$ is a regular point if given $(\mathbf{g}_\mathbf{u}, \mathbf{g}_\mathbf{w}, \mathbf{U}_0, \mathbf{W}_0) \in \mathbf{Y}$ there exists $\mathbf{t} = (\mathbf{U}, \mathbf{W}, \mathbf{F}) \in \widehat{\mathbf{X}}_\mathbf{u} \times \widehat{\mathbf{X}}_\mathbf{w} \times \mathcal{C}(\tilde{\mathbf{f}})$ such that

$$R'(\tilde{\mathbf{s}})[\mathbf{t}] = (\mathbf{g}_\mathbf{u}, \mathbf{g}_\mathbf{w}, \mathbf{U}_0, \mathbf{W}_0), \tag{3.26}$$

where $\mathcal{C}(\tilde{\mathbf{f}}) := \{\theta(\mathbf{f} - \tilde{\mathbf{f}}) \,:\, \theta \geq 0,\, \mathbf{f} \in \mathcal{U}\}$ is the conical hull of $\tilde{\mathbf{f}}$ in $\mathcal{U}$.

Lemma 3.10. *Let $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathcal{S}_{ad}$, then $\tilde{\mathbf{s}}$ is a regular point.*

Proof. Due to $\mathbf{0}$ belongs to $\mathcal{C}(\tilde{\mathbf{f}})$; then, given $(\mathbf{g}_\mathbf{u}, \mathbf{g}_\mathbf{w}, \mathbf{U}_0, \mathbf{W}_0) \in \mathbf{Y}$, it is sufficient to show the existence of $(\mathbf{U}, \mathbf{W}) \in \widehat{\mathbf{X}}_\mathbf{u} \times \widehat{\mathbf{X}}_\mathbf{w}$ such that

$$\begin{cases} \partial_t \mathbf{U} + \nu A \mathbf{U} + (\mathbf{U} \cdot \nabla)\tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla)\mathbf{U} - 2\nu_r \operatorname{curl} \mathbf{W} = \mathbf{g}_\mathbf{u} & \text{in } Q, \\ \partial_t \mathbf{W} + L\mathbf{W} + (\tilde{\mathbf{u}} \cdot \nabla)\mathbf{W} + (\mathbf{U} \cdot \nabla)\tilde{\mathbf{w}} + 4\nu_r \mathbf{W} - 2\nu_r \operatorname{curl} \mathbf{U} = \mathbf{g}_\mathbf{w} & \text{in } Q, \\ \mathbf{U}(0) = \mathbf{U}_0 & \text{in } \Omega, \\ \mathbf{W}(0) = \mathbf{W}_0 & \text{in } \Omega. \end{cases} \tag{3.27}$$

Since system (3.27) is a linear, we argue in a formal manner, proving that any regular enough solution is bounded in $\widehat{\mathbf{X}}_\mathbf{u} \times \widehat{\mathbf{X}}_\mathbf{w}$.

Testing in $(3.27)_1$ by $A\mathbf{U}$ we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{U}\|^2 + \nu_1 \|A\mathbf{U}\|^2 = & -((\mathbf{U} \cdot \nabla)\tilde{\mathbf{u}}, A\mathbf{U}) - ((\tilde{\mathbf{u}} \cdot \nabla)\mathbf{U}, A\mathbf{U}) \\ & + 2\nu_r(\operatorname{curl} \mathbf{W}, A\mathbf{U}) + (\mathbf{g}_\mathbf{u}, A\mathbf{U}). \end{aligned} \tag{3.28}$$

Now, we will bound the terms of right-side of (3.28). Using the Hölder, Poincaré and Young inequalities, and taking into account the continuous injection $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^q(\Omega)$ $(q \in [1, 6])$ we have

$$\begin{aligned} ((\mathbf{U} \cdot \nabla)\tilde{\mathbf{u}}, A\mathbf{U}) &\leq \|\mathbf{U}\|_{\mathbf{L}^3} \|\nabla \tilde{\mathbf{u}}\|_{\mathbf{L}^6} \|A\mathbf{U}\| \leq C\|\mathbf{U}\|_{\mathbf{H}^1} \|\nabla \tilde{\mathbf{u}}\|_{\mathbf{L}^6} \|A\mathbf{U} \leq C\|\nabla \mathbf{U}\| \|\nabla \tilde{\mathbf{u}}\|_{\mathbf{L}^6} \|A\mathbf{U}\| \\ &\leq \varepsilon \|A\mathbf{U}\|^2 + C_\varepsilon \|\nabla \tilde{\mathbf{u}}\|^2_{\mathbf{L}^6} \|\nabla \mathbf{U}\|^2. \end{aligned} \tag{3.29}$$

From the equivalence $\frac{1}{2\nu\sqrt{3}}\|A\mathbf{u}\| \leq \|\mathbf{u}\|_{\mathbf{H}^2} \leq C\|A\mathbf{u}\|$ (see [24, Lemma 3.1]) and the known interpolation inequality in 3D domains $\|\mathbf{u}\|_{\mathbf{L}^3} \leq C\|\mathbf{u}\|^{1/2}\|\nabla \mathbf{u}\|^{1/2}$, we obtain

$$\begin{aligned} |((\tilde{\mathbf{u}} \cdot \nabla)\mathbf{U}, A\mathbf{U})| &\leq \|\tilde{\mathbf{u}}\|_{\mathbf{L}^6} \|\nabla \mathbf{U}\|_{\mathbf{L}^3} \|A\mathbf{U}\| \leq C\|\tilde{\mathbf{u}}\|_{\mathbf{L}^6} \|\nabla \mathbf{U}\|^{1/2} \|\nabla \mathbf{U}\|^{1/2}_{\mathbf{H}^1} \|A\mathbf{U}\| \\ &\leq C\|\tilde{\mathbf{u}}\|_{\mathbf{L}^6} \|\nabla \mathbf{U}\|^{1/2} \|\mathbf{U}\|^{1/2}_{\mathbf{H}^2} \|A\mathbf{U}\| \leq C\|\tilde{\mathbf{u}}\|_{\mathbf{L}^6} \|\nabla \mathbf{U}\|^{1/2} \|A\mathbf{U}\|^{3/2} \\ &\leq \varepsilon \|A\mathbf{U}\|^2 + C_\varepsilon \|\tilde{\mathbf{u}}\|^4_{\mathbf{L}^6} \|\nabla \mathbf{U}\|. \end{aligned} \tag{3.30}$$

Again using the Hölder and Young inequalities, we have

$$\begin{aligned} 2\nu_r|(\mathrm{curl}\,\mathbf{W}, A\mathbf{U})| &\le 2\nu_r\|\mathrm{curl}\,\mathbf{W}\|\|A\mathbf{U}\| \le \varepsilon\|A\mathbf{U}\|^2 + C_\varepsilon\|\mathrm{curl}\,\mathbf{W}\|^2 \\ &\le \varepsilon\|A\mathbf{U}\|^2 + C_\varepsilon\|\nabla\mathbf{W}\|^2, \qquad (3.31) \\ |(\mathbf{g_u}, A\mathbf{U})| &\le \|\mathbf{g_u}\|\|A\mathbf{U}\| \le \varepsilon\|A\mathbf{U}\|^2 + C_\varepsilon\|\mathbf{g_u}\|^2. \qquad (3.32) \end{aligned}$$

Thus, replacing (3.29)–(3.32) in (3.28) and choosing ε suitably, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{U}\|^2 + C\|A\mathbf{U}\|^2 \le C\|\nabla\tilde{\mathbf{u}}\|^2_{\mathbf{L}^6}\|\nabla\mathbf{U}\|^2 + C\|\tilde{\mathbf{u}}\|^4_{\mathbf{L}^6}\|\nabla\mathbf{U}\|^2 + C\|\mathbf{g_u}\|^2 + C\|\nabla\mathbf{W}\|^2. \qquad (3.33)$$

Now, testing in $(3.27)_2$ by $-\Delta\mathbf{W}$ we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{W}\|^2 + \nu_2\|\Delta\mathbf{W}\|^2 + \nu_3(\nabla\mathrm{div}\,\mathbf{W}, \Delta\mathbf{W}) + 4\nu_r\|\nabla\mathbf{W}\|^2 \\ &\quad \le |((\tilde{\mathbf{u}}\cdot\nabla)\mathbf{W}, \Delta\mathbf{W})| + |((\mathbf{U}\cdot\nabla)\tilde{\mathbf{w}}, \Delta\mathbf{W})| + 2\nu_r|(\mathrm{curl}\,\mathbf{U}, \Delta\mathbf{W})| + |(\mathbf{g_w}, \Delta\mathbf{W})|. \qquad (3.34) \end{aligned}$$

Applying the Hölder and Young inequalities, we deduce

$$\begin{aligned} |((\tilde{\mathbf{u}}\cdot\nabla\mathbf{W}), \Delta\mathbf{W})| &\le \|\tilde{\mathbf{u}}\|_{\mathbf{L}^6}\|\nabla\mathbf{W}\|_{\mathbf{L}^3}\|\Delta\mathbf{W}\| \le C\|\tilde{\mathbf{u}}\|_{\mathbf{L}^6}\|\nabla\mathbf{W}\|^{1/2}\|\Delta\mathbf{W}\|^{3/2} \\ &\le \varepsilon\|\Delta\mathbf{W}\|^2 + C_\varepsilon\|\tilde{\mathbf{u}}\|^4_{\mathbf{L}^6}\|\nabla\mathbf{W}\|^2, \qquad (3.35) \\ |((\mathbf{U}\cdot\nabla)\tilde{\mathbf{w}}, \Delta\mathbf{W})| &\le \|\mathbf{U}\|_{\mathbf{L}^3}\|\nabla\tilde{\mathbf{w}}\|_{\mathbf{L}^6}\|\Delta\mathbf{W}\| \\ &\le \varepsilon\|\Delta\mathbf{W}\|^2 + C_\varepsilon\|\nabla\tilde{\mathbf{w}}\|^2_{\mathbf{L}^6}\|\nabla\mathbf{U}\|^2, \qquad (3.36) \\ 2\nu_r|(\mathrm{curl}\,\mathbf{U}, \Delta\mathbf{W})| &\le 2\nu_r\|\nabla\mathbf{U}\|\|\Delta\mathbf{W}\| \le \varepsilon\|\Delta\mathbf{W}\|^2 + C_\varepsilon\|\nabla\mathbf{U}\|^2, \qquad (3.37) \\ |(\mathbf{g_w}, \Delta\mathbf{W})| &\le \varepsilon\|\Delta\mathbf{W}\|^2 + C_\varepsilon\|\mathbf{g_w}\|^2. \qquad (3.38) \end{aligned}$$

Then, carrying (3.35)–(3.38) to (3.34) and choosing ε suitably, we can obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{W}\|^2 + C\|\Delta\mathbf{W}\|^2 + \nu_3(\nabla\mathrm{div}\,\mathbf{W}, \Delta\mathbf{W}) + 4\nu_r\|\nabla\mathbf{W}\|^2 \\ &\quad \le C\|\tilde{\mathbf{u}}\|^4_{\mathbf{L}^6}\|\nabla\mathbf{W}\|^2 + C(\|\nabla\tilde{\mathbf{w}}\|^2_{\mathbf{L}^6} + 1)\|\nabla\mathbf{U}\|^2 + C\|\mathbf{g_w}\|^2. \qquad (3.39) \end{aligned}$$

Moreover, since operator $L = -\nu_2\Delta - \nu_3\nabla\mathrm{div}$ is strongly elliptic, we have

$$(L\mathbf{W}, -\Delta\mathbf{W}) \ge C_1\|\Delta\mathbf{W}\|^2 - C_2\|\nabla\mathbf{W}\|^2, \qquad (3.40)$$

where C_1 and C_2 are positive constant which depend only on ν_2, ν_3 and $\partial\Omega$ (see [19], for more details). Then, estimates (3.39) and (3.40) implies

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{W}\|^2 + C\|\Delta\mathbf{W}\|^2 + 4\nu_r\|\nabla\mathbf{W}\|^2 &\le C(\|\tilde{\mathbf{u}}\|^4_{\mathbf{L}^6} + 1)\|\nabla\mathbf{W}\|^2 \\ &\quad + C(\|\nabla\tilde{\mathbf{w}}\|^2_{\mathbf{L}^6} + 1)\|\nabla\mathbf{U}\|^2 + C\|\mathbf{g_w}\|^2. \qquad (3.41) \end{aligned}$$

Therefore, from (3.33) and (3.41) we deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla\mathbf{U}\|^2 + \|\nabla\mathbf{W}\|^2) + (\|A\mathbf{U}\|^2 + \|\Delta\mathbf{W}\|^2) + 4\nu_r\|\nabla\mathbf{W}\|^2 \\ &\quad \le (\|\nabla\tilde{\mathbf{u}}\|^2_{\mathbf{L}^6} + \|\tilde{\mathbf{u}}\|^4_{\mathbf{L}^6} + \|\nabla\tilde{\mathbf{w}}\|^2_{\mathbf{L}^6} + 1)\|\nabla\mathbf{U}\|^2 + C(\|\tilde{\mathbf{u}}\|^4_{\mathbf{L}^6} + 1)\|\nabla\mathbf{W}\|^2 \\ &\qquad + C(\|\mathbf{g_u}\|^2 + \|\mathbf{g_w}\|^2). \qquad (3.42) \end{aligned}$$

Then, from (3.42) and Gronwall lemma, we can deduce that $(\mathbf{U}, \mathbf{W}) \in \widehat{\mathbf{X}}_{\mathbf{u}} \times \widehat{\mathbf{X}}_{\mathbf{w}}$. □

Now we are able to prove the existence of Lagrange multipliers.

Theorem 3.11. *Let* $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathcal{S}_{ad}$ *be a local optimal solution for the control problem* (3.22). *Then, there exist Lagrange multipliers* $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \boldsymbol{\lambda}_4) \in L^2(Q) \times L^2(Q) \times \mathbf{V}' \times \mathbf{H}^{-1}(\Omega)$ *such that*

$$\begin{aligned}
&\alpha \int_0^T \int_\Omega |\tilde{\mathbf{u}} - \mathbf{u}_d|^4 (\tilde{\mathbf{u}} - \mathbf{u}_d) \cdot \mathbf{U} + \beta \int_0^T \int_\Omega (\tilde{\mathbf{w}} - \mathbf{w}_d) \cdot \mathbf{W} + \gamma \int_0^T \int_\Omega \tilde{\mathbf{f}} \cdot \mathbf{F} \\
&\quad - \int_0^T \int_\Omega (\partial_t \mathbf{U} + \nu A \mathbf{U} + (\mathbf{U} \cdot \nabla) \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla) \mathbf{U} - 2\nu_r \mathrm{curl}\, \mathbf{W} - P\mathbf{F}) \cdot \boldsymbol{\lambda}_1 \\
&\quad - \int_0^T \int_\Omega (\partial_t \mathbf{W} + L\mathbf{W} + (\tilde{\mathbf{u}} \cdot \nabla) \mathbf{W} + (\mathbf{U} \cdot \nabla) \tilde{\mathbf{w}} + 4\nu_r \mathbf{W} - 2\nu_r \mathrm{curl}\, \mathbf{U}) \cdot \boldsymbol{\lambda}_2 \\
&\quad - \int_\Omega \mathbf{U}(0) \cdot \boldsymbol{\lambda}_3 - \int_\Omega \mathbf{W}(0) \cdot \boldsymbol{\lambda}_4 \geq 0, \qquad \forall \in (\mathbf{U}, \mathbf{W}, \mathbf{F}) \in \widehat{\mathbf{X}}_{\mathbf{u}} \times \widehat{\mathbf{X}}_{\mathbf{w}} \times \mathcal{C}(\tilde{\mathbf{f}}). \qquad (3.43)
\end{aligned}$$

Proof. From Lemma 3.10 we have that $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}})$ is a regular point. Therefore, from Theorem 3.6 we deduce that there exist Lagrange multipliers satisfying (3.43). □

Theorem 3.11 allows us derive an optimality system for problem (3.22), for this purpose we consider the following spaces

$$\widehat{\mathbf{X}}_{\mathbf{u}_0} = \{\mathbf{u} \in \widehat{\mathbf{X}}_{\mathbf{u}} \,:\, \mathbf{u}(0) = \mathbf{0}\}, \qquad \widehat{\mathbf{X}}_{\mathbf{w}_0} = \{\mathbf{u} \in \widehat{\mathbf{X}}_{\mathbf{w}} \,:\, \mathbf{u}(0) = \mathbf{0}\}. \qquad (3.44)$$

Corollary 3.12. *Let* $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathcal{S}_{ad}$ *be a local optimal solution of control problem* (3.22). *Then the Lagrange multipliers* $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \in L^2(Q) \times L^2(Q)$ *satisfy the system*

$$\begin{aligned}
&\int_0^T \int_\Omega (\partial_t \mathbf{U} + \nu A \mathbf{U} + (\mathbf{U} \cdot \nabla) \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla) \mathbf{U} - 2\nu_r \mathrm{curl}\, \mathbf{W}) \cdot \boldsymbol{\lambda}_1 \\
&\qquad = \alpha \int_0^T \int_\Omega |\tilde{\mathbf{u}} - \mathbf{u}_d|^4 (\tilde{\mathbf{u}} - \mathbf{u}_d) \cdot \mathbf{U}, \qquad (3.45) \\
&\int_0^T \int_\Omega (\partial_t \mathbf{W} + L\mathbf{W} + (\tilde{\mathbf{u}} \cdot \nabla) \mathbf{W} + 4\nu_r \mathbf{W} - 2\nu_r \mathrm{curl}\, \mathbf{U}) \cdot \boldsymbol{\lambda}_2 \\
&\qquad = \beta \int_0^T \int_\Omega (\tilde{\mathbf{w}} - \mathbf{w}_d) \cdot \mathbf{W}, \qquad (3.46)
\end{aligned}$$

for all $(\mathbf{U}, \mathbf{W}) \in \widehat{\mathbf{X}}_{\mathbf{u}_0} \times \widehat{\mathbf{W}}_{\mathbf{w}_0}$, *and the optimality condition*

$$\gamma \int_0^T \int_\Omega (\tilde{\mathbf{f}} + \boldsymbol{\lambda}_1) \cdot (\mathbf{f} - \tilde{\mathbf{f}}) \geq 0 \qquad \forall \mathbf{f} \in \mathcal{U}. \qquad (3.47)$$

Proof. Notice that $\widehat{\mathbf{W}}_{\mathbf{u}_0} \times \widehat{\mathbf{W}}_{\mathbf{w}_0}$ is a vector space; then, from (3.43), taking $(\mathbf{U}, \mathbf{F}) = (\mathbf{0}, \mathbf{0})$ we have (3.45). Analogously, taking $(\mathbf{W}, \mathbf{F}) = (\mathbf{0}, \mathbf{0})$ in (3.43), we deduce (3.46). Finally, taking $(\mathbf{U}, \mathbf{W}) = (\mathbf{0}, \mathbf{0})$ in (3.43) we obtain

$$\gamma \int_0^T \int_\Omega \tilde{\mathbf{f}} \cdot \mathbf{F} + \int_0^T \int_\Omega \mathbf{F} \cdot \boldsymbol{\lambda}_1 \geq 0 \qquad \forall \mathbf{F} \in \mathcal{C}(\tilde{\mathbf{f}}). \qquad (3.48)$$

Thus, choosing $F = \mathbf{f} - \tilde{\mathbf{f}} \in \mathcal{C}(\tilde{\mathbf{f}})$ in (3.48) we have (3.47). □

Remark 3.13. Problem (3.45)–(3.46) corresponds to the concept of the very weak solution of the parabolic linear problem

$$
\begin{aligned}
-\partial_t\boldsymbol{\lambda}_1 - \nu_1\Delta\boldsymbol{\lambda}_1 - \tilde{\mathbf{u}}\cdot\nabla\boldsymbol{\lambda}_1 + (\nabla\boldsymbol{\lambda}_1)^T\cdot\tilde{\mathbf{u}} + (\nabla\boldsymbol{\lambda}_2)^T\cdot\tilde{\mathbf{w}} + \nabla q&\\
= 2\nu_r \operatorname{curl}\boldsymbol{\lambda}_2 - \alpha|\tilde{\mathbf{u}}-\mathbf{u}_d|^4(\tilde{\mathbf{u}}-\mathbf{u}_d) \quad \text{in } Q,& \qquad (3.49)\\
-\partial_t\boldsymbol{\lambda}_2 - \nu_2\Delta\boldsymbol{\lambda}_2 - \nu_3\nabla\operatorname{div}\boldsymbol{\lambda}_2 - \tilde{\mathbf{u}}\cdot\nabla\boldsymbol{\lambda}_2 + 4\nu_r\boldsymbol{\lambda}_2&\\
= 2\nu_r\operatorname{curl}\boldsymbol{\lambda}_1 - \beta(\tilde{\mathbf{w}}-\mathbf{w}_d) \quad \text{in } Q,& \qquad (3.50)\\
\operatorname{div}\boldsymbol{\lambda}_1 = 0 \quad \text{in } Q,& \qquad (3.51)\\
\boldsymbol{\lambda}_1(T) = \mathbf{0},\ \boldsymbol{\lambda}_2(T) = \mathbf{0} \quad \text{in } \Omega,& \qquad (3.52)\\
\boldsymbol{\lambda}_1 = \mathbf{0},\ \boldsymbol{\lambda}_2 = \mathbf{0} \quad \text{on } \partial\Omega\times(0,T).& \qquad (3.53)
\end{aligned}
$$

Now, we will obtain some extra regularity for the Lagrange multipliers $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$ provided by Theorem 3.11.

Theorem 3.14. *Let $(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathcal{S}_{ad}$ be a local optimal solution of problem (3.22). Then, the Lagrange multipliers $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$, provided by Theorem 3.11, satisfy*

$$
\boldsymbol{\lambda}_1 \in L^\infty(0,T;\mathbf{V})\cap L^2(0,T;\mathbf{H}^2(\Omega)), \quad \partial_t\boldsymbol{\lambda}_1 \in L^2(Q), \qquad (3.54)
$$

$$
\boldsymbol{\lambda}_2 \in L^\infty(0,T;\mathbf{H}_0^1(\Omega))\cap L^2(0,T;\mathbf{H}^2(\Omega)), \quad \partial_t\boldsymbol{\lambda}_2 \in L^2(Q). \qquad (3.55)
$$

Proof. First we will show that the solution of system (3.49)–(3.53) has regularity (3.54)–(3.55). In fact, let $\tau := T-t$, with $t\in(0,T)$, and $\boldsymbol{\eta}_1(\tau) := \boldsymbol{\lambda}_1(t)$, $\boldsymbol{\eta}_2(\tau) := \boldsymbol{\lambda}_2(t)$. Then, system (3.49)–(3.53) is equivalent to

$$
\begin{cases}
\partial_\tau\boldsymbol{\eta}_1 - \nu_1\Delta\boldsymbol{\eta}_1 - \tilde{\mathbf{u}}\cdot\nabla\boldsymbol{\eta}_1 + (\nabla\boldsymbol{\eta}_1)^T\cdot\tilde{\mathbf{u}} + (\nabla\boldsymbol{\eta}_2)^T\cdot\tilde{\mathbf{w}} + \nabla q\\
\qquad = 2\nu_r\operatorname{curl}\boldsymbol{\eta}_2 - \alpha|\tilde{\mathbf{u}}-\mathbf{u}_d|^4(\tilde{\mathbf{u}}-\mathbf{u}_d) \quad \text{in } Q,\\
\partial_\tau\boldsymbol{\eta}_2 - \nu_2\Delta\boldsymbol{\eta}_2 - \nu_3\nabla\operatorname{div}\boldsymbol{\eta}_2 - \tilde{\mathbf{u}}\cdot\nabla\boldsymbol{\eta}_2 + 4\nu_r\boldsymbol{\eta}_2\\
\qquad = 2\nu_r\operatorname{curl}\boldsymbol{\eta}_1 - \beta(\tilde{\mathbf{w}}-\mathbf{w}_d) \quad \text{in } Q,\\
\quad \operatorname{div}\boldsymbol{\eta}_1 = 0 \quad \text{in } Q,\\
\quad \boldsymbol{\eta}_1(T) = \mathbf{0},\ \boldsymbol{\eta}_2(T) = \mathbf{0} \quad \text{in } \Omega,\\
\qquad \boldsymbol{\eta}_1 = \mathbf{0},\ \boldsymbol{\eta}_2 = \mathbf{0} \quad \text{on } \partial\Omega\times(0,T).
\end{cases} \qquad (3.56)
$$

Following similar arguments that in the proof of Lemma 3.10 we can obtain that the unique solution $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ of problem (3.56) satisfies

$$
\begin{aligned}
&\boldsymbol{\eta}_1 \in L^\infty(0,T;\mathbf{V})\cap L^2(0,T;\mathbf{H}^2(\Omega)), \quad \partial_t\boldsymbol{\eta}_1 \in L^2(Q),\\
&\boldsymbol{\eta}_2 \in L^\infty(0,T;\mathbf{H}_0^1(\Omega))\cap L^2(0,T;\mathbf{H}^2(\Omega)), \quad \partial_t\boldsymbol{\eta}_2 \in L^2(Q).
\end{aligned}
$$

Consequently, the unique solution of system (3.49)–(3.53) satisfies the regularity (3.54)–(3.55). Now, let $(\overline{\boldsymbol{\lambda}}_1, \overline{\boldsymbol{\lambda}}_2)$ the unique solution of (3.49)–(3.53); then, it suffices to identify $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$ with $(\overline{\boldsymbol{\lambda}}_1, \overline{\boldsymbol{\lambda}}_2)$. For this, we consider the unique solution $(\mathbf{U}, \mathbf{W}) \in \widehat{\mathbf{X}}_{\mathbf{u}} \times \widehat{\mathbf{X}}_{\mathbf{w}}$ of problem (3.27) (see the proof of Lemma 3.10 above) for $\mathbf{g}_{\mathbf{u}} := (\boldsymbol{\lambda}_1 - \overline{\boldsymbol{\lambda}}_1) \in L^2(Q)$ and $\mathbf{g}_{\mathbf{w}} := (\boldsymbol{\lambda}_2 - \overline{\boldsymbol{\lambda}}_2) \in L^2(Q)$. Then, written (3.49)-(3.52) for $(\overline{\boldsymbol{\lambda}}_1, \overline{\boldsymbol{\lambda}}_2)$ instead of $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$, and testing the first equation by $\mathbf{U}$

and the second equation by $\mathbf{W}$, we can obtain

$$\int_0^T \int_\Omega (\partial_t \mathbf{U} + \nu A\mathbf{U} + (\mathbf{U}\cdot\nabla)\tilde{\mathbf{u}} + (\tilde{\mathbf{u}}\cdot\nabla)\mathbf{U} - 2\nu_r \operatorname{curl}\mathbf{W})\cdot\overline{\boldsymbol{\lambda}}_1 = \alpha \int_0^T \int_\Omega |\tilde{\mathbf{u}} - \mathbf{u}_d|^4 (\tilde{\mathbf{u}} - \mathbf{u}_d)\cdot\mathbf{U}, \tag{3.57}$$

$$\int_0^T \int_\Omega (\partial_t \mathbf{W} + L\mathbf{W} + (\tilde{\mathbf{u}}\cdot\nabla)\mathbf{W} + 4\nu_r\mathbf{W} - 2\nu_r \operatorname{curl}\mathbf{U})\cdot\overline{\boldsymbol{\lambda}}_2 = \beta \int_0^T \int_\Omega (\tilde{\mathbf{w}} - \mathbf{w}_d)\cdot\mathbf{W}. \tag{3.58}$$

Making the difference between (3.45) for and (3.57), and between (3.46) and (3.58), and then adding the respective equations, we can deduce

$$\int_0^T \int_\Omega (\partial_t \mathbf{U} + \nu A\mathbf{U} + (\mathbf{U}\cdot\nabla)\tilde{\mathbf{u}} + (\tilde{\mathbf{u}}\cdot\nabla)\mathbf{U} - 2\nu_r \operatorname{curl}\mathbf{W})\cdot(\boldsymbol{\lambda}_1 - \overline{\boldsymbol{\lambda}}_1) + \int_0^T \int_\Omega (\partial_t \mathbf{W} + L\mathbf{W} + (\tilde{\mathbf{u}}\cdot\nabla)\mathbf{W} + 4\nu_r\mathbf{W} - 2\nu_r \operatorname{curl}\mathbf{U})\cdot(\boldsymbol{\lambda}_2 - \overline{\boldsymbol{\lambda}}_2) = 0. \tag{3.59}$$

Therefore, taking into account that $(\mathbf{U}, \mathbf{W})$ is the unique solution of (3.27) for $(\boldsymbol{\lambda}_1 - \overline{\boldsymbol{\lambda}}_1)$ and $(\boldsymbol{\lambda}_2 - \overline{\boldsymbol{\lambda}}_2)$, from (3.59) we obtain

$$\|\boldsymbol{\lambda}_1 - \overline{\boldsymbol{\lambda}}_1\|^2_{L^2(Q)} + \|\boldsymbol{\lambda}_2 - \overline{\boldsymbol{\lambda}}_2\|^2_{L^2(Q)} = 0,$$

which implies that $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = (\overline{\boldsymbol{\lambda}}_1, \overline{\boldsymbol{\lambda}}_2)$ in $L^2(Q) \times L^2(Q)$. Consequently, the regularity of $(\overline{\boldsymbol{\lambda}}_1, \overline{\boldsymbol{\lambda}}_2)$ imply that

$$\boldsymbol{\lambda}_1 \in L^\infty(0,T;\mathbf{V}) \cap L^2(0,T;\mathbf{H}^2(\Omega)), \quad \partial_t \boldsymbol{\lambda}_1 \in L^2(Q),$$
$$\boldsymbol{\lambda}_2 \in L^\infty(0,T;\mathbf{H}^1_0(\Omega)) \cap L^2(0,T;\mathbf{H}^2(\Omega)), \quad \partial_t \boldsymbol{\lambda}_2 \in L^2(Q). \qquad \square$$

Finally, we deduce the optimality system of control problem (3.22).

Corollary 3.15. *Let* $(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathcal{S}_{ad}$ *be a local optimal solution of problem* (3.22)*. Then, the Lagrange multipliers* $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$*, with*

$$\boldsymbol{\lambda}_1 \in L^\infty(0,T;\mathbf{V}) \cap L^2(0,T;\mathbf{H}^2(\Omega)), \quad \partial_t \boldsymbol{\lambda}_1 \in L^2(Q),$$
$$\boldsymbol{\lambda}_2 \in L^\infty(0,T;\mathbf{H}^1_0(\Omega)) \cap L^2(0,T;\mathbf{H}^2(\Omega)), \quad \partial_t \boldsymbol{\lambda}_2 \in L^2(Q).$$

satisfiy the following optimality system

$$\begin{cases} -\partial_t \boldsymbol{\lambda}_1 - \nu_1 \Delta\boldsymbol{\lambda}_1 - \tilde{\mathbf{u}}\cdot\nabla\boldsymbol{\lambda}_1 + (\nabla\boldsymbol{\lambda}_1)^T\cdot\tilde{\mathbf{u}} + (\nabla\boldsymbol{\lambda}_2)^T\cdot\tilde{\mathbf{w}} + \nabla q \\ \qquad = 2\nu_r \operatorname{curl}\boldsymbol{\lambda}_2 - \alpha|\tilde{\mathbf{u}} - \mathbf{u}_d|^4(\tilde{\mathbf{u}} - \mathbf{u}_d) \quad \text{in } Q, \\ -\partial_t \boldsymbol{\lambda}_2 - \nu_2\Delta\boldsymbol{\lambda}_2 - \nu_3\nabla\operatorname{div}\boldsymbol{\lambda}_2 - \tilde{\mathbf{u}}\cdot\nabla\boldsymbol{\lambda}_2 + 4\nu_r\boldsymbol{\lambda}_2 \\ \qquad = 2\nu_r\operatorname{curl}\boldsymbol{\lambda}_1 - \beta(\tilde{\mathbf{w}} - \mathbf{w}_d) \quad \text{in } Q, \\ \operatorname{div}\boldsymbol{\lambda}_1 = 0 \quad \text{in } Q, \\ \boldsymbol{\lambda}_1(T) = \mathbf{0},\ \boldsymbol{\lambda}_2(T) = \mathbf{0} \quad \text{in } \Omega, \\ \boldsymbol{\lambda}_1 = \mathbf{0},\ \boldsymbol{\lambda}_2 = \mathbf{0} \quad \text{on } \partial\Omega\times(0,T), \\ \gamma\int_0^T\int_\Omega(\tilde{\mathbf{f}} + \boldsymbol{\lambda}_1)\cdot(\mathbf{f} - \tilde{\mathbf{f}}) \geq 0 \qquad \forall \mathbf{f} \in \mathcal{U}. \end{cases} \tag{3.60}$$

Remark 3.16. If $\gamma > 0$. Then, from $(3.60)_6$, the fact that the control set $\mathcal{U}$ is closed and convex, and [2, Theorem 5.2, p. 132], we can characterizes the optimal control $\tilde{\mathbf{f}}$ as the projection of $-\frac{\lambda_1}{\gamma}$ onto $\mathcal{U}$; that is,

$$\tilde{\mathbf{f}} = \operatorname*{Proj}_{\mathcal{U}}\left(-\frac{\boldsymbol{\lambda}_1}{\gamma}\right).$$

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Conflict of interest

The authors declare that they have no conflict of interest.

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Positive solutions for $(p, 2)$-equations with superlinear reaction and a concave boundary term

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Abstract. We consider a nonlinear boundary value problem driven by the $(p, 2)$-Laplacian, with a $(p-1)$-superlinear reaction and a parametric concave boundary term (a "concave-convex" problem). Using variational tools (critical point theory) together with truncation and comparison techniques, we prove a bifurcation type theorem describing the changes in the set of positive solutions as the parameter $\lambda > 0$ varies. We also show that for every admissible parameter $\lambda > 0$, the problem has a minimal positive solution $\overline{u}_\lambda$ and determine the monotonicity and continuity properties of the map $\lambda \mapsto \overline{u}_\lambda$.

Keywords: concave boundary term, superlinear reaction, $(p, 2)$-Laplacian, nonlinear regularity, nonlinear maximum principle, positive solutions.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear parametric $(p, 2)$-equation

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) + \xi(z) u(z)^{p-1} = f(z, u(z)) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_{p2}} = \lambda u^{\tau-1} & \text{on } \partial\Omega\,, \\ u > 0,\ \lambda > 0,\ 1 < \tau < 2 < p < N. \end{cases} \tag{P_λ}$$

In this problem, Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(|Du|^{p-2} Du\right) \quad \text{for all } u \in W^{1,p}(\Omega),\ 1 < p < N.$$

The potential function $\xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$, $\xi \not\equiv 0$. The reaction term $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z, x)$ is measurable and for a.a.

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$z \in \Omega$, $x \mapsto f(z,x)$ is continuous). We assume that $f(z,\cdot)$ is $(p-1)$-superlinear satisfying the Ambrosetti–Rabinowitz condition (the *AR-condition* for short). In the boundary condition, $\frac{\partial u}{\partial n_{p2}}$ denotes the conormal derivative of u corresponding to the $(p,2)$-Laplace differential operator. This directional derivative of u, is interpreted via the nonlinear Green's identity (see Papageorgiou–Rădulescu–Repovš [21], pp. 34, 35). If $u \in C^1(\overline{\Omega})$, then

$$\frac{\partial u}{\partial n_{p2}} = \left[|Du|^{p-2}+1\right]\frac{\partial u}{\partial n}$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. Also $\lambda > 0$ is a parameter and $\tau \in (1,2)$. So, in problem (P_λ) we have the competing effects of two nonlinearities of different nature. One is the reaction term which is superlinear ("convex" nonlinearity) and the other is the parametric boundary term, which is sublinear ("concave" nonlinearity). Therefore, problem (P_λ) is a variant of the classical "concave-convex" problem, with the concave term coming from the boundary condition.

The study of "concave-convex" problems was initiated with the seminal paper of Ambrosetti–Brezis–Cerami [2] (semilinear Dirichlet equations). Their work was extended to nonlinear Dirichlet problems driven by the p-Laplacian by García Azorero–Manfredi–Peral Alonso [7] and Guo-Zhang [9]. In these works the reaction has the special form

$$x \mapsto \lambda x^{\tau-1} + x^{r-1} \quad \text{for all } x \geq 0,$$

with $\lambda > 0$ (the parameter) and $1 < \tau < p < r < p^*$,

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leq p. \end{cases}$$

Recently more general reactions and different boundary conditions were considered by Papageorgiou–Rădulescu–Repovš [18] (semilinear Robin problems), by Leonardi–Papageorgiou [12], Marano–Marino–Papageorgiou [14] (nonlinear Dirichlet problems) and by Papageorgiou–Scapellato [23] (nonlinear Robin problems). In these works the competition phenomena occur in the reaction of the equation, where we have the presence of concave and convex nonlinearities. Problems with parametric concave boundary term were considered by Hu–Papageorgiou [11] (semilinear equations), Papageorgiou–Rădulescu [16], Papageorgiou–Rădulescu–Repovš [20], Sabina de Lis–Segura de Leon [25] (nonlinear problems driven by the p-Laplacian). Finally we mention the recent work of Papageorgiou–Scapellato [22] where in the reaction we have the combined effects of linear and superlinear terms.

Our work here extends those of Hu–Papageorgiou [11] and of Sabina de Lis–Segura de Leon [25].

Using variational tools based on the critical point theory, together with truncation and comparison techniques, we prove a bifurcation-type result describing in a precise way the set of positive solutions of problem (P_λ) as the parameter $\lambda > 0$ varies. Also we show that for every admissible $\lambda > 0$, problem (P_λ) has a smallest positive solution.

We mention that boundary value problems driven by a combination of differential operators of different nature (such as $(p,2)$-equations), arise in many mathematical models of physical processes. Among the first such examples we mention the Cahn–Hilliard equation (see [4]) describing the process of separation of binary alloys. More recently, we mention the works of Benci–D'Avenia–Fortunato–Pisani [3] (quantum physics) and Cherfils–Il'yasov [5] (reaction-diffusion systems).

2 Mathematical background – hypotheses

In the study of problem (P_λ), we will use the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesgue spaces $L^s(\partial\Omega)$ $(1 \leq s < \infty)$.

By $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$, defined by

$$\|u\| = \left[\|u\|_p^p + \|Du\|_p^p\right]^{\frac{1}{p}} \quad \text{for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

We will also use another open cone in $C^1(\overline{\Omega})$ given by

$$D_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) > 0 \text{ for all } z \in \Omega, \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega \cap u^{-1}(0)} < 0\right\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using $\sigma(\cdot)$, we can define in the usual way the boundary Lebesgue spaces $L^s(\partial\Omega)$ $(1 \leq s \leq \infty)$. We know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the *trace map*, such that

$$\gamma_0(u) = u\Big|_{\partial\Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map extends the notion of boundary values to all Sobolev functions. This map is compact into $L^s(\partial\Omega)$ for all $1 \leq s < \frac{(N-1)p}{N-p}$ when $p < N$ and into $L^s(\Omega)$ for all $1 \leq s < \infty$ when $N \leq p$. Moreover, we have

$$\begin{aligned} \operatorname{im} \gamma_0 &= W^{\frac{1}{p'},p}(\partial\Omega) \qquad \left(\frac{1}{p} + \frac{1}{p'} = 1\right), \\ \ker \gamma_0 &= W_0^{1,p}(\Omega). \end{aligned}$$

In what follows for the sake of notational simplicity we drop the use of the trace map. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

If $u, v \in W^{1,p}(\Omega)$ with $u(z) \leq v(z)$ for a.a. $z \in \Omega$, then we define

$$\begin{aligned} [u, v] &= \{h \in W^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \quad \text{for a.a. } z \in \Omega\}, \\ [u) &= \{h \in W^{1,p}(\Omega) : u(z) \leq h(z) \quad \text{for a.a. } z \in \Omega\}. \end{aligned}$$

Given $g_1, g_2 \in L^\infty(\Omega)$, we write $g_1 \prec g_2$ if for every $K \subseteq \Omega$ compact we can find $c_K > 0$ such that

$$c_K \leq g_2(z) - g_1(z) \quad \text{for a.a. } z \in K.$$

Note that if $g_1, g_2 \in C(\Omega)$ and $g_1(z) < g_2(z)$ for all $z \in \Omega$, then $g_1 \prec g_2$.

We say that a set $S \subseteq W^{1,p}(\Omega)$ is *downward directed*, if given $u_1, u_2 \in S$, we can find $u \in S$ such that $u \leq u_1$, $u \leq u_2$.

Let $\langle \cdot, \cdot \rangle$ denote the duality brackets for the pair $(W^{1,p}(\Omega), W^{1,p}(\Omega)^*)$ and let $A_p : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear operator defined by

$$\langle A_p(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W^{1,p}(\Omega).$$

Proposition 2.1. *The operator $A_p(\cdot)$ is bounded (maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is,*

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty}\langle A_p(u_n), u_n - u\rangle \le 0 \quad \Rightarrow \quad u_n \to u \text{ in } W^{1,p}(\Omega).$$

If $p = 2$, then $A_2 = A \in \mathscr{L}(H^1(\Omega), H^1(\Omega)^*)$.

For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then, given $u \in W^{1,p}(\Omega)$, we define

$$u^\pm(z) = u(z)^\pm \quad \text{for all } z \in \Omega.$$

We know that

$$u^\pm \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Finally, if X is a Banach space and $\varphi \in C^1(X, \mathbb{R})$, then by K_φ we denote the critical set of $\varphi(\cdot)$, that is,

$$K_\varphi = \{u \in W^{1,p}(\Omega) : \varphi'(u) = 0\}.$$

Now we introduce our hypotheses on the data of problem (P_λ).

H(ξ): $\xi \in L^\infty(\Omega)$, $\xi(z) \ge 0$ for a.a. $z \in \Omega$, $\xi \not\equiv 0$.

H(f): $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $0 \le f(z, x) \le \eta x^{r-1}$ for a.a. $z \in \Omega$, all $x \ge 0$, with $0 < \eta$, $p < r < p^*$;

(ii) if $F(z, x) = \int_0^x f(z, s)\,ds$, then there exist $\vartheta_0 \in (p, r)$ and $M > 0$ such that

$$0 < \vartheta_0 F(z, x) \le f(z, x)x \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M,$$
$$0 < \operatorname*{ess\,inf}_{\Omega} F(\cdot, M).$$

Remark 2.2. Since we are looking for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality we may assume that

$$f(z, x) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \le 0. \tag{2.1}$$

Hypothesis H(f)(i) implies that

$$\lim_{x\to 0^+} \frac{f(z, x)}{x^{\tau-1}} = 0 \quad \text{uniformly for a.a. } z \in \Omega. \tag{2.2}$$

Hypothesis H(f)(ii) is the well known AR-condition (unilateral version due to (2.1)). The AR-condition implies that

$$\begin{aligned} & c_0 x^{\vartheta_0} \le F(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M, \text{ some } c_0 > 0 \\ \Rightarrow\ & c_0 x^{\vartheta_0 - 1} \le f(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M \\ \Rightarrow\ & f(z, \cdot) \text{ is } (p-1)\text{-superlinear (since } \vartheta_0 > p). \end{aligned}$$

It is an interesting open problem whether we can replace the AR-condition by a less restrictive one as in Papageorgiou–Rădulescu [17].

The following functions satisfy hypotheses H(f). For the sake of simplicity we drop the z-dependence:

$$f_1(x) = \begin{cases} (x^+)^{r-1} + \ln(1 + (x^+)^{q-1}) & \text{if } x \le 1 \\ x^{s-1} & \text{if } 1 < x \end{cases} \quad \text{with } p < r \le q < \infty,\ p < s < p^*,$$

$$f_2(x) = (x^+)^{r-1} \quad \text{with } p < r < p^*.$$

In the sequel, by $\gamma_p : W^{1,p}(\Omega) \to \mathbb{R}$ we denote the C^1-functional defined by

$$\gamma_p(u) = \|Du\|_p^p + \int_\Omega \xi(z)|u|^p \,\mathrm{d}z \quad \text{for all } u \in W^{1,p}(\Omega).$$

On account of hypothesis H(ξ) and Lemma 4.11 of Mugnai–Papageorgiou [15], we have

$$c_1\|u\|^p \le \gamma_p(u) \quad \text{for all } u \in W^{1,p}(\Omega), \text{ some } c_1 > 0. \tag{2.3}$$

3 Positive solutions

We introduce the following sets

$$\mathscr{L} = \{\lambda > 0 : \text{problem } (\mathrm{P}_\lambda) \text{ admits a positive solution}\},$$
$$S_\lambda = \text{ set of positive solutions of } (\mathrm{P}_\lambda).$$

Proposition 3.1. *If hypotheses* H(ξ), H(f) *hold, then* $\mathscr{L} \neq \varnothing$ *and* $S_\lambda \subseteq \operatorname{int} C_+$ *for all* $\lambda > 0$.

Proof. For every $\lambda > 0$, let $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\varphi_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z,u^+)\,\mathrm{d}z - \frac{\lambda}{\tau}\int_{\partial\Omega}(u^+)^\tau \,\mathrm{d}\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

On account of (2.2) and hypothesis H(f)(i), we see that given $\epsilon > 0$, we can find $c_2 = c_2(\epsilon) > 0$ such that

$$F(z,x) \le \epsilon x^\tau + c_2|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

Then we have

$$\varphi_\lambda(u) \ge \frac{c_1}{p}\|u\|^p - c_3\left[\epsilon\|u\|^\tau + \|u\|^r + \lambda\|u\|^\tau\right] \quad \text{for some } c_3 > 0, \text{ all } u \in W^{1,p}(\Omega). \tag{3.1}$$

Here we used (2.3) and the fact that via the trace map the Sobolev space $W^{1,p}(\Omega)$ is embedded continuously (in fact compactly) into $L^\tau(\partial\Omega)$.

For $\rho > 0$, we let $\epsilon = \frac{1}{2}\frac{c_1}{p}\frac{\rho^{p-\tau}}{c_3}$. Then we have

$$\left[\frac{c_1}{p}\rho^{p-\tau} - \epsilon c_3\right]\rho^\tau = \frac{1}{2}\frac{c_1}{p}\rho^p. \tag{3.2}$$

Using (3.2) in (3.1) we obtain

$$\varphi_\lambda(u) \ge \frac{1}{2}\frac{c_1}{p}\rho^p - c_3[\rho^r + \lambda\rho^\tau] \quad \text{for all } u \in W^{1,p}(\Omega) \text{ with } \|u\| = \rho.$$

Since $p < r$, we can choose $\rho \in (0,1)$ small such that

$$\frac{1}{2}\frac{c_1}{p}\rho^p - c_3\rho^r \ge \overline{\eta} > 0.$$

Then we choose $\lambda_0 > 0$ small so that

$$\begin{aligned}
&\overline{\eta} - \lambda_0 c_3\rho^\tau \ge \frac{1}{2}\overline{\eta} > 0 \\
\Rightarrow\quad &\overline{\eta} - \lambda c_3\rho^\tau \ge \frac{1}{2}\overline{\eta} > 0 \quad \text{for all } \lambda \in (0,\lambda_0] \\
\Rightarrow\quad &\varphi_\lambda(u) \ge \frac{1}{2}\overline{\eta} > 0 \quad \text{for all } u \in W^{1,p}(\Omega) \text{ with } \|u\| = \rho, \text{ all } 0 < \lambda \le \lambda_0.
\end{aligned} \tag{3.3}$$

We introduce the open ball

$$B_\rho = \{u \in W^{1,p}(\Omega) : \|u\| < \rho\}.$$

By the Alaoglu and Eberlein-Šmulian theorems, we have that $\overline{B_\rho}$ is sequentially weakly compact. Also, using the Sobolev embedding theorem and the compactness of the trace map, we see that $\varphi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. Invoking the Weierstrass–Tonelli theorem, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\varphi_\lambda(u_0) = \min\left[\varphi_\lambda(u) : u \in \overline{B_\rho}\right]. \tag{3.4}$$

Since $\tau < 2 < p$, we see that

$$\begin{aligned} &\varphi_\lambda(u_0) < 0 = \varphi_\lambda(0) < \frac{1}{2}\overline{\eta} \\ \Rightarrow\quad &u_0 \in B_\rho \setminus \{0\} \quad \text{(see (3.3))}. \end{aligned} \tag{3.5}$$

Then from (3.4) and (3.5) it follows that

$$\begin{aligned} &\varphi'_\lambda(u_0) = 0, \\ \Rightarrow\quad &\langle A_p(u_0), h\rangle + \langle A(u_0), h\rangle + \int_\Omega \xi(z)|u_0|^{p-2}u_0 h\,dz \\ &\quad = \int_\Omega f(x, u_0^+)h\,dz + \lambda\int_{\partial\Omega}(u_0^+)^{\tau-1}h\,d\sigma \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.6}$$

In (3.6) we choose $h = -u_0^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\gamma_p(u_0^-) + \|Du_0^-\|_2^2 = 0 \\ \Rightarrow\quad &c_1\|u_0^-\|^p \le 0 \quad \text{(see (2.3))} \\ \Rightarrow\quad &u_0 \ge 0,\ u_0 \ne 0. \end{aligned}$$

From (3.6) we see that $u_0 \in W^{1,p}(\Omega)$ is a positive solution of (P_λ) and we have

$$\begin{cases} -\Delta_p u_0(z) - \Delta u_0(z) + \xi(z)u_0(z)^{p-1} = f(z, u_0(z)) & \text{for a.a. } z \in \Omega, \\ \dfrac{\partial u_0}{\partial n_{p2}} = \lambda u_0^{\tau-1} & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Proposition 2.10 of Papageorgiou–Rădulescu [17] implies that $u_0 \in L^\infty(\Omega)$ and then from Theorem 2 of Lieberman [13], we have that $u_0 \in C_+ \setminus \{0\}$. From (3.7) we see that

$$\begin{aligned} &\Delta_p u_0(z) + \Delta u_0(z) \le \|\xi\|_\infty u_0(z)^{p-1} \quad \text{for a.a. } x \in \Omega \\ \Rightarrow\quad &u_0 \in \operatorname{int} C_+ \quad \text{(see Pucci–Serrin [24], pp. 111, 120).} \end{aligned}$$

So, we have proved that

$$\begin{aligned} &(0, \lambda_0] \subseteq \mathscr{L},\ \text{that is, } \mathscr{L} \ne \emptyset, \\ &S_\lambda \subseteq \operatorname{int} C_+ \text{ for all } \lambda > 0. \end{aligned}$$

□

Next we show that $\mathscr{L}$ is an interval.

Proposition 3.2. *If hypotheses* H(ξ), H(f) *hold,* $\lambda \in \mathscr{L}$ *and* $\mu \in (0, \lambda)$*, then* $\mu \in \mathscr{L}$*.*

Proof. Since $\lambda \in \mathscr{L}$, we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ (see Proposition 3.1). We introduce the following truncations of the data of problem (P_μ):

$$\widehat{f}(z,x) = \begin{cases} f(z,x^+) & \text{if } x \le u_\lambda(z) \\ f(z,u_\lambda(z)) & \text{if } u_\lambda(z) < x \end{cases} \quad \text{for all } (z,x) \in \Omega \times \mathbb{R}, \tag{3.8}$$

$$e_\mu(z,x) = \begin{cases} \mu(x^+)^{\tau-1} & \text{if } x \le u_\lambda(z) \\ \mu u_\lambda(z)^{\tau-1} & \text{if } u_\lambda(z) < x \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathbb{R}. \tag{3.9}$$

Both are Carathéodory functions. We set

$$\widehat{F}(z,x) = \int_0^x \widehat{f}(z,s)\,\mathrm{d}s, \qquad E_\mu(z,x) = \int_0^x e_\mu(z,s)\,\mathrm{d}s$$

and consider the C^1-functional $\psi_\mu : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\mu(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \widehat{F}(z,u)\,\mathrm{d}z - \int_{\partial\Omega} E_\mu(z,u)\,\mathrm{d}\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (2.3), (3.8) and (3.9), we see that $\psi_\mu(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. Therefore we can find $u_\mu \in W^{1,p}(\Omega)$ such that

$$\psi_\mu(u_\mu) = \inf\left[\psi_\mu(u) : u \in W^{1,p}(\Omega)\right]. \tag{3.10}$$

Let $u \in \operatorname{int} C_+$ and choose $t \in (0,1)$ small (at least so that $tu \le u_\lambda$, recall that $u_\lambda \in \operatorname{int} C_+$). Then since $\tau < 2 < p$, we will have

$$\begin{aligned} & \psi_\mu(tu) < 0 \\ \Rightarrow\quad & \psi_\mu(u_\mu) < 0 = \psi_\mu(0) \quad \text{(see (3.10))} \\ \Rightarrow\quad & u_\mu \neq 0. \end{aligned}$$

From (3.10) we have

$$\begin{aligned} & \psi'_\mu(u_\mu) = 0 \\ \Rightarrow\quad & \langle A_p(u_\mu),h\rangle + \langle A(u_\mu),h\rangle + \int_\Omega \xi(z)|u_\mu|^{p-2}u_\mu h\,\mathrm{d}z \\ & = \int_\Omega \widehat{f}(z,u_\mu)h\,\mathrm{d}z + \int_{\partial\Omega} e(z,u_\mu)h\,\mathrm{d}\sigma \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.11}$$

In (3.11) first we choose $h = -u_\mu^- \in W^{1,p}(\Omega)$. We obtain

$$\begin{aligned} & \gamma_p(u_\mu^-) + \|Du_\mu^-\|_2^2 = 0 \\ \Rightarrow\quad & c_1\|u_\mu^-\|^p \le 0 \quad \text{(see (2.3))} \\ \Rightarrow\quad & u_\mu \ge 0,\ u_\mu \neq 0. \end{aligned}$$

Next in (3.11) we choose $h = (u_\mu - u_\lambda)^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned} & \langle A_p(u_\mu),(u_\mu-u_\lambda)^+\rangle + \langle A(u_\mu),(u_\mu-u_\lambda)^+\rangle + \int_\Omega \xi(z)u_\mu^{p-1}(u_\mu-u_\lambda)^+\,\mathrm{d}z = \\ & = \int_\Omega f(x,u_\lambda)(u_\mu-u_\lambda)^+\,\mathrm{d}z + \int_{\partial\Omega} \mu u_\lambda^{\tau-1}(u_\mu-u_\lambda)^+\,\mathrm{d}\sigma \quad \text{(see (3.8), (3.9))} \\ & \le \int_\Omega f(z,u_\lambda)(u_\mu-u_\lambda)^+\,\mathrm{d}z + \int_{\partial\Omega} \lambda u_\lambda^{\tau-1}(u_\mu-u_\lambda)^+\,\mathrm{d}z \quad \text{(since } \mu < \lambda) \\ & = \langle A_p(u_\lambda),(u_\mu-u_\lambda)^+\rangle + \langle A(u_\lambda),(u_\mu-u_\lambda)^+\rangle + \int_\Omega \xi(z)u_\lambda^{p-1}(u_\mu-u_\lambda)^+\,\mathrm{d}z \\ & \text{(since } u_\lambda \in S_\lambda) \end{aligned}$$

$$\Rightarrow \quad u_\mu \leq u_\lambda \quad \text{(see Proposition 2.1).}$$

So we have proved that

$$u_\mu \in [0, u_\lambda] \setminus \{0\}. \tag{3.12}$$

From (3.11), (3.12), (3.8), (3.9) it follows that

$$\begin{aligned} & u_\mu \in S_\mu \subseteq \operatorname{int} C_+, \\ \Rightarrow \quad & \mu \in \mathscr{L}. \end{aligned}$$

□

An interesting byproduct of the above proof is the following corollary.

Corollary 3.3. *If hypotheses* H(ξ), H(f) *hold,* $\lambda \in \mathscr{L}$, $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ *and* $\mu \in (0, \lambda)$, *then* $\mu \in \mathscr{L}$ *and there exists* $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ *such that* $u_\mu \leq u_\lambda$.

We can improve this corollary, by imposing an additional mild condition on $f(z, \cdot)$. So, the new hypotheses on the reaction $f(z, x)$ are the following:

H(f)′: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, hypotheses H(f)′(i), (ii) are the same as the corresponding hypotheses H(f)(i), (ii) and

(i) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$ the function

$$x \mapsto f(z, x) + \widehat{\xi}_\rho x^{p-1}$$

is nondecreasing on $[0, \rho]$.

Remark 3.4. The extra condition is a one-sided local Lipschitz condition (recall that $p > 2$). If $f(z, \cdot)$ is differentiable for a.a. $z \in \Omega$ and for every $\rho > 0$, there exists $c_\rho > 0$ such that

$$f'_x(z, x) \geq -c_\rho x^{p-2} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0, \rho],$$

then hypothesis H(f)′(i) is satisfied.

Proposition 3.5. *If hypotheses* H(ξ), H(f)′ *hold,* $\lambda \in \mathscr{L}$, $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ *and* $\mu \in (0, \lambda)$, *then* $\mu \in \mathscr{L}$ *and we can find* $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ *such that*

$$u_\lambda - u_\mu \in D_+.$$

Proof. From Corollary 3.3 we already know that $\mu \in \mathscr{L}$ and we can find $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that

$$u_\mu \leq u_\lambda. \tag{3.13}$$

Let $a : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$a(y) = |y|^{p-2} y + y \quad \text{for all } y \in \mathbb{R}^N.$$

Evidently $a \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ (recall that $p > 2$) and

$$\begin{aligned} & \nabla a(y) = |y|^{p-2} \left[I + (p-2) \frac{y \otimes y}{|y|^2} \right] + I \\ \Rightarrow \quad & (\nabla a(y) \vartheta, \vartheta)_{\mathbb{R}^N} \geq |\vartheta|^2 \quad \text{for all } y, \vartheta \in \mathbb{R}^N. \end{aligned} \tag{3.14}$$

Observe that

$$\operatorname{div} a(Du) = \Delta_p u + \Delta u \quad \text{for all } u \in W^{1,p}(\Omega). \tag{3.15}$$

From (3.13), (3.14), (3.15) and the tangency principle of Pucci-Serrin [24], p. 35, we have

$$u_\mu(z) < u_\lambda(z) \quad \text{for all } z \in \Omega. \tag{3.16}$$

Let $\rho = \|u_\lambda\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis H$(f)'$(i). Let $\widetilde{\xi}_\rho > \widehat{\xi}_\rho$. We have

$$\begin{aligned} &-\Delta_p u_\mu - \Delta u_\mu + \left[\xi(z) + \widetilde{\xi}_\rho\right] u_\mu^{p-1} \\ &\quad = f(z,u_\mu) + \widehat{\xi}_\rho u_\mu^{p-1} + \left[\widetilde{\xi}_\rho - \widehat{\xi}_\rho\right] u_\mu^{p-1} \\ &\quad \leq f(z,u_\lambda) + \widehat{\xi}_\rho u_\lambda^{p-1} + \left[\widetilde{\xi}_\rho - \widehat{\xi}_\rho\right] u_\lambda^{p-1} \quad \text{(see (3.13) and hypothesis H}(f)'\text{(i))} \\ &\quad = -\Delta_p u_\lambda - \Delta u_\lambda + \left[\xi(z) + \widetilde{\xi}_\rho\right] u_\lambda^{p-1} \quad \text{for a.a. } z \in \Omega. \end{aligned} \tag{3.17}$$

On account of (3.16), we see that

$$\left[\widetilde{\xi}_\rho - \widehat{\xi}_\rho\right] u_\mu^{p-1} \prec \left[\widetilde{\xi}_\rho - \widehat{\xi}_\rho\right] u_\lambda^{p-1}.$$

Then from (3.17) and Proposition 3.2 of Gasiński–Papageorgiou [8] we have

$$u_\lambda - u_\mu \in D_+.$$

□

From Papageorgiou–Rădulescu–Repovš [19] (see the proof of Proposition 7), we know that S_λ is downward directed. We will use this fact to show that for every $\lambda \in \mathscr{L}$ problem (P_λ) has a smallest positive solution $\overline{u}_\lambda \in S_\lambda$, that is, $\overline{u}_\lambda \leq u$ for all $u \in S_\lambda$.

Proposition 3.6. *If hypotheses* H(ξ), H(f) *hold and* $\lambda \in \mathscr{L}$, *then problem* (P_λ) *admits a smallest positive solution*

$$\overline{u}_\lambda \in \operatorname{int} C_+.$$

Proof. Since S_λ is downward directed, using Lemma 3.10, p. 178, of Hu–Papageorgiou [10], we can find $\{u_n\}_{n\geq1} \subseteq S_\lambda$ decreasing such that

$$\inf_{n\geq1} u_n = \inf S_\lambda.$$

We have

$$\langle A_p(u_n),h\rangle + \langle A(u_n),h\rangle + \int_\Omega \xi(z)u_n^{p-1}h\,dz = \int_\Omega f(z,u_n)h\,dz + \lambda\int_{\partial\Omega} u_n^{\tau-1}h\,d\sigma \tag{3.18}$$

for all $h \in W^{1,p}(\Omega)$, all $n \in \mathbb{N}$.

In (3.18) we choose $h = u_n \in W^{1,p}(\Omega)$. Since $0 \leq u_n \leq u_1$ for all $n \in \mathbb{N}$, using (2.3) and hypothesis H(f)(i), we see that

$$\{u_n\}_{n\geq1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

From Lieberman [13] (Theorem 2), we see that there exist $\alpha \in (0,1)$ and $c_4 > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq c_4 \quad \text{for all } n \in \mathbb{N}.$$

Recall that $C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ compactly. This fact and the monotonicity of the sequence $\{u_n\}_{n\geq 1}$ imply that there exists $\overline{u}_\lambda \in C^1(\overline{\Omega})$ such that

$$u_n \to \overline{u}_\lambda \text{ in } C^1(\Omega) \text{ as } n \to \infty. \tag{3.19}$$

We need to show that $\overline{u}_\lambda \neq 0$. To this end we consider the following auxiliary boundary value problem

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) + \xi(z)u(z)^{p-1} = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_{p2}} = \lambda u^{\tau-1} & \text{on } \partial\Omega\,. \\ u > 0, \lambda > 0,\ \tau < 2 < p \end{cases} \tag{Q_λ}$$

<u>Claim 1.</u> *For every $\lambda > 0$ problem (Q_λ) admits a unique solution $\widetilde{u}_\lambda \in \operatorname{int} C_+$.*

First we show the existence of a positive solution for problem (Q_λ). For this purpose we introduce the C^1-functional $\beta_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\beta_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{2}\|Du\|_2^2 - \frac{\lambda}{\tau}\int_{\partial\Omega}(u^+)^\tau \, d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (2.3) and since $\tau < 2 < p$, we see that

$$\beta_\lambda(\cdot) \text{ is coercive.}$$

Also the Sobolev embedding theorem and the compactness of the trace map, imply that

$$\beta_\lambda(\cdot) \text{ is sequentially weakly lower semicontinuous.}$$

So, we can find $\widetilde{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\beta_\lambda(\widetilde{u}_\lambda) = \min\left[\beta_\lambda(u) : u \in W^{1,p}(\Omega)\right]. \tag{3.20}$$

Since $\tau < 2 < p$, we infer that

$$\begin{aligned} & \beta_\lambda(\widetilde{u}_\lambda) < 0 = \beta_\lambda(0) \\ \Rightarrow\ & \widetilde{u}_\lambda \neq 0. \end{aligned}$$

From (3.20) we have

$$\begin{aligned} & \beta'_\lambda(\widetilde{u}_\lambda) = 0 \\ \Rightarrow\ & \langle A_p(\widetilde{u}_\lambda), h\rangle + \langle A(\widetilde{u}_\lambda), h\rangle + \int_\Omega \xi(z)|\widetilde{u}_\lambda|^{p-2}\widetilde{u}_\lambda h \, dz = \lambda \int_{\partial\Omega} (\widetilde{u}_\lambda^+)^{\tau-1} h \, d\sigma \\ & \text{for all } h \in W^{1,p}(\Omega). \end{aligned}$$

Choosing $h = -\widetilde{u}_\lambda^- \in W^{1,p}(\Omega)$ and using (2.3), we infer that

$$\widetilde{u}_\lambda \geq 0, \quad \widetilde{u}_\lambda \neq 0.$$

Moreover, as before (see the proof of Proposition 3.1), using the nonlinear regularity theory of Lieberman [13] (Theorem 2) and the nonlinear maximum principle of Pucci–Serrin [24] (p. 120), we conclude that

$$\widetilde{u}_\lambda \in \operatorname{int} C_+. \tag{3.21}$$

Now we show the uniqueness of this positive solution of problem (Q_λ). To this end, we consider the integral functional $j_\lambda : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j_\lambda(u) = \begin{cases} \frac{1}{p}\|Du^{\frac{1}{2}}\|_p^p + \frac{1}{2}\|Du^{\frac{1}{2}}\|_2^2 + \frac{1}{p}\int_\Omega \xi(z)u^{\frac{p}{2}}\,dz, & \text{if } u \geq 0,\, u^{\frac{1}{2}} \in W^{1,p}(\Omega) \\ +\infty, & \text{otherwise.} \end{cases}$$

From Diaz–Saá [6] (Lemma 1), we know that $j_\lambda(\cdot)$ is convex.

Let $\operatorname{dom} j_\lambda = \{u \in L^1(\Omega) : j_\lambda(u) < \infty\}$ (the effective domain of $j_\lambda(\cdot)$). Let $\widetilde{v}_\lambda$ be another positive solution of (Q_λ). Reasoning as we did for $\widetilde{u}_\lambda$, we show that

$$\widetilde{v}_\lambda \in \operatorname{int} C_+. \tag{3.22}$$

Then from (3.21), (3.22) and Proposition 4.1.22, p. 274, of Papageorgiou–Rădulescu–Repovš [21], we have $\frac{\widetilde{u}_\lambda}{\widetilde{v}_\lambda}, \frac{\widetilde{v}_\lambda}{\widetilde{u}_\lambda} \in L^\infty(\Omega)$. Let $h = \widetilde{u}_\lambda^2 - \widetilde{v}_\lambda^2$. For $t \in [0,1]$ we have

$$\widetilde{u}_\lambda^2 - th \in \operatorname{dom} j_\lambda \quad \text{and} \quad \widetilde{v}_\lambda^2 + th \in \operatorname{dom} j_\lambda.$$

Then $j_\lambda(\cdot)$ is Gâteaux differentiable at $\widetilde{u}_\lambda^2$ and at $\widetilde{v}_\lambda^2$ in the direction h. Moreover, using the nonlinear Green's identity, we have

$$\begin{aligned} j_\lambda'(\widetilde{u}_\lambda^2)(h) &= \frac{\lambda}{2}\int_{\partial\Omega} \widetilde{u}_\lambda^{\tau-2}(\widetilde{u}_\lambda^2 - \widetilde{v}_\lambda^2)\,d\sigma, \\ j_\lambda'(\widetilde{v}_\lambda^2)(h) &= \frac{\lambda}{2}\int_{\partial\Omega} \widetilde{v}_\lambda^{\tau-2}(\widetilde{u}_\lambda^2 - \widetilde{v}_\lambda^2)\,d\sigma. \end{aligned}$$

Since $j_\lambda(\cdot)$ is convex, we have that $j_\lambda'(\cdot)$ is monotone. Since $\tau < 2$ we have

$$\begin{aligned} & 0 \leq \frac{\lambda}{2}\int_{\partial\Omega}\left[\frac{1}{\widetilde{u}_\lambda^{2-\tau}} - \frac{1}{\widetilde{v}_\lambda^{2-\tau}}\right](\widetilde{u}_\lambda^2 - \widetilde{v}_\lambda^2)\,d\sigma \leq 0 \\ \Rightarrow \quad & \widetilde{u}_\lambda = \widetilde{v}_\lambda. \end{aligned}$$

Therefore the positive solution $\widetilde{u}_\lambda \in \operatorname{int} C_+$ is unique. This proves Claim 1.

This solution provides a lower bound for the elements of S_λ.

Claim 2. *$\widetilde{u}_\lambda \leq u$ for all $u \in S_\lambda$.*

Let $u \in S_\lambda \subseteq \operatorname{int} C_+$. We introduce the following Carathéodory function

$$b_\lambda(z,x) = \begin{cases} \lambda(x^+)^{\tau-1} & \text{if } x \leq u(z) \\ \lambda u(z)^{\tau-1} & \text{if } u(z) < x \end{cases} \quad \text{for all } (x,z) \in \partial\Omega \times \mathbb{R}. \tag{3.23}$$

We set $B_\lambda(z,x) = \int_0^x b_\lambda(z,s)\,ds$ and consider the C^1-functional $\vartheta_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\vartheta_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{2}\|Du\|_2^2 - \int_{\partial\Omega} B_\lambda(z,u)\,d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (3.23) and (2.3) it is clear that $\vartheta_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\vartheta_\lambda(\widehat{u}_\lambda) = \inf\left[\vartheta_\lambda(u) : u \in W^{1,p}(\Omega)\right]. \tag{3.24}$$

As before (see Claim 1), since $\tau < 2 < p$, we see that

$$\begin{aligned} & \vartheta_\lambda(\widehat{u}_\lambda) < 0 = \vartheta_\lambda(0) \\ \Rightarrow \quad & \widehat{u}_\lambda \neq 0. \end{aligned}$$

From (3.24) we have

$$\begin{aligned} & \vartheta'_\lambda(\widehat{u}_\lambda) = 0 \\ \Rightarrow \quad & \langle A_p(\widehat{u}_\lambda), h\rangle + \langle A(\widehat{u}_\lambda), h\rangle + \int_\Omega \xi(z)|\widehat{u}_\lambda|^{p-2}\widehat{u}_\lambda h \,\mathrm{d}z = \int_{\partial\Omega} b_\lambda(z, \widehat{u}_\lambda)h \,\mathrm{d}\sigma \\ & \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.25}$$

As before (see the proof of Proposition 3.2), if in (3.25) we choose first $h = -\widetilde{u}_\lambda^- \in W^{1,p}(\Omega)$ and then $h = (\widehat{u}_\lambda - u)^+ \in W^{1,p}(\Omega)$ and using (3.23), we show that

$$\widehat{u}_\lambda \in [0, u] \setminus \{0\}. \tag{3.26}$$

From (3.26), (3.23), (3.25) and Claim 1, it follows that

$$\begin{aligned} & \widehat{u}_\lambda = \widetilde{u}_\lambda \\ \Rightarrow \quad & \widetilde{u}_\lambda \leq u \quad \text{for all } u \in S_\lambda \text{ (see (3.26))}. \end{aligned}$$

This proves Claim 2.

From (3.19) and Claim 2, we have

$$\begin{aligned} & \widetilde{u}_\lambda \leq \overline{u}_\lambda \\ \Rightarrow \quad & \overline{u}_\lambda \neq 0 \text{ and so } \overline{u}_\lambda \in S_\lambda \subseteq \operatorname{int} C_+,\ \overline{u}_\lambda = \inf S_\lambda. \end{aligned}$$

□

Proposition 3.7. *If hypotheses* H(ξ), H(f) *hold and* $0 < \mu < \lambda \in \mathcal{L}$, *then*

(a) $\overline{u}_\mu \leq \overline{u}_\lambda$;

(b) $\widetilde{u}_\mu \leq \widetilde{u}_\lambda$.

Proof.

(a) Let $\overline{u}_\lambda \in \operatorname{int} C_+$ be the minimal positive solution of problem (P_λ) (see Proposition 3.6). On account of Corollary 3.3, we can find $u_\mu \in S_\mu \in \operatorname{int} C_+$ such that

$$\begin{aligned} & u_\mu \leq \overline{u}_\lambda \\ \Rightarrow \quad & \overline{u}_\mu \leq \overline{u}_\lambda \quad \text{recall that } \overline{u}_\mu \leq u \text{ for all } u \in S_\mu. \end{aligned}$$

(b) Let $\widetilde{e}_\mu(z, x)$ be the Carathéodory function defined by

$$\widetilde{e}_\mu(z, x) = \begin{cases} \mu(x^+)^{\tau-1} & \text{if } x \leq \widetilde{u}_\lambda(z) \\ \mu\widetilde{u}_\lambda(z)^{\tau-1} & \text{if } \widetilde{u}_\lambda(z) < x \end{cases} \quad \text{for all } (z, x) \in \partial\Omega \times \mathbb{R}. \tag{3.27}$$

We set $\widetilde{E}_\mu(z, x) = \int_0^x \widetilde{e}_\mu(z, s)\,\mathrm{d}s$ and consider the C^1-functional $\widetilde{\varphi}_\mu : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{\varphi}_\mu(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{2}\|Du\|_2^2 - \int_{\partial\Omega} \widetilde{E}_\mu(z, u)\,\mathrm{d}z \quad \text{for all } u \in W^{1,p}(\Omega).$$

Evidently $\widetilde{\varphi}_\mu(\cdot)$ is coercive (see (3.27) and (2.3)) and sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_\mu \in W^{1,p}(\Omega)$ such that

$$\widetilde{\varphi}_\mu(\widehat{u}_\mu) = \inf\left[\widetilde{\varphi}_\mu(u) : u \in W^{1,p}(\Omega)\right] < 0 = \widetilde{\varphi}_\mu(0) \quad (\text{since } \tau < 2 < p)$$
$$\Rightarrow \quad \widehat{u}_\mu \neq 0.$$

We have

$$\langle \widetilde{\varphi}'_\mu(\widehat{u}_\mu), h\rangle = 0 \quad \text{for all } h \in W^{1,p}(\Omega).$$

Choosing $h = -\widehat{u}_\mu^- \in W^{1,p}(\Omega)$ and $h = (\widehat{u}_\mu - \widetilde{u}_\lambda)^+ \in W^{1,p}(\Omega)$, we obtain

$$\begin{aligned} & \widehat{u}_\mu \in [0, \widetilde{u}_\lambda],\ \widehat{u}_\mu \neq 0 \\ \Rightarrow \quad & \widehat{u}_\mu = \widetilde{u}_\mu \in \operatorname{int} C_+ \quad (\text{see (3.27) and Claim 1 in the proof of Proposition 3.6}) \\ \Rightarrow \quad & \widetilde{u}_\mu \leq \widetilde{u}_\lambda. \end{aligned}$$

□

Let $0 < \mu < \lambda$ and $\eta_0 = \frac{\eta}{\mu}$. Then $\eta \leq \lambda\eta_0$. Motivated by hypothesis H(f)(i), we consider the following auxiliary boundary value problem

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) + \xi(z)u(z)^{p-1} = \lambda\eta_0 u(z)^{r-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_{p2}} = \lambda u^{\tau-1} & \text{on } \partial\Omega, \\ u > 0, \lambda > 0, \tau < 2 < p < r. \end{cases} \tag{R$_\lambda$}$$

Reasoning as in the proofs of Propositions 3.1 and 3.6, we obtain the following result.

Proposition 3.8. *If hypothesis* H(ξ) *holds and* $\lambda \in \mathscr{L}$, *then problem* (R$_\lambda$) *admits a smallest positive solution* $u_\lambda^* \in \operatorname{int} C_+$ *and there exists* $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ *such that*

$$\widetilde{u}_\lambda \leq u_\lambda \leq u_\lambda^*.$$

Let $\lambda^* = \sup \mathscr{L}$.

Proposition 3.9. *If hypotheses* H(ξ), H(f) *hold, then* $\lambda^* < \infty$.

Proof. Let $\mu \in (0,\lambda)$ and set $0 < \widetilde{m}_\mu = \min_{\overline{\Omega}} \widetilde{u}_\mu$ (recall that $\widetilde{u}_\mu \in \operatorname{int} C_+$). From Propositions 3.8 and 3.7(b), we have

$$0 < \widetilde{m}_\mu \leq \widetilde{u}_\lambda \leq u_\lambda^*.$$

We have

$$\begin{cases} -\Delta_p u_\lambda^* - \Delta u_\lambda^* + \xi(z)(u_\lambda^*)^{p-1} = \lambda\eta_0 (u_\lambda^*)^{r-1} & \text{in } \Omega, \\ \dfrac{\partial u_\lambda^*}{\partial n_{p2}} = \lambda (u_\lambda^*)^{\tau-1} & \text{on } \partial\Omega, \\ \lambda > 0,\ \tau < 2 < p < r. \end{cases} \tag{3.28}$$

Let $a(z) = \eta_0(u_\lambda^*(z))^{r-2}$ and $d(z) = u_\lambda^*(z)^{\tau-2}$. Then $a \in L^\infty(\Omega)$ and $d \in C(\overline{\Omega})$. We rewrite (3.28) using $a(\cdot)$ and $d(\cdot)$. So, we have

$$\begin{cases} -\Delta_p u_\lambda^* - \Delta u_\lambda^* + \xi(z)(u_\lambda^*)^{p-1} = \lambda a(z) u_\lambda^* & \text{in } \Omega, \\ \dfrac{\partial u_\lambda^*}{\partial n_{p2}} = \lambda d(z) u_\lambda^* & \text{on } \partial\Omega, \\ \lambda > 0. \end{cases} \tag{3.29}$$

Let

$$\widehat{W}_p = \left\{ w \in W^{1,p}(\Omega) : k(w) = \int_\Omega a(z) w \, dz + \int_{\partial\Omega} d(z) w \, d\sigma = 0 \right\}.$$

We have $W^{1,p}(\Omega) = \mathbb{R} \oplus \widehat{W}_p$ (see Abreu-Madeira [1], Lemma 2.2). Then from (3.29) and Theorem 1.1 of [1], we have

$$0 < \lambda \le \widehat{c} \inf \left[\frac{\frac{1}{p}\gamma_p(w) + \frac{1}{2}\|Dw\|_2^2}{k(w)} : w \in \widehat{W}_p, \ w \neq 0 \right] < \infty \quad \text{for some } \widehat{c} > 0.$$

This fact combined with Proposition 3.8 implies that we have $\lambda^* < \infty$. □

Proposition 3.10. *If hypotheses* H(ξ), H(f)$'$ *hold and* $\lambda \in (0, \lambda^*)$, *then problem* ($P_\lambda$) *admits at least two positive solutions:*

$$u_0, \widehat{u} \in \operatorname{int} C_+, \ u_0 \le \widehat{u}, \ u_0 \neq \widehat{u}.$$

Proof. Let $\vartheta \in (\lambda, \lambda^*)$. Using Proposition 3.5 we can find $u_0 \in S_\lambda \subseteq \operatorname{int} C_+$ and $u_\vartheta \in S_\vartheta \subseteq \operatorname{int} C_+$ such that

$$u_\vartheta - u_0 \in D_+. \tag{3.30}$$

We introduce the following truncations of the data of (P_λ)

$$\widehat{\mu}(z,x) = \begin{cases} f(z, u_0(z)) & \text{if } x \le u_0(z) \\ f(z,x) & \text{if } u_0(z) < x \end{cases} \quad \text{for all } (z,x) \in \Omega \times \mathbb{R}, \tag{3.31}$$

$$\widehat{w}_\lambda(z,x) = \begin{cases} \lambda u_0(z)^{\tau-1} & \text{if } x \le u_0(z) \\ \lambda x^{\tau-1} & \text{if } u_0(z) < x \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathbb{R}. \tag{3.32}$$

These are Carathéodory functions. We set

$$\widehat{M}(z,x) = \int_0^x \widehat{\mu}(z,s) \, ds \quad \text{and} \quad \widehat{W}_\lambda(z,x) = \int_0^x \widehat{w}_\lambda(z,s) \, ds$$

and consider the C^1-functional $\widehat{d}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{d}_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \widehat{M}(z,u) \, dz - \int_{\partial\Omega} \widehat{W}_\lambda(z,u) \, d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

In addition, we introduce the following truncations of $\widehat{\mu}(z,\cdot)$ and of $\widehat{w}_\lambda(z,\cdot)$

$$\widehat{\mu}_0(z,x) = \begin{cases} \widehat{\mu}(z,x) & \text{if } x \le u_\vartheta(z) \\ \widehat{\mu}(z, u_\vartheta(z)) & \text{if } u_\vartheta(z) < x \end{cases} \quad \text{for all } (z,x) \in \Omega \times \mathbb{R}, \tag{3.33}$$

$$\widehat{w}_\lambda^0(z,x) = \begin{cases} \widehat{w}_\lambda(z,x) & \text{if } x \le u_\vartheta(z) \\ \widehat{w}_\lambda(z, u_\vartheta(z)) & \text{if } u_\vartheta(z) < x \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathbb{R}. \tag{3.34}$$

These are Carathéodory functions. We set

$$\widehat{M}_0(z,x) = \int_0^x \widehat{\mu}_0(z,s) \, ds \quad \text{and} \quad \widehat{W}_\lambda^0(z,x) = \int_0^x \widehat{w}_\lambda^0(z,s) \, ds$$

and consider the C^1-functional $\widehat{d}_\lambda^0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{d}_\lambda^0(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \widehat{M}_0(z,u) \, dz - \int_{\partial\Omega} \widehat{W}_\lambda^0(z,u) \, d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (3.31), (3.32), (3.33) and (3.34) it is clear that

$$\widehat{d}_\lambda\Big|_{[0,u_\vartheta]} = \widehat{d}^0_\lambda\Big|_{[0,u_\vartheta]} \quad \text{and} \quad \widehat{d}'_\lambda\Big|_{[0,u_\vartheta]} = \left(\widehat{d}^0_\lambda\right)'\Big|_{[0,u_\vartheta]}. \tag{3.35}$$

Moreover, we have

$$K_{\widehat{d}_\lambda} \subseteq [u_0) \cap \operatorname{int} C_+ \quad \text{(see (3.31), (3.32))} \tag{3.36}$$

$$K_{\widehat{d}^0_\lambda} \subseteq [u_0, u_\vartheta] \cap \operatorname{int} C_+ \quad \text{(see (3.33), (3.34))}. \tag{3.37}$$

From (3.35) and (3.36) we see that without any loss of generality we may assume that

$$K_{\widehat{d}_\lambda} \cap [0, u_\vartheta] = \{u_0\}. \tag{3.38}$$

Otherwise we already have a second positive smooth solution of (P_λ) bigger than u_0 (see (3.36)) and so we are done.

From (3.33), (3.34) and (2.3) it is clear that $\widehat{d}^0_\lambda(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$\begin{aligned}
&\widehat{d}^0_\lambda(\widetilde{u}_0) = \min\left[\widehat{d}^0_\lambda(u) : u \in W^{1,p}(\Omega)\right] \\
\Rightarrow\ & \widetilde{u}_0 \in [u_0, u_\vartheta] \cap \operatorname{int} C_+ \quad \text{(see (3.37))} \\
\Rightarrow\ & \widetilde{u}_0 \in K_{\widehat{d}_\lambda} \quad \text{(see (3.35))} \\
\Rightarrow\ & \widetilde{u}_0 = u_0 \quad \text{(see (3.38))}.
\end{aligned}$$

From (3.30) and (3.35) it follows that

$$\begin{aligned}
& u_0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } d_\lambda \\
\Rightarrow\ & u_0 \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } d_\lambda \\
& \text{(see Papageorgiou–Rădulescu [17], Proposition 2.12).}
\end{aligned}$$

We assume that $K_{\widehat{d}_\lambda}$ is finite or otherwise on account of (3.36) we already have an infinity of positive smooth solutions bigger than u_0 and so we are done. Invoking Theorem 5.7.6, p. 449, of Papageorgiou–Rădulescu–Repovš [21], we can find $\rho \in (0,1)$ small such that

$$\widehat{d}_\lambda(u_0) < \inf\left[d_\lambda(u) : \|u - u_0\| = \rho\right] = \widehat{m}_\lambda. \tag{3.39}$$

Moreover, on account of hypothesis H$(f)'$(ii)=H(f)(ii), we have that

$$\widehat{d}_\lambda(\cdot) \text{ satisfies the Palais–Smale condition} \tag{3.40}$$

and if $u \in \operatorname{int} C_+$, then

$$\widehat{d}_\lambda(tu) \to -\infty \text{ as } t \to +\infty. \tag{3.41}$$

Then (3.39), (3.40) and (3.41) permit the use of the mountain pass theorem. So, we can find $\widehat{u} \in W^{1,p}(\Omega)$ such that

$$\begin{aligned}
& \widehat{u} \in K_{\widehat{d}_\lambda} \quad \text{and} \quad \widehat{m}_\lambda \le d_\lambda(\widehat{u}) \\
\Rightarrow\ & u_0 \le \widehat{u} \in \operatorname{int} C_+ \text{ (see (3.36))},\ u_0 \ne \widehat{u} \text{ (see (3.39))},\ \widehat{u} \in S_\lambda \text{ (see (3.31), (3.32))}.
\end{aligned}$$ □

Proposition 3.11. *If hypotheses* H(ξ), H(f) *hold, then* $\lambda^* \in \mathscr{L}$.

Proof. Let $\lambda_n \uparrow \lambda^*$ as $n \to \infty$. We can find $u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+$, $n \in \mathbb{N}$, such that

$$\varphi_{\lambda_n}(u_n) < 0 \quad \text{for all } n \in \mathbb{N} \text{ (see the proof of Proposition 3.2)}, \tag{3.42}$$
$$\varphi'_{\lambda_n}(u_n) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.43}$$

From (3.42), (3.43) and hypothesis H(f)(ii) (the AR-condition) we deduce that

$$\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u_{\lambda^*} \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_{\lambda^*} \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.44}$$

From (3.43) we have

$$\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h \,\mathrm{d}z = \int_\Omega f(z, u_n) h \,\mathrm{d}z + \lambda_n \int_{\partial\Omega} u_n^{\tau-1} h \,\mathrm{d}\sigma \tag{3.45}$$

for all $h \in W^{1,p}(\Omega)$.

We choose $h = u_n - u_{\lambda^*} \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.44). Then

$$\begin{aligned}
&\lim_{n\to\infty} \left[\langle A_p(u_n), u_n - u_{\lambda^*}\rangle + \langle A(u_n), u_n - u_{\lambda^*}\rangle\right] = 0 \\
\Rightarrow\quad &\limsup_{n\to+\infty} \left[\langle A_p(u_n), u_n - u_{\lambda^*}\rangle + \langle A(u_{\lambda^*}), u_n - u_{\lambda^*}\rangle\right] \leq 0 \quad \text{(since } A(\cdot) \text{ is monotone)} \\
\Rightarrow\quad &\limsup_{n\to\infty} \langle A_p(u_n), u_n - u_{\lambda^*}\rangle \leq 0 \quad \text{(see (3.44))} \\
\Rightarrow\quad &u_n \to u_{\lambda^*} \quad \text{in } W^{1,p}(\Omega) \text{ (see Proposition 2.1).}
\end{aligned} \tag{3.46}$$

Passing to the limit as $n \to \infty$ in (3.45) and using (3.46), we obtain

$$\langle A_p(u_{\lambda^*}), h\rangle + \langle A(u_{\lambda^*}), h\rangle + \int_\Omega \xi(z) u_{\lambda^*}^{p-1} h \,\mathrm{d}z = \int_\Omega f(z, u_{\lambda^*}) h \,\mathrm{d}z + \lambda^* \int_{\partial\Omega} u_{\lambda^*}^{\tau-1} h \,\mathrm{d}\sigma \tag{3.47}$$

for all $h \in W^{1,p}(\Omega)$,

$$\widetilde{u}_{\lambda_1} \leq u_\lambda \quad \text{(see Claim 2 in the proof of Proposition 3.6 and Proposition 3.7(b))}. \tag{3.48}$$

From (3.47) and (3.48) we infer that

$$u_{\lambda^*} \in S_{\lambda^*}, \text{ that is, } \lambda^* \in \mathscr{L}.$$ □

Therefore we have

$$\mathscr{L} = (0, \lambda^*].$$

Next we examine the properties of the minimal solution map $\lambda \mapsto \overline{u}_\lambda$ from $\mathscr{L}$ into $C^1(\overline{\Omega})$.

Proposition 3.12. *If hypotheses* H(ξ), H(f)$'$ *hold, then the minimal solution map* $\lambda \mapsto \overline{u}_\lambda$ *from* $\mathscr{L}$ *into* $C^1(\overline{\Omega})$ *is*

(a) *strictly increasing in the sense that*

$$0 < \mu < \lambda \leq \lambda^* \quad \Rightarrow \quad \overline{u}_\lambda - \overline{u}_\mu \in D_+;$$

(b) *left continuous.*

Proof.

(a) Let $0 < \mu < \lambda \leq \lambda^*$. According to Proposition 3.5, we can find $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that

$$\begin{aligned} & \overline{u}_\lambda - u_\mu \in D_+ \\ \Rightarrow \quad & \overline{u}_\lambda - \overline{u}_\mu \in D_+ \quad (\text{since } \overline{u}_\mu \leq u \text{ for all } u \in S_\mu). \end{aligned}$$

(b) Let $\lambda_n \uparrow \lambda \in \mathscr{L}$. We have $\overline{u}_n = \overline{u}_{\lambda_n} \leq \overline{u}_{\lambda^*} \in \operatorname{int} C_+$ for all $n \in \mathbb{N}$. So, from Theorem 2 of Lieberman [13], we know that there exist $\alpha \in (0,1)$ and $c_5 > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq c_5 \quad \text{for all } n \in \mathbb{N}. \tag{3.49}$$

Exploiting the fact that $C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ compactly and the monotonicity of $\{\overline{u}_n\}_{n\geq 1}$ (see part (a)), from (3.49) we have

$$\overline{u}_n \to \widehat{u}_\lambda \quad \text{in } C^1(\overline{\Omega}). \tag{3.50}$$

If $\widehat{u}_\lambda \neq \overline{u}_\lambda$, then we can find $z_0 \in \overline{\Omega}$ such that $\overline{u}_\lambda(z_0) < \widehat{u}_\lambda(z_0)$. On account of (3.50) we have

$$\overline{u}_\lambda(z_0) < \overline{u}_n(z_0) \quad \text{for all } n \geq n_0,$$

which contradicts part (a). So, we conclude that $\lambda \mapsto \overline{u}_\lambda$ is left continuous. □

The following bifurcation-type theorem describes the dependence on the parameter $\lambda > 0$ of the set of positive solutions of (P_λ).

Theorem 3.13. *If hypotheses* $H(\xi)$, $H(f)'$ *hold, then there exists* $\lambda^* > 0$ *such that*

(a) *for all* $\lambda \in (0, \lambda^*)$ *problem* (P_λ) *admits at least two positive solutions*

$$u_0, \widehat{u} \in \operatorname{int} C_+,\ u_0 \leq \widehat{u},\ u_0 \neq \widehat{u};$$

(b) *for* $\lambda = \lambda^*$ *problem* (P_λ) *has at least one positive solution* $u_{\lambda^*} \in \operatorname{int} C_+$*;*

(c) *for all* $\lambda > \lambda^*$ *there are no positive solutions;*

(d) *for all* $\lambda \in \mathscr{L} = (0, \lambda^*]$ *problem* (P_λ) *has a smallest positive solution*

$$\overline{u}_\lambda \in \operatorname{int} C_+$$

and the map $\lambda \mapsto \overline{u}_\lambda$ *from* $\mathscr{L}$ *into* $C^1(\overline{\Omega})$ *is*

- *strictly increasing, that is,* $0 < \mu < \lambda \leq \lambda^* \Rightarrow \overline{u}_\lambda - \overline{u}_\mu \in D_+$*;*
- *left continuous.*

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Fractional integral inequalities and global solutions of fractional differential equations

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Abstract. New fractional integral inequalities are established, which generalize some famous inequalities. Then we apply these new fractional integral inequalities to study global existence results for fractional differential equations.

Keywords: fractional integral inequalities, fixed point theorem, fractional differential equations, global solutions.

2010 Mathematics Subject Classification: 34A12, 39B62, 34A08.

1 Introduction

In [9, p. 190], Henry obtained the following result about weakly singular Gronwall type inequality.

Theorem 1.1. *Let a, b, α, β be nonnegative constants with $\alpha < 1$, $\beta < 1$. Suppose that $u \in L^1[0, T]$ satisfies*

$$u(t) \le at^{-\alpha} + b\int_0^t (t-s)^{-\beta}u(s)ds, \quad a.e.\ t \in (0, T]. \tag{1.1}$$

Then there is a constant $C(b, \beta, T)$ such that

$$u(t) \le \frac{at^{-\alpha}}{1-\alpha}C(b, \beta, T), \quad a.e.\ t \in (0, T]. \tag{1.2}$$

One version of a doubly singular case of Henry is the following, cf. [9, p. 189].

Theorem 1.2. *Suppose $\beta > 0$, $\gamma > 0$, $\beta + \gamma > 1$ and $a \ge 0$, $b \ge 0$, u is nonnegative and $t^{\gamma-1}u(t)$ is locally integrable on $0 \le t < T$, and u satisfies*

$$u(t) \le a + b\int_0^t (t-s)^{\beta-1}s^{\gamma-1}u(s)ds, \quad a.e.\ t \in [0, T). \tag{1.3}$$

Then

$$u(t) \le aE_{\beta,\gamma}\left(b\Gamma(\beta)^{\frac{1}{\beta+\gamma-1}}t\right), \tag{1.4}$$

where $E_{\beta,\gamma}(z)$ is given by an infinite series related to the two-parameter Mittag-Leffler function.

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Since fractional integral inequality is a well-known tool in the study of fractional differential equations and evolution equations, Henry's work was followed by many scholars (for example, see [6,12–14,19,21–23]). Recently, by the Hölder inequality and a method introduced by Medved' [13,14], Zhu [22] considered the following inequality

Theorem 1.3. *Let $0 < T \leq \infty$, $\beta > 0$, $a(t)$, $b(t)$ and $l(t)$ be continuous, nonnegative functions on $[0, T)$, and $u(t)$ be a continuous, nonnegative function on $[0, T)$ with*

$$u(t) \leq a(t) + \frac{b(t)}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l(s) u(s) ds, \tag{1.5}$$

then

$$u(t) \leq \left(A(t) + B(t) \int_0^t L(s) A(s) \exp\left(\int_s^t L(\tau) B(\tau) d\tau \right) ds \right)^{\frac{1}{p}}, \tag{1.6}$$

where

$$A(t) = 2^{p-1} a^p(t), \qquad B(t) = 2^{p-1} \left(\frac{b(t)}{\Gamma(\beta)(q(\beta-1)+1)^{\frac{1}{q}}} t^{\beta-1+\frac{1}{q}} \right)^p, \qquad L(t) = l^p(t),$$

and $p, q \in (0, \infty)$ such that $\frac{1}{q} + \beta > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$.

By a reduction to the classical Gronwall inequality, Webb [19] studied the following Gronwall type inequality with a double singularity.

Theorem 1.4. *Let $a, b \geq 0$ and $c > 0$ be constants. Let $0 < \alpha, \beta, \gamma < 1$ with $\alpha + \gamma < 1$ and $\beta + \gamma < 1$. Suppose that $u(t)t^{\alpha} \in L^{\infty}_{+}[0, T]$ and u satisfies*

$$u(t) \leq at^{-\alpha} + b + c \int_0^t (t-s)^{-\beta} s^{-\gamma} u(s) ds, \quad a.e.\ t \in (0, T]. \tag{1.7}$$

Then we have, for a.e. $t \in (0, T]$,

$$\begin{aligned} u(t) \leq\ & at^{-\alpha} + acB_1 t^{-\alpha+1-\beta-\gamma} + ac^2 B_1 B_2 t^{-\alpha+2(1-\beta-\gamma)} + \ldots \\ & + (b + ac^m B_1 B_2 \ldots B_m t^{-\alpha+m(1-\beta-\gamma)}) \exp\left(\frac{ct_{r_1}^{-\beta}}{1-\beta-\gamma} t^{1-\gamma} \right), \end{aligned} \tag{1.8}$$

where m is the smallest positive integer such that $m(1-\beta-\gamma) - \alpha \geq 0$, $r_1 = \frac{\beta}{1-\gamma}$, and for $n \in \mathbb{N}$, $B_n = B(1-\beta, 1-\alpha-\gamma+(n-1)(1-\beta-\gamma))$. In particular, there is an explicit constant $C(b, c, \beta, \gamma, T)$ such that $u(t) \leq at^{-\alpha} C$ for a.e. $t \in (0, T]$.

In this paper, we study the following fractional integral inequalities

$$u(t) \leq a(t) + b(t) \int_0^t (t-s)^{\beta-1} s^{-\gamma} l(s) u(s) ds, \qquad t \in [0, +\infty), \tag{1.9}$$

where $\gamma \geq 0$ and $\beta \in (0, 1)$, and

$$u(t) \leq at^{-\alpha} + bt^{-\delta} \int_0^t (t-s)^{\beta-1} l(s) u(s) ds, \qquad t \in (0, +\infty), \tag{1.10}$$

where $a, b \geq 0$, $\alpha > \delta \geq 0$ and $\beta \in (0, 1)$. The special cases $b(t) \equiv C$ or $\gamma = 0$ of the inequality (1.9) are proved in Medved' [13, Theorem 2 and Theorem 3] and Zhu [22, Theorem 2.4 and

Theorem 2.6]. Medved' also studied the inequality (1.9) in [13, Theorem 4] and obtained two different results with exponential functions for different β and γ. The conclusion of Theorem 4 in [13] has a more complicated appearance. Webb [19] obtained several results of inequality (1.10) for the special case $l(t) = t^{-\gamma}$ by reducing the inequality (1.10) to the classical Gronwall inequality. In this paper, we study the inequality (1.9) under the hypothesis $\beta \in (0,1)$ and $\gamma \geq 0$. The proof is more simple than Theorem 4 in [13]. We present a new method to study a integral inequality which was first studied by Willett [20]. By this integral inequality, we study the inequality (1.9) for the special cases $b(t) = t^{1-\beta}$ and $\gamma = 1-\beta$. The conclusion and the method of proof seem to be new in this case. We also obtain some results of the inequality (1.10) and examples show our results are improvements on [19].

Fractional differential equations (FDEs) have been of great interest in the past three decades. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications in various sciences. Recently, many researchers began to investigate the existence of solutions of nonlinear fractional differential equations (for example, see [4–6,8,11, 12,18,19,21–24] and references therein). In this paper, we continue to investigate the existence and uniqueness of global solutions of the following initial value problem

$$\begin{cases} D_r^{\beta} x(t) = f(t, x(t)) \qquad t \in (0, +\infty), \qquad \beta \in (0,1), \\ \lim_{t\to 0^+} t^{1-\beta} x(t) = x_0, \end{cases} \tag{1.11}$$

where D_r^{β} is the Riemann–Liouville fractional derivative. It should be pointed out that such global existence results are fundamental in the theory of fractional differential equations and crucial in stability analysis of fractional differential equations.

The existence and uniqueness of global solutions of the fractional differential equation (1.11) have been studied by many scholars. For example, under the assumption that f satisfies an inequality of the form

$$|f(t,x)| \leq p(t)\omega\left(\frac{|x|}{1+t^2}\right) + q(t),$$

Kou et al. [11] proved the global existence of solutions of fractional differential equation (1.11) in a special Banach space

$$E = \left\{ x(t) | x(t) \in C_{1-\beta}(0, +\infty), \lim_{t\to+\infty} \frac{t^{1-\beta}x(t)}{1+t^2} = 0 \right\}.$$

Trif [18] investigated the global existence of solutions to initial value problems for nonlinear fractional differential equation (1.11) by constructing a special locally convex space which is metrizable and complete. Webb [19] proved the existence results of equation (1.11) under the assumption that nonnegative function f satisfies $f(t,x) = t^{-\gamma} g(t,x)$, where $g(t,x) \leq M(1+x)$, $M > 0$ and $0 \leq \gamma < \beta$. Unlike all the previous papers, by new fractional inequality (1.9) and fixed point theorem, we present the existence and uniqueness results of the fractional differential equation (1.11). Our result includes the main result of [18, Theorem 4.2]. Finally, examples are given to illustrate the applicability of our results and can not be solved by Theorem 4.2 in [18].

2 Preliminaries

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper.

Let $\beta \in (0,1)$, denote $C_\beta(0,T] = \{x : (0,T] \to \mathbb{R} \text{ and } x(t) = t^{-\beta}y(t) \text{ for some } y \in C[0,T]\}$. Let $\|x\|_\beta = \sup_{0<t\le T} t^\beta |x(t)|$, then $C_\beta(0,T]$ endowed with the norm $\|\cdot\|_\beta$ is a Banach space. We denote $C_\beta(0,+\infty) = \{x : (0,+\infty) \to \mathbb{R} \text{ and } x(t) = t^{-\beta}y(t) \text{ for some } y \in C[0,+\infty)\}$. $L^p_{Loc}[0,+\infty)$ $(p \ge 1)$ is the space of all real valued functions which are Lebesgue integrable over every bounded subinterval of $[0,+\infty)$.

Definition 2.1. The Riemann–Liouville fractional integral of order $\beta \in (0,1)$ of a function $f \in L^1[0,T]$ is defined by

$$(I^\beta f)(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s)}{(t-s)^{1-\beta}} ds.$$

Definition 2.2. The Riemann–Liouville fractional derivative of order $\beta \in (0,1)$ of a function f where $I^{1-\beta}f$ is absolutely continuous (AC) is defined by

$$(D_r^\beta f)(t) = \frac{d}{dt}(I^{1-\beta}f)(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^\beta} ds.$$

Remark 2.3. If $f \in L^1[0,T]$, then the integral $(I^\beta f)(t)$ exists for almost every $t \in [0,T]$ and $I^\beta f \in L^1[0,T]$. If $f \in AC[0,T]$, then $D_r^\beta f$ exists almost everywhere in $[0,T]$. If $f \in I^\beta(L^1) = \{f : f = I^\beta g, g \in L^1[0,T]\}$, then $I^{1-\beta}f \in AC[0,T]$. For more details about fractional calculus, we refer the reader to the texts [7,10,16,17].

Theorem 2.4 ([3])**.** *Let $f(t,x)$ be a function that is continuous on the set*

$$\mathbf{B} = \left\{(t,x) \in \mathbb{R}^2 : 0 < t \le T, x \in I\right\},$$

where $I \subseteq \mathbb{R}$ denotes an unbounded interval. Suppose a function $x : (0,T] \to I$ is continuous and that both $x(t)$ and $f(t,x(t))$ are absolutely integrable on $(0,T]$. Then $x(t)$ satisfies the initial value problem (1.11) *on $(0,T]$ if and only if it satisfies the Volterra integral equation*

$$x(t) = x_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s)) ds \tag{2.1}$$

on $(0,T]$.

Remark 2.5. f is absolutely integrable on $(0,T]$ if f is Riemann integrable on every closed interval $[\eta,T]$, where $\eta \in (0,T]$, and $\lim_{\eta\to 0^+} \int_\eta^T |f(t)|dt$ exists and is finite. From Proposition 2.1 in [3], if $f \in L^1[0,T]$ is continuous on $(0,T]$, then f is absolutely integrable on $(0,T]$.

Lemma 2.6 ([2,17])**.** *Suppose $\rho \in L^q[0,1]$. Then*

$$\int_0^t (t-s)^{\beta-1}\rho(s) ds$$

is continuous on $[0,1]$, where $\beta \in (0,1)$ and $q > \frac{1}{\beta}$.

Theorem 2.7 ([1])**.** *Let E be a Banach space, C a closed, convex subset of E and $0 \in C$. Let $F : C \to C$ be a continuous and completely continuous map, and let the set $\{x \in E : x = \lambda Fx \text{ for some } \lambda \in (0,1)\}$ be bounded. Then F has at least one fixed point in E.*

3 Fractional integral inequalities

In this section, we are now to prove some results concerning fractional integral inequalities (1.9) and (1.10), which can be used to study the global existence of solutions of fractional differential equation (1.11).

Theorem 3.1. *Let $\beta \in (0,1)$ and $\gamma \geq 0$, $a(t)$ and $b(t)$ be nonnegative and continuous functions on $[0,+\infty)$, $l(t)$ be a nonnegative and continuous function on $(0,+\infty)$ and $t^{-\gamma}l(t) \in L^q_{Loc}[0,+\infty)$ $(q > \frac{1}{\beta})$, and $u(t)$ be a continuous, nonnegative function on $[0,+\infty)$ with*

$$u(t) \leq a(t) + b(t)\int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)u(s)ds. \tag{3.1}$$

Then

$$u(t) \leq \left(A(t) + B(t)\int_0^t L(s)A(s)\exp\left(\int_s^t L(\tau)B(\tau)d\tau\right)ds\right)^{\frac{1}{q}}, \qquad t \in [0,+\infty), \tag{3.2}$$

where $A(t) = 2^{q-1}a^q(t)$, $B(t) = \frac{2^{q-1}b^q(t)t^{q\beta-q+\frac{q}{p}}}{(p\beta-p+1)^{\frac{q}{p}}}$, $L(t) = t^{-q\gamma}l^q(t)$ and $p \in (1,+\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since $q > \frac{1}{\beta}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\beta - 1 + \frac{1}{p} > 0$. From the inequality (3.1) and using the Hölder inequality, we have

$$\begin{aligned} u(t) &\leq a(t) + b(t)\int_0^t (t-s)^{\beta-1}s^{-\gamma}l(s)u(s)ds \\ &\leq a(t) + b(t)\left(\int_0^t (t-s)^{p(\beta-1)}ds\right)^{\frac{1}{p}}\left(\int_0^t (s^{-\gamma}l(s)u(s))^q ds\right)^{\frac{1}{q}} \\ &= a(t) + \frac{b(t)t^{\beta-1+\frac{1}{p}}}{(p\beta-p+1)^{\frac{1}{p}}}\left(\int_0^t (s^{-\gamma}l(s)u(s))^q ds\right)^{\frac{1}{q}}. \end{aligned} \tag{3.3}$$

Then

$$u^q(t) \leq 2^{q-1}a^q(t) + \frac{2^{q-1}b^q(t)t^{q\beta-q+\frac{q}{p}}}{(p\beta-p+1)^{\frac{q}{p}}}\int_0^t s^{-q\gamma}l^q(s)u^q(s)ds.$$

Let $w(t) = u^q(t)$, $A(t) = 2^{q-1}a^q(t)$, $B(t) = \frac{2^{q-1}b^q(t)t^{q\beta-q+\frac{q}{p}}}{(p\beta-p+1)^{\frac{q}{p}}}$ and $L(t) = t^{-q\gamma}l^q(t)$, then

$$w(t) \leq A(t) + B(t)\int_0^t L(s)w(s)ds.$$

By the Gronwall–Beesack inequality [15, p. 356], we obtain

$$w(t) \leq A(t) + B(t)\int_0^t L(s)A(s)\exp(\int_s^t L(\tau)B(\tau)d\tau)ds.$$

Thus, we obtain the inequality (3.2) and complete the proof. □

Theorem 3.2. *Let $a, b \geq 0$, $\alpha > \delta \geq 0$ and $\beta \in (0,1)$, $l(t)$ be a nonnegative and continuous function on $(0,+\infty)$ and $t^{-\alpha}l(t) \in L^q_{Loc}[0,+\infty)$ $(q > \frac{1}{\beta})$. Suppose that $t^\alpha u(t)$ is a continuous, nonnegative function on $[0,+\infty)$ and $u(t)$ satisfies the inequality*

$$u(t) \leq at^{-\alpha} + bt^{-\delta}\int_0^t (t-s)^{\beta-1}l(s)u(s)ds, \qquad t \in (0,+\infty). \tag{3.4}$$

Then

$$u(t) \leq t^{-\alpha} \left(2^{q-1}a^q + 2^{q-1}a^q B(t) \int_0^t L(s) \exp\left(\int_s^t L(\tau)B(\tau)d\tau \right) ds \right)^{\frac{1}{q}}, \qquad t \in (0, +\infty), \quad (3.5)$$

where $B(t) = \frac{2^{q-1}b^q t^{q\alpha - q\delta + q\beta - q + \frac{q}{p}}}{(p\beta - p + 1)^{\frac{q}{p}}}$, $L(t) = t^{-q\alpha} l^q(t)$ *and* $p \in (1, +\infty)$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $v(t) = t^\alpha u(t)$, so that $v(t)$ satisfies the inequality

$$v(t) \leq a + bt^{\alpha - \delta} \int_0^t (t-s)^{\beta - 1} s^{-\alpha} l(s) v(s) ds, \qquad t \in [0, +\infty). \quad (3.6)$$

By Theorem 3.1, we obtain the inequality (3.5) and complete the proof. □

Lemma 3.3 ([20]). *Let* $1 \leq p < \infty$, $a(t)$ *and* $b(t)$ *be nonnegative continuous on* $[0, \infty)$, $l(t)$ *be a nonnegative and continuous function on* $(0, +\infty)$ *and* $l(t) \in L^1_{Loc}[0, +\infty)$. *Suppose* $u(t)$ *is a nonnegative continuous function on* $[0, +\infty)$ *with*

$$u(t) \leq a(t) + b(t) \left(\int_0^t l(s) u^p(s) ds \right)^{\frac{1}{p}}, \qquad t \in [0, \infty). \quad (3.7)$$

Then

$$u(t) \leq a(t) + b(t) \frac{\left(\int_0^t l(s)e(s)a^p(s)ds \right)^{\frac{1}{p}}}{1 - [1 - e(t)]^{\frac{1}{p}}},$$

where $e(t) = \exp(-\int_0^t l(s) b^p(s) ds)$.

Theorem 3.4. *Let* $a, b \geq 0$, $\alpha > \delta \geq 0$ *and* $\beta \in (0, 1)$, $l(t)$ *be a nonnegative and continuous function on* $(0, +\infty)$ *and* $t^{-\alpha} l(t) \in L^q_{Loc}[0, +\infty)$ $(q > \frac{1}{\beta})$. *Suppose that* $t^\alpha u(t)$ *is a continuous, nonnegative function on* $[0, +\infty)$ *and* $u(t)$ *satisfies the inequality*

$$u(t) \leq at^{-\alpha} + bt^{-\delta} \int_0^t (t-s)^{\beta - 1} l(s) u(s) ds, \qquad t \in (0, +\infty). \quad (3.8)$$

Then

$$u(t) \leq at^{-\alpha} + at^{-\alpha} B(t) \frac{\left(\int_0^t L(s)e(s)ds \right)^{\frac{1}{q}}}{1 - [1 - e(t)]^{\frac{1}{q}}}, \qquad t \in (0, +\infty), \quad (3.9)$$

where $B(t) = \frac{bt^{\alpha - \delta + \beta - 1 + \frac{1}{p}}}{(p\beta - p + 1)^{\frac{1}{p}}}$, $L(t) = t^{-q\alpha} l^q(t)$, $e(t) = \exp(-\int_0^t L(s) B^q(s) ds)$, *and* $p \in (1, +\infty)$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $v(t) = t^\alpha u(t)$ and using the Hölder inequality, we have

$$\begin{aligned} v(t) &\leq a + bt^{\alpha - \delta} \left(\int_0^t (t-s)^{p(\beta - 1)} ds \right)^{\frac{1}{p}} \left(\int_0^t (s^{-\alpha} l(s) v(s))^q ds \right)^{\frac{1}{q}} \\ &= a + \frac{bt^{\alpha - \delta + \beta - 1 + \frac{1}{p}}}{(p\beta - p + 1)^{\frac{1}{p}}} \left(\int_0^t s^{-q\alpha} l^q(s) v^q(s) ds \right)^{\frac{1}{q}}. \end{aligned} \quad (3.10)$$

By Lemma 3.3, we get

$$v(t) \le a + aB(t)\frac{\left(\int_0^t L(s)e(s)ds\right)^{\frac{1}{q}}}{1-[1-e(t)]^{\frac{1}{q}}},$$

where $B(t) = \frac{bt^{\alpha-\delta+\beta-1+\frac{1}{p}}}{(p\beta-p+1)^{\frac{1}{p}}}$, $L(t) = t^{-q\alpha}l^q(t)$ and $e(t) = \exp(-\int_0^t L(s)B^q(s)ds)$. Then we obtain the inequality (3.9) and complete the proof. □

In [20], Willett studied the inequality (3.7) by using the Minkowski inequality. Now, we use a new method to study the inequality (3.7).

Lemma 3.5. *Let $1 \le p < \infty$, $a(t)$ and $b(t)$ be continuous and nonnegative functions on $[0,\infty)$, nonnegative function $l(t) \in L^p_{Loc}[0,+\infty)$, and $u(t)$ be a continuous and nonnegative function with*

$$u(t) \le a(t) + b(t)\left(\int_0^t l^p(s)u^p(s)ds\right)^{\frac{1}{p}}, \qquad t \in [0,\infty). \tag{3.11}$$

Then

$$u(t) \le a(t) + b(t)\left(A(t)\exp(\int_0^t L(s)ds)\right)^{\frac{1}{p}}, \qquad t \in [0,\infty), \tag{3.12}$$

where $A(t) = \int_0^t 2^{p-1}l^p(s)a^p(s)ds$ and $L(t) = 2^{p-1}l^p(t)b^p(t)$.

Proof. From (3.11), we know

$$l(t)u(t) \le l(t)a(t) + l(t)b(t)\left(\int_0^t l^p(s)u^p(s)ds\right)^{\frac{1}{p}}$$

and

$$\begin{aligned}\int_0^t l^p(s)u^p(s)ds &\le \int_0^t \left(l(s)a(s) + l(s)b(s)\left(\int_0^s l^p(\tau)u^p(\tau)d\tau\right)^{\frac{1}{p}}\right)^p ds \\ &\le \int_0^t 2^{p-1}l^p(s)a^p(s) + 2^{p-1}l^p(s)b^p(s)\int_0^s l^p(\tau)u^p(\tau)d\tau ds.\end{aligned} \tag{3.13}$$

Let $w(t) = \int_0^t l^p(s)u^p(s)ds$, $A(t) = \int_0^t 2^{p-1}l^p(s)a^p(s)ds$ and $L(t) = 2^{p-1}l^p(t)b^p(t)$, then

$$w(t) \le A(t) + \int_0^t L(s)w(s)ds.$$

Since $A(t)$ is a nondecreasing function and using Gronwall integral inequality, thus we obtain

$$w(t) \le A(t)\exp\left(\int_0^t L(s)ds\right).$$

Thus, we obtain the inequality (3.12) and complete the proof. □

If we replace $b(t)$ by $t^{1-\beta}$ and γ by $1-\beta$ in Theorem 3.1, and using Lemma 3.5, we can obtain the following conclusions under the hypotheses $l(t) \in L^q_{Loc}[0,+\infty)$.

Theorem 3.6. *Let $\beta \in (0,1)$, $a(t)$ be a nonnegative and continuous function on $[0,+\infty)$, $l(t)$ be a nonnegative and continuous function on $(0,+\infty)$ and $l(t) \in L^q_{Loc}[0,+\infty)(q > \frac{1}{\beta})$, and $u(t)$ be a continuous, nonnegative function on $[0,+\infty)$ with*

$$u(t) \le a(t) + t^{1-\beta}\int_0^t (t-s)^{\beta-1}s^{\beta-1}l(s)u(s)ds. \tag{3.14}$$

Then

$$u(t) \le a(t) + b(t)\left(A(t)\exp\left(\int_0^t L(s)ds\right)\right)^{\frac{1}{q}}, \qquad t \in [0,\infty), \tag{3.15}$$

where $b(t) = \frac{2^{\frac{1}{p}} t^{\beta-1+\frac{1}{p}}}{(p\beta-p+1)^{\frac{1}{p}}}$, $A(t) = \int_0^t 2^{q-1}l^q(s)a^q(s)ds$, $L(t) = 2^{q-1}l^q(t)b^q(t)$ and $p \in (1,+\infty)$ such that $\frac{1}{p}+\frac{1}{q} = 1$.

Proof. Since $q > \frac{1}{\beta}$ and $\frac{1}{p}+\frac{1}{q} = 1$, then $1 < p < \frac{1}{1-\beta}$. From the inequality (3.14) we have

$$\begin{aligned}
u(t) &\le a(t) + \int_0^t \left(\frac{t}{(t-s)s}\right)^{1-\beta} l(s)u(s)ds \\
&= a(t) + \int_0^t \left(\frac{1}{t-s}+\frac{1}{s}\right)^{1-\beta} l(s)u(s)ds \\
&\le a(t) + \left(\int_0^t \left(\frac{1}{t-s}+\frac{1}{s}\right)^{p(1-\beta)} ds\right)^{\frac{1}{p}} \left(\int_0^t (l(s)u(s))^q ds\right)^{\frac{1}{q}} \\
&\le a(t) + \left(\int_0^t (t-s)^{p(\beta-1)} + s^{p(\beta-1)}ds\right)^{\frac{1}{p}} \left(\int_0^t (l(s)u(s))^q ds\right)^{\frac{1}{q}} \\
&= a(t) + \frac{2^{\frac{1}{p}} t^{\beta-1+\frac{1}{p}}}{(p\beta-p+1)^{\frac{1}{p}}} \left(\int_0^t l^q(s)u^q(s)ds\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.16}$$

Let $b(t) = \frac{2^{\frac{1}{p}} t^{\beta-1+\frac{1}{p}}}{(p\beta-p+1)^{\frac{1}{p}}}$. Then by Lemma 3.5, we obtain the inequality (3.15). □

Corollary 3.7. *Let $\beta \in (0,1)$ and $u_0 > 0$, $l(t)$ be a nonnegative and continuous function on $(0,+\infty)$ and $l(t) \in L^q_{Loc}[0,+\infty)$ $(q > \frac{1}{\beta})$, and nonnegative function $u(t) \in C_{1-\beta}(0,+\infty)$ with*

$$u(t) \le u_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}l(s)u(s)ds, \qquad t \in (0,+\infty). \tag{3.17}$$

Then

$$u(t) \le u_0 t^{\beta-1} + t^{\beta-1}b(t)\left(A(t)\exp\left(\int_0^t L(s)ds\right)\right)^{\frac{1}{q}}, \qquad t \in (0,+\infty), \tag{3.18}$$

where $b(t) = \frac{2^{\frac{1}{p}} t^{\beta-1+\frac{1}{p}}}{\Gamma(\beta)(p\beta-p+1)^{\frac{1}{p}}}$, $A(t) = \int_0^t 2^{q-1}u_0^q l^q(s)ds$, $L(t) = 2^{q-1}l^q(t)b^q(t)$ and $p \in (1,+\infty)$ such that $\frac{1}{p}+\frac{1}{q} = 1$.

Proof. Since $u(t) \in C_{1-\beta}(0,+\infty)$, then $v(t) = t^{1-\beta}u(t) \in C[0,+\infty)$ and

$$v(t) \le u_0 + \frac{t^{1-\beta}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}s^{\beta-1}l(s)v(s)ds.$$

By Theorem 3.6, we obtain the inequality (3.18) and complete the proof. □

Remark 3.8. Medved′ studied the inequality (1.9) in [13, Theorem 4] for different β and γ. If $\beta > \frac{1}{2}$ and $\gamma > 1 - \frac{1}{2p}$ ($p > 1$), then Medved′ obtained the bound of the inequality (1.9). If $\beta = \frac{1}{m+1}$ and $\gamma > 1 - \frac{1}{kq}$ ($m \geq 1$, $k > 1$ and $q = m + 2$), then Medved′ obtained another bound. In Theorem 3.1, we study the inequality (1.9) under the hypothesis $\beta \in (0, 1)$ and $\gamma \geq 0$. The proof of the inequality (1.9) is more simple than Theorem 4 in [13]. Lemma 3.5 and Theorem 3.6 we now discuss seem to be new. For the special $b(t)$ and γ, the hypothesis in Theorem 3.6 is weaker than that in Theorem 3.1.

Example 3.9. Suppose that $t^{\frac{1}{2}}u(t)$ is a continuous, nonnegative function on $[0, +\infty)$ and $u(t)$ satisfies the inequality

$$u(t) \leq t^{-\frac{1}{2}} + t^{-\frac{1}{3}} \int_0^t (t-s)^{-\frac{1}{3}} \frac{\sqrt[6]{s}}{\sqrt{1+s^2}} u(s)ds, \qquad t \in (0, +\infty). \tag{3.19}$$

Let $p = q = 2$, by Theorem 3.2, then we have

$$u(t) \leq t^{-\frac{1}{2}} (2 + 12t^{\frac{2}{3}} \exp(6 \arctan t) \int_0^t \frac{s^{\frac{-2}{3}}}{1+s^2} \exp(-6 \arctan s) ds)^{\frac{1}{2}}.$$

We know

$$\begin{aligned} \int_0^t \frac{s^{\frac{-2}{3}}}{1+s^2} \exp(-6 \arctan s) ds &\leq \int_0^{+\infty} \frac{s^{\frac{-2}{3}}}{1+s^2} ds \\ &= \frac{1}{2} \int_0^1 (1-u)^{\frac{-5}{6}} u^{\frac{-1}{6}} du \\ &= \pi, \end{aligned} \tag{3.20}$$

where $u = \frac{1}{1+s^2}$. Then we obtain

$$u(t) \leq t^{-\frac{1}{2}} \left(2 + 12\pi \exp(3\pi) t^{\frac{2}{3}}\right)^{\frac{1}{2}}, \qquad t \in (0, +\infty).$$

Example 3.10. Suppose that $t^{\frac{1}{3}}u(t)$ is a continuous, nonnegative function on $[0, +\infty)$ and $u(t)$ satisfies the inequality

$$u(t) \leq t^{-\frac{1}{3}} + \int_0^t (t-s)^{-\frac{1}{3}} s^{-\frac{1}{3}} u(s) ds, \qquad t \in (0, +\infty). \tag{3.21}$$

Let $v(t) = t^{\frac{1}{3}}u(t)$, then

$$v(t) \leq 1 + t^{\frac{1}{3}} \int_0^t (t-s)^{-\frac{1}{3}} s^{-\frac{2}{3}} v(s) ds, \qquad t \in [0, +\infty).$$

Let $p = \frac{8}{3}$ and $q = \frac{8}{5}$, by Theorem 3.6, we have

$$\begin{aligned} v(t) &\leq 1 + 18^{\frac{3}{8}} t^{\frac{1}{24}} \left(\frac{15}{7} 2^{\frac{3}{5}} t^{\frac{7}{15}} \exp\left(\frac{15}{8} 36^{\frac{3}{5}} t^{\frac{8}{15}} \right) \right)^{\frac{5}{8}} \\ &= 1 + 36^{\frac{3}{8}} \left(\frac{15}{7} \right)^{\frac{5}{8}} t^{\frac{1}{3}} \exp\left(\frac{75}{64} 36^{\frac{3}{5}} t^{\frac{8}{15}} \right) \\ &\leq 1 + 7t^{\frac{1}{3}} \exp(11 t^{\frac{8}{15}}) \end{aligned} \tag{3.22}$$

and

$$u(t) \le t^{\frac{-1}{3}} + 7\exp(11t^{\frac{8}{15}}), \qquad t \in (0, +\infty).$$

We know $t^{\frac{-2}{3}} \notin L^q_{Loc}[0, +\infty)$ $(q > \frac{3}{2})$. Thus, Theorem 3.2 can not be applied to Example 3.10.

Using Theorem 3.9 in [19], we know

$$u(t) \le t^{\frac{-1}{3}} + B_1 \exp(6B_0 t^{\frac{2}{3}}), \qquad t \in (0, +\infty),$$

where $B_0 = B(\frac{2}{3}, \frac{2}{3})$ and $B_1 = B(\frac{2}{3}, \frac{1}{3})$ $(B(p,q) = \int_0^1 (1-s)^{p-1} s^{q-1} ds$ is the Beta function). Due to $\frac{8}{15} < \frac{2}{3}$, this indicates that our results are improvements on [19] as $t \to \infty$. Theorem 3.9 of [19] can also be applied to the inequality (1.10) when $l(t) = t^{-\gamma}$.

4 Global solutions of fractional differential equations

In this section, we give the existence and uniqueness results of the initial value problem (1.11).

Lemma 4.1. *Suppose $f : (0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and there exist nonnegative functions $l(t), k(t)$ with $t^{\beta-1} l(t) \in C(0, T] \bigcap L^q[0, T]$ and $k(t) \in C(0, T] \bigcap L^q[0, T]$ $(q > \frac{1}{\beta}, \beta \in (0, 1))$ such that*

$$|f(t, x)| \le l(t)|x| + k(t)$$

for all $(t, x) \in (0, T] \times \mathbb{R}$. Then the following Volterra integral equation

$$x(t) = x_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s)) ds \tag{4.1}$$

has at least one solution in $C_{1-\beta}(0, T]$.

Proof. Let $G : C_{1-\beta}(0, T] \to C_{1-\beta}(0, T]$ be the operator defined by

$$Gx(t) = x_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s)) ds \tag{4.2}$$

for all $x \in C_{1-\beta}(0, T]$.

Step 1: we show that the operator G is continuous. To see this let $x_n \to x$ in $C_{1-\beta}(0, T]$ and we will show that $Gx_n \to Gx$ in $C_{1-\beta}(0, T]$. Now $x_n \to x$ implies that there exists $r > 0$ such that $\|x_n\|_{1-\beta} \le r$ and $\|x\|_{1-\beta} \le r$. For each $s \in (0, T]$, we have

$$f(s, x_n(s)) \to f(s, x(s)).$$

Using the assumption of f, we get

$$(t-s)^{\beta-1} |f(s, x_n(s)) - f(s, x(s))| \le 2(t-s)^{\beta-1} (r s^{\beta-1} l(s) + k(s)).$$

Since $t^{\beta-1} l(t) \in C(0, T] \bigcap L^q[0, T]$ and $k(t) \in C(0, T] \bigcap L^q[0, T]$, using the Hölder inequality, then we know the function

$$s \to 2r(t-s)^{\beta-1} s^{\beta-1} l(s) + 2(t-s)^{\beta-1} k(s)$$

is integrable for $s \in [0, t]$. By means of the Lebesgue dominated convergence theorem yields

$$t^{1-\beta} \left| \int_0^t (t-s)^{\beta-1} [f(s, x_n(s)) - f(s, x(s))] ds \right| \to 0$$

as $n \to +\infty$. Therefore $t^{1-\beta}Gx_n(t) \to t^{1-\beta}Gx(t)$ pointwise on $[0, T]$ as $n \to +\infty$. If we show the convergence is uniform then of course G is continuous. Let $t_1, t_2 \in [0, T]$ with $t_2 < t_1$. Then

$$\begin{aligned} &\left|t_1^{1-\beta}Gx(t_1) - t_2^{1-\beta}Gx(t_2)\right| \\ &\quad \le \left|\frac{t_1^{1-\beta} - t_2^{1-\beta}}{\Gamma(\beta)}\right| \left|\int_0^{t_2} (t_2 - s)^{\beta-1} f(s, x(s))ds\right| \\ &\quad + \frac{t_1^{1-\beta}}{\Gamma(\beta)} \left|\int_0^{t_1} (t_1 - s)^{\beta-1} f(s, x(s))ds - \int_0^{t_2} (t_2 - s)^{\beta-1} f(s, x(s))ds\right|. \end{aligned} \tag{4.3}$$

Since

$$|f(t, x(t))| \le l(t)|x(t)| + k(t) \le t^{\beta-1}l(t)t^{1-\beta}|x(t)| + k(t),$$

from the assumptions of f, we know $f(t, x(t)) \in L^q[0, T]$ $(q > \frac{1}{\beta})$ when $x(t) \in C_{1-\beta}(0, T]$. From Lemma 2.6, we obtain

$$\int_0^t (t - s)^{\beta-1} f(s, x(s))ds$$

is continuous on $[0, T]$. As $t_1 \to t_2$, the right-hand side of the above inequality (4.3) tends to zero. Now (4.3) together with the fact that $t^{1-\beta}Gx_n(t) \to t^{1-\beta}Gx(t)$ pointwise on $[0, T]$ implies that the convergence is uniform. Consequently $G : C_{1-\beta}(0, T] \to C_{1-\beta}(0, T]$ is continuous.

Step 2: Next we claim that the operator G is completely continuous. To see this let $\Omega \in C_{1-\beta}(0, T]$ be bounded and $\|x\|_{1-\beta} \le M$ for each $x \in \Omega$, we will show that $t^{1-\beta}G(\Omega)$ is uniformly bounded and equicontinuous on $[0, T]$. The equicontinuity of $t^{1-\beta}G(\Omega)$ on $[0, T]$ follows essentially the same reasoning as that used to prove (4.3). Also $t^{1-\beta}G(\Omega)$ is uniformly bounded. Since for $t \in [0, T]$, we have

$$\begin{aligned} |t^{1-\beta}Gx(t)| &\le |x_0| + \frac{t^{1-\beta}}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} s^{\beta-1} l(s) s^{1-\beta} |x(s)| ds + \frac{t^{1-\beta}}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} k(s) ds \\ &\le |x_0| + \frac{t^{1-\beta}}{\Gamma(\beta)} \left(\int_0^t (t - s)^{p(\beta-1)} ds\right)^{\frac{1}{p}} \left(\int_0^t (M s^{\beta-1} l(s))^q ds\right)^{\frac{1}{q}} \\ &\quad + \frac{t^{1-\beta}}{\Gamma(\beta)} \left(\int_0^t (t - s)^{p(\beta-1)} ds\right)^{\frac{1}{p}} \left(\int_0^t k^q(s) ds\right)^{\frac{1}{q}} \\ &\le |x_0| + \frac{t^{\frac{1}{p}}}{\Gamma(\beta)(p(\beta - 1) + 1)^{\frac{1}{p}}} \left[\left(\int_0^t (M s^{\beta-1} l(s))^q ds\right)^{\frac{1}{q}} + \left(\int_0^t k^q(s) ds\right)^{\frac{1}{q}}\right], \end{aligned} \tag{4.4}$$

then

$$\|Gx\|_{1-\beta} \le |x_0| + \frac{T^{\frac{1}{p}}}{\Gamma(\beta)(p(\beta - 1) + 1)^{\frac{1}{p}}} \left[\left(\int_0^T (M s^{\beta-1} l(s))^q ds\right)^{\frac{1}{q}} + \left(\int_0^T k^q(s) ds\right)^{\frac{1}{q}}\right].$$

Consequently $G : C_{1-\beta}(0, T] \to C_{1-\beta}(0, T]$ is completely continuous.

Step 3: If $x \in C_{1-\beta}(0, T]$ is any solution of

$$x(t) = \lambda \left(x_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} f(s, x(s)) ds\right), \qquad t \in (0, T]$$

for $\lambda \in (0,1)$. Let $v(t) = t^{1-\beta}x(t) \in C[0,T]$, then

$$\begin{aligned} |v(t)| &\le |x_0| + \left| \frac{t^{1-\beta}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, s^{\beta-1}v(s))ds \right| \\ &\le |x_0| + \frac{t^{\frac{1}{p}}}{\Gamma(\beta)(p(\beta-1)+1)^{\frac{1}{p}}} \left(\int_0^t k^q(s)ds \right)^{\frac{1}{q}} + \frac{t^{1-\beta}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}s^{\beta-1}l(s)|v(s)|ds. \end{aligned} \tag{4.5}$$

Consequently, by Theorem 3.1, we can get

$$|v(t)| \le \left(A(t) + B(t) \int_0^t L(s)A(s) \exp\left(\int_s^t L(\tau)B(\tau)d\tau \right) ds \right)^{\frac{1}{q}}, \qquad t \in [0,T],$$

where

$$\begin{aligned} A(t) &= 2^{q-1} \left(|x_0| + \frac{t^{\frac{1}{p}}}{\Gamma(\beta)\,(p(\beta-1)+1)^{\frac{1}{p}}} \left(\int_0^t k^q(s)ds \right)^{\frac{1}{q}} \right)^q, \\ B(t) &= \frac{2^{q-1}t^{\frac{q}{p}}}{\Gamma^q(\beta)(p\beta - p + 1)^{\frac{q}{p}}}, \\ L(t) &= t^{q(\beta-1)}l^q(t) \end{aligned}$$

and $p \in (1, +\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we get

$$\|v\| = \|x\|_{1-\beta} \le \left(A(T) + B(T) \int_0^T L(s)A(s) \exp\left(\int_s^T L(\tau)B(\tau)d\tau \right) ds \right)^{\frac{1}{q}}.$$

Finally, by applying fixed point Theorem 2.7, the operator G has a fixed point $x(t) \in C_{1-\beta}(0,T]$, which is the solution of the integral equation (4.1). □

Lemma 4.2. *Let f be as in Lemma 4.1. A function $x \in C_{1-\beta}(0,T]$ is a solution of fractional differential equation (1.11) if and only if it is a solution of the Volterra integral equation*

$$x(t) = x_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s))ds, \qquad t \in (0,T]. \tag{4.6}$$

Proof. Since $x \in C_{1-\beta}(0,T]$ and

$$|f(t,x(t))| \le l(t)|x(t)| + k(t) = t^{\beta-1}l(t)t^{1-\beta}|x(t)| + k(t)$$

with $t^{\beta-1}l(t) \in C(0,T] \bigcap L^q[0,T]$ and $k(t) \in C(0,T] \bigcap L^q[0,T]$, then we have $x \in C(0,T] \bigcap L^1[0,T]$ and $f(t,x(t)) \in C(0,T] \bigcap L^1[0,T]$. By virtue of Theorem 2.4, then we complete the proof. □

Theorem 4.3. *Suppose $f : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and there exist nonnegative functions $l(t), k(t)$ with $t^{\beta-1}l(t) \in C(0,+\infty) \bigcap L^q_{Loc}[0,+\infty)$ and $k(t) \in C(0,+\infty) \bigcap L^q_{Loc}[0,+\infty)$ ($q > \frac{1}{\beta}, \beta \in (0,1)$) such that*

$$|f(t,x)| \le l(t)|x| + k(t)$$

for all $(t,x) \in (0,+\infty) \times \mathbb{R}$. Then the initial value problem (1.11) has at least one continuous solution in $C_{1-\beta}(0,+\infty)$.

Proof. From Lemma 4.1 and Lemma 4.2, We know the equation (1.11) has at least one solution in $C_{1-\beta}(0,T]$. Since T can be chosen arbitrarily constant, then the equation (1.11) has at least one global solution on $(0,+\infty)$. Thus, we complete the proof of Theorem 4.3. □

Theorem 4.4. *If $f:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$ is a continuous function, and*

$$|f(t,x)-f(t,y)|\le l(t)|x-y|$$

for all $x,y\in\mathbb{R}$ and $t\in(0,+\infty)$, where $t^{\beta-1}l(t)\in C(0,+\infty)\bigcap L^q_{Loc}[0,+\infty)$ and $|f(t,0)|\in L^q_{Loc}[0,+\infty)$ $(q>\frac{1}{\beta})$. Then equation (1.11) has a unique solution on $(0,+\infty)$.

Proof. We know

$$|f(t,x)|\le|f(t,x)-f(t,0)|+|f(t,0)|\le l(t)|x|+|f(t,0)|.$$

By Theorem 4.3, we suppose $x_1(t)$, $x_2(t)$ are two global solutions of equation (1.11). Then

$$\begin{aligned}|x_1(t)-x_2(t)|&=\left|\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}(f(s,x_1(s))-f(s,x_2(s)))ds\right|\\&\le\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}l(s)|x_1(s)-x_2(s)|ds\\&=\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}s^{\beta-1}l(s)s^{1-\beta}|x_1(s)-x_2(s)|ds.\end{aligned}\tag{4.7}$$

Let $u(t)=t^{1-\beta}|x_1(t)-x_2(t)|$, then

$$u(t)\le\frac{t^{1-\beta}}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}s^{\beta-1}l(s)u(s)ds.$$

By Theorem 3.1, we can get $x_1(t)=x_2(t)$. Thus the proof is complete. □

Remark 4.5. In [18], Trif proved that the equation (1.11) has a unique solution when continuous function $f(t,x)=p(t)x+q(t)$ for all $(t,x)\in(0,+\infty)\times\mathbb{R}$, where $p\in C_\alpha(0,+\infty)$ and $q\in C_{1-\beta}(0,+\infty)$ with $0\le\alpha<\beta$. Then under the above conclusions, Trif presented the existence result when $f(t,x)\le p(t)x+q(t)$, where $p\in C_\alpha(0,+\infty)$ and $q\in C_{1-\beta}(0,+\infty)$ with $0\le\alpha<\beta$ and $2\beta-\alpha>1$. In fact, if $p\in C_\alpha(0,+\infty)$ and $q\in C_{1-\beta}(0,+\infty)$, let $1+\alpha-\beta<\frac{1}{q}<\beta$, then $t^{\beta-1}p(t)\in C(0,+\infty)\bigcap L^q_{Loc}[0,+\infty)$ and $q(t)\in C(0,+\infty)\bigcap L^q_{Loc}[0,+\infty)$. Thus, our result includes the main result of [18, Theorem 4.2]. Theorem 4.11 of [19] also states a global existence result of the equation (1.11) but with only a sketch of the proof.

Example 4.6.

$$\begin{cases}D_r^{\frac{3}{4}}x(t)=(t^{\frac{-1}{4}}+1)\sqrt{x(t)}+t^{\frac{-1}{2}},\\ \lim_{t\to0^+}t^{\frac{3}{4}}x(t)=1.\end{cases}\tag{4.8}$$

We know

$$(t^{\frac{-1}{4}}+1)\sqrt{x(t)}+t^{\frac{-1}{2}}\le\frac{t^{\frac{-1}{4}}+1}{2}|x(t)|+\frac{t^{\frac{-1}{4}}+1}{2}+t^{\frac{-1}{2}}.\tag{4.9}$$

Let $q=\frac{5}{3}$, then $\frac{t^{\frac{-1}{4}}(t^{\frac{-1}{4}}+1)}{2}\in C(0,+\infty)\bigcap L^{\frac{5}{3}}_{Loc}[0,+\infty)$ and $\frac{t^{\frac{-1}{4}}+1}{2}+t^{\frac{-1}{2}}\in C(0,+\infty)\bigcap L^{\frac{5}{3}}_{Loc}[0,+\infty)$. From Theorem 4.3, equation (4.8) has at least one global solution on $(0,+\infty)$.

A global solution is proved in [18] under the following hypothesis $f(t,x)\le p(t)x+q(t)$, where $p\in C_\alpha(0,+\infty)$ and $q\in C_{1-\beta}(0,+\infty)$ with $0\le\alpha<\beta$ and $2\beta-\alpha>1$. From (4.9), we know $\frac{t^{\frac{-1}{4}}+1}{2}+t^{\frac{-1}{2}}\notin C_{\frac{1}{4}}(0,+\infty)$.

Example 4.7.

$$\begin{cases} D_r^{\frac{3}{4}} x(t) = t^{\frac{-1}{3}} \frac{1+x^2(t)}{1+x(t)} + t^{\frac{-1}{2}}, \\ \lim_{t\to 0^+} t^{\frac{3}{4}} x(t) = 1. \end{cases} \tag{4.10}$$

We know

$$\left|\frac{1+x^2}{1+x} - \frac{1+y^2}{1+y}\right| \le |x-y|,$$

where $x, y \in [0, +\infty)$. Since $t^{\frac{-7}{12}} \in C(0,+\infty) \bigcap L^q_{Loc}[0,+\infty)$ and $t^{\frac{-1}{3}} + t^{\frac{-1}{2}} \in C(0,+\infty) \bigcap L^q_{Loc}[0,+\infty)$ $(q > \frac{4}{3})$, then from Theorem 4.4, equation (4.10) has a unique global solution on $(0, +\infty)$.

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On quasi-periodic solutions of forced higher order nonlinear difference equations

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Abstract. Consider the following higher order difference equation

$$x(n+1) = f(n, x(n)) + g(n, x(n-k)) + b(n), \qquad n = 0, 1, \dots$$

where $f(n,x), g(n,x) : \{0,1,\dots\} \times [0,\infty) \to [0,\infty)$ are continuous functions in x and periodic functions with period ω in n, $\{b(n)\}$ is a real sequence, and k is a nonnegative integer. We show that under proper conditions, every nonnegative solution of the equation is quasi-periodic with period ω. Applications to some other difference equations derived from mathematical biology are also given.

Keywords: difference equations, quasi-periodic solutions, population models.

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1 Introduction

Consider the following nonlinear difference equation of order $k+1$ with forcing term $b(n)$

$$x(n+1) = f(n, x(n)) + g(n, x(n-k)) + b(n), \qquad n = 0, 1, \dots \tag{1.1}$$

where $f(n,x), g(n,x) : \{0,1,\dots\} \times [0,\infty) \to [0,\infty)$ are continuous functions in x and periodic functions with period ω in n, $\{b(n)\}$ is a real sequence, and k is a nonnegative integer. Our aim in the paper is to study the quasi-periodicity of solutions of Eq. (1.1) in the sense that

Definition 1.1. We say that a solution $\{x(n)\}$ of Eq. (1.1) is quasi-periodic with period ω if there exist sequences $\{p(n)\}$ and $\{q(n)\}$ such that $\{p(n)\}$ is periodic with period ω and $\{q(n)\}$ converges to zero as $n \to \infty$ and $x(n) = p(n) + q(n)$, $n = 0, 1, \dots$

By using, among others, some methods and ideas related to the linear first-order difference equation, in the next section we show that under proper conditions every solution of Eq. (1.1)

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is quasi-periodic with period ω. More specifically, we show that under proper conditions, every solution $\{x(n)\}$ of Eq. (1.1) satisfies

$$\lim_{n\to\infty}(x(n)-\tilde{y}(n))=0$$

where $\{\tilde{y}(n)\}$ is a periodic solution with period ω of the following associated difference equation of Eq. (1.1) without forcing term

$$y(n+1)=f(n,y(n))+g(n,y(n-k)), \qquad n=0,1,\ldots \tag{1.2}$$

Existence and global attractivity of periodic solutions of Eq. (1.2) and some other forms have been studied by numerous authors, see for example [1,3,13,15–17,19,20,22,23,31] and the references cited therein. While there has been much progress made in the study of the existence and global attractivity of periodic solutions of Eq. (1.2), the quasi-periodicity of solutions of Eq. (1.1) is relatively scarce. In order to study this phenomenon, we note the following recent result from [15] for the existence of a periodic solution $\tilde{y}(t)$ of Eq. (1.2) (some new results related to those in [15] have been recently presented in [26]).

Theorem A. Assume that there is a nonnegative periodic sequence $\{a(n)\}$ with period ω such that

$$\hat{a}=\prod_{j=0}^{\omega-1}a(j)<1 \quad \text{and} \quad f(n,y)\le a(n)y \quad \text{for } n=0,1,\ldots,\omega-1 \text{ and } y\ge 0$$

and that $f(n,y)-a(n)y$ is nonincreasing in y. Suppose also that $g(n,y)$ is nonincreasing in y and that there is a positive constant B such that

$$\sum_{i=n}^{n+\omega-1}\left(\prod_{j=i+1}^{n+\omega-1}a(j)\right)[f(i,B)-a(i)B+g(i,B)]\ge 0, \qquad n=0,1,\ldots,\omega-1 \tag{1.3}$$

and

$$\frac{1}{1-\hat{a}}\sum_{i=n}^{n+\omega-1}\left(\prod_{j=i+1}^{n+\omega-1}a(j)\right)g(i,0)\le B, \qquad n=0,1,\ldots,\omega-1. \tag{1.4}$$

Then Eq. (1.2) has a nonnegative periodic solution $\{\tilde{y}(n)\}$ with period ω.

We will make use of this theorem in the next section to guarantee a periodic solution of Eq. (1.2), a prerequisite for the existence of quasi-periodic solutions of Eq. (1.1). In Section 3, we show that our main results may be applied to some difference equations derived from applications.

2 Main results

For the sake of convenience, we adopt the notation $\prod_{i=m}^{n}\rho(i)=1$ and $\sum_{i=m}^{n}\rho(i)=0$ whenever $\{\rho(n)\}$ is a real sequence and $m>n$ in the following discussion.

The following lemma – which is needed in the proof of our main result – is folklore, and all the ingredients for its proof can be found in some papers dealing with the linear first-order difference equation (see, for example, [18] and [23] and the related references therein). Nevertheless, we will give a proof for the sake of completeness.

Lemma 2.1. *Assume that $\{a(n)\}$ is a nonnegative periodic sequence with period ω and $\{b(n)\}$ is a real sequence. If*

$$\prod_{i=0}^{\omega-1} a(i) < 1 \quad \text{and} \quad b(n) \to 0 \quad \text{as } n \to \infty, \tag{2.1}$$

then

$$\sum_{i=0}^{n} \left(\prod_{j=i+1}^{n} a(j) \right) b(i) \to 0 \quad \text{as } n \to \infty. \tag{2.2}$$

Proof. First we show that there is a positive constant A such that

$$\sum_{i=0}^{n} \left(\prod_{j=i+1}^{n} a(j) \right) \le A, \qquad n = 0, 1, \dots \tag{2.3}$$

Observe that for any $n \ge 0$, there are nonnegative integers m and l such that

$$n = m\omega + l, \qquad 0 \le l \le \omega - 1.$$

Then

$$\begin{aligned}
\sum_{i=0}^{n} \left(\prod_{j=i+1}^{n} a(j) \right) &= \sum_{i=0}^{m\omega+l} \left(\prod_{j=i+1}^{m\omega+l} a(j) \right) \\
&= \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{m\omega+l} a(j) \right) + \sum_{i=\omega}^{2\omega-1} \left(\prod_{j=i+1}^{m\omega+l} a(j) \right) + \cdots + \sum_{i=(m-1)\omega}^{m\omega-1} \left(\prod_{j=i+1}^{m\omega+l} a(j) \right) \\
&\quad + \sum_{i=m\omega}^{m\omega+l} \left(\prod_{j=i+1}^{m\omega+l} a(j) \right) \\
&= \prod_{j=\omega}^{m\omega+l} a(j) \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) + \prod_{j=2\omega}^{m\omega+l} a(j) \sum_{i=\omega}^{2\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) \\
&\quad + \cdots + \prod_{j=m\omega}^{m\omega+l} a(j) \sum_{i=(m-1)\omega}^{m\omega-1} \left(\prod_{j=i+1}^{m\omega-1} a(j) \right) + \sum_{i=0}^{l} \left(\prod_{j=i+1}^{l} a(j) \right) \\
&= \prod_{j=\omega}^{m\omega-1} a(j) \prod_{j=m\omega}^{m\omega+l} a(j) \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) + \prod_{j=2\omega}^{m\omega-1} a(j) \prod_{j=m\omega}^{m\omega+l} a(j) \sum_{i=\omega}^{2\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) \\
&\quad + \cdots + \prod_{j=m\omega}^{m\omega-1} a(j) \prod_{j=m\omega}^{m\omega+l} a(j) \sum_{i=(m-1)\omega}^{m\omega-1} \left(\prod_{j=i+1}^{m\omega-1} a(j) \right) + \sum_{i=0}^{l} \left(\prod_{j=i+1}^{l} a(j) \right) \\
&= \left(\prod_{i=0}^{\omega-1} a(j) \right)^{m-1} \prod_{i=0}^{l} a(j) \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) \\
&\quad + \left(\prod_{i=0}^{\omega-1} a(j) \right)^{m-2} \prod_{i=0}^{l} a(j) \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) \\
&\quad + \cdots + \prod_{i=0}^{l} a(j) \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) + \sum_{i=0}^{l} \left(\prod_{j=i+1}^{l} a(j) \right)
\end{aligned}$$

$$= \left[\left(\prod_{i=0}^{\omega-1} a(i) \right)^{m-1} + \left(\prod_{i=0}^{\omega-1} a(j) \right)^{m-2} + \cdots + 1 \right] \prod_{i=0}^{l} a(j) \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right)$$
$$+ \sum_{i=0}^{l} \left(\prod_{j=i+1}^{l} a(j) \right)$$
$$= \frac{1 - \left(\prod_{j=0}^{\omega-1} a(j) \right)^m}{1 - \prod_{j=0}^{\omega-1} a(j)} \prod_{j=0}^{l} a(j) \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) + \sum_{i=0}^{l} \left(\prod_{j=i+1}^{l} a(j) \right).$$

Thus

$$\sum_{i=0}^{n} \left(\prod_{j=i+1}^{n} a(j) \right) \leq \frac{\prod_{j=0}^{l} a(j)}{1 - \prod_{j=0}^{\omega-1} a(j)} \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) + \sum_{i=0}^{l} \left(\prod_{j=i+1}^{l} a(j) \right), \quad l = 0, 1, \ldots, \omega - 1. \tag{2.4}$$

Let

$$A_1 = \max_{0 \leq l \leq \omega-1} \prod_{j=0}^{l} a(j), \qquad A_2 = \max_{0 \leq l \leq \omega-1} \sum_{i=0}^{l} \left(\prod_{j=i+1}^{l} a(j) \right)$$

and

$$A = \frac{A_1}{1 - \prod_{j=0}^{\omega-1} a(j)} \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) + A_2.$$

Then from (2.4) we see that (2.3) holds. Next, we show that (2.2) holds. Since $b(n) \to 0$ as $n \to \infty$, there is a positive constant $C(\geq A)$ such that

$$|b(n)| \leq C, \qquad n \geq 0$$

and for each $\epsilon > 0$, there is a positive integer N_1 such that

$$|b(n)| < \frac{\epsilon}{2C}, \qquad n > N_1.$$

Hence, by noting (2.3), we see that

$$\sum_{i=N_1+1}^{n} \left(\prod_{j=i+1}^{n} a(j) \right) |b(i)| \leq \sum_{i=N_1+1}^{n} \left(\prod_{j=i+1}^{n} a(j) \right) \frac{\epsilon}{2C} \leq A \frac{\epsilon}{2C} \leq \epsilon/2, \qquad n > N_1.$$

Since for each $t = 1, 2, \ldots, N_1 + 1$, $\prod_{j=t}^{n} a(j) \to 0$ as $n \to \infty$, there is a positive integer $N_2(> N_1)$ such that

$$\prod_{j=t}^{n} a(j) < \frac{\epsilon}{2(N_1+1)C}, \qquad n > N_2, \ t = 1, 2, \ldots, N_1 + 1.$$

Hence,

$$\sum_{i=0}^{N_1} \left(\prod_{j=i+1}^{n} a(j) \right) |b(i)| \leq \sum_{i=0}^{N_1} \left(\prod_{j=i+1}^{n} a(j) \right) C \leq (N_1 + 1) \frac{\epsilon}{2(N_1+1)C} C = \epsilon/2, \qquad n > N_2.$$

Then it follows that

$$\begin{aligned}\left|\sum_{i=0}^{n}\left(\prod_{j=i+1}^{n}a(j)\right)b(i)\right| &= \left|\sum_{i=0}^{N_1}\left(\prod_{j=i+1}^{n}a(j)\right)b(i)+\sum_{i=N_1+1}^{n}\left(\prod_{j=i+1}^{n}a(j)\right)b(i)\right| \\ &\leq \sum_{i=0}^{N_1}\left(\prod_{j=i+1}^{n}a(j)\right)|b(i)|+\sum_{i=N_1+1}^{n}\left(\prod_{j=i+1}^{n}a(j)\right)|b(i)| \\ &\leq \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon, \qquad n>N_2\end{aligned}$$

which yields (2.2). The proof is complete. □

Now, consider the linear difference equation

$$u(n+1)=a(n)u(n)+b(n), \qquad n=0,1,\dots, \tag{2.5}$$

where $\{a(n)\}$ and $\{b(n)\}$ satisfy the hypotheses in Lemma 2.1. Assume that $\{u(n)\}$ is a solution of Eq. (2.5). It is known that the general solution to the equation is

$$u(n+1)=\left(\prod_{j=0}^{n}a(j)\right)u(0)+\sum_{i=0}^{n}\left(\prod_{j=i+1}^{n}a(j)\right)b(i), \qquad n=0,1,\dots,$$

which is frequently used in the literature (see, e.g., recent papers [21,23–25], as well as many related references therein, where some applications to ordinary and partial difference equations, as well as many historical facts on the equation and related solvable ones can be found). Clearly, by noting the periodicity of $\{a(n)\}$ and (2.1), we see that

$$\left(\prod_{j=0}^{n}a(j)\right)u(0)\to 0 \quad \text{as } n\to\infty.$$

Hence, the following conclusion comes from Lemma 2.1 immediately.

Corollary 2.2. *Assume that $\{a(n)\}$ and $\{b(n)\}$ satisfy the hypotheses in Lemma 2.1. Then every solution $\{u(n)\}$ of Eq. (2.5) converges to zero as $n\to\infty$.*

The following corollary is about the difference inequality

$$v(n+1)\leq a(n)v(n)+b(n), \qquad n=0,1,\dots \tag{2.6}$$

Assume that $\{v(n)\}$ is a nonnegative solution of (2.6). Clearly, $\{v(n)\}$ satisfies

$$0\leq v(n)\leq u(n),\ n=0,1,\cdots$$

where $\{u(n)\}$ is the solution of Eq. (2.5) with $u(0)=v(0)$. Hence, the following conclusion is a direct consequence of Corollary 2.2.

Corollary 2.3. *Assume that $\{a(n)\}$ and $\{b(n)\}$ satisfy the hypotheses in Lemma 2.1. Then every nonnegative solution $\{v(n)\}$ of (2.6) converges to zero as $n\to\infty$.*

The following lemma is straightforward but will be referenced multiple times in the main result.

Lemma 2.4. *Suppose $f(n,x)$, $g(n,x)$ are real functions and that $\{a(n)\}$ is a real sequence, and assume $f(n,x)-a(n)x$ and $g(n,x)$ are nonincreasing. Then for any $y \geq 0$,*

$$f(n,x+y)-f(n,x) \leq a(n)y$$

and

$$f(n,x+y)-f(n,x)+g(n,x+y)-g(n,x) \leq a(n)y.$$

Proof. Let $y \geq 0$. As $f(n,x)-a(n)x$ is nonincreasing we have

$$f(n,x+y)-a(n)(x+y) \leq f(n,x)-a(n)x.$$

Thus, $f(n,x+y)-f(n,x) \leq a(n)y$. As $g(n,x)$ is nonincreasing, we see that $g(n,x+y)-g(n,x) \leq 0$. Combining the above inequalities completes the proof. □

The following theorem is our main result.

Theorem 2.5. *Consider Eq. (1.1) and assume that $f(n,x)$ is nondecreasing in x. Suppose that $\{a(n)\}$ is a nonnegative periodic sequence with period ω, and $\{b(n)\}$ is a real sequence such that $\{a(n)\}$ and $\{b(n)\}$ satisfy (2.1), $f(n,x) \leq a(n)x$ and $f(n,x)-a(n)x$ is nonincreasing in x. Suppose also that $g(n,x)$ is nonincreasing in x and there is a positive constant B such that (1.3) and (1.4) are satisfied. Suppose there is a nonnegative sequence $\{L(n)\}$ with period ω such that*

$$|g(n,x)-g(n,y)| \leq L(n)|x-y|, \qquad n=0,1,\dots,\omega-1 \tag{2.7}$$

and that either

$$a(n) \leq 1 \text{ and } \sum_{i=n}^{n+k}\left(\prod_{j=i+1}^{n+k} a(j)\right) L(i) < 1, \qquad n=0,1,\dots,\omega-1 \tag{2.8}$$

or

$$\sum_{i=n}^{n+k+\omega-1}\left(\prod_{j=i+1}^{n+k+\omega-1} a(j)\right) L(i) < 1, \qquad n=0,1,\dots,\omega-1. \tag{2.9}$$

Then every solution $\{x(n)\}$ of Eq. (1.1) satisfies

$$\lim_{n\to\infty}(x(n)-\tilde{y}(n))=0 \tag{2.10}$$

where $\{\tilde{y}(n)\}$ is the unique periodic solution of Eq. (1.2) with period ω.

Proof. In view of Theorem A, we know that Eq. (1.2) has a unique periodic solution $\{\tilde{y}(n)\}$. Let $z(n)=x(n)-\tilde{y}(n)$. Then $\{z(n)\}$ satisfies

$$z(n+1)+\tilde{y}(n+1)=f(n,z(n)+\tilde{y}(n))+g(n,z(n-k)+\tilde{y}(n-k))+b(n), \qquad n=0,1,\dots$$

Since $\{\tilde{y}(n)\}$ is a solution of Eq. (1.2), $\tilde{y}(n+1)=f(n,\tilde{y}(n))+g(n,\tilde{y}(n-k))$. Hence, it follows that

$$\begin{aligned} z(n+1)=&f(n,z(n)+\tilde{y}(n))-f(n,\tilde{y}(n)) \\ &+g(n,z(n-k)+\tilde{y}(n-k))-g(n,\tilde{y}(n-k))+b(n), \qquad n=0,1,\dots \end{aligned} \tag{2.11}$$

Clearly, to complete the proof of the theorem and show that (2.10) holds, it suffices to show that every solution $\{z(n)\}$ of Eq. (2.11) tends to zero as $n\to\infty$. First assume that $\{z(n)\}$ is a nonoscillatory solution of Eq. (2.11). Then $\{z(n)\}$ is either eventually positive or eventually negative. We assume that $\{z(n)\}$ is eventually positive. The proof for the case that $\{z(n)\}$ is eventually negative is similar and will be omitted. Hence, there is a positive integer n_0 such that $z(n) > 0$ for $n \ge n_0$. Then by noting $f(n,x) - a(n)x$ and $g(n,x)$ are nonincreasing in x, it follows from Lemma 2.4 and (2.11) that

$$z(n+1) \le a(n)z(n) + b(n), \qquad n \ge n_0 + k$$

and so by Corollary 2.3, $z(n) \to 0$ as $n \to \infty$.

Next, assume that $\{z(n)\}$ is an oscillatory solution of Eq. (2.11). Then there is an increasing sequence $\{n_t\}$ of positive integers such that $y(n_1) \le 0$ and for $\tau = 1, 2, \dots,$

$$\begin{cases} y(n) > 0 & \text{when } n_{2\tau-1} < n \le n_{2\tau} \text{ and} \\ y(n) \le 0 & \text{when } n_{2\tau} < n \le n_{2\tau+1}. \end{cases} \tag{2.12}$$

Case 1. Assume that (2.8) holds. Then there is a positive number μ such that

$$\mu < 1 \quad \text{and} \quad \sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} a(j) \right) L(i) \le \mu, \qquad n = 0, 1, \dots$$

We show that

$$z(n) \le \mu \max_{n_1-k\le l\le n_1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)|, \; n_1 < n \le n_2. \tag{2.13}$$

In fact, from (2.12) we see that $z(n_1) \le 0$ and $z(n) > 0$, $n_1 < n \le n_2$. As $f(n,x) - a(n)x$ is nonincreasing in x, from Lemma 2.4 we see that $f(n, z(n) + \tilde{y}(n)) - f(n, \tilde{y}(n)) \le a(n)z(n)$ and (2.11) becomes

$$z(n+1) \le a(n)z(n) + g(n, z(n-k) + \tilde{y}(n-k)) - g(n, \tilde{y}(n-k)) + b(n).$$

Then by using (2.7), it follows that when $n_1 < n \le n_2$,

$$\begin{aligned} z(n) &= \left(\prod_{j=n_1}^{n-1} a(j) \right) z(n_1) + \sum_{i=n_1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) [g(i, z(i-k) + \tilde{y}(i-k)) - g(i, \tilde{y}(i-k)) + b(i)] \\ &\le \sum_{i=n_1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |g(i, z(i-k) + \tilde{y}(i-k)) - g(i, \tilde{y}(i-k))| + \sum_{i=n_1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \\ &\le \sum_{i=n_1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) L(i) |z(i-k)| + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \end{aligned} \tag{2.14}$$

Now, consider two cases $n_2 \le n_1 + k + 1$ and $n_2 > n_1 + k + 1$, respectively. When $n_2 \le n_1 + k + 1$, for any $n_1 < n \le n_2$, $n - k - 1 \le n_1$ and so (2.14) yields

$$\begin{aligned} z(n) &\le \sum_{i=n_1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) L(i) \max_{n_1-k\le l\le n_1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \\ &\le \sum_{i=n-k-1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) L(i) \max_{n_1-k\le l\le n_1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \\ &\le \mu \max_{n_1-k\le l\le n_1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)|. \end{aligned}$$

Hence, (2.13) holds in this case. Next, consider the case that $n_2 > n_1 + k + 1$. When $n_1 < n \le n_1 + k + 1$, as we have shown above, (2.13) holds. In particular,

$$z(n_1 + k + 1) \le \mu \max_{n_1 - k \le l \le n_1} \{|z(l)|\} + \sum_{i=0}^{n_1+k} \left(\prod_{j=i+1}^{n_1+k} a(j) \right) |b(i)|. \tag{2.15}$$

When $n_1 + k + 1 < n \le n_2$, by noting $z(n - k - 1) > 0$, (2.15) holds and Lemma 2.4, (2.11) yields

$$\begin{aligned}
z(n) &\le a(n-1)z(n-1) + b(n-1) \\
&= \left(\prod_{j=n_1+k+1}^{n-1} a(j) \right) z(n_1 + k + 1) + \sum_{i=n_1+k+1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) b(i) \\
&\le \left(\prod_{j=n_1+k+1}^{n-1} a(j) \right) \left(\mu \max_{n_1-k \le l \le n_1} \{|z(l)|\} + \sum_{i=0}^{n_1+k} \left(\prod_{j=i+1}^{n_1+k} a(j) \right) |b(i)| \right) \\
&\quad + \sum_{i=n_1+k+1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) b(i) \\
&\le \mu \max_{n_1-k \le l \le n_1} \{|z(l)|\} + \sum_{i=0}^{n_1+k} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| + \sum_{i=n_1+k+1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) b(i) \\
&= \mu \max_{n_1-k \le l \le n_1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)|
\end{aligned}$$

and so $z(n)$ satisfies (2.13). Hence for any case, (2.13) holds. Then by a similar argument, we may show that

$$z(n) \ge - \left[\mu \max_{n_2-k \le l \le n_2} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \right], \qquad n_2 < n \le n_3,$$

and in general,

$$|z(n)| \le \mu B(t) + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)|, \qquad n_t < n \le n_{t+1}. \tag{2.16}$$

where

$$B(t) = \max_{n_t-k \le l \le n_t} \{|z(l)|\}, \qquad t = 1, 2, \ldots$$

Since $b(n) \to 0$ as $n \to \infty$, $|b(n)| \to 0$ as $n \to \infty$. Then it follows from Lemma 2.1,

$$\sum_{i=0}^{n} \left(\prod_{j=i+1}^{n} a(i) \right) |b(i)| \to 0 \quad \text{as } n \to \infty. \tag{2.17}$$

Hence, from (2.16) we see that if $B(t) \to 0$ as $t \to \infty$, then $z(n) \to 0$ as $n \to \infty$. In the following, we assume that $B(t) \not\to 0$ as $t \to \infty$. Then there is a subsequence $\{B(t_s)\}$ of $\{B(t)\}$ such that

$$B(t_s) \ge \eta, \qquad s = 1, 2, \ldots$$

where η is a positive constant.

By noting (2.17) again, we may choose a positive number δ such that

$$\mu + \delta < 1$$

and a subsequence $\{n_{t_{s_r}}\}$ of $\{n_{t_s}\}$ such that for each $r = 1, 2, \ldots,$

$$n_{t_{s_{r+1}}} - n_{t_{s_r}} \geq 1 + 2k$$

and

$$\sum_{i=0}^{n} \left(\prod_{j=i+1}^{n} a(i) \right) |b(i)| < \eta \delta^r, \qquad n \geq n_{t_{s_r}} - 1. \tag{2.18}$$

We claim that

$$B(t) \leq B(t_{s_r}) \quad \text{for } t \geq t_{s_r},\ r = 1, 2, \ldots \tag{2.19}$$

In fact, if $n_{t_{s_r}+1} - k > n_{t_{s_r}}$, we see that when $n_{t_{s_r}+1} - k \leq n \leq n_{t_{s_r}+1}$, it follows from (2.16) and (2.18) that

$$|z(n)| \leq \mu B(t_{s_r}) + \eta \delta^r \leq (\mu + \delta^r) B(t_{s_r}) \leq B(t_{s_r}). \tag{2.20}$$

If $n_{t_{s_r}+1} - k \leq n_{t_{s_r}}$ we see that (2.20) holds when $n_{t_{s_r}} < n \leq n_{t_{s_r}+1}$; while when $n_{t_{s_r}+1} - k \leq n \leq n_{t_{s_r}}$, by noting $n_{t_{s_r}} - k < n_{t_{s_r}+1} - k$, we see that

$$|z(n)| \leq \max_{n_{t_{s_r}} - k \leq l \leq n_{t_{s_r}}} \{|z(l)|\} = B(t_{s_r}).$$

Hence, from the above discussion we see that for any case when $n_{t_{s_r}+1} - k \leq n \leq n_{t_{s_r}+1}$,

$$|z(n)| \leq B(t_{s_r})$$

and so

$$B(t_{s_r} + 1) = \max_{n_{t_{s_r}+1} - k \leq l \leq n_{t_{s_r}+1}} \{|z(l)|\} \leq B(t_{s_r}).$$

Then by a similar argument and induction, we may show that for any $l \geq 1$,

$$B(t_{s_r} + l) \leq B(t_{s_r})$$

that is, (2.19) holds. Then it follows from (2.16) and (2.19) that

$$|z(n)| \leq \mu B(t_{s_r}) + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)|, \qquad n > n_{s_r}. \tag{2.21}$$

Next, we show that

$$|z(n)| \leq (\mu + \delta)^r B(t_{s_1}), \qquad n > n_{t_{s_1}},\ r = 1, 2, \ldots \tag{2.22}$$

When $r = 1$, from (2.18) and (2.21) we see that

$$|z(n)| \leq \mu B(t_{s_1}) + \eta \delta \leq (\mu + \delta) B(t_{s_1}), \qquad n > n_{t_{s_1}}$$

which satisfies (2.22) with $r = 1$. Assume that when $r = m$, (2.22) holds, that is,

$$|z(n)| \leq (\mu + \delta)^m B(t_{s_1}), \qquad n > n_{t_{s_m}}. \tag{2.23}$$

Then from (2.21) and (2.23) we see that when $n > n_{t_{s_{m+1}}}$,

$$\begin{aligned}|z(n)| &\le \mu B(t_{s_{m+1}}) + \sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n-1} a(j)\right)|b(i)| \\ &\le \mu(\mu+\delta)^m B(t_{s_1}) + \eta\delta^{m+1} \\ &\le (\mu(\mu+\delta)^m + \delta^{m+1})B(t_{s_1}) \\ &\le (\mu+\delta)^{m+1}B(t_{s_1}),\end{aligned}$$

which satisfies (2.22) with $r = m+1$. Hence, by induction, (2.22) holds. Clearly, (2.22) implies that $z(n) \to 0$ as $n \to \infty$.

Case 2. Assume that (2.9) holds. Then there is a positive number ν such that

$$\nu < 1 \quad \text{and} \quad \sum_{i=n}^{n+k+\omega-1}\left(\prod_{j=i+1}^{n+k+\omega-1} a(j)\right)L(i) \le \nu, \qquad n = 0, 1, \dots$$

We claim that

$$z(n) \le \nu \max_{n_1-k\le l\le n_1+\omega-1}\{|z(l)|\} + \sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n-1} a(j)\right)|b(i)|, \qquad n_1 < n \le n_2. \tag{2.24}$$

First, from the proof of Case 1, we see that when $n_1 < n \le n_2$, (2.14) holds. Next, consider two cases $n_2 \le n_1+k+\omega$ and $n_2 > n_1+k+\omega$, respectively. When $n_2 \le n_1+k+\omega$, for any $n_1 < n \le n_2$, $n-k-\omega \le n_1$ and so (2.14) yields

$$\begin{aligned}z(n) &\le \sum_{i=n_1}^{n-1}\left(\prod_{j=i+1}^{n-1} a(j)\right)L(i)\max_{n_1-k\le l\le n_1+\omega-1}\{|z(l)|\} + \sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n-1} a(j)\right)|b(i)| \\ &\le \sum_{i=n-k-\omega}^{n-1}\left(\prod_{j=i+1}^{n-1} a(j)\right)L(i)\max_{n_1-k\le l\le n_1+\omega-1}\{|z(l)|\} + \sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n-1} a(j)\right)|b(i)| \\ &\le \nu \max_{n_1-k\le l\le n_1+\omega-1}\{|z(l)|\} + \sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n-1} a(j)\right)|b(i)|.\end{aligned} \tag{2.25}$$

Hence, (2.24) holds in this case. Next, consider the case that $n_2 > n_1+k+\omega$. When $n_1 < n \le n_1+k+\omega$, as we have shown above, (2.24) holds. Hence, we only need to show that (2.24) holds also when $n_1+k+\omega < n \le n_2$. In fact, by noting that when $n_1+k+1 < n \le n_2$, $z(n-k-1) > 0$, and the result of Lemma 2.4, (2.11) yields

$$z(n) \le a(n-1)z(n-1) + b(n-1),\ n_1+k+1 < n \le n_2. \tag{2.26}$$

Hence, it follows from (2.25) and (2.26) that

$$\begin{aligned}z(n_1+k+\omega+1) \le& \left(\prod_{j=n_1+k+1}^{n_1+k+\omega} a(j)\right)z(n_1+k+1) + \sum_{i=n_1+k+1}^{n_1+k+\omega}\left(\prod_{j=i+1}^{n_1+k+\omega} a(j)\right)|b(i)| \\ \le& \left(\prod_{j=n_1+k+1}^{n_1+k+\omega} a(j)\right)\left(\nu \max_{n_1-k\le l\le n_1+\omega-1}\{|z(l)|\} + \sum_{i=0}^{n_1+k}\left(\prod_{j=i+1}^{n_1+k} a(j)\right)|b(i)|\right) \\ &+ \sum_{i=n_1+k+1}^{n_1+k+\omega}\left(\prod_{j=i+1}^{n_1+k+\omega} a(j)\right)|b(i)|\end{aligned}$$

$$\leq \nu \max_{n_1-k\leq l\leq n_1+\omega-1}\{|z(l)|\} + \sum_{i=0}^{n_1+k}\left(\prod_{j=i+1}^{n_1+k+\omega} a(j)\right)|b(i)|$$
$$+ \sum_{i=n_1+k+1}^{n_1+k+\omega}\left(\prod_{j=i+1}^{n_1+k+\omega} a(j)\right)|b(i)|$$
$$= \nu \max_{n_1-k\leq l\leq n_1+\omega-1}\{|z(l)|\} + \sum_{i=0}^{n_1+k+\omega}\left(\prod_{j=i+1}^{n_1+k+\omega} a(j)\right)|b(i)|$$

and similarly,

$$z(n_1+k+\omega+2) \leq \nu \max_{n_1-k\leq l\leq n_1+\omega-1}\{|z(l)|\} + \sum_{i=0}^{n_1+k+\omega+1}\left(\prod_{j=i+1}^{n_1+k+\omega+1} a(j)\right)|b(i)|$$
$$\vdots$$
$$z(n_2) \leq \nu \max_{n_1-k\leq l\leq n_1+\omega-1}\{|z(l)|\} + \sum_{i=0}^{n_2-1}\left(\prod_{j=i+1}^{n_2-1} a(j)\right)|b(i)|.$$

Hence for any case, (2.24) holds. Then by a similar argument, we may show that

$$z(n) \geq -\left[\nu \max_{n_2-k\leq l\leq n_2+\omega-1}\{|z(l)|\} + \sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n-1} a(j)\right)|b(i)|\right], \qquad n_2 < n \leq n_3,$$

and in general,

$$|z(n)| \leq \mu C(t) + \sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n-1} a(j)\right)|b(i)|, \qquad n_t < n \leq n_{t+1}.$$

where

$$C(t) = \max_{n_t-k\leq l\leq n_t+\omega-1}\{|z(l)|\}, \qquad t = 1, 2, \ldots$$

Then by an argument similar to that for Case 1, we may show the following.

If $C(t) \to 0$ as $t \to \infty$, then $z(n) \to 0$ as $n \to \infty$; If $C(t) \not\to 0$ as $t \to \infty$, then there is a subsequence $\{C(t_s)\}$ of $\{C(t)\}$ such that

$$C(t_s) \geq \eta, \qquad s = 1, 2, \ldots$$

where η is a positive constant. A positive number δ such that

$$\nu + \delta < 1$$

and a subsequence $\{n_{t_{s_r}}\}$ of $\{n_{t_s}\}$ such that for each $r = 1, 2, \ldots,$

$$n_{t_{s_{r+1}}} - n_{t_{s_r}} \geq 1 + 2k$$

could be chosen such that

$$\sum_{i=0}^{n}\left(\prod_{j=i+1}^{n} a(i)\right)|b(i)| < \eta\delta^r, \qquad n \geq n_{t_{s_r}} - 1$$

and

$$|z(n)| \leq (\mu+\delta)^r C(t_{s_1}), \qquad n > n_{t_{s_1}},\ r = 1, 2, \ldots$$

Clearly, the above inequalities imply that $z(n) \to 0$ as $n \to \infty$. The proof is complete. □

When $g(n,x) = p(n)h(x)$, where $\{p(n)\}$ is a nonnegative periodic sequence with period ω and h is a nonnegative continuous function, Eq. (1.1) becomes

$$x(n+1) = f(n, x(n)) + p(n)h(x(n-k)) + b(n), \qquad n = 0, 1, \ldots \tag{2.27}$$

and the following result is a direct consequence of Theorem 2.5.

Corollary 2.6. *Consider Eq. (2.27) and assume that $f(n,x)$ is nondecreasing in x. Assume also that $\{a(n)\}$ is a nonnegative periodic sequence with period ω and $\{b(n)\}$ is a real sequence such that $\{a(n)\}$ and $\{b(n)\}$ satisfy (2.1), $f(n,x) \le a(n)x$ and that $f(n,x) - a(n)x$ is nonincreasing in x. Suppose that h is nonincreasing and L-Lipschitz and that there is a positive constant B such that*

$$\sum_{i=n}^{n+\omega-1} \left(\prod_{j=i+1}^{n+\omega-1} a(j) \right) [f(i,B) - a(i)B + p(i)h(B)] \ge 0, \qquad n = 0, 1, \ldots, \omega - 1 \tag{2.28}$$

and

$$\frac{1}{1 - \prod\limits_{j=0}^{\omega-1} a(j)} \sum_{i=n}^{n+\omega-1} \left(\prod_{j=i+1}^{n+\omega-1} a(j) \right) p(i)h(0) \le B, \qquad n = 0, 1, \ldots, \omega - 1. \tag{2.29}$$

Suppose also that either

$$a(n) \le 1 \quad \text{and} \quad L \sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} a(j) \right) p(i) < 1, \qquad n = 0, 1, \ldots, \omega - 1$$

or

$$L \sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) p(i) < 1, \qquad n = 0, 1, \ldots, \omega - 1.$$

Then every solution $\{x(n)\}$ of Eq. (2.27) satisfies

$$\lim_{n\to\infty} (x(n) - \tilde{y}(n)) = 0$$

where $\{\tilde{y}(n)\}$ is the unique periodic solution with period ω of the equation

$$y(n+1) = f(n, y(n)) + p(n)h(y(n-k)), \qquad n = 0, 1, \ldots$$

When $f(n,x) = a(n)x(n)$, Eq. (2.27) becomes

$$x(n+1) = a(n)x(n) + p(n)h(x(n-k)) + b(n), \qquad n = 0, 1, \ldots \tag{2.30}$$

(2.28) is satisfied for any $B > 0$ and (2.29) holds for B large enough. Thus the following result is a direct consequence of Corollary 2.6.

Corollary 2.7. *Consider Eq. (2.30) and assume that $\{a(n)\}$ is a nonnegative periodic sequence with period ω and $\{b(n)\}$ is a real sequence such that $\{a(n)\}$ and $\{b(n)\}$ satisfy (2.1). Suppose also that $h(x)$ is nonincreasing and L-Lipschitz, and that either*

$$a(n) \le 1 \quad \text{and} \quad L \sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} a(j) \right) p(i) < 1, \qquad n = 0, 1, \ldots, \omega - 1 \tag{2.31}$$

or

$$L\sum_{i=n}^{n+k+\omega-1}\left(\prod_{j=i+1}^{n+k+\omega-1}a(j)\right)p(i)<1,\qquad n=0,1,\ldots,\omega-1. \tag{2.32}$$

Then every solutions $\{x(n)\}$ of Eq. (2.30) satisfies

$$\lim_{n\to\infty}(x(n)-\tilde{y}(n))=0$$

where $\{\tilde{y}(n)\}$ is the unique periodic solution with period ω of the equation

$$y(n+1)=a(n)y(n)+p(n)h(y(n-k)),\qquad n=0,1,\ldots$$

In particular, when $h(x)\equiv 1$, Eq. (2.27) reduces to the first order linear equation

$$x(n+1)=a(n)x(n)+p(n)+b(n),\qquad n=0,1,\ldots \tag{2.33}$$

Since we may choose $L=0$, (2.31) and (2.32) hold. Hence, from Corollary 2.7, we have the following result immediately.

Corollary 2.8. *Consider Eq. (2.33) and assume that $\{a(n)\}$ is a nonnegative periodic sequence with period ω and $\{b(n)\}$ is a real sequence such that $\{a(n)\}$ and $\{b(n)\}$ satisfy (2.1). Then every solution $\{x(n)\}$ of Eq. (2.33) satisfies*

$$\lim_{n\to\infty}(x(n)-\tilde{y}(n))=0$$

where $\{\tilde{y}(n)\}$ is the unique periodic solution with period ω of the equation

$$y(n+1)=a(n)y(n)+p(n),\qquad n=0,1,\ldots \tag{2.34}$$

Remark 2.9. When $a(n)\equiv a$ and $p(n)\equiv p$ are nonnegative constants, Eqs. (2.33) and (2.34) become

$$x(n+1)=ax(n)+p+b(n),\qquad n=0,1,\ldots \tag{2.35}$$

and

$$y(n+1)=ay(n)+p,\qquad n=0,1,\ldots \tag{2.36}$$

respectively. The nonnegative periodic solution $\{\tilde{y}(n)\}$ of Eq. (2.36) becomes the nonnegative equilibrium point $\bar{y}=\frac{p}{1-a}$. Then by Corollary 2.8, when $a<1$, every nonnegative solution $\{x(n)\}$ of Eq. (2.35) converges to $\bar{y}$ as $n\to\infty$. In fact, in this case, the solution of Eq. (2.35) is

$$x(n)=a^n x(0)+p\frac{1-a^n}{1-a}+\sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n-1}a(j)\right)b(i),\qquad n=1,2,\ldots$$

By noting (2.1) and Lemma 2.1, we know that $\sum_{i=0}^{n}\left(\prod_{j=i+1}^{n}a(j)\right)b(i)\to 0$ as $n\to\infty$ and so

$$x(n)\to\frac{p}{1-a}\quad\text{as } n\to\infty.$$

Remark 2.10. Clearly, Corollary 2.8 implies that for the equation

$$x(n+1)=a(n)x(n)+q(n),\qquad n=0,1,\ldots$$

where $\{a(n)\}$ is nonnegative and periodic with period ω, and $\{q(n)\}$ is nonnegative and quasi-periodic with period ω, if $\sum_{i=0}^{\omega-1}a(j)<1$, then every nonnegative solution of the equation is quasi-periodic with period ω.

3 Applications

In this section, we apply our results obtained in Section 2 to some equations derived from mathematical biology. In applications, there are often external factors – known or unknown – that affect the mathematical model. Two such factors that have been studied in related models are migration and subsets of populations which become isolated and unchanged by density-dependent effects, see [11,27] and references cited therein.

Consider the difference equations

$$x(n+1) = \frac{a(n)x^2(n)}{x(n)+\delta(n)} + \frac{\nu(n)\rho(n)\sigma(n)}{1+e^{\beta(n)x(n-k)-\alpha(n)}} + b(n), \qquad n = 0,1,\dots, \tag{3.1}$$

$$x(n+1) = a(n)x(n) + \beta(n)e^{-\sigma(n)x(n-k)} + b(n), \qquad n = 0,1,\dots \tag{3.2}$$

and

$$x(n+1) = a(n)x(n) + \frac{\beta(n)}{1+x^{\gamma}(n-k)} + b(n), \qquad n = 0,1,\dots \tag{3.3}$$

where $\{a(n)\}$, $\{\alpha(n)\}$, $\{\beta(n)\}$, $\{\nu(n)\}$, $\{\delta(n)\}$, $\{\rho(n)\}$, $\{\sigma(n)\}$ are nonnegative periodic sequences with period ω, $\{b(n)\}$ is a real sequence, γ is a positive constant and k is a nonnegative integer. When $a(n) \equiv a$, $\alpha(n) \equiv \alpha$, $\beta(n) \equiv \beta$, $\nu(n) \equiv \nu$, $\delta(n) \equiv \delta$, $\rho(n) \equiv \rho$ and $\sigma(n) \equiv \sigma$ are nonnegative constants and $b(n) \equiv 0$, Eqs. (3.1), (3.2) and (3.3) reduce to

$$x(n+1) = \frac{ax^2(n)}{x(n)+\delta} + \frac{\nu\rho\sigma}{1+e^{\beta x(n-k)-\alpha}}, \qquad n = 0,1,\dots, \tag{3.4}$$

$$x(n+1) = ax(n) + \beta e^{-\sigma x(n-k)}, \qquad n = 0,1,\dots \tag{3.5}$$

and

$$x(n+1) = ax(n) + \frac{\beta}{1+x^{\gamma}(n-k)}, \qquad n = 0,1,\dots \tag{3.6}$$

respectively. Eq. (3.4) is derived from a model of the energy cost for new leaf growth in citrus crops, see [30]. When $b(n) \not\equiv 0$, $\{b(n)\}$ may represent defoliation that does not occur naturally or is not considered natural defoliation by the model parameters. A similar equation is given for the litter mass in perennial grasses, and the results that follow will apply directly to this model, see [28]. Eq. (3.5) is a discrete version of a model of the survival of red blood cells in an animal [29], and Eq. (3.6) is a discrete analog of a model that has been used to study blood cell production [10]. The global attractivity of positive solutions of Eqs. (3.5), (3.6) and some extensions of them has been studied by numerous authors, see for example [4–7,9,12,14] and references cited therein. When $b(n) \not\equiv 0$, $\{b(n)\}$ may represent the medical replacement of blood cells or administration of antibodies, see [2,8] and references cited therein.

Suppose $\{b(n)\}$ is quasi-periodic, that is, there exist real sequences $\{q(n)\}$ and $\{r(n)\}$ such that $\{q(n)\}$ is periodic with period ω, $\{r(n)\}$ is such that $r(n) \to 0$ as $n \to \infty$, and $b(n) = q(n) + r(n)$. Then Eqs. (3.1), (3.2) and (3.3) become

$$x(n+1) = \frac{a(n)x^2(n)}{x(n)+\delta(n)} + \frac{\gamma(n)\rho(n)\sigma(n)}{1+e^{\beta(n)x(n-k)-\alpha(n)}} + q(n) + r(n), \qquad n = 0,1,\dots, \tag{3.7}$$

$$x(n+1) = a(n)x(n) + \beta(n)e^{-\sigma(n)x(n-k)} + q(n) + r(n), \qquad n = 0,1,\dots \tag{3.8}$$

and

$$x(n+1) = a(n)x(n) + \frac{\beta(n)}{1+x^{\gamma}(n-k)} + q(n) + r(n), \qquad n = 0,1,\dots \tag{3.9}$$

respectively.

First, consider Eq. (3.7). It is of the form of Eq. (1.1) with

$$f(n,x) = \frac{a(n)x^2}{x+\delta(n)} \quad \text{and} \quad g(n,x) = \frac{\nu(n)\rho(n)\sigma(n)}{1+e^{\beta(n)x-\alpha(n)}} + q(n).$$

As

$$\frac{df}{dx} = \frac{a(n)x(x+2\delta(n))}{(x+\delta(n))^2}, \qquad x \geq 0,$$

we see that $f(n,x)$ is nondecreasing in x. We next note that

$$f(n,x) - a(n)x = \frac{-a(n)\delta(n)x}{x+\delta(n)}, \qquad x \geq 0$$

and

$$\frac{d}{dx}(f(n,x) - a(n)x) = \frac{-a(n)\delta^2(n)}{(x+\delta(n))^2}, \qquad x \geq 0,$$

thus $f(n,x) \leq a(n)x$ and $f(n,x) - a(n)x$ is nonincreasing in x. As

$$\frac{dg}{dx} = -\beta(n)\nu(n)\rho(n)\sigma(n)\frac{e^{\beta(n)x-\alpha(n)}}{(1+e^{\beta(n)x-\alpha(n)})^2}, \qquad x \geq 0$$

and

$$\frac{d^2g}{dx^2} = -\beta^2(n)\nu(n)\rho(n)\sigma(n)\frac{e^{\beta(n)x-\alpha(n)}(1-e^{\beta(n)x-\alpha(n)})}{(1+e^{\beta(n)x-\alpha(n)})^3}, \qquad x \geq 0,$$

we see that $g(n,x)$ is nonincreasing in x, and for each n, $\left|\frac{dg(n,x)}{dx}\right|$ achieves a maximum when $x = \frac{\alpha(n)}{\beta(n)}$, and

$$\left|\frac{dg(n,x)}{dx}\right|_{x=\frac{\alpha(n)}{\beta(n)}} = \frac{\beta(n)\nu(n)\rho(n)\sigma(n)}{4}.$$

Thus $g(n,x)$ is L-Lipschitz with $L(n) = \frac{\beta(n)\nu(n)\rho(n)\sigma(n)}{4}$. Hence, we have the following conclusion from Theorem 2.5.

Corollary 3.1. *Assume that*

$$\hat{a} = \prod_{j=0}^{\omega-1} a(j) < 1.$$

Suppose there exists a positive constant B such that

$$\sum_{i=n}^{n+k+\omega-1}\left(\prod_{j=i+1}^{n+k+\omega-1} a(j)\right)\left[q(i) + \frac{\nu(i)\rho(i)\sigma(i)}{1+e^{B\cdot\beta(i)-\alpha(i)}} - \frac{B^2a(i)\delta(i)}{B+\delta(i)}\right] \geq 0, \qquad n = 0,1,\ldots,\omega-1$$

and

$$\frac{1}{1-\hat{a}}\sum_{i=n}^{n+k+\omega-1}\left(\prod_{j=i+1}^{n+k+\omega-1} a(j)\right)\left(\frac{\nu(i)\rho(i)\sigma(i)}{1+e^{-\alpha(i)}} + q(i)\right) \leq B, \qquad n = 0,1,\ldots,\omega-1.$$

Suppose also that either

$$a(n) \leq 1 \text{ and } \sum_{i=n}^{n+k}\left(\prod_{j=i+1}^{n+k} a(j)\right)\beta(i)\nu(i)\rho(i)\sigma(i) < 4, \qquad n = 0,1,\ldots,\omega-1$$

or

$$\sum_{i=n}^{n+k+\omega-1}\left(\prod_{j=i+1}^{n+k+\omega-1} a(j)\right)\beta(i)\nu(i)\rho(i)\sigma(i) < 4, \qquad n = 0, 1, \ldots, \omega-1.$$

Then every solution $\{x(n)\}$ *of Eq.* (3.7) *satisfies*

$$\lim_{n\to\infty}(x(n) - \tilde{y}(n)) = 0$$

where $\{\tilde{y}(n)\}$ *is the unique periodic solution with period* ω *of the following equation*

$$y(n+1) = \frac{a(n)y^2(n)}{y(n)+\delta(n)} + \frac{\gamma(n)\rho(n)\sigma(n)}{1+e^{\beta(n)y(n-k)-\alpha(n)}} + q(n), \qquad n = 0, 1, \ldots$$

Next consider Eq. (3.8). It is in the form of Eq. (1.1) with

$$f(n,x) = a(n)x \quad \text{and} \quad g(n,x) = \beta(n)e^{-\sigma(n)x} + q(n).$$

(1.3) is satisfied for any $B > 0$ and (1.4) holds for B large enough. Observing

$$\frac{dg}{dx} = -\beta(n)\sigma(n)e^{-\sigma(n)x}, \qquad x \geq 0,$$

we see that $g(n,x)$ is nonincreasing in x and

$$\left|\frac{dg}{dx}\right| \leq \beta(n)\sigma(n) \quad \text{for } x \geq 0,$$

which implies that for each n, $g(n,x)$ is L-Lipschitz with $L(n) = \beta(n)\sigma(n)$. Hence, we have the following conclusion from Theorem 2.5.

Corollary 3.2. *Assume that*

$$\prod_{j=0}^{\omega-1} a(j) < 1$$

and that either

$$a(n) \leq 1 \textit{ and } \sum_{i=n}^{n+k}\left(\prod_{j=i+1}^{n+k} a(j)\right)\beta(i)\sigma(i) < 1, \qquad n = 0, 1, \ldots, \omega-1$$

or

$$\sum_{i=n}^{n+k+\omega-1}\left(\prod_{j=i+1}^{n+k+\omega-1} a(j)\right)\beta(i)\sigma(i) < 1, \qquad n = 0, 1, \ldots, \omega-1.$$

Then every solution $\{x(n)\}$ *of Eq.* (3.8) *satisfies*

$$\lim_{n\to\infty}(x(n) - \tilde{y}(n)) = 0$$

where $\{\tilde{y}(n)\}$ *is the unique periodic solution with period* ω *of the following equation*

$$y(n+1) = a(n)y(n) + \beta(n)e^{-\sigma(n)y(n-k)} + q(n), \qquad n = 0, 1, \ldots$$

Finally, consider Eq. (3.9). It is in the form of (1.1) with

$$f(n,x) = a(n)x \quad \text{and} \quad g(n,x) = \frac{\beta(n)}{1+x^\gamma} + q(n).$$

gain, (1.3) is satisfied for any $B > 0$ and (1.4) hold for B large enough. Observing that

$$\frac{dg}{dx} = -\beta(n)\frac{\gamma x^{\gamma-1}}{(1+x^\gamma)^2} \quad \text{and} \quad \frac{d^2g}{dx^2} = \beta(n)\frac{\gamma x^{\gamma-2}((\gamma+1)x^\gamma - (\gamma-1))}{(1+\gamma)^3}$$

we see that for each n, when $\gamma = 1$,

$$\left|\frac{dg}{dx}\right| \le \left|\frac{dg}{dx}\right|_{x=0} = \beta(n) \quad \text{for } x \ge 0$$

and when $\gamma > 1$, $\left|\frac{dg}{dx}\right|$ attains its maximum at $x^* = \left(\frac{\gamma-1}{\gamma+1}\right)^{1/\gamma}$ and

$$\left|\frac{dg}{dx}\right|_{x=x^*} = \frac{(\gamma-1)^{\frac{\gamma-1}{\gamma}}(\gamma+1)^{\frac{\gamma+1}{\gamma}}}{4\gamma}\beta(n), \qquad n = 0,1,\dots,\omega-1.$$

Hence, $g(n,x)$ is L-Lipschitz with

$$L(n) = \begin{cases} \beta(n), & \gamma = 1, \\ \frac{(\gamma-1)^{\frac{\gamma-1}{\gamma}}(\gamma+1)^{\frac{\gamma+1}{\gamma}}}{4\gamma}\beta(n), & \gamma > 1. \end{cases}$$

It follows from Theorem 2.5 that the following conclusion holds.

Corollary 3.3. *Assume that*

$$\prod_{j=0}^{\omega-1} a(j) < 1.$$

Suppose also that when $\gamma = 1$, either

$$a(n) \le 1 \quad \textit{and} \quad \sum_{i=n}^{n+k}\left(\prod_{j=i+1}^{n+k} a(j)\right)\beta(i) < 1, \qquad n = 0,1,\dots,\omega-1$$

or

$$\sum_{i=n}^{n+k+\omega-1}\left(\prod_{j=i+1}^{n+k+\omega-1} a(j)\right)\beta(i) < 1, \qquad n = 0,1,\dots,\omega-1;$$

when $\gamma > 1$, either

$$a(n) \le 1 \quad \textit{and} \quad \sum_{i=n}^{n+k}\left(\prod_{j=i+1}^{n+k} a(j)\right)\beta(i) < \frac{4\gamma}{(\gamma-1)^{\frac{\gamma-1}{\gamma}}(\gamma+1)^{\frac{\gamma+1}{\gamma}}}, \qquad n = 0,1,\dots,\omega-1$$

or

$$\sum_{i=n}^{n+k+\omega-1}\left(\prod_{j=i+1}^{n+k+\omega-1} a(j)\right)\beta(i) < \frac{4\gamma}{(\gamma-1)^{\frac{\gamma-1}{\gamma}}(\gamma+1)^{\frac{\gamma+1}{\gamma}}}, \ n = 0,1,\dots,\omega-1.$$

Then every solution $\{x(n)\}$ of Eq. (3.9) satisfies

$$\lim_{n\to\infty}(x(n) - \tilde{y}(n)) = 0$$

where $\{\tilde{y}(n)\}$ is the unique periodic solution of with period ω of the following equation

$$y(n+1) = a(n)y(n) + \frac{\beta(n)}{1+y^\gamma(n-k)} + q(n), \qquad n = 0,1,\dots$$

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Existence of solution for two classes of Schrödinger equations in $\mathbb{R}^N$ with magnetic field and zero mass

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Abstract. In this paper, we consider the existence of a nontrivial solution for the following Schrödinger equations with a magnetic potential A

$$-\Delta_A u = K(x)f(|u|^2)u, \quad \text{in } \mathbb{R}^N$$

where $N \geqslant 3$, K is a nonnegative function verifying two kinds of conditions and f is continuous with subcritical growth. We discuss the above equation with K asymptotically periodic and $K \in L^r$.

Keywords: Schrödinger equation, magnetic field, zero mass, periodic condition, asymptotically periodic condition.

2010 Mathematics Subject Classification: 35Q55, 35J60, 35J62.

1 Introduction

In this paper, we consider the existence of a nontrivial solution for the following equation

$$-\Delta_A u = K(x)f(|u|^2)u, \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

where $N \geqslant 3$, $K : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative function and $f : \mathbb{R} \to \mathbb{R}$ is continuous with subcritical growth.

Problem (1.1) is motivated by the following nonlinear Schrödinger equation

$$\left(\frac{h}{i}\nabla - A(x)\right)^2 \psi = K(x)f(|\psi|^2)\psi,$$

where $N \geqslant 3$, h is the Planck constant and A is a magnetic potential of a given magnetic field $B = \operatorname{curl} A$, and the nonlinear term f is a nonlinear coupling and K is nonnegative. The function $A : \mathbb{R}^N \to \mathbb{R}^N$ denotes a magnetic potential and the Schrödinger operator is defined by

$$-\Delta_A \psi = -\Delta\psi + |A|^2\psi - 2iA\nabla\psi - i\psi \operatorname{div} A, \quad \text{in } \mathbb{R}^N.$$

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This class of problem with the nonlinearity f verifying the condition $f'(0) = 0$ is known as zero mass.

In recent years, much attention has been paid to the nonlinear Schrödinger equations, we may refer to [6, 13, 23, 25–29]. In particular, we notice that the existence of solutions for the problems with zero mass and without magnetic field, namely, $A \equiv 0$ and $f'(0) = 0$. In [5], Alves and Souto investigated the following problem

$$-\Delta u = K(x)f(u), \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where f is a continuous function with quasicritical growth and K is nonnegative function. Using the variational method and some technical lemmas, the authors gave the existence of positive solution for problem (1.2).

In [20], Li, Li and Shi considered a nonlinear Kirchhoff type problem

$$-\Big(a + \lambda \int_{\mathbb{R}^N} |\nabla u|^2\Big)\Delta u = K(x)f(u), \qquad x \in \mathbb{R}^N,$$

where $N \geqslant 3$, a is a positive constant, $\lambda \geqslant 0$ is a parameter and K is a potential function. The authors used a priori estimate and a Pohozaev type identity in the case with constant coefficient nonlinearity. And in the problem with the variable-coefficient, a cut-off functional and Pohozaev type identity were used to find Palais–Smale sequences.

In [1], Alves studied a quasilinear equation given by

$$-\Delta u + V(x)u - k\Delta(u^2)u = K(x)f(u), \qquad x \in \mathbb{R}^N,$$

where $N \geqslant 1$, $k \in R$, $V : \mathbb{R}^N \to \mathbb{R}$ is the potential, and $f : \mathbb{R} \to \mathbb{R}$ and $K : \mathbb{R}^N \to \mathbb{R}$ are continuous. The variational methods were used to establish a Berestycki–Lions type result. For further results about the elliptic equations with zero mass, we may refer to [4,7,8,19,24].

Inspired by [1,5,20], we would like to consider Schrödinger equations in $\mathbb{R}^N$ with magnetic field and zero mass.

Due to the appearance of the magnetic field, the problem cannot be changed into a pure real-valued problem, hence we should deal with a complex-valued directly, which causes more new difficulties in employing the methods and some estimates. Thus there are a few results for the Schrödinger equations with magnetic field than ones for that without the magnetic field. In [18], Ji and Yin showed the existence of nontrivial solutions for the following Schrödinger equation

$$-\Delta_A u + V(x)u = f(|u|^2)u, \quad \text{in } \mathbb{R}^N,$$

where $N \geqslant 3$, f has subcritical growth, and the potential V is nonnegative. The solution is obtained by the variational method combined with penalization technique of del Pino and Felmer [17] and Moser iteration.

In [15], Chabrowski and Szulkin discussed the semilinear Schrödinger equation

$$-\Delta_A u + V(x)u = Q(x)|u|^{2^*-2}u, \qquad u \in H^1_{A,V^+}(\mathbb{R}^N),$$

where V changes sign. The authors considered the problem by a min-max type argument based on a topological linking. For the more results involving the magnetic Schrödinger equations, we see [2,3,9,11,12,16,25] and the references therein.

In this paper, we consider problem (1.1) with the different function K. First of all, we assume the potential A verifying

(A) $A \in L^2_{loc}(\mathbb{R}^N, \mathbb{R}^N)$.

In the first case, we propose the following assumptions for function K:

(K1) there exist $k_0 > 0$ such that

$$K(x) \geqslant k_0, \quad \text{for } \forall x \in \mathbb{R}^N,$$

(K2) there exist a positive continuous periodic function $K_p : \mathbb{R}^N \to \mathbb{R}$

$$K_p(x+y) = K_p(x), \qquad \forall x \in \mathbb{R}^N \text{ and } \forall y \in \mathbb{Z}^N,$$

such that

$$|K(x) - K_p(x)| \to 0 \quad \text{as } |x| \to +\infty.$$

(K3) K_P is defined in (K2) such that

$$K(x) \geqslant K_p(x), \qquad \forall x \in \mathbb{R}^N.$$

In addition, we assume that function f satisfies:

($f1$) there holds

$$\lim_{t\to 0^+} \frac{f(t)}{t^{\frac{2^*-2}{2}}} = \lim_{t\to +\infty} \frac{f(t)}{t^{\frac{2^*-2}{2}}} = 0,$$

where $2^* = \frac{2N}{N-2}$ and $N \geqslant 3$.

($f2$) function F is defined by $F(t) = \int_0^t f(s)ds$, and

$$\frac{F(t)}{t} \to \infty \quad \text{as } t \to +\infty,$$

($f3$) function $H(t) = tf(t) - F(t)$ is increasing in t and $H(0) = 0$.

Now we are in a position to state the first result.

Theorem 1.1. *Assume that (A), (K1)–(K3) and ($f1$)–($f3$) hold. Then, problem* (1.1) *has a nontrivial solution.*

In the second case, we involve that K is positive almost everywhere:

(K4) the Lebesgue measure of $\{x \in \mathbb{R}^N : K(x) \leqslant 0\}$ is zero.

Then, we state the second result as follows.

Theorem 1.2. *Assume that $K \in L^\infty(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$, for some $r \geqslant 1$, satisfies (K4), and (A), ($f1$)–($f3$) hold. Then, problem* (1.1) *has a ground state solution.*

Remark 1.3. In fact, we consider the second case under a weaker condition than $K \in L^r(\mathbb{R}^N)$. We only require to suppose that for all $R > 0$ and any sequence of Borel sets $\{E_n\}$ of $\mathbb{R}^N$ such that $|E_n| \leqslant R$, for every n, we have

$$\lim_{R\to+\infty} \int_{E_n \cap B^c_R(0)} K(x)dx = 0, \quad \text{uniformly in } n \in \mathbb{N}. \tag{1.3}$$

The paper is organized as follows. In the next section, we state the functional setting and give some preliminary lemmas. In Section 3, when K verifies the periodic condition, we study problem (1.1) and establish the existence of a ground state solution. In Section 4, we give the existence of a nontrivial solution for asymptotically periodic problem, proving Theorem 1.1. In the last section we consider problem (1.1) with condition (K4) and we prove Theorem 1.2.

2 Preliminaries

In this section, we outline the variational framework for problem (1.1) and give some preliminary lemmas. We write

$$\Delta_A u := (\nabla + iA)^2 u$$

and

$$\nabla_A u := (\nabla + iA)u.$$

Let $N \geqslant 3$ and $2^* = 2N/(N-2)$. We denote $D_A^{1,2}(\mathbb{R}^N)$ the Hilbert space with the scalar product

$$\langle u, v\rangle_A = \mathrm{Re} \int_{\mathbb{R}^N} (\nabla u + iA(x)u)\overline{(\nabla v + iA(x)v)} dx,$$

and the norm induced by the product $\langle \cdot, \cdot\rangle_A$ is

$$\begin{aligned}\|u\|_A &= \Big(\int_{\mathbb{R}^N} |\nabla_A u|^2 dx\Big)^{\frac{1}{2}} \\ &= \Big(\int_{\mathbb{R}^N} |\nabla u + iA(x)u|^2 dx\Big)^{\frac{1}{2}} \\ &= \Big(\int_{\mathbb{R}^N} (|\nabla u|^2 + |A(x)|^2|u|^2) dx - 2\,\mathrm{Re} \int_{\mathbb{R}^N} iA(x)u\nabla u dx\Big)^{\frac{1}{2}},\end{aligned}$$

and $C_0^\infty(\mathbb{R}^N, \mathbb{C})$ is dense in $D_A^{1,2}(\mathbb{R}^N)$ with respect to the norm $\|u\|_A$. It is easy to know that

$$D_A^{1,2}(\mathbb{R}^N) := \Big\{ u \in L^{2^*}(\mathbb{R}^N, \mathbb{C}) : \nabla_A u \in L^2(\mathbb{R}^N, \mathbb{C}) \Big\}.$$

Furthermore, the following diamagnetic inequality (see [21, Theorem 7.21]) will be used frequently:

$$\big|\nabla_A u(x)\big| \geqslant \big|\nabla |u(x)|\big|, \quad \text{for } \forall u \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C}), \tag{2.1}$$

and it implies that if $u(x) \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$, the fact that $|u(x)| \in D^{1,2}(\mathbb{R}^N, \mathbb{R})$ will holds. Therefore, by Sobolev embedding $\int_{\mathbb{R}^N} \big|\nabla |u|\big|^2 dx \geqslant S\big(\int_{\mathbb{R}^N} |u|^{2^*} dx\big)^{\frac{2}{2^*}}$, the embedding $D_A^{1,2}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^{2^*}(\mathbb{R}^N, \mathbb{C})$ is continuous for $N \geqslant 3$.

3 A periodic problem

In the section, we will discuss the existence of a ground state solution for the following equation

$$\begin{cases} -\Delta_A u = K_p(x) f(|u|^2)u, & \text{in } \mathbb{R}^N, \\ u \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C}), \end{cases} \tag{3.1}$$

where $K_p : \mathbb{R}^N \to \mathbb{R}$ is a continuous function verifying the following hypotheses

(K5) for all $x \in \mathbb{R}^N$ and $y \in \mathbb{Z}^N$,

$$K_p(x + y) = K_p(x),$$

(K6) there is a positive constant $k_1 \geqslant 0$ such that

$$K_p(x) \geqslant k_1, \qquad \forall x \in \mathbb{R}^N.$$

In this section, the main result is the following.

Theorem 3.1. *Assume that (A), (K5)–(K6) and ($f1$)–($f3$) hold. Then, problem* (3.1) *has a nontrivial solution.*

We denote by $I : D_A^{1,2}(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ the energy functional for the problem (3.1), which is defined by

$$I(u) = \frac{1}{2}\|u\|_A^2 - \frac{1}{2}\int_{\mathbb{R}^N} K_p(x)F(|u|^2)dx, \tag{3.2}$$

with derivative, for $\forall u, v \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$,

$$I'(u)v = \mathrm{Re}\int_{\mathbb{R}^N} \nabla_A u\overline{\nabla_A v}dx - \mathrm{Re}\int_{\mathbb{R}^N} K_p(x)f(|u|^2)u\bar{v}dx. \tag{3.3}$$

The weak solution for (3.1) are the critical points of I. furthermore, we can use ($f1$)–($f3$) to check that functional I satisfies the geometry of the mountain pass. There is a sequence $(u_n) \subset D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$ such that

$$I(u_n) \to c \tag{3.4}$$

and

$$\Big(1 + \|u_n\|_A\Big)\|I'(u_n)\| \to 0, \tag{3.5}$$

where c is the mountain pass level given by

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I\big(\gamma(t)\big)$$

with

$$\Gamma = \Big\{\gamma \in C\big([0,1], D_A^{1,2}(\mathbb{R}^N, \mathbb{C})\big) : \gamma(0) = 0 \text{ and } I\big(\gamma(1)\big) \leqslant 0\Big\}.$$

This sequence is called as Cerami sequence for I at level c, see [14].

Notice that from ($f3$) one obtains $H(s) \geqslant 0$ for every $s \in \mathbb{R}$. Then, we have the next estimates: by ($f1$), for $\forall\varepsilon > 0$, there exist a $\tau = \tau(\varepsilon)$ and $c_\varepsilon > 0$ such that

$$\big|s^2 f(s^2)\big| \leqslant \varepsilon|s|^{2^*} + c_\varepsilon|s|^p\chi_{\{|s|\geqslant\tau\}}(s) \tag{3.6}$$

and, by ($f3$),

$$\big|F(s^2)\big| \leqslant \varepsilon|s|^{2^*} + c_\varepsilon|s|^p\chi_{\{|s|\geqslant\tau\}}(s) \tag{3.7}$$

where χ is the characteristic function to the set $T = \{t \in \mathbb{R}^N : |t| \geqslant \tau\}$.

In the proof of Theorem 3.1, we announce a lemma which resembles a classical result in [22].

Lemma 3.2. *Let* (u_n) *be a bounded sequence in* $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$. *Then either*

(i) there are $R, \eta > 0$ and $(y_n) \subset \mathbb{R}^N$ such that $\int_{B_R(y_n)} |u_n|^2 \geqslant \eta$, for all n,

or

(ii) $\int_{\mathbb{R}^N} |\hat{u}_n|^q \to 0,$ where $\hat{u}_n = u_n\chi_{\{|s|\geqslant\tau\}}$, $\forall q \in (2, 2^*)$ and $\tau > 0$.

Proof. If (i) does not happen, going if necessary to a subsequence, we have

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}} \int_{B_R(y)} |u_n|^2 = 0.$$

Let $\psi : \mathbb{C} \to \mathbb{R}$ be a smooth function such that

$$0 \leqslant \psi(s) \leqslant 1, \quad \psi(s) = 0 \quad \text{for } |s| < \frac{\tau}{2} \quad \text{and} \quad \psi(s) = 1 \quad \text{for } |s| \geqslant \tau,$$

it is easy to check that the sequence $\tilde{u}_n = \psi(u_n)u_n$ belongs to $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$ and satisfies

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |\tilde{u}_n|^2 = 0.$$

Hence, by [22],

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |\tilde{u}_n|^p = 0, \qquad \forall q \in (2, 2^*),$$

from where it follows that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |\hat{u}_n|^p = 0, \qquad \forall q \in (2, 2^*) \text{ and } \tau > 0,$$

finishing the proof. □

The next lemma is used to prove that the Cerami sequence is bounded in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$.

Lemma 3.3. *There is a positive constant $M > 0$ such that $I(tu_n) \leqslant M$ for every $t \in [0, 1]$ and $n \in \mathbb{N}$.*

Proof. Let $t_n \in [0, 1]$ be such that $I(t_n u_n) = \max_{t\geqslant 0} I(tu_n)$. If either $t_n = 0$ or $t_n = 1$, we are done. Thereby, we can assume that $t_n \in (0, 1)$, and so $I'(t_n u_n)t_n u_n = 0$. From this

$$2I(t_n u_n) = 2I(t_n u_n) - I'(t_n u_n)t_n u_n = \int_{\mathbb{R}^N} K_p(x)H(|t_n u_n|^2).$$

Once that K_p is positive, it follows that $(f3)$

$$2I(t_n u_n) \leqslant \int_{\mathbb{R}^N} K_p(x)H(|u_n|^2) = 2I(u_n) - I'(u_n)u_n = 2I(u_n) + o_n(1).$$

Since $(I(u_n))$ converges to c, so $I(tu_n)$ is bounded. □

Lemma 3.4. *The sequence (u_n) is bounded in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$.*

Proof. Suppose by contradiction that $\|u\|_A \to \infty$ and set $w_n = \frac{u_n}{\|u_n\|_A}$. Since $\|w_n\|_A = 1$, there exists $w \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$ such that $w_n \rightharpoonup w$ in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$. Next, we will show that $w = 0$. First of all, notice that

$$o_n(1) + 1 = \int_{\mathbb{R}^N} \frac{K_p(x)F(|u_n|^2)}{\|u_n\|_A^2} = \int_{\mathbb{R}^N} \frac{K_p(x)F(|u_n|^2)}{|u_n|^2} |w_n|^2.$$

By $(f2)$, for each $M > 0$, there is $\xi > 0$ such that

$$\frac{F(s^2)}{s^2} \geqslant M, \quad \text{for } |s| \geqslant \xi,$$

hence

$$o_n(1) + 1 \geqslant \int_{\Omega\cap\{|u_n|\geqslant\xi\}} \frac{K_p(x)F(|u_n|^2)}{|u_n|^2} |w_n|^2 \geqslant Mk_1 \int_{\Omega\cap\{|u_n|\geqslant\xi\}} |w_n|^2,$$

where $\Omega = \{x \in \mathbb{R}^N : w(x) \neq 0\}$. By Fatou's Lemma

$$1 \geqslant Mk_1 \int_{\Omega} |w|^2 dx.$$

Therefore $|\Omega| = 0$, showing that $w = 0$.

Notice that for each $C > 0$, one has $\frac{C}{\|u_n\|_A} \in [0, 1]$ for n sufficiently large. Thus

$$I(t_n u_n) \geqslant I\Big(\frac{C}{\|u\|_A} u_n\Big) = I(Cw_n) = \frac{C^2}{2} - \frac{1}{2}\int_{\mathbb{R}^N} K_p(x) F\big(C^2|w_n|^2\big).$$

We claim that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K_p(x) F\big(C^2|w_n|^2\big) = 0. \tag{3.8}$$

We postpone for minutes the proof of (3.8). But if it were true, we would get

$$\lim_{n\to+\infty} I(t_n u_n) \geqslant \frac{C^2}{2}, \quad \text{for every } C > 0,$$

which is a contradiction with Lemma 3.3, since $(I(t_n u_n)) \leqslant M$.

We prove (3.8) by using Lemma 3.2, which gives two alternatives: either

$$\int_{B_R(y_n)} |w_n|^2 \geqslant \eta \quad \text{for some } \eta > 0 \text{ and } (y_n) \in \mathbb{Z}^N,$$

or

$$\int_{\mathbb{R}^N} |\hat{w}_n|^p dx \to 0, \quad \text{where } \hat{w}_n = w_n \chi_{\{|u_n| \geqslant \tau\}},\ p \in (2, 2^*) \text{ and } \tau > 0.$$

By showing the boundedness of (u_n), we will prove that the first alternative does not hold. If the first alternative occurs, we define $\tilde{u}_n = u_n(x + y_n)$ and $\tilde{w}_n = \frac{\tilde{u}_n}{\|u_n\|_A}$. These two sequences satisfy

$$I(\tilde{u}_n) \to c, \qquad \Big(1 + \|\tilde{u}_n\|_A\Big)\|I'(\tilde{u}_n)\| \to 0 \quad \text{and} \quad \tilde{w}_n \rightharpoonup \tilde{w} \neq 0,$$

which is a contraction compared to what we have written in the beginning of this proof. Hence, the second alternative holds and

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |\hat{w}_n|^p dx = 0.$$

Then

$$\big|K_p(x) F\big(C^2|w_n|^2\big)\big| \leqslant \|K_p\|_\infty \big|F\big(C^2|w_n|^2\big)\big| \leqslant \|K_p\|_\infty \Big[\varepsilon C^{2^*}|w_n|^{2^*} + c_\varepsilon C^p |w_n|^p \chi_{\{C|w_n| \geqslant \delta\}}\Big],$$

from where it follows

$$\big|K_p(x) F\big(C^2|w_n|^2\big)\big| \leqslant \|K_p\|_\infty [\varepsilon C^{2^*}|w_n|^{2^*} + c_\varepsilon C^p |w_n|^p].$$

Consequently

$$\int_{\mathbb{R}^N} \big|K_p(x) F\big(C^2|w_n|^2\big)\big| dx \leqslant \|K_p\|_\infty \Big[\varepsilon C^{2^*} \int_{\mathbb{R}^N} |w_n|^{2^*} dx + c_\varepsilon C^p \int_{\mathbb{R}^N} |w_n|^p dx\Big],$$

showing that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \big|K_p(x) F\big(C^2|w_n|^2\big)\big| dx = 0,$$

and the proof is finished. □

Proof of Theorem 3.1. Since (u_n) is bounded, by applying Lemma 3.2, we have two alternatives, either

$$(i) \qquad \text{there are } R, \eta > 0 \text{ and } (y_n) \subset \mathbb{R}^N \text{ such that } \int_{B_R(y_n)} |u_n|^2 \geqslant \eta, \text{ for all } n,$$

or

$$(ii) \qquad \int_{\mathbb{R}^N} |\hat{u}_n|^q \to 0, \quad \text{where } \hat{u}_n = u_n \chi_{\{|s| \geqslant \tau\}}, \ q \in (2, 2^*) \text{ and } \tau > 0.$$

Notice that (ii) does not occur. Otherwise, the inequality

$$\int_{\mathbb{R}^N} \left| K_p(x) f(|u_n|^2) |u_n|^2 \right| \leqslant \|K_p\|_\infty \left[\varepsilon \int_{\mathbb{R}^N} |u_n|^{2^*} + c_\varepsilon \int_{\mathbb{R}^N} |u_n|^p \right]$$

leads to

$$\limsup_{n \to +\infty} \int_{\mathbb{R}^N} \left| K_p(x) f(|u_n|^2) |u_n|^2 \right| = 0,$$

and so

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} K_p(x) f(|u_n|^2) |u_n|^2 = 0.$$

The fact that $I'(u_n)u_n = o_n(1)$ imply that $\|u_n\|_A \to 0$, constituting a contradiction. Since alternative (i) is true and K_p is periodic, the sequence $\tilde{u}_n(x) = u_n(x + y_n)$ is a Cerami sequence for I at level c, namely,

$$I(\tilde{u}_n) \to c, \qquad \left(1 + \|\tilde{u}_n\|_A\right) \|I'(u_n)\| \to 0 \quad \text{and} \quad \tilde{u}_n \rightharpoonup \tilde{u} \text{ in } D_A^{1,2}(\mathbb{R}^N, \mathbb{C}).$$

A direct computation indicates that $I'(\tilde{u}) = 0$, and $\tilde{u}$ is a nontrivial weak solution for problem (3.1). Then, we will prove that $\tilde{u}$ is a ground state solution for (3.1).we will check that $I(\tilde{u})$ accords with the mountain pass level. By Fatou's Lemma,

$$2c = \liminf_{n \to +\infty} 2I(\tilde{u}_n) = \liminf_{n \to +\infty} \left(2I(\tilde{u}_n) - I'(\tilde{u}_n)\tilde{u}_n\right) = \liminf_{n \to +\infty} \int_{\mathbb{R}^N} K_p(x) H(|\tilde{u}_n|^2) \geqslant \int_{\mathbb{R}^N} K_p(x) H(|\tilde{u}|^2).$$

Since

$$2I(\tilde{u}) = 2I(\tilde{u}) - I'(\tilde{u})\tilde{u} = \int_{\mathbb{R}^N} K_p(x) H(|\tilde{u}|^2) dx,$$

we can conclude that $I(\tilde{u}) \leqslant c$. But then, the condition $(f3)$ leads to

$$c = \inf \left\{ I(u) : u \in D_A^{1,2}(\mathbb{R}^N) \backslash \{0\} \text{ and } I'(u)u = 0 \right\}.$$

It follows that $I'(\tilde{u}) \geqslant c$, and so $I'(\tilde{u}) = c$. □

4 The proof of Theorem 1.1

In the section, we will discuss the existence of a nontrivial solution for problem (1.1), thus showing Theorem 1.1. Therefore, we need to prove Lemmas 4.1 and 4.2 below. Hence, we will presume that the condition (A), (K1)–(K3) and $(f1)$–$(f3)$ hold.

We recall that $u \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$ is a weak solution of problem (1.1), if

$$\operatorname{Re} \int_{\mathbb{R}^N} \nabla_A u \overline{\nabla_A v} dx = \operatorname{Re} \int_{\mathbb{R}^N} K(x) F(|u|^2) u \bar{v} dx,$$

for all $v \in D_A^{1,2}(\mathbb{R}^N,\mathbb{C})$.

The Energy functional associated to (1.1) is

$$J(u) = \frac{1}{2}\|u\|_A^2 - \frac{1}{2}\int_{\mathbb{R}^N} K(x)F(|u|^2)dx, \qquad \forall u \in D_A^{1,2}(\mathbb{R}^N,\mathbb{C}) \tag{4.1}$$

with derivative

$$J'(u)v = \mathrm{Re}\int_{\mathbb{R}^N} \nabla_A u\overline{\nabla_A v}dx - \mathrm{Re}\int_{\mathbb{R}^N} K(x)f(|u|^2)u\bar{v}dx, \qquad \forall u,v \in D_A^{1,2}(\mathbb{R}^N,\mathbb{C}). \tag{4.2}$$

As in the proof of the periodic case, one observes that J satisfying the geometry of the mountain pass. Therefore, there is a sequence $(v_n) \subset D_A^{1,2}(\mathbb{R}^N,\mathbb{C})$ verifying

$$J(v_n) \to d \quad \text{and} \quad \Big(1 + \|v_n\|_A\Big)\|J'(v_n)\| \to 0, \tag{4.3}$$

where d denotes the mountain pass level correlative of J.

Since $I(u) = c$, by property (K3), one obtains $d \leqslant c$. With loss of generality, we can assume that $K \not\equiv K_p$, consequently

$$d \leqslant \max_{t\geqslant 0} J(tu) = J(t_0 u) < I(t_0 u) \leqslant I(u) = c. \tag{4.4}$$

Lemma 4.1. *The sequence (u_n) is bounded in $D_A^{1,2}(\mathbb{R}^N,\mathbb{C})$.*

Proof. Let $t_n \in [0,1]$ be such that $J(t_n v_n) = \max_{t\geqslant 0} J(tv_n)$. If either $t_n = 0$ or $t_n = 1$, we are done. Thereby, we can assume $t_n \in (0,1)$, and so $J'(t_n v_n)t_n v_n = 0$. From this

$$2J(t_n v_n) = 2J(t_n v_n) - J'(t_n v_n)t_n v_n = \int_{\mathbb{R}^N} K(x)H\big(t_n^2|v_n|^2\big).$$

Since K is a nonnegative function, from $(f3)$,

$$2J(t_n v_n) \leqslant \int_{\mathbb{R}^N} K(x)H\big(|v_n|^2\big) = 2J(v_n) - J'(v_n)v_n = 2J(v_n) + o_n(1).$$

Since $(J(v_n))$ is convergent, so it is bounded.

Suppose by contradiction that $\|v_n\|_A \to \infty$. Proving as in Lemma 3.4, the sequence $w_n = \frac{v_n}{\|v_n\|_A}$ weakly converges to 0 in $D_A^{1,2}(\mathbb{R}^N,\mathbb{C})$. Since $\|w_n\|_A = 1$, by applying Lemma 3.2, we have two alternatives, either

(i) there are $R,\eta > 0$ and $(y_n) \subset \mathbb{R}^N$ such that $\displaystyle\int_{B_R(y_n)} |w_n|^2 \geqslant \eta$, for all n,

or

(ii) $\displaystyle\int_{\mathbb{R}^N} |\hat{w}_n|^q \to 0$, where $\hat{w}_n = w_n\chi_{\{|s|\geqslant\tau\}}$, $\forall q \in (2,2^*)$ and $\tau > 0$.

If that (i) occurred, we could define the functions $\tilde{v}_n(x) = v_n(x + y_n)$ and $\tilde{w}_n(x) = \frac{\tilde{v}_n(x)}{\|\tilde{(v)}_n\|_A}$. These two sequences satisfy

$$J(\tilde{v}_n) \to d, \qquad \Big(1 + \|\tilde{v}_n\|_A\Big)\|J'(\tilde{v}_n)\| \to 0 \quad \text{and} \quad \tilde{w}_n \rightharpoonup \tilde{w} \neq 0,$$

which contradicts $w_n \rightharpoonup 0$.

Suppose that (ii) is true. As in the proof of Lemma 3.4

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} K(x)F\big(C^2|w_n|^2\big)=0 \tag{4.5}$$

for each $C>0$, and one has $\frac{C}{\|v_n\|_A}\in[0,1]$ for n sufficiently large. There is a constant $M>0$ such that $J(tv_n)\leqslant M$ for every $t\in[0,1]$ and $n\in\mathbb{N}$. Thus

$$J(t_nv_n)\geqslant J\Big(\frac{C}{\|v_n\|_A}v_n\Big)=J(Cw_n)=\frac{C^2}{2}-\frac{1}{2}\int_{\mathbb{R}^N}K(x)F\big(C^2|w_n|^2\big).$$

By (4.5), one would get

$$\lim_{n\to+\infty}J(t_nv_n)\geqslant\frac{C^2}{2},\quad\text{for every } C>0,$$

which constitutes a contradiction, since $\big(J(t_nv_n)\big)$ is bounded. Consequently, the sequence (v_n) is bounded. □

From the preceding lemma, since the Hilbert space $D_A^{1,2}(\mathbb{R}^N,\mathbb{C})$ is reflexive, there exists $v\in D_A^{1,2}(\mathbb{R}^N,\mathbb{C})$ and a subsequence of (v_n), still denoted by (v_n), such that $v_n\rightharpoonup v$ in $D_A^{1,2}(\mathbb{R}^N,\mathbb{C})$.

Lemma 4.2. *The weak limit v of (v_n) is nontrivial.*

Proof. Suppose by contradiction that $v\equiv 0$. Since

$$\int_{B_R}|K(x)-K_p(x)||F(|v_n|^2)|dx\leqslant\varepsilon\int_{B_R}|K(x)-K_p(x)||v_n|^{2^*}dx+\int_{B_R}|K(x)-K_p(x)||v_n|^pdx,$$

as consequence of $v\equiv 0$, it follows that

$$\int_{B_R}|K(x)-K_p(x)||F(|v_n|^2)|dx\to 0\quad\text{as } n\to+\infty. \tag{4.6}$$

On the other hand, from (K2), given $\epsilon>0$ there exists $R=R(\epsilon)$ such that

$$\big|K(x)-K_p(x)\big|<\epsilon,\quad\text{for all } |x|>R.$$

Thus

$$\int_{B_R^c}|K(x)-K_p(x)||F(|v_n|^2)|dx\leqslant\epsilon M \tag{4.7}$$

where

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^N}|F(|v_n|^2)|dx=M.$$

From (4.6) and (4.7)

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|K(x)-K_p(x)||F(|v_n|^2)|dx=0, \tag{4.8}$$

and

$$|J(v_n)-I(v_n)|\to 0\quad\text{as } n\to+\infty.$$

A similar argument shows that

$$|J'(v_n)v_n-I'(v_n)v_n|\to 0\quad\text{as } n\to+\infty.$$

Consequently,

$$I(v_n) = d + o_n(1) \quad \text{and} \quad I'(v_n)v_n = o_n(1). \tag{4.9}$$

Let s_n be positive number verifying

$$I'(s_n v_n)v_n = 0. \tag{4.10}$$

We claim that (s_n) converges to 1 as $n \to +\infty$. We begin proving that

$$\limsup_{n\to+\infty} s_n \leqslant 1. \tag{4.11}$$

Suppose by contradiction that, going if necessary to a subsequence, $s_n \geqslant 1+\delta$ for all $n \in \mathbb{N}$, for some $\delta > 0$. From (4.9),

$$\|v_n\|_A^2 = \int_{\mathbb{R}^N} K_p(x) f(|v_n|^2)|v_n|^2 dx + o_n(1).$$

On the other hand, from (4.10),

$$s_n\|v_n\|_A^2 = \int_{\mathbb{R}^N} K_p(x) f(s_n^2|v_n|^2)s_n|v_n|^2 dx.$$

Consequently

$$\int_{\mathbb{R}^N} K_p(x)\left[f\left(s_n^2|v_n|^2\right) - f\left(|v_n|^2\right)\right]|v_n|^2 dx = o_n(1),$$

and from $(f3)$ combined with (K1)–(K3),

$$\int_{\mathbb{R}^N} \left[f\left(s_n^2|v_n|^2\right) - f\left(|v_n|^2\right)\right]|v_n|^2 dx = o_n(1). \tag{4.12}$$

Since (v_n) is bounded, by Lemma 3.2 again, we have two alternatives, either

(i) there are $R, \eta > 0$ and $(y_n) \subset \mathbb{R}^N$ such that $\displaystyle\int_{B_R(y_n)} |v_n|^2 \geqslant \eta$, for all n,

or

(ii) $\displaystyle\int_{\mathbb{R}^N} |\hat{v}_n|^q \to 0$, where $\hat{v}_n = v_n \chi_{\{|s|\geqslant\tau\}}$, $\forall q \in (2, 2*)$ and $\tau > 0$.

In case (ii), we derive

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} f\left(|v_n|^2\right)|v_n|^2 dx = 0,$$

which implies $v_n \to 0$ in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$ that is impossible.

Let (y_n) be given by (i), and define $\tilde{v}_n(x) = v_n(x+y_n)$. Since

$$\int_{B_R(0)} |\tilde{v}_n|^2 dx \geqslant \eta > 0,$$

there exists $\tilde{v} \neq 0$ in n $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$ such that (v_n) is weakly convergent to $\tilde{v}$ in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$. From (4.12) and $(f3)$, Fatou's Lemma yields,

$$0 < \int_{\mathbb{R}^N} \left[f\left((1+\delta)^2|\tilde{v}_n|^2\right) - f\left(|\tilde{v}_n|^2\right)\right]|\tilde{v}_n|^2 dx = 0,$$

which is impossible. Hence

$$\limsup_{n\to+\infty} s_n \leqslant 1.$$

From this, (s_n) is bounded. Without loss of generality, we can assume that

$$\lim_{n\to+\infty} s_n = s_0 \leqslant 1.$$

If $s_0 < 1$, we have that $s_n < 1$ for n large enough. Hence, by Fatou's Lemma

$$0 < \int_{\mathbb{R}^N} \left[f\left(|\tilde{v}_n|^2\right) - f\left(s_0^2|\tilde{v}_n|^2\right)\right] |\tilde{v}_n|^2 dx = 0, \quad \text{when } s_0 > 0,$$

and

$$0 < \int_{\mathbb{R}^N} f\left(|\tilde{v}_n|^2\right)|\tilde{v}_n|^2 dx = 0, \quad \text{when } s_0 = 0,$$

which are impossible. Therefore,

$$\lim_{n\to+\infty} s_n = 1. \tag{4.13}$$

As a consequence of (4.13),

$$\int_{\mathbb{R}^N} K_p(x)F\left(s_n^2|v_n|^2\right)dx - \int_{\mathbb{R}^N} K_p(x)F\left(|v_n|^2\right)dx = o_n(1)$$

and

$$\left(s_n^2 - 1\right)\|v_n\|_A^2 = o_n(1),$$

leading to

$$I(s_n v_n) - I(v_n) = o_n(1).$$

Then, by(4.9)

$$c \leqslant I(s_n v_n) = I(v_n) + o_n(1) = d + o_n(1).$$

Taking $n \to +\infty$, we find $c \leqslant d$, which obtain a contradiction, because, by (4.4), $d < c$. This contradiction comes from the assumption that $v \equiv 0$. □

5 The proof of Theorem 1.2

In this section, we mean to prove Theorem 1.2. As the proof in the preceding section, we can prove that the functional I satisfies the geometry of the mountain pass and there is a Cerami sequence $(u_n) \in D_A^{1,2}(\mathbb{R}^N,\mathbb{C})$ satisfying (3.4) and (3.5). Finally, we have proved Lemma 3.3. In order to check that (u_n) is bounded in $D_A^{1,2}(\mathbb{R}^N,\mathbb{C})$, we should show that the (3.8) holds and proceed as in the proof of Lemma 3.4.

Let Ω, ξ, w, M be defined as in the proof of Lemma 3.4. Notice that $\left|\Omega\right| = 0$, since

$$o_n(1) + 1 \geqslant \int_{\Omega\cap\{|u_n|\geqslant\xi\}} \frac{K(x)F(|u_n|^2)}{|u_n|^2}|w_n|^2$$

implies that

$$1 \geqslant M\int_\Omega K(x)|w|^2,$$

and from (K4), we have $w = 0$.

Let us prove the limit (3.8). From $(f1)$, for each $\varepsilon > 0$, we have $\delta > 0$ and $C_\varepsilon > 0$ such that

$$\left|s^2 f(s^2)\right| \leqslant \varepsilon |s|^{2^*} + C_\varepsilon \chi_{\{|s|\geqslant\delta\}}, \quad \text{for all } s \in \mathbb{R}^N, \tag{5.1}$$

and

$$\left|F(s^2)\right| \leqslant \varepsilon |s|^{2^*} + C_\varepsilon \chi_{\{|s|\geqslant\delta\}}, \quad \text{for all } s \in \mathbb{R}^N. \tag{5.2}$$

By Sobolev embedding and (2.1), there exists $\hat{S} > 0$ such that

$$\int_{\mathbb{R}^N} |v|^{2^*} dx \leqslant \hat{S}\Big(\int_{\mathbb{R}^N} |\nabla_A v|^2 dx\Big)^{\frac{2^*}{2}},$$

for all $v \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$. Observe that $\Delta_n = \{x \in \mathbb{R}^N : |Cw_n(x)| \geqslant \delta\}$ is such that

$$\int_{\Delta_n} |w_n|^{2^*} \leqslant \hat{S}.$$

This implies, besides (5.2), that

$$\int_{|x|\geqslant R} K(x)F\big(|Cw_n|^2\big)dx \leqslant \varepsilon C^{2^*} \|K\|_\infty \int_{B_R^c(0)} |w_n|^{2^*} dx + C_\varepsilon \int_{B_R^c(0)\cap\Delta_n} K(x)dx,$$

and from (1.3)

$$\lim_{R\to+\infty} \int_{|x|\geqslant R} K(x)F\big(|Cw_n|^2\big)dx \leqslant \varepsilon \hat{S} C^{2^*} \|K\|_\infty, \quad \text{uniformly in } n.$$

On the other hand, for any $R > 0$, from $(f1)$ and Strauss' compactness lemma (see [10])

$$\lim_{n\to+\infty} \int_{|x|\leqslant R} K(x)F\big(|Cw_n|^2\big)dx = 0,$$

which shows that (3.8) holds and (u_n) is bounded in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$.

To prove Theorem 1.2, it is important to show that (u_n) converges in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$. In this way we can see that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)f(|u_n|^2)|u_n|^2 dx = \int_{\mathbb{R}^N} K(x)f(|u|^2)|u|^2 dx. \tag{5.3}$$

To verify (5.3), consider $E_n = \big\{x \in \mathbb{R}^N : |u_n(x)| \geqslant \delta\big\}$ which satisfies $\sup_{n\in\mathbb{N}} |E_n| < \infty$. From (5.1)

$$\int_{|x|\geqslant R} K(x)f(|u_n|^2)|u_n|^2 dx \leqslant \varepsilon \|K\|_\infty \int_{B_R^c(0)} |u_n|^{2^*} dx + C_\varepsilon \int_{B_R^c(0)\cap E_n} K(x)dx$$

and from (1.3)

$$\limsup_{R\to+\infty} \int_{|x|\geqslant R} K(x)f(|u_n|^2)|u_n|^2 dx \leqslant \varepsilon \hat{S} \|K\|_\infty, \quad \text{uniformly in } n.$$

Again, from $(f1)$ and Strauss' compactness lemma

$$\lim_{n\to+\infty} \int_{|x|\leqslant R} K(x)f(|u_n|^2)|u_n|^2 dx = \int_{|x|\leqslant R} K(x)f(|u|^2)|u|^2 dx,$$

for all $r > 0$ fixed, and it shows that (5.3) holds. Since $I'(u_n)u_n \to 0$, (5.3) implies that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla_A u_n|^2 dx = \int_{\mathbb{R}^N} K(x)f(|u|^2)|u|^2 dx = \int_{\mathbb{R}^N} |\nabla_A u|^2 dx$$

finishing the proof of Theorem 1.2.

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On a class of difference equations involving a linear map with two dimensional kernel

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Abstract. We establish necessary and sufficient conditions for the existence of periodic solutions to second-order nonlinear difference equations of the form $\Delta^2 x_i + \lambda x_i + \Delta f(x_i) = e_i$, $i \in \mathbb{N}$, and for a simpler equation with difference-free nonlinearity.

The linear part of the equation has two-dimensional kernel.

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1 Introduction

The problem of finding periodic solutions for discrete semilinear systems has been studied in recent years by many authors, with emphasis in a variety of features and with recourse to several techniques. Among the extensive literature on this kind of problems, let us mention a selection of papers (see also their references) which display also a variety of methods used: Lyapunov–Schmidt reduction, Brouwer fixed point theorem [1, 11, 12], minimax methods, critical point theory, Morse theory [3, 8, 10, 13, 15], upper and lower solutions [2, 4, 5]. See also [14] for the analysis of linear eigenvalue theory.

If one considers, in particular, second order scalar difference equations, it turns out that an interesting feature of periodic problems is that they provide resonance models that may involve a linear operator whose kernel has dimension one or two. Both settings have been considered in some of the above mentioned articles. An illustration of peculiarities of such problems can found in [11].

Our purpose in this paper is to study a problem where, on one hand, we have to deal with a two-dimensional kernel and, on the other hand, the nonlinear part involves first order differences. Our motivation goes back to the paper of A. C. Lazer [9], where the existence of 2π-periodic solutions to the resonant problem

$$u'' + u + \big(F(u)\big)' = e(t) \tag{1.1}$$

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is studied. Here e is continuous, 2π-periodic, and F is C^1. Necessary and sufficient conditions for existence are found, in terms of the size of the projection of e onto the kernel of the linear part: namely, if $a \sin t + b \cos t$ appears in the Fourier series of e, then the condition for existence is found to be

$$\pi \sqrt{a^2 + b^2} < 2\big(F(\infty) - F(-\infty)\big). \tag{1.2}$$

We propose to consider the difference equation whose structure is reminiscent of (1.1). Specifically, we want to give criteria for the existence of N-periodic solutions to the second-order nonlinear difference equation

$$\Delta^2 x_i + \lambda x_i + \Delta f(x_i) = e_i, \qquad i \in \mathbb{N}, \tag{1.3}$$

where, considering the jump $h = \frac{2\pi}{N}$, we define the difference operators as

$$\Delta^2 x_i = \frac{1}{h^2}(x_{i+1} - 2x_i + x_{i-1})$$

and

$$\Delta f(x_i) = \frac{1}{h}\big(f(x_i) - f(x_{i-1})\big).$$

In addition, $f : \mathbb{R} \to \mathbb{R}$ is a given function, $\lambda = \frac{N^2}{\pi^2} \sin^2 \frac{\pi}{N}$ is the smallest positive eigenvalue of $-\Delta^2$ with N-periodic conditions (which approaches 1 as N grows larger) and $e = (e_i)$ is a N-periodic vector.

Therefore, the underlying linear operator in our discrete system has in fact two-dimensional kernel; on the other hand the nonlinear term contains first order differences. However, because it appears as a by-product of the method, we deal also with the (simpler) version in which the nonlinearity is difference-free

$$\Delta^2 x_i + \lambda x_i + f(x_i) = e_i, \qquad i \in \mathbb{N}. \tag{1.4}$$

It is our purpose to relate the existence of periodic solutions to (1.3) – or (1.4) – to some relationship between f, e and the kernel of the linear operator $\Delta^2 + \lambda$ acting on N-periodic vectors.

We shall proceed by rephrasing the Poincaré–Miranda theorem in appropriate form, so that it can be used to recover results that correspond to those given by Lazer in [9]. Our necessary or sufficient conditions for existence are a little more complicated than those in [9] because the discretization does not allow a sharp statement; they are close to the conditions in [9] when N is large, but it will be seen that we need to introduce "correcting terms" in the corresponding inequalities.

Since N-periodic sequences can be identified with vectors in $\mathbb{R}^N$, we henceforth identify the elements of $\mathbb{R}^N$ with such sequences, that may be indexed in $\mathbb{Z}$. It will be convenient to consider the following norm and the associated inner product in N-dimensional space:

$$\|x\| = \sqrt{h \sum_{i=1}^{N} x_i^2}\,.$$

It is easy to see that the kernel of the operator $\Delta^2 + \lambda$ is 2-dimensional and is spanned by $\underline{s}$ and $\underline{c}$, with

$$s_j = \sin\left(\frac{2\pi j}{N}\right) \quad \text{and} \quad c_j = \cos\left(\frac{2\pi j}{N}\right).$$

With the previous definition in mind, we have that $\underline{s}$ and $\underline{c}$ are orthogonal and $\|\underline{s}\|^2 = \|\underline{c}\|^2 = \pi$.

Another useful observation is that the linear operator Δ^2 acting on periodic vectors is symmetric. That is, we can write it in matrix form as the $N \times N$ symmetric matrix

$$\frac{N^2}{4\pi^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -2 \end{bmatrix}.$$

Hence, setting

$$\mathcal{A} = \Delta^2 + \lambda,$$

we have

$$\sum_{i=1}^{N}(\Delta^2 a_i + \lambda a_i)b_i = (\mathcal{A}a)\cdot b = a\cdot(\mathcal{A}b) = \sum_{i=1}^{N} a_i(\Delta^2 b_i + \lambda b_i).$$

From this, it also follows that the kernel and the image of the operator $\mathcal{A}$ are orthogonal ($\mathrm{Im}(\mathcal{A}) = \mathrm{Ker}(\mathcal{A})^{\perp}$) and any $x \in \mathbb{R}^N$ can be written uniquely as $x = \alpha\underline{s} + \beta\underline{c} + w$, for some $\alpha, \beta \in \mathbb{R}$ and $w \in M := \mathrm{Im}(\mathcal{A})$.

As already stated, we think of e and the solution x as N-periodic vectors, which are identified with elements of $\mathbb{R}^N$. We consider the orthogonal projection of e on $\mathrm{Ker}(\mathcal{A})$, denoted by

$$A\underline{s} + B\underline{c}$$

meaning that

$$A = \frac{h}{\pi}\sum_{i=1}^{N} e_i \sin\left(\frac{2\pi i}{N}\right), \qquad B = \frac{h}{\pi}\sum_{i=1}^{N} e_i \cos\left(\frac{2\pi i}{N}\right). \tag{1.5}$$

We also set

$$f(-\infty) = \lim_{t\to-\infty} f(t), \qquad f(\infty) = \lim_{t\to+\infty} f(t)$$

and

$$m = \sup_{t\in\mathbb{R}} |f(t)|. \tag{1.6}$$

Before stating the main results, further notation must be introduced. For $\theta \in \mathbb{R}$ consider the N-periodic vector $\sigma_j = \sigma_j(\theta) = \sin\bigl(\theta + \frac{2\pi j}{N}\bigr)$. Let $x^+ = \max\{x, 0\}$. We introduce the numbers α_N, β_N by

$$\alpha_N = \min_{\theta\in\mathbb{R}} h\sum_{j=1}^{N}\sigma_j^+, \qquad \beta_N = \max_{\theta\in\mathbb{R}} h\sum_{j=1}^{N}\sigma_j^+ \tag{1.7}$$

and we also set

$$\alpha'_N := 2\cos\frac{\pi}{N}\,\cos\frac{2\pi}{N}. \tag{1.8}$$

It is easily seen that the sequences α_N, β_N and α'_N have limit 2 as $N \to \infty$.

In order to simplify the statements and proofs, *we shall take N to be a multiple of* 4. This assumption will not appear in the statements.

Theorem 1.1. *Let $\{e_i\}_{i\in\mathbb{N}}$ be N-periodic and $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f(\infty)$ and $f(-\infty)$ are finite. Then with the notation of (1.5), (1.6) and (1.8):*

(i) Suppose that $\forall x\in\mathbb{R}$, $f(-\infty)<f(x)<f(\infty)$. Then if the equation (1.3) has a N-periodic solution, the condition

$$\pi\sqrt{A^2+B^2}<2\big(f(\infty)-f(-\infty)\big)$$

is satisfied.

(ii) Assume that

$$\pi\sqrt{A^2+B^2}+4\,m\sin\frac{\pi}{N}<\alpha_N'\big(f(\infty)-f(-\infty)\big). \tag{1.9}$$

Then equation (1.3) has a N-periodic solution.

Theorem 1.2. *Let $\{e_i\}_{i\in\mathbb{N}}$ be N-periodic and $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f(\infty)$ and $f(-\infty)$ are finite. With the notation of (1.5), (1.6) and (1.7):*

(i) Suppose that $\forall x\in\mathbb{R}$, $f(-\infty)<f(x)<f(\infty)$. Then if the equation (1.4) has a N-periodic solution, the condition

$$\pi\sqrt{A^2+B^2}<\beta_N\big(f(\infty)-f(-\infty)\big) \tag{1.10}$$

holds.

(ii) Assume that

$$\pi\sqrt{A^2+B^2}+8\,m\pi^2/N^2<\alpha_N\big(f(\infty)-f(-\infty)\big). \tag{1.11}$$

Then equation (1.4) has a N-periodic solution.

Remark 1.3. In the above conditions (1.9), (1.10), (1.11), we must use the approximations α_N, β_N, α_N', rather than the constant 2 (the integral of $\sin^+$ over a period) that appears in [9]. Moreover, we add "correcting terms" that behave as $O(1/N)$ and $O(1/N^2)$, respectively, and are not needed when one deals with a differential equation. Our conditions make sense for large values of N.

2 Auxiliary results

We shall use the following elementary formula for "summing by parts".

Lemma 2.1. *Let a_i and b_i be two N-periodic vectors. Setting $\Delta a_i=a_i-a_{i-1}$ we have:*

$$\sum_{i=1}^{N}\Delta a_i b_i=-\sum_{i=1}^{N}a_i\,\Delta b_{i+1}.$$

Let us recall the Poincaré–Miranda's theorem, stated as follows.

Theorem 2.2. *Let $L_i>0$, $i=1,\dots,N$, $\Omega=\big\{x\in\mathbb{R}^N:\ |x_i|\le L_i,\ i=1,\dots,N\big\}$ and $f:\Omega\to\mathbb{R}^N$ be continuous satisfying:*

$$f_i\big(x_1,x_2,\dots,x_{i-1},-L_i,x_{i+1},\dots,x_N\big)\ge 0 \quad \text{for } 1\le i\le N,$$
$$f_i\big(x_1,x_2,\dots,x_{i-1},+L_i,x_{i+1},\dots,x_N\big)\le 0 \quad \text{for } 1\le i\le N.$$

Then, $f(x)=0$ has a solution in Ω.

We need slight variations of this statement, where the vector field is defined on a product of intervals with a ball. Although such versions may be related to the approach of [7], we include simple proofs for completeness.

In what follows we shall denote by γ the orthogonal projection of $\mathbb{R}^N = \mathbb{R}^{N-2} \times \mathbb{R}^2$ onto the second factor $\mathbb{R}^2$.

Proposition 2.3. *Let L_i $(i=1,\dots,N)$ and R be positive numbers. Let $\Omega = \{x \in \mathbb{R}^N : |x_i| \le L_i,\ i = 1,\dots,N-2,\ x_{N-1}^2 + x_N^2 \le R^2\} = \prod_{i=1}^{N-2} [-L_i, L_i] \times \overline{B_R} \subseteq \mathbb{R}^{N-2} \times \mathbb{R}^2$ and $f : \Omega \to \mathbb{R}^N$ be a continuous function satisfying:*

$$f_i(x_1, x_2, \dots, x_{i-1}, -L_i, x_{i+1}, \dots, x_N) < 0 \quad \text{for } 1 \le i \le N-2,$$
$$f_i(x_1, x_2, \dots, x_{i-1}, +L_i, x_{i+1}, \dots, x_N) > 0 \quad \text{for } 1 \le i \le N-2$$

and

$$\forall x \in \prod_{i=1}^{N-2} [-L_i, L_i] \times \partial\overline{B_R}, \qquad f(x) \cdot \gamma x > 0.$$

Then there exists $x^* \in \Omega$ such that $f(x^*) = 0$.

Proof. We use a standard compactness argument to show that there exists $\varepsilon > 0$ such that the mapping $x \mapsto x - \varepsilon f(x)$ maps Ω into Ω. The conclusion follows from Brouwer's fixed point theorem. In fact, if the claim is not true, we find $\epsilon_n \downarrow 0$ and $x_n \in \Omega$ such that $x_n - \varepsilon_n f(x_n) \notin \Omega$. Then, considering subsequences if necessary, either there exists $i \in \{1, \dots, n-2\}$ such that, say

$$x_{ni} - \epsilon_n f_i(x_n) > L_i$$

or

$$\|\gamma x_n - \varepsilon_n \gamma f(x_n)\| > R^2.$$

We may suppose that $x_n \to x$. In the first case we obtain $x_i \ge L_i$, that is, $x_i = L_i$, and then, by the continuity of f and the assumption on f_i, the first inequality gives a contradiction for large n. In the second case, setting $M = \max_{z\in\Omega} \|f(z)\|$, we have

$$\|\gamma x_n\|^2 - 2\epsilon_n \gamma x_n \cdot \gamma f(x_n) + M^2 \epsilon_n^2 > R^2.$$

The previous argument then gives $\|\gamma x\| = R$ and, since by the assumptions $\lim_{n\to\infty} \gamma x_n \cdot \gamma f(x_n) > 0$, again a contradiction for large n is obtained. □

Proposition 2.3 is a very natural generalization of Poincaré–Miranda's theorem, as the dot product condition gives a reasonable notion of the vector field "to point outside" of the domain. Finally, we state a last version of the result, with a variation of the dot product condition.

Proposition 2.4. *Let Ω be as in the preceding proposition and $f : \Omega \to \mathbb{R}^N$ be a continuous function satisfying:*

$$f_i(x_1, x_2, \dots, x_{i-1}, -L_i, x_{i+1}, \dots, x_N) < 0 \quad \text{for } 1 \le i \le N-2,$$
$$f_i(x_1, x_2, \dots, x_{i-1}, +L_i, x_{i+1}, \dots, x_N) > 0 \quad \text{for } 1 \le i \le N-2$$

and

$$\forall x \in \left(\prod_{i=1}^{N-2} [-L_i, L_i]\right) \times \partial\overline{B_R}, \qquad f(x) \cdot \rho(\gamma x) > 0,$$

where ρ denotes a rotation of angle $\frac{\pi}{2}$ in the plane $\mathbb{R}^2$.

Then there exists $x^* \in \Omega$ such that $f(x^*) = 0$.

Proof. Define $g : \Omega \to \mathbb{R}^N$ by $g(x) = f\big(x - \gamma(x), \rho^{-1}(\gamma(x))\big)$. Then g satisfies the conditions of the previous proposition. The conclusion follows. □

Now let $Q, P : \mathbb{R}^N \to \mathbb{R}^N$ be the orthogonal projections onto $\mathrm{Ker}(\mathcal{A})$ and $M = \mathrm{Ker}(\mathcal{A})^\perp$, respectively. Let $K : M \to M$ be defined by

$$K = \left(\mathcal{A}_{|M}\right)^{-1}.$$

We now write problem (1.3) in operator form as

$$\mathcal{A}x + \mathcal{G}(x) = e$$

where $\mathcal{G} : \mathbb{R}^N \to \mathbb{R}^N$ is the nonlinear map whose i-th component is $\frac{1}{h}\big(f(x_i) - f(x_{i-1})\big)$.

Using the orthogonal decomposition $x = u + v$, with $u \in \mathrm{Ker}(\mathcal{A})$ and $v \in M$, we obtain

$$\mathcal{A}x + \mathcal{G}(x) = e \iff \mathcal{A}v + \mathcal{G}(u + v) = e$$

or equivalently

$$v - K\big(-P\mathcal{G}(u + v) + Pe\big) = 0, \qquad Q\mathcal{G}(u + v) - Qe = 0. \tag{2.1}$$

We can then define $V : M \times \mathrm{Ker}(\mathcal{A}) \to M \times \mathrm{Ker}(\mathcal{A})$ by:

$$V(v, u) = \Big(v - K\big(-P\mathcal{G}(u + v) + Pe\big), Q\mathcal{G}(u + v) - Qe\Big),$$

and conclude that:

Proposition 2.5. *The periodic problem* (1.3) *has a solution if and only if there is a solution to* $V(v, u) = 0$.

3 Proof of Theorem 1.2

We start with some simple remarks and notation. Recall the meaning of the expression

$$\sigma_i = \sigma_i(t) = \sin\left(t + \frac{2\pi i}{N}\right)$$

and set

$$S^+ = \{i : \sigma_i > 0,\ i = 1, \dots, N\}, \qquad S^- = \{i : \sigma_i < 0,\ i = 1, \dots, N\}.$$

Since N is even, there is at most an index $i* \in S^+$ such that $0 < \sigma_{i*} < \sin\frac{\pi}{N}$. In such case, there exists a (unique) $j* \in S^-$ with $|\sigma_{j*}| = \sigma_{i*} < \sin\frac{\pi}{N}$. In fact it is easy to see that, assuming without loss of generality that $-\frac{2\pi}{N} < t \le 0$, we have $i* = 1$ or $i* = \frac{N}{2}$. Let us then define

$$S^+* = S^+ \setminus i*, \qquad S^-* = S^+ \setminus j*\,.$$

Otherwise, if $\sigma_i \ge \sin\frac{\pi}{N}$ for all $i \in S^+$, put

$$S^+* = S^+, \qquad S^-* = S^-.$$

We are now ready to present the proof for the case of a difference-free nonlinearity. The abstract approach is very similar to the one described above, where we replace $\mathcal{G}$ with $\mathcal{F} : \mathbb{R}^N \to \mathbb{R}^N$ which is defined component-wise as $\mathcal{F}_i(x) = f(x_i)$. Hence we consider the operator problem

$$\mathcal{A}x + \mathcal{F}(x) = e.$$

As before, finding a periodic solution to (1.2) is equivalent to solving

$$W(v,u) := \Big(v - K\big(-P\mathcal{F}(u+v) + Pe\big), \, Q\mathcal{F}(u+v) - Qe\Big) = 0.$$

Proof. (i) Let x be a solution of (1.4) and consider the orthogonal splitting of e,

$$e = A\underline{s} + B\underline{c} + w,$$

where A, $B \in \mathbb{R}$ and $w \in M$. The inner product of equation (1.4) with $z = A\underline{s} + B\underline{c}$ yields

$$\mathcal{F}(x) \cdot z = e \cdot z = \|z\|^2 = \pi(A^2 + B^2).$$

On the other hand

$$\mathcal{F}(x) \cdot z = h \sum_{i=1}^{N} f(x_i)\, z_i$$

and there exists $\varphi \in \mathbb{R}$ such that $z_i = \sqrt{A^2 + B^2}\sin\big(\varphi + \frac{2\pi i}{N}\big)$. Hence summing separately over the sets of indices where the z_i are positive and where the z_i are negative and using the definition of β_N and the assumption of (i) we obtain

$$\pi(A^2 + B^2) < \beta_N \sqrt{A^2 + B^2}\,\big(f(\infty) - f(-\infty)\big).$$

(ii) We have to prove that $W(v,u) = 0$ has a solution, using the analogue of Proposition 2.5. Suppose that (1.11) holds.

First, we want to show that there exists an $L > 0$ such that

($*$) If $v_i = L$, then $W_i(v,u) > 0$ (respectively if $v_i = -L$, then $W_i(v,u) < 0$), for $1 \le i \le N-2$. Here of course the v_i are coordinates with respect to some basis of M.

To this purpose it suffices to prove that $K\big(-P\mathcal{F}(u+v) + Pe\big)$ is bounded.

Since K is linear there is a constant C such that:

$$\|Kx\| \le C\|x\|, \qquad \forall x \in \mathbb{R}^N.$$

Since f is bounded, so is $\mathcal{F}$ and we have

$$\| - \mathcal{F}(u+v) + e\| \le C^* \quad \text{for some } C^* \in \mathbb{R}.$$

Since P is an orthogonal projection, it follows then that

$$\begin{aligned} \big\|K\big(-P\mathcal{F}(u+v) + e\big)\big\| &\le C\big\|P\big(-\mathcal{F}(u+v) + e\big)\big\| \\ &\le CC^*. \end{aligned}$$

Therefore we can pick up a positive number L with the property ($*$).

Now fix ε such that

$$\pi\sqrt{A^2 + B^2} + 8\,m\pi^2/N^2 < \alpha_N\big(f(\infty) - f(-\infty) - 2\,\varepsilon\big).$$

Consider a ball in $\mathrm{Ker}(\mathcal{A})$ with radius R. Let u be on the boundary of the ball, with $u = \alpha\underline{s} + \beta\underline{c}$. There exists $t \in \mathbb{R}$ so that we can write

$$u = \sqrt{\alpha^2 + \beta^2}\,\sigma, \qquad \sigma_i = \sin\left(\frac{2\pi i}{N} + t\right).$$

In particular $R = \sqrt{\pi(\alpha^2 + \beta^2)}$. Let $v \in M$ with $|v_i| \le L$. Then, with the notation introduced in the beginning of this section

$$\begin{aligned}
Q\big(\mathcal{F}(u+v) - e\big)\cdot u &= \mathcal{F}(u+v)\cdot u - e\cdot u \\
&\ge h\sum_{i=1}^{N} f(u_i + v_i)u_i \;-\; \pi\sqrt{A^2+B^2}\sqrt{\alpha^2+\beta^2} \\
&= h\sum_{i\in S^+*} f\left(\frac{R}{\sqrt{\pi}}\sigma_i + v_i\right)\frac{R}{\sqrt{\pi}}\sigma_i \;+\; hf\left(\frac{R}{\sqrt{\pi}}\sigma_{i*} + v_{i*}\right)\frac{R}{\sqrt{\pi}}\sigma_{i*} \\
&\quad + h\sum_{i\in S^-*} f\left(\frac{R}{\sqrt{\pi}}\sigma_i + v_i\right)\frac{R}{\sqrt{\pi}}\sigma_i \;+\; hf\left(\frac{R}{\sqrt{\pi}}\sigma_{j*} + v_{j*}\right)\frac{R}{\sqrt{\pi}}\sigma_{j*} \\
&\quad - \pi\sqrt{A^2+B^2}\sqrt{\alpha^2+\beta^2}
\end{aligned}$$

where the summands that contain $hf\big(\frac{R}{\sqrt{\pi}}\sigma_{i*} + v_{i*}\big)$ and $hf\big(\frac{R}{\sqrt{\pi}}\sigma_{j*} + v_{j*}\big)$ appear only if $i*$ and $j*$ exist.

Let R be so large that

$$\frac{R}{\sqrt{\pi}}\sin\frac{\pi}{N} - L > T$$

where T is such that

$$f(x) > f(+\infty) - \varepsilon \quad \forall x \ge T, \qquad f(x) < f(-\infty) + \varepsilon \quad \forall x \le -T.$$

Hence, using symmetry, in any case the above expression is greater than

$$\begin{aligned}
&\frac{R}{\sqrt{\pi}}\left(\big(f(+\infty) - \varepsilon\big)\,h\sum_{i\in S^+*}\sigma_i - \big(f(-\infty) + \varepsilon\big)\,h\sum_{i\in S^-*}|\sigma_i| - 2\,hm\frac{\pi}{N} - \pi\sqrt{A^2+B^2}\right) \ge \\
&\qquad \ge \frac{R}{\sqrt{\pi}}\left(\big(f(+\infty) - f(-\infty) - 2\,\varepsilon\big)\,h\sum_{i\in S^+*}\sigma_i - 4\,m\frac{\pi^2}{N^2} - \pi\sqrt{A^2+B^2}\right) \\
&\qquad \ge \frac{R}{\sqrt{\pi}}\left(\big(f(+\infty) - f(-\infty) - 2\,\varepsilon\big)\left(\alpha_N - h\frac{\pi}{N}\right) - 4\,m\frac{\pi^2}{N^2} - \pi\sqrt{A^2+B^2}\right) \\
&\qquad \ge \frac{R}{\sqrt{\pi}}\left(\big(f(+\infty) - f(-\infty) - 2\,\varepsilon\big)\,\alpha_N - 8\,m\frac{\pi^2}{N^2} - \pi\sqrt{A^2+B^2}\right) \;>\; 0.
\end{aligned}$$

By Proposition 2.3, it follows that there is a solution to $W(v,u) = 0$ and, consequently, a solution to the periodic problem (1.2). □

4 Proof of the main result

First we list some elementary facts to be used in the sequel.

Lemma 4.1. *If* $\sigma_i(t) > \sin\frac{\pi}{N}$, *then* $\sigma_{i+1}(t+\frac{\pi}{2}) < \sigma_i(t+\frac{\pi}{2})$. *If* $0 \le \sigma_k(t) \le \sin\frac{\pi}{N}$ *then*

$$\left|\sigma_{k+1}\left(t+\frac{\pi}{2}\right) - \sigma_k\left(t+\frac{\pi}{2}\right)\right| \le 2\,\sin\frac{\pi}{N}.$$

Proof. It suffices to remark that $\sigma_{i+1}(t+\frac{\pi}{2}) - \sigma_i(t+\frac{\pi}{2}) = -2\,\sin\frac{\pi}{N}\,\sin(\frac{2\pi i}{N}+\frac{\pi}{N}+t)$. □

Lemma 4.2. $\sum_{i=1}^{N}\big(\sigma_{i+1}(t)-\sigma_i(t)\big)^+ \le 2.$

Lemma 4.3. $\sum_{i\in S^+*}\left(\sigma_{i+1}\left(t+\frac{\pi}{2}\right) - \sigma_i\left(t+\frac{\pi}{2}\right)\right)^- \ge 2\,\cos\frac{2\pi}{N}\,\cos\frac{\pi}{N}.$

Proof. Suppose first that $i*$ exists, and to fix ideas $i* = 1$. Then we may take $S^+* = \{2,\ldots,N/2\}$ and $-\frac{2\pi}{N} < t < -\frac{\pi}{N}$ (so that in fact $0 < \frac{2\pi}{N}+t < \frac{\pi}{N}$). Then, writing $N = 4p$ and using the elementary formula for $\sin x - \sin y$,

$$\begin{aligned}\sum_{i\in S^+*}\left(\sigma_{i+1}\left(t+\frac{\pi}{2}\right) - \sigma_i\left(t+\frac{\pi}{2}\right)\right)^- &= \sum_{i=2}^{N/2}\left[\sin\left(\frac{2\pi(i+1+p)}{N}+t\right) - \sin\left(\frac{2\pi(i+p)}{N}+t\right)\right]^- \\ &= \sin\left(\frac{2\pi(2+p)}{N}+t\right) - \sin\left(\frac{2\pi(\frac{N}{2}+p+1)}{N}+t\right) \\ &= \sin\left(\frac{4\pi+2\pi p}{N}+t\right) - \sin\left(\frac{N\pi+2\pi+2\pi p}{N}+t\right) \\ &= 2\cos\left(\frac{3\pi}{N}+t\right)\cos\frac{\pi}{N}.\end{aligned}$$

Since $\frac{3\pi}{N}+t \in [\frac{\pi}{N}, \frac{2\pi}{N}]$, the inequality follows.

Now suppose that $S^+* = S^+$. Then either $S^+* = \{1,\ldots,N/2\}$ with $t = -\frac{\pi}{N}$ or $S^+* = \{1,\ldots,N/2-1\}$ with $t=0$. In the first case the sum is $2 - 2\big(1-\cos\frac{\pi}{N}\big) = 2\cos\frac{\pi}{N}$. In the second case the sum is equal to $2 - \big(1-\cos\frac{2\pi}{N}\big) = 1+\cos\frac{2\pi}{N}$. In both cases the result is greater than $2\cos\frac{2\pi}{N}\cos\frac{\pi}{N}$. □

Remark 4.4. The fact that N is a multiple of 4 yields a simple formulation and proof of the above lemma.

We now prove Theorem 1.1.

Proof. (i) Let x be a solution to (1.1) and consider again the orthogonal splitting of e,

$$e = A\underline{s} + B\underline{c} + w,$$

where $A, B \in \mathbb{R}$ and $w \in M$. The inner product of equation (1.3) with $z = A\underline{s} + B\underline{c}$ yields

$$\mathcal{G}(x)\cdot z = e\cdot z = \|z\|^2 = \pi(A^2+B^2).$$

On the other hand, by Lemma 2.1,

$$\mathcal{G}(x)\cdot z = -h\sum_{i=1}^{N}\frac{f(x_i)\,(z_{i+1}-z_i)}{h}.$$

There exists $\varphi \in \mathbb{R}$ such that $z_i = \sqrt{A^2+B^2}\sin\left(\varphi + \frac{2\pi i}{N}\right)$. Hence splitting the sum into

$$-\sum_{i=1}^{N} f(x_i)\,(z_{i+1}-z_i)^+ + \sum_{i=1}^{N} f(x_i)\,(z_{i+1}-z_i)^-$$

and using the assumptions and Lemma 4.2 we obtain

$$\pi(A^2+B^2) < 2\sqrt{A^2+B^2}\,\big(f(\infty)-f(-\infty)\big).$$

(ii) By Proposition 2.5, we only need to prove that $V(v,u)=0$ has a solution, which we do using Proposition 2.4. Suppose that (1.9) holds.

First, we want to show that there exists an L such that if $v_i = L$, then $V_i(v,u) > 0$ (respectively if $v_i = -L$, then $V_i(v,u) < 0$), for $1 \le i \le N-2$. It suffices then to prove that $K\big(-P\mathcal{G}(u+v)+Pe\big)$ is bounded, and this is done the same way as given in the proof of Theorem 2.2 (note that $\mathcal{G}$ is bounded as well).

Let $\varepsilon > 0$ be such that

$$\big(f(+\infty)-f(-\infty)-2\,\varepsilon\big)\,\alpha'_N - 4\,m\sin\frac{\pi}{N} - \pi\sqrt{A^2+B^2} \;>\; 0$$

and fix $T > 0$ such that

$$f(x) > f(+\infty)-\varepsilon \quad \forall x \ge T, \qquad f(x) < f(-\infty)+\varepsilon \quad \forall x \le -T.$$

Consider now a ball in $\mathrm{Ker}(\Delta^2+\lambda)$ with radius R so that $\frac{R}{\sqrt{\pi}}\sin\frac{\pi}{N} - L > T$. Let u be on the boundary of the ball, with $u = \alpha\underline{s} + \beta\underline{c}$, meaning that $R = \sqrt{\pi(\alpha^2+\beta^2)}$. Consider the rotation ρ of angle $\pi/2$ in this two-dimensional subspace, given by

$$\rho(u) = -\beta\underline{s} + \alpha\underline{c}.$$

It is easily seen that, if $u_i = \frac{R}{\sqrt{\pi}}\sin\left(\frac{2\pi i}{N}+t\right)$, then $\rho(u)_i = \frac{R}{\sqrt{\pi}}\sin\left(\frac{2\pi i}{N}+t+\frac{\pi}{2}\right)$. Then we compute, with $|v_i| \le L$:

$$\begin{aligned} Q\big(\mathcal{G}(u+v)-e\big)\cdot\rho(u) &= \mathcal{G}(u+v)\cdot\rho(u) - e\cdot\rho(u) \\ &\ge h\sum_{i=1}^{N}\Delta f(u_i+v_i)\,\rho(u)_i \;-\; \pi\sqrt{A^2+B^2}\sqrt{\alpha^2+\beta^2} \\ &= -\sum_{i=1}^{N} f(u_i+v_i)\,\big(\rho(u)_{i+1}-\rho(u)_i\big) - \pi\sqrt{A^2+B^2}\sqrt{\alpha^2+\beta^2}. \end{aligned}$$

Noticing that the σ_i and the differences $\rho(u)_{i+1}-\rho(u)_i$ have opposite signs (as they lie in sine graphs misaligned by a translation of $\frac{\pi}{2}$) we may write

$$\begin{aligned} &-\sum_{i=1}^{N} f(u_i+v_i)\,\big(\rho(u)_{i+1}-\rho(u)_i\big) \\ &\quad= \sum_{i\in S^+} f\left(\frac{R}{\sqrt{\pi}}\sigma_i + v_i\right)\big(\rho(u)_{i+1}-\rho(u)_i\big)^- - \sum_{i\in S^-} f\left(\frac{R}{\sqrt{\pi}}\sigma_i + v_i\right)\big(\rho(u)_{i+1}-\rho(u)_i\big)^+. \end{aligned}$$

Hence

$$Q\big(\mathcal{G}(u+v)-e\big)\cdot\rho(u)$$
$$\geq \sum_{i\in S^{+*}} f\Big(\frac{R}{\sqrt{\pi}}\,\sigma_i+v_i\Big)\big(\rho(u)_{i+1}-\rho(u)_i\big)^{-}-\sum_{i\in S^{-*}} f\Big(\frac{R}{\sqrt{\pi}}\,\sigma_i+v_i\Big)\big(\rho(u)_{i+1}-\rho(u)_i\big)^{+}$$
$$-\,m\big(\rho(u)_{i*+1}-\rho(u)_{i*}\big)^{-}-m\big(\rho(u)_{j*+1}-\rho(u)_{j*}\big)^{+}-\pi\sqrt{A^2+B^2}\sqrt{\alpha^2+\beta^2}\,.$$

By Lemmas 4.1 and 4.3 and the definition of α'_N we obtain

$$Q\big(\mathcal{G}(u+v)-e\big)\cdot\rho(u)\ \geq\ \frac{R}{\sqrt{\pi}}\Big(\big(f(+\infty)-f(-\infty)-2\,\varepsilon\big)\alpha'_N-4\,m\sin\frac{\pi}{N}-\pi\sqrt{A^2+B^2}\Big)>0.$$

We then conclude that there exists a solution to $V(v,u)=0$ and therefore there exists a periodic solution to (1.3). □

A final remark is in order. The estimates for L and R obtained in the proof of Theorem 1.1 depend on N. However under natural assumptions we can show that norms of the solutions are kept below some constant. This is so because there exist a priori bounds for the solutions of (1.3) which do not depend on N. To see this, suppose that $e=e_N$ is defined for all N and that

$$E:=\sup_N\|e_N\|<\infty.$$

Keeping the notation introduced in section 2, consider a solution $x=v+u$. Let us decompose v into

$$v=(c,c,\ldots,c)+w$$

where $c\in\mathbb{R}$ and w is orthogonal to $(1,1,\ldots,1)$ (and, of course, to $\underline{s}$ and $\underline{c}$ as well). The inner product of (1.3) with $(c,c,\ldots,c)$ yields

$$\lambda|c|\leq E.$$

The next step consists in proving that w is bounded. In fact the inner product of (1.3) with w gives

$$-\frac{1}{h}\sum_{i=1}^{N}\big(w_{i+1}w_i-2\,w_i^2+w_{i-1}w_i\big)\ =\ \lambda\|w\|^2+\sum_{i=1}^{N}f(u_i+v_i)\,(w_{i+1}-w_i)-e\cdot w+2\pi\,\lambda c\|w\|.$$

Hence

$$\frac{N}{2\pi}\sum_{i=1}^{N}(w_{i+1}-w_i)^2\ \leq\ \lambda\|w\|^2+C\|w\|+m\sqrt{N\sum_{i=1}^{N}(w_{i+1}-w_i)^2}$$

where C is a constant independent of N. Recall that $\lambda=\lambda_N$ stays close to 1 for large N. Now we claim that for all w orthogonal to $(1,1,\ldots,1)$, $\underline{s}$ and $\underline{c}$ we have

$$\sum_{i=1}^{N}(w_{i+1}-w_i)^2\ \geq\ 4\,\sin^2\frac{2\pi}{N}\sum_{i=1}^{N}w_i^2. \tag{4.1}$$

Combining this with the previous inequality we conclude that the quantity

$$\sum_{i=1}^{N}|w_{i+1}-w_i|$$

is bounded independently of N and therefore (using the fact that w has components with both signs) it follows that there is a constant L such that, for all N,

$$|w_i| \leq L, \qquad \forall i = 1, \dots, N.$$

Finally we consider the boundedness of the component u. Assume in addition that there exists $\delta > 0$ such that

$$\pi\sqrt{A^2 + B^2} + \delta < 2\big(f(\infty) - f(-\infty)\big)$$

for all sufficiently large N (recall that $A = A_N$ and $B = B_N$ although we omit the subscript). If the components of u are $u_i = R\sin\big(t + \frac{2\pi i}{N}\big)$, we consider $\tilde{u}$ with $\tilde{u}_i = R\sin\big(t + \frac{\pi}{2} + \frac{2\pi i}{N}\big)$. The inner product of the second equation in (2.1) with $\tilde{u}$ gives

$$\sum_{i=1}^{N} f(u_i + v_i)\,(\tilde{u}_{i+1} - \tilde{u}_i) = Q\,e \cdot \tilde{u}$$

or equivalently

$$\sum_{i=1}^{N} f\left(R\sin\left(t + \frac{2\pi i}{N}\right) + v_i\right) 2\,\sin\frac{\pi}{N}\,\sin\left(\frac{2\pi i}{N} + \frac{\pi}{N} + t\right) = h\sum_{i=1}^{N} e_i \sin\left(t + \frac{\pi}{2} + \frac{2\pi i}{N}\right),$$

which implies

$$\xi_N\,\frac{2\pi}{N}\,\sum_{i=1}^{N} f\left(R\,\sin\left(t + \frac{2\pi i}{N}\right) + v_i\right) \sin\left(\frac{2\pi i}{N} + \frac{\pi}{N} + t\right) \leq \pi\sqrt{A^2 + B^2}$$

where $\xi_N \to 1$ as $N \to \infty$. Given the boundedness of the v_i it is not difficult to see that, for all large N and R sufficiently large, the left-hand side becomes arbitrarily close to $2\big(f(\infty) - f(-\infty)\big)$, a contradiction with the assumption.

For completeness, we provide a

Proof of (4.1). We compute the minimum of the quadratic form $\sum_{i=1}^{N}(w_{i+1} - w_i)^2$ in the unit sphere (for the standard norm of $\mathbb{R}^N$) of the subspace M' consisting of vectors orthogonal to $(1, 1, \dots, 1)$, $\underline{s}$ and $\underline{c}$. Since in the unit sphere

$$\sum_{i=1}^{N}(w_{i+1} - w_i)^2 = 2 - 2\sum_{i=1}^{N}(w_{i+1}w_i)$$

we have only to compute the maximum of $2\sum_{i=1}^{N}(w_{i+1}w_i)$ in the sphere. Now the matrix of this quadratic form

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

is symmetric and circulant, hence it shares the same eigenvectors of the matrix for Δ^2. By elementary properties of circulant matrices (see e.g. [6]), the eigenvalues corresponding to eigenvectors in M' are the numbers $2\,\cos\frac{j\pi}{N}$, $j = 4, \dots, \frac{N}{2} - 1$. The greatest of them is $2\,\cos\frac{4\pi}{N} = 2 - 4\,\sin^2\frac{2\pi}{N}$. This completes the proof. □

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Infinitely many homoclinic solutions for perturbed second-order Hamiltonian systems with subquadratic potentials

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Abstract. In this paper, we consider the following perturbed second-order Hamiltonian system

$$-\ddot{u}(t) + L(t)u = \nabla W(t, u(t)) + \nabla G(t, u(t)), \qquad \forall\, t \in \mathbb{R},$$

where $W(t,u)$ is subquadratic near origin with respect to u; the perturbation term $G(t,u)$ is only locally defined near the origin and may not be even in u. By using the variant Rabinowitz's perturbation method, we establish a new criterion for guaranteeing that this perturbed second-order Hamiltonian system has infinitely many homoclinic solutions under broken symmetry situations. Our result improves some related results in the literature.

Keywords: broken symmetry, Hamiltonian system, homoclinic solutions, subquadratic potential, Rabinowitz's perturbation method.

2010 Mathematics Subject Classification: 34C37, 37J45.

1 Introduction

Consider the following second-order Hamiltonian system

$$-\ddot{u}(t) + L(t)u(t) = \nabla W\big(t, u(t)\big) + \nabla G\big(t, u(t)\big), \qquad \forall\, t \in \mathbb{R}, \tag{1.1}$$

where $u = (u_1, u_2, \dots, u_N) \in \mathbb{R}^N$ and $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ is a symmetric matrix-valued function. As usual, a solution u of problem (1.1) is homoclinic (to 0), if $|u(t)| \to 0$ as $|t| \to +\infty$. In addition, if $u \not\equiv 0$ then u is called a nontrivial homoclinic solution.

When $G \equiv 0$, (1.1) reduces to the second-order Hamiltonian system

$$-\ddot{u}(t) + L(t)u(t) = \nabla W\big(t, u(t)\big), \qquad \forall\, t \in \mathbb{R}. \tag{1.2}$$

In the past twenty years, the existence and multiplicity of homoclinic solutions for problem (1.2) have been extensively investigated by variational methods. Next we recall some results in

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this aspect. For problem (1.2), under the assumption that $L(t)$ and $W(t,x)$ are T-periodic in t, Rabinowitz [16] proved the existence of homoclinic orbits as a limit of $2kT$-periodic solutions of problem (1.2). Then this trick has been developed to study the existence and multiplicity of homoclinic solutions for more general Hamiltonian systems (see, e.g., [8,21,28]).

When $L(t)$ and $W(t,x)$ are not periodic in t, the problem of existence of homoclinic solutions for (1.2) is quite different from the one just described, since the Sobolev embedding is no longer compact. To overcome this difficulty, Rabinowitz and Tanaka [17] introduced the following coercive condition:

(L_0) $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$ and there is a continuous function $l : \mathbb{R} \to \mathbb{R}$ such that $l(t) > 0$ for all $t \in \mathbb{R}$ and $(L(t)u, u) \geq l(t)|u|^2$, $\forall\, u \in \mathbb{R}^N$ and $l(t) \to +\infty$ as $|t| \to +\infty$.

The condition (L_0) implies that the self-adjoint operator of $-d^2/dt^2 + L(t)$ in $L^2(\mathbb{R}, \mathbb{R}^N)$ has a sequence of eigenvalues λ_n (counted with multiplicity) and

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to \infty. \tag{1.3}$$

Under this assumption on L, they obtained the existence of a nontrivial homoclinic solution for problem (1.2) by using a variant of the Mountain Pass Theorem without the Palais–Smale condition. Subsequently, Omana and Willem [13] showed that the Palais–Smale condition is satisfied under the coercive condition (L_0), and they used the usual Mountain Pass Theorem to prove the same result as in [17]. Since then, the coercive condition (L_0) and its variants have been used in a number of papers, and we refer the readers to [10,23,25–27] and the references therein.

Assume that $W(t,x)$ is of subquadratic growth as $|x| \to 0$ for all $t \in \mathbb{R}$, Ding [6] considered this case and presented the following condition

(L_0') there is a constant $\alpha < 2$ such that

$$l(t)|t|^{\alpha-2} \to +\infty \quad \text{as } |t| \to +\infty,$$

where $l(t)$ is given in (L_0). The main purpose of (L_0') is to guarantee some better properties of Sobolev embedding in the subquadratic case. If $W(t,x)$ is even in x, Ding proved a sequence of homoclinic solutions for problem (1.2). After the work of Ding [6], there are many papers concerning the existence of infinitely many homoclinic solutions in the subquadratic case (see, e.g., [20,22,34,35]). It is worth pointing out that most of these mentioned papers assumed that $W(t,x)$ is even with respect to x. Actually, the approaches used in these works depend on the notion of genus for symmetric sets. Therefore, the condition that $W(t,x)$ is even with respect to x is crucial in the application of these methods. When $W(t,x)$ is not even in x, the symmetry of the corresponding functional for problem (1.2) is broken. It is natural to ask whether an infinite number of homoclinic solutions can be maintained in broken symmetry case, and such a problem is often called perturbation from symmetry problem.

Since 1980s, many scholars have developed different methods to study the perturbation from symmetry problem for elliptic equations and Hamiltonian systems (see, e.g., [1,3,9,11,18,19,24,31–33]. If $G(t,x)$ is not even in x, problem (1.1) loses its symmetry under the assumption that $W(t,x)$ is even in x, and the authors [30] studied the perturbation from symmetry problem for (1.1). Specifically speaking, when $W(t,x)$ is locally superquadratic as

$|x| \to +\infty$, we obtained an unbounded sequence of homoclinic solutions by means of Bolle's perturbation method introduced in [3].

If $W(t,x)$ is subquadratic near origin with respect to x, i.e., $\lim_{x\to 0} W(t,x)/|x|^2 = +\infty$ for all $t \in \mathbb{R}$, an interesting question is whether the infinite number of homoclinic solutions persists under symmetry breaking situations. To the best of our knowledge, there are very few results on this topic. The main purpose of this paper is to give a positive answer to this question. To be precise, if the non-even perturbation term G is locally defined and satisfies some growth conditions near the origin, the existence of infinitely many homoclinic solutions for (1.1) can be preserved. Our tool is a variant of the perturbation method developed by Rabinowitz in [14]. The main idea of our proof is to introduce a modified functional by subtle truncation of the original functional, then the nonsymmetric part of this modified functional can be estimated. Then we can prove that the modified functional has almost the same small critical values as the original functional. Next we state the main result of this paper.

Theorem 1.1. *Let the condition (L_0) hold. Moreover, assume that the following condition hold:*

(H_1) $W(t,x) = W_1(t,x) + W_2(t,x)$, $W_1, W_2 \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ and there exist a constant $1 < p < 2$ such that

$$\left|\nabla W_1(t,x)\right| \le a(t)|x|^{p-1}, \qquad \forall\,(t,x) \in \mathbb{R}\times\mathbb{R}^N, \tag{1.4}$$

where $a:\ \mathbb{R} \to \mathbb{R}^+$ is a continuous function such that $a \in L^{\frac{2}{2-p}}(\mathbb{R})$;

(H_2) $W_1(t,0) \equiv 0$ and there exist constants $C_1 > 0$, $1 < \mu < 2$ and $\alpha_1 > 2$ such that

$$-C_1|x|^{\alpha_1} \le (\nabla W_1(t,x),x) - \mu W_1(t,x) \le 0, \qquad \forall\,(t,x) \in \mathbb{R}\times\mathbb{R}^N; \tag{1.5}$$

(H_3) there exist constants $C_2 > 0$, $1 < \alpha_2 < 2$ and $\alpha_3 > 2$ such that

$$W_1(t,x) \ge b(t)|x|^{\alpha_2} - C_2|x|^{\alpha_3}, \qquad \forall\,(t,x) \in \mathbb{R}\times\mathbb{R}^N, \tag{1.6}$$

where $b:\ \mathbb{R} \to \mathbb{R}^+$ is a continuous function such that $b \in L^{\frac{2}{2-\alpha_2}}(\mathbb{R})$;

(H_4) $W_2(t,0) \equiv 0$ and there exist constants $C_3 > 0$ and $\alpha_4 > 2$ such that

$$|\nabla W_2(t,x)| \le C_3|x|^{\alpha_4-1}, \qquad \forall\,(t,x) \in \mathbb{R}\times\mathbb{R}^N; \tag{1.7}$$

(H_5) $W_i(t,x) = W_i(t,-x)$, $i = 1,2$, $\forall\,(t,x) \in \mathbb{R}\times\mathbb{R}^N$;

(G_1) $G \in C^1(\mathbb{R}\times B_{r_0}(0),\mathbb{R})$, $G(t,0) \equiv 0$ and there exist constants $C_4 > 0$ and $\sigma > 2$ such that

$$\left|\nabla G(t,x)\right| \le C_4|x|^{\sigma-1}, \qquad \forall\,(t,x) \in \mathbb{R}\times B_{r_0}(0), \tag{1.8}$$

where $B_{r_0}(0)$ denotes the open ball in $\mathbb{R}^N$ centred at 0 with radius r_0;

(G_2) there exist constants $C_5 > 0$, $\beta > \frac{2(2-p)}{p(\sigma-2)}$ and $n_0 \in \mathbb{N}$ such that $\lambda_n \ge C_5 n^\beta$, $n \ge n_0$, where the eigenvalues λ_n are given in (1.3).

Then problem (1.1) has a sequence of homoclinic solutions $\{u_n\}$ such that $\max_{t\in\mathbb{R}}|u_n(t)| \to 0$ as $n \to \infty$.

Notation. Throughout the paper, we denote by C_n various positive constants which may vary from line to line and are not essential to the proof.

2 Variational setting and preliminaries

Let

$$E = \left\{ u \in H^1(\mathbb{R}, \mathbb{R}^N) : \int_{\mathbb{R}} \left[|\dot{u}(t)|^2 + \big(L(t)u(t), u(t)\big) \right] dt < +\infty \right\}$$

endowed with the inner product

$$(u, v) = \int_{\mathbb{R}} \Big[\big(\dot{u}(t), \dot{v}(t)\big) + \big(L(t)u(t), u(t)\big) \Big] dt.$$

Then E is a Hilbert space with this inner product and we denote by $\|\cdot\|$ the induced norm. As usual, for $1 \le \nu < +\infty$, let

$$\|u\|_\nu = \Big(\int_{\mathbb{R}} |u(t)|^\nu dt \Big)^{1/\nu}, \qquad u \in L^\nu(\mathbb{R}, \mathbb{R}^N).$$

It is evident that E is continuously embedded into $H^1(\mathbb{R}, \mathbb{R}^N)$, so E is continuously embedded into $L^\nu(\mathbb{R}, \mathbb{R}^N)$ for any $\nu \in [2, \infty]$, i.e., there exists $\tau_\nu > 0$ such that

$$\|u\|_\nu \le \tau_\nu \|u\|, \qquad \forall\, u \in E. \tag{2.1}$$

Moreover, E is compactly embedded into $L^\nu_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)$ for all $\nu \in [1, \infty]$.

Next we introduce a useful result proved in Lemma 2.3 of [21] by Tang and Xiao.

Lemma 2.1. *For any $u \in E$, the following inequalities hold:*

$$|u(t)| \le \left\{ \int_t^\infty \frac{1}{\sqrt{l(s)}} \Big[|\dot{u}(s)|^2 + \big(L(s)u(s), u(s)\big) \Big] ds \right\}^{1/2}, \qquad t \in \mathbb{R}, \tag{2.2}$$

and

$$|u(t)| \le \left\{ \int_{-\infty}^t \frac{1}{\sqrt{l(s)}} \Big[|\dot{u}(s)|^2 + \big(L(s)u(s), u(s)\big) \Big] ds \right\}^{1/2}, \qquad t \in \mathbb{R}. \tag{2.3}$$

In view of condition (G_1) in Theorem 1.1, the perturbation term G is only locally defined, so we can't apply the variational methods directly. To overcome this difficulty, we use cut-off method to modify $G(t, x)$ for x outside a neighbourhood of the origin. In detail, we have the following lemma.

Lemma 2.2. *Suppose that (G_1) is satisfied. Then there exists a new function $\widetilde{G}$ possessing the following properties:*

(i) $\widetilde{G} \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $\widetilde{G}(t, 0) \equiv 0$ *and*

$$\big|\nabla \widetilde{G}(t, x)\big| \le 16 C_4 |x|^{\sigma - 1}, \qquad \forall\, (t, x) \in \mathbb{R} \times \mathbb{R}^N; \tag{2.4}$$

(ii) there exists a positive constant $r_1 \le \min\{r_0/2, 1/2\}$ *such that*

$$\widetilde{G}(t, x) = G(t, x), \qquad \forall\, (t, x) \in \mathbb{R} \times B_{r_1}(0); \tag{2.5}$$

where $B_{r_1}(0)$ denotes the open ball in $\mathbb{R}^N$ centred at 0 with radius r_1.

Proof. Since $G(t,0) = 0$, by (1.8) and direct computation we have

$$|G(t,x)| \le C_4|x|^\sigma, \qquad \forall\ (t,x) \in \mathbb{R} \times B_{r_0}(0). \tag{2.6}$$

Choose a constant $r_1 = \min\{r_0/2, 1/2\}$ and define a cut-off function $h \in C^1(\mathbb{R},\mathbb{R})$ such that $h(t) = 1$ for $t \le 1$, $h(t) = 0$ for $t \ge 2$ and $-2 \le h'(t) < 0$ for $1 < t < 2$. Set

$$\begin{cases} \widetilde{G}(t,x) = h(|x|^2/r_1^2)G(t,x), & \forall\ (t,x) \in \mathbb{R} \times B_{\sqrt{2}r_1}(0), \\ \widetilde{G}(t,x) \equiv 0, & \forall\ (t,x) \in \mathbb{R} \times (\mathbb{R}^N \backslash B_{\sqrt{2}r_1}(0)). \end{cases} \tag{2.7}$$

In view of (2.7), for $i = 1,2,\dots,N$, we have

$$\frac{\partial \widetilde{G}}{\partial x_i} = \frac{2x_i}{r_1^2} h'\left(\frac{|x|^2}{r_1^2}\right) G(t,x) + h\left(\frac{|x|^2}{r_1^2}\right)\frac{\partial G}{\partial x_i}, \qquad \forall\ (t,x) \in \mathbb{R} \times B_{\sqrt{2}r_1}(0), \tag{2.8}$$

and $\partial \widetilde{G}/\partial x_i = 0$, $\forall (t,x) \in \mathbb{R} \times (\mathbb{R}^N \backslash B_{\sqrt{2}r_1}(0))$. By (2.7) and (2.8), $\widetilde{G} \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $\widetilde{G}(t,0) \equiv 0$ and $\widetilde{G}(t,x) = G(t,x)$, $\forall\ (t,x) \in \mathbb{R} \times B_{r_1}(0)$. Moreover, it is easy to verify (2.4) by (1.8), (2.6) and (2.8). □

Next we introduce the following modified Hamiltonian system

$$-\ddot{u}(t) + L(t)u(t) = \nabla W(t,u(t)) + \nabla \widetilde{G}(t,u(t)), \qquad \forall\ t \in \mathbb{R}. \tag{2.9}$$

Let $I : E \to \mathbb{R}$ be defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W_1(t,u)dt - \int_{\mathbb{R}} W_2(t,u)dt - \int_{\mathbb{R}} \widetilde{G}(t,u)dt. \tag{2.10}$$

Under assumptions (L_0), (H_1), (H_2), (H_4) and (G_1), $I \in C^1(E,\mathbb{R})$ and

$$\langle I'(u), v\rangle = (u,v) - \int_{\mathbb{R}} \nabla W_1(t,u)v dt - \int_{\mathbb{R}} \nabla W_2(t,u)v dt - \int_{\mathbb{R}} \nabla \widetilde{G}(t,u)v dt \tag{2.11}$$

for all $u, v \in E$. The critical points of I in E are solutions of (2.9). Moreover, by the coercivity of l, (2.2) and (2.3), these solutions are homoclinic to 0.

Next we introduce a cut-off function $\zeta_\mu \in C^\infty(\mathbb{R},\mathbb{R})$ satisfying

$$\begin{cases} \zeta_\mu(t) = 1, & t \in (-\infty, A/2], \\ 0 \le \zeta_\mu(t) \le 1, & t \in (A/2, A/4), \\ \zeta_\mu(t) = 0, & t \in [A/4, \infty), \\ |\zeta'_\mu(t)| \le -8A^{-1}, & t \in \mathbb{R}, \end{cases} \tag{2.12}$$

where $A := (4\mu)^{-1}(\mu - 2) < 0$. Setting $T_0 := \min\{T_1, T_2, T_3, 1/2\}$, where

$$T_1 = \left\{ \frac{2-\mu}{8\mu\left(C_1\tau_{\alpha_1}^{\alpha_1} + 10C_3\tau_{\alpha_4}^{\alpha_4} + 16(10 - 32A^{-1})C_4\tau_\sigma^\sigma\right)} \right\}^{\frac{1}{\alpha-2}}, \tag{2.13}$$

$$T_2 = \left\{ \frac{1}{12\left(2^{\frac{\alpha_4+4}{2}}C_3\tau_{\alpha_4}^{\alpha_4} - 2^{\frac{\sigma+12}{2}}C_4\tau_\sigma^\sigma A^{-1}\right)} \right\}^{\frac{2}{\alpha-2}} \quad \text{and} \quad T_3 = \left\{ \frac{-A}{2^{\frac{\sigma+18}{2}}C_4\tau_\sigma^\sigma} \right\}^{\frac{2}{\sigma-2}}, \tag{2.14}$$

$\alpha := \min\{\alpha_1, \alpha_4, \sigma\}$ and τ_{α_1}, τ_{α_4} and τ_σ are embedding constants given in (2.1). By the definition of T_0, T_0 is a fixed positive constant.

With the help of T_0 and the cut-off function h introduced in Lemma 2.2, define

$$k_{T_0}(u) = h\left(\frac{\|u\|^2}{T_0}\right), \qquad \forall\, u \in E. \tag{2.15}$$

Lemma 2.3. *The functional k_{T_0} defined by (2.15) is of $C^1(E, \mathbb{R})$ and*

$$|\langle k'_{T_0}(u), u\rangle| \le 8, \qquad \forall\, u \in E.$$

Proof. By (2.15) and direct calculation we have

$$\langle k'_{T_0}(u), v\rangle = 2h'\left(\frac{\|u\|^2}{T_0}\right)\frac{(u,v)}{T_0}, \qquad \forall\, u, v \in E. \tag{2.16}$$

Assume that $u_n \to u_0$ in E. In view of (2.16), for any $v \in E$, we obtain

$$\begin{aligned}
&|\langle k'_{T_0}(u_n) - k'_{T_0}(u_0), v\rangle| \\
&\quad = 2\left|h'\left(\frac{\|u_n\|^2}{T_0}\right)\frac{(u_n,v)}{T_0} - h'\left(\frac{\|u_0\|^2}{T_0}\right)\frac{(u_0,v)}{T_0}\right| \\
&\quad \le 2{T_0}^{-1}\|v\|\left[\left|h'\left(\frac{\|u_n\|^2}{T_0}\right)\right|\|u_n - u_0\| + \left|h'\left(\frac{\|u_n\|^2}{T_0}\right) - h'\left(\frac{\|u_0\|^2}{T_0}\right)\right|\|u_0\|\right],
\end{aligned}$$

which implies that $\|k'_{T_0}(u_n) - k'_{T_0}(u_0)\|_{E^*} \to 0$, $n \to \infty$. So $k_{T_0} \in C^1(E, \mathbb{R})$. By the definition of h and (2.16), we get $|\langle k'_{T_0}(u), u\rangle| \le 8$, $\forall\, u \in E$. □

With the help of this functional k_{T_0}, we define a new functional $\bar{I}_{T_0}$ on E by

$$\bar{I}_{T_0}(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W_1(t,u)dt - k_{T_0}(u)\left(\int_{\mathbb{R}} W_2(t,u)dt + \int_{\mathbb{R}} \tilde{G}(t,u)dt\right), \qquad \forall\, u \in E. \tag{2.17}$$

By (2.16), $\bar{I}_{T_0} \in C^1(E, \mathbb{R})$ and one can easily check that

$$\begin{aligned}
\langle \bar{I}'_{T_0}(u), v\rangle = (u,v) &- \int_{\mathbb{R}} \nabla W_1(t,u)v dt - k_{T_0}(u)\left(\int_{\mathbb{R}} \nabla W_2(t,u)v dt + \int_{\mathbb{R}} \nabla\tilde{G}(t,u)v dt\right) \\
&- \langle k'_{T_0}(u), v\rangle\left(\int_{\mathbb{R}} W_2(t,u)dt + \int_{\mathbb{R}} \tilde{G}(t,u)dt\right), \qquad \forall\, u, v \in E.
\end{aligned} \tag{2.18}$$

We will give some prior bounds for critical points of $\bar{I}_{T_0}$ based on the corresponding critical values in the following lemma, which is useful to introduce a modified functional.

Lemma 2.4. *Assume that (H_2), (H_4) and (G_1) are satisfied, if u is a critical point of $\bar{I}_{T_0}$, then*

$$\bar{I}_{T_0}(u) \le \frac{\mu - 2}{4\mu}\|u\|^2. \tag{2.19}$$

Proof. When u is a critical point of $\bar{I}_{T_0}$ and $\|u\|^2 > 2T_0$, by (2.16) and (2.17), $k_{T_0}(u) = 0$ and $k'_{T_0}(u) = 0$. In view of (2.18) and (2.19), we conclude that

$$\bar{I}_{T_0}(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W_1(t,u)dt, \qquad \langle \bar{I}'_{T_0}(u), u\rangle = \|u\|^2 - \int_{\mathbb{R}} (\nabla W_1(t,u), u)dt. \tag{2.20}$$

By (1.5) and (2.20), we get

$$\begin{aligned}\bar{I}_{T_0}(u) &= \bar{I}_{T_0}(u) - \mu^{-1}\langle \bar{I}'_{T_0}(u), u\rangle \\ &= \frac{\mu-2}{2\mu}\|u\|^2 + \mu^{-1}\int_{\mathbb{R}} \big((\nabla W_1(t,u),u) - \mu W_1(t,u)\big)dt \\ &\le \frac{\mu-2}{4\mu}\|u\|^2. \end{aligned} \tag{2.21}$$

If u is a critical point of $\bar{I}_{T_0}$ with $\|u\|^2 \le 2T_0$, by Lemma 2.2, Lemma 2.3, (1.5), (1.7), (2.17) and (2.18) we have

$$\begin{aligned}\bar{I}_{T_0}(u) &= \bar{I}_{T_0}(u) - \mu^{-1}\langle \bar{I}'_{T_0}(u), u\rangle \\ &\le \frac{\mu-2}{2\mu}\|u\|^2 + C_1\tau_{\alpha_1}^{\alpha_1}\|u\|^{\alpha_1} + 10(C_3\tau_{\alpha_4}^{\alpha_4}\|u\|^{\alpha_4} + 16C_4\tau_\sigma^\sigma\|u\|^\sigma). \end{aligned} \tag{2.22}$$

By the definition of T_0 and (2.13), we get

$$C_1\tau_{\alpha_1}^{\alpha_1}\|u\|^{\alpha_1} + 10C_3\tau_{\alpha_4}^{\alpha_4}\|u\|^{\alpha_4} + 16(10-32A^{-1})C_4\tau_\sigma^\sigma\|u\|^\sigma < \frac{2-\mu}{4\mu}\|u\|^2. \tag{2.23}$$

In both cases, it follows from (2.21)–(2.23) that (2.19) holds. □

By the cut-off function ζ_μ and $\bar{I}_{T_0}$, define a functional as follows

$$l_\mu(u) = \zeta_\mu\big(\|u\|^{-2}\bar{I}_{T_0}(u)\big), \qquad \forall\, u \in E\backslash\{0\}. \tag{2.24}$$

By direct computation, for any $u \in E\backslash\{0\}$ and any $v \in E$,

$$\langle l'_\mu(u), v\rangle = \zeta'_\mu(\theta_{T_0}(u))\|u\|^{-4}\Big(\|u\|^2\langle \bar{I}'_{T_0}(u), v\rangle - 2\bar{I}_{T_0}(u)(u,v)\Big), \tag{2.25}$$

where $\theta_{T_0}(u) := \|u\|^{-2}\bar{I}_{T_0}(u)$, $\forall\, u \in E\backslash\{0\}$. Under assumptions of Theorem 1.1, it is easy to check that l_μ is continuously differentiable at any $u \in E\backslash\{0\}$.

By these functionals k_{T_0} and l_μ, we can introduce a modified functional J_{T_0} as follows:

$$J_{T_0}(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W_1(t,u)dt - k_{T_0}(u)\int_{\mathbb{R}} W_2(t,u)dt - \psi(u), \qquad \forall\, u \in E, \tag{2.26}$$

where

$$\psi(u) := \begin{cases} k_{T_0}(u)\; l_\mu(u)\; Q(u), & u \in E \setminus \{0\}, \\ 0, & u = 0, \end{cases} \tag{2.27}$$

and $Q(u) := \int_{\mathbb{R}} \tilde{G}(t,u)dt$, $\forall\, u \in E$. It follows from (2.1) and (2.4) that

$$\int_{\mathbb{R}} |\tilde{G}(t,u)|dt \le 16C_4\tau_\sigma^\sigma\|u\|^\sigma, \qquad \forall\, u \in E. \tag{2.28}$$

Moreover, it is easy to check that $Q \in C^1(E,\mathbb{R})$ and

$$\langle Q'(u), v\rangle = \int_{\mathbb{R}} \nabla\tilde{G}(t,u)v dt, \qquad \forall\, u, v \in E. \tag{2.29}$$

Next we give a bound on $|\langle \psi'(u), u\rangle|$, $\forall\, u \in E$, which is used to obtain the estimate of $|J_{T_0}(u) - J_{T_0}(-u)|$, $\forall\, u \in E$. Then we show that J_{T_0} has no critical point with positive critical value on E.

Lemma 2.5. *Assume that* (L_0), (H_1), (H_2), (H_4), (H_5) *and* (G_1) *holds. Then*

(i) *the functional* ψ *defined by* (2.27) *is of class* $C^1(E,\mathbb{R})$ *and*

$$|\langle\psi'(u),u\rangle| \le 16(9-32A^{-1})C_4\tau_\sigma^\sigma\|u\|^\sigma, \qquad \forall\, u\in E; \tag{2.30}$$

(ii) $J_{T_0}\in C^1(E,\mathbb{R})$ *and there exists a constant* $C_6>0$ *independent of* u *such that*

$$|J_{T_0}(u)-J_{T_0}(-u)| \le C_6|J_{T_0}(u)|^{\frac{\sigma}{2}}, \qquad \forall\, u\in E; \tag{2.31}$$

(iii) J_{T_0} *has no critical point with positive critical value on* E *and* $K_0=\{0\}$, *where* $K_0:=\{u\in E: J_{T_0}(u)=0,\ J'_{T_0}(u)=0\}$.

Proof. For $u=0$ and any $v\in E$, by (2.4), (2.15), (2.24) and (2.27) we have

$$|\langle\psi'(0),v\rangle| = \left|\lim_{\lambda\to 0}\frac{\psi(\lambda v)-\psi(0)}{\lambda}\right| \le 16C_4\lim_{\lambda\to 0}|\lambda|^{\sigma-1}\int_{\mathbb{R}}|v(t)|^\sigma dt = 0,$$

so $\psi'(0)=0$. Combining (2.16), (2.25), (2.27) and (2.29), for $u\in E\setminus\{0\}$ and $v\in E$, we obtain

$$\langle\psi'(u),v\rangle = \langle k'_{T_0}(u),v\rangle l_\mu(u)Q(u) + k_{T_0}(u)\langle l'_\mu(u),v\rangle Q(u) + k_{T_0}(u)l_\mu(u)\langle Q'(u),v\rangle. \tag{2.32}$$

Next we prove $\psi'\in C^1(E,\mathbb{R})$. Suppose that $u_n\to u_0$ in E. We consider two possible cases.

Case 1. $u_0\ne 0$. In view of Lemma 2.3, (2.25), (2.29) and (2.32), $\psi'(u_n)\to\psi'(u_0)$, $n\to\infty$.

Case 2. $u_0=0$. Without loss of generality, we can assume $\|u_n\|^2<T_0$. It follows from (2.15) and (2.16) that $k'_{T_0}(u_n)=0$ and $k_{T_0}(u_n)=1$. Then (2.32) reduces to

$$\langle\psi'(u_n),v\rangle = \langle l'_\mu(u_n),v\rangle Q(u_n) + l_\mu(u_n)\langle Q'(u_n),v\rangle, \qquad \forall\, v\in E. \tag{2.33}$$

By (2.25), we can divide $\langle l'_\mu(u_n),v\rangle\ Q(u_n)$ into two parts as follows

$$\langle l'_\mu(u_n),v\rangle\ Q(u_n) = Q_1(u_n,v)-Q_2(u_n,v), \tag{2.34}$$

where

$$Q_1(u_n,v) = \zeta'_\mu(\theta_{T_0}(u_n))\|u_n\|^{-2}\langle\bar I'_{T_0}(u_n),v\rangle Q(u_n) \qquad \forall\, v\in E, \tag{2.35}$$

and

$$\begin{aligned} Q_2(u_n,v) &= 2\zeta'_\mu(\theta_{T_0}(u_n))\|u_n\|^{-4}\bar I_{T_0}(u_n)(u_n,v)Q(u_n)\\ &= 2\zeta'_\mu(\theta_{T_0}(u_n))\theta_{T_0}(u_n)\|u_n\|^{-2}(u_n,v)Q(u_n) \qquad \forall\, v\in E. \end{aligned} \tag{2.36}$$

In view of (2.12), (2.28), (2.35) and (2.36), we deduce that

$$|Q_1(u_n,v)| \le C_7\|\bar I'_{T_0}(u_n)\|_{E^*}\|u_n\|^{\sigma-2}\|v\|, \tag{2.37}$$

and

$$|Q_2(u_n,v)| \le C_8\|u_n\|^{\sigma-1}\|v\|. \tag{2.38}$$

Since $k'_{T_0}(u_n)=0$, $k_{T_0}(u_n)=1$ and $u_n\to 0$, by (1.4), (1.7), (2.4), (2.18) and (2.29),

$$\|\bar I'_{T_0}(u_n)\|_{E^*}\to 0 \quad\text{and}\quad \|Q'(u_n)\|_{E^*}\to 0, \qquad n\to\infty. \tag{2.39}$$

In combination with (2.24)-(2.25), (2.33), (2.34), (2.37)-(2.39), we have

$$\|\psi'(u_n) - \psi'(0)\|_{E^*} = \sup_{\|v\|\le 1} \left|\langle l'_\mu(u_n), v\rangle Q(u_n) + l_\mu(u_n)\langle Q'(u_n), v\rangle\right| \to 0, \qquad n \to \infty,$$

which implies the continuity of ψ' at 0. So we have $\psi \in C^1(E, \mathbb{R})$.

If $\|u\|^2 > 2T_0$ or $u = 0$, by (2.15), (2.16) and (2.26), it is easy to see that $\langle \psi'(u), u\rangle = 0$. Otherwise, $\|u\|^2 \le 2T_0$ and $u \neq 0$. Arguing similarly as in (2.22), we obtain

$$|\bar{I}_{T_0}(u) - \mu^{-1}\langle \bar{I}'_{T_0}(u), u\rangle| \le 2|A|\|u\|^2 + C_1\tau_{\alpha_1}^{\alpha_1}\|u\|^{\alpha_1} + 10(C_3\tau_{\alpha_4}^{\alpha_4}\|u\|^{\alpha_4} + 16C_4\tau_\sigma^\sigma\|u\|^\sigma). \tag{2.40}$$

Since $\|u\|^2 \le 2T_0$, by (2.13), (2.23) and (2.40) we get

$$|\langle \bar{I}'_{T_0}(u), u\rangle| \le \mu\big(3|A|\|u\|^2 + |\bar{I}_{T_0}(u)|\big). \tag{2.41}$$

In combination with (2.12) and (2.25), if $\theta_{T_0}(u) \notin [A/2, A/4]$, we have $l'_\mu(u) = 0$. Otherwise, $A/2 \le \theta_{T_0}(u) \le A/4$, then the definition of θ_{T_0} imply that

$$|\bar{I}_{T_0}(u)| \le |A|\|u\|^2. \tag{2.42}$$

When $\|u\|^2 \le 2T_0$ and $u \neq 0$, it follows from (2.25), (2.28), (2.41)–(2.42) that

$$\begin{aligned}\left|k_{T_0}(u)\langle l'_\mu(u), u\rangle Q(u)\right| &\le -16A^{-1}\|u\|^{-2}\big(|\bar{I}_{T_0}(u)| + |\langle \bar{I}'_{T_0}(u), u\rangle|\big)|Q(u)| \\ &\le -512A^{-1}C_4\tau_\sigma^\sigma\|u\|^\sigma.\end{aligned} \tag{2.43}$$

In view of Lemma 2.3, (2.4), (2.12), (2.15), (2.24), (2.28) and (2.29), we have

$$\left|\langle k'_{T_0}(u), u\rangle l_\mu(u)Q(u) + k_{T_0}(u)l_\mu(u)\langle Q'(u), u\rangle\right| \le 144C_4\tau_\sigma^\sigma\|u\|^\sigma, \qquad \forall\, u \in E\setminus\{0\}. \tag{2.44}$$

It follows from (2.32), (2.43) and (2.44) that (2.30) holds.

Next we prove (ii). By (1.4), (1.7), Lemma 2.3 and (i) in Lemma 2.5, we deduce that $J_{T_0} \in C^1(E, \mathbb{R})$ and

$$\begin{aligned}\langle J'_{T_0}(u), v\rangle = (u, v) - \int_{\mathbb{R}} \nabla W_1(t, u)v dt - k_{T_0}(u)\int_{\mathbb{R}} \nabla W_2(t, u)v dt \\ - \langle k'_{T_0}(u), v\rangle \int_{\mathbb{R}} W_2(t, u)dt - \langle \psi'(u), v\rangle, \qquad \forall\, u,\ v \in E.\end{aligned} \tag{2.45}$$

When $\|u\|^2 > 2T_0$ or $\theta_{T_0}(u) > A/4$, by (2.15) or (2.24) and (2.27) we have $\psi_{T_0}(u) = 0$. Then (2.31) holds by (H_5) and (2.26). If $\theta_{T_0}(u) \le A/4$, then the definition of θ_{T_0} imply that

$$|\bar{I}_{T_0}(u)| \ge \frac{|A|}{4}\|u\|^2. \tag{2.46}$$

When $\|u\|^2 \le 2T_0$ and $\theta_{T_0}(u) \le A/4$, by (2.13), (2.17), (2.26), (2.28) and (2.46) we get

$$|J_{T_0}(u)| \ge |\bar{I}_{T_0}(u)| - 2|Q(u)| \ge \frac{|A|}{4}\|u\|^2 - 32C_4\tau_\sigma^\sigma\|u\|^\sigma \ge \frac{|A|}{20}\|u\|^2. \tag{2.47}$$

In view of (H_5), (2.15), (2.24), (2.26)–(2.28), we obtain that

$$|J_{T_0}(u) - J_{T_0}(-u)| \le 32C_4\tau_\sigma^\sigma\|u\|^\sigma, \qquad \forall\, u \in E. \tag{2.48}$$

So (2.31) holds by (2.47) and (2.48).

Next we prove (iii) by contradiction. If u_0 is a critical point of J_{T_0} with $J_{T_0}(u_0) > 0$, by (H_2), (H_4), (2.26) and (2.27) we have $u_0 \neq 0$. Without loss of generality, we assume $\|u_0\|^2 \leq 2T_0$. Otherwise, (2.15)–(2.16) and (2.32) imply that $k_{T_0}(u_0) = 0$, $k'_{T_0}(u_0) = 0$ and $\psi'(u_0) = 0$. By (2.26), (2.27) and (2.45), we get

$$J_{T_0}(u_0) = \frac{1}{2}\|u_0\|^2 - \int_{\mathbb{R}} W_1(t, u_0)dt, \tag{2.49}$$

and

$$\langle J'_{T_0}(u_0), u_0\rangle = \|u_0\|^2 - \int_{\mathbb{R}} (\nabla W_1(t, u_0), u_0)dt. \tag{2.50}$$

In combination with (1.5), (2.49) and (2.50), it is easy to verify that

$$\begin{aligned} 0 < J_{T_0}(u_0) &= J_{T_0}(u_0) - \mu^{-1}\langle J'_{T_0}(u_0), u_0\rangle \\ &= 2A\|u_0\|^2 + \mu^{-1}\int_{\mathbb{R}} ((\nabla W_1(t, u_0), u_0) - \mu W_1(t, u_0))dt \\ &\leq 2A\|u_0\|^2 < 0, \end{aligned}$$

which is a contradiction, so $\|u_0\|^2 \leq 2T_0$.

It follows from Lemma 2.3, (2.26)–(2.28), (2.30) and (2.45) that

$$J_{T_0}(u_0) \leq \frac{1}{2}\|u_0\|^2 - \int_{\mathbb{R}} W_1(t, u_0)dt + C_3\tau_{\alpha_4}^{\alpha_4}\|u_0\|^{\alpha_4} + 16C_4\tau_\sigma^\sigma\|u_0\|^\sigma,$$

and

$$\langle J'_{T_0}(u_0), u_0\rangle \geq \|u_0\|^2 - \int_{\mathbb{R}} (\nabla W_1(t, u_0), u_0)dt - 9C_3\tau_{\alpha_4}^{\alpha_4}\|u_0\|^{\alpha_4} - 16(9 - 32A^{-1})C_4\tau_\sigma^\sigma\|u_0\|^\sigma.$$

Since $\|u_0\|^2 \leq 2T_0$, by (1.5), (2.13) and two inequalities above, we have

$$\begin{aligned} 0 < J_{T_0}(u_0) &= J_{T_0}(u_0) - \mu^{-1}\langle J'_{T_0}(u_0), u_0\rangle \\ &\leq 2A\|u_0\|^2 + C_1\tau_{\alpha_1}^{\alpha_1}\|u_0\|^{\alpha_1} + 10C_3\tau_{\alpha_4}^{\alpha_4}\|u_0\|^{\alpha_4} + 16(10 - 32A^{-1})C_4\tau_\sigma^\sigma\|u_0\|^\sigma \\ &< A\|u_0\|^2 < 0, \end{aligned}$$

which is also a contradiction. Moreover, by a similar proof, we have $K_0 = \{0\}$. □

3 Proofs of main results

Lemma 3.1. *Suppose that (L_0), (H_1), (H_4) and (G_1) are satisfied. Then the functional J_{T_0} satisfies the Palais–Smale condition.*

Proof. First we prove that J_{T_0} is bounded from below. From Hölder's inequality, (1.4), (2.15), (2.26) and (2.27), if $\|u\|^2 > 2T_0$,

$$J_{T_0}(u) \geq \frac{1}{2}\|u\|^2 - C_9\|u\|^p. \tag{3.1}$$

Since $1 < p < 2$, (3.1) implies that $J_{T_0}(u) \to +\infty$ as $\|u\| \to +\infty$.

Next we show that J_{T_0} satisfies the Palais–Smale condition. Let $\{u_n\}_{n\in\mathbb{N}} \subset E$ be a Palais–Smale sequence, i.e., $\{J_{T_0}(u_n)\}_{n\in\mathbb{N}}$ is bounded and $J'_{T_0}(u_n) \to 0$ as $n \to +\infty$. Since J_{T_0} is coercive, $\{u_n\}$ is bounded in E. Then there is a positive constant A such that $\|u_n\| \leq A$,

$n \in \mathbb{N}$, passing to subsequence, also denoted by $\{u_n\}$, it can be assumed that $u_n \rightharpoonup u_0$, $n \to \infty$ for some $u_0 \in E$.

Since $a \in L^{\frac{2}{2-p}}(\mathbb{R})$, for any given number $\varepsilon > 0$, we can choose $T_\varepsilon > 0$ such that

$$\left(\int_{|t|>T_\varepsilon} |a(t)|^{2/(2-p)} dt\right)^{(2-p)/2} < \varepsilon. \tag{3.2}$$

By (1.4) and the Hölder inequality, we have

$$\int_{-T_\varepsilon}^{T_\varepsilon} |\nabla W_1(t, u_n(t))||u_n(t) - u_0(t)|dt \le (\tau_2 A)^{p-1} \|a\|_{2/(2-p)} \left(\int_{-T_\varepsilon}^{T_\varepsilon} |u_n - u_0|^2 dt\right)^{1/2}. \tag{3.3}$$

By Sobolev embedding theorem, we also get

$$u_n \to u_0 \quad \text{in } L^2_{loc}(\mathbb{R}, \mathbb{R}^N),\ n \to \infty. \tag{3.4}$$

Consequently, in view of (3.3) and (3.4),

$$\int_{-T_\varepsilon}^{T_\varepsilon} |\nabla W_1(t, u_n(t))||u_n(t) - u_0(t)|dt \to 0, \qquad n \to \infty. \tag{3.5}$$

On the other hand, it follows from (1.4), (3.2) and the Hölder inequality that

$$\begin{aligned}
&\int_{|t|>T_\varepsilon} |\nabla W_1(t, u_n(t))||u_n(t) - u_0(t)|dt \\
&\quad\le \int_{|t|>T_\varepsilon} |a(t)||u_n(t)|^{p-1}\big(|u_n(t)| + |u_0(t)|\big)dt \\
&\quad\le 2\int_{|t|>T_\varepsilon} |a(t)|\big(|u_n(t)|^p + |u_0(t)|^p\big)dt \\
&\quad\le 2\tau_2^p \Big(\int_{|t|>T_\varepsilon} |a(t)|^{2/(2-p)} dt\Big)^{(2-p)/2} (\|u_n\|^p + \|u_0\|^p) \\
&\quad\le 2\tau_2^p (A^p + \|u_0\|^p)\varepsilon, \qquad n \in \mathbb{N}.
\end{aligned} \tag{3.6}$$

Note that ε is arbitrary, combining (3.5) with (3.6),

$$\int_{\mathbb{R}} |\nabla W_1(t, u_n(t))||u_n(t) - u_0(t)|dt \to 0, \qquad n \to \infty. \tag{3.7}$$

Since l is coercive, for any given number $\varepsilon > 0$, there exists $T'_\varepsilon > 0$ such that

$$\varepsilon l(t) > 1, \qquad |t| > T'_\varepsilon. \tag{3.8}$$

It follows from (1.7), (3.4) and the Hölder inequality that

$$\int_{-T'_\varepsilon}^{T'_\varepsilon} |\nabla W_2(t, u_n(t))||u_n(t) - u_0(t)|dt \to 0, \qquad n \to \infty. \tag{3.9}$$

Since E is continuously embedded into $L^\infty(\mathbb{R}, \mathbb{R}^N)$ and $\|u_n\| \le A$, we get

$$\|u_n\|_\infty \le \tau_\infty A, \qquad n \in \mathbb{N}. \tag{3.10}$$

By (L_0), (1.7), (3.8) and (3.10), we have

$$\begin{aligned}
&\int_{|t|>T'_\varepsilon} |\nabla W_2(t,u_n(t))||u_n(t)-u_0(t)|dt \\
&\le C_3(\tau_\infty A)^{\alpha_4-2}\int_{|t|>T'_\varepsilon} |u_n(t)|\big(|u_n(t)|+|u_0(t)|\big)dt \\
&\le 2C_3(\tau_\infty A)^{\alpha_4-2}\varepsilon\int_{|t|>T'_\varepsilon} l(t)\big(|u_n(t)|^2+|u_0(t)|^2\big)dt \\
&\le 2C_3(\tau_\infty A)^{\alpha_4-2}\varepsilon\int_{|t|>T'_\varepsilon} \Big[\big(L(t)u_n(t),u_n(t)\big)+\big(L(t)u_0(t),u_0(t)\big)\Big]dt \\
&\le 2C_3(\tau_\infty A)^{\alpha_4-2}(A^2+\|u_0\|^2)\varepsilon, \qquad n\in\mathbb{N}. \qquad (3.11)
\end{aligned}$$

Since ε is arbitrary, it follows from (3.9) and (3.11) that

$$\int_{\mathbb{R}} |\nabla W_2(t,u_n(t))||u_n(t)-u_0(t)|dt \to 0, \qquad n\to\infty. \qquad (3.12)$$

By a similar proof as (3.9) and (3.11), we also have

$$\int_{\mathbb{R}} |\nabla \tilde{G}(t,u_n(t))||u_n(t)-u_0(t)|dt \to 0, \qquad n\to\infty. \qquad (3.13)$$

Next we consider the following two possible cases.

Case 1. $\|u_n\|^2 > 2T_0$ or $u_n = 0$. From (2.15), (2.16) and (2.32), $k_{T_0}(u_n) = 0$, $k'_{T_0}(u_n) = 0$ and $\psi'(u_n) = 0$. Therefore, by (2.45), we have

$$|\langle J'_{T_0}(u_n), u_n-u_0\rangle| \ge \|u_n-u_0\|^2 + (u_0,u_n-u_0) - \int_{\mathbb{R}} |\nabla W_1(t,u_n)||u_n-u_0|dt. \qquad (3.14)$$

Case 2. $\|u_n\|^2 \le 2T_0$ and $u_n \ne 0$. In combination with (2.16) and (2.28), we get

$$\begin{aligned}
\big|\langle k'_{T_0}(u_n), u_n-u_0\rangle Q(u_n)\big| &\le 32C_4\tau_\sigma^\sigma h'\left(\frac{\|u_n\|^2}{T_0}\right)\frac{(u_n,u_n-u_0)}{T_0}\|u_n\|^\sigma \\
&\le 2^{\frac{\sigma+12}{2}} C_4\tau_\sigma^\sigma T_0^{\frac{\sigma-2}{2}}\big(\|u_n-u_0\|^2+(u_0,u_n-u_0)\big). \qquad (3.15)
\end{aligned}$$

In view of (2.12) and (2.24), $|l(u_n)| \le 1$. Arguing as in (3.15), we also have

$$\Big|\langle k'_{T_0}(u_n), u_n-u_0\rangle l(u_n) Q(u_n)\Big| \le 2^{\frac{\sigma+12}{2}} C_4\tau_\sigma^\sigma T_0^{\frac{\sigma-2}{2}}\big(\|u_n-u_0\|^2+(u_0,u_n-u_0)\big). \qquad (3.16)$$

It follows from (1.7) and (2.10) that

$$\begin{aligned}
\left|\langle k'_{T_0}(u_n), u_n-u_0\rangle \int_{\mathbb{R}} W_2(t,u_n)dt\right| &\le 2C_3\tau_{\alpha_4}^{\alpha_4} h'\left(\frac{\|u_n\|^2}{T_0}\right)\frac{(u_n,u_n-u_0)}{T_0}\|u_n\|^{\alpha_4} \\
&\le 2^{\frac{\alpha_4+4}{2}} C_3\tau_{\alpha_4}^{\alpha_4} T_0^{\frac{\alpha_4-2}{2}}\big(\|u_n-u_0\|^2+(u_0,u_n-u_0)\big). \qquad (3.17)
\end{aligned}$$

By (2.16) and (2.34), we have

$$\big|k'_{T_0}(u_n)\langle l'(u_n), u_n-u_0\rangle Q(u_n)\big| \le \big|Q_1(u_n,u_n-u_0)\big| + \big|Q_2(u_n,u_n-u_0)\big|. \qquad (3.18)$$

In view of (2.12), (2.28) and (2.35), we obtain

$$\begin{aligned}
\big|Q_1(u_n,u_n-u_0)\big| &= |\zeta'_\mu(\theta_{T_0}(u_n))|\|u_n\|^{-2}|\langle \bar{I}'_{T_0}(u_n), u_n-u_0\rangle||Q(u_n)| \\
&\le -2^{\frac{\sigma+12}{2}} A^{-1} C_4\tau_\sigma^\sigma T_0^{\frac{\sigma-2}{2}}\big|\langle \bar{I}'_{T_0}(u_n), u_n-u_0\rangle\big|. \qquad (3.19)
\end{aligned}$$

It follows from (2.18), (3.7), (3.12), (3.13), (3.15) and (3.17) that

$$\begin{aligned}|\langle \bar{I}'_{T_0}(u_n), u_n - u_0\rangle| &\le \|u_n - u_0\|^2 + \left|\langle k'_{T_0}(u_n), u_n - u_0\rangle \int_{\mathbb{R}} W_2(t,u_n)dt\right| \\ &\quad + \left|\langle k'_{T_0}(u_n), u_n - u_0\rangle Q(u_n)\right| + o_n(1) \\ &\le (C_{10}+1)\|u_n - u_0\|^2 + o_n(1). \end{aligned} \tag{3.20}$$

where $C_{10} = 2^{\frac{\alpha_4+4}{2}} C_3 \tau_{\alpha_4}^{\alpha_4} T_0^{\frac{\alpha_4-2}{2}} - 2^{\frac{\sigma+12}{2}} A^{-1} C_4 \tau_\sigma^\sigma T_0^{\frac{\sigma-2}{2}}$.
By (3.19) and (3.20), we obtain

$$|Q_1(u_n, u_n - u_0)| \le C_{11}\|u_n - u_0\|^2 + o_n(1), \tag{3.21}$$

where $C_{11} = -2^{\frac{\sigma+12}{2}} A^{-1} C_4 \tau_\sigma^\sigma T_0^{\frac{\sigma-2}{2}} (C_{10}+1)$. In view of (2.12), (2.28) and (2.36),

$$\begin{aligned}|Q_2(u_n, u_n - u_0)| &\le 2|\zeta'_\mu(\theta_{T_0}(u_n))||\theta_{T_0}(u_n)|\|u_n\|^{-2} Q(u_n)(u_n, u_n - u_0) \\ &\le 2^{\frac{\sigma+12}{2}} C_4 \tau_\sigma^\sigma T_0^{\frac{\sigma-2}{2}} \left(\|u_n - u_0\|^2 + (u_0, u_n - u_0)\right). \end{aligned} \tag{3.22}$$

In combination with (3.18), (3.21) and (3.22), we have

$$|k_{T_0}(u_n)\langle l'(u_n), u_n - u_0\rangle Q(u_n)| \le (C_{11} + C_{10})\|u_n - u_0\|^2 + o_n(1). \tag{3.23}$$

By (2.15), (2.24) and (2.29), we conclude that

$$|k_{T_0}(u_n) l(u_n)\langle Q'(u_n), u_n - u_0\rangle| \le \int_{\mathbb{R}} |\nabla \tilde{G}(t, u_n(t))||u_n(t) - u_0(t)|dt. \tag{3.24}$$

It follows from (2.32), (3.7), (3.16), (3.23) and (3.24) that

$$|\langle \psi'(u_n), u_n - u_0\rangle| \le (C_{11} + 2C_{10})\|u_n - u_0\|^2 + o_n(1). \tag{3.25}$$

In view of (2.45), (3.7), (3.12), (3.13), (3.17) and (3.25), we get

$$\begin{aligned}|\langle J'_{T_0}(u_n), u_n - u_0\rangle| &\ge \|u_n - u_0\|^2 - \left|\langle k'_{T_0}(u_n), u_n - u_0\rangle \int_{\mathbb{R}} W_2(t,u_n)dt\right| \\ &\quad - |\langle \psi'(u_n), u_n - u_0\rangle| + o_n(1) \\ &\ge (1 - C_{11} - 3C_{10})\|u_n - u_0\|^2 + o_n(1). \end{aligned} \tag{3.26}$$

By (2.14) and (3.26), we have

$$|\langle J'_{T_0}(u_n), u_n - u_0\rangle| \ge 2^{-1}\|u_n - u_0\|^2 + o_n(1). \tag{3.27}$$

In both cases, from (3.14) and (3.27), we get $u_n \to u_0$, $n \to \infty$. Thus J_{T_0} satisfies Palais–Smale condition. □

In view of (L_0), the self-adjoint operator of $-d^2/dt^2 + L(t)$ in $L^2(\mathbb{R}, \mathbb{R}^N)$ has a sequence of eigenvalues $\lambda_n \to \infty$. Moreover, the corresponding system of normalized eigenfunctions $\{e_n : n \in \mathbb{N}\}$ forms an orthogonal basis in E. Hereafter, set $E_n = \operatorname{span}\{e_1, \ldots, e_n\}$ and $E_n^\perp$ be the orthogonal complement of E_n in E. With the help of the normalized orthogonal sequence $\{e_n\}_{n=1}^\infty$, define some subspaces as follows:

$$B_n = \{u \in E_n;\ \|u\| \le 1\}, \qquad S^n = \{u \in E_n;\ \|u\| = 1\}$$

and

$$S_+^{n+1} = \{u = w + se_{n+1};\ \|u\| = 1,\ w \in B_n,\ 0 \le s \le 1\}.$$

By these subspaces, we can introduce some continuous maps and minimax sequences of J as follows

$$\Lambda_n = \Big\{\gamma \in C(S^n, E);\ \gamma \text{ is odd}\Big\}, \qquad \Gamma_n = \Big\{\gamma \in C(S_+^{n+1}, E);\ \gamma|_{S^n} \in \Lambda_n\Big\}, \tag{3.28}$$

and

$$b_n = \inf_{\gamma\in\Lambda_n} \max_{u\in S^n} J_{T_0}(\gamma(u)), \qquad c_n = \inf_{\gamma\in\Gamma_n} \max_{u\in S_+^{n+1}} J_{T_0}(\gamma(u)). \tag{3.29}$$

For any $\delta > 0$, set

$$\Gamma_n(\delta) = \Big\{\gamma \in \Gamma_n;\ J_{T_0}(\gamma(u)) \le b_n + \delta,\ u \in S^n\Big\}, \tag{3.30}$$

$$c_n(\delta) = \inf_{\gamma\in\Gamma_n(\delta)} \max_{u\in S_+^{n+1}} J_{T_0}(\gamma(u)). \tag{3.31}$$

By (3.28)–(3.31), it is obvious that $b_n \le c_n \le c_n(\delta)$, $n \in \mathbb{N}$. Next we give some useful estimates for minimax values b_n and $c_n(\delta)$.

Lemma 3.2. *Let (L_0), (H_1), (H_3), (H_4) and (G_1) be satisfied. Then for any $n \in \mathbb{N}$, $b_n < 0$.*

Proof. By (1.6), for any $u \in E_n$ we have

$$\int_{\mathbb{R}} W_1(t,u)dt \ge \int_{\mathbb{R}} b(t)|u|^{\alpha_2}dt - C_2 \int_{\mathbb{R}} |u|^{\alpha_3}dt. \tag{3.32}$$

By standard arguments as in [20], for any $u \in E_n \setminus \{0\}$, there exists $\varepsilon_1 > 0$ depending on E_n such that

$$\text{meas}\Big\{t \in \mathbb{R}:\ b(t)|u|^{\alpha_2} \ge \varepsilon_1\|u\|^{\alpha_2}\Big\} \ge \varepsilon_1. \tag{3.33}$$

By (1.7), (2.1), (2.15), (2.24), (2.28), (3.32)–(3.33), for any $u \in E_n \setminus \{0\}$, we get

$$\begin{aligned} J_{T_0}(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W_1(t,u)dt - k_{T_0}(u)\int_{\mathbb{R}} W_2(t,u)dt - \psi(u) \\ &\le \|u\|^2 + C_{12}\|u\|^{\alpha_3} + C_{13}\|u\|^{\alpha_4} + C_{14}\|u\|^{\sigma} - \varepsilon_1^2\|u\|^{\alpha_2}. \end{aligned} \tag{3.34}$$

In view of (3.34), there exist $\epsilon(n) > 0$ and $\kappa(n) > 0$ such that $J_{T_0}(\kappa u) < -\epsilon$, $u \in S^n$. Then we set $\gamma(u) = \kappa u$, $u \in S^n$. By (3.29), we obtain $b_n < 0$. □

Lemma 3.3. *Assume that (L_0), (H_1), (H_2), (H_3), (H_4) and (G_1) hold. Then for any $n \in \mathbb{N}$ and any $\delta > 0$, $c_n(\delta) < 0$.*

Proof. From (3.30) and (3.31), for fixed $n \in \mathbb{N}$, if $0 < \delta < \delta'$, we have $\Gamma_n(\delta) \subset \Gamma_n(\delta')$ and $c_n(\delta) \ge c_n(\delta')$. So we only need to prove $c_n(\delta) < 0$ for any $\delta \in (0, |b_n|)$. For any $\delta \in (0, |b_n|)$, by (3.29), there exists $\gamma_0 \in \Lambda_n$ such that $\max_{u\in S^n} J_{T_0}\big(\gamma_0(u)\big) \le b_n + \frac{\delta}{2}$. By the fact that $\gamma_0(S^n)$ is a compact set in E, there exists a positive integer m_0 such that

$$\max_{u\in S^n} J_{T_0}\big((P_{m_0} \circ \gamma_0)u\big) \le b_n + \delta, \tag{3.35}$$

where P_{m_0} denotes the orthogonal projective operator from E to E_{m_0}.

For any $c \in \mathbb{R}$, let $J_{T_0}^c = \{u \in E : J_{T_0}(u) \le c\}$. Choose $\bar{\varepsilon} = -(b_n + \delta)/2 > 0$. By a similar proof as in Lemma 3.2, there exists $\rho_{m_0+1} > 0$ such that if $u \in \bar{B}(0,\rho_0) \cap E_{m_0+1}$,

$J_{T_0}(u) \leq 0$, where $B(x_0,\rho)$ denotes the open ball of radius ρ centred at x_0 in E, and $\bar{B}(x_0,\rho)$ denotes the closure of $B(x_0,\rho)$ in E. Since $J_{T_0} \in C^1(E,\mathbb{R})$ and $J_{T_0}(0) = 0$, dist $(0, J_{T_0}^{-\bar{\varepsilon}}) > 0$. Set $\rho_0' = \min\{\rho_0,\ \text{dist}\,(0, J_{T_0}^{-\bar{\varepsilon}})\}$, then $\rho_0' > 0$. By Deformation Theorem (see Theorem A.4 in [15]), there exists $\varepsilon \in (0,\bar{\varepsilon})$ and a continuous map $\eta \in C([0,1]\times E, E)$ such that

$$\eta(1,u) = u, \quad \text{if } J_{T_0}(u) \notin [-\bar{\varepsilon},\bar{\varepsilon}], \tag{3.36}$$

and

$$\eta\big(1, J_{T_0}^{\varepsilon}\backslash B(0,\rho_0')\big) \subset J_{T_0}^{-\varepsilon}, \tag{3.37}$$

where $B(0,\rho_0')$ is a neighbourhood of K_0 defined by (iii) in Lemma 2.5.

By (3.28), $P_{m_0}\circ\gamma_0 \in C(S^n, E_{m_0})$. Since E_{n+1} is a metric space with the norm $\|\cdot\|$ and S^n is a closed subset in E_{n+1}, there exists an extension $\widetilde{P_{m_0}\circ\gamma_0} : E_{n+1} \to E_{m_0}$ of $(P_{m_0}\circ\gamma_0)$ by Dugundji extension theorem (see Theorem 4.1 in [7]); furthermore,

$$\big((\widetilde{P_{m_0}\circ\gamma_0})E_{n+1}\big) \subset \text{co}\big((P_{m_0}\circ\gamma_0)S^n\big), \tag{3.38}$$

where the symbol co denotes the convex hull. Since $(P_{m_0}\circ\gamma_0)S^n$ is a compact set in E_{m_0}, by the definition of convex hull, $\text{co}\big((P_{m_0}\circ\gamma_0)S^n\big)$ is a bounded set in E_{m_0}. Then there exists a constant ν such that $J_{T_0}(u) \leq \nu$, $u \in \text{co}\big((P_{m_0}\circ\gamma_0)S^n\big)$. It follows from (3.38) that

$$J_{T_0}\big((\widetilde{P_{m_0}\circ\gamma_0})u\big) \leq \nu, \qquad \forall\, u \in E_{n+1}. \tag{3.39}$$

Next we distinguish two cases.

Case 1. $\nu \leq \varepsilon$. Since $\widetilde{P_{m_0}\circ\gamma_0} \in C(E_{n+1}, E_{m_0})$, by (3.39) we have

$$(\widetilde{P_{m_0}\circ\gamma_0})u \in J_{T_0,m_0}^{\varepsilon}, \qquad \forall\, u \in E_{n+1}, \tag{3.40}$$

where $J_{T_0,m_0}^{\varepsilon} := \{u \in E_{m_0} :\ J_{T_0}(u) \leq \varepsilon\}$. Define a map χ as follows:

$$\chi(u) = \begin{cases} u, & u \notin \bar{B}(0,\rho_0') \cap E_{m_0} \\ u + (\rho_0'^2 - \|u\|^2)^{1/2} e_{m_0+1}, & u \in \bar{B}(0,\rho_0') \cap E_{m_0}. \end{cases} \tag{3.41}$$

It is clear that $\chi \in C(E_{m_0}, E_{m_0+1})$ and

$$\big(\chi\circ(\widetilde{P_{m_0}\circ\gamma_0})\big)u \notin B(0,\rho_0'), \qquad \forall\, u \in E_{n+1}. \tag{3.42}$$

If $u \in E_{n+1}$ and $\|(\widetilde{P_{m_0}\circ\gamma_0})u\| > \rho_0'$, in view of (3.40) and (3.41), we get

$$\big(\chi\circ(\widetilde{P_{m_0}\circ\gamma_0})\big)u = (\widetilde{P_{m_0}\circ\gamma_0})u \in J_{m_0}^{\varepsilon}. \tag{3.43}$$

Otherwise, when $u \in E_{n+1}$ and $\|(\widetilde{P_{m_0}\circ\gamma_0})u\| \leq \rho_0'$, by (3.41) $\|\big(\chi\circ(\widetilde{P_{m_0}\circ\gamma_0})\big)u\| = \rho_0'$. By the definition of ρ_0' and (3.43), we deduce that

$$\big(\chi\circ(\widetilde{P_{m_0}\circ\gamma_0})\big)u \in J_{T_0}^{\varepsilon}, \qquad \forall\, u \in E_{n+1}. \tag{3.44}$$

Define a map $H_0 : E_{n+1} \to E$ as follows:

$$H_0(\cdot) = \eta\Big(1, \big(\chi\circ(\widetilde{P_{m_0}\circ\gamma_0})\big)(\cdot)\Big). \tag{3.45}$$

Next we need to prove $H_0 \in \Gamma_n(\delta)$ and $\max_{u\in S_+^{n+1}} J_{T_0}(H_0(u)) < 0$. First, it is obvious that $H_0 \in C(S_+^{n+1}, E)$. Next we prove $H_0|_{S^n} \in \Lambda_n$. By Dugundji extension theorem, we obtain

$$(\widetilde{P_{m_0} \circ \gamma_0})u = (P_{m_0} \circ \gamma_0)u, \qquad \forall\, u \in S^n. \tag{3.46}$$

From (3.35), $(P_{m_0} \circ \gamma_0)\, u \in J_{T_0}^{-2\bar{\varepsilon}}$, $u \in S^n$. The definition of ρ_0' and $J_{T_0}^{-2\bar{\varepsilon}} \subset J_{T_0}^{-\bar{\varepsilon}}$ imply that

$$\|(P_{m_0} \circ \gamma_0)\, u\| \geq \rho_0', \qquad \forall\, u \in S^n. \tag{3.47}$$

It follows from (3.41), (3.46) and (3.47) that

$$\big(\chi \circ (\widetilde{P_{m_0} \circ \gamma_0})\big)\, u = \chi \circ \big((P_{m_0} \circ \gamma_0)\, u\big) = (P_{m_0} \circ \gamma_0)\, u, \qquad \forall\, u \in S^n. \tag{3.48}$$

Since $(P_{m_0} \circ \gamma_0)\, u \in J_{T_0}^{-2\bar{\varepsilon}}$, $\forall\, u \in S^n$, in view of (3.35)–(3.36), (3.45) and (3.48)

$$H_0(u) = \eta\Big(1, \big(\chi \circ (\widetilde{P_{m_0} \circ \gamma_0})\big)\, u\Big) = (P_{m_0} \circ \gamma_0)\, u, \qquad \forall\, u \in S^n. \tag{3.49}$$

which implies that $H_0|_{S^n} \in \Lambda_n$. Moreover, from (3.30), (3.35) and (3.49), we have $H_0 \in \Gamma_n(\delta)$. Since $S^{n+1} \subset E_{n+1}$, by (3.42) and (3.44), we have $\big(\chi \circ (\widetilde{P_{m_0} \circ \gamma_0})\big)u \notin B(0, \rho_0')$, $\forall\, u \in S_+^{n+1}$ and $\big(\chi \circ (\widetilde{P_{m_0} \circ \gamma_0})\big)u \in J_{T_0}^{\varepsilon}$, $\forall\, u \in S_+^{n+1}$. From (3.37) and (3.45), we deduce that $\max_{u\in S_+^{n+1}} J_{T_0}(H_0(u)) \leq -\varepsilon < 0$, which implies that $c_n(\delta) < 0$ by (3.31).

Case 2. $\nu > \varepsilon$. Let $J_{T_0}|_{E_{m_0}}$ denote the restriction of J_{T_0} on E_{m_0}. By a similar proof as in Lemma 2.5 and Lemma 3.1, we can prove that $J_{T_0}|_{E_{m_0}} \in C^1(E_{m_0}, \mathbb{R})$ and satisfies Palais–Smale condition. Moreover, $J_{T_0}|_{E_{m_0}}$ has no critical point with positive critical values on E_{m_0}. By Noncritical interval theorem (see Theorem 5.1.6 in [5]), $J^{\varepsilon}_{T_0,m_0}$ is a strong deformation retraction of $J^{\nu}_{T_0,m_0}$. So there exists a map ς such that $\varsigma \in C(J^{\nu}_{T_0,m_0}, J^{\varepsilon}_{T_0,m_0})$ and $\varsigma(u) = u$, if $u \in J^{\varepsilon}_{T_0,m_0}$. Define a map from $E_{n+1} \to E$ as follows:

$$\bar{H}_0(\cdot) = \eta\Big(1, \Big(\chi \circ \big(\varsigma \circ (\widetilde{P_{m_0} \circ \gamma_0})\big)\Big)(\cdot)\Big).$$

By a similar proof as in Case 1, we also obtain $\bar{H}_0 \in \Gamma_n(\delta)$ and $\max_{u\in S_+^{n+1}} J_{T_0}(\bar{H}_0(u)) \leq -\varepsilon < 0$, which leads to $c_n(\delta) < 0$ by (3.31). □

Lemma 3.4. *Suppose that (L_0), (H_1), (H_4) and (G_1) are satisfied. Then there exists a positive constant C_{15} independent of n such that for all n large enough*

$$b_n \geq -C_{15} n^{\frac{-\beta p}{2-p}}. \tag{3.50}$$

Proof. For any $\gamma \in \Lambda_n$ $(n \geq 2)$, when $0 \notin \gamma(S^n)$, the genus $\Pi(\gamma(S^n))$ is well defined and $\Pi(\gamma(S^n)) \geq \Pi(S^n) = n$. From Proposition 7.8 in [15], we have $\gamma(S^n) \cap E_{n-1}^{\perp} \neq \emptyset$. Otherwise, if $0 \in \gamma(S^n)$, then $0 \in \gamma(S^n) \cap E_{n-1}^{\perp}$. So for any $\gamma \in \Lambda_n$ $(n \geq 2)$, $\gamma(S^n) \cap E_{n-1}^{\perp} \neq \emptyset$. Therefore, for any $\gamma \in \Lambda_n$ $(n \geq 2)$, we get

$$\max_{u\in S^n} J_{T_0}(\gamma(u)) \geq \inf_{u\in E_{n-1}^{\perp}} J_{T_0}(u). \tag{3.51}$$

It follows from Hölder inequality, (1.4), (1.7), (2.13), (2.15) and (2.26) that

$$J_{T_0}(u) \geq \frac{1}{4}\|u\|^2 - C_{16}\|u\|_2^p, \qquad \forall\, u \in E. \tag{3.52}$$

If $u \in E_{n-1}^{\perp}$, $\lambda_n \|u\|_2^2 \le \|u\|^2$. When $u \in E_{n-1}^{\perp}$, by (3.52) we obtain

$$J_{T_0}(u) \ge \frac{1}{4}\|u\|^2 - C_{16}\lambda_n^{-\frac{p}{2}}\|u\|^p. \tag{3.53}$$

By (3.29), (3.51) and (3.53), for $n \ge 2$, we conclude that

$$\begin{aligned} b_n &\ge \inf_{t\ge 0}\left\{\frac{1}{4}t^2 - C_{16}\lambda_n^{-\frac{p}{2}}t^p\right\} \\ &= -C_{17}\lambda_n^{\frac{-p}{2-p}}, \end{aligned} \tag{3.54}$$

where C_{17} is a positive constant independent of n and λ_n. From (G_2) in Theorem 1.1 and (3.54), it is easy to verify that (3.50) holds. □

Lemma 3.5. *Suppose that $c_n = b_n$ for $n \ge n_0$, where $n_0 \in \mathbb{N}$. Then there exists a positive integer n_1 such that*

$$|b_n| \ge C_{18} n^{\frac{2}{2-\sigma}}, \qquad n \ge n_1, \tag{3.55}$$

where C_{18} is a positive constant independent of n.

Proof. For any $n \ge n_0$ and any $\varepsilon \in (0, |b_n|)$, by (3.29) there exists a map $\gamma_1 \in \Gamma_n$ such that

$$\max_{u\in S_+^{n+1}} J_{T_0}(\gamma_1(u)) < c_n + \varepsilon = b_n + \varepsilon < 0. \tag{3.56}$$

In view of $S^{n+1} = S_+^{n+1} \cup (-S_+^{n+1})$, γ_1 can be continuously extended to S^{n+1} as an odd function, also denoted by γ_1, then $\gamma_1 \in \Lambda_{n+1}$. From (3.29), we have

$$b_{n+1} \le \max_{u\in S^{n+1}} J_{T_0}(\gamma_1(u)) = J_{T_0}(\gamma_1(u_0)) \tag{3.57}$$

for some $u_0 \in S^{n+1}$. When $u_0 \in S_+^{n+1}$, in combination with (3.56) and (3.57), $b_{n+1} \le J_{T_0}(\gamma_1(u_0)) < b_n + \varepsilon$. We have

$$b_{n+1} < b_n + \varepsilon + C_6|b_{n+1}|^{\frac{\sigma}{2}}, \tag{3.58}$$

where C_6 is given in (2.31). Otherwise, $u_0 \in -S_+^{n+1}$. It follows from (2.31) and (3.56) that

$$\begin{aligned} J_{T_0}(\gamma_1(u_0)) &\le J_{T_0}(\gamma_1(-u_0)) + C_6|J_{T_0}(\gamma_1(u_0))|^{\frac{\sigma}{2}} \\ &\le b_n + \varepsilon + C_6|J_{T_0}(\gamma_1(u_0))|^{\frac{\sigma}{2}}. \end{aligned} \tag{3.59}$$

Next we consider two possible cases.

Case 1. $J_{T_0}(\gamma_1(u_0)) \le |b_{n+1}|$. By (3.57) and (3.59), we get

$$b_{n+1} \le b_n + \varepsilon + C_6|b_{n+1}|^{\frac{\sigma}{2}}. \tag{3.60}$$

Case 2. $J_{T_0}(\gamma_1(u_0)) > |b_{n+1}|$. From (3.56), there exists $u_1 \in S_+^{n+1}$ such that

$$J_{T_0}(\gamma_1(u_1)) < b_n + \varepsilon < 0. \tag{3.61}$$

In view of $J_{T_0}(\gamma_1(u_0)) > |b_{n+1}|$ and $J_{T_0}(\gamma_1(u_1)) < 0$. Since $(J_{T_0} \circ \gamma_1) \in C(S^{n+1}, \mathbb{R})$ and S^{n+1} is a connected space with the norm $\|\cdot\|$, by the Intermediate Value Theorem (see Theorem 24.3

in [12]), there exists $u_2 \in S^{n+1}$ such that $J_{T_0}(\gamma_1(u_2)) = |b_{n+1}|/2$. By (3.56), $u_2 \in -S_+^{n+1}$. From (2.31) and (3.56), we obtain

$$\begin{aligned} J_{T_0}(\gamma_1(u_2)) &\le J_{T_0}(\gamma_1(-u_2)) + C_6|J_{T_0}(\gamma_1(u_2))|^{\frac{\sigma}{2}} \\ &< b_n + \varepsilon + C_6|J_{T_0}(\gamma_1(u_2))|^{\frac{\sigma}{2}}, \end{aligned}$$

which implies that

$$b_{n+1} \le b_n + \varepsilon + C_6|b_{n+1}|^{\frac{\sigma}{2}}. \tag{3.62}$$

By Lemma 3.2, $b_n < 0$ for any $n \in \mathbb{N}$. It follows from (3.58), (3.60) and (3.62) that

$$|b_n| \le |b_{n+1}| + C_6|b_{n+1}|^{\frac{\sigma}{2}}, \qquad n \ge n_0. \tag{3.63}$$

Next we show that (3.63) implies (3.55). The proof will be done by induction. First, we introduce a useful inequality as follows:

$$(1+x)^{\alpha_0} \ge 1 + \frac{\alpha_0 x}{2}, \qquad x \in [0, \delta], \tag{3.64}$$

where α_0, δ are positive constants and δ depends on α_0. Set $\alpha_0 = 2(\sigma-2)^{-1}$. In view of (3.64), there exists $\bar{n}_0 \in \mathbb{N}$ such that

$$\left(1 + \frac{1}{n}\right)^{\frac{2}{\sigma-2}} \ge 1 + \frac{1}{(\sigma-2)n}, \qquad n \ge \bar{n}_0. \tag{3.65}$$

Set

$$C_{18} = \min\left\{ n_1^{\frac{2}{\sigma-2}}|b_{n_1}|,\ \left(\frac{1}{C_6(\sigma-2)}\right)^{\frac{2}{\sigma-2}} \right\}, \tag{3.66}$$

where $n_1 := \max\{n_0,\ \bar{n}_0\}$. We claim (3.55) holds. By (3.66), it is obvious that $|b_{n_1}| \ge C_{18} n_1^{\frac{2}{2-\sigma}}$. Assume that (3.55) holds for $j \ge n_1$. Then we only need to prove (3.55) also holds for $j+1$. If not, we have

$$|b_{j+1}| < C_{18}(j+1)^{\frac{2}{2-\sigma}}. \tag{3.67}$$

Since (3.55) holds for j, by (2.31), (3.63) and (3.67), we have

$$C_{18} j^{\frac{2}{2-\sigma}} \le |b_j| \le |b_{j+1}| + C_6|b_{j+1}|^{\frac{\sigma}{2}} < C_{18}(j+1)^{\frac{2}{2-\sigma}} + C_6 C_{18}^{\frac{\sigma}{2}}(j+1)^{\frac{\sigma}{2-\sigma}}, \tag{3.68}$$

When we divide (3.68) by $C_{18}(j+1)^{\frac{2}{2-\sigma}}$ on both sides, in view of (3.66) we get

$$\left(1 + \frac{1}{j}\right)^{\frac{2}{\sigma-2}} < 1 + C_6 C_{18}^{\frac{\sigma-2}{2}} \frac{1}{j+1} < 1 + C_6 C_{18}^{\frac{\sigma-2}{2}} \frac{1}{j} \le 1 + \frac{1}{(\sigma-2)j},$$

which contradicts (3.65). So (3.55) holds. □

By the fact that $b_n < 0$, (G_2), (3.50) and (3.55), it is impossible that $c_n = b_n$ for all large n. Next we can construct critical values of J_{T_0} as follows.

Lemma 3.6. *Suppose that $c_n > b_n$. Then for any $\delta \in (0, c_n - b_n)$, $c_n(\delta)$ given by (3.31) is a critical value of J_{T_0}.*

Proof. We prove this lemma by contradiction. For any $\delta \in (0, c_n - b_n)$, if $c_n(\delta)$ is not a critical value of J_{T_0}, define $\bar{\varepsilon} = (c_n - b_n - \delta)/2$, by Deformation Theorem, there exist a positive constant $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in C([0,1] \times E, E)$ such that

$$\eta(1, u) = u, \quad \text{if } J_{T_0}(u) \notin [c_n(\delta) - \bar{\varepsilon}, c_n(\delta) + \bar{\varepsilon}], \tag{3.69}$$

and

$$\eta\left(1, J_{T_0}^{c_n(\delta)+\varepsilon}\right) \subset J_{T_0}^{c_n(\delta)-\varepsilon}. \tag{3.70}$$

By (3.31), there exists $\gamma_2 \in \Gamma_n(\delta)$ such that

$$\max_{u \in S_+^{n+1}} J_{T_0}(\gamma_2(u)) < c_n(\delta) + \varepsilon. \tag{3.71}$$

Define

$$\bar{\gamma}_2(u) = \eta(1, \gamma_2(u)), \qquad u \in S_+^{n+1}. \tag{3.72}$$

It is evident that $\bar{\gamma}_2 \in C(S_+^{n+1}, E)$. Since $\gamma_2 \in \Gamma_n(\delta)$, by (3.30) we have

$$J_{T_0}(\gamma_2(u)) \le b_n + \delta = c_n - 2\bar{\varepsilon} \le c_n(\delta) - 2\bar{\varepsilon}, \qquad u \in S^n. \tag{3.73}$$

By (3.69), (3.72) and (3.73), we have $\bar{\gamma}_2(u) = \gamma_2(u)$, $u \in S^n$, which yields

$$\bar{\gamma}_2|_{S_n} \in \Lambda_n \quad \text{and} \quad J_{T_0}(\bar{\gamma}_2(u)) = J_{T_0}(\gamma_2(u)) \le b_n + \delta, \qquad u \in S^n. \tag{3.74}$$

In view of (3.74), we obtain $\bar{\gamma}_2 \in \Gamma_n(\delta)$. It follows from (3.70), (3.71) and (3.72) that

$$\max_{u \in S_+^{n+1}} J_{T_0}(\bar{\gamma}_2(u)) = \max_{u \in S_+^{n+1}} J_{T_0}\big(\eta(1, \gamma_2(u))\big) \le c_n(\delta) - \varepsilon,$$

which contradicts (3.31). So $c_n(\delta)$ given by (3.31) is a critical value of J_{T_0}. □

Proof of Theorem 1.1. Since it is impossible that $c_n = b_n$ for all large n, we can choose a subsequence $\{n_j\} \subset \mathbb{N}$ such that $c_{n_j} > b_{n_j}$. In combination with Lemma 3.3, Lemma 3.4 and Lemma 3.6, there exists a sequence of critical points $\{u_{n_j}\}_{j=1}^{\infty}$ of J such that

$$-C_{15} n_j^{-\frac{\beta p}{(2-p)}} \le b_{n_j} < c_{n_j} \le c_{n_j}(\delta_j) = J_{T_0}(u_{n_j}) < 0, \tag{3.75}$$

where $\delta_j \in (0, c_{n_j} - b_{n_j})$. In view of (H_2), (H_4), (2.5), (2.26) and (2.27), $u_{n_j} \neq 0$, $j \in \mathbb{N}$. Next we consider the following two possible cases.

Case 1. $\|u_{n_j}\|^2 > 2T_0$. Combining (2.15), (2.16) and (2.32), we have $k_{T_0}(u_{n_j}) = 0$, $k'_{T_0}(u_{n_j}) = 0$ and $\psi'(u_{n_j}) = 0$. It follows from (2.17) and (2.45) that

$$\bar{I}_{T_0}(u_{n_j}) = \frac{1}{2}\|u_{n_j}\|^2 - \int_{\mathbb{R}} W_1(t, u_{n_j})dt, \qquad \langle J'_{T_0}(u_{n_j}), u_{n_j}\rangle = \|u_{n_j}\|^2 - \int_{\mathbb{R}} (\nabla W_1(t, u_{n_j}), u_{n_j})dt.$$

By (1.5) and two equalities above, we get

$$\begin{aligned} \bar{I}_{T_0}(u_{n_j}) &= \bar{I}_{T_0}(u_{n_j}) - \mu^{-1}\langle J'_{T_0}(u_{n_j}), u_{n_j}\rangle \\ &= 2A\|u_{n_j}\|^2 + \mu^{-1}\int_{\mathbb{R}} \big((\nabla W_1(t, u_{n_j}), u_{n_j}) - \mu W_1(t, u_{n_j})\big)dt \\ &< A\|u_{n_j}\|^2. \end{aligned} \tag{3.76}$$

Case 2. $\|u_{n_j}\|^2 \le 2T_0$. It follows from Lemma 2.3, (2.17), (2.26)–(2.28) and (2.45) that

$$\bar{I}_{T_0}(u_{n_j}) \le \frac{1}{2}\|u_0\|^2 - \int_{\mathbb{R}} W_1(t, u_{n_j})dt + C_3\tau_{\alpha_4}^{\alpha_4}\|u_{n_j}\|^{\alpha_4} + 16C_4\tau_\sigma^\sigma\|u_{n_j}\|^\sigma,$$

and

$$\langle J'_{T_0}(u_{n_j}), u_{n_j}\rangle \ge \|u_{n_j}\|^2 - \int_{\mathbb{R}}(\nabla W_1(t, u_{n_j}), u_{n_j})dt - 9C_3\tau_{\alpha_4}^{\alpha_4}\|u_{n_j}\|^{\alpha_4} - 16(9 - 32A^{-1})C_4\tau_\sigma^\sigma\|u_{n_j}\|^\sigma.$$

By (1.5), (2.13), and two equalities above, we obtain

$$\begin{aligned}\bar{I}_{T_0}(u_{n_j}) &= \bar{I}_{T_0}(u_{n_j}) - \mu^{-1}\langle J'_{T_0}(u_{n_j}), u_{n_j}\rangle \\ &\le 2A\|u_{n_j}\|^2 + C_1\tau_{\alpha_1}^{\alpha_1}\|u_{n_j}\|^{\alpha_1} + 10C_3\tau_{\alpha_4}^{\alpha_4}\|u_{n_j}\|^{\alpha_4} + 16(10 - 32A^{-1})C_4\tau_\sigma^\sigma\|u_{n_j}\|^\sigma \\ &\le A\|u_{n_j}\|^2. \end{aligned} \tag{3.77}$$

In both cases, by (2.12), (2.24), (3.76) or (3.77), we get $l_\mu(u_{n_j}) = 1$ and $l'_\mu(u_{n_j}) = 0$. Moreover, it follows from (2.26) and (2.27) that $J_{T_0}(u_{n_j}) = \bar{I}_{T_0}(u_{n_j}) \le A\|u_{n_j}\|^2 < 0$, which implies that $\|u_{n_j}\| \to 0$, $j \to \infty$ by (3.75). So there exists $j_0 \in \mathbb{N}$ such that $\|u_{n_j}\|^2 < T_0$, $j \ge j_0$. By (2.15)-(2.16), we have $k_{T_0}(u_{n_j}) = 1$ and $k'_{T_0}(u_{n_j}) = 0$ for all $j \ge j_0$, which leads to $\{u_{n_j}\}$ are also critical points of I for all $j \ge j_0$ by (2.5) and (2.11).

Since E is continuously embedded into $L^\infty(\mathbb{R}, \mathbb{R}^N)$ and $\|u_{n_j}\| \to 0$ as $j \to \infty$, then $\max_{t\in\mathbb{R}} |u_{n_j}(t)| \to 0$ as $j \to \infty$. Thus, there exists $j_1 \in \mathbb{N}$ such that $\max_{t\in\mathbb{R}} |u_{n_j}(t)| < r_1$ for all $j \ge j_1$. Set $j_2 = \max\{j_0, j_1\}$. By (2.5) and (2.11), u_{n_j} are also homoclinic solutions of problem (1.1) for each $j \ge j_2$. This completes the proof. □

4 Examples

In this section, we give an example to illustrate our result.

Example 4.1. In problem (1.1), let $L(t) = t^2 + 1$ and $W(t, x) = a(t)\ln(1 + |x|^{3/2})$, $(t, x) \in \mathbb{R} \times \mathbb{R}$, where $a : \mathbb{R} \to \mathbb{R}^+$ is a continuous function such that $a(t) \in L^4(\mathbb{R})$. Moreover, the perturbation term G is given by

$$G(t, x) = b(t)|x|^{\sigma-1}x, \qquad (t, x) \in \mathbb{R} \times (-r_2, r_2),$$

where b is a bounded continuous function on $\mathbb{R}$ and $\sigma > 8/3$. Let $W_1(t, x) = a(t)|x|^{3/2}$ and $W_2(t, x) = a(t)\big(\ln(1 + |x|^{3/2}) - |x|^{3/2}\big)$. Then we choose $p = \mu = 3/2$ and

$$\alpha_1 = \alpha_3 = \alpha_4 = 3, \qquad \alpha_2 = 3/2, \qquad N = 1.$$

Since $-d^2/dt^2 + L(t)$ has eigenvalues $\lambda_n = 2n + 2$ with multiplicity 1 (see [2]), we can choose $\beta = 1$. By Theorem 1.1, problem (1.1) has infinitely many homoclinic solutions. Since the perturbation term G breaks the symmetry of the energy functional, the results in [20,22,34,35] cannot be applied to this example.

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