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Infinitely many weak solutions for $p(x)$-Laplacian-like problems with sign-changing potential

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Abstract. This study is concerned with the $p(x)$-Laplacian-like problems and arising from capillarity phenomena of the following type

$$\begin{cases} -\mathrm{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)=\lambda f(x,u), & \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p \in C(\overline{\Omega})$, and the primitive of the nonlinearity f of super-p^+ growth near infinity in u and is also allowed to be sign-changing. Based on a direct sum decomposition of a space $W_0^{1,p(x)}(\Omega)$, we establish the existence of infinitely many solutions via variational methods for the above equation. Furthermore, our assumptions are suitable and different from those studied previously.

Keywords: $p(x)$-Laplacian-like, variational method, multiple solutions, sign-changing potential.

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1 Introduction and main results

The present study is concerned with the existence of infinitely many nontrivial solutions for the nonlinear eigenvalue problems involving the $p(x)$-Laplacian-like operators, originated from a capillary phenomena,

$$\begin{cases} -\mathrm{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)=\lambda f(x,u), & \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega, \end{cases} \tag{P}$$

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where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p \in C(\overline{\Omega})$, $\lambda > 0$ is a parameter and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition and the primitive of the nonlinearity f is allowed to be sign-changing.

Capillarity can be briefly explained by considering the effects of two opposing forces: adhesion, i.e., the attractive (or repulsive) force between the molecules of the liquid and those of the container; and cohesion, i.e., the attractive force between the molecules of the liquid. The study of capillary phenomena has gained some attention recently. This increasing interest is motivated not only by fascination in naturally-occurring phenomena such as motion of drops, bubbles and waves but also its importance in applied fields ranging from industrial and biomedical and pharmaceutical to microfluidic systems.

Recently, problem (P) has begun to receive more and more attention, see, for example, [2,7,8,11–13,15]. Let us recall some known results on problem (P). When the the primitive F of f oscillates at infinity, Shokooh and Neirameh [12] showed the existence of infinitely many weak solutions for this problem by using Ricceri's variational principle. For the case of f is p^+-superlinear at infinity, Zhou [15] and Ge [7] both obtained the existence of nontrivial solution of problem (P) for every parameter $\lambda > 0$, under suitable conditions on f. Rodrigues in [11], by using Fountain Theorem, established the existence of sequence of high energy solutions for problem (P), by assuming the following assumptions:

(h_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, $t \to f(x,t)$ is continuous for a.e. $x \in \Omega$, and $x \to f(x,t)$ is Lebesgue measurable for all $t \in \mathbb{R}$;

(h_2) There exists a positive constant C such that

$$|f(x,t)| \le C(1 + |t|^{r(x)-1}),$$

for all $x \in \Omega$ and $t \in \mathbb{R}$, where $r \in C_+(\overline{\Omega})$ such that $1 < p^- \le p^+ < r^- \le r(x) < p^*(x)$ for all $x \in \overline{\Omega}$, $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$, $p^*(x) = +\infty$ if $p(x) \ge N$;

$(h_3)'$ there exist $M > 0$, $\mu > p^+$ such that for $|t| \ge M$ and a.e. $x \in \Omega$,

$$0 < \mu F(x,t) \le tf(x,t),$$

where $F(x,t) = \int_0^t f(x,s)ds$;

(h_4) $f(x,-t) = -f(x,t)$, for all $(x,t) \in \Omega \times \mathbb{R}$.

Specifically, the author established the following theorem in [11].

Theorem 1.1 ([11, Theorem 4.7]). *Suppose that (h_1), (h_2), $(h_3)'$ and (h_4) hold. Then the problem (P) has an unbounded sequence of weak solutions for every $0 < \lambda < \frac{2r^+}{p^+}$.*

Observe that condition $(h_3)'$ plays an important role for showing that any Palais–Smale sequence is bounded in the work. However, there are some functions which do not satisfy condition $(h_3)'$, for example,

$$f(x,u) = |u|^{p^+-2}u\ln(1+|u|).$$

In the present paper, we shall prove the same result as in [11] for problem (P) under more general assumptions on the nonlinearity, which unifies and significantly improves the result of [11]. The underlying idea for proving our main result is motivated by the argument used in [10]. In order to state the main result of this paper, we need the following assumptions:

(h_3) $\lim\limits_{|t|\to\infty} \frac{|F(x,t)|}{|t|^{p^+}} = +\infty$ uniformly in x, and there exists $r_0 > 0$ such that

$$F(x,t) \geq 0, \qquad \forall (x,t) \in \Omega \times \mathbb{R},\ |t| \geq r_0;$$

(h_5) $\mathcal{F}(x,t) := \frac{1}{p^+} f(x,t)t - F(x,t) \geq 0$, and there exist $c_0 > 0$ and $\sigma \in C_+(\Omega)$ with $\sigma^- > \max\{1, \frac{N}{p^-}\}$ such that

$$|F(x,t)|^{\sigma(x)} \leq c_0 |t|^{p^- \sigma(x)} \mathcal{F}(x,t), \qquad \forall (x,t) \in \Omega \times \mathbb{R},\ |t| \geq r_0;$$

(h_6) there exist $\mu > p^+$ and $\theta > 0$ such that

$$\mu F(x,t) \leq t f(x,t) + \theta |t|^{p^-}, \qquad \forall (x,t) \in \Omega \times \mathbb{R}.$$

We are now in the position to state our main results.

Theorem 1.2. *Suppose that (h_1)–(h_5) hold. Then for each $\lambda \in \left(0, \frac{2r^+}{p^+}\right)$, problem (P) possesses infinitely many nontrivial solutions.*

Theorem 1.3. *Suppose that (h_1)–(h_4) hold. and (h_6) hold. Then for each $\lambda \in \left(0, \frac{2r^+}{p^+}\right)$, problem (P) possesses infinitely many nontrivial solutions.*

Remark 1.4. It is easy to see that (h_3) and (h_5) are weaker than $(h_3)'$. In particular, $F(x,t)$ is allowed to be sign-changing in Theorems 1.2 and Theorems 1.3. The role of $(h_3)'$ is to ensure the boundedness of the Palais–Smale sequences of the energy functional, it is also significant to construct the variational framework. This is very crucial in applying the critical point theory. However, there are many functions which are superlinear at infinity, but do not satisfy the condition $(h_3)'$ for any $\mu > p^+$. For example, set $f(x,t) = p^+ |t|^{p^+ - 2} t \ln(1+t^2)$, then $F(x,t) = |t|^{p^+} \ln(1+t^2) - \frac{2|t|^{p^+} t}{1+t^2}$. It is easy to check that $f(x,t)$ satisfy assumptions (h_3) and (h_5).

The rest of this paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces. In Section 3, the proof of the main results is given.

2 Preliminaries

In order to discuss problem (P), we need some facts on space $W_0^{1,p(x)}(\Omega)$ which are called variable exponent Sobolev space. For this reason, we will recall some properties involving the variable exponent Lebesgue–Sobolev spaces, which can be found in [3–6,9] and references therein.

Throughout this paper, we always assume $p(x) > 1$, $p \in C(\overline{\Omega})$. Set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$, we will denote

$$h^- = \min_{x \in \overline{\Omega}} h(x), \qquad h^+ = \max_{x \in \overline{\Omega}} h(x)$$

and denote by $h_1 \ll h_2$ the fact that $\inf_{x \in \Omega} (h_2(x) - h_1(x)) > 0$.

For $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space:

$$L^{p(x)}(\Omega) = \left\{ u : u \text{ is a measurable real value function } \int_\Omega |u(x)|^{p(x)} dx < +\infty \right\},$$

with the norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \le 1\}$, and define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm $\|u\| = \|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}$.

We recall that spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.

Denote by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, then the Hölder type inequality

$$\int_\Omega |uv| dx \le \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{L^{p(x)}(\Omega)} |v|_{L^{q(x)}(\Omega)}, \qquad u \in L^{p(x)}(\Omega), \quad v \in L^{q(x)}(\Omega) \tag{2.1}$$

holds. Furthermore, if we define the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ by

$$\rho(u) = \int_\Omega |u|^{p(x)} dx,$$

then the following relations hold

$$|u|_{p(x)} < 1 (= 1, > 1) \Leftrightarrow \rho(u) < 1 (= 1, > 1), \tag{2.2}$$

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}, \tag{2.3}$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}. \tag{2.4}$$

Next, we denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Moreover, we have the following.

Proposition 2.1 ([6]).

(1) The Poincaré inequality in $W_0^{1,p(x)}(\Omega)$ holds, that is, there exists a positive constant C such that

$$|u|_{p(x)} \le C |\nabla u|_{p(x)}, \qquad \forall u \in W_0^{1,p(x)}(\Omega).$$

(2) If $q \in C(\overline{\Omega})$ and $1 < q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W_0^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^*(x) = +\infty$ if $p(x) \ge N$.

By (1) of Proposition 2.1, we know that $|\nabla u|_{p(x)}$ and $\|u\|$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. We will use $|\nabla u|_{p(x)}$ to replace $\|u\|$ in the following discussions.

Proposition 2.2 ([4]). *Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\Omega)$ and $1 \le p(x)q(x) \le \infty$ almost every where in Ω. Let $u \in L^{q(x)}(\Omega)$, $u \ne 0$. Then*

$$|u|_{p(x)q(x)} \ge 1 \Rightarrow |u|_{p(x)q(x)}^{p^-} \le \left| |u|^{p(x)} \right|_{q(x)} \le |u|_{p(x)q(x)}^{p^+},$$

$$|u|_{p(x)q(x)} \le 1 \Rightarrow |u|_{p(x)q(x)}^{p^+} \le \left| |u|^{p(x)} \right|_{q(x)} \le |u|_{p(x)q(x)}^{p^-}.$$

In particular, if $p(x) = p$ is a constant, then $\left| |u|^p \right|_{q(x)} = |u|_{pq(x)}^p$.

Consider the following function:

$$J(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + \sqrt{1 + |\nabla u|^{2p(x)}} \right) dx, \qquad \forall u \in W_0^{1,p(x)}(\Omega).$$

We know that $J \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$. If we denote $A = J' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$, then

$$\langle A(u), v \rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) (\nabla u, \nabla v)_{\mathbb{R}^N} dx,$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$.

Proposition 2.3 ([11]). *Set $E = W_0^{1,p(x)}(\Omega)$, A is as above, then*

(1) $A : E \to E^*$ is a convex, bounded and strictly monotone operator;

(2) $A : E \to E^*$ is a mapping of type $(S)_+$, i.e., $u_n \rightharpoonup u$ in E and $\limsup_{n\to\infty} \langle A(u_n), u_n - u \rangle \leq 0$, implies $u_n \to u$ in E;

(3) $A : E \to E^*$ is a homeomorphism.

3 Variational setting and proof of the main results

For each $u \in E$, we define

$$\varphi_\lambda(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + \sqrt{1 + |\nabla u|^{2p(x)}} \right) dx - \lambda \int_\Omega F(x,u) dx. \tag{3.1}$$

Then we have the following lemma.

Lemma 3.1. *If assumptions (h_1)–(h_2) hold, then $\varphi \in C^1(E, \mathbb{R})$ and*

$$\langle \varphi_\lambda'(u), v \rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) (\nabla u, \nabla v)_{\mathbb{R}^N} dx - \lambda \int_\Omega f(x,u) v dx \tag{3.2}$$

for all $u, v \in E$. Moreover, $\psi' : E \to E^*$ is weakly continuous, where $\psi(u) = \int_\Omega F(x,u) dx$.

Proof. To prove $\varphi_\lambda \in C^1(E, \mathbb{R})$ and (3.2), we only need to show that $\psi \in C^1(E, \mathbb{R})$ and

$$\langle \psi'(u), v \rangle = \int_\Omega f(x,u) v dx, \qquad \forall u, v \in E.$$

On the one hand, for any $u, v \in E$ and $0 < |t| < 1$, by condition (h_2), we obtain

$$\begin{aligned} |f(x, u + tv)v| &\leq C(1 + |u + tv|^{r(x)-1})|v| \\ &\leq C(|v| + 2^{r^+-1}|u|^{r(x)-1}|v| + 2^{r^+-1}|v|^{r(x)}). \end{aligned}$$

Note that $1 < p(x) < r(x) < p^*(x)$, the Hölder inequality implies that

$$|v| + 2^{r^+-1}|u|^{r(x)-1}|v| + 2^{r^+-1}|v|^{r(x)} \in L^1(\Omega).$$

Consequently, by the mean value theorem and the Lebesgue dominated convergence theorem, there exists $0 < \lambda < 1$ such that

$$\begin{aligned}\langle \psi'(u), v\rangle &= \lim_{t\to 0}\int_\Omega \frac{F(x,u+tv)-F(x,u)}{t}dx\\ &= \lim_{t\to 0}\int_\Omega f(x,u+\lambda tv)vdx\\ &= \int_\Omega f(x,u)vdx,\end{aligned}$$

for all $u, v \in E$. Hence ψ is Gateaux differentiable.

It remains to prove that ψ' is weakly continuous. Assume that $u_n \rightharpoonup u$ in E. By Proposition 2.1, we conclude that $u_n \to u$ in $L^{r(x)}(\Omega)$ and $u_n(x) \to u(x)$ a.e. $x \in \Omega$. Recalling

$$\begin{aligned}\|\psi'(u_n)-\psi'(u)\|_{E_*} &= \sup_{\|v\|\le 1} |\langle \psi'(u_n)-\psi'(u), v\rangle|\\ &\le \sup_{\|v\|\le 1}\int_\Omega |f(x,u_n)-f(x,u)||v|dx.\end{aligned}$$

Set $\alpha := \lim_{n\to+\infty}\sup_{\|v\|\le 1}\int_\Omega |f(x,u_n)-f(x,u)||v|dx$. We claim that $\alpha = 0$. Suppose, by contradiction, that $\alpha > 0$. Hence, there exists a sequence $\{\phi_n\} \subseteq E$ and $\|\phi_n\| = 1$ such that $|\int_\Omega |f(x,u_n)-f(x,u)||\phi_n|dx| > \frac{\alpha}{2}$ for enough large n. By (h_2), one has

$$\begin{aligned}|(f(x,u_n)-f(x,u))\phi_n| &\le C(1+|u_n|^{r(x)-1})|\phi_n| + C(1+|u|^{r(x)-1})|\phi_n|\\ &\le C(2|\phi_n| + |u_n|^{r(x)-1}|\phi_n| + |u|^{r(x)-1}|\phi_n|).\end{aligned}$$

Using again Hölder inequality, we get $2|\phi_n| + |u_n|^{r(x)-1}|\phi_n| + |u|^{r(x)-1}|\phi_n| \in L^1(\Omega)$. In view of [14, Lemma A.1], there exist $w_1 \in L^1(\Omega)$ and $\xi_1, w_2 \in L^{r(x)}(\Omega)$ such that

$$\max\{|u_n(x)|, |u(x)|\} \le |\xi_1(x)| \text{ and } |\phi_n(x)| \le \min\{|w_1(x)|, |w_2(x)|\}.$$

Therefore, it follows from the Lebesgue dominated convergence theorem that

$$\lim_{n\to+\infty}\int_\Omega |f(x,u_n)-f(x,u)||\phi_n|dx = 0,$$

which contradicts with $\alpha > 0$. Hence, $\|\psi'(u_n)-\psi'(u)\|_{E_*} \to 0$ as $n \to +\infty$. The proof is completed. □

Definition 3.2. We say that $\varphi_\lambda \in C^1(E, \mathbb{R})$ satisfies $(C)_c$-condition if any sequence $\{u_n\} \subset E$ satisfying

$$\varphi_\lambda(u_n) \to c \quad \text{and} \quad \|\varphi'_\lambda(u_n)\|_{E^*}(1+\|u_n\|) \to 0 \tag{3.3}$$

contains a convergent subsequence.

Now, we present the following theorem which will play a crucial role in the proof of Main Theorems.

Let X be a reflexive and separable Banach space. It is well known that there exist $\{e_n\} \subset X$ and $\{e_n^*\} \subset X^*$ such that

(i) $\langle e_n^*, e_m\rangle = \delta_{n,m}$, where $\delta_{n,m} = 1$ for $n = m$ and $\delta_{n,m} = 0$ for $n \ne m$;

(ii) $X = \overline{\mathrm{span}\{e_n : n \in N\}}$ and $X^* = \overline{\mathrm{span}\{e_n^* : n \in N\}}$.

Let $X_i = \mathbb{R}e_i$, then $X = \overline{\oplus_{i\geq 1} X_i}$. Now, we define

$$Y_n = \oplus_{i=1}^{n} X_i \text{ and } Z_n = \overline{\oplus_{i\geq n} X_i}. \tag{3.4}$$

Then we have the following Fountain Theorem.

Lemma 3.3 ([1,14]). *Assume that $I \in C^1(X,\mathbb{R})$ satisfies $(C)_c$-condition for all $c > 0$ and I is even. If for each sufficiently large $n \in N$, there exist $\rho_n > \delta_n > 0$ such that the following conditions hold:*

(A_1) $b_n := \inf\{I(u) : u \in Z_n, \|u\| = \delta_n\} \to +\infty$ *as* $n \to +\infty$;

(A_2) $a_n := \inf\{I(u) : u \in Y_n, \|u\| = \rho_n\} \leq 0.$

Then the functional I has an unbounded sequence of critical values, i.e., there exists a sequence $\{u_n\} \subset X$ such that $I'(u_n) = 0$ and $I(u_n) \to +\infty$ as $n \to +\infty$.

Lemma 3.4. *Assume that (h_2), (h_3) and (h_5) hold. Then any $(C)_c$ sequence is bounded.*

Proof. Let $\{u_n\} \subset E$ be a $(C)_c$ sequence. To complete our goals, arguing by contradiction, suppose that $\|u_n\| \to \infty$, as $n \to \infty$. Observe that for n large,

$$\begin{aligned}
c+1 &\geq \varphi_\lambda(u_n) - \frac{1}{p^+}\langle \varphi'_\lambda(u_n), u_n\rangle \\
&= \int_\Omega \frac{1}{p(x)}\left(|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}}\right)dx - \lambda\int_\Omega F(x,u_n)dx \\
&\quad - \frac{1}{p^+}\int_\Omega \left(|\nabla u_n|^{p(x)} + \frac{|\nabla u_n|^{2p(x)}}{\sqrt{1+|\nabla u_n|^{2p(x)}}}\right)dx + \frac{\lambda}{p^+}\int_\Omega f(x,u_n)u_n dx \\
&\geq \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+}\right)\left(|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}}\right)dx + \lambda\int_\Omega \mathcal{F}(x,u_n)dx \\
&\geq \lambda \int_\Omega \mathcal{F}(x,u_n)dx.
\end{aligned} \tag{3.5}$$

Since $\|u_n\| > 1$ for n large, using (3.3) we have

$$\begin{aligned}
0 = \lim_{n\to\infty}\frac{c+o(1)}{\|u_n\|^{p^-}} &= \lim_{n\to\infty}\frac{\varphi_\lambda(u_n)}{\|u_n\|^{p^-}} \\
&\geq \frac{1}{p^+}\frac{\int_\Omega\left(|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}}\right)dx}{\|u_n\|^{p^-}} - \lambda\int_\Omega \frac{F(x,u_n)}{\|u_n\|^{p^-}}dx \\
&\geq \frac{2}{p^+} - \lambda\int_\Omega \frac{F(x,u_n)}{\|u_n\|^{p^-}}dx,
\end{aligned}$$

which implies that

$$\frac{2}{p^+\lambda} \leq \limsup_{n\to\infty}\int_\Omega \frac{|F(x,u_n)|}{\|u_n\|^{p^-}}dx. \tag{3.6}$$

For $0 \leq \alpha < \beta$, let $\Omega_n(\alpha,\beta) = \{x \in \Omega : \alpha \leq |u_n(x)| < \beta\}$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$ and $|v_n|_{r(x)} \leq C_0\|v_n\| = C_0$ for some $C_0 > 0$. Going if necessary to a subsequence, we may assume that $v_n \rightharpoonup v$ in E and

$$v_n \to v \quad \text{in } L^{s(x)}(\Omega), \qquad 1 \leq s(x) < p^*(x) \quad \text{and} \quad v_n(x) \to v(x) \quad \text{a.e. on } \Omega. \tag{3.7}$$

Now, we consider two possible cases: $v=0$ or $v\neq 0$.

(1) If $v=0$, then we have that $v_n\to 0$ in $L^{s(x)}(\Omega)$ and $v_n(x)\to 0$ a.e. on Ω. Hence, it follows from (h_2) that

$$\int_{\Omega_n(0,r_0)}\frac{|F(x,u_n)|}{\|u_n\|^{p^-}}dx\le\frac{C(r_0+r_0^{\bar r})\text{meas}(\Omega)}{\|u_n\|^{p^-}}\to 0\quad\text{as } n\to+\infty,\tag{3.8}$$

where $\bar r=r^+$ if $r_0\ge 1$, $\bar r=r^-$ if $r_0<1$.

Set $\sigma'(x)=\frac{\sigma(x)}{\sigma(x)-1}$. Since $\sigma^->\max\{1,\frac{N}{p^-}\}$ one sees that $1<p^-\sigma'(x)<p^*(x)$. So, $v_n\to 0$ in $L^{p^-\sigma'(x)}(\Omega)$ as $n\to+\infty$. Hence, we deduce from Proposition 2.2, (h_5), (3.5) and (3.7) that

$$\begin{aligned}
&\int_{\Omega_n(r_0,+\infty)}\frac{|F(x,u_n)|}{|u_n|^{p^-}}|v_n|^{p^-}dx\\
&\le 2\left|\frac{|F(x,u_n)|}{|u_n|^{p^-}}\right|_{L^{\sigma(x)}(\Omega_n(r_0,+\infty))}\left||v_n|^{p^-}\right|_{L^{\sigma'(x)}(\Omega_n(r_0,+\infty))}\\
&\le 2\max\left\{\left(\int_{\Omega_n(r_0,+\infty)}\frac{|F(x,u_n)|^{\sigma(x)}}{|u_n|^{(p^-)\sigma(x)}}dx\right)^{\frac{1}{\sigma^+}},\left(\int_{\Omega_n(r_0,+\infty)}\frac{|F(x,u_n)|^{\sigma(x)}}{|u_n|^{(p^-)\sigma(x)}}dx\right)^{\frac{1}{\sigma^-}}\right\}\\
&\quad\times\max\left\{\left(\int_{\Omega_n(r_0,+\infty)}|v_n|^{p^-\sigma'(x)}dx\right)^{\frac{1}{(\sigma')^-}},\left(\int_{\Omega_n(r_0,+\infty)}|v_n|^{p^-\sigma'(x)}dx\right)^{\frac{1}{(\sigma')^+}}\right\}\\
&\le 2\max\left\{\left(\int_{\Omega_n(r_0,+\infty)}\mathcal{F}(x,u_n)dx\right)^{\frac{1}{\sigma^+}},\left(\int_{\Omega_n(r_0,+\infty)}\mathcal{F}(x,u_n)dx\right)^{\frac{1}{\sigma^-}}\right\}\\
&\quad\times\max\left\{\left(\int_{\Omega_n(r_0,+\infty)}|v_n|^{p^-\sigma'(x)}dx\right)^{\frac{1}{(\sigma')^-}},\left(\int_{\Omega_n(r_0,+\infty)}|v_n|^{p^-\sigma'(x)}dx\right)^{\frac{1}{(\sigma')^+}}\right\}\\
&\le 2\max\left\{\left(\frac{c_0}{\lambda}(c+1)\right)^{\frac{1}{\sigma^+}},\left(\frac{c_0}{\lambda}(c+1)\right)^{\frac{1}{\sigma^-}}\right\}\\
&\quad\times\max\left\{\left(\int_{\Omega_n(r_0,+\infty)}|v_n|^{p^-\sigma'(x)}dx\right)^{\frac{1}{(\sigma')^-}},\left(\int_{\Omega_n(r_0,+\infty)}|v_n|^{p^-\sigma'(x)}dx\right)^{\frac{1}{(\sigma')^+}}\right\}\\
&\to 0,\quad\text{as } n\to\infty.
\end{aligned}\tag{3.9}$$

Combining (3.8) with (3.9), we get

$$\begin{aligned}
\int_\Omega\frac{|F(x,u_n)|}{\|u_n\|^{p^-}}dx&=\int_{\Omega_n(0,r_0)}\frac{|F(x,u_n)|}{\|u_n\|^{p^-}}dx+\int_{\Omega_n(r_0,+\infty)}\frac{|F(x,u_n)|}{\|u_n\|^{p^-}}dx\\
&=\int_{\Omega_n(0,r_0)}\frac{|F(x,u_n)|}{\|u_n\|^{p^-}}dx+\int_{\Omega_n(r_0,+\infty)}\frac{|F(x,u_n)|}{|u_n|^{p^-}}|v_n|^{p^-}dx\\
&\to 0,\quad\text{as } n\to\infty,
\end{aligned}\tag{3.10}$$

which contradicts (3.6).

(2) If $v\neq 0$, set $\Omega_{\neq}:=\{x\in\Omega: v(x)\neq 0\}$, then $\text{meas}(\Omega_{\neq})>0$. For a.e. $x\in\Omega_{\neq}$, we have $\lim_{n\to\infty}|u_n(x)|=+\infty$. Hence, $\Omega_{\neq}\subset\Omega_n(r_0,\infty)$ for large $n\in N$. As the proof of (3.8), we also obtain that

$$\int_{\Omega_n(0,r_0)}\frac{|F(x,u_n)|}{\|u_n\|^{p^+}}dx\le\frac{C(r_0+r_0^{\bar r})\text{meas}(\Omega)}{\|u_n\|^{p^+}}\to 0,\quad\text{as } n\to+\infty.\tag{3.11}$$

It follows from (h_2), (h_3), (3.11) and Fatou's Lemma that

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|^{p^+}} = \lim_{n\to\infty} \frac{\varphi(u_n)}{\|u_n\|^{p^+}} \\
&\leq \lim_{n\to\infty} \left[\frac{1}{p^-} \frac{\int_\Omega \left(|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}} \right) dx}{\|u_n\|^{p^+}} - \lambda \int_\Omega \frac{F(x,u_n)}{\|u_n\|^{p^+}} dx \right] \\
&= \lim_{n\to\infty} \left[\frac{1}{p^-} \frac{\int_\Omega \left(|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}} \right) dx}{\|u_n\|^{p^+}} \right. \\
&\qquad \left. - \lambda \int_{\Omega_n(0,r_0)} \frac{F(x,u_n)}{\|u_n\|^{p^+}} dx - \lambda \int_{\Omega_n(r_0,+\infty)} \frac{F(x,u_n)}{\|u_n\|^{p^+}} dx \right] \\
&= \lim_{n\to\infty} \left[\frac{1}{p^-} \frac{\int_\Omega \left(|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}} \right) dx}{\|u_n\|^{p^+}} - \lambda \int_{\Omega_n(r_0,+\infty)} \frac{F(x,u_n)}{\|u_n\|^{p^+}} dx \right] \qquad (3.12) \\
&\leq \lim_{n\to\infty} \left[\frac{1}{p^-} \frac{\int_\Omega \left(|\nabla u_n|^{p(x)} + 1 + |\nabla u_n|^{p(x)} \right) dx}{\|u_n\|^{p^+}} - \lambda \int_{\Omega_n(r_0,+\infty)} \frac{F(x,u_n)}{\|u_n\|^{p^+}} dx \right] \\
&= \lim_{n\to\infty} \left[\frac{2}{p^-} \frac{\int_\Omega |\nabla u_n|^{p(x)} dx}{\|u_n\|^{p^+}} - \lambda \int_{\Omega_n(r_0,+\infty)} \frac{F(x,u_n)}{\|u_n\|^{p^+}} dx \right] \\
&\leq \limsup_{n\to\infty} \left[\frac{2}{p^-} - \lambda \int_{\Omega_n(r_0,+\infty)} \frac{F(x,u_n)}{\|u_n\|^{p^+}} dx \right] \\
&= \frac{2}{p^-} - \liminf_{n\to\infty} \lambda \int_{\Omega_n(r_0,+\infty)} \frac{F(x,u_n)}{|u_n|^{p^+}} |v_n|^{p^+} dx \\
&= \frac{2}{p^-} - \liminf_{n\to\infty} \lambda \int_\Omega \frac{F(x,u_n)}{|u_n|^{p^+}} \chi_{\Omega_n(r_0,+\infty)}(x) |v_n|^{p^+} dx \\
&\leq \frac{2}{p^-} - \lambda \int_\Omega \liminf_{n\to\infty} \frac{F(x,u_n)}{|u_n|^{p^+}} \chi_{\Omega_n(r_0,+\infty)}(x) |v_n|^{p^+} dx \\
&\to -\infty, \quad \text{as } n\to\infty,
\end{aligned}
$$

which is a contradiction. Thus $\{u_n\}$ is bounded in E. The proof is accomplished. □

Lemma 3.5. *Suppose that (h_2), (h_3) and (h_5) hold. Then any $(C)_c$-sequence of φ has a convergent subsequence in E.*

Proof. Let $\{u_n\} \subset E$ be a $(C)_c$ sequence. In view of the Lemma 3.4, the sequence $\{u_n\}$ is bounded in E. Then, up to a subsequence we have $u_n \rightharpoonup u$ in E. By Proposition 2.2, it follows that

$$
\begin{aligned}
&u_n \to u \text{ in } L^{r(x)}(\Omega), \\
&\{u_n\} \text{ is bounded in } L^{r(x)}(\Omega).
\end{aligned}
$$

It is easy to compute directly that

$$\begin{aligned}
&\int_\Omega |f(x,u_n)-f(x,u)||u_n-u|dx \\
&\quad\le \int_\Omega (|f(x,u_n)|+|f(x,u)|)|u_n-u|dx \\
&\quad\le \int_\Omega [C(1+|u_n|^{r(x)-1})+C(1+|u|^{r(x)-1})]|u_n-u|dx \\
&\quad\le 2C\int_\Omega |u_n-u|dx + C\int_\Omega |u_n|^{r(x)-1}|u_n-u|dx \\
&\qquad + \int_\Omega |u|^{r(x)-1}|u_n-u|dx \\
&\quad\le 2C|u_n-u|_1 + 2C\big||u_n|^{r(x)-1}\big|_{r'(x)}|u_n-u|_{r(x)} \\
&\qquad + 2C\big||u|^{r(x)-1}\big|_{r'(x)}|u_n-u|_{r(x)} \\
&\quad\le 2C|u_n-u|_1 + 2C\max\Big\{|u_n|_{r(x)}^{r^+-1}, |u_n|_{r(x)}^{r^--1}\Big\}|u_n-u|_{r(x)} \\
&\qquad + 2C\max\Big\{|u|_{r(x)}^{r^+-1}, |u|_{r(x)}^{r^--1}\Big\}|u_n-u|_{r(x)} \\
&\quad\to 0, \quad \text{as } n\to\infty,
\end{aligned} \tag{3.13}$$

where $\frac{1}{r(x)}+\frac{1}{r'(x)}=1$. Noting that

$$\begin{aligned}
\langle A(u_n)-A(u), u_n-u\rangle &= \langle \varphi'(u_n)-\varphi'(u), u_n-u\rangle \\
&\quad + \int_\Omega (f(x,u_n)-f(x,u))(u_n-u)dx.
\end{aligned} \tag{3.14}$$

Moreover, by (3.3), one infers

$$\lim_{n\to\infty}\langle \varphi'(u_n)-\varphi'(u), u_n-u\rangle = 0. \tag{3.15}$$

Finally, the combination of (3.13)–(3.15) implies

$$\lim_{n\to\infty}\langle A(u_n)-A(u), u_n-u\rangle = 0. \tag{3.16}$$

Since A is of type $(S)_+$ by Lemma 2.3, we obtain $u_n\to u$ in E. The proof is complete. □

Lemma 3.6. *Suppose that (h_2), (h_3) and (h_6) hold. Then any $(C)_c$-sequence of φ has a convergent subsequence in E.*

Proof. Similar to the proof of Lemma 3.5, we only prove that $\{u_n\}$ is bounded in E. Suppose by contradiction that $\|u_n\|\to\infty$ as $n\to\infty$. Let $v_n=\frac{u_n}{\|u_n\|}$, then $\|v_n\|=1$ and $|v_n|_{r(x)}\le C_0\|v_n\|=C_0$ for some $C_0>0$. Going if necessary to a subsequence, we may assume that $v_n\rightharpoonup v$ in E,

$$v_n\to v \quad \text{in } L^{r(x)}(\Omega), \qquad 1\le r(x)<p^*(x) \quad \text{and} \quad v_n(x)\to v(x) \quad \text{a.e. on } \Omega. \tag{3.17}$$

By (3.1), (3.2) and (h_6), one has

$$\begin{aligned}
c+1 &\ge \varphi_\lambda(u_n) - \frac{1}{\mu}\langle \varphi'_\lambda(u_n), u_n\rangle \\
&= \int_\Omega \frac{1}{p(x)}\left(|\nabla u_n|^{p(x)}+\sqrt{1+|\nabla u_n|^{2p(x)}}\right)dx - \lambda\int_\Omega F(x,u_n)dx \\
&\quad - \frac{1}{\mu}\int_\Omega\left(|\nabla u_n|^{p(x)}+\frac{|\nabla u_n|^{2p(x)}}{\sqrt{1+|\nabla u_n|^{2p(x)}}}\right)dx + \frac{\lambda}{\mu}\int_\Omega f(x,u_n)u_n dx
\end{aligned}$$

$$
\begin{aligned}
&\geq \left(\frac{1}{p^+}-\frac{1}{\mu}\right)\int_\Omega \left(|\nabla u_n|^{p(x)}+\sqrt{1+|\nabla u_n|^{2p(x)}}\right)dx-\frac{\lambda\theta}{\mu}\int_\Omega |u_n|^{p^-}dx \\
&\geq \frac{\mu-p^+}{p^+\mu}\|u_n\|^{p^-}-\frac{\lambda\theta}{\mu}|u_n|_{p^-}^{p^-},
\end{aligned}
$$

for $n \in N$, which implies

$$
1 \leq \frac{\lambda\theta p^+}{\mu-p^+}\limsup_{n\to\infty}|v_n|_{p^-}^{p^-}. \tag{3.18}
$$

In view of (3.17), $v_n \to v$ in $L^{p^-}(\Omega)$. Hence, we deduce from (3.18) that $v \neq 0$. By a similar reasoning as in the proof of Lemma 3.4 step (2), we can conclude a contradiction. Thus, $\{u_n\}$ is bounded in E. The rest proof is the same as that in Lemma 3.5. □

Proof of Theorem 1.2. Let $X = E$, Y_n and Z_n be defined by (3.4). Obviously, $\varphi_\lambda(u) = \varphi_\lambda(-u)$ by (h_4), and Lemma 3.5 implies that φ_λ satisfies the $(C)_c$ condition for any $\lambda > 0$. Hence, to prove Theorem 1.2, it remains to verify the conditions (A_1) and (A_2) in Lemma 3.3.

Verification of (A_1). Set $\beta_n := \sup_{u\in Z_n,\ \|u\|=1}|u|_{r(x)}$, where $p^+ < r^- \leq r(x) < p^*(x)$ and $n \in N$. We claim that $\beta_n \to 0$ as $n \to \infty$. Indeed, it is obvious that $\beta_n \geq \beta_{n+1} \geq 0$. so $\beta_n \to \beta \geq 0$ as $n \to \infty$. For each $n = 1, 2, \ldots$, taking $u_n \in Z_n$, $\|u_n\| = 1$ such that $0 \leq \beta_n - |u_n|_{r(x)} \leq \frac{1}{n}$. As E is reflexive, $\{u_n\}$ has a weakly convergent subsequence, without loss of generality, suppose $u_n \rightharpoonup u$ in E. By definition of Z_n, one knows that $u = 0$. Proposition 2.3 implies that $u_n \to 0$ in $L^{r(x)}(\Omega)$. Thus we have proved that $\beta = 0$.

By the above definition of β_n, for $u \in Z_n$ with $\|u\| > 1$, we have

$$
|u|_{r(x)} \leq \beta_n\|u\|. \tag{3.19}
$$

Moreover, we consider the real function $k : \mathbb{R} \to \mathbb{R}$, defined by

$$
k(t) = \frac{1}{p^+}t^{p^-} - \lambda C\beta_n^{r^-}t^{r^+}.
$$

Choosing $\delta_n = (2Cr^+\beta_n^{r^-})^{\frac{1}{p^- - r^+}}$ for $n \in N$, it is clear that

$$
\begin{aligned}
k(\delta_n) &= \frac{1}{p^+}\delta_n^{p^-} - \lambda C\beta_n^{r^-}\delta_n^{r^+} \\
&= (2Cr^+\beta_n^{r^-})^{\frac{p^-}{p^- - r^+}}\left[\frac{1}{p^+}-\frac{\lambda}{2r^+}\right].
\end{aligned} \tag{3.20}
$$

Therefore, since $r^- > p^+$, $\lambda < \frac{2r^+}{p^+}$ and $\beta_n \to 0$ as $n \to +\infty$, we obtain that

$$
\delta_n \to +\infty, \qquad k(\delta_n) \to +\infty, \quad \text{as } n \to +\infty. \tag{3.21}
$$

It follows from (h_2) that

$$
F(x,t) \leq C(|t| + |t|^{r(x)}) \leq 2C(1 + |t|^{r(x)})
$$

for all $(x,t) \in \Omega \times \mathbb{R}$. Then, for any $u \in Z_n$, assume that $\|u\| = \delta_n$. It follows from (h_2), (3.19), (3.20) and (3.21) that

$$\begin{aligned}\varphi_\lambda(u) &= \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}\right)dx - \lambda\int_\Omega F(x,u)dx \\ &\geq \frac{2}{p^+}\|u\|^{p^-} - 2\lambda C\mathrm{meas}(\Omega) - 2\lambda C\int_\Omega |u|^{r(x)}dx \\ &\geq \frac{2}{p^+}\|u\|^{p^-} - 2\lambda C\mathrm{meas}(\Omega) - 2\lambda C\max\left\{|u|^{r^+}_{r(x)}, |u|^{r^-}_{r(x)}\right\} \\ &\geq \frac{2}{p^+}\|u\|^{p^-} - 2\lambda C\mathrm{meas}(\Omega) - 2\lambda C\max\left\{\beta_n^{r^+}\|u\|^{r^+}, \beta_n^{r^-}\|u\|^{r^-}\right\} \\ &\geq \frac{2}{p^+}\|u\|^{p^-} - 2\lambda C\mathrm{meas}(\Omega) - 2\lambda C\beta_n^{r^-}\|u\|^{r^+} \\ &= 2k(\delta_n) - 2\lambda C\mathrm{meas}(\Omega) \to +\infty, \quad \text{as } n\to+\infty.\end{aligned} \tag{3.22}$$

This gives relation (A_1).

Verification of (A_2). Assume that (A_2) of Lemma 3.3 does not hold for some given n. Then there exists a sequence $\{u_k\}\subset Y_n$ such that

$$\|u_k\|\to+\infty \quad \text{as } k\to+\infty \quad \text{and} \quad \varphi_\lambda(u_k)\geq 0. \tag{3.23}$$

Let $w_k = \frac{u_k}{\|u_k\|}$. Then it is obvious that $\|w_k\| = 1$. Since $\dim Y_n < +\infty$, there exists $w\in Y_n\setminus\{0\}$ such that up to a subsequence, $\|w_k - w\|\to 0$ and $w_k(x)\to w(x)$ a.e. $x\in\Omega$ as $k\to+\infty$.

If $w(x)\neq 0$, then $|u_k(x)|\to+\infty$ as $k\to+\infty$. By virtue of (h_3), we get $\lim_{k\to+\infty}\frac{F(x,u_k(x))}{\|u_k\|^{p^+}} = \lim_{k\to+\infty}\frac{F(x,u_k(x))}{|u_k(x)|^{p^+}}|w_k(x)|^{p^+} = +\infty$, for all $x\in\Omega_0 := \{x\in\Omega : w(x)\neq 0\}$. The estimate in Lemma 3.4 implies that

$$\int_{\Omega_0}\frac{F(x,u_k)}{\|u_k\|^{p^+}}dx \to +\infty \text{ as } k\to+\infty.$$

Note that, $\Omega_0\subset\Omega_n(r_0,\infty)$ for large $n\in N$. Therefore, we have

$$\begin{aligned}\varphi_\lambda(u_k) &= \int_\Omega\frac{1}{p(x)}\left(|\nabla u_k|^{p(x)} + \sqrt{1+|\nabla u_k|^{2p(x)}}\right)dx - \lambda\int_\Omega F(x,u_k)dx \\ &= \int_\Omega\frac{1}{p(x)}\left(|\nabla u_k|^{p(x)} + \sqrt{1+|\nabla u_k|^{2p(x)}}\right)dx \\ &\quad -\lambda\int_{\Omega_k(0,r_0)}F(x,u_k)dx - \lambda\int_{\Omega_k(r_0,+\infty)}F(x,u_k)dx \\ &\leq \frac{1}{p^-}\|u_k\|^{p^+} + C\int_{\Omega_k(0,r_0)}(r_0+r_0^r)dx - \int_{\Omega_k(r_0,+\infty)}F(x,u_k)dx \\ &\leq \frac{1}{p^-}\|u_k\|^{p^+} + C(r_0+r_0^r)\mathrm{meas}(\Omega) - \int_{\Omega_k(r_0,+\infty)\cap\Omega_0}F(x,u_k)dx \\ &\leq \|u_k\|^{p^+}\left(\frac{1}{p^-} + \frac{C(r_0+r_0^r)\mathrm{meas}(\Omega)}{\|u_k\|^{p^+}} - \int_{\Omega_k(r_0,+\infty)\cap\Omega_0}\frac{F(x,u_k)}{\|u_k\|^{p^+}}dx\right) \\ &\to -\infty, \text{ as } k\to+\infty,\end{aligned}$$

which is contradiction to (3.23). This gives relation (A_2). Hence, all conditions of Lemma 3.3 are satisfied. Namely, for each $\lambda\in(0,\frac{2r^+}{p^+})$, problem (P) possesses infinitely many nontrivial solutions sequence $\{u_n\}$ such that $\varphi_\lambda(u_n)\to+\infty$ as $n\to+\infty$. □

Proof of Theorem 1.3. Let $X = E$, Y_n and Z_n be defined by (3.4). We know that φ_λ satisfies the $(C)_c$ condition from Lemma 3.6 and $\varphi_\lambda(u) = \varphi_\lambda(-u)$. The rest proof is the same as that of Theorem 1.2. □

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Uniform boundedness and extinction results of solutions to a predator–prey system

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Abstract. Global existence, positivity, uniform boundedness and extinction results of solutions to a system of reaction-diffusion equations on unbounded domain modeling two species on a predator–prey relationship is considered.

Keywords: reaction-diffusion, global existence, positivity, predator–prey, extinction of solutions.

2010 Mathematics Subject Classification: 35A01, 35A02, 35B09, 35B50, 35K57.

1 Introduction

We consider the qualitative theory of the Cauchy problem for a system of reaction-diffusion equations modeling two species interacting with predator–prey relationship. The system in consideration is

$$L_{a,\nu} \equiv u_t - au_{xx} - \nu u_x = -pu + quv \equiv f(u,v), \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$
$$L_{b,\mu} \equiv v_t - bv_{xx} - \mu v_x = +rv - suv \equiv g(u,v), \qquad x \in \mathbb{R},\ t > 0, \tag{1.2}$$

supplemented with the initial conditions

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \qquad x \in \mathbb{R}. \tag{1.3}$$

The functions $u = u(x,t)$ and $v = v(x,t)$ represent the densities of predators and preys in time t and at position x, respectively. The coefficient of diffusion a and b are positive constants which describe the rate of movement of predators and prey respectively. The nonnegative constants p and r are the coefficients of evolution, and the coefficients q and s are related to the increase of the density of predators, and the decrease of the density of preys due to the presence of predators, respectively. The initial conditions u_0 and v_0 are two bounded and uniformly continuous functions on $\mathbb{R}$.

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For a biomathematical discussion of these factors and for a background of the equations, see see [10] and [15].

For the modeling of this system see for example [12], page 53: if we have a lack where there are two species of fish: A, which lives on plants of which there is a plentiful supply, and B (the predator) which subsists by eating A (the prey), where $u = u(x,t)$ represents the population of B and $v = v(x,t)$ represents the population of A.
Further, we suppose the domain is unbounded without boundary and no flux boundary conditions, instead of this we can suppose that the initial species distribution are describing by functions of finite support u_0 and v_0; namely, the initial conditions are of the form

$$\begin{aligned} u(x,0) &= u_0(x) \quad \text{for } -\Delta_u < x < \Delta_u, \quad \text{otherwise } u(x,0) = 0. \\ v(x,0) &= v_0(x) \quad \text{for } -\Delta_v < x < \Delta_v, \quad \text{otherwise } v(x,0) = 0. \end{aligned}$$

where Δ_u and Δ_v give the radius of the initially invaded domain, see [16].

The problem could be treated in the realistic two spatial dimension setting, in order to simplify the mathematics we are to treat it by one dimension space.

When the initial data are continuous, uniformly bounded, and nonnegative, it is shown that (1.1)–(1.2)–(1.3) has a classical positive global solution. Under some conditions on the coefficients or on the initial data, we show that this solution is in fact globally bounded. Moreover, if

- $r = 0$, $p > q\,\|v_0\|$, then v is bounded and $u \to 0$ exponentially as $t \to \infty$.
- $p = 0$, $u_0 \geq k > r/s$ or $u_0^* = \min\left\{u_0^-, u_0^+\right\} > r/s$, where $u_0^\pm = \lim_{x\to\pm\infty} u_0(x)$ then $u(t)$ is bounded and $v \to 0$ exponentially as $t \to \infty$.

On the other hand, we study the behaviour of (u,v) when $x \to \pm\infty$ whenever u_0 and v_0 have limits at $\pm\infty$. We show that $u(\pm\infty,t)$ and $v(\pm\infty,t)$ satisfy an ordinary differential system (ODS) in t. The qualitative behaviour of solutions to (1.1)–(1.2)–(1.3), as $x \to \pm\infty$, can then be obtained from the ODS associated to it [7].

Some systems of predator–prey were studied in bounded domains, see [9, 19] and in the references therein. Also, some results about global existence of solutions for systems of reaction-diffusion systems were established in [4,5,8,13,14].

In the following, u_0 and v_0 will be taken nonnegative and are elements of the Banach space $X = (BUC(\mathbb{R}), \|\cdot\|)$, the space of bounded and uniformly continuous functions on $\mathbb{R}$ endowed with the supremum norm $\|u\| = \sup_{x\in\mathbb{R}} |u(x)|$.

Note here that every continuous function of finite support is a uniformly continuous function on $\mathbb{R}$.

2 Existence, positivity and a priori bounds

We denote by A_1 and A_2 the linear operators $a(\cdot)_{xx} + \nu(\cdot)_x$ and $b(\cdot)_{xx} + \mu(\cdot)_x$, respectively. It is well known that A_j, $j = 1,2$, generates an analytic semigroup of contractions on the Banach space X given explicitly by the expression

$$\left[S_j(t)u\right](x) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{+\infty} \left[\exp\left(-\frac{|x+\sigma t-\xi|^2}{4\alpha t}\right)\right] u(\xi)d\xi, \tag{2.1}$$

where $\alpha = a$ and $\sigma = \nu$ for $j = 1$, and $\alpha = b$ and $\sigma = \mu$ for $j = 2$.

Moreover, for any integer n there is a positive constant $c = c(n)$ such that for any $u \in X$, any positive t we have $D_n S_j(t)u \in X$ and the estimates

$$\left\| D_n S_j(t)u \right\| \leq ct^{-n/2} \left\| u \right\|, \tag{2.2}$$

where $D_n = d^n/(dx)^n$, and $j = 1, 2$, holds true [6].

Our first result provides the existence of a global positive solution.

Theorem 2.1. *Suppose that u_0, $v_0 \in X$, there exists a unique global classical nonnegative solution to the problem* (1.1)–(1.2)–(1.3).

Proof. Local existence and uniqueness follow from standard arguments of abstract parabolic theory or from fixed point arguments involving the heat kernel and the Duhamel principle; whence, there exists a $t_0 > 0$ such that the problem (1.1)–(1.2)–(1.3) has a unique local mild solution $(u, v) \in \mathcal{C}\left([0, t_0]\,; X\right) \times \mathcal{C}\left([0, t_0]\,; X\right)$, i.e.

$$\begin{aligned} u(t) &= S_1(t)u_0 + \int_0^t S_1(t-s) f\left(u(s), v(s)\right) ds, \qquad t \in [0, t_0]\,, \\ v(t) &= S_2(t)v_0 + \int_0^t S_2(t-s) g\left(u(s), v(s)\right) ds, \qquad t \in [0, t_0]\,. \end{aligned}$$

From the Lebesgue theory and the fact the functions $(x, y) \longmapsto f(x, y)$ and $(x, y) \longmapsto g(x, y)$ are of class $\mathcal{C}^\infty(\mathbb{R}^2; \mathbb{R})$ we can conclude that the solution $(u, v) \in \mathcal{C}^\infty\left(]0, T]; X\right) \times \mathcal{C}^\infty\left(]0, T]; X\right)$ for all $0 < T < T_{\max}$, and $(u(t), v(t)) \in \mathcal{C}^\infty\left(\mathbb{R}; \mathbb{R}\right) \times \mathcal{C}^\infty\left(\mathbb{R}; \mathbb{R}\right)$ for all $t \in]0, T]$; where $T_{\max}$ is the maximal time of existence of the solution.

The continuous dependence of the solution on the initial data makes use of the local existence result and the Gronwall lemma.

The nonnegativity of the solution can be proved as follows: let $\lambda_1 = \sup\{\|u(t)\|,\ 0 \leq t \leq T\}$ and $\lambda_2 = \sup\left\{\|v(t)\|,\ 0 \leq t \leq T\right\}$ where $0 < T < T_{\max}$ ($T_{\max}$ is the maximal time of existence of (u, v)), and $\lambda_0 \geq \sup\left\{r + s\lambda_1, p + q\lambda_2\right\}$. The substitutions $u = e^{\lambda_0 t}\varphi$ and $v = e^{\lambda_0 t}\psi$ transform system (1.1)–(1.2)–(1.3) into

$$\begin{aligned} \varphi_t - a\varphi_{xx} - \nu\varphi_x + (p - qv + \lambda_0)\varphi &\equiv 0, \qquad x \in \mathbb{R},\ 0 < t \leq T, \\ \psi_t - b\psi_{xx} - \mu\varphi_x + (-r + su + \lambda_0)\psi &\equiv 0, \qquad x \in \mathbb{R},\ 0 < t \leq T, \end{aligned}$$

with

$$\varphi(x, 0) = e^{-\lambda_0 t}u_0(x) \geq 0 \quad \text{and} \quad \psi(x, 0) = e^{-\lambda_0 t}v_0(x) \geq 0, \qquad x \in \mathbb{R}.$$

As u, $v \in \mathcal{C}\left([0, T]\,; X\right)$ and $p - qv + \lambda_0 \geq 0$ and $-r + su + \lambda_0 \geq 0$ for all $t \in [0, T]$, we can use Theorem 9 on page 43 of the maximum principle in [11] to get that φ and ψ are nonnegative which in turn implies the nonnegativity of u and v.

If one can establish the existence of a priori bounds for the solution components u, v on $[0, T_{\max}[$, standard continuation arguments yield global well posedness.

The solution to (1.1)–(1.2)–(1.3) can be written in integral form as follows

$$u(t) = e^{-pt}S_1(t)u_0 + \int_0^t e^{-p(t-\tau)}S_1(t-\tau)qu(\tau)v(\tau)d\tau, \tag{2.3}$$

$$v(t) = e^{+rt}S_2(t)u_0 - \int_0^t e^{+r(t-\tau)}S_2(t-\tau)su(\tau)v(\tau)d\tau. \tag{2.4}$$

Using the nonnegativity of (u, v) we get

$$\|v(t)\| \leq e^{rt} \|v_0\|, \quad \text{for all } t \geq 0. \tag{2.5}$$

Using (2.3) and (2.5) we obtain

$$\|u(t)\| \leq \|u_0\| + q \|v_0\| \int_0^t e^{r\tau} \|u(\tau)\| \, d\tau, \quad \text{for all } t \geq 0. \tag{2.6}$$

Gronwall's inequality yields

$$\|u(t)\| \leq \|u_0\| e^{q\|v_0\|k(t)}, \quad \text{for all } t \geq 0, \tag{2.7}$$

where

$$k(t) = \begin{cases} \frac{1}{r}\left(e^{rt} - 1\right), & \text{if } r > 0, \\ t, & \text{if } r = 0. \end{cases}$$

Estimates (2.5) and (2.7) imply that the solution is global, i.e., $T_{\max} = +\infty$. □

3 Boundedness and extinction results

The solution to (1.1)–(1.2)–(1.3) established in Theorem 2.1 is not always bounded as is shown in the following proposition.

Proposition 3.1. *Assume $v_0 \not\equiv 0$ (v_0 is not identically null) and r is sufficiently large, then (u, v) is unbounded. More precisely, v grows exponentially as t goes to ∞.*

Proof. Assume the contrary that (u, v) is a globally bounded solution, i.e., $\|u(t)\| \leq C$ and $\|v(t)\| \leq C$ for any $t \geq 0$ and some constant $C > 0$. As $v_0 \neq 0$, there exists a constant $\delta > 0$ such that $S_2(t)v_0 \geq \delta$ for any $t \geq 0$. Furthermore, we use (2.4) to obtain

$$v(t) \geq \left(\delta - sC^2/r\right) e^{rt} + sC^2/r, \quad \text{for all } t \geq 0.$$

Choosing $r > sC^2/\delta$ we clearly have $\|v(t)\| \longrightarrow +\infty$ as t goes to $+\infty$. Whence (u, v) could not be bounded. □

It is now clear that to get bounded solutions we have to impose some restrictions either on the coefficients of the system or on the initial data.

Theorem 3.2. *If u_0, $v_0 \in X$ then we have the estimates*

$$\|v(t)\| \leq \|v_0\| e^{rt}, \quad \textit{for all } t \geq 0, \tag{3.1}$$

$$\|u(t)\| \leq e^{\left(q\|v_0\|e^{rT} - p\right)t} \|u_0\|, \quad \textit{for all } t \in [0, T]. \tag{3.2}$$

Moreover, if $r = 0$ and $p > q \|v_0\|$ we have

$$\lim_{t \to \infty} \|u(t)\| = 0. \tag{3.3}$$

Proof. Setting

$$u = \varphi \exp(-pt), \tag{3.4}$$
$$v = \psi \exp(rt), \tag{3.5}$$

the system (1.1)–(1.2) becomes

$$\varphi_t - a\varphi_{xx} - \nu\varphi_x = qe^{rt}\varphi\psi, \tag{3.6}$$
$$\psi_t - b\psi_{xx} - \mu\psi_x = -se^{-pt}\varphi\psi, \tag{3.7}$$

with the initial data satisfying

$$\varphi_0(x) = u_0(x), \tag{3.8}$$
$$\psi_0(x) = v_0(x). \tag{3.9}$$

As $\varphi \geq 0$ and $\psi \geq 0$, we first have from (3.7) and (3.9)

$$\psi(t) = S_2(t)v_0 - s\int_0^t S_2(t-\tau)e^{-p\tau}\varphi(\tau)\psi(\tau)d\tau \leq S_2(t)v_0 \leq \|v_0\|, \tag{3.10}$$

for all $(x,t) \in \mathbb{R} \times [0,T]$. Whence $v(t) \leq \|v_0\| e^{rt}$, for all $t \geq 0$.
Substituting (3.10) into (3.6) yields

$$\varphi_t - a\varphi_{xx} - \nu\varphi_x \leq q\|v_0\| e^{rt}\varphi. \tag{3.11}$$

If we set $\varphi = e^{Mt}w$, where $M = q\|v_0\| e^{rT}$, then we have over $\mathbb{R} \times [0,T]$

$$w_t - aw_{xx} - \nu w_x \leq 0, \qquad w(x,0) = \varphi_0(x) = u_0(x). \tag{3.12}$$

Furthermore

$$w(t) = S_1(t)u_0 \leq \|u_0\|, \quad \text{for all } t \geq 0. \tag{3.13}$$

Whence $\varphi \leq e^{Mt}\|u_0\|$ and then

$$u(t) \leq e^{Mt}\|u_0\| e^{-pt} = e^{\left(q\|v_0\|e^{rT}-p\right)t}\|u_0\|, \quad \text{for all } t \in [0,T].$$

Thus we obtain (3.2).

We deduce from (3.1)–(3.2) that if $r = 0$ and $p > q\|v_0\|$ we will have

$$\|v(t)\| \leq \|v_0\|, \quad \text{for all } t \geq 0 \quad \text{and} \quad \lim_{t\to\infty}\|u(t)\| = 0. \qquad \square$$

Theorem 3.3. *If $r = 0$, $a \leq b$ and $\nu = \mu$, then the solution to (1.1)–(1.2) is globally bounded. We have the estimates*

$$\|v(t)\| \leq \|v_0\|, \quad \textit{for all } t \geq 0, \tag{3.14}$$

and

$$\|u(t)\| \leq \|u_0\| + \frac{q}{s}\sqrt{b/a}\,\|v_0\|, \quad \textit{for all } t \geq 0. \tag{3.15}$$

Proof. Let Y and Z be the solutions to

$$Y_t - aY_{xx} - \nu Y_x + pY = uv, \qquad Y(x,0) = 0, \tag{3.16}$$

and

$$Z_t - bZ_{xx} - \mu Z_x = uv, \qquad Z(x,0) = 0, \tag{3.17}$$

respectively, where (u,v) is the solution to (1.1)–(1.2)–(1.3) with $r = 0$, $a \leq b$ and $\mu = \nu$. Then (u,v) can be written in terms of (Y,Z) as follows

$$u(x,t) = e^{-pt}S_1(t)u_0(x) + qY(x,t), \qquad t \geq 0, \tag{3.18}$$

$$v(x,t) = S_2(t)v_0(x) - sZ(x,t), \qquad t \geq 0. \tag{3.19}$$

Using the positivity of $Z(x,t)$ we deduce (3.14) from (3.19). By the explicit formulas of Y and Z:

$$Y(t) = \int_0^t e^{-p(t-\tau)}S_1(t-\tau)u(\tau)v(\tau)d\tau \leq \int_0^t S_1(t-\tau)u(\tau)v(\tau)d\tau, \tag{3.20}$$

$$Z(t) = \int_0^t S_2(t-\tau)u(\tau)v(\tau)d\tau. \tag{3.21}$$

As $a \leq b$, $\nu = \mu$ and (2.1), it is easy (see [1]) to deduce that

$$\sqrt{a}S_1(t)w \leq \sqrt{b}S_2(t)w, \quad \text{for all } w \in X$$

and then

$$S_1(t)w \leq \sqrt{\frac{b}{a}}S_2(t)w, \quad \text{for all } t \geq 0. \tag{3.22}$$

From (3.20)–(3.22) we obtain

$$Y(t) \leq \sqrt{\frac{b}{a}}\int_0^t S_2(t-\tau)u(\tau)v(\tau)d\tau = \sqrt{\frac{b}{a}}Z(t), \quad \text{for all } t \geq 0. \tag{3.23}$$

As v is nonnegative, from (3.19) we get

$$Z(x,t) \leq \frac{1}{s}S_2(t)v_0, \quad \text{for all } t \geq 0. \tag{3.24}$$

Using (3.24) in (3.23) we get

$$Y(t) \leq \frac{1}{s}\sqrt{\frac{b}{a}}S_2(t)v_0, \quad \text{for all } t \geq 0. \tag{3.25}$$

Finally, from (3.25) in (3.18) we get (3.15). □

Theorem 3.4. *Assume $p = 0$ and $u_0 \geq r/s$ for all $x \in \mathbb{R}$. Then we have*

$$\|v(t)\| \leq \|v_0\|, \quad \textit{for all } t \geq 0. \tag{3.26}$$

Moreover, if there is a constant $k > r/s$ such that $u_0 > k$ for all $x \in \mathbb{R}$, then

$$\|u(t)\| \leq \left(1 + \frac{q}{ks - r}\|u_0\|\right)\|v_0\|, \quad \textit{for all } t \geq 0, \tag{3.27}$$

and

$$\|v(t)\| \leq e^{-(ks-r)t}\|v_0\|, \quad \textit{for all } t \geq 0. \tag{3.28}$$

In particular, $v \longrightarrow 0$ uniformly in $x \in \mathbb{R}$ as $t \longrightarrow \infty$.

Proof. For $p = 0$ and $u_0 \geq r/s$, from (2.1) we get

$$u(t) \geq r/s, \quad \text{for all } t \geq 0. \tag{3.29}$$

Setting $B(t) = r - su(t)$, we have

$$v_t = [A_2 + B(t)]\, v(t). \tag{3.30}$$

As the linear operator $B(t)$ is dissipative on X [18], $A_2 + B(t)$ generates for each t fixed a semigroup of contractions. Whence $A_2 + B(t)$ generates on X a system of evolution $P(t, \tau)$ of contractions [18]. Whence the solution to (3.17)–(1.3) is

$$v(t) = P(t, 0)v_0, \quad \text{for all } t \geq 0. \tag{3.31}$$

This implies (3.26).

If $u_0 \geq k > r/s$, then from (1.1) we get $u(t) \geq k$, and consequently $r - su(t) \leq r - ks < 0$ for any $t \geq 0$. Setting $\omega := ks - r$ ($\omega > 0$), equation (1.2) can be written in the form

$$v(t) = [A_2 + B(t) + \omega I]\, v(t) - \omega v(t). \tag{3.32}$$

The dissipative operator $B(t) + \omega I$ generates on X a system of evolution $G(t, \tau)$ of contractions. Consequently, $A_2 + B(t)$ generates a system of evolution $U(t, \tau)$ given by

$$U(t, \tau) = e^{-\omega(t-\tau)} G(t, \tau).$$

Hence the solution $v(t)$ of (3.32)–(1.3) can be written in the form

$$v(t) = U(t, 0)v_0 = e^{-\omega t} G(t, 0)v_0, \quad \text{for all } t \geq 0. \tag{3.33}$$

This implies estimate (3.13). Using (1.1), (3.33) and Gronwall's lemma we get (3.15). □

In what follows, we denote by $C_\pm$ the closed subspaces of X defined as follows

$$C_\pm := \left\{ u \in X \text{ such that} : \lim_{x \to \pm\infty} u(x) \text{ exists} \right\}.$$

Lemma 3.5. *Let $f \in C_\pm$ be such that $f^+, f^- > 0$. Then for any $\varepsilon > 0$ there exists $t^* > 0$ such that*

$$\left[S_j(t) f \right](x) \geq f^* - \varepsilon, \quad \textit{for all } x \in \mathbb{R},$$

where $f^* := \min\left(f^+, f^-\right)$.

Proof. The proof is similar to that of [4, Lemma 5.3]. □

In what follows we denote $u_0^\pm = \lim_{x \to \pm\infty} u_0(x)$ and $u_0^* = \min\left\{u_0^-, u_0^+\right\}$.

Theorem 3.6. *Assume $p = 0$ and $u_0 \in C_\pm$. If $u_0^* > r/s$, then there exists $t^* > 0$ and three positive constants C_1, C_2 and ω^* such that*

$$\|v(t)\| \leq C_1 e^{-\omega^*(t-t^*)}, \quad \textit{for all } t \geq t^*, \tag{3.34}$$

$$\|u(t)\| \leq C_2, \quad \textit{for all } t \geq t^*. \tag{3.35}$$

Proof. Choose $\varepsilon > 0$ such that $u_0^* - \varepsilon > r/s$, then by Lemma 3.5, there exists $t^* > 0$ such that $[S_1(t)u_0](x) \geq u_0^* - \varepsilon$, for any $x \in \mathbb{R}$. We then have $u(t) \geq u^* - \varepsilon$, for any $t \geq t^*$. Using Theorem 3.4 with initial data $(u(t^*), v(t^*))$ and $k = u_0^* - \varepsilon$, $\omega^* = ks - r$, we then have

$$\|v(t)\| \leq \|v(t^*)\| \, e^{-\omega^*(t-t^*)}, \quad \text{for all } t \geq t^*.$$

We get (3.34) by setting $C_1 = \|v(t^*)\|$.

Now, combining (2.3) and (3.34) we infer

$$\|u(t)\| \leq \|u_0\| + qC_1 e^{\omega^* t^*} \int_0^t e^{-\omega^* \tau} \|u(\tau)\| \, d\tau, \quad \text{for all } t \geq t^*.$$

The Gronwall inequality yields

$$\|u(t)\| \leq \|u_0\| \, e^{\frac{qC_1}{\omega^*} e^{\omega^* t^*}} = C_2, \text{ for all } t \geq t^*.$$

Whence (3.35). □

4 Stability of the solution

Definition 4.1. We say that the solution to the problem (1.1)–(1.2)–(1.3) is unconditionally stable on $\mathbb{R}_+$, if for all $T > 0$ and all $\varepsilon > 0$, there exist $\delta = \delta(T, \varepsilon) > 0$ such that for all solution $(\overline{u}, \overline{v})$ with initial condition $(\overline{u}_0, \overline{v}_0)$ to the same problem satisfying $\|\overline{u}_0 - u_0\| < \delta$ and $\|\overline{v}_0 - v_0\| < \delta$ we have $\|\overline{u}(t) - u(t)\| < \varepsilon$ and $\|\overline{v}(t) - v(t)\| < \varepsilon$ for all $t \in [0, T]$.

Proposition 4.2. *The solution of the problem* (1.1)–(1.2) *is unconditionally stable on* $\mathbb{R}_+$.

Proof. From the integral writin of the solution (u, v) and $(\overline{u}, \overline{v})$ we get

$$\|\overline{u}(t) - u(t)\| \leq \|\overline{u}_0 - u_0\| + \int_0^t \{p \, \|\overline{u}(\tau) - u(\tau)\| + q \, \|u(\tau)v(\tau) - \overline{u}(\tau)\overline{v}(t)\|\} \, d\tau, \tag{4.1}$$

$$\|\overline{v}(t) - v(t)\| \leq \|\overline{v}_0 - v_0\| + \int_0^t \{r \, \|\overline{v}(\tau) - v(\tau)\| + s \, \|u(\tau)v(\tau) - \overline{u}(\tau)\overline{v}(t)\|\} \, d\tau. \tag{4.2}$$

Setting $\Phi = (u, v)$, $\overline{\Phi} = (\overline{u}, \overline{v})$, $\Phi_0 = (u_0, v_0)$, $\overline{\Phi}_0 = (\overline{u}_0, \overline{v}_0)$ and define $\|\Phi(t)\| = \|(u(t), v(t))\| = \|u(t)\| + \|v(t)\|$; then from (4.1)–(4.2) we get

$$\begin{aligned} \left\|\overline{\Phi}(t) - \Phi(t)\right\| \leq \left\|\overline{\Phi}_0 - \Phi_0\right\| &+ (p+r) \int_0^t \|\overline{u}(\tau) - u(\tau)\| \, d\tau \\ &+ (q+s) \int_0^t \|\overline{u}(\tau)\overline{v}(\tau) - u(\tau)v(t)\| \, d\tau. \end{aligned} \tag{4.3}$$

Let $\varepsilon > 0$ and $T > 0$. As u, v, $\overline{u}$, $\overline{v} \in C(\mathbb{R}^+; X)$; then, they are bounded over $[0, T]$. Define

$$\|u\|_\infty = \sup_{t \in [0,T]} \|u(t)\|, \quad \text{for all } u \in \mathcal{C}\left(\mathbb{R}^+; X\right), \tag{4.4}$$

then we have

$$\|\overline{u}\overline{v} - uv\|_\infty \leq M \left\|\overline{\Phi}(t) - \Phi(t)\right\|, \quad \text{for all } t \in [0, T], \tag{4.5}$$

where $M = \|u\|_\infty + \|v\|_\infty$.

From (4.3) and (4.5) we get

$$\left\|\overline{\Phi}(t)-\Phi(t)\right\| \leq \left\|\overline{\Phi}_0-\Phi_0\right\| + [p+r+M(q+s)] \int_0^t \left\|\overline{\Phi}(\tau)-\Phi(\tau)\right\| d\tau. \tag{4.6}$$

Using Gronwall inequality we obtain

$$\left\|\overline{\Phi}(t)-\Phi(t)\right\| \leq \left\|\overline{\Phi}_0-\Phi_0\right\| e^{[p+r+M(q+s)]t}, \quad \text{for all } t\in[0,T]. \tag{4.7}$$

The estimate (4.6) gives the stability of the solution to the problem (1.1)–(1.2)–(1.3). □

5 Remarks

Remark 5.1. In turns out that if u_0, $v_0 \in \mathcal{C}_+$ then the diffusive system for x large will behave like the system of ordinary differential equations associated to it, and hence, for x large can be replaced by the latter which is simpler to analyze [7]

$$\frac{dU(t)}{dt} = -pU(t)+qU(t)V(t), \quad \text{for all } t>0,$$
$$\frac{dV(t)}{dt} = +rU(t)-sU(t)V(t), \quad \text{for all } t>0,$$

satisfying the initial data

$$U(0)=\lim_{x\to+\infty} u_0(x), \qquad V(0)=\lim_{x\to+\infty} v_0(x),$$

where

$$U(t)=\lim_{x\to+\infty} u(x,t), \qquad V(t)=\lim_{x\to+\infty} u(x,t)$$

This result is based on the fact that if $h\in\mathcal{C}_+$ with $h^+=\lim_{x\to+\infty} h(x)$, then $\lim_{x\to+\infty}\left[S_j(t)h\right](x)=h^+$, for $j=1,2$.

The same thing holds if u_0, $v_0\in\mathcal{C}_-$.

Remark 5.2. The same analysis can also be done for $x\in[0,+\infty[$. In this case, the explicit formula associated to (1.1)–(1.2)–(1.3)

$$u(t)=e^{-pt}S_1(t)u_0+\int_0^t e^{-p(t-\tau)}S_1(t-\tau)f\left(u(\tau),v(\tau)\right)d\tau,$$

$$v(t)=e^{+rt}S_2(t)u_0+\int_0^t e^{+r(t-\tau)}S_2(t-\tau)g\left(u(\tau),v(\tau)\right)d\tau,$$

will be

$$u(x,t)=\int_0^\infty N_1(x,\xi,t)u_0(\xi)d\xi+\int_0^t \frac{x}{t-\tau}K_1(x,t-\tau)u_1(\tau)d\tau$$
$$+\int_0^t\int_0^{+\infty} N_1(x,\xi,t-\tau)f(u,v)(\xi,\tau)d\xi d\tau,$$

and

$$v(x,t)=\int_0^\infty N_2(x,\xi,t)v_0(\xi)d\xi++\int_0^t \frac{x}{t-\tau}K_2(x,t-\tau)v_1(\tau)d\tau$$
$$+\int_0^t\int_0^{+\infty} N_2(x,\xi,t-\tau)g(u,v)(\xi,\tau)d\xi d\tau,$$

where

$$N_1(x,\xi,t) = K_1(x-\xi,t) - K_1(x+\xi,t),\ K_1(x,t) = \frac{1}{\sqrt{4\pi a t}} \exp\left(-\frac{|x+\nu t|^2}{4at}\right),$$
$$N_2(x,\xi,t) = K_2(x-\xi,t) - K_2(x+\xi,t),\ K_2(x,t) = \frac{1}{\sqrt{4\pi b t}} \exp\left(-\frac{|x+\mu t|^2}{4bt}\right),$$

and

$$u_1(t) = u(0,t), \qquad v_1(t) = v(0,t),$$

with u_1, v_1 bounded. These expressions can be deduced from [17, Chapter 3, Section 3].

It will be interesting to perform the same analysis for the case $x \in [0,+\infty[$ with other boundary conditions.

Remark 5.3. For $x \in \mathbb{R}^n$ $(n \geq 2)$ and replacing au_{xx} and bv_{xx} in (1.1)–(1.2) by the second order uniform elliptic operators

$$L_1 u = \sum_{i,j=1}^{n} \left(a_{ij}(x) u_{x_j}\right) u_{x_i}, \qquad L_2 u = \sum_{i,j=1}^{n} \left(b_{ij}(x) v_{x_j}\right) v_{x_i},$$

the problem deserves to be studied in appropriate functional spaces using the results in Aronson [2] and [3].

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Solitary wave of ground state type for a nonlinear Klein–Gordon equation coupled with Born–Infeld theory in $\mathbb{R}^2$

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Abstract. In this paper we prove the existence of nontrivial ground state solution for a nonlinear Klein–Gordon equation coupled with Born–Infeld theory in $\mathbb{R}^2$ involving unbounded or decaying radial potentials. The approach involves variational methods combined with a Trudinger–Moser type inequality and a symmetric criticality type result.

Keywords: Klein–Gordon equation, Born–Infeld theory, Trudinger–Moser inequality, unbounded or decaying radial potentials, critical exponential growth, Mountain-Pass Theorem.

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1 Introduction and main results

This paper was motivated by some works that had appeared in recent years concerning the following Klein–Gordon equation with Born–Infeld theory on $\mathbb{R}^3$:

$$\begin{cases} -\Delta u + [m^2 - (\omega+\phi)^2]u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\Delta_4\phi = \operatorname{div}(|\nabla\phi|^2\nabla\phi)$. Such a system deduced by coupling the Klein–Gordon equation

$$\psi_{tt} - \Delta\psi + m^2\psi - |\psi|^{p-2}\psi = 0$$

with the Born–Infeld theory

$$\mathcal{L}_{\mathrm{BI}} = \frac{b^2}{4\pi}\left(1 - \sqrt{1 - \frac{1}{b^2}\left(|\mathbb{E}|^2 - |\mathbb{B}|^2\right)}\right),$$

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where $\psi = \psi(x,t) \in \mathbb{C}$ ($x \in \mathbb{R}^3$, $t \in \mathbb{R}$), m is a real constant and $2 < p < 6$, $\mathbb{E}$ is the electric field and $\mathbb{B}$ is the magnetic induction field. For more details on the physical aspects of the problem we refer the readers to see [13] and the references therein.

A few existence results for the system (1.1) have been proved via modern variational methods under various hypotheses on the nonlinear term. We recall some of them as follows. d'Avenia and Pisani [13] was pioneered work with this system. They found the existence of infinitely many radially symmetric solutions for system (1.1) by using $\mathbb{Z}_2$-Mountain Pass Theorem, when $4 < p < 6$ and $|\omega| < |m|$. Later, in [21] the range $p \in (2,4]$ was also covered provided $\sqrt{(\frac{p}{2}-1)}|m| > \omega > 0$. Replacing $|u|^{p-2}u$ by $|u|^{p-2}u + |u|^4u$ in problem (1.1), Teng and Zhang in [26] get that problem

$$\begin{cases} -\Delta u + [m^2 - (\omega+\phi)^2]u = |u|^{p-2}u + |u|^4u, & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

has at least a nontrivial solution by using Mountain Pass Theorem, when $4 < p < 6$ and $\omega < m$. Subsequently, replacing $|u|^{p-2}u$ by $|u|^{p-2}u + h(x)$ in problem (1.1), Chen and Li in [9] get the existence of two nontrivial solutions for nonhomogeneous problem

$$\begin{cases} -\Delta u + [m^2 - (\omega+\phi)^2]u = |u|^{p-2}u + h(x), & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

by using the Ekeland variational principle and the Mountain Pass Theorem, when $|m| > \omega > 0$ and $4 < p < 6$ or $\sqrt{(\frac{p}{2}-1)}|m| > \omega > 0$ and $2 < p \le 4$. Other related results about Klein–Gordon equation coupled with Born–Infeld theory on $\mathbb{R}^3$ can be found in [28] and [29]. By the way, we should point out that if $\beta = 0$ then problem (1.1) becomes

$$\begin{cases} -\Delta u + [m^2 - (\omega+\phi)^2]u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi = 4\pi(\omega+\phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

for the well-known Klein–Gordon–Maxwell equations. Such problems have been intensively studied in recent years as for example in [6–8, 10–12, 14, 18, 19, 22].

In this paper we consider the following Klein–Gordon equation coupled with Born–Infeld theory:

$$\begin{cases} -\Delta u + \left[m^2 - (\omega+\phi)^2\right] V(|x|)u = K(|x|)f(u), & x \in \mathbb{R}^2, \\ \qquad\qquad \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)V(|x|)u^2, & x \in \mathbb{R}^2, \end{cases} \tag{1.2}$$

where ω is a positive frequency parameter, β depends on the so-called Born–Infeld parameter, m is a real constant, $\phi : \mathbb{R}^2 \to \mathbb{R}$ and V, $K : \mathbb{R}^2 \to \mathbb{R}$ are radial potentials which may be unbounded, singular at the origin or vanishing at infinity and the nonlinear term $f(s)$ is allowed to enjoy an critical exponential growth in the sense of the classical Trudinger–Moser inequality which will be stated later.

The bi-dimensional case is very special and quite delicate, because as we know for domains $\Omega \subset \mathbb{R}^2$ with finite volume, the Sobolev embedding theorem assures that $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in [1, +\infty)$, but, due to a function with a local singularity and this causes the failure of the embedding that $H_0^1(\Omega) \not\hookrightarrow L^\infty(\Omega)$. Therefore, and in order to overcome this trouble, the Trudinger–Moser inequality was established independently by Yudovič [17], Pohožaev [23] and Trudinger [27], came as a substitute of the Sobolev inequality. It asserts that the existence of a constant $\alpha > 0$ such that $H_0^1(\Omega) \hookrightarrow L_\phi(\Omega)$, where $L_\phi(\Omega)$ is the Orlicz space determined by the Young function $\phi(t) = e^{\alpha t^2} - 1$. Later, Moser in [20] sharpened this result by finding the best constant α in the embedding above. More precisely, he proved that for any $\alpha \leq 4\pi$, there exists a constant $c_0 > 0$ such that

$$\sup_{\|\nabla u\|_{L^2(\Omega)} \leq 1} \frac{1}{|\Omega|} \int_\Omega e^{\alpha u^2} \, \mathrm{d}x \leq c_0. \tag{1.3}$$

Moreover, the constant 4π is sharp in the sense that if $\alpha > 4\pi$, then the supremum above will become infinity.

Throughout this work, the potentials $V,\ K : \mathbb{R}^2 \to \mathbb{R}$ are positive, radial and continuous functions assuming the following behaviors at the origin and infinity:

(V) There exist real numbers a_0 and a_∞ with $a_0, a_\infty > -2$ such that

$$\liminf_{r \to 0^+} \frac{V(r)}{r^{a_0}} > 0 \quad \text{and} \quad \liminf_{r \to +\infty} \frac{V(r)}{r^{a_\infty}} > 0;$$

(K) there exist real numbers b_0 and b_∞ with $b_\infty < a_\infty$, $b_0 > -2$ such that

$$\limsup_{r \to 0^+} \frac{K(r)}{r^{b_0}} < \infty \quad \text{and} \quad \limsup_{r \to +\infty} \frac{K(r)}{r^{b_\infty}} < \infty.$$

Hereafter, we say that $(V, K) \in \mathcal{K}$ if the assumptions (V) and (K) hold.

As we mentioned initially and motivated by the aforementioned works, we consider system (1.2) involving unbounded, singular at the origin or decaying to zero at infinity radial potentials. Recently, much attention has been paid to the Schrödinger equations with potentials with these kinds of behaviors. For example, we can cite [2, 24]. In [24], the authors studied the existence and multiplicity of solutions for the problem

$$\begin{cases} -\Delta u + V(|x|)u = K(|x|)f(u), & x \in \mathbb{R}^N \\ |u(x)| \to 0 \quad \text{as } |x| \to \infty, \end{cases}$$

where the nonlinearity considered was $f(s) = |s|^{p-2}s$, with $2 < p < 2^* = \frac{2N}{N-2}$ for $N \geq 3$ is the limiting exponent in the Sobolev embedding and $2 < p < \infty$ if $N = 2$. Succeeding this study, Albuquerque et al. in [2] studied the above problem in the critical case suggested by the so-entitled Trudinger–Moser inequality (1.3). To our best knowledge, there are no literature addressing the system (1.2) where the potentials V and K have these features and the nonlinearity f has exponential critical growth in two dimensions. Hence, our results are new and complement the above results to some extent.

In order to state our results, we need to introduce some notations. If $1 \leq p < \infty$ we define the weighted Lebesgue spaces

$$L^p(\mathbb{R}^2; K) := \left\{ u : \mathbb{R}^2 \to \mathbb{R} : u \text{ is measurable and } \int_{\mathbb{R}^2} K(|x|)|u|^p \, \mathrm{d}x < \infty \right\},$$

equipped with the norm

$$\|u\|_{p;K} = \left(\int_{\mathbb{R}^2} K(|x|)|u|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

Similarly, we can define $L^p(\mathbb{R}^2; V)$ with its correspondent norm

$$\|u\|_{p;V} = \left(\int_{\mathbb{R}^2} V(|x|)|u|^p \,\mathrm{d}x \right)^{\frac{1}{p}}.$$

We also define the Hilbert space

$$Y := \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2} V(|x|)u^2 \,\mathrm{d}x < \infty \right\}$$

endowed with the norm $\|u\| := \sqrt{\langle u, u \rangle}$ induced by the scalar product

$$\langle u, v \rangle := \int_{\mathbb{R}^2} [\nabla u \nabla v + V(|x|)uv] \,\mathrm{d}x. \tag{1.4}$$

Let $C_0^\infty(\mathbb{R}^2)$ be the set of smooth functions with compact support. Equivalently, the functional space Y can be regarded as the completion of $C_0^\infty(\mathbb{R}^2)$ under the norm $\|\cdot\|$. Furthermore, the subspace

$$E := Y_{\mathrm{rad}} = \{u \in Y : u \text{ is radial}\},$$

which is closed in Y, and thus it is a Hilbert space itself. Also, denote by $\mathcal{D}$ the completion of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|\phi\|_{\mathcal{D}} := \left(\int_{\mathbb{R}^2} |\nabla \phi|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^2} |\nabla \phi|^4 \,\mathrm{d}x \right)^{\frac{1}{4}}.$$

Remark 1.1. Under the behavior of V at infinity in the hypothesis (V) we can show that $\|\cdot\|$ defined above is a norm in Y. In fact, we only need to show that if $\|u\| = 0$, then $u \equiv 0$. If $\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x = 0$, u is a constant, but since $\liminf_{|x|\to\infty} |x|^{-a_\infty} V(|x|) > 0$ we should have $u = 0$.

Here, we are interested in the case where the nonlinearity $f(s)$ has maximal growth on s which allows us to treat the problem (1.2) variationally. It is assumed that $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(0) = 0$ and f behaves like $e^{\alpha s^2}$ as $s \to \infty$.

In order to perform the minimax approach to the problem (1.2), we also need to make some suitable assumptions on the behavior of $f(s)$. More precisely, we shall assume the following growth conditions:

(f_0) (small order at the origin) $\lim_{s\to 0^+} \frac{f(s)}{s} = 0$;

(f_1) (critical exponential growth) there exists $\alpha_0 > 0$ such that

$$\lim_{s\to\infty} \frac{|f(s)|}{e^{\alpha s^2}} = 0, \quad \text{for any } \alpha > \alpha_0, \qquad \lim_{s\to\infty} \frac{|f(s)|}{e^{\alpha s^2}} = +\infty, \quad \text{for any } \alpha < \alpha_0;$$

(f_2) (Ambrosetti–Rabinowitz type condition) there exists $\theta > 2(\omega^2 + 1) > 2$ such that

$$0 \le \theta F(s) := \theta \int_0^s f(t) \,\mathrm{d}t \le sf(s), \qquad \forall s \in \mathbb{R};$$

(f_3) there exist $\vartheta > 2$ and $\mu > 0$ such that

$$F(s) \ge \frac{\mu}{\vartheta}|s|^\vartheta, \qquad \forall s \in \mathbb{R}.$$

In this work, we say that the pair (u, ϕ) is a weak solution of (1.2) if $(u, \phi) \in Y \times \mathcal{D}$ and it holds the equalities

$$\int_{\mathbb{R}^2} \left(\nabla u \cdot \nabla v + [m^2 - (\omega + \phi)^2] V(|x|) uv\right) \mathrm{d}x = \int_{\mathbb{R}^2} K(|x|) f(u) v \,\mathrm{d}x \tag{1.5}$$

and

$$-\int_{\mathbb{R}^2} \left(\frac{1}{4\pi}\left((1 + \beta|\nabla\phi|^2)\nabla\phi \cdot \nabla\eta\right) + V(|x|)(\phi + \omega)u^2\eta\right) \mathrm{d}x = 0, \tag{1.6}$$

for all $v \in Y$ and $\eta \in \mathcal{D}$. We point out that from (f_0) the identically zero function is the trivial solution of (1.2). We say that a pair (u, ϕ) is called a *ground state solution* of system (1.2) if (u, ϕ) is a weak solution of (1.2) which has the least energy among all nontrivial weak solutions of system (1.2).

The main results we provide in this paper is announced below.

Theorem 1.2. *Suppose that $(V, K) \in \mathcal{K}$ and (f_0)–(f_3) are satisfied. If $|m| > \omega > 0$, then there exists $\mu_0 > 0$ such that system (1.2) has a nontrivial solution (u_0, ϕ), for all $\mu > \mu_0$, with u_0 nonnegative.*

Theorem 1.3. *Under the conditions of Theorem 1.2 and supposing that $s \mapsto \frac{f(s)}{s}$ is increasing for $s > 0$, then the solution obtained in Theorem 1.2 is a ground state.*

Remark 1.4. Our interest in ground states solutions is justified by the fact that they in general exhibit some type of stability and, from a physical point of view, the stability of a standing wave is a crucial point to establish the existence of stand waves solutions.

Remark 1.5. Our existence result complements the study [4, 10] in the sense that we study a class of systems with critical exponential growth and involving unbounded, singular or decaying radial potentials.

We observe that the hypotheses (f_0)–(f_3) have been used in many papers to find solutions using the classical Mountain-Pass Theorem introduced by Ambrosetti and Rabinowitz in the celebrated paper [5], see for instance [15, 16] and references therein. It is worth pointing out that when we deal with critical nonlinearities like the exponential at infinity and in the whole space, the problem becomes much more complicated due to the possible lack of compactness. There is other considerable difficulty in dealing with systems like (1.2), which we will treat throughout the text, due to a not very good variational structure since the indefiniteness of the action associated to this set of equations.

The rest of the paper is arranged as follows. In Section 2, we introduce some auxiliary embedding results. In Section 3, we establish a variational setting of our problem. Finally, Section 4 is devoted to the proof of the main results.

2 Some useful auxiliary embedding results

To prove Theorem 1.2 and for the reader's convenience, we need review some embedding lemmas and a Trudinger–Moser type inequality built in [3] (see also [2]) where one can refer to the proofs of these results and related comments.

In the following, B_r denotes the open ball in $\mathbb{R}^2$ centered at the origin with radius r and $B_R \setminus B_r$ denotes the annulus with interior radius r and exterior radius R. Throughout the paper, we use C or C_i $(i = 0, 1, 2, \ldots)$ to denote (possibly different) positive constants.

Lemma 2.1 ([2, Lemma 2.1]). *Suppose that* (V) *holds. Then there exist $C > 0$ and $R > 1$ such that, for all $u \in E$, we have*

$$|u(x)| \leq C\|u\||x|^{-\frac{a_\infty+2}{4}}, \quad \text{for } |x| \geq R.$$

For any open set $A \subset \mathbb{R}^2$ we define $W^{1,2}_{\text{rad}}(A;V) = \{u_{|_A} : u \in E\}$.

Lemma 2.2 ([25, Lemma 3]). *Assume that $(V,K) \in \mathcal{K}$. For any fixed $0 < r < R < \infty$, the embeddings*

$$W^{1,2}_{\text{rad}}(B_R \setminus B_r; V) \hookrightarrow L^p(B_R \setminus B_r; K), \qquad 1 \leq p \leq \infty,$$

are compact.

Remark 2.3. For $R \gg 1$, the embedding

$$W^{1,2}_{\text{rad}}(B_R; V) \hookrightarrow W^{1,2}(B_R)$$

is continuous. That last result can be obtained by proceeding exactly as in [24, Lemma 4].

Using the above lemmas, the authors in [3] (see also [2]) have obtained the following crucial embedding result.

Lemma 2.4 ([3, Lemma 2.4]). *Assume that $(V,K) \in \mathcal{K}$. Then the embeddings $E \hookrightarrow L^q(\mathbb{R}^2; K)$ are compact for all $2 \leq q < \infty$.*

With the aid of classical Trudinger–Moser inequality (1.3) and that one involving singular weights obtained by Adimurthi and K. Sandeep in [1, Theorem 2.1] (this used in 2-D), by using Lemmas 2.1 and 2.4, the authors in [3] established the following Trudinger–Moser inequality in the functional space E.

Theorem 2.5 ([3, Theorem 1.3]). *Assume that $(V,K) \in \mathcal{K}$. Then, for any $u \in E$ and $\alpha > 0$, we have that $\left(e^{\alpha u^2} - 1\right) \in L^1(\mathbb{R}^2; K)$. Moreover, if $\alpha < \lambda := \min\{4\pi, 4\pi(1 + \frac{b_0}{2})\}$, there holds*

$$\sup_{u \in E:\, \|u\| \leq 1} \int_{\mathbb{R}^2} K(|x|) \left(e^{\alpha u^2} - 1\right) \mathrm{d}x < \infty. \tag{2.1}$$

An immediate consequence of Theorem 2.5 is the following:

Corollary 2.6. *Under the assumptions of Theorem 2.5, if $u \in E$ is such that $\|u\| \leq M < \sqrt{\frac{\lambda}{\alpha}}$, then there exists a constant $C = C(M, \alpha) > 0$ independent of u such that*

$$\int_{\mathbb{R}^2} K(|x|) \left(e^{\alpha u^2} - 1\right) \mathrm{d}x \leq C.$$

3 Variational formulation

Since we are interested in solutions (u, ϕ) such that u is nontrivial nonnegative, it is convenient to define $f(s) = 0$ for all $s \leq 0$. Let $\alpha > \alpha_0$ and $q \geq 2$. From (f_0) and (f_1), for any given $\varepsilon > 0$, there exists $b_1 > 0$ such that

$$|F(s)| \leq \frac{\varepsilon}{2} s^2 + b_1 |s|^q \left(e^{\alpha s^2} - 1\right), \qquad \forall s \in \mathbb{R}. \tag{3.1}$$

Given $u \in E$, by (3.1) it yields

$$\int_{\mathbb{R}^2} K(|x|) F(u)\, \mathrm{d}x \leq \frac{\varepsilon}{2} \int_{\mathbb{R}^2} K(|x|) u^2\, \mathrm{d}x + b_1 \int_{\mathbb{R}^2} K(|x|) |u|^q \left(e^{\alpha u^2} - 1\right) \mathrm{d}x.$$

From Lemma 2.4, the first integral in right-hand side is finite. Now, let $r_1, r_2 > 1$ be such that $\frac{1}{r_1} + \frac{1}{r_2} = 1$. Hölder's inequality, Lemma 2.4 and (2.1) imply that

$$\int_{\mathbb{R}^2} K(|x|)|u|^q \left(e^{\alpha u^2} - 1\right) \mathrm{d}x \le \left(\int_{\mathbb{R}^2} K(|x|)|u|^{qr_1} \mathrm{d}x\right)^{\frac{1}{r_1}} \left(\int_{\mathbb{R}^2} K(|x|) \left(e^{\alpha r_2 u^2} - 1\right) \mathrm{d}x\right)^{\frac{1}{r_2}},$$

which is finite, where we have used the elementary inequality

$$(e^s - 1)^r \le e^{rs} - 1, \tag{3.2}$$

for all $r \ge 1, s \ge 0$. Thereby, the energy functional $J : E \times \mathcal{D} \to \mathbb{R}$ associated to system (1.2) and given by

$$\begin{aligned} J(u,\phi) := &\frac{1}{2} \int_{\mathbb{R}^2} \left(|\nabla u|^2 + \left[m^2 - (\omega + \phi)^2\right] V(|x|)u^2\right) \mathrm{d}x \\ &- \frac{1}{8\pi} \int_{\mathbb{R}^2} |\nabla \phi|^2 \,\mathrm{d}x - \frac{\beta}{16\pi} \int_{\mathbb{R}^2} |\nabla \phi|^4 \,\mathrm{d}x - \int_{\mathbb{R}^2} K(|x|)F(u)\,\mathrm{d}x \end{aligned}$$

is well-defined. Using standard arguments, one can easily show that $J \in C^1(E \times \mathcal{D}, \mathbb{R})$ and with the partial derivatives given by

$$J_u(u,\phi)v = \int_{\mathbb{R}^2} \left(\nabla u \cdot \nabla v + [m^2 - (\omega+\phi)^2] V(|x|)uv - K(|x|)f(u)v\right) \mathrm{d}x$$

and

$$J_\phi(u,\phi)\eta = -\int_{\mathbb{R}^2} \left(\frac{1}{4\pi}\left((1 + \beta|\nabla\phi|^2)\nabla\phi \cdot \nabla\eta\right) + V(|x|)(\phi+\omega)u^2\eta\right) \mathrm{d}x,$$

for $v \in E$ and $\eta \in \mathcal{D}$. Consequently, the critical points $(u,\phi) \in E \times \mathcal{D}$ of J satisfy (1.5) and (1.6) for all $v \in E$ and $\eta \in \mathcal{D}$.

The functional J has got a strong indefiniteness (unbounded both from below and from above on infinite dimensional subspace). For this reason the usual tools of the critical point theory cannot be used in a direct way. So to avoid this difficulty we will need the following technical result which proof is based in the ideas introduced by [13, Lemma 3] and [21, Lemma 2.3].

Lemma 3.1. *For any fixed $u \in E$, there exists a unique critical point $\phi = \phi_u \in \mathcal{D}$ for the functional*

$$\mathcal{E}_u(\phi) := \int_{\mathbb{R}^2} \left[\frac{1}{8\pi}|\nabla\phi|^2 + \frac{\beta}{16\pi}|\nabla\phi|^4 + \left(\omega + \frac{1}{2}\phi\right) V(|x|)\phi u^2\right] \mathrm{d}x$$

defined on $\mathcal{D}$ (i.e., $\mathcal{E}_u$ is the energy functional associated to the second equation in (1.2)). Moreover:

1. *$\phi_u \le 0$ and, if $u(x) \neq 0$, $-\omega \le \phi_u(x)$;*
2. *if u is radially symmetric, then ϕ_u is radial too.*

Proof. We consider the minimizing argument on $\mathcal{E}_u$. Obviously, the functional $\mathcal{E}_u$ is well-defined on $\mathcal{D}$. Furthermore, it is strictly convex, coercive and weakly lower semi-continuous. Indeed, the coercivity of $\mathcal{E}_u$ on $\mathcal{D}$ is the following fact that

$$\begin{aligned} \mathcal{E}_u(\phi) &= \int_{\mathbb{R}^2} \left[\frac{1}{8\pi}|\nabla\phi|^2 + \frac{\beta}{16\pi}|\nabla\phi|^4 + \frac{1}{2}(\omega+\phi)^2 V(|x|)u^2 - \frac{1}{2}\omega^2 V(|x|)u^2\right] \mathrm{d}x \\ &\ge \int_{\mathbb{R}^2} \left[\frac{1}{8\pi}|\nabla\phi|^2 + \frac{\beta}{16\pi}|\nabla\phi|^4\right] \mathrm{d}x - \frac{\omega^2}{2} \int_{\mathbb{R}^2} V(|x|)u^2 \mathrm{d}x. \end{aligned}$$

The convexity and weakly lower semi-continuity of $\mathcal{E}_u$ on $\mathcal{D}$ is obviously true. Hence, there is a unique minimizer ϕ_u of the functional $\mathcal{E}_u$ on $\mathcal{D}$, concluding the first part of the lemma. For the second part, since ϕ_u is a critical point of $\mathcal{E}_u$, we get

$$-\int_{\mathbb{R}^2}\left(\frac{1}{4\pi}\left((1+\beta|\nabla\phi_u|^2)\nabla\phi_u\cdot\nabla\eta\right)+V(|x|)(\phi_u+\omega)u^2\eta\right)\mathrm{d}x=0, \tag{3.3}$$

for all $\eta\in\mathcal{D}$. Then, if we take $\eta=\phi_u^+:=\max\{\phi_u,0\}$, that is, the positive part of ϕ_u, in (3.3), we obtain

$$\int_{\mathbb{R}^2}\left(|\nabla\phi_u^+|^2+\beta|\nabla\phi_u^+|^4\right)\mathrm{d}x=-4\pi\int_{\mathbb{R}^2}(\omega+\phi_u^+)\phi_u^+V(|x|)u^2\,\mathrm{d}x\le 0,$$

which implies that $\phi_u^+\equiv 0$ and, consequently, $\phi_u\le 0$. On the other hand, if we take $\eta=(\omega+\phi_u)^-:=\max\{-(\omega+\phi_u),0\}$, that is, the negative part of $\omega+\phi_u$, in (3.3), we get

$$\begin{aligned}\int_{\{x\in\mathbb{R}^2:\phi_u(x)\le-\omega\}}|\nabla\phi_u^-|^2\,\mathrm{d}x+&\int_{\{x\in\mathbb{R}^2:\phi_u(x)\le-\omega\}}\beta|\nabla\phi_u^-|^4\,\mathrm{d}x\\ &=-4\pi\int_{\{x\in\mathbb{R}^2:\phi_u(x)\le-\omega\}}V(|x|)[(\phi_u+\omega)^-]^2u^2\,\mathrm{d}x\le 0,\end{aligned}$$

so that $(\phi_u+\omega)^-\equiv 0$ where $u\neq 0$.

Finally, let $O(2)$ denote the group of rotations in $\mathbb{R}^2$. Then for every $g\in O(2)$ and $h:\mathbb{R}^2\to\mathbb{R}$, set $T_g(h)(x):=h(gx)$. It is well-known that

$$\Delta T_g(\phi_u)=T_g(\Delta\phi_u)\quad\text{and}\quad\Delta_4T_g(\phi_u)=T_g(\Delta_4\phi_u).$$

With this in mind, it is easy to verify that $\phi_{T_g(u)}$ and $T_g(\phi_u)$ are critical point of $\mathcal{E}_{T_g(u)}$. Hence, by the uniqueness of the critical point of $\mathcal{E}_{T_g(u)}$, we infer that

$$\phi_{T_g(u)}=T_g(\phi_u),$$

for all $g\in O(2)$. In particular, if u is radially symmetric, i.e., $u\in Y$ is a fixed point for the action T_g, ϕ_u is radial too and the result follows. This concludes the proof of the lemma. □

So, we can consider a C^1 functional $I:E\to\mathbb{R}$ defined by $I(u):=J(u,\phi_u)$, that is,

$$\begin{aligned}I(u)=&\frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla u|^2+\left[m^2-(\omega+\phi_u)^2\right]V(|x|)u^2\right)\mathrm{d}x\\ &-\frac{1}{8\pi}\int_{\mathbb{R}^2}|\nabla\phi_u|^2\,\mathrm{d}x-\frac{\beta}{16\pi}\int_{\mathbb{R}^2}|\nabla\phi_u|^4\,\mathrm{d}x-\int_{\mathbb{R}^2}K(|x|)F(u)\,\mathrm{d}x\end{aligned} \tag{3.4}$$

with Gâteaux derivative given by

$$\begin{aligned}I'(u)v=&\int_{\mathbb{R}^2}\left(\nabla u\cdot\nabla v+\left(m^2-\omega^2\right)V(|x|)uv-2V(|x|)\omega\phi_uuv-V(|x|)\phi_u^2uv\right)\mathrm{d}x\\ &-\int_{\mathbb{R}^2}K(|x|)f(u)v\,\mathrm{d}x,\end{aligned} \tag{3.5}$$

for all $v\in E$.

After using (3.3) with ϕ_u and through simple computation, we deduce

$$-\int_{\mathbb{R}^2}\left(|\nabla\phi_u|^2+\beta|\nabla\phi_u|^4\right)\mathrm{d}x=4\pi\int_{\mathbb{R}^2}(\omega+\phi_u)\phi_uV(|x|)u^2\,\mathrm{d}x. \tag{3.6}$$

Therefore, the reduced functional also takes the form

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + \left(m^2 - \omega^2\right) V(|x|)u^2 + V(|x|)\phi_u^2 u^2\right) \mathrm{d}x \\ + \frac{1}{8\pi}\int_{\mathbb{R}^2} |\nabla \phi_u|^2 \,\mathrm{d}x + \frac{3\beta}{16\pi}\int_{\mathbb{R}^2} |\nabla \phi_u|^4 \,\mathrm{d}x - \int_{\mathbb{R}^2} K(|x|)F(u)\,\mathrm{d}x. \tag{3.7}$$

Throughout the rest of the paper, and according the convenience, we will use both forms (3.4) or (3.7). Now, following [6], a pair $(u, \phi) \in E \times \mathcal{D}$ is a critical point for J if and only if u is a critical point for I with $\phi = \phi_u$. Hence, we will look for its critical points. The next lemma shows that E actually is, in some sense, a natural constraint for finding weak solutions of problem (1.2). In fact, it is a symmetric criticality type result.

Lemma 3.2. *Assume that* $(V, K) \in \mathcal{K}$ *and the hypothesis* (f_1) *holds. Then, every critical point* $u \in E$ *of* $I\colon E \to \mathbb{R}$ *is a weak solution to problem* (1.2)*, that is, satisfies* (1.5) *with* $\phi = \phi_u$.

Proof. We will show that if $u \in E$ satisfies (1.5) with $\phi = \phi_u$ and for all $v \in E$, then (1.5) holds also true for all $v \in Y$. Let $u \in E$. By Hölder's inequality, Lemma 2.4 and the growth assumption (f_1) on nonlinear term f yield a positive constant $C = C(\|u\|)$ such that

$$\left|\int_{\mathbb{R}^2} K(|x|)f(u)v\,\mathrm{d}x\right| \leq C\|v\|, \qquad \forall v \in Y.$$

Thus, the linear functional $T_u : Y \to \mathbb{R}$ defined by

$$T_u(v) := \int_{\mathbb{R}^2} \left(\nabla u \cdot \nabla v + [m^2 - (\omega + \phi_u)^2]V(|x|)uv\right) \mathrm{d}x - \int_{\mathbb{R}^2} K(|x|)f(u)v\,\mathrm{d}x,$$

is well-defined and continuous on Y and so, by the Riesz Representation Theorem in the space Y with the inner product (1.4), there exists a unique $\tilde{u} \in Y$ such that $T_u(\tilde{u}) = \|\tilde{u}\|^2 = \|T_u\|_{Y'}$, where Y' denotes the dual space of Y. Then, by using change of variables, one has for each $v \in Y$

$$T_u(gv) = T_u(v) \quad \text{and} \quad \|gv\| = \|v\|, \quad \text{for all } g \in O(2),$$

whence, applying with $v = \tilde{u}$, one deduce, by uniqueness, $g\tilde{u} = \tilde{u}$, for all $g \in O(2)$, which means, $\tilde{u} \in E$. Hence, since $T_u(v) = 0$ for all $v \in E$, one has $T_u(\tilde{u}) = 0$, that is, $\|T_u\|_{Y'} = 0$ and therefore (1.5) with $\phi = \phi_u$ ensues. This concludes the proof of the lemma. □

In the next lemma, we show that the functional I satisfies the geometric conditions of the Mountain-Pass Theorem.

Lemma 3.3. *Suppose that* $(V, K) \in \mathcal{K}$ *and* (f_0)*–*(f_2) *hold. If* $|m| > \omega > 0$*, then*

1. *there exist some constants* $\tau, \rho > 0$ *such that* $I(u) \geq \tau$ *provided* $\|u\| = \rho$*;*

2. *there exists* $v \in E$ *satisfying* $\|v\| > \rho$ *and* $I(v) < 0$.

Proof. 1. From (3.1), we get

$$\int_{\mathbb{R}^2} K(|x|)F(u)\,\mathrm{d}x \leq \frac{\varepsilon}{2}\int_{\mathbb{R}^2} K(|x|)u^2\,\mathrm{d}x + b_1\int_{\mathbb{R}^2} K(|x|)|u|^q\left(e^{\alpha u^2} - 1\right)\mathrm{d}x.$$

Let $r_1, r_2 > 1$ be such that $\frac{1}{r_1} + \frac{1}{r_2} = 1$. By Hölder's inequality and (3.2), we infer

$$\int_{\mathbb{R}^2} K(|x|)|u|^q \left(e^{\alpha u^2} - 1\right) \mathrm{d}x \leq \left(\int_{\mathbb{R}^2} K(|x|)\, |u|^{qr_1} \,\mathrm{d}x\right)^{\frac{1}{r_1}} \left(\int_{\mathbb{R}^2} K(|x|) \left(e^{\alpha r_2 u^2} - 1\right) \mathrm{d}x\right)^{\frac{1}{r_2}}$$
$$\leq \|u\|^q_{qr_1;K} \left(\int_{\mathbb{R}^2} K(|x|) \left(e^{\alpha r_2 M^2 \left(\frac{u}{\|u\|}\right)^2} - 1\right) \mathrm{d}x\right)^{\frac{1}{r_2}}.$$

Choosing $r_2 > 1$ sufficiently close to 1 and $0 < M < \left(\frac{\lambda}{r_2 \alpha}\right)^{\frac{1}{2}}$, then for $\|u\| \leq M$, it follows from Corollary 2.6 that

$$\int_{\mathbb{R}^2} K(|x|) \left(e^{\alpha r_2 M^2 \left(\frac{u}{\|u\|}\right)^2} - 1\right) \mathrm{d}x \leq C.$$

Hence, from Lemma 2.4, we deduce that

$$\int_{\mathbb{R}^2} K(|x|) F(u)\, \mathrm{d}x \leq \frac{C_1 \varepsilon}{2} \|u\|^2 - C_2 \|u\|^q.$$

Consequently, since $|m| > \omega > 0$, by (3.7) we have

$$I(u) \geq \left(\frac{\min\{1, m^2 - \omega^2\}}{2} - \frac{C_1 \varepsilon}{2}\right) \|u\|^2 - C_2 \|u\|^q$$
$$= \left(\frac{\min\{1, m^2 - \omega^2\}}{2} - \frac{C_1 \varepsilon}{2}\right) \rho^2 - C_2 \rho^q$$

and, choosing $\varepsilon > 0$ sufficiently small such that $C_3 := \frac{\min\{1, m^2 - \omega^2\}}{2} - \frac{C_1 \varepsilon}{2} > 0$,

$$I(u) \geq C_3 \rho^2 - C_2 \rho^q.$$

Inasmuch $q > 2$, for $\rho > 0$ small enough, there exists $\tau > 0$ such that

$$I(u) \geq \tau, \quad \text{for any } u \in E \text{ with } \|u\| = \rho.$$

2. By the Ambrosetti–Rabinowitz type condition (f_2), for all $\delta > 0$, there exists a positive constant $C_4 = C_4(\delta)$ such that $F(s) \geq C_4 |s|^\theta - \delta s^2$, for all $s \in \mathbb{R}$. Let $\varphi \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^2)$ be such that $\mathrm{supp}(\varphi)$ is a compact set of $\mathbb{R}^2$. Thus, by (3.4) and Lemma 2.4, we have

$$I(t\varphi) \leq \frac{\max\{1, m^2\}}{2} t^2 \|\varphi\|^2 - C_4 t^\theta \int_{\mathrm{supp}(\varphi)} K(|x|) |\varphi|^\theta \,\mathrm{d}x + \delta t^2 \int_{\mathrm{supp}(\varphi)} K(|x|) \varphi^2 \,\mathrm{d}x$$
$$\leq \left(\frac{\max\{1, m^2\}}{2} + C_5 \delta\right) t^2 \|\varphi\|^2 - C_4 t^\theta \int_{\mathrm{supp}(\varphi)} K(|x|) |\varphi|^\theta \,\mathrm{d}x$$
$$\to -\infty, \quad \text{as } t \to +\infty,$$

since $\theta > 2$. Therefore, for t large enough and taking $v := t\varphi$ we conclude that $I(v) < 0$ and the lemma is proved. □

Next, we investigate the compactness conditions for the functional I. Recall that $(u_n) \subset E$ is a Palais–Smale, (P–S) for short, sequence at a level $c \in \mathbb{R}$ for the functional I if

$$I(u_n) \to c, \quad I'(u_n) \to 0, \text{ as } n \to +\infty,$$

where the second limit above occurs in the dual space E'. We say that I satisfies the Palais–Smale compactness condition if any (P–S) sequence has a convergent subsequence.

Lemma 3.4 (Boundedness). *Let $(u_n) \subset E$ be a (P–S) sequence at a level $c \in \mathbb{R}$ for the functional I. Then (u_n) is bounded in E.*

Proof. Let $(u_n) \subset E$ be a (P–S) sequence at a level $c \in \mathbb{R}$ for the functional I. In order to check that (u_n) is bounded in E, there are two cases to be considered: either $\theta > 4$ or $2 < \theta \leq 4$ and $\theta - 2 > 2\omega^2$.

Case 1: $\theta > 4$. Combining (3.5), (3.6), (3.7) and (f_2) together we can estimate

$$
\begin{aligned}
&\theta(c+1) + o_n(1)\|u_n\| \\
&\quad \geq \theta I(u_n) - I'(u_n)u_n \\
&\quad = \left(\frac{\theta}{2} - 1\right) \int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + \left(m^2 - \omega^2\right) V(|x|)u_n^2\right) \mathrm{d}x \\
&\qquad + \left(\frac{\theta}{2} + 1\right) \int_{\mathbb{R}^2} K(|x|)\phi_{u_n}^2 u_n^2 \,\mathrm{d}x + 2\int_{\mathbb{R}^2} K(|x|)\omega\phi_{u_n} u_n^2 \,\mathrm{d}x \\
&\qquad + \frac{\theta}{8\pi} \int_{\mathbb{R}^2} |\nabla\phi_{u_n}|^2 \,\mathrm{d}x + \frac{3\beta\theta}{16\pi} \int_{\mathbb{R}^2} |\nabla\phi_{u_n}|^4 \,\mathrm{d}x + \int_{\mathbb{R}^2} K(|x|)[f(u_n)u_n - \theta F(u_n)] \,\mathrm{d}x \\
&\quad \geq \left(\frac{\theta}{2} - 1\right) \int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + \left(m^2 - \omega^2\right) V(|x|)u_n^2\right) \mathrm{d}x + 2\int_{\mathbb{R}^2} K(|x|)(\phi_{u_n} + \omega)\phi_{u_n} u_n^2 \,\mathrm{d}x \\
&\qquad + \frac{\theta}{8\pi} \int_{\mathbb{R}^2} |\nabla\phi_{u_n}|^2 \,\mathrm{d}x + \frac{3\beta\theta}{16\pi} \int_{\mathbb{R}^2} |\nabla\phi_{u_n}|^4 \,\mathrm{d}x \\
&\quad = \left(\frac{\theta}{2} - 1\right) \int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + \left(m^2 - \omega^2\right) V(|x|)u_n^2\right) \mathrm{d}x + \left(\frac{\theta}{8\pi} - \frac{1}{2\pi}\right) \int_{\mathbb{R}^2} |\nabla\phi_{u_n}|^2 \,\mathrm{d}x \\
&\qquad + \left(\frac{3\beta\theta}{16\pi} - \frac{\beta}{2\pi}\right) \int_{\mathbb{R}^2} |\nabla\phi_{u_n}|^4 \,\mathrm{d}x \\
&\quad \geq \frac{\max\{\theta - 2, m^2 - \omega^2\}}{2} \|u_n\|^2.
\end{aligned}
$$

Before passing to the next case, we need first to rewrite $\theta I(u)$ as follows. By (3.4) and (3.6), we can write

$$
\begin{aligned}
\theta I(u_n) &= \frac{\theta}{2} \int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + \left(m^2 - \omega^2\right) V(|x|)u_n^2\right) \mathrm{d}x - \theta \int_{\mathbb{R}^2} V(|x|)\omega\phi_{u_n} u_n^2 \,\mathrm{d}x \\
&\quad - \frac{\theta}{2} \int_{\mathbb{R}^2} V(|x|)\phi_{u_n}^2 u_n^2 \,\mathrm{d}x - \frac{\theta}{8\pi} \int_{\mathbb{R}^2} |\nabla\phi_{u_n}|^2 \,\mathrm{d}x - \frac{\beta\theta}{16\pi} \int_{\mathbb{R}^2} |\nabla\phi_{u_n}|^4 \,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^2} K(|x|)\theta F(u_n) \,\mathrm{d}x \\
&= \frac{\theta}{2} \int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + \left(m^2 - \omega^2\right) V(|x|)u_n^2\right) \mathrm{d}x - \frac{\theta}{2} \int_{\mathbb{R}^2} V(|x|)\omega\phi_{u_n} u_n^2 \,\mathrm{d}x \\
&\quad + \frac{\beta\theta}{16\pi} \int_{\mathbb{R}^2} |\nabla\phi_{u_n}|^4 \,\mathrm{d}x - \int_{\mathbb{R}^2} K(|x|)\theta F(u_n) \,\mathrm{d}x.
\end{aligned}
$$

Now, we are able to treat the next case.

Case 2: $2 < \theta \leq 4$ and $\theta - 2 > 2\omega^2$. By using $\theta I(u_n)$ rewritten above, (3.5) and (f_2), we can estimate

$$
\begin{aligned}
&\theta(c+1) + o_n(1)\|u_n\| \\
&\quad \geq \theta I(u_n) - I'(u_n)u_n \\
&\quad = \left(\frac{\theta}{2} - 1\right) \int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + \left(m^2 - \omega^2\right) V(|x|)u_n^2\right) \mathrm{d}x + \int_{\mathbb{R}^2} V(|x|)\phi_{u_n}^2 u_n^2 \,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\frac{\theta}{2} - 2\right) \int_{\mathbb{R}^2} V(|x|)\omega\phi_{u_n} u_n^2 \,\mathrm{d}x + \frac{\beta\theta}{16\pi} \int_{\mathbb{R}^2} |\nabla \phi_{u_n}|^4 \,\mathrm{d}x \\
&+ \int_{\mathbb{R}^2} K(|x|)[f(u_n)u_n - \theta F(u_n)] \,\mathrm{d}x \\
\geq\; & \left(\frac{\theta}{2} - 1\right) \int_{\mathbb{R}^2} (|\nabla u_n|^2 + m^2 V(|x|)u_n^2) \,\mathrm{d}x - \omega^2 \int_{\mathbb{R}^2} V(|x|) u_n^2 \,\mathrm{d}x \\
\geq\; & \left(\frac{\max\{\theta - 2, m^2\}}{2} - \omega^2\right) \|u_n\|^2.
\end{aligned}
$$

In any case, we infer that (u_n) stays bounded in E, concluding the proof of the lemma. $\square$

In view of the mountain-pass geometry of I assured by Lemma 3.3, we introduce the mountain pass level

$$c_\mu := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \geq \tau > 0,$$

where the set of paths is defined as

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\}.$$

With the purpose to verify that I satisfies the Palais–Smale condition in certain levels of energy we will need the following upper bound for the mountain-pass level c_μ:

Lemma 3.5 (Level estimate)**.** *Suppose that* (f_3) *is satisfied with*

$$\mu \geq \mu_0 := \max\left\{\mu_1, \left[\frac{2\alpha_0\theta(\vartheta - 2)\|K\|_{L^1(B_1)}}{\lambda\vartheta(\theta - 2)}\right]^{\frac{\vartheta-2}{2}} \left(\frac{2\mu_1}{\vartheta}\right)^{\frac{\vartheta}{2}}\right\},$$

where $\mu_1 = \frac{\vartheta \max\{1,m^2\}\left(4\pi + \|V\|_{L^1(B_2)}\right)}{2\|K\|_{L^1(B_1)}}$. *Then*

$$c_\mu < \frac{\lambda}{2\alpha_0}\left(\frac{1}{2} - \frac{1}{\theta}\right). \tag{3.8}$$

Proof. We shall consider a cut-off function $\varphi_0 \in C_0^\infty(\mathbb{R}^2)$ verifying

$$0 \leq \varphi_0 \leq 1 \quad \text{in } \mathbb{R}^2, \qquad \varphi_0 \equiv 1 \quad \text{in } \overline{B}_1, \qquad \varphi_0 \equiv 0 \quad \text{in } B_2^c \quad \text{and} \quad |\nabla \varphi_0| \leq 1 \quad \text{in } \mathbb{R}^2.$$

From (3.4) and (f_3), we get

$$
\begin{aligned}
I(\varphi_0) &\leq \frac{\max\{1, m^2\}}{2} \int_{B_2} (|\nabla \varphi_0|^2 + V(|x|)\varphi_0^2) \,\mathrm{d}x - \frac{\mu_1}{\vartheta} \int_{B_2} K(|x|)|\varphi_0|^\vartheta \,\mathrm{d}x \\
&< \frac{\max\{1, m^2\}}{2} \left(4\pi + \|V\|_{L^1(B_2)}\right) - \frac{\mu_1}{\vartheta} \|K\|_{L^1(B_1)} = 0,
\end{aligned}
$$

since $\mu_1 = \frac{\vartheta \max\{1,m^2\}\left(4\pi + \|V\|_{L^1(B_2)}\right)}{2\|K\|_{L^1(B_1)}}$. In particular,

$$\frac{\max\{1, m^2\}}{2} \int_{B_2} (|\nabla \varphi_0|^2 + V(|x|)\varphi_0^2) \,\mathrm{d}x < \frac{\mu_1}{\vartheta} \|K\|_{L^1(B_1)}. \tag{3.9}$$

According to the definition of c_μ, (3.4), (3.9) and straightforward manipulations, we deduce that

$$\begin{aligned} c_\mu &\leq \max_{t\geq 0}\left[\frac{\max\{1,m^2\}}{2}t^2\int_{B_2}\left(|\nabla\varphi_0|^2+V(|x|)\varphi_0^2\right)\mathrm{d}x - t^\vartheta\frac{\mu}{\vartheta}\int_{B_2}K(|x|)|\varphi_0|^\vartheta\,\mathrm{d}x\right] \\ &< \max_{t\geq 0}\left[\frac{\mu_1}{\vartheta}\|K\|_{L^1(B_1)}t^2-\frac{\mu}{\vartheta}\|K\|_{L^1(B_1)}t^\vartheta\right] \\ &\leq \frac{\|K\|_{L^1(B_1)}}{\vartheta}\max_{t\geq 0}\left[\mu_1 t^2-\mu t^\vartheta\right] \\ &= \frac{\|K\|_{L^1(B_1)}}{\vartheta}(\vartheta-2)\left(\frac{2}{\mu}\right)^{\frac{2}{\vartheta-2}}\left(\frac{\mu_1}{\vartheta}\right)^{\frac{\vartheta}{\vartheta-2}}. \end{aligned} \tag{3.10}$$

Thus, if

$$\mu \geq \left[\frac{2\alpha_0\theta(\vartheta-2)\|K\|_{L^1(B_1)}}{\lambda\vartheta(\theta-2)}\right]^{\frac{\vartheta-2}{2}}\left(\frac{2\mu_1}{\vartheta}\right)^{\frac{\vartheta}{2}},$$

we immediately arrive at estimate (3.8), concluding the proof of the lemma. □

Corollary 3.6 (Behavior of the minimax level)**.** *The minimax level vanishes, i.e., $c_\mu \to 0$ as $\mu \to +\infty$.*

Proof. This can be easily checked as a byproduct from the proof of Lemma 3.5, specifically estimate (3.10). □

Taking into account Lemma 3.3, we may apply the Mountain-Pass Theorem without the Palais–Smale compactness condition (see [5]) to guarantee the existence of a (P–S) sequence (u_n) in E at the level c_μ. To obtain the existence of nontrivial solutions to (1.2), the following technical result will be useful and plays a crucial role in the proof of Theorem 1.2.

Lemma 3.7. *The sequence $(u_n) \subset E$ obtained above satisfies*

$$\sup_{n\geq 1}\|f(u_n)\|_{2;K} < +\infty. \tag{3.11}$$

Proof. We begin the proof estimating the quantity $\theta I(u_n)$. For this aim, similarly was done in the proof of Lemma 3.4, we also divide our proof into two cases about θ as follows.

Case 1: $\theta > 4$.

$$\begin{aligned} \theta I(u_n) &= \theta I(u_n) - I'(u_n)u_n + o_n(1) \\ &\geq \frac{\max\{\theta-2, m^2-\omega^2\}}{2}\|u_n\|^2 + o_n(1) \to \theta c_\mu, \quad \text{as } n\to+\infty. \end{aligned}$$

Hence, invoking the level estimate (3.8) and Corollary 3.6, for any $\mu > \mu_0$, it follows that

$$\frac{\theta c_\mu}{\frac{\max\{\theta-2,m^2-\omega^2\}}{2}} < \frac{\lambda}{2\alpha_0}.$$

Case 2: $2 < \theta \leq 4$ and $\theta - 2 > 2\omega^2$.

$$\begin{aligned} \theta I(u_n) &= \theta I(u_n) - I'(u_n)u_n + o_n(1) \\ &\geq \left(\frac{\max\{\theta-2, m^2\}}{2} - \omega^2\right)\|u_n\|^2 + o_n(1) \to \theta c_\mu, \quad \text{as } n\to+\infty. \end{aligned}$$

Again, by virtue of (3.8) and Corollary 3.6, for any $\mu > \mu_0$, it follows that

$$\frac{\theta c_\mu}{\frac{\max\{\theta-2,m^2\}}{2} - \omega^2} < \frac{\lambda}{2\alpha_0}.$$

Thereby, in any case, we deduce that

$$\limsup_{n\to+\infty} \|u_n\|^2 < \frac{\lambda}{2\alpha_0},$$

and in view of Trudinger–Moser type inequality (2.1) we conclude that

$$\sup_{n\geq 1} \int_{\mathbb{R}^2} K(|x|) \left(e^{2\alpha_0 u_n^2} - 1\right) \mathrm{d}x < +\infty. \tag{3.12}$$

On the other hand, by (f_0) and (f_1), and using the fact that $2\alpha_0 > \alpha_0$, there exists a positive constant C_1 such that

$$|f(u_n)|^2 \leq C_1 \left(u_n^2 + e^{2\alpha_0 u_n^2} - 1\right).$$

Therefore, having in mind that (u_n) is bounded in $L^2(\mathbb{R}^2;K)$ and (3.12), our lemma immediately follows. □

4 Proof of the main results

In this section, we will prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Let $(u_n) \subset E$ be the (P–S) sequence at the level c_μ. From Lemma 3.4, (u_n) is bounded in E, which implies the weak convergence $u_n \rightharpoonup u_0$ in E. We shall prove that, up to a subsequence, $u_n \to u_0$ strongly in E and $(u_0, \phi_{u_0}) \in E \times \mathcal{D}$ is a weak solution of (1.2). Set

$$\mathcal{I}_n^1 := \int_{\mathbb{R}^2} K(|x|) f(u_n)(u_n - u_0)\, \mathrm{d}x \tag{4.1}$$

and

$$\mathcal{I}_n^2 = \int_{\mathbb{R}^2} K(|x|)\phi_{u_n} u_n (u_n - u_0)\, \mathrm{d}x, \quad \mathcal{I}_n^3 = \int_{\mathbb{R}^2} K(|x|)\phi_{u_n}^2 u_n (u_n - u_0)\, \mathrm{d}x. \tag{4.2}$$

We claim that $\mathcal{I}_n^1$, $\mathcal{I}_n^2$, $\mathcal{I}_n^3 \to 0$, as $n \to +\infty$. Let us to check these convergences in the following steps:

Step 1: $\mathcal{I}_n^1 = o_n(1)$, as $n \to +\infty$. In fact, by Hölder's inequality

$$|\mathcal{I}_n^1| \leq \|f(u_n)\|_{2;K} \|u_n - u_0\|_{2;K}.$$

The compact embedding $E \hookrightarrow L^2(\mathbb{R}^2;K)$ implies that $u_n \to u_0$ strongly in $L^2(\mathbb{R}^2;K)$. Consequently,

$$\|u_n - u_0\|_{2;K} \to 0, \text{ as } n \to +\infty,$$

and from (3.11) we get the first convergence.

Step 2: $\mathcal{I}_n^2$, $\mathcal{I}_n^3 \to 0$, as $n \to +\infty$. In fact, combining Hölder's inequality, Lemmas 2.4, 3.1 and the boundedness of (u_n) in E, we have

$$\begin{aligned}
|\mathcal{I}_n^2| &\le \int_{\mathbb{R}^2} K(|x|)|\phi_{u_n}||u_n||u_n - u_0|\,\mathrm{d}x \\
&\le \left(\int_{\mathbb{R}^2} K(|x|)\phi_{u_n}^2 {u_n}^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} K(|x|)(u_n - u_0)^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\le \omega\|u_n\|_{2;K}\|u_n - u_0\|_{2;K} \le \omega C_1\|u_n\|\|u_n - u_0\|_{2;K} \\
&\le C_2\|u_n - u_0\|_{2;K} \to 0, \text{ as } n \to +\infty,
\end{aligned}$$

since, again by Lemma 2.4, $u_n \to u_0$ strongly in $L^2(\mathbb{R}^2; K)$. Analogously, $\mathcal{I}_n^3 \to 0$, as $n \to +\infty$. Thus, from (4.1), (4.2) and having in mind that

$$\lim_{n\to\infty} I'(u_n)(u_n - u_0) = 0,$$

it leads to

$$\int_{\mathbb{R}^2} \big(\nabla u_n \cdot \nabla(u_n - u_0) + (m^2 - \omega^2)V(|x|)u_n(u_n - u_0)\big)\,\mathrm{d}x = o_n(1).$$

Now, as an immediate consequence of the weak convergence $u_n \rightharpoonup u_0$ in E, we have

$$\int_{\mathbb{R}^2} \big(\nabla u_0 \cdot \nabla(u_n - u_0) + V(|x|)u_0(u_n - u_0)\big)\,\mathrm{d}x = o_n(1).$$

Combining that last identities, we conclude that $u_n \to u_0$ strongly in E. Since I and I' are continuous, then

$$I'(u_n) = o_n(1) \to I'(u_0) = 0 \text{ and } I(u_n) \to I(u_0) = c_\mu > 0,$$

proving that u_0 is a nontrivial critical point of the functional I and, consequently, (u_0, ϕ_{u_0}) is a solution of (1.2). Finally, it remains to check that u_0 is nonnegative. But, it just suffices to observe that $I'(u_0)(u_0^-) = 0$ which leads to $\|u_0^-\|^2 = 0$ and therefore $u_0 = u_0^+ \ge 0$. This completes the proof. □

To finish the paper, we give the end of our proof.

Proof of Theorem 1.3. Our goal is to show that (u_0, ϕ_{u_0}) is a ground state solution, that is, is a solution which minimizes the functional J among all the nontrivial solutions of (1.2), namely, $J(u_0, \phi_{u_0}) \le J(u, \phi)$ for any nontrivial solution (u, ϕ) of (1.2). In this direction, this aim will carry out by considering a minimization problem where the constraint is defined by the Nehari manifold. By a ground state solution of system (1.2) we mean a nontrivial solution $(\tilde{u}, \phi_{\tilde{u}}) \in E \times \mathcal{D}$ of (1.2) such that

$$I(\tilde{u}) = \min\{I(u) : u \in E \setminus \{0\} \text{ is a critical point of } I\}.$$

So, let

$$M_\mu := \min_{u\in\mathcal{N}} I(u),$$

where $\mathcal{N}$ is the Nehari manifold

$$\mathcal{N} := \{u \in E \setminus \{0\} : I'(u)u = 0\}.$$

For this aim, it is sufficient to prove that $c_\mu \leq M_\mu$. The Nehari manifold $\mathcal{N}$ is closely linked to the behavior of the function of the form $h_u : t \to I(tu)$ for $t > 0$. Such map is known as fibering map. Let $u \in \mathcal{N}$, from (3.4), we find

$$h'_u(t) = t \int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + \left(m^2 - \omega^2\right) V(|x|) u_n^2\right) \mathrm{d}x - 2t \int_{\mathbb{R}^2} V(|x|) \omega \phi_u u^2 \, \mathrm{d}x \\ - t \int_{\mathbb{R}^2} V(|x|) \phi_u^2 u^2 \, \mathrm{d}x - \int_{\mathbb{R}^2} K(|x|) f(tu) u \, \mathrm{d}x.$$

Since $I'(u)u = 0$, as a direct consequence, we obtain

$$h'_u(t) = t \int_{\mathbb{R}^2} K(|x|) \left[\frac{f(u)}{u} - \frac{f(tu)}{tu}\right] u^2 \, \mathrm{d}x,$$

for $t > 0$. Taking into account that $f(s)/s$ is increasing for $s > 0$, we infer that $h'_u(t) > 0$ for $t \in (0,1)$ and $h'_u(t) < 0$ for $t \in (1, \infty)$. Hence, after observing $h'_u(1) = 0$, we conclude that $I(u) = \max_{t \geq 0} I(tu)$. Setting $\gamma(t) := tt_0 u$, for $t \in [0,1]$, where t_0 is such that $I(t_0 u) < 0$, we have $\gamma \in \Gamma$, and so

$$c_\mu \leq \max_{t \in [0,1]} I(\gamma(t)) \leq \max_{t \geq 0} I(tu) = I(u).$$

Thereby, since $u \in \mathcal{N}$ is arbitrary $c_\mu \leq M_\mu$. This implies that (u_0, ϕ_{u_0}) is a ground state solution for (1.2) and, therefore, the proof of Theorem 1.3 is finished. □

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On the impulsive Dirichlet problem for second-order differential inclusions

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Abstract. Solutions in a given set of an impulsive Dirichlet boundary value problem are investigated for second-order differential inclusions. The method used for obtaining the existence and the localization of a solution is based on the combination of a fixed point index technique developed by ourselves earlier with a bound sets approach and Scorza-Dragoni type result. Since the related bounding (Liapunov-like) functions are strictly localized on the boundaries of parameter sets of candidate solutions, some trajectories are allowed to escape from these sets.

Keywords: impulsive Dirichlet problem, differential inclusions, topological methods, bounding functions, Scorza-Dragoni technique.

2020 Mathematics Subject Classification: 34A60, 34B15, 47H04.

1 Introduction

Let us consider the Dirichlet boundary value problem

$$\begin{cases} \ddot{x}(t) \in F(t, x(t), \dot{x}(t)), & \text{for a.a. } t \in [0, T], \\ x(T) = x(0) = 0, \end{cases} \tag{1.1}$$

where $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory multivalued mapping.

Moreover, let a finite number of points $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$, and real $n \times n$ matrices A_i, B_i, $i = 1, \ldots, p$, be given. In the paper, the solvability of the Dirichlet b.v.p. (1.1) will be investigated in the presence of the following impulse conditions

$$x(t_i^+) = A_i x(t_i), \qquad i = 1, \ldots, p, \tag{1.2}$$

$$\dot{x}(t_i^+) = B_i \dot{x}(t_i), \qquad i = 1, \ldots, p, \tag{1.3}$$

where the notation $\lim_{t\to a^+} x(t) = x(a^+)$ is used.

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By a *solution* of problem (1.1)–(1.3) we shall mean a function $x \in PAC^1([0,T],\mathbb{R}^n)$ (see Section 2 for the definition) satisfying (1.1), for almost all $t \in [0,T]$, and fulfilling the conditions (1.2) and (1.3).

Boundary value problems with impulses have been widely studied because of their applications in areas, where the parameters are subject to certain perturbations in time. For instance, in the treatment of some diseases, impulses may correspond to administration of a drug treatment or in environmental sciences, they can describe the seasonal changes or harvesting.

While the theory of single valued impulsive problems is deeply examined (see, e.g. [9,10, 22]), the theory dealing with multivalued impulsive problems has not been studied so much yet (for the overview of known results see, e.g., the monographs [11,19] and the references therein). However, it is worth to study also the multivalued case, since the multivalued problems come e.g. from single valued problems with discontinuous right-hand sides, or from control theory.

The most of mentioned results dealing with impulsive problems have been obtained using fixed point theorems, upper and lower-solutions methods, or using topological and variational approaches.

In this paper, the existence and the localization of a solution for the impulsive Dirichlet b.v.p. (1.1)–(1.3) will be studied using a continuation principle. On this purpose, it will be necessary to embed the original problem into a family of problems and to ensure that the boundary of a prescribed set of candidate solutions is fixed point free, i.e. to verify so called transversality condition. This condition can be guaranteed by a bound sets technique that was described by Gaines and Mawhin in [17] for single valued problems without impulses. Recently, in [25], a bound sets technique for the multivalued impulsive b.v.p. using non strictly localized bounding (Liapunov-like) functions has been developed. Such a non-strict localization of bounding functions makes parameter sets of candidate solutions "only" positively invariant.

In this paper, the conditions imposed on the bounding function will be strictly localized on the boundary of the set of candidate solutions, which eliminates this unpleasant handicap. Both the possible cases will be discussed – problems with an upper semicontinuous r.h.s. and also problems with an upper-Carathéodory r.h.s. More concretely, in Theorem 4.3 below, the upper semicontinuous case is considered and the transversality condition is obtained reasoning pointwise via a C^1-bounding function with a locally Lipschitzian gradient. In Theorem 5.2, the upper-Carathéodory case and a C^2-bounding function will be considered and the reasoning will be based on a Scorza-Dragoni approximation technique. In fact, even if the first kind of regularity of the r.h.s. is a special case of the second one, in the first case the stronger regularity will allow to use C^1-bounding functions, while in the second case, C^2-bounding functions will be needed. Moreover, even when using C^2-bounding functions, the more regularity of the r.h.s. allows to obtain the result under weaker conditions. Let us note that a similar approach was employed for problems with upper semicontinous r.h.s. without impulses e.g. in [3,6] and for problems with upper-Carathéodory r.h.s. without impulses e.g. in [4,24].

This paper is organized as follows. In the second section, we recall suitable definitions and statements which will be used in the sequel. Section 3 is devoted to the study of bound sets and Liapunov-like bounding functions for impulsive Dirichlet problems with an upper semicontinuous r.h.s. At first, C^1-bounding functions with locally Lipschitzian gradients are considered. Consequently, it is shown how conditions ensuring the existence of bound set change in case of C^2-bounding functions. In Section 4, the bound sets approach is combined with a continuation principle and the existence and localization result is obtained in this way for the impulsive Dirichlet problem (1.1)–(1.3). Section 5 deals with the existence and localization of a solution of the Dirichlet impulsive problem in case when the r.h.s. is an upper-Carathéodory mapping. In Section 6, the obtained result is applied to an illustrative example.

2 Some preliminaries

Let us recall at first some geometric notions of subsets of metric spaces. If (X, d) is an arbitrary metric space and $A \subset X$, by $\mathrm{Int}(A), \overline{A}$ and ∂A we mean the interior, the closure and the boundary of A, respectively. For a subset $A \subset X$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) := \{x \in X \mid \exists a \in A : d(x,a) < \varepsilon\}$, i.e. $N_\varepsilon(A)$ is an open neighborhood of the set A in X.

For a given compact real interval J, we denote by $C(J, \mathbb{R}^n)$ (by $C^1(J, \mathbb{R}^n)$) the set of all functions $x : J \to \mathbb{R}^n$ which are continuous (have continuous first derivatives) on J. By $AC^1(J, \mathbb{R}^n)$, we shall mean the set of all functions $x : J \to \mathbb{R}^n$ with absolutely continuous first derivatives on J.

Let $PAC^1([0, T], \mathbb{R}^n)$ be the space of all functions $x : [0, T] \to \mathbb{R}^n$ such that

$$x(t) = \begin{cases} x_{[0]}(t), & \text{for } t \in [0, t_1], \\ x_{[1]}(t), & \text{for } t \in (t_1, t_2], \\ \quad\vdots & \\ x_{[p]}(t), & \text{for } t \in (t_p, T], \end{cases}$$

where $x_{[0]} \in AC^1([0, t_1], \mathbb{R}^n), x_{[i]} \in AC^1((t_i, t_{i+1}], \mathbb{R}^n), x(t_i^+) = \lim_{t \to t_i^+} x(t) \in \mathbb{R}$ and $\dot{x}(t_i^+) = \lim_{t \to t_i^+} \dot{x}(t) \in \mathbb{R}$, for every $i = 1, \ldots, p$. The space $PAC^1([0, T], \mathbb{R}^n)$ is a normed space with the norm

$$\|x\| := \sup_{t \in [0,T]} |x(t)| + \sup_{t \in [0,T]} |\dot{x}(t)|. \tag{2.1}$$

In a similar way, we can define the spaces $PC([0, T], \mathbb{R}^n)$ and $PC^1([0, T], \mathbb{R}^n)$ as the spaces of functions $x : [0, T] \to \mathbb{R}^n$ satisfying the previous definition with $x_{[0]} \in C([0, t_1], \mathbb{R}^n), x_{[i]} \in C((t_i, t_{i+1}], \mathbb{R}^n)$ or with $x_{[0]} \in C^1([0, t_1], \mathbb{R}^n)$, $x_{[i]} \in C^1((t_i, t_{i+1}], \mathbb{R}^n)$, for every $i = 1, \ldots, p$, respectively. The space $PC^1([0, T], \mathbb{R}^n)$ with the norm defined by (2.1) is a Banach space (see [23, page 128]).

A subset $A \subset X$ is called a *retract* of a metric space X if there exists a retraction $r : X \to A$, i.e. a continuous function satisfying $r(x) = x$, for every $x \in A$. We say that a space X is an *absolute retract* (*AR*-space) if, for each space Y and every closed $A \subset Y$, each continuous mapping $f : A \to X$ is extendable over Y. If f is extendable only over some neighborhood of A, for each closed $A \subset Y$ and each continuous mapping $f : A \to X$, then X is called an *absolute neighborhood retract* (*ANR*-space). Let us note that X is an *ANR*-space if and only if it is a retract of an open subset of a normed space and that X is an *AR*-space if and only if it is a retract of some normed space (see, e.g. [2]). Conversely, if X is a retract (of an open subset) of a convex set in a Banach space, then it is an *AR*-space (*ANR*-space). So, the space $C^1(J, \mathbb{R}^n)$, where $J \subset \mathbb{R}$ is a compact interval, is an *AR*-space as well as its convex subsets or retracts, while its open subsets are *ANR*-spaces.

A nonempty set $A \subset X$ is called an *R_δ-set* if there exists a decreasing sequence $\{A_n\}_{n=1}^{\infty}$ of compact AR-spaces such that

$$A = \bigcap_{n=1}^{\infty} A_n.$$

The following hierarchy holds for nonempty subsets of a metric space:

$$\text{compact+convex} \subset \text{compact } AR\text{-space} \subset R_\delta\text{-set}, \tag{2.2}$$

and all the above inclusions are proper. For more details concerning the theory of retracts, see [14].

We also employ the following definitions from the multivalued analysis in the sequel. Let X and Y be arbitrary metric spaces. We say that φ is a multivalued mapping from X to Y (written $\varphi : X \multimap Y$) if, for every $x \in X$, a nonempty subset $\varphi(x)$ of Y is prescribed. We associate with F its *graph* Γ_F, the subset of $X \times Y$, defined by

$$\Gamma_F := \{(x,y) \in X \times Y \mid y \in F(x)\}.$$

Let us mention also some basic notions concerning multivalued mappings. A multivalued mapping $\varphi : X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if, for each open $U \subset Y$, the set $\{x \in X \mid \varphi(x) \subset U\}$ is open in X.

Let $F : J \times \mathbb{R}^m \multimap \mathbb{R}^n$ be an upper semicontinuous multimap and let, for all $r > 0$, exist an integrable function $\mu_r : J \to [0,\infty)$ such that $|y| \le \mu_r(t)$, for every $(t,x) \in J \times \mathbb{R}^m$, with $|x| \le r$, and every $y \in F(t,x)$. Then if we consider the composition of F with a function $q \in PC^1([0,T],\mathbb{R}^n)$, the corresponding superposition multioperator $\mathcal{P}_F(q)$ given by

$$\mathcal{P}_F(q) = \{f \in L^1([0,T];\mathbb{R}^m) \ : \ f(t) \in F(t,q(t)) \text{ a.a. } t \in [0,T]\},$$

is well defined and nonempty (see [12, Proposition 6]).

Let Y be a metric space and $(\Omega,\mathcal{U},\nu)$ be a *measurable space,* i.e. a nonempty set Ω equipped with a σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure ν on $\mathcal{U}$. A multivalued mapping $\varphi : \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega \mid \varphi(\omega) \subset V\} \in \mathcal{U}$, for each open set $V \subset Y$. Obviously, every u.s.c. mapping is measurable.

We say that mapping $\varphi : J \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $J \subset \mathbb{R}$ is a compact interval, is an *upper-Carathéodory mapping* if the map $\varphi(\cdot,x) : J \multimap \mathbb{R}^n$ is measurable, for all $x \in \mathbb{R}^m$, the map $\varphi(t,\cdot) : \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c., for almost all $t \in J$, and the set $\varphi(t,x)$ is compact and convex, for all $(t,x) \in J \times \mathbb{R}^m$.

If $X \cap Y \neq \emptyset$ and $\varphi : X \multimap Y$, then a point $x \in X \cap Y$ is called a *fixed point* of φ if $x \in \varphi(x)$. The set of all fixed points of φ is denoted by $\mathrm{Fix}(\varphi)$, i.e.

$$\mathrm{Fix}(\varphi) := \{x \in X \mid x \in \varphi(x)\}.$$

For more information and details concerning multivalued analysis, see, e.g., [2,8,18,21].

The continuation principle which will be applied in the paper requires in particular the transformation of the studied problem into a suitable family of associated problems which does not have solutions tangent to the boundary of a given set Q of candidate solutions. This will be ensured by means of Hartman-type conditions (see Section 3) and by means of the following result based on Nagumo conditions (see [27, Lemma 2.1] and [20, Lemma 5.1]).

Proposition 2.1. *Let $\psi : [0,+\infty) \to [0,+\infty)$ be a continuous and increasing function, with*

$$\lim_{s\to\infty} \frac{s^2}{\psi(s)} ds = \infty, \tag{2.3}$$

and let R be a positive constant. Then there exists a positive constant

$$B = \psi^{-1}(\psi(2R) + 2R) \tag{2.4}$$

such that if $x \in PC^1([0,T],\mathbb{R}^n)$ is such that $|\ddot{x}(t)| \le \psi(|\dot{x}(t)|)$, for a.a. $t \in [0,T]$, and $|x(t)| \le R$, for every $t \in [0,T]$, then it holds that $|\dot{x}(t)| \le B$, for every $t \in [0,T]$.

Let us note that the previous result is classically given for C^2-functions. However, it is easy to prove (see, e.g., [5]) that the statement holds also for piecewise continuously differentiable functions.

For obtaining the existence and localization result for the case of upper-Carathéodory r.h.s., we will need the following Scorza-Dragoni type result for multivalued maps (see [15, Proposition 5.1]).

Proposition 2.2. *Let $X \subset \mathbb{R}^m$ be compact and let $F : [a,b] \times X \multimap \mathbb{R}^n$ be an upper-Carathéodory map. Then there exists a multivalued mapping $F_0 : [a,b] \times X \multimap \mathbb{R}^n \cup \{\emptyset\}$ with compact, convex values and $F_0(t,x) \subset F(t,x)$, for all $(t,x) \in [a,b] \times X$, having the following properties:*

(i) if $u : [a,b] \to \mathbb{R}^m$, $v : [a,b] \to \mathbb{R}^n$ are measurable functions with $v(t) \in F(t,u(t))$, on $[a,b]$, then $v(t) \in F_0(t,u(t))$, a.e. on $[a,b]$;

(ii) for every $\epsilon > 0$, there exists a closed $I_\epsilon \subset [a,b]$ such that $\nu([a,b] \setminus I_\epsilon) < \epsilon$, $F_0(t,x) \neq \emptyset$, for all $(t,x) \in I_\epsilon \times X$ and F_0 is u.s.c. on $I_\epsilon \times X$.

3 Bound sets for Dirichlet problems with upper semicontinuous r.h.s.

In this section, we consider an u.s.c. multimap F and we are interested in introducing a Liapunov-like function V, usually called a *bounding function*, verifying suitable transversality conditions which assure that there does not exist a solution of the b.v.p. lying in a closed set $\overline{K}$ and touching the boundary ∂K of K at some point.

Let $K \subset \mathbb{R}^n$ be a nonempty open set with $0 \in K$ and $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

$(H1)$ $V|_{\partial K} = 0$,

$(H2)$ $V(x) \le 0$, for all $x \in \overline{K}$.

Definition 3.1. A nonempty open set $K \subset \mathbb{R}^n$ is called a *bound set* for problem (1.1)–(1.3) if there does not exist a solution x of (1.1)–(1.3) such that $x(t) \in \overline{K}$, for each $t \in [0,T]$, and $x(t_0) \in \partial K$, for some $t_0 \in [0,T]$.

Firstly, we show sufficient conditions for the existence of a bound set for the second-order impulsive Dirichlet problem (1.1)–(1.3) in the case of a smooth bounding function V with a locally Lipschitzian gradient.

Proposition 3.2. *Let $K \subset \mathbb{R}^n$ be a nonempty open set with $0 \in K$, $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semicontinuous multivalued mapping with nonempty, compact, convex values. Assume that there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ with a locally Lipschitzian gradient ∇V which satisfies conditions $(H1)$ and $(H2)$. Suppose moreover that, for all $x \in \partial K$, $t \in (0,T) \setminus \{t_1, \ldots, t_p\}$ and $v \in \mathbb{R}^n$ with*

$$\langle \nabla V(x), v \rangle = 0, \tag{3.1}$$

the following condition holds

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x + hv), v + hw \rangle}{h} > 0, \tag{3.2}$$

for all $w \in F(t, x, v)$. Then all solutions $x : [0,T] \to \overline{K}$ of problem (1.1) satisfy $x(t) \in K$, for every $t \in [0,T] \setminus \{t_1, \ldots, t_p\}$.

Proof. Let $x : [0,T] \to \overline{K}$ be a solution of problem (1.1). We assume by a contradiction that there exists $\bar{t} \in [0,T] \setminus \{t_1, \ldots, t_p\}$ such that $x(\bar{t}) \in \partial K$. Since $x(0) = x(T) = 0 \in K$, it must be $\bar{t} \in (0,T)$.

Let us define the function g in the following way $g(h) := V(x(\bar{t} + h))$. Then $g(0) = 0$ and there exists $\alpha > 0$ such that $g(h) \leq 0$, for all $h \in [-\alpha, \alpha]$, i.e., there is a local maximum for g at the point 0, and $g \in C^1([-\alpha, \alpha], \mathbb{R}^n)$, so $\dot{g}(0) = \langle \nabla V(x(\bar{t})), \dot{x}(\bar{t}) \rangle = 0$. Consequently, $x := x(\bar{t}), v := \dot{x}(\bar{t})$ satisfy condition (3.1).

Since ∇V is locally Lipschitzian, there exist an open set $U \subset \mathbb{R}^n$, with $x(\bar{t}) \in U$, and a constant $L > 0$ such that $\nabla V|_U$ is Lipschitzian with constant L. We can assume, without loss of generality, that $x(\bar{t} + h) \in U$ for all $h \in [-\alpha, \alpha]$.

Since $g(0) = 0$ and $g(h) \leq 0$, for all $h \in [-\alpha, 0)$, there exists an increasing sequence of negative numbers $\{h_k\}_{k=1}^{\infty}$such that $h_1 > -\alpha, h_k \to 0^-$ as $k \to \infty$, and $\dot{g}(h_k) \geq 0$, for each $k \in \mathbb{N}$. Since $x \in C^1([-\alpha, 0], \mathbb{R}^n)$, it holds, for each $k \in \mathbb{N}$, that

$$x(\bar{t} + h_k) = x(\bar{t}) + h_k[\dot{x}(\bar{t}) + b_k], \tag{3.3}$$

where $b_k \to 0$ as $k \to \infty$.

Since $x([-\alpha, 0])$ and $\dot{x}([-\alpha, 0])$ are compact sets and F is globally upper semicontinuous with compact values, $F(\cdot, x(\cdot), \dot{x}(\cdot))$ must be bounded on $[-\alpha, 0]$, by which $\dot{x}$ is Lipschitzian on $[-\alpha, 0]$. Thus, there exists a constant λ such that, for all $k \in \mathbb{N}$,

$$\left| \frac{\dot{x}(\bar{t} + h_k) - \dot{x}(\bar{t})}{h_k} \right| \leq \lambda,$$

i.e. the sequence $\left\{ \frac{\dot{x}(\bar{t}+h_k) - \dot{x}(\bar{t})}{h_k} \right\}_{k=1}^{\infty}$ is bounded. Therefore, there exist a subsequence, for the sake of simplicity denoted as the sequence, of $\left\{ \frac{\dot{x}(\bar{t}+h_k) - \dot{x}(\bar{t})}{h_k} \right\}$ and $w \in \mathbb{R}^n$ such that

$$\frac{\dot{x}(\bar{t} + h_k) - \dot{x}(\bar{t})}{h_k} \to w \tag{3.4}$$

as $k \to \infty$.

Let $\varepsilon > 0$ be given. Then, as a consequence of the regularity assumptions on F and of the continuity of both x and $\dot{x}$ on $[-\alpha, 0]$, there exists $\overline{\delta} = \overline{\delta}(\varepsilon) > 0$ such that, for each $h \in [-\alpha, 0]$, $h \geq -\overline{\delta}$, it follows that

$$F(\bar{t} + h, x(\bar{t} + h), \dot{x}(\bar{t} + h)) \subset F(\bar{t}, x(\bar{t}), \dot{x}(\bar{t})) + \varepsilon \overline{B}_n,$$

where B_n denotes the unit open ball in $\mathbb{R}^n$ centered at the origin. Subsequently, since F is convex valued, according to the Mean-Value Theorem (See [8], Theorem 0.5.3), there exists $k_\varepsilon \in \mathbb{N}$ such that, for each $k \geq k_\varepsilon$,

$$\frac{\dot{x}(\bar{t}+h_k)-\dot{x}(\bar{t})}{h_k} = \frac{1}{-h_k}\int_{\bar{t}+h_k}^{\bar{t}} \ddot{x}(s)\, ds \in F(\bar{t}, x(\bar{t}), \dot{x}(\bar{t})) + \varepsilon \overline{B}_n.$$

Since F has compact values and $\varepsilon > 0$ is arbitrary,

$$w \in F(\bar{t}, x(\bar{t}), \dot{x}(\bar{t})).$$

As a consequence of property (3.4) , there exists a sequence $\{a_k\}_{k=1}^{\infty}$, $a_k \to 0$ as $k \to \infty$, such that

$$\dot{x}(\bar{t}+h_k) = \dot{x}(\bar{t}) + h_k[w + a_k], \tag{3.5}$$

for each $k \in \mathbb{N}$. Since $h_k < 0$ and $\dot{g}(h_k) \geq 0$, in view of (3.3) and (3.5),

$$\begin{aligned} 0 \geq \frac{\dot{g}(h_k)}{h_k} &= \frac{\langle \nabla V(x(\bar{t}+h_k)), \dot{x}(\bar{t}+h_k)\rangle}{h_k} \\ &= \frac{\langle \nabla V(x(\bar{t}) + h_k[\dot{x}(\bar{t}) + b_k]), \dot{x}(\bar{t}) + h_k[w + a_k]\rangle}{h_k}. \end{aligned}$$

Since $b_k \to 0$ when $k \to +\infty$, it is possible to find $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, it holds that $x(\bar{t}) + \dot{x}(\bar{t})h_k \in U$, because U is open. By means of the local Lipschitzianity of ∇V, for all $k \geq k_0$,

$$\begin{aligned} 0 \geq \frac{\dot{g}(h_k)}{h_k} &\geq \frac{\langle \nabla V(x(\bar{t}) + h_k\dot{x}(\bar{t})), \dot{x}(\bar{t}) + h_k[w + a_k]\rangle}{h_k} - L \cdot |b_k| \cdot |\dot{x}(\bar{t}) + h_k[w + a_k]| \\ &= \frac{\langle \nabla V(x(\bar{t}) + h_k\dot{x}(\bar{t})), \dot{x}(\bar{t}) + h_k w\rangle}{h_k} - L \cdot |b_k| \cdot |\dot{x}(\bar{t}) + h_k[w + a_k]| + \langle \nabla V(x(\bar{t}) + h_k\dot{x}(\bar{t})), a_k\rangle. \end{aligned}$$

Since $\langle \nabla V(x(\bar{t}) + h_k\dot{x}(\bar{t})), a_k\rangle - L \cdot |b_k| \cdot |\dot{x}(\bar{t}) + h_k[w + a_k]| \to 0$ as $k \to \infty$,

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x(\bar{t}) + h\dot{x}(\bar{t})), \dot{x}(\bar{t}) + hw\rangle}{h} \leq 0 \tag{3.6}$$

in contradiction with (3.2). Thus $x(t) \in K$ for every $t \in [0, T] \setminus \{t_1, \dots, t_p\}$. □

Remark 3.3. It is obvious that condition (3.2) in Proposition 3.2 can be replaced by the following assumption: suppose that, for all $x \in \partial K$, $t \in (0, T) \setminus \{t_1, \dots, t_p\}$ and $v \in \mathbb{R}^n$ satisfying (3.1) the following condition holds

$$\liminf_{h \to 0^+} \frac{\langle \nabla V(x + hv), v + hw\rangle}{h} > 0, \tag{3.7}$$

for all $w \in F(t, x, v)$.

Now, let us focus our attention also to the impulsive points $t_1, \dots, t_p$.

Theorem 3.4. *Let $K \subset \mathbb{R}^n$ be a nonempty open set with $0 \in K$, $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semicontinuous multivalued mapping with nonempty, compact, convex values. Assume that there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ with a locally Lipschitzian gradient ∇V which satisfies* *conditions* $(H1)$ *and* $(H2)$*. Furthermore, assume that* A_i*,* B_i*,* $i = 1, \dots, p$*, are real* $n \times n$ *matrices such that* A_i*,* $i = 1, \dots, p$*, satisfy*

$$A_i(\partial K) = \partial K, \quad \textit{for all } i = 1, \dots, p. \tag{3.8}$$

Moreover, let, for all $x \in \partial K$*,* $t \in (0, T) \setminus \{t_1, \dots t_p\}$ *and* $v \in \mathbb{R}^n$ *satisfying* (3.1)*, condition* (3.2) *holds, for all* $w \in F(t, x, v)$*.*

At last, suppose that, for all $x \in \partial K$ *and* $v \in \mathbb{R}^n$ *with*

$$\langle \nabla V(A_i x), B_i v \rangle \le 0 \le \langle \nabla V(x), v \rangle, \quad \textit{for some } i = 1, \dots, p, \tag{3.9}$$

the following condition

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x + hv), v + hw \rangle}{h} > 0 \tag{3.10}$$

holds, for all $w \in F(t_i, x, v)$*. Then* K *is a bound set for the impulsive Dirichlet problem* (1.1)*–*(1.3)*.*

Proof. Applying Proposition 3.2, we only need to show that if $x : [0, T] \to \overline{K}$ is a solution of problem (1.1), then $x(t_i) \in K$, for all $i = 1, \dots, p$. As in the proof of Proposition 3.2, we argue by a contradiction, i.e. we assume that there exists $i \in \{1, \dots, p\}$ such that $x(t_i) \in \partial K$. Following the same reasoning as in the proof of Proposition 3.2, for $\bar{t} = t_i$, we obtain

$$\langle \nabla V(x(t_i)), \dot{x}(t_i) \rangle \ge 0,$$

because $V(x(t_i)) = 0$ and $V(x(t)) \le 0$, for all $t \in [0, T]$.

Moreover, according to the condition (3.8), $V(A_i(x(t_i))) = 0$ as well, and so we can apply the same reasoning to the function $\tilde{g}(h) = V(x(t_i + h))$ for $h > 0$ and $\tilde{g}(0) = V(x(t_i^+))$. Since $x \in PC^1([0, T], \mathbb{R}^n)$, also $\tilde{g} \in C^1([0, \alpha], \mathbb{R})$ and $\tilde{g}(h) \le 0$ for $h > 0$ and $\tilde{g}(0) = 0$ imply $\dot{\tilde{g}}(0) \le 0$, i.e.

$$0 \ge \langle \nabla V(A_i(x(t_i))), B_i \dot{x}(t_i) \rangle.$$

Therefore, $x := x(t_i), v := \dot{x}(t_i)$ satisfy condition (3.9).

Using the same procedure as in the proof of Proposition 3.2, for $\bar{t} = t_i$, we obtain the existence of a sequence of negative numbers $\{h_k\}_{k=1}^{\infty}$ and of point $w \in F(t_i, x(t_i), \dot{x}(t_i))$ such that

$$\frac{\dot{x}(t_i + h_k) - \dot{x}(t_i)}{h_k} \to w \quad \text{as } k \to \infty.$$

By the same arguments as in the previous proof, we get

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x(t_i) + h\dot{x}(t_i)), \dot{x}(t_i) + hw \rangle}{h} \le 0. \tag{3.11}$$

Inequality (3.11) is in a contradiction with condition (3.10), which completes the proof. □

Remark 3.5. If condition (3.10) holds, for some $x \in \partial K$, $v \in \mathbb{R}^n$ satisfying (3.9) and $w \in F(t_i, x, v)$, then, according to the continuity of ∇V,

$$\langle \nabla V(x), v \rangle = 0. \tag{3.12}$$

Indeed

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x + hv), v + hw \rangle}{h} = \liminf_{h \to 0^-} \left[\frac{\langle \nabla V(x + hv), v \rangle}{h} + \langle \nabla V(x + hv), w \rangle \right]$$

which, since $\langle \nabla V(x), v \rangle \ge 0$, can be positive only if (3.12) holds.

Definition 3.6. A function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying all assumptions of Theorem 3.4 is called a *bounding function* for the set K relative to (1.1)–(1.3).

For our main result concerning the existence and localization of a solution of the Dirichlet b.v.p., we need to ensure that no solution of given b.v.p lies on the boundary ∂Q of a parameter set Q of candidate solutions. In the following section, it will be shown that if the set Q is defined as follows

$$Q := \{q \in PC^1([0,T],\mathbb{R}^n) \mid q(t) \in \overline{K}, \ \text{ for all } t \in [0,T]\} \tag{3.13}$$

and if all assumptions of Theorem 3.4 are satisfied, then solutions of the b.v.p. (1.1)–(1.3) behave as indicated.

Proposition 3.7. *Let $K \subset \mathbb{R}^n$ be a nonempty open bounded set with $0 \in K$, let $Q \subset PC^1([0,T],\mathbb{R}^n)$ be defined by the formula (3.13) and let $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semicontinuous multivalued mapping with nonempty, compact, convex values. Assume that there exists a function $V \in C^1(\mathbb{R}^n,\mathbb{R})$ with a locally Lipschitzian gradient ∇V which satisfies conditions $(H1)$ and $(H2)$. Moreover, assume that A_i, B_i, $i = 1,\ldots,p$, are real $n \times n$ matrices such that A_i, $i = 1,\ldots,p$, satisfy (3.8).*

Furthermore, suppose that, for all $x \in \partial K$, $t \in (0,T) \setminus \{t_1,\ldots,t_p\}$ and $v \in \mathbb{R}^n$ satisfying (3.1), condition (3.2) holds, for all $w \in F(t,x,v)$, and that, for all $x \in \partial K$ and $v \in \mathbb{R}^n$ satisfying (3.9), the condition (3.10) holds, for all $w \in F(t_i,x,v)$. Then problem (1.1)–(1.3) has no solution on ∂Q.

Proof. One can readily check that if $x \in \partial Q$, then there exists a point $t_x \in [0,T]$ such that $x(t_x) \in \partial K$. But then, according to Theorem 3.4, x cannot be a solution of (1.1)–(1.3). □

Let us now consider the particular case when the bounding function V is of class C^2. Then conditions (3.2) and (3.10) can be rewritten in terms of gradients and Hessian matrices and the following result can be directly obtained.

Corollary 3.8. *Let $K \subset \mathbb{R}^n$ be a nonempty open bounded set with $0 \in K$, let $Q \subset PC^1([0,T],\mathbb{R}^n)$ be defined by the formula (3.13) and let $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semicontinuous multivalued mapping with nonempty, compact, convex values. Assume that there exists a function $V \in C^2(\mathbb{R}^n,\mathbb{R})$ which satisfies conditions $(H1)$ and $(H2)$. Moreover, assume that A_i, B_i, $i = 1,\ldots,p$, are real $n \times n$ matrices such that A_i, $i = 1,\ldots,p$, satisfy (3.8).*

Furthermore, suppose that, for all $x \in \partial K$ and $v \in \mathbb{R}^n$ the following holds:

$$\text{if } \langle \nabla V(x), v\rangle = 0, \quad \text{then } \langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle > 0, \tag{3.14}$$

for all $t \in (0,T) \setminus \{t_1,\ldots,t_p\}$ and $w \in F(t,x,v)$, and fixed $i = 1,\ldots,n$

$$\text{if } \langle \nabla V(A_i x), B_i v\rangle \le 0 \le \langle \nabla V(x), v\rangle \quad \text{then } \langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle > 0, \tag{3.15}$$

for all $w \in F(t_i,x,v)$. Then problem (1.1)–(1.3) has no solution on ∂Q.

Proof. The statement of Corollary 3.8 follows immediately from Remark 3.5 and the fact that if $V \in C^2(\mathbb{R}^n,\mathbb{R})$, then, for all $x \in \partial K$, $t \in (0,T)$, $v \in \mathbb{R}^n$ and $w \in F(t,x,v)$, there exists

$$\begin{aligned}\lim_{h\to 0} \frac{\langle \nabla V(x+hv), v+hw\rangle}{h} &= \lim_{h\to 0} \frac{\langle \nabla V(x+hv), v+hw\rangle - \langle \nabla V(x), v\rangle}{h} \\ &= \langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle.\end{aligned}$$

□

Remark 3.9. In conditions (3.2), (3.10), (3.14) and (3.15), the element v plays the role of the first derivative of the solution x. If x is a solution of (1.1)–(1.3) such that $x(t) \in \overline{K}$, for every $t \in [0,T]$, and there exists a continuous increasing function $\psi : [0,\infty) \to [0,\infty)$ satisfying condition (2.3) and such that

$$|F(t,c,d)| \leq \psi(|d|), \tag{3.16}$$

for a.a. $t \in [0,T]$ and every $c,d \in \mathbb{R}^n$ with $|c| \leq R := \max\{|x| : x \in \overline{K}\}$, then, according to Proposition 2.1, it holds that $|\dot{x}(t)| \leq B$, for every $t \in [0,T]$, where B is defined by (2.4). Hence, it is sufficient to require conditions (3.2), (3.10), (3.14) and (3.15) in Proposition 3.2, Theorem 3.4 and Corollary 3.8 only for all $v \in \mathbb{R}^n$ with $|v| \leq B$ and not for all $v \in \mathbb{R}^n$.

4 Existence and localization result for the impulsive Dirichlet problem with upper semi-continuous r.h.s.

In order to obtain the main existence theorem, the bound sets technique described in the previous section will be combined with the topological method which was developed by ourselves in [25] for the impulsive boundary value problems. The version of the continuation principle for problems without impulses can be found e.g. in [7].

Proposition 4.1 ([25, Proposition 2.4])**.** *Let us consider the b.v.p.*

$$\begin{cases} \ddot{x}(t) \in F(t,x(t),\dot{x}(t)), & \text{for a.a. } t \in [0,T], \\ x \in S, \end{cases} \tag{4.1}$$

where $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping and S is a subset of $PC^1([0,T],\mathbb{R}^n)$. Let $H : [0,T] \times \mathbb{R}^{4n} \times [0,1] \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping such that

$$H(t,c,d,c,d,1) \subset F(t,c,d), \quad \text{for all } (t,c,d) \in [0,T] \times \mathbb{R}^{2n}. \tag{4.2}$$

Assume that

(i) there exists a retract Q of $PC^1([0,T],\mathbb{R}^n)$, with $Q \setminus \partial Q \neq \emptyset$, and a closed subset S_1 of S such that the associated problem

$$\begin{cases} \ddot{x}(t) \in H(t,x(t),\dot{x}(t),q(t),\dot{q}(t),\lambda), & \text{for a.a. } t \in [0,T], \\ x \in S_1 \end{cases} \tag{4.3}$$

has, for each $(q,\lambda) \in Q \times [0,1]$, a non-empty and convex set of solutions $\mathfrak{T}(q,\lambda)$;

(ii) there exists a nonnegative, integrable function $\alpha : [0,T] \to \mathbb{R}$ such that

$$|H(t,x(t),\dot{x}(t),q(t),\dot{q}(t),\lambda)| \leq \alpha(t)(1 + |x(t)| + |\dot{x}(t)|), \quad \text{for a.a. } t \in [0,T],$$

for any $(q,\lambda,x) \in \Gamma_{\mathfrak{T}}$;

(iii) $\mathfrak{T}(Q \times \{0\}) \subset Q$;

(iv) there exist constants $M_0 \geq 0$, $M_1 \geq 0$ such that $|x(0)| \leq M_0$ and $|\dot{x}(0)| \leq M_1$, for all $x \in \mathfrak{T}(Q \times [0,1])$;

(v) the solution map $\mathfrak{T}(\cdot,\lambda)$ has no fixed points on the boundary ∂Q of Q, for every $\lambda \in [0,1)$.

Then the b.v.p. (4.1) *has a solution in* $S_1 \cap Q$.

Remark 4.2. The condition that Q is a retract of $PC^1([0,T],\mathbb{R}^n)$ in Proposition 4.1 can be replaced by the assumption that Q is an absolute neighborhood retract and $\operatorname{ind}(\mathfrak{T}(\cdot,0),Q,Q) \neq 0$ (see, e.g., [2]). It is therefore possible to assume alternatively that Q is a retract of a convex subset of $PC^1([0,T],\mathbb{R}^n)$ or of an open subset of $PC^1([0,T],\mathbb{R}^n)$ together with $\operatorname{ind}(\mathfrak{T}(\cdot,0),Q,Q)\neq 0$.

The solvability of (1.1) will be now proved, on the basis of Proposition 4.1. Defining namely the set Q of candidate solutions by the formula (3.13), we are able to verify, for each $(q,\lambda) \in Q \times [0,1)$, the transversality condition (v) in Proposition 4.1.

Theorem 4.3. *Let* $K \subset \mathbb{R}^n$ *be a nonempty, open, bounded and convex set with* $0 \in K$ *and let us consider the impulsive Dirichlet problem* (1.1)–(1.3), *where* $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is an upper semicontinuous multivalued mapping,* $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T, p \in \mathbb{N}$, *and* A_i, B_i, $i = 1,\ldots,p$, *are real* $n \times n$ *matrices with* $A_i\partial K = \partial K$, *for all* $i = 1,\ldots,p$. *Moreover, assume that*

(i) *there exists a function* $\beta : [0,\infty) \to [0,\infty)$ *continuous and increasing satisfying*

$$\lim_{s\to\infty} \frac{s^2}{\beta(s)} ds = \infty$$

such that

$$|F(t,c,d)| \leq \beta(|d|),$$

for a.a. $t \in [0,T]$ *and every* $c, d \in \mathbb{R}^n$ *with* $|c| \leq R := \max\{|x| : x \in \overline{K}\}$;

(ii) *the problem*

$$\begin{cases} \ddot{x}(t) = 0, & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, & \\ x(t_i^+) = A_i x(t_i), & i = 1,\ldots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1,\ldots,p, \end{cases} \tag{4.4}$$

has only the trivial solution;

(iii) *there exists a function* $V \in C^1(\mathbb{R}^n,\mathbb{R})$, *with* ∇V *locally Lipschitzian, satisfying conditions* $(H1)$ *and* $(H2)$;

(iv) *for all* $x \in \partial K$ *and* $v \in \mathbb{R}^n$ *with* $|v| \leq \beta^{-1}(\beta(2R) + 2R)$, *the inequality*

$$\liminf_{h\to 0^-} \frac{\langle \nabla V(x+hv), v + h\lambda w\rangle}{h} > 0$$

holds, for all $t \in (0,T) \setminus \{t_1,\ldots,t_p\}, \lambda \in (0,1)$ *and* $w \in F(t,x,v)$ *if* $\langle \nabla V(x), v\rangle = 0$ *and for all* $\lambda \in (0,1), w \in F(t_i,x,v)$ *if* $\langle \nabla V(A_i x), B_i v\rangle \leq 0 \leq \langle \nabla V(x), v\rangle$.

Then the Dirichlet problem (1.1)–(1.3) *has a solution* $x(\cdot)$ *such that* $x(t) \in \overline{K}$, *for all* $t \in [0,T]$.

Proof. Define

$$B = \beta^{-1}(\beta(2R) + 2R),$$

$$S = S_1 = Q := \{q \in PC^1([0,T],\mathbb{R}^n) \mid q(t) \in \overline{K},\ |\dot{q}(t)| \leq 2B,\ \text{for all } t \in [0,T]\}$$

and $H(t,c,d,e,f,\lambda) = \lambda F(t,e,f)$. Thus the associated problem (4.3) is the fully linearized problem

$$\begin{cases} \ddot{x}(t) \in \lambda F(t,q(t),\dot{q}(t)), & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, & \\ x(t_i^+) = A_i x(t_i), & i = 1,\dots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1,\dots,p. \end{cases} \tag{4.5}$$

For each $(q,\lambda) \in Q \times [0,1]$, let $\mathfrak{T}(q,\lambda)$ be the solution set of (4.5). We will check now that all the assumptions of Proposition 4.1 are satisfied.

Since the closure of a convex set is still a convex set, it follows that Q is convex, and hence a retract of $PC^1([0,T],\mathbb{R}^n)$. Moreover,

$$\operatorname{Int} Q = \{q \in PC^1([0,T],\mathbb{R}^n) \mid q(t) \in K,\ |\dot{q}(t)| < 2B,\ \text{for all } t \in [0,T]\} \neq \emptyset,$$

since K is nonempty.

Notice now that, for every $t \in [0,T], c, d \in \mathbb{R}^n$, the inequality

$$|H(t,e,f,c,d,\lambda)| = \lambda |F(t,e,f)| \le \beta(|f|) \tag{4.6}$$

holds. Hence, denoting $z = (c,d,e,f,\lambda) \in \mathbb{R}^{4n+1}$, since $|f| \le |z|$, when $|z| \le r$, the monotonicity of β implies that $|H(t,c,d,e,f,\lambda)| \le \beta(r)$, which ensures, for every $q \in Q$, the existence of $f_q \in \mathcal{P}_F(q)$. Given $q \in Q$, $\lambda \in [0,1]$, and a L^1-selection $f_q(\cdot)$ of $F(\cdot,q(\cdot),\dot{q}(\cdot))$, let us consider the corresponding single valued linear problem with linear impulses

$$\begin{cases} \ddot{x}(t) = \lambda f_q(t), & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, & \\ x(t_i^+) = A_i x(t_i), & i = 1,\dots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1,\dots,p. \end{cases} \tag{4.7}$$

Clearly, for all $q \in Q$ and $\lambda \in [0,1]$,

$$\mathfrak{T}(q,\lambda) = \{x_{\lambda f_q} \in PC^1([0,T],\mathbb{R}^n) : x_{\lambda f_q} \text{ is a solution of (4.7), for some } f_q \in \mathcal{P}_F(q)\}.$$

Using the notation

$$C := \begin{cases} B_1(T-t_1) + A_1 t_1 & \text{if } p = 1 \\ \displaystyle\prod_{l=1}^{p} B_l (T-t_p) + \prod_{k=1}^{p} A_k t_1 + \sum_{j=2}^{p} \prod_{k=j}^{p} A_k \prod_{l=1}^{j-1} B_l (t_j - t_{j-1}) & \text{if } p \ge 2, \end{cases} \tag{4.8}$$

it is easy to prove that the initial problem

$$\begin{cases} \ddot{x}(t) = 0, & \text{for a.a. } t \in [0,T], \\ x(0) = 0, & \\ x(t_i^+) = A_i x(t_i), & i = 1,\dots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1,\dots,p \end{cases}$$

has infinitely many solutions given by

$$x_0(t) = \begin{cases} \dot{x}_0(0)t & \text{if } t \in [0,t_1], \\ B_1 \dot{x}_0(0)(t-t_1) + A_1 \dot{x}_0(0) t_1 & \text{if } t \in (t_1,t_2] \\ \left[\displaystyle\prod_{l=1}^{i} B_l (t-t_i) + \prod_{k=1}^{i} A_k t_1 + \sum_{j=2}^{i} \prod_{k=j}^{i} A_k \prod_{l=1}^{j-1} B_l (t_j - t_{j-1})\right] \dot{x}_0(0) & \text{if } t \in (t_i,t_{i+1}],\ 2 \le i \le p \end{cases}$$

with $\dot{x}_0(0) \in \mathbb{R}^n$. Since $x_0(T) = 0$ if and only if $C\dot{x}_0(0) = 0$, assumption (ii) holds if and only if C is regular. Then (4.7) has a unique solution given by

$$x_{\lambda f_q}(t) = \begin{cases} \dot{x}_{\lambda f_q}(0)t + \lambda \int_0^t (t-\tau) f_q(\tau) d\tau \quad \text{if } t \in [0, t_1], \\ B_1 \dot{x}_{\lambda f_q}(0)(t - t_1) + \lambda \int_{t_1}^t (t-\tau) f_q(\tau) d\tau + B_1(t - t_1)\lambda \int_0^{t_1} f_q(\tau) d\tau + A_1 \dot{x}_{\lambda f_q}(0) t_1 \\ \qquad + A_1 \lambda \int_0^{t_1} (t_1 - \tau) f_q(\tau) d\tau \quad \text{if } t \in (t_1, t_2], \\ \prod_{l=1}^{i} B_l \dot{x}_{\lambda f_q}(0)(t - t_i) + \lambda \int_{t_i}^t (t-\tau) f_q(\tau) d\tau + \sum_{r=1}^{i} \prod_{l=r}^{i} B_l (t - t_i) \lambda \int_{t_{r-1}}^{t_r} f_q(\tau) d\tau \\ \qquad + \prod_{k=1}^{i} A_k \dot{x}_{\lambda f_q}(0) t_1 + \prod_{k=1}^{i} A_k \lambda \int_0^{t_1} (t_1 - \tau) f_q(\tau) d\tau \\ \qquad + \sum_{j=2}^{i} \prod_{k=j}^{i} A_k \Bigg[\prod_{l=1}^{j-1} B_l \dot{x}_{\lambda f_q}(0)(t_j - t_{j-1}) + \lambda \int_{t_{j-1}}^{t_j} (t_j - \tau) f_q(\tau) d\tau \\ \qquad + \sum_{r=1}^{k-1} \prod_{l=r}^{k-1} B_l (t_j - t_{j-1}) \lambda \int_{t_{r-1}}^{t_r} f_q(\tau) d\tau \Bigg] \quad \text{if } t \in (t_i, t_{i+1}],\ 2 \le i \le p \end{cases}$$

with

$$\dot{x}_{\lambda f_q}(0) = -C^{-1} \left(\lambda \int_{t_1}^T (T-\tau) f_q(\tau) d\tau + B_1(T - t_1)\lambda \int_0^{t_1} f_q(\tau) d\tau + A_1 \lambda \int_0^{t_1} (t_1 - \tau) f_q(\tau) d\tau \right) \tag{4.9}$$

if $p = 1$ and

$$\begin{aligned} \dot{x}_{\lambda f_q}(0) = -C^{-1} \Bigg(& \lambda \int_{t_p}^T (T-\tau) f_q(\tau) d\tau + \sum_{r=1}^{p} \prod_{l=r}^{p} B_l (T - t_p) \lambda \int_{t_{r-1}}^{t_r} f_q(\tau) d\tau \\ & + \prod_{k=1}^{p} A_k \lambda \int_0^{t_1} (t_1 - \tau) f_q(\tau) d\tau \\ & + \sum_{j=2}^{p} \prod_{k=j}^{p} A_k \Bigg[\lambda \int_{t_{j-1}}^{t_j} (t_j - \tau) f_q(\tau) d\tau + \sum_{r=1}^{k-1} \prod_{l=r}^{k-1} B_l (t_j - t_{j-1}) \lambda \int_{t_{r-1}}^{t_r} f_q(\tau) d\tau \Bigg] \Bigg) \end{aligned} \tag{4.10}$$

if $p \geq 2$. Therefore $\mathfrak{T}(q, \lambda) \neq \emptyset$. Moreover, given $x_1, x_2 \in \mathfrak{T}(q, \lambda)$, there exist f_q^1, f_q^2 such that $x_1 = x_{\lambda f_q^1}$ and $x_2 = x_{\lambda f_q^2}$. Since the right-hand side F has convex values, it holds that, for any $c \in [0, 1]$ and $t \in [0, T]$, $c f_q^1(t) + (1 - c) f_q^2(t) \in F(t, q(t), \dot{q}(t))$ as well. The linearity of both the equation and of the impulses yields that $cx_1 + (1 - c)x_2 = x_{c f_q^1 + (1-c) f_q^2}$, i.e. that the set of solutions of problem (4.5) is, for each $(q, \lambda) \in Q \times [0, 1]$, convex. Hence assumption (i) of Proposition 4.1 is satisfied.

Moreover, from (4.6), we obtain that, for every $\lambda \in [0, 1], q \in Q, x \in \mathfrak{T}(q, \lambda)$,

$$|H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda| \leq \beta(|\dot{q}(t)|) \leq \beta(2B) \leq \beta(2B)(1 + |x(t)| + |\dot{x}(t)|), \tag{4.11}$$

thus also assumption (ii) of the same proposition holds.

The fulfillment of condition (iii) in Proposition 4.1 follows from the fact that, for $\lambda = 0$, problems (4.7) and (4.4) coincide and the latter one has only the trivial solution. Hence, $\mathfrak{T}(q, 0) = 0 \in \operatorname{Int} Q$, because $0 \in K$.

For every $\lambda \in [0,1]$, $q \in Q$ and every solution $x_{\lambda f_q}$ of (4.7), $|x_{\lambda f_q}(0)| = 0$. Moreover, according to assumption (i) and formulas (4.9) and (4.10),

$$
\begin{aligned}
|\dot{x}_{\lambda f_q}(0)| &\leq \|C^{-1}\| \left[\beta(2B)\frac{1}{2}T^2 + T^2\|B_1\|\beta(2B) + \frac{1}{2}T^2\|A_1\|\beta(2B)\right] \\
&= T^2\|C^{-1}\| \cdot \beta(2B)\left[\frac{1}{2} + \|B_1\| + \frac{1}{2}\|A_1\|\right]
\end{aligned}
$$

if $p = 1$ and

$$
\begin{aligned}
|\dot{x}_{\lambda f_q}(0)| \leq \|C^{-1}\| &\left[\frac{1}{2}T^2\beta(2B) + T^2\prod_{l=1}^{p}\|B_l\| \cdot \beta(2B)\right. \\
&\left. + T^2\prod_{k=1}^{p}\|A_k\|\beta(2B) + T^2\prod_{l=1}^{p}\|B_l\|\prod_{k=1}^{p}\|A_k\| \cdot \beta(2B)\right] \\
= T^2\|C^{-1}\| \cdot \beta(2B) &\left[\frac{1}{2} + \prod_{l=1}^{p}\|B_l\| + \prod_{k=1}^{p}\|A_k\| + \prod_{l=1}^{p}\|B_l\|\prod_{k=1}^{p}\|A_k\|\right]
\end{aligned}
$$

if $p \geq 2$. Therefore there exists a constant M_1 such that $|\dot{x}(0)| \leq M_1$, for all solutions x of (4.5). Hence, condition (iv) in Proposition 4.1 is satisfied as well.

At last, let us assume that $q_* \in Q$ is, for some $\lambda \in [0,1)$, a fixed point of the solution mapping $\mathfrak{T}(\cdot,\lambda)$. We will show now that q_* can not lay in ∂Q. We already proved this property if $\lambda = 0$, thus we can assume that $\lambda \in (0,1)$. From (4.11), we have, for a.a. $t \in [0,T]$, that

$$
|\ddot{q}_*(t)| = \lambda|F(t,q_*(t),\dot{q}_*(t))| \leq \beta(|\dot{q}_*(t)|).
$$

Therefore, since $|q_*(t)| \leq R$, for every $t \in [0,T]$, Proposition 2.1 implies that $|\dot{q}_*(t)| \leq B < 2B$, for every $t \in [0,T]$. Moreover, according to Theorem 3.4 and Remark 3.9, hypotheses (iii) and (iv) guarantee that $q_*(t) \in K$, for all $t \in [0,T]$. Thus $q_* \in \operatorname{Int} Q$, which implies that condition (v) from Proposition 4.1 is satisfied, for all $\lambda \in [0,1)$, and the proof is completed. □

Remark 4.4. An easy example of impulses conditions guaranteeing assumption (ii) in Theorem 4.3 are the antiperiodic impulses, i.e. $A_i = B_i = -I$, for every $i = 1,\ldots,p$. In this case, the matrix $C = (-1)^pTI$ (see [25]) and it is clearly regular. If $p = 1$ condition (ii) holds also e.g. for $A_1 = -I$ and $B_1 = I$ provided $T \neq 2t_1$.

5 Existence and localization result for the impulsive Dirichlet problem with upper-Carathéodory r.h.s.

In this section, we will study the impulsive Dirichlet b.v.p. (1.1)–(1.3) with an upper-Carathéodory r.h.s. and we will develop the bounding functions method with the strictly localized bounding functions also in this more general case. The technique which will be applied for obtaining the final result consists in replacing the original problem by the sequence of problems with non-strict localized bounding functions which satisfy all the assumptions of the following result developed by ourselves recently in [25].

Proposition 5.1 ([25, Theorem 4.1 and Remark 4.3])**.** *Let $K \subset \mathbb{R}^n$ be a nonempty, open, bounded and convex set with $0 \in K$ and let us consider the impulsive Dirichlet problem (1.1)–(1.3), where $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory multivalued mapping, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T, p \in \mathbb{N}$, and A_i, B_i, $i = 1,\ldots,p$, are real $n \times n$ matrices with $A_i\partial K = \partial K$, for all $i = 1,\ldots,p$. Moreover, assume that* (*i*) *there exists a function* $\varphi : [0,\infty) \to [0,\infty)$ *continuous and increasing satisfying*

$$\lim_{s\to\infty} \frac{s^2}{\varphi(s)} ds = \infty \tag{5.1}$$

such that

$$|F(t,c,d)| \le \varphi(|d|), \tag{5.2}$$

for a.a. $t \in [0,T]$ *and every* $c,d \in \mathbb{R}^n$ *with* $|c| \le R := \max\{|x| : x \in \overline{K}\}$;

(*ii*) *the problem*

$$\begin{cases} \ddot{x}(t) = 0, & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, & \\ x(t_i^+) = A_i x(t_i), & i = 1,\dots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1,\dots,p, \end{cases} \tag{5.3}$$

has only the trivial solution;

(*iii*) *there exists a function* $V \in C^1(\mathbb{R}^n,\mathbb{R})$, *with* ∇V *locally Lipschitzian, satisfying conditions* $(H1)$ *and* $(H2)$;

(*iv*) *there exists* $\varepsilon > 0$ *such that, for all* $\lambda \in (0,1)$, $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0,T)$, *and* $v \in \mathbb{R}^n$, *with* $|v| \le \varphi^{-1}(\varphi(2R) + 2R)$, *the following condition*

$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle > 0 \tag{5.4}$$

holds, for all $w \in \lambda F(t,x,v)$;

(*v*) *for all* $i = 1,\dots,p$, $x \in \partial K$ *and* $v \in \mathbb{R}^n$, *with* $|v| \le \varphi^{-1}(\varphi(2R)+2R)$ *and* $\langle \nabla V(x), v\rangle \ne 0$, *it holds that*

$$\langle \nabla V(A_i x), B_i v\rangle \cdot \langle \nabla V(x), v\rangle > 0.$$

Then the Dirichlet problem (1.1)–(1.3) *has a solution* $x(\cdot)$ *such that* $x(t) \in \overline{K}$, *for all* $t \in [0,T]$.

Approximating the original problem by a sequence of problems satisfying conditions of Proposition 5.1 and applying the Scorza-Dragoni type result (Proposition 2.2), we are already able to state the second main result of the paper. The transversality condition is now required only on the boundary ∂K of the set K and not on the whole neighborhood $\overline{K} \cap N_\varepsilon(\partial K)$, as in Proposition 5.1.

Theorem 5.2. *Let* $K \subset \mathbb{R}^n$ *be a nonempty, open, bounded and convex set with* $0 \in K$ *and let us consider the impulsive Dirichlet problem* (1.1)–(1.3), *where* $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is an upper Carathéodory multivalued mapping,* $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T, p \in \mathbb{N}$, *and* A_i, B_i, $i = 1,\dots,p$, *are real* $n \times n$ *matrices with* $A_i \partial K = \partial K$, *for all* $i = 1,\dots,p$. *Moreover, assume that*

(*i*) *there exists a function* $\beta : [0,\infty) \to [0,\infty)$ *continuous and increasing satisfying*

$$\lim_{s\to\infty} \frac{s^2}{\beta(s)} ds = \infty$$

such that

$$|F(t,c,d)| \le \beta(|d|),$$

for a.a. $t \in [0,T]$ *and every* $c,d \in \mathbb{R}^n$ *with* $|c| \le R := \max\{|x| : x \in \overline{K}\}$;

(*ii*) *the problem*

$$\begin{cases} \ddot{x}(t) = 0, & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, & \\ x(t_i^+) = A_i x(t_i), & i = 1,\dots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1,\dots,p, \end{cases} \tag{5.5}$$

has only the trivial solution;

(*iii*) *there exists $h > 0$ and a function $V \in C^2(\mathbb{R}^n, \mathbb{R})$, with $HV(x)$ positive semidefinite in $N_h(\partial K)$, satisfying conditions $(H1), (H2)$;*

(*iv*) *for all $x \in \partial K$ and $v \in \mathbb{R}^n$, with $|v| \le \beta^{-1}(\beta(2R) + 2R)$, the inequality*

$$\langle \nabla V(x), w \rangle > 0$$

holds for all $t \in (0,T)$ and $w \in F(t,x,v)$;

(*v*) *for all $i = 1,\dots,p$, $x \in \partial K$ and $v \in \mathbb{R}^n$, with $|v| \le \beta^{-1}(\beta(2R) + 2R)$ and $\langle \nabla V(x), v \rangle \ne 0$, it holds that*

$$\langle \nabla V(A_i x), B_i v \rangle \cdot \langle \nabla V(x), v \rangle > 0.$$

Then the Dirichlet problem (1.1)–(1.3) *has a solution $x(\cdot)$ such that $x(t) \in \overline{K}$, for all $t \in [0,T]$.*

Proof. Since $V \in C^2(\mathbb{R}^n, \mathbb{R})$, the function $x \to |\nabla V(x)|$ is continuous on the compact set ∂K, and hence there exists $k > 0$ such that $|\nabla V(x)| > 0$ for every $x \in N_k(\partial K)$. Define $\delta = \min\{h,k\}$. According to Urysohn's Lemma, there exists a function $\mu \in C(\mathbb{R}^n, [0,1])$ such that $\mu \equiv 1 \in N_{\frac{\delta}{2}}(\partial K)$ and $\mu \equiv 0 \in \mathbb{R}^n \setminus N_\delta(\partial K)$. Take a sequence of positive numbers $\{\epsilon_m\}$ decreasing to zero, an open and bounded set G, with $\overline{K} \subset G$, and $L > \beta^{-1}(\beta(2R) + 2R)$. According to Proposition 2.2 there exist a monotone decreasing sequence $\{\theta_m\}$ of open subsets of $[0,T]$ and a measurable multimap $F_0 : [0,T] \times \overline{G} \times \{v \in \mathbb{R}^n : |v| \le L\} \multimap \mathbb{R}^n$ such that $\nu(\theta_m) \le \epsilon_m, F_0(t,x,v) \subset F(t,x,v)$ and F_0 is u.s.c. on $([0,T] \setminus \theta_m) \times \overline{G} \times \{v \in \mathbb{R}^n : |v| \le L\}$ for every $m \in \mathbb{N}$. Trivially $\nu(\cap_{m=1}^\infty \theta_m) = 0$ and $\lim_{m\to\infty} \chi_{\theta_m}(t) = 0$ for every $t \notin \cap_{m=1}^\infty \theta_m$.

Define, for each $m \in \mathbb{N}$, $(t,x,v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$F_m(t,x,v) = \begin{cases} F_0(t,x,v) + 2\mu(x)\beta(|v|)\chi_{\theta_m}(t)\frac{\nabla V(x)}{|\nabla V(x)|} & \text{if } x \in G \text{ and } |v| < L \\ F(t,x,v) + 2\mu(x)\beta(|v|)\chi_{\theta_m}(t)\frac{\nabla V(x)}{|\nabla V(x)|}, & \text{otherwise.} \end{cases}$$

Since $\delta \le k$, we have that $\mu(x) = 0$ for $x \in \mathbb{R}^n \setminus N_\delta(\partial K)$ and $\nabla V(x) \ne 0$ in $N_\delta(\partial K)$, hence it follows that F_m is well defined. Since μ and β are continuous, V is of class C^2, G is open, $F_0(t,x,v) \subset F(t,x,v)$, and F is an upper-Carathéodory map, F_m is a Carathéodory map as well.

Let us now prove that problem

$$\begin{cases} \ddot{x}(t) \in F_m(t,x(t),\dot{x}(t)), & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, & \\ x(t_i^+) = A_i x(t_i), & i = 1,\dots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1,\dots,p. \end{cases} \tag{5.6}$$

satisfies the assumptions of Proposition 5.1.

First of all notice that, since $0 \le \mu(x) \le 1, 0 \le \chi_{\theta_m}(t) \le 1$, for every $x \in \mathbb{R}^n, t \in [0,T]$, it holds, according to (i),

$$|F_m(t,c,d)| \le |F(t,c,d)| + 2\beta(|d|) \le 3\beta(|d|),$$

for every $(t,c,d) \in t \times \mathbb{R}^n \times \mathbb{R}^n$ with $|c| \le R$. Thus condition (i) of Proposition 5.1 is satisfied by the continuous increasing function $\varphi = 3\beta$, since it clearly holds that

$$\lim_{s\to\infty} \frac{s^2}{\varphi(s)} = \frac{1}{3} \lim_{s\to\infty} \frac{s^2}{\beta(s)} = \infty.$$

Moreover, conditions (ii) and (iii) imply the analogous conditions in Proposition 5.1.

Let us now observe that, since $\varphi(s) = 3\beta(s)$, then $\varphi^{-1}(s) = \beta^{-1}(\frac{s}{3})$, which is an increasing function, as inverse of an increasing function. Hence

$$\varphi^{-1}(\varphi(2R) + 2R) = \beta^{-1}\left(\frac{3\beta(2R) + 2R}{3}\right) = \beta^{-1}\left(\beta(2R) + \frac{2}{3}R\right) \le \beta^{-1}(\beta(2R) + 2R).$$

Therefore, condition (v) implies the analogous condition of Proposition 5.1. Moreover, for every $\lambda \in (0,1), x \in \overline{K} \cap N_{\frac{\delta}{2}}(\partial K)$, $t \in (0,T)$, and $v \in \mathbb{R}^n$, with $|v| \le \varphi^{-1}(\varphi(2R) + 2R), w_1 \in \lambda F_m(t,x,v)$,

$$\begin{aligned}\langle HV(x)v, v\rangle + \langle \nabla V(x), w_1\rangle &= \langle HV(x)v, v\rangle + \lambda[\langle \nabla V(x), w\rangle + 2\mu(x)\beta(|v|)\chi_{\theta_m}(t)|\nabla V(x)|] \\ &= \langle HV(x)v, v\rangle + \lambda[\langle \nabla V(x), w\rangle + 2\beta(|v|)\chi_{\theta_m}(t)|\nabla V(x)|].\end{aligned}$$

Then, if $t \in [0,T] \setminus \theta_m$

$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w_1\rangle = \langle HV(x)v, v\rangle + \lambda\langle \nabla V(x), w\rangle \ge \lambda\langle \nabla V(x), w\rangle,$$

with $w \in F_0(t,x,v)$, because $\overline{K} \cap N_{\frac{\delta}{2}}(\partial K) \subset \overline{K} \subset G$ and $\varphi^{-1}(\varphi(2R) + 2R) \le \beta^{-1}(\beta(2R) + 2R) < L$. Since V is of class C^2, F_0 is u.s.c. on the compact set $([0,T] \setminus \theta_m) \times \partial K \times \{v \in \mathbb{R}^n : |v| \le \varphi^{-1}(\varphi(2R) + 2R)\}$, and F_0 is compact valued, condition (iv) implies that there exists $k_1 > 0$ such that

$$\langle \nabla V(x), w\rangle > 0$$

for every $t \in [0,T] \setminus \theta_m, x \in \overline{K} \cap N_{k_1}(\partial K), v \in \mathbb{R}^n : |v| \le \varphi^{-1}(\varphi(2R) + 2R), w \in F_0(t,x,v)$. Hence,

$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w_1\rangle \ge \lambda\langle \nabla V(x), w\rangle > 0,$$

for all $\lambda \in (0,1), t \in [0,T] \setminus \theta_m, x \in \overline{K} \cap N_{k_1}(\partial K), v \in \mathbb{R}^n : |v| \le \varphi^{-1}(\varphi(2R) + 2R), w_1 \in \lambda F_m(t,x,v)$.

On the other hand, if $t \in \theta_m$, since $x \in N_{\frac{\delta}{2}}(\partial K)$ and $h \ge \delta$,

$$\begin{aligned}\langle HV(x)v, v\rangle + \langle \nabla V(x), w_1\rangle &\ge \lambda[\langle \nabla V(x), w\rangle + 2\beta(|v|)|\nabla V(x)|] \\ &\ge \lambda[-|w| + 2\beta(|v|)]|\nabla V(x)| \ge \lambda\beta(|v|)|\nabla V(x)| > 0.\end{aligned}$$

Condition (iv) in Proposition 5.1 follows taking $\epsilon = \min\{k_1, \frac{\delta}{2}\}$.

Applying Proposition 5.1 we obtain that, for every $m \in \mathbb{N}$, there exists a solution x_m of (5.6) such that $x_m(t) \in \overline{K}$ and $|\dot{x}_m(t)| \le \varphi^{-1}(\varphi(2R) + 2R)$, for every $t \in [0,T]$. Hence $|\ddot{x}_m(t)| \le \varphi(2R) + 2R$ for every $t \in [0,T]$. The Ascoli–Arzelà theorem implies that $\{x_m\} \to x$ uniformly in $C^1([0,T],\mathbb{R}^n)$ and $\ddot{x}_m \to \ddot{x}$ weakly in $L^1([0,T],\mathbb{R}^n)$. Thus $x(t) \in \overline{K}$, $|\dot{x}(t)| \le \varphi^{-1}(\varphi(2R)+2R)$ for every $t \in [0,T]$, and x satisfies (1.2)–(1.3). Moreover, since $\nu(\cap_{n=1}^{\infty}\theta_m) = 0$,

$$\lim_{m\to\infty} 2\mu(x_m(t))\beta(|\dot{x}_m(t)|)\chi_{\theta_m}(t)\frac{\nabla V(x_m(t))}{|\nabla V(x_m(t))|} = 0,$$

for a.a. $t \in [0,T]$. Consequently, a standard limiting argument (see e.g. [28, Theorem 3.1.2]) implies that x is a solution of

$$\ddot{x}(t) = F_0(t,x(t),\dot{x}(t))$$

and, since $F_0(t,x(t),\dot{x}(t)) \subset F(t,x(t),\dot{x}(t))$, a solution of the problem (1.1)–(1.3). □

Remark 5.3. Both Theorems 4.3 and 5.2 give an existence result for an impulsive Dirichlet boundary value problem with a strictly localized bounding function respectively for u.s.c. and upper-Carathéodory multimap. However Theorem 5.2 does not represent an extension of Theorem 4.3, since the first one deals with a C^2-bounding function, while the second one is related to a C^1-bounding function and can not be easily extended to the Carathéodory case.

In the case when the multivalued mapping F is u.s.c. and the bounding function V is of class C^2, i.e. when it is possible to apply both theorems, conditions of Theorem 4.3 are weaker than assumptions of Theorem 5.2. In fact, in this case, according to Corollary 3.8, condition (iv) of the first theorem reads as

$$\langle HV(x)v,v\rangle + \lambda\langle\nabla V(x),w\rangle > 0$$

for every $x \in \partial K, \lambda \in (0,1)$, $v \in \mathbb{R}^n$, with $|v| \le \beta^{-1}(\beta(2R)+2R)$, and for every $t \in [0,T] \setminus \{t_1,\dots,t_p\}, w \in F(t,x,v)$ if $\langle\nabla V(x),v\rangle \ne 0$, or for every $w \in F(t_i,x,v)$ if $\langle\nabla V(A_i x),B_i v\rangle \le 0 \le \langle\nabla V(x),v\rangle$, which are implied by assumptions (iii) and (iv) of the second theorem.

6 An application of the main result

As an application of Theorem 5.2, let us consider the second-order inclusion

$$\ddot{x}(t) \in a(t)\dot{x}(t) + h(t,x(t))), \quad \text{for a.a. } t \in [0,T], \tag{6.1}$$

together with antiperiodic impulses and Dirichlet boundary conditions

$$x(t_i^+) = -x(t_i), \qquad i = 1,\dots,p, \tag{6.2}$$
$$\dot{x}(t_i^+) = -\dot{x}(t_i), \qquad i = 1,\dots,p, \tag{6.3}$$
$$x(0) = x(T) = 0, \tag{6.4}$$

where $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$. Assume that $a \in L^\infty([0,T],\mathbb{R})$, with $\|a\|_\infty > 0$, and $h : [0,T]\times\mathbb{R} \multimap \mathbb{R}$ is an upper-Carathéodory multivalued mapping with

$$|h(t,y)| \le \alpha(t)g(y)$$

for some $\alpha \in L^\infty([0,T],\mathbb{R})$, $g \in C(\mathbb{R},\mathbb{R})$.

When h is a function, the impulsive Dirichlet boundary value problem associated to the single valued equation $\ddot{x}(t) = a(t)\dot{x}(t) + h(t,x(t))$ represents a generalization of a wide class of equations which are widely studied in literature (see, e.g., [1,13,16,26,29]) for its several applications (including biological phenomena involving thresholds, models describing population dynamics or inspection processes in operations research). Much more rare are the results concerning the multivalued case which can be e.g. used for modelling optimal control problems in economics.

We will show now that, under very general conditions, the Dirichlet multivalued problem (6.1), (6.4) together with impulse conditions (6.2), (6.3) satisfies all the assumptions of Theorem 5.2. On this purpose, let us consider the nonempty, open, bounded, convex and symmetric neighbourhood of the origin $K = (-k, k)$, with k to be specified later, and the C^2-function $V(x) = \frac{1}{2}(x^2 - k^2)$ that trivially satisfies conditions $(H1)$ and $(H2)$.

In order to verify condition (i), let us define the continuous and increasing function

$$\beta(d) = \|a\|_\infty d + \|\alpha\|_\infty \overline{g}, \quad \text{for all } d \in [0, +\infty),$$

where $\overline{g} = \max_{|x|\le k} |g(x)|$. The function β obviously satisfies (5.1) and $F(t, c, d) := a(t)d + h(t, c)$ satisfies (3.16), for all $t \in [0, T]$ and all c, $d \in \mathbb{R}$, with $|c| \le k$.

Assumption (ii) holds as well since, according to Remark 4.4, the associated homogeneous problem has only the trivial solution.

Condition (iii) follows from the fact that $\dot{V}(x) = x$ and $\ddot{V}(x) = 1$, for every $x \in \mathbb{R}$.

Notice moreover that, whenever $xv \ne 0$, then $(-x)(-v)xv = x^2v^2 > 0$, hence also condition (v) holds.

Finally, since $\beta^{-1}(d) = \frac{1}{\|a\|_\infty}(d - \|\alpha\|_\infty \overline{g})$, we easily get that

$$\beta^{-1}(\beta(2k) + 2k) = 2k\left(1 + \frac{1}{\|a\|_\infty}\right).$$

Thus condition (iv) reads as

$$a(t)xv + xw > 0 \tag{6.5}$$

for every $t \in [0, T]$, x with $|x| = k$, v with $|v| \le 2k(1 + \frac{1}{\|a\|_\infty})$ and $w \in h(t, x)$. Taking $x = k$ we then get $w > -a(t)v$, for every $w \in h(t, k)$. Since the previous condition must hold both for positive and negative values of v, $h(t, k)$ must take only positive values and the transversality condition is satisfied if

$$w > \|a\|_\infty 2k\left(1 + \frac{1}{\|a\|_\infty}\right) = 2k(\|a\|_\infty + 1) \qquad \forall\, w \in h(t, k).$$

Similarly,taking $x = -k$ we get that (6.5) is equivalent to $w < -a(t)v$, for every $w \in h(t, -k)$ which is satisfied only if w is negative. A sufficient condition then becomes

$$w < -2k(\|a\|_\infty + 1) \qquad \forall\, w \in h(t, -k).$$

Thus condition (iv) holds if there exists $k > 0$ such that for every $w_1 \in h(t, k)$, $w_2 \in h(t, -k)$,

$$w_1 > 2k(\|a\|_\infty + 1) \quad \text{and} \quad w_2 < -2k(\|a\|_\infty + 1). \tag{6.6}$$

The previous result can be stated in the form of the following theorem.

Theorem 6.1. *Assume that $a \in L^\infty([0, T], \mathbb{R})$, with $\|a\|_\infty > 0$, $h : [0, T] \times \mathbb{R} \multimap \mathbb{R}$ is an upper-Carathédory multivalued mapping with*

$$|h(t, y)| \le \alpha(t)g(y),$$

for some $\alpha \in L^\infty([0, T], \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$. Moreover, assume that there exists $k > 0$ such that, for every $t \in [0, T]$, and $w \in h(t, k)$,

$$w > 2k(\|a\|_\infty + 1)$$

and that, for every $t \in [0, T]$, *and* $w \in h(t, -k)$,

$$w < -2k(\|a\|_\infty + 1).$$

Then problem (6.1)–(6.4) *has a solution* $x(\cdot)$ *such that* $|x(t)| \le k$, *for every* $t \in [0, T]$.

Remark 6.2. Suppose that, in (6.1), $h(t, x) = \gamma(t) + \alpha(t) f(x)$, where f is an odd semicontinuous multimap and $\alpha, \gamma \in L^\infty([0, T], \mathbb{R})$. Then (6.6) is equivalent to require the existence of $k > 0$ such that, for every $t \in [0, T]$,

$$\alpha(t) f(k) > 2k(|a\|_\infty + 1) - \gamma(t).$$

If $\alpha(t) \ge \overline{\alpha} > 0$, for every $t \in [0, T]$, then (6.6) is equivalent to

$$\overline{\alpha} f(k) > 2k(\|a\|_\infty + 1) - \|\gamma^-\|_\infty,$$

where $\gamma^-(t) = \min\{0, \gamma(t)\}$, which holds, e.g., if f is superlinear at infinity, which is true in many applications. The superlinearity of f at infinity is a sufficient condition also if $\alpha(t) \le -\overline{\alpha} < 0$, for every $t \in [0, T]$. Notice that the obtained solution is a nonzero function whenever γ is a nonzero function.

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On a nonlocal nonlinear Schrödinger equation with self-induced parity-time-symmetric potential

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Abstract. We consider the Cauchy problem of a nonlocal nonlinear Schrödinger equation with self-induced parity-symmetric potential. Global existence of solution and decay estimates are obtained for suitably small initial data when the spatial dimension $d \geq 2$.

Keywords: nonlocal Schrödinger equation, global solution, decay estimate.

2020 Mathematics Subject Classification: 35Q55, 35B08.

1 Introduction

This paper is concerned with a nonlocal nolinear Schrödinger (NLS) equation which reads

$$i\psi_t(t,x) + \frac{1}{2}\Delta\psi(t,x) + g\psi(t,x)\overline{\psi}(t,\mathcal{P}x)\psi(t,x) = 0, \tag{1.1}$$

where $\psi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is unknown, $\overline{\psi}$ is the complex conjugation of ψ, and g is a real constant ($g > 0$ and $g < 0$ denote the focusing and defocusing cases, respectively). In the above equation, $\mathcal{P}$ is a $d \times d$ matrix, which denotes a parity transformation with the determinant satisfying

$$\det \mathcal{P} = -1. \tag{1.2}$$

More precisely, in odd spatial dimensions, $\mathcal{P}x = -x$, that is, the sign of all the coordinates is changed, while in even spatial dimensions, a parity transformation means that the sign of only an odd number of coordinates can be reversed. In particular, in one dimensional case, equation (1.1) reduces to

$$i\psi_t(t,x) + \frac{1}{2}\psi_{xx}(t,x) + g\psi(t,x)\overline{\psi}(t,-x)\psi(t,x) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{1.3}$$

Note that $\mathcal{P}$ is not unique in even dimensions. For example, if $d = 2$, $\mathcal{P}x$ can take as either $\mathcal{P}x = (-x_1, x_2)$ or $\mathcal{P}x = (x_1, -x_2)$.

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Equation (1.1) was first derived by Ablowitz and Musslimani [1] in one dimensional case, and by Sinha and Ghosh [9] in higher dimensional case. In the equation, the self-induced potential $V(t,x) := \psi(t,x)\overline{\psi}(t,\mathcal{P}x)$ is non-Hermitian but parity-time-symmetric ($\mathcal{PT}$-symmetric), that is, $\overline{V}(t,\mathcal{P}x) = V(t,x)$. Note that the value of the potential V at x depends not only on the information of ψ at x, but also on $\mathcal{P}x$, so it is a nonlocal potential. $\mathcal{PT}$-symmetric condition is weaker than the condition of self-adjointness, however, it was shown by Bender and Boettcher [3] that non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry may also exhibit real spectra, hence, a great deal of investigations on $\mathcal{PT}$-symmetric systems are carried out both theoretically and experimentally. Using a unified two-parameter model, equation (1.1) can be generalized to vector form [12]. If $\psi(t,-x) = \psi(t,x)$, equation (1.1) reduces to the classical NLS equation

$$i\psi_t(t,x) + \frac{1}{2}\Delta\psi(t,x) + g|\psi(t,x)|^2\psi(t,x) = 0. \tag{1.4}$$

When $d = 1$, Ablowitz and Musslimani [1] showed that the nonlocal NLS equation (1.1) is an integrable system. Exact soliton solutions were obtained in [1,2,6,8,9]. In particular, from the identity (22) in [1], we know the focusing nonlocal NLS equation (1.3) (i.e., $g > 0$) has the one-soliton solution

$$\psi^*(t,x) = \pm\frac{2(\eta_1+\eta_2)e^{i\theta_2}e^{i2g\eta_2^2 t}e^{-2\sqrt{g}\eta_2 x}}{1+e^{i(\theta_1+\theta_2)}e^{-i2g(\eta_1^2-\eta_2^2)t}e^{-2\sqrt{g}(\eta_1+\eta_2)x}},$$

where the four parameters $\eta_1,\eta_2,\theta_1,\theta_2$ are real, $\eta_1,\eta_2 > 0$ and $\eta_1 \neq \eta_2$. Note that ψ^* eventually develops a singularity in finite time T_n at $x = 0$,

$$\lim_{t\to T_n}|\psi^*(t,0)| = +\infty \ \text{ with } \ T_n = \frac{(2n+1)\pi - \theta_1 - \theta_2}{2g(\eta_2^2-\eta_1^2)}, \quad n \in \mathbb{Z}.$$

In particular, by setting $\theta_1 = \theta_2 = 0$, $\eta_1 = \epsilon$, $\eta_2 = 2\epsilon$, it can be computed that

$$\|\psi^*(0,x)\|_{L^2(\mathbb{R})} \lesssim \epsilon^{\frac{1}{2}}, \quad \|\psi_x^*(0,x)\|_{L^2(\mathbb{R})} \lesssim \epsilon^{\frac{3}{2}}.$$

This implies that solutions of (1.1) may develop finite time blow up behavior even with H^1 small initial data. Therefore, compared to the classical NLS equation (1.4) where we know global solutions exist with arbitrarily large H^1 initial data and possesses a modified scattering behavior for small initial data [5,7], the nonlocal NLS equation exhibits a completely different picture in one spatial dimension due to the presence of the nonlocal nonlinearity. So a natural question is whether such phenomenon still occurs for higher space dimensions. This is the main motivation of the present work.

In this paper, the notation $A \lesssim B$ ($A, B \geq 0$) means that there exists a constant $C > 0$ such that $A \leq CB$. For $1 \leq p \leq +\infty$, $L^p(\mathbb{R}^d)$ is the usual Lebesgue space. For $s \in \mathbb{R}$, $H^s(\mathbb{R}^d)$ denotes the inhomogeneous Sobolev space equipped with the norm

$$\|u\|_{H^s} := \|(1+|\xi|^2)^{s/2}\widehat{u}\|_{L^2},$$

where $\widehat{u} = \widehat{u}(\xi)$ is the Fourier transform of u, namely,

$$\widehat{u}(\xi) = \mathcal{F}u := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}u(x)dx.$$

Now, we state the main result of the paper, see Theorems 1.1 and 1.2 below. The initial data of the equation (1.1) is endowed as

$$\psi(0,x) = \psi_0(x). \tag{1.5}$$

Theorem 1.1. *Let $d \geq 3$, $N > \frac{d}{2}$ be an integer. Then there exists a sufficiently small constant $\epsilon_0 > 0$ such that if the initial data ψ_0 satisfies*

$$\|\psi_0\|_{H^N(\mathbb{R}^d)} + \|\psi_0\|_{L^1(\mathbb{R}^d)} \leq \epsilon_0, \tag{1.6}$$

then the nonlocal NLS equation (1.1) admits a unique global solution $\psi \in C(\mathbb{R}; H^N(\mathbb{R}^d))$. Moreover, for all $t \in \mathbb{R}$, there hold that

$$\|\psi(t,x)\|_{H^N(\mathbb{R}^d)} \lesssim \epsilon_0, \quad \|\psi(t,x)\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{\epsilon_0}{(1+|t|)^{d/2}}. \tag{1.7}$$

Theorem 1.2. *Assume $d = 2$ and the initial data ψ_0 satisfies*

$$\|\psi_0\|_{H^N(\mathbb{R}^2)} + \||x|^2\psi_0\|_{L^2(\mathbb{R}^2)} \leq \epsilon_0, \tag{1.8}$$

where the integer $N > 1$ and $\epsilon_0 > 0$ is sufficiently small. Then the Cauchy problem (1.1) and (1.5) has a unique global solution $\psi \in C(\mathbb{R}; H^N(\mathbb{R}^2))$ satisfying for all $t \in \mathbb{R}$,

$$\|\psi(t,x)\|_{H^N(\mathbb{R}^2)} + \||x|^2 f(t,x)\|_{L^2(\mathbb{R}^2)} \lesssim \epsilon_0, \quad \|\psi(t,x)\|_{L^\infty(\mathbb{R}^2)} \lesssim \frac{\epsilon_0}{1+|t|}, \tag{1.9}$$

where $f(t,x) := e^{-\frac{it\Delta}{2}}\psi(t,x)$ is the profile of $\psi(t,x)$.

From the above theorems, we observe that small initial data still leads to global solution for the nonlocal NLS equation when $d \geq 2$, which is different from one dimensional case. This shows that for long time existence, the dispersive effect dominates the nonlocal effect in higher dimensions. By using the energy norm and the decay norm, Theorems 1.1 and 1.2 are proved in Sections 3 and 4, respectively.

Finally, we remark that the total charge $\mathcal{N}$ and the Hamiltonian $\mathcal{H}$ of the equation (1.1) are conserved (see [9]), namely, $\mathcal{N}(t) = \mathcal{N}(0)$ and $\mathcal{H}(t) = \mathcal{H}(0)$ with

$$\begin{aligned}
\mathcal{N}(t) &:= \int_{\mathbb{R}^d} \psi(t,x)\overline{\psi}(t,\mathcal{P}x)dx, \\
\mathcal{H}(t) &:= \int_{\mathbb{R}^d} \big[\frac{1}{2}\nabla\psi(t,x)\cdot\nabla\overline{\psi}(t,\mathcal{P}x) - \frac{g}{2}\big(\psi(t,x)\overline{\psi}(t,\mathcal{P}x)\big)^2\big]dx.
\end{aligned}$$

Although each term in $\mathcal{N}$ and $\mathcal{H}$ is real-valued, it is not semipositive-definite. Hence, unlike the classical NLS equation, it is not known clearly how to use these conserved quantities in our mathematical analysis, especially in the study of the blow up problems for the nonlocal NLS equation (1.1). Such issues will be exploited in the further research.

2 Preliminaries

In this section, we collect preparatory materials, including some basic inequalities, linear decay estimates for the Schrödinger operator and the local well-posedness result. Firstly, from (1.2) and the definition of the parity transformation $\mathcal{P}$, it is easy to see for any function $u(x)$, there hold

$$\begin{aligned}
\|u(\mathcal{P}x)\|_{L^p(\mathbb{R}^d)} &= \|u(x)\|_{L^p(\mathbb{R}^d)}, && 1 \leq p \leq +\infty, \\
\mathcal{F}[u(\mathcal{P}x)](\xi) &= \widehat{u}(\mathcal{Q}\xi), && \mathcal{Q} := \mathcal{P}^{-1}, \\
\|u(\mathcal{P}x)\|_{H^s(\mathbb{R}^d)} &= \|u(x)\|_{H^s(\mathbb{R}^d)}, && s \geq 0.
\end{aligned} \tag{2.1}$$

Lemma 2.1. *Assume $u, v \in H^s(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with $s \geq 0$, then there holds*

$$\|uv\|_{H^s} \lesssim \|u\|_{H^s}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^s}. \tag{2.2}$$

The proof of this lemma can be found, for example, in [11, Lemma A.8].

Lemma 2.2. *There hold that*

$$\|f\|_{L^1(\mathbb{R}^2)} \lesssim \|f\|^{1/2}_{L^2(\mathbb{R}^2)}\||x|^2 f\|^{1/2}_{L^2(\mathbb{R}^2)}, \tag{2.3}$$

$$\|f\|_{L^{4/3}(\mathbb{R}^2)} \lesssim \|f\|^{1/2}_{L^2(\mathbb{R}^2)}\|xf\|^{1/2}_{L^2(\mathbb{R}^2)}. \tag{2.4}$$

Proof. Let $a > 0$ be determined later. Using the basic estimate $\int_{|x|\geq a}|x|^{-4}dx \lesssim a^{-2}$, we deduce by the Cauchy–Schwarz inequality that

$$\|f\|_{L^1} \leq \int_{|x|\leq a}|f(x)|\cdot 1dx + \int_{|x|\geq a}|x|^2|f(x)|\cdot|x|^{-2}dx \lesssim \|f\|_{L^2}a + \||x|^2 f\|_{L^2}a^{-1}.$$

Then (2.3) follows easily if we choose $a = \||x|^2 f\|^{1/2}_{L^2}\|f\|^{-1/2}_{L^2}$. Here we may assume $\|f\|_{L^2} \neq 0$, otherwise the estimate (2.3) holds obviously.

The proof for (2.4) is similar. In fact, using Hölder's inequality, we have

$$\begin{aligned}\|f\|^{4/3}_{L^{4/3}} &\leq \int_{|x|\leq b}|f(x)|^{4/3}\cdot 1dx + \int_{|x|\geq b}|xf(x)|^{4/3}\cdot|x|^{-4/3} \\ &\lesssim \|f\|^{4/3}_{L^2}b^{2/3} + \|xf\|^{4/3}_{L^2}b^{-2/3},\end{aligned}$$

which gives the desired estimate (2.4) provided that we set $b = \|xf\|_{L^2}\|f\|^{-1}_{L^2}$. □

For the Schrödinger operator $e^{\frac{it\Delta}{2}}$, it is known that (see e.g., [10])

$$\|e^{\frac{it\Delta}{2}}u\|_{L^p(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{d(\frac{1}{2}-\frac{1}{p})}}\|u\|_{L^{p'}(\mathbb{R}^d)}, \qquad \frac{1}{p}+\frac{1}{p'} = 1, \quad 2 \leq p \leq +\infty. \tag{2.5}$$

Using Duhamel's formula, the solution $\psi(t,x)$ of (1.1) can be expressed by

$$\psi(t,x) = e^{\frac{it\Delta}{2}}\psi_0(x) - ig\int_0^t e^{\frac{i(t-s)\Delta}{2}}\psi(s,x)\overline{\psi}(s,\mathcal{P}x)\psi(s,x)ds. \tag{2.6}$$

Equation (2.6) is the main identity that we will discuss later.

Finally, we end this section with a local well-posedness result.

Proposition 2.3. *For any $\psi_0 \in H^N(\mathbb{R}^d)$ with $N > \frac{d}{2}$, $d \geq 1$, there exists $T_0 = T_0(\|\psi_0\|_{H^N}) > 0$ such that the Cauchy problem (1.1) and (1.5) has a unique solution $\psi \in C([0,T_0];H^N)$ satisfying (2.6). Moreover, if $T^* < +\infty$ is the maximal existence time for this solution, then*

$$\limsup_{t\uparrow T^*}\|\psi(t,x)\|_{H^N} = +\infty. \tag{2.7}$$

This proposition can be proved by applying the Banach fixed-point theorem, since the argument is standard, we skip the details.

3 Proof of Theorem 1.1

From now on, we focus on the case $t \geq 0$ for simplicity. To prove Theorem 1.1, we first introduce the work space A_T as follows,

$$\|\psi\|_{A_T} := \sup_{t\in[0,T)} \left(\|\psi(t,x)\|_{H^N(\mathbb{R}^d)} + (1+t)^{\frac{d}{2}}\|\psi(t,x)\|_{L^\infty(\mathbb{R}^d)}\right), \tag{3.1}$$

where $T \in (0, +\infty]$. The result of Theorem 1.1 relies essentially on the following proposition.

Proposition 3.1. *Let $d \geq 3$, $N > \frac{d}{2}$ be an integer and $\psi_0 \in H^N(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Assume $\psi(t,x) \in C([0,T]; H^N(\mathbb{R}^d))$ is the solution of (1.1) and (1.5). Then we have*

$$\|\psi\|_{A_T} \lesssim \|\psi_0\|_{H^N\cap L^1} + \|\psi\|^3_{A_T}, \tag{3.2}$$

where the implicit constant is independent of T.

Proof. The start point is the identity (2.6). Using Lemma 2.1, (2.1) and the definition of $\|\cdot\|_{A_T}$, we have for any $t \in [0,T]$,

$$\begin{aligned}
\|\psi(t,x)\|_{H^N} &\leq \|\psi_0(x)\|_{H^N} + |g|\int_0^t \|\psi(s,x)\overline{\psi}(s,\mathcal{P}x)\psi(s,x)\|_{H^N}ds \\
&\lesssim \|\psi_0(x)\|_{H^N} + \int_0^t \|\psi^2(s,x)\|_{H^N}\|\overline{\psi}(s,\mathcal{P}x)\|_{L^\infty}ds \\
&\qquad + \int_0^t \|\psi^2(s,x)\|_{L^\infty}\|\overline{\psi}(s,\mathcal{P}x)\|_{H^N}ds \\
&\lesssim \|\psi_0(x)\|_{H^N} + \int_0^t \|\psi(s,x)\|_{H^N}\|\psi(s,x)\|^2_{L^\infty}ds \\
&\lesssim \|\psi_0(x)\|_{H^N} + \|\psi\|^3_{A_T}\int_0^t (1+s)^{-d}ds \\
&\lesssim \|\psi_0(x)\|_{H^N} + \|\psi\|^3_{A_T}.
\end{aligned} \tag{3.3}$$

Next, we turn to estimate the L^∞ norm of $\psi(t,x)$. Note that

$$\|e^{\frac{it\Delta}{2}}\psi_0(x)\|_{L^\infty} \lesssim \frac{1}{(1+t)^{\frac{d}{2}}}\|\psi_0(x)\|_{L^1\cap H^N}, \tag{3.4}$$

which is a consequence of (2.5) for large t and the Sobolev embedding $H^N \hookrightarrow L^\infty$ for small t. Hence, using (3.4), (2.1), Lemma 2.1 and Hölder's inequality, it follows from (2.6) that

$$\begin{aligned}
\|\psi(t,x)\|_{L^\infty} &\leq \|e^{\frac{it\Delta}{2}}\psi_0(x)\|_{L^\infty} + |g|\int_0^t \|e^{\frac{i(t-s)\Delta}{2}}(\psi^2(s,x)\overline{\psi}(s,\mathcal{P}x))\|_{L^\infty}ds \\
&\lesssim \frac{1}{(1+t)^{\frac{d}{2}}}\|\psi_0(x)\|_{L^1\cap H^N} + \int_0^t \frac{1}{(1+t-s)^{\frac{d}{2}}}\|\psi^2(s,x)\overline{\psi}(s,\mathcal{P}x)\|_{L^1\cap H^N}ds \\
&\lesssim \frac{1}{(1+t)^{\frac{d}{2}}}\|\psi_0(x)\|_{L^1\cap H^N} + \int_0^t \frac{1}{(1+t-s)^{\frac{d}{2}}}\|\psi(s,x)\|^2_{L^2}\|\psi(s,x)\|_{L^\infty}ds \\
&\qquad + \int_0^t \frac{1}{(1+t-s)^{\frac{d}{2}}}\|\psi(s,x)\|_{H^N}\|\psi(s,x)\|^2_{L^\infty}ds \\
&\lesssim \frac{1}{(1+t)^{\frac{d}{2}}}\|\psi_0(x)\|_{L^1\cap H^N} + \|\psi\|_{A^3_T}\int_0^t \frac{1}{(1+t-s)^{\frac{d}{2}}}\cdot\frac{1}{(1+s)^{\frac{d}{2}}}ds \\
&\lesssim \frac{1}{(1+t)^{\frac{d}{2}}}\|\psi_0(x)\|_{L^1\cap H^N} + \frac{1}{(1+t)^{\frac{d}{2}}}\|\psi\|_{A^3_T}.
\end{aligned} \tag{3.5}$$

Therefore, the desired estimate (3.2) follows easily from (3.3) and (3.5). $\square$

Based on Proposition 3.1, we now present the proof of Theorem 1.1.

Proof of Theorem 1.1. By Proposition 2.3, we know there exists a unique solution ψ to (1.1) and (1.5) such that $\psi \in C([0,T^*);H^N)$ with T^* the maximal existence time of the solution. In order to obtain Theorem 1.1, we shall show $T^* = +\infty$ if the initial data is small enough. Define $\phi(t) := \|\psi\|_{A_t}$ for $t \geq 0$, where A_t is given by (3.1). Then from the condition (1.6) and Proposition 3.1, we obtain

$$\phi(t) \leq C\epsilon_0 + C\phi^3(t), \quad t \in [0,T^*). \tag{3.6}$$

where $C > 1$ is independent of T^*.

The bound (1.6) implies $\phi(0) \leq \epsilon_0$, then by the continuity of the solution, there exists a time T such that $\phi(t) \leq 2C\epsilon_0$ for all $t \in [0,T]$. Here, C is the same as (3.6). Let

$$T' := \sup\{T;\ \phi(t) \leq 2C\epsilon_0 \text{ for all } t \in [0,T]\}$$

Using (3.6) and the continuity of ψ, there holds

$$\phi(T') \leq C\epsilon_0 + C\phi^3(T'). \tag{3.7}$$

Now we claim $T' = T^*$ provided that $\epsilon_0^2 \leq (16C^3)^{-1}$. Indeed, if $T' < T^*$, (3.7) gives

$$2C\epsilon_0 \leq C\epsilon_0 + 8C^4\epsilon_0^3,$$

which is a contradiction for sufficiently small ϵ_0. Therefore, we conclude that if $\epsilon_0 \leq (16C^3)^{-\frac{1}{2}}$, then $\phi(T^*) \leq 2C\epsilon_0$. This bound and the blowup criterion (2.7) in turn yield $T^* = +\infty$. Hence, we obtain $\psi \in C(\mathbb{R}^+;H^N)$ and the bound (1.7) holds for $t \geq 0$. The case $t \leq 0$ can be proved similarly. This ends the proof of Theorem 1.1. □

4 Proof of Theorem 1.2

Since the decay rate is only t^{-1} in dimension two, the argument used in Section 3 can not be applied to prove Theorem 1.2. Inspired from the work [4,7] on the method of space-time resonances, here we would like to work on the space B_T defined by

$$\|\psi\|_{B_T} := \sup_{t\in[0,T)} \left(\|\psi(t,x)\|_{H^N(\mathbb{R}^2)} + \||x|^2 f(t,x)\|_{L^2(\mathbb{R}^2)}\right), \tag{4.1}$$

where $T \in (0,+\infty]$, and

$$f(t,x) := e^{-\frac{it\Delta}{2}}\psi(t,x) \tag{4.2}$$

is the profile of a solution $\psi(t,x)$ of (1.1). Notice that (4.1) implies

$$\begin{aligned} \|xf(t,x)\|_{L^2} &\leq \|f(t,x)\|_{L^2} + \||x|^2 f(t,x)\|_{L^2} \\ &= \|\psi(t,x)\|_{L^2} + \||x|^2 f(t,x)\|_{L^2} \\ &\leq 2\|\psi\|_{B_T}. \end{aligned} \tag{4.3}$$

Moreover, using (2.3), (2.5), (4.1) and (4.2), we have

$$\|\psi(t,x)\|_{L^\infty(\mathbb{R}^2)} = \|e^{\frac{it\Delta}{2}} f(t,x)\|_{L^\infty(\mathbb{R}^2)} \lesssim \frac{1}{1+t}\|\psi\|_{B_T}, \qquad t \in [0,T], \tag{4.4}$$

which shows that the decay rate of the solution ψ is bounded by the norm $\|\psi\|_{B_T}$.

Proposition 4.1. *Assume $\psi(t,x) \in C([0,T]; H^N(\mathbb{R}^2))$ $(N > 1)$ is the solution of (1.1) with the initial data satisfying $\psi_0 \in H^N(\mathbb{R}^2)$ and $|x|^2\psi_0 \in L^2(\mathbb{R}^2)$, then we have $xf(t,x), |x|^2 f(t,x) \in C([0,T]; L^2(\mathbb{R}^2))$ and*

$$\|\psi\|_{B_T} \lesssim \|\psi_0\|_{H^N} + \||x|^2\psi_0\|_{L^2} + \|\psi\|_{B_T}^3, \tag{4.5}$$

where the implicit constant is independent of T.

Proof. We first show the continuity for $xf(t,x)$ and $|x|^2 f(t,x)$. Recall the definition (4.2), it follows from (1.1) that

$$f_t(t,x) = ige^{-\frac{it\Delta}{2}}[\psi(t,x)\overline{\psi}(t,\mathcal{P}x)\psi(t,x)]. \tag{4.6}$$

Using the identity

$$x(e^{\pm\frac{it\Delta}{2}}u(x)) = e^{\pm\frac{it\Delta}{2}}(xu(x)) \mp ite^{\pm\frac{it\Delta}{2}}\nabla u(x), \tag{4.7}$$

which can be verified by taking Fourier transform on both sides of (4.7), then we can obtain

$$(xf)_t = ige^{-\frac{it\Delta}{2}}[x\psi(t,x)\overline{\psi}(t,\mathcal{P}x)\psi(t,x)] - gte^{-\frac{it\Delta}{2}}\nabla[\psi(t,x)\overline{\psi}(t,\mathcal{P}x)\psi(t,x)].$$

Integrating this equality with respect to time over $[0,t]$ gives (using also the fact $f(0,x) = \psi_0(x)$, and $\psi_0 \in L^2$, $|x|^2\psi_0 \in L^2$ implies $x\psi_0 \in L^2$)

$$\sup_{s\in[0,t]} \|xf(s,x)\|_{L^2} \le \|x\psi_0\|_{L^2} + Ct \sup_{s\in[0,T]} \|\psi(s,x)\|_{H^N}^2 \sup_{s\in[0,t]} \|xf(s,x)\|_{L^2} + Ct^2 \sup_{s\in[0,T]} \|\psi(s,x)\|_{H^N}^3.$$

This implies $xf(t,x) \in L^2$ for $t \le T_0 := [2C\sup_{s\in[0,T]} \|\psi(s,x)\|_{H^N}^2]^{-1}$. Moreover, with the same arguments as above, it is easy to obtain

$$\|xf(t_2,x) - xf(t_1,x)\|_{L^2} \lesssim |t_2 - t_1| \sup_{s\in[0,T]} \|\psi(s,x)\|_{H^N}^3, \quad t_1, t_2 \in [0, T_0],$$

which gives $xf \in C([0,T_0]; L^2)$. Note that T_0 depends only on $\sup_{s\in[0,T]} \|\psi(s,x)\|_{H^N}$, so a standard bootstrap argument clearly yields that the continuity of xf holds in the whole interval $[0,T]$. The continuity of $|x|^2 f$ can be proved similarly but with more complicated computation, we thus omit the detailed proof for simplicity.

Next, we prove the bound (4.5). For the H^N norm part, one can use (4.4) and similar treatment as (3.3) to obtain

$$\|\psi(t,x)\|_{H^N} \lesssim \|\psi_0\|_{H^N} + \|\psi\|_{B_T}^3 \int_0^t (1+s)^{-2} ds \lesssim \|\psi_0\|_{H^N} + \|\psi\|_{B_T}^3. \tag{4.8}$$

So it remains to estimate the weighted norm. To this end, we integrate the equation (4.6) with respect to time and rewrite the resulted identity in the form of Fourier space, then we obtain (using also (4.2) and (2.1))

$$\widehat{f}(t,\xi) = \widehat{f}(0,\xi) + \frac{ig}{(2\pi)^2}\int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)}\widehat{f}(s,\xi-\eta)\widehat{f}(s,\eta-\sigma)\widehat{\overline{f}}(s,\mathcal{Q}\sigma)d\eta d\sigma ds, \tag{4.9}$$

where the phase $\Phi(\xi,\eta,\sigma)$ is given by

$$\Phi(\xi,\eta,\sigma) := \frac{1}{2}(|\xi|^2 - |\xi-\eta|^2 - |\eta-\sigma|^2 + |\sigma|^2) = \xi\cdot\eta - |\eta|^2 + \eta\cdot\sigma. \tag{4.10}$$

Using Plancharel's identity, we know $\||x|^2 f\|_{L^2} = \|\Delta_\xi \widehat{f}\|_{L^2}$. Now applying Δ_ξ to (4.9) and recalling the fact $f(0,x) = \psi_0(x)$, we have

$$\Delta_\xi \widehat{f}(t,\xi) = \Delta_\xi \widehat{\psi_0} + I_1 + I_2 + I_3 \tag{4.11}$$

with

$$\begin{aligned}
I_1 &:= ig(2\pi)^{-2} \int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)} \Delta_\xi \widehat{f}(s,\xi-\eta)\widehat{f}(s,\eta-\sigma)\widehat{\overline{f}}(s,Q\sigma)d\eta d\sigma ds,\\
I_2 &:= 2ig(2\pi)^{-2} \int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)} (is\nabla_\xi\Phi)\nabla_\xi \widehat{f}(s,\xi-\eta)\widehat{f}(s,\eta-\sigma)\widehat{\overline{f}}(s,Q\sigma)d\eta d\sigma ds,\\
I_3 &:= ig(2\pi)^{-2} \int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)} (is)^2|\nabla_\xi\Phi|^2 \widehat{f}(s,\xi-\eta)\widehat{f}(s,\eta-\sigma)\widehat{\overline{f}}(s,Q\sigma)d\eta d\sigma ds.
\end{aligned}$$

Note that both I_2 and I_3 contain growth factor of s. However, the factor will not cause any difficulty for small s such as $s \in [0,1]$. Hence, the contribution of the time integral from 0 to 1 in I_2 and I_3 can be easily estimated by using only the energy bound and the weighted norm. In the following, we mainly deal with the time integral from 1 to t (still denoted by I_2 and I_3). In order to eliminate the growth factor s in the term I_2, we use the following crucial relation for Φ (see (4.10))

$$\nabla_\xi\Phi = \eta = \nabla_\sigma\Phi \tag{4.12}$$

to integrate by part in σ, then $I_2 = I_{2,1} + I_{2,2}$ with

$$\begin{aligned}
I_{2,1} &:= -2ig(2\pi)^{-2} \int_1^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)} \nabla_\xi \widehat{f}(s,\xi-\eta)\nabla_\sigma\widehat{f}(s,\eta-\sigma)\widehat{\overline{f}}(s,Q\sigma)d\eta d\sigma ds,\\
I_{2,2} &:= -2ig(2\pi)^{-2} \int_1^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)} \nabla_\xi \widehat{f}(s,\xi-\eta)\widehat{f}(s,\eta-\sigma)\nabla_\sigma\widehat{\overline{f}}(s,Q\sigma)d\eta d\sigma ds.
\end{aligned}$$

Similarly, using (4.12) to integrate I_3 by part twice, then $I_3 = I_{3,1} + I_{3,2} + I_{3,3}$ with

$$\begin{aligned}
I_{3,1} &:= ig(2\pi)^{-2} \int_1^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)} \widehat{f}(s,\xi-\eta)\Delta_\sigma\widehat{f}(s,\eta-\sigma)\widehat{\overline{f}}(s,Q\sigma)d\eta d\sigma ds,\\
I_{3,2} &:= ig(2\pi)^{-2} \int_1^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)} \widehat{f}(s,\xi-\eta)\widehat{f}(s,\eta-\sigma)\Delta_\sigma\widehat{\overline{f}}(s,Q\sigma)d\eta d\sigma ds,\\
I_{3,3} &:= 2ig(2\pi)^{-2} \int_1^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)} \widehat{f}(s,\xi-\eta)\nabla_\sigma\widehat{f}(s,\eta-\sigma)\nabla_\sigma\widehat{\overline{f}}(s,Q\sigma)d\eta d\sigma ds.
\end{aligned}$$

Returning back to the physical space and using Hölder's inequality and (4.4), then

$$\begin{aligned}
\|I_1\|_{L^2} + \|I_{3,1}\|_{L^2} + \|I_{3,2}\|_{L^2} &\lesssim \int_0^t \|\psi(t,x)\|_{L^\infty}^2 \||x|^2 f(s,x)\|_{L^2} ds\\
&\lesssim \|\psi\|_{B_T}^3 \int_0^t (1+s)^{-2} ds\\
&\lesssim \|\psi\|_{B_T}^3.
\end{aligned} \tag{4.13}$$

For the remaining terms, we should use the inequality

$$\|e^{\frac{is\Delta}{2}}(xf(s,x))\|_{L^4} \lesssim s^{-\frac{1}{2}}\|\psi\|_{B_T}.$$

which follows from (2.4), (2.5) and (4.3), then

$$\begin{aligned}\|I_{2,1}\|_{L^2}+\|I_{2,2}\|_{L^2}+\|I_{3,3}\|_{L^2} &\lesssim \int_1^t \|\psi(t,x)\|_{L^\infty}\|e^{\frac{is\Delta}{2}}(xf(s,x))\|_{L^4}^2 ds\\ &\lesssim \|\psi\|_{B_T}^3\int_1^t (1+s)^{-2}ds\\ &\lesssim \|\psi\|_{B_T}^3. \end{aligned}\tag{4.14}$$

Now, combing (4.11), (4.13) and (4.14) together yields

$$\||x|^2 f(t,x)\|_{L^2}\lesssim \||x|^2\psi_0\|_{L^2}+\|\psi\|_{B_T}^3. \tag{4.15}$$

Therefore, the desired bound (4.5) follows immediately from (4.8) and (4.15). □

Finally, based on Proposition 4.1, one can argue analogously as the proof of Theorem 1.1 and obtain global existence of solution as stated in Theorem 1.2. The L^∞ decay bound in (1.9) follows also by using (4.4). Since the proof is similar as Theorem 1.1, we thus omit further details.

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Monotone solutions for singular fractional boundary value problems

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Abstract. In this paper, we investigate a boundary value problem of fractional differential equation. The nonlinear term includes fractional derivatives and is singular with respect to space variables. By means of Schaefer's fixed point theorem and Vitali convergence theorem, an existence result of monotone solutions is obtained. The proofs are based on regularization and sequential techniques. An example is also given to illustrate our main result.

Keywords: Caputo fractional derivative, monotone solution, boundary value problem, singularity.

2020 Mathematics Subject Classification: 34B16, 26A33.

1 Introduction

In this work, we consider the following boundary value problem (BVP for short)

$$\begin{cases} {}^CD^{\alpha}_{0^+}u(t) = f(t, u(t), u'(t), {}^CD^{\beta}_{0^+}u(t)), \\ u(0) + u(1) = 0, \ u'(0) = 0, \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$, $0 < \beta < 1$, ${}^CD^{\alpha}_{0^+}$ and ${}^CD^{\beta}_{0^+}$ are Caputo fractional derivatives, $f(t, x, y, z)$ is singular at the value 0 of its space variables x, y and z. We establish an existence result of monotone increasing and continuous solution $u(t)$ of BVP (1.1). Since $\lim_{x\to 0} f(t, x, y, z) = \infty$, it follows from the condition $u(0) + u(1) = 0$ that there exists $\xi \in (0, 1)$ such that $u(\xi) = 0$ and thus ξ is a singular point of f.

Throughout the paper, $AC[0, 1]$ and $AC^k[0, 1]$ are the set of absolutely continuous functions on $[0, 1]$ and the set of functions having absolutely continuous kth derivatives on $[0, 1]$ respectively, $AC^0[0, 1] = AC[0, 1]$ for $k = 0$. $\|x\|_p = \left(\int_0^1 |x(t)|^p dt\right)^{\frac{1}{p}}$ is the norm in $L^p[0, 1]$, $1 \le p < \infty$. The basic space used in this paper is Banach space $C^1[0, 1]$ equipped with the norm $\|x\|_* = \max\{\|x\|, \|x'\|\}$, here $\|x\| = \max_{t\in[0,1]} |x(t)|$, $\|x'\| = \max_{t\in[0,1]} |x'(t)|$. We say that a monotone increasing function $u \in C^1[0, 1]$ is a solution of BVP (1.1) if u satisfies

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the boundary conditions in (1.1), $u(\xi) = 0$ for some $\xi \in (0,1)$, ${}^{C}D^{\alpha}_{0+}u(t)$ is continuous on $(0,1] \setminus \{\xi\}$ and satisfies the equation in (1.1) for $t \in (0,1] \setminus \{\xi\}$.

In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives due to their wide range of applications in varied fields of sciences and engineering. Many research papers have appeared recently concerning the existence of positive solutions for fractional boundary value problems with singularities on time and/or space variables, see, for example, the papers [8, 10–12, 14, 21, 23] and the references therein. In [1–4, 6, 7, 17–20, 22], using techniques of nonlinear analysis such as fixed point theorems on cones and nonlinear alternatives combined with the methods of regularization and sequential approximation, the authors proved the existence of positive solutions for singular fractional boundary value problems in which the singularities are with respect to space variables.

The singular boundary value problem

$$\begin{cases} {}^{C}D^{\alpha}_{0+}u(t) + f(t, u(t), u'(t), {}^{C}D^{\mu}_{0+}u(t)) = 0, \\ u'(0) = 0,\ u(1) = 0, \end{cases}$$

is studied in [2], where $1 < \alpha < 2, 0 < \mu < 1$, $f(t,x,y,z)$ is positive and may be singular at the value 0 of its space variables x, y and z. $f(t,x,y,z)$ is a L^q-Carathéodory function on $[0,1] \times \mathcal{B}$ with $q > \frac{1}{\alpha-1}$, $\mathcal{B} = (0,\infty) \times (-\infty,0) \times (-\infty,0)$. An existence result of positive solutions in space $C^1[0,1]$ is proved by the combination of regularization and sequential techniques with the Guo–Krasnosel'skii fixed point theorem on cones.

In [17] the author discussed the existence of positive solutions for the singular fractional boundary value problem

$$\begin{cases} D^{\alpha}_{0+}u(t) + f(t, u(t), u'(t), D^{\mu}_{0+}u(t)) = 0, \\ u(0) = 0,\ u'(0) = u'(1) = 0, \end{cases}$$

where $2 < \alpha < 3, 0 < \mu < 1$, D^{α}_{0+} and D^{μ}_{0+} are the standard Riemann–Liouville fractional derivatives of order α and μ respectively. The function $f(t,x,y,z)$ is positive and may be singular at the value 0 of its arguments x, y and z, moreover, $f(t,x,y,z)$ satisfies the local Carathéodory conditions on $[0,1] \times (0,\infty) \times (0,\infty) \times (0,\infty)$. By regularization and sequential techniques and by the Guo–Krasnosel'skii fixed point theorem on cones, positive solutions in $C^1[0,1]$ are obtained.

Although the singular fractional boundary value problems have been investigated widely, the solutions allowing negative values of fractional boundary value problems with singularities on space variables are seldom considered. By Schaefer's fixed point theorem and Vitali convergence theorem, O'Regan and Staněk in [13] investigated monotone solutions in space $C[0,1]$ of the fractional boundary value problem

$$\begin{cases} {}^{C}D^{\alpha}_{0+}u(t) = f(t, u(t)), \\ u(0) + u(1) = 0,\ u'(0) = 0, \end{cases}$$

where $1 < \alpha < 2$, $f(t,x) \in C([0,1] \times (\mathbb{R}\setminus\{0\}))$. $f(t,x)$ is nonnegative and may be singular at $x = 0$.

Inspired by above works, we prove the existence of monotone increasing solutions for BVP (1.1). The main tool used in this paper is Schaefer's fixed point theorem. Our proofs are based on regularization and sequential techniques. Compared with the existing literature, this paper presents the following new features. Firstly, as far as we know, the existence results of solutions allowing negative values are even less for fractional boundary value problems with singularities on space variables. Our result compensates for this deficiency to some extent. Secondly, the significant difference with the problem discussed in [13] lies in that the nonlinear term f in BVP (1.1) is related to fractional derivatives and permits singularities on all its space variables. That is to say the problem considered in this paper performs a more general form. Moreover, the conditions on f in our paper are more general than those in [13].

2 Preliminaries

In this section, we introduce some notations and preliminary facts which are used throughout this paper.

The Riemann–Liouville fractional integral of order $\delta > 0$ of a function $f(t) \in L^1(a,b)$ is defined by (see [9, p. 69])

$$I^{\delta}_{a^+} f(t) = \frac{1}{\Gamma(\delta)} \int_a^t (t-s)^{\delta-1} f(s) ds, \qquad t > a.$$

The Riemann–Liouville fractional derivative of order $\delta > 0$ of a continuous function f on $(a,b]$ is given by (see [9, p. 70])

$$D^{\delta}_{a^+} f(t) = \left(\frac{d}{dt}\right)^n I^{n-\delta}_{a^+} f(t) = \frac{1}{\Gamma(n-\delta)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\delta-1} f(s) ds,$$

provided that the right-hand side is pointwise defined on $(a,b]$, where n is the smallest integer greater than or equal to δ. In particular, for $\delta = n$, $D^n_{a^+} f(t) = f^{(n)}(t)$.

The Caputo fractional derivative of order $\delta > 0$ of a function $f(t) \in C(a,b]$ is defined by (see [9, p. 91])

$${}^C D^{\delta}_{a^+} f(t) = D^{\delta}_{a^+} \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k \right],$$

provided that the right-hand side is pointwise defined on $(a,b]$, where n is the smallest integer greater than or equal to δ. In particular, for $\delta = n$, ${}^C D^n_{a^+} f(t) = f^{(n)}(t)$.

Remark 2.1. For a function $f(t) \in L^1(a,b)$, a sufficient condition for the existence of Riemann–Liouville fractional derivative almost everywhere is that $I^{n-\delta}_{a^+} f(t) \in AC^{n-1}[a,b]$. In this case, the function f is said to have a summable fractional derivative of order δ ([15, Definition 2.4]). In view of the definition of Caputo fractional derivative, ${}^C D^{\delta}_{a^+} f(t) = D^{\delta}_{a^+} f(t)$ for $\delta \in \mathbb{N}$ and ${}^C D^{\delta}_{a^+} f(t) = D^{\delta}_{a^+} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\delta+1)} (t-a)^{k-\delta}$ for $\delta \notin \mathbb{N}$ (see (2.4.6) in [9]), thus this is also a sufficient condition for the existence of Caputo fractional derivative. It is worth mentioning that the solution $u(t)$ in our main result not only has summable fractional derivative ${}^C D^{\alpha}_{0^+} u(t)$ on $[0,1]$ but also has continuous fractional derivative ${}^C D^{\alpha}_{0^+} u(t)$ on $(0,1] \setminus \{\xi\}$. For more details, see Step 3 in the proof of Theorem 4.1 and Remark 4.2 in Section 4.

Remark 2.2. The following properties are useful for our discussion.

(i) ([9, Lemma 2.8]) $I^{\delta}_{a^+} : C[a,b] \to C[a,b]$ for $\delta > 0$.

(ii) ([9, Lemma 2.3]) If $\delta > 0$, $\gamma > 0$, $\delta + \gamma > 1$ and $f \in L^p(a,b)$ $(1 \leq p \leq \infty)$, then $I^{\delta}_{a^+} I^{\gamma}_{a^+} f(t) = I^{\delta+\gamma}_{a^+} f(t)$, $t \in [a,b]$.

(iii) ([9, Theorem 2.2]) If $n-1<\delta\le n$ and $f(t)\in C^n[a,b]$, then ${}^{C}D^{\delta}_{a^+}f(t)=I^{n-\delta}_{a^+}f^{(n)}(t),\ t\in[a,b]$.

(iv) ([9, Lemma 2.21]) If $\delta>0$ and $f\in C[a,b]$, then ${}^{C}D^{\delta}_{a^+}I^{\delta}_{a^+}f(t)=f(t),\ t\in[a,b]$.

(v) ([17, Lemma 2.1]) $I^{\delta}_{a^+}:L^1[a,b]\to L^1[a,b]$ for $\delta\in(0,1)$ and $I^{\delta}_{a^+}:L^1[a,b]\to AC^{[\delta]-1}[a,b]$ for $\delta\ge 1$, where $[\delta]$ means the integral part of δ.

For convenience, in the following discussion we use I^{α}, ${}^{C}D^{\alpha}$ and D^{α} to denote $I^{\alpha}_{0^+}$, ${}^{C}D^{\alpha}_{0^+}$ and $D^{\alpha}_{0^+}$, respectively.

A sequence $\{\phi_n\}\subset L^1[0,1]$ is said to have uniformly absolutely continuous integrals on $[0,1]$ if for any $\epsilon>0$, there exists $\delta>0$ such that if $E\subset[0,1]$ and $\operatorname{meas}(E)<\delta$, then $\int_E|\phi_n(t)|dt<\epsilon$ for all $n\in\mathbb{N}$ (see [5, p. 178]). To prove the main result, we need the following Vitali convergence theorem and nonlinear alternative.

Lemma 2.3 ([5, pp. 178–179] Vitali convergence theorem). *Let $\{\phi_n\}\subset L^1[0,1]$, $\lim_{n\to+\infty}\phi_n(t)=\phi(t)$ for a.e. $t\in[0,1]$ and $|\phi(t)|<\infty$ for a.e. $t\in[0,1]$. Then the following statements are equivalent.*

(1) $\phi\in L^1[0,1]$ and $\lim_{n\to+\infty}\|\phi_n-\phi\|_1=0$.

(2) The sequence $\{\phi_n\}$ has uniformly absolutely continuous integrals on $[0,1]$.

Lemma 2.4 ([16, p. 29] Schaefer's fixed point theorem). *Let X be a Banach space and $T:X\to X$ be completely continuous. Then the following alternative holds. Either the equation $x=\lambda T(x)$ has a solution for every $\lambda\in[0,1]$ or the set $A=\{x\in X: x=\lambda Tx$ for some $\lambda\in(0,1)\}$ is unbounded.*

Denote $\mathbb{R}_0=\mathbb{R}\setminus\{0\}$, $\mathbb{R}^+=[0,+\infty)$ and $\mathbb{R}^+_0=(0,+\infty)$. We work with the following conditions on the function f in (1.1).

(H1) $f\in C([0,1]\times\mathbb{R}_0\times\mathbb{R}^+_0\times\mathbb{R}^+_0)$, $\lim_{x\to 0}f(t,x,y,z)=\lim_{y\to 0^+}f(t,x,y,z)=\lim_{z\to 0^+}f(t,x,y,z)=+\infty$ and $f(t,x,y,z)\ge mt^{2-\alpha}$ for $(t,x,y,z)\in[0,1]\times\mathbb{R}_0\times\mathbb{R}^+_0\times\mathbb{R}^+_0$.

(H2) $f(t,x,y,z)\le\rho(t)g(|x|,y,z)+p(|x|)+q(y,z)$ for $(t,x,y,z)\in[0,1]\times\mathbb{R}_0\times\mathbb{R}^+_0\times\mathbb{R}^+_0$, where $\rho(t)$ is nonnegative on $[0,1]$, $g(x,y,z)\in C(\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+)$ is nonnegative and nondecreasing in all its arguments, $p(x)\in C(\mathbb{R}^+_0)$ is nonnegative and nonincreasing, $q(y,z)\in C(\mathbb{R}^+_0\times\mathbb{R}^+_0)$ is nonnegative and nonincreasing in all its arguments.

(H3) $\lim_{x\to+\infty}\frac{g(x,x,x)}{x}=0$. $p(\lambda x)\le\lambda^{-\sigma}p(x)$ for some $\sigma\in(0,\frac{\alpha-1}{2})$ and for any $\lambda\in(0,1]$, $x\in\mathbb{R}^+_0$. $\rho(t)$, $p(t^2)$ and $q(m\Gamma(3-\alpha)t,\frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}t^{2-\beta})\in L^{\nu}[0,1]$ for some $\nu\in(\frac{1}{\alpha-1},\frac{1}{2\sigma})$.

Remark 2.5. In [13], the nonlinear term satisfies $f(t,x)\le g(|x|)+\frac{A}{|x|^{\nu}}$, where $A>0$ is a constant and $\nu>0$ is a suitable small number. It is easy to verify that the simple function $p(x)=\frac{1}{x^{\omega}}$ for $0<\omega<\frac{\alpha-1}{2}$ fulfils the conditions (H2) and (H3) with $\omega\le\sigma<\frac{\alpha-1}{2}$ and $\nu\in(\frac{1}{\alpha-1},\frac{1}{2\sigma})$.

Remark 2.6. By Lemma 2.1 and 2.2 in [2], for any $f(t)\in L^{\nu}[0,1]$ with $\nu>\frac{1}{\alpha-1}$, $I^{\alpha-1}f(t)\in C[0,1]$ and $\left|\int_0^t(t-s)^{\alpha-2}f(s)ds\right|\le\left(\frac{t^d}{d}\right)^{\frac{1}{\mu}}\|f\|_{\nu}$, where $d=(\alpha-2)\mu+1$ and $\mu=\frac{\nu}{\nu-1}$. Thus we can know easily $\lim_{t\to 0^+}I^{\alpha-1}f(t)=0$. Similarly, $I^{\alpha}f(t)\in C[0,1]$ and $\lim_{t\to 0^+}I^{\alpha}f(t)=0$. The continuity of $I^{\alpha}f(t)$ on $[0,1]$ can also be derived from the continuity of $I^{\alpha-1}f(t)$, Remark 2.2 (i) and (ii).

3 Auxiliary regular problem

This section deals with an auxiliary regular problem. We prove its solvability and give the properties of its solutions. We also state a necessary lemma and its useful corollary.

Consider the integral equation defined by

$$\begin{aligned} u(t) = &\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_n(s,u(s),u'(s),{}^C D^\beta u(s))ds \\ &- \frac{1}{2\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f_n(s,u(s),u'(s),{}^C D^\beta u(s))ds, \end{aligned} \tag{3.1}$$

where

$$f_n(t,x,y,z) = \begin{cases} f(t,x,\chi_n(y),\chi_n(z)), & x \ge \frac{1}{n}, \\ \frac{n}{2}\left[f(t,\frac{1}{n},\chi_n(y),\chi_n(z))(\frac{1}{n}+x) + f(t,-\frac{1}{n},\chi_n(y),\chi_n(z))(\frac{1}{n}-x)\right], & |x| \le \frac{1}{n}, \\ f(t,x,\chi_n(y),\chi_n(z)), & x \le -\frac{1}{n}, \end{cases}$$

and

$$\chi_n(\tau) = \begin{cases} \tau, & \tau \ge \frac{1}{n}, \\ \frac{1}{n}, & \tau \le \frac{1}{n}. \end{cases}$$

Then the conditions (H1) and (H2) give

(K1) $f_n \in C([0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$ and $f_n(t,x,y,z) \ge mt^{2-\alpha}$ for $(t,x,y,z) \in [0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$.

(K2) $f_n(t,x,y,z) \le \rho(t)g(|x|+1,y+1,z+1) + p(\frac{1}{n}) + q(\frac{1}{n},\frac{1}{n})$ for $(t,x,y,z) \in [0,1]\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+$, $f_n(t,x,y,z) \le \rho(t)g(|x|+1,y+1,z+1) + p(|x|) + q(y,z)$ for $(t,x,y,z) \in [0,1]\times\mathbb{R}_0\times\mathbb{R}_0^+\times\mathbb{R}_0^+$.

Define an operator T_n by the formula

$$\begin{aligned} T_n u(t) = &\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_n(s,u(s),u'(s),{}^C D^\beta u(s))ds \\ &- \frac{1}{2\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f_n(s,u(s),u'(s),{}^C D^\beta u(s))ds. \end{aligned} \tag{3.2}$$

Obviously, the fixed points of T_n are exactly the solutions of integral equation (3.1).

Lemma 3.1. *Suppose that (H1) holds. Then $T_n : C^1[0,1] \to C^1[0,1]$ is completely continuous.*

Proof. Let $u \in C^1[0,1]$. Using Remark 2.2 (i) and (iii) we have ${}^C D^\beta u(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}u'(s)ds$ and ${}^C D^\beta u(t) \in C[0,1]$. Thus, in view of (3.2), Remark 2.2 (i) and (K1) ensure $T_n u(t) \in C[0,1]$. Moreover, according to (K1), Remark 2.2 (i), (ii) and (iv), we know $(T_n u)'(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} f_n(s,u(s),u'(s),{}^C D^\beta u(s))ds$ and $(T_n u)'(t) \in C[0,1]$. So we have $T_n\colon C^1[0,1] \to C^1[0,1]$.

T_n is a continuous operator. In fact, let $\{u_k\} \subset C^1[0,1]$ be such that $\lim_{k\to+\infty}\|u_k - u\|_* = 0$, then $u(t) \in C^1[0,1]$. Since

$$\begin{aligned} |{}^C D^\beta u_k(t) - {}^C D^\beta u(t)| &\le \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}|u_k'(s) - u'(s)|ds \\ &\le \frac{\|u_k' - u'\|}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}ds \le \frac{\|u_k' - u'\|}{\Gamma(2-\beta)}, \end{aligned}$$

we get $\|^C D^\beta u_k -^C D^\beta u\| \to 0$ and thus $\|f_n(t,u_k,u_k',{}^C D^\beta u_k) - f_n(t,u,u',{}^C D^\beta u)\| \to 0$ as $k \to +\infty$. Note that

$$\begin{aligned}
&|T_n u_k(t) - T_n u(t)| \\
&\quad = \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[f_n(s,u_k(s),u_k'(s),{}^C D^\beta u_k(s)) - f_n(s,u(s),u'(s),{}^C D^\beta u(s)) \Big] ds \\
&\qquad - \frac{1}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \Big[f_n(s,u_k(s),u_k'(s),{}^C D^\beta u_k(s)) - f_n(s,u(s),u'(s),{}^C D^\beta u(s)) \Big] ds \Bigg| \\
&\quad \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big| f_n(s,u_k(s),u_k'(s),{}^C D^\beta u_k(s)) - f_n(s,u(s),u'(s),{}^C D^\beta u(s)) \Big| ds \\
&\qquad + \frac{1}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \Big| f_n(s,u_k(s),u_k'(s),{}^C D^\beta u_k(s)) - f_n(s,u(s),u'(s),{}^C D^\beta u(s)) \Big| ds \\
&\quad \le \frac{\|f_n(t,u_k,u_k',{}^C D^\beta u_k) - f_n(t,u,u',{}^C D^\beta u)\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
&\qquad + \frac{\|f_n(t,u_k,u_k',{}^C D^\beta u_k) - f_n(t,u,u',{}^C D^\beta u)\|}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds \\
&\quad = \frac{\|f_n(t,u_k,u_k',{}^C D^\beta u_k) - f_n(t,u,u',{}^C D^\beta u)\|}{\Gamma(\alpha+1)} \left(t^\alpha + \frac{1}{2} \right) \\
&\quad \le \frac{3\|f_n(t,u_k,u_k',{}^C D^\beta u_k) - f_n(t,u,u',{}^C D^\beta u)\|}{2\Gamma(\alpha+1)},
\end{aligned}$$

and

$$\begin{aligned}
&|(T_n u_k)'(t) - (T_n u)'(t)| \\
&\quad = \Bigg| \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \Big[f_n(s,u_k(s),u_k'(s),{}^C D^\beta u_k(s)) - f_n(s,u(s),u'(s),{}^C D^\beta u(s)) \Big] ds \Bigg| \\
&\quad \le \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \Big| f_n(s,u_k(s),u_k'(s),{}^C D^\beta u_k(s)) - f_n(s,u(s),u'(s),{}^C D^\beta u(s)) \Big| ds \\
&\quad \le \frac{\|f_n(t,u_k,u_k',{}^C D^\beta u_k) - f_n(t,u,u',{}^C D^\beta u)\|}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} ds \\
&\quad = \frac{\|f_n(t,u_k,u_k',{}^C D^\beta u_k) - f_n(t,u,u',{}^C D^\beta u)\|}{\Gamma(\alpha)} t^{\alpha-1} \\
&\quad \le \frac{\|f_n(t,u_k,u_k',{}^C D^\beta u_k) - f_n(t,u,u',{}^C D^\beta u)\|}{\Gamma(\alpha)}.
\end{aligned}$$

So we obtain $\lim_{k\to+\infty} \|T_n u_k - T_n u\|_* = 0$. Therefore, T_n is a continuous operator.

Furthermore, T_n is completely continuous. Suppose that $\Omega \subset C^1[0,1]$ is bounded and let $M_n = \sup\{\|f_n(t,u,u',{}^C D^\beta u)\|, u \in \Omega\}$, here M_n is well defined because ${}^C D^\beta u(t) \le \frac{\|u'\|}{\Gamma(2-\beta)}$. Then we have

$$\begin{aligned}
|T_n u(t)| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big| f_n(s,u(s),u'(s),{}^C D^\beta u(s)) \big| ds \\
&\quad + \frac{1}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \big| f_n(s,u(s),u'(s),{}^C D^\beta u(s)) \big| ds \\
&\le \frac{M_n}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{M_n}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds \le \frac{3M_n}{2\Gamma(\alpha+1)},
\end{aligned}$$

and

$$\begin{aligned}|(T_nu)'(t)| &\leq \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\big|f_n(s,u(s),u'(s),{}^CD^\beta u(s))\big|ds\\ &\leq \frac{M_n}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}ds \leq \frac{M_n}{\Gamma(\alpha)}.\end{aligned}$$

Therefore, $T_n(\Omega)$ is bounded. Now we are in the position to prove $T_n(\Omega) \subset C^1[0,1]$ is an equicontinuous set. Let $t_1, t_2 \in [0,1]$ and $t_1 < t_2$, then $|T_nu(t_2) - T_nu(t_1)| \leq \frac{M_n}{\Gamma(\alpha)}(t_2 - t_1)$ by the mean value theorem and $|(T_nu)'(t)| \leq \frac{M_n}{\Gamma(\alpha)}$. Moreover,

$$\begin{aligned}&|(T_nu)'(t_2) - (T_nu)'(t_1)|\\ &= \frac{1}{\Gamma(\alpha-1)}\bigg|\int_0^{t_2}(t_2-s)^{\alpha-2}f_n(s,u(s),u'(s),{}^CD^\beta u(s))ds\\ &\qquad - \int_0^{t_1}(t_1-s)^{\alpha-2}f_n(s,u(s),u'(s),{}^CD^\beta u(s))ds\bigg|\\ &\leq \frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}\left[(t_1-s)^{\alpha-2}-(t_2-s)^{\alpha-2}\right]\big|f_n(s,u(s),u'(s),{}^CD^\beta u(s))\big|ds\\ &\quad + \frac{1}{\Gamma(\alpha-1)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-2}\big|f_n(s,u(s),u'(s),{}^CD^\beta u(s))\big|ds\\ &\leq \frac{M_n}{\Gamma(\alpha-1)}\int_0^{t_1}\left[(t_1-s)^{\alpha-2}-(t_2-s)^{\alpha-2}\right]ds + \frac{M_n}{\Gamma(\alpha-1)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-2}ds\\ &= \frac{M_n}{\Gamma(\alpha)}\big(t_1^{\alpha-1} - t_2^{\alpha-1} + 2(t_2-t_1)^{\alpha-1}\big).\end{aligned}$$

Keeping in mind that the function $t^{\alpha-1}$ is uniformly continuous on $[0,1]$, we have $T_n(\Omega)$ is equicontinuous. Consequently, the Arzelà–Ascoli theorem guarantees that T_n is a completely continuous operator. The proof of Lemma 3.1 is finished. □

The next lemma presents the existence of fixed points for the operator T_n.

Lemma 3.2. *Assume that the conditions (H1), (H2) and (H3) are satisfied. Then T_n has a fixed point in $C^1[0,1]$ for any $n \in \mathbb{N}$.*

Proof. In view of Lemma 2.4 and Lemma 3.1, it is sufficient to prove the set $A_n = \{u \in C^1[0,1] : u = \lambda T_nu$ for some $\lambda \in (0,1)\}$ is bounded. For any $u \in A_n$, we have

$$\begin{aligned}u(t) &= \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f_n(s,u(s),u'(s),{}^CD^\beta u(s))ds\\ &\qquad - \frac{\lambda}{2\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}f_n(s,u(s),u'(s),{}^CD^\beta u(s))ds,\end{aligned} \tag{3.3}$$

$$\begin{aligned}u'(t) &= \frac{\lambda}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}f_n(s,u(s),u'(s),{}^CD^\beta u(s))ds\\ &\geq \frac{m\lambda}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}s^{2-\alpha}ds\\ &= \frac{m\lambda t}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}s^{2-\alpha}ds = m\lambda\Gamma(3-\alpha)t \geq 0\end{aligned} \tag{3.4}$$

by (K1). According to (3.3) and (3.4), one has $u(0) + u(1) = 0$, $u'(0) > 0$ on $(0, 1]$. Thus there exists $\xi \in (0, 1)$ such that $u(\xi) = 0$. It follows that $|u(t)| = |u(t) - u(\xi)| \leq \|u'\||t - \xi|$ and hence $\|u\| \leq \|u'\|$. Since ${}^C D^\beta u(t) \geq 0$ by (3.4) and ${}^C D^\beta u(t) \leq \frac{\|u'\|}{\Gamma(2-\beta)}$, applying the conditions (H2), (H3) and (K2) we can derive

$$\begin{aligned}
u'(t) &\leq \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \left[\rho(s) g(|u(s)| + 1, u'(s) + 1, {}^C D^\beta u(s) + 1) + p\left(\frac{1}{n}\right) + q\left(\frac{1}{n}, \frac{1}{n}\right) \right] ds \\
&\leq \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \left[\rho(s) g(\|u'\| + 1, \|u'\| + 1, \frac{\|u'\|}{\Gamma(2-\beta)} + 1) + p\left(\frac{1}{n}\right) + q\left(\frac{1}{n}, \frac{1}{n}\right) \right] ds \\
&\leq \frac{g(\|u'\| + 1, \|u'\| + 1, \frac{\|u'\|}{\Gamma(2-\beta)} + 1)}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \rho(s) ds + \frac{p(\frac{1}{n}) + q(\frac{1}{n}, \frac{1}{n})}{\Gamma(\alpha)} \\
&\leq C g\left(\|u'\| + 1, \|u'\| + 1, \frac{\|u'\|}{\Gamma(2-\beta)} + 1 \right) + \frac{p(\frac{1}{n}) + q(\frac{1}{n}, \frac{1}{n})}{\Gamma(\alpha)},
\end{aligned}$$

here $C = \max_{t\in[0,1]} \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \rho(s) ds$, C is well defined by Remark 2.6 and (H3). In particular, the inequality

$$1 \leq \frac{C g(\|u'\| + 1, \|u'\| + 1, \frac{\|u'\|}{\Gamma(2-\beta)} + 1)}{\|u'\|} + \frac{p(\frac{1}{n}) + q(\frac{1}{n}, \frac{1}{n})}{\|u'\| \Gamma(\alpha)}$$

is fulfilled. The condition $\lim_{x\to+\infty} \frac{g(x,x,x)}{x} = 0$ in (H3) guarantees that there exists $L > 0$ such that

$$\frac{C g(\|u'\| + 1, \|u'\| + 1, \frac{\|u'\|}{\Gamma(2-\beta)} + 1)}{\|u'\|} + \frac{p(\frac{1}{n}) + q(\frac{1}{n}, \frac{1}{n})}{\|u'\| \Gamma(\alpha)} < 1$$

for $\|u'\| > L$. Consequently, we obtain $\|u\| \leq \|u'\| \leq L$ for $u \in A_n$. Therefore, A_n is bounded and we complete the proof. □

Lemma 3.2 shows that the integral equation (3.1) admits a solution u_n in $C^1[0, 1]$ for any $n \in \mathbb{N}$. The properties of solutions to (3.1) are collected in the following lemma.

Lemma 3.3. *Let the conditions (H1), (H2) and (H3) be valid and u_n be solution of (3.1). Then*

(1) $u_n(0) + u_n(1) = 0$, $u_n'(0) = 0$, $u_n'(t) \geq m\Gamma(3-\alpha)t$ and there exists $\xi_n \in (0, 1)$ such that $u_n(\xi_n) = 0$.

(2) $|u_n(t)| \geq \frac{m\Gamma(3-\alpha)}{2} |t^2 - \xi_n^2|$.

(3) $\{u_n(t), n \in \mathbb{N}\}$ is a compact subset of $C^1[0, 1]$.

(4) There exists a constant $l \in (0, 1)$ such that $l \leq \xi_n < 1$ for any $n \in \mathbb{N}$.

Proof. **Proof of (1).** Similar to (3.4), the condition (K1) ensures $u_n'(t) \geq m\Gamma(3-\alpha)t$. Other assertions in (1) are obvious so we omit their proofs.

Proof of (2). Using (1), one has easily $|u_n(t)| = \left| \int_{\xi_n}^t u_n'(s) ds \right| \geq m\Gamma(3-\alpha) \left| \int_{\xi_n}^t s ds \right| = \frac{m\Gamma(3-\alpha)}{2} |t^2 - \xi_n^2|$.

Proof of (3). In order to apply the Arzelà–Ascoli theorem, we need to prove $\{u_n(t)\}$ is bounded in $C^1[0, 1]$ and $\{u_n'(t)\}$ is equicontinuous. Firstly, we prove $\{u_n(t)\}$ is bounded. In view of (1), we get

$$\|u_n\| \leq \|u_n'\|, \quad {}^C D^\beta u_n(t) \geq m \frac{\Gamma(3-\alpha)}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} s ds = \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)} t^{2-\beta}.$$

We also know ${}^{C}D^{\beta}u_n(t) \leq \frac{\|u_n'\|}{\Gamma(2-\beta)}$. Thus, for $t \in (0,1] \setminus \{\xi_n\}$, by (H2), (K2), (1) and (2) we derive

$$
\begin{aligned}
f_n(t, u_n(t), u_n'(t), {}^{C}D^{\beta}u_n(t)) &\leq \rho(t) g(|u_n(t)| + 1, u_n'(t) + 1, {}^{C}D^{\beta}u_n(t) + 1) \\
&\quad + p(|u_n(t)|) + q(u_n'(t), {}^{C}D^{\beta}u_n(t)) \\
&\leq \rho(t) g\left(\|u_n'\| + 1, \|u_n'\| + 1, \frac{\|u_n'\|}{\Gamma(2-\beta)} + 1 \right) \\
&\quad + p\left(\frac{m\Gamma(3-\alpha)}{2} |t^2 - \xi_n^2| \right) + q\left(m\Gamma(3-\alpha)t, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)} t^{2-\beta} \right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
u_n'(t) \leq &\frac{g(\|u_n'\| + 1, \|u_n'\| + 1, \frac{\|u_n'\|}{\Gamma(2-\beta)} + 1)}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \rho(s) ds \\
&+ \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} p\left(\frac{m\Gamma(3-\alpha)}{2} |s^2 - \xi_n^2| \right) ds \\
&+ \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} q\left(m\Gamma(3-\alpha)s, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)} s^{2-\beta} \right) ds.
\end{aligned}
\tag{3.5}
$$

Furthermore, by (H3) and Remark 2.6, we can let

$$
C_1 = \max_{t\in[0,1]} \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \rho(s) ds, \tag{3.6}
$$

$$
C_2 = \max_{t\in[0,1]} \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} q\left(m\Gamma(3-\alpha)s, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)} s^{2-\beta} \right) ds \tag{3.7}
$$

and by the Hölder inequality one has

$$
\begin{aligned}
&\int_0^t (t-s)^{\alpha-2} p\left(\frac{m\Gamma(3-\alpha)}{2} |s^2 - \xi_n^2| \right) ds \\
&\qquad \leq [(\alpha-2)\mu + 1]^{-\frac{1}{\mu}} \left(\int_0^t p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2} |s^2 - \xi_n^2| \right) ds \right)^{\frac{1}{\nu}},
\end{aligned}
\tag{3.8}
$$

here $(\alpha-2)\mu + 1 > 0$ and $\frac{1}{\mu} + \frac{1}{\nu} = 1$, μ is well defined by the choice of ν in condition (H3). Next we estimate the integral on the right side of (3.8).

$$
\begin{aligned}
&\int_0^t p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2} |s^2 - \xi_n^2| \right) ds \\
&\quad \leq \int_0^1 p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2} |s^2 - \xi_n^2| \right) ds \\
&\quad = \int_0^{\xi_n} p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2} |s^2 - \xi_n^2| \right) ds + \int_{\xi_n}^1 p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2} |s^2 - \xi_n^2| \right) ds \\
&\quad = I_1 + I_2.
\end{aligned}
\tag{3.9}
$$

In view of the monotone property of p and (H3), we get

$$
\begin{aligned}
I_1 &\leq \int_0^{\xi_n} p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2} \xi_n(\xi_n - s) \right) ds = \int_0^1 \xi_n p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2} \xi_n^2 (1-s) \right) ds \\
&\leq A\xi_n^{1-2\sigma\nu} \int_0^1 (p(1-s))^{\nu} ds \leq A \int_0^1 (p(s))^{\nu} ds \leq A \int_0^1 (p(s^2))^{\nu} ds = C_3 < +\infty,
\end{aligned}
\tag{3.10}
$$

where $A = 1$ if $\frac{m\Gamma(3-\alpha)}{2} \geq 1$, otherwise $A = \left(\frac{m\Gamma(3-\alpha)}{2}\right)^{-\sigma\nu}$.

$$\begin{aligned} I_2 &= \int_{\xi_n}^1 p^\nu\left(\frac{m\Gamma(3-\alpha)}{2}(s+\xi_n)(s-\xi_n)\right)ds \\ &= \int_0^1 (1-\xi_n)p^\nu\left(\frac{m\Gamma(3-\alpha)}{2}(1-\xi_n)s((1-\xi_n)s+2\xi_n)\right)ds \\ &\leq \int_0^1 (1-\xi_n)p^\nu\left(\frac{m\Gamma(3-\alpha)}{2}(1-\xi_n)^2 s^2\right)ds \leq A(1-\xi_n)^{1-2\sigma\nu}\int_0^1 (p(s^2))^\nu ds \\ &\leq A\int_0^1 (p(s^2))^\nu ds = C_3 < +\infty. \end{aligned} \tag{3.11}$$

As a result, the inequalities from (3.5) to (3.11) show that for any $n \in \mathbb{N}$ and $t \in [0,1]$,

$$u_n'(t) \leq C_1 g\left(\|u_n'\|+1, \|u_n'\|+1, \frac{\|u_n'\|}{\Gamma(2-\beta)}+1\right) + \frac{[(\alpha-2)\mu+1]^{-\frac{1}{\mu}}}{\Gamma(\alpha-1)}(2C_3)^{\frac{1}{\nu}} + C_2.$$

Consequently, similar to the proof in Lemma 3.2, we can conclude that $\{u_n(t)\}$ is bounded.

Now it remains to prove that $\{u_n'(t)\}$ is equicontinuous. Let $t_1, t_2 \in [0,1]$ be such that $t_1 < t_2$ and $L = \sup\{\|u_n\|_*, n \in \mathbb{N}\}$. Then

$$\begin{aligned} |u_n'(t_2) - u_n'(t_1)| \leq& \frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}((t_1-s)^{\alpha-2}-(t_2-s)^{\alpha-2})f_n(s,u_n(s),u_n'(s),{}^C D^\beta u_n(s))ds \\ &+ \frac{1}{\Gamma(\alpha-1)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-2}f_n(s,u_n(s),u_n'(s),{}^C D^\beta u_n(s))ds \\ \leq& \frac{\|\rho\|_\nu g(L+1, L+1, \frac{L}{\Gamma(2-\beta)}+1)}{\Gamma(\alpha-1)}\left(\int_0^{t_1}((t_1-s)^{\alpha-2}-(t_2-s)^{\alpha-2})^\mu ds\right)^{\frac{1}{\mu}} \\ &+ \frac{1}{\Gamma(\alpha-1)}\left(\int_0^{t_1} p^\nu\left(\frac{m\Gamma(3-\alpha)}{2}|s^2-\xi_n^2|\right)ds\right)^{\frac{1}{\nu}}\left(\int_0^{t_1}((t_1-s)^{\alpha-2}-(t_2-s)^{\alpha-2})^\mu ds\right)^{\frac{1}{\mu}} \\ &+ \frac{1}{\Gamma(\alpha-1)}\left(\int_0^{t_1} q^\nu\left(m\Gamma(3-\alpha)s, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}s^{2-\beta}\right)ds\right)^{\frac{1}{\nu}} \\ &\cdot\left(\int_0^{t_1}((t_1-s)^{\alpha-2}-(t_2-s)^{\alpha-2})^\mu ds\right)^{\frac{1}{\mu}} \\ &+ \frac{\|\rho\|_\nu g(L+1, L+1, \frac{L}{\Gamma(2-\beta)}+1)}{\Gamma(\alpha-1)}\left(\int_{t_1}^{t_2}(t_2-s)^{(\alpha-2)\mu}ds\right)^{\frac{1}{\mu}} \\ &+ \frac{1}{\Gamma(\alpha-1)}\left(\int_{t_1}^{t_2} p^\nu\left(\frac{m\Gamma(3-\alpha)}{2}|s^2-\xi_n^2|\right)ds\right)^{\frac{1}{\nu}}\left(\int_{t_1}^{t_2}(t_2-s)^{(\alpha-2)\mu}ds\right)^{\frac{1}{\mu}} \\ &+ \frac{1}{\Gamma(\alpha-1)}\left(\int_{t_1}^{t_2} q^\nu\left(m\Gamma(3-\alpha)s, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}s^{2-\beta}\right)ds\right)^{\frac{1}{\nu}}\left(\int_{t_1}^{t_2}(t_2-s)^{(\alpha-2)\mu}ds\right)^{\frac{1}{\mu}}. \end{aligned}$$

According to (3.9), (3.10), (3.11) and the condition (H3) we know $\int_0^{t_1} p^\nu(\frac{m\Gamma(3-\alpha)}{2}|s^2-\xi_n^2|)ds$, $\int_0^{t_1} q^\nu(m\Gamma(3-\alpha)s, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}s^{2-\beta})ds$, $\int_{t_1}^{t_2} p^\nu(\frac{m\Gamma(3-\alpha)}{2}|s^2-\xi_n^2|)ds$, $\int_{t_1}^{t_2} q^\nu(m\Gamma(3-\alpha)s, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}s^{2-\beta})ds$ are bounded. Furthermore, the relation $(x-y)^\eta \leq x^\eta - y^\eta$ for $x \geq y \geq 0, \eta > 1$ ensures that

$$\begin{aligned} \int_0^{t_1}((t_1-s)^{\alpha-2}-(t_2-s)^{\alpha-2})^\mu ds &\leq \int_0^{t_1}((t_1-s)^{(\alpha-2)\mu}-(t_2-s)^{(\alpha-2)\mu})ds \\ &= \frac{t_1^{(\alpha-2)\mu+1}-t_2^{(\alpha-2)\mu+1}+(t_2-t_1)^{(\alpha-2)\mu+1}}{(\alpha-2)\mu+1}. \end{aligned}$$

In addition,

$$\int_{t_1}^{t_2}(t_2-s)^{(\alpha-2)\mu}ds=\frac{(t_2-t_1)^{(\alpha-2)\mu+1}}{(\alpha-2)\mu+1}.$$

Hence we can obtain that $\{u_n'(t)\}$ is equicontinuous. Consequently, the Arzelà–Ascoli theorem implies that $\{u_n(t)\}$ is a compact subset of $C^1[0,1]$.

Proof of (4). Suppose that there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $\lim_{k\to+\infty}\xi_{n_k}=0$. Since $|u_{n_k}(0)|=|u_{n_k}(0)-u_{n_k}(\xi_{n_k})|\le\|u_{n_k}'\|\xi_{n_k}$, we have $\lim_{k\to+\infty}u_{n_k}(0)=0$. Thus, $\lim_{k\to+\infty}u_{n_k}(1)=0$ because $u_{n_k}(0)+u_{n_k}(1)=0$, which contradicts $u_{n_k}(1)-u_{n_k}(0)=\int_0^1u_{n_k}'(s)ds\ge\frac{m\Gamma(3-\alpha)}{2}$. Hence, $\inf\{\xi_n : n\in\mathbb{N}\}>0$. As a result, we arrive at $\xi_n\in[l,1)$ for $n\in\mathbb{N}$ with some $l>0$.

We complete the proof of Lemma 3.3. □

In order to apply the Vitali convergence theorem in the proof of our main theorem, we need the following result.

Lemma 3.4. *Let the conditions (H1), (H2) and (H3) be satisfied and u_n be solution of (3.1). Then $\{f_n(t,u_n(t),u_n'(t),{}^CD^\beta u_n(t)),n\in\mathbb{N}\}\subset C[0,1]$ has uniformly absolutely continuous integrals on $[0,1]$.*

Proof. Let $E\subset[0,1]$ be measurable and $L=\sup\{\|u_n\|_*,n\in\mathbb{N}\}$. Then

$$\begin{aligned}\int_E f_n(t,u_n(t),u_n'(t),{}^CD^\beta u_n(t))dt\le{}& g\left(L+1,L+1,\frac{L}{\Gamma(2-\beta)}+1\right)\int_E\rho(t)dt\\&+\int_E p\left(\frac{m\Gamma(3-\alpha)}{2}|t^2-\xi_n^2|\right)dt\\&+\int_E q\left(m\Gamma(3-\alpha)t,\frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}t^{2-\beta}\right)dt.\end{aligned}$$

Applying the Hölder inequality, we have

$$\int_E\rho(t)dt\le(\mathrm{meas}(E))^{\frac{1}{\mu}}\left(\int_E(\rho(t))^\nu dt\right)^{\frac{1}{\nu}},$$

$$\int_E p\left(\frac{m\Gamma(3-\alpha)}{2}|t^2-\xi_n^2|\right)dt\le(\mathrm{meas}(E))^{\frac{1}{\mu}}\left(\int_E p^\nu\left(\frac{m\Gamma(3-\alpha)}{2}|t^2-\xi_n^2|\right)dt\right)^{\frac{1}{\nu}},$$

$$\int_E q\left(m\Gamma(3-\alpha)t,\frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}t^{2-\beta}\right)dt\le(\mathrm{meas}(E))^{\frac{1}{\mu}}\left(\int_E q^\nu\left(m\Gamma(3-\alpha)t,\frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}t^{2-\beta}\right)dt\right)^{\frac{1}{\nu}}.$$

Noticing the condition (H3), (3.9), (3.10) and (3.11), we conclude that the sequence $\{f_n(t,u_n(t),u_n'(t),{}^CD^\beta u_n(t))\}$ has uniformly absolutely continuous integrals on $[0,1]$. □

Corollary 3.5. *Let the conditions (H1), (H2) and (H3) hold and u_n be solution of (3.1). Then $\{(t_0-t)^{\alpha-1}f_n(t,u_n(t),u_n'(t),{}^CD^\beta u_n(t)),n\in\mathbb{N}\}\subset C[0,t_0]$ has uniformly absolutely continuous integrals on $[0,t_0]$ for any $t_0\in[0,1]$.*

The assertion in Corollary 3.5 follows from Lemma 3.4 and the fact

$$(t_0-t)^{\alpha-1}f_n(t,u_n(t),u_n'(t),{}^CD^\beta u_n(t))\le f_n(t,u_n(t),u_n'(t),{}^CD^\beta u_n(t)),\qquad t\in[0,t_0].$$

4 Main result

Now we can give the existence result for the singular BVP (1.1).

Theorem 4.1. *Assume that the conditions (H1), (H2) and (H3) are valid. Then there exists at least one increasing function $u(t) \in C^1[0,1]$ solving the BVP (1.1).*

Proof. For clarity, we divide the proof into several steps.
Step 1: Firstly, Lemma 3.3 and the Bolzano–Weierstrass theorem guarantee that there exist subsequences $\{u_{n_k}\} \subset \{u_n\}$, $\{\xi_{n_k}\} \subset \{\xi_n\}$ and $u \in C^1[0,1]$, $\xi \in [l,1]$ such that $\lim_{k\to+\infty} \xi_{n_k} = \xi$ and $\lim_{k\to+\infty} \|u_{n_k} - u\|_* = 0$. Then again by Lemma 3.3, $u(\xi) = 0$, $u(0) + u(1) = 0$, $u'(0) = 0$, $u'(t) > 0$ for $t \in (0,1]$ and $|u(t)| \geq \frac{m\Gamma(3-\alpha)}{2}|t^2 - \xi^2|$. The last inequality together with $u(0) + u(1) = 0$ implies $u(1) \neq 0$, that is, $\xi \in [l,1)$.

Furthermore, since ${}^CD^\beta u_{n_k}(t) \geq \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)} t^{2-\beta}$ and $\lim_{k\to+\infty} \|{}^CD^\beta u_{n_k} - {}^CD^\beta u\| = 0$, we get ${}^CD^\beta u(t) \geq \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)} t^{2-\beta}$ and thus ${}^CD^\beta u(t) > 0$ on $(0,1]$. Hence, $f(t,u(t),u'(t),{}^CD^\beta u(t)) \in C((0,1] \setminus \{\xi\})$ and

$$\lim_{k\to+\infty} f_{n_k}(t,u_{n_k}(t),u'_{n_k}(t),{}^CD^\beta u_{n_k}(t)) = f(t,u(t),u'(t),{}^CD^\beta u(t)), \qquad t \in (0,1] \setminus \{\xi\}.$$

Also, we can know $f(t,u(t),u'(t),{}^CD^\beta u(t)) \in L^1[0,1]$ by Lemma 2.3 and Lemma 3.4. Moreover, according to Lemma 2.3 and Corollary 3.5, passing to the limit as $k \to +\infty$ on both sides of the equality

$$\begin{aligned} u_{n_k}(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_{n_k}(s,u_{n_k}(s),u'_{n_k}(s),{}^CD^\beta u_{n_k}(s))ds \\ & - \frac{1}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f_{n_k}(s,u_{n_k}(s),u'_{n_k}(s),{}^CD^\beta u_{n_k}(s))ds, \end{aligned}$$

we obtain

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s),u'(s),{}^CD^\beta u(s))ds \\ & - \frac{1}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s,u(s),u'(s),{}^CD^\beta u(s))ds, \qquad t \in [0,1]. \end{aligned} \tag{4.1}$$

Therefore, $u(t)$ is a solution of integral equation (4.1). Next we prove $u(t)$ is a solution of (1.1).
Step 2: In this step we prove that the right side integral in (4.1) belongs to $C^1[0,1]$ and satisfies the boundary value conditions in (1.1).

Let $L = \|u\|_*$. In view of for any $t \in (0,1] \setminus \{\xi\}$,

$$\begin{aligned} f(t,u(t),u'(t),{}^CD^\beta u(t)) \leq {} & g\left(L+1, L+1, \frac{L}{\Gamma(2-\beta)} + 1\right)\rho(t) + p\left(\frac{m\Gamma(3-\alpha)}{2}|t^2 - \xi^2|\right) \\ & + q\left(m\Gamma(3-\alpha)t, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)} t^{2-\beta}\right), \end{aligned}$$

this together with (3.9), (3.10), (3.11) and (H3) guarantees that $f(t,u(t),u'(t),{}^CD^\beta u(t)) \in L^\nu[0,1]$. Hence $I^\alpha f \in C[0,1]$ and $I^{\alpha-1} f \in C[0,1]$ by Remark 2.6. Furthermore, by Remark 2.2 (ii) one has for any $t \in [0,1]$,

$$\begin{aligned} D^1 I^\alpha f(t,u(t),u'(t),{}^CD^\beta u(t)) &= D^1 I^1 I^{\alpha-1} f(t,u(t),u'(t),{}^CD^\beta u(t)) \\ &= I^{\alpha-1} f(t,u(t),u'(t),{}^CD^\beta u(t)). \end{aligned}$$

Thus we obtain that the right side integral in (4.1) belongs to $C^1[0,1]$. Since by Remark 2.6

$$\lim_{t\to 0^+} I^\alpha f(t,u(t),u'(t),{}^C D^\beta u(t)) = \lim_{t\to 0^+} I^{\alpha-1} f(t,u(t),u'(t),{}^C D^\beta u(t)) = 0,$$

we can know easily the right side integral in (4.1) satisfies the boundary value conditions in BVP (1.1).

Step 3: Now it remains to prove that the Caputo derivative of order α of the right side integral in (4.1) exists and is continuous on $(0,1]\setminus\{\xi\}$ and satisfies the differential equation in (1.1) for $t\in(0,1]\setminus\{\xi\}$.

In fact, using the definitions of Caputo fractional derivative and Riemann–Liouville fractional derivative, we have

$$\begin{aligned}
{}^C D^\alpha u(t) &= D^\alpha[u(t)-u(0)-u'(0)t]\\
&= D^\alpha\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s),u'(s),{}^C D^\beta u(s))ds\right)\\
&= D^\alpha I^\alpha f(t,u(t),u'(t),{}^C D^\beta u(t)) = \left(\frac{d}{dt}\right)^2 I^{2-\alpha} I^\alpha f(t,u(t),u'(t),{}^C D^\beta u(t)).
\end{aligned}$$

Thus we need to prove that $(\frac{d}{dt})^2 I^{2-\alpha} I^\alpha f(t,u(t),u'(t),{}^C D^\beta u(t))$ exists and is continuous on $(0,1]\setminus\{\xi\}$ and is equal to $f(t,u(t),u'(t),{}^C D^\beta u(t))$ for $t\in(0,1]\setminus\{\xi\}$.

Firstly, by $f\in L^1[0,1]$ and Remark 2.2 (ii), for any $t\in[0,1]$, we have

$$\begin{aligned}
I^{2-\alpha} I^\alpha f(t,u(t),u'(t),{}^C D^\beta u(t)) &= I^2 f(t,u(t),u'(t),{}^C D^\beta u(t))\\
&= \int_0^t (t-s) f(s,u(s),u'(s),{}^C D^\beta u(s))ds.
\end{aligned}$$

Secondly, by $f\in C((0,1]\setminus\{\xi\})$, for any $t\in(0,1]\setminus\{\xi\}$, let Δt be small enough so that f is continuous on $[t-|\Delta t|, t+|\Delta t|]$ (for $t=1$, f is continuous on $[t-|\Delta t|, t]$), then applying mean value theorem for integrals, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\int_0^t f(s,u(s),u'(s),{}^C D^\beta u(s))ds\right)\\
&\quad= \lim_{\Delta t\to 0}\frac{\int_0^{t+\Delta t} f(s,u(s),u'(s),{}^C D^\beta u(s))ds - \int_0^t f(s,u(s),u'(s),{}^C D^\beta u(s))ds}{\Delta t}\\
&\quad= \lim_{\Delta t\to 0}\frac{\int_t^{t+\Delta t} f(s,u(s),u'(s),{}^C D^\beta u(s))ds}{\Delta t} = f(t,u(t),u'(t),{}^C D^\beta u(t)).
\end{aligned}$$

Similarly, $\frac{d}{dt}\big(\int_0^t s f(s,u(s),u'(s),{}^C D^\beta u(s))ds\big) = t f(t,u(t),u'(t),{}^C D^\beta u(t))$. As a result we have

$$\begin{aligned}
&\frac{d}{dt}\left(\int_0^t (t-s) f(s,u(s),u'(s),{}^C D^\beta u(s))ds\right)\\
&\quad= \frac{d}{dt}\left(t\int_0^t f(s,u(s),u'(s),{}^C D^\beta u(s))ds\right) - \frac{d}{dt}\left(\int_0^t s f(s,u(s),u'(s),{}^C D^\beta u(s))ds\right)\\
&\quad= \int_0^t f(s,u(s),u'(s),{}^C D^\beta u(s))ds,
\end{aligned}$$

and hence, $(\frac{d}{dt})^2\big(\int_0^t (t-s) f(s,u(s),u'(s),{}^C D^\beta u(s))ds\big) = f(t,u(t),u'(t),{}^C D^\beta u(t))$ for any $t\in(0,1]\setminus\{\xi\}$.

We complete the proof of our main result. □

Remark 4.2. In Theorem 4.1, by $f \in L^1[0,1]$ and Remark 2.2 (v) we can know $I^2 f(t, u(t), u'(t), {}^C D^\beta u(t)) \in AC^1[0,1]$. Thus the function $u(t)$ defined by (4.1) has summable fractional derivative and ${}^C D^\alpha u(t) = f(t, u(t), u'(t), {}^C D^\beta u(t))$ for a.e. $t \in [0,1]$. Furthermore, $f \in C((0,1] \setminus \{\xi\})$ and this ensures ${}^C D^\alpha u(t) = f(t, u(t), u'(t), {}^C D^\beta u(t))$ for any $t \in (0,1] \setminus \{\xi\}$.

5 An example

In this section we give an example to illustrate our result.

Example 5.1. Consider the boundary value problem

$$\begin{cases} {}^C D^{\frac{3}{2}} x(t) = (t+1)^2 + |\cos t| \left[\ln(1+|x(t)|) + \arctan x'(t) + ({}^C D^{\frac{1}{2}} x(t))^{\frac{1}{2}} \right] \\ \qquad\qquad + \frac{e^t}{|x(t)|^{\frac{1}{8}}} + \frac{1}{\left(x'(t) {}^C D^{\frac{1}{2}} x(t)\right)^{\frac{1}{10}}}, \\ x(0) + x(1) = 0, \ x'(0) = 0. \end{cases} \tag{5.1}$$

Clearly $\alpha = \frac{3}{2}, \beta = \frac{1}{2}$ and the nonlinear term is

$$\begin{aligned} f(t,x,y,z) &= (t+1)^2 + |\cos t| \left[\ln(1+|x|) + \arctan y + z^{\frac{1}{2}} \right] \\ &\quad + \frac{e^t}{|x|^{\frac{1}{8}}} + \frac{1}{(yz)^{\frac{1}{10}}}, \qquad (t,x,y,z) \in [0,1] \times \mathbb{R}_0 \times \mathbb{R}_0^+ \times \mathbb{R}_0^+. \end{aligned}$$

The conditions (H1), (H2) and (H3) are satisfied with $m = \min_{t \in [0,1]} (t+1)^2 = 1$, $\sigma \in [\frac{1}{8}, \frac{1}{4})$, $\nu \in (2, \frac{1}{2\sigma})$, $\rho(t) = (t+1)^2 + |\cos t|$, $g(x,y,z) = 1 + \ln(1+x) + \arctan y + z^{\frac{1}{2}}$, $p(x) = \frac{e}{x^{1/8}}$ and $q(y,z) = \frac{1}{(yz)^{1/10}}$. We only verify that $p(x)$ and $q(y,z)$ satisfy the conditions in (H3). Other conditions are easy to verify and we omit here. First of all, we have $p(\lambda x) = \lambda^{-\frac{1}{8}} \frac{e}{x^{1/8}} \leq \lambda^{-\sigma} \frac{e}{x^{1/8}} = \lambda^{-\sigma} p(x)$ for $\lambda \in (0,1]$ and $x \in \mathbb{R}_0^+$. Moreover, $\frac{\nu}{4} < 1$ and this ensures $p(x^2) \in L^\nu[0,1]$ and $q(m\Gamma(3-\alpha)x, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)} x^{2-\beta}) = (\frac{\sqrt{\pi}}{3})^{-\frac{1}{10}} \frac{1}{x^{1/4}} \in L^\nu[0,1]$. As a result, Theorem 4.1 guarantees that the problem (5.1) has an increasing solution.

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Antisymmetric solutions for a class of quasilinear defocusing Schrödinger equations

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Abstract. In this paper we consider the existence of antisymmetric solutions for the quasilinear defocusing Schrödinger equation in $H^1(\mathbb{R}^N)$:

$$-\Delta u + \frac{k}{2} u \Delta u^2 + V(x) u = g(u),$$

where $N \geq 3$, $V(x)$ is a positive continuous potential, $g(u)$ is of subcritical growth and k is a non-negative parameter. By considering a minimizing problem restricted on a partial Nehari manifold, we prove the existence of antisymmetric solutions via a deformation lemma.

Keywords: quasilinear Schrödinger equation, antisymmetric solutions, Nehari manifold.

2020 Mathematics Subject Classification: 35J20, 35J60, 35D05.

1 Introduction and main results

In this paper we are interested in the existence of antisymmetric solutions in $H^1(\mathbb{R}^N)$ for the modified quasilinear Schrödinger equation

$$-\Delta u + \frac{k}{2} u \Delta u^2 + V(x) u = g(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $V : \mathbb{R}^N \to (0, \infty)$ is a continuous and positive potential function, $g : \mathbb{R} \to \mathbb{R}$ is a continuous and subcritical function, $k \geq 0$ is a parameter. The existence of solutions for (1.1) is closely related to study of standing waves $\omega(x,t) = u(x) e^{-(iEt)/\hbar}$ for the superfluid film equation arising in the plasma physics (see [9]),

$$i\hbar \partial_t \omega = -\Delta \omega + W(x)\omega - \widetilde{h}(|\omega|^2)\omega + \frac{k}{2}\omega \Delta \omega^2, \tag{1.2}$$

where $W(x)$ is a given potential and $\widetilde{h}(u^2)u = g(u)$ is a real function. So, $\omega(x,t)$ will be a such solution of (1.2) if and only if $u(x)$ solves equation (1.1) with $V(x) = W(x) - E$.

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For the case $k = 0$, equation (1.1) becomes a semilinear Schrödinger equation. The existence of positive ground states or least action nodal solutions for the semilinear Schrödinger equation has been studied widely, we refer the readers to [3,8,24,26] and the references therein for the literature on nodal solutions of the semilinear Schrödinger equation.

For $k = -1$, the modified quasilinear Schrödinger equation has received a lot of attention. The appearance of the quasilinear part $u\Delta u^2$ makes the problem much more complicated, it is quite difficult to study the associated energy functional directly in the Sobolev space $H^1(\mathbb{R}^N)$ and requires one to develop new techniques to apply variational methods. The existence of a positive ground state solution of equation (1.1) has been proved in [16] and [25] by introducing a parameter λ in front of the nonlinear term. In [17], by a change of variables, the authors studied the quasilinear problem was transformed to a semilinear one and the existence of a positive solution was proved using the Mountain-Pass Lemma in an Orlicz space. Different from the change of variable methods, in [20] the authors introduced new perturbation techniques and also proved the existence of solutions for a new kind of critical problems for the modified quasilinear Schrödinger equation in [21].

The existence of sign-changing solution is an interesting topic i.e. looking for solutions u with $u^+, u^- \neq 0$, where $u^+(x) = \max\{u(x), 0\} \geq 0$, and $u^-(x) = \min\{u(x), 0\} \leq 0$, $x \in \mathbb{R}^N$. In [18] the authors proved the existence of sign-changing ground state solution for (1.1) with $k = -1$ and $g(s) = |s|^{p-2}s$, $s \in \mathbb{R}$ with $3 \leq p < 22^* - 1$, that is, g having subcritical growth (22^* plays the role of critical exponent here), and V is a continuous function such that $0 < V_0 = \inf_{\mathbb{R}^N} V(x) \leq \lim_{|x|\to\infty} V(x) = V_\infty$ with $V(x) \leq V_\infty - A/(1 + |x|^m)$, for $|x| \geq M$, for some real constants $A, M, m > 0$. The perturbation arguments in [21] was successfully applied to study the existence of multiple nodal solutions for a general class of sub-critical quasilinear Schrödinger equation in [19].

Also, we would also like to mention [10, 11, 13, 15, 18] and references therein for some recent progress of the study of the quasilinear Schrödinger equation for $k < 0$. However, in [12, 14], the nonlinearity g is permitted to behave in a critical way, under the more restrictive assumption that V is symmetric radially positive and differentiable continuous function with $V'(r) \geq 0$ for $r \geq 0$. Their approach was based on Mountain Pass Theorem on Nehari manifolds.

But, for the case $k > 0$, it seems that there are few work about this type of problems. The existence results of solutions, we like to mention [1] and the existence of sign-changing solutions, we like to mention [2].

The existence of τ-antisymmetric solutions, in [5] and [6], the authors proved existence of τ-antisymmetric solutions for the problem

$$-\Delta u + V(x)u = g(u) \quad \text{in } \mathbb{R}^N,$$

by considering the limit problem

$$-\Delta u + V_\infty u = g(u) \quad \text{in } \mathbb{R}^N.$$

In [7], the authors showed the existence of τ-antisymmetric solutions for the system

$$\begin{cases} -\Delta u + u = |u|^{2p-2}u + \beta(x)|v|^p|u|^{p-2}u, & \text{in } \mathbb{R}^N, \\ -\Delta v + \omega^2 v = |v|^{2p-2}v + \beta(x)|u|^p|v|^{p-2}v, & \text{in } \mathbb{R}^N \end{cases}$$

under suitable assumptions by considering the limit problem

$$\begin{cases} -\Delta u + u = |u|^{2p-2}u + \beta_\infty|v|^p|u|^{p-2}u, & \text{in } \mathbb{R}^N, \\ -\Delta v + \omega^2 v = |v|^{2p-2}v + \beta_\infty|u|^p|v|^{p-2}v, & \text{in } \mathbb{R}^N, \end{cases}$$

and other additional conditions.

However, for the case $k \neq 0$, it seems that the existence results of solutions of τ-antisymmetric solutions to equation (1.1) has not been considered yet. Thus the aim of the present paper is to study the existence of τ-antisymmetric solution for a quasilinear defocusing Schrödinger equation.

To state the main results, we may assume that the potential function V is continuous such that $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^N$, and:

(V_1) $V(\tau x) = V(x)$, where $\tau : \mathbb{R}^N \to \mathbb{R}^N$ is a nontrivial orthogonal involution that is a linear orthogonal transformation on $\mathbb{R}^N$ such that $\tau \neq \mathrm{Id}$ and $\tau^2 = \mathrm{Id}$;

(V_2) V is 1-periodic in x_i, $1 \leq i \leq N$;

(V_3) V is radially symmetric, i.e. $V(x) = V(|x|)$ and $V \in L^\infty(\mathbb{R}^N)$;

(V_4) $\lim_{|x|\to\infty} V(x) = \infty$.

The nonlinearity g is supposed to satisfy:

(G_1) $g \in C(\mathbb{R}, \mathbb{R})$ is such that $g(0) = 0$ and odd;

(G_2) $\lim_{|t|\to 0} \frac{g(t)}{t} = 0$ and $\limsup_{|t|\to\infty} \frac{g(t)}{|t|^{q-1}} < \infty$ for some $q \in (2, 2^*)$;

(G_3) $0 < \theta G(s) \leq sg(s)$, $s \neq 0$ for some $2 < \theta < 2^*$, where $G(u) = \int_0^u g(t)dt$;

(G_4) $t \longmapsto \frac{g(t)}{t^\rho}$, $t > 0$ is non-decreasing for some $\rho > 1$.

Our principal result shows the existence of a τ-antisymmetric solution, that is u satisfies (1.1) and $u(\tau x) = -u(x)$.

Theorem 1.1. *Suppose that (V_1) holds and one of (V_2), (V_3) and (V_4) is satisfied and the conditions (G_1)–(G_4) hold. Then there exists $k_0 > 0$ such that for each $k \in (0, k_0)$ equation (1.1) has at least one τ-antisymmetric solution $u \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with*

$$\max_{x\in\mathbb{R}^N} |u(x)| \leq \frac{\sigma}{\sqrt{k}}, \quad \text{where } \sigma = \left[\left(4 - \frac{1}{\rho} - \sqrt{\frac{1}{\rho^2} + \frac{8}{\rho}}\right)/8\right]^{1/2}. \tag{1.3}$$

The antisymmetric solution found in Theorem 1.1 minimizes the energy functional among all possible solutions for (1.1), and so we can call it the least action antisymmetric solution.

This work contributes to the literature of modified quasilinear defocusing Schrödinger equation in the two senses: on the hand, we found an τ-antisymmetric solution instead of a limit problem, we used several different conditions of the function V; on the other hand, we just need the function g to be continuous, so we can not use directly Ekeland's variational principle.

The paper is organized as follows. In Section 2, we introduce the variational framework for the quasilinear defocusing Schrödinger equation. In Section 3, establishing some auxiliary lemmas and build a homeomorphism between sphere and Nehari manifold. Finally in Section 4, we prove the existence of τ-antisymmetric solution for (1.1) with subcritical growth and obtaining a L^∞-estimate.

Notation

We will use the following notations frequently:
• $C, C_0, C_1, C_2, \ldots$ denote positive (possibly different) constants.
• B_R denotes the open ball centered at the origin with radius $R > 0$.
• $C_0^\infty(\mathbb{R}^N)$ denotes functions infinitely differentiable with compact support in $\mathbb{R}^N$.
• For $1 \le s \le \infty$, $L^s(\mathbb{R}^N)$ denotes the usual Lebesgue space with the norms

$$|u|_s := \Big(\int_{\mathbb{R}^N} |u|^s\Big)^{1/s}, \quad 1 \le s < \infty;$$
$$|u|_\infty := \inf\{C > 0 : |u(x)| \le C \text{ almost everywhere in } \mathbb{R}^N\}.$$

• $H^1(\mathbb{R}^N)$ denotes the Sobolev spaces with usual norm

$$\|u\|_{1,2} := \big(|\nabla u|_2^2 + |u|_2^2\big)^{1/2}.$$

• The weak convergence in $H^1(\mathbb{R}^N)$ is denoted by $\rightharpoonup$, and the strong convergence by $\to$.

2 The modified problem

Formally, this equation has a variational structure, that is, by considering

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}(1-k|u|^2)|\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 - \int_{\mathbb{R}^N} G(u),$$

a function $u \in H^1(\mathbb{R}^N)$ is said to be a weak solution of equation (1.1) if it satisfies

$$\int_{\mathbb{R}^N}(1-k|u|^2)\nabla u \nabla \varphi - k\int_{\mathbb{R}^N}|\nabla u|^2 u\varphi + \int_{\mathbb{R}^N} V(x)u\varphi = \int_{\mathbb{R}^N} g(u)\varphi$$

for all $\varphi \in H^1(\mathbb{R}^N)$, which means $\langle I'(u), \varphi\rangle = 0$ for all $\varphi \in H^1(\mathbb{R}^N)$.

First, we point out that, under the hypothesis $V(x) \ge V_0 > 0$ for all $x \in \mathbb{R}^N$, the subset

$$E = \left\{u \in H^1(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} V(x)u^2(x) < \infty\right\}$$

is a closed subspace of $H^1(\mathbb{R}^N)$. Moreover,

$$\|u\|_E^2 = \int_{\mathbb{R}^N}|\nabla u|^2 + \int_{\mathbb{R}^N} V(x)u^2(x)$$

defines a norm on E. However, the presence of the second order nonhomogeneous term $u\Delta u^2$ prevents us to work directly with the functional I, because it is not even well defined in general in $H^1(\mathbb{R}^N)$.

In order to prove the main results, we first establish the existence of nontrivial solution for a modified quasilinear Schrödinger equation. More precisely, we will show the existence of sign changing solutions for the following quasilinear Schrödinger equations

$$-\,div(l^2(u)\nabla u) + l(u)l'(u)|\nabla u|^2 + V(x)u = g(u), \qquad x \in \mathbb{R}^N \tag{2.1}$$

with $l(t) = \sqrt{1-kt^2}$ for $|t| < \sigma/\sqrt{k}$ for $k > 0$, where $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function and $\sigma > 0$ was chosen in (1.3). Clearly, when $l(t) = \sqrt{1-kt^2}$, we derive that (2.1) turns into

(1.1). Then, by using Morse type L^∞-estimate, we will prove that there exist k_0 such that for all $k \in [0, k_0)$ the solution found verifies the estimate $\max_{\mathbb{R}^N} |u| < \sigma/\sqrt{k}$. After that, we conclude that the solutions obtained are solutions of the original equation (1.1).

For the equation (2.1), we will consider $l : \mathbb{R} \to \mathbb{R}$ defined by

$$l(t) = \begin{cases} \sqrt{1-kt^2}, & \text{if } 0 \le t < \frac{\sigma}{\sqrt{k}}, \\ \dfrac{\sigma^3\sqrt{k}}{kt\sqrt{1-\sigma^2}} + \sqrt{\dfrac{1}{\rho}}, & \text{if } t \ge \frac{\sigma}{\sqrt{k}}, \end{cases}$$

and $l(t) = l(-t)$ for all $t \le 0$. So, it follows from the choice of $\sigma = \sigma(\rho) > 0$ for $\rho > 1$ in (1.3) that $l \in C^1(\mathbb{R}, (\sqrt{1/\rho}, 1))$ is an even function and it increases in $(-\infty, 0)$ and decreases in $[0, +\infty)$.

Note that (2.1) is the Euler–Lagrange equation associated to the energy functional

$$I_k(u) = \frac{1}{2}\int_{\mathbb{R}^N} l^2(u)|\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 - \int_{\mathbb{R}^N} G(u) \tag{2.2}$$

for $|u| < \sigma/\sqrt{k}$.

In the sequel, we will prove the existence of nontrivial antisymmetric critical points u of (2.2) satisfying $\sup_{x\in\mathbb{R}^N} |u(x)| \le \sigma/\sqrt{k}$. This means that it is a nontrivial antisymmetric solution of (2.1) with $l(u) = \sqrt{1-ku^2}$, and so, a nontrivial antisymmetric solution of (1.1) can be got from the function l.

In what follows, we set

$$L(t) = \int_0^t l(s)ds, \qquad t \in \mathbb{R}.$$

By a simple computation, we see that the inverse function $L^{-1}(t)$ exists and it is an odd function. Moreover, it is very important to note that $L, L^{-1} \in C^2(\mathbb{R})$. The lemma below shows some important properties of the functions l and L^{-1} that will be used in the later part of the paper.

Remark 2.1. From assumption (G_4), if $\rho_2 > \rho_1 > 1$ and $g(t)/t^{\rho_2}$ is non-decreasing, then $g(t)/t^{\rho_1}$ is non-decreasing as well. Thus, if $g(t)/t^\rho$ is non-decreasing for some $\rho > 1$, we can assume that ρ is sufficiently close to 1, satisfying

$$4 + \frac{1}{\rho} + \sqrt{\frac{1}{\rho^2} + \frac{8}{\rho}} > \frac{8}{\sqrt{\rho}} \quad \text{and} \quad 2 < 2\sqrt{\rho} < \theta. \tag{2.3}$$

Throughout the paper, we need the following lemma. Its proof can be found in [1] and [2].

Lemma 2.2. *The functions l and L^{-1} satisfy:*

(1) $\lim_{t\to 0} \frac{L^{-1}(t)}{t} = 1$;

(2) $\lim_{t\to\infty} \frac{L^{-1}(t)}{t} = \sqrt{\rho}$;

(3) $\sqrt{\frac{1}{\rho}}t \le l(t)t \le L(t) \le t$ *and* $t \le L^{-1}(t) \le \sqrt{\rho}t$, *for all* $t \ge 0$;

(4) $-\frac{\sigma^2}{1-\sigma^2} \le \frac{t}{l(t)}l'(t) \le 0$, *for all* $t \ge 0$;

(5) $\frac{[L^{-1}(t)]^\delta}{l(L^{-1}(t))t}$, $t > 0$ *is increasing for* $\delta > 1$ *and non-decreasing for* $\delta = 1$,

(6) $\frac{L^{-1}(t)}{l(L^{-1}(t))t^\rho}$, $t > 0$ *is decreasing for* $\rho > 1$ *close to* 1 *and* $\frac{L^{-1}(t)}{t}$, $t > 0$ *is non-decreasing.*

Now, changing variable by
$$v = L(u) = \int_0^u l(s)ds,$$
we can observe that the functional I_k can be rewritten in the form
$$J_k(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|L^{-1}(v)|^2 - \int_{\mathbb{R}^N} G(L^{-1}(v)).$$
From Lemma 2.2, J_k is well defined in $H^1(\mathbb{R}^N)$ and $J_k \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ with
$$\langle J_k'(v), \phi\rangle = \int_{\mathbb{R}^N} \left[\nabla v \nabla \phi + V(x)\frac{L^{-1}(v)}{l(L^{-1}(v))}\phi - \frac{g(L^{-1}(v))}{l(L^{-1}(v))}\phi\right], \tag{2.4}$$
for all $v, \phi \in H^1(\mathbb{R}^N)$.

Lemma 2.3. *If $v \in H^1(\mathbb{R}^N)$ is a critical point of J_k, then $u = L^{-1}(v) \in H^1(\mathbb{R}^N)$ and additionally it is a weak solution for (2.1) if $\sup_{x\in\mathbb{R}^N} |u(x)| \leq \sigma/\sqrt{k}$.*

Proof. See [2]. □

The following embedding result plays an important role in showing that the minimizing function on the partial Nehari manifold are non-trivial functions.

Proposition 2.4. *The function L^{-1} is such that:*

1. *the map $v \longmapsto L^{-1}(v)$ from $(E, \|\cdot\|_E)$ to $(L^s(\mathbb{R}^N), |\cdot|_s)$ is continuous for $2 \leq s \leq 2^*$.*
2. *under (V_4), the above map is compact for $2 \leq s < 2^*$, and under (V_3) with $N \geq 2$, this map is compact for $2 < s < 2^*$.*

Proof. See [2]. □

3 Auxiliary results

Before stating the auxiliary results, let us point out some consequences of our hypotheses.

Remark 3.1. From assumption (G_2), there exists $c_\epsilon > 0$ such that
$$g(t)t \leq \epsilon|t|^2 + c_\epsilon|t|^q \qquad \forall\, t \in \mathbb{R}$$
for each $\epsilon > 0$ given.

Remark 3.2. From assumption (G_3), there exists a constant $K > 0$ such that
$$G(t) \geq K|t|^\theta \quad \text{for all } |t| > \delta$$
for each $\delta > 0$ given.

After these, let us associate to the functional J_k the Nehari manifold
$$\mathcal{N} = \{v \in E\backslash\{0\} \mid \langle J_k'(v), v\rangle = 0\}.$$
In order to find τ-antisymmetric solutions, we look for critical points of the functional J_k on
$$\mathcal{N}^\tau = \{v \in \mathcal{N} \mid v(\tau x) = -v(x)\} \subset \mathcal{N}.$$

The involution τ on $\mathbb{R}^N$ induces an involution $T_\tau : E \to E$ given by

$$T_\tau(v(x)) := -v(\tau(x)).$$

We denote by $E^\tau := \{u \in E : T_\tau(v(x)) = v(x)\}$ the subspace of τ-invariant functions of E, we have

$$\mathcal{N}^\tau = \mathcal{N} \cap E^\tau.$$

Now, we are going to introduce the differentiable continuous function $h_k^v : [0,\infty) \to \mathbb{R}$ by setting $h_k^v(t) = J_k(tv)$, that is,

$$h_k^v(t) := \frac{1}{2}\int_{\mathbb{R}^N} |t\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|L^{-1}(tv)|^2 - \int_{\mathbb{R}^N} G(L^{-1}(tv)),$$

for each $v \in E$ with $v \neq 0$.

Lemma 3.3. *Assume that (G_1)–(G_3) hold. If $v \in E^\tau$ with $v \neq 0$, then there exist $\alpha > 0$ such that*

$$\langle J_k'(\alpha v), v\rangle = 0,$$

that is, $\alpha v \in \mathcal{N}^\tau$, and $\alpha \in (0,\infty)$ is a critical point of h_k^v.

Proof. It follows from the definition of h_k^v, that

$$\begin{aligned}\frac{\partial h_k^v(t)}{\partial t} &= t\int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} V(x)\frac{L^{-1}(tv)}{l(L^{-1}(tv))}v - \int_{\mathbb{R}^N}\frac{g(L^{-1}(tv))}{l(L^{-1}(tv))}v \\ &= \langle J_k'(tv), v\rangle.\end{aligned} \tag{3.1}$$

So, it follows from Remark 3.1 and (3) of Lemma 2.2, that

$$\begin{aligned}\langle J_k'(tv), tv\rangle &\geq t^2\int_{\mathbb{R}^N} |\nabla v|^2 - \int_{\mathbb{R}^N}\frac{g(L^{-1}(tv))}{l(L^{-1}(tv))}tv \\ &\geq t^2\int_{\mathbb{R}^N} |\nabla v|^2 - \int_{\mathbb{R}^N}\frac{\epsilon|L^{-1}(tv)|^2 + c_\epsilon|L^{-1}(tv)|^q}{\sqrt{1/\rho}|L^{-1}(tv)|}|tv| \\ &\geq t^2|\nabla v|_2^2 - \rho\epsilon t^2|v|_2^2 - \sqrt{\rho}^q c_\epsilon t^q|v|_q^q,\end{aligned}$$

which means there exists $t_m > 0$ sufficiently small such that

$$\langle J_k'(t_m v), t_m v\rangle > 0,$$

since $q > 2$.

On the other hand, it follows from Hypothesis (G_3) that

$$\langle J_k'(tv), tv\rangle \leq t^2\int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} V(x)\frac{L^{-1}(tv)}{l(L^{-1}(tv))}(tv) - \theta\int_{\mathbb{R}^N}\frac{G(L^{-1}(tv))}{l(L^{-1}(tv))L^{-1}(tv)}(tv).$$

Set $\delta > 0$ such that the set

$$\mathcal{A} = \{x \in \mathbb{R}^N;\ |v(x)| \geq \delta\} \subset \mathbb{R}^N$$

is not empty. By Remark 3.2; $l(t) > 1/\sqrt{\rho}$, $t > 0$; and (3) of Lemma 2.2, we get

$$\langle J_k'(tv), tv\rangle \leq t^2\int_{\mathbb{R}^N} |\nabla v|^2 + \sqrt{\rho}t^2\int_{\mathbb{R}^N} V(x)v^2 - \theta K t^\theta\int_{\mathcal{A}} |v|^\theta$$

for $t > 0$.

As a consequence, we obtain $t_M > 0$ sufficiently large such that

$$\langle J_k'(t_M v), t_M v\rangle < 0,$$

since $\theta > 2$. Hence, the lemma follows from intermediate value theorem. $\square$

Lemma 3.4. *If $v \in \mathcal{N}$ and (G_4) hold, then*

$$\frac{\partial h_k^v}{\partial t}(t) > 0 \quad \textit{for } 0 < t < 1, \qquad \frac{\partial h_k^v}{\partial t}(t) < 0 \quad \textit{for } t > 1,$$

In particular, $h_k^v(t) < h_k^v(1) = J_k(v)$ for all $t \geq 0$ such that $t \neq 1$.

Proof. By the facts of l being even and L odd, it is sufficiently to prove the case of that $v \geq 0$. First, it follows from (3.1) that

$$\frac{\partial h_k^v(t)}{\partial t} = t^\rho \left\{ \int_{\mathbb{R}^N} \frac{|\nabla v|^2}{t^{\rho-1}} - \int_{\mathbb{R}^N} \left[\frac{g(L^{-1}(tv))}{l(L^{-1}(tv))(tv)^\rho} - \frac{V(x)L^{-1}(tv)}{l(L^{-1}(tv))(tv)^\rho} \right] v^{\rho+1} \right\}.$$

Now, by using (G_4), (5), (6) of Lemma 2.2, and the monotonicity of l, L^{-1}, we obtain

$$\begin{aligned}
&\frac{g(L^{-1}(tv))}{l(L^{-1}(tv))(tv)^\rho} - \frac{V(x)L^{-1}(tv)}{l(L^{-1}(tv))(tv)^\rho} \\
&= \frac{g(L^{-1}(tv))}{(L^{-1}(tv))^\rho} \left[\frac{(L^{-1}(tv))}{tv} \right]^\rho \frac{1}{l(L^{-1}(tv))} - V(x) \frac{L^{-1}(tv)}{l(L^{-1}(tv))(tv)^\rho} \\
&< \frac{g(L^{-1}(v))}{(L^{-1}(v))^\rho} \left[\frac{(L^{-1}(v))}{v} \right]^\rho \frac{1}{l(L^{-1}(v))} - V(x) \frac{L^{-1}(v)}{l(L^{-1}(v))(v)^\rho} \\
&= \frac{g(L^{-1}(v))}{l(L^{-1}(v))(v)^\rho} - V(x) \frac{L^{-1}(v)}{l(L^{-1}(v))(v)^\rho}
\end{aligned}$$

for $0 < t < 1$, and in a similar way, we obtain

$$\frac{g(L^{-1}(tv))}{l(L^{-1}(tv))(tv)^\rho} - \frac{V(x)L^{-1}(tv)}{l(L^{-1}(tv))(tv)^\rho} > \frac{g(L^{-1}(v))}{l(L^{-1}(v))(v)^\rho} - \frac{V(x)L^{-1}(v)}{l(L^{-1}(v))(v)^\rho}$$

for $t > 1$.

So, it follows from above informations, and the hypothesis $v \in \mathcal{N}$, that

$$\frac{\partial h_k^v}{\partial t}(t) > 0 \quad \text{for } 0 < t < 1, \quad \text{and} \quad \frac{\partial h_k^v}{\partial t}(t) < 0 \quad \text{for } t > 1. \tag{3.2}$$

That is, $h_k^v(t) < h_k^v(1) = J_k(v)$. So, the lemma is proved. $\square$

It follows from above informations, that:

Remark 3.5. If $v \in \mathcal{N}$, then 1 is an unique critical point of h_k^v.

Remark 3.6. If $v \in E$ with $v \neq 0$, then the critical point $\alpha = \alpha_v \in (0, +\infty)$ of h_k^v, given by Lemma 3.3, is unique.

In fact, by Lemma 3.3 there is $\alpha > 0$ such that α is a critical point of h_k^v. Finally, assume that α_1 and α_2 are two critical points of h_k^v, then

$$\frac{\alpha_2}{\alpha_1}(\alpha_1 v) = \alpha_2 v.$$

Since $\alpha_1 v \in \mathcal{N}$, then by the Remark 3.5, we have $\alpha_2/\alpha_1 = 1$, and so $\alpha_1 = \alpha_2$.

The following two lemmas are important to prove our theorem, the proofs can be found in [2]

Lemma 3.7. *Assume that V is continuous such that $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^N$ and (G_1)–(G_3) hold. Then:*

(i) for all $v \in \mathcal{N}$, we have

$$J_k(v) \geq \frac{\theta - 2\sqrt{\rho}}{2\theta} \left(\int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} V(x)|L^{-1}(v)|^2 \right),$$

(ii) there is $\gamma > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} V(x)|L^{-1}(v)|^2 \geq \gamma, \quad \textit{for all } v \in \mathcal{N}.$$

Lemma 3.8. *Assume the same hypotheses of Lemma 3.7, and (v_n) being a sequence in $\mathcal{N}$. Then*

$$\liminf_{n \to \infty} \int_{\mathbb{R}^N} |L^{-1}(v_n)|^q dx > 0$$

for some $q \in (2, 2^*)$.

Remark 3.9. By Lemma 3.8 and (3) of Lemma 2.2, there exists a constant $\gamma_1 > 0$ such that

$$\int_{\mathbb{R}^N} |v_n|^q \geq \gamma_1 > 0.$$

Lemma 3.10. *Assume that (G_4) hold. If $\mathcal{V} \subset S^\tau$ is a compact subset of E^τ, then there exists $R > 0$ such that $J_k \leq 0$ on $(\mathbb{R}^+\mathcal{V}) \setminus B_R(0)$, where $S^\tau := \{u \in E^\tau;\ \|u\|_E = 1\}$.*

Proof. Arguing by contradiction, suppose there exits $u_n \in \mathcal{V}$ and $w_n = t_n u_n$ such that $J_k(w_n) \geq 0$ and $t_n \to \infty$ as $n \to \infty$.

By the definition of J_k and (3) of Lemma 2.2 have

$$J_k(w_n) \leq \frac{\rho}{2}\|w_n\|_E - \int_{\mathbb{R}^N} G(L^{-1}(w_n)) = \frac{\rho}{2} t_n^2 - \int_{\mathbb{R}^N} G(L^{-1}(w_n)).$$

Using (G_4), we have $t \longmapsto \frac{G(t)}{t^{\rho+1}}$, $t > 0$ is non-decreasing for some $\rho > 1$ and

$$\frac{G(L^{-1}(w))}{L^{-1}(w)^2} \to \infty \quad \text{uniformly in } x \text{ as } |w| \to \infty. \tag{3.3}$$

Passing to a subsequence, we may assume that $u_n \to u \in S^\tau$. Since $|w_n(x)| \to \infty$ if $u(x) \neq 0$, it follows from (3) of Lemma 2.2, (3.3) and Fatou's lemma that

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{G(L^{-1}(w_n))}{t_n^2} &= \int_{\mathbb{R}^N} \frac{G(L^{-1}(w_n))u_n^2}{w_n^2} \\ &= \int_{\mathbb{R}^N} \frac{G(L^{-1}(w_n))}{L^{-1}(w_n)^2} \frac{L^{-1}(w_n)^2}{w_n^2} u_n^2 \to \infty\end{aligned}$$

Hence

$$0 \leq J_k(w_n) \leq t_n^2 \left[\frac{\rho}{2} - \int_{\mathbb{R}^N} \frac{G(L^{-1}(w_n))}{t_n^2} \right] \to -\infty,$$

a contradiction. □

Recall that S is the unit sphere in E and define the mapping $m : S \to \mathcal{N}$ by setting

$$m(w) := t_w w,$$

where t_w is as α in Lemma 3.3. Moreover, $\|m(w)\|_E = t_w$.

Recall that S^τ is the unit sphere in E^τ, and consider the mapping $m^\tau : S^\tau \to \mathcal{N}^\tau$ by setting

$$m^\tau := m|_{S^\tau}.$$

We shall consider the functional

$$\psi_k^\tau(w) := J_k(m^\tau(w)).$$

By Lemma 3.3, Lemma 3.4, Remark 3.5, Lemma 3.7 and Lemma 3.10, we have the following two lemmas, similar to the results in [23].

Lemma 3.11. *The mapping m^τ is a homeomorphism between S^τ and $\mathcal{N}^\tau$, and the inverse of m^τ is given by $(m^\tau)^{-1}(u) = \frac{u}{\|u\|_E}$.*

Lemma 3.12.

(1) $\psi_k^\tau \in C^1(S^\tau, \mathbb{R})$ and

$$\langle (\psi_k^\tau)'(w), z \rangle = \|m^\tau(w)\|_E \langle J_k'(m^\tau(w)), z \rangle \quad \textit{for all } z \in T_w(S^\tau) \subset E^\tau.$$

(2) If (w_n) is a Palais–Smale sequence for ψ_k^τ, then $(m^\tau(w_n))$ is a Palais–Smale sequence for J_k. If $(u_n) \subset \mathcal{N}^\tau$ is a bounded Palais–Smale sequence for J_k, then $((m^\tau)^{-1}(u_n))$ is a Palais–Smale sequence for ψ_k^τ.

(3) w is a critical point of ψ_k^τ if and only if $m^\tau(w)$ is a nontrivial critical point of $J_k|_{E^\tau}$. Moreover, the corresponding values of ψ_k^τ and J_k coincide and $\inf_{S^\tau} \psi_k^\tau = \inf_{\mathcal{N}^\tau} J_k$.

(4) If J_k is even, then so is ψ_k^τ.

4 Proof of Theorem 1.1

Now, we are ready to prove Theorem 1.1 by applying the auxiliary results in Section 3.

Proof of Theorem 1.1. It follows from Lemma 3.7 that there exists $c_0 > 0$ such that

$$c_0 = \inf_{w \in \mathcal{N}^\tau} J_k(w).$$

Moreover, if $u_0 \in \mathcal{N}^\tau$ satisfies $J_k(u_0) = c_0$, then $(m^\tau)^{-1}(u_0) \in S^\tau$ is a minimizer of ψ_k^τ and therefore a critical point of ψ_k^τ, so that u_0 is a critical point of J_k in E^τ by Lemma 3.12. We will show that there exists a minimizer $v \in \mathcal{N}^\tau$ of $J_k|_{\mathcal{N}^\tau}$. By Ekeland's variational principle [27], there exists a sequence $(w_n) \subset S^\tau$ with $\psi_k^\tau(w_n) \to c_0$ and $(\psi_k^\tau)'(w_n) \to 0$ as $n \to \infty$. Put $u_n = m^\tau(w_n) \in \mathcal{N}^\tau$ for $n \in \mathbb{N}$. Then $J_k(u_n) \to c_0$ and $J_k'(u_n) \to 0$ as $n \to \infty$ by Lemma 3.12 (2).

<u>Claim:</u> $(u_n) \subset E^\tau$ is bounded.

In fact, assume by contradiction that $\|u_n\| \to +\infty$ up to subsequence, that is,

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(x) u_n^2 = \|u_n\|_E^2 \to \infty.$$

So, at least one of the two terms goes to infinity. If

$$\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 \right)^{1/2} \to \infty,$$

it would follow from Lemma 3.7 that

$$J_k(u_n) \geq \frac{\theta - 2\sqrt{\rho}}{2\theta} \int_{\mathbb{R}^N} |\nabla u_n|^2 \to \infty,$$

which is a contradiction, because $(J_k(u_n)) \subset \mathbb{R}$ is bounded. Now, if

$$\int_{\mathbb{R}^N} V(x) u_n^2 \to \infty,$$

then it would follow from Lemma 3.7 again and (3) of Lemma 2.2, that

$$\begin{aligned} J_k(u_n) &\geq \frac{\theta - 2\sqrt{\rho}}{2\theta} \int_{\mathbb{R}^N} V(x) |L^{-1}(u_n)|^2 \\ &\geq \frac{\theta - 2\sqrt{\rho}}{2\theta} \int_{\mathbb{R}^N} V(x) u_n^2 \to \infty, \end{aligned}$$

which is a contradiction again. Hence $u_n \rightharpoonup v$ after passing to a subsequence.

<u>Claim:</u> $v \neq 0$ and $J_k'(v) = 0$ in E^τ.

If (V_2) is fulfilled, then let $y_n \in \mathbb{R}^N$ satisfy

$$\int_{B_1(y_n)} u_n^2 dx = \max_{y \in \mathbb{R}^N} \int_{B_1(y)} u_n^2 dx.$$

Using once more that J_k and $\mathcal{N}^\tau$ are invariant under translations of the form $u \longmapsto u(\cdot - k)$ with $k \in \mathbb{Z}^N$, we may assume that (y_n) is bounded in $\mathbb{R}^N$. If

$$\int_{B_1(y_n)} u_n^2 dx \to 0 \quad \text{as } n \to \infty, \tag{4.1}$$

then $u_n \to 0$ in $L^p(\mathbb{R}^N)$, $2 < p < 2^*$, by Lemma 1.21 in [27]. From Proposition 2.4 and (G_2), we infer that

$$\int_{\mathbb{R}^N} \frac{g(L^{-1}(u_n))u_n}{l(L^{-1}(u_n))} dx = o(\|u_n\|_E)$$

as $n \to \infty$, hence

$$\begin{aligned} o(\|u_n\|_E) = J_k'(u_n)u_n &= \int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(x)\frac{L^{-1}(u_n)u_n}{l(L^{-1}(u_n))} - \int_{\mathbb{R}^N} \frac{g(L^{-1}(u_n))u_n}{l(L^{-1}(u_n))} dx \\ &= \int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(x)\frac{L^{-1}(u_n)u_n}{l(L^{-1}(u_n))} - o(\|u_n\|_E) \end{aligned}$$

and therefore $\|u_n\|_E \to 0$, contrary to Lemma 3.7. It follows that (4.1) cannot hold, so $u_n \rightharpoonup v \neq 0$ and $J_k'(v) = 0$.

Suppose that (V_3) or (V_4) is satisfied. Then it follows from Proposition 2.4, that

$$L^{-1}(u_n) \to L^{-1}(v) \quad \text{in } L^{\gamma}(\mathbb{R}^N) \text{ for all } \gamma \in (2, 2^*).$$

Then by Lemma 3.8, we conclude that $v \neq 0$ and $J_k'(v) = 0$ in E^τ.

Hence, we conclude that $v \in \mathcal{N}^\tau$ is a critical point of J_k in E^τ. Now we will show that $J_k(v) = c_0$. By Lemma 2.2, Fatou's lemma and since $(u_n) \subset E^\tau$ is bounded,

$$\begin{aligned} c_0 + o(1) &= J_k(u_n) - \frac{1}{\theta}\langle J_k'(u_n), u_n\rangle \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|L^{-1}(u_n)|^2 dx - \int_{\mathbb{R}^N} G(L^{-1}(u_n))dx \\ &\quad - \frac{1}{\theta}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \frac{1}{\theta}\int_{\mathbb{R}^N} V(x)\frac{L^{-1}(u_n)u_n}{l(L^{-1}(u_n))} dx + \frac{1}{\theta}\int_{\mathbb{R}^N} \frac{g(L^{-1}(u_n))u_n}{l(L^{-1}(u_n))} dx \\ &= \frac{\theta-2}{2\theta}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|L^{-1}(u_n)|^2 dx - \frac{\sqrt{\rho}}{\theta} V(x)|L^{-1}(u_n)|^2 dx \\ &\quad + \frac{\sqrt{\rho}}{\theta} V(x)|L^{-1}(u_n)|^2 dx - \frac{1}{\theta}\int_{\mathbb{R}^N} V(x)\frac{L^{-1}(u_n)u_n}{l(L^{-1}(u_n))} dx \\ &\quad + \int_{\mathbb{R}^N} \left[\frac{1}{\theta}\frac{g(L^{-1}(u_n))u_n}{l(L^{-1}(u_n))} - G(L^{-1}(u_n))\right] dx \\ &= \frac{\theta-2}{2\theta}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{\theta-2\sqrt{\rho}}{2\theta}\int_{\mathbb{R}^N} V(x)|L^{-1}(u_n)|^2 dx \\ &\quad + \frac{1}{\theta}\int_{\mathbb{R}^N} V(x)\left[\sqrt{\rho}|L^{-1}(u_n)|^2 - \frac{L^{-1}(u_n)u_n}{l(L^{-1}(u_n))}\right] dx \\ &\quad + \int_{\mathbb{R}^N} \left[\frac{1}{\theta}\frac{g(L^{-1}(u_n))u_n}{l(L^{-1}(u_n))} - G(L^{-1}(u_n))\right] dx \\ &\geq \frac{\theta-2}{2\theta}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{\theta-2\sqrt{\rho}}{2\theta}\int_{\mathbb{R}^N} V(x)|L^{-1}(v)|^2 dx \\ &\quad + \frac{1}{\theta}\int_{\mathbb{R}^N} V(x)\left[\sqrt{\rho}|L^{-1}(v)|^2 - \frac{L^{-1}(v)v}{l(L^{-1}(v))}\right] dx \\ &\quad + \int_{\mathbb{R}^N} \left[\frac{1}{\theta}\frac{g(L^{-1}(v))v}{l(L^{-1}(v))} - G(L^{-1}(v))\right] dx + o(1) \\ &= J_k(v) - \frac{1}{\theta}\langle J_k'(v), v\rangle + o(1) = J_k(v) + o(1). \end{aligned}$$

On the other hand, since $J_k(v) \geq c_0$, hence $J_k(v) = c_0$.

Now, by using a quantitative deformation lemma and adapting the arguments in [4, 11], we are going to show $J_k'(v) = 0$ in E.

Suppose, by contradiction, that $J_k'(v) \neq 0$. Then there exist $\delta > 0$ and $\nu > 0$ such that

$$\|J_k'(w)\| \geq \nu \quad \text{for every } w \in E \ \text{ with } \|w - v\| \leq 2\delta.$$

Since $v \neq 0$, we can take $L = \|v\|_E > 0$ and, without loss of generality, we may assume $6\delta < L$.

Let $I = \left[\frac{1}{2}, \frac{3}{2}\right]$. Since, $\langle J_k'(v), v\rangle = 0$ and by Lemma 3.4,

$$J_k(tv) < J_k(v) = c_0,$$

holds for $t \in I$ with $t \neq 1$, we obtain that

$$\tilde{c} = \max_{\partial I} J_k(tv) < c_0.$$

Applying Theorem A.4 in [28] with $\epsilon = \min\{(c_0 - \tilde{c})/2, \nu\delta/8\}$ and $S = B(v, \delta)$, there exists $\eta \in C([0,1] \times E, E)$ such that

(i) $\eta(\theta, u) = u$ if $\theta = 0$ or if $u \notin J_k^{-1}[c_0 - 2\epsilon, c_0 + 2\epsilon] \cap B(v, 2\delta)$;

(ii) $\eta(1, J_k^{c_0+\epsilon}) \cap B(v, \delta) \subset J_k^{c_0-\epsilon}$;

(iii) $J_k(\eta(1, w)) \leq J_k(w)$ for every $w \in E$, where $J_k^a = \{w \in E;\ J_k(w) \leq a\}$,

(iv) $\eta(t, u)$ is odd in u.

Consequently, we have

$$\max_{t \in I} J_k(\eta(1, tv)) < c_0. \tag{4.2}$$

On the other hand, we claim that there exists $t_0 \in I$ such that

$$\eta(1, t_0 v) \in \mathcal{N}^\tau.$$

In fact, by (iv) for η, we know $\eta(1, tv) \in E^\tau$ for each t. Now we will prove that there exists $t_0 \in I$ such that $t_0 v \in \mathcal{N}$. Define $\varphi(t) = \eta(1, tv)$ and

$$\Psi(t) = \langle J_k'(\varphi(t)), \varphi(t)\rangle$$

for $t > 0$. Since,

$$\|v - tv\|_E = |1 - t|\|v\|_E = |1 - t|L \geq 6\delta|1 - t| > 2\delta \tag{4.3}$$

if only if $t < \frac{2}{3}$ or $t > \frac{4}{3}$. It follows from property (i) for η and inequality (4.3) that $\varphi(t) = \eta(1, tv) = tv \in E^\tau$ if $t \in [\frac{1}{2}, \frac{2}{3}) \cup (\frac{4}{3}, \frac{3}{2}]$.

Thus,

$$\Psi(\tfrac{1}{2}) = \left\langle J_k'\left(\varphi\left(\tfrac{1}{2}\right)\right), \varphi(\tfrac{1}{2})\right\rangle = \left\langle J_k'\left(\tfrac{1}{2}v\right), \tfrac{1}{2}v\right\rangle,$$

and it follows from (3.2) that

$$\left\langle J_k'\left(\tfrac{1}{2}v\right), \tfrac{1}{2}v\right\rangle = \frac{1}{2}\frac{\partial h_k^v}{\partial t}\left(\frac{1}{2}\right) > 0. \tag{4.4}$$

On the other hand,

$$\Psi(\tfrac{3}{2}) = \langle J_k'(\varphi(\tfrac{3}{2})), \varphi(\tfrac{3}{2})\rangle = \langle J_k'(\tfrac{3}{2}v), \tfrac{3}{2}v\rangle,$$

and it follows from (3.2) that

$$\left\langle J_k'\left(\tfrac{3}{2}v\right), \tfrac{3}{2}v\right\rangle = \tfrac{3}{2}\tfrac{\partial h_k^v}{\partial t}\left(\tfrac{3}{2}\right) < 0. \tag{4.5}$$

Noting that the function Ψ is continuous on I and taking (4.4) and (4.5) into account, we can apply the intermediate value theorem again to conclude that there exists $t_0 \in I$ such that $\Psi(t_0) = 0$. This and (4.2) lead to a contradiction. Hence, we conclude that v is a critical point of J_k. So, by Lemma 2.3, we just need to show that $|u|_\infty = |L^{-1}(v)|_\infty \le \sigma/\sqrt{k}$ holds to conclude that u is a solution of problem (1.1).

Now, set $\varphi = L^{-1}(v)l(L^{-1}(v))$. It follows from Lemma 2.2 that

$$|\varphi| = |L^{-1}(v)l(L^{-1}(v))| \le |v|, \quad \text{and} \quad |\nabla\varphi| = \left|1 + \frac{L^{-1}(v)l'(L^{-1}(v))}{l(L^{-1}(v))}\right| |\nabla v| \le |\nabla v|,$$

that is, $\varphi \in H^1(\mathbb{R}^N)$. So, by taking φ as a test function in (2.4), we obtain

$$\int_{\mathbb{R}^N} \left[1 + \frac{L^{-1}(v)l'(L^{-1}(v))}{l(L^{-1}(v))}\right] |\nabla v|^2 + V(x)|L^{-1}(v)|^2 - g(L^{-1}(v))L^{-1}(v) = 0.$$

As a consequence of (4) of Lemma 2.2, we have

$$\int_{\mathbb{R}^N} |\nabla v|^2 + V(x)|L^{-1}(v)|^2 - g(L^{-1}(v))L^{-1}(v) \ge 0.$$

Since v is a critical point of J_k, it follows that

$$\begin{aligned} \theta c_0 &= \theta J_k(v) - \langle J_k'(v), L^{-1}(v)l(L^{-1}(v))\rangle \\ &\ge \frac{\theta - 2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + V(x)|L^{-1}(v)|^2. \end{aligned}$$

Then, by (3) of Lemma 2.2,

$$\|v\|_E^2 \le \frac{2\theta c_0}{\theta - 2}. \tag{4.6}$$

For each $m \in \mathbb{N}$ and $\beta > 1$ given, define

$$A_m = \{x \in \mathbb{R}^N;\ |v|^{\beta-1} \le m\} \text{ and } B_m = \mathbb{R}^N \backslash A_m,$$

and

$$v_m = \begin{cases} v|v|^{2(\beta-1)} & \text{in } A_m, \\ m^2 v & \text{in } B_m. \end{cases}$$

We know $v_m \in H^1(\mathbb{R}^N)$, $v_m \le v_{m+1}$, $v_m \le |v|^{2\beta-1}$, and

$$\nabla v_m = \begin{cases} (2\beta-1)|v|^{2(\beta-1)}\nabla v & \text{in } A_m, \\ m^2\nabla v & \text{in } B_m, \end{cases}$$

that is, v_m can be used as a test function. Besides this, we have

$$\int_{\mathbb{R}^N} \nabla v \nabla v_m = (2\beta - 1)\int_{A_m} |v|^{2(\beta-1)}|\nabla v|^2 + m^2 \int_{B_m} |\nabla v|^2. \tag{4.7}$$

Letting

$$w_m = \begin{cases} v|v|^{\beta-1} & \text{in } A_m, \\ mv & \text{in } B_m, \end{cases}$$

we obtain $w_m^2 = vv_m \le |v|^{2\beta}$, $w_m \le w_{m+1}$, and

$$\nabla w_m = \begin{cases} \beta |v|^{\beta-1}\nabla v & \text{in } A_m, \\ m\nabla v & \text{in } B_m. \end{cases}$$

So,

$$\int_{\mathbb{R}^N} |\nabla w_m|^2 = \beta^2 \int_{A_m} |v|^{2(\beta-1)}|\nabla v|^2 + m^2 \int_{B_m} |\nabla v|^2. \tag{4.8}$$

As a consequence of (4.7) and (4.8), we obtain

$$\int_{\mathbb{R}^N} [|\nabla w_m|^2 - \nabla v \nabla v_m] = (\beta-1)^2 \int_{A_m} |v|^{2(\beta-1)}|\nabla v|^2. \tag{4.9}$$

Taking v_m as a test function, it follows from (4.7) and (4.9) that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla w_m|^2 \quad & +\beta^2 \int_{\mathbb{R}^N} V(x) \frac{L^{-1}(v)}{l(L^{-1}(v))} v_m \\
&= (\beta-1)^2 \int_{A_m} |v|^{2(\beta-1)}|\nabla v|^2 + \int_{\mathbb{R}^N} \nabla v \nabla v_m + \beta^2 \int_{\mathbb{R}^N} V(x) \frac{L^{-1}(v)}{l(L^{-1}(v))} v_m \\
&\le \left[\frac{(\beta-1)^2}{2\beta-1} + 1\right] \int_{\mathbb{R}^N} \nabla v \nabla v_m + +\beta^2 \int_{\mathbb{R}^N} V(x) \frac{L^{-1}(v)}{l(L^{-1}(v))} v_m \\
&\le \beta^2 \int_{\mathbb{R}^N} [\nabla v \nabla v_m + V(x) \frac{L^{-1}(v)}{l(L^{-1}(v))} v_m] \\
&= \beta^2 \int_{\mathbb{R}^N} \frac{g(L^{-1}(v))}{l(L^{-1}(v))} v_m.
\end{aligned}$$

Now, it follows from Remark 3.1 that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla w_m|^2 + \beta^2 \int_{\mathbb{R}^N} V(x) \frac{L^{-1}(v)}{l(L^{-1}(v))} v_m \\
&\quad \le \beta^2 \int_{\mathbb{R}^N} \frac{\epsilon |L^{-1}(v)|^2}{|l(L^{-1}(v))L^{-1}(v)|} |v_m| + \beta^2 \int_{\mathbb{R}^N} \frac{c_\epsilon |L^{-1}(v)|^q}{|l(L^{-1}(v))L^{-1}(v)|} |v_m|,
\end{aligned}$$

that is,

$$\beta^2 \int_{\mathbb{R}^N} V(x) \frac{L^{-1}(v)}{l(L^{-1}(v))} v_m \ge \beta^2 \int_{\mathbb{R}^N} \frac{\epsilon |L^{-1}(v)|^2}{|l(L^{-1}(v))L^{-1}(v)|} |v_m|,$$

for $\epsilon > 0$ sufficiently small. So, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla w_m|^2 &\le \beta^2 \int_{\mathbb{R}^N} \frac{c_\epsilon |L^{-1}(v)|^q}{|l(L^{-1}(v))L^{-1}(v)|} |v_m| \\
&\le \beta^2 \int_{\mathbb{R}^N} c_\epsilon \rho^{\frac{q}{2}} |v|^{q-2} w_m^2.
\end{aligned}$$

Then, it follows from the Sobolev inequality that

$$\begin{aligned}
\left(\int_{A_m} |w_m|^{2^*}\right)^{\frac{N-2}{N}} &\le S \int_{\mathbb{R}^N} |\nabla w_m|^2 \\
&\le S\beta^2 \int_{\mathbb{R}^N} c_\epsilon \rho^{\frac{q}{2}} |v|^{q-2} w_m^2.
\end{aligned}$$

The Hölder inequality implies that

$$\left(\int_{A_m} |w_m|^{2^*}\right)^{\frac{N-2}{N}} \le c_\epsilon \rho^{\frac{q}{2}} S\beta^2 |v|_{2^*}^{q-2} \left(\int_{\mathbb{R}^N} |w_m|^{2r_1}\right)^{1/r_1},$$

where $1/r_1 + (q-2)/2^* = 1$.

Since, $|w_m| \le |v|^\beta$ in $\mathbb{R}^N$ and $|w_m| = |v|^\beta$ in A_m, we have

$$\left(\int_{A_m} |v|^{\beta 2^*}\right)^{\frac{N-2}{N}} \le c_\epsilon \rho^{\frac{q}{2}} S\beta^2 |v|_{2^*}^{q-2} \left(\int_{\mathbb{R}^N} |v|^{2\beta r_1}\right)^{1/r_1},$$

which implies, by the Monotone Convergence Theorem, that

$$|v|_{\beta 2^*} \le \beta^{1/\beta} (c_\epsilon \rho^{\frac{q}{2}} S |v|_{2^*}^{q-2})^{1/2\beta} |v|_{2\beta r_1} \tag{4.10}$$

So, taking $\sigma = 2^*/(2r_1)$ and set $\beta = \sigma^i$, $i = 1, 2, \dots$, in an iterative way in (4.10), we get

$$|v|_{\sigma^i 2^*} \le \sigma^{(\sum_{j=1}^{i} j/\sigma^j)} (c_\epsilon \rho^{\frac{q}{2}} S |v|_{2^*}^{q-2})^{(1/2 \sum_{j=1}^{i} 1/\sigma^j)} |v|_{2^*},$$

that is, by doing $i \to \infty$ and using the limitation of $\|v\|_E$, given by (4.6), together with the Sobolev inequality, we get $|v|_\infty \le C_0$, where $C_0 > 0$ is a real constant independent of $k > 0$.

Now, it follows from Lemma 2.2-(3) that

$$|u|_\infty = |L^{-1}(v)|_\infty \le \sqrt{\rho}|v|_\infty \le \sqrt{\rho} C_0 \le \sigma/\sqrt{k}$$

holds for all $k \in (0, k_0)$, where $k_0 > 0$ is such that $\sqrt{\rho} C_0 \le \sigma/\sqrt{k_0}$. Thus, Lemma 2.3 implies that problem (1.1) admits a solution. □

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A study of logistic growth models influenced by the exterior matrix hostility and grazing in an interior patch

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Abstract. We will analyze the symmetric positive solutions to the two-point steady state reaction-diffusion equation:

$$\begin{aligned} -u'' &= \begin{cases} \lambda\left[u - \dfrac{1}{K}u^2 - \dfrac{cu^2}{1+u^2}\right]; & x \in [L, 1-L], \\ \lambda\left[u - \dfrac{1}{K}u^2\right]; & x \in (0, L) \cup (1-L, 1), \end{cases} \\ -u'(0) &+ \sqrt{\lambda}\gamma u(0) = 0, \\ u'(1) &+ \sqrt{\lambda}\gamma u(1) = 0, \end{aligned}$$

where λ, c, K, and γ are positive parameters and the parameter $L \in (0, \frac{1}{2})$. The steady state reaction-diffusion equation above occurs in ecological systems and population dynamics. The above model exhibits logistic growth in the one-dimensional habitat $\Omega_0 = (0, 1)$, where grazing (type of predation) is occurring on the subregion $[L, 1-L]$. In this model, u is the population density and c is the maximum grazing rate. λ is a parameter which influences the equation as well as the boundary conditions, and γ represents the hostility factor of the surrounding matrix. Previous studies have shown the occurrence of S-shaped bifurcation curves for positive solutions for certain parameter ranges when the boundary condition is Dirichlet ($\gamma \longrightarrow \infty$). Here we discuss the occurrence of S-shaped bifurcation curves for certain parameter ranges, when γ is finite, and their evolutions as γ and L vary.

Keywords: differential equations, boundary value problems, logistic growth, exterior matrix hostility, interior grazing, positive solutions.

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1 Introduction

First, we briefly discuss the history of grazing type models. Recently in [5], authors discussed the following boundary value problem:

$$\begin{cases} -\Delta u = \lambda \left(u - \dfrac{u^2}{K} - \dfrac{cu^2}{1+u^2} \right); & \Omega, \\ \dfrac{\partial u}{\partial \eta} + \sqrt{\lambda} u = 0; & \partial\Omega, \end{cases} \tag{1.1}$$

where $\frac{\partial u}{\partial \eta}$ is the outward normal derivative of u, $\lambda > 0$, $K > 0$, $0 < c < 2$, and Ω is a bounded domain in $\mathbb{R}^N$; $N \geq 1$ with smooth boundary $\partial\Omega$. Here, u is the population density, λ is a positive parameter, and c is the maximum grazing rate. The term $u - \frac{1}{K}u^2$ represents a logistic growth, which means the per capita growth rate is a linear depreciation. The term $\frac{cu^2}{1+u^2}$ represents the rate of grazing by a constant number of grazers (see Figure 1.2). The authors established the occurrence of S-shaped bifurcation curves when parameters c and K satisfy certain conditions. Grazing type models apply to many ecological systems arising in population dynamics such as the dynamics of salmon fish and spruce budworms (see [9] and [12]).

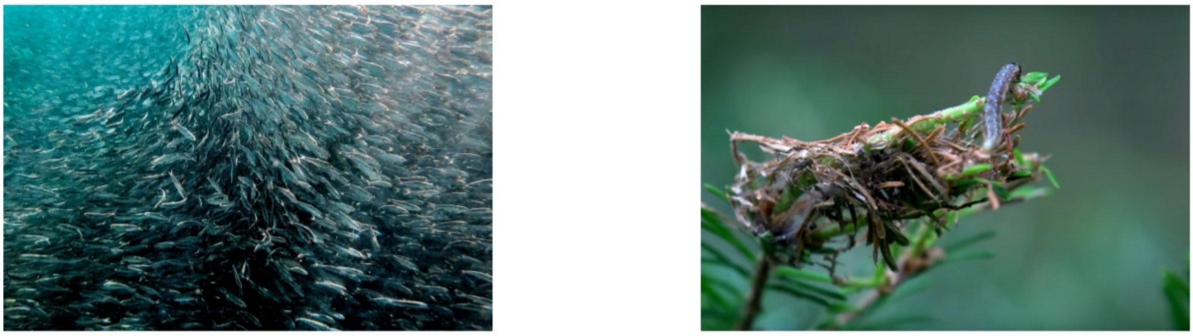


Figure 1.1: Examples of salmon and spruce budworms

However, it turns out that the grazing presents itself only in an interior patch in many real-world situations. We refer the reader to [1] for a study in this direction where the authors studied the following Dirichlet boundary value problem:

$$\begin{aligned} -u'' &= \begin{cases} \lambda \tilde{f}(u); & x \in [L, 1-L], \\ \lambda f(u); & x \in (0, L) \cup (1-L, 1), \end{cases} \\ u(0) &= u(1) = 0, \end{aligned} \tag{1.2}$$

where $\tilde{f}(u) = u - \frac{1}{K}u^2 - \frac{cu^2}{1+u^2}$ and $f(u) = u - \frac{1}{K}u^2$, which corresponds to the case where $\gamma \to \infty$ (see (1.5)). Now, λ, c, and K are positive parameters and the parameter $L \in (0, \frac{1}{2})$. The authors showed the occurrence of S-shaped bifurcation curves for certain parameter ranges and numerically obtained the evolution of the bifurcation curves over a range of L-values and K-values, for a fixed value of c. In particular, for $c = 1.5$ they showed that occurrence of S-shaped bifurcation persists for any value of L, if K is chosen to be large enough.

Biologists have recently observed that in the study of grazing models, to better predict the behavior of the ecological system, it is vital to take the exterior matrix hostility factor into

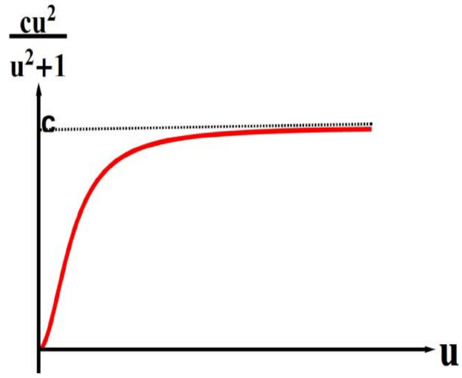


Figure 1.2: Grazing.

account. In this paper, we extend the study in [1] to the case when the exterior matrix hostility is incorporated into the model. We obtain our results via a modified quadrature method and Mathematica computations.

We now briefly discuss the modeling aspect of the problem. We consider the domain $\Omega_0 = \{lx \mid x \in \Omega\}$, where $\Omega = (0, 1)$ and l is a parameter representing the size of the habitat. We assume that the diffusion rate in the patch Ω_0 is D. In the matrix $\mathbb{R} \setminus \overline{\Omega}_0$, we assume that the diffusion rate is D_0, and the death rate is S_0.

We will further assume that the population exhibits density dependent dispersal (DDD) on the boundary $\partial\Omega_0$. Defining $\alpha(u)$ as the probability of the population remaining in Ω_0 when it reaches the boundary, the resulting model is (see [2, 6, 10, 11]):

$$\begin{cases} u_t = Du_{xx} + h(u); & x \in \Omega_0, t > 0, \\ u(0, x) = u_0(x); & x \in \Omega_0, \\ D\alpha(u)\dfrac{\partial u}{\partial \eta} + \dfrac{\sqrt{S_0 D_0}}{k}[1 - \alpha(u)]u = 0; & x \in \partial\Omega_0, t > 0 \end{cases} \tag{1.3}$$

with the corresponding steady state equation:

$$\begin{cases} -u'' = \dfrac{1}{D}h(u); & x \in \Omega_0, \\ D\alpha(u)\dfrac{\partial u}{\partial \eta} + \dfrac{\sqrt{S_0 D_0}}{k}[1 - \alpha(u)]u = 0; & x \in \partial\Omega_0, \end{cases}$$

or equivalently

$$\begin{cases} -u'' = \dfrac{l^2}{D}h(u); & x \in \Omega, \\ \dfrac{\partial u}{\partial \eta} + \dfrac{\sqrt{S_0 D_0}\,l}{kD}\left[\dfrac{1 - \alpha(u)}{\alpha(u)}\right]u = 0; & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where k is a positive parameter related to the movement behavior of the species (see [2], [3]). Here $h(u)$ represents the reaction term. More precisely, $h(u) = u - \frac{1}{K}u^2$ in the case of logistic population growth, whereas in the case of logistic growth with grazing $h(u) = u - \frac{1}{K}u^2 - \frac{cu^2}{1+u^2}$. Let $\lambda = \frac{l^2}{D}$ and $\gamma = \frac{\sqrt{S_0 D_0}}{k\sqrt{D}}$. Here γ represents the matrix hostility factor. Then (1.4) reduces to

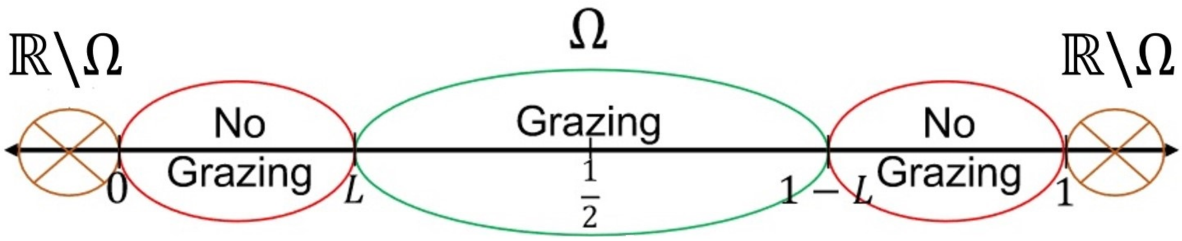


Figure 1.3: Grazing region, non grazing regions and exterior matrix.

$$\begin{cases} -u'' = \lambda h(u); \quad x \in (0,1), \\ -u'(0) + \gamma\sqrt{\lambda} g(u(0))u(0) = 0, \\ u'(1) + \gamma\sqrt{\lambda} g(u(1))u(1) = 0, \end{cases} \tag{1.5}$$

where $g(s) = \frac{1-\alpha(s)}{\alpha(s)}$.

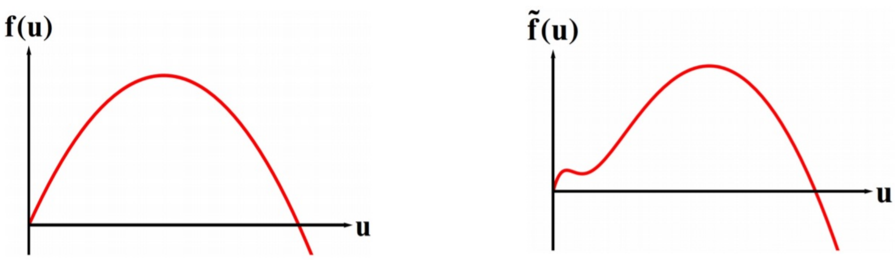
In this paper, we will study positive solutions of (1.5) which are symmetric about $x = \frac{1}{2}$, when $\alpha(s) = \frac{1}{2}$ and

$$h(u) = \begin{cases} \lambda \tilde{f}(u); \ x \in [L, 1-L], \\ \lambda f(u); \ x \in (0, L) \cup (1-L, 1) \end{cases}$$

via a quadrature method. Namely, when $K = 10$ and $c = 1.5$ we will study positive solutions of:

$$\begin{aligned} -u'' &= \begin{cases} \lambda \tilde{f}(u); \ x \in [L, 1-L], \\ \lambda f(u); \ x \in (0, L) \cup (1-L, 1), \end{cases} \\ -u'(0) &+ \gamma\sqrt{\lambda} u(0) = 0, \\ u'(1) &+ \gamma\sqrt{\lambda} u(1) = 0, \end{aligned} \tag{1.6}$$

such that $u(L^-) = u(L^+)$ and $u'(L^-) = u'(L^+)$ where γ is a parameter related to the matrix hostility.



(a) $f(u) = u - \frac{1}{K}u^2$ (b) $\tilde{f}(u) = u - \frac{1}{K}u^2 - \frac{cu^2}{1+u^2}$

Figure 1.4: Shapes of f and $\tilde{f}$.

In particular, we study the evolution of these steady states of (1.6) with respect to L when the hostility parameter γ is fixed and vice-versa.

Now we present the following theorem which describes the structure of such positive solutions.

Let $\|u\|_\infty = \rho$, $u(L) = \sigma$, and $u(0) = u(1) = q$, $\tilde{F}(s) := \int_0^s \tilde{f}(t)dt$ and $F(s) := \int_0^s f(t)dt$.

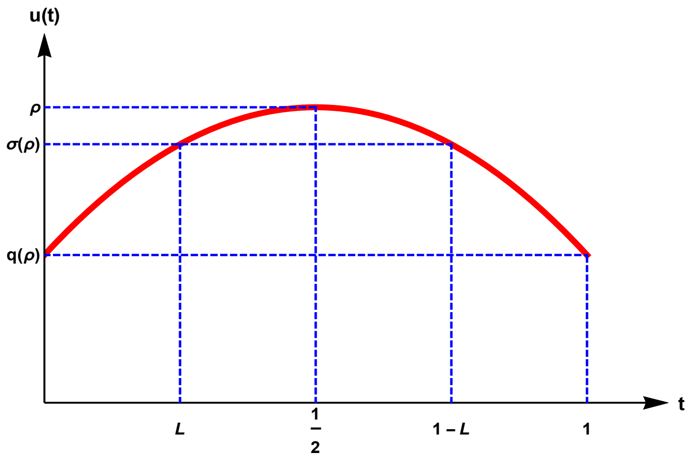


Figure 1.5: Graph of a symmetric solution u to (1.6).

Theorem 1.1. *A symmetric solution (as in Figure 1.5) of (1.6) exists if and only if λ, ρ, σ and q satisfy:*

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}L}\int_q^\sigma \frac{dv}{\sqrt{F(q)+\frac{\gamma^2q^2}{2}-F(v)}} = \frac{1}{\sqrt{2}(\frac{1}{2}-L)}\int_\sigma^\rho \frac{dv}{\sqrt{\tilde{F}(\rho)-\tilde{F}(v)}},$$
$$F(q)+\frac{\gamma^2q^2}{2}-F(\sigma) = \tilde{F}(\rho)-\tilde{F}(\sigma).$$

In Section 2, we detail the proof of Theorem 1.1. In Section 3, we provide biological implications and numerical results.

2 Proof of Theorem 1.1

Suppose $u > 0$ is a solution of (1.6). We first focus on the region $(L, \frac{1}{2})$. Multiply both sides of (1.6) by u' and obtain

$$\left[\frac{-(u'(x))^2}{2}\right]' = \lambda\left[\tilde{F}(u(x))\right]'.$$

Next, by integrating, we obtain

$$u'(x) = \sqrt{2\lambda\left[\tilde{F}(\rho)-\tilde{F}(u(x))\right]}; \qquad x\in\left[L,\tfrac{1}{2}\right],$$

and further integration leads to

$$\int_x^{\frac{1}{2}} \frac{u'(s)}{\sqrt{\tilde{F}(\rho)-\tilde{F}(u(s))}}\,ds = \int_x^{\frac{1}{2}}\sqrt{2\lambda}\,ds; \qquad x\in\left[L,\tfrac{1}{2}\right).$$

Now using the substitution $v = u(s)$ we obtain

$$\int_{u(x)}^{u(\frac{1}{2})} \frac{1}{\sqrt{\tilde{F}(\rho)-\tilde{F}(v)}}\,dv = \sqrt{2\lambda}\left[\frac{1}{2}-x\right]; \qquad x\in\left[L,\tfrac{1}{2}\right).$$

Setting $x = L$ we have

$$\int_\sigma^\rho \frac{1}{\sqrt{\tilde{F}(\rho) - \tilde{F}(v)}}\, dv = \sqrt{2\lambda}\left[\frac{1}{2} - L\right].$$

Further, solving for λ we obtain

$$\lambda = \left[\frac{1}{\sqrt{2}(\frac{1}{2} - L)} \int_\sigma^\rho \frac{1}{\sqrt{\tilde{F}(\rho) - \tilde{F}(v)}}\, dv\right]^2. \tag{2.1}$$

We next focus on the region $(0, L)$. Again by the above quadrature method, letting $u(0) = q$, by the boundary conditions we get

$$u'(x) = \sqrt{2\lambda\left[F(q) + \frac{\gamma^2 q^2}{2} - F(u(x))\right]}; \qquad x \in [0, L].$$

Integrating on $(0, x)$ we have

$$\int_q^{u(x)} \frac{1}{\sqrt{F(q) + \frac{\gamma^2 q^2}{2} - F(v)}}\, dv = \sqrt{2\lambda}x; \qquad x \in [0, L].$$

Hence substituting $x = L$ and solving for λ yields

$$\lambda = \left[\frac{1}{\sqrt{2}L} \int_q^\sigma \frac{1}{\sqrt{F(q) + \frac{\gamma^2 q^2}{2} - F(v)}}\, dv\right]^2. \tag{2.2}$$

Now using $u'(L^-) = u'(L^+)$, (2.1) and (2.2), we obtain:

$$\frac{1}{\sqrt{2}L} \int_q^\sigma \frac{dv}{\sqrt{F(q) + \frac{\gamma^2 q^2}{2} - F(v)}} = \frac{1}{\sqrt{2}(\frac{1}{2} - L)} \int_\sigma^\rho \frac{dv}{\sqrt{\tilde{F}(\rho) - \tilde{F}(v)}}, \tag{2.3}$$

$$F(q) + \frac{\gamma^2 q^2}{2} - F(\sigma) = \tilde{F}(\rho) - \tilde{F}(\sigma). \tag{2.4}$$

In fact, given ρ, q and σ satisfy (2.3) and (2.4), we can back track and use the Implicit Function Theorem to obtain a solution as described in Figure 1.5 with

$$\lambda = \left[\frac{1}{\sqrt{2}L} \int_q^\sigma \frac{1}{\sqrt{F(q) + \frac{\gamma^2 q^2}{2} - F(v)}}\, dv\right]^2.$$

Hence the proof is complete.

We provide our computational results in the next section.

3 Computational results and biological implications

In [1], authors showed the occurrence of an S-shaped bifurcation curve for (1.2) for certain parameter ranges when grazing is confined to an interior region of $(0, 1)$. Indeed, they numerically showed that for a fixed $c = 1.5$, occurrence of an S-shaped bifurcation curve for (1.2) always happens if K is chosen to be large enough. Namely, they showed that for $K \gg 1$ there exist m_1, m_2, and m_3 such that (1.2) has (see Figure 3.1):

- no positive solution for $\lambda \in (0, m_1]$
- exactly one positive solution for $\lambda \in (m_1, m_2)$
- exactly two positive solutions for $\lambda = m_2$
- exactly three positive solutions for $\lambda \in (m_2, m_3)$
- exactly two positive solutions for $\lambda = m_3$
- exactly one positive solution for $\lambda \in (m_3, \infty)$

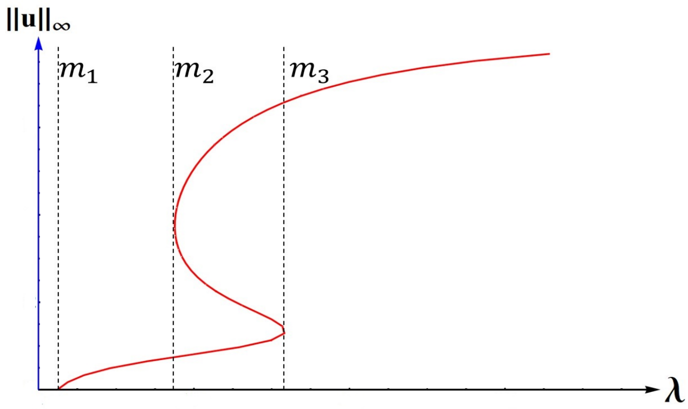


Figure 3.1: Occurrence of S-shaped bifurcation for (1.2).

We will obtain similar results when grazing is restricted to an interior patch, namely for (1.6). Moreover, we investigate the λ region where multiplicity of positive solutions occurs. In particular, we fix all parameters with the exception of L and γ, where variations are implemented. First, we consider fixed values of L, namely $L = 0.05$, 0.30, and 0.45, and we demonstrate the evolution of the bifurcation diagrams for positive solutions when γ varies. Next, for $\gamma = 50$ (fixed), we demonstrate the evolution of the bifurcation diagrams for positive solutions when L varies.

We briefly explain how we obtain numerical bifurcation diagrams. Let $\gamma > 0$, $L > 0$, and $M > 0$ be fixed, and let $x_i = \frac{i}{n+1}$; $i = 1, \ldots, n+1$ for some $n \geq 1$. Letting $\rho = x_1$, we numerically solve the equations (2.3) and (2.4) simultaneously for σ and q using the FindRoot command in Mathematica. The values of σ and q are substituted into (2.2) to find the corresponding value of λ. Repeating this procedure for $\rho = x_i$, $i = 2, \ldots, n+1$, we obtain (λ, ρ) points for the bifurcation diagram.

Our research shows the following four cases:

1) For small values of L, multiplicity of positive solutions persists for certain ranges of λ irrespective of the value of hostility factor.

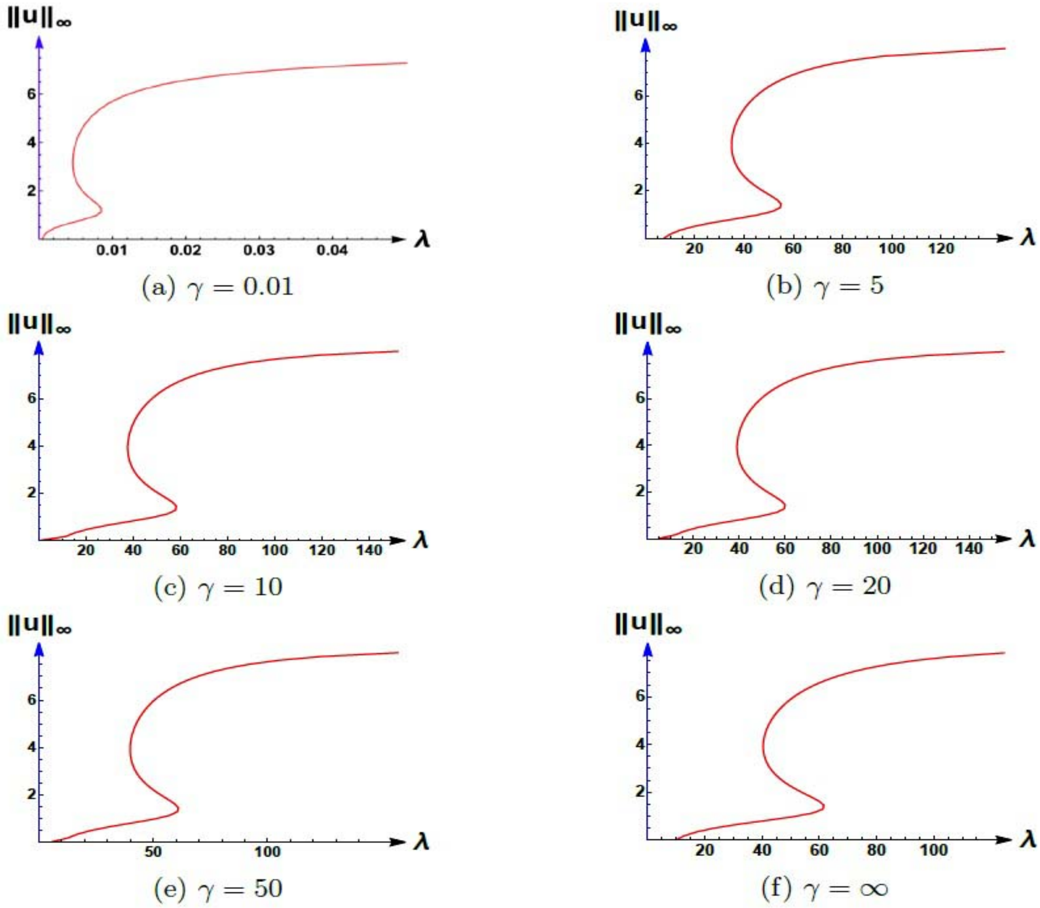
2) For large values of L, for no ranges of λ multiplicity occurs, regardless of the value of hostility factor.

3) For intermediate values of L, attainment or elimination of multiplicity regions is possible depending on the value of hostility factor.

4) For a fixed $\gamma > 0$, multiplicity regions persist for small L and multiplicity regions are lost for large L.

3.1 Bifurcation diagrams for fixed values of L as γ varies

We closely examine our solutions via extracting the value $E(\gamma)$, where the non-trivial positive solution bifurcates from the trivial branch of solutions, as well as the interval $(A(\gamma, L), B(\gamma, L))$ corresponding to the λ region where multiplicity of positive solutions occurs.
For $L = 0.05$:

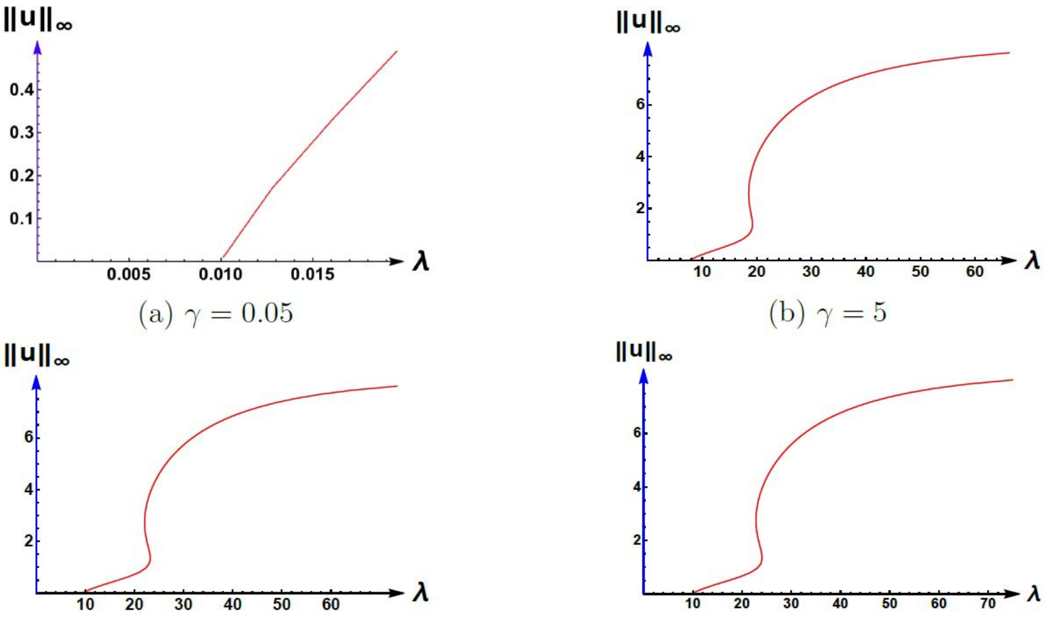


(a) $\gamma = 0.01$ (b) $\gamma = 5$ (c) $\gamma = 10$ (d) $\gamma = 20$ (e) $\gamma = 50$ (f) $\gamma = \infty$

Figure 3.2: Bifurcation diagrams for (1.6) where $K = 10$, $c = 1.5$, and $L = 0.05$.

γ	$E(\gamma)$	$A(\gamma, L)$	$B(\gamma, L)$	$B(\gamma, L) - A(\gamma, L)$
0.01	0.000411825	0.00459959	0.00848834	0.00388875
5	7.66329	34.9839	54.9993	20.0154
10	8.78401	37.6855	58.2939	20.6084
20	9.38331	39.0397	59.946	20.9063
50	9.75404	39.8512	60.937	21.0858
∞	10.0055	40.3913	61.597	21.2057

Table 3.1: Varying γ while $L = 0.05$.

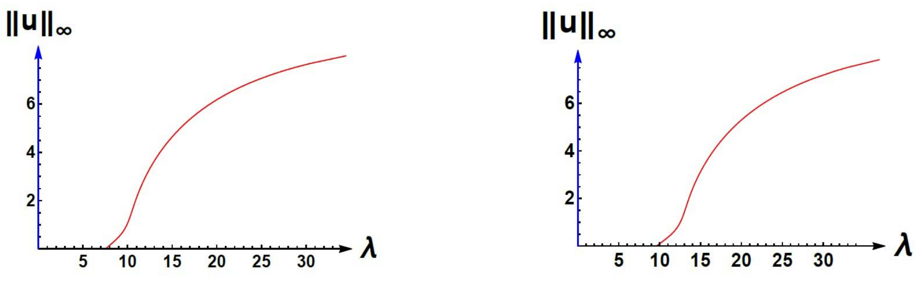


(a) $\gamma = 0.05$ (b) $\gamma = 5$ (c) $\gamma = 20$ (d) $\gamma = 50$

Figure 3.3: Bifurcation diagrams for (1.6) where $K = 10$, $c = 1.5$, and $L = 0.30$.

γ	$E(\gamma)$	$A(\gamma, L)$	$B(\gamma, L)$	$B(\gamma, L) - A(\gamma, L)$
5	7.63138	18.5239	19.2104	0.6865
20	9.35392	22.082	23.2109	1.1289
50	9.72529	22.8384	24.0726	1.2342

Table 3.2: Varying γ while $L = 0.30$.



(a) $\gamma = 5$ (b) $\gamma = 50$

Figure 3.4: Bifurcation diagrams for (1.6) where $K = 10$, $c = 1.5$, and $L = 0.45$.

Remark 3.1. Our research concludes that when $K = 10$ and $c = 1.5$ there exists $L_*, L^* \in (0, \frac{1}{2})$ with $L_* < L^*$, such that when $L < L_*$ (grazing in a large subregion), the occurrence of multiple steady states for a range of λ persists for any hostility factor γ, and when $L > L^*$ (grazing in a small subregion), for any hostility factor γ, multiplicity of steady states does not occur for any λ. However, for $L \in (L_*, L^*)$, there exists a $\gamma^*(L) > 0$ such that multiplicity of steady states for a range of λ does occur for any hostility factor $\gamma > \gamma^*(L)$.

3.2 Bifurcation diagrams for a fixed value of γ as L varies

For $\gamma = 50$:

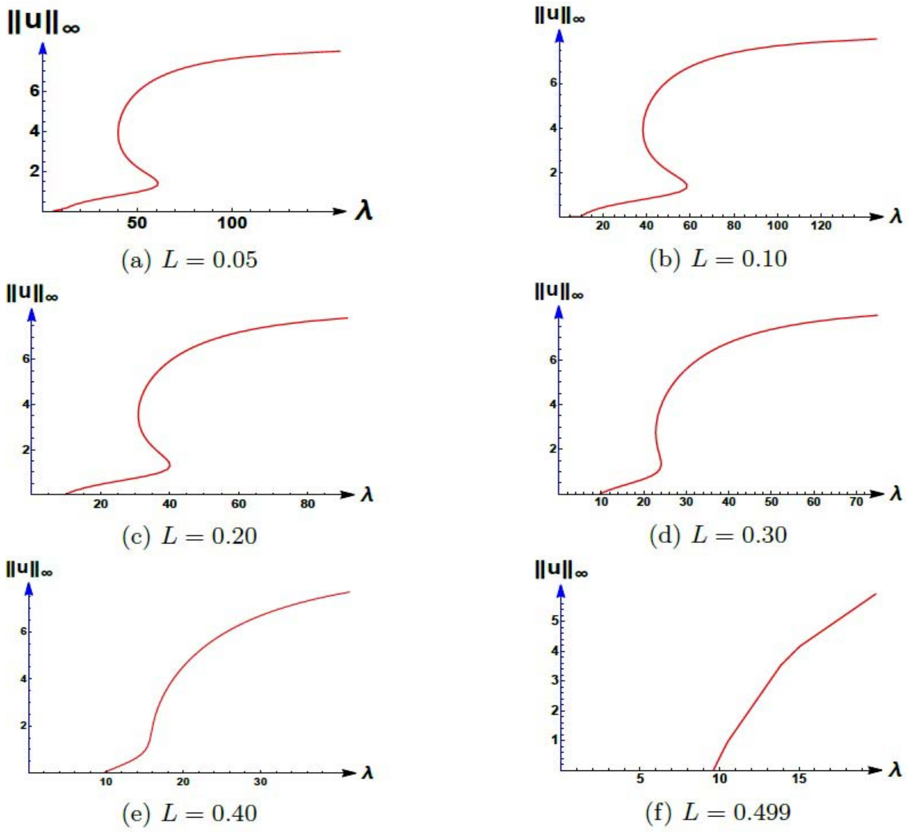


Figure 3.5: Bifurcation diagrams for (1.6) where $K = 10$, $c = 1.5$, and $\gamma = 50$.

L	$E(\gamma)$	$A(\gamma, L)$	$B(\gamma, L)$	$B(\gamma, L) - A(\gamma, L)$
0.01	12.4772	40.0324	59.6438	19.6114
0.10	9.75354	38.3015	58.4055	20.104
0.20	9.74708	30.9087	40.1478	9.2391
0.30	9.72529	22.8384	24.0726	1.2342

Table 3.3: Varying L while $\gamma = 50$.

Remark 3.2. Note that for $\gamma = 50$, when $K = 10$ and $c = 1.5$ the occurrence of multiple positive steady states for a range of λ is lost when L is large (grazing in a small subregion). Furthermore, for any fixed $\gamma > 0$, occurrence of multiple positive steady states for a range of λ are observed for $L \approx 0$ and occurrence of multiple positive steady states for any λ is lost for L large.

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A remark on Philos-type oscillation criteria for differential equations

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Abstract. The purpose of this short note is to call attention to expressions of Philos-type criteria for the oscillation of solutions of a simple differential equation. A perfect square expression is used to obtain an evaluation that plays an essential role in the proof of Philos-type oscillation theorems. The required condition is pointed out when using the perfect square expression. To simplify the discussion, here we deal with two second-order linear differential equations, but its content is also applied to a variety of equations.

Keywords: oscillation of solutions, perfect square expression, Riccati transformation, integral averaging technique.

2020 Mathematics Subject Classification: Primary 34C10, 34C29, 34K11; Secondary 35B05, 39A21.

1 Introduction

Oscillation problems represent one of the main themes of qualitative theory of differential equations. Over many years, a large number of sufficient (or necessary) conditions have been reported by numerous researchers for the oscillation of solutions of the second-order linear differential equation

$$x'' + c(t)x = 0, \qquad t \geq t_0, \tag{1.1}$$

and equations that generalize it to various directions. Here, the coefficient c is a continuous real-valued function on $[t_0, \infty)$. Since equation (1.1) is linear, all solutions are guaranteed to exist until an infinite amount of time. For this reason, all nontrivial solutions of (1.1) can be classified into two groups. A nontrivial solution x of (1.1) is said to be *oscillatory* if there exists a divergence sequence $\{t_n\}$ such that $x(t_n) = 0$, and otherwise, it said to be *nonoscillatory*. Sturm's separation theorem ensures that if there is an oscillatory solution of (1.1), then all nontrivial solutions of (1.1) are oscillatory. Equation (1.1) is often called oscillatory if all nontrivial solutions of (1.1) are oscillatory.

As an example of the many superior conditions to ensure that equation (1.1) is oscillatory, we can cite Philos's criterion as follows.

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Theorem A. *Let* $H\colon D \stackrel{\text{def}}{=} \{(t,s) : t \geq s \geq t_0\} \to \mathbb{R}$ *be a continuous function, which is such that*

$$H(t,t) = 0 \quad \textit{for} \ \ t \geq t_0 \quad \textit{and} \quad H(t,s) > 0 \quad \textit{for} \ \ t > s \geq t_0$$

and has a continuous and nonpositive partial derivative on D *with respect to the second variable. Moreover, let* $h\colon D \to [0,\infty)$ *be a continuous function with*

$$-\frac{\partial}{\partial s}H(t,s) = h(t,s)\sqrt{H(t,s)} \quad \textit{for all} \ \ (t,s) \in D. \tag{1.2}$$

Then equation (1.1) *is oscillatory if*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)c(s) - \frac{1}{4}h^2(t,s)\right) ds = \infty. \tag{1.3}$$

The feature of Theorem A is to use an auxiliary function H that is not directly related to equation (1.1) in order to examine the oscillation of solutions of (1.1). One thing to note here is that the domain of h is the same D as that of H. When we choose $(t-s)^\alpha$ as the auxiliary function H, from condition (1.2), the function h becomes $\alpha(t-s)^{(\alpha-2)/2}$. Hence, in order for the function h to be continuous on D, the exponent α must be greater than or equal to 2. As long as $\alpha > 1$, we can easily confirm that $\int_{t_0}^{t} h^2(t,s)ds/H(t,t_0)$ tends to 0 as $t \to \infty$ and $t^\alpha/H(t,t_0)$ converges to 1 as $t \to \infty$. From Theorem A and these facts, we see that equation (1.1) is oscillatory if

$$\limsup_{t\to\infty} \frac{1}{t^\alpha} \int_{t_0}^{t} (t-s)^\alpha c(s) ds = \infty, \tag{1.4}$$

where $\alpha \geq 2$. In other words, Theorem A can be said to be a partial extension of the criterion was given by Kamenev [3]:

Theorem B. *Equation* (1.1) *is oscillatory if condition* (1.4) *holds for some* $\alpha > 1$.

2 Additional condition for generalization

In order for Theorem A to completely cover Theorem B, it needs to change the domain D of h to $D_0 \stackrel{\text{def}}{=} \{(t,s) : t > s \geq t_0\}$ and rewrite condition (1.2) to

$$-\frac{\partial}{\partial s}H(t,s) = h(t,s)\sqrt{H(t,s)} \quad \text{for all} \ \ (t,s) \in D_0. \tag{2.1}$$

The reason is that the function $\alpha(t-s)^{(\alpha-2)/2}$ becomes nonnegative and continuous on D_0 provided that $\alpha \geq 0$. Of course, even when α is 1, the function h satisfying condition (2.1) is nonnegative and continuous on D_0. From such a consideration, if the domain of h is changed as described previously, Theorem A may seem to hold in the form as follows:

Proposition C. *Let* H *be the same function as in Theorem A. Suppose that there exists a continuous function* $h\colon D_0 \to [0,\infty)$ *satisfying condition* (2.1)*. Then equation* (1.1) *is oscillatory if condition* (1.3) *is satisfied.*

However, there is an issue to be discussed here. Philos [6] used the method of proof by contradiction together with the Riccati transformation and integral averaging techniques to obtain Theorem A. Let x be a nonoscillatory solution of (1.1) and let

$$w(t) = \frac{x'(t)}{x(t)} \quad \text{for} \ \ t \geq T,$$

where T is a sufficiently large number. Then equation (1.1) becomes

$$c(t) = -w'(t) - w^2(t) \quad \text{for every} \quad t \geq T.$$

Using this Riccati equation and condition (1.2), we obtain

$$\int_T^t H(t,s)c(s)ds = H(t,T)w(T) - \int_T^t h(t,s)\sqrt{H(t,s)}w(s)ds - \int_T^t H(t,s)w^2(s)ds$$

for $t \geq T$. The right-hand side of this evaluation is rewritten as follows:

$$\begin{aligned}
&H(t,T)w(T) - \int_T^t h(t,s)\sqrt{H(t,s)}w(s)ds - \int_T^t H(t,s)w^2(s)ds \\
&\quad = H(t,T)w(T) + \frac{1}{4}\int_T^t h^2(t,s)ds - \frac{1}{4}\int_T^t h^2(t,s)ds - \int_T^t h(t,s)\sqrt{H(t,s)}w(s)ds \\
&\qquad - \int_T^t H(t,s)w^2(s)ds \\
&\quad = H(t,T)w(T) + \frac{1}{4}\int_T^t h^2(t,s)ds - \int_T^t \left(\frac{1}{2}h(t,s) + \sqrt{H(t,s)}w(s)\right)^2 ds.
\end{aligned}$$

In Theorem A, the function h was assumed to be continuous on D; thus, the integral value

$$\int_{t_0}^t h^2(t,s)ds \left(= \int_T^t h^2(t,s)ds + \int_{t_0}^T h^2(t,s)ds\right)$$

exists for each fixed $t \geq t_0$. Hence, the previous perfect square expression is correct. Even if the function h is continuous only on D_0 included in D, there is a possibility that the integral value of h^2 exists for each fixed value $t \geq t_0$. For example, consider $h(t,s) = \alpha(t-s)^{(\alpha-2)/2}$ with $\alpha > 1$. Then we have

$$\int_{t_0}^t h^2(t,s)ds = \frac{\alpha^2}{\alpha-1}(t-t_0)^{\alpha-1} < \infty \quad \text{for each fixed value} \quad t \geq t_0.$$

However, in the case where $0 \leq \alpha \leq 1$, this integral value does not exist for each fixed value $t \geq t_0$. Hence, in this case, it is not possible to use a perfect square expression to obtain the evaluation as described previously.

We can therefore conclude that Theorems A and B are correct, but Proposition C cannot be proved simply by changing condition (1.2) to condition (2.1). Proposition C lacks an important condition that is unnoticeable and it needs to be modified as follows:

Theorem 2.1. *Let H be the same function as in Theorem A. Suppose that there exists a continuous function $h\colon D_0 \to [0,\infty)$ satisfying condition (2.1) and*

$$\int_{t_0}^t h^2(t,s)ds < \infty \quad \text{for each fixed value} \quad t \geq t_0. \tag{2.2}$$

Then equation (1.1) is oscillatory if condition (1.3) is satisfied.

Proof. As mentioned above, we use the contradiction method. Suppose that equation (1.1) has a nonoscillatory solution x. We may assume without loss of generality that there exists a $T \geq t_0$ such that $x(t) > 0$ for $t \geq T$. Using the Riccati transformation

$$w(t) = \frac{x'(t)}{x(t)} \quad \text{for} \quad t \geq T,$$

we can rewrite equation (1.1) to

$$c(t) = -w'(t) - w^2(t) \quad \text{for } t \geq T.$$

Note that $H(t,t) = 0$. Applying integration by parts together with condition (2.1), we obtain

$$\begin{aligned}\int_T^t H(t,s)c(s)ds &= -\int_T^t H(t,s)w'(s)ds - \int_T^t H(t,s)w^2(s)ds \\ &= H(t,T)w(T) + \int_T^t \left(\frac{\partial}{\partial s}H(t,s)\right)w(s)ds - \int_T^t H(t,s)w^2(s)ds \\ &= H(t,T)w(T) - \int_T^t \left(h(t,s)\sqrt{H(t,s)}w(s) + H(t,s)w^2(s)\right)ds \qquad (2.3)\end{aligned}$$

for $t \geq T$. Using a perfect square expression, we have

$$\begin{aligned}h(t,s)\sqrt{H(t,s)}w(s) + H(t,s)w^2(s) &= -\frac{1}{4}h^2(t,s) + \left(\frac{1}{2}h(t,s) + \sqrt{H(t,s)}w(s)\right)^2 \\ &\geq -\frac{1}{4}h^2(t,s)\end{aligned}$$

for all $(t,s) \in D_0$. The function h is continuous on D_0, but not necessarily continuous on D. For this reason, the integral of h on D may be an improper integral. However, condition (2.2) guarantees that this integral always converges to a finite value. Hence, we can obtain the inequality

$$-\int_T^t \left(h(t,s)\sqrt{H(t,s)}w(s) + H(t,s)w^2(s)\right)ds \leq \frac{1}{4}\int_T^t h^2(t,s)ds < \infty$$

for each fixed $t \geq T$. Combining (2.3) and this inequality, we get

$$\int_T^t \left(H(t,s)c(s) - \frac{1}{4}h^2(t,s)\right)ds \leq H(t,T)w(T).$$

Hence, we have

$$\begin{aligned}\int_{t_0}^t \left(H(t,s)c(s) - \frac{1}{4}h^2(t,s)\right)ds &= \int_{t_0}^T \left(H(t,s)c(s) - \frac{1}{4}h^2(t,s)\right)ds \\ &\quad + \int_T^t \left(H(t,s)c(s) - \frac{1}{4}h^2(t,s)\right)ds \\ &\leq \int_{t_0}^T H(t,s)c(s)ds + H(t,T)w(T).\end{aligned}$$

From the assumption of $\partial H(t,s)/\partial s$, we see that

$$H(t,t_0) \geq H(t,s) > 0 \quad \text{for } t > s \geq t_0.$$

Hence, we have

$$\begin{aligned}\frac{1}{H(t,t_0)}\int_{t_0}^t \left(H(t,s)c(s) - \frac{1}{4}h^2(t,s)\right)ds &\leq \int_{t_0}^T \frac{H(t,s)}{H(t,t_0)}c(s)ds + \frac{H(t,T)}{H(t,t_0)}w(T) \\ &\leq \int_{t_0}^T c(s)ds + w(T) < \infty\end{aligned}$$

for $t > t_0$. This contradicts condition (1.3). Thus, the proof of Theorem 2.1 is complete. □

3 Discussion

As the auxiliary function H, we choose $t-s$; namely, the case where the power index α used in Sections 1 and 2 corresponds to 1. In this case, as already mentioned, all assumptions of Proposition C are satisfied. Also, condition (1.3) becomes

$$\limsup_{t\to\infty}\frac{1}{t}\left(\int_{t_0}^{t}(t-s)c(s)ds-\frac{1}{4}\int_{t_0}^{t}\frac{1}{t-s}ds\right)=\infty \tag{3.1}$$

because $H(t,s)=t-s$ and $h^2(t,s)=1/(t-s)$, and

$$\frac{t}{H(t,t_0)}=\frac{t}{t-t_0}\to 1 \quad \text{as } t\to\infty.$$

There are two integrals in evaluation (3.1). The former is a proper (or normal) integral but the latter is an improper integral. The latter improper integral diverges to infinity for each fixed value $t>t_0$ because

$$\int_{t_0}^{t}\frac{1}{t-s}ds=\lim_{\varepsilon\to 0^+}\int_{t_0}^{t-\varepsilon}\frac{1}{t-s}ds=\lim_{\varepsilon\to 0^+}\big(\ln(t-t_0)-\ln\varepsilon\big)=\infty$$

for each fixed value $t>t_0$. Hence, this improper integral cannot be defined for any $t>t_0$. For this reason, the evaluation (3.1) has no meaning.

If the above expression is meaningful and has a finite value, condition (3.1) is identical with the assumption

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}(t-s)c(s)ds=\infty.$$

This assumption is equivalent to

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}c(\tau)d\tau ds=\infty. \tag{3.2}$$

Wintner [10] proved that equation (1.1) is oscillatory if

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}c(\tau)d\tau ds=\infty.$$

Three years later, Hartman [2] reported that equation (1.1) is oscillatory if

$$-\infty<\liminf_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}c(\tau)d\tau ds<\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}c(\tau)d\tau ds\le\infty.$$

Even after that, many researchers have continued to improve sufficient conditions for equation (1.1) to be oscillatory. However, it is not yet settled whether equation (1.1) is oscillatory or not in the case where condition (3.2) alone is satisfied.

As described in Section 2, Proposition C cannot be proved using a perfect square expression. The proof becomes incomplete. The above discussion shows that if Proposition C holds, condition (3.2) is a sufficient condition for equation (1.1) to be oscillatory. But, the author thinks that it is not possible to judge whether equation (1.1) is oscillatory or not without adding another condition to condition (3.2), as Hartman [2] showed.

4 Attention

Li [4] considered a second-order linear differential equation of the self-adjoint form

$$\big(r(t)x'\big)' + c(t)x = 0, \qquad t \geq t_0. \tag{4.1}$$

Here, the coefficients r and c are continuous real-valued functions on $[t_0, \infty)$ and it is assumed that $r(t) > 0$ for all $t \geq t_0$. He gave the following Philos-type oscillation criterion.

Theorem D. *Let H and h be the same functions as in Proposition C. Suppose that there exists a continuous differentiable function $f\colon [t_0, \infty) \to \mathbb{R}$ such that*

$$\int_{t_0}^{t} a(s)r(s)h^2(t,s)ds < \infty \quad \textit{for all} \ \ t \geq t_0 \tag{4.2}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)\psi(s) - \frac{1}{4}a(s)r(s)h^2(t,s) \right) ds = \infty,$$

where $a(s) = \exp\{-2\int^s f(\xi)d\xi\}$ and $\psi(s) = a(s)\{c(s)r(s)f^2(s) - \big(r(s)f(s)\big)'\}$. Then, equation (4.1) is oscillatory.

Note that if the coefficient r is smooth enough (continuously differentiable more than twice), then equations (1.1) and (4.1) can be transformed into the form of each other. In fact, by letting $x = \sqrt{r(t)}u$ for any positive and smooth enough function r, equation (1.1) becomes

$$\big(r(t)u'\big)' + \left(c(t)r(t) - \frac{\big(r'(t)\big)^2}{4r(t)} + \frac{1}{2}r''(t) \right) u = 0.$$

Conversely, by changing $u = \sqrt{r(t)}x$, equation (4.1) becomes

$$u'' + \left(\frac{c(t)}{r(t)} + \frac{\big(r'(t)\big)^2}{4r^2(t)} - \frac{r''(t)}{2r(t)} \right) u = 0.$$

Rogovchenko [7] observed that condition (4.2) appears to be superfluous. Certainly, the expression of condition (4.2) is incorrect (there is the same mistake in [5]). However, Theorem D does not hold only by deleting condition (4.2), because equation (4.1) contains equation (1.1). It is necessary to assume the condition

$$\int_{t_0}^{t} a(s)r(s)h^2(t,s)ds < \infty \quad \text{for each fixed value} \ \ t \geq t_0, \tag{4.3}$$

which plays the same role as condition (2.2), instead of condition (4.2). By using condition (4.3), a perfect square expression will have the correct meaning in the proof of Theorem D. We omit the proof of the result of changing condition (4.2) to condition (4.3).

Philos's criterion has been improved, extending its applicability to a variety of equations. For example, those results can be found in studies on nonlinear differential equations including the Emden–Fowler equation, half-linear differential equations with and without the self-adjoint form, damped differential equations with and without time-delay, higher-order differential equations, matrix differential systems, elliptic partial differential equations, Hamiltonian systems, difference equations, dynamic equations and others. When the function h is continuous on D, there seems to be no problem with the result. However, one needs to pay close attention to an additional condition when defining the domain of h to D_0 and assuming a condition such as (2.1). Unfortunately, there are mistakes arising from this carelessness in some previous research papers. For example, Proposition C which is wrong is included in [1, Corollary 3.3], [8, Theorem 2.1], [9, Theorem 2.1], [11, Theorem 3.4] and [12, Theorem 2.8].

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Crossing limit cycles for piecewise linear differential centers separated by a reducible cubic curve

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Abstract. As for the general planar differential systems one of the main problems for the piecewise linear differential systems is to determine the existence and the maximum number of crossing limits cycles that these systems can exhibit. But in general to provide a sharp upper bound on the number of crossing limit cycles is a very difficult problem. In this work we study the existence of crossing limit cycles and their distribution for piecewise linear differential systems formed by linear differential centers and separated by a reducible cubic curve, formed either by a circle and a straight line, or by a parabola and a straight line.

Keywords: discontinuous piecewise linear differential centers, limit cycles, conics.

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1 Introduction and statement of the main results

The discontinuous piecewise differential systems arose from the study of nonlinear oscillations by Andronov, Vitt and Khaikin in [1]. And nowadays the qualitative theory of the discontinuous piecewise differential systems is a matter of great interest in the mathematical community because these systems arise naturally in the modeling of several real phenomena and processes for instance in electronics, mechanics, economy, biology, neuroscience etc., see [3,4,10,13,21,23] and references quoted therein.

One of the main problems in the qualitative theory of the discontinuous piecewise differential systems is to determine the maximum number of crossing limits cycles that these systems can have and their distribution. In this work we study the *crossing limit cycles* which are periodic orbits isolated in the set of all periodic orbits of the piecewise linear differential system, which only have isolated points of intersection with the discontinuity curve.

We recall that the 16th Hilbert's problem requests for the maximum number of limit cycles that can have a polynomial differential system in $\mathbb{R}^2$ in function of the degree of the system,

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see [11,12]. Then the problem of establishing a sharp upper bound on the number of crossing limits cycles for the class of planar piecewise linear differential systems can be considered as an extension of the 16th Hilbert's problem to this class and is in general a very difficult problem, because there are few developed techniques. In the plane the class of piecewise linear differential systems separated by a straight line is apparently the simplest class to study, and has been studied in several papers, see [2,5–9,16,19,22] but it is still an open problem to know if three is the maximum number of crossing limit cycles that this class can have.

In particular when the class of piecewise linear differential systems separated by a straight line is formed by linear differential centers we know that these systems have no crossing limit cycles, see [15]. However, there are more recent works which study planar discontinuous piecewise linear differential centers where the curve of discontinuity is not a straight line, see [18,20], there it was proved that there are crossing limit cycles in those systems. Moreover in the paper [14] it was provided the maximum number of crossing limit cycles for piecewise linear differential centers separated by any conic, then the objective of this work is to study the existence of crossing limit cycles of the discontinuous piecewise linear differential centers in $\mathbb{R}^2$ separated a reducible cubic curve, formed either by a circle and a straight line, or by a parabola and a straight line.

In this paper we study the crossing limit cycles of the discontinuous piecewise linear differential centers separated by such reducible cubic curves which intersect either in two, or in four, or in six points the discontinuity curve. First we have the crossing limit cycles which intersect in two points the discontinuity curve. In [15] was proved that the class of linear differential centers separated by a straight line have no crossing limit cycles, then we can consider that those two intersection points on the discontinuity curve are on the circle or on the parabola and these two options were considered in the paper [14]. Second the crossing limit cycles intersect the discontinuity curve in exactly four points, here we consider that at least one of the four points is on the straight line, because the case which the four points are only on the circle or on the parabola was studied in [14]. Finally we have the crossing limit cycles such that intersect the discontinuity curve in six points.

In this paper we study the crossing limit cycles with four points on discontinuity curve. In subsection 1.1 we consider the piecewise linear differential systems formed by linear differential centers separated by the cubic

$$\Sigma_k = \left\{(x,y) \in \mathbb{R}^2 : (x-k)(x^2+y^2-1) = 0,\ k \in \mathbb{R},\ k \geq 0\right\}.$$

And in subsection 1.2 we consider the piecewise linear differential systems formed by linear differential centers separated by the cubic

$$\tilde{\Sigma}_k = \left\{(x,y) \in \mathbb{R}^2 : (y-k)(y-x^2) = 0,\ k \in \mathbb{R}\right\}.$$

1.1 Crossing limit cycles intersecting the discontinuity curve Σ_k

Let $\mathcal{F}_1$ be the family of piecewise linear differential centers separated by Σ_k with $k > 1$. Let $\mathcal{F}_2$ be the family of piecewise linear differential centers separated by Σ_k with $k = 1$. In these two cases we have the following regions in the plane

$$\begin{aligned} R_1 =& \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}, \\ R_2 =& \{(x,y) \in \mathbb{R}^2 : x^2+y^2 > 1 \text{ and } x < k\}, \\ R_3 =& \{(x,y) \in \mathbb{R}^2 : x^2+y^2 > 1 \text{ and } x > k\}. \end{aligned}$$

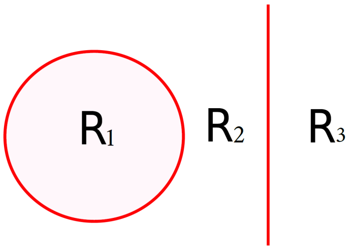

Figure 1.1: The regions for the familes $\mathcal{F}_1$ and $\mathcal{F}_2$.

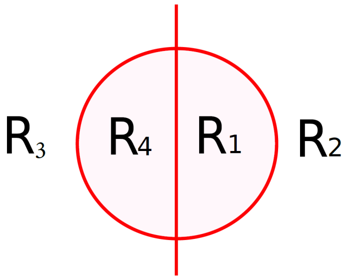

Figure 1.2: The regions for the familes $\mathcal{F}_3$.

And finally let $\mathcal{F}_3$ be the family of piecewise linear differential centers separated by Σ_k with $0 \leq k < 1$. Here we have the following regions in the plane

$$\begin{aligned} R_1 =& \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1, \text{ and } x > k\}, \\ R_2 =& \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1, \text{ and } x > k\}, \\ R_3 =& \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1 \text{ and } x < k\}, \\ R_4 =& \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \text{ and } x < k\}. \end{aligned}$$

In the family $\mathcal{F}_3$ we have three types of crossing limit cycles. First crossing limit cycles such that are formed by parts of orbits of the four linear differential centers considered, namely crossing limit cycles of type 1, see Figure 2.3, second we have crossing limit cycles which intersect the regions R_1, R_2 and R_4 or crossing limit cycles that intersect the regions R_1, R_4 and R_3, namely crossing limit cycles of type 2^+ or crossing limit cycles of type 2^-, respectively, see Figure 2.4. Without loss of generality we only study the crossing limit cycles of type 2^+ because the analysis for the crossing limit cycles of type 2^- is the same, moreover we observe that these two cases can not occur simultaneously, because the orbits of linear differential system in the region R_4 which are pieces of ellipses would have these ellipses not nested in contradiction with the fact that the ellipses of a linear center are nested. And finally we have the crossing limit cycles such that are formed by parts of orbits of the three linear differential centers in the regions R_1, R_2 and R_3, or crossing limit cycles formed by parts of orbits of the three linear differential centers in the regions R_2, R_3 and R_4, namely crossing limit cycles of type 3^+ and crossing limit cycles of type 3^-, respectively, see Figure 2.5. Without loss of generality in Theorem 1.1 we study the crossing limit cycles of type 3^+ because the study by the crossing limit cycles of type 3^- is the same. We observe that these types of crossing limit cycles can not appear simultaneously, because the orbits of linear differential system in the region R_3 which are pieces of ellipses would have these ellipses not nested in contradiction with the fact that the ellipses of a linear center are nested. If we study the piecewise linear differential centers in the family $\mathcal{F}_3$ which have simultaneously two types of crossing limit cycles we observe we would have three possible combinations between the three different crossing limit cycles types, $1, 2^+$ and 3^+, but we observe that the crossing limit cycles of types 2^+ and 3^+ can not appear simultaneously, because the orbits of linear differential system in the region R_1 which are pieces of ellipses would have these ellipses not nested in contradiction with the fact that the ellipses of a linear center are nested. For this same reason there are no piecewise linear differential centers in $\mathcal{F}_3$ with three types of crossing limit cycles simultaneously. Then in the following theorem we provide examples of piecewise linear differential centers in $\mathcal{F}_3$ with crossing limit cycles of types 1, 2^+ and 3^+ separately and piecewise linear differential centers in $\mathcal{F}_3$ such that have simultaneously crossing limit cycles of types 1 and 2^+ or of types 1 and 3^+.

Theorem 1.1. *The following statements hold.*

(a) There are piecewise linear differential systems in $\mathcal{F}_1$ and in $\mathcal{F}_2$ formed by three linear differential centers that have four crossing limit cycles, see Figures 2.1 and 2.2.

(b) There are piecewise linear differential systems in $\mathcal{F}_3$ that have five crossing limit cycles of type 1*, see Figure 2.3.*

(c) There are piecewise linear differential systems in $\mathcal{F}_3$ that have four crossing limit cycles of type 2^+*, see Figure 2.4.*

(d) There are piecewise linear differential systems in $\mathcal{F}_3$ that have three crossing limit cycles of type 3^+*, see Figure 2.5.*

(e) There are piecewise linear differential systems in $\mathcal{F}_3$ that have four crossing limit cycles of type 1 *and two crossing limit cycles of type* 2^+*, see Figure 2.6.*

(f) There are piecewise linear differential systems in $\mathcal{F}_3$ that have four crossing limit cycles of type 1 *and one crossing limit cycle of type* 3^+*, see Figure 2.7.*

Theorem 1.1 is proved in Section 2.

By the numerical computations made for the families $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ and the illustrated examples of Theorem 1.1 we propose the following problem.

Open problem 1. *The numbers of crossing limit cycles determined in Theorem 1.1 for the families $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are the maximum numbers of crossing limit cycles in each family.*

1.2 Crossing limit cycles intersecting the discontinuity curve $\tilde{\Sigma}_k$

Let $\mathcal{F}_4$ be the family of piecewise linear differential centers separated by $\tilde{\Sigma}_k$ with $k < 0$. In this case, we have following three regions in the plane

$$
\begin{aligned}
R_1 =& \{(x,y) \in \mathbb{R}^2 : y > x^2\}, \\
R_2 =& \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y > k\}, \\
R_3 =& \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y < k\}.
\end{aligned}
$$

For this family we have the following Theorem.

Theorem 1.2. *There are piecewise linear differential systems in $\mathcal{F}_4$ that have four crossing limit cycles with four points on $\tilde{\Sigma}_k$, see Figure 3.1.*

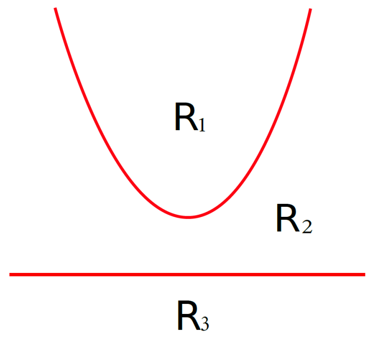

Figure 1.3: The regions for the familes $\mathcal{F}_4$.

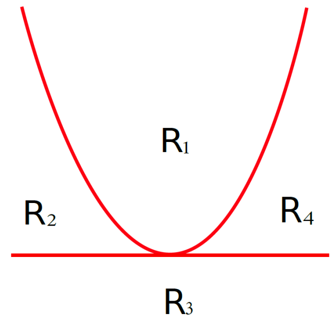

Figure 1.4: The regions for the familes $\mathcal{F}_5$.

Theorem 1.2 is proved in Section 3.

Let $\mathcal{F}_5$ be the family of piecewise linear differential centers separated by $\tilde{\Sigma}_k$ with $k = 0$. When the discontinuity curve is $\tilde{\Sigma}_0$ we have following four regions in the plane

$$\begin{aligned} R_1 =& \{(x, y) \in \mathbb{R}^2 : y > x^2 \}, \\ R_2 =& \{(x, y) \in \mathbb{R}^2 : y < x^2 \text{ and } y > 0,\ x < 0\}, \\ R_3 =& \{(x, y) \in \mathbb{R}^2 : y < x^2 \text{ and } y < 0\}, \\ R_4 =& \{(x, y) \in \mathbb{R}^2 : y < x^2 \text{ and } y > 0,\ x > 0\}. \end{aligned}$$

Here we have two types of crossing limit cycles, first crossing limit cycles formed by parts of orbits of the four linear differential centers considered, namely crossing limits cycles of type 4, see Figure 4.1. Second crossing limit cycles of type 5, see Figure 4.2, which intersect only three regions, in this case we have two options, first we have the case where the crossing limit cycles are formed by parts of orbits of the linear differential centers in the regions R_1, R_3 and R_4 and second the crossing limit cycles that intersect only the three regions $R_1,\ R_2$ and R_3, without loss of generality we can consider the first case because the study for the second type of crossing limit cycles is the same. Here we observe that it is not possible to have crossing limit cycles of type 5 that satisfy those two cases simultaneously, because the orbits of linear differential system in the region R_3 which are pieces of ellipses would have these ellipses not nested in contradiction with the fact that the ellipses of a linear center are nested. Therefore in the following Theorem we study the piecewise linear differential centers in $\mathcal{F}_5$ which have crossing limit cycles of types 4 and 5 separately, and crossing limit cycles of types 4 and 5 simultaneously.

Theorem 1.3. *The following statements hold.*

(a) There are piecewise linear differential systems in $\mathcal{F}_5$ that have four crossing limit cycles of type 4*, see Figure* 4.1.

(b) There are piecewise linear differential systems in $\mathcal{F}_5$ that have three crossing limit cycles of type 5*, see Figure* 4.2.

(c) There are piecewise linear differential systems in $\mathcal{F}_5$ that have simultaneously four crossing limit cycles of type 4 *and two crossing limit cycles of type* 5*, see Figure* 4.3.

Theorem 1.3 is proved in Section 4.

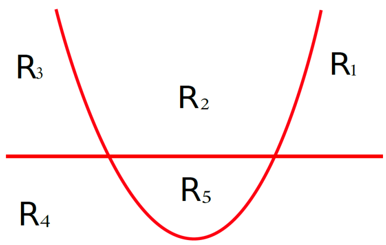

Figure 1.5: The regions for the familes $\mathcal{F}_6$.

Let $\mathcal{F}_6$ be the family of piecewise linear differential centers separated by $\tilde{\Sigma}_k$ with $k > 0$, in this case we have the following five regions in the plane

$$\begin{aligned}
R_1 =& \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y > k,\ x > \sqrt{k}\},\\
R_2 =& \{(x,y) \in \mathbb{R}^2 : y > x^2 \text{ and } y > k\},\\
R_3 =& \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y > k,\ x < -\sqrt{k}\},\\
R_4 =& \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y < k\},\\
R_5 =& \{(x,y) \in \mathbb{R}^2 : x^2 < y < k\}.
\end{aligned}$$

Here we have six types of crossing limit cycles. First we have crossing limit cycles such that are formed by parts of orbits of the four linear differential centers in the regions R_1, R_2, R_5 and R_4, or crossing limit cycles formed by parts of orbits of the four linear differential centers in the regions R_2, R_3, R_4 and R_5, namely crossing limit cycles of type 6^+ and crossing limit cycles of type 6^-, respectively, see Figure 6.1. In Theorem 1.4 we study the crossing limit cycles of type 6^+ because the study for the case of crossing limit cycles of type 6^- is the same. Second we have crossing limit cycles type 7, see Figure 5.2, which intersect the three regions R_2, R_5 and R_4. Third we have the crossing limit cycles of type 8, see Figure 5.3, which intersect the regions R_1, R_2, R_3 and R_4. And finally we have the crossing limit cycles such that are formed by parts of orbits of the three linear differential centers in the regions R_1, R_2 and R_4, or crossing limit cycles formed by parts of orbits of the three linear differential centers in the regions R_2, R_3 and R_4, namely crossing limit cycles of type 9^+ and crossing limit cycles of type 9^-, respectively, see Figure 5.4. Without loss of generality in Theorem 1.4 we study the crossing limit cycles of type 9^+ because the study by the crossing limit cycles of type 9^- is the same. Then in Theorem 1.4 we study the crossing limit cycles of types 6^+, 7, 8 and 9^+. In Theorem 1.5 we study the piecewise linear differential centers in the family $\mathcal{F}_6$ which have two types of crossing limit cycles simultaneously. And in Theorem 1.6 we study the piecewise linear differential centers in the family $\mathcal{F}_6$ which have three types of crossing limit cycles simultaneously.

Theorem 1.4. *The following statements hold.*

(a) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have five crossing limit cycles of type 6^+, see Figure 5.1.*

(b) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have three crossing limit cycles of type 7, see Figure 5.2.*

(c) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have four crossing limit cycles of type 8, see Figure 5.3.*

(d) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have three crossing limit cycles of type 9^+, see Figure 5.4.*

Theorem 1.4 is proved in Section 5.

In Theorem 1.5 we would have fifteen possible combinations of pairs between the six different crossing limit cycles types, namely types $6^+, 6^-, 7, 8, 9^+$ and 9^-, we will analyze each one. Piecewise linear differential centers with crossing limit cycles of types 6^+ and 6^- are analyzed in statement (a) of Theorem. The study for piecewise linear differential centers with crossing limit cycles of types 6^+ and 7, or 6^+ and 8, or 6^+ and 9^+ is the same for piecewise linear differential centers with crossing limit cycles of types 6^- and 7, or 6^- and 8, or 6^- and 9^-, respectively, and they are in statements (b), (c) and (d) of Theorem 1.5, respectively. The crossing limit cycles of types 6^- and 9^+ can not appear simultaneously because the orientation of these crossing limit cycles in region R_4 would not be well defined, similarly happens with the crossing limit cycles of types 6^+ and 9^-. Piecewise linear differential centers with crossing limit cycles of types 7 and 8 are analyzed in statement (e) of Theorem 1.5. It is not possible to have crossing limit cycles of type 7 and 9^+, or 7 and 9^- simultaneously, because the orbits of linear differential system in the region R_2 would not be nested. Piecewise linear differential centers with crossing limit cycles of types 8 and 9^+ are analyzed in statement (f) of Theorem 1.5, the case where appear crossing limit cycles of types 8 and 9^-, simultaneously is the same. Finally we observe that it is not possible to have simultaneously crossing limit cycles of types 9^+ and 9^-, because the orbits of linear differential system in the region R_4 would not be nested. Then we only have six cases analyzed in the following Theorem.

Theorem 1.5. *The following statements hold.*

(a) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have simultaneously four crossing limit cycles of type 6^+ and four crossing limit cycles of type 6^-, see Figure 6.1.*

(b) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have simultaneously four crossing limit cycles of type 6^+ and two crossing limit cycles of type 7, see Figure 6.2.*

(c) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have simultaneously three crossing limit cycles of type 6^+ and four crossing limit cycle of type 8, see Figure 6.3.*

(d) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have simultaneously four crossing limit cycles of type 6^+ and two crossing limit cycles of type 9^+, see Figure 6.4.*

(e) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have simultaneously three crossing limit cycles of type* 7 *and four crossing limit cycle of type* 8*, see Figure* 6.5*.*

(f) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have simultaneously four crossing limit cycles of type* 8 *and two crossing limit cycles of type* 9^+*, see Figure* 6.6*.*

Theorem 1.5 is proved in Section 6.

In Theorem 1.6 we would have twenty possible combinations of triplets between the six different crossing limit cycles types above, but we have fourteen combinations that include couples 6^+ and 9^-, 6^- and 9^+, 7 and $9^{\pm}$, or 9^+ and 9^- and as it was said before these combinations are not possible. Therefore we have six options, first we observed that crossing limit cycles of types 6^+, 6^- and 7, or 6^+, 6^- and 8 can not appear simultaneously because the orientation of these crossing limit cycles in region R_4 would not be well defined. Second we have that there are piecewise linear differential centers with crossing limit cycles of types 6^+, 7 and 8, this case is in statement (a) of Theorem 1.6, the case where appear crossing limit cycles of types 6^-, 7 and 8 is the same. Finally we have the piecewise linear differential centers with crossing limit cycles of types 6^+, 8 and 9^+, this case is in statement (b) of Theorem 1.6 and the case where appear crossing limit cycles of types 6^-, 7 and 9^- is the same.

Theorem 1.6. *The following statements hold.*

(a) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have simultaneously two crossing limit cycles of type* 6^+*, two crossing limit cycles of type* 7 *and four crossing limit cycles of type* 8*, see Figure* 7.1*.*

(b) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have simultaneously four crossing limit cycles of type* 6^+*, three crossing limit cycles of type* 8 *and two crossing limit cycles of type* 9^+*, see Figure* 7.2*.*

Theorem 1.6 is proved in Section 7.

Similar to the previous case and by the illustrated examples in previous theorems we propose the following problem.

Open problem 2. *The numbers of crossing limit cycles determined in Theorems* 1.2, 1.3, 1.4, 1.5 *and* 1.6 *for the families $\mathcal{F}_4$, $\mathcal{F}_5$ and $\mathcal{F}_6$ are the maximum numbers of crossing limit cycles in each family.*

By the previous analysis we observed that it is not possible to have piecewise linear differential centers in $\mathcal{F}_6$ with four, five or six types of crossing limit cycles simultaneously.

2 Proof of Theorem 1.1

In the proof of the theorems will use the following lemma which provides a normal form for an arbitrary linear differential linear differential center, see a proof in [17].

Lemma 2.1. *Through a linear change of variables and a rescaling of the independent variable every center in $\mathbb{R}^2$ can be written*

$$\dot{x} = -bx - \frac{4b^2 + \omega^2}{4a} y + d, \qquad \dot{y} = ax + by + c, \tag{2.1}$$

with $a \neq 0$ and $\omega > 0$. This linear differential system has the first integral

$$H_1(x, y) = 4(ax + by)^2 + 8a(cx - dy) + y^2\omega^2. \tag{2.2}$$

Remark. As we shall see in the proofs of our results in order to find the crossing limit cycles we must solve equations of the form

$$H_1(x_1, y_1) - H_1(x_2, y_2) = 0, \tag{2.3}$$

where H_1 is defined in (2.2). Since $a \neq 0$ the solutions of equation (2.3) do not change if in it we change the function H_1 by the function

$$\bar{H}(x,y) = 4\left(x + \frac{b}{a}y\right)^2 + 8\left(\frac{c}{a}x - \frac{d}{a}y\right) + \left(\frac{\omega}{a}\right)^2 y^2,$$

because this is equivalent to divide equation (2.3) by the positive constant $a^2 > 0$. Defininig

$$\frac{b}{a} = \bar{b}, \quad \frac{c}{a} = \bar{c}, \quad \frac{d}{a} = \bar{d}, \quad \frac{\omega}{a} = \bar{\omega},$$

the function $\bar{H}(x,y)$ is a first integral of the differential system

$$\dot{x} = -\bar{b}x - \frac{4\bar{b}^2 + \bar{\omega}^2}{4}y + \bar{d}, \qquad \dot{y} = x + \bar{b}y + \bar{c}. \tag{2.4}$$

Note that system (2.4) is essentially system (2.1) with $a = 1$. So in what follows we always will work with systems (2.1) with $a = 1$. In this way we shall work with systems having one parameter less and this will simplify a little the computations that will come.

Proof of statement (a) for the family $\mathcal{F}_1$ of Theorem 1.1. By Lemma 2.1 we can consider the following piecewise linear differential center

$$\begin{aligned}
\dot{x} &= -b_1x - \frac{4b_1^2 + \omega_1^2}{4}y + d_1, \qquad \dot{y} = x + b_1y + c_1, \text{ in } R_1,\\
\dot{x} &= -b_2x - \frac{4b_2^2 + \omega_2^2}{4}y + d_2, \qquad \dot{y} = x + b_2y + c_2, \text{ in } R_2,\\
\dot{x} &= -b_3x - \frac{4b_3^2 + \omega_3^2}{4}y + d_3, \qquad \dot{y} = x + b_3y + c_3, \text{ in } R_3.
\end{aligned} \tag{2.5}$$

And the linear differential centers in (2.5) have the first integrals

$$H_i(x,y) = 4(x + b_iy)^2 + 8(c_ix - d_iy) + y^2\omega_i^2, \text{ with } i = 1,2,3,$$

respectively. In order to have a crossing limit cycle, which intersects Σ_k in four different points $p_1 = (k, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (x_3, y_3)$ and $p_4 = (k, y_4)$, with $p_2, p_3 \in \mathbb{S}^1$, where $\mathbb{S}^1 = \{(x,y) : x^2 + y^2 = 1\}$. These points must satisfy the *closing equations*

$$\begin{aligned}
e_1 = H_2(k,\ y_1) - H_2(x_2, y_2) &= 0,\\
e_2 = H_1(x_2, y_2) - H_1(x_3, y_3) &= 0,\\
e_3 = H_2(x_3, y_3) - H_2(\ k, y_4) &= 0,\\
e_4 = H_3(k,\ y_4) - H_3(\ k, y_1) &= 0,\\
x_2^2 + y_2^2 &= 1,\\
x_3^2 + y_3^2 &= 1.
\end{aligned} \tag{2.6}$$

For to build the example, we will impose the existence of periodic solutions and we will determine the parameters in (2.5) with the established conditions. First we fix the constant

$k = 2$ and we assume that there is a real solution, namely $q^1 = (y_1^1, x_2^1, y_2^1, x_3^1, y_3^1, y_4^1) = (3, \cos(\pi/2), \sin(\pi/2), \cos(-\pi/3), \sin(-\pi/3), -5/2)$, then by equations e_i with $i = 1,2,3,4$ in (2.6) we have the parameters

$$\begin{aligned} d_2 &= 1 + b_2(3+2b_2) + c_2 + \frac{\omega_2^2}{2};\\ d_1 &= -\frac{1}{16}(-2+\sqrt{3})(-4+4b_1(2\sqrt{3}+b_1) - 16c_1 + \omega_1^2);\\ c_2 &= \frac{70 - 8\sqrt{3} + 4b_2(10 - 5\sqrt{3} + 31b_2 - 4\sqrt{3}b_2) + (31 - 4\sqrt{3})\omega_2^2}{8(-8+\sqrt{3})};\\ d_3 &= \frac{1}{16}(4b_3(8+b_3) + \omega_3^2), \end{aligned}$$

respectively. Now by the equation e_4 we have

$$y_4 = \frac{1}{2}(1 - 2y_1),$$

then we suppose that the point $q^2 = (y_1^2, x_2^2, y_2^2, x_3^2, y_3^2, y_4^2) = (3, \cos(\pi/2), \sin(\pi/2), \cos(-\pi/3), \sin(-\pi/3), -5/2)$ is also a real solution of system (2.6), then by the equations e_1, e_2 and e_3 in (2.6) we obtain the following parameters

$$\begin{aligned} \omega_2 = &-\frac{2}{\sqrt{3894 - 523\sqrt{3} + 225\sqrt{15} + 50\sqrt{2(5+\sqrt{5})}}}\\ &\times\Bigg(-635 + 25\sqrt{3} + 675\sqrt{5} - 75\sqrt{15} + 75\sqrt{2(5+\sqrt{5})}\\ &\quad + 5\Big(1468 + 34\sqrt{3} + 100\sqrt{5} - 50\sqrt{15} + 5\sqrt{2(5+\sqrt{5})}(-68 + \sqrt{3} - 8\sqrt{5} + \sqrt{15})\Big)b_2\\ &\quad + (-3894 + 523\sqrt{3} + 25\sqrt{5}(70 - 9\sqrt{3}) - 50\sqrt{2(5+\sqrt{5})})b_2^2\Bigg)^{\frac{1}{2}};\\ c_1 = &\frac{(-2+\sqrt{3})\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}(-4 + 8\sqrt{3}b_1 + 4b_1^2 + \omega_1^2)}{8(-1+\sqrt{5} - 2\sqrt{2(5+\sqrt{5})}} + \sqrt{6(5+\sqrt{5})}));\\ b_2 = &\ 3.119845.., \end{aligned}$$

respectively. Now we fix the points $x_2 = \cos(4\pi/7)$, $y_2 = \sin(4\pi/7)$ and by equation e_6 we have

$$y_3 = -\sqrt{1 - x_3^2},$$

then by the equations e_1, e_2 and e_3 we have $y_1 = 3.144465..; \omega_1 = -9.702226..\sqrt{0.042492.. + 0.031501..b_1 - 0.042492..b_1^2}$; $x_3 = 0.365470..$, respectively. These conditions generate the real solution $q^3 = (3.144465.., \cos(4\pi/7), \sin(4\pi/7), 0.365470.., -0.930823.., -2.644465..)$. We build a fourth solution fixing the points $x_2 = -0.018219..$; $y_2 = 0.999834..$; therefore by the equations e_1, e_2 and e_3 we obtain $y_1 = 3.012016..$; $x_3 = 0.489429..$; $b_1 = 0.608380..$, respectively. With these conditions we have the real solution $q^4 = (3.012016.., -0.018219.., 0.999834.., 0.489429.., -0.872042.., -2.512016..)$. With these four real solutions we determined all the parameters that appear in system (2.6), even more in this particular case the parameters $b_3, c_3,\ \omega_3 \in \mathbb{R}$, then we fix them, $b_3 = 1;\ c_3 = 1/4;\ \omega_3 = 1$. Therefore we obtain the following piecewise linear differential center

$$\begin{aligned}
&\dot{x} = 0.977474.. - 0.608380..x - 1.451017..y, && \dot{y} = -3.008357.. + x + 0.608380..y, \text{ in } R_1,\\
&\dot{x} = 9.710162.. - 3.119845..x - 10.075224..y, && \dot{y} = -20.799821.. + x + 3.119845..y, \text{ in } R_2,\\
&\dot{x} = \frac{37}{16} - x - \frac{5}{4}y, && \dot{y} = \frac{1}{4} + x + y, \text{ in } R_3.
\end{aligned} \tag{2.7}$$

The linear differential centers in (2.7) have the first integrals

$$\begin{aligned}
H_1(x,y) &= x^2 + x(-6.016714.. + 1.216760..y) + y(-1.954949.. + 1.451017..y),\\
H_2(x,y) &= x^2 + x(-41.599643.. + 6.239690..y) + y(-19.420324.. + 10.075224..y),\\
H_3(x,y) &= 2x + 4x^2 - \frac{37}{2}y + 8xy + 5y^2,
\end{aligned}$$

respectively.

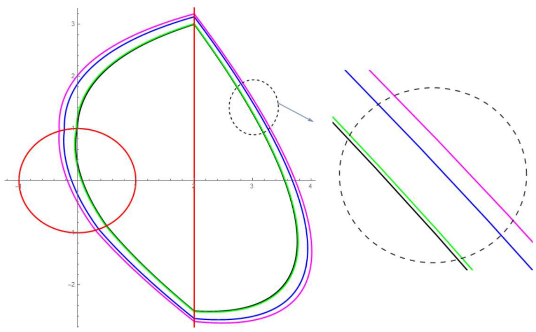

Figure 2.1: Four crossing limit cycles of the discontinuous piecewise linear differential (2.7). These limit cycles are traveled in counterclockwise.

In this case system (2.6) is equivalent to system

$$\begin{aligned}
79.199286.. + x_2^2 + 6.940944..y_1 - 10.075224..y_1^2 - 19.420324..y_2 + 10.075224..y_2^2&\\
+x_2(-41.599643.. + 6.239690..y_2) &= 0,\\
x_2^2 - x_3^2 - 1.954949..y_2 + 1.451017..y_2^2 + x_2(-6.016714.. + 1.216760..y_2)&\\
+x_3(6.016714.. - 1.216760..y_3) + 1.954949..y_3 - 1.451017..y_3^2) &= 0,\\
79.199286.. + x_3^2 - 19.420324..y_3 + 10.075224..y_3^2&\\
+x_3(-41.599643.. + 6.239690..y_3) + 6.940944..y_4 - 10.075224..y_4^2 &= 0,\\
(y_1 - y_4)\left(-\frac{5}{2} + 5y_1 + 5y_4\right) &= 0,\\
x_2^2 + y_2^2 = 1, \qquad x_3^2 + y_3^2 &= 1.
\end{aligned} \tag{2.8}$$

Taking into account that the solutions $q^i = (y_1^i, x_2^i, y_2^i, x_3^i, y_3^i, y_4^i)$ of system (2.8) must satisfy $y_4^i < y_1^i$ we have that the unique reals solutions are the points q^1, q^2, q^3 and q^4 which provide four crossing limit cycles of the piecewise linear differential center (2.7). See these crossing limit cycles in Figure 2.1. □

Proof of statement (a) for the family $\mathcal{F}_2$ of Theorem 1.1. Following the steps illustrated in the previous case we obtain a discontinuous piecewise linear differential system which is formed by the following linear differential centers in each region. First in the region R_1 we have

$$\dot{x} = 2.185588.. - \frac{3}{20}x - 6.201094..y, \quad \dot{y} = -6.726549.. + x + \frac{3}{20}y. \tag{2.9}$$

This linear differential center has the first integral $H_1(x,y) = x^2 + x(-13.453098.. + 3y/10) + y(-4.371176.. + 6.201094..y)$. In the region R_2 we consider the linear differential center

$$\dot{x} = -0.263120.. - 0.874044..x - 4.914345..y, \qquad \dot{y} = -23.305757.. + x + 0.874044..y, \tag{2.10}$$

which has the first integral $H_2(x,y) = x^2 + x(-46.611514.. + 1.748088..y) + y(0.526241.. + 4.914345..y)$. And in the region R_3 we have the linear differential center

$$\dot{x} = \frac{21}{16} - x - \frac{5}{4}y, \qquad \dot{y} = \frac{1}{4} + x + y, \tag{2.11}$$

which has the first integral $H_3(x,y) = 2x + 4x^2 - 21y/2 + 8xy + 5y^2$. In order to have a

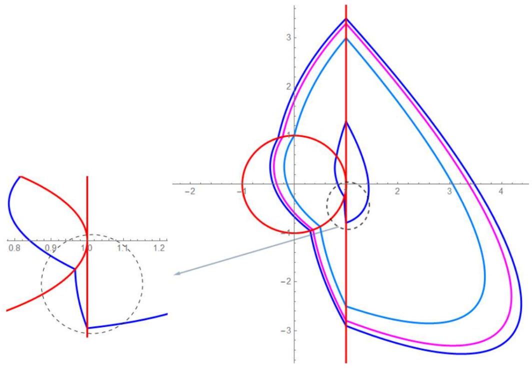

Figure 2.2: Four crossing limit cycles of the discontinuous piecewise linear differential formed by (2.9), (2.10) and (2.11) and separated by Σ_1. These limit cycles are traveled in counterclockwise.

crossing limit cycle, which intersects Σ_1 in four different points $p_1 = (1, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (x_3, y_3)$ and $p_4 = (1, y_4)$, with $p_2, p_3 \in \mathbb{S}^1$, these points must satisfy the *closing equations* given in (2.6). Then for the piecewise linear differential system formed by the centers (2.9), (2.10) and (2.11) we have that system (2.6) is equivalent to system

$$\begin{aligned}
&45.611514.. + x_2^2 - 2.274330..y_1 - 4.914345..y_1^2 + 0.526241..y_2 \\
&\qquad +4.914345..y_2^2 + x_2(-46.611514.. + 1.748088..y_2) = 0, \\
&x_2^2 - x_3^2 + x_2\left(-13.453098.. + \frac{3}{10}y_2\right) - 4.371176..y_2 \\
&+6.201094..y_2^2 + x_3\left(13.453098.. - \frac{3}{10}y_3\right) + 4.371176..y_3 - 6.201094..y_3^2 = 0, \\
&45.611514.. + x_3^2 + 0.526241..y_3 + 4.914345..y_3^2 \\
&+x_3(-46.611514.. + 1.748088..y_3) - 2.274330..y_4 - 4.914345..y_4^2 = 0, \\
&(y_1 - y_4)\left(-\frac{5}{2} + 5y_1 + 5y_4\right) = 0, \\
&x_2^2 + y_2^2 = 1, \qquad x_3^2 + y_3^2 = 1,
\end{aligned} \tag{2.12}$$

Therefore the unique real solutions $q^i = (y_1^i, x_2^i, y_2^i, x_3^i, y_3^i, y_4^i)$ for system (2.12) that satisfy the condition $y_4^i < y_1^i$, are the points $q^1 = (3, \cos(\pi/2), \sin(\pi/2), \cos(-\pi/3), \sin(-\pi/3), -5/2)$; $q^2 = (17/5, \cos(3\pi/5), \sin(3\pi/5), \cos(-2\pi/5), \sin(-2\pi/5), -29/10)$; $q^3 = (3.294676.., \cos(4\pi/7), \sin(4\pi/7), 0.362651.., -0.931924.., -2.794676..)$ and $q^4 = (1.287554.., 0.814865.., 0.579649.., 0.966364.., -0.257177.., -0.787554)$, which generated four crossing limit cycles. See these crossing limit cycles of the piecewise linear differential center formed by (2.9), (2.10) and (2.11) in Figure 2.2. □

Proof of statement (b) of Theorem 1.1. We consider the piecewise linear differential center such that in the region R_1 it has the linear differential center

$$\dot{x} = 0.309248.. - 0.237408..x - 0.439335..y, \qquad \dot{y} = -0.478770.. + x + 0.237408..y, \tag{2.13}$$

this system has the first integral $H_1(x,y) = x^2 + x(-0.957540.. + 0.474817..y) + (-0.618496.. + 0.439335..y)y$. In the region R_2 we have the linear differential center

$$\dot{x} = 0.396090.. - 0.335276..x - 0.180370..y, \qquad \dot{y} = -0.861570.. + x + 0.335276..y, \tag{2.14}$$

which has the first integral $H_2(x,y) = x^2 + x(-1.723140.. + 0.670553..y) + (-0.792181.. + 0.180370..y)y$. In the region R_3 we have the linear differential center

$$\dot{x} = 0.242967.. + 0.112091..x - 0.194871..y, \qquad \dot{y} = 0.375114.. + x - 0.112091..y, \tag{2.15}$$

this system has the first integral $H_3(x,y) = x^2 + x(0.750229.. - 0.224182..y) + (-0.485935.. + 0.194871..y)y$. And in the region R_4 we have the linear differential center

$$\dot{x} = 0.394133.. + 0.278957..x - 0.25146..y, \qquad \dot{y} = 0.516804.. + x - 0.278957..y, \tag{2.16}$$

which has the first integral $H_4(x,y) = x^2 + x(1.033609.. - 0.557914..y) - (0.788267.. - 0.251469..y)y$. In order to have a crossing limit cycle of type 1, which intersects the dis-

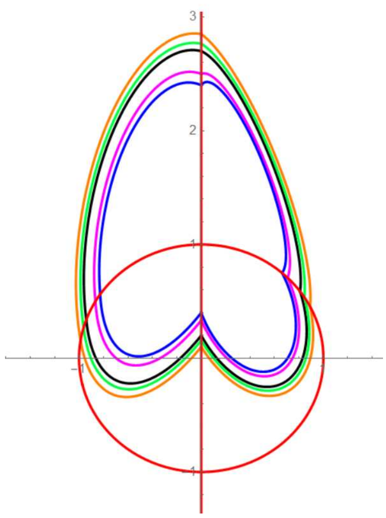

Figure 2.3: Five crossing limit cycles of type 1 of the discontinuous piecewise linear differential system formed by the centers (2.13), (2.14), (2.15) and (2.16). These limit cycles are traveled in counterclockwise.

continuity curve Σ_k in four different points $p_1 = (k, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (k, y_3)$ and $p_4 = (x_4, y_4)$, with $p_2, p_4 \in \mathbb{S}^1$, then these points must satisfy the system

$$\begin{aligned} H_1(k,\ y_1) &= H_1(x_2, y_2),\\ H_2(x_2, y_2) &= H_2(k,\ y_3),\\ H_3(k,\ y_3) &= H_3(x_4, y_4),\\ H_4(x_4, y_4) &= H_4(k, y_1),\\ x_2^2 + y_2^2 &= 1,\\ x_4^2 + y_4^2 &= 1, \end{aligned} \tag{2.17}$$

Considering $k = 0$ and the previous piecewise linear differential center, system (2.17) is equivalent to system

$$\begin{aligned} x_2^2 + 0.618497..y_1 - 0.439336..y_1^2 + x_2(-0.957541.. + 0.474817..y_2) - 0.618497..y_2&\\ +0.439336..y_2^2 = 0,&\\ 4x_2^2 - 3.168726..y_2 + 0.721481..y_2^2 + x_2(-6.892562.. + 2.682214..y_2) + 3.168726..y_3&\\ -0.721481..y_3^2 = 0,&\\ x_4^2 + 0.485936..y_3 - 0.194871..y_3^2 + x_4(0.750229.. - 0.224183..y_4) - 0.485936..y_4&\\ +0.194871..y_4^2 = 0,&\\ 4x_4^2 + 3.153071..y_1 - 1.005879..y_1^2 + x_4(4.134439.. - 2.231658..y_4) - 3.153071..y_4&\\ +1.005879..y_4^2 = 0,&\\ x_2^2 + y_2^2 = 1, \qquad x_4^2 + y_4^2 = 1.& \end{aligned} \tag{2.18}$$

Therefore discontinuous piecewise differential system formed by the linear differential centers (2.13), (2.14), (2.15) and (2.16) has five crossing limit cycles of type 1, because system (2.18) has five real solutions $q^i = (y_1^i, x_2^i, y_2^i, y_3^i, x_4^i, y_4^i)$, for $i = 1, 2, 3, 4, 5$ that satisfy the conditions $-1 < y_1^i < 1 < y_3^i$; $x_2^i > 0$ and $x_4^i < 0$. Where $q^1 = (1/3,\ \cos(\pi/4),\ \sin(\pi/4),\ 5/2,\ \cos(5\pi/6),\ \sin(5\pi/6))$; $q^2 = (\ 2/5,\ \cos(27\pi/10),\ \sin(27\pi/10),\ 12/5,\ \cos(81\pi/100), \cos(81\pi/100))$; $q^3 = (1/5, \cos(\pi/5),\ \sin(\pi/5), 27/10, \cos(89\pi/100), \sin(89\pi/100))$; $q^4 = (1/10, \cos(3\pi/20), \sin(3\pi/20), 57/20,\ \cos(19\pi/20),\ \sin(19\pi/20))$ and $q^5 = (0.157052..,\ 0.843891.., 0.536513.., 2.764619.., -0.962848.., 0.270041..)$. See these five crossing limit cycles of type 1 in Figure 2.3. □

Proof of statement (c) of Theorem 1.1. We consider the following discontinuous piecewise linear differential system

$$\begin{aligned} &\dot{x} = -0.045605.. + 0.048166..x - 0.671455..y, && \dot{y} = -0.418364.. + x - 0.048166..y, && \text{in } R_1,\\ &\dot{x} = 0.058276.. + \frac{x}{100} - 0.178664..y, && \dot{y} = -0.763833.. + x - \frac{y}{100}, && \text{in } R_2,\\ &\dot{x} = \frac{901}{50000} - \frac{x}{50} - \frac{901}{2500}y, && \dot{y} = \frac{1}{10} + x + \frac{y}{50}, && \text{in } R_4. \end{aligned} \tag{2.19}$$

The linear differential centers in (2.19) have the first integrals

$$\begin{aligned} H_1(x,y) &= x^2 + x(-0.836729.. - 0.096332..y) + (0.091210.. + 0.671455..y)y,\\ H_2(x,y) &= x^2 + x\left(-1.527667.. - \frac{y}{50}\right) + (-0.116553.. + 0.178664..y)y,\\ H_4(x,y) &= 4x^2 + \frac{4}{25}x(5+y) + \frac{901y(-1+10y)}{6250}, \end{aligned}$$

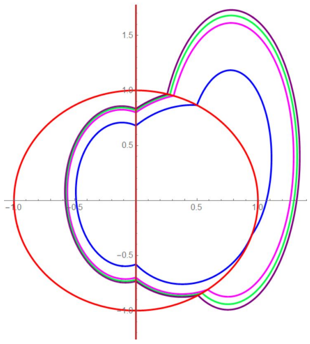

Figure 2.4: Four crossing limit cycles of type 2^+ of the discontinuous piecewise linear differential center (2.19). These limit cycles are traveled in counterclockwise.

respectively. In order to have a crossing limit cycle of type 2^+, which intersects Σ_k in four different points $p_1 = (x_1, y_1)$, $p_2 = (k, y_2)$, $p_3 = (k, y_3)$ and $p_4 = (x_4, y_4)$, with $p_1, p_4 \in \mathbb{S}^1$, these points must satisfy the system

$$\begin{aligned} H_1(x_1,y_1) &= H_1(\,k,y_2),\\ H_4(\,k,y_2) &= H_4(\,k,y_3),\\ H_1(\,k,y_3) &= H_1(x_4,y_4),\\ H_2(x_4,y_4) &= H_2(x_1,y_1),\\ x_1^2+y_1^2 &= 1,\\ x_4^2+y_4^2 &= 1. \end{aligned} \tag{2.20}$$

Then for the piecewise linear differential system (2.19) we have that system (2.20) becomes

$$\begin{aligned} &4x_1^2 + x_1(-3.346917.. - 0.385331..y_1) + y_1(0.364840.. + 2.685822..y_1)\\ &\qquad\qquad +(-0.364840.. - 2.685822..y_2)y_2 = 0,\\ &(y_2 - y_3)\left(-\frac{901}{6250} + \frac{901}{625}(y_2+y_3)\right) = 0,\\ &-4x_4^2 + y_3(0.364840.. + 2.685822..y_3) + x_4(3.346917.. + 0.385331..y_4)\\ &\qquad\qquad +(-0.364840.. - 2.685822..y_4)y_4 = 0,\\ &-4x_1^2 + 4x_4^2 + x_1\left(6.110671.. + \frac{2}{25}y_1\right) + (0.466214.. - 0.714659..y_1)y_1\\ &\qquad +x_4\left(-6.110671.. - \frac{2}{25}y_4\right) + (-0.466214.. + 0.714659..y_4)y_4 = 0,\\ &\qquad\qquad x_1^2 + y_1^2 = 1, \quad x_4^2 + y_4^2 = 1, \end{aligned} \tag{2.21}$$

where $k = 0$. Therefore the unique real solutions $q^i = (x_1^i, y_1^i, y_2^i, y_3^i, x_4^i, y_4^i)$ for system (2.21) that satisfy the conditions $-1 < y_3^i < y_2^i < 1$; $x_1^i > 0$ and $x_4^i > 0$ are the points $q^1 = (\cos(2\pi/5), \sin(2\pi/5), 8/10, -7/10, \cos(-3\pi/10), \sin(-3\pi/10))$; $q^2 = (\cos(\pi/3), \sin(\pi/3), 17/25, -29/50, \cos(-\pi/10), \sin(-\pi/10))$; $q^3 = (\cos(41\pi/100), \sin(41\pi/100), 0.819235.., -0.719235.., 0.541860.., -0.840468..)$ and $q^4 = (0.256532.., 0.966535.., 0.833667.., -0.733667.., 0.508672.., -0.860960..)$. These four real solutions generated four crossing limit cycles of type 2^+. See these crossing limit cycles of the piecewise linear differential center (2.19) in Figure 2.4.

□

Proof of statement (d) of Theorem 1.1. We consider the following discontinuous piecewise linear differential system

$$
\begin{aligned}
\dot{x} &= 1.018312.. + \frac{51}{40}x + 9.463668..y, & \dot{y} &= -5.008011.. - x - \frac{51}{40}y, \text{ in } R_1,\\
\dot{x} &= 0.712799.. - 0.278320..x - 0.250791..y, & \dot{y} &= -1.026464.. + x + 0.278320..y, \text{ in } R_2,\\
\dot{x} &= \frac{969}{1280} + \frac{x}{8} - \frac{17}{64}y, & \dot{y} &= \frac{1}{8} + x - \frac{y}{8}, \text{ in } R_3.
\end{aligned}
\tag{2.22}
$$

The linear differential centers in (2.22) have the first integrals

$$
\begin{aligned}
H_1(x,y) &= 4x^2 + x\left(40.064090.. + \frac{51}{5}y\right) + y(8.146500.. + 37.854675..y),\\
H_2(x,y) &= x^2 + x(-2.052928.. + 0.556641..y) + (-1.425599.. + 0.250791..y)y,\\
H_3(x,y) &= x + 4x^2 - xy + \frac{17}{160}y(-57 + 10y).
\end{aligned}
$$

respectively. In order to have a crossing limit cycle of type 3^+, which intersects the discontinuity curve Σ_k in four different points $p_1 = (k, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (k, y_3)$ and $p_4 = (x_4, y_4)$, with $p_2, p_4 \in \mathbb{S}^1$, these points must satisfy the system

$$
\begin{aligned}
H_2(x_1, y_1) &= H_2(k,\ y_2),\\
H_3(k,\ y_2) &= H_3(k,\ y_3),\\
H_2(k,\ y_3) &= H_2(x_4, y_4),\\
H_1(x_4, y_4) &= H_1(x_1, y_1),\\
x_1^2 + y_1^2 &= 1,\\
x_4^2 + y_4^2 &= 1,
\end{aligned}
\tag{2.23}
$$

Considering $k = 0$, system (2.23) is equivalent to system

$$
\begin{aligned}
4x_1^2 + y_1(-5.702397.. + 1.003165..y_1) + x_1(-8.211712.. + 2.226564..y_1)&\\
+5.702397..y_2 - 1.003165..y_2^2 &= 0,\\
(y_2 - y_3)(-57 + 10y_2 + 10y_3) &= 0,\\
x_4^2 + (1.425599.. - 0.250791..y_3)y_3 + x_4(-2.052928.. + 0.556641..y_4)&\\
-1.425599..y_4 + 0.250791..y_4^2 &= 0,\\
x_1^2 - x_4^2 + x_1\left(10.016022.. + \frac{51}{20}y_1\right) + y_1(2.036625.. + 9.463668..y_1)&\\
+x_4\left(-10.016022.. - \frac{51}{20}y_4\right) + (-2.036625.. - 9.463668..y_4)y_4 &= 0,\\
x_1^2 + y_1^2 = 1, \quad x_4^2 + y_4^2 &= 1.
\end{aligned}
\tag{2.24}
$$

Therefore discontinuous piecewise differential (2.22) has three crossing limit cycles of type 3^+, because system (2.24) has three real solutions $q^i = (x_1^i, y_1^i, y_2^i, y_3^i, x_4^i, y_4^i)$, for $i = 1, 2, 3$ that satisfy the conditions $0 < x_4^i < x_1^i$ and $1 < y_3^i < y_2^i$. Where $q^1 = (\cos(\pi/5),\ \sin(\pi/5),\ 43/10,\ 7/5\cos(2\pi/5), \sin(2\pi/5))$; $q^2 = (\cos(16\pi/125),\ \sin(16\pi/125), 447/100,\ 123/100\cos(9\pi/50),\ \cos(9\pi/50))$ and $q^3 = (\cos(17\pi/100),\ \sin(17\pi/100),\ 4.366812..,\ 1.333187..,\ 0.242211..,\ 0.970223..)$. See these three crossing limit cycles of type 3^+ in Figure 2.5. □

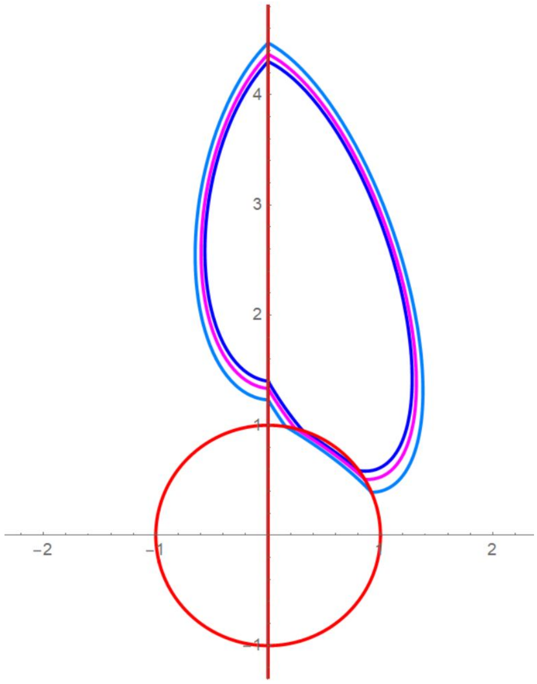

Figure 2.5: Three crossing limit cycles of type 3^+ of the discontinuous piecewise linear differential center (2.22). These limit cycles are traveled in counterclockwise.

Proof of statement (e) of Theorem 1.1. We consider the following discontinuous piecewise linear differential system

$$\begin{aligned}
\dot{x} &= 0.244909.. - 0.132672..x - 0.724279..y, \quad & \dot{y} &= -0.471887.. + x + 0.132672..y, \text{ in } R_1,\\
\dot{x} &= 0.668802.. - 0.514522..x - 0.636209..y, \quad & \dot{y} &= -0.985653.. + x + 0.514522..y, \text{ in } R_2,\\
\dot{x} &= -0.081198.. - 0.207828..x - 0.061343..y, \quad & \dot{y} &= -0.124956.. + x + 0.207828..y, \text{ in } R_3,\\
\dot{x} &= 0.211524.. - 0.634777..x - 0.705080..y, \quad & \dot{y} &= -0.356652.. + x + 0.634777..y, \text{ in } R_4.
\end{aligned} \tag{2.25}$$

The linear differential centers in (2.25) have the first integrals

$$\begin{aligned}
H_1(x,y) &= x^2 + x(-0.943775.. + 0.265344..y) + (-0.489818.. + 0.724279..y)y,\\
H_2(x,y) &= x^2 + (-1.337605.. + 0.636209..y)y + x(-1.971307.. + 1.029044..y),\\
H_3(x,y) &= x^2 + x(-0.249913.. + 0.415657..y) + (0.162397.. + 0.061343..y)y,\\
H_4(x,y) &= x^2 + (-0.423048.. + 0.705080..y)y + x(-0.713304.. + 1.269555..y),
\end{aligned}$$

respectively. In order to have crossing limit cycles of types 1 and 2^+, simultaneously, such that the crossing limit cycles of type 1 intersect the discontinuity curve Σ_0 in four different points $p_1 = (0, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (0, y_3)$ and $p_4 = (x_4, y_4)$, with $-1 < y_1 < 1 < y_3$ and $x_4 < 0 < x_2$ and $p_2, p_4 \in \mathbb{S}^1$; and the crossing limit cycles of type 2^+ intersect the discontinuity curve Σ_0 in four different points $p_5 = (x_5, y_5)$, $p_6 = (0, y_6)$, $p_7 = (0, y_7)$ and $p_8 = (x_8, y_8)$, with $-1 < y_7 < y_6 < 1$ and $x_5, x_8 > 0$, with $p_5, p_8 \in \mathbb{S}^1$. These points must satisfy systems (2.17) and (2.20), respectively. Considering the piecewise linear differential center (2.25) systems

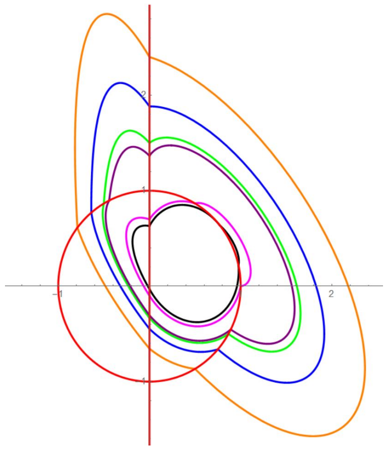

Figure 2.6: Four crossing limit cycles of type 1 and two crossing limit cycles of type 2^+ (black and magenta) of the discontinuous piecewise linear differential center (2.25). These limit cycles are traveled in counterclockwise.

(2.17) and (2.20) become

$$
\begin{aligned}
x_2^2 + x_2(-0.943775.. + 0.265344..y_2) - 0.489818..y_2 + 0.724279..y_2^2 & \\
+0.489818..y_1 - 0.724279..y_1^2 &= 0,\\
4x_2^2 - 5.350421..y_2 + 2.544838..y_2^2 + x_2(-7.885229.. + 4.116178..y_2) & \\
+5.350421..y_3 - 2.544838..y_3^2 &= 0,\\
x_4^2 - 0.162397..y_3 - 0.061343..y_3^2 + x_4(-0.249913.. + 0.415657..y_4) & \\
+0.162397..y_4 + 0.061343..y_4^2 &= 0,\\
4x_4^2 - 1.692192..y_4 + 2.820321..y_4^2 + x_4(-2.853217.. + 5.078222..y_4) & \\
+1.692192..y_1 - 2.820321..y_1^2 &= 0,\\
4x_5^2 - 1.959275..y_5 + 2.897117..y_5^2 + x_5(-3.775101.. + 1.061377..y_5) & \\
+1.959275..y_6 - 2.897117..y_6^2 &= 0,\\
(y_6 - y_7)(-1.692192.. + 2.820321..(y_6 + y_7)) &= 0,\\
x_8^2 + 0.489818..y_7 - 0.724279..y_7^2 + x_8(-0.943775.. + 0.265344..y_8) & \\
-0.489818..y_8 + 0.724279..y_8^2) &= 0,\\
x_5^2 - x_8^2 - 1.337605..y_5 + 0.636209..y_5^2 + x_5(-1.971307.. + 1.029044..y_5) & \\
+x_8(1.971307.. - 1.029044..y_8) + 1.337605..y_8 - 0.636209..y_8^2 &= 0,\\
x_2^2 + y_2^2 = 1, \quad x_4^2 + y_4^2 = 1, \quad x_5^2 + y_5^2 = 1, \quad x_8^2 + y_8^2 &= 1.
\end{aligned}
\tag{2.26}
$$

We have four real solutions $q^i = (y_1^i, x_2^i, y_2^i, y_3^i, x_4^i, y_4^i, x_5^i, y_5^i, y_6^i, y_7^i, x_8^i, y_8^i)$ with $i = 1, 2, 3, 4$, for system (2.26) that satisfy the above conditions, namely $q^1 = (\,-1/3,\ \cos(-\pi/6),\ \sin(-\pi/6),\ 3/2,\ \cos(2\pi/3),\ \sin(2\pi/3),\ \cos(\pi/3),\ \sin(\pi/3),\ 7/10,\ -1/10,\ 1, 0)$; $q^2 = (-0.654342..,\ \cos(-\pi/3),\ \sin(-\pi/3),\ 12/5,\ \cos(79\pi/100),\ \sin(79\pi/100),\ \cos(11\pi/50),\ \sin(11\pi/50),\ 63/100,\ -3/100,\ 0.975733..,\ 0.216981..)$; $q^3 = (-0.447098..,\ \cos(-23\pi/100),\ \sin(-23\pi/100),\ 1.882264..,\ \cos(18\pi/25),\ \sin(18\pi/25),\ -0.654342..,\ \cos(11\pi/50),\ \sin(11\pi/50),\ 63/100,\ -3/100, 0.975733.., 0.216981..)$; $q^4 = (\,-0.305568..,\ \cos(-3\pi/20),\ \sin(-3\pi/20),\ 1.365012..,\ -0.441883..,\ 0.897073..,\ \cos(11\pi/50),\ \sin(11\pi/50), 63/100, -3/100, 0.975733.., 0.216981..)$, these four solutions generated four crossing limit cycles of type 1 and two crossing limit cycles of type 2^+. See these crossing limit cycles of the piecewise linear differential center (2.25) in Figure 2.6.

Here we observed that we obtain a total of six crossing limit cycles between limit cycles of type 1 and of type 2^+, moreover these six crossing limit cycles have the configuration $(4,2)$, this is, 4-crossing limit cycle of type 1 and 2-crossing limit cycles of type 2^+. Clearly this lower bound for the maximum number of crossing limit cycles of types 1 and 2^+ simultaneously, could be also obtained with the configurations $(3,3)$ or $(2,4)$. But after several numeric computations we could not build a third limit cycle of type 2^+, previously fixing two limit cycles of type 1, so we only get those lower bound with the configuration $(4,2)$. □

Proof of statement (f) of Theorem 1.1. We consider the following discontinuous piecewise linear differential system

$$\begin{aligned}
\dot{x} &= 0.078341.. + 0.855624..x + 1.571418..y, & \dot{y} &= -0.065526.. - x - 0.855624..y, \text{ in } R_1,\\
\dot{x} &= 0.496667.. + 0.078616..x - 0.193136..y, & \dot{y} &= -0.471461.. + x - 0.078616..y, \text{ in } R_2,\\
\dot{x} &= 5.276135.. + 0.212817..x - 1.851275..y, & \dot{y} &= -5.383865.. + x - 0.212817..y, \text{ in } R_3,\\
\dot{x} &= 0.484115.. + 0.548314..x - 0.303113..y, & \dot{y} &= 0.569064.. + x - 0.548314..y, \text{ in } R_4.
\end{aligned} \tag{2.27}$$

The linear differential centers in (2.27) have the first integrals

$$\begin{aligned}
H_1(x,y) &= x^2 + y(0.156682.. + 1.571418..y) + x(0.131053.. + 1.711249..y),\\
H_2(x,y) &= x^2 + x(-0.942922.. - 0.157232..y) + (-0.993334.. + 0.193136..y)y,\\
H_3(x,y) &= x^2 + x(-10.767731.. - 0.425635..y) + y(-10.552270.. + 1.851275..y),\\
H_4(x,y) &= x^2 + x(1.138128.. - 1.096628..y) + (-0.968231.. + 0.303113..y)y,
\end{aligned}$$

respectively. In order to have crossing limit cycles of types 1 and 3^+, simultaneously, such that the crossing limit cycles of type 1 intersect the discontinuity curve Σ_0 in four different points $p_1 = (0,y_1)$, $p_2 = (x_2,y_2)$, $p_3 = (0,y_3)$ and $p_4 = (x_4,y_4)$,with $-1 < y_1 < 1 < y_3$ and $x_4 < 0 < x_2$ and $p_2, p_4 \in \mathbb{S}^1$; and the crossing limit cycles of type 3^+ intersect the discontinuity curve Σ_0 in four different points $p_5 = (x_5,y_5)$, $p_6 = (0,y_6)$, $p_7 = (0,y_7)$ and $p_8 = (x_8,y_8)$, with $1 < y_7 < y_6$, $x_5, x_8 > 0$ and $p_5, p_8 \in \mathbb{S}^1$, these points must satisfy systems (2.17) and (2.23), respectively. Considering the piecewise linear differential center (2.27) systems (2.17) and (2.23) become

$$\begin{aligned}
-0.524214..x_2 - 4x_2^2 - 6.844999..x_2y_2 + (y_1 - y_2)(0.626729.. + 6.285673..(y_1 + y_2)) &= 0,\\
-3.771690..x_2 + 4x_2^2 - 0.628929..x_2y_2 + (y_2 - y_3)(-3.973339.. + 0.772547..(y_2 + y_3)) &= 0,\\
43.070926..x_4 - 4x_4^2 + 1.702543..x_4y_4 + (y_3 - y_4)(-42.209082.. + 7.405102..(y_3 + y_4)) &= 0,\\
4.552513..x_4 + 4x_4^2 - 4.386514..x_4y_4 - (y_1 - y_4)(-3.872927.. + 1.212454..(y_1 + y_4)) &= 0,\\
-3.771690..x_5 + 4x_5^2 - 0.628929..x_5y_5 + (y_5 - y_6)(-3.973339.. + 0.772547..(y_5 + y_6)) &= 0,\\
(y_6 - y_7)(-42.209082.. + 7.405102..(y_6 + y_7)) &= 0,\\
3.771690..x_8 - 4x_8^2 + 0.628929..x_8y_8 + (y_7 - y_8)(-3.973339.. + 0.772547..(y_7 + y_8)) &= 0,\\
-4x_5^2 + 4x_8^2 + x_5(-0.524214.. - 6.844999..y_5) + (-0.626729..&\\
-6.285673..y_5)y_5 + y_8(0.626729.. + 6.285673..y_8) + x_8(0.524214.. + 6.844999..y_8) &= 0,\\
x_2^2 + y_2^2 = 1, \quad x_4^2 + y_4^2 = 1, \quad x_5^2 + y_5^2 = 1, \quad x_8^2 + y_8^2 &= 1.
\end{aligned} \tag{2.28}$$

We have four real solutions $q^i = (y_1^i, x_2^i, y_2^i, y_3^i, x_4^i, y_4^i, x_5, y_5, y_6, y_7, x_8, y_8)$ with $i = 1,2,3,4$, for system (2.28) that satisfy the above conditions, namely $q^1 = (4/5,\ 1,\ 0,\ 26/5,\ \cos(3\pi/5),\ \sin(3\pi/5),\ \cos(\pi/5),\ \sin(\pi/5),\ 43/10,\ 7/5,\ \cos(2\pi/5), \sin(2\pi/5))$; $q^2 = (53/100,$

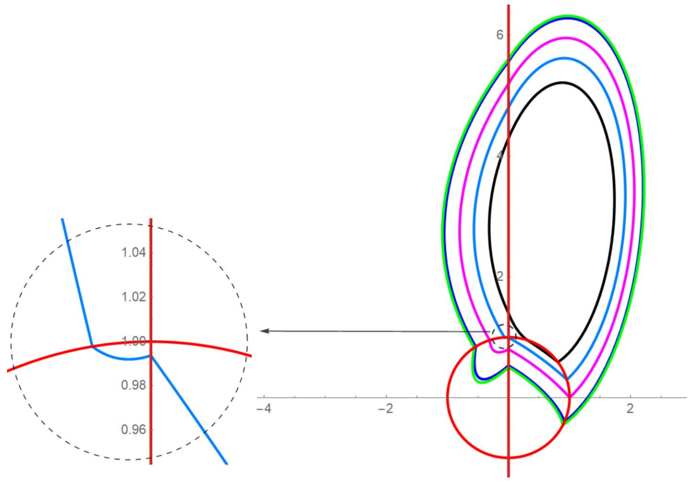

Figure 2.7: Four crossing limit cycles of type 1 and one crossing limit cycle of type 3^+ (black) of the discontinuous piecewise linear differential center (2.27). These limit cycles are traveled in counterclockwise.

$\cos(-13\pi/100), \sin(-13\pi/100),\ 557/100, \cos(17\pi/25),\ \sin(17\pi/25), \cos(\pi/5), \sin(\pi/5), 43/10,\ 7/5,\ \cos(2\pi/5),\ \sin(2\pi/5));\ q^3 = (1/2,\ \cos(-3\pi/20),\ \sin(-3\pi/20),\ 5.611962..,$ $\cos(17239\pi/25000),\ \sin(17239\pi/25000),\ \cos(\pi/5),\ \sin(\pi/5),\ 43/10,\ 7/5,\ \cos(2\pi/5),$ $\sin(2\pi/5));\ q^4 = (0.993727.., \cos(12\pi/125), \sin(12\pi/125), 4.808026.., -0.066301.., 0.997799..,$ $\cos(\pi/5), \sin(\pi/5), 43/10, 7/5, \cos(2\pi/5), \sin(2\pi/5))$, these four solutions generated four crossing limit cycles of type 1 and one crossing limit cycle of type 3^+. See these crossing limit cycles of the piecewise linear differential center (2.27) in Figure 2.7.

Here we observed that we obtain a total of five crossing limit cycles between limit cycles of type 1 and of type 3^+, moreover these five crossing limit cycles have the configuration $(4, 1)$, this is, 4-crossing limit cycle of type 1 and 1-crossing limit cycles of type 3^+. In order to obtain a result similar to the previous statement, this is, an example with a configuration $(4, 2)$, we tried to build a second cycle of type 3^+ but when building this second cycle we lost a cycle of type 1, so we only got a configuration $(3, 2)$. If we consider the piecewise linear system

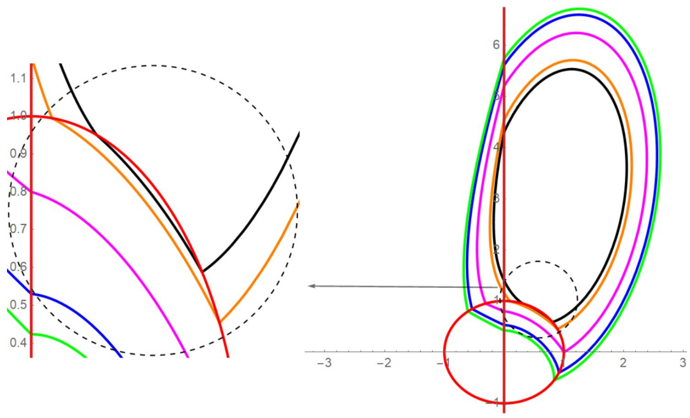

Figure 2.8: Three crossing limit cycles of type 1 and two crossing limit cycle of type 3^+ (black and orange) of the discontinuous piecewise linear differential center (2.29). These limit cycles are traveled in counterclockwise.

$$
\begin{aligned}
&\dot{x} = -0.128852.. - 0.332114..x - 0.791281..y, && \dot{y} = -0.143708.. + x + 0.332114..y, \text{ in } R_1,\\
&\dot{x} = 0.597908.. + 0.108856..x - 0.227688..y, && \dot{y} = -0.530777.. + x - 0.108856..y, \text{ in } R_2,\\
&\dot{x} = 0.716356.. + 0.457342..x - 0.251353..y, && \dot{y} = -0.189975.. + x - 0.457342..y, \text{ in } R_3,\\
&\dot{x} = 1.857676.. - \frac{4}{5}x - 0.688147..y, && \dot{y} = -1.219907.. + x + \frac{4}{5}y, \text{ in } R_4.
\end{aligned}
\tag{2.29}
$$

It is possible verify that we obtain the configuration $(3, 2)$, see Figure 2.8. But after several numeric computations we could not build a third limit cycle of type 3^+, previously fixing two limit cycles of type 1, so we only get those lower bound by the maximum number of types 1 and 3^+, simultaneously, with the configurations $(4, 1)$ and $(3, 2)$. □

3 Proof of Theorem 1.2

We consider the following piecewise linear differential center

$$
\begin{aligned}
&\dot{x} = -124.644504.. + \frac{111}{50}x - 6.045715..y, && \dot{y} = -148.901657.. + x - \frac{111}{50}y, \text{ in } R_1,\\
&\dot{x} = 0.236087.. + 0.003662..x - 0.009243..y, && \dot{y} = -0.402647.. + x - 0.003662..y, \text{ in } R_2,\\
&\dot{x} = 1 + \frac{x}{5} - 0.102500..y, && \dot{y} = -\frac{9}{20} + x - \frac{y}{5}, \text{ in } R_3.
\end{aligned}
\tag{3.1}
$$

The linear differential centers in (3.1) have the first integrals

$$
\begin{aligned}
H_1(x, y) &= x^2 + x\left(-297.803314.. - \frac{111}{25}y\right) + y(249.289008.. + 6.045715..y),\\
H_2(x, y) &= x^2 + x(-0.805295.. - 0.007324..y) + (-0.472175.. + 0.009243..y)y,\\
H_3(x, y) &= x^2 + x\left(-\frac{9}{10} - \frac{2}{5}y\right) + (-2 + 0.102500..y)y,
\end{aligned}
$$

respectively.

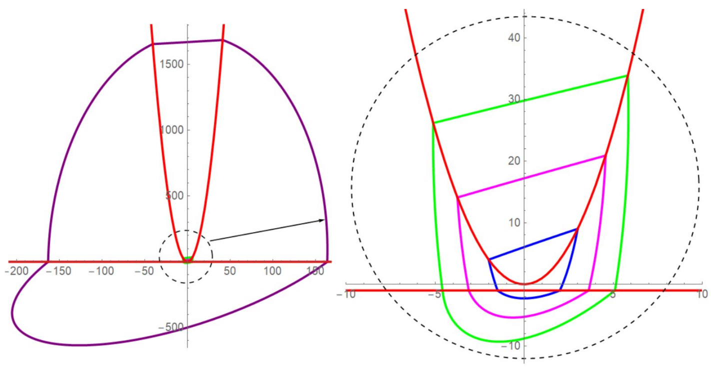

Figure 3.1: Four crossing limit cycles of the discontinuous piecewise linear differential system (3.1). These limit cycles are traveled in counterclockwise.

For piecewise linear differential systems in the family $\mathcal{F}_4$ we have crossing limit cycles which intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, x_2^2)$,

$p_3 = (x_3, k)$ and $p_4 = (x_4, k)$, if these points satisfy the system

$$\begin{aligned} H_1(x_1, x_1^2) &= H_1(x_2, x_2^2), \\ H_2(x_2, x_2^2) &= H_2(x_3, k), \\ H_3(x_3,\ k) &= H_3(x_4, k), \\ H_2(x_4,\ k) &= H_2(x_1, x_1^2). \end{aligned} \tag{3.2}$$

Then for the piecewise linear differential centers (3.1) and $\tilde{\Sigma}_k$ considering $k = -1$, system (3.2) becomes

$$\begin{aligned} x_1(-1191.213259.. + x_1(1001.156032.. + x_1\,(-44425 + 24.182863..x_1))) \\ +x_2(1191.213259.. + x_2(-1001.156032.. + (-44425 - 24.182863..x_2)\,x_2)) &= 0, \\ -1.925675.. + x_2(-3.221182.. + x_2(2.111297.. + (-0.029297.. + 0.036973..x_2)x_2)) \\ +(3.191885.. - 4x_3)x_3 &= 0, \\ (x_3 - x_4)\left(-\frac{1}{2} + x_3 + x_4\right) &= 0, \\ 1.925675.. + x_1(3.221182.. + x_1(-2.111297.. + (0.029297.. - 0.036973..x_1)x_1)) \\ +x_4(-3.191885.. + 4x_4) &= 0. \end{aligned} \tag{3.3}$$

Taking into account that the solutions (x_1, x_2, x_3, x_4) must satisfy $x_2 < x_1$ and $x_3 < x_4$, system (3.3) has four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i)$, with $i = 1, 2, 3, 4$. Namely, $q^1 = (3, -2, -3/2, 2)$; $q^2 = (457/100,\ -3.753677..,\ -3.116713..,\ 3.616713..)$; $q^3 = (5.820000..,\ -5.115260..,\ -4.592690..,\ 5.092690..)$ and $q^4 = (41.045251..,\ -40.667957..,\ -162.945374..,\ 163.445374..)$. Which provide four crossing limit cycles of the piecewise linear differential center (3.1). See these four crossing limit cycles in Figure 3.1.

Here we observe that there is a duality between the crossing limit cycles that intersect the discontinuity curve $\tilde{\Sigma}_{-1}$ and the crossing limit cycles that intersect the discontinuity curve Σ_2 for the family $\mathcal{F}_1$ studied in statement (a) of Theorem 1.1, where we also got four crossing limit cycles, see Figures 2.1 and 3.1.

4 Proof of Theorem 1.3

Proof of statement (a) of Theorem 1.3. We consider the following piecewise linear differential center

$$\begin{aligned} \dot{x} &= \frac{11}{10} + \frac{4}{5}x - \frac{4}{5}y, & \dot{y} &= 1 + x - \frac{4}{5}y, \text{ in } R_1, \\ \dot{x} &= \frac{17}{75} - \frac{3}{10}x - \frac{17}{150}y, & \dot{y} &= -\frac{61}{20} + x + \frac{3}{10}y, \text{ in } R_2, \\ \dot{x} &= \frac{1}{6} + x - \frac{25}{16}y, & \dot{y} &= -\frac{1}{4} + x - y, \text{ in } R_3, \\ \dot{x} &= \frac{133}{36} + \frac{x}{10} - \frac{7}{45}y, & \dot{y} &= \frac{543}{20} + x - \frac{y}{10}, \text{ in } R_4. \end{aligned} \tag{4.1}$$

The linear differential centers in (4.1) have the first integrals

$$\begin{aligned} H_1(x, y) &= 5x(2 + x) - (11 + 8x)y + 4y^2, \\ H_2(x, y) &= 150x^2 + 17(-4 + y)y + 15x(-61 + 6y), \\ H_3(x, y) &= 4x^2 - 2x(1 + 4y) + \frac{y}{12}(-16 + 75y), \\ H_4(x, y) &= 90x^2 + 9x(543 - 2y) + 7y(-95 + 2y), \end{aligned}$$

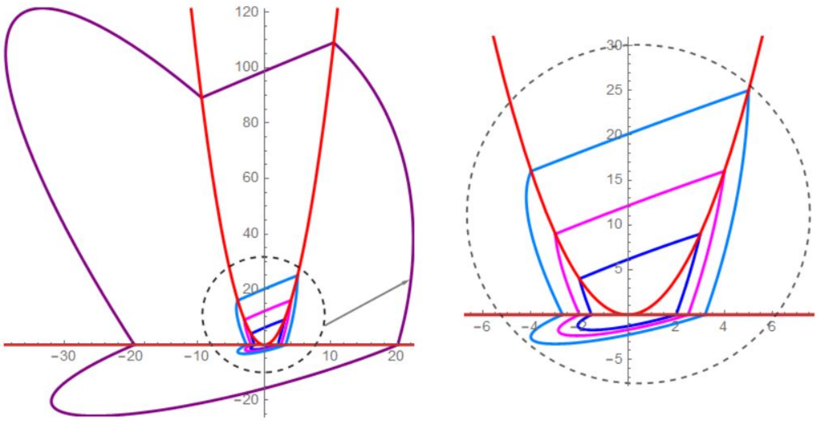

Figure 4.1: Four crossing limit cycles of type 4 of the discontinuous piecewise linear differential system (4.1). These limit cycles are traveled in counterclockwise.

respectively. In order to have a crossing limit cycle of type 4, which intersects the discontinuity curve $\tilde{\Sigma}_0$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, x_2^2)$, $p_3 = (x_3, 0)$ and $p_4 = (x_4, 0)$, these points must satisfy system

$$\begin{aligned} H_1(x_1, x_1^2) &= H_1(x_2, x_2^2), \\ H_2(x_2, x_2^2) &= H_2(x_3, 0), \\ H_3(x_3,\ 0) &= H_3(x_4, 0), \\ H_4(x_4,\ 0) &= H_4(x_1, x_1^2). \end{aligned} \tag{4.2}$$

Considering the piecewise linear differential center (4.1) system (4.2) becomes

$$\begin{aligned} (x_1 - x_2)(-1 + x_1 + x_2)(-5 + 2(-1 + x_1)x_1 + 2(-1 + x_2)x_2) &= 0, \\ 2x_2(-915 + x_2(82 + x_2(90 + 17x_2))) + 30(61 - 10x_3)x_3 &= 0, \\ 4(x_3 - x_4)\left(-\frac{1}{2} + x_3 + x_4\right) &= 0, \\ 2x_1(-4887 + x_1(575 + 2(9 - 7x_1)x_1)) + 18x_4(543 + 10x_4) &= 0. \end{aligned} \tag{4.3}$$

In this case we have that the solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i)$ must satisfy $x_2^i < 0 < x_1^i$ and $x_3^i < 0 < x_4^i$ then we have four real solutions $q^1 = (3, -2, -3/2, 2)$; $q^2 = (4, -3, -2, 5/2)$; $q^3 = (5, -4, 27/10, 16/5)$ and $q^4 = (10.440607.., -9.440607.., -19.555603.., 20.055606..)$ of system (4.3), which provide four crossing limit cycles of type 4 of the piecewise linear differential center (4.1). See these four crossing limit cycles in Figure 4.1.

Here we observe that there is a duality between the crossing limit cycles of type 4 that intersect the discontinuity curve $\tilde{\Sigma}_0$ and the crossing limit cycles that intersect the discontinuity curve Σ_1 for the family $\mathcal{F}_2$ studied in statement (a) of Theorem 1.1, where we also got four crossing limit cycles, see Figures 2.2 and 4.1. □

Proof of statement (b) of Theorem 1.3. In this case we consider the following piecewise linear differential center

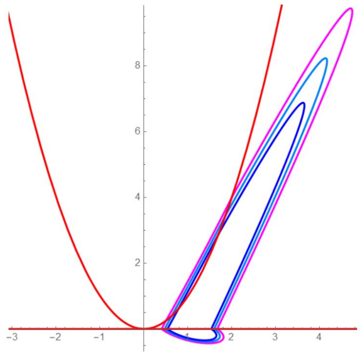

Figure 4.2: Three crossing limit cycles of type 5 of the discontinuous piecewise linear differential system (4.4). These limit cycles are traveled in counterclockwise.

$$
\begin{aligned}
&\dot{x} = 0.100318.. - \frac{2}{5}x + 0.161744..y && \dot{y} = 0.260062.. - x + \frac{2}{5}y, \text{ in } R_1,\\
&\dot{x} = 1 - x - \frac{13}{4}y, && \dot{y} = -\frac{31}{30} + x + y, \text{ in } R_3,\\
&\dot{x} = -0.399222.. + 0.378090..x - 0.144616..y, && \dot{y} = -1.020635.. + x - 0.378090..y, \text{ in } R_4.
\end{aligned} \tag{4.4}
$$

The linear differential centers in (4.4) have the first integrals

$$
\begin{aligned}
H_1(x,y) &= x^2 + x\left(-0.520124.. - \frac{4}{5}y\right) + (0.200636.. + 0.161744..y)y,\\
H_3(x,y) &= -\frac{124}{15}x - 8y + 9y^2 + 4(x+y)^2,\\
H_4(x,y) &= 4(x - 0.378090..y)^2 + 8(-1.020635..x + 0.399222..y) + 0.006657..y^2,
\end{aligned}
$$

respectively. In order to have a crossing limit cycle of type 5, which intersects the discontinuity curve $\tilde{\Sigma}_0$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, x_2^2)$, $p_3 = (x_3, 0)$ and $p_4 = (x_4, 0)$, with $0 < x_2 < x_1$ and $0 < x_3 < x_4$, these points must satisfy system

$$
\begin{aligned}
H_1(x_1, x_1^2) &= H_1(x_2, x_2^2),\\
H_4(x_2, x_2^2) &= H_4(x_3, 0),\\
H_3(x_3,\ 0) &= H_3(x_4, 0),\\
H_4(x_4,\ 0) &= H_4(x_1, x_1^2).
\end{aligned} \tag{4.5}
$$

Considering the piecewise linear differential center (4.4) system (4.5) becomes

$$
\begin{aligned}
&-2.080498..x_1 + 4.802546..x_1^2 - 3.199999..x_1^3 + 0.646977..x_1^4\\
&\quad + x_2(2.080498.. - 4.802546..x_2 + 3.199999..x_2^2 - 0.646977..x_2^3) = 0,\\
&x_2(-2591625737556 + x_2(2283329836763 + 50x_2(-19201143493 + 3672147700x_2)))\\
&\quad - 324x_3(-7998844869 + 3918560960x_3) = 0,\\
&4(x_3 - x_4)\left(-\frac{31}{15} + x_3 + x_4\right) = 0,\\
&x_1(2591625737556 + x_1(-2283329836763 + 50(19201143493 - 3672147700x_1)x_1))\\
&\quad + 324x_4(-7998844869 + 3918560960x_4) = 0.
\end{aligned} \tag{4.6}
$$

In this case system (4.6) has three real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i)$, where $q^1 = (2, 1/2, 2/5, 5/3)$; $q^2 = (93/50, 63/100, 47/100, 479/300)$ and $q^3 = (17/10, 0.785691.., 0.534387.., 1.532279..)$ which provide three crossing limit cycles of type 5 of the piecewise linear differential center (4.4). See these three crossing limit cycles in Figure 4.2. □

Proof of statement (c) of Theorem 1.3. We consider the following piecewise linear differential center

$$
\begin{aligned}
&\dot{x} = 45.736851.. - \frac{x}{2} - 7.515818..y, && \dot{y} = -1146.321640.. + x + \frac{y}{2}, \text{ in } R_1,\\
&\dot{x} = -0.320594.. - 0.199436..x - 0.051960..y, && \dot{y} = 0.460058.. + x + 0.199436..y, \text{ in } R_2,\\
&\dot{x} = 2 + \frac{x}{20} - \frac{13}{200}y, && \dot{y} = -\frac{23}{4} + x - \frac{y}{20}, \text{ in } R_3,\\
&\dot{x} = -0.457007.. + 0.276952..x - 0.076768..y, && \dot{y} = -4.377702.. + x - 0.276952..y, \text{ in } R_4.
\end{aligned}
\tag{4.7}
$$

The linear differential centers in (4.7) have the first integrals

$$
\begin{aligned}
H_1(x,y) &= x^2 + x(-2292.643280.. + y) + y(-91.473702.. + 7.515818..y),\\
H_2(x,y) &= x^2 + x(0.920117.. + 0.398872..y) + (0.641188.. + 0.051960..y)yx^2\\
&\quad + x(0.920117.. + 0.398872..y) + (0.641188.. + 0.051960..y)y,\\
H_3(x,y) &= 2x(-23 + 2x) - \frac{2}{5}(40 + x)y + \frac{13}{50}y^2,\\
H_4(x,y) &= x^2 + x(-8.755405.. - 0.553904..y) + (0.914014.. + 0.076768..y)y,
\end{aligned}
$$

respectively In order to have crossing limit cycles of type 4 and 5, simultaneously, such that the crossing limit cycles of type 4 intersect the discontinuity curve $\tilde{\Sigma}_0$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, x_2^2)$, $p_3 = (x_3, 0)$ and $p_4 = (x_4, 0)$, with $x_2 < 0 < x_1$ and $x_3 < 0 < x_4$, and the crossing limit cycles of type 5 intersect the discontinuity curve $\tilde{\Sigma}_0$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, x_6^2)$, $p_7 = (x_7, 0)$ and $p_8 = (x_8, 0)$, with $0 < x_6 < x_5$ and $0 < x_7 < x_8$, these points must satisfy systems (4.2) and (4.5), respectively. Considering the piecewise linear differential center (4.7) systems (4.2) and (4.5) become

$$
\begin{aligned}
&x_1(-9170.573120.. + x_1(-361.894811.. + x_1(3.999999.. + 30.063275..x_1)))\\
&\quad + x_2(9170.573120.. + x_2(361.894811.. + (-3.999999.. - 30.063275..x_2)x_2)) = 0,\\
&x_2(3.680468.. + x_2(6.564754.. + (1.595489.. + 0.207843..x_2)x_2)) - 3.680468..x_3 - 4x_3^2 = 0,\\
&(x_3 - x_4)(-23 + 2x_3 + 2x_4) = 0,\\
&x_1(35.021620.. + x_1(-7.656056.. + (2.215618.. - 0.307072..x_1)x_1)) - 35.021620..x_4 + 4x_4^2 = 0,\\
&x_5(-9170.573120.. + x_5(-361.894811.. + x_5(3.999999.. + 30.063275..x_5)))\\
&\quad + x_6(9170.573120.. + x_6(361.894811.. + (-3.999999.. - 30.063275..x_6)x_6)) = 0,\\
&x_6(-35.021620.. + x_6(7.656056.. + (-2.215618.. + 0.307072..x_6)x_6)) + 35.021620..x_7 - 4x_7^2 = 0,\\
&(x_7 - x_8)(-23 + 2x_7 + 2x_8) = 0,\\
&x_5(35.021620.. + x_5(-7.656056.. + (2.215618.. - 0.307072..x_5)x_5)) - 35.021620..x_8 + 4x_8^2 = 0.
\end{aligned}
\tag{4.8}
$$

In this case system (4.8) has four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5, x_6, x_7, x_8)$, that satisfy the necessary conditions to have crossing limit cycles of types 4 and 5. Namely, $q^1 = (8, -16/5, -3, 29/2, 6, 3, 16/5, 83/10)$; $q^2 = (823/100, -413/100, -96/25, 767/50, 6, 3, 16/5,$

83/10); $q^3 = (841/100, -4.737905.., -4.516438.., 16.016438.., 6.040228.., 2.934482.., 3.093430.., 8.406569..)$ and $q^4 = (429/50, -5.236369.., -5.170738.., 16.670738.., 6.040228.., 2.934482.., 3.093430.., 8.406569..)$. These solutions provide four crossing limit cycles of type 4 and two crossing limit cycles of type 5 of the piecewise linear differential center (4.7). See these crossing limit cycles in Figure 4.3. Here we observed that we obtain a total of six crossing limit cycles between limit cycles of type 4 and of type 5, moreover these six crossing limit cycles have the configuration $(4,2)$, this is, 4-crossing limit cycle of type 4 and 2-crossing limit cycles of type 5. We know that this lower bound for the maximum number of crossing limit cycles of types 4 and 5 simultaneously, could be also obtained with the configuration $(3,3)$. But if we previously fixing two limit cycles of each type after several numeric computations we could not build a third limit cycle of type 5, then we only get those lower bound with the configuration $(4,2)$. $\square$

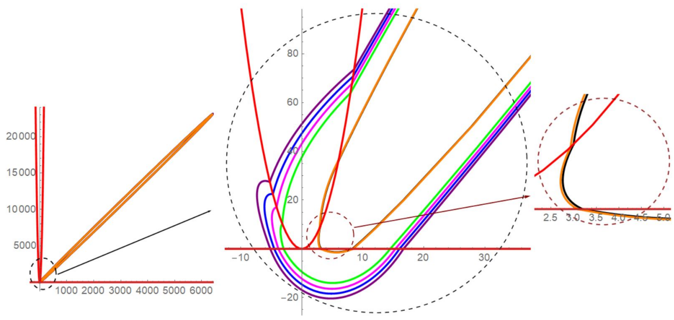

Figure 4.3: Four crossing limit cycles of type 4 and two crossing limit cycles of type 5 (black and orange) of the discontinuous piecewise linear differential system (4.7). These limit cycles are traveled in counterclockwise.

5 Proof of Theorem 1.4

Proof of statement (a) of Theorem 1.4. We consider the following piecewise linear differential center

$$\begin{aligned}
&\dot{x} = -0.678037.. + 0.111302..x - 0.025436..y, && \dot{y} = -3.106005.. + x - 0.111302..y, \text{ in } R_1,\\
&\dot{x} = -0.133244.. + 0.232759..x - 0.058573..y, && \dot{y} = -0.290609.. + x - 0.232759..y, \text{ in } R_2,\\
&\dot{x} = 3.074032.. + 0.434135..x - 2.713559..y, && \dot{y} = -3.035258.. + x - 0.434135..y, \text{ in } R_4,\\
&\dot{x} = 1.427543.. + 0.059092..x - 0.651180..y, && \dot{y} = -1.450367.. + x - 0.059092..y, \text{ in } R_5.
\end{aligned} \tag{5.1}$$

The linear differential centers in (5.1) have the first integrals

$$\begin{aligned}
H_1(x,y) &= x^2 + x(-6.212010.. - 0.222604..y) + (1.356074.. + 0.025436..y)y,\\
H_2(x,y) &= x^2 + x(-0.581218.. - 0.465518..y) + (0.266488.. + 0.058573..y)y,\\
H_4(x,y) &= x^2 + x(-6.070516.. - 0.868271..y) + y(-6.148064.. + 2.713559..y),\\
H_5(x,y) &= x^2 + x(-2.900734.. - 0.118185..y) + (-2.855087.. + 0.651180..y)y,
\end{aligned}$$

respectively. In order to have a crossing limit cycle of type 6^+, which intersects the discontinu-

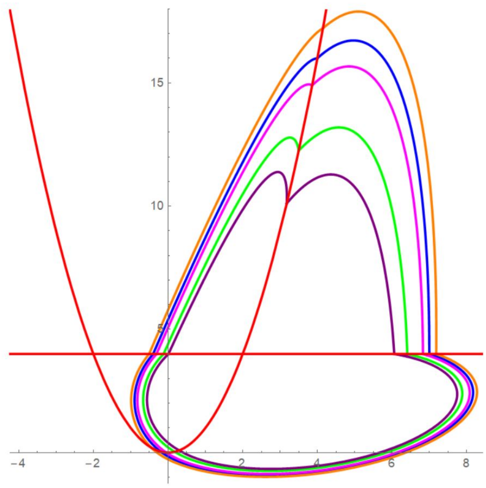

Figure 5.1: Five crossing limit cycles of type 6^+ of the discontinuous piecewise linear differential system (5.1). These limit cycles are traveled in counterclockwise.

ity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, these points must satisfy system

$$\begin{aligned} H_1(x_4,\ k) &= H_1(x_1, x_1^2), \\ H_2(x_1, x_1^2) &= H_2(x_2, k), \\ H_5(x_2,\ k) &= H_5(x_3, x_3^2), \\ H_4(x_3, x_3^2) &= H_4(x_4, k). \end{aligned} \tag{5.2}$$

Considering piecewise linear differential center (5.1) and $k = 4$, system (5.2) becomes

$$\begin{aligned} &-8.012495.. + x_1(-2.324875.. + x_1(5.065954.. + (-1.862075.. + 0.234292..x_1)x_1)) \\ &\qquad + (9.773178.. - 3.999999..x_2)x_2 = 0, \\ &-1.001459.. + (-3.373476.. + x_2)x_2 + x_3(2.900734.. + x_3(1.855087.. + (0.118185.. \\ &\qquad -0.651180..x_3)x_3)) = 0, \\ &-75.298768.. + x_3(-24.282066.. + x_3(-20.592258.. + x_3(-3.473086.. \\ &\qquad +10.854237..x_3))) + (38.174413.. - 4x_4)x_4 = 0, \\ &23.325149.. + x_1(24.848040.. + x_1(-9.424297.. + (0.890418.. - 0.101747..x_1)x_1)) \\ &\qquad + x_4(-28.409714.. + 4x_4) = 0, \end{aligned} \tag{5.3}$$

In this case system (5.3) has five real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i)$ that satisfy the conditions $-2 < x_2^i < 2 < x_1^i$ and $-2 < x_3^i < 2 < x_4^i$. We have $q^1 = (4, -2/5, -1/5, 7)$; $q^2 = (193/50, -31/100, -1/20, 683/100)$; $q^3 = (7/2, -3/25, 9/50, 641/100)$; $q^4 = (159/50, 1/100, 3/10, 303/50)$ and $q^5 = (4.149236.., -0.507154.., -0.449658.., 7.185104..)$, which provide five crossing limit cycles of type 6^+ of the piecewise linear differential center (5.1). See these crossing limit cycles in Figure 5.1.

Here we observe that there is a duality between the crossing limit cycles of type 6^+ that intersect the discontinuity curve $\tilde{\Sigma}_4$ and the crossing limit cycles of type 1 for the family $\mathcal{F}_3$ that intersect the discontinuity curve Σ_0 studied in statement (b) of Theorem 1.1, where we also got five crossing limit cycles, see Figures 2.3 and 5.1. □

Proof of statement (b) of Theorem 1.4. We consider the following piecewise linear differential center

$$
\begin{aligned}
&\dot{x} = 3 + \frac{x}{4} - \frac{17}{16}y, && \dot{y} = \frac{21}{20} + x - \frac{y}{4}, \text{ in } R_2,\\
&\dot{x} = 3.601959.. - x - 5.323060..y, && \dot{y} = -\frac{36}{25} + x + y, \text{ in } R_4,\\
&\dot{x} = \frac{11827667}{24434928} - \frac{91445}{6205696}x - \frac{8433175}{977739712}y, && \dot{y} = \frac{26369}{1108160} + x + \frac{91445}{6205696}y, \text{ in } R_5.
\end{aligned} \tag{5.4}
$$

The linear differential centers in (5.4) have the first integrals

$$
\begin{aligned}
H_2(x,y) &= \frac{2}{5}x(21+10x) - 2(12+x)y + \frac{17}{4}y^2,\\
H_4(x,y) &= x^2 + x\left(-\frac{72}{25} + 2y\right) + y(-7.203918.. + 5.323060..y),\\
H_5(x,y) &= 977397120x^2 + 63x(738332 + 457225y) + 10y(-94621336 + 8433175y),
\end{aligned}
$$

respectively. In order to have a crossing limit cycle of type 7, which intersects the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, k)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, x_4^2)$, these points must satisfy system

$$
\begin{aligned}
H_2(x_1,\ k) &= H_2(x_2,\ k),\\
H_5(x_2,\ k) &= H_5(x_3, x_3^2),\\
H_4(x_3, x_3^2) &= H_4(x_4, x_4^2),\\
H_5(x_4, x_4^2) &= H_5(x_1,\ k).
\end{aligned} \tag{5.5}
$$

In this case considering $k = 4$, system (5.5) becomes

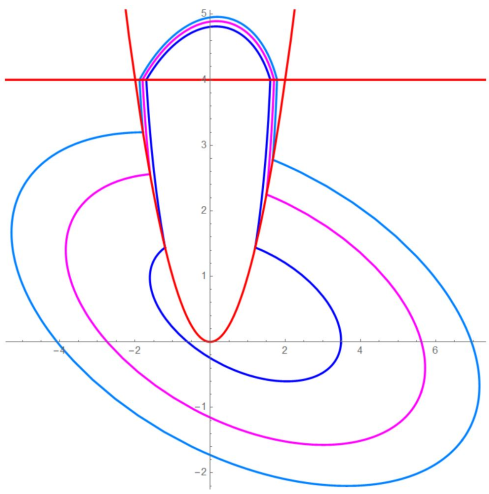

Figure 5.2: Three crossing limit cycles of type 7 of the discontinuous piecewise linear differential system (5.4). These limit cycles are traveled in counterclockwise.

$$\begin{aligned}
4(x_1-x_2)\left(\frac{1}{10}+x_1+x_2\right)&=0,\\
-2435545440+4032x_2(40113+242410x_2)-x_3(46514916+5x_3(6236752&\\
+5x_3(1152207+3373270x_3)))&=0,\\
x_3\left(-\frac{288}{25}+x_3(-24.815674..+x_3(8+21.292240..x_3))\right)&\\
+x_4\left(-\frac{288}{25}+x_4(24.815674..+(-8-21.292240..x_4)x_4)\right)&=0,\\
2435545440-4032x_1(40113+242410x_1)+x_4(46514916+5x_4(6236752&\\
+5x_4(1152207+3373270x_4)))&=0.
\end{aligned} \tag{5.6}$$

System (5.6) has three real solutions $q^i=(x_1^i,x_2^i,x_3^i,x_4^i)$ that satisfy the conditions $-2<x_2^i<x_1^i<2$ and $-2<x_3^i<x_4^i<2$. They are $q^1=(17/10,-9/5,-8/5,3/2)$; $q^2=(8/5,-17/10,-6/5,6/5)$ and $q^3=(89/50,-47/25,-1.788665..,1.667136..)$, which provide three crossing limit cycles of type 7 of the piecewise linear differential center (5.4). See these three crossing limit cycles in Figure 5.2. □

Proof of statement (c) of Theorem 1.4. We consider the following piecewise linear differential center

$$\begin{aligned}
\dot{x}&=-0.228658..+0.153388..x-0.043263..y, & \dot{y}&=-1.233713..+x-0.153388..y, \text{ in } R_1,\\
\dot{x}&=\frac{52}{5}+x-5y, & \dot{y}&=2+x-y, \text{ in } R_2,\\
\dot{x}&=-0.208786..-0.135584..x-0.040106..y, & \dot{y}&=1.549735+x+0.135584..y, \text{ in } R_3,\\
\dot{x}&=2-\frac{x}{2}-\frac{5}{4}y, & \dot{y}&=-\frac{41}{20}+x+\frac{y}{2}, \text{ in } R_4.
\end{aligned} \tag{5.7}$$

The linear differential centers in (5.7) have the first integrals

$$\begin{aligned}
H_1(x,y)&=15298879995x^2+5y(1399284923+132375500y)-6x(6291478429+782226050y),\\
H_2(x,y)&=4x(4+x)-\frac{8}{5}(52+5x)y+20y^2,\\
H_3(x,y)&=57070082030x^2+15y(1588730299+152593500y)+x(176887019081\\
&\quad+15475638300y),\\
H_4(x,y)&=4x^2+x\left(-\frac{82}{5}+4y\right)+y(-16+5y),
\end{aligned}$$

respectively. In order to have a crossing limit cycle of type 8, which intersects the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1=(x_1,k)$, $p_2=(x_2,x_2^2)$, $p_3=(x_3,x_3^2)$ and $p_4=(x_4,k)$, these points must satisfy system

$$\begin{aligned}
H_2(x_1,x_1^2)&=H_2(x_2,x_2^2),\\
H_3(x_2,x_2^2)&=H_3(x_3,\ k),\\
H_4(x_3,\ k)&=H_4(x_4,\ k),\\
H_1(x_4,\ k)&=H_1(x_1,x_1^2).
\end{aligned} \tag{5.8}$$

In this case considering $k=4$, system (5.8) becomes

$$
\begin{aligned}
(x_1-x_2)(-1+5x_1+5x_2)(-20+x_1(-1+5x_1)+x_2(-1+5x_2))&=0,\\
\frac{2}{28535041015}(-131946257940+x_2(176887019081+5x_2(16180207303&\\
+60x_2(51585461+7629675x_2))))-\frac{854345518x_3}{51046585}-4x_3^2&=0,\\
4(x_3-x_4)\left(-\frac{1}{10}+x_3+x_4\right)&=0,\\
\frac{8}{15298879995}(19287869230+x_1(18874435287-5x_1(2229530461&\\
+10x_1(-46933563+6618775x_1))))-\frac{1196239064x_4}{80946455}+4x_4^2&=0.
\end{aligned}
\tag{5.9}
$$

System (5.9) has four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i)$ that satisfy the conditions $x_3^i < -2 < 2 < x_4^i$ and $x_2^i < -2 < 2 < x_1^i$. They are $q^1 = (5/2, -23/10, -13/5, 27/10)$; $q^2 = (29/10, -27/10, -3, 31/10)$; $q^3 = (17/5, -16/5, -7/2, 18/5)$ and $q^4 = (98/25, -93/25, -203/50, 104/25)$ which provide four crossing limit cycles of type 8 of the piecewise linear differential center (5.7). See these four crossing limit cycles in Figure 5.3.

Here we observe that there is a duality between the crossing limit cycles for family $\mathcal{F}_4$ studied in Theorem 1.2, the crossing limit cycles of type 4 for the family $\mathcal{F}_5$ studied in statement (a) of Theorem 1.3 and crossing limit cycles of type 8 for the family $\mathcal{F}_6$ studied in statement (c) of Theorem 1.4. In these three cases we got four crossing limit cycles. See Figures 3.1, 4.1 and 5.3. □

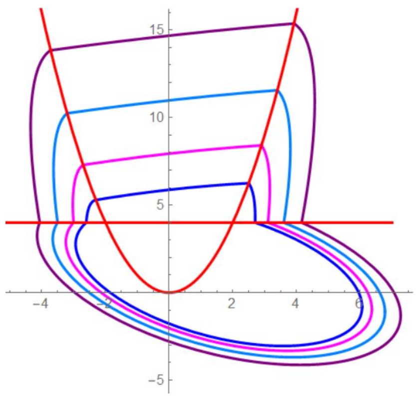

Figure 5.3: Four crossing limit cycles of type 8 of the discontinuous piecewise linear differential system (5.7). These limit cycles are traveled in counterclockwise.

Proof of statement (d) of Theorem 1.4. We consider the following piecewise linear differential center

$$
\begin{aligned}
\dot{x}&=\frac{243469}{1620885}+\frac{1826}{77185}x-\frac{9088}{324177}y, & \dot{y}&=-\frac{614289}{154370}+x-\frac{1826}{77185}y, \text{ in } R_1,\\
\dot{x}&=-0.229652..+\frac{7}{5}x-0.020472..y, & \dot{y}&=-1.718896..+x-\frac{7}{5}y, \text{ in } R_2,\\
\dot{x}&=1+\frac{9}{10}x-\frac{53}{50}y, & \dot{y}&=-\frac{1}{2}+x-\frac{9}{10}y, \text{ in } R_4.
\end{aligned}
\tag{5.10}
$$

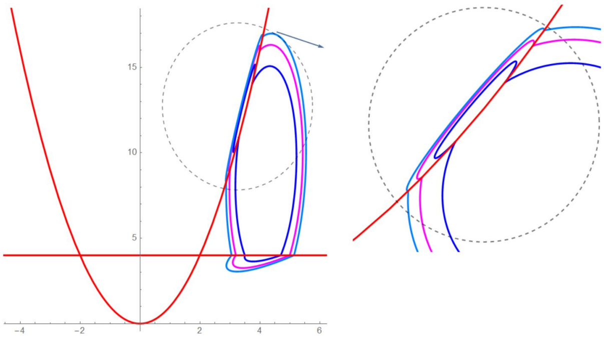

Figure 5.4: Three crossing limit cycles of type 9^+ of the discontinuous piecewise linear differential system (5.10). These limit cycles are traveled in counterclockwise.

The linear differential centers in (5.10) have the first integrals

$$\begin{aligned} H_1(x,y) &= 21x(-614289+77185x)-2(243469+38346x)y+45440y^2,\\ H_2(x,y) &= x^2+x\left(-3.437793..-\frac{14}{5}y\right)+(0.459305..+0.020472..y)y,\\ H_4(x,y) &= 4\left(x-\frac{9}{10}y\right)^2+y^2-4(x+2y), \end{aligned}$$

respectively. In order to have a crossing limit cycle of type 9^+, which intersects the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1=(x_1,x_1^2)$, $p_2=(x_2,x_2^2)$, $p_3=(x_3,k)$ and $p_4=(x_4,k)$, these points must satisfy system

$$\begin{aligned} H_2(x_1,x_1^2) &= H_2(x_2,x_2^2),\\ H_1(x_2,x_2^2) &= H_1(x_3,\ k),\\ H_4(x_3,\ k) &= H_4(x_4,\ k),\\ H_1(x_4,\ k) &= H_1(x_1,x_1^2). \end{aligned} \tag{5.11}$$

Considering $k=4$, system (5.11) becomes

$$\begin{aligned} x_1\left(-13.751172..+x_1\left(5.837222..+\left(-\frac{23}{25}+0.081889..x_1\right)x_1\right)\right)+x_2\,(13.751172..&\\ +x_2\left(-5.837222..+\left(-\frac{23}{25}-0.081889..x_2\right)x_2\right)\right) &= 0,\\ x_2(12900069-x_2(1133947-76692x_2+45440x_2^2))-3(406904+7(628897-77185x_3)x_3) &= 0,\\ 4(x_3-x_4)\left(-\frac{41}{5}+x_3+x_4\right) &= 0,\\ x_1(12900069-x_1(1133947-76692x_1+45440x_1^2))-3(406904+7(628897-77185x_4)x_4) &= 0, \end{aligned} \tag{5.12}$$

And we have that system (5.12) has three real solutions $q^i=(x_1^i,x_2^i,x_3^i,x_4^i)$ that satisfy the conditions $2<x_2^i<x_1^i$ and $2<x_3^i<x_4^i$. They are $q^1=(4,3,16/5,5)$; $q^2=(15/4,33/10,7/2,47/10)$ and $q^3=(41/10,2.879320..,3.058075..,5.141924..)$ which provide three crossing limit cycles of type 9^+ of the piecewise linear differential center (5.10). See these three crossing limit cycles in Figure 5.4.

Here we observe that there is a duality between the crossing limit cycles of type 3^+ for family $\mathcal{F}_3$ studied in statement (d) of Theorem 1.1, the crossing limit cycles of type 5 for the family $\mathcal{F}_5$ studied in statement (b) of Theorem 1.3 and crossing limit cycles of type 9^+ for the family $\mathcal{F}_6$ studied in statement (d) of Theorem 1.4. In these three cases we got three crossing limit cycles. See Figures 2.5, 4.2 and 5.4. □

6 Proof of Theorem 1.5

Proof of statement (a) of Theorem 1.5. We consider the following discontinuous piecewise linear differential system

$$
\begin{aligned}
&\dot{x} = 0.751960.. - 0.008805..x - 0.043938..y, && \dot{y} = -1.117055.. + x + 0.008805..y, \text{ in } R_1,\\
&\dot{x} = -\frac{4701043}{7161144} - \frac{122761}{156650025}x + \frac{91946}{31330005}y, && \dot{y} = -\frac{42715283}{313300050} - x + \frac{122761}{156650025}y, \text{ in } R_2,\\
&\dot{x} = 0.041424.. - 0.228644..x - 0.115044..y, && \dot{y} = 2.030027.. + x + 0.228644..y, \text{ in } R_3,\\
&\dot{x} = 6.094659.. - 0.970562..x - 1.475325..y, && \dot{y} = -4.066695 + x + 0.970562..y, \text{ in } R_4,\\
&\dot{x} = -0.014046.. - 0.011408..x + 0.000796..y, && \dot{y} = -0.900270.. - x + 0.011408..y, \text{ in } R_5.
\end{aligned} \tag{6.1}
$$

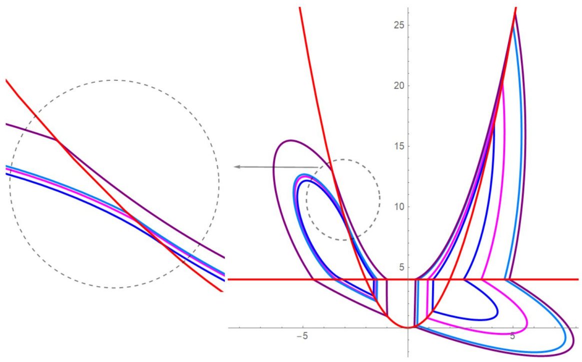

Figure 6.1: Four crossing limit cycles of type 6^+ in the right hand side and four crossing limit cycles of type 6^- in the left hand side, of the discontinuous piecewise linear differential system (6.1). These limit cycles are traveled in counterclockwise.

The linear differential centers in (6.1) have the first integrals

$$
\begin{aligned}
H_1(x,y) &= x^2 + x(-2.234111.. + 0.017610..y) + (-1.503920.. + 0.043938..y)y,\\
H_2(x,y) &= 626600100x^2 + x(170861132 - 982088y) + 5y(-164536505 + 367784y),\\
H_3(x,y) &= x^2 + x(4.060055.. + 0.457288..y) + (-0.082848.. + 0.115044..y)y,\\
H_4(x,y) &= x(-5448004792428006890183 + 6698319382773302134 20x) - 160y\\
&\quad (5102943483431243 6627 - 8126422570764957500x) + 988220002292252000000y^2,\\
H_5(x,y) &= 17172023317192110696x^2 + x(30918934250652233287 - 391817091205831000y)\\
&\quad + 6y(-80400672913407451 + 2279188834700000y),
\end{aligned}
$$

respectively. In order to have simultaneously crossing limit cycles of types 6^+ and 6^-, such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, with $-2 < x_2 < 2 < x_1$ and $-2 < x_3 < 2 < x_4$, and the crossing limit cycles of type 6^- intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, k)$, $p_7 = (x_7, x_7^2)$ and $p_8 = (x_8, k)$, with $x_5 < -2 < x_7 < 2$ and $x_6 < -2 < x_8 < 2$, these points must satisfy systems (5.2) and

$$
\begin{aligned}
H_3(x_5, x_5^2) &= H_3(x_6,\ k),\\
H_4(x_6,\ k) &= H_4(x_7, x_7^2),\\
H_5(x_7, x_7^2) &= H_5(x_8,\ k),\\
H_2(x_8,\ k) &= H_2(x_5, x_5^2),
\end{aligned}
\tag{6.2}
$$

respectively. Considering the piecewise linear differential center (6.1) and $k = 4$, systems (5.2) and (6.2) become

$$
\begin{aligned}
&170861132x_1 - 196082425x_1^2 - 982088x_1^3 + 1838920x_1^4 - 60(-54355123 + 2782213x_2\\
&\qquad +10443335x_2^2) = 0,\\
&-1710814021790578824 + 2935166585828909287x_2 + 1717202331719211069 6x_2^2\\
&\qquad -3091893425065223328 7x_3 - 16689619279711665990x_3^2 + 3918170912058310 00x_3^3\\
&\qquad -136751330082000 00x_3^4 = 0,\\
&-5448004792428006890183x_3 - 7494877635212659646900x_3^2 +\\
&\quad 1300227611322393200000x_3^3 + 988220002292252000000x_3^4 + 21(802253250346853687680\\
&\qquad -117663974827825757 23x_4 + 3189675896558715302 0x_4^2) = 0,\\
&-21.250638.. + 8.936444..x_1 + 2.015680..x_1^2 - 0.070440..x_1^3\\
&\qquad -0.175755..x_1^4 - 8.654682..x_4 + 4x_4^2 = 0,\\
&-6.037269.. + 16.240221..x_5 + 3.668606..x_5^2 + 1.829154..x_5^3\\
&\qquad +0.460177..x_5^4 - 23.556840..x_6 - 4x_6^2 = 0,\\
&1684731825728392744 1280 + 2470943471384340901 83x_6 - 669831938277330213420x_6^2\\
&\qquad -5448004792428006890183x_7 - 7494877635212659646900x_7^2\\
&\qquad +1300227611322393200000x_7^3 + 988220002292252000000x_7^4 = 0,\\
&3091893425065223328 7x_7 + 16689619279711665990x_7^2 - 3918170912058310 00x_7^3\\
&\qquad +136751330082000 00x_7^4 - 21(-81467334370979944 + 1397698375515662347x_8\\
&\qquad +817715396056767176x_8^2) = 0,\\
&-170861132x_5 + 196082425x_5^2 + 982088x_5^3 - 1838920x_5^4 + 60(-54355123 + 2782213x_8\\
&\qquad +10443335x_8^2) = 0.
\end{aligned}
\tag{6.3}
$$

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5^i, x_6^i, x_7^i, x_8^i)$ with $i = 1, 2, 3, 4$, for system (6.3) that satisfy the above conditions namely $q^1 = (5, 1/2, 9/50, 23/5, -18/5, -9/2, -49/50, -1)$; $q^2 = (9/2,\ 19/20,\ 91/100,\ 7/5,\ 3, -17/5, -303/200,\ -3/2)$; $q^3 = (41/10, 1.208958.., 1.176604.., 2.657283.., -2.816357.., -31/10, -1.626433.., -1.613770..)$, and $q^4 = (51/10, 0.368157.., 0.315951.., 4.829311.., -3.059352.., -7/2, -1.475955.., -1.460360..)$, these four solutions generated four crossing limit cycles of type 6^+ and four crossing limit cycles of type 6^-. See these crossing limit cycles of the piecewise linear differential center (6.1) in Figure 6.1.

Here we obtain a total of eight crossing limit cycles of types 6^+ and 6^- simultaneously, with a configuration $(4, 4)$. And observed that it is possible obtain this lower bound with the configurations $(5, 3)$ or $(3, 5)$, but here we only present the example with the configuration $(4, 4)$. □

Proof of statement (b) of Theorem 1.5. We consider the following discontinuous piecewise linear differential system

$$
\begin{aligned}
&\dot{x} = 1.717686.. + 0.650612..x - 0.423688..y, && \dot{y} = 0.850546.. + x - 0.650612..y, \text{ in } R_1,\\
&\dot{x} = 0.516832.. + 0.082481..x - 0.038759..y, && \dot{y} = 0.179926.. + x - 0.082481..y, \text{ in } R_2,\\
&\dot{x} = 1.470269.. + 0.406982..x - 3.640154..y, && \dot{y} = -0.122065.. + x - 0.406982..y, \text{ in } R_4,\\
&\dot{x} = 0.685228.. + 0.043300..x - 0.293631..y, && \dot{y} = 0.017396.. + x - 0.043300..y, \text{ in } R_5.
\end{aligned}
\tag{6.4}
$$

The linear differential centers in (6.4) have the first integrals

$$
\begin{aligned}
H_1(x,y) &= x^2 + x(1.701093.. - 1.301224..y) + (-3.435373.. + 0.423688..y)y,\\
H_2(x,y) &= x^2 + x(0.359853.. - 0.164963..y) + (-1.033664.. + 0.038759..y)y,\\
H_4(x,y) &= x^2 + x(-0.244130.. - 0.813965..y) + y(-2.940538.. + 3.640154..y),\\
H_5(x,y) &= x^2 + x(0.034792.. - 0.086601..y) + (-1.370456.. + 0.293631..y)y,
\end{aligned}
$$

respectively. In order to have simultaneously crossing limit cycles of types 6^+ and 7, such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, with $-2 < x_2 < 2 < x_1$ and $-2 < x_3 < 2 < x_4$, and the crossing limit cycles of type 7 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, k)$, $p_6 = (x_6, k)$, $p_7 = (x_7, x_7^2)$ and $p_8 = (x_8, x_8^2)$, with $-2 < x_6 < x_5 < 2$ and $-2 < x_7 < x_8 < 2$ these points must satisfy systems (5.2) and (5.5), respectively. Considering the piecewise linear differential center (6.4) and $k = 4$, systems (5.2) and (5.5) become

$$
\begin{aligned}
&14.058034.. + 1.439414..x_1 - 0.134656..x_1^2 - 0.659853..x_1^3 + 0.155036..x_1^4\\
&\qquad + \frac{6}{5}x_2 - 4x_2^2 = 0,\\
&-0.783728.. - 0.311613..x_2 + x_2^2 - 0.034792..x_3 + 0.370456..x_3^2 + 0.086601..x_3^3\\
&\qquad -0.293631..x_3^4 = 0,\\
&-185.921253.. - 0.976522..x_3 - 7.762153..x_3^2 - 3.255860..x_3^3 + 14.560616..x_3^4\\
&\qquad +13.999964..x_4 - 4x_4^2 = 0,\\
&-27.849933.. - 6.804375..x_1 + 9.741494..x_1^2 + 5.204898..x_1^3 - 1.694752..x_1^4\\
&\qquad -14.015217..x_4 + 4x_4^2 = 0,\\
&4(x_5 - x_6)\left(-\frac{3}{10} + x_5 + x_6\right) = 0,\\
&-0.783728.. - 0.311613..x_6 + x_6^2 - 0.034792..x_7 + 0.370456..x_7^2\\
&\qquad +0.086601..x_7^3 - 0.293631..x_7^4 = 0,\\
&-0.976522..x_7 - 7.762153..x_7^2 - 3.255860..x_7^3 + 14.560616..x_7^4\\
&\qquad +x_8(0.976522.. + 7.762153..x_8 + 3.255860..x_8^2 - 14.560616..x_8^3) = 0,\\
&-0.783728.. - 0.311613..x_5 + x_5^2 - 0.034792..x_8 + 0.370456..x_8^2\\
&\qquad +0.086601..x_8^3 - 0.293631..x_8^4 = 0.
\end{aligned}
\tag{6.5}
$$

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5, x_6, x_7, x_8)$ with $i = 1,2,3,4$, for system (6.5) that satisfy the above conditions. We have $q^1 = (4, -9/5, -19/10, 7/2, 1, -7/10, -9/10, 11/10)$; $q^2 = (106/25, -39/20, -1.975633.., 51/10, 1, -7/10, -9/10, 11/10)$; $q^3 = (413/100, -469/250, -1.938820.., 4.420122.., 101/100, -71/100, -941/1000, 1.132764..)$ and $q^4 =$

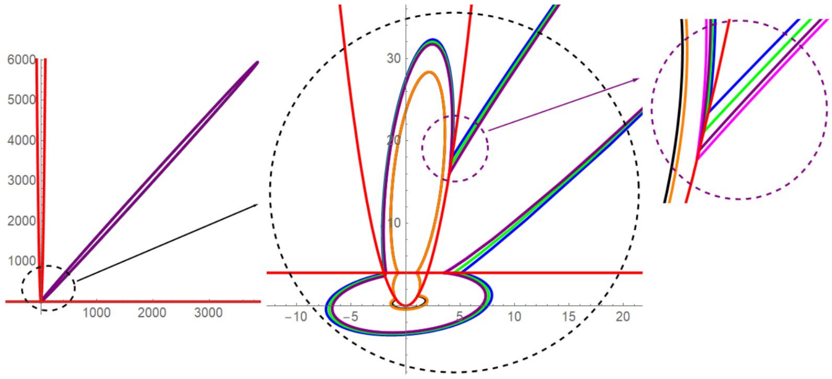

Figure 6.2: Four crossing limit cycles of type 6^+ and two crossing limit cycles of type 7 (black and orange) of the discontinuous piecewise linear differential system (6.4). These limit cycles are traveled in counterclockwise.

$(401/100, -1.805407.., -1.902798.., 3.579564.., 101/100, -71/100, -941/1000, 1.132764..)$. These four real solutions generated four crossing limit cycles of type 6^+ and two crossing limit cycles of type 7. See these crossing limit cycles of the piecewise linear differential center (6.4) in Figure 6.2.

Here we observed that we obtain a total of six crossing limit cycles between limit cycles of type 6^+ and of type 7, moreover these six crossing limit cycles have the configuration $(4, 2)$. We observe that this lower bound for the maximum number of crossing limit cycles of types 6^+ and 7 simultaneously, could be also obtained with the configuration $(3, 3)$. But if we previously fixing two limit cycles of type 6^+ after several numeric computations we could not build a third limit cycle of type 7, then we only get those lower bound with the configuration $(4, 2)$.

We can also observe that there is a duality between the case studied in statement (e) of Theorem 1.1, where we have studied simultaneously crossing limit cycles of types 1 and 2^+ and this case, where study the crossing limit cycles of types 6^+ and 7, simultaneously. In these two cases we got the configuration $(4, 2)$. See Figures 2.6 and 6.2. $\square$

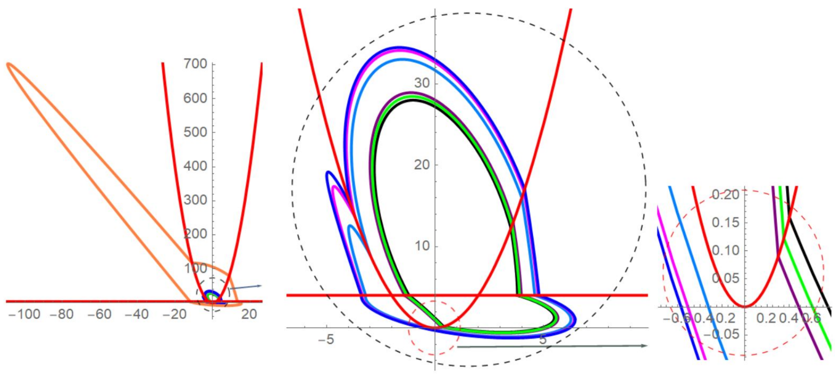

Figure 6.3: Three crossing limit cycles of type 6^+ (purple, green and black) and four crossing limit cycles of type 8 (orange, blue, magenta and light blue) of the discontinuous piecewise linear differential system (6.6). These limit cycles are traveled in counterclockwise.

Proof of statement (c) of Theorem 1.5. We consider the following discontinuous piecewise linear differential system

$$
\begin{aligned}
&\dot{x} = 0.212208.. - 0.051128..x - 0.004724..y, &&\dot{y} = -3.713538.. + x + 0.051128..y, \text{ in } R_1,\\
&\dot{x} = 0.592855.. - 0.098217..x - 0.044462..y, &&\dot{y} = -1.739750.. + x + 0.098217..y, \text{ in } R_2,\\
&\dot{x} = -0.324307.. - 0.152006..x - 0.023227..y, &&\dot{y} = 2.010345.. + x + 0.152006y, \text{ in } R_3,\\
&\dot{x} = 5.173755.. - 0.530837..x - 1.789344..y, &&\dot{y} = -2.823348.. + x + 0.530837..y, \text{ in } R_4,\\
&\dot{x} = 0.905547.. + \frac{9}{50}x + 0.037591..y, &&\dot{y} = -2.213772.. - x - \frac{9}{50}y, \text{ in } R_5.
\end{aligned}
\tag{6.6}
$$

The linear differential centers in (6.6) have the first integrals

$$
\begin{aligned}
H_1(x,y) &= 92350000x^2 + 2y(-19597489 + 218145y) + x(-685890524 + 9443461y),\\
H_2(x,y) &= x(-2350427721 + 675507095x) + 2(-400478067 + 66346510x)y + 30034700y^2,\\
H_3(x,y) &= x^2 + x(4.020691.. + 0.304014..y) + (0.648615.. + 0.023227..y)y,\\
H_4(x,y) &= 2.248715.. \times 10^{16}x^2 - 5x(2.539563.. \times 10^{16} - 4.774807.. \times 10^{15}y)\\
&\quad + y(-2.326860.. \times 10^{17} + 4.023727.. \times 10^{16}y),\\
H_5(x,y) &= -5.437818.. \times 10^{22}x^2 + 6x(-4.012698.. \times 10^{22} - 3.262691.. \times 10^{21}y)\\
&\quad + 5(-1.969681.. \times 10^{22} - 4.088345.. \times 10^{20}y)y,
\end{aligned}
$$

respectively. In order to have crossing limit cycles of types 6^+ and 8, simultaneously, such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, with $-2 < x_2 < 2 < x_4$ and $-2 < x_3 < 2 < x_1$, and the crossing limit cycles of type 8 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, x_6^2)$, $p_7 = (x_7, k)$ and $p_8 = (x_8, k)$, with $x_7 < -2 < 2 < x_8$ and $x_6 < -2 < 2 < x_5$, these points must satisfy systems (5.2) and (5.8), respectively. Considering the piecewise linear differential center (6.6) and $k = 4$, systems (5.2) and (5.8) become

$$
\begin{aligned}
&16.125777.. - 13.918004..x_1 - 0.742843..x_1^2 + 0.785738..x_1^3 + 0.177849..x_1^4\\
&\qquad +10.775049..x_2 - 4x_2^2 = 0,\\
&31.383400.. + 23.470181..x_2 + 4x_2^2 - 17.710181..x_3 - 11.244381..x_3^2 - \frac{36}{25}x_3^3\\
&\qquad -0.150367..x_3^4 = 0,\\
&51.042105.. - 22.586789..x_3 - 37.390043..x_3^2 + 4.246697..x_3^3 + 7.157379..x_3^4\\
&\qquad +5.599999..x_4 - 4x_4^2 = 0,\\
&-6.488327.. + 29.708306..x_1 - 2.302329..x_1^2 - 0.409029..x_1^3 - 0.018897..x_1^4\\
&\qquad -28.072189..x_4 + 4x_4^2 = 0,\\
&-149799272 - 648116680x_8 + 92350000x_8^2 + 685890524x_5 - 53155022x_5^2\\
&\qquad -9443461x_5^3 - 436290x_5^4 = 0,\\
&-2350427721x_5 - 125449039x_5^2 + 132693020x_5^3 + 30034700x_5^4\\
&\qquad +x_6(2350427721 + 125449039x_6 - 132693020x_6^2 - 30034700x_6^3) = 0,\\
&-11.864396.. + 16.082766..x_6 + 6.594461..x_6^2 + 1.216054..x_6^3 + 0.092909..x_6^4\\
&\qquad -20.946982..x_7 - 4x_7^2 = 0,\\
&(x_8 - x_7)(-7 + 5x_8 + 5x_7) = 0.
\end{aligned}
\tag{6.7}
$$

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5^i, x_6^i, x_7^i, x_8^i)$ with $i = 1, 2, 3, 4$, for system (6.7) that satisfy the above conditions. We have $q^1 = (7/2, -6/5, 2/5, 19/5, 4, -3, -16/5, 23/5)$;

$q^2 = (18/5, -7/5, 3/10, 199/50, 41/10, -37/10, -3351/1000, 4751/1000)$; $q^3 = (71/20, -1.299400.., 7/20, 3.893976.., 4.132430.., -3.871790.., -17/5, 24/5)$ and $q^4 = (71/20, -1.299400.., 7/20, 3.893976.., 178349/20000, 108083/10000, -119/10, 133/10)$. These four real solutions generated three crossing limit cycles of type 6^+ and four crossing limit cycle of type 8. See these crossing limit cycles of the piecewise linear differential center (6.6) in Figure 6.3.

Here we observed that we obtain a total of seven crossing limit cycles between limit cycles of type 6^+ and of type 8, moreover in this example, the seven crossing limit cycles have the configuration $(3, 4)$. We observe that this lower bound for the maximum number of crossing limit cycles of types 6^+ and 8 simultaneously, could be also obtained with the configurations $(4, 3)$. And we obtain a example with this configuration in the proof of statement (b) of Theorem 1.6 with piecewise linear differential center (7.3), see Figure 7.2. □

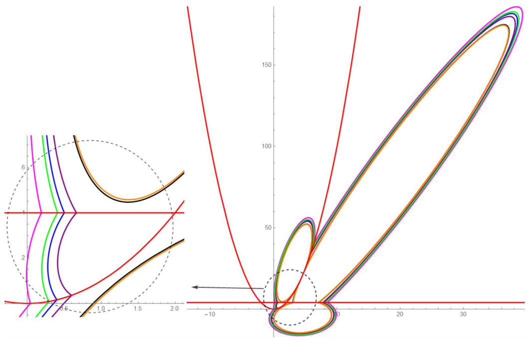

Figure 6.4: Four crossing limit cycles of type 6^+ and two crossing limit cycles of type 9^+ (black and orange) of the discontinuous piecewise linear differential system (6.8). These limit cycles are traveled in counterclockwise.

Proof of statement (d) of Theorem 1.5. We consider the following discontinuous piecewise linear differential system

$$\begin{aligned}
\dot{x} &= -0.478750.. + 0.183274..x - 0.037189..y, & \dot{y} &= -4.300673.. + x - 0.183274..y, \text{ in } R_1,\\
\dot{x} &= 0.122511.. + 0.079715..x - 0.013506..y, & \dot{y} &= -1.007263.. + x - 0.079715..y, \text{ in } R_2,\\
\dot{x} &= -1.261810.. + 0.053348..x - 0.212413..y, & \dot{y} &= -4.836606.. + x - 0.053348..y, \text{ in } R_4,\\
\dot{x} &= 0.060157.. + 0.062627..x - 0.047729..y, & \dot{y} &= -0.739728.. + x - 0.062627..y, \text{ in } R_5.
\end{aligned} \tag{6.8}$$

The linear differential centers in (6.8) have the first integrals

$$\begin{aligned}
H_1(x, y) &= x^2 + x(-8.601346.. - 0.366548..y) + (0.957501401147845' + 0.037189..y)y,\\
H_2(x, y) &= x^2 + x(-2.014527.. - 0.159430..y) + (-0.245022.. + 0.013506..y)y,\\
H_4(x, y) &= x^2 + x(-9.673213.. - 0.106696..y) + (2.523620.. + 0.212413..y)y,\\
H_5(x, y) &= x^2 + x(-1.479456.. - 0.125255..y) + (-0.120314.. + 0.047729..y)y,
\end{aligned}$$

respectively. In order to have simultaneously crossing limit cycles of types 6^+ and 9^+, such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, with $-2 < x_2 < 2 < x_4$ and $-2 < x_3 < 2 < x_1$, and the crossing limit cycles of type 9^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, x_6^2)$, $p_7 = (x_7, k)$ and $p_8 = (x_8, k)$, with $2 < x_6 < x_5$ and $2 < x_7 < x_8$, these points must satisfy systems (5.2) and (5.11), respectively.

Considering the piecewise linear differential center (6.8) and $k=4$, systems (5.2) and (5.11) become

$$\begin{aligned}
&3.055923..+x_1(-8.058108..+x_1(3.019909..+(-0.637722..+0.054027..x_1)x_1))\\
&\qquad+(10.608997..-4x_2)x_2=0,\\
&0.282412..+(-1.980480..+x_2)x_2+x_3(1.479456..+x_3(-0.879685..+(0.125255..\\
&\qquad-0.047729..x_3)x_3))=0,\\
&-53.972411..+x_3(-38.692854..+x_3(14.094480..+(-0.426786..+0.849655..x_3)x_3))\\
&\qquad+(40.4000000..-3.999999..x_4)x_4=0,\\
&17.700131..+x_1(34.405384..+x_1(-7.8300056..+(1.466193..-0.148756..x_1)x_1))\\
&\qquad+x_4(-40.270159..+4x_4)=0,\\
&-8.058108..x_5+3.019909..x_5^2-0.637722..x_5^3+0.054027..x_5^4+x_6(8.058108..\\
&\qquad-3.019909..x_6+0.637722..x_6^2-0.054027..x_6^3)=0,\\
&-17.700131..-34.405384..x_6+7.830005..x_6^2-1.466193..x_6^3+0.148756..x_6^4\\
&\qquad+40.270159..x_7-4x_7^2=0,\\
&4(x_7-x_8)(-10.100000..+x_7+x_8)=0,\\
&17.700131..+34.405384..x_5-7.830005..x_5^2+1.466193..x_5^3-0.148756..x_5^4\\
&\qquad-40.270159..x_8+4x_8^2=0.
\end{aligned}\tag{6.9}$$

We have four real solutions $q^i=(x_1^i,x_2^i,x_3^i,x_4^i,x_5,x_6,x_7,x_8)$ with $i=1,2,3,4$, for system (6.9) that satisfy the above conditions. We have $q^1=(6,1/2,4/10,8,5,14/5,3,71/10)$; $q^2=(317/50,\ 19/100,\ 1/25,\ 423/50,\ 5,\ 14/5,\ 3,\ 71/10)$; $q^3=(291/50,\ 0.664193..,\ 3/5, 7.554404..,$ $487/100, 3.986608.., 3.058022.., 7.041977..)$ and $q^4=(61/10, 0.409425.., 0.293958.., 8.128324..,$ $487/100, 3.986608.., 3.058022..,\ 7.041977..)$ These four real solutions generated four crossing limit cycles of type 6^+ and two crossing limit cycles of type 9^+. See these crossing limit cycles of the piecewise linear differential center (6.8) in Figure 6.4.

Here we obtain a total of six crossing limit cycles between limit cycles of type 6^+ and of type 9^+, moreover these six crossing limit cycles have the configuration $(4,2)$. We observed that this lower bound for the maximum number of crossing limit cycles of types 6^+ and 9^+ simultaneously, could be also obtained with the configuration $(3,3)$. But if we build two crossing limit cycles of type 6^+ and two of type 9^+, simultaneously, we have that all the parameters that appear in system (5.11) are determined, where this system is such that generated limit cycles of type 9^+, then it is no possible to build a third crossing limit cycle of type 9^+ and therefore we can not obtain the configuration $(3,3)$. □

Proof of statement (e) of Theorem 1.5. We consider the following discontinuous piecewise linear differential system

$$\begin{aligned}
&\dot{x}=-0.147861..+0.083875..x-0.018000..y, && \dot{y}=-3.106437..+x-0.083875..y,\ \text{in } R_1,\\
&\dot{x}=\frac{7769951}{9492348}+\frac{176465}{2373087}x-\frac{204250}{2373087}y, && \dot{y}=\frac{6997939}{47461740}+x-\frac{176465}{2373087}y,\ \text{in } R_2,\\
&\dot{x}=-0.284659..-0.174915..x-0.046689..y, && \dot{y}=1.660380..+x+0.174915..y,\ \text{in } R_3,\\
&\dot{x}=-\frac{3871251}{31913000}+\frac{3}{10}x-\frac{4335}{31913}y, && \dot{y}=-\frac{19}{20}+x-\frac{3}{10}y,\ \text{in } R_4,\\
&\dot{x}=0.206531..+0.150466..x-0.054352..y, && \dot{y}=0.451143..+x-0.150466..y,\ \text{in } R_5.
\end{aligned}\tag{6.10}$$

The linear differential centers in (6.10) have the first integrals

$$\begin{aligned}
H_1(x,y) &= (58546435625x^2 + 4y(4328392296 + 263466775y) - 15x(24249448597 \\
&\quad + 654747306y), \\
H_2(x,y) &= x(6997939 + 23730870x) - 5(7769951 + 705860x)y + 2042500y97^2, \\
H_3(x,y) &= 1.054579.. \times 10^{-58}(3.792980.. \times 10^{58}x^2 + y(2.15949717.. \times 10^{58} \\
&\quad + 1.770939.. \times 10^{57}y) + x(1.259558..10^{59} + 1.326899.. \times 10^{58}y)), \\
H_4(x,y) &= 4x^2 + \frac{2}{5}x(19-6y) + \frac{3y(129041797 + 722500y)}{3989125}, \\
H_5(x,y) &= 16x(472818597 + 524021995x) - 75(46176919 + 33641680x)y + 45571250097y^2,
\end{aligned}$$

respectively. In order to have crossing limit cycles of types 7 and 8, simultaneously, such that the crossing limit cycles of type 7 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, k)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, x_4^2)$, with $-2 < x_2 < x_1 < 2$ and $-2 < x_3 < x_4 < 2$, and the crossing limit cycles of type 8 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, x_6^2)$, $p_7 = (x_7, k)$ and $p_8 = (x_8, k)$, with $x_6 < -2 < 2 < x_5$ and $x_7 < -2 < 2 < x_8$, these points must satisfy systems (5.5) and (5.8), respectively. Considering the piecewise linear differential center (6.10) and $k = 4$, systems (5.5) and (5.8) become

$$\begin{aligned}
4(x_1 - x_2)\left(-\frac{3}{10} + x_1 + x_2\right) &= 0, \\
-6561675700 - 2527406448x_2 + 8384351920x_2^2 - 7565097552x_3 & \\
-4921082995x_3^2 + 2523126000x_3^3 - 455712500x_3^4 &= 0, \\
30317350x_3 + 19827751x_3^2 - 9573900x_3^3 + 2167500x_3^4 & \\
-x_4(30317350 + 19827751x_4 - 9573900x_4^2 + 2167500x_4^3) &= 0, \\
6561675700 + 2527406448x_1 - 8384351920x_1^2 + 7565097552x_4 & \\
+4921082995x_4^2 - 2523126000x_4^3 + 455712500x_4^4 &= 0, \\
86116150336 - 403026567315x_8 + 58546435625x_8^2 + 363741728955x_5 & \\
-75860004809x_5^2 + 9821209590x_5^3 - 1053867100x_5^4 &= 0, \\
6997939x_5 - 15118885x_5^2 - 3529300x_5^3 + 2042500x_5^4 + x_6(-6997939 & \\
+15118885x_6 + 3529300x_6^2 - 2042500x_6^3) &= 0, \\
-1.030050.. + 8(1.660379.. + 0.284660..x_6)x_6 + 4(1 + 0.174915..x_6)^2x_6^2 & \\
+0.064378..x_6^4 + 8(-1.138640.. - 1.660379..x_7) - 4(0.699661.. + x_7)^2 &= 0, \\
-4(x_8 - x_7)\left(-\frac{1}{2} + x_8 + x_7\right) &= 0.
\end{aligned} \tag{6.11}$$

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5^i, x_6^i, x_7^i, x_8^i)$ with $i = 1, 2, 3, 4$, for system (6.11) that satisfy the above conditions. We have $q^1 = (1,\ -7/10, -9/10, -1/10,\ 37/10,\ -5/2, -3, 7/2)$; $q^2 = (1, -7/10, -9/10, -1/10, 4,\ -29/10,\ -33/10,\ 19/5)$; $q^3 = (11/10,\ -8/10, -26/25,\ 1/10,\ 21/5,\ -157/100,\ -7/2, 4)$ and $q^4 = (1.194602..,\ -0.894602.., -1.147986.., 0.273096.., 87/20, -3.312719.., -3653/1000, 4153/1000)$. These four real solutions generated three crossing limit cycles of type 7 and four crossing limit cycle of type 8. See these crossing limit cycles of the piecewise linear differential center (6.10) in Figure 6.5.

Here we obtain a total of seven crossing limit cycles between limit cycles of type 7 and of type 8, moreover these seven crossing limit cycles have the configuration $(3, 4)$. By our numer-

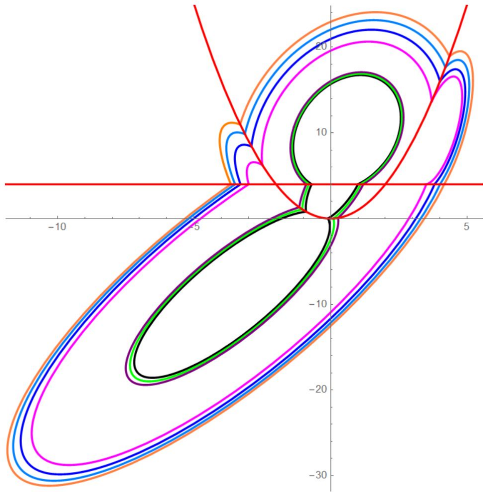

Figure 6.5: Three crossing limit cycles of type 7 (purple, green and black) and four crossing limit cycles of type 8 of the discontinuous piecewise linear differential system (6.10). These limit cycles are traveled in counterclockwise.

ical computations we observed that this lower bound for the maximum number of crossing limit cycles of types 7 and 8 simultaneously, could not be obtained with the configuration $(4,3)$, because in the statement (b) of Theorem 1.4 we only got three crossing limit cycle of type 7. □

Proof of statement (f) of Theorem 1.5. We consider the following discontinuous piecewise linear differential system

$$
\begin{aligned}
&\dot{x} = -0.224106.. + 0.256615..x - 0.075244..y, && \dot{y} = -3.489877.. + x - 0.256615..y, \text{ in } R_1,\\
&\dot{x} = 33.031408.. - \frac{x}{2} - 5.321982..y, && \dot{y} = -816.418879.. + x + \frac{y}{2}, \text{ in } R_2,\\
&\dot{x} = -0.151463.. - 0.173662..x - 0.047290..y, && \dot{y} = 0.297861.. + x + 0.173662..y, \text{ in } R_3,\\
&\dot{x} = 2 + \frac{x}{20} - \frac{13}{200}y, && \dot{y} = -\frac{111}{20} + x - \frac{y}{20}, \text{ in } R_4.
\end{aligned}
\tag{6.12}
$$

The linear differential centers in (6.12) have the first integrals

$$
\begin{aligned}
H_1(x,y) &= x^2 + x(-6.979755.. - 0.513231..y) + (0.448213.. + 0.075244..y)y,\\
H_2(x,y) &= x^2 + x(-1632.837759.. + y) + y(-66.062816.. + 5.321982..y),\\
H_3(x,y) &= x^2 + x(0.595723.. + 0.347324..y) + (0.302926.. + 0.047290..y)y,\\
H_4(x,y) &= 4x^2 - 16y + \frac{13}{50}y^2 - \frac{2}{5}x(111 + y),
\end{aligned}
$$

respectively. In order to have simultaneously crossing limit cycles of types 8 and 9^+, such that the crossing limit cycles of type 8 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, x_2^2)$, $p_3 = (x_3, k)$ and $p_4 = (x_4, k)$, with $x_2 < -2 < 2 < x_1$ and $x_3 < -2 < 2 < x_4$, and the crossing limit cycles of type 9^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, x_6^2)$, $p_7 = (x_7, k)$ and $p_8 = (x_8, k)$, with $2 < x_6 < x_5$ and $2 < x_7 < x_8$, these points must satisfy systems (5.8) and (5.11), respectively. Considering the piecewise linear differential center (6.12) and $k = 4$, systems (5.8) and (5.11) become

$$
\begin{aligned}
-6531.351039..x_1 - 260.251264..x_1^2 + 4x_1^3 + 21.287931..x_1^4 + x_2(6531.351039..&\\
+260.251264..x_2 - 4x_2^2 - 21.287931..x_2^3) &= 0,\\
-7.873414.. + 2.382895..x_2 + 5.211706..x_2^2 + 1.389297..x_2^3 + 0.189161..x_2^4&\\
-7.940084..x_3 - 4x_3^2 &= 0,\\
4(x_3 - x_4)\left(-\frac{23}{2} + x_3 + x_4\right) &= 0,\\
11.987037.. + 27.919023..x_1 - 5.792854..x_1^2 + 2.052924..x_1^3 - 0.300976..x_1^4&\\
-36.130722..x_4 + 4x_4^2 &= 0\\
x_5(-6531.351039.. + x_5(-260.251264.. + x_5(4 + 21.287931..x_5))) + x_6(6531.351039..&\\
+x_6(260.251264.. + (-4 - 21.287931..x_6)x_6)) &= 0,\\
-11.987037.. + x_6(-27.919023.. + x_6(5.792854.. + (-2.052924.. + 0.300976..x_6)x_6))&\\
+(36.130722.. - 4x_7)x_7 &= 0,\\
4(x_7 - x_8)\left(-\frac{23}{2} + x_7 + x_8\right) &= 0,\\
11.987037.. + x_5(27.919023.. + x_5(-5.792854.. + (2.052924.. - 0.300976..x_5)x_5))&\\
+x_8(-36.130722.. + 4x_8) &= 0,
\end{aligned}
\tag{6.13}
$$

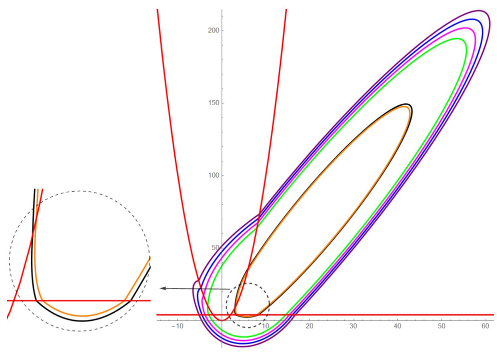

Figure 6.6: Four crossing limit cycles of type 8 and two crossing limit cycles of type 9^+ (black and orange) of the discontinuous piecewise linear differential system (6.12). These limit cycles are traveled in counterclockwise.

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5, x_6, x_7, x_8)$ with $i = 1, 2, 3, 4$, for system (6.13) that satisfy the above conditions. We have $q^1 = (8,\ -16/5,\ -3,\ 29/2,\ 6,\ 3, 16/5, 83/10)$; $q^2 = (823/100,\ -4.136449..,\ -3.840062..,\ 15.340062..,\ 6,\ 3,\ 16/5,\ 83/10)$; $q^3 = (\ 841/100,$ $-\ 4.748093.., -4.516514.., 16.016514..587/100, 3.203924.., 177/50, 199/25)$ and $q^4 = (429/50,$ $-\ 5.249123.., -5.170790.., 16.670790.., 587/100, 3.203924.., 177/50, 199/25)$. These four real solutions generated four crossing limit cycles of type 8 and two crossing limit cycles of type 9^+. See these crossing limit cycles of the piecewise linear differential center (6.12) in Figure 6.6.

Here we obtain a total of six crossing limit cycles between limit cycles of type 8 and of type 9^+, moreover these six crossing limit cycles have the configuration $(4, 2)$. We observed that this lower bound for the maximum number of crossing limit cycles of types 8 and 9^+ simultaneously, could be also obtained with the configurations $(3, 3)$. But if we build two crossing limit cycles of type 8 and two of type 9^+, simultaneously, we have that all the parameters that appear in system (5.11) are determined, where this system is such that generated limit cycles of type 9^+, then it is no possible to build a third crossing limit cycle of type 9^+ and therefore we can not obtain the configurations $(3,3)$.

We can also observe that there is a duality between the case studied in statement (c) of Theorem 1.3, where we have studied simultaneously crossing limit cycles of types 4 and 5 and this case, where study the crossing limit cycles of types 8 and 9^+, simultaneously. In these two cases we got the configuration $(4,2)$. See Figures 4.3 and 6.6. $\square$

7 Proof of Theorem 1.6

Proof of statement (a) of Theorem 1.6. We consider the following discontinuous piecewise linear differential system

$$
\begin{aligned}
\dot{x} &= -0.107128.. + 0.268308..x - 0.095415..y, & \dot{y} &= -2.390037.. + x - 0.268308..y, \text{ in } R_1,\\
\dot{x} &= 0.492346.. + 0.144928..x - 0.061289..y, & \dot{y} &= 0.429713.. + x - 0.144928..y, \text{ in } R_2,\\
\dot{x} &= 1.394400.. + 0.300769..x - 0.091362..y, & \dot{y} &= 2.707746.. + x - 0.300769..y, \text{ in } R_3,\\
\dot{x} &= 0.976917.. + 0.400189..x - 4.241691..y, & \dot{y} &= -0.349243.. + x - 0.400189..y, \text{ in } R_4,\\
\dot{x} &= 0.685228.. + 0.043300..x - 0.293631..y, & \dot{y} &= 0.017396.. + x - 0.043300..y, \text{ in } R_5.
\end{aligned}
\tag{7.1}
$$

The linear differential centers in (7.1) have the first integrals

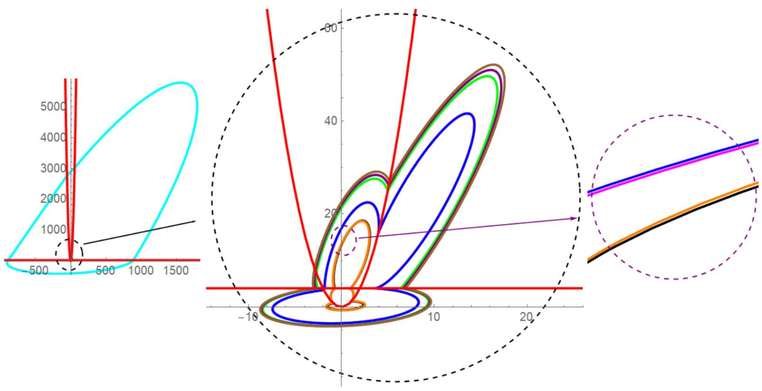

Figure 7.1: Two crossing limit cycle of type 6^+ (magenta and blue), two crossing limit cycles of type 7 (black and orange) and four crossing limit cycles of type 8 (green, purple, brown and cyan) of the discontinuous piecewise linear differential system (7.1). These limit cycles are traveled in counterclockwise.

$$
\begin{aligned}
H_1(x,y) &= x^2 + x(-4.780074.. - 0.536616..y) + (0.214257.. + 0.095415..y)y,\\
H_2(x,y) &= x^2 + x(0.859427.. - 0.289856..y) + (-0.984693.. + 0.061289..y)y,\\
H_3(x,y) &= x^2 + x(5.415492.. - 0.601538..y) + (-2.788801.. + 0.091362..y)y,\\
H_4(x,y) &= x^2 + x(-0.698486.. - 0.800378..y) + y(-1.953834.. + 4.241691..y),\\
H_5(x,y) &= x^2 + x(0.034792.. - 0.086601..y) + (-1.370456.. + 0.293631..y)y,
\end{aligned}
$$

respectively. In order to have crossing limit cycles of types 6^+, 7 and 8 simultaneously, such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, with $-2 < x_2 < 2 < x_1$ and $-2 < x_3 < 2 < x_4$, the crossing limit cycles of type 7 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, k)$, $p_6 = (x_6, k)$, $p_7 = (x_7, x_7^2)$ and $p_8 = (x_8, x_8^2)$, with $x_5 < -2 < x_7 < 2$ and $x_6 < -2 < x_8 < 2$ and the crossing limit cycles of type 8 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_9 = (x_9, x_9^2)$, $p_{10} = (x_{10}, x_{10}^2)$, $p_{11} = (x_{11}, k)$ and $p_{12} = (x_{12}, k)$, with $x_{10} < -2 < 2 < x_9$ and $x_{11} < -2 < 2 < x_{12}$ these points must satisfy systems (5.2), (5.5) and (5.8) respectively. Considering the piecewise linear differential center (7.1) and $k = 4$, systems (5.2), (5.5) and (5.8) become

$$
\begin{aligned}
&11.832571.. + 3.437710..x_1 + 0.061227..x_1^2 - 1.159427..x_1^3 + 0.245157..x_1^4 \\
&\qquad +1.200000..x_2 - 3.999999..x_2^2 = 0, \\
&-0.783728.. - 0.311613..x_2 + x_2^2 - 0.034792..x_3 + 0.370456..x_3^2 + 0.086601..x_3^3 \\
&\qquad -0.293631..x_3^4 = 0, \\
&-240.206876.. - 2.793946..x_3 - 3.815339..x_3^2 - 3.201513..x_3^3 + 16.966764..x_3^4 \\
&\qquad +15.600000..x_4 - 4x_4^2 = 0, \\
&9.534728.. + 19.120296..x_1 - 4.857030..x_1^2 + 2.146465..x_1^3 - 0.381662..x_1^4 \\
&\qquad -27.706159..x_4 + 4x_4^2 = 0, \\
&4(x_5 - x_6)(-0.300000.. + x_5 + x_6) = 0, \\
&-0.783728.. - 0.311613..x_6 + x_6^2 - 0.034792..x_7 + 0.370456..x_7^2 + 0.086601..x_7^3 \\
&\qquad -0.293631..x_7^4 = 0, \\
&-2.793946..x_7 - 3.815339..x_7^2 - 3.201513..x_7^3 + 16.966764..x_7^4 + x_8(2.793946.. \\
&\qquad +3.815339..x_8 + 3.201513..x_8^2 - 16.966764..x_8^3) = 0, \\
&-0.783728.. - 0.311613..x_5 + x5^2 - 0.034792..x_8 + 0.370456..x_8^2 \\
&\qquad +0.086601..x_8^3 - 0.293631..x_8^4 = 0, \\
&-3.437710..x_{10} - 0.061227..x_{10}^2 + 1.159427..x_{10}^3 - 0.245157..x_{10}^4 \\
&\qquad +x_9(3.437710.. + 0.061227..x_9 - 1.159427..x_9^2 + 0.245157..x_9^3) = 0, \\
&38.773655.. + 21.661968..x_{10} - 7.155207..x_{10}^2 - 2.406152..x_{10}^3 + 0.365448..x_{10}^4 \\
&\qquad -12.037359..x_{11} - 4x_{11}^2 = 0, \\
&4(x_{11} - x_{12})(-3.900000.. + x_{11} + x_{12}) = 0, \\
&2.383682.. - 6.926539..x_{12} + x_{12}^2 + 4.780074..x_9 - 1.214257..x_9^2 \\
&\qquad +0.536616..x_9^3 - 0.095415..x_9^4 = 0.
\end{aligned}
\tag{7.2}
$$

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5^i, x_6^i, x_7^i, x_8^i, x_9^i, x_{10}^i, x_{11}^i, x_{12}^i)$ with $i = 1, 2, 3, 4$, for system (7.2) that satisfy the above conditions, namely $q^1 = (4, -9/5, -19/10, 7/2, 1, -7/10, -9/10, 11/10, 5, -27/10, -5/2, 32/5)$; $q^2 = (2007/500, -181/100, -1.905170.., 3.692535.., 101/100, -71/100, -941/1000, 1.132764.., 511/100, -2.805313.., -139/50, 167/25)$; $q^3 = (2007/500, -181/100, -1.905170.., 3.692535.., 101/100, -71/100, -941/1000, 1.132764.., 26/5, -2.891869.., -3.012824.., 6.912824..)$ and $q^4 = (2007/500, -181/100, -1.905170.., 3.692535.., 101/100, -71/100, -941/1000, 1.132764.., 549/10, -52.535582.., -883.528310.., 887.428310..)$. These four real solutions generated two crossing limit cycles of type 6^+, two crossing limit cycles of type 7 and four crossing limit cycles of type 8. See these crossing limit cycles of the piecewise linear differential center (7.1) in Figure 7.1.

Here we obtain a total of eight crossing limit cycles between limit cycles of types 6^+, 7 and 8, moreover these eight crossing limit cycles have the configuration $(2, 2, 4)$, this is 2-crossing limit cycles of type 6^+, 2-crossing limit cycles of type 7 and 4-crossing limit of type 8. We observed that this lower bound for the maximum number of crossing limit cycles of types

6^+, 7 and 8 simultaneously, could be also obtained with other configurations. But if we build two crossing limit cycles of each type we obtain that all parameters of systems (5.2) and (5.5) are determined, and these systems are such that generated the limit cycles of types 6^+ and 7, then we can not build more than two crossing limit cycles of types 6^+ or 7 when we have previously fixed two crossing limit cycles of each type. Then we only obtain the configuration obtained here, namely $(2, 2, 4)$. □

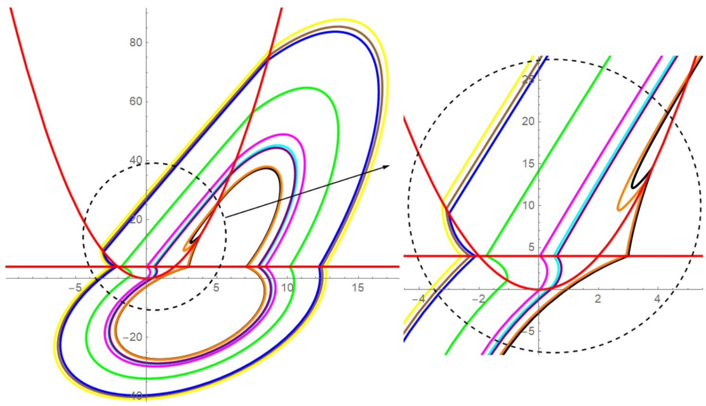

Figure 7.2: Four crossing limit cycles of type 6^+ (green, magenta, cyan and purple), three crossing limit cycles of type 8 (yellow, brown and blue) and two crossing limit cycles of type 9^+ (black and orange) of the discontinuous piecewise linear differential system (7.3). These limit cycles are traveled in counterclockwise.

Proof of statement (b) of Theorem 1.6. We consider the following discontinuous piecewise linear differential system

$$\begin{aligned}
&\dot{x} = -0.312756.. + 0.105676..x - 0.022483..y, && \dot{y} = -4.523476.. + x - 0.105676..y, \text{ in } R_1,\\
&\dot{x} = -0.158662.. + 0.176712..x - 0.031977..y, && \dot{y} = -1.018470.. + x - 0.176712..y, \text{ in } R_2,\\
&\dot{x} = 0.893671.. + \frac{x}{10} - 0.055338..y, && \dot{y} = 1.647781.. + x - \frac{y}{10}, \text{ in } R_3,\\
&\dot{x} = -1.521810.. + 0.129660..x - 0.102089..y, && \dot{y} = -4.531357.. + x - 0.129660..y, \text{ in } R_4,\\
&\dot{x} = 2.392166.. + 0.863445..x - 1.210282..y, && \dot{y} = 11.457801.. + x - 0.863445..y, \text{ in } R_5.
\end{aligned} \tag{7.3}$$

The linear differential centers in (7.3) have the first integrals

$$\begin{aligned}
H_1(x,y) &= x^2 + x(-9.046952.. - 0.211353..y) + (0.625512.. + 0.022483..y)y,\\
H_2(x,y) &= x^2 + x(-2.03694.. - 0.353424..y) + (0.317325.. + 0.031977..y)y,\\
H_3(x,y) &= x^2 + x\left(3.295563.. - \frac{y}{5}\right) + (-1.787342.. + 0.055338..y)y,\\
H_4(x,y) &= x^2 + x(-9.062715.. - 0.259321..y) + (3.043621.. + 0.102089..y)y,\\
H_5(x,y) &= x^2 + x(22.915603.. - 1.726890..y) + y(-4.784333.. + 1.210282..y),
\end{aligned}$$

respectively. In order to have crossing limit cycles of types 6^+, 8 and 9^+ simultaneously, such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, with $-2 < x_2 < 2 < x_1$ and $-2 < x_3 < 2 < x_4$, the crossing limit cycles of type 8 intersect the discontinuity curve

$\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, x_6^2)$, $p_7 = (x_7, k)$ and $p_8 = (x_8, k)$, with $x_6 < -2 < 2 < x_5$ and $x_7 < -2 < 2 < x_8$ and the crossing limit cycles of type 9^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_9 = (x_9, x_9^2)$, $p_{10} = (x_{10}, x_{10}^2)$, $p_{11} = (x_{11}, k)$ and $p_{12} = (x_{12}, k)$, with $2 < x_{10} < x_9$ and $2 < x_{11} < x_{12}$ these points must satisfy systems (5.2), (5.8) and (5.11) respectively. Considering the piecewise linear differential center (7.3) and $k = 4$, systems (5.2), (5.8) and (5.11) become

$$
\begin{aligned}
-7.123782.. + x_1(-8.147767.. + x_1(5.269300.. + (-1.413698.. + 0.127911..x_1)x_1))&\\
+(13.802561.. - 4x_2)x_2 &= 0,\\
0.227189.. + x_2(16.008041.. + x_2) + x_3(-22.915603.. + x_3(3.784333.. + (1.726890..&\\
-1.210282..x_3)x_3)) &= 0,\\
-55.231640.. + x_3(-36.250863.. + x_3(16.174485.. + (-1.037284.. + 0.408356..x_3)x_3))&\\
+\left(\frac{202}{5} - 4x_4\right)x_4 &= 0,\\
11.447141.. + x_1(36.187810.. + x_1(-6.502051.. + (0.845414.. - 0.089933..x_1)x_1))&\\
+x_4(-39.569467.. + 4x_4) &= 0,\\
x_5(-8.147767.. + x_5(5.269300.. + (-1.413698.. + 0.127911..x_5)x_5)) + x_6(8.147767..&\\
+x_6(-5.269300.. + (1.413698.. - 0.127911..x_6)x_6)) &= 0,\\
25.055786.. + x_6\left(13.182255.. + x_6\left(-3.149369.. + \left(-\frac{4}{5} + 0.221355..x_6\right)x_6\right)\right)&\\
+(-9.982255.. - 4x_7)x_7 &= 0,\\
4(x_7 - x_8)\left(-\frac{101}{10} + x_7 + x_8\right) &= 0,\\
11.447141.. + x_5(36.187810.. + x_5(-6.502051.. + (0.845414.. - 0.089933..x_5)x_5))&\\
+x_8(-39.569467.. + 4x_8) &= 0,\\
x_{10}(8.147767.. + x_{10}(-5.269300.. + (1.413698.. - 0.127911..x_{10})x_{10}))&\\
+x_9(-8.147767.. + x_9(5.269300.. + (-1.413698.. + 0.127911..x_9)x_9)) &= 0,\\
-11.447141.. + x_{10}(-36.187810.. + x_{10}(6.502051.. + (-0.845414..&\\
+0.089933..x_{10})x_{10})) + (39.569467.. - 4x_{11})x_{11} &= 0,\\
4(x_{11} - x_{12})\left(-\frac{101}{10} + x_{11} + x_{12}\right) &= 0,\\
2.861785.. + (-9.892366.. + x_{12})x_{12} + x_9(9.046952.. + x_9(-1.625512.. + (0.211353..&\\
-0.022483..x_9)x_9)) &= 0.
\end{aligned}
\tag{7.4}
$$

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5^i, x_6^i, x_7^i, x_8^i, x_9^i, x_{10}^i, x_{11}^i, x_{12}^i)$ with $i = 1, 2, 3, 4$, for system (7.4) that satisfy the above conditions, namely $q^1 = (6, 1/2, 2/5, 8, 87/10, -31/10, -23/10, 62/5, 5, 19/5, 3, 71/10)$; $q^2 = (317/50, 0.042569.., 1/25, 8.417274.., 861/100, -3.007479.., -2.117234.., 12.217234.., 5, 19/5, 3, 71/10)$; $q^3 = (1479/250, -0.610424.., -1/2, 7.904488.., 883/100, -3.233408.., -2.568105.., 12.668105.., 51/10, 3.582979.., 2.936322.., 7.163677..)$, and $q^4 = (15/2, -1.752776.., -1.049779.., 10.157706.., 883/100, -3.233408.., -2.568105.., 12.668105.., 51/10, 3.582979.., 2.936322.., 7.163677..)$ these four solutions generated four crossing limit cycles of type 6^+, three crossing limit cycles of type 8 and two crossing limit cycle of type 9^+. See these crossing limit cycles of the piecewise linear differential center (7.3) in Figure 7.2.

Here we obtain a total of nine crossing limit cycles between limit cycles of types 6^+, 8 and 9^+, moreover these nine crossing limit cycles have the configuration $(4, 3, 2)$. We observed that this lower bound for the maximum number of crossing limit cycles of types 6^+, 8 and

9^+ simultaneously, could be also obtained with other configurations. When we build two crossing limit cycles of each type we obtain that system (5.11) has all parameters determined, and therefore we can not build a third crossing limit cycle of type 9^+. Systems (5.2), (5.8) which generated the limit cycles of types 8 and 9^+ would still have free parameters and it is possible verify that we can have the configurations $(4, 3, 2)$ or $(3, 4, 2)$. Here we have illustrated the configuration $(4, 3, 2)$. □

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A model for spatial spreading and dynamics of fox rabies on a growing domain

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Abstract. In order to explore the impact of the growth rate of the habitat on the transmission of rabies, we consider a SEI model for fox rabies on a growing spatial domain. The basic reproduction number is introduced using the next infection operator, spectral analysis and the corresponding eigenvalue problem. The stability of equilibria is also established using the upper and lower solutions method in terms of this number. Our results show that a large growth rate of the domain has a negative impact on the prevention and control of rabies. Numerical simulations are presented to verify our theoretical results.

Keywords: SEI model, fox rabies, growing domain, basic reproduction number, stability.

2020 Mathematics Subject Classification: 35K57, 37L15, 92D25.

1 Introduction

Rabies, an acute infectious disease caused by virus infecting the central nervous system, is mainly transmitted by direct contact such as biting [3]. Most mammals are susceptible to the disease, and although only very few human fatalities occur every year, rabies is still a considerable threat to human beings on account of inefficient treatment and a nearly 100% mortality rate once it reaches the clinical stage [11]. In order to develop public policies for prevention and control of rabies, various mathematical models have been established to study the transmission mechanism of rabies.

The red fox is the main carrier of rabies in Europe [2]. The following SEI model for fox rabies was proposed and studied by Murray et al. in [17]:

$$\begin{cases} E_t = \beta IS - \sigma E - \left[b + (a-b)\frac{N}{K}\right] E, \\ I_t = D\Delta I + \sigma E - \alpha I - \left[b + (a-b)\frac{N}{K}\right] I, \\ S_t = (a-b)S\left(1 - \frac{N}{K}\right) - \beta IS, \end{cases} \tag{1.1}$$

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where $S(x,t)$, $E(x,t)$ and $I(x,t)$ are the densities of susceptible foxes, infected but non-infectious foxes and rabid foxes at location x and time t, respectively. $N = E + I + S$ is the total fox population. On account of the random wandering of the rabid foxes, the diffusion coefficient D is introduced in the equation for I. α represents the mortality rate of the rabid foxes and β is the disease transmission coefficient. We assume that infected foxes become infectious at the per capita rate σ. a is the birth rate, b is the intrinsic death rate and K is the environmental carrying capacity. The term $(a-b)\frac{N}{K}$ denotes the depletion of the food supply by all foxes, where $a > b$ ensures a sustainable population size. All coefficients in the model (1.1) are nonnegative constants.

Letting $W = K - S$, model (1.1) becomes

$$\begin{cases} E_t = \beta I(K-W) - \sigma E - \left[b + (a-b)\frac{N}{K}\right] E, \\ I_t = D\Delta I + \sigma E - \alpha I - \left[b + (a-b)\frac{N}{K}\right] I, \\ W_t = -(a-b)(K-W)\left(1 - \frac{N}{K}\right) + \beta I(K-W), \end{cases} \tag{1.2}$$

where $N = E + I + K - W$ is the total fox population.

Problems describing ecological models on fixed spatial domains have been extensively investigated in the literature. However, the habitats of species in nature are not invariable. Some habitats are affected by climate, temperature and rainfall, and the shifting boundaries are known, for example the area of Dongting Lake in China changes by season, that is, Dongting lake covers an average area of 1814 square kilometres in summer while it covers only 568 square kilometres in winter in the period 1996 to 2016, see [12,15,16,18,22,26] and references therein. Some habitats are influenced by the species itself and the boundaries are moving and unknown. Such boundaries have recently been described by free boundaries, which have been studied in [9,13,23] and [24] for invasive species and in [14] for the transmission of disease. Domain growth, as one possibility for domain evolution, plays an important role in the formation of living patterns.

Inspired by the aforementioned works, we consider a SEI model (1.2) on a growing domain as in [7] and [8]. Let $\Omega_t \subset \mathbb{R}^2$ be a bounded growing domain at time t, and its growing boundary is denoted $\partial\Omega_t$. Also we assume that $E(x(t),t)$, $I(x(t),t)$ and $W(x(t),t)$ are the densities of the three kinds of fox population at location $x(t) \in \Omega_t$ and time t. Additionally, the growth of the domain Ω_t generates a flow velocity $\mathbf{a} = \dot{x}(t)$, that is, the flow velocity is identical to the domain velocity. According to the principle of mass conservation and the Reynolds transport theorem [1], we can formulate the problem on a growing domain related to (1.2) as

$$\begin{cases} E_t + \mathbf{a}\cdot\nabla E + E(\nabla\cdot\mathbf{a}) = \beta I(K-W) - \sigma E - \left[b + (a-b)\frac{N}{K}\right] E & \text{in } \Omega_t, \\ I_t - D\Delta I + \mathbf{a}\cdot\nabla I + I(\nabla\cdot\mathbf{a}) = \sigma E - \alpha I - \left[b + (a-b)\frac{N}{K}\right] I & \text{in } \Omega_t, \\ W_t + \mathbf{a}\cdot\nabla W + W(\nabla\cdot\mathbf{a}) = -(a-b)(K-W)\left(1 - \frac{N}{K}\right) + \beta I(K-W) & \text{in } \Omega_t, \\ E(x(t),t) = I(x(t),t) = W(x(t),t) = 0 & \text{on } \partial\Omega_t, \\ E(x(0),0) = E_0(x), I(x(0),0) = I_0(x), W(x(0),0) = W_0(x) & \text{in } \overline{\Omega}_0. \end{cases} \tag{1.3}$$

Here $\mathbf{a}\cdot\nabla E$, $\mathbf{a}\cdot\nabla I$ and $\mathbf{a}\cdot\nabla W$ are called advection terms related to the transport of material across $\partial\Omega_t$ with the flow $\mathbf{a}$, and other extra terms introduced by the growth of the domain Ω_t are the dilution terms $E(\nabla\cdot\mathbf{a})$, $I(\nabla\cdot\mathbf{a})$ and $W(\nabla\cdot\mathbf{a})$ due to the local volume expansion [5]. The null Dirichlet boundary conditions mean that there is no infection outside the growing domain and on the boundary.

In order to simplify problem (1.3), we assume that the growth of the domain Ω_t is uniform and isotropic. Biologically, the infected domain Ω_t is supposed to grow at the same rate $\rho(t)$ in all directions as time t increases. Mathematically, we can formulate this as

$$x(t) = \rho(t)y \quad \text{for all } x(t) \in \Omega_t \text{ and } (y,t) \in \Omega_0 \times [0,+\infty),$$

where $\rho(t) \in C^1[0,+\infty)$ is called the growth function and satisfies

$$\rho(0) = 1, \qquad \dot{\rho}(t) > 0, \qquad \lim_{t\to\infty} \rho(t) = \rho_\infty > 1 \quad \text{and} \quad \lim_{t\to\infty} \dot{\rho}(t) = 0.$$

By Lagrangian transformations (see e.g. [4]), we define $E(x(t),t) = u_1(y,t)$, $I(x(t),t) = u_2(y,t)$ and $W(x(t),t) = u_3(y,t)$. Then we have

$$\begin{aligned} &u_{1t} = E_t + \mathbf{a}\cdot\nabla E,\ u_{2t} = I_t + \mathbf{a}\cdot\nabla I,\ u_{3t} = W_t + \mathbf{a}\cdot\nabla W,\\ &\mathbf{a} = \dot{x}(t) = \dot{\rho}(t)y = \frac{\dot{\rho}(t)}{\rho(t)}x(t),\\ &\nabla\cdot\mathbf{a} = \frac{n\dot{\rho}(t)}{\rho(t)},\ \Delta I = \frac{1}{\rho^2(t)}\Delta u_2 \end{aligned}$$

and problem (1.3) can be transformed into the following reaction-diffusion model on the fixed domain Ω_0

$$\begin{cases} u_{1t} = \beta u_2(K-u_3) - \sigma u_1 - \left[b + (a-b)\frac{N}{K}\right]u_1 - \frac{n\dot{\rho}(t)}{\rho(t)}u_1, & y\in\Omega_0,\ t>0,\\ u_{2t} - \frac{D}{\rho^2(t)}\Delta u_2 = \sigma u_1 - \alpha u_2 - \left[b + (a-b)\frac{N}{K}\right]u_2 - \frac{n\dot{\rho}(t)}{\rho(t)}u_2, & y\in\Omega_0,\ t>0,\\ u_{3t} = -(a-b)(K-u_3)\left(1-\frac{N}{K}\right) + \beta u_2(K-u_3) - \frac{n\dot{\rho}(t)}{\rho(t)}u_3, & y\in\Omega_0,\ t>0,\\ u_1(y,t) = u_2(y,t) = u_3(y,t) = 0, & y\in\partial\Omega_0,\ t>0,\\ u_1(y,0) := \eta_1(y), u_2(y,0) := \eta_2(y), u_3(y,0) := \eta_3(y), & y\in\overline{\Omega}_0, \end{cases} \tag{1.4}$$

where $N = u_1 + u_2 + K - u_3$ is the total fox population.

The rest of the paper is organized as follows: Section 2 is devoted to the basic reproduction number of problem (1.4) as well as its analytic properties. In Section 3, we investigate the stability of the disease-free steady state. Numerical simulations and the discussion are finally presented in Sections 4 and 5, respectively.

2 The basic reproduction number

In this section, we first present the principal eigenvalue R_0^* of the linearized system of problem (1.4) at $(0,0,0)$, then define the basic reproduction number R_0 and analyze its properties. Epidemiologically, the basic reproduction number is a critical threshold that reflects whether the disease will be spread or disappear.

Problem (1.4) admits a disease-free steady state $(0,0,0)$. Linearizing system (1.4) at $(0,0,0)$ and recalling that $\dot{\rho}(t) \to 0$ as $t\to\infty$, we are led to consider the system

$$\begin{cases} u_t = \beta K v - (\sigma + a)u, & y\in\Omega_0,\ t>0,\\ v_t - \frac{D\Delta v}{\rho_\infty^2} = \sigma u - (\alpha + a)v, & y\in\Omega_0,\ t>0,\\ w_t = (a-b)(u+v-w) + \beta K v, & y\in\Omega_0,\ t>0. \end{cases} \tag{2.1}$$

Since the first two equations of (2.1) are decoupled from the last equation, we consider the following eigenvalue problem

$$\begin{cases} 0 = \frac{\beta K \psi}{R_0^*} - (\sigma + a)\phi, & y \in \Omega_0, \\ -\frac{D\Delta\psi}{\rho_\infty^2} = \frac{\sigma\phi}{R_0^*} - (\alpha + a)\psi, & y \in \Omega_0, \\ \phi(y) = \psi(y) = 0, & y \in \partial\Omega_0, \end{cases} \tag{2.2}$$

which is equivalent to the eigenvalue problem

$$\begin{cases} -\frac{D\Delta\psi}{\rho_\infty^2} = \frac{\sigma\beta K\psi}{(\sigma+a)(R_0^*)^2} - (\alpha + a)\psi, & y \in \Omega_0, \\ \phi(y) = 0, & y \in \partial\Omega_0. \end{cases} \tag{2.3}$$

Direct calculation shows that the principal eigenvalue of problem (1.4)

$$R_0^* = \sqrt{\frac{\sigma\beta K}{(\sigma + a)(\frac{D}{\rho_\infty^2}\lambda_1 + \alpha + a)}}, \tag{2.4}$$

where $(\lambda_1, \zeta(y))$ is the principal eigen-pair of the eigenvalue problem

$$\begin{cases} -\Delta\zeta = \lambda_1\zeta, & y \in \Omega_0, \\ \zeta(y) = 0, & y \in \partial\Omega_0. \end{cases} \tag{2.5}$$

Now we define the basic reproduction number R_0. Similarly as in [25] and [27], we write the first two equations of (2.1) as the following equivalent single equation:

$$\begin{cases} U_t = d\Delta U + FU - VU, & y \in \Omega_0,\ t > 0, \\ u = v = 0, & y \in \partial\Omega_0,\ t > 0, \end{cases}$$

where $U = (u, v)^T$, $d = (0, D)^T$,

$$F = \begin{pmatrix} 0 & \beta K \\ 0 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} \sigma + a & 0 \\ -\sigma & \alpha + a \end{pmatrix}.$$

Let $X_1 = C(\overline{\Omega}_0, \mathbb{R}^2)$ and $X_1^+ := C(\overline{\Omega}_0, \mathbb{R}_+^2)$, and let $T(t)$ be the solution semigroup of the following system on X_1

$$\begin{cases} U_t = d\Delta U - VU, & y \in \Omega_0,\ t > 0, \\ u = v = 0, & y \in \partial\Omega_0,\ t > 0, \end{cases}$$

and let $\phi(y)$ be the density of the initial infectious fox population. Define the next infection operator L by

$$L(\phi)(y) := \int_0^\infty F(y)[T(t)\phi](y)dt = F(y)\int_0^\infty [T(t)\phi](y)dt.$$

Then $R_0 = r(L)$, where $r(L)$ is the spectral radius of L. We have the following result, we refer to Theorem 11.3.3 in [27] for more details:

Lemma 2.1. *$R_0 = R_0^*$ and $\operatorname{sign}(1 - R_0) = \operatorname{sign}\lambda^*$, where λ^* is the principal eigenvalue of the following eigenvalue problem*

$$\begin{cases} 0 = \beta K\psi - (\sigma + a)\phi + \lambda\phi, & y \in \Omega_0, \\ -\frac{D\Delta\psi}{\rho_\infty^2} = \sigma\phi - (\alpha + a)\psi + \lambda\psi, & y \in \Omega_0, \\ \phi(y) = \psi(y) = 0, & y \in \partial\Omega_0. \end{cases} \tag{2.6}$$

According to the explicit expression of R_0, we can list some properties of R_0.

Theorem 2.2. *The following assertions hold.*

(i) $R_0(\rho_\infty, \Omega)$ is a positive and strictly increasing function with respect to Ω, that is, $R_0(\rho_\infty, \Omega_1) \le R_0(\rho_\infty, \Omega_2)$ provided that $\Omega_1 \subseteq \Omega_2$, with strict inequality if $\Omega_2 \backslash \overline{\Omega_1}$ is a non-empty open set;

(ii) $R_0(\rho_\infty, \Omega)$ is a monotonically increasing function with respect to ρ_∞, in the sense that $R_0(\rho_\infty, \Omega) < R_0(\rho_\infty^*, \Omega)$ provided that $\rho_\infty < \rho_\infty^*$.

Proof. The proof of the monotonicity in (i) is similar to Corollary 2.3 in [6]. The proof of (ii) follows directly from (2.4). □

Remark 2.3. The basic reproduction number is used as a threshold parameter for the transmission mechanism of the disease and plays a central role in mathematical epidemiology. Biologically, R_0 is the average number of new infections produced by a typical infective individual over its infection period. R_0 can be obtained by the second generation matrix method [10] for epidemic models described by spatially-independent systems, and it can be calculated as the spectral radius of the next-generation operator for models in a constant environment [25] or in a periodic environment [27].

3 The stability of the disease-free equilibrium

In this section we will investigate the stability of the disease-free equilibrium $(0, 0, 0)$ in terms of the threshold R_0. First we introduce the definition of the pair of coupled upper and lower solutions.

Definition 3.1. Let $(\tilde{u}_1(y,t), \tilde{u}_2(y,t), \tilde{u}_3(y,t)), (\hat{u}_1(y,t), \hat{u}_2(y,t), \hat{u}_3(y,t))$ be a pair of (triplets of) functions in $C^{2,1}(\Omega_0 \times (0, +\infty)) \bigcap C(\bar{\Omega}_0 \times [0, +\infty))$, satisfying $(0, 0, 0) \le (\hat{u}_1, \hat{u}_2, \hat{u}_3) \le (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \le (K, K, K)$. The pair (of triplets) is called coupled upper and lower solutions of (1.4), if the following relations are satisfied:

$$\begin{cases} \hat{u}_{1t} \le \beta\hat{u}_2(K - \tilde{u}_3) - \sigma\hat{u}_1 - \left[b + (a-b)\frac{\hat{u}_1 + \tilde{u}_2 + K - \hat{u}_3}{K}\right]\hat{u}_1 - \frac{n\dot{\rho}(t)}{\rho(t)}\hat{u}_1, \\ \hat{u}_{2t} - \frac{D}{\rho^2(t)}\Delta\hat{u}_2 \le \sigma\hat{u}_1 - \alpha\hat{u}_2 - \left[b + (a-b)\frac{\tilde{u}_1 + \hat{u}_2 + K - \hat{u}_3}{K}\right]\hat{u}_2 - \frac{n\dot{\rho}(t)}{\rho(t)}\hat{u}_2, \\ \hat{u}_{3t} \le -(a-b)(K - \hat{u}_3)\left(1 - \frac{\hat{u}_1 + \hat{u}_2 + K - \hat{u}_3}{K}\right) + \beta\hat{u}_2(K - \hat{u}_3) - \frac{n\dot{\rho}(t)}{\rho(t)}\hat{u}_3, \\ \tilde{u}_{1t} \ge \beta\tilde{u}_2(K - \hat{u}_3) - \sigma\tilde{u}_1 - \left[b + (a-b)\frac{\tilde{u}_1 + \hat{u}_2 + K - \tilde{u}_3}{K}\right]\tilde{u}_1 - \frac{n\dot{\rho}(t)}{\rho(t)}\tilde{u}_1, \\ \tilde{u}_{2t} - \frac{D}{\rho^2(t)}\Delta\tilde{u}_2 \ge \sigma\tilde{u}_1 - \alpha\tilde{u}_2 - \left[b + (a-b)\frac{\hat{u}_1 + \tilde{u}_2 + K - \tilde{u}_3}{K}\right]\tilde{u}_2 - \frac{n\dot{\rho}(t)}{\rho(t)}\tilde{u}_2, \\ \tilde{u}_{3t} \ge -(a-b)(K - \tilde{u}_3)\left(1 - \frac{\tilde{u}_1 + \tilde{u}_2 + K - \tilde{u}_3}{K}\right) + \beta\tilde{u}_2(K - \tilde{u}_3) - \frac{n\dot{\rho}(t)}{\rho(t)}\tilde{u}_3, & y \in \Omega_0,\ t > 0, \\ \hat{u}_1(y,t) = 0 \le \tilde{u}_1(y,t), \hat{u}_2(y,t) = 0 \le \tilde{u}_2(y,t), \hat{u}_3(y,t) = 0 \le \tilde{u}_3(y,t), & y \in \partial\Omega_0,\ t > 0, \\ \hat{u}_1(y,0) \le \eta_1(y), \hat{u}_2(y,0) \le \eta_2(y), \hat{u}_3(y,0) \le \eta_3(y), & y \in \overline{\Omega}_0, \\ \tilde{u}_1(y,0) \ge \eta_1(y), \tilde{u}_2(y,0) \ge \eta_2(y), \tilde{u}_3(y,0) \ge \eta_3(y), & y \in \overline{\Omega}_0. \end{cases} \tag{3.1}$$

R_0 is a threshold value for the local stability of the disease-free equilibrium [25]. In the following we investigate the local stability of the disease-free equilibrium $(0,0,0)$ in the two cases $R_0 < 1$ and $R_0 > 1$.

Theorem 3.2. *If $R_0 < 1$, then the disease-free steady state $(0,0,0)$ is a locally asymptotically stable equilibrium for problem (1.4).*

Proof. The upper and lower solutions method is used to prove this theorem. Let

$$(\hat{u}_1, \hat{u}_2, \hat{u}_3)(y,t) = (0,0,0), \qquad (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)(y,t) = (\varepsilon\phi(y), \varepsilon\psi(y), \varepsilon\xi(y)), \tag{3.2}$$

where ε is sufficiently small, $\phi(y)$ and $\psi(y)$ are the normalized positive eigenfunctions in problem (2.2), and $\xi(y)$ satisfies

$$0 = \frac{(a-b)\phi + (a-b+\beta K)\psi}{R_0} - (a-b)\xi. \tag{3.3}$$

Plugging (3.2) back into (3.1), it is easy to verify that the first three inequalities in (3.1) hold. The fourth inequality becomes

$$0 \geq \beta K\psi - \sigma\phi - \left[b + (a-b)\frac{\varepsilon\phi + K - \varepsilon\xi}{K}\right]\phi - \frac{n\dot{\rho}(t)}{\rho(t)}\phi.$$

According to the first equation in (2.2), we only need to prove that

$$b + (a-b)\frac{\varepsilon\phi + K - \varepsilon\xi}{K} + \sigma + \frac{n\dot{\rho}(t)}{\rho(t)} \geq R_0(\sigma + a). \tag{3.4}$$

Since $R_0 < 1$ and ε is sufficiently small, (3.4) holds and the fourth inequality in (3.1) holds. The fifth inequality becomes

$$-\frac{D\Delta\psi}{\rho(t)^2} \geq \sigma\phi - \alpha\psi - \left[b + (a-b)\frac{\varepsilon\psi + K - \varepsilon\xi}{K}\right]\psi - \frac{n\dot{\rho}(t)}{\rho(t)}\psi. \tag{3.5}$$

It is easy to check that $\psi(y) = \zeta(y)$, where $\zeta(y)$ satisfies (2.5). We have $-\frac{D\Delta\psi}{\rho^2(t)} \geq -\frac{D\Delta\psi}{\rho_\infty^2}$ due to $\Delta\psi = \Delta\zeta = -\lambda_1\zeta \leq 0$. We have that (3.5) is satisfied if

$$-\frac{D\Delta\psi}{\rho_\infty^2} \geq \sigma\phi - \alpha\psi - \left[b + (a-b)\frac{\varepsilon\psi + K - \varepsilon\xi}{K}\right]\psi - \frac{n\dot{\rho}(t)}{\rho(t)}\psi \tag{3.6}$$

holds. From the second equation in (2.2), (3.5) becomes

$$\left(\frac{1}{R_0} - 1\right)\sigma\phi \geq \left\{a - \left[b + (a-b)\frac{\varepsilon\psi + K - \varepsilon\xi}{K}\right]\right\}\psi - \frac{n\dot{\rho}(t)}{\rho(t)}\psi. \tag{3.7}$$

Since $R_0 < 1$ and that the right of (3.7) tends to 0 as $\varepsilon \to 0$, the fifth inequality in (3.1) holds for sufficiently small ε. The sixth inequality in (3.1) is equivalent to

$$0 \geq (a-b)(K - \varepsilon\xi)\frac{\phi + \psi - \xi}{K} + \beta\phi(K - \varepsilon\xi) - \frac{n\dot{\rho}(t)}{\rho(t)}. \tag{3.8}$$

Due to (3.3), (3.8) becomes

$$(a-b)(1 - R_0) + \frac{n\dot{\rho}(t)}{\rho(t)} \geq -\varepsilon\left[(a-b)\frac{\phi + \psi - \xi}{K} + \beta\psi\right]. \tag{3.9}$$

Since $R_0 < 1$ and $\dot{\rho}(t) > 0$, (3.8) is also true for sufficiently small ε.

Therefore, the function-pairs

$$(\hat{u}_1, \hat{u}_2, \hat{u}_3)(y,t) = (0,0,0), \qquad (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)(y,t) = (\varepsilon\phi(y), \varepsilon\psi(y), \varepsilon\xi(y))$$

are the upper and lower solutions of problem (1.4). This implies that the solutions of problem (1.4) lies between the lower solutions and the upper solutions as long as the initial values belong to the prescribed intervals. Therefore, given the condition $R_0 < 1$, we can conclude local stability of the disease-free equilibrium $(0,0,0)$. □

The next result shows that the disease-free equilibrium $(0,0,0)$ is unstable if $R_0 > 1$.

Theorem 3.3. *If $R_0 > 1$, then there exists a $\delta_0 > 0$ such that any positive solution of problem* (1.4) *satisfies* $\limsup_{t\to\infty} \| (u_1(\cdot,t), u_2(\cdot,t), u_3(\cdot,t)) - (0,0,0) \| \geq \delta_0$.

Proof. We argue by contradiction and assume that for any $\delta \in (0,K)$, there exists a $T_\delta > 0$ such that

$$0 < u_1(y,t), u_2(y,t), u_3(y,t) < \delta \quad \text{for all } y \in \Omega_0, t \geq T_\delta. \tag{3.10}$$

We consider the following eigenvalue problem:

$$\begin{cases} 0 = \beta u_2(K-\delta) - \sigma u_1 - \left[b + (a-b)\frac{K+3\delta}{K}\right] u_1 - \delta u_1 + \lambda u_1, & y \in \Omega_0, \\ -\frac{D\Delta u_2}{(\rho_\infty - \delta)^2} = \sigma u_1 - \alpha u_2 - \left[b + (a-b)\frac{K+3\delta}{K}\right] u_2 - \delta u_2 + \lambda u_2, & y \in \Omega_0, \\ u_1 = u_2 = 0, & y \in \partial\Omega_0. \end{cases} \tag{3.11}$$

Problem (3.11) has a principal eigenvalue λ^*_δ and a pair of positive corresponding eigenfunctions $(\phi^*_\delta(y), \psi^*_\delta(y))$. It is easy to check that $\psi^*_\delta(y) = \zeta(y)$, where $\zeta(y)$ satisfies (2.5). By Lemma 2.1, $R_0 > 1$ implies that $\lambda^* < 0$. Therefore, $\lim_{\delta\to 0} \lambda^*_\delta = \lambda^* < 0$. We can fix a small $\delta_0 \in (0,K)$ such that $\lambda^*_{\delta_0} < 0$. Then there exists a $T_1 > 0$ such that

$$0 < u_1(y,t), u_2(y,t), u_3(y,t) < \delta_0 \quad \text{for all } y \in \Omega_0, t \geq T_1.$$

Since $\lim_{t\to\infty} \rho(t) = \rho_\infty$, there exists a $T_2 > 0$ such that

$$\rho_\infty - \delta_0 < \rho(t) \leq \rho_\infty \quad \text{for } t \geq T_2.$$

Similarly, the limit $\lim_{t\to\infty} \frac{n\dot{\rho}(t)}{\rho(t)} = 0$ implies that there exists a $T_3 > 0$ such that

$$\frac{n\dot{\rho}(t)}{\rho(t)} < \delta_0 \quad \text{for } t \geq T_3.$$

Now choose a large $T^* = \max\{T_1, T_2, T_3\}$. Note that any positive solution (u_1, u_2, u_3) of the problem (1.4) satisfies

$$\begin{cases} u_{1t} \geq \beta u_2(K-\delta_0) - \sigma u_1 - \left[b + (a-b)\frac{K+3\delta_0}{K}\right] u_1 - \delta_0 u_1, \\ u_{2t} - \frac{D\Delta u_2}{\rho(t)^2} \geq \sigma u_1 - \alpha u_2 - \left[b + (a-b)\frac{K+3\delta_0}{K}\right] u_2 - \delta_0 u_2, \end{cases}$$

for all $y \in \Omega_0, t \geq T^*$. Define $(\underline{u}_1(y,t), \underline{u}_2(y,t))$ to be a positive solution of the problem

$$\begin{cases} \underline{u}_{1t} = \beta \underline{u}_2(K-\delta_0) - \sigma \underline{u}_1 - \left[b + (a-b)\frac{K+3\delta_0}{K}\right] \underline{u}_1 - \delta_0 \underline{u}_1, & y \in \Omega_0, t \geq T^*, \\ \underline{u}_{2t} - \frac{D\Delta \underline{u}_2}{\rho^2(t)} = \sigma \underline{u}_1 - \alpha \underline{u}_2 - \left[b + (a-b)\frac{K+3\delta_0}{K}\right] \underline{u}_2 - \delta_0 \underline{u}_2, & y \in \Omega_0, t \geq T^*, \\ \underline{u}_1 = \underline{u}_2 = 0, & y \in \partial\Omega_0, t \geq T^*, \\ \underline{u}_1(y,T^*) = u_1(y,T^*), \underline{u}_2(y,T^*) = u_2(y,T^*), & y \in \overline{\Omega}_0. \end{cases} \tag{3.12}$$

It then follows from the comparison principle that

$$(u_1(y,t),u_2(y,t)) \geq (\underline{u}_1(y,t),\underline{u}_2(y,t)) > (0,0) \quad \text{for all } y \in \Omega_0, t \geq T^*. \tag{3.13}$$

Now we conclude that $(\underline{u}_1(y,T^*),\underline{u}_2(y,T^*)) \geq (\mu\phi^*_{\delta_0}(y),\mu\psi^*_{\delta_0}(y))$ in $\overline{\Omega}_0$ for sufficiently small μ. In fact, since $\underline{u}_1(y,T^*)$, $\underline{u}_2(y,T^*)$, $\phi^*_{\delta_0}(y)$ and $\psi^*_{\delta_0}(y)$ are all >0 for $y \in \Omega_0$, we have $\frac{\partial \underline{u}_1(y,T^*)}{\partial\eta}\big|_{\partial\Omega_0}$, $\frac{\partial \underline{u}_2(y,T^*)}{\partial\eta}\big|_{\partial\Omega_0}$, $\frac{\partial\phi^*_{\delta_0}(y)}{\partial\eta}\big|_{\partial\Omega_0}$ and $\frac{\partial\psi^*_{\delta_0}(y)}{\partial\eta}\big|_{\partial\Omega_0} < 0$ by the strong maximum principle [19], where η is the outer unit normal on $\partial\Omega_0$. For $y_0 \in \partial\Omega_0$, there exists a small $\varepsilon(y_0) > 0$ such that

$$\frac{\partial \underline{u}_1(y,T^*)}{\partial\eta} < \frac{1}{2}\frac{\partial \underline{u}_1(y,T^*)}{\partial\eta}\bigg|_{\partial\Omega_0} < 0, \qquad \frac{\partial \underline{u}_2(y,T^*)}{\partial\eta} < \frac{1}{2}\frac{\partial \underline{u}_2(y,T^*)}{\partial\eta}\bigg|_{\partial\Omega_0} < 0,$$
$$\frac{\partial\phi^*_{\delta_0}(y)}{\partial\eta} < \frac{1}{2}\frac{\partial\phi^*_{\delta_0}(y)}{\partial\eta}\bigg|_{\partial\Omega_0} < 0, \qquad \frac{\partial\psi^*_{\delta_0}(y)}{\partial\eta} < \frac{1}{2}\frac{\partial\psi^*_{\delta_0}(y)}{\partial\eta}\bigg|_{\partial\Omega_0} < 0$$

for $y \in \overline{B(y_0,\varepsilon(y_0))\bigcap\Omega_0}$. Set $\mu_1 = \min\left\{\frac{\partial \underline{u}_1(y,T^*)}{\partial\eta}/\frac{\partial\phi^*_{\delta_0}(y)}{\partial\eta}, \frac{\partial \underline{u}_2(y,T^*)}{\partial\eta}/\frac{\partial\psi^*_{\delta_0}(y)}{\partial\eta}\right\}$, $y \in \overline{B(y_0,\varepsilon(y_0))\bigcap\Omega_0}$, then

$$\frac{\partial \underline{u}_1(y,T^*)}{\partial\eta} \geq \mu_1\frac{\partial\phi^*_{\delta_0}(y)}{\partial\eta}, \qquad \frac{\partial \underline{u}_2(y,T^*)}{\partial\eta} \geq \mu_1\frac{\partial\psi^*_{\delta_0}(y)}{\partial\eta} \quad \text{for } y \in \overline{B(y_0,\varepsilon(y_0))\bigcap\Omega_0}.$$

By the mean value theorem, we have

$$\underline{u}_1(y,T^*) \geq \mu_1\phi^*_{\delta_0}(y), \qquad \underline{u}_2(y,T^*) \geq \mu_1\psi^*_{\delta_0}(y) \quad \text{for } y \in \overline{B(y_0,\varepsilon(y_0))\bigcap\Omega_0}.$$

Since $\partial\Omega_0$ is bounded, we can find finitely many points $y_0^i \in \partial\Omega_0$, radii $\varepsilon(y_0^i) > 0$ $(i = 1,\ldots,N)$ such that $\partial\Omega_0 \subset \bigcup_{i=1}^N B(y_0^i,\varepsilon(y_0^i))$, hence there exists a small $h = \min_i \varepsilon(y_0^i) > 0$ such that

$$\underline{u}_1(y,T^*) \geq \mu_1\phi^*_{\delta_0}(y), \qquad \underline{u}_2(y,T^*) \geq \mu_1\psi^*_{\delta_0}(y) \quad \text{for } y \in \{y \in \Omega_0 |\ \mathrm{dist}(y,\partial\Omega_0) \leq h\}.$$

Meanwhile, for any $y \in \{y \in \Omega_0 \mid \mathrm{dist}(y,\partial\Omega_0) > h\}$, since $\underline{u}_1(y,T^*)$, $\underline{u}_2(y,T^*)$, $\phi^*_{\delta_0}(y)$ and $\psi^*_{\delta_0}(y)$ are all > 0 , there exists a small $\mu_2 > 0$ such that $\frac{\underline{u}_1(y,T^*)}{\phi^*_{\delta_0}(y)}$ and $\frac{\underline{u}_2(y,T^*)}{\psi^*_{\delta_0}(y)} \geq \mu_2$ for $y \in \{y \in \Omega_0 \mid \mathrm{dist}(y,\partial\Omega_0) > h\}$. Therefore, a sufficiently small $\mu > 0$ satisfying $\mu \leq \min\{\mu_1,\mu_2\}$ can be chosen to make sure $(\underline{u}_1(y,T^*),\underline{u}_2(y,T^*)) \geq (\mu\phi^*_{\delta_0}(y),\mu\psi^*_{\delta_0}(y))$ in $\overline{\Omega}_0$.

Set

$$U_1 = \mu e^{-\lambda^*_{\delta_0}(t-T^*)}\phi^*_{\delta_0}(y) \quad \text{and} \quad U_2 = \mu e^{-\lambda^*_{\delta_0}(t-T^*)}\psi^*_{\delta_0}(y).$$

It is easy to verify that $(U_1(y,t),U_2(y,t))$ is a positive solution of the problem

$$\begin{cases} U_{1t} = \beta U_2(K-\delta_0) - \sigma U_1 - \left[b + (a-b)\frac{K+3\delta_0}{K}\right]U_1 - \delta_0 U_1, & y \in \Omega_0,\ t \geq T^*, \\ U_{2t} = \frac{D\Delta U_2}{(\rho_\infty-\delta_0)^2} + \sigma u_1 - \alpha U_2 - \left[b + (a-b)\frac{K+3\delta_0}{K}\right]U_2 - \delta_0 U_2, & y \in \Omega_0,\ t \geq T^*, \\ U_1 = U_2 = 0, & y \in \partial\Omega_0,\ t \geq T^*, \\ U_1(y,T^*) = \mu\phi^*_{\delta_0}(y), U_2(y,T^*) = \mu\psi^*_{\delta_0}(y), & y \in \overline{\Omega}_0. \end{cases}$$

Recalling that $\Delta\psi^*_{\delta_0}(y) = \Delta\zeta(y) = -\lambda_1\zeta(y) \leq 0$ yields

$$U_{2t} \leq \frac{D\Delta U_2}{\rho^2(t)} + \sigma u_1 - \alpha U_2 - \left[b + (a-b)\frac{K+3\delta_0}{K}\right]U_2 - \delta_0 U_2 \quad \text{for all } y \in \Omega_0, t \geq T^*,$$

which means that $(U_1(y,t),U_2(y,t))$ is a lower solution of problem (3.12), so by the comparison principle we have that

$$(\underline{u}_1(y,t),\underline{u}_2(y,t)) \geq (U_1(y,t),U_2(y,t)) \quad \text{for all } y \in \Omega_0, t \geq T^*,$$

which together with (3.13) gives

$$(u_1(y,t), u_2(y,t)) \geq (U_1(y,t), U_2(y,t)) = (\mu e^{-\lambda^*_{\delta_0}(t-T^*)}\phi^*_{\delta_0}(y), \mu e^{-\lambda^*_{\delta_0}(t-T^*)}\psi^*_{\delta_0}(y)),$$

for all $y \in \overline{\Omega}_0, t \geq T^*$. But since $\lambda^*_{\delta_0} < 0$, $u_1(y,t)$ and $u_2(y,t)$ tends to ∞ as t goes to ∞, for any fixed $y \in \overline{\Omega}_0$ which contradicts (3.10). The proof is now completed. $\square$

4 Numerical simulations

In this section we carry out some numerical simulations in one space dimension to illustrate our theoretical analysis.

Regarding the domain growth, we choose $\Omega(t) = (0, x(t)) = (0, \rho(t)y)$, where $\rho(t) = \frac{e^t}{1+\frac{1}{m}(e^t-1)}$ and $y \in \Omega_0 = (0,1)$. Then, the domain grows like $\rho(t)$ from initial rate $\rho(0) = 1$ to the final rate $\rho_\infty = m$ with $m > 1$. To highlight the impacts of the domain growth on the transmission of rabies, we first fix the following parameters

$$D = 1, \quad a = 1, \quad b = 0.2, \quad K = 1000, \quad \alpha = 0.01, \quad \beta = 0.08, \quad \sigma = 0.05$$

and subsequently obtain $\lambda_1 = \pi^2$. Next, we choose a different growth rate $\rho(t)$ for the domain and study the asymptotic behavior of the solution to the problem (1.4).

Example 4.1. Set $m = 1.2$ and we have

$$R_0 = \sqrt{\frac{\sigma\beta K}{(\sigma + a)(\frac{D}{\rho^2_{1\infty}}\lambda_1 + \alpha + a)}} = 0.64 < 1.$$

By Theorem 3.2, we know that the disease-free equilibrium of problem (1.4) is stable. One can see from Fig. 4.1 that the solution $(u_1(y,t), u_2(y,t), u_3(y,t))$ decays to zero, which consists with the result of Theorem 3.2.

Example 4.2. Set $m = 4$ and a direct calculation shows that

$$R_0 = \sqrt{\frac{\sigma\beta K}{(\sigma + a)(\frac{D}{\rho^2_{2\infty}}\lambda_1 + \alpha + a)}} = 1.05 > 1.$$

Theorem 3.3 shows that the disease-free equilibrium $(0,0,0)$ is now unstable. It is easily seen from Fig. 4.2 that (u_1, u_2, u_3) stabilizes to a positive steady state.

Comparing the above two cases, it can be seen that the infected but non-infectious population u_1 and rabid population u_2 vanish at small growth rate, but spread at large growth rate.

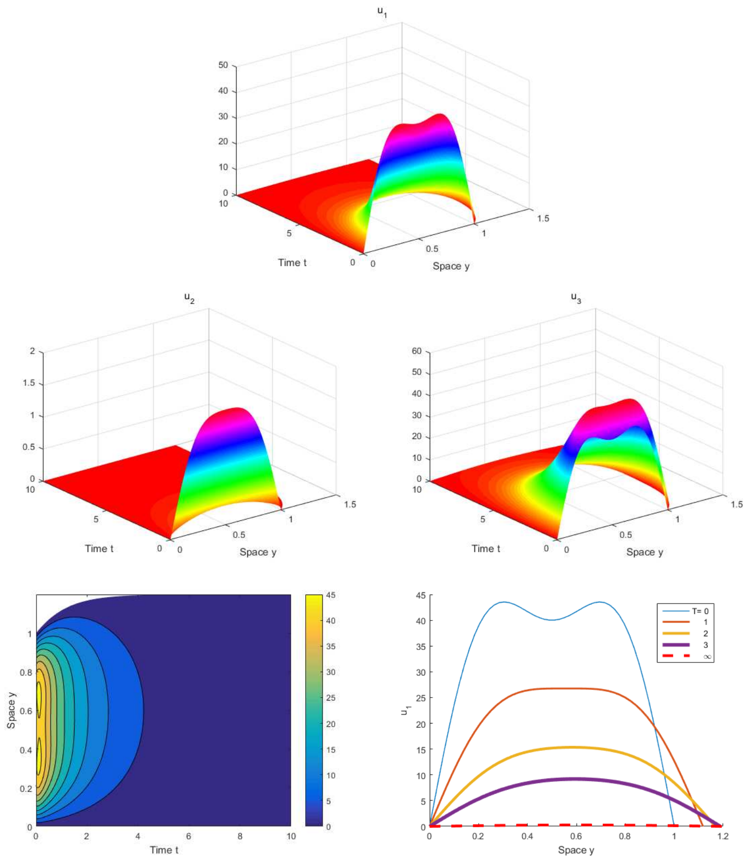

Figure 4.1: $\rho_1(t) = \frac{e^t}{1+\frac{1}{1.2}(e^t-1)}$. For small growth rate $\rho_1(t)$, we have $R_0 < 1$. The first three graphs show that (u_1, u_2, u_3) decays to zero quickly. The last two graphs in line 3 are the cross-sectional view (the left) and contour map (the right) of species u_1, respectively. The color bar in the graph of the cross-sectional view shows the density of the species u_1. The contour map shows the convergence of the temporal solution u_1 to the trivial solution (red dashed line).

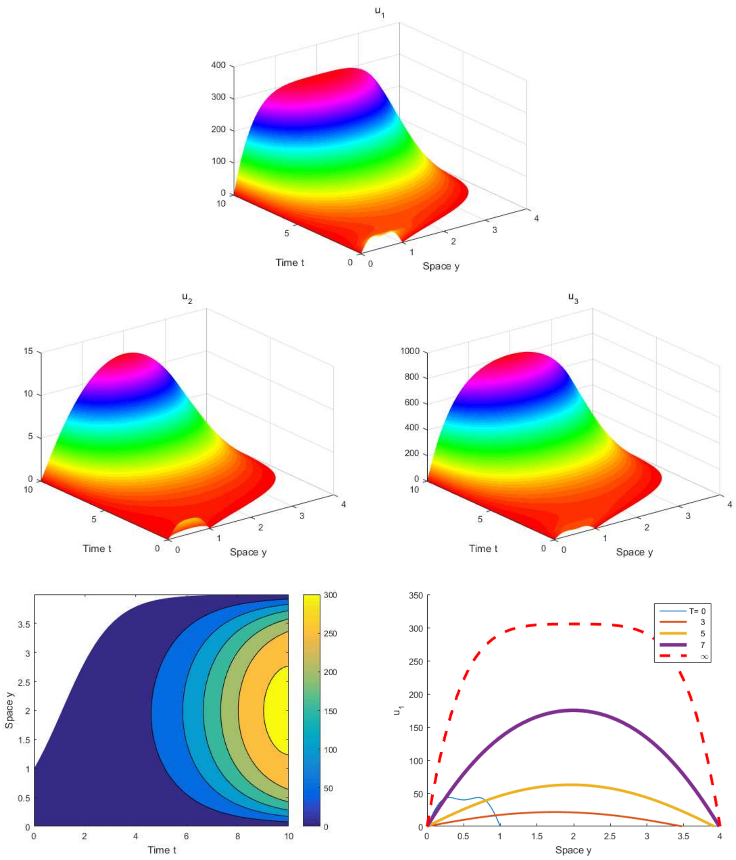

Figure 4.2: $\rho_2(t) = \frac{e^t}{1+\frac{1}{4}(e^t-1)}$. In this case, the growth rate $\rho_2(t)$ is now large enough to give that $R_0 > 1$. (u_1, u_2, u_3) tends to a positive steady state from the first three graphs. The last two graphs present the growth of the domain. The color bar in the graph of the cross-sectional view shows the density of the species u_1. The contour map shows the convergence of the temporal solution u_1 to the positive solution (red dashed line).

5 Discussion

Domain growth plays a significant role in the evolution of a biological population, and this has drawn much attention recently. In order to explore the impact of the domain growth on the transmission of fox rabies, we investigate a SEI model for fox rabies with uniform and isotropic domain growth.

We first transform the SEI model on the growing domain into a reaction-diffusion system on a fixed domain, and the basic reproduction number R_0 is introduced by spectral analysis and the so-called next infection operator. The relationship between R_0 and ρ_∞ directly follows by the explicit expression of R_0 which is determined by the variational method. Then, the stability of the disease-free equilibrium in terms of the threshold value R_0 is investigated by the upper and lower solutions method. It is proved in Theorem 3.2 that if $R_0 < 1$, the disease-free steady state $(0,0,0)$ for the problem (1.4) is locally asymptotically stable, while if $R_0 > 1$, the disease-free equilibrium $(0,0,0)$ is unstable according to Theorem 3.3. Finally our analytical results are clearly supported by numerical simulations. When $R_0 < 1$, the solution of (1.4) decays to zero when the domain growth is small (see Fig. 4.1) while when $R_0 > 1$, the disease-free equilibrium is unstable at a large domain growth (see Fig. 4.2). Our results show that a large growth of the domain has a negative effect on the stability of disease-free equilibrium, in the sense that it works against the prevention and control of rabies.

However, we can not derive the existence and uniqueness of the positive equilibrium. Moreover, all coefficients except $\rho(t)$ are constants in the problem (1.4), but in fact rabies is mainly affected by spatial heterogeneity and spatial distribution of habitats [20, 21], which implies that the diffusion coefficient D and the disease transmission coefficient β (and other constants) depend on the location x. We plan to investigate these problems in the future.

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Stability of positive equilibrium of a Nicholson blowflies model with stochastic perturbations

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Abstract. This paper is concerned with the stability problem of the positive equilibrium of a Nicholson's blowflies model with nonlinear density-dependent mortality rate subject to stochastic perturbations. More specifically, the existence of a unique positive equilibrium of a Nicholson's blowflies model described by the delay differential equation

$$N'(t) = -\left(a - be^{-N(t)}\right) + \beta N(t-\tau)e^{-\gamma N(t-\tau)}$$

is first quoted. It is assumed that the underlying model in noisy environments is exposed to stochastic perturbations, which are proportional to the derivation of the state from the equilibrium point. Then, by utilizing a stability criterion formulated for linear stochastic differential delay equations, explicit stability conditions are obtained. An extension to models with multiple delays is also presented.

Keywords: Nicholson's blowflies model, nonlinear mortality rate, stochastic perturbations, asymptotic stability.

2010 Mathematics Subject Classification: 34C25, 34K11, 34K25.

1 Introduction

Delay differential equations (DDEs) are typically used to describe dynamics of biology and ecology systems [3,4]. For example, Gurney et al. [5] proposed the following DDE

$$N'(t) = -\alpha N(t) + \beta N(t-\tau)e^{-\gamma N(t-\tau)} \tag{1.1}$$

to model the laboratory population of the Australian sheep-blowfly, where $N(t)$ represents the population size at time t, α is the per capita daily adult mortality rate, β is the maximum per capita daily egg production rate, $\frac{1}{\gamma}$ is the size at which the population reproduces at its

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maximum rate and $\tau > 0$ is the generation time (i.e. the time taken from birth to maturity). This equation is known as the celebrated Nicholson's blowflies equation.

In the past four decades, Nicholson equation and its extensions have been extensively studied (see, for example, [2, 7, 12, 14] and the references therein). In particular, Wang et al. [13] considered a stochastic variant of model (1.1) where the mortality rate α is affected by environmental noises, $\alpha \rightsquigarrow \alpha - \sigma dB(t)$, which is presented by the following Itô-type differential equation

$$dN(t) = \left[-\alpha N(t) + \beta N(t-\tau)e^{-\gamma N(t-\tau)}\right] dt + \sigma N(t)dB(t) \tag{1.2}$$

with initial condition $N(s) = \phi(s)$, $s \in [-\tau, 0]$, $\phi \in C([-\tau, 0], [0, \infty))$ and $\phi(0) > 0$. Finite ultimate estimations for $\limsup_{t\to\infty} E[N(t)]$ and $\limsup_{t\to\infty} \frac{1}{t} E\left[\int_0^t N(s)ds\right]$ were obtained under condition $\alpha > \sigma^2/2$. The results of [13] were later extended to stochastic Nicholson's blowflies differential equations with regime switching

$$dN(t) = \left[-\alpha_{r_t} N(t) + \beta_{r_t} N(t-\tau_{r_t})e^{-\gamma_{r_t} N(t-\tau_{r_t})}\right] dt + \sigma_{r_t} N(t)dB(t) \tag{1.3}$$

in [17], where $(r_t)_{t\geq 0}$ is a finite state continuous-time Markov chain. An extension of (1.2) to include a patch structure was also investigated in recent work [6].

However, the aforementioned works only dealt with stochastic Nicholson-type models with linear density-dependent mortality rates of the form $D(N) = \alpha N$ with some positive constant α. As discussed in [2], a model of linear density-dependent mortality rate will only be most accurate for populations at low densities. In addition, according to marine ecologists, many models in fishery such as marine protected areas or models of B-cell chronic lymphocytic leukemia dynamics are described by Nicholson-type delay differential equations of the form

$$N'(t) = -D(N(t)) + \beta N(t-\tau)e^{-\gamma N(t-\tau)}, \tag{1.4}$$

where the mortality rate function $D(N)$ is of the forms $D(N) = a - be^{-N}$ (type-I) or $D(N) = \frac{aN}{b+N}$ (type-II) with positive constants a and b. In the past few years, significant research attention has been devoted to studies of model (1.4) and its extensions. For example, by utilizing some reasoning techniques of the so-called fluctuation lemma combining with the method of using differential and integral inequalities, the problems of existence and global convergence of positive periodic/almost periodic solutions of Nicholson-type models with nonlinear mortality rates of type-I and type-II were investigated in [15] and [16], respectively. In [11], a novel approach based on comparison techniques via differential and integral inequalities and extended Lyapunov functions was developed to establish the existence, uniqueness and global attractivity of a positive periodic solution of Nicholson-type models with type-I mortality rate function. The proposed approach of [11] can also be utilized to derive conditions ensuring the global convergence of a unique positive equilibrium of autonomous (constant coefficients) Nicholson-type models with type-I mortality rates. However, up to date the study of Nicholson-type models as (1.4) subject to certain types of stochastic noises has received considerably less attention. It is noted that in population models, characteristic quantities as growth rates, environmental capacity, competition coefficients and some other parameters are always affected by environmental noises due to which model (1.4) is more suitable to be described by stochastic DDEs [8, 13]. Thus, it is relevent to study model (1.4) and its variants subject to certain type of stochastic noises. This motivates us for the present investigation.

In this paper, we study the problem of asymptotic stability in probability of a stochastic extension of model (1.4). Specifically, we consider Nicholson-type model (1.4) with nonlinear mortality rate function $D(N) = a - be^{-N}$ for positive scalars a, b and apply the method of Son et al. [11] to establish the existence of a unique positive equilibrium namely N^*. We then consider the case that model (1.4) is exposed to stochastic perturbations which are proportional to the derivation of its state from the equilibrium point N^*. This will be represented in the form of an Itô stochastic differential equation. Based on the linearization method and by utilizing a stability criterion established for linear stochastic differential delay equations [9, Lemma 2.1], explicit delay-dependent stability conditions are obtained. The presented result is then also extended to models with multiple delays.

2 Preliminaries

Consider the following Nicholson-type delay differential equation

$$N'(t) = -\left(a - be^{-N(t)}\right) + \beta N(t-\tau)e^{-\gamma N(t-\tau)}, \qquad t > 0, \tag{2.1}$$

with initial condition

$$N(s) = \phi(s) \quad \text{for } s \in [-\tau, 0] \quad \text{and} \quad \phi \in C([-\tau, 0], [0, \infty)), \phi(0) > 0, \tag{2.2}$$

where a, b, β, γ and τ are positive constants. It was shown in [11, Theorem 3.1] that if $b > a$ the initial value problem (IVP) governed by (2.1)-(2.2) has a unique solution $N(t, \phi)$ which is strictly positive on $[0, \infty)$ and satisfies $\liminf_{t\to\infty} N(t, \phi) \geq \ln(\frac{b}{a})$. Moreover, if $\frac{\beta}{\gamma e} < a < b$ then, for any solution $N(t, \phi)$ of (2.1)-(2.2), it holds that [11, Proposition 5.1]

$$\ln\left(\frac{b}{a}\right) \leq \liminf_{t\to\infty} N(t, \phi) \leq \limsup_{t\to\infty} N(t, \phi) \leq \ln\left(\frac{b}{a - \frac{\beta}{\gamma e}}\right). \tag{2.3}$$

2.1 Positive equilibrium

By substituting $N(t) = N^*$, a positive equilibrium point of (2.1) is defined by the following algebraic equation

$$-a + be^{-N^*} + \beta N^* e^{-\gamma N^*} = 0. \tag{2.4}$$

Assume that the parameters β, γ, a and b of model (2.1) satisfy the following condition

$$\beta\left(\frac{1}{\gamma e} + \max\left\{\frac{1}{e^2}, \frac{1 - \gamma\ln(\frac{b}{a})}{e^{\gamma\ln(\frac{b}{a})}}\right\}\right) < a < b. \tag{2.5}$$

Then, by (2.3), any positive equilibrium point of (2.1) is confined within the range $[r_1, r_2]$, where $r_1 = \ln(\frac{b}{a})$ and $r_2 = \ln\left(\frac{b}{a - \frac{\beta}{\gamma e}}\right)$.

Lemma 2.1. *Assume that $\frac{\beta}{\gamma e} < a < b$. Then, for any $x \in [r_1, r_2]$, where $r_1 = \ln(\frac{b}{a})$, $r_2 = \ln\left(\frac{b}{a - \frac{\beta}{\gamma e}}\right)$, it holds that*

$$|1 - \gamma x|e^{-\gamma x} \leq \max\left\{\frac{1}{e^2}, \frac{1 - \gamma\ln(\frac{b}{a})}{e^{\gamma\ln(\frac{b}{a})}}\right\}.$$

Proof. Let $\varphi(x) = |1-\gamma x|e^{-\gamma x}$, $-\infty < x < \infty$. Note that $\varphi(x) = (1-\gamma x)e^{-\gamma x}$ for $x < 1/\gamma$ and $\varphi'(x) = \gamma(\gamma x - 2)e^{-\gamma x} < 0$. Thus, the function $\varphi(x)$ is strictly deceasing on the interval $(-\infty, 1/\gamma)$. On the other hand, for $x > 1/\gamma$, we have $\varphi'(x) = \gamma(2-\gamma x)e^{-\gamma x}$, $\varphi'(2/\gamma) = 0$, $\varphi'(x) > 0$ for $x \in (1/\gamma, 2/\gamma)$ and $\varphi'(x) < 0$ for $x > 2/\gamma$. Therefore, $\varphi(x) \le \varphi(2/\gamma) = \frac{1}{e^2}$ for any $x \ge 1/\gamma$. This shows that for any $x \in [r_1, r_2]$, we have

$$\varphi(x) \le \max\left\{\frac{1}{e^2}, \varphi(r_1)\right\} = \max\left\{\frac{1}{e^2}, \frac{1-\gamma\ln(\frac{b}{a})}{e^{\gamma\ln(\frac{b}{a})}}\right\}.$$

The proof of this lemma is now completed. □

Lemma 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = xe^{-\gamma x}$, $\gamma > 0$. Then, $f(x) \le (\gamma e)^{-1}$ for all $x \in \mathbb{R}$. Moreover, $f(x) = (\gamma e)^{-1}$ if and only if $x = 1/\gamma$.*

Proof. The derivative $f'(x)$ of $f(x)$ is given by

$$f'(x) = (1-\gamma x)\, e^{-\gamma x}.$$

Thus, $f'(1/\gamma) = 0$, $f'(x) > 0$ for $x < 1/\gamma$ and $f'(x) < 0$ for $x > 1/\gamma$. Therefore, the function $f(x)$ is strictly increasing on the interval $(-\infty, 1/\gamma)$ and decreasing on the interval $(1/\gamma, \infty)$. This shows that $f(x)$ attains its maximum $f(1/\gamma) = (\gamma e)^{-1}$ at $x = 1/\gamma$. Consequently, $f(x) \le (\gamma e)^{-1}$. The proof is completed. □

It is clear that the function $\Psi(N) = -a + be^{-N} + \beta N e^{-\gamma N}$ is continuous on $[r_1, r_2]$, $\Psi(r_1) = \beta r_1 e^{-\gamma r_1} > 0$ and $\Psi(r_2) = \beta(r_2 e^{-\gamma r_2} - \frac{1}{\gamma e}) < 0$ according to Lemma 2.2 and the fact $r_2 < 1/\gamma$. Thus, there exists an $N^* \in (r_1, r_2)$ such that $\Psi(N^*) = 0$, which is a positive equilibrium of (2.1). On the other hand, for any $N \in [r_1, r_2]$, by Lemma 2.1, we have $be^{-N} \ge be^{-r_2} = a - \frac{\beta}{\gamma e}$ and $|1-\gamma N|e^{-\gamma N} \le \max\left\{\frac{1}{e^2}, \frac{1-\gamma\ln(\frac{b}{a})}{e^{\gamma\ln(\frac{b}{a})}}\right\}$. Therefore,

$$\Psi'(N) = -be^{-N} + \beta(1-\gamma N)e^{-\gamma N} < 0, \qquad \forall N \in [r_1, r_2],$$

which implies that the function $\Psi(N)$ is strictly decreasing on $[r_1, r_2]$. By this, we can conclude under condition (2.5) that model (2.1) has a unique positive equilibrium point N^* which is defined by equation (2.4).

2.2 Stochastic perturbations

Considering that equation (2.1) is affected by some white noise of the environment, which is proportional to the derivation of $N(t)$ from the equilibrium N^* [1]. Then, model (2.1) can be represented by the following Itô stochastic differential equation [9]

$$dN(t) = \left[-D(N(t)) + \beta N(t-\tau)e^{-\gamma N(t-\tau)}\right] dt + \sigma(N(t) - N^*)dB(t), \tag{2.6}$$

where $D(N) = a - be^{-N}$, σ denotes the intensity of the white noise and $B(t)$ is an one-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$. Note that the equilibrium point N^* is also a stationary solution of the stochastic differential equation (2.6). We now define $N(t) = N^* + x(t)$ then, by (2.6), we have

$$dx(t) = \left[-D(x(t) + N^*) + \beta(N^* + x(t-\tau))e^{-\gamma(N^* + x(t-\tau))}\right] dt + \sigma x(t)dB(t). \tag{2.7}$$

Remark 2.3. By similar arguments of [17, Theorem 2.1] and [11, Theorem 3.1], it can be verified that for any initial function $\phi \in C([-\tau, 0], \mathbb{R})$, Eq. (2.7) possesses a unique solution $x(t, \phi)$ defined on the interval $[-\tau, \infty)$.

According to (2.4), we have $\beta N^* e^{-\gamma N^*} = a - be^{-N^*}$. Therefore,

$$\beta N^* e^{-\gamma(N^* + x(t-\tau))} = \left(a - be^{-N^*}\right) e^{-\gamma x(t-\tau)}.$$

This, together with (2.7), leads to

$$\begin{aligned} dx(t) = \Big[& -a + be^{-N^*} e^{-x(t)} + (a - be^{-N^*}) e^{-\gamma x(t-\tau)} \\ & + \beta e^{-\gamma N^*} x(t-\tau) e^{-\gamma x(t-\tau)} \Big] dt + \sigma x(t) dB(t). \end{aligned} \tag{2.8}$$

The asymptotic stability of the equilibrium N^* of (2.6) is equivalent to that of the zero solution $x = 0$ of (2.8) [10]. Thus, together with (2.8), we consider the following linearized equation at the zero point

$$d\tilde{x}(t) = [-\delta \tilde{x}(t) + p\tilde{x}(t-\tau)]dt + \sigma \tilde{x}(t) dB(t), \tag{2.9}$$

where $\delta = be^{-N^*}$ and

$$p = \beta e^{-\gamma N^*} - \gamma(a - be^{-N^*}) = \beta(1 - \gamma N^*) e^{-\gamma N^*}.$$

Note also that $N^* \le r_2 < 1/\gamma$, thus δ, p are positive coefficients.

2.3 Auxiliary results

In this section, we present some definitions of stability and auxiliary results which will be used to derive stability conditions of the positive equilibrium point N^* of (2.1).

Definition 2.4 ([9]). The zero solution $x = 0$ of (2.7) is said to be stable in probability if for any $\epsilon > 0$, $\eta > 0$, there exists a $\delta > 0$ such that $\mathbb{P}\{\sup_{t \ge 0} |x(t, \phi)| > \epsilon | \mathcal{F}_0\} < \eta$ for any initial function $\phi \in C([-\tau, 0], \mathbb{R})$ with $\mathbb{P}\{\sup_{s \in [-\tau, 0]} |\phi(s)| < \delta\} = 1$.

Definition 2.5 ([9]). The linearized Eq. (2.9) is said to be (i) mean square stable (MSS) if for any given $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that for any initial function ϕ with $\sup_{s \in [-\tau, 0]} \mathbf{E}|\phi(s)|^2 < \delta$, it holds that $\mathbf{E}|\tilde{x}(t, \phi)|^2 < \epsilon$ for all $t \ge 0$, where $\mathbf{E}\{\cdot\}$ denotes the mathematical expectation on $(\Omega, \mathcal{F}, \mathbb{P})$; and (ii) asymptotically mean square stable (AMSS) if it is MSS and any solution $\tilde{x}(t, \phi)$ of (2.9) satisfies $\lim_{t \to \infty} \mathbf{E}|\tilde{x}(t, \phi)|^2 = 0$.

Remark 2.6. As mentioned in [9,10], the AMSS property of (2.9) implies stability in probability of the zero solution of nonlinear equation (2.7). This fact will be used to derive stability conditions for the equilibrium N^*.

In the remaining of this section, let us reformulate an auxiliary result on asymptotic mean square stability of linear stochastic differential equations from [9]. Consider the following linear stochastic differential equation

$$dx = [Ax(t) + Bx(t-\tau)]dt + \sigma x(t) dB(t) \tag{2.10}$$

where A, B, σ, $\tau \ge 0$ are known constants.

Lemma 2.7 ([9, Lemma 2.1, p. 44]). *The zero solution of* (2.10) *is asymptotically mean square stable if and only if*

$$A + B < 0, \quad G^{-1} > \frac{\sigma^2}{2},$$

where

$$G = \frac{2}{\pi} \int_0^\infty \frac{dt}{(A + B\cos\tau t)^2 + (t + B\sin\tau t)^2}.$$

Moreover,

$$G = \begin{cases} \frac{Bq^{-1}\sin(q\tau)-1}{A+B\cos(q\tau)} & \text{for } B + |A| < 0,\ q = \sqrt{B^2 - A^2}, \\ \frac{1+|A|\tau}{2|A|} & \text{for } B = |A| < 0, \\ \frac{Bq^{-1}\sinh(q\tau)-1}{A+B\cosh(q\tau)} & \text{for } A + |B| < 0,\ q = \sqrt{A^2 - B^2}, \end{cases}$$

where $\sinh(\cdot)$ *and* $\cosh(\cdot)$ *are the hyperbolic sine and hyperbolic cosine functions, respectively.*

3 Stability conditions

For given scalars a, b, β, γ and τ, which satisfy condition (2.5), let N^* be the unique positive root of (2.4) in the interval $[r_1, r_2]$. We denote the following positive constants

$$\delta = be^{-N^*} \quad \text{and} \quad p = \beta(1 - \gamma N^*)e^{-\gamma N^*}. \tag{3.1}$$

We have the following result.

Theorem 3.1. *Assume that the condition given in Eq.* (2.5) *holds. Then, the linearized equation* (2.9) *is AMSS if and only if the following condition holds*

$$\frac{p\cosh\left(\tau\sqrt{\delta^2 - p^2}\right) - \delta}{\frac{p}{\sqrt{\delta^2 - p^2}}\sinh\left(\tau\sqrt{\delta^2 - p^2}\right) - 1} > \sigma^2/2, \tag{3.2}$$

where δ, p *are positive constants given in Eq.* (3.1).

Proof. As shown in the preceding section, under condition (2.5), the positive root N^* of (2.4) exists and is unique. Moreover, we have

$$-be^{-N^*} + \beta(1 - \gamma N^*)e^{-\gamma N^*} < 0.$$

Therefore, Eq. (2.9) is AMSS if and only if (see, Lemma 2.7)

$$G^{-1} > \sigma^2/2, \tag{3.3}$$

where

$$G = \frac{2}{\pi}\int_0^\infty \frac{dt}{(p\cos\tau t - \delta)^2 + (t + p\sin\tau t)^2}. \tag{3.4}$$

Moreover, the exact value of the constant G can be calculated via elementary functions as

$$G = \frac{q + \delta + pe^{-q\tau}}{q\,(q + \delta - pe^{-q\tau})},$$

where $q = \sqrt{\delta^2 - p^2}$. Using the fact that $\cosh(q\tau) = \sinh(q\tau) + e^{-q\tau}$, $q^2 - \delta^2 = -p^2$, we have

$$\begin{aligned}
&\left(q + \delta + pe^{-q\tau}\right)\left(p\cosh(q\tau) - \delta\right) \\
&\quad = \left(q + \delta + pe^{-q\tau}\right)\left(p\sinh(q\tau) + pe^{-q\tau} - \delta\right) \\
&\quad = p(q+\delta)\sinh(q\tau) + p^2 e^{-q\tau}\left(\sinh(q\tau) + e^{-q\tau}\right) + pqe^{-q\tau} - \delta(q+\delta) \\
&\quad = (q+\delta)(p\sinh(q\tau) - q) + pe^{-q\tau}\left(q - p\sinh(q\tau)\right) \\
&\quad = \left(q + \delta - pe^{-q\tau}\right)\left(p\sinh(q\tau) - q\right).
\end{aligned}$$

Therefore,

$$G = \frac{\frac{p}{q}\sinh(q\tau) - 1}{p\cosh(q\tau) - \delta}.$$

This, together with (3.3), leads to condition (3.2). The proof is completed. □

Remark 3.2. In a more restrictive case, we assume that

$$2\beta(1-\gamma N^*)e^{-\gamma N^*} < be^{-N^*} \quad \text{i.e. } \delta > 2p, \tag{3.5}$$

then the equality (3.4) can be estimated as follows

$$\begin{aligned}
G &\le \frac{2}{\pi}\int_0^\infty \frac{dt}{(\delta^2 - 2\delta p) + (t-p)^2} \\
&= \frac{1}{\sqrt{\delta^2 - 2\delta p}}\left(1 + \frac{2}{\pi}\arctan\frac{p}{\sqrt{\delta^2 - 2\delta p}}\right).
\end{aligned} \tag{3.6}$$

By (3.3) and (3.6), a sufficient condition for the AMSS of Eq. (2.9) is

$$\frac{\sqrt{\delta^2 - 2\delta p}}{1 + \frac{2}{\pi}\arctan\frac{p}{\sqrt{\delta^2 - 2\delta p}}} > \sigma^2/2. \tag{3.7}$$

For Nicholson-type DDEs with multiple delays

$$N'(t) = -\left(a - be^{-N(t)}\right) + \sum_{k=1}^{m}\beta_k N(t-\tau_k)e^{-\gamma_k N(t-\tau_k)}, \tag{3.8}$$

condition (2.5) is extended to (see [11], Theorem 5.2)

$$\sum_{k=1}^{m}\beta_k\left(\frac{1}{e\gamma_k} + \max\left\{\frac{1}{e^2}, \frac{1 - \gamma_k\ln\left(\frac{b}{a}\right)}{e^{\gamma_k\ln\left(\frac{b}{a}\right)}}\right\}\right) < a < b \tag{3.9}$$

and the positive root N^* of the equation

$$-a + b^{-N^*} + \left(\sum_{k=1}^{m}\beta_k e^{-\gamma_k N^*}\right)N^* = 0 \tag{3.10}$$

exists and is unique. By a similar process, Eq. (2.9) is now given as

$$d\tilde{x}(t) = \left[-\delta\tilde{x}(t) + \sum_{k=1}^{m}p_k\tilde{x}(t-\tau_k)\right]dt + \sigma\tilde{x}(t)dB(t), \tag{3.11}$$

where

$$\delta = be^{-N^*} \quad \text{and} \quad p_k = \beta_k(1-\gamma_k N^*)e^{-\gamma_k N^*}, \qquad k = 1, 2, \ldots, m. \tag{3.12}$$

Similar to Theorem 3.1, Eq. (3.11) is AMSS if and only if $G_m^{-1} > \sigma^2/2$, where

$$G_m = \frac{2}{\pi}\int_0^\infty \frac{dt}{\left(\sum_{k=1}^m p_k \cos\tau_k t - \delta\right)^2 + \left(t + \sum_{k=1}^m p_k \sin\tau_k t\right)^2}. \tag{3.13}$$

Unfortunately, the computation of exact value of G_m in (3.13) is still an unsolved problem [9]. To derive sufficient conditions, we use the estimating method as (3.7). More specifically, assume that

$$\Delta^2 = \delta^2 - 2\delta\sum_{k=1}^m p_k - 4\sum_{1\le i<j\le m} p_i p_j > 0. \tag{3.14}$$

Then, we have

$$\begin{aligned}
&\left(-\delta + \sum_{k=1}^m p_k\cos\tau_k t\right)^2 + \left(t + \sum_{k=1}^m p_k \sin\tau_k t\right)^2\\
&= t^2 + 2t\sum_{k=1}^m p_k\sin\tau_k t + \delta^2 - 2\delta\sum_{k=1}^m p_k\cos\tau_k t\\
&\quad + \left(\sum_{k=1}^m p_k\sin\tau_k t\right)^2 + \left(\sum_{k=1}^m p_k\cos\tau_k t\right)^2\\
&\ge t^2 - 2t\sum_{k=1}^m p_k + \delta^2 - 2\delta\sum_{k=1}^m p_k + \sum_{k=1}^m p_k^2\\
&\quad + 2\sum_{1\le i<j\le m} p_i p_j\cos(\tau_i - \tau_j)t\\
&\ge \left(t - \sum_{k=1}^m p_k\right)^2 + \Delta^2.
\end{aligned}$$

Therefore,

$$G_m \le \frac{2}{\pi}\int_0^\infty \frac{dt}{\left(t - \sum_{k=1}^m p_k\right)^2 + \Delta^2} = \frac{1}{\Delta}\left(1 + \frac{2}{\pi}\arctan\frac{\sum_{k=1}^m p_k}{\Delta}\right).$$

In summary, we have the following result.

Proposition 3.3. *Consider model (3.8) and assume that the derived conditions in Eqs. (3.9) and (3.14) are fulfilled, where δ and p_k, $k = 1, 2, \ldots, m$, are positive constants defined in (3.12). Then, the linearized equation (3.11) is AMSS if the following condition holds*

$$\frac{\sqrt{\delta^2 - 2\delta\sum\limits_{k=1}^m p_k - 4\sum\limits_{1\le i<j\le m} p_i p_j}}{1 + \frac{2}{\pi}\arctan\dfrac{\sum\limits_{k=1}^m p_k}{\sqrt{\delta^2 - 2\delta\sum\limits_{k=1}^m p_k - 4\sum\limits_{1\le i<j\le m} p_i p_j}}} > \frac{\sigma^2}{2}. \tag{3.15}$$

Remark 3.4. Clearly, conditions (3.2), (3.7) and (3.15) hold for sufficiently small σ. In other words, the positive equilibrium N^* of model (2.1) or (3.8) is stable in probability under small stochastic perturbations. In this regard, the result of Proposition 3.3 in this paper extends that of Theorem 5.2 in [11].

4 Simulations

Consider model (2.1) with $\beta = 1$. It can be seen that condition (2.5) holds if and only if

$$\frac{1}{\gamma e} + \max\left\{\frac{1}{e^2}, \frac{1-\ln\kappa}{\kappa}\right\} < a < b, \tag{4.1}$$

where $\kappa = \left(\frac{b}{a}\right)^{\gamma}$. Since the equation $\frac{1-\ln\kappa}{\kappa} = \frac{1}{e^2}$ has a unique positive root $\kappa_* \simeq 2.0576$, condition (4.1) holds if and only if

$$a > \begin{cases} \frac{1}{\gamma e} + \frac{1-\ln\kappa}{\kappa} & \text{if } \kappa \in (1, \kappa_*) \\ \frac{1}{\gamma e} + \frac{1}{e^2} & \text{if } \kappa \geq \kappa_*. \end{cases} \tag{4.2}$$

For $\gamma = 0.5$, $\kappa = 1.1$, $a = 1.6$ and $b = 1.936$, Eq. (2.4) has a unique positive root $N^* = 0.4399$. Then, we have $\delta = 1.247$ and $p = 0.626$. With the delay $\tau = 2$, by condition (3.2), the linearized equation (2.9) is AMSS if and only if $\sigma^2 < 2.0266$. Simulation results given in Figure 4.1 are taken with $\sigma = 1.42$ and various initial conditions. It can be seen that all sample trajectories converge to N^*, which supports the conclusion.

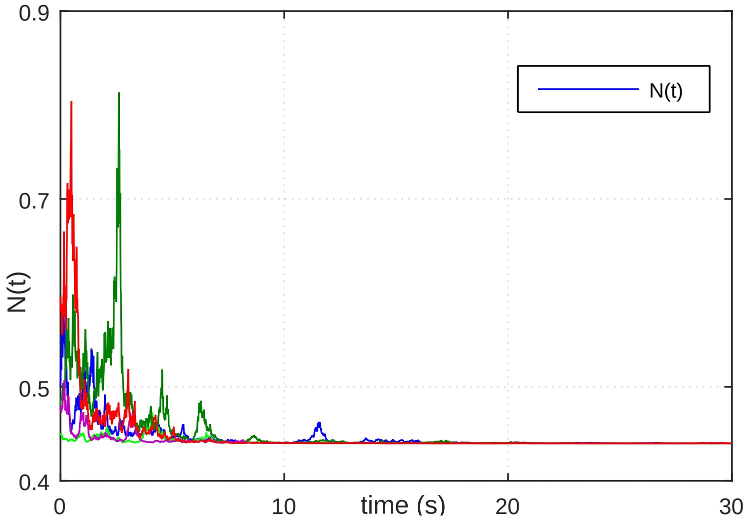

Figure 4.1: Sample trajectories of $N(t)$

5 Conclusions

In this paper, a stochastic Nicholson-type blowflies model with nonlinear density-dependent mortality rate has been investigated. Sufficient conditions have been derived to ensure the existence of a unique positive equilibrium which is stable in probability subject to stochastic perturbations of the white noise type. Numerical simulations have been given to illustrate the effectiveness of the derived stability conditions.

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Analysis of singular one-dimensional linear boundary value problems using two-point Taylor expansions

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Abstract. We consider the second-order linear differential equation $(x^2-1)y''+f(x)y'+g(x)y=h(x)$ in the interval $(-1,1)$ with initial conditions or boundary conditions (Dirichlet, Neumann or mixed Dirichlet–Neumann). The functions f, g and h are analytic in a Cassini disk $\mathcal{D}_r$ with foci at $x=\pm 1$ containing the interval $[-1,1]$. Then, the two end points of the interval may be regular singular points of the differential equation. The two-point Taylor expansion of the solution $y(x)$ at the end points ± 1 is used to study the space of analytic solutions in $\mathcal{D}_r$ of the differential equation, and to give a criterion for the existence and uniqueness of analytic solutions of the boundary value problem. This method is constructive and provides the two-point Taylor approximation of the analytic solutions when they exist.

Keywords: second-order linear differential equations, regular singular point, boundary value problem, Frobenius method, two-point Taylor expansions.

2020 Mathematics Subject Classification: 34A25, 34B05, 41A58.

1 Introduction

In [6] we considered the second-order linear equation $y''+f(x)y'+g(x)y=h(x)$ in the interval $(-1,1)$ with initial conditions or boundary conditions of the type Dirichlet, Neumann or mixed Dirichlet–Neumann. The functions f, g and h are analytic in a Cassini disk with foci at $x=\pm 1$ containing the interval $[-1,1]$. Then, the end points of the interval, where the boundary data are given, are regular points of the differential equation. The two-point Taylor expansion of the solution $y(x)$ at the end points ± 1 was used to give a criterion for the existence and uniqueness of analytic solutions of the initial or boundary value problem and approximate the solutions when they exist. In [1] we have considered problems that have an extra difficulty: one of the end points of the interval is a regular singular point of the differential equation, that is, we have considered the equation $(x+1)y''+f(x)y'+g(x)y=h(x)$.

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In this paper we continue our investigation considering problems where both end points of the interval are regular singular points of the differential equation. We consider initial or boundary value problems of the form

$$\begin{cases} (x^2-1)y''+f(x)y'+g(x)y=h(x) \quad \text{in } (-1,1), \\ B\begin{pmatrix} y(-1) \\ y(1) \\ y'(-1) \\ y'(1) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \end{cases} \tag{1.1}$$

where f, g and h are analytic in a Cassini disk with foci at $x=\pm 1$ containing the interval $[-1,1]$ (we give more details in the next section), $\alpha,\beta\in\mathbb{C}$ and B is a 2×4 matrix of rank two which defines the initial conditions or the boundary conditions (Dirichlet, Neumann or mixed).

The consideration of the interval $(-1,1)$ is not a restriction, as any real interval (a,b) can be transformed into the interval $(-1,1)$ by means of an affine change of the independent variable. The form of the differential equation in (1.1) is not a restriction either: consider the differential equation $(x^2-1)^2u''(x)+(x^2-1)F(x)u'(x)+G(x)u(x)=0$, with F and G analytic at $x=\pm 1$. After the change of the dependent variable $u=(x-1)^{\lambda}(x+1)^{\mu}y$, with λ a solution of the equation $4\lambda(\lambda-1)+2F(1)\lambda+G(1)=0$ and μ a solution of the equation $4\mu(\mu-1)-2F(-1)\mu+G(-1)=0$, the equation may be written in the form $(x^2-1)y''+f(x)y'+g(x)y=0$, with f and g analytic at $x=\pm 1$. On the other hand, the points $x=\pm 1$ are both indeed regular singular points of the differential equation $(x^2-1)y''+f(x)y'+g(x)y=h(x)$ when $|f(\pm 1)|+|g(\pm 1)|+|h(\pm 1)|\neq 0$; if $f(\pm 1)=g(\pm 1)=h(\pm 1)=0$, then both, $x=\pm 1$, are regular points, and problem (1.1) is the regular problem analyzed in [6]. If $f(1)=g(1)=h(1)=0$ and $|f(-1)|+|g(-1)|+|h(-1)|\neq 0$, then only one end point is a regular singular point of the equation, and problem (1.1) has been analyzed in [1]. We omit these restrictions here and then, the regular case studied in [6] or the cases studied in [1] may be considered particular cases of the more general one analyzed in this paper.

A standard theorem for the existence and uniqueness of solution of (1.1) is based on the knowledge of the two-dimensional linear space of solutions of the homogeneous equation $(x^2-1)y''+f(x)y'+g(x)y=0$ [2, Chapter 4, Section 1]. When f are g are constants or in some other particular situation, it is possible to find the general solution of the equation (sometimes via the Green function [2, Chapter 4], [7, Chapters 1 and 3])). But this is not possible in general situations and that standard criterion for the existence and uniqueness of solution of (1.1) is not practical. Other well-known criterion for the existence and uniqueness of solution of (1.1) is based on the Lax–Milgram theorem when (1.1) is an elliptic problem [3]. In any case, the determination of the existence and uniqueness of solution of (1.1) requires a non-systematic detailed study of the problem, like for example the study of the eigenvalue problem associated to (1.1) [2, Chapter 4, Section 2], [7, Chapter 7].

When f, g and h are analytic in a disk with center at $x=0$ and containing the interval $[-1,1]$, we may consider the initial value problem

$$\begin{cases} (x^2-1)y''+f(x)y'+g(x)y=h(x), \quad x\in(-1,1), \\ y(0)=y_0, \quad y'(0)=y_0', \end{cases} \tag{1.2}$$

with $y_0, y_0'\in\mathbb{C}$. Using the Frobenius method we can approximate the solution of this problem by its Taylor polynomial of degree $N\in\mathbb{N}$ at $x=0$, $y_N(x)=\sum_{n=0}^{N}c_k x^k$, where the coefficients

c_k are affine functions of $c_0 = y_0$ and $c_1 = y'_0$. By imposing the boundary conditions given in (1.1) over $y_N(x)$, we obtain an algebraic linear system for y_0 and y'_0. The existence and uniqueness of solution to this algebraic linear system gives us information about the existence and uniqueness of solution of (1.1). This procedure, although theoretically possible, has a difficult practical implementation since the data of the problem are given at $x = \pm 1$, not at $x = 0$ (see [6] for further details).

In [6] we improved the ideas of the previous paragraph for the regular case (when $f(-1) = g(-1) = h(-1) = 0$) using, not the standard Taylor expansion in the associated initial value problem (1.2), but a two-point Taylor expansion [4] at the end points $x = \pm 1$ directly in the differential equation and in the boundary conditions. The convergence region for a two-point Taylor expansion is a Cassini disk (see Figure 2.1), and this Cassini disk avoids the possible singularities of the coefficient functions located near the interval $[-1, 1]$ more efficiently than the standard Taylor disk [5].

In this paper we investigate if a two-point Taylor expansion at the end points $x = \pm 1$ also works for the more general problem (1.1), in particular when both, $x = -1$ and $x = 1$ are regular singular points of the equation. Thus, we use the two-point Taylor expansion of the solution $y(x)$ to give a criterion for the existence and uniqueness of analytic solutions based on the data of the problem, not based on the knowledge of the general solution of the differential equation.

The paper is organized as follows. In the next section we introduce some elements of the theory of two-point Taylor expansions and study the space S of analytic solutions of the differential equation $(x^2 - 1)y'' + f(x)y' + g(x)y = h(x)$. In Section 3 we derive the two-point Taylor expansion at the end points $x = \pm 1$ of the functions of S (when S is nonempty). In Section 4 we give an algebraic characterization of S that we use, in Section 5, to formulate a criterion of existence and uniqueness of analytic solutions of problem (1.1). Section 6 includes some illustrative examples and Section 7 a few final remarks. The analysis of this paper paper follows the same pattern as the analysis of [5].

2 Global analytic solutions of the differential equation

Assume that the coefficient functions f, g and h in (1.1) are analytic in the Cassini disk $\mathcal{D}_r = \{z \in \mathbb{C} \mid |z^2 - 1| < r\}$ with foci at $z = \pm 1$ and Cassini's radius r, with $r > 1$ (see [4]). The requirement $r > 1$ assures that the interval $[-1, 1]$ is contained into the Cassini disk $\mathcal{D}_r$ (see Figure 2.1). Then, the three functions f, g and h, admit a two-point Taylor series in $\mathcal{D}_r$ of the form [4],

$$f(z) = \sum_{n=0}^{\infty} [f_n^0 + f_n^1 z](z^2 - 1)^n, \quad g(z) = \sum_{n=0}^{\infty} [g_n^0 + g_n^1 z](z^2 - 1)^n, \quad h(z) = \sum_{n=0}^{\infty} [h_n^0 + h_n^1 z](z^2 - 1)^n, \tag{2.1}$$

where the coefficients of the expansions of f are [4]

$$\begin{aligned}
f_0^0 &:= \frac{f(1) + f(-1)}{2}, \quad f_0^1 := \frac{f(1) - f(-1)}{2}, \\
f_n^0 &:= \sum_{k=0}^{n} \frac{(n+k-1)!}{(n-k-1)!} \frac{(-1)^k f^{(n-k)}(1) + (-1)^n f^{(n-k)}(-1)}{n!\, k!\, 2^{n+k+1}}, \quad n = 1, 2, 3, \ldots, \\
f_n^1 &:= \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!} \frac{(-1)^k f^{(n-k)}(1) - (-1)^n f^{(n-k)}(-1)}{n!\, k!\, 2^{n+k+1}}, \quad n = 1, 2, 3, \ldots
\end{aligned} \tag{2.2}$$

The coefficients g_n^0 and g_n^1 of the expansion of g and the coefficients h_n^0 and h_n^1 of the expansion of h are defined by means of similar formulas. The three expansions in (2.1) converge absolute and uniformly to the respective functions f, g and h in $\mathcal{D}_r$ (see [4]). The regular case analyzed in [6] corresponds to the particular situation $f_0^0 = f_0^1 = g_0^0 = g_0^1 = h_0^0 = h_0^1 = 0$ (that is equivalent to $f(\pm 1) = g(\pm 1) = h(\pm 1) = 0$).

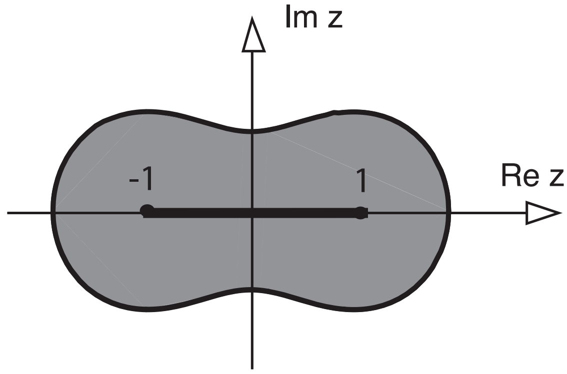

Figure 2.1: The Cassini disk $\mathcal{D}_r = \{z \in \mathbb{C} \mid |z^2 - 1| < r\}$ with foci at $z = \pm 1$ and radius $r > 1$ contains the real interval $[-1, 1]$.

As it is argued in [6], when $f(\pm 1) = g(\pm 1) = h(\pm 1) = 0$, any solution of the differential equation is analytic in $\mathcal{D}_r$. But the situation is different when $|f(1)| + |g(1)| + |h(1)| \neq 0$ and/or $|f(-1)| + |g(-1)| + |h(-1)| \neq 0$ (see [1]) and we need to introduce the following definition.

Definition 2.1. Denote by S_h the linear space of solutions of the homogeneous equation $(z^2 - 1)y'' + f(z)y' + g(z)y = 0$ that are analytic in $\mathcal{D}_r$. Denote by S the affine space of solutions of the complete equation $(z^2 - 1)y'' + f(z)y' + g(z)y = h(z)$ that are analytic in $\mathcal{D}_r$.

From Frobenius theory we know that the critical exponents of the homogeneous differential equation $(z^2 - 1)y'' + f(z)y' + g(z)y = 0$ at $z = -1$ are $\mu_1(-1) = 0$ and $\mu_2(-1) = 1 + f(-1)/2$. When $\mu_2(-1) \notin \mathbb{Z}$ ($f(-1) \notin 2\mathbb{Z}$), one independent solution of the homogeneous equation is analytic in $\mathcal{D}_r \setminus \{1\}$ and the other one is not, as it is of the form $(z+1)^{\mu_2(-1)}a(z)$ with $a(z)$ analytic in $\mathcal{D}_r$. When $\mu_2(-1) = 0, -1, -2, \ldots$, ($f(-1) \in -2\mathbb{N}$), one independent solution of $(z^2 - 1)y'' + f(z)y' + g(z)y = 0$ is analytic in $\mathcal{D}_r \setminus \{1\}$ and the other one is not, as it is of the form $a_1(z)\log(z+1) + (z+1)^{\mu_2(-1)}a_2(z)$ with $a_1(z)$ and $a_2(z)$ analytic in $\mathcal{D}_r \setminus \{1\}$. When $\mu_2(-1) \in \mathbb{N}$ ($f(-1) \in 2\mathbb{N} \cup \{0\}$), one independent solution of the homogeneous equation is analytic in $\mathcal{D}_r \setminus \{1\}$ (and it is canceled $\mu_2(-1)$ times at $z = -1$) and the other one is of the form $(z+1)^{\mu_2(-1)}a_1(z)\log(z+1) + a_2(z)$ with $a_1(z)$ and $a_2(z)$ analytic in $\mathcal{D}_r \setminus \{1\}$. Therefore, when $\mu_2(-1) \in \mathbb{N}$, may be only one or may be two independent solutions of $(z^2 - 1)y'' + f(z)y' + g(z)y = 0$ analytic at $z = -1$.

The discussion is similar at the point $z = 1$ replacing $f(-1)$ by $-f(1)$, that is, $\mu_1(1) = 0$ and $\mu_2(1) = 1 - f(1)/2$: when $f(1) \notin 2\mathbb{Z}$, one independent solution of the homogeneous equation is analytic in $\mathcal{D}_r \setminus \{-1\}$ and the other one is not. When $f(1) \in 2\mathbb{N}$, one independent solution of $(z^2 - 1)y'' + f(z)y' + g(z)y = 0$ is analytic in $\mathcal{D}_r \setminus \{-1\}$ and the other one is not. When $f(1) \in -2\mathbb{N} \cup \{0\}$, one independent solution of the homogeneous equation is analytic in $\mathcal{D}_r \setminus \{-1\}$ (and it is canceled $\mu_2(1)$ times at $z = 1$) and the other one may be or may be not analytic in $\mathcal{D}_r \setminus \{-1\}$. Therefore, when $\mu_2(1) \in \mathbb{N}$, may be only one or may be two independent solutions of $(z^2 - 1)y'' + f(z)y' + g(z)y = 0$ analytic at $z = 1$.

Then, all the possibilities may be summarized as follows: When $f(-1) \neq 0, 2, 4, \ldots$ or $f(1) \neq 0, -2, -4, \ldots$, then the homogeneous equation has only the null solution or a one-dimensional space of analytic solutions in $\mathcal{D}_r$. When $f(-1) = 0, 2, 4, \ldots$ and $f(1) = 0, -2, -4, \ldots$ then everything is possible: the homogeneous equation has only the null solution, it has a one-dimensional space or it has a two-dimensional space of analytic solutions in $\mathcal{D}_r$.

From the above discussion we conclude that

$$\dim(S_h) = \begin{cases} 0 \text{ or } 1 & \text{when} \quad f(-1) \neq 0, 2, 4, \ldots \text{ or } f(1) \neq 0, -2, -4, \ldots \\ 0, 1 \text{ or } 2 & \text{when} \quad f(-1) = 0, 2, 4, \ldots \text{ and } f(1) = 0, -2, -4, \ldots \end{cases}$$

On the other hand, it is clear that $S = y_p + S_h$, where $y_p(z)$ is a particular solution of $(z^2-1)y'' + f(z)y' + g(z)y = h(z)$ that is analytic in $\mathcal{D}_r$. The existence of that particular solution $y_p(z)$ is not guaranteed *a priori*; then, either $\dim(S) = \dim(S_h)$ or S is empty. (For example, the general solution of the equation $(z^2-1)y'' = 1$ is $y(z) = c_1 + c_2 z + z\log(\sqrt{(1-z)/(z+1)}) - \log(\sqrt{z^2-1})$, c_1, $c_2 \in \mathbb{C}$, then $\dim(S_h) = 2$ and S is empty. The general solution of the equation $(z^2-1)y'' - y' = 1$ is $y(z) = c_1 + c_2(\arcsin z + \sqrt{1-z^2}) - z$, c_1, $c_2 \in \mathbb{C}$, then $\dim(S_h) = \dim(S) = 1$.)

Once we have a picture of the spaces S and S_h in relation to the values of $f(\pm 1)$, we introduce the key point in the discussion of the paper. Any function $y(z) \in S$ or $y(z) \in S_h$ can be written in the form of a two-point Taylor expansion at the base points $z = \pm 1$ (see [4]),

$$y(z) = \sum_{n=0}^{\infty} [a_n + b_n z](z^2-1)^n, \quad z \in \mathcal{D}_r, \tag{2.3}$$

where the coefficients a_n and b_n are related to the values of the derivatives of $y(z)$ at $z = \pm 1$ in the same form as the coefficients f_n^0 and f_n^1 of f are related to the derivatives of f at $z = \pm 1$ in (2.2). If we can derive the coefficients a_n and b_n from $(z^2-1)y'' + f(z)y' + g(z)y = h(z)$, we will obtain the functions $y \in S$ in the form of a two-point Taylor series (2.3), when the space S is nonempty. This fact is not guaranteed *a priori* from the data of the problem. In the regular case $f(\pm 1) = g(\pm 1) = 0$, it is guaranteed that the dimension of S_h is two [6]. When only one of the end points is a regular singular point, then it is guaranteed that the dimension of S_h is, at least, one (see [1]).

In the more general case analyzed in this paper it is not guaranteed *a priori* that S_h or S are nonempty. Then, the existence of one analytic solution in $\mathcal{D}_r$ of the initial or boundary value problem (1.1) is not guaranteed *a priori* either; nor even when $h = 0$ (homogeneous case) or in the regular case $f(\pm 1) = g(\pm 1) = h(\pm 1) = 0$. In this paper we analyze the size of the spaces S_h and S and then, the existence and uniqueness of analytic solutions in $\mathcal{D}_r$ of the problem (1.1). We accomplish this task using that any function in S may be written in the form (2.3): in the remaining of the paper we replace the formal two-point Taylor series (2.3) in (1.1) and study if it is possible to obtain the coefficients a_n and b_n from the differential equation and the boundary conditions given in (1.1).

For any function $y(z)$ analytic in $\mathcal{D}_r$, the series (2.3) is absolute and uniformly convergent in the interval $[-1, 1]$, and we also have [6]

$$\begin{aligned} y'(z) &= \sum_{k=0}^{\infty} \left\{ [(2k+1)b_k + 2(k+1)b_{k+1}] + 2(k+1)a_{k+1}z \right\} (z^2-1)^k, \\ y''(z) &= \sum_{k=0}^{\infty} 2(k+1) \left\{ [(2k+1)a_{k+1} + 2(k+2)a_{k+2}] + [(2k+3)b_{k+1} + 2(k+2)b_{k+2}]z \right\} (z^2-1)^k, \end{aligned} \tag{2.4}$$

where the convergence of the series is absolute and uniform in the interval $[-1,1]$.

3 Two-point Taylor expansion representation of the functions of S

As it happens in the standard Frobenius method for initial value problems, when we replace f, g and h by their two-point Taylor expansions (2.1) in the differential equation $(z^2-1)y''+f(z)y'+g(z)y=h(z)$, and the solution $y(z)$ and its derivatives by their two-point Taylor expansions (2.3) and (2.4), we find that the coefficients a_n and b_n satisfy, for $n=0,1,2,\ldots$, a linear system of two recurrences

$$\begin{aligned}
&2(n+1)[(2n+f_0^1)a_{n+1}+f_0^0b_{n+1}]+2n(2n-1)a_n+2\sum_{k=0}^{n-1}(k+1)(f_{n-k}^0b_{k+1}+f_{n-k}^1a_{k+1})\\
&\quad+\sum_{k=0}^{n}\Big[(2k+1)f_{n-k}^0b_k+2(k+1)f_{n-k-1}^1a_{k+1}+g_{n-k}^0a_k+(g_{n-k}^1+g_{n-k-1}^1)b_k\Big]=h_n^0,\\
&2(n+1)[(2n+f_0^1)b_{n+1}+f_0^0a_{n+1}]+2n(2n+1)b_n+2\sum_{k=0}^{n-1}(k+1)(f_{n-k}^0a_{k+1}+f_{n-k}^1b_{k+1})\\
&\quad+\sum_{k=0}^{n}\Big[(2k+1)f_{n-k}^1b_k+g_{n-k}^0b_k+g_{n-k}^1a_k\Big]=h_n^1,
\end{aligned}\tag{3.1}$$

with $f_{-1}^0=g_{-1}^0=f_{-1}^1=g_{-1}^1:=0$. Then, in general, as it happens in the standard Frobenius method or in the particular regular boundary problem analyzed in [6], the computation of the coefficients a_n and b_n involve the previous coefficients a_0, $b_0,\ldots,a_{n-1}$ and b_{n-1}. But we find here a particularity that we do not find in the standard Frobenius method nor in the regular problem solved in [6]: in general, for a given $n=0,1,2,\ldots$, we can solve the linear system (3.1) for a_{n+1} and b_{n+1} if and only if

$$\begin{vmatrix}2n+f_0^1 & f_0^0\\ f_0^0 & 2n+f_0^1\end{vmatrix}\neq 0 \Leftrightarrow \begin{cases} f(-1)\equiv f_0^0-f_0^1\neq 2n,\\ f(1)\equiv f_0^0+f_0^1\neq -2n.\end{cases}$$

Then, if $f(-1)/2$ and $-f(1)/2\notin\mathbb{N}\cup\{0\}$, we can solve the linear system (3.1) for a_{n+1} and b_{n+1} for any $n=0,1,2,\ldots$ But if $f(-1)/2$ or $-f(1)/2\equiv n_0\in\mathbb{N}\cup\{0\}$, then we can solve the system (3.1) for a_{n+1} and b_{n+1} for any $n=0,1,2,\ldots$, except for $n=n_0$. If $f(-1)/2$ and $-f(1)/2\in\mathbb{N}\cup\{0\}$, then we define $n_0\equiv\max\{f(-1)/2,-f(1)/2\}$. For convenience, when $f(-1)/2\notin\mathbb{N}\cup\{0\}$ and $-f(1)/2\notin\mathbb{N}\cup\{0\}$ we define $n_0=-1$.

Therefore, in any case, we can solve the linear system (3.1) for a_{n+1} and b_{n+1} for $n=n_0+1,n_0+2,n_0+3,\ldots$ This means that we obtain all the coefficients a_n and b_n needed in (2.3) for $n=n_0+2,n_0+3,n_0+4,\ldots$, as a function of the first $2(n_0+2)$ coefficients $a_0,b_0,a_1,b_1,\ldots,a_{n_0+1},b_{n_0+1}$. But these $2(n_0+2)$ first coefficients are not totally free, as they must satisfy the equations (3.1) for $n=0,1,2,\ldots,n_0$. These facts impose $2(n_0+1)$ linear restrictions (not all of them necessarily independent) to the $2(n_0+2)$ first coefficients $a_0,b_0,a_1,b_1,\ldots,a_{n_0+1},b_{n_0+1}$. Let us denote these equations by $L_k[a_0,b_0,a_1,b_1,\ldots,a_{n_0+1},b_{n_0+1}]=0$, $k=1,2,3,\ldots,2n_0+2$. In general, these equations are non homogeneous; they are homogeneous when $h(z)=0$.

In the particular case of the regular problem analyzed in [6] we have that $n_0=0$, since $f(\pm1)=0$. Then, we can obtain from system (3.1) all the coefficients a_n and b_n for $n\geq 2$ as a function of the first four coefficients a_0, b_0, a_1 and b_1. In this case, the above mentioned set of restrictions consists of the equations (3.1) for $n = 0$. But using that $f(\pm 1) = g(\pm 1) = h(\pm 1) = 0$ we see that these equations are the tautology $0 = 0$ and then, they do not introduce any linear dependence between the coefficients a_0, b_0, a_1 and b_1.

As a difference with the Frobenius method where we only have one recurrence relation for the sequence of standard Taylor coefficients, here we have a system of two recurrences (3.1). But moreover, the computation of the coefficients a_n, b_n for $n \geq n_0 + 2$ requires the initial seed $a_0, b_0, a_1, b_1, \ldots, a_{n_0+1}, b_{n_0+1}$. These $2n_0 + 4$ coefficients satisfy the above mentioned $2n_0 + 2$ equations $L_k = 0$. This does not mean that the linear space S_h or the affine space S may have dimension two or more, these spaces have, of course, dimension at most two. It is happening that, apart from the affine space S of (true) solutions of $(z^2 - 1)y'' + f(z)y' + g(z)y = h(z)$, there is a bigger space of formal solutions W defined by

$$W := \left\{ y(z) = \sum_{n=0}^{\infty} [a_n + b_n z](z^2-1)^n \;\middle|\; a_n, b_n \text{ given in (3.1) for } n \geq n_0 + 2; \right.$$
$$(a_0, b_0, a_1, b_1, \ldots, a_{n_0+1}, b_{n_0+1}) \in \mathbb{C}^{2n_0+4}$$
$$\left. \text{with } L_k[a_0, b_0, \ldots, a_{n_0+1}, b_{n_0+1}] = 0,\ k = 1, 2, 3, \ldots, 2n_0 + 2 \right\}.$$

Formally, all the two-point Taylor series in W are solutions of $(z^2 - 1)y'' + f(z)y' + g(z)y = h(z)$. But not all of them are convergent, only a subset: the affine space S of (true) solutions of $(z^2 - 1)y'' + f(z)y' + g(z)y = h(z)$, that may be written in the form

$$S = \left\{ y \in W \mid \sum_{n=0}^{\infty} [a_n + b_n z](z^2-1)^n \text{ is uniformly convergent in } [-1, 1] \right\}.$$

In the following section we derive a more practical characterization of the space S in the form of two extra linear equations for the coefficients $a_0, b_0, a_1, b_1, \ldots, a_{n_0+1}, b_{n_0+1}$. This characterization allows us to give some more precise information about the size of the space S.

4 Algebraic characterization of the space S

From (3.1) and the discussion below that formula, we see that we may solve (3.1) for (a_n, b_n) for $n \geq n_0 + 2$ in the schematic form

$$\begin{aligned} a_n &= \sum_{k=0}^{n-1} [A_{n,k} a_k + B_{n,k} b_k] + E_n, \\ b_n &= \sum_{k=0}^{n-1} [C_{n,k} a_k + D_{n,k} b_k] + F_n, \end{aligned} \tag{4.1}$$

where the coefficients $A_{n,k}$, $B_{n,k}$, $C_{n,k}$ and $D_{n,k}$ are functions of f_k^0, f_k^1, g_k^0 and g_k^1. The coefficients $E_{n,k}$ and $F_{n,k}$ are functions of h_k^0 and h_k^1, $k = 0, 1, 2, \ldots, n-1$. For simplicity, we do not detail here these functions, as the precise value of these coefficients is not needed in the theoretical discussion. It is not needed either in computation in the particular examples, as it is more convenient the use of an algebraic manipulator to compute (a_n, b_n), $n \geq n_0 + 2$, directly from (3.1).

For a fixed $m \in \mathbb{N}$, $m \geq 2n_0 + 2$, and $n = 0, 1, 2, \ldots, m - n_0 - 1$, we define the vectors

$$v_n := (a_{n+n_0+2-m}, b_{n+n_0+2-m}, a_{n+n_0+3-m}, b_{n+n_0+3-m}, \ldots, a_{n+n_0}, b_{n+n_0}, a_{n+n_0+1}, b_{n+n_0+1}) \in \mathbb{C}^{2m},$$

with $a_{-k} = b_{-k} = 0$ for $k \in \mathbb{N}$. In particular, we have

$$v_{m-n_0-2} = (a_0, b_0, a_1, b_1, \ldots, a_{m-1}, b_{m-1})$$

and

$$v_0 = (0, 0, \ldots, 0, 0, a_0, b_0, a_1, b_1, \ldots, a_{n_0+1}, b_{n_0+1}).$$

For $n = 0, 1, 2, \ldots, m - n_0 - 2$, define the $(2m) \times (2m)$ matrix

$$M_n := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & \ldots & 0 & A_{n+n_0+2,0} & B_{n+n_0+2,0} & \ldots & \ldots & A_{n+n_0+2,n+n_0+1} & B_{n+n_0+2,n+n_0+1} \\ 0 & \ldots & 0 & C_{n+n_0+2,0} & D_{n+n_0+2,0} & \ldots & \ldots & C_{n+n_0+2,n+n_0+1} & D_{n+n_0+2,n+n_0+1} \end{pmatrix}. \tag{4.2}$$

The only non-null elements of this matrix are the corresponding to the entries $m_{i,i+2} = 1$, $i = 1, 2, 3, \ldots, 2m - 2$, and to the entries $m_{2m-1,k}$, $m_{2m,k}$, $k = 0, 1, 2, \ldots, n + n_0 + 1$. In particular we have

$$M_0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & \ldots & 0 & A_{n_0+2,0} & B_{n_0+2,0} & \ldots & \ldots & A_{n_0+2,n_0+1} & B_{n_0+2,n_0+1} \\ 0 & \ldots & 0 & C_{n_0+2,0} & D_{n_0+2,0} & \ldots & \ldots & C_{n_0+2,n_0+1} & D_{n_0+2,n_0+1} \end{pmatrix}$$

and

$$M_{m-n_0-2} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\ A_{m,0} & B_{m,0} & A_{m,1} & B_{m,1} & \ldots & \ldots & \ldots & A_{m,m-1} & B_{m,m-1} \\ C_{m,0} & D_{m,0} & C_{m,1} & D_{m,1} & \ldots & \ldots & \ldots & C_{m,m-1} & D_{m,m-1} \end{pmatrix}.$$

We also need, for $n = 0, 1, 2, \ldots, m - n_0 - 2$, the definition of the vector

$$c_n := (0, 0, \ldots, 0, 0, E_{n+2}, F_{n+2}) \in \mathbb{C}^{2m}.$$

Then, the system of recurrences (4.1) (that indeed represents (3.1)) can be written in matrix form

$$v_n = M_{n-1} v_{n-1} + c_{n-1}, \quad n = 1, 2, 3, \ldots, m - n_0 - 1.$$

To find the solution of this linear recurrence for the vector v_n, we define recurrently the following matrices

$$\begin{aligned} \mathcal{M}_0 :=& M_0, \quad \mathcal{M}_n := M_n \mathcal{M}_{n-1}, \\ \mathcal{C}_0 :=& c_0, \qquad \mathcal{C}_n := M_n \mathcal{C}_{n-1} + c_n, \quad n = 1, 2, 3, \ldots, m - n_0 - 2, \end{aligned}$$

or equivalently,

$$\mathcal{M}_n = \prod_{k=0}^{n} M_{n-k}, \quad \mathcal{C}_n = c_n + \sum_{k=0}^{n-1} [M_n \cdot M_{n-1} \cdots M_{k+1}] c_k, \quad n = 0, 1, 2, 3, \ldots, m - n_0 - 2.$$

Then, we find

$$v_{m-n_0-1} = \mathcal{M}_{m-n_0-2} v_0 + \mathcal{C}_{m-n_0-2},$$

or, in an extended form

$$\begin{pmatrix} \star \\ \star \\ \cdot \\ \cdot \\ \cdot \\ \star \\ \star \\ a_m \\ b_m \end{pmatrix} = \begin{pmatrix} \star \\ \star \\ \cdot \\ \cdot \\ \cdot \\ \star \\ \star \\ \mathcal{B}_{2m-1} \\ \mathcal{B}_{2m} \end{pmatrix} + \begin{pmatrix} \star & \star & \star & \star & \star & \cdots & \star & \star \\ \star & \star & \star & \star & \star & \cdots & \star & \star \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \star & \star & \star & \star & \star & \cdots & \star & \star \\ \star & \star & \star & \star & \star & \cdots & \star & \star \\ \star & \cdots & \star & \mathcal{M}_{2m-1,2m-2n_0-3} & \mathcal{M}_{2m-1,2m-2n_0-2} & \cdots & \mathcal{M}_{2m-1,2m-1} & \mathcal{M}_{2m-1,2m} \\ \star & \cdots & \star & \mathcal{M}_{2m,2m-2n_0-3} & \mathcal{M}_{2m,2m-2n_0-2} & \cdots & \mathcal{M}_{2m,2m-1} & \mathcal{M}_{2m,2m} \end{pmatrix} \begin{pmatrix} 0 \\ \cdot \\ 0 \\ a_0 \\ b_0 \\ \cdot \\ \cdot \\ a_{n_0+1} \\ b_{n_0+1} \end{pmatrix},$$

where $\mathcal{M}_{i,j}$ are the entrances of the last two rows and last $2n_0 + 4$ columns of the matrix $\mathcal{M}_{m-n_0-2}$, $\mathcal{B}_i$ are the last two components of the vector $\mathcal{C}_{m-n_0-2}$ and the $\star$ denote complex (unspecified) numbers. The two-point Taylor series of an analytic function in $\mathcal{D}_r$ converges in $[-1, 1]$ if it converges at $z = 0$ [4]. And it converges at $z = 0$ if and only if $\lim_{m\to\infty}(a_m, b_m) = (0, 0)$. Then, taking the limit $m \to \infty$ into the above equation we find

$$\begin{pmatrix} \star \\ \star \\ \cdot \\ \cdot \\ \cdot \\ \star \\ \star \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \star & \star & \star & \star & \star & \cdots & \star & \star \\ \star & \star & \star & \star & \star & \cdots & \star & \star \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \star & \star & \star & \star & \star & \cdots & \star & \star \\ \star & \star & \star & \star & \star & \cdots & \star & \star \\ \star & \cdots & \star & H_{1,1} & H_{1,2} & \cdots & H_{1,2n_0+3} & H_{1,2n_0+4} \\ \star & \cdots & \star & H_{2,1} & H_{2,2} & \cdots & H_{2,2n_0+3} & H_{2,2n_0+4} \end{pmatrix} \begin{pmatrix} 0 \\ \cdot \\ 0 \\ a_0 \\ b_0 \\ \cdot \\ \cdot \\ a_{n_0+1} \\ b_{n_0+1} \end{pmatrix} + \begin{pmatrix} \star \\ \star \\ \cdot \\ \cdot \\ \cdot \\ \star \\ \star \\ \gamma_1 \\ \gamma_2 \end{pmatrix},$$

where we have denoted

$$\begin{aligned} H_{i,j} &:= \lim_{m\to\infty} \mathcal{M}_{2m+i-2,2m-2n_0+j-4}, \quad i = 1, 2, \quad j = 1, 2, 3, \ldots, 2n_0 + 4, \\ \gamma_1 &= \lim_{m\to\infty} \mathcal{B}_{2m-1}, \quad \gamma_2 = \lim_{m\to\infty} \mathcal{B}_{2m}. \end{aligned} \tag{4.3}$$

Then, the two equations that we were looking for are, for $k = 1, 2$

$$H_k[a_0, b_0, \ldots, a_{n_0+1}, b_{n_0+1}] := H_{k,1} a_0 + H_{k,2} b_0 + \cdots + H_{k,2n_0+3} a_{n_0+1} + H_{k,2n_0+4} b_{n_0+1} + \gamma_k = 0. \tag{4.4}$$

Therefore, at this moment, we have found the more practical characterization of the space S of true solutions of $(z^2 - 1)y'' + f(z)y' + g(z)y = h(z)$ that we were looking for

$$\begin{aligned} S := \Bigg\{ y(z) = \sum_{n=0}^{\infty} [a_n + b_n z](z^2 - 1)^n \;\Bigg|\; & a_n, b_n \text{ given in (3.1) for } n \geq n_0 + 2; \\ & (a_0, b_0, a_1, b_1, \ldots, a_{n_0+1}, b_{n_0+1}) \in \mathbb{C}^{2n_0+4} \text{ with } L_k[a_0, b_0, a_1, b_1, \ldots, a_{n_0+1}, b_{n_0+1}] = 0 \\ & \text{for } k = 1, 2, 3, \ldots, 2n_0 + 2, \text{ and } H_k[a_0, b_0, a_1, b_1, \ldots, a_{n_0+1}, b_{n_0+1}] = 0 \text{ for } k = 1, 2 \Bigg\}. \end{aligned} \tag{4.5}$$

In other words, the $2n_0+4$ coefficients $a_0, b_0, a_1, b_1, \dots, a_{n_0+1}, b_{n_0+1}$ of the two-point Taylor expansion of any function in S must be a solution of the following linear system of $2n_0+4$ equations

$$\begin{cases} L_k[a_0, b_0, a_1, b_1, \dots, a_{n_0+1}, b_{n_0+1}] = 0, & k = 1, 2, 3, \dots, 2n_0+2, \\ H_k[a_0, b_0, a_1, b_1, \dots, a_{n_0+1}, b_{n_0+1}] = 0, & k = 1, 2. \end{cases} \tag{4.6}$$

This system is homogeneous when $h = 0$ (when $h_n^0 = h_n^1 = 0$) and non-homogeneous when $h \neq 0$. Let's denote $(4.6)_h$ the system (4.6) when $h = 0$. We know that $\dim(S_h) = 0$, 1 or 2. This means that $\text{rank}[(4.6)_h] = 2n_0+2$, $2n_0+3$ or $2n_0+4$ and then, the homogeneous system has a one or two-dimensional space of solutions or $S_h = \{0\}$. On the other hand, we know that $\dim(S) = 1$ or 2, or S is empty. This means that there are five possibilities:

(i) $\text{rank}[(4.6)] = \text{rank}[(4.6)_h] = 2n_0+2$; then $\dim(S) = \dim(S_h) = 2$,

(ii) $\text{rank}[(4.6)] = \text{rank}[(4.6)_h] = 2n_0+3$; then $\dim(S) = \dim(S_h) = 1$,

(iii) $\text{rank}[(4.6)] = \text{rank}[(4.6)_h] = 2n_0+4$; then $S_h = \{0\}$ and $S = \{y_p\}$,

(iv) $\text{rank}[(4.6)] = 2n_0+3$ or $2n_0+4$ and $\text{rank}[(4.6)_h] = 2n_0+2$; then $\dim(S_h) = 2$ and S is empty,

(v) $\text{rank}[(4.6)] = 2n_0+4$ and $\text{rank}[(4.6)_h] = 2n_0+3$; then $\dim(S_h) = 1$ and S is empty.

Therefore,

- When $\text{rank}[(4.6)_h] = 2n_0+4$, the unique analytic solution in $\mathcal{D}_r$ of the homogeneous equation $(z^2-1)y'' + f(z)y' + g(z)y = 0$ is the null solution and the complete equation $(z^2-1)y'' + f(z)y' + g(z)y = h(z)$ has a unique solution analytic in $\mathcal{D}_r$.
- When $\text{rank}[(4.6)_h] = 2n_0+2$ then either, $\dim(S) = 2$ or S is empty.
- When $\text{rank}[(4.6)_h] = 2n_0+3$ then either, $\dim(S) = 1$ or S is empty.

In the regular case we know that $\dim(S) = 2$ (it is proved in [6] that the only two equations $H_k = 0$ that define S in this case are linearly independent). But, in general, we need to compute the above ranks in order to obtain some information about the sizes of S and S_h.

4.1 Polynomial coefficients

When the coefficient functions f and g are polynomials, we can simplify the formulation of the above existence and uniqueness criterion. In general, the computation of the coefficients (a_n, b_n) requires a matrix M_n of size $(2m) \times (2m)$ with $m \geq n + n_0 + 2$. This means that we need matrices of increasing size to compute the coefficients when n increases. In the case of polynomial coefficients, the situation is different. The recurrences (3.1) are of constant order s independent of n and the computation of the coefficients a_n and b_n involves only the previous $2s$ coefficients a_{n-s}, $b_{n-s}, \dots, a_{n-1}$ and b_{n-1}. Thus, in this case, we do not need matrices of increasing size, but matrices of constant size $(2s) \times (2s)$.

The recurrence system (3.1) for polynomial coefficients is of the form

$$a_n = \sum_{k=n-s}^{n-1} [A_{n,k} a_k + B_{n,k} b_k] + E_n,$$

$$b_n = \sum_{k=n-s}^{n-1} [C_{n,k} a_k + D_{n,k} b_k] + F_n,$$

for a certain $s \in \mathbb{N}$, $n = n_0, n_0+1, n_0+2, \ldots$, with $a_{-k} = b_{-k} = 0$, $k \in \mathbb{N}$. The discussion is identical to the one developed in the general case analyzed above, but now we can eliminate the restriction $n \le m - n_0 - 2$. Moreover, we can simplify the computations because now, the size of the matrices M_n does not depend on n. We can now define the matrices M_n of fixed size $(2s) \times (2s)$ in the form

$$M_n := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\ A_{n+2,n+2-s} & B_{n+2,n+2-s} & \ldots & A_{n+2,0} & B_{n+2,0} & \ldots & \ldots & A_{n+2,n+1} & B_{n+2,n+1} \\ C_{n+2,n+2-s} & D_{n+2,n+2-s} & \ldots & C_{n+2,0} & D_{n+2,0} & \ldots & \ldots & C_{n+2,n+1} & D_{n+2,n+1} \end{pmatrix}$$

instead of the form (4.2), with $A_{n,-k} = B_{n,-k} = C_{n,-k} = D_{n,-k} = 0$ for $k \in \mathbb{N}$. The computation of the system (4.6) is identical. The only difference is that now, the matrices $\mathcal{M}_m$ are of size $(2s) \times (2s)$ $\forall m \in \mathbb{N}$ and the vectors $\mathcal{C}_m \in \mathbb{R}^{2s}$ $\forall m \in \mathbb{N}$.

5 Existence and uniqueness criterion for the boundary value problem (1.1)

Once we have the algebraic description (4.5) of the space S of solutions analytic in $\mathcal{D}_r$ of the equation $(z^2-1)y'' + f(z)y' + g(z)y = h(z)$, we focus our attention on the boundary value problem (1.1) stated in the introduction. We introduce now the two boundary conditions in order to find an algebraic description of the solutions of (1.1). From (2.3) and (2.4) we have

$$\begin{pmatrix} y(-1) \\ y(1) \\ y'(-1) \\ y'(1) \end{pmatrix} = T \begin{pmatrix} a_0 \\ b_0 \\ a_1 \\ b_1 \end{pmatrix},$$

where T is the regular matrix

$$T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix}.$$

(The first four coefficients a_0, b_0, a_1, b_1 of the two-point Taylor expansion (2.3) are related to $y(-1)$, $y(1)$, $y'(-1)$, $y'(1)$ by the matrix T^{-1}). Write the matrix BT, where B is the 2×4 matrix defining the boundary condition in (1.1), in the form

$$BT = \begin{pmatrix} R_{1,1} & R_{1,2} & R_{1,3} & R_{1,4} \\ R_{2,1} & R_{2,2} & R_{2,3} & R_{2,4} \end{pmatrix}.$$

Then, the boundary value problem (1.1) may be written in the following equivalent form that stresses the role of the first four coefficients of the two-point Taylor expansion of $y(x)$ in the boundary value equations

$$\begin{cases} (x^2-1)y''+f(x)y'+g(x)y = h(x) \ \text{ in } \ (-1,1), \\ R_1[a_0,b_0,a_1,b_1] := R_{1,1}a_0+R_{1,2}b_0+R_{1,3}a_1+R_{1,4}b_1-\alpha = 0, \\ R_2[a_0,b_0,a_1,b_1] := R_{2,1}a_0+R_{2,2}b_0+R_{2,3}a_1+R_{2,4}b_1-\beta = 0. \end{cases} \tag{5.1}$$

When we add the above two algebraic equations R_1 and R_2 to the set of equations (4.6) that describe the space S of solutions of $(x^2-1)y''+f(x)y'+g(x)y = h(x)$, we find that the coefficients $a_0, b_0, \ldots, a_{n_0+1}, b_{n_0+1}$ of the two-point Taylor solutions $y(x)$ of (5.1) (if any) are solutions of the algebraic linear system

$$\begin{cases} L_k[a_0,b_0,a_1,b_1,\ldots,a_{n_0+1},b_{n_0+1}] = 0, & k=1,2,3,\ldots,2n_0+2, \\ H_k[a_0,b_0,a_1,b_1,\ldots,a_{n_0+1},b_{n_0+1}] = 0, & k=1,2, \\ R_k[a_0,b_0,a_1,b_1] = 0, & k=1,2. \end{cases} \tag{5.2}$$

The remaining coefficients a_n, b_n for $n \geq n_0+2$ are obtained recursively from (3.1). The system (5.2) is a linear system of $2n_0+6$ equations with $2n_0+4$ unknowns (in the regular case, the system reduces to the last 4 equations). The existence and uniqueness of solutions of the system (5.2) is equivalent to the existence and uniqueness of solution of the problem (5.1). Then, we can finally formulate the following existence and uniqueness criterion for the boundary value problem (1.1).

Existence and uniqueness criterion

(i) If the system (5.2) has not a solution, then problem (1.1) has not an analytic solution in $\mathcal{D}_r$.

(ii) If the system (5.2) has a unique solution, then problem (1.1) has a unique analytic solution in $\mathcal{D}_r$.

(iii) If the system (5.2) has a one-dimensional space of solutions, then problem (1.1) has a one-dimensional family of analytic solutions in $\mathcal{D}_r$.

(iv) If the system (5.2) has a two-dimensional space of solutions, then problem (1.1) has a two-dimensional family of analytic solutions in $\mathcal{D}_r$.

Remark 5.1. According to the ranks of (4.6) and $(4.6)_h$ we have that

1. If $\text{rank}[(4.6)] = \text{rank}[(4.6)_h] = 2n_0+3$, then (iv) is not possible.

2. If $\text{rank}[(4.6)] = \text{rank}[(4.6)_h] = 2n_0+4$, then (iii) and (iv) are not possible.

3. If $\text{rank}[(4.6)] \neq \text{rank}[(4.6)_h]$, then only (i) is possible.

Remark 5.2. In practice, the coefficients of the two equations H_k in (5.2) are computed approximately, as the limits involved in their computation can be computed only approximately (see (4.3) and (4.4)). Therefore, the above existence and uniqueness criterion for solution of (1.1) is useful when system (5.2) is well conditioned. In order to determine the rank of system (5.2) and then, the dimension of the space of solutions, it is convenient to compute the limits of the determinants of the principal minors. On the other hand, the criterion is constructive as it provides an approximation to the solution of the form (2.3) once the coefficients $(a_0, b_0, \ldots, a_{n_0+1}, b_{n_0+1})$ are computed from (5.2).

Remark 5.3. When $-f(1)$ or $f(-1) \neq 0, 2, 4, \ldots$, the rank of the first $2n_0 + 4$ equations in (5.2) is, at least, $2n_0 + 3$ and (iv) is not possible. When $f(\pm 1) = g(\pm 1) = h(\pm 1) = 0$ (regular case), the system (5.2) only consists of the four last equations and the rank of the two equations $H_k = 0$, $k = 1, 2$, is 2. In any other case, the rank of the first $2n_0 + 4$ equations in the system (5.2) is not known *a priori*; it is calculated once we have computed the first $2n_0 + 4$ equations of system (5.2).

The key point in the discussion of the dimensions of S and S_h is system (4.6), and the key point in the discussion of the existence and uniqueness of problem (1.1) is system (5.2). In the examples of the following section we show how these systems are computed in practice and how the above criterium of existence and uniqueness may be implemented.

6 Examples

In the examples of this section we explore all the possible situations in relation to the values of $f(1)$ and $f(-1)$ and the sizes of the spaces S and S_h:

(i) $f(-1)/2$ and $-f(1)/2 \in \mathbb{N} \cup \{0\}$ and $\dim(S_h) = 2$, S is empty. Example 6.1.

(ii) $f(-1)/2$ and $-f(1)/2 \in \mathbb{N} \cup \{0\}$ and $\dim(S_h) = \dim(S) = 2$. Example 6.2.

(iii) $f(-1)/2$ or $-f(1)/2 \notin \mathbb{N} \cup \{0\}$ and $\dim(S_h) = \dim(S) = 1$. Example 6.3.

(iv) $f(-1)/2$ and $-f(1)/2 \in \mathbb{N} \cup \{0\}$ and $\dim(S_h) = \dim(S) = 1$. Example 6.4.

(v) $f(-1)/2$ or $-f(1)/2 \notin \mathbb{N} \cup \{0\}$ and $\dim(S_h) = 1$, S empty. Example 6.5.

(vi) $f(-1)/2$ and $-f(1)/2 \in \mathbb{N} \cup \{0\}$ and $\dim(S_h) = 1$, S empty. Example 6.6.

(vii) $f(-1)/2$ or $-f(1)/2 \notin \mathbb{N} \cup \{0\}$ and $S_h = \{0\}$, S non empty. Example 6.7.

(viii) $f(-1)/2$ and $-f(1)/2 \in \mathbb{N} \cup \{0\}$ and $S_h = \{0\}$, S non empty. Example 6.8.

In all the examples below, the parameters $a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, C, \alpha$ and β are arbitrary complex numbers. The limits in the m index of the sequences (4.3) are approximated by the value of the sequences at $m = 10$. We have selected examples for which the general solution of the differential equation is known; in this way we may check the validity of the existence and uniqueness criterion of Section 5.

Example 6.1. Consider the boundary value problem

$$\begin{cases} (x^2 - 1)y'' - (x+1)y' = 1 \quad \text{in } (-1, 1), \\ ay(-1) + by(1) + cy'(-1) + dy'(1) = \alpha, \\ \tilde{a}y(-1) + \tilde{b}y(1) + \tilde{c}y'(-1) + \tilde{d}y'(1) = \beta. \end{cases} \tag{6.1}$$

We have $f(x) = -(x+1)$, $g(x) = 0$ and $h(x) = 1$. As $f(-1) = 0$ and $f(1) = -2$, the critical exponents at the points $x = \pm 1$ are $\mu_2(-1) = 1$ and $\mu_2(1) = 2$ respectively and $n_0 = 1$. For this example, the recurrence relations (3.1) may be written in the form $v_{n+1} = M_n v_n + c_n$ with $v_n = (a_n, b_n)$, $c_n = (0, 0)$, $n = 2, 3, \ldots$, and

$$M_n = \begin{pmatrix} \frac{1-2n}{2(n+1)} & 0 \\ -\frac{1}{2(n+1)} & -\frac{2n+1}{2(n+1)} \end{pmatrix}.$$

System (4.6) is given by

$$\begin{cases} b_0 + 2a_1 + 2b_1 = -1, \\ b_0 + 2a_1 + 2b_1 = 0, \\ -3b_1 + 4a_2 - 4b_2 = 0, \\ 3b_1 - 4a_2 + 4b_2 = 0, \\ 0.028031a_2 = 0, \\ 0.165074a_2 + 0.358179b_2 = 0, \end{cases}$$

which has no solution. The solution to the homogeneous system $(4.6)_h$ is $(a_1, b_1, a_2, b_2) = (-b_0/2, 0, 0, 0)$ with $a_0, b_0 \in \mathbb{C}$ free parameters. Then, $\dim(S_h) = 2$, but S is empty. This conclusion is the same one that we obtain from the knowledge of the general solution of the differential equation in (6.1)

$$y(x) = c_1 + c_2 x(x-2) + \frac{1}{8}[(x^2 - 2x - 3)\log(x+1) - (x-1)^2 \log(x-1) - 2x].$$

Example 6.2. Consider the boundary value problem

$$\begin{cases} (x^2-1)y'' - 2x^3 y' + 2(x^2+1)y = 0 \quad \text{in } (-1,1), \\ ay(-1) + by(1) + cy'(-1) + dy'(1) = \alpha, \\ \widetilde{a}y(-1) + \widetilde{b}y(1) + \widetilde{c}y'(-1) + \widetilde{d}y'(1) = \beta. \end{cases} \tag{6.2}$$

We have $f(x) = -2x^3$, $g(x) = 2(x^2+1)$ and $h(x) = 0$. As $f(-1) = 2$ and $f(1) = -2$, the critical exponents at the points $x = \pm 1$ are $\mu_2(-1) = \mu_2(1) = 2$ respectively and $n_0 = 1$. For this example, the recurrence relations (3.1) may be written in the form $v_{n+1} = M_n v_n + c_n$ with $v_n = (a_{n-1}, b_{n-1}, a_n, b_n)$, $c_n = (0,0,0,0)$, $n = 2, 3, \ldots$, and

$$M_n = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2n-3}{2(n-1)(n+1)} & 0 & -\frac{(n-2)(2n-1)}{2(n-1)(n+1)} & 0 \\ 0 & \frac{1}{n+1} & 0 & -\frac{2n-1}{2(n+1)} \end{pmatrix}.$$

System (4.6)=$(4.6)_h$ is given by

$$\begin{cases} a_0 - a_1 = 0, \\ b_0 - 2b_1 = 0, \\ a_0 - a_1 = 0, \\ 0.015263a_1 - 0.030525a_2 = 0, \\ 0.150515b_1 - 0.35839b_2 = 0, \end{cases} \tag{6.3}$$

whose solution is $(a_1, b_1, a_2, b_2) = (a_0, b_0/2, a_0/2, 0.209988b_0)$ with $a_0, b_0 \in \mathbb{C}$ free parameters. As $\dim(S_h) = \dim(S) = 2$, the differential equation in (6.2) has a two-dimensional family of analytic solutions in $[-1,1]$, which agrees with the fact that the differential equation has two independent solutions e^{x^2} and $\sqrt{\pi}e^{x^2}\mathrm{erf}(x) + 2x$, both of them analytic in $[-1,1]$.

Now we apply the existence and uniqueness criterion of Section 5: the existence and uniqueness of solution of (6.2) is equivalent to the existence and uniqueness of solution of the linear system given by equations (6.3) and the boundary value conditions written in terms of the coefficients a_k and b_k

$$\begin{cases} (a+b)a_0 + (-a+b+c+d)b_0 + (-2c+2d)a_1 + (2c+2d)b_1 = \alpha, \\ (\widetilde{a}+\widetilde{b})a_0 + (-\widetilde{a}+\widetilde{b}+\widetilde{c}+\widetilde{d})b_0 + (-2\widetilde{c}+2\widetilde{d})a_1 + (2\widetilde{c}+2\widetilde{d})b_1 = \beta, \end{cases} \tag{6.4}$$

that, for this example, are given by

$$\begin{cases} (a+b-2c+2d)a_0 + (-a+b+2c+2d)b_0 = \alpha, \\ (\widetilde{a}+\widetilde{b}-2\widetilde{c}+2\widetilde{d})a_0 + (-\widetilde{a}+\widetilde{b}+2\widetilde{c}+2\widetilde{d})b_0 = \beta. \end{cases}$$

Then, problem (6.2) has a unique solution if and only if

$$\begin{pmatrix} a+b-2c+2d & -a+b+2c+2d \\ \widetilde{a}+\widetilde{b}-2\widetilde{c}+2\widetilde{d} & -\widetilde{a}+\widetilde{b}+2\widetilde{c}+2\widetilde{d} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{6.5}$$

The existence and uniqueness condition obtained with this criterion coincides with the one provided by the knowledge of the family of analytic solutions of the differential equation given in (6.2)

$$y(x, C_1, C_2) = C_1 e^{x^2} + C_2(\sqrt{\pi} e^{x^2} \operatorname{erf}(x) + 2x).$$

The standard criterion of existence and uniqueness of solution of problem (6.2) depends on the existence of two complex numbers C_1 and C_2 that make $y(x, C_1, C_2)$ compatible with the boundary conditions in (6.2), that is,

$$\begin{pmatrix} (a+b-2c+2d)e & (-a+b+2(c+d))\left(2+e\sqrt{\pi}\operatorname{erf}(1)\right) \\ (\widetilde{a}+\widetilde{b}-2\widetilde{c}+2\widetilde{d})e & (-\widetilde{a}+\widetilde{b}+2(\widetilde{c}+\widetilde{d}))\left(2+e\sqrt{\pi}\operatorname{erf}(1)\right) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{6.6}$$

It can be checked that (6.5) and (6.6) are equivalent.

Example 6.3. Consider the boundary value problem

$$\begin{cases} (x^2-1)y'' + x(1-2x^2)y' + 2y = 0 \quad \text{in } (-1,1), \\ ay(-1) + by(1) + cy'(-1) + dy'(1) = \alpha, \\ \widetilde{a}y(-1) + \widetilde{b}y(1) + \widetilde{c}y'(-1) + \widetilde{d}y'(1) = \beta. \end{cases} \tag{6.7}$$

In this problem, $f(x) = x(1-2x^2)$, $g(x) = 2$ and $h(x) = 0$. We have $f(-1) = 1$ and $f(1) = -1$, so the critical exponents at the points $x = \pm 1$ are $\mu_2(-1) = \mu_2(1) = 3/2$ respectively and $n_0 = -1$.

For this example, the recurrence relations (3.1) may be written in the form $v_{n+1} = M_n v_n + c_n$ with $v_n = (a_{n-1}, b_{n-1}, a_n, b_n)$, $c_n = (0,0,0,0)$, $n = 2, 3, \ldots$, and

$$M_n = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2(n-1)}{(n+1)(2n-1)} & 0 & -\frac{2n^2-4n+1}{(n+1)(2n-1)} & 0 \\ 0 & \frac{1}{n+1} & 0 & -\frac{2n-1}{2(n+1)} \end{pmatrix}.$$

System (4.6) = $(4.6)_h$ is given by

$$\begin{cases} a_0 - a_1 = 0, \\ b_0 - 2b_1 = 0, \\ -a_1 + 2a_2 = 0, \\ -2b_0 + b_1 + 4b_2 = 0, \\ 0.009033a_1 - 0.018067a_2 = 0, \\ 0.111897b_1 - 0.266438b_2 = 0, \end{cases} \tag{6.8}$$

whose solution is $(b_0, a_1, b_1, a_2, b_2) = (0, a_0, 0, a_0/2, 0)$, with $a_0 \in \mathbb{C}$ a free parameter. As $\dim(S_h) = \dim(S) = 1$, the differential equation in (6.7) has a one-dimensional family of analytic solutions in $[-1, 1]$, which agrees with the fact that the differential equation has two independent solutions, e^{x^2-1} and $e^{x^2-1}\int^x e^{-t^2}\sqrt{1-t^2}dt$, and just one of them is analytic in $[-1, 1]$.

Now we apply the existence and uniqueness criterion of Section 5: the existence and uniqueness of solution of (6.7) is equivalent to the existence and uniqueness of solution of the linear system given by equations (6.8) and (6.4), that, for this example, are given by

$$\begin{cases} (a+b)a_0 + (-2c+2d)a_0 = \alpha, \\ (\tilde{a}+\tilde{b})a_0 + (-2\tilde{c}+2\tilde{d})a_0 = \beta. \end{cases}$$

Then, problem (6.7) has a unique solution if and only if

$$\frac{\alpha}{a+b-2c+2d} = \frac{\beta}{\tilde{a}+\tilde{b}-2\tilde{c}+2\tilde{d}} \tag{6.9}$$

with $a+b-2c+2d \neq 0$ and $\tilde{a}+\tilde{b}-2\tilde{c}+2\tilde{d} \neq 0$. The existence and uniqueness condition obtained with this criterion coincides with the one provided by the knowledge of the family of analytic solutions of the differential equation given in (6.7)

$$y(x, C) = Ce^{x^2-1}.$$

The standard criterion of existence and uniqueness of solution of problem (6.7) depends on the existence of a complex number C that makes $y(x, C)$ compatible with the boundary conditions in (6.7), that is,

$$\begin{cases} aC + bC - 2cC + 2dC = \alpha, \\ \tilde{a}C + \tilde{b}C - 2\tilde{c}C + 2\tilde{d}C = \beta. \end{cases} \tag{6.10}$$

It can be checked that conditions (6.9) and (6.10) are the same.

Example 6.4. Consider the boundary value problem

$$\begin{cases} (x^2-1)y'' - 2y = -2 \quad \text{in } (-1, 1), \\ ay(-1) + by(1) + cy'(-1) + dy'(1) = \alpha, \\ \tilde{a}y(-1) + \tilde{b}y(1) + \tilde{c}y'(-1) + \tilde{d}y'(1) = \beta. \end{cases} \tag{6.11}$$

We have $f(x) = 0$, $g(x) = -2$ and $h(x) = -2$. As $f(-1) = f(1) = 0$, the critical exponents at the points $x = \pm 1$ are $\mu_2(-1) = \mu_2(1) = 1$ respectively and $n_0 = 0$. For this example, the recurrence relations (3.1) may be written in the form $v_{n+1} = M_n v_n + c_n$ with $v_n = (a_n, b_n)$, $c_n = (0,0)$, $n = 1, 2, \ldots$, and

$$M_n = \begin{pmatrix} -\frac{(n-1)(2n+1)}{2n(n+1)} & 0 \\ 0 & -\frac{2n-1}{2n} \end{pmatrix}.$$

System (4.6) and $(4.6)_h$ are given, respectively, by

$$\begin{cases} a_0 = 1, \\ b_0 = 0, \\ 0.176197 b_1 = 0, \end{cases} \qquad \begin{cases} a_0 = 0, \\ b_0 = 0, \\ 0.176197 b_1 = 0, \end{cases} \tag{6.12}$$

whose respective solutions are $(a_0, b_0, b_1) = (1,0,0)$ and $(a_0, b_0, b_1) = (0,0,0)$ with $a_1 \in \mathbb{C}$ a free parameter. As $\dim(S_h) = \dim(S) = 1$, the differential equation in (6.11) has a one-dimensional family of analytic solutions in $[-1,1]$, which agrees with the fact that the homogeneous differential equation has two independent solutions, $x^2 - 1$ and $(x^2-1)\log((x+1)/(1-x)) - 2x$, and just one is analytic in $[-1,1]$.

Now we apply the existence and uniqueness criterion of Section 5: the existence and uniqueness of solution of (6.11) is equivalent to the existence and uniqueness of solution of the linear system given by equations (6.12) and (6.4). Then, problem (6.11) has a unique solution if and only if

$$\frac{\alpha - a - b}{d - c} = \frac{\beta - \tilde{a} - \tilde{b}}{\tilde{d} - \tilde{c}} \tag{6.13}$$

with $c \neq d$ and $\tilde{c} \neq \tilde{d}$.

The existence and uniqueness condition obtained with this criterion coincides with the one provided by the knowledge of the family of analytic solutions of the differential equation given in (6.11)

$$y(x, C) = C(x^2 - 1) + 1.$$

The standard criterion of existence and uniqueness of solution of problem (6.11) depends on the existence of a complex number C that makes $y(x, C)$ compatible with the boundary conditions in (6.11), that is,

$$a + b - 2cC + 2dC = \alpha, \quad \tilde{a} + \tilde{b} - 2\tilde{c}C + 2\tilde{d}C = \beta \tag{6.14}$$

It can be checked that equations (6.14) and (6.13) are equivalent.

Example 6.5. Consider the boundary value problem

$$\begin{cases} (x^2-1)y'' + y' = x \quad \text{in } (-1,1), \\ ay(-1) + by(1) + cy'(-1) + dy'(1) = \alpha, \\ \tilde{a}y(-1) + \tilde{b}y(1) + \tilde{c}y'(-1) + \tilde{d}y'(1) = \beta. \end{cases}$$

We have $f(x) = 1$, $g(x) = 0$ and $h(x) = x$. As $f(-1) = f(1) = 1$, the critical exponents at the points $x = \pm 1$ are $\mu_2(-1) = 3/2$ and $\mu_2(1) = 1/2$ respectively and $n_0 = -1$. For this example, the recurrence relations (3.1) may be written in the form $v_{n+1} = M_n v_n + c_n$ with $v_n = (a_n, b_n)$, $c_n = (0,0)$, $n = 1, 2, \ldots$, and

$$M_n = \begin{pmatrix} -\frac{2n^2}{2n^2+3n+1} & 0 \\ \frac{n}{2n^2+3n+1} & -\frac{2n+1}{2n+2} \end{pmatrix}.$$

Systems (4.6) and $(4.6)_h$ are given, respectively, by

$$\begin{cases} b_0 + 2b_1 = 0, \\ 2a_1 = 1, \\ 0.024569a_1 = 0, \\ -0.132232a_1 + 0.336376b_1 = 0, \end{cases} \qquad \begin{cases} b_0 + 2b_1 = 0, \\ 2a_1 = 0, \\ 0.024569a_1 = 0, \\ -0.132232a_1 + 0.336376b_1 = 0. \end{cases}$$

The first system has no solution; the solution of the second one is $(b_0, a_1, b_1) = (0, 0, 0)$ and $a_0 \in \mathbb{C}$ a free parameter. For this example, $\dim(S_h) = 1$ but S is empty, which agrees with the fact that the solution to the differential equation is

$$\begin{aligned} y(x) = c_1 \left[\sqrt{1-x^2} + 2\arctan\left(\frac{x}{\sqrt{1-x^2}}\right)\right] + c_2 \\ - 2x + \sqrt{1-x^2}\arctan\left(\frac{x}{\sqrt{1-x^2}}\right) - \frac{1}{2}\arctan\left(\frac{x}{\sqrt{1-x^2}}\right)^2, \end{aligned}$$

that is not analytic in $[-1, 1]$ for any value of (c_1, c_2).

Example 6.6. Consider the boundary value problem

$$\begin{cases} (x^2-1)y'' + (1-x)y' + y = x \quad \text{in } (-1,1), \\ ay(-1) + by(1) + cy'(-1) + dy'(1) = \alpha, \\ \widetilde{a}y(-1) + \widetilde{b}y(1) + \widetilde{c}y'(-1) + \widetilde{d}y'(1) = \beta. \end{cases}$$

We have $f(x) = 1 - x$, $g(x) = 1$ and $h(x) = x$. As $f(-1) = 2$ and $f(1) = 0$, the critical exponents at the points $x = \pm 1$ are $\mu_2(-1) = 2$ and $\mu_2(1) = 1$ respectively, and $n_0 = 1$. For this example, the recurrence relations (3.1) may be written in the form $v_{n+1} = M_n v_n + c_n$ with $v_n = (a_n, b_n)$, $c_n = (0, 0)$, $n = 2, 3, \ldots$, and

$$M_n = \begin{pmatrix} -\frac{(2n-1)^3}{8n(n^2-1)} & \frac{1}{8n(n^2-1)} \\ \frac{(1-2n)^2}{8n(n^2-1)} & \frac{1+2n+4n^2-8n^3}{8n(n^2-1)} \end{pmatrix}.$$

System (4.6) is given by

$$\begin{cases} a_0 - 2a_1 + b_0 + 2b_1 = 0, \\ 2a_1 - 2b_1 = 1, \\ a_2 + b_1 + b_2 = 0, \\ a_1 + 4a_2 + 3b_1 + 4b_2 = 0, \\ -0.080292a_2 + 0.005407b_2 = 0, \\ 0.236021a_2 - 0.527175b_2 = 0. \end{cases}$$

This system has no solution. For this example, the solution to the homogeneous system $(4.6)_h$ is $(b_0, a_1, b_1, a_2, b_2) = (-a_0, 0, 0, 0, 0)$ with $a_0 \in \mathbb{C}$ a free parameter. Then, $\dim(S_h) = 1$ but S is empty, which agrees with the fact that the solution of the differential equation is

$$\begin{aligned} y(x) = c_1(1-x) + c_2[2 + (1-x)\log(x-1)] - (x-1)\mathrm{Li}_2\left(\frac{1-x}{2}\right) \\ + (x-1)\log(1-x) + \log(2)(x-1)\log(x-1) - (x+1)\log(x+1) - 1, \end{aligned}$$

that is not analytic in $[-1, 1]$ for any value of (c_1, c_2). (Here $\mathrm{Li}_2(z)$ is the polylogarithmic function.)

Example 6.7. Consider the boundary value problem

$$\begin{cases} (x^2-1)y''+4xy'+2y=e^x \quad \text{in } (-1,1), \\ ay(-1)+by(1)+cy'(-1)+dy'(1)=\alpha, \\ \widetilde{a}y(-1)+\widetilde{b}y(1)+\widetilde{c}y'(-1)+\widetilde{d}y'(1)=\beta. \end{cases} \tag{6.15}$$

We have $f(x)=4x$, $g(x)=2$ and $h(x)=e^x$. As $f(-1)=-4$ and $f(1)=4$, the critical exponents at the points $x=\pm 1$ are $\mu_2(-1)=\mu_2(1)=-1$ respectively and $n_0=-1$. For this example, the recurrence relations (3.1) may be written in the form $v_{n+1}=M_nv_n+c_n$ with $v_n=(a_n,b_n)$, $c_n=(A_n,B_n)$, $n=1,2,\ldots,$

$$M_n=\begin{pmatrix} -\frac{2n+1}{2(n+2)} & 0 \\ 0 & -\frac{2n+3}{2(n+2)} \end{pmatrix},$$

$$A_n=\frac{1}{4(n+1)(n+2)}\sum_{k=0}^{n}\frac{(n+k-1)}{k!n!(n-k-1)!2^{n+k+1}}((-1)^k e-(-1)^{n+1}e^{-1}),$$

and

$$B_n=\frac{1}{4(n+1)(n+2)}\sum_{k=0}^{n}\frac{(n+k-1)}{k!n!(n-k-1)!2^{n+k+1}}((-1)^k e+(-1)^{n+1}e^{-1}).$$

System (4.6) is given by

$$\begin{cases} 2a_0+8a_1=\cosh 1, \\ 6b_0+8b_1=\sinh 1, \\ 0.056062a_1=0.002946, \\ 0.429814b_1=0.004012, \end{cases} \tag{6.16}$$

whose solution is $(a_0,b_0,a_1,b_1)=(0.561323, 0.183421, 0.0525542, 0.00933429)$. In this case, the solution to the system $(4.6)_h$ is $S_h=\{0\}$ and S is non empty.

This conclusion is the same that we obtain from the knowledge of the general solution of the differential equation in (6.15),

$$y(x)=-\frac{e^x}{1-x^2}+\frac{c_1}{1-x^2}+\frac{c_2x}{1-x^2}.$$

There is only one analytic solution obtained for $(c_1,c_2)=(1,1)$.

Now we apply the existence and uniqueness criterion of Section 5: the existence and uniqueness of solution of (6.15) is equivalent to the existence and uniqueness of solution of the linear system given by equations (6.16) and (6.4). Then, problem (6.15) has a unique solution if and only if

$$\begin{cases} 0.377902a+0.744745b+0.0969813c+0.307198d=\alpha, \\ 0.377902\widetilde{a}+0.744745\widetilde{b}+0.0969813\widetilde{c}+0.307198\widetilde{d}=\beta. \end{cases}$$

The same conditions may be obtained from the exact solution.

Example 6.8. Consider the boundary value problem

$$\begin{cases} (x^2-1)y''+\dfrac{1}{4}y=0 \quad \text{in } (-1,1), \\ ay(-1)+by(1)+cy'(-1)+dy'(1)=\alpha, \\ \widetilde{a}y(-1)+\widetilde{b}y(1)+\widetilde{c}y'(-1)+\widetilde{d}y'(1)=\beta. \end{cases} \tag{6.17}$$

We have $f(x) = 0$, $g(x) = 1/4$ and $h(x) = 0$. As $f(-1) = f(1) = 0$, the critical exponents at the points $x = \pm 1$ are $\mu_2(-1) = \mu_2(1) = 1$ respectively and $n_0 = 0$. For this example, the recurrence relations (3.1) may be written in the form $v_{n+1} = M_n v_n + c_n$ with $v_n = (a_n, b_n)$, $c_n = (0,0)$, $n = 1, 2, \ldots$, and

$$M_n = \begin{pmatrix} -\frac{(1-4n)^2}{16n(1+n)} & 0 \\ 0 & -\frac{(1+4n)^2}{16n(1+n)} \end{pmatrix}.$$

System (4.6) = $(4.6)_h$ is given by

$$\begin{cases} a_0 = 0, \\ b_0 = 0, \\ 0.018792 a_1 = 0, \\ 0.360749 b_1 = 0, \end{cases} \tag{6.18}$$

whose solution is $(a_0, b_0, a_1, b_1) = (0,0,0,0)$. Then, $S_h = S = \{0\}$ and the unique analytic solution in $(-1,1)$ of the differential equation in problem (6.17) is the null solution. This conclusion is the same one that we obtain from the knowledge of the general solution of the differential equation in (6.17), since two independent solutions, none of them analytic in $[-1,1]$ are

$${}_2F_1\left(-\frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, x^2\right), \quad x\,{}_2F_1\left(\frac{1}{4}, \frac{1}{4}, \frac{3}{2}, x^2\right).$$

Now we apply the existence and uniqueness criterion of Section 5: the existence and uniqueness of solution of (6.17) is equivalent to the existence and uniqueness of solution of the linear system given by equations (6.18) and (6.4). Then, problem (6.17) has a unique solution if and only if $\alpha = \beta = 0$.

7 Final remarks

In Section 2 we have detailed the dimensionality of the space S_h of analytic solutions in $\mathcal{D}_r$ of the homogeneous differential equation $(z^2 - 1)y'' + f(z)y' + g(z)y = 0$. The dimension of S_h is: (i) zero or one when $f(-1) \neq 0, 2, 4, \ldots$ or $f(1) \neq 0, -2, -4, \ldots$; (ii) zero, one or two when $f(-1) = 0, 2, 4, \ldots$ and $f(1) = 0, -2, -4, \ldots$; (iii) two when $f(\pm 1) = g(\pm 1) = 0$ (regular case). We have included the regular case analyzed in [6] as a particular case of the more general situation analyzed in this paper. The dimension of the space S of analytic solutions in $\mathcal{D}_r$ of the complete differential equation is either, the same as the dimension of S_h, or it is empty. A complete characterization of this space is given at the end of Section 4 from the study of the ranks of the algebraic linear systems (4.6) and $(4.6)_h$.

In Section 3 we have derived an algorithm to obtain the two-point Taylor expansion of the solutions of (1.1) (if any). In Section 5 we have given a straightforward and systematic criterion for the existence and uniqueness of analytic solutions of the boundary value problem (1.1). The criterion is very simple and establishes that the existence and uniqueness of solution of the boundary value problem (1.1) is equivalent to the existence and uniqueness of solution of the algebraic linear system (5.2). Two equations of this algebraic system are defined by the limits (4.3), whose exact computation is, in general, difficult. Then, in practice, the entrances of two of the equations of this algebraic system must be computed approximately and then, the solution is computed in an approximated form. Also, in practice, we must apply the above existence and uniqueness criterion for the solution of (1.1) using the approximate linear system. Then, the conclusions about the existence and uniqueness of solution are exact unless the system is ill-conditioned. In this case, the ranks of the coefficient matrix and/or of the augmented matrix of the system (5.2) sensibly depend on the precision in the computation of the approximate limits.

Formally, the criterion proposed in this paper is similar to the standard criterion based on the knowledge of the space of solutions: both criteria relate the existence and uniqueness of solution of the boundary value problem (1.1) to the existence and uniqueness of a solution of an algebraic linear system. As a difference with that standard criterion, our criterion does not require the knowledge of the general solution of the differential equation. This qualitative difference is essential when the general solution of the equation is not known. In this case, the standard criterion is not useful, whereas our criterion can always be applied (except in the case of ill-conditioning before discussed), as we have shown in the examples analyzed in Section 6.

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Strong solutions for the steady incompressible MHD equations of non-Newtonian fluids

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Abstract. In this paper we deal with a system of partial differential equations describing a steady motion of an incompressible magnetohydrodynamic fluid, where the extra stress tensor is induced by a potential with p-structure ($p = 2$ corresponds to the Newtonian case). By using a fixed point argument in an appropriate functional setting, we proved the existence and uniqueness of strong solutions for the problem in a smooth domain $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ under the conditions that the external force is small in a suitable norm.

Keywords: strong solutions, existence and uniqueness, incompressible magnetohydrodynamics, non-Newtonian fluids.

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1 Introduction and main result

Magnetohydrodynamics (MHD) concerns the interaction of electrically conductive fluids and electromagnetic fields. The system of partial differential equations in MHD are basically obtained through the coupling of the dynamical equations of the fluids with the Maxwell's equations which is used to take into account the effect of the Lorentz force due to the magnetic field, it has spanned a very large range of applications [21, 24, 25]. By neglecting the displacement current term, a commonly used simplified MHD system could be described by

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \operatorname{div}\boldsymbol{\tau}(\mathcal{D}\boldsymbol{u}) + \nabla p = \dfrac{1}{\mu}(\nabla\times\boldsymbol{b})\times\boldsymbol{b} + \boldsymbol{f}, & \text{in } Q_T, \\ \boldsymbol{b}_t + \dfrac{1}{\mu}\operatorname{curl}\left(\dfrac{1}{\sigma}\operatorname{curl}\boldsymbol{b}\right) = \operatorname{curl}(\boldsymbol{u}\times\boldsymbol{b}), & \text{in } Q_T, \\ \operatorname{div}\boldsymbol{u} = 0, \qquad \operatorname{div}\boldsymbol{b} = 0, & \text{in } Q_T, \end{cases} \tag{1.1}$$

where $Q_T = \Omega \times (0, T)$, the unknown functions $\boldsymbol{u} = (u_1(x,t), u_2(x,t), \dots, u_n(x,t))$ denotes the velocity of the fluid, $\boldsymbol{b} = (b_1(x,t), b_2(x,t), \dots, b_n(x,t))$ the magnetic field, $p = p(x,t)$ the pressure and $\boldsymbol{f} = (f_1(x,t), f_2(x,t), \dots, f_n(x,t))$ the external force applied to the fluid.

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Also, $\boldsymbol{\tau} = (\boldsymbol{\tau}_{ij})$ is the stress tensor depending on the strain rate tensor $\mathcal{D}\boldsymbol{u} = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$, $\mu > 0$ and $\sigma > 0$ denotes the permeability coefficient and the electric conductivity coefficient respectively. For the sake of simplicity, in this work, we take $\mu = 1$ and $\sigma = 1$.

Due to the conventional belief that the Navier–Stokes equations are an accurate model for the motion of incompressible fluids in many practical situations, the majority of the known work have assumed that the stress tensor $\boldsymbol{\tau}(\mathcal{D}\boldsymbol{u})$ is a linear function of the strain rate $\mathcal{D}\boldsymbol{u}$. In this way we obtain the conventional system for MHD, and this classical model has been extensively studied. For instance, Duvaut and Lions [7] established the local existence and uniqueness of a solution in the Sobolev space $H^s(R^N)(s \geq N)$. They also proved the global existence of a solutions to this system with small initial data. Sermange and Temam [28] proved the existence of a unique global solution in the two space dimensions. For the zero magnetic diffusion case, Lin, Xu and Zhang [22] and Xu and Zhang [29] established the global well-posedness in two and three dimensional space, respectively, under the assumption that the initial data are sufficiently close to the equilibrium state. The global existence of smooth solutions was proved by Lei [18] for the ideal MHD with axially symmetric initial datum in $H^s(R^3)$ with $s \geq 2$. For more details, one can also refer [3–5,8,9,11,13–16,23] and the reference cited therein.

In recent years, the flow of non-Newtonian fluids (i.e. the stress tensor $\boldsymbol{\tau}(\mathcal{D}\boldsymbol{u})$ being a non-linear function of $\mathcal{D}\boldsymbol{u}$) has gained much importance in numerous technological applications. Further, the motion of the non-Newtonian fluids in the presence of a magnetic field in different contexts has been studied by several authors (see [2,6,26]). A typical form of the stress tensor $\boldsymbol{\tau}(\mathcal{D}\boldsymbol{u})$ is of some p– structure with $\mathcal{D}\boldsymbol{u}$ which were firstly proposed by Ladyzhenskaya in [19,20]. For the MHD equations of non-Newtonian type (1.1), the known results are limited and here we only recall two results closely related to ours. In case that $\boldsymbol{\tau}(\mathcal{D}\boldsymbol{u}) = |\mathcal{D}\boldsymbol{u}|^{p-2}\mathcal{D}\boldsymbol{u}$ for $p \geq \frac{5}{2}$, Samokhin proved in [27] the existence of weak solutions by using Galerkin method and the monotone theory, which solve the equations in the sense of distributions and satisfy the following energy inequality

$$\sup_{0\leq t\leq T} (\|u(t)\|_2^2 + \|b(t)\|_2^2) + 2\int_0^T (\|\nabla u(t)\|_p^p + \|\nabla b(t)\|_2^2)dt \leq (\|u_0\|_2^2 + \|b_0\|_2^2).$$

Later on, Gunzburger and his collaborators considered (1.1) with $\boldsymbol{\tau}(\mathcal{D}\boldsymbol{u}) = (1 + |\mathcal{D}\boldsymbol{u}|^{p-2})\mathcal{D}\boldsymbol{u}$ for the case of bounded or periodic domains, and they showed the existence and uniqueness of a weak solutions, see [12] for more details.

In this paper, in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3), we consider a steady incompressible MHD equations of non-Newtonian fluids described by

$$\begin{cases} -\operatorname{div}\left[2\mu(1+|D\boldsymbol{u}|^2)^{\frac{p-2}{2}}D\boldsymbol{u}\right] + \nabla p = \boldsymbol{f} - \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) + (\nabla\times\boldsymbol{b})\times\boldsymbol{b}, & x\in\Omega, \\ -\Delta\boldsymbol{b} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{b}, & x\in\Omega, \\ \operatorname{div}\boldsymbol{u} = 0, \qquad \operatorname{div}\boldsymbol{b} = 0, & x\in\Omega, \end{cases} \tag{1.2}$$

supplemented by the boundary conditions

$$\boldsymbol{u}|_{\partial\Omega} = 0, \quad \boldsymbol{b}\cdot\boldsymbol{n}|_{\partial\Omega} = 0, \qquad (\nabla\times\boldsymbol{b})\times\boldsymbol{n}|_{\partial\Omega} = \boldsymbol{0}, \tag{1.3}$$

where $p > 1$, $\boldsymbol{n}$ is the unit outward normal vector of $\partial\Omega$.

Remark 1.1. Since $\boldsymbol{u}$ and $\boldsymbol{b}$ are divergence free (i.e. $\operatorname{div}\boldsymbol{u} = 0$, $\operatorname{div}\boldsymbol{b} = 0$), an elementary computations leads to the formulas

$$\operatorname{curl}\operatorname{curl}\boldsymbol{b} = -\Delta\boldsymbol{b}, \qquad \operatorname{curl}(\boldsymbol{u}\times\boldsymbol{b}) = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{b}. \tag{1.4}$$

The aim of this paper is to prove the existence and uniqueness of strong solutions to system (1.2)–(1.3) under the assumption that the L^q-norm of the external force field $\boldsymbol{f}$ is small in a suitable sense. Our approach is based on regularity results for the Stokes problem and magnetic equation, and a fixed-point argument.

Throughout the paper, for $m \in \mathbb{N}$, the standard Lebesgue spaces are denoted by $\mathbf{L}^q(\Omega)$ and their norms by $\|\cdot\|_q$, the standard Sobolev spaces are denoted by $\mathbf{W}^{m,q}(\Omega)$ and their norms by $\|\cdot\|_{m,q}$. We also denote by $\mathbf{W}_0^{m,q}(\Omega)$ the closure in $\mathbf{W}^{m,q}(\Omega)$ of $C_0^\infty(\Omega)$. $W^{-1,q}(\Omega)$ denotes the dual of $W_0^{1,q}(\Omega)$ and their norms by $\|\cdot\|_{-1,q;\Omega}$. For $x, y \in \mathbb{R}$ we denote $(x,y)^+ = \max\{x,y\}$, $x^+ = \max\{x,0\}$. We introduce the constants

$$S_p := (|p-2|,2)^+, \qquad r_p := \frac{1+(p-3)^+ - (p-4)^+}{2}, \qquad \gamma_p := \frac{[(p,3)^+ - 2]^{(p,3)^+-2}}{[(p,3)^+ - 1]^{(p,3)^+-1}}. \tag{1.5}$$

We also introduce the space

$$\begin{aligned}
\mathcal{V} &:= \{\boldsymbol{u} \in C_0^\infty(\Omega),\ \operatorname{div} \boldsymbol{u} = 0\};\\
\mathbf{V}_p &:= \{\boldsymbol{u} \in \mathbf{W}_0^{1,p}(\Omega) : \operatorname{div} \boldsymbol{u} = 0\};\\
\mathbf{V}_{m,p} &:= \{\boldsymbol{v} \in \mathbf{W}_0^{1,p}(\Omega) \cap \mathbf{W}^{m,p}(\Omega) : \operatorname{div} \boldsymbol{v} = 0\ \};\\
\mathbf{W} &:= \{\boldsymbol{b} \in \mathbf{W}^{1,2}(\Omega) : \operatorname{div} \boldsymbol{b} = 0\ , \boldsymbol{b} \cdot \boldsymbol{n}|_{\partial\Omega} = 0\}.
\end{aligned}$$

Also, for $q > r > n$ and $\delta > 0$, let us denote by B_δ the convex set defined by

$$B_\delta := \Big\{(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbf{V}_{2,q} \times (\mathbf{W}^{2,r}(\Omega) \cap \mathbf{W}) : C_E\|\nabla\boldsymbol{\xi}\|_{1,q} \le \delta, \quad C_{\widetilde{E}}\|\nabla\boldsymbol{\eta}\|_{1,r} \le \delta\Big\}, \tag{1.6}$$

where C_E is the norm of the embedding of $W^{1,q}(\Omega)$ into $L^\infty(\Omega)$ and $C_{\widetilde{E}}$ is the norm of the embedding of $W^{1,r}(\Omega)$ into $L^\infty(\Omega)$, also C_p denotes the Poincaré constant corresponding to the general Poincaré inequality $\|\cdot\|_s \le C_p\|\nabla(\cdot)\|_s$. We consider the space $\mathbf{V}_{2,q} \times \mathbf{W}^{2,r}(\Omega)$ endowed with the norm

$$\|(\boldsymbol{\xi}, \boldsymbol{\eta})\|_{1,q,r} := \max\{\|\nabla\boldsymbol{\xi}\|_{1,q}, \ \|\nabla\boldsymbol{\eta}\|_{1,r}\}.$$

Now, we formulate the main theorem of this paper.

Theorem 1.2. *Assume that $q > r > n$, $p > 1$, $\mu > 0$, and let $\boldsymbol{f} \in \mathbf{L}^q(\Omega)$. There exist positive constant $\overline{C} = \overline{C}(C_0, C_p, C_E, C_{\widetilde{E}}, C_{-1}, c_2)$ such that if*

$$\overline{C}\left[\Big(1+\frac{1}{\mu}\Big)\frac{\overline{C}\|\boldsymbol{f}\|_q}{\mu} + S_p\Big(\frac{\overline{C}\|\boldsymbol{f}\|_q}{\mu}\Big)^{2r_p}\Big(1+\frac{\overline{C}\|\boldsymbol{f}\|_q}{\mu}\Big)^{(p-4)^+}\right] < \frac{1}{4^{(p-2,1)^+}}, \tag{1.7}$$

then, problem (1.2)–(1.3) has a unique strong solution $(\boldsymbol{u}, \boldsymbol{b}) \in \mathbf{V}_{2,q} \times \mathbf{W}^{2,r}(\Omega)$.

Remark 1.3. As usual, the pressure π has disappeared from the notion of solution. Actually, the pressure may be recovered by de Rham Theorem at least in $L^2(\Omega)$, such that the triple $(\boldsymbol{u}, \pi, \boldsymbol{b})$ satisfies equations (1.2)–(1.3) almost everywhere (see [11]).

The rest of our paper is organized as follows: in Section 2, we review some known results and Section 3 is devoted to proving the main theorem to problem (1.2)–(1.3).

2 Preliminary lemmas

In this section, we recall some basic facts which will be used later.

Lemma 2.1 ([10, Theorem 6.1, pp. 225]). *Let $m \geq -1$ be an integer and let Ω be a bounded domain in $\mathbb{R}^n$ $(n = 2, 3)$ with boundary $\partial\Omega$ of class $\mathcal{C}^k$ with $k = (m+2, 2)^+$. Then for any $\boldsymbol{\psi} \in \mathbf{W}^{m,\rho}(\Omega)$, the following system*

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{\psi}, & x \in \Omega, \\ \operatorname{div} \boldsymbol{u} = 0, & x \in \Omega, \\ \boldsymbol{u}|_{\partial\Omega} = 0, \end{cases}$$

admits a unique solution $[\boldsymbol{u}, \pi] \in \mathbf{W}^{m+2,\rho}(\Omega) \times W^{m+1,\rho}(\Omega)$. Moreover, the following estimate holds

$$\|\nabla \boldsymbol{u}\|_{m+1,\rho} + \|\pi\|_{m+1,\rho/\mathbb{R}} \leq C_m \|\boldsymbol{\psi}\|_{m,\rho},$$

where $C_m = C_m(n, \rho, \Omega)$ is a positive constant.

Lemma 2.2 ([1]). *Let r_p, γ_p are given by (1.5) and let $G : \mathbb{R}^+ \to \mathbb{R}$ be defined by*

$$G(\delta) = A\delta^2 - \delta + E\delta\mathcal{H}(\delta) + D,$$

where A, E, D are positive constants and $\mathcal{H}(x) = x^{2r_p}(1+x)^{(p-4)^+}$. Thus, if the following assertion holds

$$AD + ED^{2r_p}(1+D)^{(p-4)^+} \leq \gamma_p,$$

then G possesses at least one root δ_0. Moreover, $\delta_0 > D$ and for every $\beta \in [1, 2]$ the following estimate holds

$$\frac{\beta-1}{\beta}\delta_0 + \frac{2-\beta}{\beta}A\delta_0^2 + \frac{2r_p+1-\beta}{\beta}E\delta_0\mathcal{H}(\delta_0) + \frac{E(p-4)^+}{\beta}\delta_0^{2r_p+2}(1+\delta_0)^{(p-4)^+-1} \leq D.$$

Lemma 2.3 ([17]). *Let X and Y be Banach spaces such that X is reflexive and $X \hookrightarrow Y$. Let B be a non-empty, closed, convex and bounded subset of X and let $T : B \to B$ be a mapping such that*

$$\|T(u) - T(v)\|_Y \leq K\|u - v\|_Y, \qquad \forall\, u, v \in B \ \ (0 < K < 1),$$

then T has a unique fixed point in B.

3 Proof of Theorem 1.2

Our proof relies on a Banach fixed point theorem. Toward this aim, we first reformulate the problem as follows

$$\begin{cases} -\mu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} - \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) + (\nabla\times\boldsymbol{b})\times\boldsymbol{b} + \operatorname{div}[2\mu\sigma(|D\boldsymbol{u}|^2)D\boldsymbol{u}], & x \in \Omega, \\ -\Delta\boldsymbol{b} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{b}, & x \in \Omega, \\ \operatorname{div}\boldsymbol{u} = 0, \qquad \operatorname{div}\boldsymbol{b} = 0, & x \in \Omega, \\ \boldsymbol{u}|_{\partial\Omega} = 0, \qquad \boldsymbol{b}\cdot\boldsymbol{n}|_{\partial\Omega} = 0, \qquad (\nabla\times\boldsymbol{b})\times\boldsymbol{n}|_{\partial\Omega} = 0, \end{cases} \tag{3.1}$$

where $\sigma(x) = (1+x)^{\frac{p-2}{2}} - 1$.

Given $(\boldsymbol{\xi},\boldsymbol{\eta}) \in \mathbf{V}_{2,q} \times \mathbf{W}^{2,r}(\Omega)$, we consider the following problem

$$\begin{cases} -\mu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} - \operatorname{div}(\boldsymbol{\xi}\otimes\boldsymbol{\xi}) + (\nabla\times\boldsymbol{\eta})\times\boldsymbol{\eta} + \operatorname{div}[2\mu\sigma(|D\boldsymbol{\xi}|^2)D\boldsymbol{\xi}], & x\in\Omega,\\ -\Delta \boldsymbol{b} = (\boldsymbol{\eta}\cdot\nabla)\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\eta}, & x\in\Omega,\\ \operatorname{div}\boldsymbol{u} = 0, \qquad \operatorname{div}\boldsymbol{b}=0, & x\in\Omega,\\ \boldsymbol{u}|_{\partial\Omega}=0, \qquad \boldsymbol{b}\cdot\boldsymbol{n}|_{\partial\Omega}=0, \qquad (\nabla\times\boldsymbol{b})\times\boldsymbol{n}|_{\partial\Omega}=0. \end{cases} \tag{3.2}$$

From Lemma 2.1 and Proposition 2.30 in [11], there exists a unique solution $(\boldsymbol{u},\boldsymbol{b}) \in \mathbf{V}_{2,q}\times \mathbf{W}^{2,r}(\Omega)$ to (3.2). We define the mapping

$$T : (\boldsymbol{\xi},\boldsymbol{\eta}) \to (\boldsymbol{u},\boldsymbol{b}).$$

Our purpose now is to prove that $T_{B_{\delta_0}}$ is a contraction from B_{δ_0} to itself for some $\delta_0>0$. Here B_{δ_0} is the closed ball defined in (1.6).

Proposition 3.1. *Let $q>r>n$, $p>1$, $\mu>0$, and let $\boldsymbol{f}\in \mathbf{L}^q(\Omega)$. There exists a positive constant $M_1 = M_1(C_0, C_p, C_E, C_{\widetilde{E}})$ such that if*

$$\frac{M_1^2\|\boldsymbol{f}\|_q}{\mu^2} + M_1 S_p\left(\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{2r_p}\left(1+\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{(p-4)^+} \le \gamma_p, \tag{3.3}$$

then $T(B_{\delta_0}) \subseteq B_{\delta_0}$ for some $\delta_0>0$.

Proof. Let $(\boldsymbol{\xi},\boldsymbol{\eta})\in B_\delta$. From Lemma 2.1, $\boldsymbol{u}\in\mathbf{V}_{2,q}$ and it satisfies

$$\|\nabla\boldsymbol{u}\|_{1,q} \le \frac{C_0}{\mu}\big(\|\boldsymbol{f}\|_q + \|\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}\|_q + \|(\nabla\times\boldsymbol{\eta})\times\boldsymbol{\eta}\|_q + \|\operatorname{div}[2\mu\sigma(|D\boldsymbol{\xi}|^2)D\boldsymbol{\xi}]\,\|_q\big). \tag{3.4}$$

Notice that

$$\begin{aligned}\|(\nabla\times\boldsymbol{\eta})\times\boldsymbol{\eta}\|_q &\le \|\boldsymbol{\eta}\|_\infty\|\nabla\boldsymbol{\eta}\|_q \le C_{\widetilde{E}}\|\boldsymbol{\eta}\|_{1,r}\|\nabla\boldsymbol{\eta}\|_{1,r}\\ &\le \delta(C_p+1)\|\nabla\boldsymbol{\eta}\|_r \le \delta(C_p+1)\|\nabla\boldsymbol{\eta}\|_{1,r}\\ &\le \frac{(C_p+1)}{C_{\widetilde{E}}}\delta^2,\end{aligned} \tag{3.5}$$

reasoning as in [1], we could obtain

$$\|\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}\|_q + \|\operatorname{div}[2\mu\sigma(|D\boldsymbol{\xi}|^2)D\boldsymbol{\xi}]\,\|_q \le \frac{C_p}{C_E}\delta^2 + \frac{4\mu S_p}{C_E}\delta\mathcal{H}(\delta). \tag{3.6}$$

Combining (3.4), (3.5) and (3.6), we get

$$\|\nabla\boldsymbol{u}\|_{1,q} \le \frac{M_1}{\mu}\big(\|\boldsymbol{f}\|_q + \delta^2 + \mu S_p\delta\mathcal{H}(\delta)\big),$$

where $M_1 = C_0\max\{1, \frac{C_p}{C_E} + \frac{(C_p+1)}{C_{\widetilde{E}}}, \frac{4}{C_E}\}$.

On the other hand, by Proposition 2.30 in [11], there exists a constant $c_1>0$ such that

$$\begin{aligned}\|\nabla\boldsymbol{b}\|_{1,r} &\le c_1\left[\|\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}\|_r + \|\boldsymbol{\xi}\cdot\nabla\boldsymbol{\eta}\|_r\right]\\ &\le c_1\left[C_{\widetilde{E}}\|\boldsymbol{\eta}\|_{1,r}\|\nabla\boldsymbol{\xi}\|_{1,q} + C_E\|\boldsymbol{\xi}\|_{1,q}\|\nabla\boldsymbol{\eta}\|_{1,r}\right]\\ &\le c_1\left[C_{\widetilde{E}}(C_p+1)\|\nabla\boldsymbol{\eta}\|_r\|\nabla\boldsymbol{\xi}\|_{1,q} + C_E(C_p+1)\|\nabla\boldsymbol{\xi}\|_q\|\nabla\boldsymbol{\eta}\|_{1,r}\right]\\ &\le c_1\left[C_{\widetilde{E}}(C_p+1)\|\nabla\boldsymbol{\eta}\|_{1,r}\frac{\delta}{C_E} + C_E(C_p+1)\|\nabla\boldsymbol{\xi}\|_{1,q}\frac{\delta}{C_{\widetilde{E}}}\right]\\ &\le c_1\left[\frac{(C_p+1)}{C_E}\delta^2 + \frac{(C_p+1)}{C_{\widetilde{E}}}\delta^2\right]\\ &\le 2M_2\delta^2,\end{aligned} \tag{3.7}$$

where $M_2 = c_1 \max\{\frac{(C_p+1)}{C_E}, \frac{(C_p+1)}{C_{\tilde{E}}}\}$. In order to ensure that $T(B_\delta) \subseteq B_\delta$, it is enough to show that

$$\begin{aligned} \|\nabla \boldsymbol{u}\|_{1,q} &\leq \frac{M_1}{\mu}\big(\|\boldsymbol{f}\|_q + \delta^2 + \mu S_p \delta \mathcal{H}(\delta)\big) \leq \delta, \\ \|\nabla \boldsymbol{b}\|_{1,r} &\leq 2M_2\delta^2 \leq \delta. \end{aligned} \tag{3.8}$$

Using Lemma 2.2 with $A = \frac{M_1}{\mu}$, $E = M_1 S_p$ and $D = \frac{M_1\|\boldsymbol{f}\|_q}{\mu}$, there exists $\delta_1 > \frac{M_1\|\boldsymbol{f}\|_q}{\mu}$ such that

$$\frac{M_1}{\mu}\big(\|\boldsymbol{f}\|_q + \delta_1^2 + \mu S_p \delta_1 \mathcal{H}(\delta_1)\big) \leq \delta_1,$$

provided that

$$AD + ED^{2r_p}(1+D)^{(p-4)^+} \leq \gamma_p,$$

which holds from the hypothesis (3.3). Also, it holds ($\beta = 2$ in Lemma 2.2) that

$$\delta_1 \leq \frac{2M_1\|\boldsymbol{f}\|_q}{\mu}.$$

On the other hand, we reformulate the inequality $(3.8)_2$ as

$$2M_2\delta^2 - \delta \leq 0. \tag{3.9}$$

Due to

$$\Delta = 1 > 0,$$

we deduce that for some δ, the inequality (3.9) is valid.

Take the constant D to satisfy $\delta^- < D < 2D < \delta^+$, where

$$\delta^\pm = \frac{1}{4M_2} \pm \sqrt{1} = \frac{1 \pm 4M_2}{4M_2}.$$

Moreover, given that for every $\delta \in [\delta^-, \delta^+]$, the inequality (3.9) is valid, we can choose $\delta_2 \in (\delta^-, D)$ such that

$$2M_2\delta_2^2 \leq \delta_2.$$

In conclusion, we obtain

$$\delta_2 < \frac{M_1\|\boldsymbol{f}\|_q}{\mu} < \delta_1 \leq \frac{2M_1\|\boldsymbol{f}\|_q}{\mu}.$$

Thus, taking $\delta_0 = \delta_1$ we obtain that $T(B_{\delta_0}) \subseteq B_{\delta_0}$. □

Proposition 3.2. *There is a positive constant $m = m(C_{-1}, C_p, c_2, C_E, C_{\tilde{E}})$ such that if*

$$m\left[\left(1+\frac{1}{\mu}\right)\frac{M_1\|\boldsymbol{f}\|_q}{\mu} + S_p\left(\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{2r_p}\left(1+\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{(p-4)^+}\right] < \frac{1}{4^{(p-2,1)^+}}, \tag{3.10}$$

then $T : B_{\delta_0} \to B_{\delta_0}$ is a contraction in $\mathbf{W}_0^{1,q}(\Omega) \times \mathbf{W}^{1,r}(\Omega)$.

Proof. Let $(\boldsymbol{\xi},\boldsymbol{\eta})$, $(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}}) \in B_{\delta_0}$ and let $(\boldsymbol{u},\boldsymbol{b})$, $(\hat{\boldsymbol{u}},\hat{\boldsymbol{b}})$ be their respective images under T. Then, from (3.2) we obtain

$$\begin{cases} -\mu\Delta(\boldsymbol{u}-\hat{\boldsymbol{u}}) + \nabla(p-\hat{p}) = \boldsymbol{F}, & x \in \Omega, \\ -\Delta(\boldsymbol{b}-\hat{\boldsymbol{b}}) = \boldsymbol{G}, & x \in \Omega, \\ \operatorname{div}(\boldsymbol{u}-\hat{\boldsymbol{u}}) = 0, \qquad \operatorname{div}(\boldsymbol{b}-\hat{\boldsymbol{b}}) = 0, & x \in \Omega, \\ (\boldsymbol{u}-\hat{\boldsymbol{u}})|_{\partial\Omega} = 0, \qquad (\boldsymbol{b}-\hat{\boldsymbol{b}})\cdot\boldsymbol{n}|_{\partial\Omega} = 0, \qquad (\nabla\times(\boldsymbol{b}-\hat{\boldsymbol{b}}))\times\boldsymbol{n}|_{\partial\Omega} = 0, & \end{cases}$$

where

$$\begin{aligned} &\boldsymbol{F} := \operatorname{div}(\hat{\boldsymbol{\xi}}\otimes\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}\otimes\boldsymbol{\xi}) + (\nabla\times\boldsymbol{\eta})\times\boldsymbol{\eta} - (\nabla\times\hat{\boldsymbol{\eta}})\times\hat{\boldsymbol{\eta}} + 2\mu\operatorname{div}[\sigma(|D\boldsymbol{\xi}|^2)D\boldsymbol{\xi} - \sigma(|D\hat{\boldsymbol{\xi}}|^2)D\hat{\boldsymbol{\xi}}], \\ &\boldsymbol{G} := (\boldsymbol{\eta}\cdot\nabla)\boldsymbol{\xi} - (\hat{\boldsymbol{\eta}}\cdot\nabla)\hat{\boldsymbol{\xi}} + (\hat{\boldsymbol{\xi}}\cdot\nabla)\hat{\boldsymbol{\eta}} - (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\eta}. \end{aligned}$$

Applying Lemma 2.1 with $\boldsymbol{\psi} = \boldsymbol{F}$ we obtain

$$\begin{aligned} \|\nabla(\boldsymbol{u}-\hat{\boldsymbol{u}})\|_q \le \frac{C_{-1}}{\mu}(&\|\operatorname{div}(\hat{\boldsymbol{\xi}}\otimes\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}\otimes\boldsymbol{\xi})\|_{-1,q} + \|(\nabla\times\boldsymbol{\eta})\times\boldsymbol{\eta} - (\nabla\times\hat{\boldsymbol{\eta}})\times\hat{\boldsymbol{\eta}}\|_{-1,q} \\ &+ 2\mu\|\operatorname{div}[\sigma(|D\boldsymbol{\xi}|^2)D\boldsymbol{\xi} - \sigma(|D\hat{\boldsymbol{\xi}}|^2)D\hat{\boldsymbol{\xi}}]\,\|_{-1,q}). \end{aligned} \tag{3.11}$$

Notice that

$$\begin{aligned} &\|(\nabla\times\boldsymbol{\eta})\times\boldsymbol{\eta} - (\nabla\times\hat{\boldsymbol{\eta}})\times\hat{\boldsymbol{\eta}}\|_{-1,q} \\ &\quad\le \|(\nabla\times\boldsymbol{\eta})\times\boldsymbol{\eta} - (\nabla\times\hat{\boldsymbol{\eta}})\times\hat{\boldsymbol{\eta}}\|_r \\ &\quad= \|(\nabla\times\boldsymbol{\eta})\times\boldsymbol{\eta} - (\nabla\times\hat{\boldsymbol{\eta}})\times\boldsymbol{\eta} + (\nabla\times\hat{\boldsymbol{\eta}})\times\boldsymbol{\eta} - (\nabla\times\hat{\boldsymbol{\eta}})\times\hat{\boldsymbol{\eta}}\|_r \\ &\quad\le \|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\|\boldsymbol{\eta}\|_\infty + \|\nabla\hat{\boldsymbol{\eta}}\|_r\|\boldsymbol{\eta}-\hat{\boldsymbol{\eta}}\|_\infty \\ &\quad\le C_{\widetilde{E}}\|\boldsymbol{\eta}\|_{1,r}\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r + \|\nabla\hat{\boldsymbol{\eta}}\|_{1,r}C_{\widetilde{E}}\|\boldsymbol{\eta}-\hat{\boldsymbol{\eta}}\|_{1,r} \\ &\quad\le C_{\widetilde{E}}(C_p+1)\|\nabla\boldsymbol{\eta}\|_r\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r + \delta_0(C_p+1)\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r \\ &\quad\le C_{\widetilde{E}}(C_p+1)\|\nabla\boldsymbol{\eta}\|_{1,r}\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r + \delta_0(C_p+1)\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r \\ &\quad\le \delta_0(C_p+1)\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r + \delta_0(C_p+1)\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r, \\ &\quad= 2\delta_0(C_p+1)\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r, \end{aligned} \tag{3.12}$$

reasoning as in [1], we obtain

$$\begin{aligned} \|\operatorname{div}(\hat{\boldsymbol{\xi}}\otimes\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}\otimes\boldsymbol{\xi})\|_{-1,q} &\le C\|(\hat{\boldsymbol{\xi}}\otimes\hat{\boldsymbol{\xi}} - \boldsymbol{\xi}\otimes\boldsymbol{\xi})\|_q \\ &\le CC_p(C_p^q+1)^{\frac{1}{q}}\delta_0\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q, \end{aligned} \tag{3.13}$$

$$\begin{aligned} 2\mu\|\operatorname{div}[\sigma(|D\boldsymbol{\xi}|^2)D\boldsymbol{\xi} - \sigma(|D\hat{\boldsymbol{\xi}}|^2)D\hat{\boldsymbol{\xi}}]\,\|_{-1,q} &\le C\mu\|\,[\sigma(|D\boldsymbol{\xi}|^2)D\boldsymbol{\xi} - \sigma(|D\hat{\boldsymbol{\xi}}|^2)D\hat{\boldsymbol{\xi}}]\,\|_q \\ &\le C\mu S_p\mathcal{H}(2\delta_0)\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q. \end{aligned} \tag{3.14}$$

From (3.11)–(3.14) we obtain

$$\|\nabla(\boldsymbol{u}-\hat{\boldsymbol{u}})\|_q \le M_3\left[\frac{2\delta_0}{\mu} + S_p\mathcal{H}(2\delta_0)\right]\max\{\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q, \|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\}, \tag{3.15}$$

where $M_3 = C_{-1}\max\{CC_p(C_p^q+1)^{\frac{1}{q}}, 2(C_p+1), C\}$.

On the other hand, again by Proposition 2.30 in [11], there exists a constant $c_2 > 0$ such that

$$
\begin{aligned}
\|\nabla(\boldsymbol{b}-\hat{\boldsymbol{b}})\|_r \leq\ & \|\nabla(\boldsymbol{b}-\hat{\boldsymbol{b}})\|_{1,r} \\
\leq\ & c_2\big[\|(\boldsymbol{\eta}\cdot\nabla)\boldsymbol{\xi}-(\hat{\boldsymbol{\eta}}\cdot\nabla)\hat{\boldsymbol{\xi}}\|_r+\|(\hat{\boldsymbol{\xi}}\cdot\nabla)\hat{\boldsymbol{\eta}}-(\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\eta}\|_r\big] \\
=\ & c_2\Big[\|(\boldsymbol{\eta}\cdot\nabla)\boldsymbol{\xi}-(\hat{\boldsymbol{\eta}}\cdot\nabla)\boldsymbol{\xi}+(\hat{\boldsymbol{\eta}}\cdot\nabla)\boldsymbol{\xi}-(\hat{\boldsymbol{\eta}}\cdot\nabla)\hat{\boldsymbol{\xi}}\|_r \\
& +\|(\hat{\boldsymbol{\xi}}\cdot\nabla)\hat{\boldsymbol{\eta}}-(\hat{\boldsymbol{\xi}}\cdot\nabla)\boldsymbol{\eta}+(\hat{\boldsymbol{\xi}}\cdot\nabla)\boldsymbol{\eta}-(\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\eta}\|_r\Big] \\
\leq\ & c_2\Big[\|\boldsymbol{\eta}-\hat{\boldsymbol{\eta}}\|_\infty\|\nabla\boldsymbol{\xi}\|_r+\|\hat{\boldsymbol{\eta}}\|_\infty\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_r \\
& +\|\hat{\boldsymbol{\xi}}\|_\infty\|\nabla(\hat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r+\|\hat{\boldsymbol{\xi}}-\boldsymbol{\xi}\|_\infty\|\nabla\boldsymbol{\eta}\|_r\Big] \\
\leq\ & c_2\Big[C_{\widetilde{E}}\|\boldsymbol{\eta}-\hat{\boldsymbol{\eta}}\|_{1,r}\|\nabla\boldsymbol{\xi}\|_r+C_{\widetilde{E}}\|\hat{\boldsymbol{\eta}}\|_{1,r}\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_r \\
& +C_E\|\hat{\boldsymbol{\xi}}\|_{1,q}\|\nabla(\hat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r+C_E\|\hat{\boldsymbol{\xi}}-\boldsymbol{\xi}\|_{1,q}\|\nabla\boldsymbol{\eta}\|_r\Big] \\
\leq\ & c_2\Big[C_{\widetilde{E}}(C_p+1)\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\|\nabla\boldsymbol{\xi}\|_{1,q}+C_{\widetilde{E}}(C_p+1)\|\nabla\hat{\boldsymbol{\eta}}\|_r\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q \\
& +C_E(C_p+1)\|\nabla\hat{\boldsymbol{\xi}}\|_q\|\nabla(\hat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r+C_E(C_p+1)\|\nabla(\hat{\boldsymbol{\xi}}-\boldsymbol{\xi})\|_q\|\nabla\boldsymbol{\eta}\|_r\Big] \\
\leq\ & c_2\Big[C_{\widetilde{E}}(C_p+1)\|\nabla\boldsymbol{\xi}\|_{1,q}\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r+C_{\widetilde{E}}(C_p+1)\|\nabla\hat{\boldsymbol{\eta}}\|_{1,r}\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q \\
& +C_E(C_p+1)\|\nabla\hat{\boldsymbol{\xi}}\|_{1,q}\|\nabla(\hat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r+C_E(C_p+1)\|\nabla\boldsymbol{\eta}\|_{1,r}\|\nabla(\hat{\boldsymbol{\xi}}-\boldsymbol{\xi})\|_q\Big] \\
\leq\ & c_2\left[\frac{C_{\widetilde{E}}(C_p+1)}{C_E}\delta_0\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r+(C_p+1)\delta_0\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q\right. \\
& \left.+(C_p+1)\delta_0\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r+\frac{C_E(C_p+1)}{C_{\widetilde{E}}}\delta_0\|\nabla(\hat{\boldsymbol{\xi}}-\boldsymbol{\xi})\|_q\right] \\
\leq\ & 4M_4\delta_0\max\{\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q,\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\},
\end{aligned} \tag{3.16}
$$

where $M_4 = c_2\max\{\frac{C_{\widetilde{E}}(C_p+1)}{C_E},(C_p+1),\frac{C_E(C_p+1)}{C_{\widetilde{E}}}\}$.

Combining (3.15) and (3.16), we deduce that

$$
\begin{aligned}
&\max\{\|\nabla(\boldsymbol{u}-\hat{\boldsymbol{u}})\|_q,\|\nabla(\boldsymbol{b}-\hat{\boldsymbol{b}})\|_r\} \\
&\qquad\leq\left(\frac{2M_3\delta_0}{\mu}+4M_4\delta_0+M_3S_p\mathcal{H}(2\delta_0)\right)\cdot\max\{\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q,\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\}.
\end{aligned}
$$

From here, and taking into account that $\delta_0\leq\frac{2M_1\|\boldsymbol{f}\|_q}{\mu}$, $\mathcal{H}$ is nondecreasing, $\mathcal{H}(4y)\leq 4^{(p-2,1)^+}\mathcal{H}(y)$ and defining $m=\max\{2M_3,4M_4\}$, we get

$$
\begin{aligned}
&\max\{\|\nabla(\boldsymbol{u}-\hat{\boldsymbol{u}})\|_q,\|\nabla(\boldsymbol{b}-\hat{\boldsymbol{b}})\|_r\} \\
&\quad\leq m\left[\frac{\delta_0}{\mu}+\delta_0+S_p\mathcal{H}(2\delta_0)\right]\max\{\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q,\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\} \\
&\quad\leq m\left[\frac{2M_1\|\boldsymbol{f}\|_q}{\mu^2}+\frac{2M_1\|\boldsymbol{f}\|_q}{\mu}+S_p4^{(p-2,1)^+}\mathcal{H}\left(\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)\right] \\
&\qquad\cdot\max\{\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q,\|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\}
\end{aligned}
$$

$$
\begin{aligned}
&= m\left[\left(1+\frac{1}{\mu}\right)\frac{2M_1\|\boldsymbol{f}\|_q}{\mu} + 4^{(p-2,1)^+}S_p\left(\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{2r_p}\left(1+\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{(p-4)^+}\right] \\
&\quad \cdot \max\{\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q, \|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\} \\
&\le 4^{(p-2,1)^+} m\left[\left(1+\frac{1}{\mu}\right)\frac{M_1\|\boldsymbol{f}\|_q}{\mu} + S_p\left(\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{2r_p}\left(1+\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{(p-4)^+}\right] \\
&\quad \cdot \max\{\|\nabla(\boldsymbol{\xi}-\hat{\boldsymbol{\xi}})\|_q, \|\nabla(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})\|_r\}.
\end{aligned} \tag{3.17}
$$

Considering the space $Y := \mathbf{W}_0^{1,q}(\Omega) \times \mathbf{W}^{1,r}(\Omega)$, with norm $\max\{\|\nabla\cdot\|_q, \|\nabla\cdot\|_r\}$, the inequality (3.17) implies that

$$
\begin{aligned}
\|T\left(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}}\right) - T\left(\boldsymbol{\xi},\boldsymbol{\eta}\right)\|_Y \le 4^{(p-2,1)^+} m &\left[\left(1+\frac{1}{\mu}\right)\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right. \\
&\left. + S_p\left(\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{2r_p}\left(1+\frac{M_1\|\boldsymbol{f}\|_q}{\mu}\right)^{(p-4)^+}\right] \left\|\left(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}}\right) - (\boldsymbol{\xi},\boldsymbol{\eta})\right\|_Y.
\end{aligned}
$$

From which and hypothesis (3.10), we obtain $T : B_{\delta_0} \to B_{\delta_0}$ is a contraction in $\mathbf{W}_0^{1,q}(\Omega) \times \mathbf{W}^{1,r}(\Omega)$. □

Proof of Theorem 1.2. Notice that for $p \le 3$, $\gamma_p = 1/4 = 1/4^{(p-2,1)^+}$ and for $p > 3$, $\gamma_p > 1/4^{(p-2,1)^+}$. Thus, by taking $\overline{C} = (M_1, m)^+$ and because of (1.7) implies (3.3) and (3.10), Propositions 3.1 and Propositions 3.2 yield that the mapping $T : B_{\delta_0} \to B_{\delta_0}$ is a contraction in $\mathbf{W}_0^{1,q}(\Omega) \times \mathbf{W}^{1,r}(\Omega)$.

Applying Lemma 2.3 with $X = \mathbf{V}_{2,q} \times \mathbf{W}^{2,r}(\Omega)$, $Y = \mathbf{W}_0^{1,q}(\Omega) \times \mathbf{W}^{1,r}(\Omega)$ and $B = B_{\delta_0}$, we could obtain that T has a unique fixed point in B_{δ_0} and this implies the original problem (1.2)–(1.3) has a unique strong solution $(\boldsymbol{u}, \boldsymbol{b}) \in \mathbf{V}_{2,q} \times \mathbf{W}^{2,r}(\Omega)$.

The proof of Theorem 1.2 is finished. □

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Strong solutions to the nonhomogeneous Boussinesq equations for magnetohydrodynamics convection without thermal diffusion

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Abstract. We are concerned with the Cauchy problem of nonhomogeneous Boussinesq equations for magnetohydrodynamics convection in $\mathbb{R}^2$. We show that there exists a unique local strong solution provided the initial density, the magnetic field, and the initial temperature decrease at infinity sufficiently quickly. In particular, the initial data can be arbitrarily large and the initial density may contain vacuum states.

Keywords: nonhomogeneous Boussinesq-MHD system, strong solutions, Cauchy problem.

2020 Mathematics Subject Classification: 35Q35, 76D03.

1 Introduction

Consider the following nonhomogeneous Boussinesq system for magnetohydrodynamic convection (Boussinesq-MHD) in $\mathbb{R}^2$:

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} + \nabla P = \mathbf{b} \cdot \nabla \mathbf{b} + \rho \theta \mathbf{e}_2, \\ \theta_t + \mathbf{u} \cdot \nabla \theta = 0, \\ \mathbf{b}_t - \nu \Delta \mathbf{b} + \mathbf{u} \cdot \nabla \mathbf{b} - \mathbf{b} \cdot \nabla \mathbf{u} = \mathbf{0}, \\ \operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{b} = 0, \end{cases} \tag{1.1}$$

where $t \geq 0$ is time, $x = (x_1, x_2) \in \mathbb{R}^2$ is the spatial coordinate, and $\rho = \rho(x,t)$, $\mathbf{u} = (u^1, u^2)(x,t)$, $\mathbf{b} = (b^1, b^2)(x,t)$, $\theta = \theta(x,t)$, and $P = P(x,t)$ denote the density, velocity, magnetic field, temperature, and pressure of the fluid, respectively. The coefficients μ and ν are positive constants. $\mathbf{e}_2 = (0,1)^T$, where T is the transpose.

We consider the Cauchy problem for (1.1) with the far field behavior

$$(\rho, \mathbf{u}, \theta, \mathbf{b}) \to (0, \mathbf{0}, 0, \mathbf{0}), \quad \text{as } |x| \to \infty, \tag{1.2}$$

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and the initial condition

$$\rho(x,0)=\rho_0(x),\quad \rho\mathbf{u}(x,0)=\rho_0\mathbf{u}_0(x),\quad \theta(x,0)=\theta_0(x),\quad \mathbf{b}(x,0)=\mathbf{b}_0(x),\quad x\in\mathbb{R}^2, \tag{1.3}$$

for given initial data ρ_0, $\mathbf{u}_0$, θ_0, and $\mathbf{b}_0$.

The system (1.1) is a combination of the nonhomogeneous Boussinesq equations of fluid dynamics and Maxwell's equations of electromagnetism, where the displacement current can be neglected. The Boussinesq-MHD system models the convection of an incompressible flow driven by the buoyant effect of a thermal or density field, and the Lorenz force, generated by the magnetic field of the fluid and the Lorentz force. Specifically, it closely relates to a natural type of the Rayleigh-Bénard convection, which occurs in a horizontal layer of conductive fluid heated from below, with the presence of a magnetic field. For more physics background, one may refer to [7, 14, 16] and references therein.

When ρ is constant, the system (1.1) reduces to the homogeneous Boussinesq-MHD system. Recently, the well-posedness issue of solutions has attracted much attention. Bian [3] studied the initial boundary value problem of two-dimensional (2D) viscous Boussinesq-MHD system and obtained a unique classical solution for H^3 initial data. Without smallness assumption on the initial data, Bian and Gui [4] proved the global unique solvability of 2D Boussinesq-MHD system with the temperature-dependent viscosity, thermal diffusivity, and electrical conductivity. Later on, the authors [5] established the global existence of weak solutions with H^1 initial data. By imposing a higher regularity assumption on the initial data, they also obtained a unique global strong solution. In [10], Larios and Pei proved the local well-posedness of solutions to the fully dissipative 3D Boussinesq-MHD system, and also the fully inviscid, irresistive, non-diffusive Boussinesq-MHD system. Moreover, they also provided a Prodi–Serrin-type global regularity condition for the 3D Boussinesq-MHD system without thermal diffusion, in terms of only two velocity and two magnetic components. By Fourier localization techniques, Zhai and Chen [20] investigated well-posedness to the Cauchy problem of the Boussinesq-MHD system with the temperature-dependent viscosity in Besov spaces. Very recently, Liu et al. [13] showed the global existence and uniqueness of strong and smooth large solutions to the 3D Boussinesq-MHD system with a damping term. Meanwhile, Bian and Pu [6] proved global axisymmetric smooth solutions for the 3D Boussinesq-MHD equations without magnetic diffusion and heat convection.

If the fluid is not affected by the Lorentz force (i.e., $\mathbf{b}=\mathbf{0}$), then the system (1.1) becomes the nonhomogeneous Boussinesq system. The authors [9, 21] studied regularity criteria for 3D nonhomogeneous incompressible Boussinesq equations, while Qiu and Yao [17] showed the local existence and uniqueness of strong solutions of multi-dimensional nonhomogeneous incompressible Boussinesq equations in Besov spaces. A blow-up criterion was also obtained in [17]. We should point out here that the results in [9,17,21] always require the initial density is bounded away from zero. For the initial density allowing vacuum states, Zhong [22] recently showed local existence of strong solutions of the Cauchy problem in $\mathbb{R}^2$ by making use of weighted energy estimate techniques. In this paper, we will investigate the local existence of strong solutions to the problem (1.1)–(1.3) with zero density at infinity. The initial density is allowed to vanish and the spatial measure of the set of vacuum can be arbitrarily large, in particular, the initial density can even have compact support.

Before stating our main result, we first explain the notations and conventions used throughout this paper. For $r>0$, set

$$B_r \triangleq \left\{x\in\mathbb{R}^2 \mid |x|<r\right\}.$$

For $1 \le p \le \infty$ and integer $k \ge 0$, the standard Sobolev spaces are denoted by:

$$L^p = L^p(\mathbb{R}^2), \quad W^{k,p} = W^{k,p}(\mathbb{R}^2), \quad H^k = H^{k,2}(\mathbb{R}^2), \quad D^{k,p} = \{u \in L^1_{\mathrm{loc}} \mid \nabla^k u \in L^p\}.$$

Our main result can be stated as follows:

Theorem 1.1. *Let η_0 be a positive constant and*

$$\bar{x} \triangleq (3 + |x|^2)^{\frac{1}{2}} \log^{1+\eta_0}(3 + |x|^2). \tag{1.4}$$

For constants $q > 2$ and $a > 1$, we assume that the initial data $(\rho_0 \ge 0, \mathbf{u}_0, \theta_0 \ge 0, \mathbf{b}_0)$ satisfy

$$\begin{cases} \rho_0 \bar{x}^a \in L^1 \cap H^1 \cap W^{1,q}, \ \theta_0 \in H^1 \cap W^{1,q}, \\ \sqrt{\rho_0}\mathbf{u}_0 \in L^2, \ \nabla \mathbf{u}_0 \in L^2, \ \operatorname{div} \mathbf{u}_0 = 0, \\ \mathbf{b}_0 \bar{x}^{\frac{a}{2}} \in L^2, \ \nabla \mathbf{b}_0 \in L^2, \ \operatorname{div} \mathbf{b}_0 = 0. \end{cases} \tag{1.5}$$

Then there exists a positive time $T_0 > 0$ such that the problem (1.1)–(1.3) has a strong solution $(\rho \ge 0, \mathbf{u}, \theta \ge 0, \mathbf{b})$ on $\mathbb{R}^2 \times (0, T_0]$ satisfying

$$\begin{cases} \rho \in C([0,T_0]; L^1 \cap H^1 \cap W^{1,q}), \\ \rho \bar{x}^a \in L^\infty(0,T_0; L^1 \cap H^1 \cap W^{1,q}), \\ \sqrt{\rho}\mathbf{u}, \ \nabla \mathbf{u}, \ \sqrt{t}\sqrt{\rho}\mathbf{u}_t, \ \sqrt{t}\nabla^2 \mathbf{u} \in L^\infty(0,T_0; L^2), \\ \theta \in C([0,T_0]; H^1 \cap W^{1,q}), \\ \mathbf{b}, \mathbf{b}\bar{x}^{\frac{a}{2}}, \nabla \mathbf{b}, \sqrt{t}\mathbf{b}_t, \ \sqrt{t}\nabla^2 \mathbf{b} \in L^\infty(0,T_0; L^2), \\ \nabla \mathbf{u} \in L^2(0,T_0; H^1) \cap L^{\frac{q+1}{q}}(0,T_0; W^{1,q}), \\ \nabla \mathbf{b} \in L^2(0,T_0; H^1), \ \mathbf{b}_t, \ \nabla \mathbf{b}\bar{x}^{\frac{a}{2}} \in L^2(0,T_0; L^2), \\ \sqrt{t}\nabla \mathbf{u} \in L^2(0,T_0; W^{1,q}), \\ \sqrt{\rho}\mathbf{u}_t, \ \sqrt{t}\nabla \mathbf{b}\bar{x}^{\frac{a}{2}}, \ \sqrt{t}\nabla \mathbf{u}_t, \ \sqrt{t}\nabla \mathbf{b}_t \in L^2(\mathbb{R}^2 \times (0,T_0)), \end{cases} \tag{1.6}$$

and

$$\inf_{0 \le t \le T_0} \int_{B_{N_1}} \rho(x,t)dx \ge \frac{1}{4} \int_{\mathbb{R}^2} \rho_0(x)dx, \tag{1.7}$$

for some positive constant N_1. Moreover, if $\theta_0 \bar{x}^a \in H^1 \cap W^{1,q}$, then the strong solution just established is unique.

Remark 1.2. When there is no electromagnetic field effect, that is $\mathbf{b} = \mathbf{0}$, (1.1) turns to be the nonhomogeneous Boussinesq equations, and Theorem 1.1 is the same as that of in [22]. Hence we generalize the main result of [22] to the nonhomogeneous Boussinesq-MHD system (1.1). However, compared with [22], for the system (1.1) treated here, the strong coupling between the velocity field and the magnetic field, such as $\mathbf{u} \cdot \nabla \mathbf{b}$, as well as strong nonlinearity $\mathbf{b} \cdot \nabla \mathbf{b}$, will bring out some new difficulties. To this end, we require $\mathbf{b}_0 \bar{x}^{\frac{a}{2}} \in L^2$ and $\nabla \mathbf{b}_0 \in L^2$ beyond the typical hypothesis of $\mathbf{b}_0 \in H^1$. This additional hypothesis is needed in order to obtain the estimate (3.10), which plays a crucial role in dealing with coupling between the velocity field and the magnetic field.

The rest of the paper is organized as follows. In Section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Sections 3 is devoted to the a priori estimates which are needed to obtain the local existence of strong solutions. The main result Theorem 1.1 is proved in Section 4.

2 Preliminaries

In this section, we will recall some known facts and elementary inequalities which will be used frequently later. First of all, if the initial density is strictly away from vacuum, the following local existence theorem on bounded balls can be shown by similar arguments as in [19].

Lemma 2.1. *For $R>0$ and $B_R=\{x\in\mathbb{R}^2 \mid |x|<R\}$, assume that $(\rho_0,\mathbf{u}_0,\theta_0\geq 0,\mathbf{b}_0)$ satisfies*

$$(\rho_0,\mathbf{u}_0,\theta_0,\mathbf{b}_0)\in H^2(B_R),\quad \inf_{x\in B_R}\rho_0(x)>0,\quad \operatorname{div}\mathbf{u}_0=\operatorname{div}\mathbf{b}_0=0. \tag{2.1}$$

Then there exists a small time $T_R>0$ and a unique classical solution $(\rho,\mathbf{u},P,\theta,\mathbf{b})$ to the following initial-boundary-value problem

$$\begin{cases}\rho_t+\operatorname{div}(\rho\mathbf{u})=0,\\ (\rho\mathbf{u})_t+\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})-\mu\Delta\mathbf{u}+\nabla P=\mathbf{b}\cdot\nabla\mathbf{b}+\rho\theta\mathbf{e}_2,\\ \theta_t+\mathbf{u}\cdot\nabla\theta=0,\\ \mathbf{b}_t-\nu\Delta\mathbf{b}+\mathbf{u}\cdot\nabla\mathbf{b}-\mathbf{b}\cdot\nabla\mathbf{u}=\mathbf{0},\\ \operatorname{div}\mathbf{u}=\operatorname{div}\mathbf{b}=0,\\ (\rho,\mathbf{u},\theta,\mathbf{b})(x,t=0)=(\rho_0,\mathbf{u}_0,\theta_0,\mathbf{b}_0), & x\in B_R,\\ \mathbf{u}(x,t)=\mathbf{b}(x,t)=\mathbf{0}, & x\in\partial B_R,\ t>0,\end{cases} \tag{2.2}$$

on $B_R\times(0,T_R]$ such that

$$\begin{cases}(\rho,\theta)\in C\left([0,T_R];H^2\right),\\ (\mathbf{u},\mathbf{b})\in C\left([0,T_R];H^2\right)\cap L^2\left(0,T_R;H^3\right),\\ P\in C\left([0,T_R];H^1\right)\cap L^2\left(0,T_R;H^2\right),\end{cases} \tag{2.3}$$

where we denote $H^k=H^k(B_R)$ for positive integer k.

Next, for $\Omega\subset\mathbb{R}^2$, the following weighted L^m-bounds for elements of the Hilbert space $\tilde{D}^{1,2}(\Omega)\triangleq\{v\in H^1_{\mathrm{loc}}(\Omega)|\nabla v\in L^2(\Omega)\}$ can be found in [12, Theorem B.1].

Lemma 2.2. *For $m\in[2,\infty)$ and $s\in(1+\frac{m}{2},\infty)$, there exists a positive constant C such that for either $\Omega=\mathbb{R}^2$ or $\Omega=B_R$ with $R\geq 1$ and for any $v\in\tilde{D}^{1,2}(\Omega)$,*

$$\left(\int_\Omega\frac{|v|^m}{3+|x|^2}(\log(3+|x|^2))^{-s}dx\right)^{\frac{1}{m}}\leq C\|v\|_{L^2(B_1)}+C\|\nabla v\|_{L^2(\Omega)}. \tag{2.4}$$

A useful consequence of Lemma 2.2 is the following crucial weighted bounds for elements of $\tilde{D}^{1,2}(\Omega)$, which have been proved in [11, Lemma 2.3].

Lemma 2.3. *Let $\bar{x}$ and η_0 be as in (1.4) and Ω be as in Lemma 2.2. Assume that $\rho\in L^1(\Omega)\cap L^\infty(\Omega)$ is a non-negative function such that*

$$\int_{B_{N_1}}\rho dx\geq M_1,\quad \|\rho\|_{L^1(\Omega)\cap L^\infty(\Omega)}\leq M_2, \tag{2.5}$$

for positive constants M_1,M_2, and $N_1\geq 1$ with $B_{N_1}\subset\Omega$. Then for $\varepsilon>0$ and $\eta>0$, there is a positive constant C depending only on $\varepsilon,\eta,M_1,M_2,N_1$, and η_0 such that every $v\in\tilde{D}^{1,2}(\Omega)$ satisfies

$$\|v\bar{x}^{-\eta}\|_{L^{(2+\varepsilon)/\tilde{\eta}}(\Omega)}\leq C\|\sqrt{\rho}v\|_{L^2(\Omega)}+C\|\nabla v\|_{L^2(\Omega)} \tag{2.6}$$

with $\tilde{\eta}=\min\{1,\eta\}$.

Next, the following L^p-bound for elliptic systems, whose proof is similar to that of [8, Lemma 12], is a direct result of the combination of the well-known elliptic theory [1,2] and a standard scaling procedure.

Lemma 2.4. *For $p>1$ and $k\geq 0$, there exists a positive constant C depending only on p and k such that*

$$\|\nabla^{k+2}v\|_{L^p(B_R)}\leq C\|\Delta v\|_{W^{k,p}(B_R)}, \tag{2.7}$$

for every $v\in W^{k+2,p}(B_R)$ satisfying

$$v=0 \quad on\ B_R.$$

3 A priori estimates

Throughout this section, for $r\in[1,\infty]$ and $k\geq 0$, we denote

$$\int\cdot\,dx=\int_{B_R}\cdot\,dx,\quad L^r=L^r(B_R),\quad W^{k,r}=W^{k,r}(B_R),\quad H^k=W^{k,2}.$$

Moreover, for $R>4N_0\geq 4$ with N_0 fixed, assume that $(\rho_0,\mathbf{u}_0,\theta_0,\mathbf{b}_0)$ satisfies, in addition to (2.1), that

$$\frac{1}{2}\leq\int_{B_{N_0}}\rho_0(x)dx\leq\int_{B_R}\rho_0(x)dx\leq 1. \tag{3.1}$$

Thus Lemma 2.1 yields that there exists some $T_R>0$ such that the initial-boundary-value problem (1.1) and (2.2) has a unique classical solution $(\rho,\mathbf{u},P,\theta,\mathbf{b})$ on $B_R\times[0,T_R]$ satisfying (2.3).

Let $\bar{x},\eta_0,a,$ and q be as in Theorem 1.1, the main aim of this section is to derive the following key a priori estimate on ψ defined by

$$\psi(t)\triangleq 1+\|\sqrt{\rho}\mathbf{u}\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}+\|\theta\|_{H^1\cap W^{1,q}}+\|\nabla\mathbf{b}\|_{L^2}+\|\bar{x}^{\frac{a}{2}}\mathbf{b}\|_{L^2}+\|\bar{x}^a\rho\|_{L^1\cap H^1\cap W^{1,q}}. \tag{3.2}$$

Proposition 3.1. *Assume that $(\rho_0,\mathbf{u}_0,\theta_0,\mathbf{b}_0)$ satisfies (2.1) and (3.1). Let $(\rho,\mathbf{u},P,\theta,\mathbf{b})$ be the solution to the initial-boundary-value problem (1.1) and (2.2) on $B_R\times(0,T_R]$ obtained by Lemma 2.1. Then there exist positive constants T_0 and M both depending only on $\mu,\nu,\eta_0,q,\ a,\ N_0,$ and E_0 such that*

$$\begin{aligned}&\sup_{0\leq t\leq T_0}\left[\psi(t)+\sqrt{t}\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}+\|\nabla^2\mathbf{u}\|_{L^2}+\|\mathbf{b}_t\|_{L^2}+\|\nabla^2\mathbf{b}\|_{L^2}+\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\right)\right]\\&+\int_0^{T_0}\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\nabla^2\mathbf{u}\|_{L^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2+\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\right)dt\\&+\int_0^{T_0}\left(\|\nabla^2\mathbf{u}\|_{L^q}^{\frac{q+1}{q}}+\|\nabla P\|_{L^q}^{\frac{q+1}{q}}+t\|\nabla^2\mathbf{u}\|_{L^q}^2+t\|\nabla P\|_{L^q}^2\right)dt\\&+\int_0^{T_0}\left(t\|\nabla\mathbf{u}_t\|_{L^2}^2+t\|\nabla\mathbf{b}_t\|_{L^2}^2+t\|\nabla^2\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\right)dt\leq M,\end{aligned} \tag{3.3}$$

where

$$E_0\triangleq\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}+\|\nabla\mathbf{u}_0\|_{L^2}+\|\theta_0\|_{H^1\cap W^{1,q}}+\|\nabla\mathbf{b}_0\|_{L^2}+\|\bar{x}^{\frac{a}{2}}\mathbf{b}_0\|_{L^2}+\|\bar{x}^a\rho_0\|_{L^1\cap H^1\cap W^{1,q}}.$$

To show Proposition 3.1, whose proof will be postponed to the end of this subsection, we begin with the following standard energy estimate for $(\rho,\mathbf{u},P,\theta,\mathbf{b})$ and the estimate on the L^p-norm of the density.

Lemma 3.2. *Under the conditions of Proposition 3.1, let $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ be a smooth solution to the initial-boundary-value problem (1.1) and (2.2). Then for any $t \in (0, T_1]$,*

$$\sup_{0\le s\le t}\left(\|\rho\|_{L^1\cap L^\infty} + \|\theta\|_{L^2\cap L^\infty} + \|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2\right) + \int_0^t \left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{L^2}^2\right) ds \le C, \tag{3.4}$$

where (and in what follows) C denotes a generic positive constant depending only on $\mu, \nu, q, a, N_0, \eta_0$ and E_0. T_1 is as that of Lemma 3.3.

Proof. 1. Since $\operatorname{div} \mathbf{u} = 0$, we deduce from $(1.1)_1$ that

$$\rho_t + \mathbf{u} \cdot \nabla \rho = 0. \tag{3.5}$$

Define particle path

$$\begin{cases} \frac{d}{dt}\mathbf{X}(x,t) = \mathbf{u}(\mathbf{X}(x,t),t), \\ \mathbf{X}(x,0) = x. \end{cases}$$

Thus, along particle path, we obtain from (3.5) that

$$\frac{d}{dt}\rho(\mathbf{X}(x,t),t) = 0,$$

which implies

$$\rho(\mathbf{X}(x,t),t) = \rho_0. \tag{3.6}$$

Similarly, one derives from $(1.1)_3$ that

$$\theta(\mathbf{X}(x,t),t) = \theta_0. \tag{3.7}$$

2. Multiplying $(1.1)_2$ by $\mathbf{u}$ and then integrating the resulting equation over B_R, we have

$$\frac{1}{2}\frac{d}{dt}\int \rho|\mathbf{u}|^2 dx + \mu\int |\nabla \mathbf{u}|^2 dx = \int \mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}dx + \int \rho\theta\mathbf{e}_2\cdot\mathbf{u}dx. \tag{3.8}$$

Multiplying $(1.1)_4$ by $\mathbf{b}$ and integrating by parts, we arrive at

$$\frac{1}{2}\frac{d}{dt}\int |\mathbf{b}|^2 dx + \nu\int |\nabla \mathbf{b}|^2 dx + \int \mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}dx = 0,$$

which combined with (3.8) and (3.7) implies that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2\right) + \left(\mu\|\nabla\mathbf{u}\|_{L^2}^2 + \nu\|\nabla\mathbf{b}\|_{L^2}^2\right) &= \int \rho\theta\mathbf{u}\cdot\mathbf{e}_2 dx \\ &\le \|\rho\|_{L^\infty}^{\frac{1}{2}}\|\sqrt{\rho}\mathbf{u}\|_{L^2}\|\theta\|_{L^2} \\ &\le C\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + C. \end{aligned} \tag{3.9}$$

Thus, Gronwall's inequality leads to

$$\sup_{0\le s\le t}\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2\right) + \int_0^t \left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{L^2}^2\right) ds \le C,$$

which together with (3.6) and (3.7) yields (3.4) and completes the proof of Lemma 3.2. □

Next, we will give some spatial weighted estimates on the density and the magnetic.

Lemma 3.3. *Under the conditions of Proposition 3.1, let $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ be a smooth solution to the initial-boundary-value problem (1.1) and (2.2). Then there exists a $T_1 = T_1(N_0, E_0) > 0$ such that for all $t \in (0, T_1]$,*

$$\sup_{0\le s\le t}\left(\|\rho\bar{x}^a\|_{L^1} + \|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\right) + \int_0^t \|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 ds \le C. \tag{3.10}$$

Proof. 1. For $N > 1$, let $\varphi_N \in C_0^\infty(B_N)$ satisfy

$$0 \le \varphi_N \le 1, \quad \varphi_N(x) = 1, \quad \text{if } |x| \le \frac{N}{2}, \ |\nabla\varphi_N| \le CN^{-1}. \tag{3.11}$$

It follows from $(1.1)_1$ and (3.4) that

$$\begin{aligned}\frac{d}{dt}\int \rho\varphi_{2N_0}dx &= \int \rho\mathbf{u}\cdot\nabla\varphi_{2N_0}dx \\ &\ge -CN_0^{-1}\left(\int\rho dx\right)^{\frac{1}{2}}\left(\int\rho|\mathbf{u}|^2dx\right)^{\frac{1}{2}} \ge -\tilde{C}(E_0).\end{aligned} \tag{3.12}$$

Integrating (3.12) and using (3.1) give rise to

$$\inf_{0\le t\le T_1}\int_{B_{2N_0}}\rho dx \ge \inf_{0\le t\le T_1}\int\rho\varphi_{2N_0}dx \ge \int\rho_0\varphi_{2N_0}dx - \tilde{C}T_1 \ge \frac{1}{4}. \tag{3.13}$$

Here, $T_1 \triangleq \min\{1, (4\tilde{C})^{-1}\}$. From now on, we will always assume that $t \le T_1$. The combination of (3.13), (3.4), and (2.6) implies that for $\varepsilon > 0$ and $\eta > 0$, every $v \in \tilde{D}^{1,2}(B_R)$ satisfies

$$\|v\bar{x}^{-\eta}\|^2_{L^{\frac{2+\varepsilon}{\tilde{\eta}}}} \le C(\varepsilon,\eta)\|\sqrt{\rho}v\|_{L^2}^2 + C(\varepsilon,\eta)\|\nabla v\|_{L^2}^2, \tag{3.14}$$

with $\tilde{\eta} = \min\{1, \eta\}$.

2. Noting that

$$|\nabla\bar{x}| \le (3+2\eta_0)\log^{1+\eta_0}(3+|x|^2) \le C(a,\eta_0)\bar{x}^{\frac{4}{8+a}},$$

multiplying $(1.1)_1$ by $\bar{x}^a$ and integrating by parts imply that

$$\begin{aligned}\frac{d}{dt}\|\rho\bar{x}^a\|_{L^1} &= \int\rho(\mathbf{u}\cdot\nabla)\bar{x}a\bar{x}^{a-1}dx \\ &\le C\int\rho|\mathbf{u}|\bar{x}^{a-1+\frac{4}{8+a}}dx \\ &\le C\|\rho\bar{x}^{a-1+\frac{8}{8+a}}\|_{L^{\frac{8+a}{7+a}}}\|\mathbf{u}\bar{x}^{-\frac{4}{8+a}}\|_{L^{8+a}} \\ &\le C\|\rho\|_{L^\infty}^{\frac{1}{8+a}}\|\rho\bar{x}^a\|_{L^1}^{\frac{7+a}{8+a}}(\|\sqrt{\rho}\mathbf{u}\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}) \\ &\le C\left(1+\|\rho\bar{x}^a\|_{L^1}\right)\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\end{aligned}$$

due to (3.4) and (3.14). This combined with Gronwall's inequality and (3.4) leads to

$$\sup_{0\le s\le t}\|\rho\bar{x}^a\|_{L^1} \le C\exp\left\{C\int_0^t\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)ds\right\} \le C. \tag{3.15}$$

3. Multiplying $(1.1)_3$ by $\mathbf{b}\bar{x}^a$ and integrating by parts yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{b}\bar{x}^{a/2}\|_{L^2}^2 + \nu\|\nabla\mathbf{b}\bar{x}^{a/2}\|_{L^2}^2 &= \frac{\nu}{2}\int|\mathbf{b}|^2\Delta\bar{x}^a dx + \int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\bar{x}^a dx + \frac{1}{2}\int|\mathbf{b}|^2\mathbf{u}\cdot\nabla\bar{x}^a dx \\ &\triangleq \bar{I}_1 + \bar{I}_2 + \bar{I}_3,\end{aligned} \tag{3.16}$$

where

$$
\begin{aligned}
|\bar{I}_1| &\le C\int |\mathbf{b}|^2\bar{x}^a\bar{x}^{-2}\log^{2(1-\eta_0)}(3+|x|^2)dx \le C\int |\mathbf{b}|^2\bar{x}^a dx,\\
|\bar{I}_2| &\le C\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^4}^2\\
&\le C\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}(\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}+\|\mathbf{b}\nabla\bar{x}^{\frac{a}{2}}\|_{L^2})\\
&\le C(\|\nabla\mathbf{u}\|_{L^2}^2+1)\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+\frac{\nu}{4}\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2,\\
|\bar{I}_3| &\le C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^4}\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\|\mathbf{u}\bar{x}^{-\frac{3}{4}}\|_{L^4}\\
&\le C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^4}^2+C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}^2\right)\\
&\le C\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+\frac{\nu}{4}\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2,
\end{aligned}
\tag{3.17}
$$

due to Gagliardo–Nirenberg inequality, (3.4), and (3.14). Putting (3.17) into (3.16), we get after using Gronwall's inequality and (3.4) that

$$
\sup_{0\le s\le t}\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+\int_0^t\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2ds \le C\exp\left\{C\int_0^t\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)ds\right\}\le C,
\tag{3.18}
$$

which together with (3.15) gives (3.10) and finishes the proof of Lemma 3.3. □

Lemma 3.4. *Let T_1 be as in Lemma 3.3. Then there exists a positive constant $\alpha > 1$ such that for all $t \in (0, T_1]$,*

$$
\begin{aligned}
&\sup_{0\le s\le t}\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2\right)+\int_0^t\left(\|\sqrt{\rho}\mathbf{u}_s\|_{L^2}^2+\|\nabla^2\mathbf{u}\|_{L^2}^2+\|\mathbf{b}_s\|_{L^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2\right)ds\\
&\quad\le C+C\int_0^t\psi^{\alpha}(s)ds.
\end{aligned}
\tag{3.19}
$$

Proof. 1. It follows from (3.4), (3.10), and (3.14) that for any $\varepsilon > 0$ and any $\eta > 0$,

$$
\begin{aligned}
\|\rho^{\eta}v\|_{L^{\frac{2+\varepsilon}{\tilde{\eta}}}} &\le C\|\rho^{\eta}\bar{x}^{\frac{3\tilde{\eta}a}{4(2+\varepsilon)}}\|_{L^{\frac{4(2+\varepsilon)}{3\tilde{\eta}}}}\|v\bar{x}^{-\frac{3\tilde{\eta}a}{4(2+\varepsilon)}}\|_{L^{\frac{4(2+\varepsilon)}{\tilde{\eta}}}}\\
&\le C\left(\int\rho^{\frac{4(2+\varepsilon)\eta}{3\tilde{\eta}}-1}\rho\bar{x}^a dx\right)^{\frac{3\tilde{\eta}}{4(2+\varepsilon)}}\|v\bar{x}^{-\frac{3\tilde{\eta}a}{4(2+\varepsilon)}}\|_{L^{\frac{4(2+\varepsilon)}{\tilde{\eta}}}}\\
&\le C\|\rho\|_{L^\infty}^{\frac{4(2+\varepsilon)\eta-3\tilde{\eta}}{4(2+\varepsilon)}}\|\rho\bar{x}^a\|_{L^1}^{\frac{3\tilde{\eta}}{4(2+\varepsilon)}}\left(\|\sqrt{\rho}v\|_{L^2}+\|\nabla v\|_{L^2}\right)\\
&\le C\|\sqrt{\rho}v\|_{L^2}+C\|\nabla v\|_{L^2},
\end{aligned}
\tag{3.20}
$$

where $\tilde{\eta} = \min\{1,\eta\}$ and $v \in \tilde{D}^{1,2}(B_R)$. In particular, this together with (3.4) and (3.14) yields

$$
\|\rho^{\eta}\mathbf{u}\|_{L^{\frac{2+\varepsilon}{\tilde{\eta}}}}+\|\mathbf{u}\bar{x}^{-\eta}\|_{L^{\frac{2+\varepsilon}{\tilde{\eta}}}}\le C\left(1+\|\nabla\mathbf{u}\|_{L^2}\right),
\tag{3.21}
$$

$$
\|\rho^{\eta}\theta\|_{L^{\frac{2+\varepsilon}{\tilde{\eta}}}}+\|\theta\bar{x}^{-\eta}\|_{L^{\frac{2+\varepsilon}{\tilde{\eta}}}}\le C\left(1+\|\nabla\theta\|_{L^2}\right).
\tag{3.22}
$$

2. Multiplying $(1.1)_2$ by $\mathbf{u}_t$ and integrating by parts, one has

$$
\mu\frac{d}{dt}\int|\nabla\mathbf{u}|^2dx+\int\rho|\mathbf{u}_t|^2dx\le C\int\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2dx+\int\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}_t dx+\int\rho\theta|\mathbf{u}_t|dx.
\tag{3.23}
$$

We derive from (3.21), Hölder's inequality, and Gagliardo–Nirenberg inequality that

$$\begin{aligned}\int \rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2dx &\le C\|\sqrt{\rho}\mathbf{u}\|_{L^8}^2\|\nabla\mathbf{u}\|_{L^{\frac{8}{3}}}^2\\ &\le C\|\sqrt{\rho}\mathbf{u}\|_{L^8}^2\|\nabla\mathbf{u}\|_{L^2}^{\frac{3}{2}}\|\nabla\mathbf{u}\|_{H^1}^{\frac{1}{2}}\\ &\le C\psi^\alpha+\varepsilon\|\nabla^2\mathbf{u}\|_{L^2}^2, \end{aligned}\tag{3.24}$$

where (and in what follows) we use $\alpha > 1$ to denote a genetic constant, which may be different from line to line. For the second term on the right-hand side of (3.23), integration by parts together with $(1.1)_5$ and Gagliardo–Nirenberg inequality indicates that for any $\varepsilon > 0$,

$$\begin{aligned}\int \mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}_t dx &= -\frac{d}{dt}\int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\int \mathbf{b}_t\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}_t dx\\ &\le -\frac{d}{dt}\int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2+C\|\mathbf{b}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2\\ &\le -\frac{d}{dt}\int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2+C\|\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}\\ &\le -\frac{d}{dt}\int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2+\varepsilon\|\nabla^2\mathbf{u}\|_{L^2}^2+C\psi^\alpha. \end{aligned}\tag{3.25}$$

From Cauchy–Schwarz inequality and (3.4), we have

$$\int \rho\theta|\mathbf{u}_t|dx \le \frac{1}{2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\frac{1}{2}\|\rho\|_{L^\infty}\|\theta\|_{L^2}^2 \le \frac{1}{2}\int \rho|\mathbf{u}_t|^2dx+C. \tag{3.26}$$

Thus, inserting (3.24)–(3.26) into (3.23) gives

$$\frac{d}{dt}B(t)+\frac{1}{2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 \le \varepsilon\|\nabla^2\mathbf{u}\|_{L^2}^2+\frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2+C\psi^\alpha, \tag{3.27}$$

where

$$B(t) \triangleq \mu\|\nabla\mathbf{u}\|_{L^2}^2+\int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx$$

satisfies

$$\frac{\mu}{2}\|\nabla\mathbf{u}\|_{L^2}^2-C_1\|\nabla\mathbf{b}\|_{L^2}^2 \le B(t) \le C\|\nabla\mathbf{u}\|_{L^2}^2+C\|\nabla\mathbf{b}\|_{L^2}^2, \tag{3.28}$$

owing to Hölder's inequality, Gagliardo–Nirenberg inequality, and (3.4).

3. It follows from $(1.1)_3$ that

$$\begin{aligned}&\nu\frac{d}{dt}\|\nabla\mathbf{b}\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2+\nu^2\|\Delta\mathbf{b}\|_{L^2}^2\\ &\le C\||\mathbf{b}||\nabla\mathbf{u}|\|_{L^2}^2+C\||\mathbf{u}||\nabla\mathbf{b}|\|_{L^2}^2\\ &\le C\|\mathbf{b}\|_{L^2}\|\nabla^2\mathbf{b}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^2+C\|\bar{x}^{-\frac{a}{4}}\mathbf{u}\|_{L^8}^2\|\bar{x}^{\frac{a}{2}}\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^4}\\ &\le \frac{\nu^2}{2}\|\Delta\mathbf{b}\|_{L^2}^2+C\psi^\alpha+C\|\bar{x}^{\frac{a}{2}}\nabla\mathbf{b}\|_{L^2}^2 \end{aligned}\tag{3.29}$$

due to (2.7), (3.21), and Gagliardo–Nirenberg inequality. Multiplying (3.29) by $\nu^{-1}(C_1+1)$ and adding the resulting inequality to (3.27) imply

$$\begin{aligned}&\frac{d}{dt}\left(B(t)+(C_1+1)\|\nabla\mathbf{b}\|_{L^2}^2\right)+\frac{1}{2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2+\frac{\nu}{2}\|\Delta\mathbf{b}\|_{L^2}^2\\ &\qquad\le C\psi^\alpha+C\|\bar{x}^{\frac{a}{2}}\nabla\mathbf{b}\|_{L^2}^2+\varepsilon\|\nabla^2\mathbf{u}\|_{L^2}^2. \end{aligned}\tag{3.30}$$

Since $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ satisfies the following Stokes system

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla P = -\rho\mathbf{u}_t - \rho\mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{b} + \rho\theta\mathbf{e}_2, & x \in B_R, \\ \operatorname{div}\mathbf{u} = 0, & x \in B_R, \\ \mathbf{u}(x) = 0, & x \in \partial B_R, \end{cases} \tag{3.31}$$

applying regularity theory of Stokes system to (3.31) (see [18]) yields that for any $p \in [2, \infty)$,

$$\|\nabla^2\mathbf{u}\|_{L^p} + \|\nabla P\|_{L^p} \le C\|\rho\mathbf{u}_t\|_{L^p} + C\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^p} + C\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^p} + C\|\rho\theta\|_{L^p}. \tag{3.32}$$

Hence, we infer from (3.32), (3.4), (3.21), and Gagliardo–Nirenberg inequality that

$$\begin{aligned} &\|\nabla^2\mathbf{u}\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 \\ &\quad\le C\|\rho\mathbf{u}_t\|_{L^2}^2 + C\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2 + C\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^2}^2 + C\|\rho\theta\|_{L^2}^2 \\ &\quad\le C\|\rho\|_{L^\infty}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + C\|\rho\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2 + C\|\mathbf{b}\|_{L^4}^2\|\nabla\mathbf{b}\|_{L^4}^2 + C\|\rho\|_{L^\infty}^2\|\theta\|_{L^2}^2 \\ &\quad\le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + C\|\rho\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1} + C\|\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^2}^2\|\nabla\mathbf{b}\|_{H^1} + C \\ &\quad\le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \frac{1}{4}\|\nabla^2\mathbf{b}\|_{L^2}^2 + \frac{1}{2}\|\nabla^2\mathbf{u}\|_{L^2}^2 + C\left(1 + \|\nabla\mathbf{b}\|_{L^2}^4 + \|\nabla\mathbf{u}\|_{L^2}^6\right) \\ &\quad\le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \frac{1}{4}\|\nabla^2\mathbf{b}\|_{L^2}^2 + \frac{1}{2}\|\nabla^2\mathbf{u}\|_{L^2}^2 + C\psi^\alpha. \end{aligned} \tag{3.33}$$

Substituting (3.33) into (3.30) and choosing ε suitably small, one gets

$$\frac{d}{dt}\left(B(t) + (C_1+1)\|\nabla\mathbf{b}\|_{L^2}^2\right) + \frac{1}{4}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2 + \frac{\nu}{4}\|\Delta\mathbf{b}\|_{L^2}^2 \le C\psi^\alpha + C\|\bar{x}^{\frac{a}{2}}\nabla\mathbf{b}\|_{L^2}^2.$$

Integrating the above inequality over $(0, t)$, then we obtain (3.19) from (2.7), (3.28), (3.10), and (3.33). The proof of Lemma 3.4 is finished. □

Lemma 3.5. *Let T_1 be as in Lemma 3.3. Then there exists a positive constant $\alpha > 1$ such that for all $t \in (0, T_1]$,*

$$\sup_{0\le s\le t} s\left(\|\sqrt{\rho}\mathbf{u}_s\|_{L^2}^2 + \|\mathbf{b}_s\|_{L^2}^2\right) + \int_0^t s\left(\|\nabla\mathbf{u}_s\|_{L^2}^2 + \|\nabla\mathbf{b}_s\|_{L^2}^2\right) ds \le C\exp\left\{C\int_0^t \psi^\alpha ds\right\}. \tag{3.34}$$

Proof. 1. Differentiating $(1.1)_2$ with respect to t gives

$$\rho\mathbf{u}_{tt} + \rho\mathbf{u}\cdot\nabla\mathbf{u}_t - \mu\Delta\mathbf{u}_t = -\rho_t(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \rho\mathbf{u}_t\cdot\nabla\mathbf{u} - \nabla P_t + (\mathbf{b}\cdot\nabla\mathbf{b})_t + (\rho\theta\mathbf{e}_2)_t. \tag{3.35}$$

Multiplying (3.35) by $\mathbf{u}_t$ and integrating the resulting equality by parts over B_R, we obtain after using $(1.1)_1$ and $(1.1)_5$ that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\rho|\mathbf{u}_t|^2dx + \mu\int|\nabla\mathbf{u}_t|^2dx \\ &\quad\le C\int\rho|\mathbf{u}||\mathbf{u}_t|\left(|\nabla\mathbf{u}_t| + |\nabla\mathbf{u}|^2 + |\mathbf{u}||\nabla^2\mathbf{u}|\right)dx + C\int\rho|\mathbf{u}|^2|\nabla\mathbf{u}||\nabla\mathbf{u}_t|dx \\ &\qquad + C\int\rho|\mathbf{u}_t|^2|\nabla\mathbf{u}|dx + \int\mathbf{b}_t\cdot\nabla\mathbf{b}\cdot\mathbf{u}_tdx + \int\mathbf{b}\cdot\nabla\mathbf{b}_t\cdot\mathbf{u}_tdx \\ &\qquad + \int\rho_t\theta\mathbf{e}_2\cdot\mathbf{u}_tdx + \int\rho\theta_t\mathbf{e}_2\cdot\mathbf{u}_tdx \triangleq \sum_{i=1}^7 \hat{I}_i. \end{aligned} \tag{3.36}$$

It follows from (3.20), (3.21), and Gagliardo–Nirenberg inequality that

$$
\begin{aligned}
\hat{I}_1 &\le C\|\sqrt{\rho}\mathbf{u}\|_{L^6}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\mathbf{u}_t\|_{L^6}^{\frac{1}{2}}\left(\|\nabla\mathbf{u}_t\|_{L^2}+\|\nabla\mathbf{u}\|_{L^4}^2\right)\\
&\quad + C\|\rho^{\frac{1}{4}}\mathbf{u}\|_{L^{12}}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\mathbf{u}_t\|_{L^6}^{\frac{1}{2}}\|\nabla^2\mathbf{u}\|_{L^2}\\
&\le C(1+\|\nabla\mathbf{u}\|_{L^2}^2)\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{1}{2}}(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}+\|\nabla\mathbf{u}_t\|_{L^2})^{\frac{1}{2}}\\
&\quad \times\left(\|\nabla\mathbf{u}_t\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2}+\|\nabla^2\mathbf{u}\|_{L^2}\right)\\
&\le \frac{\mu}{8}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\psi^\alpha\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\psi^\alpha+C\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\nabla^2\mathbf{u}\|_{L^2}^2. \qquad (3.37)
\end{aligned}
$$

Hölder's inequality combined with (3.20) and (3.21) leads to

$$
\begin{aligned}
\hat{I}_2+\hat{I}_3 &\le C\|\sqrt{\rho}\mathbf{u}\|_{L^8}^2\|\nabla\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}_t\|_{L^2}+C\|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^6}^{\frac{3}{2}}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{1}{2}}\\
&\le \frac{\mu}{8}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\psi^\alpha\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\left(\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^2}^2\right). \qquad (3.38)
\end{aligned}
$$

Integration by parts together with $(1.1)_5$, Hölder's and Gagliardo–Nirenberg inequalities indicates that

$$
\begin{aligned}
\hat{I}_4+\hat{I}_5 &= -\int \mathbf{b}_t\cdot\nabla\mathbf{u}_t\cdot\mathbf{b}dx-\int \mathbf{b}\cdot\nabla\mathbf{u}_t\cdot\mathbf{b}_t dx\\
&\le \frac{\mu}{8}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{b}\|_{L^4}^2\|\mathbf{b}_t\|_{L^4}^2\\
&\le \frac{\mu}{8}\|\nabla\mathbf{u}_t\|_{L^2}^2+\frac{\mu\nu}{4(C_2+1)}\|\nabla\mathbf{b}_t\|_{L^2}^2+C\psi^\alpha\|\mathbf{b}_t\|_{L^2}^2. \qquad (3.39)
\end{aligned}
$$

Integration by parts together with $(1.1)_1$, $(1.1)_5$, Hölder's inequality, Gagliardo–Nirenberg inequality, and (3.7) indicates that

$$
\begin{aligned}
\hat{I}_6 &= \int \rho\mathbf{u}\cdot\nabla(\theta\mathbf{e}_2\cdot\mathbf{u}_t)dx\\
&\le \int \rho|\mathbf{u}||\nabla\theta||\mathbf{u}_t|dx+\int\rho|\mathbf{u}|\theta|\nabla\mathbf{u}_t|dx\\
&\le \|\sqrt{\rho}\mathbf{u}_t\|_{L^2}\|\sqrt{\rho}\mathbf{u}\|_{L^{\frac{2q}{q-2}}}\|\nabla\theta\|_{L^q}+\|\nabla\mathbf{u}_t\|_{L^2}\|\rho\mathbf{u}\|_{L^4}\|\theta\|_{L^4}\\
&\le \frac{\mu}{6}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\psi^\alpha\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\psi^\alpha. \qquad (3.40)
\end{aligned}
$$

We get from Hölder's inequality, (3.4), and (3.21) that

$$
\begin{aligned}
\hat{I}_7 &\le \int \rho|\mathbf{u}||\nabla\theta||\mathbf{u}_t|dx\\
&\le \|\sqrt{\rho}\mathbf{u}_t\|_{L^2}\|\sqrt{\rho}\mathbf{u}\|_{L^{\frac{2q}{q-2}}}\|\nabla\theta\|_{L^q}\\
&\le C\psi^\alpha\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\psi^\alpha. \qquad (3.41)
\end{aligned}
$$

Substituting (3.37)–(3.41) into (3.36), we obtain after using (3.33) that

$$
\begin{aligned}
\frac{d}{dt}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\mu\|\nabla\mathbf{u}_t\|_{L^2}^2 &\le C\psi^\alpha\left(1+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right)\\
&\quad+\frac{\mu\nu}{2(C_2+1)}\|\nabla\mathbf{b}_t\|_{L^2}^2+C\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\nabla^2\mathbf{b}\|_{L^2}^2. \qquad (3.42)
\end{aligned}
$$

2. Differentiating $(1.1)_3$ with respect to t shows

$$
\mathbf{b}_{tt}-\mathbf{b}_t\cdot\nabla\mathbf{u}-\mathbf{b}\cdot\nabla\mathbf{u}_t+\mathbf{u}_t\cdot\nabla\mathbf{b}+\mathbf{u}\cdot\nabla\mathbf{b}_t=\nu\Delta\mathbf{b}_t. \qquad (3.43)
$$

Multiplying (3.43) by $\mathbf{b}_t$ and integrating the resulting equality over B_R yield that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\mathbf{b}_t|^2dx+\nu\int|\nabla\mathbf{b}_t|^2dx\\
&=\int\mathbf{b}\cdot\nabla\mathbf{u}_t\cdot\mathbf{b}_tdx-\int\mathbf{u}_t\cdot\nabla\mathbf{b}\cdot\mathbf{b}_tdx+\int\mathbf{b}_t\cdot\nabla\mathbf{u}\cdot\mathbf{b}_tdx-\int\mathbf{u}\cdot\nabla\mathbf{b}_t\cdot\mathbf{b}_tdx\\
&\triangleq\sum_{i=1}^4 S_i.
\end{aligned}\tag{3.44}$$

On the one hand, we deduce from (3.14) and (3.18) that

$$\begin{aligned}
\sum_{i=1}^2 S_i&\le C\|\nabla\mathbf{u}_t\|_{L^2}\|\mathbf{b}_t\|_{L^4}\|\mathbf{b}\|_{L^4}+C\|\nabla\mathbf{b}_t\|_{L^2}\||\mathbf{u}_t||\mathbf{b}|\|_{L^2}\\
&\le C\|\mathbf{b}_t\|_{L^4}^2+C\|\nabla\mathbf{u}_t\|_{L^2}^2+\frac{\nu}{8}\|\nabla\mathbf{b}_t\|_{L^2}^2+C\||\mathbf{u}_t||\mathbf{b}|\|_{L^2}^2\\
&\le\frac{\nu}{4}\|\nabla\mathbf{b}_t\|_{L^2}^2+C\|\mathbf{b}_t\|_{L^2}^2+C\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{u}_t\bar{x}^{-\frac{a}{4}}\|_{L^8}^2\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\|\mathbf{b}\|_{L^4}\\
&\le\frac{\nu}{4}\|\nabla\mathbf{b}_t\|_{L^2}^2+C\|\mathbf{b}_t\|_{L^2}^2+C\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2,
\end{aligned}\tag{3.45}$$

where one has used the following estimate

$$\sup_{0\le s\le t}\||\mathbf{b}|^2\|_{L^2}^2+\int_0^t\||\nabla\mathbf{b}||\mathbf{b}|\|_{L^2}^2ds\le C.\tag{3.46}$$

Indeed, multiplying $(1.1)_3$ by $\mathbf{b}|\mathbf{b}|^2$ and integrating by parts lead to

$$\begin{aligned}
&\frac{1}{4}\left(\||\mathbf{b}|^2\|_{L^2}^2\right)_t+\nu\||\nabla\mathbf{b}||\mathbf{b}|\|_{L^2}^2+\frac{\nu}{2}\|\nabla|\mathbf{b}|^2\|_{L^2}^2\\
&\le C\|\nabla\mathbf{u}\|_{L^2}\||\mathbf{b}|^2\|_{L^4}^2\le C\|\nabla\mathbf{u}\|_{L^2}\||\mathbf{b}|^2\|_{L^2}\|\nabla|\mathbf{b}|^2\|_{L^2}\\
&\le\frac{\nu}{4}\|\nabla|\mathbf{b}|^2\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2\||\mathbf{b}|^2\|_{L^2}^2,
\end{aligned}\tag{3.47}$$

which together with Gronwall's inequality and (3.4) gives (3.46).

On the other hand, integration by parts combined with $(1.1)_5$ and Gagliardo–Nirenberg inequality yields

$$\sum_{i=3}^4 S_i=\int\mathbf{b}_t\cdot\nabla\mathbf{u}\cdot\mathbf{b}_tdx\le C\|\mathbf{b}_t\|_{L^2}\|\nabla\mathbf{b}_t\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}\le\frac{\nu}{4}\|\nabla\mathbf{b}_t\|_{L^2}^2+C\psi^\alpha\|\mathbf{b}_t\|_{L^2}^2.\tag{3.48}$$

Inserting (3.45) and (3.48) into (3.44), one has

$$\frac{d}{dt}\|\mathbf{b}_t\|_{L^2}^2+\nu\|\nabla\mathbf{b}_t\|_{L^2}^2\le C\psi^\alpha\left(\|\mathbf{b}_t\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)+C_2\|\nabla\mathbf{u}_t\|_{L^2}^2.\tag{3.49}$$

3. From (3.42) multiplied by $\mu^{-1}(C_2+1)$ and (3.49), we get

$$\begin{aligned}
&\frac{d}{dt}\left(\mu^{-1}(C_2+1)\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right)+\|\nabla\mathbf{u}_t\|_{L^2}^2+\frac{\nu}{2}\|\nabla\mathbf{b}_t\|_{L^2}^2\\
&\le C\psi^\alpha\left(1+\|\mathbf{b}_t\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)+C\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\nabla^2\mathbf{b}\|_{L^2}^2.
\end{aligned}\tag{3.50}$$

Multiplying (3.50) by t, we obtain (3.34) after using Gronwall's inequality and (3.19). The proof of Lemma 3.5 is finished. □

Lemma 3.6. *Let T_1 be as in Lemma 3.3. Then there exists a positive constant $\alpha > 1$ such that for all $t \in (0, T_1]$,*

$$\begin{aligned}&\sup_{0\le s\le t} s\left(\|\nabla^2\mathbf{u}\|_{L^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2+\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\right)+\int_0^t s\|\nabla^2\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 ds\\&\quad\le C\exp\left\{C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\right\}.\end{aligned}\tag{3.51}$$

Proof. 1. Multiplying $(1.1)_4$ by $\Delta\mathbf{b}\bar{x}^a$ and integrating by parts lead to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|\nabla\mathbf{b}|^2\bar{x}^a dx+\nu\int|\Delta\mathbf{b}|^2\bar{x}^a dx\\&\quad\le C\int|\nabla\mathbf{b}||\mathbf{b}||\nabla\mathbf{u}||\nabla\bar{x}^a|dx+C\int|\nabla\mathbf{b}|^2|\mathbf{u}||\nabla\bar{x}^a|dx+C\int|\nabla\mathbf{b}||\Delta\mathbf{b}||\nabla\bar{x}^a|dx\\&\qquad+C\int|\mathbf{b}||\nabla\mathbf{u}||\Delta\mathbf{b}|\bar{x}^a dx+C\int|\nabla\mathbf{u}||\nabla\mathbf{b}|^2\bar{x}^a dx\triangleq\sum_{i=1}^5 J_i.\end{aligned}\tag{3.52}$$

Applying (3.10), (3.14), Hölder's inequality, and Gagliardo–Nirenberg inequality, one gets by some direct calculations that

$$\begin{aligned}J_1&\le C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^4}\|\nabla\mathbf{u}\|_{L^4}\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\\&\le C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^{\frac12}\left(\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}+\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\right)^{\frac12}\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\mathbf{u}\|_{H^1}^{\frac12}\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\\&\le C\psi^\alpha+C\|\nabla^2\mathbf{u}\|_{L^2}^2+C\psi^\alpha\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2,\\J_2&\le C\||\nabla\mathbf{b}|^{2-\frac{2}{3a}}\bar{x}^{a-\frac13}\|_{L^{\frac{6a}{6a-2}}}\|\mathbf{u}\bar{x}^{-\frac13}\|_{L^{6a}}\||\nabla\mathbf{b}|^{\frac{2}{3a}}\|_{L^{6a}}\\&\le C\psi^\alpha\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^{\frac{6a-2}{3a}}\|\nabla\mathbf{b}\|_{L^4}^{\frac{2}{3a}}\le C\psi^\alpha\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+C\|\nabla\mathbf{b}\|_{L^4}^2\\&\le C\psi^\alpha\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+\frac{\nu}{4}\|\Delta\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2,\\J_3+J_4&\le\frac{\nu}{4}\|\Delta\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+C\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2\\&\le\frac{\nu}{4}\|\Delta\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+C\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\\&\quad+C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\left(\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}+\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\right)\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}\\&\le\varepsilon\|\Delta\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+C\psi^\alpha\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+C\psi^\alpha+C\|\nabla^2\mathbf{u}\|_{L^2}^2,\\J_5&\le C\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\le C\left(\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^q}^{\frac{q+1}{q}}\right)\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2.\end{aligned}$$

Substituting the above estimates into (3.52) and noting the following fact

$$\begin{aligned}\int|\nabla^2\mathbf{b}|^2\bar{x}^a dx&=\int|\Delta\mathbf{b}|^2\bar{x}^a dx-\int\partial_i\partial_k\mathbf{b}\cdot\partial_k\mathbf{b}\partial_i\bar{x}^a dx+\int\partial_i\partial_i\mathbf{b}\cdot\partial_k\mathbf{b}\partial_k\bar{x}^a dx\\&\le\int|\Delta\mathbf{b}|^2\bar{x}^a dx+\frac12\int|\nabla^2\mathbf{b}|^2\bar{x}^a dx+C\int|\nabla\mathbf{b}|^2\bar{x}^a dx,\end{aligned}$$

we derive that

$$\begin{aligned}&\frac{d}{dt}\int|\nabla\mathbf{b}|^2\bar{x}^a dx+\frac{\nu}{2}\int|\nabla^2\mathbf{b}|^2\bar{x}^a dx\\&\quad\le C\left(\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^q}^{\frac{q+1}{q}}\right)\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+C\left(\|\nabla^2\mathbf{u}\|_{L^2}^2+\psi^\alpha\right).\end{aligned}\tag{3.53}$$

2. We now claim that

$$\int_0^t \left(\|\nabla^2 \mathbf{u}\|_{L^q}^{\frac{q+1}{q}} + \|\nabla P\|_{L^q}^{\frac{q+1}{q}} + s\|\nabla^2 \mathbf{u}\|_{L^q}^2 + s\|\nabla P\|_{L^q}^2 \right) ds \le C \exp\left\{ C \int_0^t \psi^\alpha(s) ds \right\}, \tag{3.54}$$

whose proof will be given at the end of this proof. Thus, multiplying (3.53) by t, we infer from (3.10), (3.4), (3.54), and Gronwall's inequality that

$$\sup_{0\le s\le t} \left(s\|\nabla \mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 \right) + \int_0^t s\|\nabla^2 \mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 ds \le C \exp\left\{ C \exp\left\{ C \int_0^t \psi^\alpha ds \right\}\right\}. \tag{3.55}$$

3. It deduces from $(1.1)_4$, (2.7), (3.4), (3.21), Hölder's inequality, and Gagliardo–Nirenberg inequality that

$$\begin{aligned} \|\nabla^2 \mathbf{b}\|_{L^2}^2 &\le C\|\mathbf{b}_t\|_{L^2}^2 + C\||\mathbf{u}||\nabla \mathbf{b}|\|_{L^2}^2 + C\||\mathbf{b}||\nabla \mathbf{u}|\|_{L^2}^2 \\ &\le C\|\mathbf{b}_t\|_{L^2}^2 + C\|\mathbf{u}\bar{x}^{-\frac{a}{4}}\|_{L^8}^2 \|\nabla \mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2} \|\nabla \mathbf{b}\|_{L^4} + C\|\mathbf{b}\|_{L^2}\|\nabla^2 \mathbf{b}\|_{L^2}\|\nabla \mathbf{u}\|_{L^2}^2 \\ &\le C\|\mathbf{b}_t\|_{L^2}^2 + C\|\nabla \mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 + C\|\mathbf{u}\bar{x}^{-\frac{a}{4}}\|_{L^8}^4 \|\nabla \mathbf{b}\|_{L^4}^2 + C\|\nabla^2 \mathbf{b}\|_{L^2}\|\nabla \mathbf{u}\|_{L^2}^2 \\ &\le C\|\mathbf{b}_t\|_{L^2}^2 + C\|\nabla \mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 + \frac{1}{4}\|\nabla^2 \mathbf{b}\|_{L^2}^2 + C\left(1 + \|\nabla \mathbf{u}\|_{L^2}^8\right)\left(1 + \|\nabla \mathbf{b}\|_{L^2}^2\right), \end{aligned} \tag{3.56}$$

which together with (3.33) gives that

$$\begin{aligned} \|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 + \|\nabla^2 \mathbf{b}\|_{L^2}^2 &\le C\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2 + \|\nabla \mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 \right) \\ &\quad + C\left(1 + \|\nabla \mathbf{u}\|_{L^2}^8\right)\left(1 + \|\nabla \mathbf{b}\|_{L^2}^4\right). \end{aligned} \tag{3.57}$$

Then, multiplying (3.57) by s, one gets from (3.19), (3.34), and (3.55) that

$$\begin{aligned} &\sup_{0\le s\le t} \left(s\|\nabla^2 \mathbf{u}\|_{L^2}^2 + s\|\nabla P\|_{L^2}^2 + s\|\nabla^2 \mathbf{b}\|_{L^2}^2 \right) \\ &\le C \exp\left\{ C \exp\left\{ C \int_0^t \psi^\alpha ds \right\}\right\} + C\left(1 + \int_0^t \psi^\alpha(s) ds\right)^{12} \\ &\le C \exp\left\{ C \exp\left\{ C \int_0^t \psi^\alpha ds \right\}\right\}. \end{aligned} \tag{3.58}$$

4. To finish the proof of Lemma 3.6, it suffices to show (3.54). Indeed, choosing $p = q$ in (3.32), we deduce from (3.19), (3.20), and Gagliardo–Nirenberg inequality that

$$\begin{aligned} &\|\nabla^2 \mathbf{u}\|_{L^q} + \|\nabla P\|_{L^q} \\ &\le C\left(\|\rho \mathbf{u}_t\|_{L^q} + \|\rho \mathbf{u}\cdot\nabla \mathbf{u}\|_{L^q} + \||\mathbf{b}||\nabla \mathbf{b}|\|_{L^q} + \|\rho\theta\|_{L^q} \right) \\ &\le C\left(\|\rho \mathbf{u}_t\|_{L^q} + \|\rho \mathbf{u}\|_{L^{2q}}\|\nabla \mathbf{u}\|_{L^{2q}} + \|\mathbf{b}\|_{L^{2q}}\|\nabla \mathbf{b}\|_{L^{2q}} + \|\sqrt{\rho}\theta\|_{L^2} + \|\nabla\theta\|_{L^2} \right) \\ &\le C\|\rho \mathbf{u}_t\|_{L^2}^{\frac{2(q-1)}{q^2-2}} \|\rho \mathbf{u}_t\|_{L^{q^2}}^{\frac{q^2-2q}{q^2-2}} + C\psi^\alpha \left(1 + \|\nabla^2 \mathbf{u}\|_{L^2}^{1-\frac{1}{q}} + \|\nabla^2 \mathbf{b}\|_{L^2}^{1-\frac{1}{q}}\right) \\ &\le C\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{2(q-1)}{q^2-2}} \|\nabla \mathbf{u}_t\|_{L^2}^{\frac{q^2-2q}{q^2-2}} + \|\sqrt{\rho}\mathbf{u}_t\|_{L^2} \right) \\ &\quad + C\psi^\alpha \left(1 + \|\nabla^2 \mathbf{u}\|_{L^2}^{1-\frac{1}{q}} + \|\nabla^2 \mathbf{b}\|_{L^2}^{1-\frac{1}{q}}\right), \end{aligned} \tag{3.59}$$

which together with (3.19) and (3.34) implies that

$$
\begin{aligned}
&\int_0^t \left(\|\nabla^2 \mathbf{u}\|_{L^q}^{\frac{q+1}{q}} + \|\nabla P\|_{L^q}^{\frac{q+1}{q}} \right) ds \\
&\le C \int_0^t s^{-\frac{q+1}{2q}} \left(s\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)^{\frac{q^2-1}{q(q^2-2)}} \left(s\|\nabla \mathbf{u}_t\|_{L^2}^2\right)^{\frac{(q-2)(q+1)}{2(q^2-2)}} ds \\
&\quad + C\int_0^t \|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{q+1}{q}} ds + C\int_0^t \psi^\alpha \left(1 + \|\nabla^2\mathbf{u}\|_{L^2}^{\frac{q^2-1}{q^2}} + \|\nabla^2 \mathbf{b}\|_{L^2}^{\frac{q^2-1}{q^2}}\right) ds \\
&\le C \sup_{0\le s\le t} \left(s\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)^{\frac{q^2-1}{q(q^2-2)}} \int_0^t s^{-\frac{q+1}{2q}} \left(s\|\nabla \mathbf{u}_t\|_{L^2}^2\right)^{\frac{(q-2)(q+1)}{2(q^2-2)}} ds \\
&\quad + C\int_0^t \left(\psi^\alpha + \|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\nabla^2\mathbf{u}\|_{L^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2\right) ds \\
&\le C \exp\left\{C\int_0^t \psi^\alpha ds\right\} \left(1 + \int_0^t \left(s^{-\frac{q^3+q^2-2q-2}{q^3+q^2-2q}} + s\|\nabla \mathbf{u}_t\|_{L^2}^2\right) ds\right) \\
&\le C \exp\left\{C\int_0^t \psi^\alpha ds\right\}
\end{aligned}
\tag{3.60}
$$

and

$$
\begin{aligned}
&\int_0^t \left(s\|\nabla^2\mathbf{u}\|_{L^q}^2 + s\|\nabla P\|_{L^q}^2\right) ds \\
&\le C\int_0^t s\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 ds + C\int_0^t \left(s\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)^{\frac{2(q-1)}{q^2-2}} \left(s\|\nabla\mathbf{u}_t\|_{L^2}^2\right)^{\frac{q^2-2q}{q^2-2}} ds \\
&\quad + C\int_0^t s\psi^\alpha \left(1 + \|\nabla^2\mathbf{u}\|_{L^2}^{1-\frac{1}{q}} + \|\nabla^2\mathbf{b}\|_{L^2}^{1-\frac{1}{q}}\right)^2 ds \\
&\le C\int_0^t s\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 ds + C\int_0^t s\|\nabla\mathbf{u}_t\|_{L^2}^2 ds + C\int_0^t \left(\psi^\alpha + s\|\nabla^2\mathbf{u}\|_{L^2}^2 + s\|\nabla^2\mathbf{b}\|_{L^2}^2\right) ds \\
&\le C\exp\left\{C\int_0^t \psi^\alpha ds\right\}.
\end{aligned}
\tag{3.61}
$$

One thus obtains (3.54) from (3.60)–(3.61) and finishes the proof of Lemma 3.6. □

Lemma 3.7. *Let T_1 be as in Lemma 3.3. Then there exists a positive constant $\alpha > 1$ such that for all $t \in (0, T_1]$,*

$$
\sup_{0\le s\le t} \left(\|\rho\bar{x}^a\|_{H^1\cap W^{1,q}} + \|\nabla\theta\|_{L^2\cap L^q}\right) \le \exp\left\{C\exp\left\{C\int_0^t \psi^\alpha ds\right\}\right\}. \tag{3.62}
$$

Proof. 1. It follows from Sobolev's inequality and (3.21) that for $0 < \delta < 1$,

$$
\begin{aligned}
\|\mathbf{u}\bar{x}^{-\delta}\|_{L^\infty} &\le C(\delta)\left(\|\mathbf{u}\bar{x}^{-\delta}\|_{L^{\frac{4}{\delta}}} + \|\nabla(\mathbf{u}\bar{x}^{-\delta})\|_{L^3}\right) \\
&\le C(\delta)\left(\|\mathbf{u}\bar{x}^{-\delta}\|_{L^{\frac{4}{\delta}}} + \|\nabla\mathbf{u}\|_{L^3} + \|\mathbf{u}\bar{x}^{-\delta}\|_{L^{\frac{4}{\delta}}}\|\bar{x}^{-1}\nabla\bar{x}\|_{L^{\frac{12}{4-3\delta}}}\right) \\
&\le C(\delta)\left(\psi^\alpha + \|\nabla^2\mathbf{u}\|_{L^2}\right).
\end{aligned}
\tag{3.63}
$$

One derives from $(1.1)_1$ and $(1.1)_4$ that $\rho\bar{x}^a$ satisfies

$$
(\rho\bar{x}^a)_t + \mathbf{u}\cdot\nabla(\rho\bar{x}^a) - a\rho\bar{x}^a\mathbf{u}\cdot\nabla\log\bar{x} = 0, \tag{3.64}
$$

which along with (3.63) gives that for any $r \in [2, q]$,

$$\begin{aligned}
\frac{d}{dt}\|\nabla(\rho\bar{x}^a)\|_{L^r} &\le C\left(1+\|\nabla \mathbf{u}\|_{L^\infty}+\|\mathbf{u}\cdot\nabla\log\bar{x}\|_{L^\infty}\right)\|\nabla(\rho\bar{x}^a)\|_{L^r} \\
&\quad + C\|\rho\bar{x}^a\|_{L^\infty}\left(\||\nabla\mathbf{u}||\nabla\log\bar{x}|\|_{L^r}+\||\mathbf{u}||\nabla^2\log\bar{x}|\|_{L^r}\right) \\
&\le C\left(\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^2\cap L^q}\right)\|\nabla(\rho\bar{x}^a)\|_{L^r} \\
&\quad + C\|\rho\bar{x}^a\|_{L^\infty}\left(\|\nabla\mathbf{u}\|_{L^r}+\|\mathbf{u}\bar{x}^{-\frac{2}{5}}\|_{L^{4r}}\|\bar{x}^{-\frac{3}{2}}\|_{L^{\frac{4r}{3}}}\right) \\
&\le C\left(\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^2\cap L^q}\right)\left(1+\|\nabla(\rho\bar{x}^a)\|_{L^r}+\|\nabla(\rho\bar{x}^a)\|_{L^q}\right).
\end{aligned} \tag{3.65}$$

Then we derive from (3.65), (3.54), and Gronwall's inequality that

$$\sup_{0\le s\le t}\|\rho\bar{x}^a\|_{H^1\cap W^{1,q}} \le \exp\left\{C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\right\}. \tag{3.66}$$

2. Operating ∇ to $(1.1)_3$ and then multiplying $|\nabla\theta|^{r-2}\nabla\theta$ for $r\in[2,q]$ gives that

$$\begin{aligned}
\frac{d}{dt}\|\nabla\theta\|_{L^r} &\le C\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\theta\|_{L^r}+C\|\theta\|_{L^\infty}\|\nabla^2\mathbf{u}\|_{L^r} \\
&\le C\left(\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^2\cap L^q}\right)\|\nabla\theta\|_{L^r}+C\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^2\cap L^q}^{\frac{q+1}{q}} \\
&\le C\left(\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^2\cap L^q}^{\frac{q+1}{q}}\right)\left(1+\|\nabla\theta\|_{L^r}\right),
\end{aligned} \tag{3.67}$$

which along with Gronwall's inequality leads to

$$\sup_{0\le s\le t}\|\nabla\theta\|_{L^2\cap L^q} \le \exp\left\{C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\right\}. \tag{3.68}$$

Hence the desired (3.62) follows from (3.66) and (3.68). □

Now, Proposition 3.1 is a direct consequence of Lemmas 3.2–3.7.

Proof of Proposition 3.1. It follows from (3.4), (3.19), and (3.62) that

$$\psi(t) \le \exp\left\{C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\right\}.$$

Standard arguments yield that for $M \triangleq e^{Ce}$ and $T_0 \triangleq \min\{T_2, (CM^\alpha)^{-1}\}$,

$$\sup_{0\le t\le T_0}\psi(t)\le M,$$

which together with (3.62), (3.19), (3.34), and (3.54) gives (3.3). The proof of Proposition 3.1 is thus completed. □

4 Proof of Theorem 1.1

With the a priori estimates in Section 3 at hand, it is a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $(\rho_0, \mathbf{u}_0, \theta_0, \mathbf{b}_0)$ be as in Theorem 1.1. Without loss of generality, we assume that the initial density ρ_0 satisfies

$$\int_{\mathbb{R}^2} \rho_0 dx = 1,$$

which implies that there exists a positive constant N_0 such that

$$\int_{B_{N_0}} \rho_0 dx \geq \frac{3}{4} \int_{\mathbb{R}^2} \rho_0 dx = \frac{3}{4}. \tag{4.1}$$

We construct $\rho_0^R = \hat{\rho}_0^R + R^{-1} e^{-|x|^2}$, where $0 \leq \hat{\rho}_0^R \in C_0^\infty(\mathbb{R}^2)$ satisfies

$$\begin{cases} \int_{B_{N_0}} \hat{\rho}_0^R dx \geq 1/2, \\ \bar{x}^a \hat{\rho}_0^R \to \bar{x}^a \rho_0 \quad \text{in } L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2), \text{ as } R \to \infty. \end{cases} \tag{4.2}$$

Due to $\mathbf{b}_0 \bar{x}^{\frac{a}{2}} \in L^2(\mathbb{R}^2)$ and $\nabla \mathbf{b}_0 \in L^2(\mathbb{R}^2)$, we choose $\mathbf{b}_0^R \in \{\mathbf{w} \in C_0^\infty(B_R) \mid \operatorname{div} \mathbf{w} = 0\}$ satisfying

$$\mathbf{b}_0^R \bar{x}^{\frac{a}{2}} \to \mathbf{b}_0 \bar{x}^{\frac{a}{2}}, \quad \nabla \mathbf{b}_0^R \to \nabla \mathbf{b}_0 \quad \text{in } L^2(\mathbb{R}^2), \quad \text{as } R \to \infty. \tag{4.3}$$

Noting that $\theta_0 \in H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2)$, we choose $\theta_0^R \in C_0^\infty(B_R)$ such that

$$\theta_0^R \to \theta_0 \quad \text{in } H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2), \quad \text{as } R \to \infty. \tag{4.4}$$

Since $\nabla \mathbf{u}_0 \in L^2(\mathbb{R}^2)$, we select $\mathbf{v}_i^R \in C_0^\infty(B_R)$ $(i = 1, 2)$ such that for $i = 1, 2$,

$$\lim_{R \to \infty} \|\mathbf{v}_i^R - \partial_i \mathbf{u}_0\|_{L^2(\mathbb{R}^2)} = 0. \tag{4.5}$$

We consider the unique smooth solution $\mathbf{u}_0^R$ of the following elliptic problem:

$$\begin{cases} -\Delta \mathbf{u}_0^R + \rho_0^R \mathbf{u}_0^R + \nabla P_0^R = \sqrt{\rho_0^R} \mathbf{h}^R - \partial_i \mathbf{v}_i^R, & \text{in } B_R, \\ \operatorname{div} \mathbf{u}_0^R = 0, & \text{in } B_R, \\ \mathbf{u}_0^R = \mathbf{0}, & \text{on } \partial B_R, \end{cases} \tag{4.6}$$

where $\mathbf{h}^R = (\sqrt{\rho_0} \mathbf{u}_0) * j_{\frac{1}{R}}$ with j_δ being the standard mollifying kernel of width δ.

Extending $\mathbf{u}_0^R$ to $\mathbb{R}^2$ by defining $\mathbf{0}$ outside B_R and denoting it by $\tilde{\mathbf{u}}_0^R$, we claim that

$$\lim_{R \to \infty} \left(\|\nabla (\tilde{\mathbf{u}}_0^R - \mathbf{u}_0)\|_{L^2(\mathbb{R}^2)} + \|\sqrt{\rho_0^R} \tilde{\mathbf{u}}_0^R - \sqrt{\rho_0} \mathbf{u}_0\|_{L^2(\mathbb{R}^2)} \right) = 0. \tag{4.7}$$

In fact, it is easy to find that $\tilde{\mathbf{u}}_0^R$ is also a solution of (4.6) in $\mathbb{R}^2$. Multiplying (4.6) by $\tilde{\mathbf{u}}_0^R$ and integrating the resulting equation over $\mathbb{R}^2$ lead to

$$\begin{aligned} &\int_{\mathbb{R}^2} \rho_0^R |\tilde{\mathbf{u}}_0^R|^2 dx + \int_{\mathbb{R}^2} |\nabla \tilde{\mathbf{u}}_0^R|^2 dx \\ &\quad \leq \|\sqrt{\rho_0^R} \tilde{\mathbf{u}}_0^R\|_{L^2(B_R)} \|\mathbf{h}^R\|_{L^2(B_R)} + C \|\mathbf{v}_i^R\|_{L^2(B_R)} \|\partial_i \tilde{\mathbf{u}}_0^R\|_{L^2(B_R)} \\ &\quad \leq \frac{1}{2} \|\nabla \tilde{\mathbf{u}}_0^R\|_{L^2(B_R)}^2 + \frac{1}{2} \int_{B_R} \rho_0^R |\tilde{\mathbf{u}}_0^R|^2 dx + C \left(\|\mathbf{h}^R\|_{L^2(B_R)}^2 + \|\mathbf{v}_i^R\|_{L^2(B_R)}^2 \right), \end{aligned}$$

which implies

$$\int_{\mathbb{R}^2} \rho_0^R |\tilde{\mathbf{u}}_0^R|^2 dx + \int_{\mathbb{R}^2} |\nabla \tilde{\mathbf{u}}_0^R|^2 dx \le C \tag{4.8}$$

for some C independent of R. This together with (4.2) yields that there exist a subsequence $R_j \to \infty$ and a function $\tilde{\mathbf{u}}_0 \in \{\tilde{\mathbf{u}}_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^2) | \sqrt{\rho_0}\tilde{\mathbf{u}}_0 \in L^2(\mathbb{R}^2), \nabla\tilde{\mathbf{u}}_0 \in L^2(\mathbb{R}^2)\}$ such that

$$\begin{cases} \sqrt{\rho_0^{R_j}}\tilde{\mathbf{u}}_0^{R_j} \rightharpoonup \sqrt{\rho_0}\tilde{\mathbf{u}}_0 \text{ weakly in } L^2(\mathbb{R}^2), \\ \nabla\tilde{\mathbf{u}}_0^{R_j} \rightharpoonup \nabla\tilde{\mathbf{u}}_0 \text{ weakly in } L^2(\mathbb{R}^2). \end{cases} \tag{4.9}$$

Next, we will show

$$\tilde{\mathbf{u}}_0 = \mathbf{u}_0. \tag{4.10}$$

Indeed, multiplying (4.6) by a test function $\boldsymbol{\pi} \in C_0^\infty(\mathbb{R}^2)$ with $\operatorname{div}\boldsymbol{\pi} = 0$, it holds that

$$\int_{\mathbb{R}^2} (\partial_i \tilde{\mathbf{u}}_0^{R_j} - \mathbf{v}_i^{R_j}) \cdot \partial_i \boldsymbol{\pi} dx + \int_{\mathbb{R}^2} \sqrt{\rho_0^{R_j}} (\sqrt{\rho_0^{R_j}}\tilde{\mathbf{u}}_0^{R_j} - \mathbf{h}^{R_j}) \cdot \boldsymbol{\pi} dx = 0. \tag{4.11}$$

Let $R_j \to \infty$, it follows from (4.2), (4.5), and (4.9) that

$$\int_{\mathbb{R}^2} \partial_i (\tilde{\mathbf{u}}_0 - \mathbf{u}_0) \cdot \partial_i \boldsymbol{\pi} dx + \int_{\mathbb{R}^2} \rho_0 (\tilde{\mathbf{u}}_0 - \mathbf{u}_0) \cdot \boldsymbol{\pi} dx = 0, \tag{4.12}$$

which implies (4.10).

Furthermore, multiplying (4.6) by $\tilde{\mathbf{u}}_0^{R_j}$ and integrating the resulting equation over $\mathbb{R}^2$, by the same arguments as (4.12), we have

$$\lim_{R_j\to\infty} \int_{\mathbb{R}^2} \left(|\nabla \tilde{\mathbf{u}}_0^{R_j}|^2 + \rho_0^{R_j} |\tilde{\mathbf{u}}_0^{R_j}|^2 \right) dx = \int_{\mathbb{R}^2} \left(|\nabla \mathbf{u}_0|^2 + \rho_0 |\mathbf{u}_0|^2 \right) dx,$$

which combined with (4.9) leads to

$$\lim_{R_j\to\infty} \int_{\mathbb{R}^2} |\nabla \tilde{\mathbf{u}}_0^{R_j}|^2 dx = \int_{\mathbb{R}^2} |\nabla \tilde{\mathbf{u}}_0|^2 dx, \ \lim_{R_j\to\infty} \int_{\mathbb{R}^2} \rho_0^{R_j} |\tilde{\mathbf{u}}_0^{R_j}|^2 dx = \int_{\mathbb{R}^2} \rho_0 |\tilde{\mathbf{u}}_0|^2 dx.$$

This, along with (4.10) and (4.9), gives (4.7).

Hence, by virtue of Lemma 2.1, the initial-boundary-value problem (2.2) with the initial data $(\rho_0^R, \mathbf{u}_0^R, \theta_0^R, \mathbf{b}_0^R)$ has a classical solution $(\rho^R, \mathbf{u}^R, P^R, \theta^R, \mathbf{b}^R)$ on $B_R \times [0, T_R]$. Moreover, Proposition 3.1 shows that there exists a T_0 independent of R such that (3.3) holds for $(\rho^R, \mathbf{u}^R, P^R, \theta^R, \mathbf{b}^R)$.

For simplicity, in what follows, we denote

$$L^p = L^p(\mathbb{R}^2), \ W^{k,p} = W^{k,p}(\mathbb{R}^2).$$

Extending $(\rho^R, \mathbf{u}^R, P^R, \theta^R, \mathbf{b}^R)$ by zero on $\mathbb{R}^2 \setminus B_R$ and denoting it by

$$\left(\tilde{\rho}^R \triangleq \varphi_R \rho^R, \tilde{\mathbf{u}}^R, \tilde{P}^R, \tilde{\theta}^R, \tilde{\mathbf{b}}^R \right)$$

with φ_R satisfying (3.11). First, (3.3) leads to

$$\begin{aligned} &\sup_{0\le t\le T_0} \left(\|\sqrt{\tilde{\rho}^R}\tilde{\mathbf{u}}^R\|_{L^2} + \|\nabla\tilde{\mathbf{u}}^R\|_{L^2} + \|\nabla\tilde{\theta}^R\|_{L^2\cap L^q} + \|\nabla\tilde{\mathbf{b}}^R\|_{L^2} + \|\tilde{\mathbf{b}}^R\bar{x}^{\frac{a}{2}}\|_{L^2} \right) \\ &\le \sup_{0\le t\le T_0} \Big(\|\sqrt{\rho^R}\mathbf{u}^R\|_{L^2(B_R)} + \|\nabla\mathbf{u}^R\|_{L^2(B_R)} \\ &\qquad + \|\nabla\theta^R\|_{L^2(B_R)\cap L^q(B_R)} + \|\nabla\mathbf{b}^R\|_{L^2(B_R)} + \|\mathbf{b}^R\bar{x}^{\frac{a}{2}}\|_{L^2(B_R)} \Big) \\ &\le C, \end{aligned} \tag{4.13}$$

and

$$\sup_{0\le t\le T_0}\|\tilde\rho^R\bar x^a\|_{L^1\cap L^\infty}\le C. \tag{4.14}$$

Similarly, it follows from (3.3) that for $q>2$,

$$\begin{aligned}&\sup_{0\le t\le T_0}\sqrt{t}\left(\|\sqrt{\tilde\rho^R}\tilde{\mathbf{u}}_t^R\|_{L^2}+\|\nabla^2\tilde{\mathbf{u}}^R\|_{L^2}+\|\nabla^2\tilde{\mathbf{b}}^R\|_{L^2}+\|\tilde{\mathbf{b}}_t^R\|_{L^2}\right)\\&\quad+\int_0^{T_0}\left(\|\sqrt{\tilde\rho^R}\tilde{\mathbf{u}}_t^R\|_{L^2}^2+\|\nabla^2\tilde{\mathbf{u}}^R\|_{L^2}^2+\|\nabla^2\tilde{\mathbf{b}}^R\|_{L^2}^2+\|\nabla\tilde{\mathbf{b}}^R\bar x^{\frac a2}\|_{L^2}^2\right)dt\\&\quad+\int_0^{T_0}\left(\|\nabla^2\tilde{\mathbf{u}}^R\|_{L^q}^{\frac{q+1}{q}}+t\|\nabla^2\tilde{\mathbf{u}}^R\|_{L^q}^2+t\|\nabla\tilde{\mathbf{u}}_t^R\|_{L^2}^2+t\|\nabla\tilde{\mathbf{b}}_t^R\|_{L^2}^2\right)dt\le C.\end{aligned} \tag{4.15}$$

Next, for $p\in[2,q]$, we obtain from (3.3) and (3.62) that

$$\begin{aligned}\sup_{0\le t\le T_0}\|\nabla(\tilde\rho^R\bar x^a)\|_{L^p}&\le C\sup_{0\le t\le T_0}\left(\|\nabla(\rho^R\bar x^a)\|_{L^p(B_R)}+R^{-1}\|\rho^R\bar x^a\|_{L^p(B_R)}\right)\\&\le C\sup_{0\le t\le T_0}\|\rho^R\bar x^a\|_{H^1(B_R)\cap W^{1,p}(B_R)}\le C,\end{aligned} \tag{4.16}$$

which together with (3.63) and (3.3) yields

$$\begin{aligned}\int_0^{T_0}\|\bar x\tilde\rho_t^R\|_{L^p}^2dt&\le C\int_0^{T_0}\|\bar x|\mathbf{u}^R||\nabla\rho^R|\|_{L^p(B_R)}^2)dt\\&\le C\int_0^{T_0}\|\bar x^{1-a}\mathbf{u}^R\|_{L^\infty(B_R)}^2\|\bar x^a\nabla\rho^R\|_{L^p(B_R)}^2dt\\&\le C.\end{aligned} \tag{4.17}$$

With the estimates (4.13)–(4.17) together with $(2.2)_1$ and $(2.2)_3$, we find that the sequence $(\tilde\rho^R,\tilde{\mathbf{u}}^R,\tilde P^R,\tilde\theta^R,\tilde{\mathbf{b}}^R)$ converges, up to the extraction of subsequences, to some limit $(\rho,\mathbf{u},P,\theta,\mathbf{b})$ in the obvious weak sense, that is, as $R\to\infty$, we have

$$\tilde\rho^R\bar x\to\rho\bar x,\ \tilde\theta^R\to\theta,\ \text{in } C(\overline{B_N}\times[0,T_0]),\ \text{for any } N>0, \tag{4.18}$$
$$\tilde\rho^R\bar x^a\rightharpoonup\rho\bar x^a,\ \text{weakly * in } L^\infty(0,T_0;H^1\cap W^{1,q}), \tag{4.19}$$
$$\nabla\tilde\theta^R\rightharpoonup\nabla\theta,\ \text{weakly * in } L^\infty(0,T_0;L^2\cap L^q), \tag{4.20}$$
$$\tilde{\mathbf{b}}^R\bar x^{\frac a2}\rightharpoonup\mathbf{b}\bar x^{\frac a2},\ \text{weakly * in} L^\infty(0,T_0;L^2), \tag{4.21}$$
$$\tilde{\mathbf{b}}_t^R\rightharpoonup\mathbf{b}_t,\ \nabla\tilde{\mathbf{b}}^R\bar x^{\frac a2}\rightharpoonup\nabla\mathbf{b}\bar x^{\frac a2},\ \nabla^2\tilde{\mathbf{b}}^R\rightharpoonup\nabla^2\mathbf{b},\ \text{weakly in } L^2(\mathbb{R}^2\times(0,T_0)), \tag{4.22}$$
$$\sqrt{\tilde\rho^R}\tilde{\mathbf{u}}^R\rightharpoonup\sqrt{\rho}\mathbf{u},\ \nabla\tilde{\mathbf{u}}^R\rightharpoonup\nabla\mathbf{u},\ \nabla\tilde{\mathbf{b}}^R\rightharpoonup\nabla\mathbf{b},\ \text{weakly * in } L^\infty(0,T_0;L^2), \tag{4.23}$$
$$\nabla^2\tilde{\mathbf{u}}^R\rightharpoonup\nabla^2\mathbf{u},\ \text{weakly in } L^{\frac{q+1}{q}}(0,T_0;L^q)\cap L^2(\mathbb{R}^2\times(0,T_0)), \tag{4.24}$$
$$\sqrt{t}\nabla^2\tilde{\mathbf{u}}^R\rightharpoonup\sqrt{t}\nabla^2\mathbf{u},\ \text{weakly in } L^2(0,T_0;L^q),\ \text{weakly * in } L^\infty(0,T_0;L^2), \tag{4.25}$$
$$\sqrt{t}\tilde{\mathbf{b}}_t^R\rightharpoonup\sqrt{t}\mathbf{b}_t,\ \sqrt{t}\nabla^2\tilde{\mathbf{b}}^R\rightharpoonup\sqrt{t}\nabla^2\mathbf{b},\ \text{weakly * in } L^\infty(0,T_0;L^2), \tag{4.26}$$
$$\sqrt{t}\sqrt{\tilde\rho^R}\tilde{\mathbf{u}}_t^R\rightharpoonup\sqrt{t}\sqrt{\rho}\mathbf{u}_t,\ \text{weakly * in } L^\infty(0,T_0;L^2), \tag{4.27}$$
$$\sqrt{t}\nabla\tilde{\mathbf{u}}_t^R\rightharpoonup\sqrt{t}\nabla\mathbf{u}_t,\ \sqrt{t}\nabla\tilde{\mathbf{b}}_t^R\rightharpoonup\sqrt{t}\nabla\mathbf{b}_t,\ \text{weakly in } L^2(\mathbb{R}^2\times(0,T_0)), \tag{4.28}$$

with

$$\rho\bar x^a\in L^\infty(0,T_0;L^1),\quad \inf_{0\le t\le T_0}\int_{B_{2N_0}}\rho(x,t)dx\ge\frac14. \tag{4.29}$$

Next, for any function $\phi \in C_0^\infty(\mathbb{R}^2 \times [0, T_0))$, we take $\phi\varphi_R$ as test function in the initial-boundary-value problem (2.2) with the initial data $(\rho_0^R, \mathbf{u}_0^R, \theta_0^R, \mathbf{b}_0^R)$. Then, letting $R \to \infty$, standard arguments together with (4.18)–(4.29) show that $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ is a strong solution of (1.1)–(1.3) on $\mathbb{R}^2 \times (0, T_0]$ satisfying (1.6) and (1.7). Indeed, the existence of a pressure P follows immediately from the $(1.1)_2$ and $(1.1)_4$ by a classical consideration. The proof of the existence part of Theorem 1.1 is finished.

It remains only to prove the uniqueness of the strong solutions provided that $\theta_0\bar{x}^a \in H^1 \cap W^{1,q}$. Let $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ and $(\bar{\rho}, \bar{\mathbf{u}}, \bar{P}, \bar{\theta}, \bar{\mathbf{b}})$ be two strong solutions satisfying (1.6) and (1.7) with the same initial data, and denote

$$\Theta \triangleq \rho - \bar{\rho}, \ \mathbf{U} \triangleq \mathbf{u} - \bar{\mathbf{u}}, \ \Psi \triangleq \theta - \bar{\theta}, \ \mathbf{\Phi} \triangleq \mathbf{b} - \bar{\mathbf{b}}.$$

First, subtracting the mass equation satisfied by $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ and $(\bar{\rho}, \bar{\mathbf{u}}, \bar{P}, \bar{\theta}, \bar{\mathbf{b}})$ gives

$$\Theta_t + \bar{\mathbf{u}} \cdot \nabla\Theta + \mathbf{U} \cdot \nabla\rho = 0. \tag{4.30}$$

Multiplying (4.30) by $2\Theta\bar{x}^{2r}$ for $r \in (1, \tilde{a})$ with $\tilde{a} = \min\{2, a\}$ and integrating by parts yield

$$\begin{aligned}\frac{d}{dt}\int |\Theta\bar{x}^r|^2 dx &\le C\|\bar{\mathbf{u}}\bar{x}^{-\frac{1}{2}}\|_{L^\infty}\|\Theta\bar{x}^r\|_{L^2}^2 + C\|\Theta\bar{x}^r\|_{L^2}\|\mathbf{U}\bar{x}^{-(\tilde{a}-r)}\|_{L^{\frac{2q}{(q-2)(\tilde{a}-r)}}}\|\bar{x}^{\tilde{a}}\nabla\rho\|_{L^{\frac{2q}{q-(q-2)(\tilde{a}-r)}}} \\ &\le C\left(1 + \|\nabla\bar{\mathbf{u}}\|_{W^{1,q}}\right)\|\Theta\bar{x}^r\|_{L^2}^2 + C\|\Theta\bar{x}^r\|_{L^2}\left(\|\nabla\mathbf{U}\|_{L^2} + \|\sqrt{\rho}\mathbf{U}\|_{L^2}\right)\end{aligned}$$

due to Sobolev's inequality, (1.7), (3.14), and (3.63). This combined with Gronwall's inequality shows that for all $0 \le t \le T_0$,

$$\|\Theta\bar{x}^r\|_{L^2} \le C\int_0^t \left(\|\nabla\mathbf{U}\|_{L^2} + \|\sqrt{\rho}\mathbf{U}\|_{L^2}\right) ds. \tag{4.31}$$

Similarly to (4.31), one has

$$\|\Psi\bar{x}^r\|_{L^2} \le C\int_0^t \left(\|\nabla\mathbf{U}\|_{L^2} + \|\sqrt{\rho}\mathbf{U}\|_{L^2}\right) ds. \tag{4.32}$$

Next, subtracting $(1.1)_2$ and $(1.1)_4$ satisfied by $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ and $(\bar{\rho}, \bar{\mathbf{u}}, \bar{P}, \bar{\theta}, \bar{\mathbf{b}})$ leads to

$$\begin{aligned}\rho\mathbf{U}_t + \rho\mathbf{U} \cdot \nabla\mathbf{U} - \mu\Delta\mathbf{U} = &- \rho\mathbf{U} \cdot \nabla\bar{\mathbf{u}} - \Theta(\bar{\mathbf{u}}_t + \bar{\mathbf{u}} \cdot \nabla\bar{\mathbf{u}}) - \nabla(P - \bar{P}) \\ &+ \mathbf{b} \cdot \nabla\mathbf{\Phi} + \mathbf{\Phi} \cdot \nabla\bar{\mathbf{b}} + \Theta\theta\mathbf{e}_2 + \bar{\rho}\Psi\mathbf{e}_2\end{aligned} \tag{4.33}$$

and

$$\mathbf{\Phi}_t - \nu\Delta\mathbf{\Phi} = \mathbf{b} \cdot \nabla\mathbf{U} + \mathbf{\Phi} \cdot \nabla\bar{\mathbf{u}} - \mathbf{u} \cdot \nabla\mathbf{\Phi} - \mathbf{U} \cdot \nabla\bar{\mathbf{b}}, \tag{4.34}$$

Multiplying (4.33) by $\mathbf{U}$ and (4.34) by $\mathbf{\Phi}$ respectively, and adding the resulting equations together, we obtain after integration by parts that

$$\begin{aligned}&\frac{d}{dt}\int \left(\rho|\mathbf{U}|^2 + |\mathbf{\Phi}|^2\right) dx + \int \left(\mu|\nabla\mathbf{U}|^2 + \nu|\nabla\mathbf{\Phi}|^2\right) dx \\ &\le C\|\nabla\bar{\mathbf{u}}\|_{L^\infty}\int \left(\rho|\mathbf{U}|^2 + |\mathbf{\Phi}|^2\right) dx + C\int |\Theta||\mathbf{U}|\left(|\bar{\mathbf{u}}_t| + |\bar{\mathbf{u}}||\nabla\bar{\mathbf{u}}|\right) dx \\ &\quad + C\int |\mathbf{U}|\left(|\Theta|\theta + \bar{\rho}|\Psi|\right) dx - \int \mathbf{\Phi} \cdot \nabla\mathbf{U} \cdot \bar{\mathbf{b}} dx - \int \mathbf{U} \cdot \nabla\bar{\mathbf{b}} \cdot \mathbf{\Phi} dx \\ &\triangleq C\|\nabla\bar{\mathbf{u}}\|_{L^\infty}\int \left(\rho|\mathbf{U}|^2 + |\mathbf{\Phi}|^2\right) dx + \sum_{i=1}^4 K_i.\end{aligned} \tag{4.35}$$

We first estimate K_1. Hölder's inequality combined with (1.7), (2.6), (3.3), (4.31), and Young's inequality yields that for $r \in (1, \tilde{a})$,

$$\begin{aligned} K_1 &\le C\|\Theta\bar{x}^r\|_{L^2}\|\mathbf{U}\bar{x}^{-\frac{r}{2}}\|_{L^4}\left(\|\bar{\mathbf{u}}_t\bar{x}^{-\frac{r}{2}}\|_{L^4} + \|\nabla\bar{\mathbf{u}}\|_{L^\infty}\|\bar{\mathbf{u}}\bar{x}^{-\frac{r}{2}}\|_{L^4}\right) \\ &\le C(\varepsilon)\left(\|\sqrt{\bar{\rho}}\bar{\mathbf{u}}_t\|_{L^2}^2 + \|\nabla\bar{\mathbf{u}}_t\|_{L^2}^2 + \|\nabla\bar{\mathbf{u}}\|_{L^\infty}^2\right)\|\Theta\bar{x}^r\|_{L^2}^2 \\ &\quad + \varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2 + \|\nabla\mathbf{U}\|_{L^2}^2\right) \\ &\le C(\varepsilon)\left(1 + t\|\nabla\bar{\mathbf{u}}_t\|_{L^2}^2 + t\|\nabla^2\bar{\mathbf{u}}\|_{L^q}^2\right)\int_0^t\left(\|\nabla\mathbf{U}\|_{L^2}^2 + \|\sqrt{\rho}\mathbf{U}\|_{L^2}^2\right)ds \\ &\quad + \varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2 + \|\nabla\mathbf{U}\|_{L^2}^2\right). \end{aligned} \tag{4.36}$$

For the term K_2, we derive from Hölder's inequality, (3.3), and (4.32) that

$$\begin{aligned} K_2 &\le C\|\Theta\bar{x}^r\|_{L^2}\|\mathbf{U}\bar{x}^{-\frac{r}{2}}\|_{L^4}\|\theta\|_{L^4}\|\bar{x}^{-\frac{r}{2}}\|_{L^\infty} + C\|\sqrt{\bar{\rho}}\|_{L^\infty}\|\sqrt{\bar{\rho}}\mathbf{U}\|_{L^2}\|\Psi\|_{L^2} \\ &\le \varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2 + \|\nabla\mathbf{U}\|_{L^2}^2\right) + C(\varepsilon)\|\Theta\bar{x}^r\|_{L^2}^2 + C\|\Psi\|_{L^2}^2 \\ &\le \varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2 + \|\nabla\mathbf{U}\|_{L^2}^2\right) + C(\varepsilon)\|\Theta\bar{x}^r\|_{L^2}^2 + C\|\Psi\bar{x}^r\|_{L^2}^2\|\bar{x}^{-r}\|_{L^\infty}^2 \\ &\le \varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2 + \|\nabla\mathbf{U}\|_{L^2}^2\right) + C(\varepsilon)\int_0^t\left(\|\nabla\mathbf{U}\|_{L^2}^2 + \|\sqrt{\rho}\mathbf{U}\|_{L^2}^2\right)ds. \end{aligned} \tag{4.37}$$

We derive from Gagliardo–Nirenberg inequality and (3.46) that

$$K_3 \le C\|\bar{\mathbf{b}}\|_{L^4}\|\mathbf{\Phi}\|_{L^4}\|\nabla\mathbf{U}\|_{L^2} \le \varepsilon\|\nabla\mathbf{U}\|_{L^2}^2 + \varepsilon\|\nabla\mathbf{\Phi}\|_{L^2}^2 + C(\varepsilon)\|\mathbf{\Phi}\|_{L^2}^2. \tag{4.38}$$

Owing to (1.7), (2.6), and (3.3), K_4 can be estimated as follows

$$\begin{aligned} K_4 \le& C\|\mathbf{U}\bar{x}^{-a}\|_{L^4}\||\nabla\bar{\mathbf{b}}|^{\frac{1}{2}}\bar{x}^a\|_{L^4}\||\nabla\bar{\mathbf{b}}|^{\frac{1}{2}}\|_{L^4}\|\mathbf{\Phi}\|_{L^4} \\ \le& C\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2} + \|\nabla\mathbf{U}\|_{L^2}\right)\|\nabla\bar{\mathbf{b}}\bar{x}^{\frac{a}{2}}\|_{L^2}^{\frac{1}{2}}\|\mathbf{\Phi}\|_{L^4} \\ \le& \varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2 + \|\nabla\mathbf{U}\|_{L^2}^2\right) + C(\varepsilon)\|\nabla\bar{\mathbf{b}}\bar{x}^{\frac{a}{2}}\|_{L^2}\|\mathbf{\Phi}\|_{L^4}^2 \\ \le& \varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2 + \|\nabla\mathbf{U}\|_{L^2}^2\right) + \varepsilon\|\nabla\mathbf{\Phi}\|_{L^2}^2 + C(\varepsilon)\|\nabla\bar{\mathbf{b}}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\|\mathbf{\Phi}\|_{L^2}^2. \end{aligned} \tag{4.39}$$

Denoting

$$G(t) \triangleq \|\sqrt{\rho}\mathbf{U}\|_{L^2}^2 + \|\mathbf{\Phi}\|_{L^2}^2 + \int_0^t\left(\|\nabla\mathbf{U}\|_{L^2}^2 + \|\nabla\mathbf{\Phi}\|_{L^2}^2 + \|\sqrt{\rho}\mathbf{U}\|_{L^2}^2\right)ds,$$

then substituting (4.36)–(4.39) into (4.35) and choosing ε suitably small lead to

$$G'(t) \le C\left(1 + \|\nabla\bar{\mathbf{u}}\|_{L^\infty} + \|\nabla\bar{\mathbf{b}}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 + t\|\nabla\bar{\mathbf{u}}_t\|_{L^2}^2 + \|\nabla\bar{\mathbf{u}}\|_{L^2}^2 + t\|\nabla^2\mathbf{u}\|_{L^q}^2\right)G(t),$$

which together with Gronwall's inequality and (1.6) implies $G(t) = 0$. Hence, $(\mathbf{U}, \mathbf{\Phi})(x,t) = (\mathbf{0}, \mathbf{0})$ for almost everywhere $(x,t) \in \mathbb{R}^2 \times (0,T)$. Finally, one can deduce from (4.31)–(4.32) that $\Theta(x,t) = 0$ and $\Psi(x,t) = 0$ for almost everywhere $(x,t) \in \mathbb{R}^2 \times (0,T)$.

The proof of Theorem 1.1 is completed. □

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On the limit cycles for a class of discontinuous piecewise cubic polynomial differential systems

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Abstract. This paper presents new results on the bifurcation of medium and small limit cycles from the periodic orbits surrounding a cubic center or from the cubic center that have a rational first integral of degree 2 respectively, when they are perturbed inside the class of all discontinuous piecewise cubic polynomial differential systems with the straight line of discontinuity $y = 0$.

We obtain that the maximum number of medium limit cycles that can bifurcate from the periodic orbits surrounding the cubic center is 9 using the first order averaging method, and the maximum number of small limit cycles that can appear in a Hopf bifurcation at the cubic center is 6 using the fifth order averaging method. Moreover, both of the numbers can be reached by analyzing the number of simple zeros of the obtained averaged functions. In some sense, our results generalize the results in [*Appl. Math. Comput.* **250**(2015), 887–907], Theorems 1 and 2 to the piecewise systems class.

Keywords: averaging method, center, piecewise differential systems, limit cycle, periodic orbits.

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1 Introduction and main results

One of the main open problems in the qualitative theory of real planar differential systems is the determination and distribution of limit cycles. There are several methods for studying the bifurcation of limit cycles. One of the methods is by perturbing a differential system which has a center. In this case the perturbed system displays limit cycles that bifurcate, either from some of the periodic orbits surrounding the center, or from the center (having the so-called Hopf bifurcation), see the book of Christopher–Li [4], and references cited therein.

The problem of bounding the number of limit cycles for planar smooth differential systems has been exhaustively studied in the last century and is closed related to the 16th Hilbert's problem [10,13]. Solving this problem even for the quadratic case seems to be out of reach at the present state of knowledge. In the last few years there has been an increasing interest in

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the study of discontinuous piecewise differential systems, see [3,7,11,14,18,21] for instance. This interest has been mainly motivated by their wider range of application in various fields of science (e.g., control theory, biology, chemistry, engineering, physics, etc.).

Our goal in this paper is to study the bifurcation of limit cycles for a class of cubic polynomial differential systems having a rational first integral of degree 2. We remark that the classification of all cubic polynomial differential systems having a center at the origin and a rational first integral of degree 2 can be found in [17]. Later on, the authors in [16] summarized this classification in six families of cubic polynomial differential systems. In particular they obtained the class

$$\dot{x} = 2y(x+\alpha)^2, \quad \dot{y} = -2(x+\alpha)(\alpha x - y^2), \tag{1.1}$$

where $\alpha \neq 0$. System (1.1) called class P_6 in [16], which has $H(x,y) = \frac{x^2+y^2}{(\alpha+x)^2}$ as its first integral with the integrating factor $\mu(x,y) = 1/(\alpha+x)^4$. See [16] for the phase portraits of system (1.1) in the Poincaré disk.

A natural question is: *What happens with the periodic orbits (or the center) of the system* (1.1) *when it is perturbed inside the class of all smooth cubic polynomial differential systems, or inside the class of all discontinuous piecewise cubic polynomial differential systems with a straight line of discontinuity?*

In this article we say a *medium limit cycle* is one which bifurcates from a periodic orbit surrounding a center, and a *small limit cycle* is one which bifurcates from a center equilibrium point. Remark that, for the piecewise cubic polynomial vector fields there are two recent works, see [8,9], obtaining at least 18 and 24 small limit cycles, respectively. Our objective in this paper is to study the maximal number of medium and small limit cycles for the cubic center (1.1), when they are perturbed inside the class of all discontinuous piecewise cubic polynomial differential systems with the straight line of discontinuity $y = 0$. The main results are based on the averaging method. We remark that the method of averaging is a classic and mature tool for studying the behaviour of nonlinear differential systems in the presence of a small parameter. For more details about this method see the book of Sanders, Verhulst and Murdock [24] and Llibre, Moeckel and Simó [19].

More precisely, we consider the following discontinuous piecewise polynomial differential systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2y(x+\alpha)^2 \\ -2(x+\alpha)(\alpha x - y^2) \end{pmatrix} + \varepsilon \begin{cases} \begin{pmatrix} p_1(x,y) \\ q_1(x,y) \end{pmatrix}, & y > 0, \\ \begin{pmatrix} p_2(x,y) \\ q_2(x,y) \end{pmatrix}, & y < 0, \end{cases} \tag{1.2}$$

where

$$\begin{aligned} p_1(x,y) &= \sum_{0\le i+j\le 3} a_{i,j}x^iy^j, \quad q_1(x,y) = \sum_{0\le i+j\le 3} b_{i,j}x^iy^j, \\ p_2(x,y) &= \sum_{0\le i+j\le 3} c_{i,j}x^iy^j, \quad q_2(x,y) = \sum_{0\le i+j\le 3} d_{i,j}x^iy^j. \end{aligned} \tag{1.3}$$

Moveover, we consider the following smooth polynomial differential systems

$$\begin{cases} \dot{x} = 2y(x+\alpha)^2 + \displaystyle\sum_{s=1}^{5} \varepsilon^s \mu_s(x,y), \\ \dot{y} = -2(x+\alpha)(\alpha x - y^2) + \displaystyle\sum_{s=1}^{5} \varepsilon^s \nu_s(x,y), \end{cases} \tag{1.4}$$

and the discontinuous piecewise cubic polynomial differential systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2y(x+\alpha)^2 \\ -2(x+\alpha)(\alpha x - y^2) \end{pmatrix} + \sum_{s=1}^{5} \varepsilon^s \begin{cases} \begin{pmatrix} \mu_s(x,y) \\ \nu_s(x,y) \end{pmatrix}, & y > 0, \\ \begin{pmatrix} \psi_s(x,y) \\ \phi_s(x,y) \end{pmatrix}, & y < 0, \end{cases} \tag{1.5}$$

where

$$\mu_s(x,y) = \sum_{0 \le i+j \le 3} a_{s,i,j} x^i y^j, \quad \nu_s(x,y) = \sum_{0 \le i+j \le 3} b_{s,i,j} x^i y^j,$$
$$\psi_s(x,y) = \sum_{0 \le i+j \le 3} c_{s,i,j} x^i y^j, \quad \phi_s(x,y) = \sum_{0 \le i+j \le 3} d_{s,i,j} x^i y^j.$$

The main results of this paper are stated as follows.

Theorem 1.1. *For $|\varepsilon| > 0$ sufficiently small the maximum number of medium limit cycles of the discontinuous piecewise differential system* (1.2) *is 9 using the first order averaging method, and this number can be reached.*

If $a_{i,j} = c_{i,j}$ and $b_{i,j} = d_{i,j}$ (see (1.3)), then the perturbed system (1.2) is smooth. In this case, we obtain the following corollary of Theorem 1.1.

Corollary 1.2. *When $a_{i,j} = c_{i,j}$ and $b_{i,j} = d_{i,j}$, the maximum number of medium limit cycles of system* (1.2) *that bifurcate using the first order averaging method is 3 and it is reached.*

Remark 1.3. Theorem 1.1 gives the exact upper bound of the number of limit cycles bifurcated from the periodic orbits of the cubic center (1.1), which is challenging. Theorem 1.1 and Corollary 1.2 show that the maximum number of limit cycles for the piecewise case is 6 more than the smooth one. We note that the smooth case of system (1.2) has been studied in [16, Section 3.3] under the condition $a_{0,0} = b_{0,0} = c_{0,0} = d_{0,0} = 0$. Corollary 1.2 shows that the non-zero constant terms provide no more information on the limit cycles. However, in the piecewise case, with the non-zero constant terms the perturbed system (1.2) can produce at least one more limit cycle than the case without them (see Remark 3.1 in Section 3). This phenomenon coincides with the well-known pseudo-Hopf bifurcation (see [2,6]).

Theorem 1.4. *For $|\varepsilon| > 0$ sufficiently small using the fifth order averaging method, we obtain that*

(a) for any real constants $a_{s,i,j}$ and $b_{s,i,j}$ ($s = 1, \ldots, 5, 0 \le i+j \le 3$) with $a_{1,0,0} = b_{1,0,0} = 0$, system (1.4) *has at most 2 small limit cycles bifurcating from the center* (1.1)*, and this number can be reached;*

(b) system (1.5) *has at most 6 small limit cycles bifurcating from the center* (1.1) *under the condition $a_{1,0,0} = b_{1,0,0} = c_{1,0,0} = d_{1,0,0} = 0$, and this number can be reached.*

More concretely, we provide in Table 1.1 the maximum number of limit cycles for systems (1.4) and (1.5) up to the i-th order averaging method for $i = 1, \ldots, 5$.

The outline of this paper is as follows. In Section 2, we introduce the basic theory of the averaging method for discontinuous piecewise planar differential systems. The averaged function associated to system (1.2) is obtained in Section 3. Section 4 focuses on the analysis of the exact upper bound for the number of zeros of the averaged function, and the theory of Chebyshev systems is used to prove Theorem 1.1. The objective of Section 5 is to study the small limit cycles of systems (1.4) and (1.5). Finally, we present the explicit formulae of the i-th order averaged function up to $i = 5$ in Appendix A for reference.

Averaging order	System (1.4)	System (1.5)
1	0	1
2	0	2
3	1	4
4	1	6
5	2	6

Table 1.1: Number of small limit cycles for systems (1.4) and (1.5).

2 Preliminary results

In this section we introduce the basic theory of the averaging method that we shall use in our study of the cubic center (1.1). The following result is due to Itikawa, Llibre and Novaes [14].

Consider the discontinuous piecewise differential systems of the form

$$\frac{dr}{d\theta} = r' = \begin{cases} F^+(\theta, r, \varepsilon), & \text{if } 0 \le \theta \le \gamma, \\ F^-(\theta, r, \varepsilon), & \text{if } \gamma \le \theta \le 2\pi, \end{cases} \tag{2.1}$$

where

$$F^\pm(\theta, r, \varepsilon) = \sum_{i=1}^{k} \varepsilon^i F_i^\pm(\theta, r) + \varepsilon^{k+1} R^\pm(\theta, r, \varepsilon),$$

and ε is a real small parameter. The set of discontinuity of system (2.1) is $\sum = \{\theta = 0\} \cup \{\theta = \gamma\}$ if $0 < \gamma < 2\pi$. Here $F_i^\pm : \mathbb{S}^1 \times D \to \mathbb{R}$ for $i = 1, \dots, k$, and $R^\pm : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ are $\mathcal{C}^k$ functions, being D an open and bounded interval of $(0, \infty)$, ε_0 is a small parameter, and $\mathbb{S}^1 \equiv \mathbb{R}/(2\pi)$. This last convention implies that the functions involved in system (2.1) are 2π-periodic in the variable θ.

The averaging method consists in defining a collection of functions $f_i : D \to \mathbb{R}$, called the i-th order averaged function, for $i = 1, 2, \dots, k$, which control (their simple zeros control), for $|\varepsilon| > 0$ sufficiently small, the isolated periodic solutions of the differential system (2.1). In Itikawa–Llibre–Novaes [14] it has been established that

$$f_i(z) = \frac{y_i^+(\gamma, z) - y_i^-(\gamma - 2\pi, z)}{i!}, \tag{2.2}$$

where $y_i^\pm : \mathbb{S}^1 \times D \to \mathbb{R}$, for $i = 1, 2, \dots, k$, are defined recurrently by the following integral equations

$$\begin{aligned} y_1^\pm(\theta, z) &= \int_0^\theta F_1^\pm(\varphi, z) d\varphi, \\ y_i^\pm(\theta, z) &= i! \int_0^\theta \Big(F_i^\pm(\varphi, z) + \sum_{\ell=1}^{i} \sum_{S_\ell} \frac{1}{b_1! b_2! 2!^{b_2} \cdots b_\ell! \ell!^{b_\ell}} \cdot \partial^L F_{i-\ell}^\pm(\varphi, z) \prod_{j=1}^{\ell} y_j^\pm(\varphi, z)^{b_j} \Big) d\varphi, \end{aligned} \tag{2.3}$$

where S_ℓ is the set of all ℓ-tuples of non-negative integers $[b_1, b_2, \dots, b_\ell]$ satisfying $b_1 + 2b_2 + \cdots + \ell b_\ell = \ell$ and $L = b_1 + b_2 + \cdots + b_\ell$. Here, $\partial^L F(\varphi, z)$ denotes the Fréchet's derivative with respect to the variable z. We remark that, the investigation in this paper is restricted to $F_0 = 0$ in expression (2.3). For the general form of the averaged functions see [20].

We point out that taking $\gamma = 2\pi$ system (2.1) becomes smooth. So the averaging method described above can also apply to smooth differential systems. In practical terms, the evaluation of the recurrence (2.3) is a computational problem that requires powerful computerized resources. Usually, the higher the averaging order is, the more complex are the computational operations to calculate the averaged function (2.2). Recently in [22] the Bell polynomials were used to provide a relatively simple alternative formula for the recurrence (2.3). And based on this new formula, an algorithmic approach to revisit the averaging method was introduced in [12] for the analysis of bifurcation of small limit cycles of planar differential systems. Moreover, we provide an upper bound of the number of zeros of the averaged functions for the general class of perturbed differential systems (see Theorem 3.1 in [12]).

The following k-th order averaging theorem gives a criterion for the existence of limit cycles. Its proof can be found in Section 2 of [14].

Theorem 2.1 ([14]). *Assume that, for some $j \in \{1,2,\dots,k\}$, $f_i = 0$ for $i = 1,2,\dots,j-1$ and $f_j \neq 0$. If there exists $z^* \in D$ such that $f_j(z^*) \neq 0$, then for $|\varepsilon| > 0$ sufficiently small, there exists a 2π-periodic solution $r(\theta,\varepsilon)$ of (2.1) such that $r(0,\varepsilon) \to z^*$ when $\varepsilon \to 0$.*

The following theorem (see Theorem 5.2 of [1] for a proof) provides an approach to transform a perturbed differential system into the standard form (2.1), which can be used to apply the first order averaging method.

Theorem 2.2 ([1]). *Consider the differential system*

$$\begin{aligned} \dot{x} &= P(x,y) + \varepsilon p(x,y), \\ \dot{y} &= Q(x,y) + \varepsilon q(x,y), \end{aligned} \tag{2.4}$$

where P, Q, p and q are continuous functions in the variables x and y, and ε is a small parameter. Suppose that system $(2.4)_{\varepsilon=0}$ has a continuous family of ovals $\{\Gamma_h\} \subset \{(x,y)|H(x,y) = h, h \in (h_1,h_2)\}$, where $H(x,y)$ is a first integral of $(2.4)_{\varepsilon=0}$, and h_1 and h_2 correspond to the center and the separatrix polycycle, respectively. For a given first integral $H = H(x,y)$, assume that $xQ(x,y) - yP(x,y) \neq 0$ for all (x,y) in the periodic annulus formed by the ovals $\{\Gamma_h\}$. Let $\rho : (\sqrt{h_1}, \sqrt{h_2}) \times [0,2\pi) \to [0,+\infty)$ be a continuous function such that

$$H(\rho(R,\varphi)\cos\varphi, \rho(R,\varphi)\sin\varphi) = R^2,$$

for all $R \in (\sqrt{h_1}, \sqrt{h_2})$ and all $\varphi \in [0,2\pi)$. Then the differential equation which describes the dependence between the square root of energy $R = \sqrt{h}$ and the angle φ for system (2.4) is

$$\frac{dR}{d\varphi} = \varepsilon \frac{\mu(x^2+y^2)(Qp - Pq)}{2R(Qx - Py)} + \mathcal{O}(\varepsilon^2), \tag{2.5}$$

where $\mu = \mu(x,y)$ is the integrating factor of system $(2.4)_{\varepsilon=0}$ corresponding to the first integral H, and $x = \rho(R,\varphi)\cos\varphi$ and $y = \rho(R,\varphi)\sin\varphi$.

In general, it is not an easy thing to deal with zeros of the averaged function (2.2). The techniques and arguments to tackle this kind of problem are usually very long and technical. In what follows we present some effective results on obtaining the lower bound and the upper bound of the number of zeros for a complicated function. The next result is used to obtain a lower bound of the number of simple zeros for an averaged function.

Lemma 2.3 ([5]). *Consider $n+1$ linearly independent analytical functions $f_i(x) : A \to \mathbb{R}, i = 0, 1, \ldots, n$, where $A \subset \mathbb{R}$ is an interval. Suppose that there exists $k \in \{0, 1, \ldots, n\}$ such that $f_k(x)$ has constant sign. Then there exist $n+1$ constants $c_i, i = 0, 1, \ldots, n$ such that $c_0 f_0(x) + c_1 f_1(x) + \cdots + c_n f_n(x)$ has at least n simple zeros in A.*

It is important to point out that the classical theory of Chebyshev systems is useful to provide an upper bound for the number of zeros. Let $\mathcal{F} = [f_0, \ldots, f_n]$ be an ordered set of functions of class C^n defined in the closed interval $[a, b]$. We consider only elements in Span $(\mathcal{F})$, that is, functions such as $f = a_0 f_0 + a_1 f_1 + \cdots + a_n f_n$ where a_i, for $i = 0, 1, \ldots, n$, are real numbers. When the maximum number of zeros, taking into account its multiplicity, is n, the set $\mathcal{F}$ is called an Extended Chebyshev system (ET-system) in $[a, b]$. We say that $\mathcal{F}$ is an Extended Complete Chebyshev system (ECT-system) in $[a, b]$, if any set $[f_0, f_1, \ldots, f_k]$, for $k = 0, \ldots, n$ is an ET-system. When all the Wronskians, $W_k := W(f_0, f_1, \ldots, f_k) \neq 0$ for $0 \leq k \leq n$ in $[a, b]$ the set $\mathcal{F}$ is an ECT-system. For more details on ET-systems and ECT-system, see [15] for instance.

We remark that not always the standard study of ET-systems can be applied to bound the number of zeros of elements in Span $(\mathcal{F})$. Here we use an extension of this theory (see [23]). The following result provides an effective estimation for the number of isolated zeros of elements in Span $(\mathcal{F})$ when some Wronskians vanish.

Theorem 2.4 ([23]). *Let $\mathcal{F} = [f_0, f_1, \ldots, f_n]$ be an ordered set of analytic functions in $[a, b]$. Assume that all the ν_i zeros of the Wronskian W_i are simple for $i = 0, 1, \ldots, n$. Then the number of isolated zeros for every element of Span $(\mathcal{F})$ does not exceed*

$$n + \nu_n + \nu_{n-1} + 2(\nu_{n-2} + \cdots + \nu_0) + \lambda_{n-1} + \cdots + \lambda_3,$$

where $\lambda_i = \min(2\nu_i, \nu_{i-3} + \cdots + \nu_0)$, for $i = 3, \ldots, n-1$.

3 Averaged function associated to system (1.2)

In this section we will get the first order averaged function associated to system (1.2) by using Theorem 2.1. We remark that the period annulus of the differential system (1.1) is formed by the ovals $\{\Gamma_h\} \subset \{(x, y) | H(x, y) = h, h \in (0, 1)\}$. By solving implicitly the equation $H(\rho(R, \varphi) \cos \varphi, \rho(R, \varphi) \sin \varphi) = R^2$ we obtain the positive function $\rho(R, \varphi)$ given by

$$\rho(R, \varphi) = -\frac{\alpha R(\text{signum}(\alpha) + R \cos \varphi)}{R^2 \cos^2 \varphi - 1}$$

for $\varphi \in [0, 2\pi)$ and $R \in (0, 1)$, where signum(α) is the sign function defined by

$$\text{signum}(\alpha) = \begin{cases} 1, & \alpha > 0, \\ -1, & \alpha < 0. \end{cases}$$

Using Theorem 2.2, we can transform system (1.2) into the form

$$\frac{dR}{d\varphi} = \begin{cases} \varepsilon \frac{-(Qp_1 - Pq_1)}{4\alpha R(x+\alpha)^5}\Big|_{x=\rho(R,\varphi)\cos\varphi, y=\rho(R,\varphi)\sin\varphi} + \mathcal{O}(\varepsilon^2), & 0 \leq \varphi \leq \pi, \\ \varepsilon \frac{-(Qp_2 - Pq_2)}{4\alpha R(x+\alpha)^5}\Big|_{x=\rho(R,\varphi)\cos\varphi, y=\rho(R,\varphi)\sin\varphi} + \mathcal{O}(\varepsilon^2), & \pi \leq \varphi \leq 2\pi. \end{cases} \tag{3.1}$$

Now the discontinuous piecewise differential system (3.1) is under the assumptions of Theorem 2.1. Next, we will study the zeros of the averaged function $f : (0,1) \to \mathbb{R}$ given by

$$
\begin{aligned}
f(R) = & \int_0^{\pi} \frac{-(Qp_1 - Pq_1)}{4\alpha R(x+\alpha)^5}\Big|_{x=\rho(R,\varphi)\cos\varphi, y=\rho(R,\varphi)\sin\varphi} d\varphi \\
& + \int_{\pi}^{2\pi} \frac{-(Qp_2 - Pq_2)}{4\alpha R(x+\alpha)^5}\Big|_{x=\rho(R,\varphi)\cos\varphi, y=\rho(R,\varphi)\cos\varphi} d\varphi \\
= & \int_0^{\pi} \frac{A(\varphi;a,b)\cos\varphi + B(\varphi;a,b)}{2\alpha^3(\operatorname{signum}(\alpha)\cdot R\cos\varphi - 1)} d\varphi + \int_{\pi}^{2\pi} \frac{A(\varphi;c,d)\cos\varphi + B(\varphi;c,d)}{2\alpha^3(\operatorname{signum}(\alpha)\cdot R\cos\varphi - 1)} d\varphi,
\end{aligned}
$$

where

$$
\begin{aligned}
A(\varphi;a,b) = & -R^3\big[\alpha^3(a_{0,3} - a_{2,1} - b_{3,0} + b_{1,2}) + \alpha^2(-b_{0,2} + b_{2,0} + a_{1,1}) \\
& + \alpha(-b_{1,0} - a_{0,1}) + b_{0,0}\big]S^3 + \operatorname{signum}(\alpha)\cdot R^2\big[\alpha^3 R^2(a_{1,2} - a_{3,0}) \\
& + \alpha^2(R^2(a_{2,0} - a_{0,2}) - a_{0,2} + a_{2,0} - b_{1,1}) + \alpha(-R^2 a_{1,0} - 2a_{1,0} + 2b_{0,1}) \\
& + (R^2+3)a_{0,0}\big]S^2 - R\big[\alpha^3 R^2(a_{2,1} + b_{3,0}) - \alpha^2 R^2(2a_{1,1} + b_{2,0}) \\
& + \alpha(R^2(3a_{0,1} + b_{1,0}) + a_{0,1} + b_{1,0}) - (R^2+3)b_{0,0}\big]S + \operatorname{signum}(\alpha) \\
& \cdot \big[\alpha^3 R^4 a_{3,0} - \alpha^2 R^2(R^2+1)a_{2,0} + \alpha R^2(R^2+3)a_{1,0} - (R^4 + 6R^2 + 1)a_{0,0}\big], \\
B(\varphi;a,b) = & R^3[\alpha^3(-b_{0,3} + b_{2,1} + a_{1,2} - a_{3,0}) + \alpha^2(-b_{1,1} + a_{2,0} - a_{0,2}) \\
& + \alpha(-a_{1,0} + b_{0,1}) + a_{0,0}]S^4 + \operatorname{signum}(\alpha)\cdot R^2[\alpha^3 R^2(a_{0,3} - a_{2,1}) \\
& + \alpha^2((R^2+1)a_{1,1} - b_{0,2} + b_{2,0}) - \alpha((R^2+2)a_{0,1} + 2b_{1,0}) + 3b_{0,0}]S^3 \\
& - R[\alpha^3 R^2(a_{1,2} - 2a_{3,0} + b_{2,1}) + \alpha^2 R^2(-2a_{0,2} + 3a_{2,0} - b_{1,1}) \\
& + \alpha(R^2(-4a_{1,0} + b_{0,1}) - a_{1,0} + b_{0,1}) + (5R^2+3)a_{0,0}]S^2 \\
& + \operatorname{signum}(\alpha)\cdot[\alpha^3 R^4 a_{2,1} - \alpha^2 R^2((R^2+1)a_{1,1} + b_{2,0}) \\
& + \alpha R^2((R^2+3)a_{0,1} + 2b_{1,0}) - (3R^2+1)b_{0,0}]S \\
& - R[\alpha^3 R^2 a_{3,0} - 2\alpha^2 R^2 a_{2,0} + \alpha(3R^2+1)a_{1,0} - 4(R^2+1)a_{0,0}]
\end{aligned}
$$

with $S = \sin\varphi$, and $a = (a_{i,j})$, $b = (b_{i,j})$, $c = (c_{i,j})$ and $d = (d_{i,j})$, with $a_{i,j}, b_{i,j}, c_{i,j}$ and $d_{i,j}$ are parameters appearing in the perturbed polynomials (1.3).

Computing the above integrals and making the transformation $R = \frac{2\omega}{1+\omega^2}$ for $0 < \omega < 1$ we obtain

$$
f(R) \overset{R=\frac{2\omega}{1+\omega^2}}{=} \frac{\tilde{f}(\omega)}{6\alpha^3\omega(\omega^2+1)^3} = \frac{\sum_{i=0}^{8} k_i f_i(\omega)}{6\alpha^3\omega(\omega^2+1)^3}, \tag{3.2}
$$

where

$$
\begin{aligned}
& f_0(\omega) = \omega^2, \quad f_1(\omega) = \omega^4, \quad f_2(\omega) = \omega^6, \\
& f_3(\omega) = \omega^8, \quad f_4(\omega) = \omega^5 + \omega^3, \quad f_5(\omega) = \omega^7 + \omega, \\
& f_6(\omega) = \omega^4 \ln\left(\frac{1+\omega}{1-\omega}\right), \quad f_7(\omega) = (\omega^8+1)\ln\left(\frac{1+\omega}{1-\omega}\right), \\
& f_8(\omega) = (\omega^6+\omega^2)\ln\left(\frac{1+\omega}{1-\omega}\right),
\end{aligned} \tag{3.3}
$$

and

$$\begin{aligned}
k_0 &= -3\pi\Big(-\alpha(a_{1,0}+c_{1,0})-\alpha(b_{0,1}+d_{0,1})+4(a_{0,0}+c_{0,0})\Big),\\
k_1 &= -3\pi\Big(-3\alpha^3(a_{3,0}+c_{3,0})-3\alpha^3(b_{0,3}+d_{0,3})-\alpha^3(a_{1,2}+c_{1,2})\\
&\quad -\alpha^3(b_{2,1}+d_{2,1})+4\alpha^2(a_{0,2}+c_{0,2})+4\alpha^2(a_{2,0}+c_{2,0})-6\alpha(a_{1,0}+c_{1,0})\\
&\quad -2\alpha(b_{0,1}+d_{0,1})+12(a_{0,0}+c_{0,0})\Big),\\
k_2 &= -3\pi\Big(2\alpha^3(a_{1,2}+c_{1,2})+2\alpha^3(a_{3,0}+c_{3,0})-2\alpha^3(b_{0,3}+d_{0,3})-2\alpha^3(b_{2,1}+d_{2,1})\\
&\quad -\alpha(a_{1,0}+c_{1,0})-\alpha(b_{0,1}+d_{0,1})+4(a_{0,0}+c_{0,0})\Big),\\
k_3 &= 3\pi\alpha^3\Big((a_{1,2}+c_{1,2})-(a_{3,0}+c_{3,0})-(b_{0,3}+d_{0,3})+(b_{2,1}+d_{2,1})\Big),\\
k_4 &= \operatorname{signum}(\alpha)\cdot\Big[-2\alpha^3(a_{2,1}-c_{2,1})-22\alpha^3(a_{0,3}-c_{0,3})+2\alpha^3(b_{1,2}-d_{1,2})\\
&\quad -26\alpha^3(b_{3,0}-d_{3,0})+8\alpha^2(b_{2,0}-d_{2,0})+8\alpha^2(a_{1,1}-c_{1,1})+16\alpha^2(b_{0,2}-d_{0,2})\\
&\quad -32\alpha(a_{0,1}-c_{0,1})-8\alpha(b_{1,0}-d_{1,0})+26(b_{0,0}-d_{0,0})\Big],\\
k_5 &= \operatorname{signum}(\alpha)\cdot\Big[6\alpha^3(a_{0,3}-c_{0,3})+6\alpha^3(b_{1,2}-d_{1,2})-6\alpha^3(b_{3,0}-d_{3,0})\\
&\quad -6\alpha^3(a_{2,1}-c_{2,1})+6(b_{0,0}-d_{0,0})\Big],\\
k_6 &= -\operatorname{signum}(\alpha)\cdot 6\alpha^3\Big(3(a_{0,3}-c_{0,3})+(a_{2,1}-c_{2,1})-(b_{1,2}-d_{1,2})-3(b_{3,0}-d_{3,0})\Big),\\
k_7 &= -\operatorname{signum}(\alpha)\cdot 3\alpha^3\Big((a_{0,3}-c_{0,3})-(a_{2,1}-c_{2,1})+(b_{1,2}-d_{1,2})-(b_{3,0}-d_{3,0})\Big),\\
k_8 &= \operatorname{signum}(\alpha)\cdot 12\alpha^3\Big((a_{0,3}-c_{0,3})+(b_{3,0}-d_{3,0})\Big).
\end{aligned}$$

It follows directly from

$$\frac{\partial(k_0,k_1,k_2,k_3,k_4,k_5,k_6,k_7,k_8)}{\partial(b_{0,0},a_{3,0},a_{1,2},a_{1,0},a_{1,1},a_{2,0},b_{3,0},a_{2,1},a_{0,3})} = \operatorname{signum}(\alpha)\cdot 107495424\pi^4\alpha^{20}\neq 0$$

that the constants $k_0, k_1, \cdots, k_8$ are independent. That is to say, the coefficients of the functions $f_i(\omega)$, $i=0,1,\ldots,8$ can be chosen arbitrarily. Moreover, the functions $f_0(\omega),\ldots,f_8(\omega)$ are linearly independent. In fact, we obtain the following Taylor expansions in the variable ω around $\omega=0$ for these functions:

$$\begin{aligned}
&f_0(\omega)=\omega^2,\quad f_1(\omega)=\omega^4,\quad f_2(\omega)=\omega^6,\\
&f_3(\omega)=\omega^8,\quad f_4(\omega)=\omega^5+\omega^3,\quad f_5(\omega)=\omega^7+\omega,\\
&f_6(\omega)=2\omega^5+\frac{2}{3}\omega^7+\frac{2}{5}\omega^9+\mathcal{O}(\omega^{11}),\\
&f_7(\omega)=2\omega+\frac{2}{3}\omega^3+\frac{2}{5}\omega^5+\frac{2}{7}\omega^7+\frac{20}{9}\omega^9+\mathcal{O}(\omega^{11}),\\
&f_8(\omega)=2\omega^3+\frac{2}{3}\omega^5+\frac{12}{5}\omega^7+\frac{20}{21}\omega^9+\mathcal{O}(\omega^{11}).
\end{aligned}\tag{3.4}$$

The determinant of the coefficient matrix of the variables $\omega,\omega^2,\ldots,\omega^9$ is equal to $8388608/496125$. Hence, by Lemma 2.3 it follows that there exists a linear combination of $f_i(\omega)$, $i=0,1,\ldots,8$ with at least 8 simple zeros, which means that system (1.2) has at least 8 limit cycles bifurcating from the period orbits surrounding the origin.

Remark 3.1. We notice that when the constant terms $a_{0,0}$, $b_{0,0}$, $c_{0,0}$, $d_{0,0}$ are identically zeros. In a similar way, we can prove that system (1.2) has at least 7 limit cycles bifurcating from the period orbits surrounding the origin. In fact, $k_5 + 2k_7 = 0$ in this case, and the function $\tilde{f}(\omega)$ in (3.2) is a linear combination of 8 linearly independent functions $f_0, \ldots, f_4, f_6, f_7 - 2f_5, f_8$. Therefore, by Lemma 2.3, the perturbed system (1.2) with the non-zero constant terms can produce at least one more limit cycle than the case without them.

Proof of Corollary 1.2. Obviously, when $a_{i,j} = c_{i,j}$ and $b_{i,j} = d_{i,j}$, the coefficients $k_4, k_5, \ldots, k_8$ are identically zeros. It is easy to check that $(f_0, \ldots, f_3)$ is an ECT-system. Therefore, the averaged function f in this case has at most 3 simple zeros and this number can be reached. Hence, by Theorem 2.1, Corollary 1.2 is proved. □

In what follows, we first provide an upper bound of the number of zeros of the function $\tilde{f}(\omega)$ in (3.2). We eliminate the logarithmic function by taking the ninth derivative of $\tilde{f}(\omega)$ and obtain

$$\tilde{f}^{(9)}(\omega) = \text{signum}(\alpha) \cdot \frac{110592\alpha^3}{(1+\omega)^9(-1+\omega)^9}(H_1\omega^8 + H_2\omega^6 + H_3\omega^4 + H_2\omega^2 + H_1),$$

where

$$\begin{aligned} H_1 &= -14(a_{2,1} - c_{2,1}) + 14(b_{1,2} - d_{1,2}) + 8(a_{0,3} - c_{0,3}) - 83(b_{3,0} - d_{3,0}), \\ H_2 &= -24(a_{2,1} - c_{2,1}) + 24(b_{1,2} - d_{1,2}) - 32(a_{0,3} - c_{0,3}) - 1988(b_{3,0} - d_{3,0}), \\ H_3 &= 76(a_{2,1} - c_{2,1}) - 76(b_{1,2} - d_{1,2}) + 48(a_{0,3} - c_{0,3}) - 4818(b_{3,0} - d_{3,0}). \end{aligned}$$

As a result of the symmetry of coefficients of the function $\tilde{f}^{(9)}(\omega)$ with respect to ω, it is easy to know that the zeros of the function $\tilde{f}^{(9)}(\omega)$ appear in pairs. Recalling this property, we obtain that $\tilde{f}^{(9)}(\omega)$ has at most 2 zeros in $(0, 1)$. Thus, by using Rolle's theorem and noting the fact that $\tilde{f}(0) = 0$, we conclude that $\tilde{f}(\omega)$ has at most $2 + 9 - 1 = 10$ zeros in $(0, 1)$, which means that system (1.2) has at most 10 limit cycles bifurcating from the period orbits surrounding the origin. In next section, we will show that the bound of the number of limit cycles can be reduced to 9 by Theorem 2.4. Moreover, this number can be reached.

4 Proof of Theorem 1.1

In this section we will study the maximum number of simple zeros of the averaged function (3.2). The main effort is based largely on algebraic calculations with the theory of Chebyshev systems used to deal with the Wronskian determinants.

First, we denote by $W_i(\omega)$ the Wronskian for the functions $f_0, f_1, \ldots, f_i$ depending on ω:

$$W_i(\omega) = W(f_0, \ldots, f_i), \quad i = 0, 1, \ldots, 8.$$

Next, we will show that all the Wronskians have no zeros except $W_7(\omega)$ which vanishes at a unique zero in $(0, 1)$, which is simple. Using the expressions in (3.3), we perform the calculation and obtain

$$\begin{aligned}
&W_0(\omega) = \omega^2, \quad W_1(\omega) = 2\omega^5, \quad W_2(\omega) = 16\omega^9,\\
&W_3(\omega) = 768\omega^{14}, \quad W_4(\omega) = 2304\omega^{13}(3\omega^2-5),\\
&W_5(\omega) = 69120\omega^9(1-\omega^2)(3\omega^6-7\omega^4-7\omega^2+35),\\
&W_6(\omega) = -\frac{3317760\omega^8(\omega^2+1)}{(1-\omega^2)^5}T_6(\omega),\\
&W_7(\omega) = -\frac{133772083200\omega(\omega^2+1)^3T_{7,0}(\omega)}{(1-\omega^2)^4}\left(\ln\left(\frac{1+\omega}{1-\omega}\right) - \frac{2\omega T_{7,1}(\omega)}{105(1-\omega^2)^6T_{7,0}(\omega)}\right),\\
&W_8(\omega) = \frac{821895679180800(\omega^2+1)^6}{(1-\omega^2)^{10}}\left(T_{8,0}(\omega)\cdot\ln\left(\frac{1+\omega}{1-\omega}\right) + \frac{2\omega T_{8,1}(\omega)}{105(1-\omega^2)^4}\right),
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
T_6(\omega) &= 15\omega^{14} - 195\omega^{12} - 89\omega^{10} + 1149\omega^8 + 421\omega^6 - 4305\omega^4 + 805\omega^2 - 105 < 0,\\
T_{7,0}(\omega) &= 15\omega^8 - 140\omega^6 + 1018\omega^4 - 140\omega^2 + 15 > 0,\\
T_{7,1}(\omega) &= 160\omega^{20} - 8569\omega^{18} + 105687\omega^{16} - 547324\omega^{14} + 1437092\omega^{12} - 2101414\omega^{10}\\
&\quad + 1752730\omega^8 - 839580\omega^6 + 210980\omega^4 - 23625\omega^2 + 1575,\\
T_{8,0}(\omega) &= 35\omega^8 - 1100\omega^6 + 2898\omega^4 - 1100\omega^2 + 35,\\
T_{8,1}(\omega) &= 45477\omega^{14} - 444465\omega^{12} + 1433397\omega^{10} - 2210985\omega^8 + 1803095\omega^6\\
&\quad - 745675\omega^4 + 128975\omega^2 - 3675,
\end{aligned} \tag{4.2}$$

by Sturm's theorem. It is easy to judge that $W_i(\omega)$ for $i = 0, \ldots, 6$ does not vanish in the open interval $(0,1)$. The difficulties mainly focus on the determination of $W_7(\omega)$ and $W_8(\omega)$.

Proposition 4.1. *$W_7(\omega)$ has a unique zero in $\omega \in (0,1)$ and this zero is simple.*

Proof. Denote the function in the parentheses of $W_7(\omega)$ by $Q_7(\omega)$, then

$$Q_7'(\omega) = \frac{64\omega^6(\omega^2+1)(5\omega^8+172\omega^6-1122\omega^4+172\omega^2+5)T_6(\omega)}{105(1-\omega^2)^7T_{7,0}^2(\omega)}$$

has a unique simple zero ω^* in $(0,1)$ and can be easily isolated (e.g. by using the command realroot(%,1/10000) in Maple) as $\omega^* \in \left[\frac{112087}{262144}, \frac{14011}{32768}\right]$. It follows that $Q_7(\omega)$ decreases in $(0,\omega^*)$ and increases in $(\omega^*,1)$. Note also that $\lim_{\omega\to0^+} Q_7(\omega) = 0$ and $\lim_{\omega\to1^-} Q_7(\omega) = +\infty$. Thus, $Q_7(\omega)$ has a unique simple zero in $(0,1)$, equivalently, $W_7(\omega)$ has a simple zero in $(0,1)$. This ends the proof. □

Proposition 4.2. *$W_8(\omega)$ does not vanish in $\omega \in (0,1)$.*

Proof. First, using Sturm's theorem, we get that $T_{8,0}(\omega)$ has two simple zeros ω_1 and ω_2 in $(0,1)$ and $T_{8,1}(\omega)$ has three simple zeros ω_3, ω_4 and ω_5 in $(0,1)$, and these zeros can be respectively isolated as

$$\begin{aligned}
0.18709157 &\approx \omega_1 \in \left[\tfrac{6277751}{33554432}, \tfrac{784719}{4194304}\right],\\
0.64417845 &\approx \omega_2 \in \left[\tfrac{337735}{524288}, \tfrac{5403761}{8388608}\right],\\
0.18709131 &\approx \omega_3 \in \left[\tfrac{3138871}{16777216}, \tfrac{6277743}{33554432}\right],\\
0.66278355 &\approx \omega_4 \in \left[\tfrac{5559831}{8388608}, \tfrac{694979}{1048576}\right],\\
0.75595958 &\approx \omega_5 \in \left[\tfrac{792681}{1048576}, \tfrac{6341449}{8388608}\right].
\end{aligned}$$

We denote the function in the parenthesis of $W_8(\omega)$ by $Q_8(\omega)$, it is easy to verify that $Q_8(\omega_1) \neq 0$ and $Q_8(\omega_2) \neq 0$. In order to study the number of zeros of $Q_8(\omega)$ in $(0,1)$ we define a function $Z_8(\omega)$ as follows

$$Z_8(\omega) := \frac{Q_8(\omega)}{T_{8,0}(\omega)} = \ln\left(\frac{1+\omega}{1-\omega}\right) + \frac{2\omega T_{8,1}(\omega)}{105(1-\omega^2)^4 T_{8,0}(\omega)}, \quad \omega \in (0,1)\backslash\{\omega_1, \omega_2\}.$$

It is obvious that the function $Z_8(\omega)$ has the following properties (see Fig. 4.1):

$$\lim_{\omega\to\omega_1^-} Z_8(\omega) = +\infty, \quad \lim_{\omega\to\omega_1^+} Z_8(\omega) = -\infty,$$
$$\lim_{\omega\to\omega_2^-} Z_8(\omega) = -\infty, \quad \lim_{\omega\to\omega_2^+} Z_8(\omega) = +\infty.$$

A direct calculation shows that

$$Z_8'(\omega) = \frac{32768\omega^8(\omega^2+1)H_8(\omega)}{35(1-\omega^2)^5 T_{8,0}^2(\omega)},$$

where

$$H_8(\omega) = 35\omega^{14} + 85\omega^{12} - 129\omega^{10} - 503\omega^8 - 119\omega^6 + 1855\omega^4 - 875\omega^2 + 35.$$

Obviously, $H_8(\omega)$ has two simple zeros ω_1^* and ω_2^* in $(0,1)$ and can be respectively isolated as

$$\begin{aligned} 0.21002672 &\approx \omega_1^* \in \left[\tfrac{451028943}{2147483648}, \tfrac{902057887}{4294967296}\right], \\ 0.69221454 &\approx \omega_2^* \in \left[\tfrac{185814925}{268435456}, \tfrac{743259701}{1073741824}\right]. \end{aligned} \tag{4.3}$$

Therefore $Z_8(\omega)$ increases when $\omega \in (0,\omega_1) \cup (\omega_1, \omega_1^*)$ and $\omega \in (\omega_2^*, 1)$; decreases when $\omega \in (\omega_1^*, \omega_2) \cup (\omega_2, \omega_2^*)$ (see Fig. 4.1). Notice that

$$\lim_{\omega\to 0^+} Z_8(\omega) = 0, \quad \lim_{\omega\to 1^-} Z_8(\omega) = +\infty.$$

It follows from (4.3) that

$$Z_8(\omega_1^*) \approx -0.0000126678 < 0, \quad Z_8(\omega_2^*) \approx 1.126483743 > 0.$$

Taking into account the above results, we conclude that $Z_8(\omega)$ does not vanish for $\omega \in (0,1)\backslash\{\omega_1,\omega_2\}$. Thus the desired result follows. □

Proof of Theorem 1.1. It follows from equation (4.1), Propositions 4.1 and 4.2 that $W_i(\omega)$, $i = 0,1,\ldots 6$ and $W_8(\omega)$ do not vanish in the interval $(0,1)$, and $W_7(\omega)$ has exactly 1 simple zero in $(0,1)$. Thus $\mathcal{F} = [f_0, f_1, \ldots, f_8]$ defined in (3.3) satisfies the assumptions of Theorem 2.4, which implies that any linear combination of $f_0, f_1, \ldots, f_8$ can possess at most 9 zeros in $(0,1)$, counting with multiplicities. But the authors in [23] do not prove that the upper bound can be reached in the general cases. In what follows we will show that the upper bound 9 can be reached in our system.

Following the ideas of [23], we first look for an element in Span $(\mathcal{F})$ with a zero of the highest multiplicity, then we perturb it inside Span $(\mathcal{F})$ in order to have the prescribed configuration of zeros. We remark that since the Wronskian determinant $W_8(\omega)$ does not vanish,

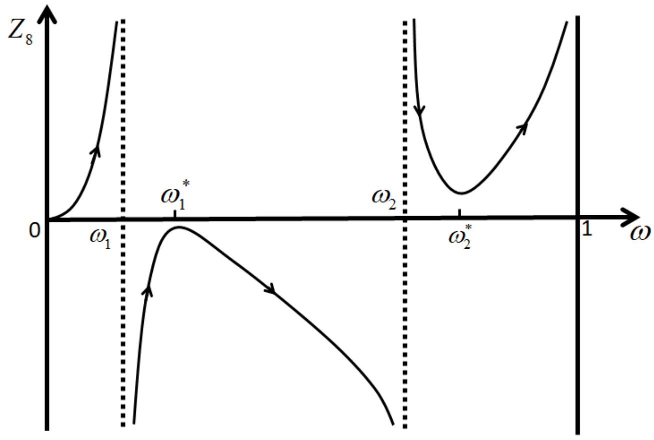

Figure 4.1: The curve $Z_8(\omega)$ does not vanish for $\omega \in (0,1)\backslash\{\omega_1, \omega_2\}$.

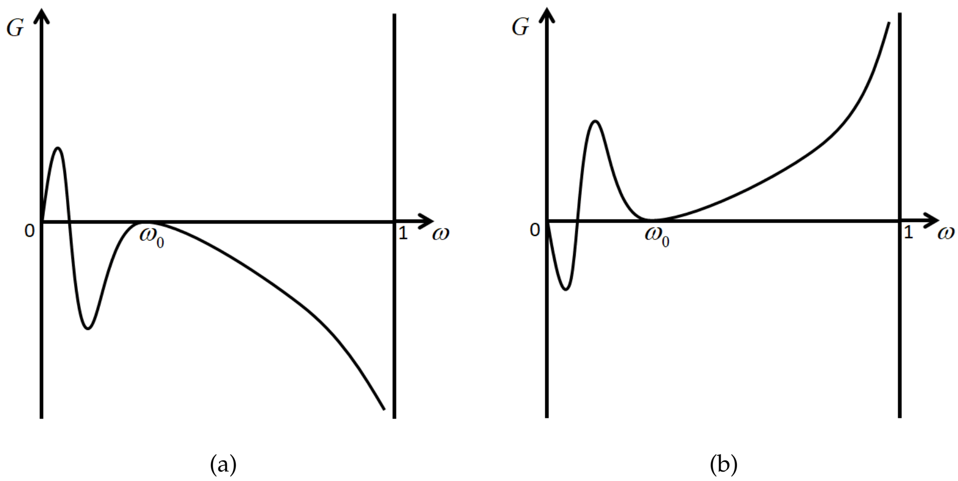

Figure 4.2: Two cases for $G(\omega)$ having 9 zeros in $(0,1)$ taking into account multiplicity. In particular ω_0 has multiplicity 8.

the averaged function (an element in Span $(\mathcal{F})$) can not have a zero in $(0,1)$ with multiplicity 9. Then we try to find an element $G(\omega) = \sum_{i=0}^{7} a_i f_i + k f_8 \in \text{Span}(\mathcal{F})$, of which has a zero $\omega_0 \in (0,1)$ with multiplicity 8. Note that $G(\omega)$ has 9 zeros in $(0,1)$ with ω_0 of multiplicity 8 may have two cases as shown in Fig. 4.2. For the generation of such ω_0 we provide an algorithm (Maple program) in Appendix B.

Now let $\omega_0 = 781/10001$, $K_0 = \ln\left(\frac{1+\omega_0}{1-\omega_0}\right)$ and $k = 10^8$. Consider the function

$$G(\omega) = a_0 f_0(\omega) + a_1 f_1(\omega) + \cdots + a_7 f_7(\omega) + k f_8(\omega), \quad \omega \in (0,1). \tag{4.4}$$

By direct calculation we get the power series of G around the point ω_0:

$$G(\omega) = e_0 + e_1(\omega - \omega_0) + \cdots + e_7(\omega - \omega_0)^7 + e_8(\omega - \omega_0)^8 + \cdots,$$

where e_i is the linear combination of $a_0, a_1, \ldots, a_7$. We solve the equations

$$e_0 = 0, \quad e_1 = 0, \quad \ldots, \quad e_7 = 0,$$

and find the values of $a_0, a_1, \ldots, a_7$ which have the form

$$a_i = \frac{\sum_{j=0}^{j_i} L_{i,j} K_0^j}{k_1 K_0 + k_2}, \quad i = 0, \ldots, 7, \tag{4.5}$$

where

$$\begin{aligned} k_1 =& \, 58539740887107254008913937583199370569724597 1 \\ & 4530223785342149243285359836224000000 0, \\ k_2 =& -916164764498521481287490087182092157549 \\ & 2097096776449170037730387965998807150160399, \end{aligned}$$

and

$$j_i = \begin{cases} 2, & i \in \{0, 1, 2\}, \\ 1, & i \in \{3, 4, 5, 6\}, \\ 0, & i \in \{7\}, \end{cases}$$

and each $L_{i,j}$ in (4.5) is an integer or rational with high number of digits in numerators and denominators. We will not write down here the explicit expression of a_i for the sake of brevity. It turns out that

$$G(\omega) = e_8 (\omega - \omega_0)^8 + \mathcal{O}((\omega - \omega_0)^9), \quad \omega \to \omega_0, \tag{4.6}$$

where

$$e_8 = -\frac{k_3 \cdot (B_1 K_0 + B_0)}{62567868120796985594771648240 1 \cdot (k_1 K_0 + k_2)},$$

with

$$\begin{aligned} k_3 &= 63739604097053650639687564229517470011768404297587090705 00, \\ B_1 &= 237183311483985729849441288215600575098623437626475734875280000 0, \\ B_0 &= -3711990906023283237845823733402369984240054504329347489316377 59, \end{aligned}$$

and $e_8 \approx 6.468110730 \times 10^7$. On the other hand, the Taylor expansion of $G(\omega)$ near $\omega = 0$ is

$$G(\omega) = C_1 \omega + \mathcal{O}(\omega^2), \tag{4.7}$$

where

$$C_1 = \frac{k_4 \cdot \big(k_5 K_0 - k_6\big)}{55588252797009 \cdot (k_1 K_0 + k_2)} \approx -3.242325599$$

with

$$\begin{aligned} k_4 &= 2270963709751407336612542323048546733131040681 00000, \\ k_5 &= 8643599130552840735000333895656822566694873780 00, \\ k_6 &= 135274953622915880496646897785052547295533923181. \end{aligned}$$

By the way, we would like to point out that our purpose of choosing such a k in (4.4) is to make the expressions of the numbers e_8 and C_1 to be relative simple. Equations (4.6) and (4.7) mean that (i) G has a zero at ω_0 with multiplicity 8, (ii) there exists an ε_0 with $0 < \varepsilon_0 < \omega_0$ such that $G(\omega)$ is positive in $[\varepsilon_0, \omega_0)$, and (iii) $G(\omega)$ is negative near $\omega = 0$. Moreover, $G(\omega)$

is positive in $(\omega_0, 1)$ because $\lim_{\omega\to 1^-} G(\omega) = +\infty$ (otherwise $G(\omega)$ would has 10 zeros in $(0,1)$ counting multiplicity, which leads to a contradiction).

Fixing the numbers $a_0, a_1, \ldots, a_7$ and k, we consider the function

$$G_\varepsilon(\omega) = G(\omega) + \sum_{i=0}^{8} \varepsilon_i f_i(\omega), \quad \omega \in (0,1). \tag{4.8}$$

We note that f_i can be extended analytically to $[0, 1)$. Thus there exists a small number $M > 0$ such that

$$\begin{aligned} G_\varepsilon(\varepsilon_0) &> \frac{1}{2} G(\varepsilon_0) > 0, \\ G_\varepsilon(\omega) &< \frac{1}{2} C_1 \omega < 0, \quad \text{when } \omega \to 0^+, \\ \lim_{\omega\to 1^-} G_\varepsilon(\omega) &= +\infty, \end{aligned}$$

for all $|\varepsilon_i| < M$, $i = 0, 1, \ldots, 8$. Moreover, near ω_0 we find

$$\sum_{i=0}^{8} \varepsilon_i f_i(\omega) = \mu_0 + \mu_1(\omega - \omega_0) + \cdots + \mu_8(\omega - \omega_0)^8 + \cdots,$$

where $\mu_i = \mu_i(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_8)$ is linear combination of ε_0, ε_1, ..., ε_8. One can directly check that the matrix of the coefficients of $\mu_0, \mu_1, \ldots, \mu_8$ with respect to $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_8$ has rank 9, and hence $\mu_0, \mu_1, \ldots, \mu_8$ are independent.

Consequently, since f_i is analytic at ω_0 and $G(\omega)$ has a zero at ω_0 with multiplicity 8, it follows that there exists some small $|\varepsilon_i| \ll M$ $(i = 0, 1, \ldots 8)$ (and hence μ_i is small) such that G_ε has exactly 8 simple zeros in a small enough neighborhood of ω_0. In view of (4.8) $G(\omega)$ has an extra zero in $(0, \varepsilon_0)$. According to the result of [23], this zero is simple. That is to say, G_ε has 9 simple zeros.

Finally, taking into account the above analysis, we see that system (1.2), up to the first order averaging method, has at most 9 limit cycles, and the upper bound can be reached. The proof of Theorem 1.1 is finished. □

Remark 4.3. If $\bar{R}$ is a simple zero of the averaged function $f(R)$ (see (3.2)), by Theorem 2.1 we have a limit cycle $R(\varphi, \varepsilon)$ of the differential system (3.1) such that $R(0, \varepsilon) = \bar{R} + \mathcal{O}(\varepsilon)$. Then, going back through the changes of variables (see (3.1)) we have for the discontinuous piecewise differential system (1.2) the medium limit cycle $(x(t,\varepsilon), y(t,\varepsilon)) = \big(\rho(\bar{R}, \cos\theta), \rho(\bar{R}, \sin\theta)\big) + \mathcal{O}(\varepsilon)$.

5 Proof of Theorem 1.4

In this section, we will present the k-th order averaged functions up to $k = 5$ associated to systems (1.4) and (1.5) respectively, and then we use them to prove Theorem 1.4.

5.1 Proof of Theorem 1.4 (a)

In order to analyze the Hopf bifurcation for system (1.4), applying Theorem 2.1, we set $\gamma = 2\pi$ in (2.2) and we introduce a small parameter ε doing the change of coordinates $x = \varepsilon X$, $y = \varepsilon Y$. After that we perform the polar change of coordinates $X = r\cos\theta$, $Y = r\sin\theta$, and by doing a

Taylor expansion truncated at the 5-th order in ε we obtain the following expression for $dr/d\theta$ of the form (2.1):

$$\frac{dr}{d\theta} = \sum_{i=0}^{5} \varepsilon^i F_i(\theta, r) + \mathcal{O}(\varepsilon^6), \tag{5.1}$$

where

$$F_0(\theta, r) = \frac{r(a_{1,0,0}\cos\theta + b_{1,0,0}\sin\theta)}{b_{1,0,0}\cos\theta - a_{1,0,0}\sin\theta - 2r\alpha^2}. \tag{5.2}$$

The explicit expressions of $F_i(\theta, r)$ for $i = 1, \ldots, 5$ are quite large so we omit them. To make $F_0(\theta, r) = 0$ we take $a_{1,0,0} = b_{1,0,0} = 0$. From now on, for each $j = 1, 2, \ldots, 5$, we will perform the calculation of the averaged function f_j under the hypothesis $f_k \equiv 0$ for $k = 1, \ldots, j-1$.

Now computing f_1 we obtain

$$f_1(r) = -\frac{\pi r}{2\alpha^2}(a_{1,1,0} + b_{1,0,1}).$$

Clearly equation $f_1(r) = 0$ has no positive zeros. Then the first order averaging theorem does not provide information about the limit cycles of system (1.4).

To apply the second order averaging theorem we take $b_{1,0,1} = -a_{1,1,0}$. Computing f_2 we obtain

$$f_2(r) = -\frac{\pi r}{2\alpha^3}\Big(\alpha(a_{2,1,0} + b_{2,0,1}) - 4a_{2,0,0}\Big).$$

As for the first function f_1, the second one also does not provide information on the bifurcating limit cycles.

Setting $a_{2,0,0} = \alpha(a_{2,1,0} + b_{2,0,1})/4$ we get $f_2(r) = 0$, and then we can apply the third order averaging theorem, and its corresponding function f_3 is

$$f_3(r) = -\frac{\pi r}{16\alpha^5}\Big(D_{3,2}r^2 + D_{3,0}\Big),$$

where

$$\begin{aligned}
D_{3,2} &= 2\alpha\Big(\alpha^2(a_{1,1,2} + 3a_{1,3,0} + 3b_{1,0,3} + b_{1,2,1}) - 4\alpha(a_{1,0,2} + a_{1,2,0}) + 4a_{1,1,0}\Big),\\
D_{3,0} &= 8\alpha^3(a_{3,1,0} + b_{3,0,1}) - \alpha^2(a_{1,1,1}a_{2,1,0} + a_{1,1,1}b_{2,0,1} + 2a_{2,1,0}b_{1,0,2} + 2b_{1,0,2}b_{2,0,1} + 32a_{3,0,0})\\
&\quad + 4\alpha(a_{1,0,1}a_{2,1,0} + a_{1,0,1}b_{2,0,1} + 2a_{1,2,0}b_{2,0,0} + b_{1,1,1}b_{2,0,0}) - 16a_{1,1,0}b_{2,0,0}.
\end{aligned}$$

Then there exists at most one positive simple zero of f_3. From Theorem 2.1 it follows that the third order averaging provides the existence of at most one small limit cycle of system (1.4) and this number can be reached by Lemma 2.3 ($D_{3,2}$ and $D_{3,0}$ are linearly independent constants). In order to apply the fourth order averaging theorem, we need to have $f_3(r) = 0$ so we let $a_{1,0,2} = D_{3,2}/8\alpha^2 + a_{1,0,2}$ and $a_{3,0,0} = D_{3,0}/32\alpha^2 + a_{3,0,0}$. The resulting fourth order averaged function is

$$f_4(r) = -\frac{\pi r}{128\alpha^7}\Big(D_{4,2}r^2 + D_{4,0}\Big),$$

where

$$\begin{aligned} D_{4,2} = 2\alpha \Big(& \alpha^4 (8a_{2,1,2} + 8b_{2,2,1} + 24a_{2,3,0} + 24b_{2,0,3}) + \alpha^3 (-a_{1,1,1}a_{1,1,2} - 3a_{1,1,1}a_{1,3,0} - 3a_{1,1,1}b_{1,0,3} \\ & - a_{1,1,1}b_{1,2,1} - 2b_{1,0,2}a_{1,1,2} - 6b_{1,0,2}a_{1,3,0} - 6b_{1,0,2}b_{1,0,3} - 2b_{1,0,2}b_{1,2,1} - 32a_{2,0,2} - 32a_{2,2,0}) \\ & + \alpha^2 (4a_{1,0,1}a_{1,1,2} + 24a_{1,0,1}a_{1,3,0} + 12a_{1,0,1}b_{1,0,3} + 8a_{1,0,1}b_{1,2,1} - 8a_{1,1,0}a_{1,2,1} - 8a_{1,1,0}b_{1,1,2} \\ & + 8b_{1,0,2}a_{1,2,0} + 8a_{1,2,0}b_{1,2,0} + 12a_{1,3,0}b_{1,1,0} + 4b_{1,0,2}b_{1,1,1} + 4b_{1,1,0}b_{1,2,1} + 4b_{1,1,1}b_{1,2,0} \\ & + 24a_{2,1,0} - 8b_{2,0,1}) + \alpha(-24a_{1,0,1}a_{1,2,0} - 4a_{1,0,1}b_{1,1,1} + 20a_{1,1,0}a_{1,1,1} - 8a_{1,1,0}b_{1,0,2} \\ & - 16a_{1,1,0}b_{1,2,0} - 24a_{1,2,0}b_{1,1,0} - 4b_{1,1,0}b_{1,1,1}) + 32a_{1,1,0}b_{1,1,0} \Big), \\ D_{4,0} = 64\alpha^5 & (b_{4,0,1} + a_{4,1,0}) + \alpha^4 (-8a_{1,1,1}a_{3,1,0} - 8a_{1,1,1}b_{3,0,1} - 8a_{2,1,1}a_{2,1,0} - 16b_{2,0,2}a_{2,1,0} \\ & - 8a_{2,1,1}b_{2,0,1} - 16b_{1,0,2}a_{3,1,0} - 16b_{1,0,2}b_{3,0,1} - 16b_{2,0,2}b_{2,0,1} - 256a_{4,0,0}) \\ & + \alpha^3 (a_{1,1,1}^2 a_{2,1,0} + a_{1,1,1}^2 b_{2,0,1} + 4a_{1,1,1}b_{1,0,2}a_{2,1,0} + 4a_{1,1,1}b_{1,0,2}b_{2,0,1} + 4b_{1,0,2}^2 a_{2,1,0} \\ & + 4b_{1,0,2}^2 b_{2,0,1} + 32a_{1,0,1}a_{3,1,0} + 32a_{1,0,1}b_{3,0,1} + 64a_{1,2,0}b_{3,0,0} + 32a_{2,0,1}a_{2,1,0} + 32a_{2,0,1}b_{2,0,1} \\ & + 64a_{2,2,0}b_{2,0,0} + 32b_{1,1,1}b_{3,0,0} + 32b_{2,0,0}b_{2,1,1}) + \alpha^2 (-4a_{1,0,1}a_{1,1,1}a_{2,1,0} - 4a_{1,0,1}a_{1,1,1}b_{2,0,1} \\ & - 8a_{1,0,1}b_{1,0,2}a_{2,1,0} - 8a_{1,0,1}b_{1,0,2}b_{2,0,1} + 8a_{1,2,0}a_{1,1,0}a_{2,1,0} + 8a_{1,2,0}a_{1,1,0}b_{2,0,1} \\ & + 4a_{1,1,0}b_{1,1,1}a_{2,1,0} + 4a_{1,1,0}b_{1,1,1}b_{2,0,1} - 8a_{1,1,1}a_{1,2,0}b_{2,0,0} - 4a_{1,1,1}b_{1,1,1}b_{2,0,0} \\ & - 16b_{1,0,2}a_{1,2,0}b_{2,0,0} - 8b_{1,0,2}b_{1,1,1}b_{2,0,0} - 128a_{1,1,0}b_{3,0,0} - 96a_{2,1,0}b_{2,0,0} \\ & + 32b_{2,0,0}b_{2,0,1}) + \alpha(32a_{1,0,1}a_{1,2,0}b_{2,0,0} + 16a_{1,0,1}b_{1,1,1}b_{2,0,0} - 16a_{1,1,0}^2 a_{2,1,0} \\ & - 16a_{1,1,0}^2 b_{2,0,1} + 32a_{1,2,0}b_{1,1,0}b_{2,0,0} + 16b_{1,1,0}b_{1,1,1}b_{2,0,0}) - 64a_{1,1,0}b_{1,1,0}b_{2,0,0}. \end{aligned}$$

Then there exists at most one positive simple zero of f_4. From Theorem 2.1 it follows that the fourth order averaging provides the existence of at most one small limit cycle of system (1.4) and this number can be reached.

Letting $a_{2,0,2} = D_{4,2}/64\alpha^4 + a_{2,0,2}$ and $a_{4,0,0} = D_{4,0}/256\alpha^4 + a_{4,0,0}$ we obtain $f_4(r) = 0$, so we can apply the fifth order averaging theorem, and its corresponding function is of the form

$$f_5(r) = \frac{\pi r}{1024\alpha^9} \left(D_{5,4}r^4 + D_{5,2}r^2 + D_{5,0} \right),$$

where $D_{5,4} = 64\alpha^5(a_{1,1,2} + a_{1,3,0} - b_{1,0,3} - b_{1,2,1})$. Here we do not explicitly provide the expressions of $D_{5,2}$ and $D_{5,0}$, because they are very long. Moreover $D_{5,4}$, $D_{5,2}$ and $D_{5,0}$ are linearly independent constants. In fact only $D_{5,2}$ has the parameter $a_{3,0,2}$, and $D_{5,0}$ is the only one with parameters $a_{5,0,0}$ and $b_{5,0,1}$. We claim that $D_{5,4}$ is also linearly independent of the other coefficients. Suppose that this is false. Then there exist real numbers m_1, m_2 not all zero such that $D_{5,4} = m_1 D_{5,0} + m_2 D_{5,2}$. But $D_{5,0}$ is the only one with the variables $a_{5,0,0}$ and $b_{5,0,1}$, so in order to $D_{5,4}$ does not present these variables we must set $m_1 = 0$. Since the other function $D_{5,2}$ also has variable which uniquely appears in its expression, the same argument holds so $m_2 = 0$. But then $D_{5,4} \equiv 0$, which is a contradiction. Therefore $D_{5,4}$, $D_{5,2}$ and $D_{5,0}$ are linearly independent constants. Hence f_5 has at most two positive simple zeros. From Theorem 2.1 it follows that the fifth order averaging provides the existence of at most two small limit cycle of system (1.4) and this number can be reached by Lemma 2.3.

5.2 Proof of Theorem 1.4 (b)

In order to analyze the Hopf bifurcation for this case, applying Theorem 2.1, we set $\gamma = \pi$ in (2.2). By using similar arguments to those presented for the proof of Theorem 1.4 (a), we can transform system (1.5) into the form

$$\frac{dr}{d\theta} = \begin{cases} \sum_{i=1}^{5} \varepsilon^i F_i^+(\theta, r) + \mathcal{O}(\varepsilon^6), & \text{if } 0 \le \theta \le \pi, \\ \sum_{i=1}^{5} \varepsilon^i F_i^-(\theta, r) + \mathcal{O}(\varepsilon^6), & \text{if } \pi \le \theta \le 2\pi, \end{cases} \tag{5.3}$$

where

$$\begin{aligned} F_1^+(\theta, r) = &-\frac{1}{2\alpha^2}\Big[\big(r(a_{1,0,1}+b_{1,1,0})\sin\theta + a_{2,0,0}\big)\cos\theta \\ &+ r(-a_{1,1,0}+b_{1,0,1})\sin^2\theta + (2\alpha r^2 + b_{2,0,0})\sin\theta + r a_{1,1,0}\Big], \end{aligned} \tag{5.4}$$

and $F_1^-(\theta, r)$ is an expression by taking $a = c$, $b = d$ in $F_1^+(\theta, r)$. The explicit expressions of $F_i^{\pm}(\theta, r)$ for $i = 2, \ldots, 5$ are quite large so we omit them here for brevity. We remark that we have used the condition $a_{1,0,0} = b_{1,0,0} = c_{1,0,0} = d_{1,0,0} = 0$ for the vanishing of the unperturbed terms $F_0^+(\theta, r)$ and $F_0^-(\theta, r)$.

Now applying Theorem 2.1 we obtain the first order averaged function

$$f_1(r) = -\frac{1}{4\alpha^2}\left(Y_{1,1} r + Y_{1,0}\right),$$

where

$$Y_{1,1} = \pi(a_{1,1,0} + c_{1,1,0} + b_{1,0,1} + d_{1,0,1}), \quad Y_{1,0} = 4(b_{2,0,0} - d_{2,0,0}).$$

It is obvious that the coefficients $Y_{1,1}$ and $Y_{1,0}$ are independent. Thus $f_1(r)$ can have one positive zero. From Theorem 2.1 it follows that the first order averaging provides the existence of at most one small limit cycle of system (1.5) and this number can be reached.

To consider the second order averaging theorem we take $d_{1,0,1} = -Y_{1,1}/\pi + d_{1,0,1}$ and $d_{2,0,0} = Y_{1,0}/4 + d_{2,0,0}$. Computing f_2 we obtain

$$f_2(r) = -\frac{1}{48\alpha^4}\left(Y_{2,2} r^2 + Y_{2,1} r + Y_{2,0}\right),$$

where

$$\begin{aligned} Y_{2,2} &= 16\alpha\Big((a_{1,1,1} - c_{1,1,1} + 2b_{1,0,2} - 2d_{1,0,2} + b_{1,2,0} - d_{1,2,0})\alpha - 4(a_{1,0,1} - c_{1,0,1}) - (b_{1,1,0} - d_{1,1,0})\Big), \\ Y_{2,1} &= -3\pi\Big(-4(a_{2,1,0} + c_{2,1,0} + b_{2,0,1} + d_{2,0,1})\alpha^2 + 16(a_{2,0,0} + c_{2,0,0})\alpha \\ &\quad + a_{1,1,0}(a_{1,0,1} - c_{1,0,1}) + b_{1,0,1}(a_{1,0,1} - c_{1,0,1}) - a_{1,1,0}(b_{1,1,0} - d_{1,1,0}) - b_{1,0,1}(b_{1,1,0} - d_{1,1,0})\Big), \\ Y_{2,0} &= 24\Big(2(b_{3,0,0} - d_{3,0,0})\alpha^2 - a_{1,1,0}(a_{2,0,0} + c_{2,0,0}) - b_{1,0,1}(a_{2,0,0} + c_{2,0,0}) + b_{2,0,0}(b_{1,1,0} - d_{1,1,0})\Big). \end{aligned}$$

Since $f_2(r)$ can have at most two positive zeros, we conclude that system (1.5) has at most two small limit cycles and this number can be reached.

To consider the third order averaging theorem we take $d_{1,0,2} = Y_{2,2}/32\alpha^2 + d_{1,0,2}$, $d_{2,0,1} = -Y_{2,1}/12\pi\alpha^2 + d_{2,0,1}$ and $d_{3,0,0} = Y_{2,0}/48\alpha^2 + d_{3,0,0}$. Computing f_3 we obtain

$$r f_3(r) = -\frac{1}{1152\alpha^6}\left(Y_{3,4} r^4 + Y_{3,3} r^3 + Y_{3,2} r^2 + Y_{3,1} r + Y_{3,0}\right),$$

where

$$Y_{3,4} = 72\pi\alpha^2\Big((a_{1,1,2} + c_{1,1,2} + 3a_{1,3,0} + 3c_{1,3,0} + 3b_{1,0,3} + 3d_{1,0,3} + b_{1,2,1} + d_{1,2,1})\alpha^2 - 4(a_{1,0,2} + c_{1,0,2} + a_{1,2,0} + c_{1,2,0})\alpha + 4(a_{1,1,0} + c_{1,1,0})\Big),$$
$$Y_{3,0} = 72\pi(a_{2,0,0} - c_{2,0,0})(a_{2,0,0} + c_{2,0,0})(a_{1,1,0} + b_{1,0,1}).$$

We do not explicitly provide the expressions of $Y_{3,i}$ for $i = 1, 2, 3$, since they are very long. Since $f_3(r)$ can have at most four positive zeros, we conclude that system (1.5) has at most four small limit cycles and this number can be reached.

To consider the fourth order averaging theorem, we need to have $f_3(r) = 0$ so we let $d_{1,0,3} = -Y_{3,4}/216\pi\alpha^4 + d_{1,0,3}$, $d_{2,0,2} = Y_{3,3}/768\alpha^4 + d_{2,0,2}$, $d_{3,0,1} = -Y_{3,2}/288\pi\alpha^4 + d_{3,0,1}$, $d_{4,0,0} = Y_{3,1}/1152\alpha^4 + d_{4,0,0}$. Note that in order to make $Y_{3,0} = 0$, we consider the following three cases.

CASE 1. $a_{2,0,0} = c_{2,0,0}$, $a_{2,0,0} \neq -c_{2,0,0}$ and $a_{1,1,0} \neq -b_{1,0,1}$.

In this case, computing f_4 we obtain

$$r^2 f_4(r) = -\frac{1}{23040\alpha^8}\Big(Y^1_{4,6}r^6 + Y^1_{4,5}r^5 + Y^1_{4,4}r^4 + Y^1_{4,3}r^3 + Y^1_{4,2}r^2 + Y^1_{4,1}r + Y^1_{4,0}\Big),$$

where

$$Y^1_{4,6} = -1536\alpha^5\Big(8(a_{1,0,3} - c_{1,0,3}) + 2(a_{1,2,1} - c_{1,2,1}) - 2(b_{1,1,2} - d_{1,1,2}) - 3(b_{1,3,0} - d_{1,3,0})\Big),$$
$$Y^1_{4,1} = -720\pi(a_{1,1,0} + b_{1,0,1})c_{2,0,0}\Big(-4(a_{3,0,0} - c_{3,0,0})\alpha^2 + (a_{1,0,1} - c_{1,0,1})c_{2,0,0} - 2(a_{1,1,0} - c_{1,1,0})b_{2,0,0} - (b_{1,1,0} - d_{1,1,0})c_{2,0,0}\Big),$$
$$Y^1_{4,0} = -1920(a_{1,1,0} + b_{1,0,1})c^3_{2,0,0}.$$

We do not explicitly provide the expressions of $Y^1_{4,i}$ for $i = 2, 3, \ldots, 5$, since they are very long. Then $f_4(r)$ can have at most six positive zeros, we conclude that system (1.5) has at most six small limit cycles and this number can be reached.

To consider the fifth order averaging theorem, we need to have $f_4(r) = 0$ so we let $d_{1,3,0} = Y^1_{4,6}/4608\alpha^5 + d_{1,3,0}$, $d_{2,0,3} = -Y^1_{4,5}/4320\pi\alpha^6 + d_{2,0,3}$, $d_{3,0,2} = Y^1_{4,4}/15360\alpha^6 + d_{3,0,2}$, $d_{4,0,1} = -Y^1_{4,3}/5760\pi\alpha^6 + d_{4,0,1}$, $d_{5,0,0} = Y^1_{4,2}/23040\alpha^6 + d_{5,0,0}$, $c_{2,0,0} = 0$. Computing f_5 we obtain

$$r f_5(r) = -\frac{1}{5529600\alpha^{10}}\Big(Y^1_{5,6}r^6 + Y^1_{5,5}r^5 + Y^1_{5,4}r^4 + Y^1_{5,3}r^3 + Y^1_{5,2}r^2 + Y^1_{5,1}r + Y^1_{5,0}\Big),$$

where

$$Y^1_{5,6} = 115200\pi\alpha^4\Big((-2a_{1,1,2} - 2c_{1,1,2} - 3a_{1,3,0} - 3c_{1,3,0} + b_{1,2,1} + d_{1,2,1})\alpha^2 + 2(a_{1,0,2} + c_{1,0,2} + a_{1,2,0} + c_{1,2,0})\alpha - 2(a_{1,1,0} + c_{1,1,0})\Big),$$
$$Y^1_{5,0} = 86400\pi(a_{1,1,0} + b_{1,0,1})\Big(2(a_{3,0,0} + c_{3,0,0})\alpha^2 + b_{2,0,0}(a_{1,1,0} + c_{1,1,0})\Big) \cdot \Big(2(a_{3,0,0} - c_{3,0,0})\alpha^2 + b_{2,0,0}(a_{1,1,0} - c_{1,1,0})\Big).$$

We do not explicitly provide the expressions of $Y^1_{5,i}$ for $i = 1, 2, \ldots, 5$, since they are very long. Then $f_5(r)$ can have at most six positive zeros, we conclude that system (1.5) has at most six small limit cycles and this number can be reached.

CASE 2. $a_{2,0,0} = -c_{2,0,0}$, $a_{2,0,0} \neq c_{2,0,0}$ and $a_{1,1,0} \neq -b_{1,0,1}$.

In this case, computing f_4 we obtain

$$rf_4(r) = -\frac{1}{23040\alpha^8}\left(Y_{4,5}^2 r^5 + Y_{4,4}^2 r^4 + Y_{4,3}^2 r^3 + Y_{4,2}^2 r^2 + Y_{4,1}^2 r + Y_{4,0}^2\right),$$

where

$$\begin{aligned} Y_{4,5}^2 =& -1536\alpha^5\Big(8(a_{1,0,3} - c_{1,0,3}) + 2(a_{1,2,1} - c_{1,2,1}) - 2(b_{1,1,2} - d_{1,1,2}) - 3(b_{1,3,0} - d_{1,3,0})\Big),\\ Y_{4,0}^2 =& -720\pi(a_{1,1,0} + b_{1,0,1})c_{2,0,0}\Big(4(a_{3,0,0} + c_{3,0,0})\alpha^2 + c_{2,0,0}(a_{1,0,1} - c_{1,0,1})\\ &+ 2b_{2,0,0}(a_{1,1,0} + c_{1,1,0}) - c_{2,0,0}(b_{1,1,0} - d_{1,1,0})\Big). \end{aligned}$$

We do not explicitly provide the expressions of $Y_{4,i}^2$ for $i = 1, 2, \ldots, 4$, since they are very long. Then $f_4(r)$ can have at most five positive simple zeros, we conclude that system (1.5) has at most five small limit cycles and this number can be reached.

To apply the fifth order averaging theorem, we need to have $f_4(r) = 0$ so we let $d_{1,3,0} = Y_{4,5}^2/4608\alpha^5 + d_{1,3,0}$, $d_{2,0,3} = -Y_{4,4}^2/4320\pi\alpha^6 + d_{2,0,3}$, $d_{3,0,2} = Y_{4,3}^2/15360\alpha^6 + d_{3,0,2}$, $d_{4,0,1} = -Y_{4,2}^2/5760\pi\alpha^6 + d_{4,0,1}$, $d_{5,0,0} = Y_{4,1}^2/23040\alpha^6 + d_{5,0,0}$. Note that in order to make $Y_{4,0}^2 = 0$, we consider two subcases.

Subcase 1. $c_{2,0,0} = 0$ and $a_{3,0,0} \neq -\frac{1}{4\alpha^2}(c_{2,0,0}(a_{1,0,1} - c_{1,0,1}) + 2b_{2,0,0}(a_{1,1,0} + c_{1,1,0}) - c_{2,0,0}(b_{1,1,0} - d_{1,1,0})) - c_{3,0,0}$.

In this subcase, computing f_5 we obtain

$$rf_5(r) = -\frac{1}{5529600\alpha^{10}}\left(Y_{5,6}^{2,1} r^6 + Y_{5,5}^{2,1} r^5 + Y_{5,4}^{2,1} r^4 + Y_{5,3}^{2,1} r^3 + Y_{5,2}^{2,1} r^2 + Y_{5,1}^{2,1} r + Y_{5,0}^{2,1}\right),$$

where

$$\begin{aligned} Y_{5,6}^{2,1} =& 115200\pi\alpha^4\Big((-2a_{1,1,2} - 2c_{1,1,2} - 3a_{1,3,0} - 3c_{1,3,0} + b_{1,2,1} + d_{1,2,1})\alpha^2\\ &+ (2a_{1,0,2} + 2c_{1,0,2} + 2a_{1,2,0} + 2c_{1,2,0})\alpha - 2(a_{1,1,0} + c_{1,1,0})\Big),\\ Y_{5,0}^{2,1} =& 86400\pi(a_{1,1,0} + b_{1,0,1})\Big(2(a_{3,0,0} + c_{3,0,0})\alpha^2 + b_{2,0,0}(a_{1,1,0} + c_{1,1,0})\Big)\\ &\cdot\Big(2(a_{3,0,0} - c_{3,0,0})\alpha^2 + b_{2,0,0}(a_{1,1,0} - c_{1,1,0})\Big). \end{aligned}$$

We do not explicitly provide the expressions of $Y_{5,i}^{2,1}$ for $i = 1, 2, \ldots, 5$, since they are very long. Then $f_5(r)$ can have at most six positive simple zeros, we conclude that system (1.5) has at most six small limit cycles and this number can be reached.

Subcase 2. $c_{2,0,0} \neq 0$ and $a_{3,0,0} = -\frac{1}{4\alpha^2}(c_{2,0,0}(a_{1,0,1} - c_{1,0,1}) + 2b_{2,0,0}(a_{1,1,0} + c_{1,1,0}) - c_{2,0,0}(b_{1,1,0} - d_{1,1,0})) - c_{3,0,0}$.

As in the **Subcase 1**, one can compute the expression of f_5 as follows:

$$rf_5(r) = -\frac{1}{5529600\alpha^{10}}\left(Y_{5,6}^{2,2} r^6 + Y_{5,5}^{2,2} r^5 + Y_{5,4}^{2,2} r^4 + Y_{5,3}^{2,2} r^3 + Y_{5,2}^{2,2} r^2 + Y_{5,1}^{2,2} r + Y_{5,0}^{2,2}\right),$$

where

$$
\begin{aligned}
Y_{5,6}^{2,2} &= Y_{5,6}^{2,1},\\
Y_{5,0}^{2,2} &= 21600\pi(a_{1,1,0}+b_{1,0,1})c_{2,0,0}\Big[-32(a_{4,0,0}+c_{4,0,0})\alpha^4+8\Big(-2b_{2,0,0}(a_{2,1,0}+c_{2,1,0})\\
&\quad +c_{3,0,0}(b_{1,1,0}-d_{1,1,0}-a_{1,0,1}+c_{1,0,1})+c_{2,0,0}(-a_{2,0,1}+c_{2,0,1}+b_{2,1,0}-d_{2,1,0})\\
&\quad -2b_{3,0,0}(a_{1,1,0}+c_{1,1,0})\Big)\alpha^2+\Big(-4b_{2,0,0}\big(c_{1,1,0}(a_{1,0,1}+b_{1,1,0}-c_{1,0,1}+d_{1,1,0})\\
&\quad +2a_{1,1,0}b_{1,1,0}\big)+c_{2,0,0}\big(a_{1,0,1}^2+2a_{1,0,1}b_{1,1,0}+2a_{1,0,1}c_{1,0,1}-2a_{1,0,1}d_{1,1,0}\\
&\quad -4a_{1,1,0}b_{1,0,1}-4a_{1,1,0}c_{1,1,0}-4b_{1,0,1}c_{1,1,0}+b_{1,1,0}^2-2b_{1,1,0}c_{1,0,1}\\
&\quad +2b_{1,1,0}d_{1,1,0}-3c_{1,0,1}^2+2c_{1,0,1}d_{1,1,0}-4c_{1,1,0}^2-3d_{1,1,0}^2\big)\Big)\Big].
\end{aligned}
$$

We do not explicitly provide the expressions of $Y_{5,i}^{2,2}$ for $i=1,2,\dots,5$, since they are very long. Then $f_5(r)$ can have at most six positive simple zeros, we conclude that system (1.5) has at most six small limit cycles and this number can be reached.

CASE 3. $a_{1,1,0}=-b_{1,0,1}$ and $a_{2,0,0}^2-c_{2,0,0}^2\neq 0$.

Since the calculations and arguments are quite similar to those used in the **CASE 1**, we just provide the expressions of f_4 and f_5 as follows:

$$
\begin{aligned}
rf_4(r) &= -\frac{1}{5760\alpha^8}\Big(Y_{4,5}^3r^5+Y_{4,4}^3r^4+Y_{4,3}^3r^3+Y_{4,2}^3r^2+Y_{4,1}^3r+Y_{4,0}^3\Big),\\
rf_5(r) &= -\frac{1}{23040\alpha^{10}}\Big(Y_{5,6}^3r^6+Y_{5,5}^3r^5+Y_{5,4}^3r^4+Y_{5,3}^3r^3+Y_{5,2}^3r^2+Y_{5,1}^3r+Y_{5,0}^3\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
Y_{4,5}^3 &= -384\alpha^5\Big(8(a_{1,0,3}-c_{1,0,3})+2(a_{1,2,1}-c_{1,2,1})-2(b_{1,1,2}-d_{1,1,2})-3(b_{1,3,0}-d_{1,3,0})\Big),\\
Y_{4,0}^3 &= -360\alpha\pi(a_{2,0,0}^2-c_{2,0,0}^2)\big(-\alpha(a_{2,1,0}+b_{2,1,0})+4a_{2,0,0}\big),\\
Y_{5,6}^3 &= 480\pi\alpha^4\Big((-2a_{1,1,2}-2c_{1,1,2}-3a_{1,3,0}-3c_{1,3,0}+b_{1,2,1}+d_{1,2,1})\alpha^2\\
&\qquad +2(a_{1,0,2}+c_{1,0,2}+a_{1,2,0}+c_{1,2,0})\alpha+2(b_{1,0,1}-c_{1,1,0})\Big),\\
Y_{5,0}^3 &= 720\pi\alpha(a_{2,0,0}^2-c_{2,0,2}^2)\Big(2(a_{3,1,0}+b_{3,0,1})\alpha^3-8a_{3,0,0}\alpha^2+(-a_{1,1,1}a_{2,0,0}+2a_{1,2,0}b_{2,0,0}\\
&\qquad -2a_{2,0,0}b_{1,0,2}+b_{1,1,1}b_{2,0,0})\alpha+4(a_{1,0,1}a_{2,0,0}+b_{1,0,1}b_{2,0,0})\Big).
\end{aligned}
$$

We do not explicitly provide the expressions of $Y_{4,i}^3$ for $i=1,2,\dots,4$ and $Y_{5,j}^3$ for $j=1,2,\dots,5$, since they are very long. Then $f_5(r)$ can have at most six positive simple zeros, we conclude that system (1.5) has at most six small limit cycles and this number can be reached.

Using the results of Sections 5.1 and 5.2, we complete the proof of Theorem 1.4.

In summary, we give a remark for the averaging method that we are using in Section 5. We know that if the averaged functions $f_j=0$ for $j=1,\dots,k-1$ and $f_k\neq 0$, and $\bar r$ is a simple zero of f_k, then by Theorem 2.1 there is a limit cycle $r(\theta,\varepsilon)$ of the differential system (5.3) such that $r(0,\varepsilon)=\bar r+\mathcal{O}(\varepsilon)$. Then, going back through the changes of variables ($x=\varepsilon r\cos\theta$, $y=\varepsilon r\sin\theta$) we have for the discontinuous piecewise differential system (1.5) the limit cycle $(x(t,\varepsilon),y(t,\varepsilon))=\varepsilon\big(\bar r\cos\theta,\bar r\sin\theta\big)+\mathcal{O}(\varepsilon^2)$, which tends to the origin of system (1.5) when the parameter $\varepsilon\to 0$. In other words, this limit cycle is a small limit cycle bifurcating from the origin, i.e., is a limit cycle coming by a Hopf bifurcation.

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A Fifth order averaging formulae

$$f_i(z) = \frac{y_i^+(\gamma, z) - y_i^-(\gamma - 2\pi, z)}{i!}, \quad \text{for } i = 1, \ldots, 5,$$

where

$$\begin{aligned}
y_1^\pm(\theta, z) = & \int_0^\theta F_1^\pm(\varphi, z) d\varphi, \\
y_2^\pm(\theta, z) = & \int_0^\theta \Big(2F_2^\pm(\varphi, z) + 2\partial F_1^\pm(\varphi, z) y_1^\pm(\varphi, z)\Big) d\varphi, \\
y_3^\pm(\theta, z) = & \int_0^\theta \Big(6F_3^\pm(\varphi, z) + 6\partial F_2^\pm(\varphi, z) y_1^\pm(\varphi, z) \\
& + 3\partial^2 F_1^\pm(\varphi, z) y_1^\pm(\varphi, z)^2 + 3\partial F_1^\pm(\varphi, z) y_2^\pm(\varphi, z)\Big) d\varphi, \\
y_4^\pm(\theta, z) = & \int_0^\theta \Big(24F_4^\pm(\varphi, z) + 24\partial F_3^\pm(\varphi, z) y_1^\pm(\varphi, z) + 12\partial^2 F_2^\pm(\varphi, z) y_1^\pm(\varphi, z)^2 \\
& + 12\partial F_2^\pm(\varphi, z) y_2^\pm(\varphi, z) + 12\partial^2 F_1^\pm(\varphi, z) y_1^\pm(\varphi, z) y_2^\pm(\varphi, z) \\
& + 4\partial^3 F_1^\pm(\varphi, z) y_1^\pm(\varphi, z)^3 + 4\partial F_1^\pm(\varphi, z) y_3^\pm(\varphi, z)\Big) d\varphi, \\
y_5^\pm(\theta, z) = & \int_0^\theta \Big(120F_5^\pm(\varphi, z) + 120\partial F_4^\pm(\varphi, z) y_1^\pm(\varphi, z) + 60\partial^2 F_3^\pm(\varphi, z) y_1^\pm(\varphi, z)^2 \\
& + 60\partial F_3^\pm(\varphi, z) y_2^\pm(\varphi, z) + 60\partial^2 F_2^\pm(\varphi, z) y_1^\pm(\varphi, z) y_2^\pm(\varphi, z) \\
& + 20\partial^3 F_2^\pm(\varphi, z) y_1^\pm(\varphi, z)^3 + 20\partial F_2^\pm(\varphi, z) y_3^\pm(\varphi, z) \\
& + 20\partial^2 F_1^\pm(\varphi, z) y_1^\pm(\varphi, z) y_3^\pm(\varphi, z) + 15\partial^2 F_1^\pm(\varphi, z) y_2^\pm(\varphi, z)^2 \\
& + 30\partial^3 F_1^\pm(\varphi, z) y_1^\pm(\varphi, z)^2 y_2^\pm(\varphi, z) + 5\partial^4 F_1^\pm(\varphi, z) y_1^\pm(\varphi, z)^4 \\
& + 5\partial F_1^\pm(\varphi, z) y_4^\pm(\varphi, z)\Big) d\varphi.
\end{aligned}$$

B Algorithm for generating ω_0

Algorithm 1

Input: a function $G = \sum_{i=0}^{7} a_i f_i(\omega) + k f_8(\omega)$

Output: a zero ω_0 of G with multiplicity 8

with(RandomTools);
Ω:=Generate(list(rational(range=0..1,denominator=10001),500));
for ω_0 **in** Ω **do**
  $G_1 := \mathrm{subs}(\omega - \omega_0 = s, \mathrm{convert}(\mathrm{series}(G, \omega = \omega_0, 9), \mathrm{polynom}))$;
  $e_0 := \mathrm{tcoeff}(G_1, s)$;
  for i **from** 1 **to** 8 **do**
    $e_i := \mathrm{coeff}(G_1, s^i)$;
  $S_0 := \mathrm{solve}(\{\mathrm{seq}(e_j = 0, j = 0..7)\}, \{\mathrm{seq}(a_j, j = 0..7)\})$;
  $A := \mathrm{normal}(\mathrm{subs}(S_0, e_8)/k)$;
  $G_2 := \mathrm{convert}(\mathrm{series}(\mathrm{subs}(S_0, G), \omega = 0, 2), \mathrm{polynom})$;
  $B := \mathrm{normal}(\mathrm{coeff}(G_2, \omega)/k)$;
  if $\mathrm{signum}(\mathrm{evalf}(A)) - \mathrm{signum}(\mathrm{limit}(\mathrm{subs}(S_0, G), \omega = 1, \mathrm{left})/\mathrm{signum}(k)) = 0$ **and** $\mathrm{signum}(\mathrm{evalf}(AB)) < 0$ **then**
return ω_0;

The following result is one output of Algorithm 1:

$$\frac{781}{10001}, \quad \frac{834}{10001}, \quad \frac{515}{10001}, \quad \frac{878}{10001}, \quad \frac{622}{10001}, \quad \frac{740}{10001}.$$

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Fractional eigenvalue problems on $\mathbb{R}^N$

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Abstract. Let $N \geq 2$ be an integer. For each real number $s \in (0,1)$ we denote by $(-\Delta)^s$ the corresponding fractional Laplace operator. First, we investigate the eigenvalue problem $(-\Delta)^s u = \lambda V(x) u$ on $\mathbb{R}^N$, where $V : \mathbb{R}^N \to \mathbb{R}$ is a given function. Under suitable conditions imposed on V we show the existence of an unbounded, increasing sequence of positive eigenvalues. Next, we perturb the above eigenvalue problem with a fractional (t,p)-Laplace operator, when $t \in (0,1)$ and $p \in (1,\infty)$ are such that $t < s$ and $s - N/2 = t - N/p$. We show that when the function V is nonnegative on $\mathbb{R}^N$, the set of eigenvalues of the perturbed eigenvalue problem is exactly the unbounded interval (λ_1, ∞), where λ_1 stands for the first eigenvalue of the initial eigenvalue problem.

Keywords: fractional Laplacian, eigenvalue problem, weak solution, minimization problem, Nehari manifold.

2020 Mathematics Subject Classification: 45A05, 45C05, 47A75, 45G99, 46E35.

1 Introduction

Let $N \geq 2$ be an integer. For each real numbers $p \in (1,\infty)$ and $s \in (0,1)$ and each function $u : \mathbb{R}^N \to \mathbb{R}$ we define the nonlocal operator

$$(-\Delta_p)^s u(x) := 2 \lim_{\epsilon \searrow 0} \int_{|x-y| \geq \epsilon} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+sp}}\, dy, \ x \in \mathbb{R}^N. \tag{1.1}$$

For $p = 2$ the above definition reduces to the linear *fractional Laplacian* denoted by $(-\Delta)^s$. For that reason we will refer to $(-\Delta_p)^s$ as being a *fractional* (s,p)*-Laplacian operator* which is a nonlinear operator when $p \in (1,\infty) \setminus \{2\}$.

1.1 Statement of the problem and motivation

The main goal of this paper is to study an eigenvalue problem for the fractional Laplacian operator on $\mathbb{R}^N$ and a perturbed version of this problem when we perturb the fractional Laplacian by a nonlinear fractional (t,p)-Laplacian . More precisely, first we will study the eigenvalue problem

$$(-\Delta)^s u(x) = \mu V(x) u(x), \qquad \forall\, x \in \mathbb{R}^N, \tag{1.2}$$

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where $s \in (0,1)$ is a given real number, μ is a real parameter and $V : \mathbb{R}^N \to \mathbb{R}$ is a function that may change sign and which satisfies the hypothesis

($\widetilde{\mathrm{V}}$) $V \in L^1_{\mathrm{loc}}\left(\mathbb{R}^N\right)$, $V^+ = V_1 + V_2 \neq 0$, $V_1 \in L^{\frac{N}{2s}}\left(\mathbb{R}^N\right)$ and $\lim_{x\to y} |x-y|^{2s} V_2(x) = 0$, for all $y \in \mathbb{R}^N$ and $\lim_{|x|\to\infty} |x|^{2s} V_2(x) = 0$.

Remark 1.1. Note that there exists functions $V : \mathbb{R}^N \to \mathbb{R}$ such that $V \notin L^{\frac{N}{2s}}\left(\mathbb{R}^N\right)$ but $\lim_{x\to y} |x-y|^{2s} V(x) = 0$, for all $y \in \mathbb{R}^N$ and $\lim_{|x|\to\infty} |x|^{2s} V(x) = 0$. Indeed, simple computations show that we can take $V(x) = |x|^{-2s}(1+|x|^{2s})^{-1}[\ln(2+|x|^{-2s})]^{-(2s)/N}$, if $x \neq 0$ and $V(0) = 1$.

Next, we will study a perturbation of problem (1.2), namely

$$(-\Delta)^s u(x) + \left(-\Delta_p\right)^t u(x) = \lambda V(x) u(x), \qquad \forall\, x \in \mathbb{R}^N, \tag{1.3}$$

under the assumption

$$0 < t < s < 1 \quad \text{and} \quad s - \frac{N}{2} = t - \frac{N}{p}, \tag{1.4}$$

where λ is a real parameter and $V : \mathbb{R}^N \to [0,\infty)$ is a function satisfying the hypothesis ($\widetilde{\mathrm{V}}$). Note that in the case of problem (1.3) we have $V = V^+$.

A first motivation in studying problems of type (1.2) comes from the paper by Szulkin & Willem [21] where a similar equation was investigated in the case when the fractional Laplacian $(-\Delta)^s$ is replaced by the classical Laplace operator Δ. In particular, we note that assumption ($\widetilde{\mathrm{V}}$) imposed here to the weight function V is suggested by condition (H) from [21]. At the same time we recall that some generalizations of the results from [21] to the case when the Laplace operator Δ is replaced by a more general class of degenerate elliptic operators of type $\operatorname{div}(|x|^\alpha \nabla)$, with $\alpha \in (0,2)$, was studied by Mihăilescu & Repovš in [18]. In the case of nonlocal operators, problems of type (1.2) were mainly investigated on bounded domains under the homogeneous Dirichlet boundary condition. Among the results obtained in this direction we recall the recent articles by Franzina & Palatucci [13], Lindgren & Lindqvist [15], Brasco, Parini & Squassina [3], Del Pezzo & Quass [5], Ferreira & Pérez-Llanos [11], Fărcăşeanu [8], Del Pezzo, Ferreira & Rossi [4], Ercole, Pereira, & Sanchis [7]. Much less papers were devoted to the study of problem (1.2) on the whole Euclidian space $\mathbb{R}^N$. Here we just recall the study by Frank, Lenzmann, & Silvestre from [12] where the issue of the existence and uniqueness of bounded radial solutions which vanishes at infinity for problems of type (1.2) was considered. More precisely, in [12, Theorem 2.1] it is showed that if $u(x) = u(|x|)$ is a radial and bounded solution of (1.2) which vanishes at infinity then $u(0) = 0$ implies $u \equiv 0$, provided that the weight function V is radial and non-decreasing on $\mathbb{R}^N$ and $V \in C^{0,\gamma}(\mathbb{R}^N)$ for some real number $\gamma > \max\{0, 1-2s\}$.

Regarding the problem (1.3) we recall that it was studied on bonded domains form the Euclidian space $\mathbb{R}^N$ under the homogeneous Dirichlet boundary condition by Fărcăşeanu, Mihăilescu, & Stancu-Dumitru in [10], in the case when $V \equiv 1$. In particular, we note that assumption (1.4) imposed here is suggested by condition (3) from [10]. We point out that in the case when the nonlocal operators from equation (1.3) are replaced by the corresponding differential operators (Laplacian and p-Laplacian) the resulting problem was analysed by Mihăilescu & Stancu-Dumitru in [19], while in the case of bounded domains similar results were obtained in [1,9,16,17] under different boundary conditions. Thus, in particular, the results from this paper complement to the case of nonlocal operators some earlier results obtained in the case of differential operators.

The rest of the paper is organized as follows: in the next two subsections we introduce the natural function space setting where problems (1.2) and (1.3) will be studied and we point out the main results of the paper; in Section 2 we state and prove an auxiliary result that will be useful for the analysis of the main results; the last two sections are devoted to the proofs of the main results.

1.2 Fractional Sobolev spaces

In this subsection we introduce the natural function spaces where we will study equations (1.2) and (1.3) and we will recall some of their properties which will be useful in our analysis. For more details we refer the reader to the book by Grisvard [14] and to the papers [2,3,5,6].

First, by [3, p. 1814] we recall that the natural setting for equations involving the operator $(-\Delta_p)^t$ is the fractional Sobolev space $\mathcal{D}_0^{t,p}(\mathbb{R}^N)$ defined as the closure of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{t,p} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+tp}}\,dxdy\right)^{1/p}.$$

The above function space is a reflexive Banach space. Moreover, in the particular case when $p=2$ the function space $\mathcal{D}_0^{t,2}(\mathbb{R}^N)$ is a Hilbert space.

From the above discussion it follows easily that the natural function space where we will study equation (1.2) will be the Hilbert space $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$. On the other hand, we note that in equation (1.3) are involved two nonlocal operators, $(-\Delta)^s$ and $(-\Delta_p)^t$, respectively. The natural function space where we analyse problems involving $(-\Delta)^s$ is the fractional Sobolev space $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$, while the function space where we study problems involving $\mathcal{D}_0^{t,p}(\mathbb{R}^N)$ is the fractional Sobolev space $\mathcal{D}_0^{t,p}(\mathbb{R}^N)$. Thus, in the case of equation (1.3) we should decide which of the spaces $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and $\mathcal{D}_0^{t,p}(\mathbb{R}^N)$ is the natural function space where we can seek solutions for the problem. A key condition in this case is assumption (1.4), which in view of [14, Theorem 1.4.4.1] assures that

$$\mathcal{D}_0^{s,2}(\mathbb{R}^N) \subset \mathcal{D}_0^{t,p}(\mathbb{R}^N). \tag{1.5}$$

Thus, the natural function space where we should study problem (1.3) is again the Hilbert space $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$.

Next, note that by [6, Theorem 6.5] there exists a positive constant $C = C(N,s)$ such that

$$\|u\|_{L^{2_s^*}(\mathbb{R}^N)} \le C\,\|u\|_{s,2}\,, \tag{1.6}$$

where $2_s^* := \frac{2N}{N-2s}$ is the so called *fractional critical exponent*. Consequently, the space $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ is continuously embedded in $L^{2_s^*}(\mathbb{R}^N)$.

Further, we point out that a Hardy-type inequality can be established on the fractional Sobolev spaces. More precisely, by [2, Theorem 6.3] (see also [20]) we know that there exists a positive constant $C = C(N,s)$ such that

$$C\int_{\mathbb{R}^N} \frac{u(x)^2}{|x|^{2s}}\,dx \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy, \qquad \forall\, u \in C_0^\infty\left(\mathbb{R}^N\right). \tag{1.7}$$

1.3 The main results

In this subsection we make precise the concept of *eigenvalue* for the equations (1.2) and (1.3) and we present the main results of this paper.

Definition 1.2. We say that $\mu \in \mathbb{R}$ is an eigenvalue of problem (1.2), if there exists $u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N) \setminus \{0\}$ such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\, dxdy = \mu \int_{\mathbb{R}^N} V(x)u(x)\varphi(x)\, dx, \tag{1.8}$$

for all $\varphi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Furthermore, u from the above relation will be called an eigenfunction corresponding to the eigenvalue μ.

The main result concerning problem (1.2) is given by the following theorem

Theorem 1.3. *Assume that condition $(\widetilde{V})$ is fulfilled. Then problem (1.2) has an unbounded, increasing sequence of positive eigenvalues.*

Definition 1.4. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.3), if there exists $u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N) \setminus \{0\}$ such that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} &\frac{(1+|u(x)-u(y)|^{p-2})(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+tp}}\, dxdy \\ &= \lambda \int_{\mathbb{R}^N} V(x)u(x)\varphi(x)\, dx\,,\end{aligned} \tag{1.9}$$

for all $\varphi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Furthermore, u from the above relation will be called an eigenfunction corresponding to the eigenvalue λ.

Assume that $V : \mathbb{R}^N \to [0,\infty)$ is a function which satisfies condition $(\widetilde{\mathrm{V}})$ and define

$$\lambda_1 := \inf_{u \in C_0^\infty(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|_{s,2}^2}{\displaystyle\int_{\mathbb{R}^N} V(x)u^2\, dx}. \tag{1.10}$$

The main result regarding problem (1.3) is given by the following theorem.

Theorem 1.5. *Assume that $V : \mathbb{R}^N \to [0,\infty)$ is a function which satisfies condition $(\widetilde{V})$. Under assumption (1.4), the set of eigenvalues of problem (1.3) is the open interval (λ_1,∞). Moreover, the corresponding eigenfunctions can be chosen to be non-negative.*

Remark. A simple analysis of the proof of Theorem 1.3 shows that in the case when function V satisfies $V(x) \geq 0$, for all $x \in \mathbb{R}^N$, then λ_1 defined in relation (1.10) is the smallest eigenvalue of problem (1.2).

2 An auxiliary result

In this section we prove an auxiliary result which will play an important role in our subsequent analysis. More precisely, we prove the following lemma.

Lemma 2.1. *Assume that condition* $(\widetilde{\mathrm{V}})$ *holds true. Then the functional* $T : \mathcal{D}_0^{s,2}(\mathbb{R}^N) \to \mathbb{R}$,

$$T(u) := \int_{\mathbb{R}^N} V^+(x) u^2 \, dx$$

is weakly continuous.

Proof. First, we show that the mapping $\mathcal{D}_0^{s,2}(\mathbb{R}^N) \ni u \to \int_{\mathbb{R}^N} V_1(x) u^2 \, dx$ is weakly continuous.

Let $\{u_n\} \subset \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ be a sequence which converges weakly to $u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Using the fact that $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ is continuously embedded in $L^{2_s^*}(\mathbb{R}^N)$, we find that $\{u_n\}$ converges weakly to u in $L^{2_s^*}(\mathbb{R}^N) = L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$. We infer that $\{u_n^2\}$ converges weakly to u^2 in $L^{\frac{N}{N-2s}}(\mathbb{R}^N)$.

Define $W : L^{\frac{N}{N-2s}}(\mathbb{R}^N) \to \mathbb{R}$ by

$$W(\xi) := \int_{\mathbb{R}^N} V_1(x) \xi \, dx, \qquad \forall \xi \in L^{\frac{N}{N-2s}}(\mathbb{R}^N).$$

Clearly, W is linear. Since $V_1 \in L^{\frac{N}{2s}}(\mathbb{R}^N)$ by Hölder's inequality we deduce that W is also continuous. Using the above pieces of information we find that

$$\lim_{n \to \infty} W(u_n) = W(u),$$

meaning that the mapping $\mathcal{D}_0^{s,2}(\mathbb{R}^N) \ni u \to \int_{\mathbb{R}^N} V_1(x) u^2 \, dx$ is weakly continuous.

In order to finish the proof, we shall prove that the mapping $\mathcal{D}_0^{s,2}(\mathbb{R}^N) \ni u \to \int_{\mathbb{R}^N} V_2(x) u^2 \, dx$ is also weakly continuous. Again, let $\{u_n\} \subset \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ be a sequence which converges weakly to $u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Let $\epsilon > 0$ arbitrary but fixed.

By hypothesis $(\widetilde{\mathrm{V}})$ we deduce that there exists $R > 0$ such that

$$|x|^{2s} V_2(x) \leq \epsilon, \qquad \forall\, x \in \mathbb{R}^N \setminus B_R(0), \tag{2.1}$$

where $B_R(0)$ is the open ball centered at the origin of radius R.

Since $\{u_n\}$ converges weakly to u in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ we deduce that $\{u_n\}$ is bounded in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Thus,

$$d := C \max \left\{ \sup_n \|u_n\|_{s,2} \, , \|u\|_{s,2} \right\} < +\infty \, ,$$

where C is the constant given by relation (1.7).

Using relations (1.7) and (2.1) we find

$$\int_{\mathbb{R}^N \setminus B_R(0)} V_2(x) u_n^2 \, dx \leq \epsilon \int_{\mathbb{R}^N \setminus B_R(0)} \frac{u_n^2}{|x|^{2s}} \, dx \leq \frac{\epsilon}{C} \|u_n\|_{s,2}^2 \leq \epsilon d^2 \, . \tag{2.2}$$

Analogously,

$$\int_{\mathbb{R}^N \setminus B_R(0)} V_2(x) u^2 \, dx \leq \frac{\epsilon}{C} \|u\|_{s,2}^2 \leq \epsilon d^2 \, . \tag{2.3}$$

Recalling again hypothesis $(\widetilde{\mathrm{V}})$ and using a compactness argument we find that $\overline{B}_R(0)$ is covered by a finite number of closed balls $\overline{B}_{r_1}(x_1), \overline{B}_{r_2}(x_2), \ldots, \overline{B}_{r_k}(x_k)$ such that for each $j \in \{1, \ldots, k\}$ we have

$$|x - x_j|^{2s} V_2(x) \leq \epsilon, \qquad \forall\, x \in \overline{B}_{r_j}(x_j). \tag{2.4}$$

Next, we see that there exists $r > 0$ such that for each $j \in \{1, \ldots, k\}$ the following relation holds

$$|x - x_j|^{2s} V_2(x) \leq \frac{\epsilon}{k}, \qquad \forall\, x \in \overline{B}_r(x_j).$$

Again, by relation (1.7) we get

$$\int_\Omega V_2(x)u_n^2\,dx \le \epsilon d^2 \quad \text{and} \quad \int_\Omega V_2(x)u^2\,dx \le \epsilon d^2\,, \tag{2.5}$$

where $\Omega := \cup_{i=1}^k B_r(x_j)$. Finally, by relation (2.4) we infer that $V_2 \in L^\infty(\overline{B}_R(0)\setminus\Omega)$. Since $\overline{B}_R(0)\setminus\Omega$ is bounded we deduce that $V_2 \in L^{\frac{N}{2s}}(\overline{B}_R(0)\setminus\Omega)$. Repeating the same arguments used in the first part of the proof we get

$$\lim_{n\to\infty}\int_{B_R(0)\setminus\Omega} V_2(x)u_n^2\,dx = \int_{B_R(0)\setminus\Omega} V_2(x)u^2\,dx. \tag{2.6}$$

By (2.2), (2.3), (2.5) and (2.6) we deduce that the mapping $\mathcal{D}_0^{s,2}(\mathbb{R}^N)\ni u \to \int_{\mathbb{R}^N} V_2(x)u^2\,dx$ is weakly continuous. Thus, the proof of the lemma is complete. □

3 Proof of Theorem 1.3

The conclusion of Theorem 1.3 will follow from the results of Propositions 3.1 and 3.2 below.

First, we consider the following minimization problem

$$(P_1) \qquad \underset{u\in\mathcal{D}_0^{s,2}(\mathbb{R}^N)}{\text{minimize}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy, \text{ under restriction } \int_{\mathbb{R}^N} V(x)u^2\,dx = 1.$$

Proposition 3.1. *Under the hypothesis* $(\widetilde{\mathrm{V}})$, *problem* (P_1) *has a solution* $e_1 \ge 0$. *Moreover,* e_1 *is an eigenfunction of problem* (1.2) *having its corresponding eigenvalue*

$$\mu_1 := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e_1(x)-e_1(y)|^2}{|x-y|^{N+2s}}\,dxdy. \tag{3.1}$$

Proof. Let $\{u_n\}_n \subset \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ be a minimizing sequence of problem (P_1), i.e.,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dxdy = \inf_{w\in\mathcal{D}_0^{s,2}(\mathbb{R}^N)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,dxdy$$

and

$$\int_{\mathbb{R}^N} V(x)u_n^2\,dx = 1, \qquad \forall\, n\ge 1.$$

It follows that $\{u_n\}$ is bounded in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and consequently there exists $u\in\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ such that u_n converges weakly to u in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Since $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ is a Hilbert space by the weakly lower semicontinuity of the norm $\|\cdot\|_{s,2}$ we get

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy &\le \liminf_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dxdy\\ &= \inf_{w\in\mathcal{D}_0^{s,2}(\mathbb{R}^N)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,dxdy.\end{aligned}$$

On the other hand, using the fact that $V(x) = V^+(x) - V^-(x)$ we deduce that

$$\int_{\mathbb{R}^N} V^-(x)u_n^2\,dx = \int_{\mathbb{R}^N} V^+(x)u_n^2\,dx - 1, \qquad \forall\, n\ge 1.$$

Fatou's lemma and Lemma 2.1 yield

$$\int_{\mathbb{R}^N} V^-(x)u^2\,dx \le \liminf_{n\to\infty}\int_{\mathbb{R}^N} V^-(x)u_n^2\,dx = \int_{\mathbb{R}^N} V^+(x)u^2\,dx - 1,$$

or

$$1 \le \int_{\mathbb{R}^N} V(x)u^2\,dx. \tag{3.2}$$

Define

$$e_1 := \frac{u}{\left(\int_{\mathbb{R}^N} V(x)u^2\,dx\right)^{1/2}}.$$

It is easy to check that

$$\int_{\mathbb{R}^N} V(x)e_1^2\,dx = 1.$$

Furthermore, using relation (3.2) we get

$$\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|e_1(x)-e_1(y)|^2}{|x-y|^{N+2s}}\,dxdy &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|\frac{u(x)}{\left(\int_{\mathbb{R}^N} V(z)u^2\,dz\right)^{1/2}} - \frac{u(y)}{\left(\int_{\mathbb{R}^N} V(z)u^2\,dz\right)^{1/2}}\right|^2}{|x-y|^{N+2s}}\,dxdy \\
&= \frac{1}{\int_{\mathbb{R}^N} V(z)u^2\,dz}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&\le \inf_{w\in\mathcal{D}_0^{s,2}(\mathbb{R}^N)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,dxdy.
\end{aligned}$$

This shows that e_1 is a solution of problem (P_1). Moreover, it is easy to see that $|e_1|$ is also a solution of problem (P_1) and consequently we can assume that $e_1 \ge 0$. Next, for each $\varphi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ we define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(\epsilon) = \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e_1(x)-e_1(y)+\epsilon(\varphi(x)-\varphi(y))|^2}{|x-y|^{N+2s}}\,dxdy}{\displaystyle\int_{\mathbb{R}^N} V(x)\left(e_1(x)+\epsilon\varphi(x)\right)^2\,dx}.$$

Clearly, f is of class C^1 and $f(0) \le f(\epsilon)$, for all $\epsilon \in \mathbb{R}$. Hence, 0 is a minimum point of f and thus,

$$f'(0) = 0,$$

or

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(e_1(x)-e_1(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy\int_{\mathbb{R}^N} V(x)e_1(x)^2\,dx \\
&\qquad = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e_1(x)-e_1(y)|^2}{|x-y|^{N+2s}}\,dxdy\int_{\mathbb{R}^N} V(x)e_1(x)\varphi(x)\,dx.
\end{aligned}$$

Since $\varphi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ has been chosen arbitrarily we deduce that the above relation holds true for each $\varphi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Taking into account that $\int_{\mathbb{R}^N} V(x)e_1^2\,dx = 1$ it follows that μ_1 defined in (3.1) is an eigenvalue of problem (1.2) with the corresponding eigenfunction e_1. Thus, the proof is complete. $\square$

Next, in order to find other eigenvalues of problem (1.2) we solve the following minimization problems

$$(P_n)\quad \begin{aligned}&\underset{u\in\mathcal{D}_0^{s,2}(\mathbb{R}^N)}{\text{minimize}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy, \text{ under restrictions } \int_{\mathbb{R}^N}V(x)u^2\,dx=1 \quad\text{and}\\ &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(e_k(x)-e_k(y))(u(x)-u(y))}{|x-y|^{N+2s}}\,dxdy=0,\ \forall\, k\in\{1,\dots,n-1\},\end{aligned}$$

where e_k represents the solution of problem (P_k), for $k\in\{1,\dots,n-1\}$.

Proposition 3.2. *Assume that the hypothesis $(\widetilde{\mathrm{V}})$ is fulfilled. Then, for every $n\geq 2$ problem (P_n) has a solution e_n. Moreover, e_n is an eigenvector of problem (1.2) corresponding to the eigenvalue*

$$\mu_n := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e_n(x)-e_n(y)|^2}{|x-y|^{N+2s}}\,dxdy.$$

Furthermore, $\lim_{n\to\infty}\mu_n=\infty$.

Proof. The existence of e_n can be obtained in the same manner as in proof of Theorem 1.3, but replacing $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ with its closed subspace

$$X_n := \left\{u\in\mathcal{D}_0^{s,2}(\mathbb{R}^N):\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(e_k(x)-e_k(y))(u(x)-u(y))}{|x-y|^{N+2s}}\,dxdy=0, \text{ for } k\in\{1,\dots,n-1\}\right\}.$$

Next, following the lines of the proof of Theorem 1.3 we find the existence of $e_n\in X_n$ which verifies

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(e_n(x)-e_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy=\mu_n\int_{\mathbb{R}^N}V(x)e_n(x)\varphi(x)\,dx,\qquad\forall\,\varphi\in X_n, \tag{3.3}$$

where

$$\mu_n := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e_n(x)-e_n(y)|^2}{|x-y|^{N+2s}}\,dxdy$$

and

$$\int_{\mathbb{R}^N}V(x)e_n^2\,dx=1.$$

We note that for each $u\in X_n$ we have

$$\int_{\mathbb{R}^N}V(x)ue_k\,dx=0,\qquad\forall\,k\in\{1,\dots,n-1\}.$$

and

$$\int_{\mathbb{R}^N}V(x)e_je_k\,dx=\delta_{j,k},\qquad\forall\,j,k\in\{1,\dots,n-1\}.$$

Hence, for each $v\in\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N}V(x)\left[v-\sum_{j=1}^{n-1}\left(\int_{\mathbb{R}^N}V(x)ve_j\,dx\right)e_j\right]e_k\,dx=0,\qquad\forall\,k\in\{1,\dots,n-1\},$$

or

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(e_k(x)-e_k(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dxdy=0,\qquad\forall\,k\in\{1,\dots,n-1\},$$

where $\psi(x) := v(x) - \sum_{j=1}^{n-1} \left(\int_{\mathbb{R}^N} V(y) v e_j \, dy\right) e_j(x)$. This implies that $\psi \in X_n$.

Thus, for each $v \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ relation (3.3) holds true for $\varphi = \psi$. On the other hand,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(e_n(x) - e_n(y))(e_k(x) - e_k(y))}{|x-y|^{N+2s}} \, dxdy = \mu_k \int_{\mathbb{R}^N} V(x) e_n e_k \, dx = \mu_n \int_{\mathbb{R}^N} V(x) e_n e_k \, dx = 0,$$

for all $k \in \{1, \ldots, n-1\}$. The above pieces of information yield

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(e_n(x) - e_n(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dxdy = \mu_n \int_{\mathbb{R}^N} V(x) e_n(x) v(x) \, dx, \qquad \forall\, v \in \mathcal{D}_0^{s,2}(\mathbb{R}^N),$$

which implies that

$$\mu_n := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|e_n(x) - e_n(y)|^2}{|x-y|^{N+2s}} \, dxdy$$

is an eigenvalue of problem (1.2) with the corresponding eigenfunction e_n.

Next, we point out that by construction $\{e_n\}_n$ is an orthonormal sequence in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and $\{\mu_n\}_n$ is an increasing sequence of positive real numbers. We prove that $\lim_{n\to\infty} \mu_n = \infty$.

Indeed, let the sequence $f_n := \frac{e_n}{\sqrt{\mu_n}}$. Then $\{f_n\}_n$ is an orthonormal sequence in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and

$$\|f_n\|_{s,2}^2 = \frac{1}{\mu_n} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|e_n(x) - e_n(y)|^2}{|x-y|^{N+2s}} \, dxdy = 1, \quad \forall\, n.$$

Consequently, $\{f_n\}_n$ is bounded in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and, therefore, there exists $f \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ such that $\{f_n\}_n$ converges weakly to f in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$.

Let m be a positive integer. For each $n > m$ we have

$$\langle f_n, f_m \rangle_{s,2} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(f_n(x) - f_n(y))(f_m(x) - f_m(y))}{|x-y|^{N+2s}} \, dxdy = 0.$$

Passing to the limit as $n \to \infty$ we find that

$$\langle f, f_m \rangle_{s,2} = 0, \qquad \forall\, m.$$

Since the above relation holds for each positive integer m, we can pass to the limit as $m \to \infty$ and we find that $\|f\|_{s,2} = 0$. This means that $f = 0$ and thus, $\{f_n\}_n$ converges weakly to 0 in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Lemma 2.1 assures us that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V^+(x) f_n^2 \, dx = 0. \tag{3.4}$$

On the other hand, for each n we have

$$\frac{1}{\mu_n} = \frac{1}{\mu_n} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f_n(x) - f_n(y)|^2}{|x-y|^{N+2s}} \, dxdy = \int_{\mathbb{R}^N} V(x) f_n^2 \, dx \le \int_{\mathbb{R}^N} V^+(x) f_n^2 \, dx.$$

Combining the above estimate with relation (3.4) we find that $\lim_{n\to\infty} \mu_n = +\infty$.

The proof of Proposition 3.2 is complete. $\square$

4 Proof of Theorem 1.5

The proof of Theorem 1.5 will be a simple consequence of Propositions 4.1, 4.2, 4.3 and 4.8 stated below in this section.

We recall that through this section we will assume that $V(x) \geq 0$, for all $x \in \mathbb{R}^N$, and conditions (1.4) and $(\widetilde{\mathrm{V}})$ hold true. Simple computations show that condition (1.4) implies $p > 2$. For each $0 < t < s < 1$ and $p > 2$ we define

$$\nu_1 := \inf_{u \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}} \frac{\frac{1}{2}\|u\|_{s,2}^2 + \frac{1}{p}\|u\|_{t,p}^p}{\frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\,dx}. \tag{4.1}$$

Proposition 4.1. $\lambda_1 = \nu_1$.

Proof. First, it is clear that $\lambda_1 \leq \nu_1$. Next, for each $u \in C_0^\infty(\mathbb{R}^N)$ and each $\theta > 0$ we have

$$\nu_1 \leq \frac{\frac{1}{2}\|\theta u\|_{s,2}^2 + \frac{1}{p}\|\theta u\|_{t,p}^p}{\frac{1}{2}\int_{\mathbb{R}^N} V(x)(\theta u)^2\,dx} = \frac{\frac{1}{2}\|u\|_{s,2}^2 + \frac{\theta^{p-2}}{p}\|u\|_{t,p}^p}{\frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\,dx}. \tag{4.2}$$

Letting $\theta \to 0^+$ and passing to the infimum over $u \in C_0^\infty(\mathbb{R}^N)$ in the right hand-side of the above relation we deduce that $\nu_1 \leq \lambda_1$. The proof of this proposition is complete. □

Proposition 4.2. *For each $\lambda \in (-\infty, \lambda_1]$, problem (1.3) has no nontrivial solutions.*

Proof. First, note that if we assume that for some $\lambda \leq 0$ problem (1.3) has a nontrivial solution denoted by u, then testing in relation (1.9) with $\varphi = u$ we get a contradiction. Thus, for any $\lambda \in (-\infty, 0]$ problem (1.3) does not have nontrivial weak solutions.

Next, let $\lambda \in (0, \lambda_1)$. Assume by contradiction that there exists $u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N) \setminus \{0\}$ a weak solution of problem (1.3). Taking $\varphi = u$ in (1.9) and by the definition of λ_1 we get

$$\lambda \int_{\mathbb{R}^N} V(x)u(x)^2\,dx = \|u\|_{s,2}^2 + \|u\|_{t,p}^p \geq \lambda_1 \int_{\mathbb{R}^N} V(x)u(x)^2\,dx,$$

a contradiction. It follows that problem (1.3) does not posses nontrivial weak solutions for any parameter $\lambda \in (0, \lambda_1)$.

In order to complete the proof of the proposition, we shall show that λ_1 cannot be an eigenvalue of problem (1.3). Again, if we assume by contradiction that there exists $u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N) \setminus \{0\}$ such that (1.9) holds with $\lambda = \lambda_1$, then letting $\varphi = u$ in (1.9) and by the definition of λ_1 we get

$$\|u\|_{s,2}^2 + \|u\|_{t,p}^p = \lambda_1 \int_{\mathbb{R}^N} V(x)u(x)^2\,dx \leq \|u\|_{s,2}^2,$$

which is equivalent with $u \equiv 0$, a contradiction. Thus, for $\lambda = \lambda_1$ problem (1.3) does not have nontrivial solutions and thus, the proof of this proposition is now complete. □

Proposition 4.3. *For each $\lambda \in (\lambda_1, \infty)$ problem (1.3) has a nontrivial solution.*

In order to prove Proposition 4.3, for each $\lambda > \lambda_1$ we define the energy functional corresponding to problem (1.3) as $J_\lambda : \mathcal{D}_0^{s,2}(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}$ given by

$$J_\lambda(u) := \frac{1}{2} \|u\|_{s,2}^2 + \frac{1}{p} \|u\|_{t,p}^p - \frac{\lambda}{2} \int_{\mathbb{R}^N} V(x)u(x)^2 \, dx.$$

Using standard arguments one can deduce that $J_\lambda \in C^1(\mathcal{D}_0^{s,2}(\mathbb{R}^N), \mathbb{R})$ with the derivative given by

$$\begin{aligned} \langle J'_\lambda(u), w\rangle = & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+2s}} \, dxdy - \lambda \int_{\mathbb{R}^N} V(x)u(x)w(x) \, dx. \\ & + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+tp}} \, dxdy. \end{aligned}$$

We note that problem (1.3) possesses a nontrivial weak solution for a certain λ if and only if J_λ possesses a non-trivial critical point. Since we cannot establish the coercivity of J_λ on $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ we cannot apply the *Direct Method in the Calculus of Variations* in order to find critical points for this functional. For that reason we will study the functional J_λ on a subset of $\mathcal{D}^{s,2}(\mathbb{R}^N)$, the so-called *Nehari manifold* defined by

$$\begin{aligned} \mathcal{N}_\lambda &:= \left\{ u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N) \setminus \{0\} : \langle J'_\lambda(u), u\rangle = 0 \right\} \\ &= \left\{ u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N) \setminus \{0\} : \|u\|_{s,2}^2 + \|u\|_{t,p}^p = \lambda \int_{\mathbb{R}^N} V(x)u(x)^2 \, dx \right\}. \end{aligned}$$

Note that if $u \in \mathcal{N}_\lambda$ then

$$J_\lambda(u) = \left(\frac{1}{p} - \frac{1}{2}\right) \|u\|_{t,p}^p < 0 \tag{4.3}$$

and

$$\lambda \int_{\mathbb{R}^N} V(x)u(x)^2 \, dx > \|u\|_{s,2}^2 . \tag{4.4}$$

Lemma 4.4. $\mathcal{N}_\lambda \neq \emptyset$.

Proof. Since $\lambda > \lambda_1$, we infer that there exists $\varphi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N) \setminus \{0\}$ for which

$$\|\varphi\|_{s,2}^2 < \lambda \int_{\mathbb{R}^N} V(x)\varphi(x)^2 \, dx.$$

Then there exists $\theta > 0$ such that $\theta\varphi \in \mathcal{N}_\lambda$, i.e.

$$\theta^2 \|\varphi\|_{s,2}^2 + \theta^p \|\varphi\|_{t,p}^p = \lambda\theta^2 \int_{\mathbb{R}^N} V(x)\varphi(x)^2 \, dx,$$

which holds true with

$$\theta = \left(\frac{\lambda \int_{\mathbb{R}^N} V(x)\varphi(x)^2 \, dx - \|\varphi\|_{s,2}^2}{\|\varphi\|_{t,p}^p} \right)^{\frac{1}{p-2}},$$

which completes the proof. $\square$

Set

$$m_\lambda := \inf_{v\in\mathcal{N}_\lambda} J_\lambda(v).$$

Note that by (4.3) we know that $m_\lambda < 0$. We show that m_λ can be achieved on $\mathcal{N}_\lambda$.

Lemma 4.5. *Every minimizing sequence of functional J_λ on $\mathcal{N}_\lambda$ is bounded in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and $\mathcal{D}_0^{t,p}(\mathbb{R}^N)$.*

Proof. Let $\{u_n\}_n \subset \mathcal{N}_\lambda$ be a minimizing sequence J_λ on $\mathcal{N}_\lambda$. We prove that $\{\,\|u_n\|_{s,2}^2\,\}_n$ is a bounded sequence. Assume the contrary that $\|u_n\|_{s,2}^2 \to \infty$, as $n\to\infty$. Next, let $w_n := \frac{u_n}{\|u_n\|_{s,2}}$. Therefore $\|w_n\|_{s,2} = 1$ for each n, which means that $\{w_n\}_n$ is bounded in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Thus, there exists $w \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ such that w_n converges weakly to w in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$.

Since $u_n \in \mathcal{N}_\lambda$, for each n, by (4.4) we deduce that $\lambda \int_{\mathbb{R}^N} V(x)w_n^2\,dx > 1$. Passing to the limit as $n\to\infty$ and taking into account Lemma 2.1, we obtain that

$$\lambda \int_{\mathbb{R}^N} V(x)w^2\,dx \ge 1. \tag{4.5}$$

On the other hand, since $u_n \in \mathcal{N}_\ge$ and $p>2$, we get

$$\|w_n\|_{t,p}^p = \|u_n\|_{s,2}^{2-p}\left(\lambda\int_{\mathbb{R}^N} V(x)w_n^2\,dx - 1\right) \to 0, \quad \text{as } n\to\infty.$$

The above relation implies that w_n converges strongly to 0 in $\mathcal{D}_0^{t,p}(\mathbb{R}^N)$ and, consequently $w=0$, which represents a contradiction with (4.5). It follows that $\{\,\|u_n\|_{s,2}\,\}_n$ is bounded. Since $u_n \in \mathcal{N}_\lambda$, by relation (4.3) we deduce that

$$J_\lambda(u_n) = \left(\frac{1}{p}-\frac{1}{2}\right)\|u_n\|_{t,p}^p = \left(\frac{1}{2}-\frac{1}{p}\right)\left(\|u_n\|_{s,2}^2 - \int_{\mathbb{R}^N} V(x)u_n^2\,dx\right).$$

Since $\{\,\|u_n\|_{s,2}\,\}_n$ is a bounded sequence and using the weak continuity of the mapping $\mathcal{D}_0^{s,2}(\mathbb{R}^N) \ni u \to \int_{\mathbb{R}^N} V(x)u^2\,dx$ given by Lemma 2.1, we deduce that $\{\,\|u_n\|_{t,p}\,\}_n$ is also a bounded sequence, and thus, the proof is complete. □

Lemma 4.6. $m_\lambda \in (-\infty, 0)$.

Proof. We already know that $m_\lambda < 0$. Let $\{u_n\}_n \subset \mathcal{N}_\lambda$ be a minimizing sequence J_λ on $\mathcal{N}_\lambda$ (in other words, $\{u_n\}_n$ is a minimizer of m_λ). Using the previous lemma we deduce the existence of a positive constant C such that $\|u_n\|_{s,2}^2 \le C$ and $\|u_n\|_{t,p}^p \le C$, for each positive integer n. Since $p>2$ we have

$$J_\lambda(u_n) = \left(\frac{1}{p}-\frac{1}{2}\right)\|u_n\|_{t,p}^p \ge \left(\frac{1}{p}-\frac{1}{2}\right)C > -\infty.$$

Thus, m_λ is bounded from below by the constant $(\frac{1}{p}-\frac{1}{2})C$, which implies that $m_\lambda \in (-\infty, 0)$. This completes the proof of this lemma. □

Lemma 4.7. *There exists $u \in \mathcal{N}_\lambda$ such that $J_\lambda(u) = m_\lambda$.*

Proof. Let $\{u_n\}_n \subset \mathcal{N}_\lambda$ be a minimizing sequence for J_λ on $\mathcal{N}_\lambda$, i.e.

$$J_\lambda(u_n) = \left(\frac{1}{p} - \frac{1}{2}\right) \|u_n\|_{t,p}^p \to m_\lambda \quad \text{as} \quad n \to \infty.$$

By Lemma 4.5, we have that $\mathcal{N}_\lambda$ is bounded in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and $\mathcal{D}_0^{t,p}(\mathbb{R}^N)$. We deduce that there exists a function $u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ such that u_n converges weakly to u in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and also in $\mathcal{D}_0^{t,q}(\mathbb{R}^N)$. Then

$$\|u\|_{s,2}^2 \leq \liminf_{n\to\infty} \|u_n\|_{s,2}^2 .$$

By Lemma 1 we deduce that

$$\lambda \int_{\mathbb{R}^N} V(x) u_n(x)^2 \, dx \to \lambda \int_{\mathbb{R}^N} V(x) u(x)^2 \, dx \quad \text{as } n \to \infty.$$

Using the above pieces of information we obtain

$$\begin{aligned} J_\lambda(u) &= \left(\frac{1}{2} - \frac{1}{p}\right) \left(\|u\|_{s,2}^2 - \lambda \int_{\mathbb{R}^N} V(x) u^2 \, dx \right) \\ &\leq \left(\frac{1}{2} - \frac{1}{p}\right) \liminf_{n\to\infty} \left(\|u_n\|_{s,2}^2 - \lambda \int_{\mathbb{R}^N} V(x) u_n^2 \, dx \right) \\ &= \liminf_{n\to\infty} J_\lambda(u_n) = m_\lambda < 0. \end{aligned} \tag{4.6}$$

By the above calculus we deduce that

$$\|u\|_{s,2}^2 < \lambda \int_{\mathbb{R}^N} V(x) u^2 \, dx,$$

which implies that certainly $u \not\equiv 0$. Since $u_n \in \mathcal{N}_\lambda$ for every n, we have

$$\|u_n\|_{s,2}^2 + \|u_n\|_{t,p}^p = \lambda \int_{\mathbb{R}^N} V(x) u_n^2 \, dx.$$

Passing to the limit as $n \to \infty$ in the above relation and by weakly convergence of u_n to u in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and $\mathcal{D}_0^{t,p}(\mathbb{R}^N)$ and also by Lemma 2.1, we get

$$\|u\|_{s,2}^2 + \|u\|_{t,p}^p \leq \lambda \int_{\mathbb{R}^N} V(x) u(x)^2 \, dx. \tag{4.7}$$

In order to finish the proof, we show that the above relation is actually an equality. Assume by contradiction that the inequality in (4.7) is strict, i.e.

$$\|u\|_{s,2}^2 + \|u\|_{t,p}^p < \lambda \int_{\mathbb{R}^N} V(x) u^2 \, dx. \tag{4.8}$$

Set

$$\theta := \left(\frac{\lambda \int_{\mathbb{R}^N} V(x) u^2 \, dx - \|u\|_{s,2}^2}{\|u\|_{t,p}^p} \right)^{\frac{1}{p-2}} .$$

Since $u \in \mathcal{N}_\lambda$ we have that $\theta u \in \mathcal{N}_\lambda$. By (4.8) it is clear that $\theta > 1$. Since $p > 2$ we deduce that

$$\begin{aligned} J_\lambda(\theta u) &= \left(\frac{1}{2} - \frac{1}{p}\right) \theta^2 \left(\|u\|_{s,2}^2 - \lambda \int_{\mathbb{R}^N} V(x) u^2 \, dx \right) \\ &< \left(\frac{1}{2} - \frac{1}{p}\right) \left(\|u\|_{s,2}^2 - \lambda \int_{\mathbb{R}^N} V(x) u^2 \, dx \right) = J_\lambda(u) \\ &\leq \liminf_{n\to\infty} J_\lambda(u_n) = m_\lambda, \end{aligned}$$

a contradiction. Thus, relation (4.8) cannot hold true. Therefore, relation (4.7) holds as an equality which implies that $u \in \mathcal{N}_\lambda$. By relation (4.6) we know that $J_\lambda(u) \leq m_\lambda$, and thus $J_\lambda(u) = m_\lambda$. Thus, the proof is complete. □

We are now ready to complete the proof of Proposition 4.1. Let u_λ be the minimizer of J_λ over $\mathcal{N}_\lambda$ given by Lemma 4.7, i.e.

$$J_\lambda(u_\lambda) = m_\lambda.$$

Since $u_\lambda \in \mathcal{N}_\lambda$, we have

$$\|u_\lambda\|_{s,2}^2 + \|u_\lambda\|_{t,p}^p = \lambda \int_{\mathbb{R}^N} V(x) u_\lambda^2 \, dx$$

and

$$\|u_\lambda\|_{s,2}^2 < \lambda \int_{\mathbb{R}^N} V(x) u_\lambda^2 \, dx.$$

We consider $\varphi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ is arbitrary but fixed, and $\delta > 0$ is sufficiently small such that for each $\epsilon \in (-\delta, \delta)$ the function $u_\lambda + \epsilon\varphi \not\equiv 0$ in $\mathbb{R}^N$ and

$$\|u_\lambda + \epsilon\varphi\|_{s,2}^2 < \lambda \int_{\mathbb{R}^N} V(x) |u_\lambda + \epsilon\varphi|^2 \, dx.$$

Define $\theta : (-\delta, \delta) \to (0, \infty)$ as

$$\theta(\epsilon) := \left(\frac{\lambda \displaystyle\int_{\mathbb{R}^N} V(x)|u_\lambda + \epsilon\varphi|^2 \, dx - \|u_\lambda + \epsilon\varphi\|_{s,2}^2}{\|u_\lambda + \epsilon\varphi\|_{t,p}^p} \right)^{\frac{1}{p-2}}.$$

We observe that $\theta(\epsilon)(u_\lambda + \epsilon\varphi) \in \mathcal{N}_\lambda$ and θ is a differentiable as a composition of some differentiable functions. Since $u_\lambda \in \mathcal{N}_\lambda$ we infer that $\theta(0) = 1$. Next, let $\gamma : (-\delta, \delta) \to \mathbb{R}$ be given by $\gamma(\epsilon) := J_\lambda(\theta(\epsilon)(u_\lambda + \epsilon\varphi))$. Clearly, $\gamma \in C^1(-\delta, \delta)$ and $m_\lambda = \gamma(0) \leq \gamma(\epsilon)$, for each $\epsilon \in (-\delta, \delta)$. Thus, we have

$$\begin{aligned} 0 = \gamma'(0) &= \langle J'(\theta(0)u_\lambda), \theta'(0)u_\lambda + \theta(0)\varphi \rangle \\ &= \theta'(0)\langle J'(u_\lambda), u_\lambda \rangle + \langle J'(u_\lambda), \varphi \rangle \\ &= \langle J'(u_\lambda), \varphi \rangle, \end{aligned}$$

where the last equality holds because $u_\lambda \in \mathcal{N}_\lambda$.

Since $\varphi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ was arbitrarily chosen we deduce that the last relation holds true for each $\varphi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and thus, u_λ is a nontrivial critical point of J_λ, and consequently a nontrivial weak solution of equation (1.3). The proof of Proposition 4.1 is now complete.

Proposition 4.8. *If $u \in \mathcal{N}_\lambda$ is the minimizer of J_λ over $\mathcal{N}_\lambda$, given by Lemma 4.7, then $|u|$ is also a minimizer of I_λ over $\mathcal{N}_\lambda$.*

Proof. For each $\xi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and for any $x, y \in \mathbb{R}^N$ we have

$$|\xi(y) - \xi(x)| \geq |\, |\xi(y)| - |\xi(y)| \,|,$$

and

$$|\xi(y) - \xi(x)| > |\, |\xi(y)| - |\xi(y)| \,|, \quad \text{if } \xi(x)\xi(y) < 0.$$

Using this, it follows that

$$\|\, |\xi| \,\|_{s,2}^2 \leq \|\xi\|_{s,2}^2 \quad \text{and} \quad \|\, |\xi| \,\|_{t,p}^p \leq \|\xi\|_{t,p}^p \qquad \forall \xi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N).$$

By the above relation we deduce that

$$J_\lambda(|u|) \leq J_\lambda(u). \tag{4.9}$$

In what follows we will prove that $J_\lambda(|u|) \geq J_\lambda(u)$. We distinguish two cases. First, if $|u| \in \mathcal{N}_\lambda$ then taking into account that $p > 2$ we get

$$J_\lambda(|u|) = \left(\frac{1}{p} - \frac{1}{2}\right) \| |u| \|_{t,p}^p \geq \left(\frac{1}{p} - \frac{1}{2}\right) \|u\|_{t,p}^p = J_\lambda(u).$$

The above estimate and relation (4.9) yield $J_\lambda(|u|) = J_\lambda(u) = m_\lambda$ and everything is done.

Next, let us assume that $|u| \notin \mathcal{N}_\lambda$. Then

$$\| |u| \|_{s,2}^2 + \| |u| \|_{t,p}^p < \lambda \int_{\mathbb{R}^N} V(x) u^2 \, dx.$$

Set

$$\theta := \left(\frac{\lambda \int_{\mathbb{R}^N} V(x) u^2 \, dx - \| |u| \|_{s,2}^2}{\| |u| \|_{t,p}^p} \right)^{\frac{1}{p-2}}.$$

Since $p > 2$ we have that $\theta \in (1, \infty)$ and also $\theta|u| \in \mathcal{N}_\lambda$. We have that

$$\begin{aligned}
m_\lambda \leq J_\lambda(\theta|u|) &= \left(\frac{1}{p} - \frac{1}{2}\right) \| |u| \|_{t,p}^p \, \theta^p \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) \left(\| |u| \|_{s,2}^2 - \lambda \int_{\mathbb{R}^N} V(x) u^2 \, dx \right) \theta^2 \\
&< \left(\frac{1}{2} - \frac{1}{p}\right) \left(\| |u| \|_{s,2}^2 - \lambda \int_{\mathbb{R}^N} V(x) u^2 \, dx \right) \\
&\leq \left(\frac{1}{2} - \frac{1}{p}\right) \left(\|u\|_{s,2}^2 - \lambda \int_{\mathbb{R}^N} V(x) u^2 \, dx \right) = J_\lambda(u) = m_\lambda,
\end{aligned}$$

which is a contradiction. Thus, $|u| \in \mathcal{N}_\lambda$. It follows that $|u|$ is also a minimizer of J_λ over $\mathcal{N}_\lambda$. □

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Existence of infinitely many radial nodal solutions for a Dirichlet problem involving mean curvature operator in Minkowski space

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Abstract. In this paper, we show the existence of infinitely many radial nodal solutions for the following Dirichlet problem involving mean curvature operator in Minkowski space

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla y}{\sqrt{1-|\nabla y|^2}}\right) = \lambda h(y) + g(|x|, y) \quad \text{in } B, \\ y = 0 \quad \text{on } \partial B, \end{cases}$$

where $B = \{x \in \mathbb{R}^N : |x| < 1\}$ is the unit ball in $\mathbb{R}^N$, $N \geq 1$, $\lambda \geq 0$ is a parameter, $h \in C(\mathbb{R})$ and $g \in C(\mathbb{R}^+ \times \mathbb{R})$. By bifurcation and topological methods, we prove the problem possesses infinitely many component of radial solutions branching off at $\lambda = 0$ from the trivial solution, each component being characterized by nodal properties.

Keywords: infinitely many radial solutions, mean curvature operator, Minkowski space, topological method, bifurcation.

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1 Introduction

The purpose of this paper is to deal with radial nodal solutions for the following 0-Dirichlet problem with mean curvature operator in the Minkowski space

$$\begin{aligned} -\mathrm{div}\left(\frac{\nabla y}{\sqrt{1-|\nabla y|^2}}\right) &= \lambda h(y) + g(|x|, y) \quad \text{in } B, \\ y &= 0 \quad \text{on } \partial B, \end{aligned} \tag{1.1}$$

where $B = \{x \in \mathbb{R}^N : |x| < 1\}$ is the unit ball in $\mathbb{R}^N$, $N \geq 1$, $\lambda \geq 0$ is a parameter, $h(y) \simeq |y|^{q-2}y$, $1 < q < 2$ near $y = 0$ and g is of higher order with respect to h at $y = 0$. This kind of problems are originated from differential geometry or classical relativity.

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For example, let

$$\mathbb{L}^{N+1} := \{(x,t) : x \in \mathbb{R}^N, t \in \mathbb{R}\}$$

be the flat Minkowski space, endowed with the Lorentzian metric

$$\sum_{j=1}^{N} dx_j^2 - dt^2.$$

It is known (see [4,28]) that the study of spacelike submanifolds of codimension one in $\mathbb{L}^{N+1}$ with prescribed mean extrinsic curvature leads to Dirichlet problems of the type

$$\begin{aligned} -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) &= H(x,u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and the nonlinearity $H : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous.

There are a large amount of papers in the literature on the existence, multiplicity and qualitative properties of solutions for this type of problems, see [1–3,7,11,12,14,16,25,26,31]. It is worth pointing out that the starting point of this type of problems is the seminal paper [9] which prove the Bernstein's property for entire solutions of the maximal (i.e., zero mean curvature) hypersurface equation. Bartnik and Simon [4] proved the existence of one strictly spacelike solution when $\lambda = 1$ and H is bounded, this always can be seen as an important universal existence result of (1.2). For the case $N = 1$, the existence and multiplicity of positive solutions of the Dirichlet problem for the quasilinear ordinary differential equation

$$\begin{aligned} -\left(\frac{u'}{\sqrt{1-u'^2}}\right)' &= H(x,u), \qquad x \in (0,1), \\ u(0) = u(1) &= 0 \end{aligned}$$

have been extensively studied by Coelho et al. [10] via variational or topological methods. For the special case Ω is a ball, by using upper and lower solutions, Leray–Schauder degree arguments and critical point theory for convex, lower semicontinuous perturbations of C^1-functionals, Bereanu, Jebelean, and Torres [5,6] obtained some nonexistence, existence and multiplicity results of classical positive radial solutions of (1.2). Ma, Gao and Lu [24] concerned with the global structure of radial positive solutions of (1.2) by using global bifurcation techniques, and extended the results of [5,6] to more general cases, all results, depending on the behavior of nonlinear term H near 0. Later, Ma and Xu [27] studied the global behavior of positive solutions of (1.2) with Ω is a general domain in $\mathbb{R}^N$.

However, few results on the existence of radial nodal solutions [15], even positive solutions, have been established for problem with mean curvature operator on general domain. In this paper, we will show an existence result of infinitely many radial nodal solutions for Dirichlet problem (1.1) by bifurcation and topological methods. For the applications of nodal solutions, see Kurth [20] and Lazer and McKenna [21].

Our study is motivated by some recent works on one-dimensional prescribed mean curvature problems with concave-convex nonlinearities, see [19,34].

Setting, as usual $|x| = r$ and $y(x) = u(r)$, the problem (1.1) reduces to the mixed boundary value problem

$$\begin{aligned} Au &= \lambda h(u) + g(r,u), \qquad r \in (0,1), \\ u'(0) &= u(1) = 0, \end{aligned} \tag{1.3}$$

where

$$Au = -\frac{1}{r^{N-1}}(r^{N-1}\phi_1(u'))', \tag{1.4}$$

and

$$\phi_1(s) = \frac{s}{\sqrt{1-s^2}}, \qquad s \in \mathbb{R},$$

note that $\phi_1 : (-1,1) \to \mathbb{R}$ is an odd, increasing homeomorphism and $\phi_1(0) = 0$. Throughout we assume $\lambda \geq 0$, $h \in C(\mathbb{R})$, $g \in C(\mathbb{R}^+ \times \mathbb{R})$ and satisfy the following conditions:

(A1) $h \in C(\mathbb{R}, \mathbb{R})$ with $sh(s) > 0$ for $s \neq 0$, $\lim\limits_{s\to 0} \frac{h(s)}{s} = \infty$;

(A2) $\lim\limits_{s\to 0} \frac{g(r,s)}{s} = 0$ uniformly for $r \in [0,1]$.

Let $X = \{u \in C^1[0,1] : u'(0) = u(1) = 0\}$ with the norm $\|u\| := \|u'\|_\infty$, and let $E = \mathbb{R} \times X$. In the sequel by a solution of (1.1) we mean a pair $(\lambda, u) \in E$, such that $u \in C^1[0,1]$, $\max_{r\in[0,1]} |u'(r)| < 1$, $r^{N-1}\phi_1(u') \in C^1[0,1]$, and satisfies (1.1). These are strong strictly space-like solutions of (1.1) according to the terminology of [4,9,18,31].

The main result of this paper is the following.

Theorem 1.1. *Let (A1) and (A2) hold. Then the point $(\lambda, u) = (0,0)$ is a bifurcation point for problem (1.1). More precisely, there are infinitely many unbounded component (i.e., closed connected sets) $\Gamma_k \subset E$ of solutions of (1.1) branching off from $(0,0)$, such that*

(i) If $(\lambda, u) \in \Gamma_k$ and $\lambda > 0$, then $u \not\equiv 0$.

(ii) If $(\lambda, u) \in \Gamma_k$, then u has exactly $k-1$ simple zeros in the interval $(0,1)$.

(iii) There exists a constant $\rho_0 \in (0, 1/2)$ such that if $\rho \in (0, \rho_0]$, and $(\lambda, u) \in \Gamma_k$ with $\|u\| = \rho$, then $\lambda > \lambda(\rho) > 0$.

As an immediate consequence we get:

Corollary 1.2. *There exists $\lambda_* > 0$ such that problem (1.1) has infinitely many radial nodal solutions for any $\lambda \in (0, \lambda_*)$.*

Remark 1.3. It is easy to find that (A2) yields that

$$g(r,0) = 0 \quad \text{uniformly for } r \in [0,1].$$

Otherwise, from the continuity of g, we get $\lim_{s\to 0} \frac{g(r,s)}{s} = \infty$ for some $r \in [0,1]$, this is a contradiction.

Remark 1.4. Let (λ, u) be a solution of (1.3), then it follows from $|u'(r)| < 1$ that

$$\|u\|_\infty < 1.$$

This leads to the bifurcation diagrams mainly depend on the behavior of $h = h(s)$ and $g = g(r,s)$ near $s = 0$. This is a significant difference between the Minkowski-curvature problems and the p-Laplacian problems.

Remark 1.5. If $g(r,s) \equiv 0$ for all $r \in [0,1]$, then

$$\lim_{s\to 0} \frac{g(r,s)}{s} = 0 \quad \text{uniformly for } r \in [0,1].$$

Clearly, Theorem 1.1 improves some well-known existence results of positive solutions [5] and radial nodal solutions [15] for related problems.

The rest of the paper is arranged as follows. In Section 2, we show the property of the superior limit of a sequence of components and obtain a topological degree jumping result. Finally in Section 3, we prove our main result and give an example to illustrate our main result.

2 Some preliminary results

2.1 Superior limit and component

The following results are somewhat scattered in Ma and An [22,23].

Definition 2.1 ([22,23])**.** Let X be a Banach space and $\{C_n : n = 1,2,\dots\}$ be a family of subsets of X. Then the the *superior limit* $\mathcal{D}$ of $\{C_n\}$ is defined by

$$\mathcal{D} := \limsup_{n\to\infty} C_n = \{x \in X : \text{ there exist } \{n_i\} \subset \mathbb{N} \text{ and } x_{n_i} \in C_{n_i} \text{ such that } x_{n_i} \to x\}.$$

Definition 2.2 ([22,23])**.** A component of a set M means a maximal connected subset of M.

Lemma 2.3 ([22, Lemma 2.4], [23, Lemma 2.2])**.** *Assume that*

(i) there exist $z_n \in C_n$, $n = 1,2,\dots$, and $z^* \in X$, such that $z_n \to z^*$;

(ii) $\lim_{n\to\infty} r_n = \infty$, where $r_n = \sup\{\|x\| : x \in C_n\}$;

(iii) for every $R > 0$, $\left(\cup_{n=1}^{\infty} C_n\right) \cap B_R$ is a relative compact set of X, where

$$B_R = \{x \in X : \|x\| \le R\}.$$

Then there exists an unbounded component $\mathcal{C}$ in $\mathcal{D}$ with $z^* \in \mathcal{C}$.

2.2 Topological degree jumping result

Let us introduce the eigenvalue problem

$$\begin{aligned} -(r^{N-1}u')' &= \lambda r^{N-1}u, \qquad r \in (0,1), \\ u'(0) &= u(1) = 0. \end{aligned} \tag{2.1}$$

From [29] with $p = 2$ or [32, p. 269], we have the following result.

Lemma 2.4. *Problem* (2.1) *has infinitely many simple real eigenvalues, which can be arranged in the increasing order*

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to +\infty \quad \text{as } k \to +\infty,$$

and no other eigenvalues. Moreover, the algebraic multiplicity of λ_k is 1, and the eigenfunction φ_k has exactly $k-1$ simple zeros in $(0,1)$.

For any $t \in (0,1]$, we consider the following auxiliary problem

$$\begin{aligned} -\frac{1}{r^{N-1}}\left(r^{N-1}\frac{u'}{\sqrt{1-tu'^2}}\right)' &= f(r), \qquad r \in (0,1), \\ u'(0) = u(1) &= 0 \end{aligned} \tag{2.2}$$

for a given $f \in C[0,1]$. Letting $v = \sqrt{t}u$, problem (2.2) is equivalent to

$$\begin{aligned} -\frac{1}{r^{N-1}}\left(r^{N-1}\frac{v'}{\sqrt{1-v'^2}}\right)' &= \sqrt{t}f(r), \qquad r \in (0,1), \\ v'(0) = v(1) &= 0. \end{aligned} \tag{2.3}$$

By Theorem 3.6 of [4], we know that there exists a unique strictly spacelike solution $v \in C^1[0,1]$ to problem (2.3) which is denoted by $\psi(\sqrt{t}f)$. So $u = \frac{v}{\sqrt{t}}$ is the unique solution of problem (2.2).

For a given $b \in C[0,1]$, we also consider the following auxiliary problem

$$\begin{aligned} -\frac{1}{r^{N-1}}\left(r^{N-1}u'\right)' &= b(r), \qquad r \in (0,1), \\ u'(0) = u(1) &= 0. \end{aligned} \tag{2.4}$$

It is well known that problem (2.4) has a solution u for every given $b \in C[0,1]$. Let $\phi(b)$ denote the unique solution to problem (2.4). It is easy to check that $\phi : C[0,1] \to X$ is linear and completely continuous.

Therefore, for any given $f \in C[0,1]$, let us define $G : [0,1] \times C[0,1] \to X$ by

$$G(t,f) = \begin{cases} \frac{\psi(\sqrt{t}f)}{\sqrt{t}}, & t \in (0,1], \\ \phi(f), & t = 0. \end{cases} \tag{2.5}$$

From the Lemma 2.3 of [14], we have G is completely continuous.

For any fixed λ, consider the following problem

$$\begin{aligned} -\frac{1}{r^{N-1}}\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)' &= \lambda u, \qquad r \in (0,1), \\ u'(0) = u(1) &= 0. \end{aligned} \tag{2.6}$$

Clearly, problem (2.6) is equivalent to the operator equation

$$u = \psi(\lambda u) := \psi_\lambda(u).$$

From Lemma 2.3 of [14], we see that $\psi_\lambda : X \to X$ is completely continuous. And we can also obtain the following topological degree jumping result.

Lemma 2.5. *For any $r > 0$, we have that*

$$\deg(I - \psi_\lambda, B_r(0), 0) = \begin{cases} 1, & \text{if } \lambda \in (0, \lambda_1), \\ (-1)^k, & \text{if } \lambda \in (\lambda_k, \lambda_{k+1}),\ k \in \mathbb{N}. \end{cases}$$

Proof. It is not difficult to show that $I-\psi_\lambda$ is a nonlinear compact perturbation of the identity. Thus, the Leray–Schauder degree $\deg(I-\psi_\lambda, B_r(0), 0)$ is well defined for arbitrary r-ball $B_r(0)$ and $\lambda \neq \lambda_k$. From the invariance of the degree under homotopies we obtain that

$$\begin{aligned}\deg(I-\psi_\lambda, B_r(0), 0) &= \deg(I-G(1,\lambda\cdot), B_r(0), 0)\\ &= \deg(I-G(0,\lambda\cdot), B_r(0), 0)\\ &= \deg(I-\lambda\phi, B_r(0), 0).\end{aligned}$$

Since ϕ is compact and linear, by [13, Lemma 3.1] or [17, Theorem 8.10], we have that

$$\deg(I-\lambda\phi, B_r(0), 0) = \begin{cases}1 & \text{if } \lambda \in (0,\lambda_1),\\ (-1)^k & \text{if } \lambda \in (\lambda_k, \lambda_{k+1}),\ k \in \mathbb{N},\end{cases}$$

and accordingly,

$$\deg(I-\psi_\lambda, B_r(0), 0) = \begin{cases}1 & \text{if } \lambda \in (0,\lambda_1),\\ (-1)^k & \text{if } \lambda \in (\lambda_k, \lambda_{k+1}),\ k \in \mathbb{N}.\end{cases}$$

□

3 Proof of the main result

Before proving the Theorem 1.1, we state the following lemmas.

Lemma 3.1. *Assume that (A1) and (A2). Let (λ, u) be a solution of problem (1.3). If u has a double zero, then $u \equiv 0$.*

Proof. Assume on the contrary that there exists a solution (λ, u), $\lambda > 0$, of (1.3) and u has a double zero. Let $\tau \in [0,1]$ be a double zero of u. Integrating the equation of (1.3) over $[\tau, r]$, we have

$$\frac{u'(r)}{\sqrt{1-(u'(r))^2}} = -\frac{1}{r^{N-1}}\int_\tau^r s^{N-1}(\lambda h(u(s)) + g(s,u(s)))ds.$$

If $\tau = 0$, then for $r \in [0,1]$, from (A1) and the fact

$$|u'(r)| < 1,$$

it follows that

$$|u'(r)| \le \frac{1}{r^{N-1}}\int_0^r s^{N-1}|g(s,u)|ds \le \frac{r}{N}|g(s,u)|.$$

Recalling (A2), there exists a constant $M > 0$ such that $|g(s,u)| \le M|u|$ for any $s \in [0,1]$ and $u \in [-1,1]$. Using the boundary conditions $u'(0) = u(1) = 0$, we get

$$|u'(r)| \le \frac{Mr}{N}|u| \le \frac{Mr}{N}\int_1^r |u'(s)|ds.$$

By the Gronwall–Bellman inequality [8], we obtain $u'(r) \equiv 0$ on $[0,1]$. Therefore, $u(r) \equiv 0$ on $[0,1]$.

If $\tau > 0$, we first assume that $r \in [0,\tau]$. Since

$$u(r) = -\int_\tau^r \phi_1^{-1}\left(\frac{1}{t^{N-1}}\int_\tau^t s^{N-1}(\lambda h(u(s)) + g(s,u(s)))ds\right)dt$$

for all $r \in [0,\tau]$, where ϕ_1^{-1} is the inverse function of ϕ_1, namely

$$\phi_1^{-1}(s) = \frac{s}{\sqrt{1+s^2}}, \qquad s \in \mathbb{R}.$$

It is easy to check that ϕ_1^{-1} is increasing. Hence, by (A1), we have

$$\begin{aligned} u(r) &= \int_r^\tau \phi_1^{-1}\left(\frac{1}{t^{N-1}}\int_\tau^t s^{N-1}\left(\lambda\, h(u(s)) + g(s,u(s))\right) ds\right) dt \\ &= \int_r^\tau \phi_1^{-1}\left(\frac{1}{t^{N-1}}\int_t^\tau s^{N-1}\left(-\lambda\, h(u(s)) - g(s,u(s))\right) ds\right) dt \\ &\le \int_r^\tau \phi_1^{-1}\left(\frac{1}{t^{N-1}}\int_\tau^t s^{N-1} g(s,u(s)) ds\right) dt \\ &= \int_r^\tau \frac{\frac{1}{t^{N-1}}\int_\tau^t s^{N-1} g(s,u(s)) ds}{\sqrt{1+\left(\frac{1}{t^{N-1}}\int_\tau^t s^{N-1} g(s,u(s)) ds\right)^2}} dt, \end{aligned}$$

since $0 \le \frac{r}{t} \le 1$ and $N \ge 1$, this implies

$$r^{N-1}|u(r)| \le \int_r^\tau \int_\tau^t s^{N-1}|g(s,u(s))| ds dt \le M\int_r^\tau s^{N-1}|u(s)| ds.$$

By Gronwall–Bellman inequality, we have $r^{N-1}|u(r)| \equiv 0$ on $[0,\tau]$. And accordingly, $u(r) \equiv 0$ on $(0,\tau]$. This fact together with the continuity of u, we conclude that $u(r) \equiv 0$ on $[0,\tau]$.

Similarly, if $\tau > 0$ and $r \in [\tau, 1]$, then by Gronwall–Bellman inequality again, we can get $u(r) \equiv 0$ on $[\tau, 1]$ and the proof is completed. □

Lemma 3.2. *There exists $\rho_0 > 0$ such that any nontrivial solution u of*

$$\begin{aligned} Au &= g(r,u), \qquad r \in (0,1), \\ u'(0) &= u(1) = 0 \end{aligned} \tag{3.1}$$

satisfies $\|u\| > \rho_0$.

Proof. Assume, by contradiction, that there is a sequence $\{u_n\}$ of solutions of (3.1) and such that $u_n \neq 0$ and $\|u_n\| \to 0$. For all $n \in \mathbb{N}$, let $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = \|v_n'\|_\infty = 1$, consequently, $\|v_n\|_\infty$ is bounded. By the Ascoli–Arzelà theorem, there exists a subsequence of $\{v_n\}$ which uniformly converges to $v \in C[0,1]$. We again denote the subsequence by $\{v_n\}$. For any u_n, we have

$$\begin{aligned} -\frac{1}{r^{N-1}}\left(r^{N-1}\frac{u_n'}{\sqrt{1-u_n'^2}}\right)' &= g(r,u_n), \qquad r \in (0,1), \\ u_n'(0) = u_n(1) &= 0. \end{aligned} \tag{3.2}$$

Multiplying both sides of (3.2) by $\|u_n\|^{-1}$, we have

$$\begin{aligned} -\frac{1}{r^{N-1}}\left(r^{N-1}\frac{v_n'}{\sqrt{1-u_n'^2}}\right)' &= \frac{g(r,u_n)}{u_n}v_n, \qquad r \in (0,1), \\ v_n'(0) = v_n(1) &= 0. \end{aligned}$$

Since $\|u_n\| \to 0$ implies $\|u_n\|_\infty \to 0$. From (A2) and Lebesgue's dominated convergence theorem, we conclude that

$$-\frac{1}{r^{N-1}}\left(r^{N-1}v'\right)' = 0, \qquad r \in (0,1),$$
$$v'(0) = v(1) = 0,$$

which means that $v \equiv 0$ contradicting with $\|v\| = 1$. □

Proof of Theorem 1.1. Theorem 1.1 cannot be proved using standard bifurcation techniques by linearization. Actually, from (A1), we have known the nonlinear term h has infinite derivative at $u = 0$. To overcome this problem we shall employ a limiting procedure. Let us define a function $\tilde{h} : \mathbb{R} \to \mathbb{R}$ by setting

$$\tilde{h}(s) = \begin{cases} h(s), & 0 \le |s| \le 1, \\ \text{linear}, & 1 < |s| < 2, \\ 0, & |s| \ge 2, \end{cases}$$

and define a function $\tilde{g} : [0,1] \times \mathbb{R} \to \mathbb{R}$ by setting, for $r \in [0,1]$,

$$\tilde{g}(r,s) = \begin{cases} g(r,s), & 0 \le |s| \le 1, \\ \text{linear}, & 1 < |s| < 2, \\ 0, & |s| \ge 2. \end{cases}$$

Observe that, within the context of positive solutions, problem (1.3) is equivalent to the same problem with h, g replaced by $\tilde{h}, \tilde{g}$. Indeed, if u is a positive solution, then $\|u'\|_\infty < 1$ and hence $\|u\|_\infty < 1$. Clearly, $\tilde{h}$ and $\tilde{g}$ satisfy all the properties assumed in the statement of the theorem. In the sequel, we shall replace h, g with $\tilde{h}$ and $\tilde{g}$, however, for the sake of simplicity, the modified functions $\tilde{h}$, $\tilde{g}$ will still be denoted by h, g. Next, for any $\delta \in (0,1)$, let us define h_δ by setting

$$h_\delta(s) = \begin{cases} \dfrac{h(\delta)}{\delta} s, & 0 \le |s| \le \delta, \\ h(s), & |s| > \delta. \end{cases}$$

Obviously,

$$\lim_{\delta \to 0} h_\delta(s) = h(s), \qquad (h_\delta)_0 = \lim_{s \to 0} \frac{h_\delta(s)}{s} = \frac{h(\delta)}{\delta} > 0. \tag{3.3}$$

This together with (A1) implies that

$$\lim_{\delta \to 0} (h_\delta)_0 = \infty. \tag{3.4}$$

Let us consider the approximated problems

$$\begin{aligned} Au &= \lambda h_\delta(u) + g(r,u), \qquad r \in (0,1), \\ u'(0) &= u(1) = 0, \end{aligned} \tag{3.5}$$

where A is given by (1.4).

Define

$$F_\delta(\lambda, u) = \lambda h_\delta(u) + g(r,u) + \frac{1}{r^{N-1}}\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)'$$

for any $(\lambda, u) \in \mathbb{R} \times X$ and fixed $\delta > 0$. Then, from Remark 1.3, and by a simple calculation, we have that

$$(F_\delta)_u(\lambda, 0)v = \lim_{t \to 0} \frac{F_\delta(\lambda, tv) - F_\delta(\lambda, 0)}{t} \\ = \lambda \frac{h(\delta)}{\delta} v + \frac{1}{r^{N-1}} \left(r^{N-1} v'\right)'. \tag{3.6}$$

Let $\lambda_{k,\delta} = \lambda_k \cdot \frac{\delta}{h(\delta)}$. Then from (3.6), it follows that if $(\lambda_{k,\delta}, 0)$ is a bifurcation point of problem (3.5), then λ_k is an eigenvalue of problem (2.1).

For any $\gamma \in [0,1]$, we consider the following problem

$$\begin{aligned} Au &= \lambda h_\delta(u) + \gamma g(r, u), \qquad r \in (0,1), \\ u'(0) &= u(1) = 0. \end{aligned} \tag{3.7}$$

Then problem (3.7) is equivalent to

$$u = \psi(\lambda h_\delta(u) + \gamma g(r, u)) := F_{\delta,\lambda}(\gamma, u).$$

From [14, Lemma 2.3], it follows that $F_{\delta,\lambda} : [0,1] \times X \to X$ is completely continuous. In particular, $H_{\delta,\lambda} := F_{\delta,\lambda}(1, \cdot) : X \to X$ is completely continuous.

By (A2) and an argument similar to that of Lemma 2.5, we can show that the Leray–Schauder degree $\deg(I - F_{\delta,\lambda}(\gamma, \cdot), B_r(0), 0)$ is well defined for $\lambda \in (0, \infty) \setminus \{\lambda_k\}$. From the invariance of the degree under homotopies we obtain that

$$\begin{aligned} \deg(I - H_{\delta,\lambda}, B_r(0), 0) &= \deg(I - F_{\delta,\lambda}(1, \cdot), B_r(0), 0) \\ &= \deg(I - F_{\delta,\lambda}(0, \cdot), B_r(0), 0) \\ &= \deg\left(I - \psi\left(\lambda \frac{h(\delta)}{\delta} \cdot \right), B_r(0), 0\right). \end{aligned}$$

So by Lemma 2.5, we have that

$$\deg(I - H_{\delta,\lambda}, B_r(0), 0) = \begin{cases} 1, & \text{if } \lambda \in \left(0, \dfrac{\delta}{h(\delta)} \lambda_1\right), \\ (-1)^k, & \text{if } \lambda \in \left(\dfrac{\delta}{h(\delta)} \lambda_k, \dfrac{\delta}{h(\delta)} \lambda_{k+1}\right), \ k \in \mathbb{N}. \end{cases}$$

Denote

$$\mathcal{F}_\delta = \overline{\{(\lambda, u) : (\lambda, u) \in [0, \infty) \times X, \ u \text{ is a solution of } (3.5)\}}^{\mathbb{R} \times X}.$$

Then by a variant of the global bifurcation theorem of Rabinowitz [30], or index jump principle of Zeidler [33], for any $\delta > 0$, there exists a maximal closed connected set $S_{k,\delta}$ in $\mathcal{F}_\delta$ such that $(\lambda_{k,\delta}, 0) \in S_{k,\delta}$ and at least one of the following conditions holds:

(i) $S_{k,\delta}$ is unbounded in $\mathbb{R} \times X$;

(ii) $S_{k,\delta} \cap (\mathbb{R} \setminus \{\lambda_{k,\delta}\} \times \{0\}) \neq \emptyset$.

Since $(0, 0)$ is the only solution of (3.5) for $\lambda = 0$ and 0 is not the eigenvalue of eigenvalue problem (2.1), therefore $S_{k,\delta} \cap (\mathbb{R}\backslash\{\lambda_{k,\delta}\} \times \{0\}) = \emptyset$. Recalling Remark 1.4, we get $S_{k,\delta}$ is unbounded in λ-direction for each fixed δ.

Combining this and (3.3) and (3.4) and using Lemma 2.3, it follows that for each $k \in \mathbb{N}$, there exists a component Γ_k in $\limsup S_{k,\delta}$ which joins $(0, 0)$ to infinity in λ-direction.

In the following, we will prove the properties (i)–(iii) of Theorem 1.1, respectively.

(i) Let δ_0 be a positive constant such that $\lambda \frac{h(\delta_0)}{\delta_0} > \lambda_1$. Let us consider $(\lambda, u) \in S_{1,\delta}$, with $\lambda > 0$ and $\delta \in (0, \delta_0]$.

Fixing $\varepsilon > 0$ small, from (A1) and (A2), we obtain there exists $c = c(\lambda) > 0$ such that

$$\lambda h_\delta(s) + g(r, s) > (\lambda_1 + \varepsilon)s, \qquad \forall s \in (0, c].$$

Hence, we obtain if $\|u_1\|_\infty \le c$, then u_1 satisfies

$$Au_1 > (\lambda_1 + \varepsilon)u_1.$$

From [6], we have u_1 is an upper solution of the eigenvalue problem

$$Au = (\lambda_1 + \varepsilon)s. \tag{3.8}$$

On the other hand, it is easy to verify that $u_2 \equiv 0$ is a lower solution of (3.8). Therefore, [6, Proposition 1] yields the existence of a positive solution $u \in X$ of the eigenvalue problem (3.8). However, this is a contradiction, because $\lambda_1 + \varepsilon$ is not the first eigenvalue of (2.1).

This shows that if $(\lambda, u) \in S_{1,\delta}$, with $\lambda > 0$ and $\delta \in (0, \delta_0]$, then $\|u\|_\infty > c(\lambda)$. Passing to the limit as $\delta \to 0$ it follows that if $(\lambda, u) \in \Gamma_1$ then $\|u\|_\infty \ge c(\lambda)$.

When we consider Γ_k with $k > 1$ the argument is similar. If $(\lambda, u) \in S_{k,\delta}$, then there exists at least one interval I_k with length $1/k$ where u has constant sign. Therefore if we restrict the discussion to the interval I_k and replace λ_1 by the first eigenvalue of (2.1) on the interval I_k, then we can get the same contradiction as before.

(ii) From (i), we have for any $(\lambda, u) \in \Gamma_k$, if $\lambda > 0$, then $u \ne 0$.

Let $\{(\lambda_n, u_n)\} \subseteq S_{k,n}$ be a sequence, converging to (λ, u) in $\mathbb{R} \times X$. First, if $k = 1$, then we have $u_n > 0$ in $[0, 1)$, therefore $u \ge 0$, moreover, the strong Maximum Principle yields that $u > 0$ in $[0, 1)$.

Next, if $k > 1$, then let $\{x_n\}$ and $\{y_n\}$ be two consecutive zeros of u_n with $x_n \to \xi$ and $y_n \to \eta$. Obviously, $u(\xi) = u(\eta) = 0$. We claim that $\xi \ne \eta$. Otherwise, there exists a third sequence $\{z_n\}$ such that $u_n'(z_n) = 0$ and $\lim_{n\to\infty} z_n = \xi$. Therefore, we can find a u, it is a solution of

$$Au = \lambda h(u) + g(r, u),$$

and satisfies

$$u(\xi) = u'(\xi) = 0.$$

However, from Lemma 3.1, we know this is impossible. Therefore, we conclude that for any $(\lambda, u) \in \Gamma_k$ and $\lambda > 0$, u has exactly $k - 1$ simple zeros in the interval $(0, 1)$.

(iii) Suppose on the contrary that there exists a sequence $\{(\lambda_n, u_n)\} \subseteq S_{k,n}$ such that $\lambda_n \to 0$, $u_n \to u$ and $\|u_n\| = \rho \leq \rho_0$. Passing to the limit we find that $u \neq 0$ is a solution of (3.1) and u satisfies $\|u\| \leq \rho_0$, however this contradicts Lemma 3.2. □

Example 3.3. Let us consider the following Dirichlet problem with mean curvature operator in the Minkowski space

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \lambda h(u) + g(r,u), & r = |x| < 1, \\ u = 0, & r = |x| = 1, \end{cases} \tag{3.9}$$

where

$$h(u) = \begin{cases} \sqrt{u}, & u \geq 0, \\ -\sqrt{-u}, & u < 0, \end{cases}$$

and

$$g(r,u) = \begin{cases} u^2, & u \geq 0, \\ -u^2, & u < 0. \end{cases}$$

Obviously, $q = \frac{3}{2}$ and all assumptions of Theorem 1.1 are valid. Therefore, from Theorem 1.1, we know there are infinitely many unbounded component of radial nodal solutions of (3.9) branching off from $(0,0)$.

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On the existence and multiplicity of eigenvalues for a class of double-phase non-autonomous problems with variable exponent growth

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Abstract. We study the following class of double-phase nonlinear eigenvalue problems

$$-\operatorname{div}\left[\phi(x,|\nabla u|)\nabla u+\psi(x,|\nabla u|)\nabla u\right]=\lambda f(x,u)$$

in Ω, $u=0$ on $\partial\Omega$, where Ω is a bounded domain from $\mathbb{R}^N$ with smooth boundary and the potential functions ϕ and ψ have $(p_1(x);p_2(x))$ variable growth. The main results of this paper are to prove the existence of a continuous spectrum consisting in a bounded interval in the near proximity of the origin, the fact that the multiplicity of every eigenvalue located in this interval is at least two and to establish the existence of infinitely many solutions for our problem. The proofs rely on variational arguments based on the Ekeland's variational principle, the mountain pass theorem, the fountain theorem and energy estimates.

Keywords: double-phase differential operator, continuous bounded spectrum, variable exponent, multiplicity of eigenvalues, multiple types of solutions.

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1 Introduction

The recent study of various mathematical models described by variational problems with non-standard variable growth conditions is motivated by many phenomena that arise in applied sciences. For instance, in some cases, to describe the behavior of some materials which are not homogeneous the classical theory of $L^p(\Omega)$ and $W^{1,p}(\Omega)$ Lebesgue and Sobolev spaces has proven its limitation.

An example of such type of materials are the thermorheological and electrorheological fluids. For a good description from the partial differential equations point of view of these types of materials we refer to V. Rădulescu [23] and V. Rădulescu, D. Repovš [24]. We remark also that the variable exponent analysis for some nonlinear problems plays a crucial role in the development of robotics, aircraft and airspace and the image restoration.

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In this paper we are interested in the study of a class of non-autonomous eigenvalue problems with a variable $(p_1(x); p_2(x))$-grow rate condition, which are described by the fact that the associated energy density changes its ellipticity and growth properties according to the point.

Our study is based on some new type of non-homogeneous differential operators developed by I. H. Kim and Y. H. Kim [12], which allow us to analyze some problems that imply the possibility of lack of uniform convexity. In this paper we extend the results of I. H. Kim and Y. H. Kim by studying a double-phase problem. Moreover, for the best of our knowledge for this type of operators it is not established yet the possibility of existence and multiplicity for some eigenvalues in the near proximity of the origin, even in the simpler case when the differential operator is driven by only one potential function. This paper also aim to extend the spectral analysis for this kind of problems made by S. Baraket, S. Chebbi, N. Chorfi, V. Rădulescu in [2].

Therefore we consider the following double-phase nonlinear eigenvalue problem:

$$\begin{cases} -\operatorname{div}\left[\phi(x,|\nabla u|)\nabla u + \psi(x,|\nabla u|)\nabla u\right] = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary and $\lambda \in \mathbb{R}$ is a real parameter.

The study of these types of problems was motivated by the fact that we may need to model a composite that changes its hardening exponent according to the point. For more details about integral functionals with nonstandard (p,q)-growth conditions, we refer to P. Marcellini [13, 14]. These types of problems was also studied by G. Mingione *et al.* [3, 6, 7], where the associated energies are of type

$$u \mapsto \int_\Omega \left(|\nabla u|^{p_1(x)} + a(x)|\nabla u|^{p_2(x)}\right) dx \tag{1.1}$$

and

$$u \mapsto \int_\Omega \left[|\nabla u|^{p_1(x)} + a(x)|\nabla u|^{p_2(x)} \log(e+|x|)\right] dx, \tag{1.2}$$

where $p_1(x) \le p_2(x)$, $p_1 \neq p_2$, for all $x \in \Omega$ and $a(x) \ge 0$.

These problems describe the behavior of two materials with variable power hardening exponents $p_1(x)$ and $p_2(x)$ and the coefficient $a(x)$ dictates the geometry of a composite of the two materials.

As we mentioned before our nonhomogeneous differential operator corresponds to the type of double-phase operators, fact that is induced by the presence of the potential functions ϕ and ψ. In order to make a better connection with the work of Mingione *et al.*, we remark that our potential functions ϕ and ψ may behave as it follows

- $\phi(x,t) = t^{p(x)-2}$, case in which we can also embed the description given by (1.1) for the fact that our operators extends the case when

$$-\operatorname{div}\left[\phi(x,|\nabla u|)\nabla u + \psi(x,\nabla u)\nabla u\right] = -\operatorname{div}\left[a(x)|\nabla u|^{p_1(x)-2}\nabla u + b(x)|\nabla u|^{p_2(x)-2}\nabla u\right],$$

  for some functions $a(x), b(x) \in L^\infty(\Omega)_+$;

- $\phi(x,t) = (1+|t|^2)^{\frac{p(x)-2}{2}}$, case in which we obtain the generalized mean curvature operator;

- $\phi(x,t) = \left(1 + \frac{t^{p(x)}}{\sqrt{1+t^{2p(x)}}}\right) t^{p(x)-2}$, case in which we obtain the corresponding differential operator that describe the capillary phenomenon.

For this cases, in order to obtain the description given by (1.2) we have to analyze the following differential operator:

$$-\operatorname{div}\left[\phi(x,|\nabla u|)\nabla u + a(x)\psi(x,|\nabla u|)\log(e+|x|)\nabla u\right].$$

As we mentioned before the main results of this paper is to establish the fact that for every $\lambda > 0$ small enough we have two different solutions and the fact that our problem (P_λ) admits a sequence of solutions with higher and higher energies provided only by the restriction $\lambda > 0$. The first solution is obtained as a local minimum near the origin. To this end we refer to [9,17] and [24, Chapter 2] for more details about the method used to point out this type of solutions. Our second solution is obtained as a mountain pass critical point. For a comprehensive study of this type of solutions we refer to the following works of P. Pucci, J. Serrin [21,22], P. Pucci, V. Rădulescu [19]. The third type of solutions is obtained as high energy solutions by employing the fountain theorem. For more details about this critical point technique we refer to the following works: [10,12,25,28].

Also more details about existence and nonexistence results related to variable exponent equations can be found in the following works [4,11], while more critical point techniques and qualitative analysis for double-phase operators can be found in [1,5,20].

Moreover, we make a parallel between the techniques used to point out our results and between our methods and some other techniques used so far to describe some spectral properties of these types of operators. For more details we mention the following works [2,12,26,27].

Also in the final part of this paper are given some examples and remarks in order to illustrate the validity of the general results obtained throughout this work.

2 The functional framework

Throughout this section we will introduce the necessary information about the functional framework that we will need in the study of problem (P_λ). To this end we will give a brief description of variable exponent Lebesgue and Sobolev spaces. Most of the following properties and results can be found in the following books by J. Musielak [18], L. Diening, P. Hästö, P. Harjulehto, M. Růžička [8], V. Rădulescu, D. Repovš [24].

First we assume that $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary. Let

$$C_+(\Omega) = \left\{ p \in C(\overline{\Omega}) : \min_{x\in\Omega} p(x) > 1 \right\},$$

and for any continuous function $p : \overline{\Omega} \to (1,+\infty)$, we have

$$p^- = \inf_{x\in\Omega} p(x) \qquad \text{and} \qquad p^+ = \sup_{x\in\Omega} p(x).$$

For any $p \in C_+(\overline{\Omega})$, with $p < +\infty$ we define the variable exponent Lebesgue space as if follows

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ a measurable function} : \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

which endowed with the following Luxemburg norm

$$|u|_{p(x)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}$$

becomes a Banach space. For any $1 < p(x) < +\infty$ as defined before, $L^{p(x)}(\Omega)$ is reflexive, uniformly convex Banach space, and moreover for any measurable bounded exponent p, $L^{p(x)}(\Omega)$ is separable.

Remark 2.1. This space is a special case of an Orlicz–Musielak space and its dual space is defined as $L^{p'(x)}(\Omega)$, where $p'(x)$ is the conjugate exponent of $p(x)$, in the sense that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

If p and q are two variable exponents and $p(x) \le q(x)$ for almost all $x \in \Omega$, with $|\Omega| < \infty$, then there exists the following continuous embedding

$$L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega),$$

where by $|\Omega|$ we denote the Lebesgue measure of Ω.

Let $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ then the following Hölder type inequality occurs:

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(x)}|v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)}. \tag{2.1}$$

A crucial role in manipulating the variable exponent Lebesgue spaces is played by the modular function associated to these types of spaces. We define the modular of $L^{p(x)}(\Omega)$ by the function $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ such that

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

If $p(x) \not\equiv$ constant in Ω and $u, (u_n)_n \in L^{p(x)}(\Omega)$, then the following relations hold true:

$$|u|_{p(x)} < 1 \Rightarrow\ |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}, \tag{2.2}$$

$$|u|_{p(x)} > 1 \Rightarrow\ |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}, \tag{2.3}$$

$$|u|_{p(x)} = 1 \Rightarrow\ \rho_{p(x)}(u) = 1, \tag{2.4}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow\ \rho_{p(x)}(u_n - u) \to 0. \tag{2.5}$$

We define in what follows the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}.$$

On $W^{1,p(x)}(\Omega)$ we can define the following equivalent norms:

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$$

and

$$\|u\| = \inf\left\{\mu > 0 : \int_\Omega \left(\left|\frac{\nabla u(x)}{\mu}\right|^{p(x)} + \left|\frac{u(x)}{\mu}\right|^{p(x)}\right) dx \le 1\right\}.$$

Since our problem necessitates that the function $u = 0$ on $\partial\Omega$, we define the associated space $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{p(x)}$ as it follows

$$W_0^{1,p(x)}(\Omega) = \left\{u; u|_{\partial\Omega} = 0, u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}.$$

Taking account of [12] for $p \in C_+(\overline{\Omega})$ it holds true the following Poincaré type inequality

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)}, \tag{2.6}$$

for $C > 0$ a constant which depends on p and Ω.

Remark 2.2. If $\Omega \subset \mathbb{R}^N$ is a bounded domain, and the function p which dictates the variable exponent is global log-Hölder continuous the norm $|\nabla u|_{p(x)}$ is equivalent with $\|u\|_{p(x)}$ on $W_0^{1,p(x)}(\Omega)$.

Remark 2.3. If $p^- > 1$, the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are reflexive, uniformly convex Banach spaces. Furthermore if p is measurable and bounded then our spaces are separable.

Remark 2.4 ([24]). If $p, q, r \in C_+(\Omega)$ with $p^+ < N$, and $p(x) < r(x) < q(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$, for any $x \in \Omega$, then the following embeddings hold true

$$\begin{aligned} &W_0^{1,r(x)}(\Omega) \hookrightarrow W_0^{1,p(x)}(\Omega) \quad \text{(continuous embedding)},\\ &W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega) \qquad \text{(continuous and compact embedding)}. \end{aligned}$$

3 Basic hypotheses and auxiliary results

In this section we will establish the main conditions imposed on the potential functions ϕ and ψ which drive us to our double-phase differential operator from the problem (P_λ) and some auxiliary results that will help us pointing out our solutions.

We assume that:

(S_1) $\phi, \psi : \Omega \times [0,\infty) \to [0,\infty)$ and

- $\phi(\cdot, t)$, $\psi(\cdot, t)$ are measurable on Ω for all $t \geq 0$;
- $\phi(x, \cdot)$, $\psi(x, \cdot)$ are locally absolutely continuous on $[0,\infty)$ for almost all $x \in \Omega$.

(S_2) There exist some functions v_1 and v_2 such that $v_1 \in L^{p_1'(x)}(\Omega)$ and $v_2 \in L^{p_2'(x)}(\Omega)$ and a constant $\xi > 0$ such that

- $|\phi(x,|t|)t| \leq v_1(x) + \xi|t|^{p_1(x)-1}$,
- $|\psi(x,|t|)t| \leq v_2(x) + \xi|t|^{p_2(x)-1}$

for almost all $x \in \Omega$, and all $t \in \mathbb{R}^N$.

(S_3) There is a strictly positive constant c such that the following statements are verified for almost all $x \in \Omega$ and all $t > 0$:

- $\phi(x,t) \geq ct^{p_1(x)-2}$ and $t\frac{\partial\phi}{\partial t} + \phi(x,t) \geq ct^{p_1(x)-2}$,
- $\psi(x,t) \geq ct^{p_2(x)-2}$ and $t\frac{\partial\psi}{\partial t} + \psi(x,t) \geq ct^{p_2(x)-2}$.

Let us now impose some conditions on the reaction term (right-hand side) of the problem (P_λ). We define $f : \Omega \times \mathbb{R} \to \mathbb{R}$ as a Carathéodory function (i.e. $f(\cdot, z)$ is measurable for all $z \in \mathbb{R}$ and $f(x, \cdot)$ is continuous for almost all $x \in \Omega$) satisfying the following hypotheses:

(R_1) $zf(x,z) \geq 0$ for almost all $(x,z) \in \Omega \times \mathbb{R}$, and there exists a function $m \in L^\infty(\Omega) \setminus \{0\}$, $m(x) \geq m^- > 0$, where m^- is a constant, for all $x \in \Omega$ such that

$$|f(x,z)| \leq m(x)|z|^{q(x)-1} \quad \text{for almost all } x \in \Omega, \text{ all } z \in \mathbb{R}.$$

(R_2) There exist some strictly positive constants A and η such that

$$0 < \eta F(x,z) \leq zf(x,z) \quad \text{for almost all } x \in \Omega, z \in \mathbb{R} \setminus \{0\},$$

where $F(x,z) = \int_0^z f(x,t)dt$, $\eta > p_2^+$ and $|z| > A$.

By hypothesis (R_1) we obtain that

(R_3) $F(x,z) \leq \dfrac{m(x)}{q(x)}|z|^{q(x)}$ for all $(x,z) \in \Omega \times \mathbb{R}$.

(R_4) There exists a constant $C_F > 0$ such that

$$|z|^{q(x)} \leq C_F F(x,z), \text{ for all } (x,z) \in \Omega \times \mathbb{R}.$$

Now we assume that $p_1, p_2, q \in C_+(\Omega)$. Our variable exponents exhibits the following behavior

$$\begin{cases} 1 < q^- < p_1^- \leq p_1(x) \leq p_1^+ < p_2^- \leq p_2(x) \leq p_2^+, \\ p_2^+ < p_1^*(x) \text{ and } q^+ < p_1^*(x), \end{cases} \tag{3.1}$$

where $p_1^*(x) = \frac{Np_1(x)}{N-p_1(x)}$ is the critical Sobolev exponent, for all $x \in \overline{\Omega}$.

Remark 3.1. At this point we do not have any information on the behavior of the quantity $\sup_{x\in\Omega} q(x)$, beside the fact that it is a subcritical exponent.

Remark 3.2. Taking account on the relation (3.1) and the embedding theorems for variable exponent Lebesgue and Sobolev spaces we will choose $W = W_0^{1,p_2(x)}(\Omega)$ as functional space for the solutions of problem (P_λ), and for the simplicity of the writing by $\|\cdot\|$ we will denote the norm associated to $W_0^{1,p_2(x)}(\Omega)$ ($\|\cdot\|_{p_2(x)}$).

Definition 3.3. We say that $u \in W \setminus \{0\}$ is a weak solution of the problem (P_λ) if

$$\int_\Omega [\phi(x, |\nabla u|)\nabla u \nabla \varphi + \psi(x, |\nabla u|)\nabla u \nabla \varphi]\, dx = \lambda \int_\Omega f(x,u)\varphi dx$$

for all $\varphi \in W$.

In order to establish the desired spectral properties for our problem we define the energy functional associated to the problem (P_λ) as it follows

$$T_\lambda : W \to \mathbb{R},$$
$$T_\lambda(u) = S(u) - \lambda R(u),$$

where

$$S(u) = \int_\Omega S_0(x, |\nabla u|)dx, \quad \text{with} \quad S_0(x,t) = \int_0^t \phi(x,s)sds + \int_0^t \psi(x,s)sds$$

and

$$R(u) = \int_\Omega F(x,u)dx.$$

An important role in the analysis made by using the energy functional T_λ is played by the fact that the part of the functional driven by our double-phase operator (left-hand side of the problem) satisfy the following hypothesis

(S_4) For all $x \in \overline{\Omega}$, all $t \in \mathbb{R}^N$, the following estimate holds true:

$$0 \leq [\phi(x,|t|) + \psi(x,|t|)]\,|t|^2 \leq \omega S_0(x,|t|),$$

for a constant $\omega > 1$.

Remark 3.4. We can observe that the functional T_λ is of class $C^1(W,\mathbb{R})$ (for more details we refer to [12, Lemmas 3.2, 3.4] and [2, Section 4]).

In order to reveal the eigenvalues associated to our differential operator we will point out that the critical points of the energy functional T_λ are weak solutions for the problem (P_λ), so they are eigenfunctions to their corresponding eigenvalues denoted by λ.

Firstly we need to prove some useful properties related by the geometry of the energy functional T_λ.

Proposition 3.5. *There exists $\lambda_{\phi,\psi} > 0$ such that for any $0 < \lambda < \lambda_{\phi,\psi}$ there exist two strictly positive constants r and δ such that $T_\lambda(u) \geq \delta > 0$ for any $u \in W$ with $\|u\| = r$.*

Proof. We will compute first the part of the energy functional driven by the differential operator in the left-hand side of the problem (P_λ).

$$\begin{aligned} S(u) &= \int_\Omega S_0(x,|\nabla u|)dx \\ &\geq \frac{1}{\omega}\int_\Omega \phi(x,|\nabla u|)|\nabla u|^2 + \psi(x,|\nabla u|)|\nabla u|^2 dx \\ &\geq \frac{1}{\omega}\int_\Omega c\left(|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)}\right)dx \\ &\geq \frac{c}{\omega}\left(\int_\Omega |\nabla u|^{p_1(x)}dx + \int_\Omega |\nabla u|^{p_2(x)}dx\right). \end{aligned} \tag{3.2}$$

Taking account of the relation (3.1) we have the following continuous embeddings

$$\begin{aligned} W = W_0^{1,p_2(x)}(\Omega) &\hookrightarrow W_0^{1,p_1(x)}(\Omega) \\ W_0^{1,p_1(x)}(\Omega) &\hookrightarrow L^{q(x)}(\Omega). \end{aligned}$$

Therefore we have the following inequalities

$$|u|_{q(x)} < C_1 \|u\|_{p_1(x)} \tag{3.3}$$

$$\|u\|_{p_1(x)} < C_2 \|u\|, \tag{3.4}$$

where $C_1 > 0$, $C_2 > 0$ are some constants.

Combining (3.3) and (3.4) we obtain

$$|u|_{q(x)} < C_1\|u\|_{p_1(x)} < C_1 \cdot C_2\|u\|.$$

Now, let $r \in (0,1)$ be fixed such that $r < \min\left\{\frac{1}{C_1C_2}, \frac{1}{C_1}\right\}$, therefore we have that

$$\begin{aligned} \|u\|_{p_1(x)} &< 1, \\ |u|_{q(x)} &< 1, \end{aligned} \quad \text{for all } u \in W, \text{ with } \|u\| = r. \tag{3.5}$$

Moreover, using the properties described by relations (2.2) and (3.2), we obtain that

$$\begin{aligned} S(u) &\geq \frac{c}{\omega}\left(\|u\|_{p_1(x)}^{p_1^+} + \|u\|^{p_2^+}\right) \\ &\geq \frac{c}{\omega}\|u\|^{p_2^+}. \end{aligned} \tag{3.6}$$

We proceed now to compute the second part of our energy functional, driven by the reaction term, using assumptions (R_1) and (R_3) we obtain that:

$$\begin{aligned} R(u) &= \int_\Omega F(x,u)dx \\ &\leq \int_\Omega \frac{m(x)}{q(x)}|u|^{q(x)}dx \\ &\leq \frac{\|m\|_\infty}{q^-}\int_\Omega |u|^{q(x)}dx. \end{aligned} \tag{3.7}$$

Taking account of relation (3.5) and the property described by (2.2) we have that

$$\int_\Omega |u|^{q(x)}dx < |u|_{q(x)}^{q^-}.$$

Using the continuous embedding for variable exponent Lebesgue and Sobolev spaces dictated by hypothesis (3.1) and relation (3.7) we obtain that

$$R(u) \leq \frac{\|m\|_\infty}{q^-}(C_1 \cdot C_2)^{q^-}\|u\|^{q^-}. \tag{3.8}$$

Hence taking account of (3.6) and (3.8) we have that:

$$\begin{aligned} T_\lambda(u) &= S(u) - \lambda R(u) \\ &\geq \frac{c}{\omega}\|u\|^{p_2^+} - \lambda\frac{\|m\|_\infty}{q^-}(C_1 \cdot C_2)^{q^-}\|u\|^{q^-} \\ &= \frac{c}{\omega}r^{p_2^+} - \lambda\frac{\|m\|_\infty}{q^-}C_3^{q^-}r^{q^-} \\ &= r^{q^-}\left(\frac{c}{\omega}r^{p_2^+-q^-} - \lambda\frac{\|m\|_\infty}{q^-}C_3^{q^-}\right), \end{aligned} \tag{3.9}$$

where $C_3 = C_1 \cdot C_2$.

Using the inequality (3.9) we find that for every

$$\lambda \in \left(0, \frac{c}{\omega}r^{p_2^+-q^-} \cdot \frac{q^-}{C_3^{q^-}\|m\|_\infty}\right)$$

we can find a constant $\delta = \delta\left(\frac{c}{\omega} r^{p_2^+ - q^-} \cdot \frac{q^-}{C_3^{q^-} \|m\|_\infty}\right) > 0$ such that

$$T_\lambda(u) \geq \delta > 0$$

for any $u \in W$, with $\|u\| = r$.

Hence the proposition is proved. □

Remark 3.6. So, further on we will denote $\lambda_{\phi,\psi}$ by the quantity

$$\lambda_{\phi,\psi} = \frac{c}{\omega} r^{p_2^+ - q^-} \cdot \frac{q^-}{C_3^{q^-} \|m\|_\infty}. \tag{3.10}$$

Remark 3.7. We also can observe that our energy functional satisfies one of the geometric hypotheses of the mountain pass theorem, that is the existence of a mountain near the origin.

Proposition 3.8. *There exists $h \in W$, with $h > 0$ such that*

$$T_\lambda(th) < 0,$$

provided by a $t > 0$ sufficiently small.

Proof. We proceed first to compute the part of the energy functional which is driven by the double-phase operator from the left-hand side of the problem (P_λ).

Using (S_2), Hölder's inequality for variable exponent Lebesgue and Sobolev spaces and the fact that $t \in (0,1)$ is sufficiently small, we have that

$$\begin{aligned} S(th) &\leq 2C_\phi |v_1|_{p_1'(x)} \|th\|_{p_1(x)}^{p_1^-} + \frac{\xi}{p_1^-} \|th\|_{p_1(x)}^{p_1^-} + 2C_\psi |v_2|_{p_2'(x)} \|th\|^{p_2^-} + \frac{\xi}{p_2^-} \|th\|^{p_2^-} \\ &\leq t^{p_1^-} \tilde{C}_1, \end{aligned} \tag{3.11}$$

where $C_\phi, C_\psi > 0$ are two constants that depend on the potential functions ϕ, ψ and on the continuous embeddings

$$W_0^{1,p_1(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$$
$$W_0^{1,p_2(x)}(\Omega) \hookrightarrow W_0^{1,p_1(x)}(\Omega),$$

and

$$\tilde{C}_1 = \left(2C_\phi |v_1|_{p_1'(x)} + \frac{\xi}{p_1^-}\right) \|h\|_{p_1(x)}^{p_1^-} + \left(2C_\psi |v_2|_{p_2'(x)} + \frac{\xi}{p_2^-}\right) \|h\|^{p_2^-}. \tag{3.12}$$

In what follows we will compute the second part of the energy functional.

Using hypotheses (R_1), (R_3) and (R_4) there exists a constant $C_F > 0$ such that $F(x,u) \geq \frac{1}{C_F}|u|^{q(x)}$, with $C_F \geq \frac{q^+}{m^-}$, where $m^- = \min\{m(x) : x \in \overline{\Omega}, m(x) \neq 0\}$.

Let us consider $C_F = \frac{q^+}{m^-} + 1$, and so we have that

$$F(x,u) \geq \frac{m^-}{q^+ + m^-} |u|^{q(x)}. \tag{3.13}$$

Hypothesis (3.1) implies the fact that $q^- < p_1^-$. Let $\alpha_0 > 0$ be such that

$$q^- + \alpha_0 < p_1^-.$$

Since $q \in C(\overline{\Omega})$ we obtain the fact that there exists an open set $\Omega_0 \subset \Omega$ such that

$$|q(x) - q^-| < \alpha_0 \quad \text{for all } x \in \Omega_0,$$

therefore we can say that

$$q(x) < q^- + \alpha_0 < p_1^- \quad \text{for all } x \in \Omega_0.$$

Consider $h \in C_0^\infty(\Omega)$ be such that $\operatorname{supp}(h) \supset \overline{\Omega}_0$, $h(x) = 1$ for all $x \in \overline{\Omega}_0$ and $0 \leq h \leq 1$ in Ω.

Now taking account of relation (3.13) one have that

$$\begin{aligned} R(th) &= \int_\Omega F(x, th)dx \\ &\geq \frac{m^-}{q^+ + m^-} \int_\Omega t^{q(x)} |h|^{q(x)} dx \\ &\geq \frac{m^-}{q^+ + m^-} t^{q^- + \alpha_0} \int_{\Omega_0} |h|^{q(x)} dx. \end{aligned} \tag{3.14}$$

Now combining relations (3.11) and (3.14) we obtain that

$$T_\lambda(th) \leq \tilde{C}_1 t^{p_1^-} - \lambda t^{q^- + \alpha_0} \frac{m^-}{q^+ + m^-} \int_{\Omega_0} |h|^{q(x)} dx. \tag{3.15}$$

Hence, taking account of relation (3.15) we obtain that

$$T_\lambda(th) < 0$$

provided by $t < s^{\frac{1}{p_1^- - q^- - \alpha_0}}$, where

$$0 < s < \min\left\{1, \frac{\lambda \tilde{C}_2}{\tilde{C}_1}\right\}$$

with $\tilde{C}_2 = \frac{m^-}{q^+ + m^-} \int_{\Omega_0} |h|^{q(x)} dx$ and $\tilde{C}_1$ as defined by relation (3.12).

Now taking account of the fact that

$$\int_{\Omega_0} |h|^{q(x)} dx \leq \int_\Omega |h|^{q(x)} dx \leq \int_\Omega |h|^{q^-} dx,$$

and by the continuous embedding $W \hookrightarrow L^{q^-}(\Omega)$, and the properties of the modular function for variable exponent Lebesgue space (relations (2.2)–(2.5)) we can affirm that

$$\|h\| > 0 \quad \text{and} \quad \int_\Omega |\nabla h|^{p_1(x)} dx > 0, \qquad \int_\Omega |\nabla h|^{p_2(x)} dx > 0,$$

and this completes the proof of our proposition. □

Remark 3.9. We can observe that our energy functional does not satisfy the second geometrical condition of the mountain pass theorem, in the sense that there exists a valley near the origin, but it is not as far away as required. Hence the mountain pass theorem can not be applied at this moment, but it can be applied if we impose some additional conditions on the growing behavior of the reaction term. We will analyze this fact later on this paper.

4 Multiple types of solutions

We can state now our first result.

Theorem 4.1. *Assume that condition* (3.1) *is satisfied and hypotheses* (S_1)–(S_4), $(R_1), (R_3), (R_4)$ *hold true. Then for* $p_2^+ < N$, *for all* $x \in \overline{\Omega}$, *there exists* $\lambda_{\phi,\psi} > 0$ *such that any* λ *with* $0 < \lambda < \lambda_{\phi,\psi}$ *is an eigenvalue for problem* (P_λ).

Proof. We proceed now to prove our first result. Let $\lambda_{\phi,\psi}$ be as declared in relation (3.10) and consider $\lambda \in (0, \lambda_{\phi,\psi})$. In what follows we will denote by $B(0,r) = \{u \in W : \|u\| < r\}$ the ball centered in the origin with r radius from W.

Using Proposition 3.5, we have that

$$\inf_{u \in \partial B(0,r)} T_\lambda(u) > 0. \tag{4.1}$$

Also by Proposition 3.8 we have that there exists $h \in W$ such that $T_\lambda(th) < 0$, provided by $t > 0$ sufficiently small. Furthermore by relation (3.3), (3.4) and (2.2) we have that

$$T_\lambda(u) \geq \frac{c}{\omega}\|u\|^{p_2^+} - \lambda \frac{\|m\|_\infty}{q^-} C_3^{q^-} \|u\|^{q^-}.$$

Therefore we can say that there exists a constant c_0 such that

$$-\infty < c_0 := \inf_{B(0,r)} T_\lambda < 0.$$

Taking account of the above relations let $\varepsilon > 0$ be such that $\varepsilon < \inf_{\partial B(0,r)} T_\lambda - \inf_{B(0,r)} T_\lambda$, by applying the Ekeland's variational principle ([9]) to the functional $T_\lambda : \overline{B(0,r)} \to \mathbb{R}$ we obtain the existence of a function $u_\varepsilon \in \overline{B(0,r)}$ such that

$$T_\lambda(u_\varepsilon) \leq \inf_{B(0,r)} T_\lambda + \varepsilon$$
$$T_\lambda(u_\varepsilon) \leq T_\lambda(u) + \varepsilon\|u - u_\varepsilon\|, \quad u \neq u_\varepsilon.$$

Therefore we have that

$$T_\lambda(u_\varepsilon) \leq \inf_{B(0,r)} T_\lambda + \varepsilon \leq \inf_{B(0,r)} T_\lambda + \varepsilon < \inf_{\partial B(0,r)} T_\lambda,$$

thus we have obtained that $\|u_\varepsilon\| < r$. Now, let E be the energy functional defined on $\overline{B(0,r)}$ as it follows

$$E : \overline{B(0,r)} \to \mathbb{R}$$
$$E(u) = T_\lambda(u) + \varepsilon\|u - u_\varepsilon\|. \tag{4.2}$$

Now using relation (4.2) we have that

$$E(u_\varepsilon) = T_\lambda(u_\varepsilon) < T_\lambda(u) + \varepsilon\|u - u_\varepsilon\| = E(u), \qquad u \neq u_\varepsilon. \tag{4.3}$$

So far, taking a look at relation (4.3) it turns out that u_ε is a minimum point for E, therefore, using arguments from [2, 12, 17] we have that

$$\frac{E(u_\varepsilon + t\varphi) - E(u_\varepsilon)}{t} \geq 0 \tag{4.4}$$

for $t > 0$ small and every φ, with $\|\varphi\| < 1$.

Relation (4.4) yields the fact that

$$\frac{T_\lambda(u_\varepsilon + t\varphi) - T_\lambda(u_\varepsilon)}{t} + \varepsilon\|\varphi\| \geq 0.$$

We let $t \to 0$ and we obtain that

$$\begin{aligned} \langle T'_\lambda(u_\varepsilon), \varphi \rangle &> -\varepsilon\|\varphi\| \\ \langle T'_\lambda(u_\varepsilon), \varphi \rangle &> -\varepsilon \end{aligned}$$

which yields to the fact that $\|T'_\lambda(u_\varepsilon)\| \leq \varepsilon$.

Therefore we get the existence of a sequence $(v_n)_n \subset B(0, r)$ such that

$$T_\lambda(v_n) \to c_0 \quad \text{and} \quad T'_\lambda(v_n) \to 0. \tag{4.5}$$

Since $(v_n)_n \subset B(0, r)$ it yields that

$$\|v_n\| \leq r, \quad \text{for every } n \in \mathbb{N}, \tag{4.6}$$

hence the sequence $(v_n)_n$ is bounded in W. As a consequence we can find an element v_0 such that (passing eventually to a subsequence)

$$v_n \rightharpoonup v_0 \quad \text{in } W.$$

By the fact that W is compactly embedded in $L^{q(x)}(\Omega)$ we get that $v_n \to v_0$ in $L^{q(x)}(\Omega)$. Using [24, Lemma 21, Chapter 3] and some arguments from the proof of [12, Lemma 3.5] we have that $R'(u)$ is compact therefore we have that

$$\begin{aligned} \lim_{n\to\infty} R(v_n) &= R(v_0) \\ \lim_{n\to\infty} \langle R'(v_n), v_n - v_0 \rangle &= 0 \end{aligned} \tag{4.7}$$

It only remains to show that

$$\lim_{n\to\infty} S(v_n) = S(v_0).$$

Using relation (4.5) we have that

$$\lim_{n\to\infty} \langle T'_\lambda(v_n), v_n - v_0 \rangle = 0. \tag{4.8}$$

Using (4.7) and (4.8) we can obtain that

$$\lim_{n\to\infty} \langle S'(v_n) - S'(v_0), v_n - v_0 \rangle \leq \lim_{n\to\infty} \langle T'_\lambda(v_n), v_n - v_0 \rangle = 0,$$

thus using [12, Lemma 3.4] we get that

$$v_n \to v_0 \quad \text{in } W. \tag{4.9}$$

Hence by relations (4.9) and (4.7) combined with relation (4.5) we obtain the fact that

$$T_\lambda(v_0) = c_0 < 0 \quad \text{and} \quad T'_\lambda(v_0) = 0.$$

We conclude by pointing out that v_0 is a nontrivial weak solution of problem (P_λ) and every $\lambda \in (0, \lambda_{\phi,\psi})$ is an eigenvalue of our problem. □

Let us assume now that the hypotheses of Theorem 4.1 are fulfilled and moreover we have more knowledge about the variable growth of the reaction term; namely the following relation holds true:

$$1 < q^- < p_1^- \leq p_1(x) \leq p_1^+ < p_2^- \leq p_2(x) \leq p_2^+ < q^+ < p_1^*(x), \tag{4.10}$$

for all $x \in \Omega$.

Remark 4.2. Taking account of the relation (4.10) we still can not prove the fact that our energy functional T_λ is coercive, so we can not apply the so called Direct Method in the Calculus of Variations in order to point out our eigenvalues. This method have been applied on this types of operators in the following works: [2,12,27].

Using the new information given by relation (4.10) about the growth behavior of the reaction term we can obtain the following property for our energy functional.

Proposition 4.3. *Suppose that hypotheses (S_1)–(S_4), (R_1)–(R_4) and (4.10) hold true, then we can find some element $\theta \in W$ such that*

$$T_\lambda(t\theta) < 0,$$

provided by t sufficiently large.

Proof. Using similar arguments as in the proof of Proposition 3.8 and keeping in mind that t is sufficiently large we obtain that

$$\begin{aligned} S(t\theta) &= \int_\Omega S_0(x, |\nabla(t\theta)|)dx \\ &\leq 2\overline{C}_\phi |v_1|_{p_1'(x)} \|t\theta\|_{p_1(x)}^{p_1^+} + \frac{\xi}{p_1^-}\|t\theta\|_{p_1(x)}^{p_1^+} + 2\overline{C}_\psi |v_2|_{p_2'(x)} \|t\theta\|^{p_2^+} + \frac{\xi}{p_2^-}\|t\theta\|^{p_2^+} \\ &\leq \tilde{C}_\theta t^{p_2^+}, \end{aligned} \tag{4.11}$$

where $\tilde{C}_\theta = \left(2\overline{C}_\phi |v_1|_{p_1'(x)} + \frac{\xi}{p_1^-}\right)\|\theta\|_{p_1(x)}^{p_1^+} + \left(2\overline{C}_\psi |v_2|_{p_2'(x)} + \frac{\xi}{p_2^-}\right)\|\theta\|^{p_2^+}$.

Hypothesis (4.10) implies that $p_2^+ < q^+$. Thinking similarly as in the proof of Proposition 3.8 we obtain the existence of a constant $\alpha_1 > 0$ such that $p_2^+ + \alpha_1 < q^+$. By the fact that $p_2, q \in C(\overline{\Omega})$ it follows that there exists an open set $\Omega_1 \subset \Omega$ such that $|q^+ - q(x)| < \alpha_1$ for all $x \in \Omega_1$. Therefore we obtain that

$$p_2^+ < q^+ - \alpha_1 < q(x) \tag{4.12}$$

for all $x \in \Omega_1$.

Now let $\theta \in C_0^\infty(\Omega)$ by such that $\operatorname{supp}(\theta) \supset \overline{\Omega}_1$, $\theta(x) = 1$ for all $x \in \overline{\Omega}_1$ and $0 \leq \theta \leq 1$ in Ω, taking account of relation (3.13) combined with hypothesis (R_2) we have that

$$F(x, t\theta) \geq \frac{m^-}{\eta + m^-}|t\theta|^{q(x)}.$$

Therefore by relation (4.12) and the properties of θ described before we obtain that

$$\begin{aligned} R(t\theta) &\geq \frac{m^-}{\eta + m^-}\int_\Omega t^{q(x)}|\theta|^{q(x)}dx \\ &\geq \frac{m^-}{\eta + m^-} t^{q^+ - \alpha_1} \int_{\Omega_1} |\theta|^{q(x)}dx. \end{aligned} \tag{4.13}$$

Hence taking use of relations (4.11) and (4.13) we obtain that

$$T_\lambda(t\theta) \leq t^{p_2^+}\tilde{C}_\theta - \frac{m^-}{\eta + m^-} t^{q^+ - \alpha_1} \int_{\Omega_1} |\theta|^{q(x)} dx.$$

Letting $t \to \infty$ and keeping in mind that $p_2^+ < q^+ - \alpha_1$ we have that

$$\lim_{t\to\infty} T_\lambda(t\theta) = -\infty.$$

Reasoning as in the end of the proof of Proposition 3.8 we have that $\|\theta\| > 0$, $\|\theta\|_{p_1(x)} > 0$ and so our proof is complete. □

Remark 4.4. Comparing the results of Proposition 3.8 and Proposition 4.3, we can observe that for the new growth conditions imposed by relation (4.10) the energy functional T_λ fulfills the second geometrical condition of the mountain pass theorem, namely we can find a valley far away of the origin as required.

In order to obtain our second result we need to require a slightly more restrictive condition (S_4), namely:

(S_4') $0 \leq [\phi(x,|t|) + \psi(x,|t|)]\,|t|^2 \leq p_2^+ S_0(x,|t|)$, for all $x \in \overline{\Omega}$, all $t \in \mathbb{R}^N$.

Of course we can observe that (S_4') implies (S_4).

We state now our second result.

Theorem 4.5. *Assume that condition* (4.10) *holds true and hypotheses* (S_1)–(S_3), (S_4'), (R_1)–(R_4) *are fulfilled. Then for every* $\lambda \in (0, \lambda_{\phi,\psi})$ *the problem* (P_λ) *has a mountain pass type solution.*

Proof. Taking account of Propositions 3.5 and 4.3, we have that our energy functional has a mountain pass geometry.

Since $T_\lambda(0) = 0$, employing the mountain pass theorem we obtain the existence of a sequence $(w_n)_n \subset W$ such that

$$T_\lambda(w_n) \to c_1 > 0 \quad \text{and} \quad T_\lambda'(w_n) \to 0 \quad \text{in} \quad W^{-1,p_2'(x)}(\Omega) \quad \text{as } n \to \infty, \tag{4.14}$$

namely a Palais–Smale sequence for the energy level c_1.

By the fact that R' is compact and S' is of type (S_+), using the fact that the space W is reflexive it suffices to prove that $(w_n)_n$ is bounded in W. To this end we argue by contradiction and suppose that $\|w_n\| \to \infty$ (passing eventually to a subsequence).

Using hypotheses (S_4'), (R_2) and the fact that we assumed $\|w_n\| \to \infty$ we obtain that

$$\begin{aligned} T_\lambda(w_n) - \frac{1}{\eta}\langle T_\lambda'(w_n), w_n\rangle &= \int_\Omega S_0(x,|\nabla w_n|) - \frac{1}{\eta}\left[\phi(x,|\nabla w_n|)\nabla w_n + \psi(x,|\nabla w_n|)\nabla w_n\right]\nabla w_n dx \\ &\quad + \lambda \int_\Omega \left[\frac{1}{\eta} f(x,w_n)w_n - F(x,w_n)\right] dx \\ &\geq \int_\Omega \left(1 - \frac{p_2^+}{\eta}\right) S_0(x,|\nabla w_n|)dx + \lambda \int_\Omega \left[\frac{1}{\eta} f(x,w_n)w_n - F(x,w_n)\right] dx. \end{aligned}$$

Let us define now

$$C_A = \sup\left\{\left|\frac{1}{\eta} f(x,z)z - F(x,z)\right| : \, x \in \overline{\Omega}, |z| \leq A\right\}.$$

Hence by assumption (R_2) we have that

$$\begin{aligned}\left(1-\frac{p_2^+}{\eta}\right)\int_\Omega S_0(x,|\nabla w_n|)dx &\le T_\lambda(w_n)-\frac{1}{\eta}\langle T'_\lambda(w_n),w_n\rangle\\ &\quad-\lambda\int_{\{x\in\Omega:\,|w_n(x)|>A\}}\left[\frac{1}{\eta}f(x,w_n)w_n-F(x,w_n)\right]dx+\lambda C_A|\Omega|\\ &\le T_\lambda(w_n)-\frac{1}{\eta}\langle T'_\lambda(w_n),w_n\rangle+\lambda C_A|\Omega|,\end{aligned}$$

where by $|\Omega|$ we denotes the Lebesgue measure of the domain Ω.

Since we supposed that $\|w_n\|\to\infty$, for a sufficiently large n we get that $\|w_n\|>1$, and by assumptions (S_3), (S'_4) and relation (2.3) we have that

$$\left(1-\frac{p_2^+}{\eta}\right)\frac{c}{p_2^+}\|w_n\|^{p_2^-}\le T_\lambda(w_n)+\frac{1}{\eta}\|T'_\lambda(w_n)\|_{W^{-1,p'_2(x)}(\Omega)}\cdot\|w_n\|+\lambda C_A|\Omega|.$$

Now by the fact that $\eta>p_2^+$ and $p_2^->1$ we obtain a contradiction.

Therefore we have proved that there exists a Palais–Smale sequence for the energy level $c_1>0$, which is bounded. So passing eventually to a subsequence (labeled for the ease of writing with the same notation) $(w_n)_n$ and taking account the fact that the space W is reflexive we can find an element w_0 such that $w_n\rightharpoonup w_0$ in W. Now, with the same arguments as in the final part of the proof for Theorem 4.1 we have that

$$T_\lambda(w_0)=c_1>0\quad\text{and}\quad T'_\lambda(w_0)=0.$$

Hence for every $\lambda\in(0,\lambda_{\phi,\psi})$ we can find a mountain pass type solution of the problem (P_λ). ◻

In the final part of this section we will present our last existence result, that is, the existence of infinitely many high-energy weak solutions of the problem (P_λ).

In order to prove our last result we first remind the following result.

Lemma 4.6 ([10]). *Let W be a reflexive and separable Banach space, then there are $\{e_j\}\subset W$ and $\{e_j^*\}\subset W^*$ such that*

$$W=\overline{\operatorname{span}\{e_j:\,j=1,2,\dots\}}\quad\textit{and}\quad W^*=\overline{\operatorname{span}\{e_j^*:\,j=1,2,\dots\}}$$

with

$$\langle e_i^*,e_j\rangle=\begin{cases}1, & \text{if } i=j\\ 0, & \text{if } i\neq j.\end{cases}$$

For the simplicity of the notation we will take use of the following:

$$W_j=\operatorname{span}\{e_j\},\qquad Y_k=\bigoplus_{j=1}^{k}W_j,\qquad Z_k=\bigoplus_{j=k}^{\infty}W_j.$$

We state now our multiplicity result.

Theorem 4.7. *Suppose that hypotheses (S_1)–(S_3), (S'_4), (R_1)–(R_4) and relation (4.10) hold true. If $f(x,-z)=-f(x,z)$ for almost all $x\in\Omega$, all $z\in\mathbb{R}$ and $\lambda>0$, then the problem (P_λ) admits a sequence of solutions $(\pm u_n)_n$ such that $T_\lambda(u_n)\to\infty$ as $n\to\infty$.*

Proof. In order to point out the sequence of solutions for the problem (P_λ) we will reveal the fact that our energy functional T_λ possesses a sequence $(\pm u_n)_n \subset W$ of critical points with higher and higher energies. To this end we have to prove the fact that functional T_λ is an even functional, and there are some constants $\gamma_k > \vartheta_k > 0$ such that for $k \in \mathbb{N}$ large enough:

(i) $\inf\{T_\lambda(u) : u \in Z_k, \|u\| = \vartheta_k\} \to \infty$ as $k \to \infty$

(ii) $\max\{T_\lambda(u) : u \in Y_k, \|u\| = \gamma_k\} \leq 0$

(iii) T_λ satisfies the Palais–Smale condition for every $c > 0$.

As the energy functional T_λ is even and with the same arguments as in the proof of Theorem 4.5 we can prove that T_λ satisfies the Palais–Smale condition for $c > 0$, it only remains to verify condition (i) and (ii).

Verification of (i): Let $a_k := \sup\{|u|_{q(x)} : \|u\| = 1, u \in Z_k\}$. From a straightforward computation, taking use of [25, proof of Theorem 3.2] we obtain that $a_k \to 0$ as $k \to \infty$.

Let $u \in Z_k$ with $\|u\| = \vartheta_k > 1$, where ϑ_k will be specified later. By hypothesis (S_3), (S_4') and (2.3) we obtain that

$$\begin{aligned}
T_\lambda(u) &= \int_\Omega S_0(x, |\nabla u|)dx - \lambda \int_\Omega F(x,u)dx \\
&\geq \frac{c}{p_2^+}\left(\int_\Omega |\nabla u|^{p_1(x)}dx + \|u\|^{p_2^-}\right) - \lambda \int_\Omega F(x,u)dx \\
&\geq \frac{c}{p_2^+}\|u\|^{p_2^-} - \frac{\lambda\|m\|_\infty}{q^-}\int_\Omega |u|^{q(x)}dx \\
&\text{(using hypothesis } (R_3)) \\
&\geq \frac{c}{p_2^+}\|u\|^{p_2^-} - \frac{\lambda\|m\|_\infty}{q^-}\max\left\{|u|^{q^-}_{q(x)}, |u|^{q^+}_{q(x)}\right\}.
\end{aligned}$$

Taking account of the continuous embedding $W \hookrightarrow L^{q(x)}(\Omega)$, we obtain that:

$$\begin{aligned}
T_\lambda(u) &\geq \frac{c}{p_2^+}\|u\|^{p_2^-} - \frac{\lambda\|m\|_\infty}{q^-}\max\left\{C_3^{q^-}\|u\|^{q^-}, C_3^{q^+}\|u\|^{q^+}\right\} \\
&\geq \frac{c}{p_2^+}\|u\|^{p_2^-} - \frac{\lambda\|m\|_\infty C_3^q}{q^-}\|u\|^{q^+} \\
&\left(\text{where } C_3^q = \max\left\{C_3^{q^-}, C_3^{q^+}\right\}\right) \\
&\geq \frac{c}{p_2^+}\|u\|^{p_2^-} - \frac{\lambda\|m\|_\infty C_3^q}{q^-}a_k^{q^+}\|u\|^{q^+}.
\end{aligned}$$

Due to a straightforward computation, we can choose

$$\vartheta_k = \left(\frac{\lambda\|m\|_\infty C_3^q}{q^-}\cdot\frac{p_2^+}{c}a_k^{p_2^+}\right)^{\frac{1}{p_2^- - q^+}}. \tag{4.15}$$

It is easy to remark that by the fact that $p_2^- < q^+$ and $a_k \to 0$ as $k \to \infty$ we obtain $\vartheta_k \to \infty$. Now taking ϑ_k as defined in relation (4.15) we obtain that

$$T_\lambda(u) \to \infty \quad \text{as } k \to \infty,$$

and so condition (i) is verified.

Verification of (ii): Let $u \in Y_k$ and $\|u\| = \gamma_k > 1$, where γ_k will be defined later. Using hypothesis (S_2) we get that

$$T_\lambda(u) \le 2C_{\gamma,\phi}|v_1|_{p_1'(x)} \max\left\{\|u\|^{p_1^-}_{p_1(x)}, \|u\|^{p_1^+}_{p_1(x)}\right\} + \frac{\xi}{p_1^-} \max\left\{\|u\|^{p_1^-}_{p_1(x)}, \|u\|^{p_1^+}_{p_1(x)}\right\}$$
$$+ 2C_{\gamma,\psi}|v_2|_{p_2'(x)}\|u\|^{p_2^+} + \frac{\xi}{p_2^-}\|u\|^{p_2^+} - \lambda \int_\Omega F(x,u)dx,$$

where $C_{\gamma,\phi} > 0$ and $C_{\gamma,\psi} > 0$ are some constants.

Taking account of relation (3.4) we obtain that

$$T_\lambda(u) \le \tilde{C}_\gamma \|u\|^{p_2^+} - \lambda \int_\Omega F(x,u)dx, \tag{4.16}$$

where $\tilde{C}_\gamma = \left(2C_{\gamma,\phi}|v_1|_{p_1'(x)}C_{p_1} + \frac{\xi}{p_1^-}C_{p_1}\right) + \left(2C_{\gamma,\psi}|v_2|_{p_2'(x)} + \frac{\xi}{p_2^-}\right)$ and $C_{p_1} = \max\left\{C_2^{p_1^-}, C_2^{p_1^+}\right\}$.

Using hypothesis (R_2) we can find two constants $C_4 > 0$ and $C_5 > 0$ such that

$$F(x,z) \ge C_4|z|^\eta - C_5. \tag{4.17}$$

In what follows using relations (4.16) and (4.17) we obtain that

$$T_\lambda(u) \le \tilde{C}_\gamma\|u\|^{p_2^+} - \lambda C_4 \int_\Omega |u|^\eta dx + \lambda C_5|\Omega|.$$

Taking use by the fact that we work on a finite dimensional space ($\dim Y_k < \infty$), by the fact that the assumption (R_2) implies that $\eta > p_2^+$ and $|\Omega| < \infty$ we obtain:

$$T_\lambda(u) \le \tilde{C}_\gamma\|u\|^{p_2^+} + C_6|\Omega| - C_7\|u\|^\eta$$

for some constants $C_6 > 0$, $C_7 > 0$.

Now letting $\|u\| \to \infty$ we have that

$$\lim_{\|u\|\to\infty} T_\lambda(u) = -\infty. \tag{4.18}$$

Choosing $\gamma_k > \vartheta_k > 0$ and keeping in mind relation (4.18) we obtain that

$$\max\{T_\lambda(u) : u \in Y_k, \|u\| = \gamma_k\} \le 0,$$

for every γ_k large enough.

In order to complete our proof we only have to apply the fountain theorem (for more details we refer to [10, Theorem 4.8], [25, Theorem 6.1], [28, Lemma 3.3]) and the proof is fulfilled. □

As the definition of our double-phase differential operator is general, in what follows we will give some specific examples in order to illustrate our results.

Example 4.8. Consider the following weight coefficient functions $a, b : \Omega \to \mathbb{R}$, with $a, b \in L^\infty(\Omega)_+$ for all $x \in \Omega$. Suppose there exist a constant $C_{a,b} > 0$ such that $a(x), b(x) \geq C_{a,b}$ for all $x \in \Omega$. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function which satisfy the assumptions $(R_1) - (R_4)$, (4.10) then the results of Theorems 4.1, 4.5 hold true for the following class of Dirichlet problems:

$$\begin{cases} -\operatorname{div}\left[a(x)|\nabla u|^{p_1(x)-2}\nabla u + b(x)|\nabla u|^{p_2(x)-2}\nabla u\right] = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{cases}$$

It is easy to check the fact that our differential operator satisfy hypotheses (S_1)–(S_3), (S_4'). Moreover if the reaction function f is odd in respect to the second argument (that is, $f(x,-z) = -f(x,z)$) then Theorem 4.7 holds also true.

Example 4.9. As we stated in the first section of this paper our potential functions ϕ and ψ generalize the following type of differential operator

$$A(x,|z|) = \left(1 + \frac{|z|^{p(x)}}{\sqrt{1+|z|^{2p(x)}}}\right)|z|^{p(x)-2} \tag{4.19}$$

corresponding to the differential operator which describes the capillary phenomenon, so we obtain the following class of double-phase problems:

$$\begin{cases} -\operatorname{div}\left[\left(|\nabla u|^{p_1(x)-2} + \frac{|\nabla u|^{2p_1(x)-2}}{\left(1+|\nabla u|^{2p_1(x)}\right)^{1/2}}\right)\nabla u \right. \\ \qquad\qquad \left. + \left(|\nabla u|^{p_2(x)-2} + \frac{|\nabla u|^{2p_2(x)-2}}{\left(1+|\nabla u|^{2p_2(x)}\right)^{1/2}}\right)\nabla u\right] = \lambda f(x,u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{cases}$$

If hypotheses (4.10), (R_1)–(R_4) hold true, then the results of Theorems 4.1 and 4.5 hold true for this class of problems. Moreover if the reaction term is odd in respect with the second argument (that is, $f(x,-z) = -f(x,z)$) for all $(x,z) \in \Omega \times \mathbb{R}$ then this class of problems admits infinitely many nontrivial weak solutions with higher and higher energies.

By simple computations we could verify that the potential function of type A from relation (4.19) satisfies the assumptions (S_1)–(S_3), (S_4'). For a thorough proof of the validity of our example we can associate the following energy functional to our problem $E_\lambda : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} E_\lambda(u) = &\int_\Omega \frac{1}{p_1(x)}\left[|\nabla u|^{p_1(x)} + \left(1+|\nabla u|^{2p_1(x)}\right)^{1/2}\right]dx \\ &+ \int_\Omega \frac{1}{p_2(x)}\left[|\nabla u|^{p_2(x)} + \left(1+|\nabla u|^{2p_2(x)}\right)^{1/2}\right]dx - \lambda\int_\Omega F(x,u)dx \end{aligned}$$

and recalculate the computations for this functional energy.

In what follows we will construct an example of a reaction function for our problems. As we can observe by relation (4.10) the reaction term variable growth is very general and in order to use an explicit defined nonlinearity in the right-hand side of the problem (P_λ) we have to impose some eloquent conditions.

By relation (4.10) we have that $1 < q(x) < p_1^*(x)$ for all $x \in \overline{\Omega}$. Similarly with the details used in the proof of Proposition 3.8 and 4.3 we may find some functions $r_1 : \overline{\Omega}_0 \to (1,\infty)$ such that $r_1(x) = q(x)$ for all $x \in \overline{\Omega}_0$ and $r_2 : \overline{\Omega}_1 \to (1,\infty)$ such that $r_2(x) = q(x)$ for all $x \in \overline{\Omega}_1$ (where $\overline{\Omega}_0 \cap \overline{\Omega}_1 = \emptyset$). So by relation (4.10) we can state that

$$1 < r_1^- \le r_1^+ < p_1^- \le p_1^+ < p_2^- \le p_2^+ < r_2^- \le r_2^+ < p_1^*(x)$$

for all $x \in \overline{\Omega}$, where

$$r_1^- = \min_{x\in\overline{\Omega}_0} r_1(x) \text{ and } r_1^+ = \max_{x\in\overline{\Omega}_0} r_1(x)$$

and

$$r_2^- = \min_{x\in\overline{\Omega}_1} r_2(x) \text{ and } r_2^+ = \max_{x\in\overline{\Omega}_1} r_2(x).$$

So our reaction function may be defined as

$$f(x,z) = \begin{cases} m(x)|z|^{r_1(x)-2}z, & \text{if } x \in \overline{\Omega_0}, \\ m(x)|z|^{r_2(x)-2}z, & \text{if } x \in \overline{\Omega_1}. \end{cases}$$

We can deduce the fact that $f(x,z)$ has a $|z|^{r_1(x)-1}$ growth near the origin and $|z|^{r_2(x)-1}$ growth near $+\infty$. For more details we refer to the proof of Proposition 3.8 and 4.3 and to [15, Lemma 2]. Also some good examples of this type of reaction nonlinearity can be find in [16]. This last restriction is necessary in order that our function to satisfy the Ambrosetti–Rabinowitz type condition (R_2) (for example we could take $\eta = r_2^-$).

We also need to impose some particular conditions on the weight function $m : \overline{\Omega} \to [0,\infty)$:

(m_1) $m \in L^\infty(\Omega)$;

(m_2) there exist a constant m^- such that $m(x) \ge m^- > 0$ for all $x \in \overline{\Omega}_0 \cup \overline{\Omega}_1$ and $m(x) = 0$ for all $x \in \Omega \setminus (\overline{\Omega}_0 \cup \overline{\Omega}_1)$.

Remark 4.10. For this particular restrictions we can observe that function f defined as above satisfies hypotheses (R_1)–(R_4) and so, our existence theorems hold true.

5 Final remarks

(i) For every $\lambda \in (0, \lambda_{\phi,\psi})$ problem (P_λ) has at least two different solutions. Indeed, suppose that solutions given by Theorem 4.1 and Theorem 4.5 coincide ($v_0 = w_0$), we get that

$$T_\lambda(w_0) = c_1 > 0 > c_0 = T_\lambda(v_0),$$

which is a contradiction. So the multiplicity of every eigenvalue $\lambda \in (0, \lambda_{\phi,\psi})$ is at least two.

(ii) We point out that hypothesis (R_2) plays a crucial role in the proof of our results. This hypothesis is an Ambrosetti–Rabinowitz type condition which implies that our reaction function $f(x,\cdot)$ has at least a $(\eta - 1)$-polynomial growth near $+\infty$.

(iii) Theorems 4.5 and 4.7 have a strong dependency on hypothesis (R_2) whilst the results of Theorem 4.1 hold true using only the weaker hypothesis (R_3).

We may consider the following functions:

- $f_1(x,z) = |z|^{\eta-1}$
- $f_2(x,z) = |z|^{p_2^+-1}\ln(1+|z|)$
- $f_3(x,z) = |z|^{q(x)-2}z$

Only the function f_1 satisfy the Ambrosetti–Rabinowitz condition. We can also remark the fact that the results of Theorem 4.1 hold true if $f(x,u) = m(x)|u|^{q(x)-2}u$, with m defined as in relation (R_1).

(iv) For our results to hold true we can not use superlinear nonlinearities with slower growth near $+\infty$. This type on nonlinearity is represented by function f_2.

(v) It is easy to observe the fact that we have a strong connection between the first and the second type of solutions, whilst the third type of solutions (high-energy solutions) does not depend on the condition that parameter $\lambda \in (0, \lambda_{\phi,\psi})$ but only on the fact that $\lambda > 0$.

(vi) Moreover using hypothesis (4.10) instead of (3.1) we can find some information about the existence of some ground state solutions of problem (P_λ) (that is, solutions which minimizes the functional of the action in the set of all weak solutions), which lie on some Nehari manifold. To this end we refer to [1,26].

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Global bifurcation and nodal solutions for homogeneous Kirchhoff type equations

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Abstract. In this paper, we shall study unilateral global bifurcation phenomenon for the following homogeneous Kirchhoff type problem

$$\begin{cases} -\left(\int_0^1 |u'|^2\,dx\right)u'' = \lambda u^3 + h(x,u,\lambda) & \text{in } (0,1),\\ u(0)=u(1)=0. \end{cases}$$

As application of bifurcation result, we shall determine the interval of λ in which there exist nodal solutions for the following homogeneous Kirchhoff type problem

$$\begin{cases} -\left(\int_0^1 |u'|^2\,dx\right)u'' = \lambda f(x,u) & \text{in } (0,1),\\ u(0)=u(1)=0, \end{cases}$$

where f is asymptotically cubic at zero and infinity. To do this, we also establish a complete characterization of the spectrum of a homogeneous nonlocal eigenvalue problem.

Keywords: bifurcation, spectrum, nonlocal problem, nodal solution, regularity results.
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1 Introduction

Consider the following problem

$$\begin{cases} -\left(\int_0^1 |u'|^2\,dx\right)u'' = \lambda u^3 + h(x,u,\lambda) & \text{in } (0,1),\\ u(0)=u(1)=0, \end{cases} \tag{1.1}$$

where λ is a nonnegative parameter and $h:(0,1)\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function satisfying

$$\lim_{s\to 0}\frac{h(x,s,\lambda)}{s^3}=0 \tag{1.2}$$

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uniformly for all $x \in (0,1)$ and λ on bounded sets.

The problem (1.1) is related to the stationary problem of a model introduced by Kirchhoff in 1883 to describe the transversal oscillations of a stretched string [16]. Some important and interesting results can be found, for example, in [1, 4, 12, 13, 15, 19, 25]. Recently, there are many mathematicians studying the problem (1.1), see [5, 6, 8, 17, 20, 21, 22, 24, 26] and the references therein. A distinguishing feature of problem (1.1) is that the first equation contains a nonlocal coefficient $\int_0^1 |u'|^2\,dx$, and hence the equation is no longer a pointwise identity, which raises some essential difficulties to the study of this kind of problems. In particular, the bifurcation theory of [11, 23] does not work on it.

As shown in [3], the following problem

$$\begin{cases} -\left(\int_0^1 |u'|^2\,dx\right) u'' = \lambda u^3 & \text{in } (0,1), \\ u(0) = u(1) = 0 \end{cases} \tag{1.3}$$

possesses infinitely many eigenvalues $0 < \mu_1 < \mu_2 < \cdots < \mu_k \to +\infty$, all of which are simple. The eigenfunction φ_k corresponding to μ_k has exactly $k-1$ simple zeros in $(0,1)$. Let S_k^+ denote the set of functions in $E := C_0^1[0,1]$ which have exactly $k-1$ interior nodal (i.e. non-degenerate) zeros in (0,1) and are positive near $x = 0$, and set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. It is clear that S_k^+ and S_k^- are disjoint and open in E. Finally, let $\Phi_k^\pm = \mathbb{R} \times S_k^\pm$ and $\Phi_k = \mathbb{R} \times S_k$ under the product topology. The first main result of this paper is the following theorem.

Theorem 1.1. *The pair $(\mu_k, 0)$ is a bifurcation point of (1.1). Moreover, there are two distinct unbounded continua in $\mathbb{R} \times H_0^1(0,1)$, $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$, consisting of the bifurcation branch $\mathscr{C}_k$ emanating from $(\mu_k, 0)$, such that $\mathscr{C}_k^\nu \subseteq \left(\{(\mu_k,0)\} \cup \Phi_k^\nu\right)$, $\nu \in \{+,-\}$.*

It is well known that the index formula of an isolated zero is very important in the study of bifurcation phenomena for semi-linear differential equations. However, problem (1.1) is nonlinear. In order to overcome this difficulty, we study the following auxiliary homogeneous eigenvalue problem

$$\begin{cases} -\left(\int_0^1 |u'|^2\,dx\right)^{p/2} u'' = \lambda |u|^p u & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.4}$$

where $p \in [0,2]$. We study the spectral structure, and establish an index formula via a suitable homotopic deformation from a general $p \in [0,2]$ to $p = 0$ for problem (1.4). Let $\lambda_1(p)$ denote the first eigenvalue of (1.4). As shown in [9], $\lambda_1(p) > 0$ is simple, isolated, the unique principal eigenvalue of (1.4), and is continuous with respect to p. Our second main result is the following theorem.

Theorem 1.2. *The set of all eigenvalues of (1.4) is formed by a sequence*

$$0 < \lambda_1(p) < \lambda_2(p) < \cdots < \lambda_k(p) \to +\infty.$$

Every $\lambda_k(p)$ is simple, continuous with respect to p and the corresponding one dimensional space of solutions of (1.4) with $\lambda = \lambda_k(p)$ is spanned by a function having precisely k bumps in $(0,1)$. Each k-bump solution is constructed by the reflection and compression of the eigenfunction φ_1 associated with $\lambda_1(p)$.

Based on Theorem 1.1, we study the existence of nodal solutions for the following problem

$$\begin{cases} -\left(\int_0^1 |u'|^2\,dx\right) u'' = \lambda f(x,u) & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1.5}$$

We assume that f satisfies the following conditions

(f1) $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x,s)s > 0$ for all $x \in (0,1)$ and any $s \neq 0$.

(f2) there exist f_0, $f_\infty \in (0,+\infty)$ such that

$$\lim_{s\to 0^+} \frac{f(x,s)}{s^3} = f_0, \qquad \lim_{s\to +\infty} \frac{f(x,s)}{s^3} = f_\infty$$

uniformly with respect to all $x \in (0,1)$.

The last main theorem of this paper is the following result.

Theorem 1.3. *Assume that f satisfies (f1)–(f2). Then the pair $(\mu_k/f_0, 0)$ is a bifurcation point of (1.5) and there are two distinct unbounded continua in $\mathbb{R} \times H_0^1(0,1)$, $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$, emanating from $(\mu_k/f_0, 0)$, such that $\mathscr{C}_k^\nu \subseteq \left(\{(\mu_k/f_0, 0)\} \cup \Phi_k^\nu\right)$ and links $(\mu_k/f_0, 0)$ to $(\mu_k/f_\infty, \infty)$.*

The rest of this paper is arranged as follows. In Section 2, we establish the spectrum of problem (1.4). In Section 3 and 4, we give the proofs of Theorem 1.1 and 1.3, respectively.

2 Spectrum of (1.4)

Let X be the usual Sobolev space $H_0^1(0,1)$ with the norm $\|u\| = \left(\int_0^1 |u'|^2\, dx\right)^{1/2}$. For any $\alpha \in (0,1]$, we use $C^\alpha[0,1]$ to denote all the real functions such that

$$\|u\|_\alpha := \sup_{x,y\in[0,1], x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < +\infty.$$

Firstly, we have the following regularity result.

Proposition 2.1. *Any weak solution $u \in X$ of problem (1.4) is also a classical solution, i.e., $u \in C^2[0,1]$ satisfying (1.4).*

Proof. Let u be a nontrivial weak solution of problem (1.4) and

$$f(x) = \frac{\lambda |u(x)|^p u(x)}{\|u\|^p}.$$

Note that

$$H_0^1(0,1) = \left\{u \in AC[0,1] : u' \in L^2(0,1) \text{ and } u(0) = u(1) = 0\right\}.$$

Then it is obvious that $f \in L^2(0,1)$, in fact continuous by the compact embedding $X \hookrightarrow C^{1/2}[0,1]$. According to the definition of weak solution, we have

$$-\left(\int_0^1 |u'|^2\, dx\right)^{\frac{p}{2}} u'' = \lambda |u|^p u$$

in the sense of distribution. It follows that

$$u'(x) = u'(0) - \int_0^x f(t)\, dt.$$

Note that

$$u(x) = \int_0^x u'(t)\,dt.$$

So, we have that

$$u(x) = \int_0^x \left(u'(0) - \int_0^t f(\tau)\,d\tau \right) dt = u'(0)\,x - \int_0^x \int_0^t f(\tau)\,d\tau\,dt.$$

Then, in view of $f \in C[0,1]$, we get that $u \in C^2[0,1]$ and satisfies (1.4). □

Lemma 2.2. *If (λ, u) is a solution of (1.4) and u has a double zero, then $u \equiv 0$.*

Proof. Let u be a solution of (1.4) and $x^* \in [0,1]$ be a double zero. If $\|u\| = 0$, the conclusion is obvious. Next, we assume that $\|u\| \neq 0$. We note that

$$u(x) = -\frac{\lambda}{\|u\|^p} \int_{x^*}^x \int_{x^*}^s |u|^p u\,d\tau\,ds.$$

Firstly, we consider $x \in [0, x^*]$. Then

$$\begin{aligned} |u(x)| &= \left| -\frac{\lambda}{\|u\|^p} \int_{x^*}^x \int_{x^*}^s |u|^p u\,d\tau\,ds \right| \leq \left| \frac{\lambda}{\|u\|^p} \int_{x^*}^x \int_{x^*}^x |u|^p u\,d\tau\,ds \right| \\ &= \left| \frac{\lambda}{\|u\|^p} (x - x^*) \int_{x^*}^x |u|^p u\,d\tau \right| \\ &\leq \frac{\lambda}{\|u\|^p} \int_x^{x^*} |u|^{p+1}\,d\tau \leq \frac{\lambda \|u\|_\infty^p}{\|u\|^p} \int_x^{x^*} |u|\,d\tau \leq \lambda \int_x^{x^*} |u|\,d\tau. \end{aligned}$$

By the Gronwall–Bellman inequality [7, Lemma 2.2], we get $u \equiv 0$ on $[0, x^*]$. Similarly, we can get $u \equiv 0$ on $[x^*, 1]$ and the proof is completed. □

Lemma 2.3. *Each nontrivial solution (λ, u) of (1.4) has a finite number of zeros.*

Proof. Suppose, on the contrary, that u has a sequence zeros x_n. Since $[0,1]$ is compact, up to a subsequence, there exists $x_0 \in [0,1]$ such that $\lim_{n\to+\infty} x_n = x_0$. By the continuity of u, we have that $u(x_0) = \lim_{n\to+\infty} u(x_n) = 0$. So, we have that

$$u'(x_0) = \lim_{n\to+\infty} \frac{u(x_n) - u(x_0)}{x_n - x_0} = 0.$$

Thus, x_0 is a double zero of u. By Lemma 2.2, we get that $u \equiv 0$, which is a contradiction. □

Let J be a strict sub-interval of I. Let $\lambda_1(J)$ denote the first eigenvalue

$$\begin{cases} -\left(\int_0^1 |u'|^2\,dx\right)^{p/2} u'' = \lambda |u|^p u & \text{in } J, \\ u(x) = 0 & \text{on } \partial J, \end{cases}$$

where $p \in [0,2]$.

Lemma 2.4. *$\lambda_1(I)$ verifies the strict monotonicity property with respect to the domain I, i.e. if J is a strict subinterval of I, then $\lambda_1(I) < \lambda_1(J)$.*

Proof. Let φ_1 with $\|\varphi_1\| = 1$ be the eigenfunction of (1.4) on J corresponding to $\lambda_1(J)$, and denote by $\widetilde{\varphi}_1$ the extension by zero on I. Then we have that

$$\frac{1}{\lambda_1(J)} = \int_J |\varphi_1|^{p+2}\,dx = \int_I |\widetilde{\varphi}_1|^{p+2}\,dx < \sup_{u\in X,\|u\|=1} \int_0^1 |u|^{p+2}\,dx = \frac{1}{\lambda_1(I)}.$$

The last strict inequality holds from the fact that $\widetilde{\varphi}_1$ vanishes in $I \setminus J$ so cannot be an eigenfunction corresponding to the principal eigenvalue $\lambda_1(I)$. □

Proof of Theorem 1.2. Let φ_1 be a positive eigenfunction corresponding to $\lambda_1(p)$. It follows from the symmetry of (1.4) and Theorem 3.1 of [9] (or Theorem 2.4 of [18]) that $\varphi_1(x) = \varphi_1(1-x)$ for $x \in [0,1]$, i.e. φ_1 is even with respect to $1/2$. For any $k \geq 2$, set

$$\varphi_k(x) = \begin{cases} \varphi_1(kx), & x \in \left[0, \frac{1}{k}\right], \\ -\varphi_1(kx-1), & x \in \left[\frac{1}{k}, \frac{2}{k}\right], \\ \quad\vdots & \quad\vdots \\ (-)^k \varphi_1(kx-k+1), & x \in \left[\frac{k-1}{k}, 1\right]. \end{cases}$$

Then φ_k is an eigenfunction of (1.4) associated with the eigenvalue $\lambda_k(p) = k^{p+2}\lambda_1(p)$. Clearly, the continuity of $\lambda_1(p)$ implies that $\lambda_k(p)$ is continuous with respect to p.

On the other hand, let $u = u(x)$ be an eigenfunction of (1.4) associated with some eigenvalue $\lambda_* > \lambda_1(p)$. According to Theorem 3.1 of [9], u changes sign in $(0,1)$. Lemmas 2.2 and 2.3 imply that $u \in S_k$ for some $k \geq 2$. Without loss of generality, we may assume that $u'(0) > 0$. Let

$$0 < \tau_1 < \tau_2 < \cdots < \tau_{k-1} < 1$$

denote the zeros of u in $(0,1)$. Without loss of generality, we may assume that $\tau_1 \leq 1/k$. Applying Lemma 2.4 on $[0,1/k]$, we have that $\lambda_* \geq \lambda_k$. By Lemma 2 of [2], there exist integers p and q, $1 \leq p \leq k-1$, $1 \leq q \leq k-1$, such that

$$\tau_p \leq \frac{1}{q+1} < \frac{1}{q} \leq \tau_{p+1}.$$

Applying Lemma 2.4 on $[\tau_p, \tau_{p+1}]$, we have that $\lambda_* \leq \lambda_k$. So we have that $\lambda_* = \lambda_k$. Furthermore, if $\tau_1 < 1/k$, we have that $\lambda_* > \lambda_k$; if $\tau_1 > 1/k$, we have that $\lambda_* < \lambda_k$. Thus we have $\tau_1 = 1/k$ and $u = c_1\varphi_k(x)$ for $x \in [0,1/k]$. Similarly, we can obtain that $\tau_i = i/k$ and $u = c_i\varphi_k(x)$ for $x \in [(i-1)/k, i/k]$, $2 \leq i \leq k-1$. Let us normalize u as $u'(0) = \varphi_k'(0)$. It follows that $c_1 = 1$. Hence $\varphi_k'\left(\frac{1}{k}\right) = c_2\varphi_k'\left(\frac{1}{k}\right)$. So we have $c_2 = 1$. Similarly, one has $c_i = 1$ for all $3 \leq i \leq k-1$. Therefore, we have that $u(x) = \varphi_k(x)$, $x \in [0,1]$. □

3 Global bifurcation

Consider the following auxiliary problem

$$\begin{cases} -\left(\int_0^1 |u'|^2\,dx\right)^{p/2} u'' = f(x) & \text{in } (0,1), \\ u(0) = u(1) = 0 \end{cases} \tag{3.1}$$

for any $p \in [0,2]$ and a given $f \in X^*$. We have shown in [9] that problem (3.1) has a unique weak solution. Let us denote by $R_p(f)$ the unique weak solution of (3.1). Then $R_p : X^* \to X$

is a continuous operator. Since the embedding of $X \hookrightarrow L^\infty(0,1)$ is compact, the restriction of R_p to $L^1(0,1)$ is a completely continuous (i.e., continuous and compact) operator. From the obvious modification of Lemma 4.2 of [9], we can get the following compactness and continuity of the operator R_p with respect to p and f.

Lemma 3.1. *The operator $R : [0,2] \times L^1(0,1) \to L^\infty(0,1)$ defined by $R(p,f) = R_p(f)$ is completely continuous.*

Now, we consider (1.4) again. Clearly, u is a weak solution of (1.4) if and only if $u \in X$, $\lambda \in [0,+\infty)$ satisfy

$$u = R_p\left(\lambda |u|^p u\right) = \lambda^{\frac{1}{p+1}} R_p\left(|u|^p u\right) := T_p^\lambda(u).$$

For any $u \in X$, we define

$$K_p(u) = |u|^p u.$$

Then we see that $K_p(u) \in L^1(0,1)$. We claim that $K_p : X \hookrightarrow L^1(0,1)$ is continuous. Assume that $u_n \to u$ in X. Since embedding $X \hookrightarrow C[0,1]$ is compact, we have $u_n \to u$ in $C[0,1]$. It follows that $u_n(x) \to u(x)$ for any $x \in [0,1]$. So, we have that $K_p\left(u_n\right) \to K_p(u)$ in $L^1(0,1)$. Since $R_p : L^1(0,1) \to X$ is a compact, we have that $T_p^\lambda = \lambda^{\frac{1}{p+1}} R_p \circ K_p : X \to X$ is completely continuous. Thus the Leray–Schauder degree

$$\deg_X\left(I - T_p^\lambda, B_r(0), 0\right)$$

is well-defined for arbitrary r-ball $B_r(0)$ and $\lambda \neq \lambda_k(p)$. It is well known that

$$\deg_X\left(I - T_0^\lambda, B_r(0), 0\right) = (-1)^\beta,$$

where β is the number of eigenvalues of problem (1.4) with $p = 0$ less than λ. As far as the general p, we can compute it through the deformation along p.

Proposition 3.2. *Let $r > 0$ and $\overline{p} \in [0,2]$. Then*

$$\deg_X\left(I - T_{\overline{p}}^\lambda, B_r(0), 0\right) = \begin{cases} 1, & \text{if } \lambda \in \left(0, \lambda_1(\overline{p})\right), \\ (-1)^k, & \text{if } \lambda \in \left(\lambda_k(\overline{p}), \lambda_{k+1}(\overline{p})\right). \end{cases}$$

Proof. If $\lambda \in \left(0, \lambda_1\left(\overline{p}\right)\right)$, the conclusion has done in [9]. So we only need to prove the case $\lambda \in \left(\lambda_k\left(\overline{p}\right), \lambda_{k+1}\left(\overline{p}\right)\right)$. Since $p \to \lambda_k(p)$ is continuous, we can define a continuous function $\chi : [0,2] \to \mathbb{R}$ such that $\lambda_k(p) < \chi(p) < \lambda_{k+1}(p)$ and $\lambda = \chi\left(\overline{p}\right)$. Set

$$d(p) = \deg_X\left(I - T_p^{\chi(p)}, B_r(0), 0\right).$$

We shall show that $d(p)$ is constant in $[0,2]$.

Define $S_p : L^\infty(0,1) \to X$ by $S_p(u) = R_p(\chi(p)|u|^p u)$. We see that $S_p(u) = \chi^{\frac{1}{p+1}}(p) R_p \circ K_p(u)$, where $K_p(u) = |u|^p u$. By the definition of K_p, we can easily verify that $K_p : L^\infty(0,1) \to L^1(0,1)$ is continuous. Since $R_p : L^1(0,1) \to X$ is a compact, we get that $S_p : L^\infty(0,1) \to X$ is completely continuous. Also we have that $T_p^{\chi(p)} = S_p \circ i$ where $i : X \to L^\infty(0,1)$ is the usual inclusion. From Lemma 2.4 of [14], we obtain that

$$d(p) = \deg_{L^\infty}\left(I - i \circ S_p, \Omega_s, 0\right) \quad \text{for } p \in [0,2],$$

where Ω_s is any open bounded set in $L^\infty(0,1)$ containing 0. It is not difficult to verify that the operator $\varphi : [0,2] \times L^\infty(0,1) \to L^1(0,1)$ defined by $\varphi(p,u) = |u|^p u$ is continuous. This fact, the continuity of $\chi(p)$ and Lemma 3.1 imply that $(p,u) \mapsto R_p(\chi(p)|u|^p u) = (i \circ S_p)(u) : [0,2] \times L^\infty(0,1) \to L^\infty(0,1)$ is completely continuous. Since $\lambda_k(p) < \chi(p) < \lambda_{k+1}(p)$ for any $p \in [0,2]$, we have that $u - R_p(\chi(p)|u|^p u) \neq 0$ on $\partial\Omega_s$. The invariance of the Leray–Schauder degree under a compact homotopy follows that $d(p) \equiv$ *constant* for $p \in [0,2]$. So, $d(\overline{p}) = d(0) = (-1)^k$, as desired. □

In particular, we have the following corollary.

Corollary 3.3. *Let $r > 0$. Then*

$$\deg_X\left(I - T_2^\lambda, B_r(0), 0\right) = \begin{cases} 1, & \text{if } \lambda \in (0,\mu_1), \\ (-1)^k, & \text{if } \lambda \in (\mu_k,\mu_{k+1}), \end{cases}$$

where μ_k is the k-th eigenvalue of (1.3).

Clearly, the pair (λ,u) is a solution of (1.1) if and only if (λ,u) satisfies

$$u = R_2\left(\lambda u^3 + h(x,u,\lambda)\right) := G_\lambda(u).$$

It is easy to see that $G_\lambda : X \to X$ is completely continuous and $G_\lambda(0) = 0$, $\forall\lambda \in [0,+\infty)$. μ_k is the λ_k. Let X_0 be any complement of span $\{\varphi_k\}$ in X.

Theorem 3.4. *The pair $(\mu_k,0)$ is a bifurcation point of* (1.1)*. Moreover, there are two distinct continua in $\mathbb{R} \times X$, $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$, consisting of the bifurcation branch $\mathscr{C}_k$ emanating from $(\mu_k,0)$, which contain $\{(\mu_k,0)\}$ and each of them satisfies one of the following non-excluding alternatives:*

1. *it is unbounded in $\mathbb{R} \times X$;*
2. *it contains a pair $(\mu_j,0)$ with $j \neq k$;*
3. *it contains a point $(\lambda,y) \in \mathbb{R} \times (X_0 \setminus \{0\})$.*

Proof. We use the abstract bifurcation result of [10] to prove this theorem. An operator L defined on X is called homogeneous if $L(cu) = cL(u)$ for any $c \in \mathbb{R}$ and $u \in X$. It is not difficult to verify that $L(\lambda) := T_2^\lambda : X \to X$ is homogeneous and completely continuous. Let $\widetilde{h}(x,u,\lambda) = \max_{0\le|s|\le u}|h(x,s,\lambda)|$ for all $x \in (0,1)$ and λ on bounded sets, then $\widetilde{h}$ is nondecreasing with respect to u and

$$\lim_{u\to 0^+}\frac{\widetilde{h}(x,u,\lambda)}{u^3} = 0. \tag{3.2}$$

Further it follows from (3.2) that

$$\frac{h(x,u,\lambda)}{\|u\|^3} \le \frac{\widetilde{h}(x,|u|,\lambda)}{\|u\|_\infty^3} \le \frac{\widetilde{h}(x,\|u\|_\infty,\lambda)}{\|u\|_\infty^3} \to 0 \quad \text{as } \|u\| \to 0 \tag{3.3}$$

uniformly for $x \in (0,1)$ and λ on bounded sets. Let

$$H(\lambda,u) = G_\lambda(u) - L(\lambda)u.$$

By (3.3), we can easily verify that $H : \mathbb{R} \times X \to X$ is completely continuous with $H = o(\|u\|)$ near $u = 0$ uniformly on bounded λ intervals. Noting Corollary 3.3, the desired conclusions can be obtained by applying Theorem 1 of [10]. □

By an argument similar to that of Proposition 2.1, we can get the following regularity result.

Proposition 3.5. *Any weak solution $u \in X$ of problem (1.1) is also a classical solution, i.e., $u \in C^2(0,1) \cap C^{1,\alpha}[0,1]$ satisfying (1.1) and $u(0) = u(1) = 0$.*

Lemma 3.6. *If (λ, u) is a solution of (1.1) and u has a double zero, then $u \equiv 0$.*

Proof. Let u be a solution of (1.1) and $x^* \in [0,1]$ be a double zero. If $\|u\| = 0$, the conclusion is done. Next, we assume that $\|u\| \neq 0$. We note that

$$u(x) = \frac{-1}{\|u\|^2} \int_{x^*}^{x} \int_{x^*}^{s} \left(\lambda u^3 + h(x,u,\lambda)\right) d\tau\, ds.$$

Firstly, we consider $x \in [0, x^*]$. Then

$$\begin{aligned} |u(x)| &\leq \frac{1}{\|u\|^2} \int_{x}^{x^*} \left|\lambda u^3 + h(x,u,\lambda)\right| d\tau, \\ &\leq \frac{\|u\|_\infty^2}{\|u\|^2} \int_{x}^{x^*} \left(|\lambda| + \left|\frac{h(\tau,u(\tau),\lambda)}{u(\tau)}\right|\right) |u(\tau)|\, d\tau. \end{aligned}$$

In view of (1.2), for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that

$$|h(x,s,\lambda)| \leq \varepsilon |s|$$

uniformly with respect to all $x \in (0,1)$ and fixed λ when $|s| \in [0,\delta]$. Hence,

$$|u(x)| \leq \int_{x}^{x^*} \left(|\lambda| + \varepsilon + \max_{s \in [\delta, \|u\|_\infty]} \left|\frac{h(\tau,s,\lambda)}{s^3}\right|\right) |u(\tau)|\, d\tau.$$

By the Gronwall–Bellman inequality [7], we get $u \equiv 0$ on $[0, x^*]$. Similarly, we can get $u \equiv 0$ on $[x^*, 1]$ and the proof is complete. $\square$

Proof of Theorem 1.1. Lemma 3.1 of [10] implies that there exists a bounded open neighborhood $\mathscr{O}_k$ of $(\mu_k, 0)$ such that $\left(\mathscr{C}_k^\nu \cap \mathscr{O}_k\right) \subseteq \left(\Phi_k^\nu \cup \{(\mu_k,0)\}\right)$ or $\left(\mathscr{C}_k^\nu \cap \mathscr{O}_k\right) \subseteq \left(\Phi_k^{-\nu} \cup \{(\mu_k,0)\}\right)$. Without loss of generality, we assume that $\left(\mathscr{C}_k^\nu \cap \mathscr{O}_k\right) \subseteq \left(\Phi_k^\nu \cup \{(\mu_k,0)\}\right)$.

Next, we show that $\mathscr{C}_k^\nu \subseteq \left(\Phi_k^\nu \cup \{(\mu_k,0)\}\right)$. Suppose $\mathscr{C}_k^\nu \nsubseteq \left(\Phi_k^\nu \cup \{(\mu_k,0)\}\right)$. Then there exists $(\mu, u) \in \mathscr{C}_k^\nu \cap \left(\mathbb{R} \times \partial S_k^\nu\right)$ such that $(\mu, u) \neq (\mu_k, 0)$ and $(\lambda_n, u_n) \to (\mu, u)$ with $(\lambda_n, u_n) \in \mathscr{C}_k^\nu \cap \left(\mathbb{R} \times S_k^\nu\right)$. Since $u \in \partial S_k^\nu$, by Lemma 3.6, $u \equiv 0$. Let $v_n := u_n / \|u_n\|$, then v_n should be a solution of the following problem

$$v = R_2\left(\lambda_n v^3 + \frac{h\left(x, u_n, \lambda_n\right)}{\|u_n(x)\|^3}\right). \tag{3.4}$$

By (3.3), (3.4) and the compactness of R_2 we obtain that for some convenient subsequence $v_n \to v_0 \neq 0$ as $n \to +\infty$. Now v_0 verifies the equation

$$-\int_0^1 \left|v'\right|^2 dx v'' = \mu v^3$$

and $\|v_0\| = 1$. Hence $\mu = \mu_j$, for some $j \neq k$. Hence $v_0 \in S_j$ which is an open set in X, and as a consequence for some n large enough, $u_n \in S_j$, and this is a contradiction. Thus, we have that

$$\mathscr{C}_k^\nu \subseteq \left(\Phi_k^\nu \cup \{(\mu_k,0)\}\right).$$

Furthermore, by an argument similar to the above, we can easily show that $\mathscr{C}_k \cap (\mathbb{R} \times \{0\}) = \{(\mu_k, 0)\}$. So Theorem 1 of [10] implies that $\mathscr{C}_k$ is unbounded.

We claim that both $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$ are unbounded. Introduce the following auxiliary problem

$$\begin{cases} -\left(\int_0^1 |u'|^2\, dx\right) u'' = \lambda u^3 + \widetilde{h}(x,u,\lambda) & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where $\widetilde{h}$ is defined by

$$\widetilde{h}(x,u,\lambda) = \begin{cases} h(x,u,\lambda), & \text{if } u'(0) > 0, \\ -h(x,-u,\lambda), & \text{if } u'(0) < 0. \end{cases}$$

The previous argument shows that an unbounded continuum $\widetilde{\mathscr{C}_k}$ bifurcates from $(\mu_k, 0)$ and can be split into $\widetilde{\mathscr{C}_k^+}$ and $\widetilde{\mathscr{C}_k^-}$ with $\widetilde{\mathscr{C}_k^\nu}$ connected, $\widetilde{\mathscr{C}_k^\nu} \subseteq \left(\{(\mu_k,0)\} \cup \left(\mathbb{R} \times S_k^\nu\right)\right)$. It is easy to see that $\widetilde{\mathscr{C}_k^-} = -\widetilde{\mathscr{C}_k^+}$. It follows that both $\widetilde{\mathscr{C}_k^+}$ and $\widetilde{\mathscr{C}_k^-}$ are unbounded. It is clear that $\widetilde{\mathscr{C}_k^+} \subseteq \mathscr{C}_k^+$. Therefore $\mathscr{C}_k^+$ must be unbounded. A symmetric argument shows that $\mathscr{C}_k^-$ is also unbounded. □

4 Nodal solutions

In this section, we apply Theorem 1.1 to study the existence of nodal solutions for (1.5).

Proof of Theorem 1.3. Let $g : (0,1) \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$f(x,s) = f_0 s^3 + g(x,s)$$

with

$$\lim_{s \to 0} \frac{g(x,s)}{s^3} = 0 \quad \text{uniformly with respect to all } x \in (0,1). \tag{4.1}$$

From (4.1), we can see that λg satisfies the assumptions of (1.2). Now, using Theorem 1.1, we have that there are two distinct unbounded continua, $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$ emanating from $(\mu_k/f_0, 0)$, such that

$$\mathscr{C}_k^\nu \subset \left(\{(\mu_k/f_0, 0)\} \cup \Phi_k^\nu\right).$$

It is sufficient to show that $\mathscr{C}_k^\nu$ joins $(\mu_k/f_0, 0)$ to $(\mu_k/f_\infty, \infty)$. Let $(\xi_n, u_n) \in \mathscr{C}_k^\nu$ where $u_n \not\equiv 0$ satisfies $|\xi_n| + \|u_n\| \to +\infty$. Proposition 5.1 of [8] implies that (0,0) is the only solution of (1.5) for $\lambda = 0$, we have $\mathscr{C}_k^\nu \cap (\{0\} \times X) = \emptyset$. It follows that $\xi_n > 0$ for all $n \in \mathbb{N}$.

Next we show that u_n is one-signed in some interval $(\alpha, \beta) \subseteq (0,1)$ with $\alpha < \beta$. Let

$$0 < \tau(1,n) < \tau(2,n) < \cdots < \tau(k-1,n) < 1$$

denote the zeros of u_n in $(0,1)$. Let $\tau(0,n) = 0$ and $\tau(k,n) = 1$. Then, after taking a subsequence if necessary,

$$\lim_{n \to +\infty} \tau(l,n) = \tau(l,\infty), \qquad l \in \{0,1,\ldots,k\}.$$

We claim that there exists $l_0 \in \{0,1,\ldots,k\}$ such that

$$\tau(l_0, \infty) < \tau(l_0 + 1, \infty).$$

Otherwise, we have that

$$1 = \Sigma_{l=0}^{k-1}(\tau(l+1,n) - \tau(l,n)) \to \Sigma_{l=0}^{k-1}(\tau(l+1,\infty) - \tau(l,\infty)) = 0.$$

This is a contradiction. Let $(\alpha,\beta) \subset (\tau(l_0,\infty), \tau(l_0+1,\infty))$ with $\alpha < \beta$. For all n sufficiently large, we have $(\alpha,\beta) \subset (\tau(l_0,n), \tau(l_0+1,n))$. So u_n does not change its sign in (α,β).

We claim that there exists a constant M such that $\xi_n \in (0,M]$ for $n \in \mathbb{N}$ large enough. On the contrary, we suppose that $\lim_{n\to+\infty} \xi_n = +\infty$. Since $(\xi_n, u_n) \in \mathscr{C}_k^\nu$, it follows that

$$\|u_n\|^2 u_n'' + \xi_n a_n(x) u_n^3 = 0 \quad \text{in } (0,1),$$

where

$$a_n(x) = \begin{cases} \frac{f(x,u_n)}{u_n^3}, & \text{if } u_n(x) \neq 0, \\ f_0, & \text{if } u_n(x) = 0. \end{cases}$$

From (f1)–(f2), we can see that $\frac{f(x,u_n)}{u_n} \geq \sigma$ for some $\sigma > 0$ and all $x \in (0,1)$, $n \in \mathbb{N}$. So, we have that $\xi_n a_n(x) = +\infty$ for all $x \in (0,1)$. Applying Theorem 4.1 of [3] on $[\alpha,\beta]$ with $g(x) \equiv \mu_1$, we have that u_n must change its sign in (α,β) for n large enough. This is a contradiction.

Therefore, we get that

$$\|u_n\| \to +\infty \quad \text{as } n \to +\infty.$$

Let $h : (0,1) \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$f(x,s) = f_\infty s^3 + h(x,s)$$

with

$$\lim_{|s|\to+\infty} \frac{h(x,s)}{s^3} = 0, \qquad \lim_{|s|\to 0} \frac{h(x,s)}{s^3} = f_0 - f_\infty \quad \text{uniformly with respect to all } x \in (0,1).$$

Then (ξ_n, u_n) satisfies

$$u_n = R_2\left(\xi_n f_\infty u_n^3 + h(x,u_n)\right).$$

Dividing the above equation by $\|u_n\|$ and letting $\overline{u}_n = u_n / \|u_n\|$, we get that

$$\overline{u}_n = R_2\left(\xi_n f_\infty \overline{u}_n^3 + \frac{h(x,u_n)}{\|u_n\|^3}\right).$$

Let

$$\widetilde{h}(x,u) = \max_{0\leq|s|\leq u} |h(x,s)| \quad \text{for any } x \in (0,1),$$

then $\widetilde{h}$ is nondecreasing with respect to u. Define

$$\overline{h}(x,u) = \max_{u/2\leq|s|\leq u} |h(x,s)| \quad \text{for any } x \in (0,1).$$

Then we can see that

$$\lim_{u\to+\infty} \frac{\overline{h}(x,u)}{u^3} = 0 \quad \text{and} \quad \widetilde{h}(x,u) \leq \widetilde{h}\left(x, \frac{u}{2}\right) + \overline{h}(x,u).$$

It follows that

$$\limsup_{u\to+\infty} \frac{\widetilde{h}(x,u)}{u^3} \leq \limsup_{u\to+\infty} \frac{\widetilde{h}\left(x,\frac{u}{2}\right)}{u^3} = \limsup_{u/2\to+\infty} \frac{\widetilde{h}\left(x,\frac{u}{2}\right)}{8\left(\frac{u}{2}\right)^3}.$$

So we have

$$\lim_{u\to+\infty}\frac{\widetilde{h}(x,u)}{u^3}=0. \tag{4.2}$$

Further it follows from (4.2) that

$$\frac{h\left(x,u_n\right)}{\|u_n\|^3}\le\frac{\widetilde{h}\left(x,|u_n|\right)}{\|u_n\|^3}\le\frac{\widetilde{h}\left(x,\|u_n\|_\infty\right)}{\|u_n\|^3}\le c^3\frac{\widetilde{h}\left(x,c\,\|u_n\|\right)}{c^3\,\|u_n\|^3}\to 0\quad\text{as } n\to+\infty$$

uniformly for $x\in(0,1)$.

By the compactness of R_2 we obtain that

$$-\|\overline{u}\|^2\,\overline{u}''=\overline{\mu}f_\infty\overline{u}^3,$$

where $\overline{u}=\lim_{n\to+\infty}\overline{u}_n$ and $\overline{\mu}=\lim_{n\to+\infty}\xi_n$, again choosing a subsequence and relabeling it if necessary. It follows from $\overline{u}=\lim_{n\to+\infty}\overline{u}_n$ and the triangle inequality that $\|\overline{u}\|=\lim_{n\to+\infty}\|\overline{u}_n\|$. Since $\|\overline{u}_n\|\equiv 1$, we obtain that $\|\overline{u}\|=1$. It is clear that $\overline{u}\in\mathscr{C}_k^\nu$. Theorem 1.2 of [3] shows that $\overline{\mu}=\mu_k/f_\infty$. Therefore, $\mathscr{C}$ joins $(\mu_k/f_0,0)$ to $(\mu_k/f_\infty,\infty)$. □

From Theorem 1.3, we can easily get the following corollary.

Corollary 4.1. *Assume that f satisfies (f1)–(f2). Then for*

$$\lambda\in\left(\frac{\mu_k}{f_0},\frac{\mu_k}{f_\infty}\right)\cup\left(\frac{\mu_k}{f_\infty},\frac{\mu_k}{f_0}\right),$$

problem (1.5) possesses at least two solutions u_k^+ and u_k^- such that u_k^+ has exactly $k-1$ simple zeros in $(0,1)$ and is positive near 0, and u_k^- has exactly $k-1$ simple zeros in $(0,1)$ and is negative near 0.

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A positive solution of asymptotically periodic Schrödinger equations with local superlinear nonlinearities

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Abstract. In this paper, we investigate the following Schrödinger equation

$$-\Delta u + V(x)u = \lambda f(u) \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\lambda > 0$, V is an asymptotically periodic potential and the nonlinearity term $f(u)$ is only locally defined for $|u|$ small and satisfies some mild conditions. By using Nehari manifold and Moser iteration, we obtain the existence of positive solutions for the equation with sufficiently large λ.

Keywords: Schrödinger equation, positive solution, locally defined nonlinearity, asymptotically periodic potential.

2020 Mathematics Subject Classification: 35J50, 35A01, 35B09, 35D30.

1 Introduction

In recent years, many researchers consider the following Schrödinger equation

$$-\Delta u + V(x)u = f(x,u), \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, V is a given potential and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. Knowledge of the solutions of Eq. (1.1) has a great importance for studying standing wave solutions for

$$ih\frac{\partial \Psi}{\partial t} = -h^2\Delta\Psi + W(x)\Psi - f(x,\Psi), \quad \text{for all } x \in \Omega, \tag{NLS}$$

where $h > 0$, W is the real-valued potential and Ω is a domain in $\mathbb{R}^N$. Eq. (NLS) is one of the main objects of the quantum physics, because it appears in problems involving nonlinear optics, plasma physics and condensed matter physics.

Eq. (1.1) has been researched intensively, see [1,3,5,7,10,11,13,14,19,21,22,28] and references therein. In the above works, we observe that many interesting conditions on f have been studied. Notice that, it seems necessary that the condition can be assumed on f at infinity, that is, f is assumed to be subcritical (or critical) at infinity, i.e.,

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(f_0) $0 \leq \lim_{|s|\to+\infty} \frac{f(x,s)s}{|s|^{2^*}} < +\infty$ uniformly for $x \in \mathbb{R}^N$,

where the number 2^* is denoted by $\frac{2N}{N-2}$ and called the critical Sobolev exponent for the embedding of $H^1(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$. This aim is to ensure that the associated energy functional would be well defined and of class C^1 on $H^1(\mathbb{R}^N)$, and then its critical points are precisely the solutions of Eq. (1.1) by using variational methods. Certainly, many researchers tried to seek some suitable conditions to replace (f_0). If there does not exist an assumption on f at infinity, can it be proved that there exists a nontrivial solution for Eq. (1.1)? Mathematically this problem is interesting. Accordingly, Costa and Wang [9] have considered the following equation

$$-\Delta u = \lambda f(u), \quad \text{in } \Omega,$$

where $\lambda > 0$ is a parameter, Ω is a bounded smooth domain in $\mathbb{R}^N (N \geq 3)$, and $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 satisfying the following conditions:

(f_1) $f(-u) = -f(u)$ for any $|u| \leq \delta$ (for some $\delta > 0$);

(f_2) there exists $\gamma \in (2, 2^*)$ such that $\limsup_{|s|\to 0} \frac{f(s)s}{|s|^\gamma} = 0$;

(f_3) there exists $\beta \in (2, 2^*)$ such that $\liminf_{|s|\to 0} \frac{f(s)s}{|s|^\beta} > 0$;

(f_4) there exists $\mu \in (2, 2^*)$ such that $sf(s) \geq \mu F(s) > 0$ for all $|s|$ small, where $F(s) = \int_0^s f(t)dt$.

Motivated by Costa and Wang [9], do Ó et al. [12] have studied the following equation

$$-\Delta u + V(x)u = \lambda f(u) \quad \text{in } \mathbb{R}^N, \tag{$\mathcal{P}$}$$

where V satisfies (V_1)–[(V_2) or (V_3)],

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) > 0$,

(V_2) $V(x) \to \infty$ as $|x| \to \infty$, or more generally, for every $M > 0$, $\text{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < +\infty$,

(V_3) the function $[V(x)]^{-1}$ belongs to $L^1(\mathbb{R}^N)$,

and $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 satisfying $(f_1') - (f_2')$ and (f_4),

(f_1') there exists $p \in (2, 2^*)$ such that $\limsup_{|s|\to 0} \frac{f(s)s}{|s|^p} < +\infty$,

(f_2') there exists $q \in (2, 2^*)$ such that $\liminf_{|s|\to 0} \frac{F(s)}{|s|^q} > 0$, where $F(s) = \int_0^s f(t)dt$.

Further results for related problems can be found in [8,15,23,24] and references therein.

Inspired by the above works, we are concerned with the existence of positive solutions for asymptotically periodic Eq. ($\mathcal{P}$) with a locally defined nonlinearity term, namely V satisfies (V_4),

(V_4) there exists a 1-periodic function $V_\infty(x) \in L^\infty(\mathbb{R}^N)$ such that $0 \leq V(x) \leq V_\infty(x)$, $\inf_{x\in\mathbb{R}^N} V_\infty(x) > 0$ and $V(x) - V_\infty(x) \in \mathcal{F}_1$, where

$$\mathcal{F}_1 := \{h(x) : \text{for any } \varepsilon > 0,\ \text{meas}\{x \in B_1(y) : |h(x)| \geq \varepsilon\} \to 0 \text{ as } |y| \to \infty\},$$

and f satisfies (f_5)–(f_6),

(f_5) $f \in C(\mathbb{R}, \mathbb{R})$ and there exist $p > 2, \delta \in (0, 1)$ such that the function $s \mapsto \frac{f(s)}{s^{p-1}}$ is nondecreasing and $f(s) > 0$ on $(0, \delta]$,

(f_6) there exists $q \in (2, 2^*)$ such that $\liminf_{s \to 0^+} \frac{F(s)}{s^q} > 0$, where $F(s) = \int_0^s f(t)dt$.

As is well known, if f were assumed to be superlinear and subcritical (or critical) at infinity, then the associated energy functional

$$\mathcal{I}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx - \lambda \int_{\mathbb{R}^N} F(u)dx$$

would be of class C^1 on $H^1(\mathbb{R}^N)$ and has the mountain pass geometry. Classically, it is a minimax principle that shows the mountain pass level is a critical level of the functional (see [4,5,26]). Here, the assumptions (f_5)–(f_6) we make on the nonlinearity $f(u)$ refer solely to its behavior in a neighborhood of $u = 0$, and we will show that they suffice for the existence of a positive solution of Eq. $(\mathcal{P})$ when λ is large enough. Exactly we give our main result.

Theorem 1.1. *Assume that $N \geq 3$, (V_4) and (f_5)–(f_6) hold. Then there exists $\lambda_1 > 0$ such that Eq. $(\mathcal{P})$ has a positive solution for $\lambda \geq \lambda_1$.*

Remark 1.2. In this paper, we study the existence of positive solutions for Schrödinger equations with the assumptions of Theorem 1.1 that has never been investigated. For the case where the nonlinear term is only locally defined for $|u|$ small, we should point out that we refer [8,9,12,15] for references in this direction. Costa and Wang [9] considered Eq. $(\mathcal{P})$ in bound domain. do Ó et al. [12] considered Eq. $(\mathcal{P})$ when V was coercive potential or satisfied that $[V(x)]^{-1}$ belongs to $L^1(\mathbb{R}^N)$. Li and Zhong [15] studied the Kirchhoff equation when the nonlinearity term was sub-linear growth. Chu and Liu [8] investigated quasi-linear Schrödinger equations in the radial space. In these papers, they have the compactness and get certain solutions easily. However, in our cases we do not have compact embedding, which is the main difficulty in this paper. Due to this difficult, the methods in [8,9,12,15] fail in our case, so we will use a different way to overcome the lack of compactness.

We now make some comments on the key ingredients of the analysis in this paper. Following the idea of [8,9,12,15], we first extend the nonlinear term f and introduce a modified nonlinear Schrödinger equation. Next, we show by variational methods that the modified nonlinear Schrödinger equation possesses a positive ground state solution. Finally, our approach is inspired by the results of [2,6,9,12,26] and is based on the fact that we can show a priori bound of the form

$$|u|_\infty < C\lambda^{-\beta}, \qquad \beta > 0,$$

for a class of solutions for the modified nonlinear Schrödinger equation.

The organization of this paper is as follows. In the next section we reserve for setting the framework and establishing some preliminary results. Theorem 1.1 is proved in Section 3.

2 Preliminaries

From now on, we will use the following notations.

- $H^1(\mathbb{R}^N)$ is the usual Sobolev space endowed with the usual norm

$$\|u\|_H^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)dx.$$

- $L^p(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the norm

$$|u|_p^p = \int_{\mathbb{R}^N} |u|^p dx \quad \text{and} \quad |u|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |u(x)| \quad \text{for all } p \in [1, \ +\infty).$$

- $E := \{u \in L^2(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} V(x)u^2 dx < +\infty\}$ has the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx.$$

- $\operatorname{meas}\Omega$ denotes the Lebesgue measure of the set Ω.
- $u^\pm := \max\{\pm u, 0\}$ and $K := \{u \in E : u^+ \neq 0\}$.
- $\langle \cdot, \cdot \rangle$ denotes action of dual.
- $B_r(y) := \{x \in \mathbb{R}^N : |x - y| \leq r\}$ and $B_r := \{x \in \mathbb{R}^N : |x| \leq r\}$.
- C denotes a positive constant and is possibly various in different places.

We work in the space E and recall some facts that the norms $\|\cdot\|$ and $\|\cdot\|_H$ are equivalent and $E \hookrightarrow L^s(\mathbb{R}^N)$ for any $s \in [2, 2^*]$ is continuous. The proof can be done similarly to that in [19] and details are omitted here. We start by observing that $(f_5) - (f_6)$ imply that $p \leq q$ and

$$|f(s)s| \leq C|s|^p, \quad \text{for any } |s| \leq \delta.$$

In order to prove our main result via variational methods, we need to modify and extend $f(u)$ for outside a neighborhood of $u = 0$ to get $\widetilde{f}(u)$. We set

$$\widetilde{f}(s) := \begin{cases} 0, & s \leq 0, \\ f(s), & 0 < s \leq \delta, \\ C_1 s^{p-1}, & \delta < s, \end{cases}$$

and fix $C_1 > 0$ such that $\widetilde{f} \in C(\mathbb{R}, \mathbb{R}^+)$. Combining with the definition of $\widetilde{f}$, one can easily obtain the following lemma.

Lemma 2.1. *Suppose that (f_5) hold. Then*

(a) $\lim_{s\to+\infty} \frac{\widetilde{F}(s)}{s^2} = +\infty$, *where* $\widetilde{F}(s) = \int_0^s \widetilde{f}(t)dt$,

(b) there exists $C > 0$ such that $|\widetilde{f}(s)s| \leq C|s|^p$ and $|\widetilde{F}(s)| \leq C|s|^p$ for all $s \in \mathbb{R}$,

(c) there exists $\mu \in (2, p)$ such that the function $s \mapsto \frac{\widetilde{f}(s)}{s^{\mu-1}}$ is strictly increasing on $(0, +\infty)$,

Now let us consider the modified equation of Eq. ($\mathcal{P}$) given by

$$\begin{cases} -\Delta u + V(x)u = \lambda \widetilde{f}(u), \\ u \in E. \end{cases} \tag{$\widetilde{\mathcal{P}}$}$$

The corresponding energy functional

$$\widetilde{\mathcal{I}}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx - \lambda \int_{\mathbb{R}^N} \widetilde{F}(u)dx$$

is of class C^1 by a standard argument and whose derivative is given by

$$\langle \widetilde{\mathcal{I}}'(u), v\rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)dx - \lambda \int_{\mathbb{R}^N} \widetilde{f}(u)v dx, \qquad v \in E.$$

Formally, critical points of $\widetilde{\mathcal{I}}$ are solutions of Eq. ($\widetilde{\mathcal{P}}$). We note that critical points of $\widetilde{\mathcal{I}}$ with L^∞-norm less than or equal to δ are also solutions of the original Eq. ($\mathcal{P}$). We recall the Nehari manifold

$$\mathcal{N} := \left\{ u \in E \setminus \{0\} : \langle \widetilde{\mathcal{I}}'(u), u\rangle = 0 \right\} = \left\{ u \in K : \langle \widetilde{\mathcal{I}}'(u), u\rangle = 0 \right\},$$

and set

$$c := \inf_{u \in \mathcal{N}} \widetilde{\mathcal{I}}(u).$$

Lemma 2.2. *Suppose that* (V_4) *and* (f_5) *hold. Then*

(a) for any $u \in K$ *, there exists a unique* $t_u > 0$ *such that* $t_u u \in \mathcal{N}$*. Moreover, the maximum of* $\widetilde{\mathcal{I}}(tu)$ *for* $t > 0$ *is achieved at* t_u*,*

(b) there exists $\rho > 0$ *such that* $\|u\| \geq \rho$ *for all* $u \in \mathcal{N}$*,*

(c) the functional $\mathcal{I}$ *is bounded from below on* $\mathcal{N}$ *by a positive constant.*

Proof. (a) For any $u \in K$, we define

$$\Psi(t) := \widetilde{\mathcal{I}}(tu) = \frac{t^2}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx - \lambda \int_{\mathbb{R}^N} \widetilde{F}(tu)dx, \qquad t \in (0, +\infty).$$

It follows from (b) of Lemma 2.1 and the Sobolev inequality that

$$\int_{\mathbb{R}^N} \widetilde{F}(tu)dx \leq C \int_{\mathbb{R}^N} |tu|^p dx \leq Ct^p\|u\|^p.$$

Thus one has

$$\Psi(t) \geq \frac{t^2}{2}\|u\|^2 - \lambda C t^p \|u\|^p.$$

Then there exists $t_0 > 0$ such that $\Psi(t_0) > 0$. We set $\Omega = \{x \in \mathbb{R}^N : u(x) > 0\}$. Combining (a) in Lemma 2.1 with Fatou's lemma, we have

$$\liminf_{t\to\infty} \int_\Omega \frac{\widetilde{F}(tu)}{(tu)^2} u^2 dx = +\infty.$$

Hence

$$\limsup_{t\to\infty} \frac{\Psi(t)}{t^2} = \frac{1}{2}\|u\|^2 - \lambda \liminf_{t\to\infty} \int_\Omega \frac{\widetilde{F}(tu)}{t^2} dx = \frac{1}{2}\|u\|^2 - \lambda \liminf_{t\to\infty} \int_\Omega \frac{\widetilde{F}(tu)}{(tu)^2} u^2 dx = -\infty.$$

One could deduce $\Psi(t) \to -\infty$ as $t \to +\infty$. So there exists $t_u > 0$ such that $\Psi(t_u) = \max_{t>0} \Psi(t)$ and $\Psi'(t_u) = 0$, i.e., $\widetilde{\mathcal{I}}(t_u u) = \max_{t>0} \widetilde{\mathcal{I}}(tu)$ and $t_u u \in \mathcal{N}$. Suppose that there exists $t_1 > t_2 > 0$ such that $t_i u \in \mathcal{N}, i = 1, 2$, one has

$$\int_\Omega \frac{\widetilde{f}(t_1 u) u^2}{t_1 u} dx = \int_\Omega \frac{\widetilde{f}(t_2 u) u^2}{t_2 u} dx,$$

which contradicts (c) of Lemma 2.1. Thus we can conclude that t_u is unique.

(b) For any $u \in \mathcal{N}$, combining the Sobolev embedding and (b) of Lemma 2.1, one obtains

$$\|u\|^2 = \lambda \int_{\mathbb{R}^N} \widetilde{f}(u) u dx \le C\lambda \int_{\mathbb{R}^N} |u|^p dx \le C\lambda \|u\|^p. \tag{2.1}$$

It follows from (2.1) that there exists $\rho > 0$ independent of u such that

$$\rho \le \|u\|.$$

(c) Also from (b) of Lemma 2.1 and the Sobolev inequality, we have

$$\widetilde{\mathcal{I}}(u) \ge \frac{1}{2}\|u\|^2 - C\lambda \|u\|^p.$$

Since $p > 2$, there exists $\sigma > 0$ such that $\widetilde{\mathcal{I}}(u) \ge \frac{\sigma^2}{4} > 0$ for $\|u\| = \sigma > 0$. For any $v \in \mathcal{N}$, there exists $t' > 0$ such that $t'\|v\| = \sigma$. Combining with (a)-(b) of Lemma 2.2, one obtains

$$\widetilde{\mathcal{I}}(v) \ge \widetilde{\mathcal{I}}(t'v) \ge \frac{\sigma^2}{4}.$$

This completes the proof. □

From Lemmas 2.1–2.2, one can easily know (see also [19,26])

$$c = \inf_{u \in \mathcal{N}} \widetilde{\mathcal{I}}(u) = \inf_{u \in K} \sup_{t>0} \widetilde{\mathcal{I}}(tu) = \min_{\gamma \in \Gamma} \max_{t \in [0,1]} \widetilde{\mathcal{I}}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \widetilde{\mathcal{I}}(\gamma(t)) < 0\}$. Notice that, $c > 0$ from (c) of Lemma 2.2. In order to prove our results, we introduce the following equation

$$-\Delta u + V_\infty(x) u = \lambda \widetilde{f}(u), \tag{$\mathcal{P}_\infty$}$$

and it follows from [16,19,26,27] that Eq. ($\mathcal{P}_\infty$) has a positive ground state solution ω. From Lemma 2.2 and (V_4), there exists a unique $t_\omega > 0$ such that $t_\omega \omega \in \mathcal{N}$ and

$$c \le \widetilde{\mathcal{I}}(t_\omega \omega) \le \widetilde{\mathcal{I}}_\infty(t_\omega \omega) \le \widetilde{\mathcal{I}}_\infty(\omega) := c_\infty, \tag{2.2}$$

where $\widetilde{\mathcal{I}}_\infty$ is the energy functional associated with Eq. ($\mathcal{P}_\infty$).

Lemma 2.3. *Suppose that (V_4) and (f_5) hold. If $u \in \mathcal{N}$ and $\widetilde{\mathcal{I}}(u) = c$, then u is a nontrivial solution of Eq. ($\widetilde{\mathcal{P}}$).*

Proof. Inspired by the method in [18], one supposes by contradiction that u is not a nontrivial solution of Eq. ($\widetilde{\mathcal{P}}$). Then there exists $\phi \in E$ such that

$$\langle \widetilde{\mathcal{I}}'(u), \phi \rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla \phi + V(x) u \phi) dx - \lambda \int_{\mathbb{R}^N} \widetilde{f}(u) \phi dx < -1.$$

Let $\varepsilon \in (0,1)$ be small enough. Then

$$\langle \widetilde{\mathcal{I}}'(tu+s\phi), \phi\rangle \le -\frac{1}{2}, \quad \text{for any } |t-1|\le\varepsilon,\ |s|\le\varepsilon. \tag{2.3}$$

We set a curve

$$\gamma(t) = tu + s\tau(t)\phi, \qquad t>0,$$

where $\tau \in C(\mathbb{R},[0,1])$ is a smooth cut-off function such that $\tau(t)=1$ for $|t-1|\le\frac{\varepsilon}{2}$, $\tau(t)=0$ for $|t-1|\ge\varepsilon$. Obviously, γ is a continuous. We can claim that $\widetilde{\mathcal{I}}(\gamma(t))<c$ for any $t\in(0,+\infty)$.

Indeed, it follows from Lemma 2.2 that $\widetilde{\mathcal{I}}(\gamma(t))=\widetilde{\mathcal{I}}(tu)<\widetilde{\mathcal{I}}(u)=c$ for $|t-1|\ge\varepsilon$. When $|t-1|<\varepsilon$, owing to $\Phi(s):=\widetilde{\mathcal{I}}(tu+s\tau(t)\phi)$ is of C^1 on $[0,\varepsilon]$, there exists $\bar{s}\in(0,\varepsilon)$ such that

$$\widetilde{\mathcal{I}}(tu+s\tau(t)\phi)=\widetilde{\mathcal{I}}(tu)+\langle\widetilde{\mathcal{I}}'(tu+\bar{s}\tau(t)\phi),\varepsilon\tau(t)\phi\rangle\le\widetilde{\mathcal{I}}(tu)-\frac{1}{2}\varepsilon\tau(t)<c,$$

where the inequality holds from (2.3). Hence $\widetilde{\mathcal{I}}(\gamma(t))<c$ for any $t\in(0,+\infty)$.

We denote $\mathcal{J}(u)=\langle\widetilde{\mathcal{I}}'(u),u\rangle$. According to Lemma 2.2 and the definition of γ , we have $\mathcal{J}(\gamma(1-\varepsilon))=\mathcal{J}((1-\varepsilon)u)>0$ and $\mathcal{J}(\gamma(1+\varepsilon))=\mathcal{J}((1+\varepsilon)u)<0$. By the continuity of $t\mapsto\mathcal{J}(\gamma(t))$ there exists $t'\in(1-\varepsilon,1+\varepsilon)$ such that $\mathcal{J}(\gamma(t'))=0$. Thus $\gamma(t')\in\mathcal{N}$ and $\widetilde{\mathcal{I}}(\gamma(t'))<c$, which is a contradiction. This completes the proof. □

Lemma 2.4. *Suppose that (V_4) and (f_5) hold. Then the Cerami sequence for $\widetilde{\mathcal{I}}$ at level $m>0$ (shortly: $(Ce)_m$ sequence) is bounded in E.*

Proof. We recall the $(Ce)_m$ sequence $\{u_n\}$, that is,

$$\widetilde{\mathcal{I}}(u_n)\to m, \qquad \|\widetilde{\mathcal{I}}'(u_n)\|(1+\|u_n\|)\to 0.$$

Then

$$o(1)=\langle\widetilde{\mathcal{I}}'(u_n),u_n^-\rangle=-\|u_n^-\|^2.$$

Consequently we could deduce that $\{u_n^+\}$ is also a $(Ce)_m$ sequence. For the sake of convenience, we denote u_n^+ by u_n. By a contradiction, we assume that $\|u_n\|\to+\infty$ and set $v_n=\frac{u_n}{\|u_n\|}$. Obviously up to a subsequence, there exists a nonnegative function $v\in E$ such that $v_n\rightharpoonup v\in E, v_n\to v\in L^2_{\text{loc}}(\mathbb{R}^N)$ and $v_n(x)\to v(x)$ a.e. in $\mathbb{R}^N$. We denote $\Omega_1=\{x\in\mathbb{R}^N: v(x)>0\}$. If $\operatorname{meas}\Omega_1>0$, Fatou's lemma and (a) of Lemma 2.1 imply

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}\frac{\widetilde{F}(u_n)}{u_n^2}v_n^2dx\ge\liminf_{n\to\infty}\int_{\Omega_1}\frac{\widetilde{F}(u_n)}{u_n^2}v_n^2dx=+\infty.$$

Then

$$0=\limsup_{n\to\infty}\frac{\widetilde{\mathcal{I}}(u_n)}{\|u_n\|^2}=\frac{1}{2}-\lambda\liminf_{n\to\infty}\int_{\mathbb{R}^N}\frac{\widetilde{F}(u_n)}{u_n^2}v_n^2dx=-\infty,$$

which is a contradiction. Thus $v=0$. We denote

$$\alpha:=\lim_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_1(z)}v_n^2dx. \tag{2.4}$$

If $\alpha=0$, we have $v_n\to0$ in $L^p(\mathbb{R}^N)$ from the Lions lemma [17,26]. Combining with (b) of Lemma 2.1, we obtain $\int_{\mathbb{R}^N}\widetilde{F}(2\sqrt{m}v_n)dx=o(1)$. By the continuity of $\widetilde{\mathcal{I}}$, there exists $t_n\in[0,1]$

such that $\widetilde{\mathcal{I}}(t_n u_n) = \max_{t\in[0,1]} \widetilde{\mathcal{I}}(tu_n)$. Since $\|u_n\| \to +\infty$, one has $\frac{2\sqrt{m}}{\|u_n\|} \leq 1$ as n large enough. We observe that

$$\begin{aligned}\widetilde{\mathcal{I}}(t_n u_n) + o(1) &\geq \widetilde{\mathcal{I}}\left(\frac{2\sqrt{m}}{\|u_n\|}u_n\right) + o(1) = 2m\|v_n\|^2 - \lambda \int_{\mathbb{R}^N} \widetilde{F}(2\sqrt{m}v_n)dx + o(1)\\ &= 2m + o(1).\end{aligned}$$

In view of $\widetilde{\mathcal{I}}(u_n) \to m$ and (a) of Lemma 2.2, we can see that $t_n \in (0,1)$ and $\langle \widetilde{\mathcal{I}}'(t_n u_n), t_n u_n\rangle = 0$ as n large enough. Hence by Lemma 2.3 in [20], one has

$$\begin{aligned} m &= \widetilde{\mathcal{I}}(u_n) + o(1)\\ &= \widetilde{\mathcal{I}}(u_n) - \frac{1}{\mu}\langle \widetilde{\mathcal{I}}'(u_n), u_n\rangle + o(1)\\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + V(x)u_n^2\right) dx + \lambda \int_{\mathbb{R}^N} \left(\frac{1}{\mu}\widetilde{f}(u_n)u_n - \widetilde{F}(u_n)\right) dx + o(1)\\ &\geq \frac{\mu-2}{2\mu}t_n^2 \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + V(x)u_n^2\right) dx + \lambda \int_{\mathbb{R}^N} \frac{1}{\mu}\widetilde{f}\left(t_n u_n\right) t_n u_n - \widetilde{F}\left(t_n u_n\right) dx + o(1)\\ &= \widetilde{\mathcal{I}}(t_n u_n) - \frac{1}{\mu}\langle \widetilde{\mathcal{I}}'(t_n u_n), t_n u_n\rangle + o(1)\\ &= \widetilde{\mathcal{I}}(t_n u_n) + o(1)\\ &\geq 2m + o(1),\end{aligned}$$

which is a contradiction.

If $\alpha > 0$, there exists $\{z_n\} \subset \mathbb{R}^N$ such that

$$\frac{\alpha}{2} \leq \int_{B_1(z_n)} v_n^2 dx.$$

If $\{z_n\}$ is bounded, there exists $R > 0$ such that

$$\frac{\alpha}{2} \leq \int_{B_R} v_n^2 dx.$$

which is a contradiction with $v_n \to 0$ in $L^2_{loc}(\mathbb{R}^N)$. Then $\{z_n\}$ is unbounded, up to a subsequence, $|z_n| \to \infty$. We set $w_n(x) := v_n(x + z_n)$, where w_n satisfies

$$\frac{\alpha}{2} \leq \int_{B_1} w_n^2 dx,$$

up to a subsequence, there exists $w \in E$ such that $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^2_{loc}(\mathbb{R}^N)$ and $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$. Evidently, $\operatorname{meas}\Omega_2 > 0$ where $\Omega_2 = \{x \in \mathbb{R}^N : w(x) > 0\}$. In fact $w_n(x) = \frac{u_n(x+z_n)}{\|u_n\|}$. Also from Fatou's lemma and (a) of Lemma 2.1, one obtains

$$\begin{aligned}\liminf_{n\to\infty}\left[\frac{1}{\|u_n\|^2}\int_{\mathbb{R}^N} \widetilde{F}(u_n)dx\right] &= \liminf_{n\to\infty}\left[\frac{1}{\|u_n\|^2}\int_{\mathbb{R}^N} \widetilde{F}\left(u_n\left(x + z_n\right)\right)dx\right]\\ &\geq \liminf \int_{\Omega_2} \frac{\widetilde{F}\left(u_n\left(x+z_n\right)\right)}{[u_n(x+z_n)]^2} w_n^2 dx\\ &= +\infty.\end{aligned}$$

Hence

$$0 = \limsup_{n\to\infty} \frac{\widetilde{\mathcal{I}}(u_n)}{\|u_n\|} = \frac{1}{2} - \lambda \liminf_{n\to\infty}\left[\frac{1}{\|u_n\|^2}\int_{\mathbb{R}^N} \widetilde{F}(u_n)dx\right] = -\infty,$$

which is a contradiction. In a word, the $(Ce)_m$ sequence $\{u_n\}$ is bounded in E. □

Proposition 2.5. *Suppose that* (V_4) *and* (f_5) *hold. Then Eq.* $(\widetilde{\mathcal{P}})$ *has a positive ground state solution.*

Proof. Notice that $0 < c \le c_\infty$. Therefore, one of the two cases occurs:

Case 1. $c = c_\infty$. It follows from (2.2) that

$$c_\infty \le \widetilde{\mathcal{I}}(t_\omega \omega) \le \widetilde{\mathcal{I}}_\infty(t_\omega \omega) \le \widetilde{\mathcal{I}}_\infty(\omega) = c_\infty.$$

Then ω is also a positive ground state solution of Eq. $(\widetilde{\mathcal{P}})$ from Lemma 2.3.

Case 2. $0 < c < c_\infty$. We see easily $\widetilde{\mathcal{I}}$ satisfies the mountain pass geometry. From the mountain pass theorem [25, 26] and Lemma 2.4, there exists a nonnegative and bounded sequence $\{u_n\} \in E$ such that

$$\widetilde{\mathcal{I}}(u_n) \to c, \qquad \|\widetilde{\mathcal{I}}'(u_n)\|(1 + \|u_n\|) \to 0.$$

Then there exists a nonnegative function $u \in E$ such that up to a subsequence, $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. For any $\varphi \in C_0^\infty(\mathbb{R}^N)$, one has $0 = \langle \widetilde{\mathcal{I}}'(u_n), \varphi \rangle + o(1) = \langle \widetilde{\mathcal{I}}'(u), \varphi \rangle$, i.e., u is a nonnegative solution of Eq. $(\widetilde{\mathcal{P}})$. If $u \ne 0$ in E, combining Lemma 2.3 in [20] with Fatou's lemma one obtains

$$\begin{aligned}
c &= \widetilde{\mathcal{I}}(u_n) + o(1) \\
&= \widetilde{\mathcal{I}}(u_n) - \frac{1}{\mu}\langle \widetilde{\mathcal{I}}'(u_n), u_n \rangle + o(1) \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + V(x) u_n^2\right) dx + \lambda \int_{\mathbb{R}^N} \left(\frac{1}{\mu}\widetilde{f}(u_n) u_n - \widetilde{F}(u_n)\right) dx + o(1) \\
&\ge \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x) u^2\right) dx + \lambda \int_{\mathbb{R}^N} \left(\frac{1}{\mu}\widetilde{f}(u) u - \widetilde{F}(u)\right) dx + o(1) \\
&= \widetilde{\mathcal{I}}(u) - \frac{1}{\mu}\langle \widetilde{\mathcal{I}}'(u), u \rangle + o(1) \\
&= \widetilde{\mathcal{I}}(u) + o(1).
\end{aligned} \tag{2.5}$$

At the same time, one knows $c \le \widetilde{\mathcal{I}}(u)$ from the definition of c and $u \in \mathcal{N}$. Applying the strongly maximum principle, we could deduce that u is a positive ground state solution of Eq. $(\widetilde{\mathcal{P}})$.

We assume that $u = 0$ (otherwise we complete the proof). Then there exists $\alpha \ge 0$ such that

$$\lim_{n\to\infty} \sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |u_n|^2 dx = \alpha.$$

Indeed, if $\alpha = 0$, applying the Lions lemma [17,26] we obtain

$$u_n \to 0 \quad \text{in } L^p(\mathbb{R}^N). \tag{2.6}$$

Hence $\widetilde{\mathcal{I}}(u_n) \to 0$ as $n \to \infty$ from (b) in Lemma 2.1, which contradicts $c > 0$. Then there exists $\{z_n\} \subset \mathbb{R}^N$ such that $\int_{B_1(z_n)} |u_n|^2 dx \ge \frac{\alpha}{2} > 0$.

If $\{z_n\}$ is bounded, there exists $R > 0$ such that $\int_{B_R(0)} |u_n|^2 dx \ge \frac{\alpha}{2} > 0$, which is a contradiction with $u_n \to 0$ in $L^2_{loc}(\mathbb{R}^N)$. Then $\{z_n\}$ is unbounded. After extracting a subsequence if necessary, we have

(i) $|z_n| \to +\infty$,

(ii) $u_n(\cdot + z_n) \rightharpoonup v \neq 0$ in E.

From Lemma 2.4 in [19], we have

$$\begin{aligned}
0 &= \langle \widetilde{\mathcal{I}}'(u_n), \varphi(\cdot - z_n)\rangle + o(1) \\
&= \int_{\mathbb{R}^N} [\nabla u_n \cdot \nabla \varphi(\cdot - z_n) + V(x) u_n \varphi(x - z_n)]\, dx - \lambda \int_{\mathbb{R}^N} \widetilde{f}(u_n)\varphi(x - z_n) dx + o(1) \\
&= \int_{\mathbb{R}^N} [\nabla u_n \cdot \nabla \varphi(\cdot - z_n) + V_\infty(x) u_n \varphi(x - z_n)]\, dx - \lambda \int_{\mathbb{R}^N} \widetilde{f}(u_n)\varphi(x - z_n) dx + o(1) \\
&= \langle \widetilde{\mathcal{I}}'_\infty(v), \varphi\rangle + o(1).
\end{aligned}$$

Then v is a nontrivial solution of Eq. ($\mathcal{P}_\infty$). Notice that, also from [19], we obtain

$$\begin{aligned}
c &= \widetilde{\mathcal{I}}(u_n) - \frac{1}{\mu}\langle \widetilde{\mathcal{I}}'(u_n), u_n\rangle + o(1) \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + V(x) u_n^2\right) dx + \lambda \int_{\mathbb{R}^N} \left(\frac{1}{\mu}\widetilde{f}(u_n) u_n - \widetilde{F}(u_n)\right) dx + o(1) \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + V_\infty(x) u_n^2\right) dx \\
&\quad + \lambda \int_{\mathbb{R}^N} \left[\frac{1}{\mu}\widetilde{f}(u_n(\cdot + z_n)) u_n(\cdot + z_n) - \widetilde{F}(u_n(\cdot + z_n))\right] dx + o(1) \\
&= \widetilde{\mathcal{I}}_\infty\left(u_n(\cdot + z_n)\right) - \frac{1}{\mu}\langle \widetilde{\mathcal{I}}'_\infty\left(u_n(\cdot + z_n)\right), u_n(\cdot + z_n)\rangle + o(1) \\
&= \widetilde{\mathcal{I}}_\infty(v) + o(1) \\
&\geq c_\infty + o(1),
\end{aligned}$$

which is a contradiction.

In conclusion, whether Case 1 occurs or Case 2 occurs, we can prove Proposition 2.5. □

3 Proof of Theorem 1.1

Lemma 3.1. *Suppose that* (V_4) *and* (f_5) *hold. If* u *is a critical point of* $\widetilde{\mathcal{I}}$, *then* $u \in L^\infty(\mathbb{R}^N)$. *Furthermore, there exists a positive constant* C *independent of* λ *such that*

$$|u|_\infty \leq C\lambda^{\frac{1}{2^*-p}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{2^*-2}{2(2^*-p)}}.$$

Proof. We prove the result by using the Moser iteration. For each $k > 0$, we define

$$u_k(x) = \begin{cases} u(x), & \text{if } |u(x)| \leq k, \\ \pm k, & \text{if } \pm u(x) > k. \end{cases}$$

For $\beta > 1$, we use $\varphi_k = |u_k|^{2(\beta-1)} u$ as a test function in $\langle \widetilde{\mathcal{I}}'(u), \varphi_k\rangle$ to obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |u_k|^{2(\beta-1)} |\nabla u|^2 dx + 2(\beta - 1) \int_{\mathbb{R}^N} |u_k|^{2(\beta-2)} u u_k \nabla u \cdot \nabla u_k dx \\
+ \int_{\mathbb{R}^N} V(x) |u_k|^{2(\beta-1)} u^2 dx = \lambda \int_{\mathbb{R}^N} \widetilde{f}(u) |u_k|^{2(\beta-1)} u dx. \quad (3.1)
\end{aligned}$$

Then we use the Sobolev inequality to yield

$$
\begin{aligned}
&\beta^2 \int_{\mathbb{R}^N} \left(|u_k|^{2(\beta-1)}|\nabla u|^2 dx + 2(\beta-1)|u_k|^{2(\beta-2)} u u_k \nabla u \cdot \nabla u_k\right) dx \\
&\geq \int_{\mathbb{R}^N} |u_k|^{2(\beta-1)}|\nabla u|^2 + (\beta-1)^2 |u_k|^{2(\beta-2)} u^2 |\nabla u_k|^2 + 2(\beta-1)|u_k|^{2(\beta-2)} u u_k \nabla u \cdot \nabla u_k dx \\
&\geq \int_{\mathbb{R}^N} \left|\nabla\left(|u_k|^{\beta-1} u\right)\right|^2 dx \\
&\geq C\left(\int_{\mathbb{R}^N} \left||u_k|^{\beta-1} u\right|^{2^*} dx\right)^{\frac{2}{2^*}}, \qquad (3.2)
\end{aligned}
$$

where we also have used the facts that $u^2|\nabla u_k|^2 \leq u_k^2|\nabla u|^2$ and $\beta > 1$. From (b) in Lemma 2.1, we deduce

$$
\int_{\mathbb{R}^N} \widetilde{f}(u)|u_k|^{2(\beta-1)} u dx \leq C \int_{\mathbb{R}^N} |u|^p |u_k|^{2(\beta-1)} dx. \qquad (3.3)
$$

Combining (3.1), (3.2) and (3.3), we obtain

$$
\begin{aligned}
\left(\int_{\mathbb{R}^N} \left||u_k|^{\beta-1} u\right|^{2^*} dx\right)^{\frac{2}{2^*}} &\leq C\beta^2\lambda \int_{\mathbb{R}^N} |u|^{p-2} |u_k|^{2(\beta-1)} u^2 dx \\
&\leq C\beta^2\lambda \left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{\frac{p-2}{2^*}} \left(\int_{\mathbb{R}^N} \left||u_k|^{2(\beta-1)} u^2\right|^{\frac{2^*}{2^*-p+2}} dx\right)^{\frac{2^*-p+2}{2^*}}.
\end{aligned}
$$

Letting $k \to \infty$, we have

$$
|u|_{\beta \cdot 2^*} \leq (C\beta^2\lambda)^{\frac{1}{2\beta}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{p-2}{4\beta}} |u|_{\frac{2\cdot 2^*\beta}{2^*-p+2}}. \qquad (3.4)
$$

To carry out an iteration process, we set

$$
\beta_m = \left(\frac{2^*-p+2}{2}\right)^{m+1}, \qquad m = 0, 1, \ldots
$$

Then we have

$$
\frac{2\cdot 2^*\beta_m}{2^*-p+2} = 2^*\beta_{m-1}.
$$

By (3.4), one obtains

$$
\begin{aligned}
|u|_{\beta_m\cdot 2^*} &\leq (C\beta_m^2\lambda)^{\frac{1}{2\beta_m}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{p-2}{4\beta_m}} |u|_{\frac{2\cdot 2^*\beta_m}{2^*-p+2}} \\
&= (C\lambda)^{\frac{1}{2\beta_m}} \beta_m^{\frac{1}{\beta_m}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{p-2}{4\beta_m}} |u|_{\beta_{m-1}\cdot 2^*}.
\end{aligned}
$$

By the Moser iteration, we have

$$
|u|_{\beta_m\cdot 2^*} \leq (C\lambda)^{\sum_{i=0}^m \frac{1}{2\beta_i}} \prod_{i=0}^m \beta_i^{\frac{1}{\beta_i}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{p-2}{4}\sum_{i=0}^m \frac{1}{\beta_i}} |u|_{2^*}. \qquad (3.5)
$$

Since $\beta_0 = \left(\frac{2^*-p+2}{2}\right) > 1$ and $\beta_i = \beta_0^{i+1}$, we observe that

$$\sum_{i=0}^{m} \frac{1}{\beta_i} = \sum_{i=0}^{m} \frac{1}{\beta_0^{i+1}}, \qquad \prod_{i=0}^{m} \beta_i^{\frac{1}{\beta_i}} = \prod_{i=0}^{m} \left(\beta_0^{i+1}\right)^{\frac{1}{\beta_0^{i+1}}} = (\beta_0)^{\sum_{i=0}^{m} \frac{i+1}{\beta_0^{i+1}}}.$$

One can easily see

$$\sum_{i=0}^{\infty} \frac{i+1}{\beta_0^{i+1}} = \beta^* < +\infty, \qquad \sum_{i=0}^{\infty} \frac{1}{\beta_0^{i+1}} = \frac{2}{2^*-p}.$$

Letting $m \to \infty$ in (3.5), we conclude that $u \in L^\infty(\mathbb{R}^N)$ and

$$|u|_\infty \le C\lambda^{\frac{1}{2^*-p}} \beta_0^{\beta^*} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{p-2}{2(2^*-p)}} |u|_{2^*} \le C\lambda^{\frac{1}{2^*-p}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{2^*-2}{2(2^*-p)}}. \tag{3.6}$$

This completes the proof. □

Proof of Theorem 1.1. By proposition 2.5, Eq. $(\widetilde{\mathcal{P}})$ has a positive ground solution u. Combining the Sobolev embedding and (b) of Lemma 2.1, one obtains

$$c = \widetilde{\mathcal{I}}(u) - \frac{1}{\mu}\langle \widetilde{\mathcal{I}}'(u), u\rangle \ge \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u\|^2. \tag{3.7}$$

We can see that there exists $v \in K \cap L^\infty(\mathbb{R}^N)$ such that $|v|_\infty < 1$. Since (f_6), there exists $C > 0$ independent of λ such that

$$\widetilde{F}(tv) \ge C|tv|^q, \qquad t \in [0,1].$$

At the same time there exists $\lambda_0 > 0$ such that $\widetilde{\mathcal{I}}(v) < 0$ for $\lambda \ge \lambda_0$. Then from the definition of c, we have

$$\begin{aligned} c &\le \max_{t\in[0,1]} \widetilde{\mathcal{I}}(tv) \\ &= \max_{t\in[0,1]} \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)v^2)dx - \lambda \int_{\mathbb{R}^N} \widetilde{F}(tv)dx \\ &\le \max_{t\in[0,1]} \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)v^2)dx - Ct^q\lambda \int_{\mathbb{R}^N} |v|^q dx \\ &\le C\lambda^{-\frac{2}{q-2}}. \end{aligned} \tag{3.8}$$

Combining (3.6), (3.7) and (3.8), we have

$$|u|_\infty \le C\lambda^{\frac{1}{2^*-p}} \|u\|^{\frac{2^*-2}{2^*-p}} \le C\lambda^{\frac{1}{2^*-p}} \lambda^{\frac{1}{2-q}\cdot\frac{2^*-2}{2^*-p}}.$$

Since $p, q \in (2, 2^*)$, there exists $\lambda_1 \ge \lambda_0$ such that

$$|u|_\infty \le C\lambda_1^{\frac{2^*-q}{(2^*-p)(2-q)}} \le \delta.$$

Therefore, from the definition of $\widetilde{f}$, we can conclude that u is also a positive solution of Eq. $(\mathcal{P})$ for $\lambda \ge \lambda_1$. This completes the proof of Theorem 1.1. □

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Multiple small solutions for Schrödinger equations involving the p-Laplacian and positive quasilinear term

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Abstract. We consider the multiplicity of solutions of a class of quasilinear Schrödinger equations involving the p-Laplacian:

$$-\Delta_p u + V(x)|u|^{p-2}u + \Delta_p(u^2)u = K(x)f(x,u), \qquad x \in \mathbb{R}^N,$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, $N \geq 3$, V, K belong to $C(\mathbb{R}^N)$ and f is an odd continuous function without any growth restrictions at large. Our method is based on a direct modification of the indefinite variational problem to a definite one. Even for the case $p = 2$, the approach also yields new multiplicity results.

Keywords: quasilinear Schrödinger equations, variational methods, Brezis–Kato type estimates.

2020 Mathematics Subject Classification: 35J20, 35J62, 35B45.

1 Introduction

In this study, the multiplicity of solutions for the quasilinear elliptic problem

$$-\Delta_p u + V(x)|u|^{p-2}u + \tau\Delta_p(u^2)u = K(x)f(x,u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

will be analyzed, where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $1 < p < N$, $\tau \in \mathbb{R}$, f is a continuous function and is only p-sublinear in a neighborhood of $u = 0$, V and K belong to $C(\mathbb{R}^N)$, satisfying

(VK) for all $x \in \mathbb{R}^N$, $0 < V_0 \leq V(x)$, $0 < K(x) \leq K_1$ and

$$W(x) := K(x)^{p/(p-q)}V(x)^{q/(q-p)} \in L^1(\mathbb{R}^N) \qquad (q \text{ will be defined in } (f_1)).$$

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For $p = 2$, quasilinear Schrödinger equations (QSE) are widely used in non-Newtonian fluids, reaction-diffusion problems and other physical phenomena. It should be noted that the solutions of problem (1.1) are closely related to solutions of the nonlinear Schrödinger equations:

$$i\partial_t z = -\Delta z + \widetilde{V}(x)z - l(x,|z|^2)z + \tau[\Delta\rho(|z|^2)]\rho'(|z|^2)z, \tag{1.2}$$

where $z : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, $K : \mathbb{R}^N \to \mathbb{R}$ is a given potential, τ is a real constant, ρ is a real function and $l : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$. They have been derived as models of many physical phenomena corresponding to various types of the function ρ. For example, when $\rho(s) = 1$, one has the classical stationary semilinear Schrödinger equation [3,12]. If $\rho(s) = s$, the equations of fluid mechanics, plasma physics and dissipative quantum mechanics are established [4,11]. When $\rho(s) = (1+s)^{1/2}$, the equation models the propagation of a high-irradiance laser in a plasma and the self-channeling of a high-power ultrashort laser in matter [13]; problem (1.2) is related to condensed matter theory. For more information on the physical background, please refer to [4,5,18].

In what follows, we discuss the case of $\rho(s) = s$ and $p = 2$. A standing wave of problem (1.2) is a solution of the form $z(x,t) = \exp(-iEt)u(x)$ where $E \in \mathbb{R}$. It is also called stationary waves. It is generally known that z is a standing wave solution for problem (1.2) when and only when u is a solution for the quasilinear elliptic problem (1.1), where $V(x) = \widetilde{V}(x) - E$ indicates the new potential.

When $\tau = 0$, equation (1.1) degenerates into a semilinear equation (i.e., the nonlinear Schrödinger equation), which has been widely studied using the variational method for the past 30 years, see [14]. Obviously, if $\tau \neq 0$, the energy functional of the quasilinear term $\tau\int_{\mathbb{R}^N} u^2|\nabla u|^2dx$ is not well defined in $H^1(\mathbb{R}^N)$. Therefore, the energy functional I of (1.1) is not a C^1 functional.

When $\tau < 0$, scholars have obtained a large number of existence and multiplicity results for equation (1.1) based on variational methods. For instance, Poppenberg, Schmitt and Wang proved the existence of positive solutions with a constrained minimization argument in [19] for the first time. By utilizing variable substitution and converting the quasilinear problem (1.1) into a semilinear one in an Orlicz space framework, Liu *et al*. in [15] obtained a general existence result. Colin *et al*. in [6] adopted the same method of variable substitution but chose the classical Sobolev space $H^1(\mathbb{R}^N)$. For further results, please refer to [8,16,21,22,25].

When $\tau > 0$, in [1], Alves *et al*. introduced a substitution of variables $u = G^{-1}(v)$, where

$$g(t) = \begin{cases} \sqrt{1-\tau t^2} & \text{if } 0 \le t < \frac{1}{\sqrt{3\tau}}, \\ \frac{1}{3\sqrt{2\tau}t} + \frac{1}{\sqrt{6}} & \text{if } t \ge \frac{1}{\sqrt{3\tau}}, \end{cases}$$

$g(t) = g(-t)$ for all $t \le 0$ and $G(s) = \int_0^s g(t)dt$. Given a sufficiently small $\tau > 0$, the authors proved that there exists a solution of

$$-\Delta u + V(x)u + \tau\Delta(u^2)u = |u|^{q-2}u, \qquad x \in \mathbb{R}^N,$$

where $2 < q < 2^*$. Wang et al. [23] investigated the existence of solutions for QSE with critical growth nonlinearities. [2] with potential V vanishing at infinity and the superlinear nonlinearities, [24] with $f(t) = \lambda|t|^{q-2}t + |t|^{i-2}t$ for $q \ge 22^*$, $4 < i < 22^*$ and $\lambda > 0$ small enough, [20] with potential V being large at infinity and nonlinearities being superlinear or asymptotically linear at infinity.

Now, from [1], two natural questions arise:

(Q_1) Can the appropriate variational framework for problem (1.1) with $\tau = 1$ (not small enough) be established?

(Q_2) When $\tau = 1$, if the nonlinearity $|t|^{q-2}t$ with $q > 2$ is replaced by $q < 2$ or a more general sublinear term $f(x,t)$ in problem (1.1), will this problem possess infinitely many solutions?

Regarding the question (Q_1), our earlier work [9] studied the existence of a positive solution for problem (1.1) with $\tau = 1$ under a local superlinear growth condition. Our aim in this work is to seek clear answers to question (Q_2). Therefore, we will be mainly interested in the existence of infinitely many solutions for the following general QSE involving local p-sublinear nonlinearities:

$$-\Delta_p u + V(x)|u|^{p-2}u + \Delta_p(u^2)u = K(x)f(x,u), \qquad x \in \mathbb{R}^N, \tag{1.3}$$

where $1 < p < N$, $N \geq 3$, V and K satisfy condition (VK). We remark that our results are new also in the case $p = 2$. Next, we suppose that the nonlinearity f is continuous and meets the following conditions that describe its behavior only in a neighborhood of the origin:

(f_1) there exist $\delta > 0$, $1 \leq q < p$ and $C > 0$ such that $f \in C(\mathbb{R}^N \times [-\delta,\delta],\mathbb{R})$, f is odd in t and
$$|f(x,t)| \leq C|t|^{q-1}, \quad \text{uniformly in } x \in \mathbb{R}^N;$$

(f_2) there exist $x_0 \in \mathbb{R}^N$ and $r_0 > 0$ such that
$$\liminf_{t\to 0}\left(\inf_{x\in B_{r_0}(x_0)} \frac{F(x,t)}{|t|^p}\right) > -\infty$$
and
$$\limsup_{t\to 0}\left(\inf_{x\in B_{r_0}(x_0)} \frac{F(x,t)}{|t|^p}\right) = +\infty,$$
where $B_{r_0}(x_0) \subset \mathbb{R}^N$ and
$$F(x,t) = \int_0^t f(x,s)ds.$$

Remark 1.1. We do not need any growth condition on f at infinity. There exist many functions satisfying (f_1) and (f_2), for example

(i) $f(x,u) = |u|^{q-1}\mathrm{sgn}\, u$ with $q \in (1,p)$;

(ii) $f(x,u) = Q(x)|u|^{q-1}\mathrm{sgn}\, u + P(x)|u|^{i-1}\mathrm{sgn}\, u$, where $1 < q < p$, $i \geq p^* := \frac{pN}{N-p}$, $Q(x)$ and $P(x)$ are bounded Hölder continuous functions on $\mathbb{R}^N$ and $Q(x_0) > 0$ at some $x_0 \in \mathbb{R}^N$.

Remark 1.2. Although problem (1.3) is not a standard elliptic equation, we can still give the definition of the weak solution of problem (1.3). Suppose that conditions (VK), (f_1) and (f_2) are satisfied. A weak solution of problem (1.3) is a function $u \in X$ (X will be defined in Section 2) such that

$$\int_{\mathbb{R}^N}(1-2^{p-1}|u|^p)|\nabla u|^{p-2}\nabla u\nabla\varphi dx - 2^{p-1}\int_{\mathbb{R}^N}|\nabla u|^p|u|^{p-2}u\varphi dx + \int_{\mathbb{R}^N}V(x)|u|^{p-2}u\varphi dx$$
$$= \int_{\mathbb{R}^N}K(x)f(x,u)\varphi dx, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N).$$

From a variational perspective, we give a formally Lagrangian functional of (1.3):

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^N}(1-2^{p-1}|u|^p)|\nabla u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \int_{\mathbb{R}^N} K(x)F(x,u)dx,$$

which is not well defined in $W^{1,p}(\mathbb{R}^N)$. For this reason, conventional variational methods cannot be applied directly. Problems such as (1.3) become interesting and challenging in this dilemma. First, because of our lack of information about the function f at infinity, the term $\int_{\mathbb{R}^N} K(x)F(x,u)dx$ may not be well defined. Second, the presence of $\int_{\mathbb{R}^N}(1-2^{p-1}|u|^p)|\nabla u|^p dx$ makes us unable to work in a classical Sobolev space. Third, ensuring the positiveness of the principal part, i.e., $\int_{\mathbb{R}^N}(1-2^{p-1}|u|^p)|\nabla u|^p dx > 0$, is also difficult.

Drawing lessons from the work of Costa and Wang [7], our earlier work [9] and the variant symmetric mountain pass lemma [10, 17], we can obtain infinitely many solutions for a modified functional with modifications made on the nonlinearity and the principal part of the Lagrangian functional J. Then, we obtain Brezis–Kato type estimates for these critical points of the modified functional. After fine estimates of the solutions for the modified problems we can show that some of these solutions for the modified problems give rise to solutions of problem (1.3) with desired properties.

We now proceed to present our main result.

Theorem 1.3. *Suppose that conditions (VK), (f_1) and (f_2) are satisfied. Then problem* (1.3) *possesses a sequence of weak solutions $u_n \in X$ with $u_n \to 0$ strongly in X, $u_n \to 0$ strongly in $L^\infty(\mathbb{R}^N)$ and $J(u_n) \to 0$.*

Remark 1.4. Since problem (1.3) is not a standard elliptic equation, conventional critical point theory is not directly applicable. Hence, some fundamental results for elliptic equations are not expected. For instance, without the symmetric condition regarding nonlinearity, the existence of solutions for problem (1.3) may not be proved.

The remainder of this paper is arranged as follows. In Section 2, the problem is reformulated. We provide the variational framework for the reformulated problem in Section 3. Section 4 is devoted to discussing the reformulated problem in detail via a cut-off technique, Morse L^∞-estimation and proving Theorem 1.3.

In what follows, C denotes positive generic constants.

2 Reformulation

Define $X = \{u \in W^{1,p}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x)|u|^p dx < \infty\}$ endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N}(|\nabla u|^p + V(x)|u|^p)dx\right)^{1/p}.$$

As usual, the norms of $L^s(\mathbb{R}^N)(s \geq 1)$ are denoted by $\|\cdot\|_s$.

For fixed $\delta > 0$ in (f_1), set $d(t) \in C(\mathbb{R})$ as a cut-off function satisfying :

$$d(t) = \begin{cases} 1, & \text{if } |t| \leq \frac{\delta}{2}, \\ 0, & \text{if } |t| \geq \delta, \end{cases}$$

$d(-t) = d(t)$ and $0 \leq d(t) \leq 1$ for $t \in \mathbb{R}$. Define

$$\widetilde{f}(x,t) = d(t)f(x,t), \quad \text{for all } x \in \mathbb{R}^N,\ t \in \mathbb{R}$$

and

$$\widetilde{F}(x,t) = \int_0^t \widetilde{f}(x,s)ds.$$

Inspired by [7,9], a modified QSE can be established:

$$-\operatorname{div}(h^p(u)|\nabla u|^{p-2}\nabla u) + h^{p-1}(u)h'(u)|\nabla u|^p + V(x)|u|^{p-2}u = K(x)\widetilde{f}(x,u), \quad x \in \mathbb{R}^N, \tag{2.1}$$

where $h(s) : [0,+\infty) \to \mathbb{R}$ satisfying

$$h(s) = \begin{cases} (1-2^{p-1}s^p)^{1/p} & \text{if } 0 \le s < (\frac{2^{1-p}}{3})^{1/p}, \\ \frac{1}{s}(\frac{2^{1-p}}{3})^{2/p} + (\frac{2^{1-p}}{3})^{1/p} & \text{if } s \ge (\frac{2^{1-p}}{3})^{1/p}, \end{cases}$$

and $h(s) = h(-s)$ for $s < 0$. It deduces that $h(s) \in C^1(\mathbb{R}, ((\frac{2^{1-p}}{3})^{1/p}, 1])$ and decreases in $[0,+\infty)$. And then, we define

$$H(t) = \int_0^t h(s)ds.$$

Obviously, $H(t)$ is an odd function, and there exists an inverse function $H^{-1}(t)$. Moreover, $H(t)$ has the following attributes, the similar proofs of which can be found in [9].

Lemma 2.1.

(i) $\lim\limits_{t\to 0} \frac{H^{-1}(t)}{t} = 1$;

(ii) $\lim\limits_{t\to +\infty} \frac{H^{-1}(t)}{t} = (\frac{3}{2^{1-p}})^{1/p}$;

(iii) $|t| \le |H^{-1}(t)| \le (\frac{3}{2^{1-p}})^{1/p}|t|$, *for all* $t \in \mathbb{R}$;

(iv) $\frac{t}{h(t)}h'(t) \le 0$, *for all* $t \in \mathbb{R}$.

Our goal is proving that (2.1) has a sequence of weak solutions $\{u_n\}$ satisfying $\|u_n\|_{L^\infty} < \min\{\delta/2, (\frac{2^{1-p}}{3})^{1/p}\}$, in this situation

$$h(u_n) = (1-2^{p-1}|u_n|^p)^{1/p} \quad \text{and} \quad \widetilde{f}(x,u_n) = f(x,u_n).$$

Thus, they are also the weak solutions of (1.3).

To find the weak solutions of (2.1) with desired properties, we focus on a Lagrangian functional defined by

$$\widetilde{J}(u) = \frac{1}{p}\int_{\mathbb{R}^N} h^p(u)|\nabla u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \int_{\mathbb{R}^N} K(x)\widetilde{F}(x,u)dx. \tag{2.2}$$

Taking the change of variable

$$v = H(u),$$

it is clear that functional $\widetilde{J}$ can be written as follows:

$$I(v) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|H^{-1}(v)|^p dx - \int_{\mathbb{R}^N} K(x)\widetilde{F}(x,H^{-1}(v))dx. \tag{2.3}$$

From the definition of $\widetilde{F}(x,t)$, we deduce

$$|\widetilde{F}(x,t)| \le C|t|^q, \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R},$$

where $1 \leq q < p$. This together with Lemma 2.1 implies that

$$\begin{aligned}\left|\int_{\mathbb{R}^N} K(x)\widetilde{F}(x, H^{-1}(v))dx\right| &\leq C\int_{\mathbb{R}^N} K(x)|H^{-1}(v)|^q dx \\ &\leq C\left(\int_{\mathbb{R}^N} W(x)dx\right)^{\frac{(p-q)}{p}}\left(\int_{\mathbb{R}^N} V(x)|v|^p dx\right)^{\frac{q}{p}} \\ &\leq C\|v\|^q.\end{aligned} \tag{2.4}$$

From the above estimation and Lemma 2.1, we obtain

$$I(v) \text{ is well defined in } X.$$

Then, it is standard to see that $I \in C^1(X, \mathbb{R})$ and for all $\varphi \in X$

$$\begin{aligned}I'(v)\varphi = &\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v\nabla\varphi dx + \int_{\mathbb{R}^N} V(x)|H^{-1}(v)|^{p-2}\frac{H^{-1}(v)}{h(H^{-1}(v))}\varphi dx \\ &- \int_{\mathbb{R}^N} K(x)\frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))}\varphi dx.\end{aligned}$$

Lemma 2.2. *Suppose that conditions (VK) and (f_1) are satisfied. If $v \in X$ is a critical point of I, then $u = H^{-1}(v) \in X$ and u is a weak solution for (2.1).*

Proof. From $v \in X$ and Lemma 2.1, we have $u = H^{-1}(v) \in X$. By v being a critical point for I, we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v\nabla\varphi dx + \int_{\mathbb{R}^N} V(x)|H^{-1}(v)|^{p-2}\frac{H^{-1}(v)}{h(H^{-1}(v))}\varphi dx \\ &- \int_{\mathbb{R}^N} K(x)\frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))}\varphi dx, \quad \text{for all } \varphi \in X.\end{aligned}$$

Taking $\varphi = h(u)\psi$ as the text function, where $u = H^{-1}(v)$ and $\psi \in C_0^\infty(\mathbb{R}^N)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v\nabla u h'(u)\psi dx + \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v\nabla\psi h(u)dx + \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\psi dx \\ - \int_{\mathbb{R}^N} K(x)\widetilde{f}(x, u)\psi dx = 0.\end{aligned}$$

or

$$\int_{\mathbb{R}^N}\left(-\mathrm{div}(h^p(u)|\nabla u|^{p-2}\nabla u) + h^{p-1}(u)h'(u)|\nabla u|^p + V(x)|u|^{p-2}u - K(x)\widetilde{f}(x, u)\right)\psi dx = 0.$$

This ends the proof. □

Therefore, for the weak solutions of (2.1), we only need to discuss the existence of the weak solutions of the following problem:

$$-\Delta_p v + V(x)|H^{-1}(v)|^{p-2}\frac{H^{-1}(v)}{h(H^{-1}(v))} = K(x)\frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))}, \qquad x \in \mathbb{R}^N. \tag{2.5}$$

3 Clark's theorem

Denote

$$\Gamma = \{A \subset X \setminus \{0\} \mid A \text{ is closed}, -A = A\}.$$

Let $A \in \Gamma$, define

$$\gamma(A) = \min\{n \in \mathbb{N} \mid \text{there exists an odd, continuous } \phi: \; A \to \mathbb{R}^n \setminus \{0\}\},$$

If such a minimum does not exist, then we define $\gamma(A) = +\infty$. Moreover, set $\gamma(\emptyset) = 0$. In order to prove Theorem 1.3, we introduce the following Clark's theorem due to [10].

Proposition 3.1. *Let X be a Banach space and $\Phi \in C^1(X, \mathbb{R})$ be an even functional with $\Phi(0) = 0$. Assume that Φ satisfies the following.*

(i) Φ is bounded from below and satisfies the Palais–Smale condition.

(ii) For all $k \in \mathbb{N}$, $\Gamma_k = \{A \in \Gamma | \gamma(A) \geq k\}$, there exists a $A_k \in \Gamma_k$ such that $\sup_{v \in A_k} \Phi(v) < 0$.

Then, at least one of the following conclusions holds.

(i) There exists a critical point sequence $\{v_k\}$ such that $\Phi(v_k) < 0$ and $v_k \to 0$ strongly in X.

(ii) There exist two critical point sequences $\{v_k\}$ and $\{w_k\}$ such that $\Phi(v_k) = 0$, $v_k \neq 0$, $v_k \to 0$ strongly in X, $\Phi(w_k) < 0$, $\lim_{k\to\infty} \Phi(w_k) = 0$ and $\{w_k\}$ converges to a non-zero limit.

The following lemma plays a fundamental role in verifying Proposition 3.1. In the proof of this lemma, we adapt some arguments of dealing with the Schrödinger–Poisson systems in [26] and the elliptic problem in [10].

Lemma 3.2. *Suppose that (VK), (f_1) and (f_2) hold. Then for all $k \in \mathbb{N}$, there exists $A_k \in \Gamma$ such that genus $\gamma(A_k) = k$ and $\sup_{v \in A_k} I(v) < 0$.*

Proof. Without loss of generality, we may assume that $x_0 = 0$ in condition (f_2). Let $\mathcal{Q}$ be the cube

$$\mathcal{Q} := \{x = (x_1, x_2, \ldots, x_N) \mid |x_i| \leq r_0/2, \; i = 1, 2, \ldots, N\},$$

where r_0 is chosen in condition (f_2). Obviously, $\mathcal{Q} \subset B_{r_0}(0)$. From (f_2) and Lemma 2.1-(iii), we can find two sequences $\delta_n \to 0$, $M_n \to \infty (\delta_n, M_n > 0)$ and a positive constant α such that

$$\frac{F(x,t)}{|t|^p} \geq -\alpha, \quad \text{for all } x \in \mathcal{Q} \text{ and } |t| \leq \delta \tag{3.1}$$

and

$$\frac{F(x, H^{-1}(\delta_n))}{|H^{-1}(\delta_n)|^p} \geq M_n \quad \text{for all } x \in \mathcal{Q} \text{ and } n \in \mathbb{N}. \tag{3.2}$$

Next, for any $k \in \mathbb{N}$ fixed, we shall construct a $A_k \in \Gamma$ which satisfies genus $\gamma(A_k) = k$ and $\sup_{v \in A_k} I(v) < 0$.

Firstly, let $k \in \mathbb{N}$ be fixed and $m \in \mathbb{N}$ is the smallest integer that satisfies $m^N \geq k$. Then, by planes parallel to each face of $\mathcal{Q}$, we can equally divide cube $\mathcal{Q}$ into m^N small cubes. Set them by $\mathcal{Q}_i$ with $1 \leq i \leq m^N$. It is well known that the length of the edge of $\mathcal{Q}_i$ is $d = r_0/m$. Furthermore, for each $1 \leq i \leq k$, let $\mathcal{U}_i$ be a cube in $\mathcal{Q}_i$ such that $\mathcal{U}_i$ has the same center as that of $\mathcal{Q}_i$, the faces of $\mathcal{U}_i$ and $\mathcal{Q}_i$ are parallel, and the length of the edge of $\mathcal{U}_i$ is $\frac{d}{2}$.

Define a cut-off function $\mu \in C_0^\infty(\mathbb{R})$ such that $0 \le \mu \le 1$, $\mu(x) = 1$ for $s \in [-\frac{d}{4}, \frac{d}{4}]$ and $\mu(x) = 0$ for $s \in \mathbb{R} \setminus [-\frac{d}{2}, \frac{d}{2}]$. Denote

$$\nu(x) := \mu(x_1)\mu(x_2)\dots\mu(x_N), \quad \text{for all } x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N.$$

For each $1 \le i \le k$, let

$$\nu_i(x) = \nu(x - y_i), \quad \text{for all } x \in \mathbb{R}^N,$$

where $y_i \in \mathbb{R}^N$ is the center of both $\mathcal{Q}_i$ and $\mathcal{U}_i$. Obviously, for all $1 \le i \le k$, we have

$$\operatorname{supp} \nu_i \subset \mathcal{Q}_i \tag{3.3}$$

and

$$\nu_i(x) = 1, \quad \text{for all } x \in \mathcal{U}_i, \qquad 0 \le \nu_i(x) \le 1, \quad \text{for all } x \in \mathbb{R}^N. \tag{3.4}$$

Denote

$$D_k := \left\{ (e_1, e_2, \dots, e_k) \in \mathbb{R}^k \mid \max_{1 \le i \le k} |e_i| = 1 \right\}$$

and

$$L_k := \left\{ \sum_{i=1}^k e_i \nu_i \mid (e_1, e_2, \dots, e_k) \in D_k \right\}.$$

It is well known that using an odd mapping, D_k is homeomorphic to the unit sphere in $\mathbb{R}^k$. Thus, $\gamma(D_k) = k$. And then, since the mapping $\mathcal{L} : D_k \to L_k$ defined by

$$\mathcal{L}(e_1, e_2, \dots, e_k) = \sum_{i=1}^k e_i \nu_i, \quad \text{for all } (e_1, e_2, \dots, e_k) \in D_k,$$

is an odd homeomorphism, this deduces $\gamma(D_k) = \gamma(L_k) = k$. Due to the compactness of L_k, there exits a constant $C_k > 0$ such that

$$\|v\| \le C_k, \quad \text{for all } v \in L_k. \tag{3.5}$$

For $v = \sum_{i=1}^k e_i \nu_i \in L_k$ and any $t \in (0, \frac{1}{2}(\frac{2^{1-p}}{3})^{1/p}\delta)$, by Lemma 2.1-(iii), the definition of $\widetilde{F}$ and the fact that $|H^{-1}(te_i\nu_i)| < \frac{\delta}{2}$ for all $1 \le i \le k$, we have

$$\begin{aligned} I(tv) &\le \frac{t^p}{p} \int_{\mathbb{R}^N} |\nabla v|^p dx + \frac{3 \cdot 2^{p-1} t^p}{p} \int_{\mathbb{R}^N} V(x)|v|^p dx - \sum_{i=1}^k \int_{\mathcal{Q}_i} K(x) \widetilde{F}(x, H^{-1}(te_i\nu_i)) dx \\ &\le \frac{3 \cdot 2^{p-1} t^p}{p} \|v\|^p - \sum_{i=1}^k \int_{\mathcal{Q}_i} K(x) F(x, H^{-1}(te_i\nu_i)) dx. \end{aligned} \tag{3.6}$$

From the definition of D_k, there exists $i_v \in [1, k]$ such that $|e_{i_v}| = 1$. Then, we rewrite the term $\sum_{i=1}^k \int_{\mathcal{Q}_i} K(x)F(x, H^{-1}(te_i\nu_i))dx$ in (3.6) as follows:

$$\int_{\mathcal{U}_{i_v}} K(x)F(x, H^{-1}(te_{i_v}\nu_{i_v}))dx + \int_{\mathcal{Q}_{i_v} \setminus \mathcal{U}_{i_v}} K(x)F(x, H^{-1}(te_{i_v}\nu_{i_v}))dx + \sum_{i \ne i_v} \int_{\mathcal{Q}_i} K(x)F(x, H^{-1}(te_i\nu_i))dx. \tag{3.7}$$

From Lemma 2.1-(2), (3.1) and (3.4), we deduce

$$\int_{\mathcal{Q}_{i_v} \setminus \mathcal{U}_{i_v}} K(x)F(x, H^{-1}(te_{i_v}\nu_{i_v}))dx + \sum_{i \ne i_v} \int_{\mathcal{Q}_i} K(x)F(x, H^{-1}(te_i\nu_i))dx \ge -\frac{3}{2^{1-p}} \alpha r_0^N K_1 t^p. \tag{3.8}$$

Choosing $t = \delta_n \in (0, \frac{1}{2}(\frac{2^{1-p}}{3})^{1/p}\delta)$ in (3.6), by using $F(x,t)$ is even for $|t| \leq \delta$, Lemma 2.1-(iii), (3.2) and (3.5)–(3.8), we obtain

$$\begin{aligned} I(\delta_n v) &\leq \frac{3 \cdot 2^{p-1}}{p} C_k^p \delta_n^p + \frac{3}{2^{1-p}} \alpha r_0^N K_1 \delta_n^p - \int_{U_{i_v}} K(x) F(x, H^{-1}(\delta_n e_{i_v} \nu_{i_v})) dx \\ &\leq \frac{3 \cdot 2^{p-1}}{p} C_k^p \delta_n^p + \frac{3}{2^{1-p}} \alpha r_0^N K_1 \delta_n^p - C \frac{d^N M_n}{2^N} |H^{-1}(\delta_n)|^p \\ &\leq \delta_n^p \left(\frac{3 \cdot 2^{p-1}}{p} C_k^p + \frac{3}{2^{1-p}} \alpha r_0^N K_1 - C \frac{d^N M_n}{2^N} \right). \end{aligned} \tag{3.9}$$

Note that $M_n \to \infty$ as $n \to \infty$, there exists an $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, we obtain

$$\frac{3 \cdot 2^{p-1}}{p} C_k^p + \frac{3}{2^{1-p}} \alpha r_0^N K_1 - C \frac{d^N M_n}{2^N} < 0.$$

Choosing

$$A_k := \{\delta_{n_0} v \mid v \in L_k\},$$

we deduce that A_k satisfies

$$\gamma(A_k) = \gamma(L_k) = k \quad \text{and} \quad \sup_{v \in A_k} I(v) < 0. \qquad \square$$

Next, we show a compactness result for the functional I.

Lemma 3.3. *Provided that assumptions (VK) and (f_1) hold, then I is bounded from below and satisfies the Palais–Smale condition.*

Proof. Let $v \in X$. Then, from (2.4), we have

$$\left| \int_{\mathbb{R}^N} K(x) \widetilde{F}(x, H^{-1}(v)) dx \right| \leq C \|v\|^q.$$

Therefore, we obtain

$$\begin{aligned} I(v) &= \frac{1}{p} \int_{\mathbb{R}^N} |\nabla v|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} V(x) |H^{-1}(v)|^p dx - \int_{\mathbb{R}^N} K(x) \widetilde{F}(x, H^{-1}(v)) dx \\ &\geq \frac{1}{p} \|v\|^p - C \|v\|^q. \end{aligned}$$

Note that $1 < q < p$, we can derive that I is bounded from below and I is coercive.

Next, we shall prove that I satisfies the Palais–Smale conditions. For $\{v_n\} \subset X$, such that

$$|I(v_n)| \leq c \quad \text{and} \quad I'(v_n) \to 0.$$

By I being coercive, we have the sequence $\{v_n\}$ bounded in X. Up to subsequence, we obtain

$$v_n \rightharpoonup v \text{ weakly in } X, \quad v_n \to v \text{ strongly in } L^q_{\text{loc}}(\mathbb{R}^{\mathbb{N}}) \quad \text{and} \quad v_n \to v \text{ a.e. on } \mathbb{R}^N.$$

Consider

$$\begin{aligned}
&\langle I'(v_n) - I'(v), v_n - v\rangle \\
&\quad = \int_{\mathbb{R}^N} (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v)(\nabla v_n - \nabla v)dx \\
&\qquad + \int_{\mathbb{R}^N} V(x)\left(|H^{-1}(v_n)|^{p-2}\frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} - |H^{-1}(v)|^{p-2}\frac{H^{-1}(v)}{h(H^{-1}(v))}\right)(v_n - v)dx \\
&\qquad - \int_{\mathbb{R}^N} K(x)\left(\frac{\widetilde{f}(x, H^{-1}(v_n))}{h(H^{-1}(v_n))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))}\right)(v_n - v)dx \\
&\quad \geq C\int_{\mathbb{R}^N} |\nabla v_n - \nabla v|^p dx \\
&\qquad + \int_{\mathbb{R}^N} V(x)\left(|H^{-1}(v_n)|^{p-2}\frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} - |H^{-1}(v)|^{p-2}\frac{H^{-1}(v)}{h(H^{-1}(v))}\right)(v_n - v)dx \\
&\qquad - \int_{\mathbb{R}^N} K(x)\left(\frac{\widetilde{f}(x, H^{-1}(v_n))}{h(H^{-1}(v_n))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))}\right)(v_n - v)dx \\
&\quad := C\int_{\mathbb{R}^N} |\nabla v_n - \nabla v|^p dx + I_1 - I_2,
\end{aligned} \tag{3.10}$$

where we use the elementary inequalities:

$$(|a|^{p-2}a - |b|^{p-2}b)(a-b) \geq \begin{cases} C(|a| + |b|)^{p-2}|a-b|^2, & \text{for } a, b \in \mathbb{R}^N \text{ if } 1 < p < 2, \\ C|a-b|^p, & \text{for } a, b \in \mathbb{R}^N \text{ if } p \geq 2. \end{cases}$$

Firstly, we will show that

$$I_1 \geq 0. \tag{3.11}$$

In fact, a direct computation shows that second derivative of the function

$$G(t) = |H^{-1}(t)|^p \quad \text{for } t \in \mathbb{R}$$

satisfies the equality

$$G''(t) = \frac{\left((p-1)g(H^{-1}(t)) - \frac{g'(H^{-1}(t))H^{-1}(t)}{g(H^{-1}(t))}\right)|H^{-1}(t)|^{p-2}}{g^2(H^{-1}(t))} > 0 \quad \text{for } t \in \mathbb{R} \setminus \{0\},$$

which implies that G is a convex function. From this, we obtain

$$(G'(t) - G'(s))(t - s) \geq 0, \quad \text{for all } t, s \in \mathbb{R},$$

that is

$$I_1 = \int_{\mathbb{R}^N} V(x)\left(|H^{-1}(v_n)|^{p-2}\frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} - |H^{-1}(v)|^{p-2}\frac{H^{-1}(v)}{h(H^{-1}(v))}\right)(v_n - v)dx \geq 0.$$

Secondly, for any $R > 0$, we estimate I_2 as follows:

$$
\begin{aligned}
&\int_{\mathbb{R}^N} K(x) \left| \frac{\widetilde{f}(x, H^{-1}(v_n))}{h(H^{-1}(v_n))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right| |v_n - v| dx \\
&\quad \le C \int_{\mathbb{R}^N \setminus B_R(0)} K(x) \left(|H^{-1}(v_n)|^{q-1} + |H^{-1}(v)|^{q-1} \right) (|v_n| + |v|) dx \\
&\qquad + C \int_{B_R(0)} \left(|v_n|^{q-1} + |v|^{q-1} \right) |v_n - v| dx \\
&\quad \le C \int_{\mathbb{R}^N \setminus B_R(0)} K(x) \left(|v_n|^q + |v|^q \right) dx + C \int_{B_R(0)} \left(|v_n|^{q-1} + |v|^{q-1} \right) |v_n - v| dx \\
&\quad \le C \|W(x)\|_{L^1(\mathbb{R}^N \setminus B_R(0))}^{(p-q)/p} \left(\|V(x) v_n^p\|_{L^1(\mathbb{R}^N \setminus B_R(0))}^{q/p} + \|V(x) v^p\|_{L^1(\mathbb{R}^N \setminus B_R(0))}^{q/p} \right) \\
&\qquad + C \left(\|v_n\|_{L^q(B_R(0))}^{q-1} + \|v\|_{L^q(B_R(0))}^{q-1} \right) \|v_n - v\|_{L^q(B_R(0))} \\
&\quad \le C \|W(x)\|_{L^1(\mathbb{R}^N \setminus B_R(0))}^{(p-q)/p} + C \|v_n - v\|_{L^q(B_R(0))},
\end{aligned}
$$

it follows that

$$
\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x) \left(\frac{\widetilde{f}(x, H^{-1}(v_n))}{h(H^{-1}(v_n))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right) (v_n - v) dx = 0.
$$

From the above estimate, (3.10) and (3.11), we get

$$
\int_{\mathbb{R}^N} |\nabla v_n - \nabla v|^p dx = o_n(1) \tag{3.12}
$$

and

$$
\begin{aligned}
I_1 = \int_{\mathbb{R}^N} V(x) \left(|H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} - |H^{-1}(v)|^{p-2} \frac{H^{-1}(v)}{h(H^{-1}(v))} \right) (v_n - v) dx \\
= o_n(1).
\end{aligned} \tag{3.13}
$$

It is easy to say (3.13) can also be expressed as

$$
\begin{aligned}
\int_{\mathbb{R}^N} V(x) |H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} v_n dx = &\int_{\mathbb{R}^N} V(x) |H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} v dx \\
&+ \int_{\mathbb{R}^N} V(x) |H^{-1}(v)|^{p-2} \frac{H^{-1}(v)}{h(H^{-1}(v))} (v_n - v) dx \\
&+ o_n(1).
\end{aligned}
$$

Since $v_n \rightharpoonup v$ weakly in X,

$$
\int_{\mathbb{R}^N} V(x) |H^{-1}(v)|^{p-2} \frac{H^{-1}(v)}{h(H^{-1}(v))} (v_n - v) dx = o_n(1),
$$

and so,

$$
\begin{aligned}
\int_{\mathbb{R}^N} V(x) |H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} v_n dx = \int_{\mathbb{R}^N} V(x) |H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} v dx \\
+ o_n(1).
\end{aligned} \tag{3.14}
$$

Recalling that

$$|H^{-1}(t)| \le \left(\frac{3}{2^{1-p}}\right)^{1/p} |t| \quad \text{and} \quad \left(\frac{2^{1-p}}{3}\right)^{1/p} < h(H^{-1}(v_n)) \le 1 \quad \text{for all } t \in \mathbb{R}.$$

From this, we know $V(x)^{\frac{p-1}{p}}|H^{-1}(v_n)|^{p-2}\frac{H^{-1}(v_n)}{h(H^{-1}(v_n))}$ is bounded sequence in $L^{\frac{p}{p-1}}(\mathbb{R}^N)$. Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^N} V(x)|H^{-1}(v_n)|^{p-2}\frac{H^{-1}(v_n)}{h(H^{-1}(v_n))}v dx\\
&\quad = \int_{\mathbb{R}^N} V(x)^{(p-1)/p}|H^{-1}(v_n)|^{p-2}\frac{H^{-1}(v_n)}{h(H^{-1}(v_n))}V(x)^{1/p}v dx\\
&\quad = \int_{\mathbb{R}^N} V(x)|H^{-1}(v)|^{p-2}\frac{H^{-1}(v)}{h(H^{-1}(v))}v dx + o_n(1).
\end{aligned} \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\begin{aligned}
\int_{\mathbb{R}^N} V(x)|H^{-1}(v_n)|^{p-2}\frac{H^{-1}(v_n)}{h(H^{-1}(v_n))}v_n dx = &\int_{\mathbb{R}^N} V(x)|H^{-1}(v)|^{p-2}\frac{H^{-1}(v)}{h(H^{-1}(v))}v dx\\
&+ o_n(1).
\end{aligned} \tag{3.16}$$

By Lemma 2.1, we have

$$V(x)|H^{-1}(v_n)|^p \le \left(\frac{3}{2^{1-p}}\right)^{1/p} V(x)|H^{-1}(v_n)|^{p-2}\frac{H^{-1}(v_n)}{h(H^{-1}(v_n))}v_n.$$

Then, using the above discussions together with Lebesgue's Theorem, we obtain

$$\int_{\mathbb{R}^3} V(x)|H^{-1}(v_n)|^p dx = \int_{\mathbb{R}^3} V(x)|H^{-1}(v)|^p dx + o_n(1). \tag{3.17}$$

On the other hand, by $(H^{-1}(t))' \le \left(\frac{3}{2^{1-p}}\right)^{1/p}$ (for all $t \in \mathbb{R}$), we obtain

$$\begin{aligned}
|H^{-1}(v_n - v)| &= H^{-1}(|v_n - v|)\\
&\le H^{-1}(|v_n| + |v|)\\
&\le H^{-1}(|v_n|) + \left(\frac{3}{2^{1-p}}\right)^{1/p} |v|,
\end{aligned}$$

which implies

$$V(x)|H^{-1}(v_n - v)|^p \le 2^p V(x)\left(|H^{-1}(v_n)|^p + \left(\frac{3}{2^{1-p}}\right)^{1/p} |v|^p\right).$$

From the last inequality, (3.17) and Lebesgue's Theorem, we get

$$\int_{\mathbb{R}^N} V(x)|H^{-1}(v_n - v)|^p dx = o_n(1). \tag{3.18}$$

Finally, combing (3.12) and (3.18), we have

$$\begin{aligned}
\int_{\mathbb{R}} (|\nabla(v_n - v)|^p + V(x)|v_n - v|^p)\, dx &\le \int_{\mathbb{R}} \left(|\nabla(v_n - v)|^p + V(x)|H^{-1}(v_n - v)|^p\right) dx\\
&= o_n(1),
\end{aligned}$$

which concludes the proof of the lemma. □

4 Proof of Theorem 1.3

In this section, we firstly study Brezis–Kato type estimates of the critical points of I.

Lemma 4.1. *Assume that $\{v_k\} \subset X$ is a critical point sequence of I satisfying $v_k \to 0$ strongly in X. Then, $v_k \to 0$ strongly in $L^\infty(\mathbb{R}^N)$.*

Proof. Let $v \in X$ be a weak solution of (2.5), i.e.,

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla \varphi dx + \int_{\mathbb{R}^N} V(x)|H^{-1}(v)|^{p-2}\frac{H^{-1}(v)}{h(H^{-1}(v))}\varphi dx \\ - \int_{\mathbb{R}^N} K(x)\frac{\widetilde{f}(x,H^{-1}(v))}{h(H^{-1}(v))}\varphi dx, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N). \tag{4.1}$$

Set $T > 0$, and denote

$$v_T = \begin{cases} -T, & \text{if } v \le -T, \\ v, & \text{if } -T < v < T, \\ T, & \text{if } v \ge T. \end{cases}$$

Taking $\varphi = |v_T|^{p(\eta-1)}v_T$ as the text function, where $\eta > 1$ to be determined later, we obtain

$$\int_{\mathbb{R}^N} |v_T|^{p(\eta-1)}|\nabla v|^{p-2}\nabla v \nabla v_T dx + p(\eta-1)\int_{\mathbb{R}^N} |v_T|^{p(\eta-1)-1}|\nabla v|^{p-2}\nabla v\nabla v_T dx \\ + \int_{\mathbb{R}^N} V(x)|H^{-1}(v)|^{p-2}\frac{H^{-1}(v)}{h(H^{-1}(v))}|v_T|^{p(\eta-1)}v_T dx = \int_{\mathbb{R}^N} K(x)\frac{\widetilde{f}(x,H^{-1}(v))}{h(H^{-1}(v))}|v_T|^{p(\eta-1)}v_T dx.$$

By using the facts

$$p(\eta-1)\int_{\mathbb{R}^N} |v_T|^{p(\eta-1)-1}|\nabla v|^{p-2}\nabla v\nabla v_T dx \ge 0,$$
$$\int_{\mathbb{R}^N} V(x)|H^{-1}(v)|^{p-2}\frac{H^{-1}(v)}{h(H^{-1}(v))}|v_T|^{p(\eta-1)}v_T dx \ge 0$$

and Lemma 2.1-(iii), we have

$$\frac{1}{\eta^p}\int_{\mathbb{R}^N} |\nabla |v_T|^\eta|^p dx \le C\int_{\mathbb{R}^N} \widetilde{f}(x,H^{-1}(v))|v|^{p(\eta-1)+1}dx \le C\int_{\mathbb{R}^N} |v|^{p\eta+q-p}dx. \tag{4.2}$$

On the other hand, it follows from the Sobolev inequality that

$$\frac{S}{\eta^p}\|v_T\|_{\eta p^*}^{p\eta} \le \frac{1}{\eta^p}\int_{\mathbb{R}^N} |\nabla |v_T|^\eta|^p dx, \tag{4.3}$$

where $S = \inf\{\int_{\mathbb{R}^N} |\nabla v|^p dx \mid \int_{\mathbb{R}^N} |v|^{p^*}dx = 1\}$. In what follows, by (4.2) and (4.3), we get

$$\frac{1}{\eta^p}\|v_T\|_{\eta p^*}^{p\eta} \le C\int_{\mathbb{R}^N} |v|^{p\eta+q-p}dx. \tag{4.4}$$

From Fatou's lemma, sending $T \to \infty$ in (4.4), it follows that

$$\|v\|_{\eta p^*} \le (C\eta)^{1/\eta}\, \|v\|_{p\eta+q-p}^{(p\eta+q-p)/p\eta}. \tag{4.5}$$

Let us define $\eta_k = \frac{p^*\eta_{k-1}+p-q}{p}$, where $k = 1, 2, \ldots$ and $\eta_0 = \frac{p^*+p-q}{p}$. We show the first step of Moser's iteration as follows:

$$\begin{aligned} \|v\|_{\eta_1 p^*} &\le (C\eta_1)^{1/\eta_1} \|v\|_{p\eta_1+q-p}^{(p\eta_1+q-p)/p\eta_1} \\ &\le (C\eta_1)^{1/\eta_1} (C\eta_0)^{1/\eta_0\cdot(p\eta_1+q-p)/p\eta_1} \|v\|_{p\eta_0+q-p}^{(p\eta_0+q-p)/p\eta_0\cdot(p\eta_1+q-p)/p\eta_1}. \end{aligned} \tag{4.6}$$

Without loss of generality, we may assume $C > 1$. For $i < j$, we have

$$(C\eta_i)^{(p\eta_j+q-p)/(p\eta_j)} \le C\eta_i. \tag{4.7}$$

From (4.6) and (4.7), we have

$$\|v\|_{\eta_1 p^*} \le (C\eta_1)^{1/\eta_1} (C\eta_0)^{1/\eta_0} \|v\|_{p\eta_0+q-p}^{(p\eta_0+q-p)/p\eta_0\cdot(p\eta_1+q-p)/p\eta_1}.$$

Then by Moser's iteration method we get

$$\|v\|_{p\eta_{k+1}+q-p} \le \exp\left(\sum_{i=0}^{k} \frac{\ln(C\eta_i)}{\eta_i}\right) \|v\|_{p^*}^{\mu_k},$$

where $\mu_k = \prod_{i=0}^{k} \frac{p\eta_i+q-p}{p\eta_i}$. Sending $k \to \infty$, we deduce that

$$\|v\|_\infty \le \exp\left(\sum_{i=0}^{\infty} \frac{\ln(C\eta_i)}{\eta_i}\right) \|v\|_{p^*}^{\mu},$$

where $\mu = \prod_{i=0}^{\infty} \frac{p\eta_i+q-p}{p\eta_i} (0 < \mu < 1)$ and $\exp\left(\sum_{i=0}^{\infty} \frac{\ln(C\eta_i)}{\eta_i}\right)$ is a positive constant. This, together with the Sobolev embedding theorem, shows that if $\{v_k\}$ is a critical point sequence of J satisfying $v_k \to 0$ strongly in X as $k \to \infty$, then $v_k \to 0$ strongly in $L^\infty(\mathbb{R}^N)$. This completes the proof. □

Proof of Theorem 1.3. It is well known that I is an even functional with $I(0) = 0$. In addition, by Lemma 3.3, Lemma 3.2 and Proposition 3.1, the functional I possesses a sequence of critical points $\{v_n\}$ such that $I(v_n) \to 0$ and $v_n \to 0$ strongly in X. Recall that the weak solutions of (2.1) with an L^∞-norm not more than $\min\{\delta/2, (\frac{2^{1-p}}{3})^{1/p}\}$ are also weak solutions of problem (1.3). Then, by Lemma 4.1, this $\{v_n\}$ is a sequence of weak solutions for (2.5) with $v_n \to 0$ strongly in $L^\infty(\mathbb{R}^N)$. Letting $u_n = H^{-1}(v_n)$, from Lemma 2.2, there exists $n^* \in \mathbb{N}$ such that u_n is a weak solution of (1.3) for each $n \ge n^*$. This ends the proof. □

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On qualitative behavior of multiple solutions of quasilinear parabolic functional equations

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Abstract. We shall consider weak solutions of initial-boundary value problems for semilinear and nonlinear parabolic differential equations for $t \in (0, \infty)$ with certain nonlocal terms. We shall prove theorems on the number of solutions and certain qualitative properties of the solutions. These statements are based on arguments for fixed points of some real functions and operators, respectively, and theorems on the existence, uniqueness and qualitative properties of the solutions of partial differential equations (without functional terms).

Keywords: partial functional differential equations, multiple solutions, qualitative properties.

2010 Mathematics Subject Classification: 35R10, 35R09.

1 Introduction

It is well known that mathematical models in several applications are functional differential equations of one variable (e.g. delay equations). In the monograph by Jianhong Wu [7] semilinear evolutionary partial functional differential equations and applications are considered, where the book is based on the theory of semigroups and generators. In the monograph by A. L. Skubachevskii [6] linear elliptic functional differential equations (equations with nonlocal terms and nonlocal boundary conditions) and applications are considered. A nonlocal boundary value problem, arising in plasma theory, was considered by A. V. Bitsadze and A. A. Samarskii in [1].

It turned out that the theory of pseudomonotone operators is useful to study nonlinear (quasilinear) partial functional differential equations (both stationary and evolutionary equations) and to prove existence of weak solutions (see [2,4]).

In [5] we considered some nonlinear parabolic functional differential equations for $t \in (0, T)$ $(T < \infty)$ and proved existence of several weak solutions of initial-boundary boundary value problems.

In the present work we shall prove existence of weak solutions of some parabolic functional equations for $t \in (0, \infty)$ and show certain qualitative properties of the solutions (boundedness and stabilization as $t \to \infty$).

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First we remind the reader of the definition of weak solutions of initial-boundary value problems of nonlinear parabolic (functional) differential equation for $t \in (0, T)$ and $t \in (0, \infty)$ with zero initial and boundary conditions.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary, $1 < p < \infty$. Denote by $W^{1,p}(\Omega)$ the usual Sobolev space of real valued functions with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left[\int_\Omega (|Du|^p + |u|^p)\right]^{1/p}.$$

Further, let $V \subset W^{1,p}(\Omega)$ be a closed linear subspace containing $C_0^\infty(\Omega)$, $V^\star$ the dual space of V, the duality between $V^\star$ and V will be denoted by $\langle \cdot, \cdot \rangle$.

Denote by $L^p(0, T; V)$ the Banach space of functions $u : (0, T) \to V$ ($V \subset W^{1,p}(\Omega)$ is a closed linear subspace) with the norm

$$\|u\|_{L^p(0,T;V)} = \left[\int_0^T \|u(t)\|_V^p dt\right]^{1/p} \quad (1 < p < \infty).$$

The dual space of $L^p(0, T; V)$ is $L^q(0, T; V^\star)$ where $1/p + 1/q = 1$. (See, e.g. [8].) Let $A : L^p(0, T; V) \to L^q(0, T; V^\star)$ be a given (nonlinear) operator and $F \in L^q(0, T; V^\star)$.

Weak solutions of

$$D_t u + A(u) = F \tag{1.1}$$

for $t \in (0, T)$ with zero initial and boundary condition is a function $u \in L^p(0, T; V)$ satisfying $D_t u \in L^q(0, T; V^\star)$, (1.1) and $u(0) = 0$. (For $p \geq 2$, $u \in L^p(0, T; V)$ and $D_t u \in L^q(0, T; V^\star)$ imply $u \in C([0, T]; L^2(\Omega))$ thus the initial condition makes sense.)

Consider first the particular case (without functional terms) $A = \tilde{A}$ where

$$\langle [\tilde{A}(u)](t), v \rangle = \int_\Omega \left[\sum_{j=1}^n a_j(t, x, u, Du) D_j v + a_0(t, x, u, Du) v\right] dx \tag{1.2}$$

for all $v \in V$, almost all $t \in [0, T]$. By using the theory of monotone operators the following existence and uniqueness theorem is proved. (See, e.g., [3,4,8].)

($C1$) The functions $a_j : (0, T) \times \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$ ($j = 0, 1, \ldots, n$) satisfy the Carathéodory conditions, i.e. $(t, x) \mapsto a_j(t, x, \xi)$ is measurable for all $\xi \in \mathbb{R}^{n+1}$ and $\xi \mapsto a_j(t, x, \xi)$ is continuous for a.a. (t, x).

($C2$) There exist a constant c_1 and a function $k_1 \in L^q((0, T) \times \Omega)$ ($1/p + 1/q = 1$, $p \geq 2$) such that

$$|a_j(t, x, \xi)| \leq c_1[1 + |\xi|^{p-1}] + k_1(t, x),$$

$j = 0, 1, \ldots, n$, for a.a. $(t, x) \in (0, T) \times \Omega$, each $\xi \in \mathbb{R}^{n+1}$.

($C3$) The inequality

$$\sum_{j=0}^n [a_j(t, x, \xi) - a_j(t, x, \xi^\star)](\xi_j - \xi_j^\star) \geq c_2 |\xi - \xi^\star|^p$$

holds with come constant $c_2 > 0$.

Theorem 1.1. *Assume ($C1$)–($C3$). Then for any $F \in L^q(0, T; V^\star)$ there exists a unique $u \in L^p(0, T; V)$ weak solution of* (1.1) *with $A = \tilde{A}$ which depends on F continuously.*

A more general case is when $[A(u)](t)$ is depending not only on $u(t)$ and $(Du)(t)$, then (1.1) is a functional equation. By using the theory of pseudomonotone operators, one can prove existence of solutions for $t \in [0, T]$ in this more general case. (See, e.g., [4].)

Now we formulate a theorem on weak solutions of (1.1) for $t \in (0, \infty)$. The set $L^p_{\text{loc}}(0, \infty; V)$ consists of all functions $f : (0, \infty) \to V$ for which the restriction $f|_{(0,T)}$ belongs to $L^p(0, T; V)$ for each finite $T > 0$. Furthermore, by using the notations $Q_T = (0, T) \times \Omega$, $Q_\infty = (0, \infty) \times \Omega$ denote by $L^p_{\text{loc}}(Q_\infty)$ the set of functions $f : Q_\infty \to \mathbb{R}$ for which $f|_{Q_T} \in L^p(Q_T)$ with arbitrary $T > 0$. Assume that

(C_∞1) Functions $a_j : Q_\infty \times \mathbb{R}^{n+1}$ satisfy the Carathéodory conditions.

(C_∞2) There exist a constant c_1 and a function $k_1 \in L^q(\Omega)$ such that

$$|a_j(t, x, \xi)| \le c_1 |\xi|^{p-1} + k_1(x).$$

(C_∞3) For a.a. $(t, x) \in Q_\infty$, all $\xi, \xi^\star \in \mathbb{R}^{n+1}$

$$\sum_{j=0}^{n} [a_j(t, x, \xi) - a_j(t, x, \xi^\star)](\xi_j - \xi^\star) \ge c_2 |\xi - \xi^\star|^p$$

with some constant $c_2 > 0$.

Theorem 1.2. *Assume (C_∞1)–(C_∞3). Then for arbitrary $F \in L^q_{\text{loc}}(0, \infty; V^\star)$ there is a unique $u \in L^p_{\text{loc}}(0, \infty; V)$ such that $u' \in L^q_{\text{loc}}(0, \infty; V^\star)$ and*

$$D_t u(t) + [\tilde{A}(u)](t) = F(t) \quad \textit{for a.a. } t \in (0, \infty), \quad u(0) = 0$$

with the operator $\tilde{A}$ defined in (1.2).

If $\|F(t)\|_{V^\star}$ is bounded for a.a. $t \in (0, \infty)$ then for a solution u, $\|u(t)\|_{L^2(\Omega)}$ is bounded and

$$\int_{T_1}^{T_2} \|u(t)\|_V^p dt \le c_3 (T_2 - T_1) \quad \textit{with some constant } c_3. \tag{1.3}$$

Now we formulate a theorem on the stabilization of $u(t)$ as $t \to \infty$.

Theorem 1.3. *Assume that the assumptions of the above theorem are satisfied. Further, there exist Carathéodory functions $a_{j,\infty} : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$, a continuous function $\Phi : (0, \infty) \to (0, \infty)$ and $F_\infty \in V^\star$ such that*

$$|a_j(t, x, \xi) - a_{j,\infty}(x, \xi)| \le \Phi(t)(|\xi|^{p-1} + 1), \quad \textit{where } \lim_{\infty} \Phi = 0, \tag{1.4}$$

$$\|F(t) - F_\infty\|_{V^\star} \le \Phi(t) \quad \textit{for a.a. } t > 0. \tag{1.5}$$

Then

$$\lim_{t \to 0} \|u(t) - u_\infty\|_{L^2(\Omega)} = 0, \qquad \lim_{T \to \infty} \int_{T-a}^{T+a} \|u(t) - u_\infty\|_V^p dt = 0 \tag{1.6}$$

for arbitrary fixed $a > 0$ where $u_\infty \in V$ is the unique solution $z \in V$ to

$$\sum_{j=1}^{n} \int_\Omega a_{j,\infty}(x, z, Dz) D_j v dx + \int_\Omega a_{0,\infty}(x, z, Dz) v dx = \langle F_\infty, v \rangle, \qquad v \in V.$$

(For the proofs, see, e.g., [4].)

By using the above results, we shall consider parabolic functional equations (equations containing some nonlocal terms) of certain particular type. In Section 2 equations with real valued functionals and in Section 3 equations with certain operators will be studied.

2 Parabolic equations with real valued functionals, applied to the solution

Case 1. First consider a semilinear parabolic functional equation for $t \in (0, \infty)$

$$D_t u + \tilde{B} u = D_t u - \sum_{j,k=1}^{n} D_j [a_{jk}(t,x) D_k u] + a_0(t,x) u = k(M(u)) F_1 + F_2 \tag{2.1}$$

(i.e. the elliptic operator $\tilde{A}$ in (1.2) is linear), where $M : L^2(0, T_0; V) \to \mathbb{R}$ is a given linear continuous functional ($T_0 < \infty$), $V \subset W^{1,2}(\Omega)$, $k : \mathbb{R} \to \mathbb{R}$ is a given continuous function, $F_1, F_2 \in L^2_{\text{loc}}(0, \infty; V^\star)$. Further, $a_{jk}, a_0 \in L^2_{\text{loc}}((0, \infty) \times \Omega)$, $a_{jk} = a_{kj}$ and the functions a_{jk} satisfy the uniform ellipticity condition

$$c_1 |\xi|^2 \le \sum_{j,k=1}^{n} a_{jk}(t,x) \xi_j \xi_k + a_0(t,x) \xi_0^2 \le c_2 |\xi|^2$$

for all $\xi = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, $x \in \Omega$, $t \in (0, \infty)$ with some positive constants c_1, c_2.

Remark 2.1. The linear continuous functional $M : L^2(0, T_0; V) \to \mathbb{R}$ may have the form

$$M(u) = \int_0^{T_0} \int_\Omega \left[K_0(t,x) u(t,x) + \sum_{j=1}^{n} K_j(t,x) D_j u(t,x) \right] dxdt \tag{2.2}$$

where $K_0, K_1 \in L^2((0, T_0) \times \Omega)$.

According to Theorem 1.2, for arbitrary $F \in L^2_{\text{loc}}(0, \infty; V^\star)$ there is a unique solution $u \in L^2_{\text{loc}}(0, \infty; V)$ of

$$D_t u + \tilde{B} u = F,$$

denoted by $u = (D_t + \tilde{B})^{-1} F$.

Theorem 2.2. *A function $u \in L^2_{\text{loc}}(0, \infty; V)$ is a weak solution of (2.1) if and only if $\lambda = Mu$ satisfies the equation*

$$\lambda = k(\lambda) M[(D_t + \tilde{B})^{-1} F_1] + M[(D_t + \tilde{B})^{-1} F_2]. \tag{2.3}$$

and

$$u = k(\lambda)(D_t + \tilde{B})^{-1} F_1 + (D_t + \tilde{B})^{-1} F_2. \tag{2.4}$$

Proof. By Theorem 1.2 function $u \in L^2_{\text{loc}}(0, \infty; V)$ is a weak solution of (2.1) if and only if

$$u = k(M(u))(D_t + \tilde{B})^{-1} F_1 + (D_t + \tilde{B})^{-1} F_2,$$

thus

$$M(u) = k(M(u)) M[(D_t + \tilde{B})^{-1} F_1 + (D_t + \tilde{B})^{-1} F_2]$$

which implies the theorem. □

Corollary 2.3. *The number of weak solutions of (2.1) (with zero initial-boundary conditions) equals the number of solutions λ of equation (2.3). Consequently, it is easy to show that for any natural number N or for $N = \infty$ one can choose functions k such that (2.1) has exactly N solutions.*

Remark 2.4. If we know the values of $M[(D_t+B)^{-1}F_1]$ and $M[(D_t+B)^{-1}F_2]$ then by using some numerical procedure one can calculate the λ roots of (2.3). Further, it is easy to show simple sufficient conditions on $M[(D_t+B)^{-1}F_1]$, $M[(D_t+B)^{-1}F_2]$ and the function k which imply that (2.3) has zero, exactly one (two or three) roots.

From Theorem 1.3 it directly follows

Theorem 2.5. *If there exist measurable functions $a_{j,k,\infty}, a_{0,\infty} \in L^\infty(\Omega)$ and $F_{1,\infty}, F_{2,\infty} \in V^\star$ such that*

$$|a_0(t,x) - a_{0,\infty}(x)| \le \Phi(t), \qquad |a_{j,k}(t,x) - a_{j,k,\infty}(x)| \le \Phi(t), \quad \text{where } \lim_\infty \Phi = 0,$$

$$\|F_1(t) - F_{1,\infty}\|_{V^\star} \le \Phi(t), \qquad \|F_2(t) - F_{2,\infty}\|_{V^\star} \le \Phi(t) \quad \text{for a.a. } t > 0$$

then we have (1.6) where $u_\infty \in V$ is the unique solution $z \in V$ to

$$\sum_{j,k=1}^n \int_\Omega a_{j,k,\infty}(x)(D_j z)(D_k v)dx + \int_\Omega a_{0,\infty}(x) z v dx = \langle k(M(u))F_{1,\infty}, v\rangle + \langle F_{2,\infty}, v\rangle, \qquad v \in V.$$

Case 2. Now consider *nonlinear* parabolic functional equations of the form

$$D_t u + [lM(u))]^\gamma \tilde{A}(u) = [lM(u))]^\beta F, \qquad t \in (0,\infty), \quad u(0) = 0 \tag{2.5}$$

where the nonlinear operator $\tilde{A}$ has the form (1.2) and has the property

$$\tilde{A}(\mu u) = \mu^{p-1}\tilde{A}(u), \quad \text{for all } \mu > 0 \text{ with some } p \ge 2 \tag{2.6}$$

(e.g. $\tilde{A}(u) = -\triangle_p u + c_0 u|u|^{p-2}$ with $c_0 > 0$ has this property), further, $M : L^p(0,T_0;V) \to \mathbb{R}$ ($V \subset W^{1,p}(\Omega)$) is (homogeneous) functional with the property

$$M(\mu u) = \mu^\sigma M(u) \quad \text{for all } \mu > 0 \text{ with some } \sigma > 0; \tag{2.7}$$

l is a given positive continuous function and the numbers β, γ satisfy

$$\gamma = \beta(2-p), \qquad \beta > 0.$$

A simple calculation shows

Theorem 2.6. *A function $u \in L^p_{\text{loc}}(0,\infty;V)$ satisfies (2.5) in weak sense if and only if*

$$\tilde{u} = [l(M(u))]^{-\beta} u \quad \text{satisfies} \quad D_t\tilde{u} + \tilde{A}(\tilde{u}) = F.$$

This theorem implies

Theorem 2.7. *A function $u \in L^p_{\text{loc}}(0,\infty;V)$ is a weak solution of (2.5) with zero initial and boundary condition if and only if $\lambda = M(u)$ satisfies the equation*

$$\lambda = [l(\lambda)]^{\beta\sigma} M[B_0^{-1}(F)] \quad \text{and} \quad u = [l(\lambda)]^\beta B_0^{-1}(F) \tag{2.8}$$

where B_0 is defined by $B_0(u) = D_t u + \tilde{A}(u)$, i.e. $B_0^{-1}(F)$ is the unique weak solution of (1.1) (with $A = \tilde{A}$ and zero initial and boundary condition). If $F \in L^\infty(0,\infty;V^\star)$ then $\|u(t)\|_{L^2(\Omega)}$ is bounded and (1.3) holds.

Corollary 2.8. *The number of weak solutions of* (2.5) *equals the number of roots of* (2.8)*. Further, assuming* $M[B_0^{-1}(F)] > 0$*, for arbitrary* $N = 1, 2, \dots, \infty$ *one can construct a continuous positive function* l *such that* (2.5) *has exactly* N *solutions, in the following way. Let* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $g(\lambda) + \lambda > 0$ *for all* $\lambda \in \mathbb{R}$ *and* g *has* N *real roots. Then for*

$$l(\lambda) = \left[\frac{g(\lambda) + \lambda}{M(B_0^{-1}(F))} \right]^{1/(\beta\sigma)}$$

(2.5) *has* N *weak solutions.*

Remark 2.9. An example for functional M with property (2.7) is integral operator

$$M(u) = \int_0^T \int_\Omega K(t,x)|u(t,x)|^\sigma dt dx.$$

By Theorems 1.3 and 2.6 one obtains

Theorem 2.10. *If the assumptions* (1.4), (1.5) *are satisfied then we have* (1.6) *where* $u_\infty \in V$ *is the unique solution* $z \in V$ *to*

$$\begin{aligned} &\sum_{j=1}^n \int_\Omega a_{j,\infty}(x, z, Dz) D_j v dx + \int_\Omega a_{0,\infty}(x, z, Dz) v dx \\ &\quad = (l(\lambda))^\beta \langle F_\infty, v \rangle = [l(M(u))]^\beta \langle F_\infty, v \rangle, \qquad v \in V. \end{aligned}$$

3 Parabolic equations with nonlocal operators

Now consider partial functional equations of the form

$$D_t u + \tilde{A}(u) = C(u) \tag{3.1}$$

where $\tilde{A}$ is nonlinear differential operator (1.2) satisfying ($C_\infty 1$)–($C_\infty 3$) (or $\tilde{A} = \tilde{B}$ is a uniformly elliptic linear differential operator (see (2.1)) and $C : L^p_{\text{loc}}(0,\infty; V) \to L^p_{\text{loc}}(0,\infty; V^\star)$ is a given (possibly nonlinear) operator. Clearly, $u \in L^p_{\text{loc}}(0,\infty; V)$ satisfies (3.1) if and only if

$$u = (D_t + \tilde{A})^{-1}[C(u)] =: G(u) \tag{3.2}$$

where $G : L^p_{\text{loc}}(0,\infty; V) \to L^p_{\text{loc}}(0,\infty; V)$ is a given (possibly nonlinear) operator, i.e. u is a fixed point of G. Then

$$C(u) = (D_t + \tilde{A})[G(u)]. \tag{3.3}$$

Now we consider three particular cases for G.

Case 1. The operator G is defined by

$$[G(u)](t,x) = (Lu)(t,x) + F(t,x) = \int_0^\infty \int_\Omega K(t,\tau,x,y) u(\tau,y) d\tau dy + F(t,x) \tag{3.4}$$

where $K \in L^2((0,\infty) \times (0,\infty) \times \Omega \times \Omega)$; $u, F \in L^2((0,\infty) \times \Omega)$.

By using (3.1) and (3.3) we find

Theorem 3.1. *If K and F are sufficiently smooth and "good" then the solution $u \in L^2(0,\infty) \times \Omega)$ of* (3.2) *with the operator* (3.4) *belongs to $L^p_{\text{loc}}(0,\infty;V)$, $D_t u$ belongs to $L^q_{\text{loc}}(0,\infty;V^\star)$ (in the linear case $\tilde{A} = \tilde{B}$, $p = q = 2$), $u(0) = 0$ and the equation* (3.1) *has the form*

$$\begin{aligned} D_t u + (\tilde{A}(u))(t,x) = & \int_0^\infty \int_\Omega D_t K(t,\tau,x,y) u(\tau,y) dx dy + D_t F(t,x) \\ & + \tilde{A}_x \left[\int_0^\infty \int_\Omega K(t,\tau,x,y) u(\tau,y) d\tau dy + F(t,x) \right]. \end{aligned} \tag{3.5}$$

In the linear case $\tilde{A} = \tilde{B}$

$$\begin{aligned} D_t u + (\tilde{B}u)(t,x) = & \int_0^\infty \int_\Omega D_t K(t,\tau,x,y) u(\tau,y) dx dy + D_t F(t,x) \\ & + \int_0^\infty \int_\Omega \tilde{B}_x K(t,\tau,x,y) u(\tau,y) d\tau dy + \tilde{B}_x F(t,x). \end{aligned} \tag{3.6}$$

($\tilde{A}_x K(t,\tau,x,y)$ denotes the differential operator $\tilde{A}$ applied to $x \mapsto K(t,\tau,x,y)$ and $\tilde{B}_x F(t,x)$ denotes the differential operator $\tilde{B}$ applied to $x \mapsto F(t,x)$.)

Further, if 1 is an eigenvalue of the linear integral operator $L : L^2((0,\infty)\Omega) \to L^2((0,\infty)\Omega)$ with multiplicity N then (for certain functions F) (3.6) *may have N "linearly independent" solutions.*

The proof is similar to the previous ones.

Remark 3.2. The value of solution u at some time t is connected with the values of u for all $t \in (0,\infty)$ (and for all $t \in [0,T_0]$ if $K(t,\tau,x,y) = 0$ for $\tau > T_0$).

By using (3.2), (3.4) and the Cauchy–Schwarz inequality, one obtains

Theorem 3.3. *Assume that there exist sufficiently smooth $K_\infty \in L^2((0,\infty) \times \Omega \times \Omega) = L^2(Q)$ and $F_\infty \in L^2(\Omega)$ such that*

$$\lim_{t\to\infty} \|K(t,\tau,x,y) - K_\infty(\tau,x,y)\|_{L^2(Q)} = 0,$$
$$\lim_{t\to\infty} \|F(t,x) - F_\infty(x)\|_{L^2(\Omega)} = 0.$$

Then

$$\lim_{t\to\infty} \|u(t,x) - u_\infty(x)\|_{L^2(\Omega)} = 0,$$

where

$$u_\infty(x) = \int_0^\infty \int_\Omega K_\infty(\tau,x,y) u(\tau,y) d\tau dy + F_\infty(x)$$

and u_∞ satisfies

$$[\tilde{A}(u_\infty)](x) = \tilde{A}_x \left[\int_0^\infty \int_\Omega K_\infty(\tau,x,y) u(\tau,y) d\tau dy + F_\infty(x) \right].$$

Case 2. Now consider operators G of the form

$$G(u) = Lu + h(Pu)F + H, \qquad t \in (0,\infty) \tag{3.7}$$

where operator L is defined by

$$(Lu)(t,x) = \int_0^t \int_\Omega K(t,\tau,x,y) u(\tau,y) d\tau dy,$$

$K \in L^2((0,\infty) \times (0,\infty) \times \Omega \times \Omega)$, $u \in L^2((0,\infty) \times \Omega)$ and the kernel K has the same smoothness property as in Theorem 3.1, $P : L^2(0, T_0; V) \to \mathbb{R}$ is a linear continuous functional $(T_0 < \infty)$, $h : \mathbb{R} \to \mathbb{R}$ is a given continuous function and $F, H \in L^2((0,\infty) \times \Omega)$, $D_t F, D_t H \in L^2((0,\infty) \times \Omega)$. In this case the integral operator L is of Volterra type and so $(I - L)^{-1} : L^2((0,\infty) \times \Omega) \to L^2((0,\infty) \times \Omega)$ exists.

Theorem 3.4. *If $\tilde{A} = \tilde{B}$ (i.e. $\tilde{A}$ is linear) then equation (3.1) has the form*

$$\begin{aligned} D_t u + \tilde{B}u = & \int_0^t \int_\Omega [D_t K(t,\tau,x,y) + \tilde{B}_x K(t,\tau,x,y)] u(\tau,y) d\tau dy \\ & + \int_\Omega K(t,t,x,y) u(t,y) dy + h(Pu)(D_t + \tilde{B})F + (D_t + \tilde{B})H, \qquad u(0,x) = 0. \end{aligned} \tag{3.8}$$

Further, $u \in L^2((0,\infty) \times \Omega)$ is a weak solution of (3.8) if and only if $u = h(\lambda)[(I-L)^{-1}F] + (I-L)^{-1}H$ where λ is a root of the equation

$$\lambda = h(\lambda) P[(I-L)^{-1}F] + P[(I-L)^{-1}H]. \tag{3.9}$$

Thus the number of solutions of (3.8) equals the number of the roots of (3.9).

Proof. Equation (3.8) is fulfilled if and only if

$$u(t,x) = \int_0^t \int_\Omega K(t,\tau,x,y) u(\tau,y) d\tau dy + h(Pu)F(t,x) + H(t,x), \tag{3.10}$$

i.e.

$$(I-L)u = h(Pu)F + H, \qquad u = h(Pu)[(I-L)^{-1}F] + (I-L)^{-1}H. \tag{3.11}$$

Let $u_\lambda = h(\lambda)(I-L)^{-1}F + (I-L)^{-1}H$ then

$$P(u_\lambda) = h(\lambda) P[(I-L)^{-1}F] + P[(I-L)^{-1}H].$$

Consequently, (3.11) (and so (3.8)) is satisfied if and only if $\lambda = Pu$ satisfies (3.9). $\square$

Corollary 3.5. *If $P[(I-L)^{-1}F] \neq 0$ then for arbitrary N $(= 0, 1, \ldots, \infty)$ we can construct h such that (3.8) has N solutions, in the following way. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous functions having N zeros. Then (3.8) has N solutions if*

$$h(\lambda) = \frac{g(\lambda) + \lambda - P[(I-L)^{-1}H]}{P[(I-L)^{-1}F]}.$$

Remark 3.6. The linear functional $P : L^2(0, T_0; V) \to \mathbb{R}$ may have the form (2.2).

By (3.10) and the Cauchy–Schwarz inequality we obtain

Theorem 3.7. *Assume that there exist sufficiently smooth $F_\infty, H_\infty \in L^2(\Omega)$ and $K_\infty \in L^2((0,\infty) \times \Omega \times \Omega)$ such that*

$$\lim_{t \to \infty} \|F(t,x) - F_\infty(x)\|_{L^2(\Omega)} = 0, \qquad \lim_{t \to \infty} \|H(t,x) - H_\infty(x)\|_{L^2(\Omega)} = 0,$$

$$\lim_{t \to \infty} \int_\Omega \left[\int_0^t \int_\Omega [K(t,\tau,x,y) - K_\infty(\tau,x,y)]^2 d\tau dy \right] dx = 0.$$

Then

$$\lim_{t \to \infty} \|u(t,x) - u_\infty(x)\|_{L^2(\Omega)} = 0,$$

where

$$u_\infty(x) = \int_0^\infty \int_\Omega K_\infty(\tau, x, y)u(\tau, y)d\tau dy + h(\lambda)F_\infty(x) + H_\infty(x),$$

$\lambda = P(u)$ *and* u_∞ *satisfies*

$$(\tilde{B}u_\infty)(x) = \int_0^\infty \int_\Omega \tilde{B}_x[K_\infty(\tau, x, y)]u(\tau, y)d\tau dy + h(\lambda)(\tilde{B}F_\infty)(x) + (\tilde{B}H_\infty)(x).$$

Case 3. Finally, consider the case

$$[G(u)](t, x) = \hat{P}(\hat{M}u(t))F(t, x), \qquad (t, x) \in (0, \infty) \times \Omega$$

where

$$(\hat{M}u))(t) = \int_0^t \int_\Omega \tilde{M}(\tau, y)u(\tau, y)d\tau dy, \qquad \tilde{M} \in C([0, \infty] \times \overline{\Omega}),$$

$\hat{P} : \mathbb{R} \to \mathbb{R}$ is a given continuously differentiable function, $\hat{P}(0) = 0$, F is sufficiently smooth, $F(0, x) = 0$, $F(t, x) = 0$ for $x \in \partial\Omega$.

Theorem 3.8. *In this case the partial functional equation (with possibly nonlinear operator $\tilde{A}$) (1.2) has the form*

$$\begin{aligned} D_t u + \tilde{A}(u) = \hat{P}'(\hat{M}u(t))F \int_\Omega \tilde{M}(t, y)u(t, y)dy + \hat{P}(\hat{M}u(t))D_t F \\ + \tilde{A}_x[\hat{P}(\hat{M}u(t))F], \quad u(0, x) = 0, \quad u(t, x) = 0 \quad \textit{for } x \in \partial\Omega \end{aligned} \tag{3.12}$$

which is satisfied if and only if

$$u(t, x) = \hat{P}(\hat{M}u(t))F(t, x). \tag{3.13}$$

Then $v(t) = \hat{M}u(t)$ satisfies the separable differential equation

$$v'(t) = \int_\Omega \tilde{M}(t, y)u(t, y)dy = \hat{P}(v(t)) \int_\Omega \tilde{M}(t, y)F(t, y)dy \quad \textit{and} \quad v(0) = 0. \tag{3.14}$$

Conversely, if v satisfies (3.14) *then $u(t, x) = \hat{P}(v(t))F(t, x)$ satisfies* (3.13).

Proof. Clearly, (3.12) is equivalent with (3.13). If u satisfies (3.13) then for

$$v(t) = (\hat{M}u)(t) = \int_0^t \int_\Omega \tilde{M}(\tau, y)u(\tau, y)d\tau dy \tag{3.15}$$

we have by (3.13)

$$\begin{aligned} v'(t) &= \int_\Omega \tilde{M}(t, y)u(t, y)dy = \hat{P}((\hat{M}u)(t)) \int_\Omega \tilde{M}(t, y)F(t, y)dy \\ &= \hat{P}(v(t)) \int_\Omega \tilde{M}(t, y)F(t, y)dy \quad \text{and, clearly,} \quad v(0) = 0. \end{aligned}$$

Conversely, if v satisfies (3.14) then for

$$u(t, x) = \hat{P}(v(t))F(t, x) \tag{3.16}$$

we have $u(x, 0) = 0$, $u(t, x) = 0$ for $x \in \Omega$ and by $v(0) = 0$

$$\begin{aligned} (\hat{M}u)(t) &= \int_0^t \int_\Omega \tilde{M}(\tau, y)u(\tau, y)d\tau dy \\ &= \hat{P}(v(t)) \int_0^t \int_\Omega \tilde{M}(\tau, y)F(\tau, y)d\tau dy = \int_0^t v'(\tau)d\tau = v(t), \end{aligned}$$

thus by (3.16)

$$u(t, x) = \hat{P}((\hat{M}u)(t))F(t, x).$$

□

Theorem 3.9. *Assume that $\hat{P}(w) > 0$ for $w > 0$ and $\hat{P}(0) = 0$, further,*

$$\hat{Q}(v) = \int_0^v \frac{dw}{\hat{P}(w)} < \infty, \qquad \lim_{v\to\infty} \hat{Q}(v) = \infty;$$

$$F(0,y) = 0 \quad \text{for all } y \in \Omega, \qquad \int_\Omega \tilde{M}(t,y)F(t,y)dy > 0 \quad \text{for all } t > 0.$$

Then we obtain for the solution of (3.14) $v = 0$ (v identically 0) and

$$v(t) = \hat{Q}^{-1}\left[\int_0^t \int_\Omega \tilde{M}(\tau,y)F(\tau,y)dyd\tau\right]$$

and, consequently, we have solutions $u = 0$ and

$$u(t,x) = \hat{P}(v(t))F(t,x) = \hat{P}\left\{\hat{Q}^{-1}\left[\int_0^t \int_\Omega \tilde{M}(\tau,y)F(\tau,y)dyd\tau\right]\right\}F(t,x). \tag{3.17}$$

Proof. By the assumptions on $\hat{P}$, $\hat{Q}$ is strictly monotone increasing, $\hat{Q}$ maps from $\mathbb{R}$ to $\mathbb{R}$, $\hat{Q}(0) = 0$, $\lim_{v\to\infty} \hat{Q}(v) = \infty$, thus

$$v(t) = \hat{Q}^{-1}\left[\int_0^t \int_\Omega \tilde{M}(\tau,y)F(\tau,y)dyd\tau\right], \qquad t \geq 0$$

is a solution of (3.14). By the previous theorem, function u, defined by (3.17) and $u = 0$ are solutions of (3.13) and (3.12). □

By using the continuity of functions $\hat{P}$ and $\hat{Q}^{-1}$, we obtain

Theorem 3.10. *Assume that there exist $F_\infty \in L^2(\Omega)$ and $c_0 \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} \int_\Omega |F(t,y) - F_\infty(y)|^2 dy = 0, \tag{3.18}$$

$$\lim_{t\to\infty} \int_0^t \int_\Omega |\tilde{M}(\tau,y)F(\tau,y)dyd\tau = c_0. \tag{3.19}$$

Then for the nonzero solution u we have

$$\lim_{t\to\infty} \|u(t,x) - u_\infty(x)\|_{L^2(\Omega)} = 0$$

where

$$u_\infty(x) = \hat{P}(\hat{Q}^{-1}(c_0))F_\infty(x).$$

Remark 3.11. If there exists $\tilde{M}_\infty \in L^2(\Omega)$ such that

$$\lim_{t\to\infty} \int_\Omega \left[\int_0^t \tilde{M}(\tau,y)d\tau - \tilde{M}_\infty(y)\right] dy = 0$$

then (3.18) implies (3.19) with $c_0 = \int_\Omega \tilde{M}_\infty(y)F_\infty(y)dy$.

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Minimal sets and chaos in planar piecewise smooth vector fields

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Abstract. Some aspects concerning chaos and minimal sets in discontinuous dynamical systems are addressed. The orientability dependence of trajectories sliding trough some variety is exploited and new phenomena emerging from this situation are highlighted. In particular, although chaotic flows and nontrivial minimal sets are not allowed for smooth vector fields in the plane, the existence of such objects for some classes of vector fields is verified. A characterization of chaotic flows in terms of orientable minimal sets is also provided. The main feature of the dynamical systems under study is related to the non uniqueness of trajectories in some zero measure region as well as the orientation of orbits reaching such region.

Keywords: vector fields, piecewise smooth vector fields, chaos, minimal sets.

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1 Introduction

Dynamical systems have become one of the most promising areas of mathematics since its strong development started by Poincaré (see [22]). The main reason for this is due to the fact that several applied sciences from economy and biology to engineering and statistical mechanics benefited of dynamical systems' tools. In the last case, for instance, ergodic theory plays an important role, we mention for short Poincaré recurrence theorem as well as the concepts of chaos and entropy. In fact, while a mathematical object models a concrete phenomena, such modeling is in fact no more than an theoretical approximation of an real event and invariably ignores some important features of it. Is therefore mandatory to search for news methods and tools that are not only more realistic but also feasible in theory.

In this direction have emerged within the theory of dynamical system a set of methods which is now widely known by *piecewise smooth vector fields* (PSVFs, for short). For a formal introduction to PSVFs see [14]. The main advantage of PSVFs over the classical theory of

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dynamical system is the fact that they provide a more accurate approach by allowing non smoothness or discontinuities of the vector field defining the system. Indeed, several problems involving impact, friction or abrupt changes of certain regime can be modeled or at least approximated by PSVFs, in the sense that the transition from one kind of behavior to another one can be idealized as a discrete and instantaneous transition. A non exhaustive list of applications of such theory involves the relay systems, the control theory, the stick-slip process, the dynamics of a bouncing ball and the antilock braking system (ABS), see those and other applications in [2–4, 11, 12, 15, 16, 18, 19] and references therein.

The main aspect of PSVFs concerns non uniqueness of solutions on some zero measure variety and consequent amalgamation of orbits under such region, which split the phase portrait into two or more pieces. That leads to the behavior known as *sliding motion*, characterized by the collapse of distinct trajectories which combine to slide on the common frontier of each dynamic. Under this scenario some behavior strange to the classical theory of dynamical systems may occur, so the study of new objects and the validation of known results is mandatory when one investigate PSVFs. For instance, we mention the Peixoto's Theorem (see [21]), the Closing Lemma (see [8]) and the Poincaré–Bendixson Theorem (see [5]), which posses analogues version in the context of PSVFs (see also [10, 13, 17]). We also mention that the study of PSVFs may take into account orientability of trajectories. This is because the collision of any particular trajectory to the boundary region and subsequent sliding occurs in different ways when considering forward or backward time.

This paper is addressed to some particular features of PSVFs. Indeed, we take into account aspects of chaotic PSVFs and how this concept relates to minimal sets. To do this, the definitions of both chaos e minimal sets are refined to consider the role of orientation and we provide a definitive characterization of chaotic PSVFs involving such objects.

Let $\mathrm{med}(W)$ be the Lebesgue measure of a set W. The first main result of the paper states that *a PSVF Z is chaotic on the set W if, and only if, Z is positive chaotic and negative chaotic on W.* The second main result of the paper states that *if Z is chaotic on the set W and* $\mathrm{med}(W) > 0$ *then W is positive minimal and negative minimal.* In order to prove these results we present and prove some other results which are indispensable to main results but also important on their own. For instance, we provide a sufficient condition for a Lebesgue measure subset of $\mathbb{R}^2$ to be chaotic, which elucidates the richness of PSVFs. Other considerations and results are presented timely throughout the text.

The paper is organized as follows: in Section 2 we provide the first statements around the subject of PSVFs, particularly considering minimal sets for PSVFs and their chaotic behavior. In Section 3 we state and prove the main results of the paper and some consequences of them. In Section 4 we provide a discussion around the results of the paper and present some examples and counterexamples contextualizing the results.

2 Preliminaries

2.1 Piecewise smooth vector fields

Consider two smooth vector fields X and Y and a codimension one manifold $\Sigma \subset \mathbb{R}^2$ that separates the plane in two regions $\Sigma+$ and $\Sigma-$. A PSVF Z is a vector field defined in $\mathbb{R}^2$ and given by

$$Z(x,y) = \begin{cases} X(x,y), & \text{for } (x,y) \in \Sigma^+, \\ Y(x,y), & \text{for } (x,y) \in \Sigma^-. \end{cases} \tag{2.1}$$

Since Σ is a codimension one manifold, there exists a function f such that $\Sigma = f^{-1}(0)$ and 0 is a regular value of f. As consequence, $\Sigma^+ = \{q \in \mathbb{R}^2 \,|\, f(q) \geq 0\}$ and $\Sigma^- = \{q \in \mathbb{R}^2 \,|\, f(q) \leq 0\}$. The trajectories of Z are solutions of $\dot{q} = Z(q)$ and we accept it to be multi-valued at points of Σ. We will call Ω the set of all PSVFs defined in $\mathbb{R}^2$. The basic results of differential equations in this context were stated by Filippov in [14], that we summarize next. Indeed, consider the Lie derivatives $X.f(p) = \langle \nabla f(p), X(p) \rangle$ and $X^i.f(p) = \langle \nabla X^{i-1}.f(p), X(p) \rangle$, $i \geq 2$, where $\langle .,. \rangle$ is the usual inner product in $\mathbb{R}^2$. We distinguish the following regions on the discontinuity set Σ:

(i) $\Sigma^c \subseteq \Sigma$ is the *sewing region* if $(X.f)(Y.f) > 0$ on Σ^c. Moreover, when $X.f(p) > 0$ and $Y.f(p) > 0$, we say that $p \in \Sigma^{c+}$ and when $X.f(p) < 0$ and $Y.f(p) < 0$, we say that $p \in \Sigma^{c-}$.

(ii) $\Sigma^e \subseteq \Sigma$ is the *escaping region* if $(X.f) > 0$ and $(Y.f) < 0$ on Σ^e.

(iii) $\Sigma^s \subseteq \Sigma$ is the *sliding region* if $(X.f) < 0$ and $(Y.f) > 0$ on Σ^s.

The *sliding vector field* associated to $Z \in \Omega$ is the vector field Z^s tangent to Σ^s and defined at $q \in \Sigma^s$ by $Z^s(q) = m - q$ with m being the point of the segment joining $q + X(q)$ and $q + Y(q)$ such that $m - q$ is tangent to Σ^s. It is clear that if $q \in \Sigma^s$ then $q \in \Sigma^e$ for $(-Z)$ and we can define the *escaping vector field* Z^e on Σ^e associated to Z by $Z^e = -(-Z)^s$. We will use the notation Z^Σ to both, Z^s and Z^e.

We say that $q \in \Sigma$ is a *Σ-regular point* if it is a sewing point or a regular point of the Filippov vector field. Lastly, any point $q \in \Sigma^p$ is called a *pseudo-equilibrium of* Z and it is characterized by $Z^\Sigma(q) = 0$. Any $q \in \Sigma^t$ is called a *tangential singularity* (or also *tangency point*) and it is characterized by $(X.f(q))(Y.f(q)) = 0$. If there exist an orbit of the vector field $X|_{\Sigma^+}$ (respec. $Y|_{\Sigma^-}$) reaching $q \in \Sigma^t$ in a finite time, then such tangency is called a *visible tangency* for X (resp. Y); otherwise we call q an *invisible tangency* for X (resp. Y).

We may also distinguish a particular tangential singularity called *two-fold*, which is a common tangency q of both X and Y (that is, $X.f(q) = Y.f(q) = 0$) satisfying $X^2.f(q)), Y^2.f(q)) \neq 0$. A two-fold is called visible if it is a visible tangency for X and Y. A visible two fold singularity is called a *singular tangency point* and all other $p \in \Sigma^t$ is called a *regular tangency point*.

Definition 2.1. The **local trajectory (orbit)** $\phi_Z(t,p)$ of a PSVF given by (2.1) through $p \in \mathbb{R}^2$ is defined as follows:

(i) For $p \in \Sigma^+ \backslash \Sigma$ and $p \in \Sigma^- \backslash \Sigma$ the trajectory is given by $\phi_Z(t,p) = \phi_X(t,p)$ and $\phi_Z(t,p) = \phi_Y(t,p)$ respectively, where $t \in I$: the maximal interval of existence of the corresponding trajectory before it hits Σ.

(ii) For $p \in \Sigma^{c+}$ and taking the origin of time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_Y(t,p)$ for $t \in I \cap \{t \leq 0\}$ and $\phi_Z(t,p) = \phi_X(t,p)$ for $t \in I \cap \{t \geq 0\}$. For the case $p \in \Sigma^{c-}$ the definition is the same reversing time. Again, I is the maximal interval of existence of the corresponding trajectory before it hits Σ again.

(iii) For $p \in \Sigma^e$ and taking the origin of time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_{Z^\Sigma}(t,p)$ for $t \in I \cap \{t \leq 0\}$ and $\phi_Z(t,p)$ is either $\phi_X(t,p)$ or $\phi_Y(t,p)$ or $\phi_{Z^\Sigma}(t,p)$ for $t \in I \cap \{t \geq 0\}$. For $p \in \Sigma^s$ the definition is the same reversing time. Here, I is the maximal interval of existence of the corresponding trajectory of $\phi_X(t,p)$ or $\phi_Y(t,p)$ before it hits Σ again or $\phi_{Z^\Sigma}(t,p)$ before it leaves Σ.

(iv) For p a regular tangency point and taking the origin of time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_1(t,p)$ for $t \in I \cap \{t \leq 0\}$ and $\phi_Z(t,p) = \phi_2(t,p)$ for $t \in I \cap \{t \geq 0\}$, where each ϕ_1, ϕ_2 is either ϕ_X or ϕ_Y or ϕ_{Z^T}. Here, I is the maximal interval of existence of the corresponding trajectory of $\phi_X(t,p)$ or $\phi_Y(t,p)$ before it hits Σ again or $\phi_{Z^\Sigma}(t,p)$ before it leaves Σ.

(v) For p a singular tangency point, $\phi_Z(t,p) = p$ for all $t \in \mathbb{R}$.

Definition 2.2. Let ϕ_Z^1 and ϕ_Z^2 two distinct local trajectories. Suppose that there exists a common point $q \in \phi_Z^1 \cap \phi_Z^2$. We say that $\phi_Z^1 \cup \phi_Z^2$ **preserves orientation** if there exists an interval I, with $0 \in I$, such that: (i) $\phi_Z^1(0,q) = \phi_Z^2(0,q)$, (ii) $\phi_Z^1(t,.)$ is well defined for $t \in I \cap \{t \leq 0\}$ and (iii) $\phi_Z^2(t,.)$ is well defined for $t \in I \cap \{t \geq 0\}$.

Remark 2.3. Note that the point q of the previous definition is such that $q \in \Sigma$. In fact, it is enough to observe that there is uniqueness of trajectories in points belonging to $\mathbb{R}^2 \backslash \Sigma$.

Definition 2.4. A **global trajectory (orbit)** $\Gamma_Z(t,p_0)$ of $Z \in \Omega$ passing through p_0 when $t = 0$, is a union $\Gamma_Z(t,p_0) = \cup_{i \in \Theta}\{\sigma_i(t,p_i); t_i \leq t \leq t_{i+1}\}$ of preserving-orientation local trajectories $\sigma_i(t,p_i)$ satisfying $\sigma_i(t_i,p_i) = p_i \in \Sigma$ and $\sigma_i(t_{i+1},p_i) = p_{i+1} \in \Sigma$, here $\Theta \subset \mathbb{Z}$. A **maximal trajectory** $\Gamma_Z(t,p_0)$ is a maximal trajectory that cannot be extended to any others global trajectories by joining local ones, that is, if $\widetilde{\Gamma}_Z$ is a global trajectory containing Γ_Z then $\widetilde{\Gamma}_Z = \Gamma_Z$. In this case, we call $I = (\tau^-(p_0), \tau^+(p_0))$ the **maximal interval of the solution** Γ_Z. A maximal trajectory is a **positive** (respectively, **negative**) **maximal trajectory** if we restrict the previous definition to $t \geq 0$ (resp. $t \leq 0$).

Definition 2.5. A maximal trajectory $\Gamma_Z(t,p_0)$ has a **positive** (respectively, **negative**) **periodic trajectory** passing through p_0 if there exists $T_+ > 0$ (respectively, $T_- > 0$) such that $\phi_Z(t + k\,T_+, p_0) = \phi_Z(t,p_0)$ for all integer $k > 0$ (respectively, $\phi_Z(t + k\,T_-, p_0) = \phi_Z(t,p_0)$ for all integer $k < 0$). A maximal trajectory $\Gamma_Z(t,p_0)$ has a **periodic trajectory** passing through p_0 if it has coincident positive and negative periodic trajectories passing through p_0 in such a way that $T_+ = T_-$.

Definition 2.6. Consider $Z = (X,Y) \in \Omega$. A closed (connected) union of trajectories Δ of Z is a:

(i) **pseudo-cycle** if $\Delta \cap \Sigma \neq \emptyset$ and it does not contain neither equilibrium nor pseudo-equilibrium.

(ii) **pseudo-graph** if $\Delta \cap \Sigma \neq \emptyset$ and it is a union of equilibria, pseudo equilibria and orbit-arcs of Z joining these points.

2.2 Minimal sets and chaotic PSVFs

One of the most important facts concerning PSVFs is the orientation of its trajectories. Indeed, it is very important, for instance, for the concept of invariance or defining the flow associated to the Filippov vector field. In the smooth theory of vector fields this distinction does not play an important role since we have uniqueness of trajectories. In this direction, we should verify if such distinction is also necessary when defining minimal sets and chaotic PSVFs. Indeed, these concepts do not play the same role by considering positive and negative times. As far as the authors know, the role of orientability under this context have not be treated in literature about PSVFs, although the concept of chaos and minimality have been discussed before, for instance, in [5], [6] and [10]. We start doing some adaptations to the definitions of invariance and minimality.

Definition 2.7. A set $A \subset \mathbb{R}^2$ is **positive invariant** (respectively, **negative invariant**) if for each $p \in A$ and all positive maximal trajectory $\Gamma^+_Z(t,p)$ (respectively, negative maximal trajectory $\Gamma^-_Z(t,p)$) passing through p it holds $\Gamma^+_Z(t,p) \subset A$ (respectively, $\Gamma^-_Z(t,p) \subset A$). A set $A \subset \mathbb{R}^2$ is **invariant** for Z if it is positive and negative invariant.

Definition 2.8. Consider $Z \in \Omega$. A non-empty set $M \subset \mathbb{R}^2$ is **minimal** (respectively, either **positive minimal** or **negative minimal**) for Z if it is compact, invariant (respectively, either positive invariant or negative invariant) for Z and does not contain proper compact invariant (respectively, either does not contain proper compact positive invariant or proper compact negative invariant) subsets.

Next we present the definitions concerning chaotic PSVFs. As commented before, we need to distinguish between forward and backward time or assuming both possibilities. The notion of chaos we take into account is that based on Devaney. So, the first aspect to be considered is related to topological transitivity.

Definition 2.9. System (2.1) is **topologically transitive** on an invariant set W if for every pair of nonempty, open sets U and V in W, there exist $q^+, q^- \in U$, $\Gamma^+_Z(.,q^+), \Gamma^-_Z(.,q^-)$ maximal trajectories and $t_0^+ > 0 > t_0^-$ such that $\Gamma^+_Z(t_0^+,q^+)$ and $\Gamma^-_Z(t_0^-,q^-) \in V$.

Definition 2.10. System (2.1) is **topologically positive transitive** (respectively, **topologically negative transitive**) on a positive invariant (respectively, negative invariant) set W if for every pair of nonempty, open sets U and V in W, there exist $q \in U$, $\Gamma^+_Z(t,q)$ a positive (respectively, $\Gamma^-_Z(t,q)$ a negative) maximal trajectory and $t_0 > 0$ (resp., $t_0 < 0$) such that $\Gamma^+_Z(t_0,q) \in V$ (resp., $\Gamma^-_Z(t_0,q) \in V$).

Remark 2.11. A direct consequence of the two previous definitions is that:

Z is topologically transitive on W if, and only if, Z is simultaneously topologically positive transitive and topologically negative transitive on W.

Analogously to the definition of topologically transitive systems, the definition of sensitive dependence for PSVFs is inspired in the classical Devaney concept of chaos.

Definition 2.12. System (2.1) exhibits **sensitive dependence** on a compact invariant set W if there is a fixed $r > 0$ satisfying $r < \mathrm{diam}(W)$ such that for each $x \in W$ and $\varepsilon > 0$ there exist $y^+, y^- \in B_\varepsilon(x) \cap W$ and maximal trajectories Γ^+_x, Γ^-_x, $\Gamma^+_{y^+}$ and $\Gamma^-_{y^-}$ passing through x, y^+ and y^-, respectively, satisfying

$$d_H(\Gamma^+_x(t), \Gamma^+_{y^+}(t)) = \sup_{a \in \Gamma^+_x(t), b \in \Gamma^+_{y^+}(t)} d(a,b) > r,$$

$$d_H(\Gamma^-_x(t), \Gamma^-_{y^-}(t)) = \sup_{a \in \Gamma^-_x(t), b \in \Gamma^-_{y^-}(t)} d(a,b) > r,$$

where $\mathrm{diam}(W)$ is the diameter of W and d is the Euclidean distance.

Associated to the previous definition we give the next one, where the orientation of the trajectories of Z is also considered:

Definition 2.13. System (2.1) exhibits **sensitive positive dependence** (resp., **sensitive negative dependence**) on a compact positive invariant (resp., negative invariant) set W if there is a fixed $r > 0$ satisfying $r < \text{diam}(W)$ such that for each $x \in W$ and $\varepsilon > 0$ there exist a $y \in B_\varepsilon(x) \cap W$ and positive (resp., negative) maximal trajectories Γ_x^+ and Γ_y^+ (resp., Γ_x^- and Γ_y^-) passing through x and y, respectively, satisfying

$$d_H(\Gamma_x^+(t), \Gamma_y^+(t)) = \sup_{a \in \Gamma_x^+(t), b \in \Gamma_y^+(t)} d(a,b) > r,$$

$$(\text{resp., } d_H(\Gamma_x^-(t), \Gamma_y^-(t)) = \sup_{a \in \Gamma_x^-(t), b \in \Gamma_y^-(t)} d(a,b) > r),$$

where $\text{diam}(W)$ is the diameter of W and d is the Euclidean distance.

Remark 2.14. A direct consequence of the two previous definitions is that:

Z exhibits sensitive dependence on W if, and only if, Z exhibits simultaneously sensitive positive dependence and sensitive negative dependence on W.

In this paper we will consider the notations stated in the following table.

Table of abbreviations	
Topologically transitive	TT
Topologically positive transitive	TPT
Topologically negative transitive	TNT
Sensitive dependence	SD
Sensitive positive dependence	SPD
Sensitive negative dependence	SND

We should mention, as observed in [10], that Definitions 2.9 and 2.12 coincide with the definitions of topological transitivity and sensible dependence of smooth vector fields for single-valued flows, so these definitions are natural extension for a set-valued flow. Lastly, in what follows we introduce the definition of chaos and orientable chaos in the piecewise smooth context. Note that the concept for chaos in the paper is inspired by Devaney for a deterministic flow, but the systems of differential equations discussed in the article define non-deterministic flows:

Definition 2.15. System (2.1) is **chaotic** (resp., either **positive chaotic** or **negative chaotic**) on a compact invariant (resp., either positive invariant or negative invariant) set W if it is TT and exhibits SD (resp., either TPT and exhibits SPD or TNT and exhibits SND) on W.

Remark 2.16. A direct consequence of the previous definition is that:

A PSVF Z is chaotic on W if, and only if, Z is positive chaotic and negative chaotic on W.

3 Main results

In this Section we present and prove the main results of the paper.

Proposition 3.1. *Let $\mathcal{A}$ be the set of pseudo cycles Γ of $Z = (X, Y)$ such that $\Gamma \cap (\overline{\Sigma^e} \cup \overline{\Sigma^s}) = \emptyset$ and Γ has at least a visible two-fold singularity. The elements Γ of $\mathcal{A}$ are chaotic for Z.*

In Figure 4.5 we exhibit an element $\Gamma \subset \mathcal{A}$. In fact, the elements $\Gamma \subset \mathcal{A}$ are obtained by the concatenation of orbits of X and Y, without using orbits of Z^{Σ}.

Proof of Proposition 3.1. Let A, B open sets relative to Γ. Since Γ is a pseudo-cycle, given points $p_A \in A$ and $p_B \in B$, there exists a trajectory of Z connecting them (for positive and negative times). So Γ is topologically transitive.

On the other hand, given $x, y \in A$, there exists a trajectory passing through x and another trajectory passing through y such that each one of them follows a distinct path after the visible two-fold singularity of Γ. So Γ has sensitive dependence.

Therefore, Γ is chaotic. □

Remark 3.2. By the previous proposition, we conclude the existence of trivial minimal sets presenting chaotic behavior.

Remark 3.3. An analogous of result of Remarks 2.11, 2.14 and 2.16 does not hold for minimal sets. Indeed, while sets which are both positive and negative minimal are also minimal, the converse is not true. The Example 2 of [6] exemplify this situation.

The most part of the results obtained in [5] and [6] takes into account sets having positive Lebesgue measure. Indeed, in almost every approach concerning ergodic aspects of PSVFs, this is the interesting case. We cite, for instance, the existence of non-trivial minimal sets and planar chaotic PSVFs, as shown in the papers cited previously. In this direction we state the next result.

Lemma 3.4. *Let $K \subset \mathbb{R}^2$ be a compact invariant set and Z a PSVF presenting a finite number of critical points and a finite number of tangency points with Σ in K. If $\mathrm{med}(K) = 0$ and $K \notin \mathcal{A}$ then Z is not chaotic on K.*

We recall that $\mathcal{A}$ is the set of pseudo-cycles having a visible two-fold singularity which does not connect to any sliding or escaping segment (see Proposition 3.1). Also, the *saturation* of a set M by a vector field W is the set

$$W(M) = \{\phi_W(t, p) \mid p \in M \text{ and } t \in I\}$$

where I is the maximal interval of existence of the W-trajectory passing through p.

Proof. First, suppose that $K \cap \Sigma \subset \Sigma^c \cup \Sigma^t$ and take $p \in K$. Consequently, $\phi_Z(t, p) \xrightarrow{t \to \infty} L \in \omega(p) \subset K$, since K is compact. Here $\omega(p)$ denotes the ω-limit set of the point p. Thus, by using the Poincaré–Bendixson Theorem for PSVFs (see [5]) we get that L is a (pseudo-)equilibrium, a (pseudo-)graph or (pseudo-)cycle which does not belongs to $\mathcal{A}$ since $L \subset K$ and $K \notin \mathcal{A}$ by hypothesis. In any case, it is trivial to see that Z is not chaotic on K since Z does not exhibits SD on K.

Now consider the case where $K \cap (\Sigma^s \cup \Sigma^e) \neq \emptyset$ and suppose that there exist a PSVF Z which is chaotic on K. Take $p \in K \cap (\Sigma^s \cup \Sigma^e)$ and $V_p \subset \mathbb{R}^2$ a neighborhood of p. Consider the sets $V_p^+ = \{\phi_t^+(p) \cap V_p \,|\, \phi_t^+$ is a positive trajectory of Z passing through $p\}$ and V_p^- defined analogously for the negative trajectory. Observe that $\text{med}(V_p^+ \cup V_p^-) > 0$, since using the Definition 2.1, in this case the saturation of $K \cap (\Sigma^s \cup \Sigma^e)$ (for either positive or negative times) contain an open set $U \subset V_p$ satisfying $0 < \text{med}(U) < \text{med}(V_p^+ \cup V_p^-)$. Consequently there exist a point $q \in V_p^+ \cup V_p^-$ such that $q \notin K$, because otherwise $V_p^+ \cup V_p^- \subset K$ and then $\text{med}(K) > \text{med}(V_p^+ \cup V_p^-) > 0$ (see Figure 3.1). As consequence, K is not invariant, producing a contradiction. □

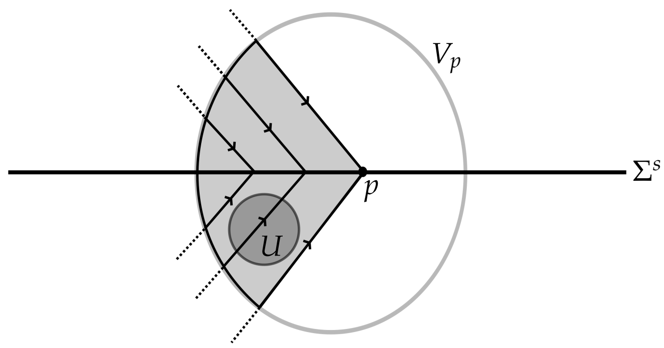

Figure 3.1: The neighborhood V_p of p. The filled region correspond to V_p^-, and in this case $V_p^+ = V_p \cap \Sigma$. Observe that it has positive Lebesgue measure.

In the proof of the next Theorem 3.6 we will use the following remark.

Remark 3.5. A direct consequence of Definition 2.15 is that

Let Z a chaotic PSVF on W. Then Z is chaotic on every compact invariant proper subset $\widetilde{W} \subset W$.

In [6], among other results, the authors prove that, if a compact invariant set W satisfying $\text{med}(W) > 0$ is simultaneously positive and negative minimal for a PSVF Z, then Z is chaotic on W. Now, we prove the converse of this important theorem. Observe that, due to Lemma 3.4, we must impose a condition demanding the positive Lebesgue measure of the considered set.

Theorem 3.6. *If Z is chaotic on the compact invariant set W, $\text{med}(W) > 0$, Z has a finite number of critical points and a finite number of tangency points with Σ in W, then W is positive minimal and negative minimal for Z.*

Proof. According to Remark 2.16, Z is positive chaotic on W. So, W is compact, non-empty and positive invariant. Suppose that W is not positive minimal. In this case, there exists a proper subset $\widetilde{W}$ of W with the previous three properties. Moreover, by Remark 3.5 and Lemma 3.4, we get $\text{med}(\widetilde{W}) > 0$ or $\text{med}(\widetilde{W}) = 0$ and $\widetilde{W} \subset \mathcal{A}$. Of course $\widetilde{W}$ is not dense in W since $\widetilde{W}$ is compact and $\widetilde{W} \neq W$. Therefore there exists an open set $A \subset W$ such that $A \cap \widetilde{W} = \emptyset$. First suppose that $\text{med}(\widetilde{W}) > 0$ and let $B \subset \widetilde{W}$ be an open set of W. In this case, using the open sets

A and B, we have that Z is not TPT. But this is a contradiction with the fact that Z is chaotic on W. On the other hand, if $\text{med}(\widetilde{W}) = 0$ we get $\widetilde{W} \subset \mathcal{A}$ and therefore $\widetilde{W}$ is a curve on W. Let $I(\widetilde{W})$ the region delimited by $\widetilde{W}$ which is clearly invariant and notice that $\text{med}(I(\widetilde{W})) > 0$ since $\text{med}(\widetilde{W}) = 0$. So we can take open sets $B \subset I(\widetilde{W})$ and $A \subset W \setminus \big(\widetilde{W} \cup I(\widetilde{W})\big)$ to lead again to a contradiction with the fact that Z is chaotic on W. Therefore, W is positive minimal for Z.

An analogous argument proves that W is negative minimal for Z. □

Next corollary is a straightforward consequence of Theorem 3.6, but it is very important once it provides a ultimate answer about the relation between chaotic systems and minimal sets.

Corollary 3.7. *If Z is chaotic on W, $\text{med}(W) > 0$, Z has a finite number of critical points and a finite number of tangency points with Σ in W, then W is minimal for Z.*

Proof. It is enough to use Theorem 3.6 and Definition 2.8. □

We remark that the converse is not true, as observed in [6].

The next two corollaries are also consequences of Theorem 3.6. Their proofs, analogously, are quite trivial although the results can find applications.

Corollary 3.8. *If $\text{med}(W) > 0$, Z has a pseudo equilibria on W and a finite number of tangency points with Σ in W then Z is not chaotic on W.*

Proof. It is not difficult to see that a pseudo equilibria is neither positive nor negative minimal for Z since there exists trajectories of X and Y hitting it in finite (positive or negative) time. So, W is not positive or negative minimal. Therefore the proof follows straightforward from Theorem 3.6. □

Remark 3.9. A consequence of the proof of Theorem 3.6 is that

If Z is positive (resp. negative) chaotic on W, $\text{med}(W) > 0$, Z has a finite number of tangency points with Σ in W then W is positive (resp. negative) minimal.

The next result provide a sufficient condition in order to a PSVF Z be chaotic on an invariant compact set W. Additionally, it guarantee that under suitable hypotheses the periodic trajectories of Z are dense in W.

Theorem 3.10. *Let Z be a PSVF and W a compact positive (resp. negative) invariant set satisfying $\text{med}(W) > 0$. Given $x, y \in W$, assume that there exist a positive (resp. negative) trajectory ϕ_t^+ (resp. ϕ_t^-) connecting x and y. Then Z is positive (resp. negative) chaotic on W and the positive (resp. negative) periodic trajectories of Z are dense in W.*

We shall prove the last result in forward time, obtaining positive chaos and dense trajectories. The proof for trajectories in backward time is completely similar.

Proof. Since $\text{med}(W) > 0$, let U and V be nonempty open sets in W and p_U, p_V points of U and V, respectively. By hypotheses there exist a positive trajectory ϕ_t^+ connecting p_U and p_V in forward time. Since U and V are arbitrary it follows that W is topologically positive transitive. On the other hand, let d_W be the diameter of W and take $r = d_W/2$, so clearly there exists $a, b \in W$ such that $d(a, b) > r$. Now consider $x \in W$, $\varepsilon > 0$ and fix $y \in B_\varepsilon(x) \cap W$. Again, by hypotheses there exists positive trajectories $\phi_a^+(t, x)$ and $\phi_b^+(t, x)$ satisfying $\phi_a^+(0, x) =$

$\phi_b^+(0,x) = x$ and values $t_a, t_b > 0$ such that $\phi_a^+(t_a,x) = a$ and $\phi_b^+(t_b,x) = b$ so Z exhibits sensitive positive dependence on W. At last, the density of positive periodic trajectories is straightforward from the fact that any point $x \in W$ can be connected to itself by a positive trajectory. □

Theorem 3.10 leads to the next corollary.

Corollary 3.11. *Let Z be a PSVF and W satisfying $med(W) > 0$ a compact invariant set on which any two points can be connected simultaneously by positive and negative trajectories. Then Z is chaotic on W and its periodic trajectories are dense in W.*

Proof. Since every pair of points in W can be connected simultaneously by positive and negative trajectories of Z, by Theorem 3.10, the PSVF Z is both positive and negative chaotic on Z. So, by Remark 2.16, we get that Z is chaotic on W. Moreover, since the positive and negative periodic trajectories of Z are dense in W, the density of the periodic trajectories of Z on W is straightforward. □

4 Discussions

We observed throughout the paper a closed relation between PSVFs presenting minimal sets or chaotic behavior. However, in order to observe the richness of such relation we introduced new concepts by considering the orientation of the trajectories in time. By one hand, according to Theorem 14 of [6], every PSVF having a positive and negative non trivial minimal set K is chaotic on K. On the other hand, in this paper, due to Remark 2.16 and 3.6 we get the equivalence. Putting those and other results of this paper together, we get the following diagram:

$$\begin{array}{c} Z \text{ is pos. and} \\ \text{neg. chaotic on } W \end{array} \Leftrightarrow \begin{array}{c} Z \text{ is chaotic} \\ \text{on } W \end{array} \Leftrightarrow \begin{array}{c} W \text{ is pos. and} \\ \text{neg. min. for } Z \end{array} \Rightarrow \begin{array}{c} W \text{ is min.} \\ \text{for } Z \end{array}$$

We note by observing the previous diagram that it could exist some minimal set which is not chaotic for the PSVF, as the authors observed in [5]. Other aspects of that diagram are presented in what follows:

Orientable chaotic sets which are not chaotic: Consider the PSVF:

$$\begin{aligned} ccZ_\epsilon(x,y) = (\dot{x},\dot{y}) = \frac{1}{2}\Big(&(-1,-2x - x^2(4x+3) + (1+\epsilon)x(3x+2)) \\ &+ \operatorname{sgn}(y)\,(3,-2x + x^2(4x+3) - (1+\epsilon)x(3x+2))\Big) \end{aligned} \tag{4.1}$$

or, equivalently,

$$Z_\epsilon(x,y) = \begin{cases} X(x,y) = (1,-2x) & \text{if } y \geq 0, \\ Y_\epsilon(x,y) = (-2,-x^2(4x+3) + (1+\epsilon)x(3x+2)) & \text{if } y \leq 0, \end{cases} \tag{4.2}$$

with $\epsilon \in \mathbb{R}$ an arbitrarily small parameter. In [6] the authors proved that Z_0 has a chaotic set given (see Figure 4.1) by

$$\Lambda = \{(x,y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1 \;\text{ and }\; x^4/2 - x^2/2 \leq y \leq 1 - x^2\}. \tag{4.3}$$

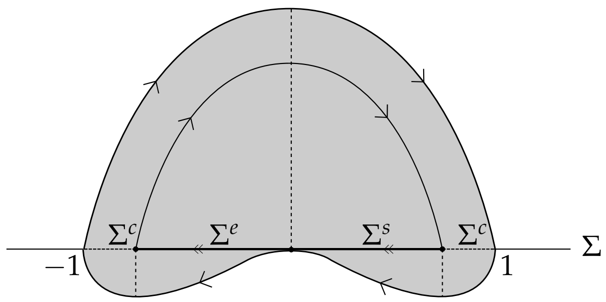

Figure 4.1: Chaotic set Λ.

Taking $\epsilon < 0$ (resp., $\epsilon > 0$) in (4.2) the PSVF Z_ϵ has a negative chaotic (resp., positive chaotic) set $\widetilde{\Lambda}$. We construct such a set for the case $\varepsilon < 0$. Indeed call p_2 the two-fold located at the origin and p_1 the first intersection of the backward trajectory of p_2 with Σ. From p_1 it can be concatenated a regular arc of trajectory of X which again intersects Σ in backward time at a point p_4. Finally, call p_3 the continuation of p_4 through the trajectory of Y until reaching Σ. Hence the set $\widetilde{\Lambda}$ is the region bounded by $\widehat{p_1\, p_2} \cup \widehat{p_2\, p_3} \cup \widehat{p_3\, p_4} \cup \widehat{p_4\, p_1}$, where $\widehat{a\, b}$ is the orbit-arc connecting the points a and b, see the shadowed region in Figure 4.2 (resp., Figure 4.3). Moreover, when $\epsilon \neq 0$, $\widetilde{\Lambda}$ is not a chaotic set. This happens because $\widetilde{\Lambda}$ is not an invariant set; it is only negative invariant (resp., positive invariant).

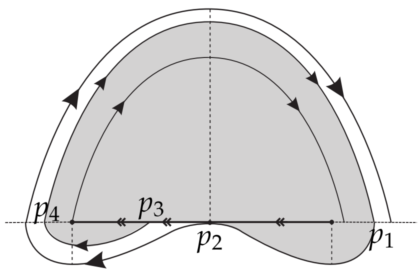

Figure 4.2: Negative chaotic set $\widetilde{\Lambda}$.

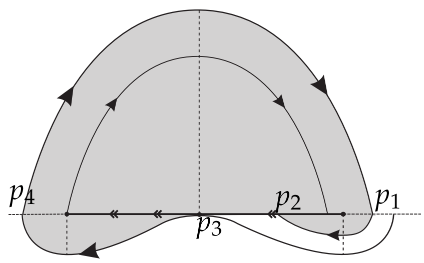

Figure 4.3: Positive chaotic set $\widetilde{\Lambda}$.

Remark 4.1. The previous paragraph remains true if we change the word chaotic by the word minimal. A complete bifurcation analysis of the family (4.2) is given in [8].

The sets given in Figures 4.2 and 4.3 are orientable chaotic and orientable minimal sets. Despite of this, it is easy to exhibit examples of orientable minimal sets that are not orientable chaotic.

Orientable chaotic sets and orientable minimality: Consider the PSVF

$$Z(x,y) = (X(x,y), Y(x,y)) = ((-1, 3x^2 - 3), (1, -(9/4) + 3(-1+x)x)).$$

Such PSVF has a periodic orbit (see Figure 4.4) which is a negative minimal set. However, Z is not a negative chaotic PSVF on the periodic orbit since it does not present SPD.

Observe that, in the last example the Lebesgue measure of the periodic orbit is null. However, it is not difficult to exhibit a minimal set W for some PSVF, with $\mathrm{med}(W) > 0$, in such way that W is neither positive chaotic nor negative chaotic. Indeed, Example 2 of [6] satisfies these properties. In other words, in general minimality does not imply chaoticity. The converse, on the other hand, is true, as proved in Section 3.

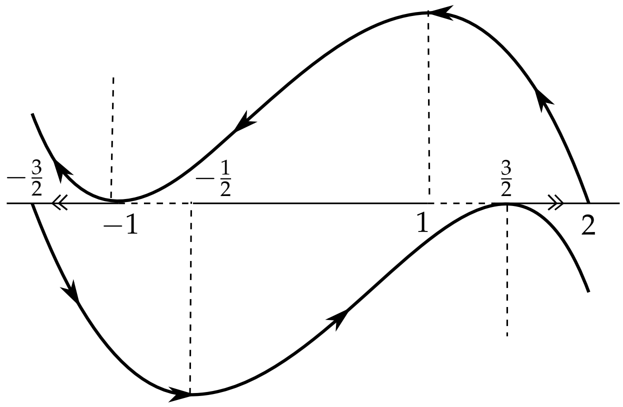

Figure 4.4: Periodic orbit (for positive time).

Trivial chaos: In PSVFs the route to chaos is not hard. In fact, here we show that a chaotic behavior can be achieved by trivial minimal sets.

Consider the PSVF $Z = (X, Y)$ where $X(x, y) = (1, 4x(1 - x^2))$ and $Y(x, y) = (-1, 4x(1 - x^2))$. The phase portrait is pictured in Figure 4.5. Take $\Lambda = \Lambda_1 \cup \Lambda_2$, where Λ_1 (respectively, Λ_2) is the trajectory of X (respectively, Y) passing through $p_1 = (-\sqrt{2}, 0)$. It is easy to see that Λ is a trivial minimal set (a pseudo-cycle) and it is a chaotic set for Z.

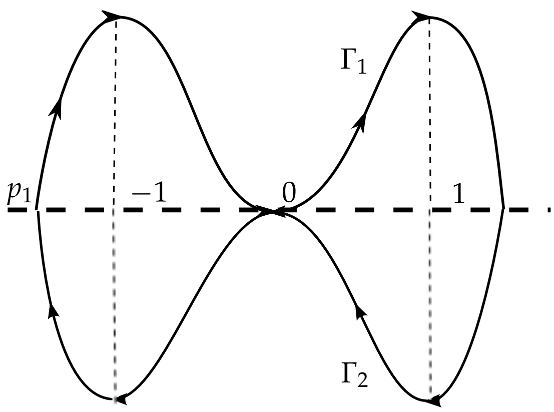

Figure 4.5: Trivial minimal set which is chaotic for Z.

The previous example illustrates a more general result, stated in Proposition 3.1.

We finish this section highlighting two particular conclusions from the results of the paper:

(i) although the chaoticity of a PSVF Z under a set W implies that W is minimal for Z, the converse is false according to Example 2 of [6];

(ii) if Z is positive chaotic on W then W is positive minimal for Z (see Remark 3.9), but the converse is false since we can exhibit positive minimal sets that are not positive chaotic (see Example 4 in [5]). Analogously for negative chaotic/minimal.

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Polynomial differential systems with hyperbolic algebraic limit cycles

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Abstract. For a given algebraic curve of degree n, we exhibit differential systems of degree greater than or equal to n, by introducing functions which are solutions of certain partial differential equations. These systems admit precisely the bounded components of the curve as limit cycles.

Keywords: sixteenth problem of Hilbert, planar differential system, invariant curve, periodic solution, hyperbolic limit cycle.

2020 Mathematics Subject Classification: 34C25, 34C05, 34C07.

1 Introduction

The second part of the sixteenth problem of Hilbert still persists as a research area. It aims to find the maximum number of limit cycles of the differential system:

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = P(x,y), \\ \dot{y} &= \frac{dy}{dt} = Q(x,y), \end{aligned} \tag{1.1}$$

where P and Q are polynomials.

Several articles and books have been published on the analysis of the existence, number and stability of limit cycles of equation (1.1) (see for instance [5,6,8,9,15,18]).

Generally, the exact analytical expressions of limit cycles for a given differential system are unknown, except in specific cases.

This paper is a contribution in the direction of determining the number of limit cycles and giving their explicit form.

Motivated by some publications [1–4,7,11–14,16], we will exhibit polynomial vector fields, where just by choosing the components of the system satisfying certain conditions, we can conclude directly the number and the explicit form of limit cycles.

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2 Introductory concepts

Let us recall some useful notions.

For $U \in \mathbb{R}[x,y]$, the algebraic curve $U = 0$ is called an invariant curve of the polynomial system (1.1), if for some polynomial $K \in \mathbb{R}[x,y]$, called the cofactor of the algebraic curve, we have

$$P(x,y)\frac{\partial U}{\partial x} + Q(x,y)\frac{\partial U}{\partial y} = KU. \tag{2.1}$$

Simple analysis of equation (2.1) shows that when $\max(\deg P, \deg Q) = n$, the degree of the cofactor K is at most $n-1$ and that the curve $U = 0$ is formed by trajectories of the system (1.1).

The curve $\Omega = \left\{(x,y) \in \mathbb{R}^2, U(x,y) = 0\right\}$ is a non-singular curve of system (1.1), if the equilibrium points of the system that satisfy

$$\begin{aligned} P(x,y) &= 0, \\ Q(x,y) &= 0 \end{aligned} \tag{2.2}$$

are not contained on the curve Ω.

A limit cycle $\Gamma = \{(x(t), y(t)), t \in [0,T]\}$ is a T-periodic solution isolated with respect to all other possible periodic solutions of the system.

A T-periodic solution Γ is a hyperbolic limit cycle if $\int_0^T \operatorname{div}(\Gamma)dt$ is different from zero.

By using the method of characteristics to solve partial differential equations, we conclude that, the solution of equation

$$\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y} = 0 \tag{2.3}$$

is

$$f(x,y) = \Phi(\beta x - \alpha y), \tag{2.4}$$

where α, β are nonzero reals and Φ is an arbitrary function.

The solution of the equation

$$\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y} = \gamma \tag{2.5}$$

is the function f solving the equation

$$\Psi(\beta x - \alpha y, \gamma x - \alpha f) = 0, \tag{2.6}$$

where α, β, γ are nonzero reals and Ψ is an arbitrary function. In the polynomial case

$$f(x,y) = \frac{\gamma}{\alpha}x + \sum_{k=0}^{n} c_k\left(\beta x - \alpha y\right)^k \tag{2.7}$$

or

$$f(x,y) = \frac{\gamma}{\beta}y + \sum_{k=0}^{n} c_k\left(\beta x - \alpha y\right)^k \tag{2.8}$$

the solution of the equation

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f \tag{2.9}$$

is the function f solving the equation

$$\Psi\left(\frac{x}{f}, \frac{y}{f}\right) = 0. \tag{2.10}$$

In the polynomial case it can be taken as

$$f(x,y) = ax + by. \tag{2.11}$$

Colin Christopher in his article [7] gives the following theorem.

Theorem 2.1. *Let $U = 0$ be a non-singular algebraic curve of degree m, and D a first degree polynomial, chosen so that the line $D = 0$ lies outside all bounded components of $U = 0$. Choose the constants α and β so that $\alpha D_x + \beta D_y \neq 0$, then the polynomial vector field of degree m,*

$$\begin{aligned} \dot{x} &= \alpha U + DU_y, \\ \dot{y} &= \beta U - DU_x \end{aligned} \tag{2.12}$$

has all the bounded components of $U = 0$ as hyperbolic limit cycles. Furthermore, the vector field has no other limit cycles.

Our contribution is a generalization, which consists in introducing polynomial functions to system (2.12) and in the study of the existence of limit cycles.

3 The main result

We start by adding a polynomial function of any degree to system (2.12), which becomes,

$$\begin{aligned} \dot{x} &= \alpha U + (ax + by + \Phi(\beta x - \alpha y))U_y, \\ \dot{y} &= \beta U - (ax + by + \Phi(\beta x - \alpha y))U_x \end{aligned} \tag{3.1}$$

and we show that system (3.1) has all the bounded components of $U = 0$ as hyperbolic limit cycles if the conditions of Theorem 1 of [7] are satisfied.

Theorem 3.1. *Let $U = 0$ be a non-singular algebraic curve of degree m, and Φ a polynomial function of degree n, chosen so that the curve $ax + by + \Phi(\beta x - \alpha y) = 0$ lies outside all bounded components of $U = 0$. Choose the constants a and b so that $a\alpha + b\beta \neq 0$, then the polynomial vector field of degree $m + n - 1$,*

$$\begin{cases} \dot{x} = \alpha U + (ax + by + \Phi(\beta x - \alpha y))U_y, \\ \dot{y} = \beta U - (ax + by + \Phi(\beta x - \alpha y))U_x \end{cases}$$

has all the bounded components of $U = 0$ as hyperbolic limit cycles.

Proof. Let Γ be the curve of $U = 0$.

Note that Γ is a non-singular curve of system (3.1) and the curve $ax + by + \Phi(\beta x - \alpha y) = 0$ lies outside all bounded components of Γ.

To show that all the bounded components of Γ are hyperbolic limit cycles of system (3.1), we will prove that Γ is an invariant curve of the system (3.1), and $\int_0^T \operatorname{div}(\Gamma)dt \neq 0$ (see for instance Perko [17]).

i) Γ is an invariant curve of the system (3.1):

$$\begin{aligned}\frac{dU}{dt} &= U_x\left(\alpha U + (ax+by+\Phi(\beta x-\alpha y))U_y\right) + U_y\left(\beta U - (ax+by+\Phi(\beta x-\alpha y))U_x\right)\\ &= \left(\alpha U_x + \beta U_y\right)U\end{aligned}$$

where the cofactor is $K(x,y) = \alpha U_x + \beta U_y$.

ii) $\int_0^T \operatorname{div}(\Gamma)dt$ is nonzero.

To see this, first note that

$$\int_0^T \operatorname{div}(\Gamma)dt = \int_0^T K(x(t),y(t))dt, \tag{3.2}$$

see for instance Giacomini & Grau [10]. Then one has

$$\begin{aligned}\int_0^T K(x(t),y(t))dt &= \oint_\Gamma \frac{\alpha U_x}{-(ax+by+\Phi(\beta x-\alpha y))U_x}dy + \oint_\Gamma \frac{\beta U_y}{(ax+by+\Phi(\beta x-\alpha y))U_y}dx\\ &= \oint_\Gamma \frac{\alpha}{-(ax+by+\Phi(\beta x-\alpha y))}dy + \oint_\Gamma \frac{\beta}{(ax+by+\Phi(\beta x-\alpha y))}dx.\end{aligned}$$

Let $\omega = \beta x - \alpha y$. By applying Green's formula we obtain

$$\begin{aligned}&\oint_\Gamma \frac{\beta}{(ax+by+\Phi(\omega))}dx - \oint_\Gamma \frac{\alpha}{(ax+by+\Phi(\omega))}dy\\ &= \int\int_{\operatorname{int}(\Gamma)}\left(\frac{\partial\left(\frac{\beta}{(ax+by+\Phi(\omega))}\right)}{\partial y} + \frac{\partial\left(\frac{\alpha}{(ax+by+\Phi(\omega))}\right)}{\partial x}\right)dxdy\\ &= \int\int_{\operatorname{int}(\Gamma)}\left(\frac{-\beta\left(b+\frac{\partial\Phi}{\partial\omega}(-\alpha)\right)}{(ax+by+\Phi(\omega))^2} + \frac{-\alpha\left(a+\frac{\partial\Phi}{\partial\omega}(\beta)\right)}{(ax+by+\Phi(\omega))^2}\right)dxdy\\ &= -\int\int_{\operatorname{int}(\Gamma)}\left(\frac{\beta\left(b+\frac{\partial\Phi}{\partial\omega}(-\alpha)\right)}{(ax+by+\Phi(\omega))^2} + \frac{\alpha\left(a+\frac{\partial\Phi}{\partial\omega}(\beta)\right)}{(ax+by+\Phi(\omega))^2}\right)dxdy\\ &= -\int\int_{\operatorname{int}(\Gamma)}\left(\frac{\beta b+\alpha a}{(ax+by+\Phi(\omega))^2}\right)dxdy,\end{aligned}$$

where $\operatorname{int}(\Gamma)$ denotes the interior of Γ.

As $\alpha a + \beta b \neq 0$, $\int_0^T K(x(t),y(t))dt$ is nonzero. □

Remark 3.2. When $\Phi(\beta x - \alpha y)$ is constant, we find ourselves in the case of Cristopher's theorem (i.e. Theorem 2.1).

When $\Phi(\beta x - \alpha y)$ is of first degree, the line $ax + by + c = 0$ in Christopher's theorem will be replaced by the line $(a+\beta)x + (b-\alpha)y + d = 0$.

Example 3.3 (Quintic system with exactly one limit cycle)**.** Let $\alpha = 1, \beta = 2, a = 1, b = 2$, $\Phi(\beta x - \alpha y) = \Phi(2x - y) = (2x-y)^2 + 1$.

The system

$$\begin{aligned}\dot{x} &= x^4 + y^2 - 4y - 3x + 5 + \left(x + 2y + (2x-y)^2 + 1\right)(2y-4),\\ \dot{y} &= 2\left(x^4 + y^2 - 4y - 3x + 5\right) - \left(x + 2y + (2x-y)^2 + 1\right)\left(4x^3 - 3\right)\end{aligned} \tag{3.3}$$

admits one hyperbolic limit cycle represented by the curve $x^4 + y^2 - 4y - 3x + 5 = 0$. See Figure 3.1.

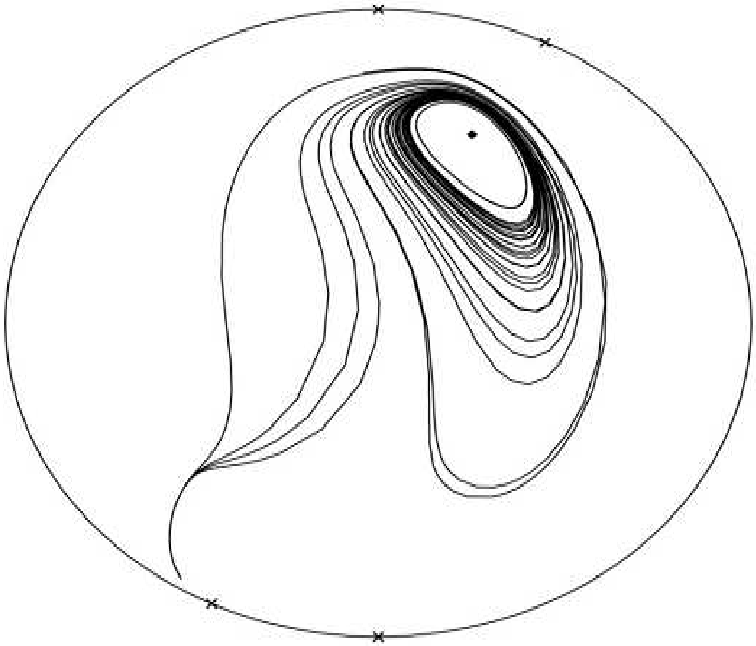

Figure 3.1: Limit cycle of system (3.3).

Remark 3.4. Let us consider the system

$$\begin{aligned} \dot{x} &= \alpha U + f(x,y)U_y, \\ \dot{y} &= \beta U - f(x,y)U_x, \end{aligned} \tag{3.4}$$

where U and f are C^1 functions on an open subset V of $\mathbb{R}^2$. To have all the bounded components of $U = 0$ as limit cycles it is necessary that f satisfies the partial differential equation

$$\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} = \gamma, \quad \text{where } \gamma \neq 0. \tag{3.5}$$

In the polynomial case $f(x,y) = \frac{\gamma}{\alpha}x + \Phi(\beta x - \alpha y)$ or $f(x,y) = \frac{\gamma}{\beta}y + \Phi(\beta x - \alpha y)$, which are just particular cases of Theorem 3.1.

Example 3.5 (Quintic system with exactly two limit cycles)**.** Let $\alpha = 1$, $\beta = -1$, $\gamma = 3$, $f(x,y) = 3x + (x+y)^2$.

The system

$$\begin{aligned} \dot{x} &= x^3 - 2xy^2 + 10xy - 15x + y^4 - 10y^3 + 35y^2 - 50y + 30 \\ &\quad + \left((x+y)^2 + 3x\right)\left(4y^3 - 30y^2 - 4xy + 10x + 70y - 50\right), \\ \dot{y} &= 2\left(x^3 - 2xy^2 + 10xy - 15x + y^4 - 10y^3 + 35y^2 - 50y + 30\right) \\ &\quad - \left((x+y)^2 + 3x\right)\left(3x^2 - 2y^2 + 10y - 15\right) \end{aligned} \tag{3.6}$$

admits two hyperbolic limit cycles represented by the curve $x^3 - 2xy^2 + 10xy - 15x + y^4 - 10y^3 + 35y^2 - 50y + 30 = 0$. See Figure 3.2.

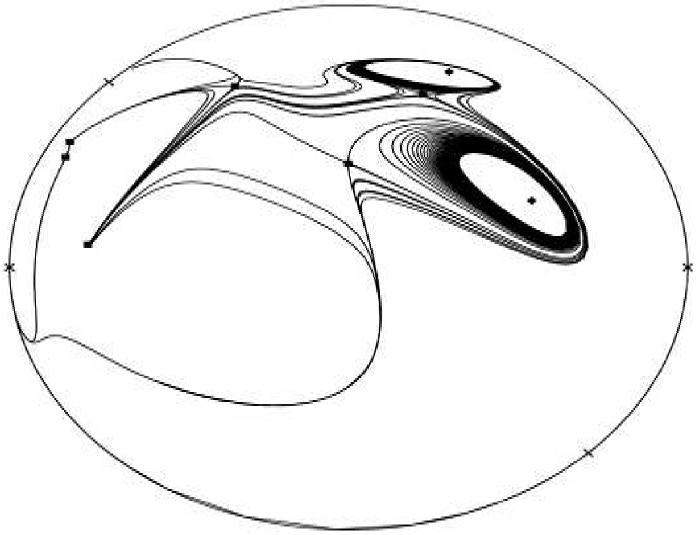

Figure 3.2: Limit cycles of system (3.6).

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Ground-state solutions to a class of modified Kirchhoff-type transmissiom problems with critical perturbation

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Abstract. This paper discusses a class of modified Kirchhoff-type transmission problems with critical perturbation. We establish an existence result of the ground-state solutions by using perturbation methods. Meanwhile, the limit properties of solution sequence are investigated.

Keywords: modified Kirchhoff-type transmission problem, critical perturbation, ground-state solution.

2010 Mathematics Subject Classification: 35J20, 35J60.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\Gamma := \partial\Omega$, $\Omega_1 \subset \mathbb{R}^3$ be a subdomain of Ω with a smooth boundary $\Sigma := \partial\Omega_1$ and $\overline{\Omega}_1 \subset \Omega$. Assume that $\Omega_2 = \Omega \setminus \overline{\Omega}_1$ is connected. Obviously, $\Gamma \cap \Sigma = \emptyset$ and $\partial\Omega_2 = \Gamma \cup \Sigma$. In the present paper we study the existence of solutions for the following Kirchhoff-type transmission problem

$$\begin{cases} \alpha\left(\int_{\Omega_1} g^2(u)|\nabla u|^2\right)\left[-\operatorname{div}\left(g^2(u)\nabla u\right) + g(u)g'(u)|\nabla u|^2\right] = f(u) + \lambda\phi(u), & \text{in } \Omega_1,\\ \beta\left(\int_{\Omega_2} g^2(v)|\nabla v|^2\right)\left[-\operatorname{div}\left(g^2(v)\nabla v\right) + g(v)g'(v)|\nabla v|^2\right] = h(v) + \lambda\psi(v), & \text{in } \Omega_2,\\ v = 0, & \text{on } \Gamma,\\ u = v, & \text{on } \Sigma,\\ \alpha\left(\int_{\Omega_1} g^2(u)|\nabla u|^2\right)\dfrac{\partial u}{\partial \nu} = \beta\left(\int_{\Omega_2} g^2(v)|\nabla v|^2\right)\dfrac{\partial v}{\partial \nu}, & \text{on } \Sigma, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}_+ := [0, \infty)$ and ν is the unit outward normal vector to $\partial\Omega_1$. This system is a modified version of Kirchhoff-type transmission problem because the appearance of nonlocal terms $\int_{\Omega_1} g^2(u)|\nabla u|^2$ and $\int_{\Omega_2} g^2(v)|\nabla v|^2$.

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There are two motivations for studying equation (1.1). The first one is the generalized quasilinear Schrödinger equations. The second one is the classical Kirchhoff-type transmission problem.

In 2015, Deng, Peng, and Yan in [9] researched the generalized quasilinear Schrödinger equations

$$-\operatorname{div}\left(g^2(u)\nabla u\right)+g(u)g'(u)|\nabla u|^2+V(x)u=f(x,u),\qquad x\in\mathbb{R}^N,\tag{1.2}$$

where $N\geqslant 3$, the potential function $V\in C(\mathbb{R}^N)$ and $f\in C(\mathbb{R}^N\times\mathbb{R})$. If we take $g^2(t)=1+\left[(l(t^2))'\right]^2/2$ for $t\in\mathbb{R}$ and l being a suitable function defined on $\mathbb{R}_+$, then the equation (1.2) turns into

$$-\Delta u+V(x)u-\Delta[l(u^2)]l'(u^2)u=f(x,u),\qquad x\in\mathbb{R}^N.\tag{1.3}$$

Solutions of (1.3) is related to the existence of solitary wave solutions for the following quasilinear Schrödinger equation

$$i\partial_t z=-\Delta z+V(x)z-f(x,z)-\Delta[l(|z|^2)]l'(|z|^2)z,\qquad x\in\mathbb{R}^N.\tag{1.4}$$

This quasilinear version of Schrödinger equations is derived from several models of various physical phenomena. The equation (1.4) is called the superfluid film equation in plasma physics when $l(t)=t$ for $t\in\mathbb{R}_+$, see [13] or [14, 15]. If $l(t)=(1+t)^{1/2}$ for $t\in\mathbb{R}_+$, the equation (1.4) was used for the self-channeling of a high-power ultrashort laser in matter, see [4,5,7,24]. In mathematics, many results about the equation (1.3) with $l(t)=t^\alpha$ for some $\alpha\geqslant 1$ have been obtained, see [1, 2, 6, 8, 10, 18–20, 22, 23, 29–31] and the references therein. Equation (1.3) with a general l was studied in the recent papers [9, 25]. We can see that the equation (1.2) is more general and more practical than the equation (1.3).

If we choose $g(t)=1$ for $t\in\mathbb{R}$ and $\lambda=0$, then the equation (1.1) becomes the classical Kirchhoff-type transmission problem

$$\begin{cases}-\alpha\left(\int_{\Omega_1}|\nabla u|^2\right)\Delta u=f(u), & \text{in } \Omega_1,\\ -\beta\left(\int_{\Omega_2}|\nabla v|^2\right)\Delta v=h(v), & \text{in } \Omega_2,\\ v=0, & \text{on } \Gamma,\\ u=v, & \text{on } \Sigma,\\ \alpha\left(\int_{\Omega_1}|\nabla u|^2\right)\dfrac{\partial u}{\partial \nu}=\beta\left(\int_{\Omega_2}|\nabla v|^2\right)\dfrac{\partial v}{\partial \nu}, & \text{on } \Sigma.\end{cases}\tag{1.5}$$

It is well known that this problem is related to the stationary analogue of the problem

$$\begin{cases}u_{tt}-\alpha\left(\int_{\Omega_1}|\nabla u|^2\right)\Delta u=f(u), & x\in\Omega_1, t>0,\\ v_{tt}-\beta\left(\int_{\Omega_2}|\nabla v|^2\right)\Delta v=g(v), & x\in\Omega_2, t>0,\\ v=0, & \text{on } \Gamma,\\ u=v, & \text{on } \Sigma,\\ \alpha\left(\int_{\Omega_1}|\nabla u|^2\right)\dfrac{\partial u}{\partial \nu}=\beta\left(\int_{\Omega_2}|\nabla v|^2\right)\dfrac{\partial v}{\partial \nu}, & \text{on } \Sigma,\\ u(0)=u_0, u_t(0)=u_1, & x\in\Omega_1,\\ v(0)=v_0, v_t(0)=v_1, & x\in\Omega_2,\end{cases}\tag{1.6}$$

which models the transverse vibrations of a membrane composed of two different materials in Ω_1 and Ω_2. According to [21], we call the problem (1.6) a transmission problem because the boundary conditions $u = v$ and $\alpha\big(\int_{\Omega_1} |\nabla u|^2\big)\frac{\partial u}{\partial \nu} = \beta\big(\int_{\Omega_2} |\nabla v|^2\big)\frac{\partial v}{\partial \nu}$ on Σ. This transmission problem (1.6) arises in physics and biology phenomena, such as in the study of electromagnetic processes in ferromagnetic media with different dielectric constants [3], and in thinking about the population distribution of subjects living in an environment composed of different ecological media. In 2003, Ma and Muñoz Rivera [21] discussed the existence and nonexistence of positive solution to the Kirchhoff-type transmission problem (1.5) by using minimization arguments with f and g having subcritical growth. In [16], Li, Zhang, Zhu, and Liang investigated the existence of the ground-state solutions to the following Kirchhoff-type transmission problem with critical perturbation

$$\begin{cases} -\alpha\left(\displaystyle\int_{\Omega_1} |\nabla u|^2\right)\Delta u = f(u) + \lambda u^5, & \text{in } \Omega_1, \\ -\beta\left(\displaystyle\int_{\Omega_2} |\nabla v|^2\right)\Delta v = g(v) + \lambda v^5, & \text{in } \Omega_2, \\ v = 0, & \text{on } \Gamma, \\ u = v, & \text{on } \Sigma, \\ \alpha\left(\displaystyle\int_{\Omega_1} |\nabla u|^2\right)\dfrac{\partial u}{\partial \nu} = \beta\left(\displaystyle\int_{\Omega_2} |\nabla v|^2\right)\dfrac{\partial v}{\partial \nu}, & \text{on } \Sigma. \end{cases} \tag{1.7}$$

Here, we will establish the existence of ground-state solutions to Kirchhoff-type transmission problem with more general g and more general perturbation terms ϕ and ψ. To obtain the existence of ground-state solutions to the more general Kirchhoff-type transmission problem (1.1), we assume that four pairs of functions (α, g, f), (β, g, h), (α, g, ϕ), and (β, g, ψ) belong to the set $\mathcal{A}$, where a pair of functions (α, g, f) is said to belongs to $\mathcal{A}$, if (α, g, f) satisfies the following assumptions

(A_0) $\alpha \in C^1(\mathbb{R}_+)$ is an increasing function and $\alpha(0) > 0$;

(A_1) there exists $\gamma \in (0, 2)$ such that $[\alpha(s) - \alpha(0)]/s^\gamma$ is decreasing on $(0, \infty)$;

(G) $g \in C^1(\mathbb{R}, \mathbb{R}_+)$ is even with $g'(s) \geqslant 0$ for $s \in \mathbb{R}_+$ and $g(0) = 1$;

(F_0) $f \in C^1(\mathbb{R}, \mathbb{R})$ and $\lim_{s\to 0} f(s)/s = 0$;

(F_1) there exists $l_f \in \mathbb{R}$ such that

$$\lim_{|s|\to\infty} \frac{f(s)}{g(s)G^5(s)} = l_f,$$

where $G(s) = \int_0^s g(t)\mathrm{d}t$ for $s \in \mathbb{R}$. And if $l_f = 0$, we call that f has a quasicritical growth; if $l_f \neq 0$, we call that f has a critical growth;

(F_2) $f(s)/(g(s)|G(s)|^{2\gamma}G(s))$ is nondecreasing on $(0, \infty)$ and nonincreasing on $(-\infty, 0)$, and $\lim_{|s|\to\infty} F(s)/|G(s)|^{2\gamma+2} = \infty$, where $F(s) = \int_0^s f(t)\mathrm{d}t$ for $s \in \mathbb{R}$ and γ is as in (A_1).

Remark 1.1. Assuming that g satisfies (G) and $\gamma \in (0, 2)$, let $f(s) = g(s)|G(s)|^{2\gamma}G(s)\ln|G(s)|$ and $\phi(s) = g(s)(G(s))^5$ for $s \in \mathbb{R}$. Then f and ϕ satisfy (F_0), (F_1), and (F_2).

Example 1.2. Let $\alpha(s) = 1 + s^2$ for $s \in \mathbb{R}_+$, and for $\gamma \in (0, 2)$, define $g(s) = s^2 + 1$, $f(s) = (s^2+1)\left|s^3/3+s\right|^{2\gamma}(s^3/3+s)\ln\left|s^3/3+s\right|$, $\phi(s) = (s^2+1)\left(s^3/3+s\right)^5$ for $s \in \mathbb{R}$. Then (α, g, f) and (α, g, ϕ) belong to $\mathcal{A}$.

Example 1.3. For $a, b > 0$, let $\alpha(s) = a + bs$ for $s \in \mathbb{R}_+$, and for $\gamma \in (0,2)$, define $g(s) = \sqrt{2s^2+1}$,

$$f(s) = \frac{\sqrt{2}}{4}\sqrt{2s^2+1}\left|\sqrt{2}s\sqrt{2s^2+1} + \ln\left(\sqrt{2}s + \sqrt{2s^2+1}\right)\right|^{2\gamma}\left(\sqrt{2}s\sqrt{2s^2+1}\right. \\ \left. + \ln\left(\sqrt{2}s + \sqrt{2s^2+1}\right)\right) \times \ln\left|\sqrt{2}s\sqrt{2s^2+1} + \ln\left(\sqrt{2}s + \sqrt{2s^2+1}\right)\right|,$$

$\phi(s) = \frac{\sqrt{2}}{4}\sqrt{2s^2+1}\left[\sqrt{2}s\sqrt{2s^2+1} + \ln\left(\sqrt{2}s + \sqrt{2s^2+1}\right)\right]^5$ for $s \in \mathbb{R}$. Then (α, g, f) and (α, g, ϕ) belong to $\mathcal{A}$.

Remark 1.4. We know that the critical exponent of equation (1.7) is 6 which has a significant influence on the properties of the solution. The critical exponent of equation (1.1) is different for different g and the critical exponent depends on G^6. This is an interesting phenomenon. For example, when $g(s) = \sqrt{2s^2+1}$ for $s \in \mathbb{R}$, the critical exponent is 12; when $g(s) = s^2 + 1$ for $s \in \mathbb{R}$, the critical exponent is 18.

For any given subdomain D of $\mathbb{R}^3$, the standard norm on $L^p(D)$ is denoted by $|\cdot|_{p,D}$ for $p \in [1, \infty)$. Let $H^1(\Omega_1)$ and $H^1(\Omega_2)$ be the usual Sobolev spaces. Then $H^1(\Omega_1) \times H^1(\Omega_2)$ is also a Sobolev space with the norm

$$\|(u,v)\| = \left(|\nabla u|^2_{2,\Omega_1} + |u|^2_{2,\Omega_1} + |\nabla v|^2_{2,\Omega_2} + |v|^2_{2,\Omega_2}\right)^{1/2}, \qquad (u,v) \in H^1(\Omega_1) \times H^1(\Omega_2). \tag{1.8}$$

Our analysis is based on the following Sobolev space

$$E = \{(u,v) \in H^1(\Omega_1) \times H^1_\Gamma(\Omega_2) : u = v \text{ on } \Sigma\},$$

where

$$H^1_\Gamma(\Omega_2) = \{v \in H^1(\Omega_2) : v = 0 \text{ on } \Gamma\}.$$

In [21] Ma and Muñoz Rivera established the following lemma which gave the definition of norm for the Sobolev space E.

Lemma 1.5 ([21, Lemma 1])**.** *E is a closed subspace of $H^1(\Omega_1) \times H^1(\Omega_2)$, and*

$$\|(u,v)\|_E = \left(|\nabla u|^2_{2,\Omega_1} + |\nabla v|^2_{2,\Omega_2}\right)^{1/2}, \qquad (u,v) \in E,$$

defines also a norm on E, which is equivalent to the standard norm (1.8).

Remark 1.6. From Lemma 1.5, we know that the space E is embedded into $L^p(\Omega_1) \times L^q(\Omega_2)$ for all $p, q \in [1,6]$, and these embeddings are compact for all $p, q \in [1,6)$. In particular, for each $p = q \in [1,6]$, there exists $\nu_p > 0$ such that

$$|(u,v)|_p := \left(|u|^p_{p,\Omega_1} + |v|^p_{p,\Omega_2}\right)^{1/p} \leqslant \nu_p \|(u,v)\|_E, \qquad (u,v) \in E. \tag{1.9}$$

In order to solve the transmission problem (1.1), due to the appearance of nonlocal terms $\int_{\Omega_1} g^2(u)|\nabla u|^2$ and $\int_{\Omega_2} g^2(v)|\nabla v|^2$, the potential working space seems to be

$$E_0 = \left\{(u,v) \in E : \int_{\Omega_1} g^2(u)|\nabla u|^2 < \infty, \int_{\Omega_2} g^2(v)|\nabla v|^2 < \infty\right\}.$$

Obviously, E_0 may not be a linear space under the assumed condition of (G). To avoid this drawback, we gave a change of variables,

$$(u,v) = \left(G^{-1}(u_1), G^{-1}(v_1)\right), \qquad (u_1,v_1) \in E,$$

which is motivated by [9, 25]. According to the properties of g, G, and G^{-1} which will be given in Section 2, if $(u_1,v_1) \in E$, then $(u,v) = (G^{-1}(u_1), G^{-1}(v_1)) \in E$ (see Remark 2.3), $\int_{\Omega_1} g^2(u)|\nabla u|^2 = \int_{\Omega_1} g^2(G^{-1}(u_1))|\nabla G^{-1}(u_1)|^2 = |\nabla u_1|_2^2 < \infty$, and $\int_{\Omega_2} g^2(v)|\nabla v|^2 = \int_{\Omega_2} g^2(G^{-1}(v_1))|\nabla G^{-1}(v_1)|^2 = |\nabla v_1|_2^2 < \infty$. Thus, it follows from the change of variables that E can be used as the working space and the transmission problem (1.1) turns into

$$\begin{cases} -\alpha\left(\int_{\Omega_1} |\nabla u_1|^2\right) g(G^{-1}(u_1))\Delta u_1 = f(G^{-1}(u_1)) + \lambda\phi(G^{-1}(u_1)), & \text{in } \Omega_1, \\ -\beta\left(\int_{\Omega_2} |\nabla v_1|^2\right) g(G^{-1}(v_1))\Delta v_1 = h(G^{-1}(v_1)) + \lambda\psi(G^{-1}(v_1)), & \text{in } \Omega_2, \\ v_1 = 0, & \text{on } \Gamma, \\ u_1 = v_1, & \text{on } \Sigma, \\ \alpha\left(\int_{\Omega_1} |\nabla u_1|^2\right)\dfrac{\partial u_1}{\partial \nu} = \beta\left(\int_{\Omega_2} |\nabla v_1|^2\right)\dfrac{\partial v_1}{\partial \nu}, & \text{on } \Sigma. \end{cases} \tag{1.10}$$

Furthermore, we can prove that if $(u_1,v_1) \in E \cap \left(H^2_{\mathrm{loc}}(\Omega_1) \times H^2_{\mathrm{loc}}(\Omega_2)\right)$ is a strong solution to the equation (1.10), then $(u,v) = (G^{-1}(u_1), G^{-1}(v_1)) \in E \cap \left(H^2_{\mathrm{loc}}(\Omega_1) \times H^2_{\mathrm{loc}}(\Omega_2)\right)$ is a strong solution to the equation (1.1). Here, we call that $(u,v) \in E \cap \left(H^2_{\mathrm{loc}}(\Omega_1) \times H^2_{\mathrm{loc}}(\Omega_2)\right)$ is a strong solution to the transmission problem (1.10) or (1.1) if the first two equations in (1.10) or (1.1) hold in the sense of almost everywhere. Actually, we only need to verify that for any an open bounded set $D \subset \mathbb{R}^3$ if $u_1 \in H^2(D)$, then $G^{-1}(u_1) \in H^2(D)$(see Lemma 4.2). Moreover, because of the continuity of g, G, and G^{-1}, to obtain a strong solution to the transmission problem (1.10), it suffices to seek for the weak solution to the following transmission problem

$$\begin{cases} -\alpha\left(\int_{\Omega_1} |\nabla u_1|^2\right)\Delta u_1 = \dfrac{f(G^{-1}(u_1))}{g(G^{-1}(u_1))} + \lambda\dfrac{\phi(G^{-1}(u_1))}{g(G^{-1}(u_1))}, & \text{in } \Omega_1, \\ -\beta\left(\int_{\Omega_2} |\nabla v_1|^2\right)\Delta v_1 = \dfrac{h(G^{-1}(v_1))}{g(G^{-1}(v_1))} + \lambda\dfrac{\psi(G^{-1}(v_1))}{g(G^{-1}(v_1))}, & \text{in } \Omega_2, \\ v_1 = 0, & \text{on } \Gamma, \\ u_1 = v_1, & \text{on } \Sigma, \\ \alpha\left(\int_{\Omega_1} |\nabla u_1|^2\right)\dfrac{\partial u_1}{\partial \nu} = \beta\left(\int_{\Omega_2} |\nabla v_1|^2\right)\dfrac{\partial v_1}{\partial \nu}, & \text{on } \Sigma. \end{cases} \tag{1.11}$$

In fact, if $(u_1,v_1) \in E$ is a weak solution to the transmission problem (1.11), then it should satisfy, for all $(w_1,z_1) \in E$,

$$\begin{aligned} &\alpha\left(|\nabla u_1|^2_{2,\Omega_1}\right)\int_{\Omega_1} \nabla u_1 \cdot \nabla w_1 + \beta\left(|\nabla v_1|^2_{2,\Omega_2}\right)\int_{\Omega_2} \nabla v_1 \cdot \nabla z_1 \\ &\quad = \int_{\Omega_1} \frac{f(G^{-1}(u_1))}{g(G^{-1}(u_1))}w_1 + \int_{\Omega_2} \frac{h(G^{-1}(v_1))}{g(G^{-1}(v_1))}z_1 + \lambda\int_{\Omega_1} \frac{\phi(G^{-1}(u_1))}{g(G^{-1}(u_1))}w_1 + \lambda\int_{\Omega_2} \frac{\psi(G^{-1}(v_1))}{g(G^{-1}(v_1))}z_1. \end{aligned}$$

Hence, $u_1 \in H^1(\Omega_1)$ weakly solves the equation

$$-\alpha\left(|\nabla u_1|^2_{2,\Omega_1}\right)\Delta u_1 = a(x)(1+u_1), \quad \text{in } \Omega_1,$$

with

$$a(x) = \frac{1}{1+u_1(x)}\left(\frac{f(G^{-1}(u_1))}{g(G^{-1}(u_1))} + \lambda\frac{\phi(G^{-1}(u_1))}{g(G^{-1}(u_1))}\right) =: \frac{1}{1+u_1(x)}\left(\widetilde{f}(u_1)+\lambda\widetilde{\phi}(u_1)\right),$$

where $\widetilde{f}(s) := \frac{f(G^{-1}(s))}{g(G^{-1}(s))}$ and $\widetilde{\phi}(s) := \frac{\phi(G^{-1}(s))}{g(G^{-1}(s))}$ for $s \in \mathbb{R}$. The condition (F_2) implies that $a \in L^{3/2}_{\mathrm{loc}}(\Omega_1)$. By the Brézis–Kato theorem, see also [26, Lemma B.3, p. 244] , we know that $u_1 \in L^q_{\mathrm{loc}}(\Omega_1)$ for any $q \in [1,\infty)$. Theorem 8.8 in [11, p. 183] shows that $u_1 \in H^1(\Omega_1) \cap H^2_{\mathrm{loc}}(\Omega_1)$ and

$$-\alpha\left(|\nabla u_1|^2_{2,\Omega_1}\right)\Delta u_1 = \widetilde{f}(u_1)+\lambda\widetilde{\phi}(u_1), \quad \text{a.e. } x \in \Omega_1.$$

Similarly, we can prove that $v_1 \in H^1_\Gamma(\Omega_2) \cap H^2_{\mathrm{loc}}(\Omega_1)$ such that

$$-\beta\left(|\nabla v_1|^2_{2,\Omega_2}\right)\Delta v_1 = \widetilde{h}(v_1)+\lambda\widetilde{\psi}(v_1), \quad \text{a.e. } x \in \Omega_2,$$

where $\widetilde{h}(s) = \frac{h(G^{-1}(s))}{g(G^{-1}(s))}$ and $\widetilde{\psi}(s) = \frac{\psi(G^{-1}(s))}{g(G^{-1}(s))}$ for $s \in \mathbb{R}$. So the problem (1.11) holds in the sense of almost everywhere and $(u_1, v_1) \in \left(H^1(\Omega_1) \cap H^2_{\mathrm{loc}}(\Omega_1)\right) \times \left(H^1_\Gamma(\Omega_2) \cap H^2_{\mathrm{loc}}(\Omega_2)\right)$ is a strong solution to the equation. Here, let $(u,v) = (G^{-1}(u_1), G^{-1}(v_1))$. Then (u,v) is a strong solution to the transmission problem (1.1). For the convenience, removing the subscripts of u_1, v_1 , we rewrite (1.11) as the following transmission problem

$$\begin{cases} -\alpha\left(\int_{\Omega_1}|\nabla u|^2\right)\Delta u = \dfrac{f(G^{-1}(u))}{g(G^{-1}(u))} + \lambda\dfrac{\phi(G^{-1}(u))}{g(G^{-1}(u))}, & \text{in } \Omega_1, \\ -\beta\left(\int_{\Omega_2}|\nabla v|^2\right)\Delta v = \dfrac{h(G^{-1}(v))}{g(G^{-1}(v))} + \lambda\dfrac{\psi(G^{-1}(v))}{g(G^{-1}(v))}, & \text{in } \Omega_2, \\ v = 0, & \text{on } \Gamma, \\ u = v, & \text{on } \Sigma, \\ \alpha\left(\int_{\Omega_1}|\nabla u|^2\right)\dfrac{\partial u}{\partial \nu} = \beta\left(\int_{\Omega_2}|\nabla v|^2\right)\dfrac{\partial v}{\partial \nu}, & \text{on } \Sigma. \end{cases} \tag{1.12}$$

In the following, we make our efforts to find the weak solution to the transmission problem (1.12). To this end, we define the energy functional $I : E \to \mathbb{R}$ associated with the transmission problem (1.12)

$$\begin{aligned} I_\lambda(u,v) = {} & \frac{1}{2}A\left(|\nabla u|^2_{2,\Omega_1}\right) + \frac{1}{2}B\left(|\nabla v|^2_{2,\Omega_2}\right) - \int_{\Omega_1} F(G^{-1}(u)) - \int_{\Omega_2} H(G^{-1}(v)) \\ & - \lambda\int_{\Omega_1}\Phi(G^{-1}(u)) - \lambda\int_{\Omega_2}\Psi(G^{-1}(v)), \qquad (u,v) \in E, \end{aligned}$$

where $A(s) = \int_0^s \alpha(t)\mathrm{d}t, B(s) = \int_0^s \beta(t)\mathrm{d}t$ for $s \in \mathbb{R}_+$, and $H(s) = \int_0^s h(t)\mathrm{d}t, \Phi(s) = \int_0^s \phi(t)\mathrm{d}t$, $\Psi(s) = \int_0^s \psi(t)\mathrm{d}t$ for $s \in \mathbb{R}$. It can be verified that I_λ is of class C^1. And for all $(u,v), (w,z) \in E$,

$$\begin{aligned} \langle I'_\lambda(u,v),(w,z)\rangle = {} & \alpha(|\nabla u|^2_{2,\Omega_1})\int_{\Omega_1}\nabla u\cdot\nabla w + \beta(|\nabla v|^2_{2,\Omega_2})\int_{\Omega_2}\nabla v\cdot\nabla z - \int_{\Omega_1}\frac{f(G^{-1}(u))}{g(G^{-1}(u))}w \\ & - \int_{\Omega_2}\frac{h(G^{-1}(v))}{g(G^{-1}(v))}z - \lambda\int_{\Omega_1}\frac{\phi(G^{-1}(u))}{g(G^{-1}(u))}w - \lambda\int_{\Omega_2}\frac{\psi(G^{-1}(v))}{g(G^{-1}(v))}z. \end{aligned}$$

Let $\widetilde{F}(s) = F(G^{-1}(s)), \widetilde{H}(s) = H(G^{-1}(s)), \widetilde{\Phi}(s) = \Phi(G^{-1}(s))$, and $\widetilde{\Psi}(s) = \Psi(G^{-1}(s))$ for $s \in \mathbb{R}$. Then, for all $(u,v), (w,z) \in E$, we have that

$$I_\lambda(u,v) = \frac{1}{2}A\left(|\nabla u|^2_{2,\Omega_1}\right) + \frac{1}{2}B\left(|\nabla v|^2_{2,\Omega_2}\right) - \int_{\Omega_1}\widetilde{F}(u) - \int_{\Omega_2}\widetilde{H}(v) - \lambda\int_{\Omega_1}\widetilde{\Phi}(u) - \lambda\int_{\Omega_2}\widetilde{\Psi}(v),$$

and

$$\langle I'_\lambda(u,v),(w,z)\rangle = \alpha\left(|\nabla u|^2_{2,\Omega_1}\right)\int_{\Omega_1}\nabla u\cdot\nabla w + \beta\left(|\nabla v|^2_{2,\Omega_2}\right)\int_{\Omega_2}\nabla v\cdot\nabla z$$
$$-\int_{\Omega_1}\widetilde{f}(u)w - \int_{\Omega_2}\widetilde{h}(v)z - \lambda\int_{\Omega_1}\widetilde{\phi}(u)w - \lambda\int_{\Omega_2}\widetilde{\psi}(v)z. \tag{1.13}$$

Then we say that $(u,v)\in E$ is a weak solution to the transmission problem (1.12) if and only if (u,v) is a critical point of the functional I_λ in E, i.e., $I'_\lambda(u,v)=0$. To sum up, it suffices to seek a critical point of the functional I_λ in E to achieve a strong solution to the transmission problem (1.1).

Now, we state our main results through the following theorems.

Theorem 1.7. *Assume that $(\alpha,g,f),(\beta,g,h)\in\mathcal{A}$ with $l_f=l_h=0$, $(\alpha,g,\phi),(\beta,g,\psi)\in\mathcal{A}$ with $l_\phi,l_\psi\neq 0$, and $\phi(s)s>0,\psi(s)s>0$ for $s\neq 0$. Then there exists $\lambda_0>0$ such that both the problem (1.12) and (1.1) have a ground-state solution (u_λ,v_λ) for all $\lambda\in[0,\lambda_0)$. Furthermore, it holds that $(u_\lambda,v_\lambda)\to(u_0,v_0)$ in E as $\lambda\to 0$, where (u_0,v_0) is a ground-state solution to the problem (1.1) with $\lambda=0$.*

Corollary 1.8. *Let $\Omega_2=\emptyset,\alpha(s)=1,g(s)=\sqrt{1+2s^2},f(s)=|s|^{q-2}$, and $\phi(s)=|s|^{10}s$ for $s\in\mathbb{R}$. Then the following equation has a ground-state solution u_λ for all $\lambda\in[0,\lambda_0)$,*

$$\begin{cases}-\Delta u-\Delta(u^2)u=|u|^{q-2}u+\lambda|u|^{10}u, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\end{cases} \tag{1.14}$$

where $q\in(4,12)$. Furthermore, it holds that $u_\lambda\to u_0$ in $H^1_0(\Omega)$ as $\lambda\to 0$, where u_0 is a ground-state solution to the above problem with $\lambda=0$.

Remark 1.9. According to [8], for a single quasilinear Schrödinger equation (1.14) in a bounded domain in $\mathbb{R}^3$, there exists a suitable energy level c^* such that if $c(\lambda)<c^*$, then the associated energy functional satisfies the $(\mathrm{PS})_{c(\lambda)}$ condition, where $c^*=S^3/6$ and S is the best Sobolev constant for $\mathcal{D}^{1,2}(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$. However, a large amount of calculations is required to prove that $c(\lambda)<c^*$ by verifying

$$\sup_{t\in\mathbb{R}_+} I_\lambda(tu_\epsilon)<c^*,$$

where u_ϵ is a modification of U and U attains the best Sobolev constant S. In this paper to avoid this difficulty, we adopt the perturbation method from [12,32].

Remark 1.10. Let $g(s)=1$ and $\phi(s)=\psi(s)=s^5$ for $s\in\mathbb{R}$. Then by Theorem 1.7, we have that the transmission problem (1.1) also has a ground-state solution, which has been achieved in [16]. Thus, Theorem 1.7 could be regarded as a generalization of Theorem 1.1 in [16].

This paper is organized as follows. We give some preliminaries in Section 2. Theorem 1.7 is proved in Section 3. Throughout this paper we denote C_i for $i\in\mathbb{N}:=\{1,2,\dots\}$ as constants which can be different from line to line.

2 Preliminaries

In this section we first give some properties of the functions $\alpha,g,\widetilde{f}$, and $A,G,G^{-1},\widetilde{F}$ via the following lemmas.

Lemma 2.1.

(i) Assume that α satisfies the condition (A_0). Then $A(s) \geqslant \alpha(0)s$ for $s \in \mathbb{R}_+$.

(ii) Assume that α satisfies the conditions (A_0) and (A_1). Then $[A(s) - \alpha(0)s]/s^{\gamma+1}$, $\alpha(s)s - (\gamma+1)A(s) + \gamma\alpha(0)s$, and $A(s)/s^{\gamma+1}$ are decreasing on $(0,\infty)$. Furthermore, we have that

$$(\gamma+1)A(s) - \alpha(s)s \geqslant \gamma\alpha(0)s, \qquad s \in \mathbb{R}_+, \tag{2.1}$$

and

$$\alpha'(s)s \leqslant \gamma[\alpha(s) - \alpha(0)] < \gamma\alpha(s), \qquad s \in \mathbb{R}_+. \tag{2.2}$$

Lemma 2.2. *The functions g, G, and G^{-1} have the following properties under the assumption of (G):*

(i) G and G^{-1} are both odd, and

$$t \leqslant G(t) \leqslant g(t)t, \quad t \in \mathbb{R}_+, \qquad s/g(G^{-1}(s)) \leqslant G^{-1}(s) \leqslant s, \quad s \in \mathbb{R}_+;$$

(ii) $\lim_{s\to 0} G^{-1}(s)/s = 1$ and $\lim_{s\to\infty} G^{-1}(s)/s = 1/g(\infty)$, where $g(\infty) = \lim_{s\to\infty} g(s)$;

(iii) $G^{-1}(s)/\left[|s|^{2\gamma} s g(G^{-1}(s))\right]$ is nonincreasing on $(0,\infty)$ and nondecreasing on $(-\infty,0)$;

(iv) $[G^{-1}(s)]^2 - G^{-1}(s)s/g(G^{-1}(s))$ is nondecreasing on $(0,\infty)$ and nonincreasing on $(-\infty,0)$;

(v) if f is a continuous function and (F_2) holds, then $f(G^{-1}(s))s/[(2\gamma+2)g(G^{-1}(s))] - F(G^{-1}(s))$ is increasing on $(0,\infty)$ and decreasing on $(-\infty,0)$.

Proof. (i), (ii), and (iv) can be derived from [17, (1), (2), and (4) of Lemma 2.2]. As for (iii), because g is even, we need only to prove that the conclusion holds on $(0,\infty)$. In fact, since $[G(t)/t]^{2\gamma+1}g(t)$ is nondecreasing on $(0,\infty)$, $[G(t)]^{2\gamma+1}g(t)/t$ is also nondecreasing on $(0,\infty)$, and then $G^{-1}(s)/[s^{2\gamma+1}g(G^{-1}(s))]$ is nonincreasing on $(0,\infty)$.

Finally, we prove that (v) holds. Indeed, since $f(t)/[g(t)|G(t)|^{2\gamma}G(t)]$ is nondecreasing on $(0,\infty)$ and nonincreasing on $(-\infty,0)$, according to [17, Lemma A.1], $f(t)G(t)/[(2\gamma+2)g(t)] - F(t)$ is nondecreasing on $(0,\infty)$ and nonincreasing on $(-\infty,0)$, and then $f(G^{-1}(s))s/[(2\gamma+2)g(G^{-1}(s))] - F(G^{-1}(s))$ is nondecreasing on $(0,\infty)$ and nonincreasing on $(-\infty,0)$, that is, (v) holds. The proof is complete. □

Remark 2.3. Let $(u,v) \in E$. Then it follows from $g(t) \geqslant 1$ for $t \in \mathbb{R}_+$, and (i) of Lemma 2.2 that $(G^{-1}(u), G^{-1}(v)) \in E$.

Lemma 2.4. *Assume that g satisfies (G) and f satisfies (F_0), (F_1), and (F_2). Let $\widetilde{f}(s) = \frac{f(G^{-1}(s))}{g(G^{-1}(s))}$ for $s \in \mathbb{R}$. Then $\widetilde{f}$ has the following properties:*

(F_0') *$\widetilde{f} \in C^1(\mathbb{R})$ and $\lim_{s\to 0} \widetilde{f}(s)/s = 0$;*

(F_1')

$$\lim_{|s|\to\infty} \frac{\widetilde{f}(s)}{s^5} = l_f;$$

(F_2') *$\widetilde{f}(s)/(|s|^{2\gamma}s)$ is nondecreasing on $(0,\infty)$ and nonincreasing on $(-\infty,0)$, and*

$$\lim_{|s|\to\infty} \widetilde{F}(s)/|s|^{2\gamma+2} = \infty.$$

From [16], the function $\widetilde{f}$ possesses some other properties as mentioned in the following Remark 2.5. With those properties, we know that Lemmas 2.6–2.8 hold.

Remark 2.5. It follows from (F_2') and [17, Lemma A.1] that $\widetilde{f}(s)s - 2(\gamma+1)\widetilde{F}(s)$ is nondecreasing on $\mathbb{R}_+$ and nonincreasing $(-\infty, 0]$, and then

$$\widetilde{f}(s)s - 2(\gamma+1)\widetilde{F}(s) \geqslant 0, \qquad s \in \mathbb{R}, \tag{2.3}$$

and

$$\widetilde{f}'(s)s - (2\gamma+1)\widetilde{f}(s) \geqslant 0, \qquad s \in \mathbb{R}_+. \tag{2.4}$$

Lemma 2.6. *Suppose that f satisfies the conditions (F_0) and (F_1) and g satisfies the conditions (G). Then for each $u \in H^1(\Omega)$, one has that*

$$\lim_{t\to 0} \int_{\Omega_1} \frac{\widetilde{f}(tu)u}{t} = 0.$$

Lemma 2.7. *Suppose that f satisfies the conditions (F_0) and (F_1) and g satisfies the conditions (G). If $u_n \rightharpoonup u \neq 0$ in $H^1(\Omega)$ and $|t_n| \to \infty$, then*

$$\lim_{n\to\infty} \int_{\Omega} \frac{\widetilde{f}(t_n u_n)u_n}{|t_n|^{2\gamma} t_n} = \infty.$$

Lemma 2.8. *Suppose that f satisfies the conditions (F_0) and (F_1) and g satisfies the conditions (G). Then for each $u \in H^1(\Omega)$ and $u \neq 0$, it holds that*

$$\lim_{|t|\to\infty} \int_{\Omega_1} \frac{\widetilde{f}(tu)u}{|t|^{2\gamma} t} = \infty.$$

3 Existence and convergence of ground-state solutions

In this section, assuming that the all conditions of Theorem 1.7 hold, we will establish the existence of ground-state solutions to the problems (1.12) and complete the proof of Theorem 1.7. First, we verify that the functional I_λ has a mountain pass geometric structure and the functional I_0 satisfies the Palais–Smale (PS for short) condition.

For each $\lambda \in \mathbb{R}_+$, let

$$\Gamma_\lambda = \{\gamma \in C([0,1], E) : \gamma(0) = 0, I_\lambda(\gamma(1)) < 0\}$$

and define

$$c(\lambda) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)).$$

Lemma 3.1. $\Gamma_\lambda \neq \emptyset$ *and* $c(\lambda) > 0$ *for* $\lambda \in \mathbb{R}_+$.

Proof. For any given $\varepsilon \in \left(0, \left[2(1+\lambda)\nu_2^2\right]^{-1} \min\{\alpha(0), \beta(0)\}\right)$ and $p \in (2\gamma+2, 6]$, we obtain from (F_0') and (F_1') that there exists $C_{\varepsilon,p}, C_\varepsilon > 0$ such that

$$|\widetilde{f}(s)|, |\widetilde{h}(s)| \leqslant \varepsilon \left[|s| + |s|^5\right] + C_{\varepsilon,p}|s|^{p-1}, \qquad s \in \mathbb{R}, \tag{3.1}$$

$$|\widetilde{F}(s)|, |\widetilde{H}(s)| \leqslant \varepsilon \left(s^2 + s^6\right) + C_{\varepsilon,p}|s|^p, \qquad s \in \mathbb{R},$$

$$|\widetilde{\varphi}(s)|, |\widetilde{\psi}(s)| \leqslant \varepsilon|s| + C_\varepsilon|s|^5, \qquad s \in \mathbb{R}, \tag{3.2}$$

$$|\widetilde{\Phi}(s)|, |\widetilde{\Psi}(s)| \leqslant \varepsilon s^2 + C_\varepsilon s^6, \qquad s \in \mathbb{R},$$

where $\widetilde{F}(s) = \int_0^s \widetilde{f}(t)\mathrm{d}t$, $\widetilde{H}(s) = \int_0^s \widetilde{h}(t)\mathrm{d}t$, $\widetilde{\Phi}(s) = \int_0^s \widetilde{\varphi}(t)\mathrm{d}t$, and $\widetilde{\Psi}(s) = \int_0^s \widetilde{\psi}(t)\mathrm{d}t$ for $s \in \mathbb{R}$. Then it follows from the Sobolev inequality (1.11) that for $(u,v) \in E$,

$$\left| \int_{\Omega_1} \widetilde{F}(u) + \int_{\Omega_2} \widetilde{H}(v) \right| \leqslant \varepsilon \nu_2^2 \|(u,v)\|_E^2 + \varepsilon \nu_6^6 \|(u,v)\|_E^6 + \nu_p^p C_{\varepsilon,p} \|(u,v)\|_E^p,$$

and

$$\left| \int_{\Omega_1} \widetilde{\Phi}(u) + \int_{\Omega_2} \widetilde{\Psi}(v) \right| \leqslant \varepsilon \nu_2^2 \|(u,v)\|_E^2 + \nu_6^6 C_\varepsilon \|(u,v)\|_E^6.$$

Thus, combining this and (i) of Lemma 2.1, we have that for $(u,v) \in E$,

$$\begin{aligned}
&I_\lambda(u,v) \\
&\quad = \frac{1}{2} A\left(|\nabla u|_{2,\Omega_1}^2\right) + \frac{1}{2} B\left(|\nabla v|_{2,\Omega_2}^2\right) - \int_{\Omega_1} \widetilde{F}(u) - \int_{\Omega_2} \widetilde{H}(v) - \lambda \left[\int_{\Omega_1} \widetilde{\Phi}(u) + \int_{\Omega_2} \widetilde{\Psi}(v) \right] \\
&\quad \geqslant \frac{1}{2} \left[\alpha(0) |\nabla u|_{2,\Omega_1}^2 + \beta(0) |\nabla v|_{2,\Omega_2}^2 \right] - (1+\lambda)\varepsilon \nu_2^2 \|(u,v)\|_E^2 \\
&\qquad - \nu_p^p C_{\varepsilon,p} \|(u,v)\|_E^p - (\varepsilon + \lambda C_\varepsilon) \nu_6^6 \|(u,v)\|_E^6 \\
&\quad \geqslant \left(\frac{1}{2} \min\{\alpha(0), \beta(0)\} - (1+\lambda)\varepsilon \nu_2^2 \right) \|(u,v)\|_E^2 - \nu_p^p C_{\varepsilon,p} \|(u,v)\|_E^p - (\varepsilon + \lambda C_\varepsilon) \nu_6^6 \|(u,v)\|_E^6.
\end{aligned}$$

Hence, letting $\rho > 0$ small enough, it is easy to see that $\inf\{I_\lambda(u,v) : \|(u,v)\|_E = \rho\} > 0$.

Next, for each $(u,v) \in E \setminus \{0\}$, according to (ii) of Lemma 2.1, the following limits exist

$$a_\infty := \lim_{t\to\infty} \frac{A\left(t^2 |\nabla u|_{2,\Omega_1}^2\right)}{2t^{2\gamma+2}} \in \mathbb{R}_+, \qquad b_\infty := \lim_{t\to\infty} \frac{B\left(t^2 |\nabla v|_{2,\Omega_2}^2\right)}{2t^{2\gamma+2}} \in \mathbb{R}_+.$$

For any given $M > (a_\infty + b_\infty) \left[(1+\lambda) \left(|u|_{2\gamma+2,\Omega_1}^{2\gamma+2} + |v|_{2\gamma+2,\Omega_2}^{2\gamma+2} \right) \right]^{-1}$, it follows from ($\mathrm{F}_2'$) and ($\mathrm{F}_0'$) that there exists $C > 0$ such that

$$\widetilde{F}(s), \widetilde{H}(s), \widetilde{\Phi}(s), \widetilde{\Psi}(s) \geqslant M|s|^{2\gamma+2} - C, \qquad s \in \mathbb{R}.$$

Thus, we have that

$$\begin{aligned}
I_\lambda(t(u,v)) &\leqslant \frac{1}{2} A\left(t^2 |\nabla u|_{2,\Omega_1}^2\right) + \frac{1}{2} B\left(t^2 |\nabla v|_{2,\Omega_2}^2\right) - M(1+\lambda) t^{2\gamma+2} \left[|u|_{2\gamma+2,\Omega_1}^{2\gamma+2} + |v|_{2\gamma+2,\Omega_2}^{2\gamma+2} \right] \\
&\quad + C(1+\lambda)[|\Omega_1| + |\Omega_2|] \\
&= t^{2\gamma+2} \left[\frac{A\left(t^2 |\nabla u|_{2,\Omega_1}^2\right)}{2t^{2\gamma+2}} + \frac{B\left(t^2 |\nabla v|_{2,\Omega_2}^2\right)}{2t^{2\gamma+2}} - M(1+\lambda) \left[|u|_{2\gamma+2,\Omega_1}^{2\gamma+2} + |v|_{2\gamma+2,\Omega_2}^{2\gamma+2} \right] \right. \\
&\qquad \left. + \frac{C(1+\lambda)}{t^{2\gamma+2}} [|\Omega_1| + |\Omega_2|] \right] \\
&\to -\infty, \quad t \to \infty.
\end{aligned}$$

The proof is complete. □

Lemma 3.2. *For each $\lambda \in \mathbb{R}_+$, any PS sequence of the functional I_λ is always bounded. Particularly, for $\lambda = 0$, the functional I_0 satisfies the PS condition.*

Proof. As for the boundedness of PS sequence, one only needs to observe that (2.1) and (2.3) imply the AR condition. Here for the completeness, we sketch out the proof. Assume that $\lambda \in \mathbb{R}_+, c \in \mathbb{R}$, and $\{(u_n, v_n)\}$ is a $(\mathrm{PS})_c$ sequence of I_λ. Then according to (2.1) and (2.3), for sufficiently large n we have that

$$\begin{aligned} c + 1 + \|(u_n, v_n)\|_E &\geqslant I_\lambda(u_n, v_n) - \frac{1}{2\gamma+2}\langle I'_\lambda(u_n, v_n), (u_n, v_n)\rangle \\ &= \frac{1}{2}A(|\nabla u_n|^2_{2,\Omega_1}) - \frac{1}{2\gamma+2}\alpha(|\nabla u_n|^2_{2,\Omega_1})|\nabla u_n|^2_{2,\Omega_1} \\ &\quad + \frac{1}{2}B(|\nabla v_n|^2_{2,\Omega_2}) - \frac{1}{2\gamma+2}\beta(|\nabla v_n|^2_{2,\Omega_2})|\nabla v_n|^2_{2,\Omega_2} \\ &\quad + \int_{\Omega_1}\left[\frac{1}{2\gamma+2}\widetilde{f}(u_n)u_n - \widetilde{F}(u_n)\right] + \int_{\Omega_2}\left[\frac{1}{2\gamma+2}\widetilde{h}(v_n)v_n - \widetilde{H}(v_n)\right] \\ &\quad + \lambda\int_{\Omega_1}\left[\frac{1}{2\gamma+2}\widetilde{\phi}(u_n)u_n - \widetilde{\Phi}(u_n)\right] + \lambda\int_{\Omega_2}\left[\frac{1}{2\gamma+2}\widetilde{\psi}(v_n)v_n - \widetilde{\Psi}(v_n)\right] \\ &\geqslant \frac{\gamma\alpha(0)}{2\gamma+2}|\nabla u_n|^2_{2,\Omega_1} + \frac{\gamma\beta(0)}{2\gamma+2}|\nabla v_n|^2_{2,\Omega_2} \\ &\geqslant \frac{\gamma}{2\gamma+2}\min\{\alpha(0), \beta(0)\}\|(u_n, v_n)\|^2_E. \end{aligned} \tag{3.3}$$

It follows that $\{(u_n, v_n)\}$ is bounded in E.

Now, we illustrate that the functional I_0 satisfies the PS condition. In fact, let $\{(u_n, v_n)\}$ be a PS sequence of I_0. First, from the above conclusion we can get the boundedness of $\{(u_n, v_n)\}$ in E . Without loss of generality, there exists $(u, v) \in E$ such that $(u_n, v_n) \rightharpoonup (u, v)$ as $n \to \infty$. Owing to (3.1) and the compact embedding $E \hookrightarrow L^p(\Omega_1) \times L^p(\Omega_2)$ for $p \in [1, 6)$, we can derive that

$$\lim_{n\to\infty}\int_{\Omega_1}\widetilde{f}(u_n)(u_n - u) = 0, \qquad \lim_{n\to\infty}\int_{\Omega_2}\widetilde{h}(v_n)(v_n - v) = 0. \tag{3.4}$$

Thus, similarly to Lemma 3.2 in [16], we can prove that $\|(u_n - u, v_n - v)\|^2_E \to 0$. The proof is complete. □

It follows from the mountain pass theorem that the following corollary holds.

Corollary 3.3.

$$K_{c(0)} := \{(u, v) \in E : I'_0(u, v) = 0, I_0(u, v) = c(0)\} \neq \emptyset. \tag{3.5}$$

Define

$$N_\lambda = \left\{(u, v) \in E \setminus \{0\} : \langle I'_\lambda(u, v), (u, v)\rangle = 0\right\}, \qquad d(\lambda) = \inf_{N_\lambda} I_\lambda. \tag{3.6}$$

We now prove that $N_\lambda \neq \emptyset$ and provide some properties of the mapping $d(\cdot)$.

Lemma 3.4. *Let $(u, v) \in E\setminus\{0\}$.*

(i) For each $\lambda \in \mathbb{R}_+$, there exists a unique $t(\lambda) > 0$ such that $t(\lambda)(u, v) \in N_\lambda$, $\langle I'_\lambda(t(u, v)), t(u, v)\rangle > 0$ for $t \in (0, t(\lambda))$, $\langle I'_\lambda(t(u, v)), t(u, v)\rangle < 0$ for $t \in (t(\lambda), \infty)$, and $I_\lambda(t(\lambda)(u, v)) = \max_{t\in\mathbb{R}_+} I_\lambda(t(u, v))$.

(ii) The function $t(\cdot) : \mathbb{R}_+ \to (0, \infty)$ is continuously differentiable and

$$t'(\lambda) = \frac{\int_{\Omega_1}\widetilde{\phi}(t(\lambda)u)t(\lambda)u + \int_{\Omega_2}\widetilde{\psi}(t(\lambda)v)t(\lambda)v}{W_1(t(\lambda), (u, v))}, \tag{3.7}$$

where W_1 is defined by

$$\begin{aligned}W_1(t,(u,v)) = & -2\gamma t\left[\alpha\left(t^2|\nabla u|^2_{2,\Omega_1}\right)|\nabla u|^2_{2,\Omega_1}+\beta(t^2|\nabla v|^2_{2,\Omega_2})|\nabla v|^2_{2,\Omega_2}\right]\\ & +2t^3\left[\alpha'(t^2|\nabla u|^2_{2,\Omega_1})|\nabla u|^4_{2,\Omega_1}+\beta'(t^2|\nabla v|^2_{2,\Omega_2})|\nabla v|^4_{2,\Omega_2}\right]\\ & +(2\gamma+1)\left[\int_{\Omega_1}\widetilde{f}(tu)u+\int_{\Omega_2}\widetilde{h}(tv)v\right]-\int_{\Omega_1}\widetilde{f}'(tu)tu^2-\int_{\Omega_2}\widetilde{h}'(tv)tv^2\\ & +(2\gamma+1)\lambda\left[\int_{\Omega_1}\widetilde{\varphi}(tu)u+\int_{\Omega_2}\widetilde{\psi}(tv)v\right]-\lambda\int_{\Omega_1}\widetilde{\varphi}'(tu)tu^2-\lambda\int_{\Omega_2}\widetilde{\psi}'(tv)tv^2.\end{aligned}$$

Particularly, $t(\cdot)$ is decreasing on $\mathbb{R}_+$.

Proof. (i) Let $(u,v)\in E\backslash\{0\}$ and $\lambda\in\mathbb{R}_+$ be fixed, and let $w(t)=I_\lambda(t(u,v))$ for $t\in\mathbb{R}_+$. Then $w\in C^1(\mathbb{R}_+)$ and we have that for $t>0$,

$$\begin{aligned}w'(t) &= \langle I'_\lambda(t(u,v)),(u,v)\rangle\\ &= t\alpha\left(t^2|\nabla u|^2_{2,\Omega_1}\right)|\nabla u|^2_{2,\Omega_1}+t\beta\left(t^2|\nabla v|^2_{2,\Omega_2}\right)|\nabla v|^2_{2,\Omega_2}-\int_{\Omega_1}\widetilde{f}(tu)u-\int_{\Omega_2}\widetilde{h}(tv)v\\ &\quad -\lambda\left[\int_{\Omega_1}\widetilde{\varphi}(tu)u+\int_{\Omega_2}\widetilde{\psi}(tv)v\right].\end{aligned}\tag{3.8}$$

By applying (A_0) and Lemma 2.6, we obtain that $w'(t)>0$ for small $t>0$. And by applying (ii) of Lemma 2.1 and Lemma 2.8, we obtain that $w'(t)<0$ for t large. Thus, there must be some $t(\lambda)>0$ such that $w'(t(\lambda))=0$. Therefore, $t(\lambda)(u,v)\in N_\lambda$.

Furthermore, we can also derive the uniqueness of $t(\lambda)$. In fact, suppose by contradiction there are $t_1,t_2\in(0,\infty)$ with $t_1<t_2$ such that $w'(t_1)=w'(t_2)=0$. Then we have that

$$\begin{aligned}&\left[\frac{\alpha\left(t_1^2|\nabla u|^2_{2,\Omega_1}\right)}{t_1^{2\gamma}}-\frac{\alpha\left(t_2^2|\nabla u|^2_{2,\Omega_1}\right)}{t_2^{2\gamma}}\right]|\nabla u|^2_{2,\Omega_1}+\left[\frac{\beta\left(t_1^2|\nabla v|^2_{2,\Omega_2}\right)}{t_1^{2\gamma}}-\frac{\beta\left(t_2^2|\nabla v|^2_{2,\Omega_2}\right)}{t_2^{2\gamma}}\right]|\nabla v|^2_{2,\Omega_2}\\ &\quad=\int_{\Omega_1}\left[\frac{\widetilde{f}(t_1u)}{t_1^{2\gamma+1}}-\frac{\widetilde{f}(t_2u)}{t_2^{2\gamma+1}}\right]u+\int_{\Omega_2}\left[\frac{\widetilde{h}(t_1v)}{t_1^{2\gamma+1}}-\frac{\widetilde{h}(t_2v)}{t_2^{2\gamma+1}}\right]v\\ &\qquad+\lambda\int_{\Omega_1}\left[\frac{\widetilde{\varphi}(t_1u)}{t_1^{2\gamma+1}}-\frac{\widetilde{\varphi}(t_2u)}{t_2^{2\gamma+1}}\right]u+\lambda\int_{\Omega_2}\left[\frac{\widetilde{\psi}(t_1v)}{t_1^{2\gamma+1}}-\frac{\widetilde{\psi}(t_2v)}{t_2^{2\gamma+1}}\right]v,\end{aligned}$$

which is absurd in view of (A_1), (F'_2), and $t_1<t_2$.

(ii) Let us define a function $W(t,\lambda)=\langle I'_\lambda(t(u,v)),(u,v)\rangle$ for $(t,\lambda)\in(-1,\infty)^2$. Then $W(t(\lambda),\lambda)=0$ for $\lambda\in\mathbb{R}_+$ and by calculation we know that for $(t,\lambda)\in(-1,\infty)^2$,

$$\begin{aligned}\frac{\partial W}{\partial t}(t,\lambda) &= \alpha\left(t^2|\nabla u|^2_{2,\Omega_1}\right)|\nabla u|^2_{2,\Omega_1}+\beta\left(t^2|\nabla v|^2_{2,\Omega_2}\right)|\nabla v|^2_{2,\Omega_2}\\ &\quad+2t^2\left[\alpha'\left(t^2|\nabla u|^2_{2,\Omega_1}\right)|\nabla u|^4_{2,\Omega_1}+\beta'\left(t^2|\nabla v|^2_{2,\Omega_2}\right)|\nabla v|^4_{2,\Omega_2}\right]\\ &\quad-\int_{\Omega_1}\widetilde{f}'(tu)u^2-\int_{\Omega_2}\widetilde{h}'(tv)v^2-\lambda\left[\int_{\Omega_1}\widetilde{\varphi}'(tu)u^2+\int_{\Omega_2}\widetilde{\psi}'(tv)v^2\right]\end{aligned}\tag{3.9}$$

and

$$\frac{\partial W}{\partial\lambda}(t,\lambda)=-\int_{\Omega_1}\widetilde{\varphi}(tu)u-\int_{\Omega_2}\widetilde{\psi}(tv)v.$$

Moreover, it follows from (3.9), (3.8), (2.2), and (2.4) that for $\lambda \in \mathbb{R}_+$,

$$\begin{aligned}
\frac{\partial W}{\partial t}(t(\lambda),\lambda) &= \frac{\partial W}{\partial t}(t(\lambda),\lambda) - \frac{2\gamma+1}{t(\lambda)}W(t(\lambda),\lambda) \\
&= -2\gamma\left[\alpha\left(t^2(\lambda)|\nabla u|_{2,\Omega_1}^2\right)|\nabla u|_{2,\Omega_1}^2 + \beta\left(t^2(\lambda)|\nabla v|_{2,\Omega_2}^2\right)|\nabla v|_{2,\Omega_2}^2\right] \\
&\quad + 2t^2\left[\alpha'\left(t^2|\nabla u|_{2,\Omega_1}^2\right)|\nabla u|_{2,\Omega_1}^4 + \beta'\left(t^2|\nabla v|_{2,\Omega_2}^2\right)|\nabla v|_{2,\Omega_2}^4\right] \\
&\quad + \frac{1}{t(\lambda)}\int_{\Omega_1}\left[(2\gamma+1)\widetilde{f}(t(\lambda)u) - \widetilde{f}'(t(\lambda)u)t(\lambda)u\right]u \\
&\quad + \frac{1}{t(\lambda)}\int_{\Omega_2}\left[(2\gamma+1)\widetilde{h}(t(\lambda)v) - \widetilde{h}'(t(\lambda)u)t(\lambda)v\right]v \\
&\quad + \frac{\lambda}{t(\lambda)}\int_{\Omega_1}\left[(2\gamma+1)\widetilde{\phi}(t(\lambda)u) - \widetilde{\phi}'(t(\lambda)u)t(\lambda)u\right]u \\
&\quad + \frac{\lambda}{t(\lambda)}\int_{\Omega_2}\left[(2\gamma+1)\widetilde{\psi}(t(\lambda)v) - \widetilde{\psi}'(t(\lambda)u)t(\lambda)v\right]v \\
&< 0.
\end{aligned}$$

Hence, the implicit function theorem and (i) imply that $t(\cdot) : \mathbb{R}_+ \to (0,\infty)$ is continuously differentiable and (3.7) holds. Particularly, recall that $\phi(s)s > 0$ and $\psi(s)s > 0$ for $s \neq 0$, so $t'(\lambda) < 0$ for $\lambda \in \mathbb{R}_+$. Thus, $t(\cdot)$ is decreasing on $\mathbb{R}_+$. □

Lemma 3.5. *For each $\mu > 0$, it holds that $\rho_\mu := \inf_{\lambda\in[0,\mu]} \mathrm{dist}(0, N_\lambda) > 0$.*

Proof. Let $\lambda \in [0,\mu]$ and $(u,v) \in N_\lambda$. Then for each $\varepsilon \in \left(0, \left[(1+\mu)\nu_2^2\right]^{-1}\min\{\alpha(0),\beta(0)\}\right)$, it follows from (3.6), (3.1), and (3.2) with $p = 5$, and the Sobolev embedding theorem that

$$\begin{aligned}
&\min\{\alpha(0),\beta(0)\}\|(u,v)\|_E^2 \\
&\quad\leqslant \alpha(0)|\nabla u|_{2,\Omega_1}^2 + \beta(0)|\nabla v|_{2,\Omega_2}^2 \\
&\quad\leqslant \alpha\left(|\nabla u|_{2,\Omega_1}^2\right)|\nabla u|_{2,\Omega_1}^2 + \beta\left(|\nabla v|_{2,\Omega_2}^2\right)|\nabla v|_{2,\Omega_2}^2 \\
&\quad= \int_{\Omega_1}\widetilde{f}(u)u + \int_{\Omega_2}\widetilde{h}(v)v + \lambda\left[\int_{\Omega_1}\widetilde{\phi}(u)u + \int_{\Omega_2}\widetilde{\psi}(v)v\right] \\
&\quad\leqslant (1+\lambda)\varepsilon\left[|u|_{2,\Omega_1}^2 + |v|_{2,\Omega_2}^2\right] + (\varepsilon+\lambda C_\varepsilon)\left[|u|_{6,\Omega_1}^6 + |v|_{6,\Omega_2}^6\right] + C_{\varepsilon,p}\left[|u|_{p,\Omega_1}^p + |v|_{p,\Omega_2}^p\right] \\
&\quad\leqslant (1+\mu)\varepsilon\nu_2^2\|(u,v)\|_E^2 + (\varepsilon+\lambda C_\varepsilon + C_{\varepsilon,6})\nu_6^6\|(u,v)\|_E^6.
\end{aligned}$$

Thus, there exists a positive number σ independent of λ such that $\|(u,v)\|_E \geqslant \sigma$ for $(u,v) \in N_\lambda$. Hence, $\rho_\mu \geqslant \sigma$. □

Subsequently, we will obtain a minimax characterization of $d(\cdot)$ given by the following lemma.

Lemma 3.6. $d(\lambda) = c(\lambda) = \inf_{(u,v)\in E\setminus\{0\}} \max_{t\in\mathbb{R}_+} I_\lambda(t(u,v))$ *for $\lambda \in \mathbb{R}_+$.*

This lemma can be achieved from (i) of Lemma 3.4 and Lemma 3.1. Here we omit the proof, and for the concrete process readers can refer Lemma 3.6 in [16].

According to the above lemma, since $c(\cdot)$ is nonincreasing on $\mathbb{R}_+$, we know that $d(\cdot)$ is nonincreasing on $\mathbb{R}_+$ and $d(\lambda) \leqslant d(0)$ for $\lambda \in \mathbb{R}_+$. Similarly, to establish the right continuity of $c(\cdot)$ at $\lambda = 0$, it suffices to prove that $d(\cdot)$ is continuous at $\lambda = 0$ from the right.

Lemma 3.7. $\lim_{\lambda\to 0} d(\lambda) = d(0)$.

Proof. Let $\{\lambda_n\} \subset (0, \mu]$ satisfy $\lambda_n \to 0$ as $n \to \infty$. Then for any given $\varepsilon \in (0, d(0))$ it follows from the definition of $d(\lambda_n)$ that there exists $(u_n, v_n) \in N_{\lambda_n}$ such that for all n,

$$I_{\lambda_n}(u_n, v_n) \leqslant d(\lambda_n) + \varepsilon. \tag{3.10}$$

We note that as in (3.3), for fixed $\lambda \in [0, \mu]$,

$$\frac{\gamma}{2\gamma+2}\min\{\alpha(0), \beta(0)\}\|(u,v)\|_E^2 \leqslant I_\lambda(u,v), \quad (u,v) \in N_\lambda.$$

Then it follows from (3.10) that for all n,

$$\|(u_n, v_n)\|_E^2 \leqslant \frac{2\gamma+2}{\gamma\min\{\alpha(0),\beta(0)\}}(d(\lambda_n)+\varepsilon) < \frac{4(\gamma+1)d(0)}{\gamma\min\{\alpha(0),\beta(0)\}}.$$

Hence, there exist $(u, v) \in E$ and a subsequence of $\{(u_n, v_n)\}$, still denoted by $\{(u_n, v_n)\}$, satisfying that $(u_n, v_n) \rightharpoonup (u, v)$. Particularly, it holds that $(u, v) \neq 0$. Otherwise, by (1.13), (3.4), and the fact that $\lambda_n \to 0$, one can conclude that

$$\begin{aligned}
&\min\{\alpha(0),\beta(0)\}\|(u_n,v_n)\|_E^2\\
&\quad\leqslant \alpha\left(|\nabla u_n|_{2,\Omega_1}^2\right)|\nabla u_n|_{2,\Omega_1}^2 + \beta\left(|\nabla v_n|_{2,\Omega_2}^2\right)|\nabla v_n|_{2,\Omega_2}^2\\
&\quad= \int_{\Omega_1}\widetilde{f}(u_n)u_n + \int_{\Omega_2}\widetilde{h}(v_n)v_n + \lambda_n\left[\int_{\Omega_1}\widetilde{\varphi}(u_n)u_n + \int_{\Omega_2}\widetilde{\psi}(v_n)v_n\right] \to 0.
\end{aligned}$$

This contradicts the fact that $\{\|(u_n, v_n)\|_E\}$ has a positive lower bound which can be derived from Lemma 3.5.

For $(u_n, v_n) \in N_{\lambda_n}$ chosen above, by $\langle I_0'(u_n, v_n), (u_n, v_n)\rangle > \langle I_{\lambda_n}'(u_n, v_n), (u_n, v_n)\rangle = 0$ and (i) of Lemma 3.4, there exists a unique $t_n(0) > 1$ such that $t_n(0)(u_n, v_n) \in N_0$. Therefore,

$$0 \leqslant d(0) - d(\lambda_n) \leqslant I_0(t_n(0)(u_n, v_n)) - I_{\lambda_n}(u_n, v_n) + \varepsilon. \tag{3.11}$$

It follows from Lemma 3.4 that there exists $t_n(\lambda) > 0$ such that $t_n(\lambda)(u_n, v_n) \in N_\lambda$. Let us define $g_n(\lambda) = I_\lambda(t_n(\lambda)(u_n, v_n))$ for $\lambda \in \mathbb{R}_+$. Then the fact $t_n(\lambda)(u_n, v_n) \in N_\lambda$ implies that

$$\begin{aligned}
g_n'(\lambda) &= \langle I_\lambda'(t_n(\lambda)(u_n,v_n)), (u_n,v_n)\rangle t_n'(\lambda) - \left[\int_{\Omega_1}\widetilde{\Phi}(t_n(\lambda)u_n) + \int_{\Omega_2}\widetilde{\Psi}(t_n(\lambda)v_n)\right]\\
&= -\left[\int_{\Omega_1}\widetilde{\Phi}(t_n(\lambda)u_n) + \int_{\Omega_2}\widetilde{\Psi}(t_n(\lambda)v_n)\right], \qquad \lambda \in \mathbb{R}_+.
\end{aligned}$$

Thus, it follows from (ii) of Lemma 3.4 that

$$\begin{aligned}
&I_0(t_n(0)(u_n,v_n)) - I_{\lambda_n}(u_n,v_n)\\
&\quad= g_n(0) - g_n(\lambda_n)\\
&\quad= -\int_0^{\lambda_n} g_n'(s)\mathrm{d}s\\
&\quad= \int_0^{\lambda_n}\left[\int_{\Omega_1}\widetilde{\Phi}(t_n(s)u_n) + \int_{\Omega_2}\widetilde{\Psi}(t_n(s)v_n)\right]\mathrm{d}s\\
&\quad\leqslant \lambda_n\left(t_n^2(0)\left[|u_n|_{2,\Omega_1}^2 + |v_n|_{2,\Omega_2}^2\right] + C_\varepsilon t_n^6(0)\left[|u_n|_{6,\Omega_1}^6 + |v_n|_{6,\Omega_2}^6\right]\right).
\end{aligned} \tag{3.12}$$

By (3.11), (3.12), and the Sobolev embedding theorem, to establish that $d(\lambda) \to d(0)$ as $\lambda \to 0$, it suffices to prove that $\{t_n(0)\}$ is bounded. We assume toward a contradiction that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $s_i := t_{n_i}(0) \to \infty$ as $i \to \infty$. Then by the fact that $t_{n_i}(0)(u_{n_i}, v_{n_i}) \in N_0$ for all i and (1.13), we have that

$$\frac{\alpha\left(s_i^2|\nabla u_{n_i}|^2_{2,\Omega_1}|\right)}{s_i^{2\gamma}}|\nabla u_{n_i}|^2_{2,\Omega_1} + \frac{\beta\left(s_i^2|\nabla v_{n_i}|^2_{2,\Omega_2}|\right)}{s_i^{2\gamma}}|\nabla v_{n_i}|^2_{2,\Omega_2} = \int_{\Omega_1}\frac{\widetilde{f}(s_i u_{n_i})}{s_i^{2\gamma+1}}u_{n_i} + \int_{\Omega_2}\frac{\widetilde{h}(s_i v_{n_i})}{s_i^{2\gamma+1}}v_{n_i}. \quad (3.13)$$

Moreover, it follows from $(u_{n_i}, v_{n_i}) \rightharpoonup (u, v) \neq 0$ and Lemma 2.7 that the right-hand side of (3.13) converges to infinity. This contradicts the fact that the limit superior of the left-hand side is finite by (A_1). Hence, $\{t_n(0)\}$ is bounded. The proof is complete. □

We now establish the existence of ground-state solutions to the problem (1.1). Motivated by [16,32], we first study the distance between any $(PS)_{c(\lambda)}$ sequence of I_λ and a compact set $K_{c(0)}$ defined in (3.5). Here, the existence of a $(PS)_{c(\lambda)}$ sequence can be derived from Lemma 3.1 and a general minimax principle [28, Theorem 2.8, p. 41]. The compactness of $K_{c(0)}$ follows directly from the fact that I_0 satisfies the PS condition.

Lemma 3.8. *For each $\lambda \in \mathbb{R}_+$, let $\{(u_n^\lambda, v_n^\lambda)\}$ be any $(PS)_{c(\lambda)}$ sequence of I_λ. Then*

$$\lim_{\lambda\to 0}\limsup_{n\to\infty} \operatorname{dist}\left((u_n^\lambda, v_n^\lambda), K_{c(0)}\right) = 0.$$

Proof. It just needs to repeat the proof of Lemma 3.8 in [16]. □

Finally, we prove Theorem 1.7.

Proof of Theorem 1.7. For each $\lambda \in \mathbb{R}_+$, let $\{(u_n^\lambda, v_n^\lambda)\}$ be a $(PS)_{c(\lambda)}$ sequence of I_λ. We note that $\{(u_n^\lambda, v_n^\lambda)\}$ is bounded by Lemma 3.2. Then there exist a subsequence of $\{(u_n^\lambda, v_n^\lambda)\}$, still denoted by $\{(u_n^\lambda, v_n^\lambda)\}$, and $(u_\lambda, v_\lambda) \in E$ such that $(u_n^\lambda, v_n^\lambda) \rightharpoonup (u_\lambda, v_\lambda)$ as $n \to \infty$. We will try to find a $\lambda_0 > 0$ such that $(u_\lambda, v_\lambda) \neq 0$ for $\lambda \in [0, \lambda_0)$. In fact, since $c(0) \neq 0$ and $K_{c(0)}$ is compact, it holds that

$$\delta_0 := \operatorname{dist}(0, K_{c(0)}) = \min_{(u,v)\in K_{c(0)}} \|(u, v)\|_E > 0.$$

Moreover, according to Lemma 3.8, for any given $\varepsilon_0 \in [0, \delta_0)$, there exists a $\lambda_0 = \lambda(\varepsilon_0)$ such that when $\lambda \in (0, \lambda_0)$, there is some $n_\lambda := n(\lambda)$ such that $\operatorname{dist}((u_n^\lambda, v_n^\lambda), K_{c(0)}) \leqslant \varepsilon_0$ for all $n > n_\lambda$. Fixing $\lambda \in (0, \lambda_0)$ and by the compactness of $K_{c(0)}$, there exists a sequence $\{(w_n^\lambda, z_n^\lambda)\} \subset K_{c(0)}$ such that $\|(u_n^\lambda, v_n^\lambda) - (w_n^\lambda, z_n^\lambda)\|_E \leqslant \varepsilon_0$ for all $n > n_\lambda$. Furthermore, for a subsequence of $\{(w_n^\lambda, z_n^\lambda)\}$, still denoted by $\{(w_n^\lambda, z_n^\lambda)\}$, and some $(w_\lambda, z_\lambda) \in K_{c(0)}$, it holds that $(w_n^\lambda, z_n^\lambda) \to (w_\lambda, z_\lambda)$ as $n \to \infty$. Hence, we have that $(u_n^\lambda, v_n^\lambda) \in B_{\varepsilon_0}(w_\lambda, z_\lambda)$ for sufficiently large n. Thus, $(u_\lambda, v_\lambda) \in \overline{B}_{\varepsilon_0}(w_\lambda, z_\lambda)$ because $\overline{B}_{\varepsilon_0}(w_\lambda, z_\lambda)$ is weakly closed. Therefore, $\|(u_\lambda, v_\lambda)\|_E \geqslant \|(w_\lambda, z_\lambda)\|_E - \varepsilon_0 \geqslant \delta_0 - \varepsilon_0 > 0$, that is, $(u_\lambda, v_\lambda) \neq 0$.

We now prove that $I'_\lambda(u_\lambda, v_\lambda) = 0$ and $I_\lambda(u_\lambda, v_\lambda) = d(\lambda)$, that is, (u_λ, v_λ) is a ground-state solution to the problem (1.1). Without loss of generality, we may assume that the sequence $\{(u_n^\lambda, v_n^\lambda)\}$ satisfies that $\left(|\nabla u_n^\lambda|^2_{2,\Omega_1}, |\nabla v_n^\lambda|^2_{2,\Omega_2}\right) \to (a, b)$ as $n \to \infty$ for some $(a, b) \in \mathbb{R}^2_+ \setminus \{0\}$. For all $(u, v) \in E$, let

$$I_{\lambda,(a,b)}(u, v) = \frac{1}{2}\left[\alpha(a)|\nabla u|^2_{2,\Omega_1} + \beta(b)|\nabla v|^2_{2,\Omega_2}\right] - \int_{\Omega_1}\widetilde{F}(u) - \int_{\Omega_2}\widetilde{H}(v) - \lambda\left[\int_{\Omega_1}\widetilde{\Phi}(u) + \int_{\Omega_1}\widetilde{\Psi}(v)\right].$$

Then $I'_{\lambda,(a,b)}(u_n^\lambda, v_n^\lambda) \to 0$. Hence, $I'_{\lambda,(a,b)}(u_\lambda, v_\lambda) = 0$. We claim that $\left(|\nabla u_\lambda|^2_{2,\Omega_1}, |\nabla v_\lambda|^2_{2,\Omega_2}\right) = (a,b)$. In fact, it would follow from $(u_n^\lambda, v_n^\lambda) \rightharpoonup (u_\lambda, v_\lambda)$ that $|\nabla u_\lambda|^2_{2,\Omega_1} \leqslant a$ and $|\nabla v_\lambda|^2_{2,\Omega_2} \leqslant b$, and then

$$\langle I'_\lambda(u_\lambda, v_\lambda), (u_\lambda, v_\lambda)\rangle \leqslant \langle I'_{\lambda,(a,b)}(u_\lambda, v_\lambda), (u_\lambda, v_\lambda)\rangle = 0.$$

Since $(u_\lambda, v_\lambda) \neq 0$, which is obtained on the above paragraph, by (i) of Lemma 3.4 there exists a unique $t(\lambda) \in (0,1]$ such that $t(\lambda)(u_\lambda, v_\lambda) \in N_\lambda$. Furthermore, the monotonicity obtained in Lemma 2.1 and Remark 2.5, the weak lower continuity of norm, Fatou's lemma, and the choice of $\{(u_n^\lambda, v_n^\lambda)\}$ imply that

$$\begin{aligned}
d(\lambda) &\leqslant I_\lambda(t(\lambda)(u_\lambda, v_\lambda)) - \frac{1}{2\gamma+2}\langle I'_\lambda(t(\lambda)(u_\lambda, v_\lambda)), t(\lambda)(u_\lambda, v_\lambda)\rangle \\
&= \frac{1}{2}A\left(|t(\lambda)\nabla u_\lambda|^2_{2,\Omega_1}\right) - \frac{1}{2\gamma+2}\alpha\left(|t(\lambda)\nabla u_\lambda|^2_{2,\Omega_1}\right)|t(\lambda)\nabla u_\lambda|^2_{2,\Omega_1} \\
&\quad + \frac{1}{2}B\left(|t(\lambda)\nabla v_\lambda|^2_{2,\Omega_2}\right) - \frac{1}{2\gamma+2}\beta\left(|t(\lambda)\nabla v_\lambda|^2_{2,\Omega_2}\right)|t(\lambda)\nabla v_\lambda|^2_{2,\Omega_2} \\
&\quad + \int_{\Omega_1}\left[\frac{1}{2\gamma+2}\widetilde{f}(t(\lambda)u_\lambda)t(\lambda)u_\lambda - \widetilde{F}(t(\lambda)u_\lambda)\right] \\
&\quad + \int_{\Omega_2}\left[\frac{1}{2\gamma+2}\widetilde{h}(t(\lambda)v_\lambda)t(\lambda)v_\lambda - \widetilde{H}(t(\lambda)v_\lambda)\right] \\
&\quad + \lambda\int_{\Omega_1}\left[\frac{1}{2\gamma+2}\widetilde{\varphi}(t(\lambda)u_\lambda)t(\lambda)u_\lambda - \widetilde{\Phi}(t(\lambda)u_\lambda)\right] \\
&\quad + \lambda\int_{\Omega_2}\left[\frac{1}{2\gamma+2}\widetilde{\psi}(t(\lambda)v_\lambda)t(\lambda)v_\lambda - \widetilde{\Psi}(t(\lambda)v_\lambda)\right] \\
&\leqslant \frac{1}{2}A\left(|\nabla u_\lambda|^2_{2,\Omega_1}\right) - \frac{1}{2\gamma+2}\alpha\left(|\nabla u_\lambda|^2_{2,\Omega_1}\right)|\nabla u_\lambda|^2_{2,\Omega_1} \\
&\quad + \frac{1}{2}B\left(|\nabla v_\lambda|^2_{2,\Omega_2}\right) - \frac{1}{2\gamma+2}\beta\left(|\nabla v_\lambda|^2_{2,\Omega_2}\right)|\nabla v_\lambda|^2_{2,\Omega_2} \\
&\quad + \int_{\Omega_1}\left[\frac{1}{2\gamma+2}\widetilde{f}(u_\lambda)u_\lambda - \widetilde{F}(u_\lambda)\right] + \int_{\Omega_2}\left[\frac{1}{2\gamma+2}\widetilde{h}(v_\lambda)v_\lambda - \widetilde{H}(v_\lambda)\right] \\
&\quad + \lambda\int_{\Omega_1}\left[\frac{1}{2\gamma+2}\widetilde{\varphi}(u_\lambda)u_\lambda - \widetilde{\Phi}(u_\lambda)\right] + \lambda\int_{\Omega_2}\left[\frac{1}{2\gamma+2}\widetilde{\psi}(v_\lambda)v_\lambda - \widetilde{\Psi}(v_\lambda)\right] \\
&\leqslant \liminf_{n\to\infty}\left[\frac{1}{2}A\left(|\nabla u_n^\lambda|^2_{2,\Omega_1}\right) - \frac{1}{2\gamma+2}\alpha\left(|\nabla u_n^\lambda|^2_{2,\Omega_1}\right)|\nabla u_n^\lambda|^2_{2,\Omega_1}\right] \\
&\quad + \liminf_{n\to\infty}\left[\frac{1}{2}B\left(|\nabla v_n^\lambda|^2_{2,\Omega_2}\right) - \frac{1}{2\gamma+2}\beta\left(|\nabla v_n^\lambda|^2_{2,\Omega_2}\right)|\nabla v_n^\lambda|^2_{2,\Omega_2}\right] \\
&\quad + \liminf_{n\to\infty}\int_{\Omega_1}\left[\frac{1}{2\gamma+2}\widetilde{f}(u_n^\lambda)u_n^\lambda - \widetilde{F}(u_n^\lambda)\right] + \liminf_{n\to\infty}\int_{\Omega_2}\left[\frac{1}{2\gamma+2}\widetilde{h}(v_n^\lambda)v_n^\lambda - \widetilde{H}(v_n^\lambda)\right] \\
&\quad + \lambda\liminf_{n\to\infty}\int_{\Omega_1}\left[\frac{1}{2\gamma+2}\widetilde{\varphi}(u_n^\lambda)u_n^\lambda - \widetilde{\Phi}(u_n^\lambda)\right] + \lambda\liminf_{n\to\infty}\int_{\Omega_2}\left[\frac{1}{2\gamma+2}\widetilde{\psi}(v_n^\lambda)v_n^\lambda - \widetilde{\Psi}(v_n^\lambda)\right] \\
&\leqslant \lim_{n\to\infty}\left[I_\lambda(u_n^\lambda, v_n^\lambda) - \frac{1}{2\gamma+2}\langle I'_\lambda(u_n^\lambda, v_n^\lambda), (u_n^\lambda, v_n^\lambda)\rangle\right] \\
&= c(\lambda) = d(\lambda).
\end{aligned}$$

Thus, there exists a subsequence $\{(u^\lambda_{n_i}, v^\lambda_{n_i})\}$ of $\{(u^\lambda_n, v^\lambda_n)\}$ such that

$$\begin{aligned}
&\frac{1}{2}A\left(|\nabla u_\lambda|^2_{2,\Omega_1}\right) - \frac{1}{2\gamma+2}\alpha\left(|\nabla u_\lambda|^2_{2,\Omega_1}\right)|\nabla u_\lambda|^2_{2,\Omega_1}\\
&\quad = \lim_{i\to\infty}\left[\frac{1}{2}A\left(|\nabla u^\lambda_{n_i}|^2_{2,\Omega_1}\right) - \frac{1}{2\gamma+2}\alpha\left(|\nabla u^\lambda_{n_i}|^2_{2,\Omega_1}\right)|\nabla u^\lambda_{n_i}|^2_{2,\Omega_1}\right]\\
&\quad = \frac{1}{2}A(a) - \frac{1}{2\gamma+2}\alpha(a)a
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}B\left(|\nabla v_\lambda|^2_{2,\Omega_2}\right) - \frac{1}{2\gamma+2}\beta\left(|\nabla v_\lambda|^2_{2,\Omega_2}\right)|\nabla v_\lambda|^2_{2,\Omega_2}\\
&\quad = \lim_{i\to\infty}\left[\frac{1}{2}B\left(|\nabla v^\lambda_{n_i}|^2_{2,\Omega_2}\right) - \frac{1}{2\gamma+2}\beta\left(|\nabla v^\lambda_{n_i}|^2_{2,\Omega_2}\right)|\nabla v^\lambda_{n_i}|^2_{2,\Omega_2}\right]\\
&\quad = \frac{1}{2}B(b) - \frac{1}{2\gamma+2}B(b)b.
\end{aligned}$$

It follows from the monotonicity of $(\gamma+1)A(s)-\alpha(s)s$ and $(\gamma+1)B(s)+\beta(s)s$ that $(|\nabla u_\lambda|^2_{2,\Omega_1}, |\nabla v_\lambda|^2_{2,\Omega_2}) = (a,b)$ holds, and then $(u^\lambda_n, v^\lambda_n) \to (u_\lambda, v_\lambda)$ in E. Moreover, since I_λ is continuously differentiability, one can also conclude that $I'_\lambda(u_\lambda, v_\lambda) = 0$ and $I_\lambda(u_\lambda, v_\lambda) = c(\lambda) = d(\lambda)$. Thus, (u_λ, v_λ) is a ground-state solution to the problem (1.1).

Finally, we will end the proof of Theorem 1.7 by proving that $(u_\lambda, v_\lambda) \to (u_0, v_0)$ as $\lambda \to 0$, where (u_0, v_0) is a ground-state solution to (1.1) with $\lambda = 0$. Actually, let $\{\lambda_n\} \subset [0, \lambda_0)$ such that $\lambda_n \to 0$ as $n \to \infty$. Then as a consequence of the fact that $(u_{\lambda_n}, v_{\lambda_n})$ is a ground-station solution to (1.1) with $\lambda = \lambda_n$, it hold that $I'_{\lambda_n}(u_{\lambda_n}, v_{\lambda_n}) = 0$ and $I_{\lambda_n}(u_{\lambda_n}, v_{\lambda_n}) = c(\lambda_n)$. By Lemma 3.7, similar to (3.3), we have that as $n \to \infty$,

$$\begin{aligned}
c(0) + o(1) &= c(\lambda_n)\\
&= I_{\lambda_n}(u_{\lambda_n}, v_{\lambda_n}) - \frac{1}{2\gamma+2}\langle I'_{\lambda_n}(u_{\lambda_n}, v_{\lambda_n}), (u_{\lambda_n}, v_{\lambda_n})\rangle\\
&\geqslant \frac{\gamma}{2\gamma+2}\min\{\alpha(0), \beta(0)\}\|(u_{\lambda_n}, v_{\lambda_n})\|^2_E.
\end{aligned}$$

This yields that $\{(u_{\lambda_n}, v_{\lambda_n})\}$ is bounded in E. Hence, it follows from the Sobolev embedding theorem that as $n \to \infty$,

$$I_0(u_{\lambda_n}, v_{\lambda_n}) = I_{\lambda_n}(u_{\lambda_n}, v_{\lambda_n}) + \lambda_n\left[\int_{\Omega_1}\widetilde{\Phi}(u_{\lambda_n}) + \int_{\Omega_1}\widetilde{\Psi}(v_{\lambda_n})\right] \to c(0),$$

and

$$\begin{aligned}
\left\langle I'_0(u_{\lambda_n}, v_{\lambda_n}), (w,z)\right\rangle &= \left\langle I'_{\lambda_n}(u_{\lambda_n}, v_{\lambda_n}), (w,z)\right\rangle + \lambda_n\left[\int_{\Omega_1}\widetilde{\varphi}(u_{\lambda_n})w + \int_{\Omega_1}\widetilde{\psi}(v_{\lambda_n})z\right]\\
&= \lambda_n\left[\int_{\Omega_1}\widetilde{\varphi}(u_{\lambda_n})w + \int_{\Omega_1}\widetilde{\psi}(v_{\lambda_n})z\right]\\
&= o(1)\|(w,z)\|_E, \qquad (w,z) \in E.
\end{aligned}$$

Thus, $\{(u_{\lambda_n}, v_{\lambda_n})\}$ is a $(\mathrm{PS})_{c(0)}$ sequence of I_0 in E, and then by Lemma 3.2, there exists a subsequence of $\{(u_{\lambda_n}, v_{\lambda_n})\}$, still denoted by $\{(u_{\lambda_n}, v_{\lambda_n})\}$, and $(u_0, v_0) \in E$ such that $(u_{\lambda_n}, v_{\lambda_n}) \to (u_0, v_0)$ and $I_0(u_0, v_0) = c(0)$, that is, (u_0, v_0) is a ground-state solution to (1.1) with $\lambda = 0$. The proof is complete. □

4 Appendix

In this section, some regular properties of compound functions will be proved.

Lemma 4.1. *Assume that* $f \in C^2(\mathbb{R})$ *and* $f', f'' \in L^\infty(\mathbb{R})$. *If* $u \in W^{2,p}(\Omega)$ *and* $2(N-p) < N$, *then* $f(u) \in W^{2,p}(\Omega)$ *and*

$$D^\alpha(f(u)) = \begin{cases} f'(u)D^\alpha u, & |\alpha| \leqslant 1, \\ f''(u)(D_1 u)^2 + f'(u)D^\alpha u, & \alpha = (2,0,0,\dots), \\ \dots \end{cases}$$

Proof. According to [27, Theorem 2.5.1, p. 70], we have that $f(u) \in W^{1,p}(\Omega)$ and $D(f(u)) = f'(u)Du$. When $|\alpha| \leqslant 1$, we have that $D^\alpha(f(u)) \in L^p(\Omega)$. Assume that $|\alpha| = 2$, without loss of generality, let $\alpha = (2,0,\dots,0)$. Then we could calculate $D_1(f(u))$ as follows. Actually, for any given $\phi \in C_0^\infty(\Omega)$, because $f(u) \in W^{1,p}(\Omega)$ and $D_1\phi \in C_0^\infty(\Omega)$, we have that

$$\int_\Omega f(u)D^\alpha\phi = -\int_\Omega D_1(f(u))D_1\phi = -\int_\Omega f'(u)D_1 u D_1\phi. \tag{4.1}$$

Let $g(s) = f'(s)$ for $s \in \mathbb{R}$. Since $f \in C^2(\mathbb{R})$ and $f'' \in L^\infty(\mathbb{R})$, then $g \in C^1(\mathbb{R})$ and $g' \in L^\infty(\mathbb{R})$. Thus, $g(u) \in W^{1,p}(\Omega)$. It follows from the weak derivative product formula that $g(u)D_1 u \in W^{1,p}(\Omega)$ and $D_1(g(u)D_1 u) = D_1(g(u))D_1 u + g(u)D^\alpha u = g'(u)(D_1 u)^2 + g(u)D^\alpha u = f''(u)(D_1 u)^2 + f'(u)D^\alpha u$. Moreover, (4.1) can be written

$$\int_\Omega f(u)D^\alpha\phi = -\int_\Omega (g(u)D_1 u)D_1\phi = \int_\Omega D_1(g(u)D_1 u)\phi = \int_\Omega \left[f''(u)(D_1 u)^2 + f'(u)D^\alpha u\right]\phi.$$

Thus, $D^\alpha(f(u)) = f''(u)(D_1 u)^2 + f'(u)D^\alpha u$.

Next, we prove $D^\alpha(f(u)) \in L^p(\Omega)$. In fact, because $f', f'' \in L^\infty(\mathbb{R}^N)$ and $D^\alpha u \in L^p(\Omega)$, we need only illustrate $(D_1 u)^2 \in L^p(\Omega)$, that is, $D_1 u \in L^{2p}(\Omega)$. In fact, since $D_1 u \in W^{1,p}(\Omega)$ and $2p < Np/(N-p)$, it follows from the Sobolev embedding theorem that $W^{1,p}(\Omega) \hookrightarrow L^{2p}(\Omega)$, and then $D_1 u \in L^{2p}(\Omega)$. The proof is complete. □

Lemma 4.2. *Assume that there exists* $M > 0$ *such that* $|g'(s)/g^3(s)| \leqslant M$ *for* $s \in \mathbb{R}$. *If* $u \in H^2(D)$, *then* $G^{-1}(u) \in H^2(D)$, *where* $D \subset \mathbb{R}^3$ *is an open domain with* $\partial D \in C^1$.

Proof. Let $f(s) = G^{-1}(s)$ for $s \in \mathbb{R}$. Then the conclusion holds by Lemma 4.1. □

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Precise Morrey regularity of the weak solutions to a kind of quasilinear systems with discontinuous data

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Abstract. We consider the Dirichlet problem for a class of quasilinear elliptic systems in domain with irregular boundary. The principal part satisfies componentwise coercivity condition and the nonlinear terms are Carathéodory maps having Morrey regularity in x and verifying controlled growth conditions with respect to the other variables. We have obtained boundedness of the weak solution to the problem that permits to apply an iteration procedure in order to find optimal Morrey regularity of its gradient.

Keywords: quasilinear elliptic systems, controlled growth conditions, componentwise coercivity, Reifenberg-flat domain, Morrey spaces.

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1 Introduction

We are interested in the regularity properties of a kind of quasilinear elliptic operators with discontinuous data acting in a bounded domain Ω, with irregular boundary $\partial\Omega$. Precisely, we consider the following Dirichlet problem

$$\begin{cases} \operatorname{div}\big(\mathbf{A}(x)D\mathbf{u} + \mathbf{a}(x,\mathbf{u})\big) = \mathbf{b}(x,\mathbf{u},D\mathbf{u}), & x \in \Omega \\ \mathbf{u}(x) = 0, & x \in \partial\Omega\,. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is a bounded *Reifenberg-flat domain*, the matrix $\mathbf{A} = \{A_{ij}^{\alpha\beta}(x)\}_{i,j\leq N}^{\alpha,\beta\leq n}$ of the coefficients is essentially bounded in Ω, and the non linear terms

$$\mathbf{a}(x,\mathbf{u}) = \{a_i^\alpha(x,\mathbf{u})\}_{i\leq N}^{\alpha\leq n} \quad \text{and} \quad \mathbf{b}(x,\mathbf{u},\mathbf{z}) = \{b_i(x,\mathbf{u},\mathbf{z})\}_{i\leq N}$$

are Carathéodory maps, i.e., they are measurable in $x \in \Omega$ for all $\mathbf{u} \in \mathbb{R}^N$, $\mathbf{z} \in \mathbb{M}^{N\times n}$ and continuous in $(\mathbf{u},\mathbf{z})$ for almost all $x \in \Omega$. Since we are going to study the weak solutions of

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(1.1) we need to impose *controlled growth conditions* on the nonlinear terms in order to ensure convergence of the integrals in the definition (2.6). For this aim we suppose that (cf. [17,33])

$$a_i^\alpha(x,\mathbf{u}) = \mathcal{O}(\varphi_1(x) + |\mathbf{u}|^{\frac{n}{n-2}}),$$
$$b_i(x,\mathbf{u},\mathbf{z}) = \mathcal{O}(\varphi_2(x) + |\mathbf{u}|^{\frac{n+2}{n-2}} + |\mathbf{z}|^{\frac{n+2}{n}})$$

for $n > 2$. In the particular case $n = 2$, the powers of $|\mathbf{u}|$ could be arbitrary positive numbers, while the growth of $|\mathbf{z}|$ is subquadratic.

Our aim is to study the dependence of the solution from the regularity of the data and to obtain Calderón–Zygmund type estimate in an optimal Morrey space.

There are various papers dealing with the integrability and regularity properties of different kind of quasilinear and nonlinear differential operators. Namely, it is studied the question *how the regularity of the data influences on the regularity of the solution.* In the scalar case $N = 1$ the celebrated result of De Giorgi and Nash asserts that *the weak solution of linear elliptic and parabolic equations with only L^∞ coefficients is Hölder continuous* [12].

Better integrability can be obtained also by the result of Gehring [16] relating to *functions satisfying the inverse Hölder inequality.* Later Giaquinta and Modica [18] noticed that certain power of the gradient of a function $u \in W^{1,p}$ satisfies locally the reverse Hölder inequality. Modifying Gehring's lemma they obtained *better integrability for the weak solutions of some quasilinear elliptic equations.* Their pioneer works have been followed by extensive research dedicated to the regularity properties of various partial differential operators using the Gehring–Giaquinta–Modica technique, called also a *"direct method"* (cf. [3, 27, 28] and the references therein.) Recently *the method of A-harmonic approximation* permits to study the regularity without using Gehring's lemma (see for example [1]).

The theory for linear divergence form operators defined in Reifenberg's domain was developed firstly in [8, 10]. In [4, 5] the authors extend this technique to quasilinear uniformly elliptic equations in the Sobolev–Morrey spaces. Making use of the Adams inequality [2] and the Hartmann–Stampacchia maximum principal they obtain Hölder regularity of the solution while in [7] it is obtained generalized Hölder regularity for regular and nonregular nonlinear elliptic equations.

Concerning nonlinear nonvariational operators we can mention the results of Campanato [11] related to basic systems of the form $F(D^2u) = 0$ in the Morrey spaces. Afterwards Marino and Maugeri in [24] have contributed to this theory with their own research on the boundary regularity of the solutions of basic systems. Imposing differentiability of the operator F they obtain, via immersion theorems, Morrey regularity of the second derivatives $D^2u \in L^{2,2-\frac{2}{q}}, q > 2$. These studies have been extended in [15] to nonlinear equations of a kind $F(x, D^2u)$ without any differentiability assumptions on F. It is obtained global Morrey regularity via the *Korn trick* and the *near operators theory* of Campanato. Moreover, in the variational case it is established a Caccioppoli-type inequality for a second-order degenerate elliptic systems of p-Laplacian type [14]. Exploiting the classical Campanato's approach and the *hole-filling technique* due to Widman, it is proved a global regularity result for the gradient of $\mathbf{u}$ in the Morrey and Lebesgue spaces.

In the present work we consider quasilinear systems in divergence form with a principal part satisfying *componentwise coercivity condition.* This condition permits to apply the results of [29,33] that gives L^∞ estimate of the weak solution. In addition the *controlled growth conditions* imposed on the nonlinear terms allow to apply the integrability result from [31]. Making use of step-by-step technique we show optimal Morrey regularity of the gradient depending explicitly on the regularity of the data.

In what follows we use the standard notation:

- Ω is a bounded domain in $\mathbb{R}^n$, with a Lebesgue measure $|\Omega|$ and boundary $\partial\Omega$;
- $\mathcal{B}_\rho(x) \subset \mathbb{R}^n$ is a ball, $\Omega_\rho(x) = \Omega \cap \mathcal{B}_\rho(x)$ with $\rho \in (0, \operatorname{diam}\Omega]$, $x \in \Omega$;
- $\mathbb{M}^{N\times n}$ is the set of $N \times n$-matrices;
- $\mathbf{u} = (u^1, \dots, u^N) : \Omega \to \mathbb{R}^N$, $\quad D_\alpha u^j = \partial u^j / \partial x_\alpha$,

  $$|\mathbf{u}|^2 = \sum_{j\le N} |u^j|^2, \qquad D\mathbf{u} = \{D_\alpha u^j\}_{j\le N}^{\alpha\le n} \in \mathbb{M}^{N\times n}, \qquad |D\mathbf{u}|^2 = \sum_{\substack{\alpha\le n\\ j\le N}} |D_\alpha u^j|^2;$$
- For $\mathbf{u} \in L^p(\Omega;\mathbb{R}^N)$ we write $\|\mathbf{u}\|_{p,\Omega}$ instead of $\|\mathbf{u}\|_{L^p(\Omega;\mathbb{R}^N)}$;
- The spaces $W^{1,p}(\Omega;\mathbb{R}^N)$ and $W_0^{1,p}(\Omega;\mathbb{R}^N)$ are the classical Sobolev spaces as they are defined in [19].

Throughout the paper the standard summation convention on repeated upper and lower indexes is adopted. The letter C is used for various positive constants and may change from one occurrence to another.

2 Definitions and auxiliary results

In [34] Reifenberg introduced a class of domains with rough boundary that can be approximated locally by hyperplanes.

Definition 2.1. The domain Ω is (δ, R) Reifenberg-flat if there exist positive constants R and $\delta < 1$ such that for each $x \in \partial\Omega$ and each $\rho \in (0, R)$ there is a local coordinate system $\{y_1, \dots, y_n\}$ with the property

$$\mathcal{B}_\rho(x) \cap \{y_n > \delta\rho\} \subset \Omega_\rho(x) \subset \mathcal{B}_\rho(x) \cap \{y_n > -\delta\rho\}. \tag{2.1}$$

Reifenberg arrived at this concept of flatness in his studies on the Plateau problem in higher dimensions and he proved that such a domain is locally a topological disc when δ is small enough, say $\delta < 1/8$. It is easy to see that a C^1-domain is a Reifenberg flat with $\delta \to 0$ as $R \to 0$. A domain with Lipschitz boundary with a Lipschitz constant less than δ also verifies the condition (2.1) if δ is small enough, say $\delta < 1/8$, (see [10, Lemma 5.1]). But the class of Reifenberg's domains is much more wider and contains domains with fractal boundaries. For instance, consider a self-similar snowflake S_β. It is a flat version of the Koch snowflake $S_{\pi/3}$ but with angle of the spike β such that $\sin\beta \in (0, 1/8)$. This kind of flatness exhibits minimal geometrical conditions necessary for some natural properties from the analysis and potential theory to hold. For more detailed overview of these domains we refer the reader to [35] (see also [8, 27] and the references therein).

In addition (2.1) implies the (A)-property (cf. [17, 28]). Precisely, there exists a positive constant $A(\delta) < 1/2$ such that

$$A(\delta)|\mathcal{B}_\rho(x)| \le |\Omega_\rho(x)| \le (1 - A(\delta))|\mathcal{B}_\rho(x)| \tag{A}$$

for any fixed $x \in \partial\Omega$, $\rho \in (0, R)$ and $\delta \in (0, 1)$. This condition excludes that Ω may have sharp outward and inward cusps. As consequence, the Reifenberg domain is $W^{1,p}$-extension domain, $1 \le p \le \infty$, hence the usual extension theorems, the Sobolev and Sobolev-Poincaré inequalities are still valid in Ω up to the boundary.

Definition 2.2. A real valued function $f \in L^p(\Omega)$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ with $p \in [1,\infty)$, $\lambda \in (0,n)$, if

$$\|f\|_{p,\lambda;\Omega} = \left(\sup_{\mathcal{B}_\rho(x)} \frac{1}{\rho^\lambda} \int_{\Omega_\rho(x)} |f(y)|^p \, dy \right)^{1/p} < \infty$$

where $\mathcal{B}_\rho(x)$ ranges in the set of all balls with radius $\rho \in (0, \operatorname{diam}\Omega]$ and $x \in \Omega$.

In [25] Morrey obtained local Hölder regularity of the solutions to second order elliptic equations. His new approach consisted in estimating the growth of the integral function $g(\rho) = \int_{\mathcal{B}_\rho} |Du(y)|^p dy$ via a power of the radius of the same ball, i.e., $C\rho^\lambda$ with $\lambda > 0$. Although he did not talk about function spaces, his paper is considered as the starting point for the theory of the *Morrey spaces* $L^{p,\lambda}$.

The family of the $L^{p,\lambda}$ spaces is partially ordered (cf. [30]).

Lemma 2.3. *For $1 \le r' \le r'' < \infty$ and $\sigma', \sigma'' \in [0,n)$ the following embedding holds*

$$L^{r''\sigma''}(\Omega) \hookrightarrow L^{r',\sigma'}(\Omega) \quad \textit{iff} \quad \frac{n-\sigma'}{r'} \ge \frac{n-\sigma''}{r''} .$$

Furthermore, we have the continuous inclusion

$$L^{\frac{nr'}{n-\sigma'}}(\Omega) \hookrightarrow L^{r',\sigma'}(\Omega) .$$

For $x \in \mathbb{R}^n$, I_α is the *Riesz potential operator* whose convolution kernel is $|x|^{\alpha-n}$, $0 < \alpha < n$. Suppose that f is extended as zero in $\mathbb{R}^n$ and consider its Riesz potential

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

In [2] Adams obtained the following inequality.

Lemma 2.4. *Let $f \in L^{r,\sigma}(\mathbb{R}^n)$, then $I_\alpha : L^{r,\sigma} \to L^{r^*_\sigma,\sigma}$ is continuous and*

$$\|I_\alpha f\|_{L^{r^*_\sigma,\sigma}(\mathbb{R}^n)} \le C\|f\|_{L^{r,\sigma}(\mathbb{R}^n)}, \tag{2.2}$$

where C depends on $n, r, \sigma, |\Omega|$, and r^*_σ is the Sobolev–Morrey conjugate

$$r^*_\sigma = \begin{cases} \frac{(n-\sigma)r}{n-\sigma-r} & \textit{if } r+\sigma < n \\ \textit{arbitrary large number} & \textit{if } r+\sigma \ge n. \end{cases} \tag{2.3}$$

The nonlinear terms $\mathbf{a}(x,\mathbf{u})$ and $\mathbf{b}(x,\mathbf{u},\mathbf{z})$ satisfy *controlled growth conditions*

$$|\mathbf{a}(x,\mathbf{u})| \le \Lambda\big(\varphi_1(x) + |\mathbf{u}|^{\frac{2^*}{2}}\big), \tag{2.4}$$
$$\varphi_1 \in L^{p,\lambda}(\Omega), \quad p > 2, \quad p+\lambda > n, \quad \lambda \in [0,n),$$
$$|\mathbf{b}(x,\mathbf{u},\mathbf{z})| \le \Lambda\big(\varphi_2(x) + |\mathbf{u}|^{2^*-1} + |\mathbf{z}|^{2\frac{(2^*-1)}{2^*}}\big), \tag{2.5}$$
$$\varphi_2 \in L^{q,\mu}(\Omega), \quad q > \frac{2^*}{2^*-1}, \quad 2q+\mu > n, \quad \mu \in [0,n)$$

with a positive constant Λ. Here 2^* is te Sobolev conjugate of 2, i.e. $2^* = \frac{2n}{n-2}$ if $n > 2$ and it is arbitrary large number if $n = 2$ (cf. [17,22,31,33]).

A *weak solution* to (1.1) is a function $\mathbf{u} \in W_0^{1,2}(\Omega;\mathbb{R}^N)$ satisfying

$$\begin{aligned}\int_\Omega A_{ij}^{\alpha\beta}(x)D_\beta u^j(x)D_\alpha\chi^i(x)dx + \int_\Omega a_i^\alpha(x,\mathbf{u}(x))D_\alpha\chi^i(x)dx \\ + \int_\Omega b_i(x,\mathbf{u}(x),D\mathbf{u}(x))\chi^i(x)dx = 0, \qquad j=1,\dots,N\end{aligned} \tag{2.6}$$

for all $\chi \in W_0^{1,2}(\Omega;\mathbb{R}^N)$ where the convergence of the integrals is ensured by (2.4) and (2.5).

3 Main result

The general theory of elliptic systems does not ensure boundedness of the solution if we impose only growth conditions as (2.4) and (2.5) (see for example [21, 23]). For this goal we need some additional structural restrictions on the operator as *componentwise coercivity* similar to that imposed in [23, 29, 32, 33].

Suppose that $\|\mathbf{A}\|_{\infty,\Omega} \le \Lambda_0$ and for each fixed $i \in \{1,\dots,N\}$ there exist positive constants θ_i and $\gamma(\Lambda_0)$ such that for $|u^i| \ge \theta_i$ we have

$$\begin{cases}\gamma|\mathbf{z}^i|^2 - \Lambda|\mathbf{u}|^{2^*} - \Lambda\varphi_1(x)^2 \le \sum_{\alpha=1}^n \left(A_{ij}^{\alpha\beta}(x)z_\beta^j + a_i^\alpha(x,\mathbf{u})\right) z_\alpha^i \\ b_i(x,\mathbf{u},\mathbf{z})\,\mathrm{sign}\,u^i(x) \ge -\Lambda\left(\varphi_2(x) + |\mathbf{u}|^{2^*-1} + |\mathbf{z}^i|^{2\frac{2^*-1}{2^*}}\right)\end{cases} \tag{3.1}$$

for a.a. $x \in \Omega$ and for all $\mathbf{z} \in \mathbb{M}^{N\times n}$. The functions φ_1 and φ_2 are as in (2.4) and (2.5).

Theorem 3.1. *Let $\mathbf{u} \in W_0^{1,2}(\Omega;\mathbb{R}^N)$ be a weak solution of the problem (1.1) under the conditions (2.1), (2.4), (2.5) and (3.1). Then*

$$\mathbf{u} \in W_0^{1,r} \cap L^\infty(\Omega;\mathbb{R}^N) \quad \textit{with} \quad r = \min\{p, q_\mu^*\}.$$

Moreover

$$|D\mathbf{u}| \in L^{r,\nu}(\Omega) \quad \textit{with} \quad \nu = \min\left\{n + \frac{r(\lambda-n)}{p}, n + \frac{r(\mu-n)}{q_\mu^*}\right\} \tag{3.2}$$

where q_μ^* is the Sobolev–Morrey conjugate of q (see (2.3)).

Remark 3.2. *If we take a bounded weak solution of (1.1) $\mathbf{u} \in W_0^{1,r} \cap L^\infty(\Omega;\mathbb{R}^N)$ we can substitute the coercivity condition (3.1) with a uniform ellipticity condition. In this case we may suppose the principal coefficients to be discontinuous with small discontinuity controlled by their BMO modulus. Precisely, we suppose that*

$$\sup_{0<\rho\le R}\sup_{y\in\Omega} ⨍_{\Omega_\rho(y)} |A_{ij}^{\alpha\beta}(x) - \overline{A_{ij}^{\alpha\beta}}_{\Omega_\rho(y)}|^2\,dx \le \delta^2,$$
$$\overline{A_{ij}^{\alpha\beta}}_{\Omega_\rho(y)} = ⨍_{\Omega_\rho(y)} A_{ij}^{\alpha\beta}(x)\,dx,$$

where $\delta \in (0,1)$ is the same parameter as in (2.1). The small BMO successfully substitute the VMO in the study of PDEs with discontinuous coefficients, harmonic analysis and integral operators studying, geometric measure analysis and differential geometry (see [4, 6, 8, 20, 28, 33] and the references therein). A higher integrability result for such kind of operators can be found in [13, 28, 31] for equations and systems, respectively.

Proof. The essential boundedness of the solution follows by [29] (see also [32,33]). Precisely, there exists a constant depending on n, Λ, p, q, $\|\varphi_1\|_{L^p(\Omega)}$, $\|\varphi_2\|_{L^q(\Omega)}$ and $\|D\mathbf{u}\|_{L^2(\Omega)}$ such that

$$\|\mathbf{u}\|_{\infty,\Omega} \le M. \tag{3.3}$$

Let the solution and the functions φ_1 and φ_2 be extended as zero outside Ω. By the Definition 2.2 we have that $\varphi_1 \in L^p(\Omega)$ and $\varphi_2 \in L^q(\Omega)$. In [17] Giaquinta show that there exists an exponent $\widetilde{r} > 2$ such that $\mathbf{u} \in W^{1,\widetilde{r}}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$. His approach is based on the reverse Hölder inequality and a version of Gehring's lemma. Since the Cacciopoli-type inequalities hold up to the boundary, this method can be carried out up to the boundary and it is done in [17, Chapter 5] for the Dirichlet problem in Lipschitz domain (see also [3,11,13,31]). In [9] the authors have shown that an inner neighborhood of (δ,R)-Reifenberg flat domain is a Lipschitz domain with the (δ,R)-Reifenberg flat property.

Lemma 3.3. *([9]) Let Ω be a (δ,R)-Reifenberg flat domain for sufficiently small $\delta>0$. Then for any $0<\varepsilon<\frac{R}{5}$ the set $\Omega_\varepsilon = \{x\in\Omega : \operatorname{dist}(x,\partial\Omega)>\varepsilon\}$ is a Lipschitz domain satisfying* (2.1)*.*

This lemma permits us to extend the results of [17, Chapter 5] in Reifenberg-flat domains. Further $|D\mathbf{u}|$ belongs at least to $L^{r_0}(\Omega)$ with $r_0 = \min\{p,q^*\} > \frac{n}{n+2}$ (cf. [31]).

Let $\boxed{n>2}$ and $\mathbf{u} \in W^{1,r_0}_0(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)$ be a solution to (1.1). Our *first step* is to improve its integrability. Fixing that solution in the nonlinear terms we obtain linear problem

$$\begin{cases} D_\alpha\big(A^{\alpha\beta}_{ij}(x)D_\beta u^j(x)\big) = f_i(x) - D_\alpha A^\alpha_i(x), & x\in\Omega\\ \mathbf{u}(x)=0, & x\in\partial\Omega \end{cases} \tag{3.4}$$

where we have used the notion

$$f_i(x) = b_i(x,\mathbf{u},D\mathbf{u}), \qquad A^\alpha_i(x) = a^\alpha_i(x,\mathbf{u}).$$

By (2.4), (2.5) and (3.3) we get

$$|A^\alpha_i(x)| \le \Lambda\left(\varphi_1(x) + |\mathbf{u}(x)|^{\frac{n}{n-2}}\right) \tag{3.5}$$

that gives $A^\alpha_i(x)\in L^{p,\lambda}(\Omega)$ with $p>2$ and $p+\lambda>n$. Analogously

$$|f_i(x)| \le \Lambda\left(\varphi_2(x) + |\mathbf{u}|^{\frac{n+2}{n-2}} + |D\mathbf{u}|^{\frac{n+2}{n}}\right). \tag{3.6}$$

Since $|D\mathbf{u}|\in L^{r_0}(\Omega)$ we get $|D\mathbf{u}|^{\frac{n+2}{n}} \in L^{\frac{r_0 n}{n+2}}(\Omega)$ that gives $f_i\in L^{q_1}(\Omega)$ where $q_1 = \min\{q, \frac{r_0 n}{n+2}\}$.

Let Γ be the fundamental solution of the Laplace operator. Recall that the *Newtonian potential* of $f_i(x)$ is given by

$$\mathcal{N}f_i(x) = \int_\Omega \Gamma(x-y)f_i(y)\,dy, \qquad \Delta\mathcal{N}f_i(x) = f_i(x) \ \text{ for a.a. } x\in\Omega$$

and by [19, Theorem 9.9] we have that $\mathcal{N}f_i\in W^{2,q_1}(\Omega)$. Denote by

$$F^\alpha_i(x) = D_\alpha\mathcal{N}f_i(x) = C(n)\int_\Omega \frac{(x-y)_\alpha f_i(y)}{|x-y|^n}\,dy \quad \text{for a.a. } x\in\Omega$$

and $\mathbb{F}_i = (F^1_i,\dots,F^n_i) = \operatorname{grad}\mathcal{N}f_i$. Hence $\operatorname{div}\mathbb{F}_i = f_i$ and

$$\begin{cases} D_\alpha\big(A^{\alpha\beta}_{ij}(x)D_\beta u^j(x)\big) = D_\alpha(F^\alpha_i(x) - A^\alpha_i(x)), & x\in\Omega\\ \mathbf{u}(x)=0, & x\in\partial\Omega. \end{cases} \tag{3.7}$$

By (3.5) and (3.6) we get

$$
\begin{aligned}
|F_i^\alpha(x) - A_i^\alpha(x)| &\le C(n,\Lambda)\int_\Omega \frac{\varphi_2(y) + |\mathbf{u}(y)|^{\frac{n+2}{n-2}} + |D\mathbf{u}(y)|^{\frac{n+2}{n}}}{|x-y|^{n-1}}\,dy \\
&\quad + \Lambda\left(\varphi_1(x) + |\mathbf{u}(x)|^{\frac{n}{n-2}}\right) \\
&\le C\left(1 + \varphi_1(x) + I_1\varphi_2(x) + I_1|D\mathbf{u}(x)|^{\frac{n+2}{n}}\right)
\end{aligned} \tag{3.8}
$$

with a constant depending on n, Λ, and $\|\mathbf{u}\|_{\infty,\Omega}$. By (2.2) we get

$$
\begin{aligned}
&\|I_1\varphi_2\|_{L^{q^*_\mu,\mu}(\Omega)} \le C\|\varphi_2\|_{L^{q,\mu}(\Omega)} \\
&\|I_1|D\mathbf{u}|^{\frac{n+2}{n}}\|_{L^{(\frac{r_0 n}{n+2})^*}(\Omega)} \le C\|\,|D\mathbf{u}|^{\frac{n+2}{n}}\,\|_{L^{\frac{r_0 n}{n+2}}(\Omega)} \le C\|D\mathbf{u}\|^{\frac{n+2}{n}}_{L^{r_0}(\Omega)}
\end{aligned}
$$

where q^*_μ is the Sobolev–Morrey conjugate of q and

$$
\left(\frac{r_0 n}{n+2}\right)^* = \begin{cases} \dfrac{r_0 n}{n+2-r_0} & \text{if } r_0 < n+2\,, \\ \text{arbitrary large number} & \text{if } r_0 \ge n+2\,. \end{cases}
$$

Hence $F_i^\alpha - A_i^\alpha \in L^{r_1}(\Omega)$ with $r_1 = \min\{p, q^*_\mu, (\frac{r_0 n}{n+2})^*\}$. If $r_1 = \min\{p, q^*_\mu\}$ then we have the assertion, otherwise $r_1 = (\frac{r_0 n}{n+2})^*$ and we consider two cases:

1. $r_0 = p$ that leads to $p > (\frac{pn}{n+2})^*$ which is impossible;
2. $r_0 = q^*$ and we consider two sub-cases:
   2a) $q^* \ge n+2$ which means that r_1 is arbitrary large number and we arrive to contradiction with the assumption $r_1 < \min\{p, q^*_\mu\}$;
   2b) $q^* < n+2$ hence $r_1 = \frac{q^* n}{n+2-q^*}$.

Applying [10, Theorem 1.7] to the linearized system (3.7) we get that for each matrix function $\mathbb{F} - \mathbb{A} \in L^{r_1}(\Omega; \mathbb{M}^{N\times n})$, with $r_1 = \frac{q^* n}{n+2-q^*}$ holds $\mathbf{u} \in W_0^{1,r_1} \cap L^\infty(\Omega; \mathbb{R}^N)$ with the estimate

$$
\|D\mathbf{u}\|_{r_1,\Omega} \le C\|\mathbb{F} - \mathbb{A}\|_{r_1,\Omega}.
$$

Here $\mathbb{A}(x) = \{A_i^\alpha(x)\}_{i\le N}^{\alpha\le n}$ and $\mathbb{F}(x) = \{F_i^\alpha(x)\}_{i\le N}^{\alpha\le n}$. Let us note that this estimate is valid for each solution of (3.7) including $\mathbf{u}$.

Repeating the above procedure for $\mathbf{u} \in W^{1,r_1}(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$ we get that

$$
|D\mathbf{u}| \in L^{r_2}(\Omega) \qquad r_2 = \min\left\{p, q^*_\mu, \left(\frac{r_1 n}{n+2}\right)^*\right\}.
$$

If $r_2 = \min\{p, q^*_\mu\}$ then we have the assertion, otherwise $r_2 = (\frac{r_1 n}{n+2})^* > r_1$ and we repeat the arguments of the previous case. In such a way we get an increasing sequence of indexes $\{r_k\}_{k\ge 0}$. After k' iterations we obtain $r_{k'} \ge \min\{p, q^*_\mu\}$ and

$$
\|D\mathbf{u}\|_{r,\Omega} \le C\|\mathbb{F} - \mathbb{A}\|_{r,\Omega} \quad \text{with} \quad r = \min\{p, q^*_\mu\}. \tag{3.9}
$$

The *second step* consists of showing that the gradient lies in a suitable Morrey space. Suppose that $|D\mathbf{u}| \in L^{r,\theta}(\Omega)$ with *arbitrary* $\theta \in [0, n)$. Direct calculations give $|D\mathbf{u}|^{\frac{n+2}{n}} \in L^{\frac{rn}{n+2},\theta}(\Omega)$

$$
\left(\frac{1}{\rho^\theta}\int_{\mathcal{B}_\rho} |D\mathbf{u}|^{\frac{n+2}{n}\frac{rn}{n+2}}\,dx\right)^{\frac{n+2}{rn}} = \left(\frac{1}{\rho^\theta}\int_{\mathcal{B}_\rho} |D\mathbf{u}|^{r}\,dx\right)^{\frac{n+2}{rn}} \le \|D\mathbf{u}\|^{\frac{n+2}{n}}_{r,\theta;\Omega}.
$$

Keeping in mind (3.8) and (2.2) we get

$$I_1|D\mathbf{u}|^{\frac{n+2}{n}} \in L^{(\frac{nr}{n+2})^*_\theta,\theta}(\Omega)$$

while $\varphi_1 \in L^{p,\lambda}(\Omega)$ and $I_1\varphi_2 \in L^{q^*_\mu,\mu}(\Omega)$.

Further by the Hölder inequality we get the estimates

$$\left(\frac{1}{\rho^{n-\frac{n-\lambda}{p}r}}\int_{\mathcal{B}_\rho}\varphi_1(x)^r\,dx\right)^{\frac{1}{r}} \le C(n)\|\varphi_1\|_{p,\lambda;\Omega},$$

$$\left(\frac{1}{\rho^{n-\frac{n-\mu}{q^*_\mu}r}}\int_{\mathcal{B}_\rho}(I_1\varphi_2(x))^r\,dx\right)^{\frac{1}{r}} \le C(n)\|I_1\varphi_2\|_{q^*_\mu,\mu;\Omega}$$

that implies $\varphi_1 \in L^{r,n-\frac{n-\lambda}{p}r}(\Omega)$ and $I_1\varphi_2 \in L^{r,n-\frac{n-\mu}{q^*_\mu}r}(\Omega)$.

Concerning the potential $I_1|D\mathbf{u}|^{\frac{n+2}{n}}$ we consider two cases:

1. $n-\theta \le \frac{rn}{n+2}$ then $\left(\frac{nr}{n+2}\right)^*_\theta$ is arbitrary large number and we can take it such that

$$I_1|D\mathbf{u}|^{\frac{n+2}{n}} \in L^r(\Omega);$$

2. $n-\theta > \frac{rn}{n+2}$ then by the embedding between the Morrey spaces we have

$$L^{(\frac{nr}{n+2})^*_\theta,\theta}(\Omega) \subset L^{r,r-2+\theta\frac{n+2}{n}}(\Omega)\,.$$

Then

$$|F^\alpha_i - A^\alpha_i| \in L^{r,\min\{r-2+\theta\frac{n+2}{n},n-\frac{n-\lambda}{p}r,n-\frac{n-\mu}{q^*_\mu}r\}}(\Omega)$$

which implies via [6, Theorem 5.1] that the gradient of the solution of the linearized problem satisfies

$$|D\mathbf{u}| \in L^{r,\min\{r-2+\theta\frac{n+2}{n},n-\frac{n-\lambda}{p}r,n-\frac{n-\mu}{q^*_\mu}r\}}(\Omega)\,.$$

In order to determine the optimal θ we use step-by-step arguments starting with the result obtained in the first step and taking as $\theta_0 = 0$. Suppose that

$$r-2 < \min\left\{n-\frac{n-\lambda}{p}r, n-\frac{n-\mu}{q^*_\mu}r\right\},$$

otherwise we have the assertion.

Repeating the above procedure with $\mathbf{u}$ such that $|D\mathbf{u}| \in L^{r,\theta_1}(\Omega)$ with $\theta_1 = r-2$ we obtain

$$|D\mathbf{u}| \in L^{r,\theta_2}(\Omega)$$

with

$$\theta_2 = \min\left\{r-2+\theta_1\frac{n+2}{n}, n-\frac{n-\lambda}{p}r, n-\frac{n-\mu}{q^*_\mu}r\right\}.$$

If $\theta_2 = \min\{n-\frac{n-\lambda}{p}r, n-\frac{n-\mu}{q^*_\mu}r\}$ we have the assertion, otherwise we take

$$\theta_2 = r-2+\theta_1\frac{n+2}{n} = (r-2)\left(1+\frac{n+2}{n}\right).$$

Iterating we obtain an increasing sequence $\{\theta_k = (r-2)\sum_{i=0}^{k-1}(\frac{n+2}{n})^i\}_{k\geq 1}$. Then there exists an index k'' for which

$$r-2+\theta_{k''}\frac{n+2}{n} \geq \min\left\{n-\frac{n-\lambda}{p}r, n-\frac{n-\mu}{q^*_\mu}r\right\}$$

that gives the assertion.

If $\boxed{n=2}$ then the growth conditions have the form

$$\begin{aligned} &|\mathbf{a}(x,\mathbf{u})| \leq \Lambda(\varphi_1(x)+|\mathbf{u}|^{\varkappa}),\\ &\qquad \varphi_1 \in L^{p,\lambda}(\Omega), \quad p>2, \quad p+\lambda>n, \quad \lambda\in[0,n), \end{aligned} \tag{3.10}$$

$$\begin{aligned} &|\mathbf{b}(x,\mathbf{u},\mathbf{z})| \leq \Lambda\big(\varphi_2(x)+|\mathbf{u}|^{\varkappa-1}+|\mathbf{z}|^{2-\epsilon}\big),\\ &\qquad \varphi_2 \in L^{q,\mu}(\Omega), \quad q>1, \quad 2q+\mu>n, \quad \mu\in[0,n) \end{aligned} \tag{3.11}$$

with $\varkappa>1$ arbitrary large number and $\epsilon>0$ arbitrary small.

Fixing again the solution $\mathbf{u}\in W_0^{1,r_0}(\Omega;\mathbb{R}^N)\cup L^\infty(\Omega;\mathbb{R}^N)$ in the nonlinear terms and using the Lemma 2.3 and Lemma 2.4 we obtain

$$F_i^\alpha - A_i^\alpha \in L^{r_1}(\Omega), \qquad r_1 = \min\left\{p, q^*_\mu, \left(\frac{r_0}{2-\epsilon}\right)^*\right\}.$$

If $r_1 = \left(\frac{r_0}{2-\epsilon}\right)^*$ then the only possible value for r_0 is $r_0 = q^*$ and hence $r_1 = \frac{2q^*}{2(2-\epsilon)-q^*}$, otherwise we rich to contradiction. Then by [10] we get $|D\mathbf{u}|\in L^{r_1}(\Omega)$.

Repeating the above procedure with $\mathbf{u}\in W_0^{1,r_1}\cap L^\infty(\Omega;\mathbb{R}^N)$ we obtain that

$$|D\mathbf{u}|\in L^{r_2}(\Omega), \qquad r_2 = \min\left\{p, q^*_\mu, \left(\frac{r_1}{2-\epsilon}\right)^*\right\}.$$

If

$$r_2 = \left(\frac{r_1}{2-\epsilon}\right)^* < \min\{p, q^*_\mu\}$$

we repeat the same procedure obtaining an increasing sequence $\{r_k\}_{k\geq 0}$. Hence there exist an index k_0 such that $r_{k_0}\leq \min\{p,q^*_\mu\}$ that gives the assertion.

To obtain Morrey's regularity we take $|D\mathbf{u}|\in L^{r,\theta}(\Omega)$ with *arbitrary* $\theta\in[0,2)$. Hence $|D\mathbf{u}|^{2-\epsilon}\in L^{\frac{r}{2-\epsilon},\theta}(\Omega)$. By Lemma 2.3 and Lemma 2.4 we obtain

$$\begin{gathered} \varphi_1 \in L^{p,\lambda}(\Omega)\subset L^{r,2-\frac{2-\lambda}{p}r}(\Omega)\\ I_1\varphi_2 \in L^{q^*_\mu,\mu}(\Omega)\subset L^{r,2-\frac{2-\mu}{q^*_\mu}r}(\Omega)\\ I_1|D\mathbf{u}|^{2-\epsilon}\in L^{(\frac{r}{2-\epsilon})^*_\theta,\theta}(\Omega)\subset L^{r,r-2(1-\epsilon)+\theta(2-\epsilon)}(\Omega). \end{gathered}$$

Hence the Calderón–Zygmund estimate for the linearized problem (see [6]) gives

$$|D\mathbf{u}|\in L^{r,\min\{2-\frac{2-\lambda}{p}r,2-\frac{2-\mu}{q^*_\mu}r,r-2(1-\epsilon)+\theta(2-\epsilon)\}}(\Omega).$$

To determine the precise Morrey space we apply the step-by-step procedure.

1. Since the last term is minimal when $\theta=0$ than we start with an this initial value $\theta_0=0$. Suppose that

$$r-2(1-\epsilon) < \min\left\{2-\frac{2-\lambda}{p}r, 2-\frac{2-\mu}{q^*_\mu}r\right\} < 2$$

(otherwise we have the assertion) and denote $\theta_1 = r-2(1-\epsilon)$.

2. Take $|D\mathbf{u}| \in L^{r,\theta_1}(\Omega)$. The above procedure gives $|D\mathbf{u}| \in L^{r,\theta_2}(\Omega)$ with

$$\theta_2 = \min\left\{2 - \frac{2-\lambda}{p}r, 2 - \frac{2-\mu}{q_\mu^*}, r - 2(1-\epsilon) + \theta_1(2-\epsilon)\right\}.$$

If $\theta_2 = r - 2(1-\epsilon) + \theta_1(2-\epsilon)$ (otherwise we have the assertion) then we continue with the same procedure obtaining the sequence defined by recurrence

$$\theta_0 = 0, \qquad \theta_k = r - 2(1-\epsilon) + \theta_{k-1}(2-\epsilon).$$

3. Since $r > 2$, hence the sequence is increasing and there exists an index $\bar{k}$ such that

$$\theta_{\bar{k}} \geq \min\left\{2 - \frac{2-\lambda}{p}r, 2 - \frac{2-\mu}{q_\mu^*}r\right\}$$

which is the assertion. □

Corollary 3.4. *Let the conditions of Theorem 3.1 hold. Then*

$$u^i \in C^{0,\alpha}(\Omega) \quad \textit{with} \quad \alpha = \min\left\{1 - \frac{n-\lambda}{p}, 1 - \frac{n-\mu}{q_\mu^*}\right\},$$

and for any ball $\mathcal{B}_\rho(z) \subset \Omega$ we have

$$\operatorname*{osc}_{\mathcal{B}_\rho(z)} u^i \leq C\rho^\alpha \qquad \forall\, i = 1, \dots, N.$$

Proof. By (3.2) we have that for each ball $\mathcal{B}_\rho(z) \subset \Omega$

$$\int_{\mathcal{B}_\rho(z)} |Du^i(y)|\, dy \leq C\rho^{n - \frac{n-\nu}{r}}.$$

Then for any $x, y \in \mathcal{B}_\rho(z)$ and for each fixed $i = 1, \dots, N$ we have

$$\begin{aligned} |u^i(x) - u^i(y)| &\leq 2|u^i(x) - u^i_{\mathcal{B}_\rho(z)}| \leq C\int_{\mathcal{B}_\rho(z)} \frac{Du^i(y)}{|x-y|^{n-1}}\, dy \\ &\leq C\int_0^\rho \int_{\mathcal{B}_t(z)} |Du^i(y)|\, dy\, \frac{dt}{t^n} \leq C\rho^{1-\frac{n-\nu}{r}}. \end{aligned}$$ □

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Reduction of order in the oscillation theory of half-linear differential equations

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Abstract. Oscillation of solutions of even order half-linear differential equations of the form

$$D(\alpha_n,\dots,\alpha_1)x + q(t)|x|^\beta \operatorname{sgn} x = 0, \qquad t \ge a > 0, \tag{1.1}$$

where α_i, $1 \le i \le n$, and β are positive constants, q is a continuous function from $[a,\infty)$ to $(0,\infty)$ and the differential operator $D(\alpha_n,\dots,\alpha_1)$ is defined by

$$D(\alpha_1)x = \frac{d}{dt}\left(|x|^{\alpha_1} \operatorname{sgn} x\right)$$

and

$$D(\alpha_i,\dots,\alpha_1)x = \frac{d}{dt}\left(|D(\alpha_{i-1},\dots,\alpha_1)x|^{\alpha_i} \operatorname{sgn} D(\alpha_{i-1},\dots,\alpha_1)x\right), \qquad i = 2,\dots,n,$$

is proved in the case where $\alpha_1 \cdots \alpha_n = \beta$ through reduction to the problem of oscillation of solutions of some lower order differential equations associated with (1.1).

Keywords: half-linear differential equation, oscillation test.

2020 Mathematics Subject Classification: 34C10.

1 Introduction

Consider differential equations of the form

$$D(\alpha_n,\dots,\alpha_1)x + q(t)|x|^\beta \operatorname{sgn} x = 0, \qquad t \ge a > 0, \tag{1.1}$$

where $n \ge 2$ is an even integer, $\alpha_1, \alpha_2, \dots, \alpha_n$ and β are positive constants, $q : [a,\infty) \to (0,\infty)$, $a > 0$, is a continuous function and the differential operator $D(\alpha_n,\dots,\alpha_1)x$ is defined recursively by

$$D(\alpha_1)x = \frac{d}{dt}\left(|x|^{\alpha_1} \operatorname{sgn} x\right)$$

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and

$$D(\alpha_i,\ldots,\alpha_1)x = \frac{d}{dt}\big(|D(\alpha_{i-1},\ldots,\alpha_1)x|^{\alpha_i}\operatorname{sgn} D(\alpha_{i-1},\ldots,\alpha_1)x\big), \qquad i = 2,\ldots,n.$$

It is convenient to denote by $C(\alpha_j,\ldots,\alpha_1)[t_0,\infty), 1 \le j \le n$, the set of continuous functions $x : [t_0,\infty) \to \mathbb{R}$ such that $D(\alpha_i,\ldots,\alpha_1)x,\ i = 1,\ldots,j$, exist and are continuous on $[t_0,\infty)$.

A function $x(t)$ from $C(\alpha_n,\ldots,\alpha_1)[t_0,\infty)$ is called a solution of equation (1.1) on $[t_0,\infty)$ if it satisfies (1.1) at each $t \in [t_0,\infty)$. We restrict our consideration to the so called proper solutions of (1.1), i.e., solutions which are not trivial in any neighborhood of infinity. Such a solution is called *oscillatory* if it has an unbounded set of zeros, and it is called *nonoscillatory* otherwise.

It is known that for any nonoscillatory solution $x(t)$ of (1.1) there exist a $t_0 \ge a$ and an odd integer $l,\ 1 \le l \le n-1$, such that for $t \ge t_0$

$$x(t)D(\alpha_j,\ldots,\alpha_1)x(t) > 0 \quad \text{for } j = 1,\ldots,l, \tag{1.2}$$

and

$$(-1)^{n+j}x(t)D(\alpha_j,\ldots,\alpha_1)x(t) < 0 \quad \text{for } j = l+1,\ldots,n, \tag{1.3}$$

(see Naito [19]). Functions belonging to $C(\alpha_n,\ldots,\alpha_1)[t_0,\infty)$ and satisfying (1.2) and (1.3) for $t \ge t_0$, will be called *nonoscillatory functions of Kiguradze's degree l*. We denote by $\mathcal{N}_l$ the set of all nonoscillatory solutions of equation (1.1) which are of degree l. The elements of $\mathcal{N}_1$ (resp. $\mathcal{N}_{n-1}$) will be called nonoscillatory solutions of the *minimal* (resp. *maximal*) Kiguradze's degree.

Existence and asymptotic behavior of positive solutions of nonlinear differential equations of the form (1.1) in the case where the exponents satisfied either $\beta < \alpha_1\cdots\alpha_n$ or $\beta > \alpha_1\cdots\alpha_n$ were studied by Naito in [18,19] (for some particular cases see also [7,8,11–13,16,17,20–22,24,25]), but the important special case in which $\beta = \alpha_1\cdots\alpha_n$ seems to remain untouched until now. As far as we know, the paper by Došlý et al. [4] devoted to the study of nonoscillation of solutions of higher order half-linear differential equations of the form

$$\sum_{k=0}^{n}(-1)^k\Big(r_k(t)|x^{(k)}|^{\alpha}\operatorname{sgn} x^{(k)}\Big)^{(k)} = 0,$$

where $r_k,\ 0 \le k \le n$, are continuous functions with $r_n(t) > 0$ in the interval under consideration, is the only work on the subject.

Recently, the present author in [6] gave an oscillation criterion which (when specialized to equation (1.1)) says that all solutions of (1.1) are oscillatory if there exists an $\varepsilon \in (0,1]$ such that

$$\int_a^\infty t^{\alpha_2\cdots\alpha_n+\alpha_3\cdots\alpha_n+\cdots+(1-\varepsilon)\alpha_n}q(t)dt = \infty. \tag{1.4}$$

The result is sharp in the sense that if $\varepsilon = 0$ in (1.1)), then equation (1.1) may have nonoscillatory solutions. On the other hand, the above criterion does not apply to such an important special case of (1.1) as the nonlinear Euler-type differential equation

$$D(\alpha_n,\ldots,\alpha_1)x + \frac{\gamma}{t^{\alpha_2\cdots\alpha_n+\alpha_3\cdots\alpha_n+\cdots+\alpha_n+1}}|x|^{\alpha_1\cdots\alpha_n}\operatorname{sgn} x = 0, \qquad t \ge a > 0, \tag{1.5}$$

where $\gamma > 0$ is a constant.

Thus, our main purpose here is to obtain criteria which would be more sensitive to oscillatory behaviour of solutions of equations of the form (1.1) and would apply also to higher order half-linear equations of the Euler type. Our approach is based on reduction of the problem of oscillation of equation (1.1) to the problem of oscillation of solutions of some lower order equations and inequalities. In the linear case this approach was used successfully by various authors in [1,2,5,9,10,14,15,23].

2 Preliminaries

We begin with some preparatory results which will be needed in the sequel.

Lemma 2.1. *Let $\alpha > 0$ and $y \in C(\alpha)[t_0, \infty)$ be such that either*

$$y(t)D(\alpha)y(t) > 0 \quad \textit{for } t \geq t_0, \tag{2.1}$$

or

$$y(t)D(\alpha)y(t) < 0 \quad \textit{for } t \geq t_0 \tag{2.2}$$

and

$$\int_{t_0}^{\infty} |D(\alpha)y(t)|dt < \infty. \tag{2.3}$$

Then $y \in C^1[t_0, \infty)$, i.e., the usual derivative $y'(t)$ exists and is continuous on $[t_0, \infty)$.

Proof. We will assume that $y(t) > 0$ on $[t_0, \infty)$. (The proof in the case $y(t) < 0$ for $t \geq t_0$ is similar and is omitted.)

If y satisfies (2.1), then we can integrate $D(\alpha)y(t)$ from t_0 to t and raise the result to the power $1/\alpha$ to get

$$y(t) = \left[y(t_0)^{\alpha} + \int_{t_0}^{t} D(\alpha)y(s)ds \right]^{\frac{1}{\alpha}}, \qquad t \geq t_0. \tag{2.4}$$

Similarly, if y satisfies (2.2) and (2.3), then $D(\alpha)y(t) < 0$ for $t \geq t_0$ implies that $y(\infty)^{\alpha} = \lim_{t\to\infty} y(t)^{\alpha}$ exists as a nonnegative finite number and after integration of $D(\alpha)y(t)$ from $t(\geq t_0)$ to ∞ we arrive at

$$y(t) = \left[y(\infty)^{\alpha} - \int_{t}^{\infty} D(\alpha)y(s)ds \right]^{\frac{1}{\alpha}}, \qquad t \geq t_0. \tag{2.5}$$

From (2.4) (resp. (2.5)) it is clear that in both cases the function $y(t)$ is continuously differentiable on $[t_0, \infty)$. □

Remark 2.2. Repeated application of Lemma 2.1 shows that if y is a nonoscillatory solution of equation (1.1) on an interval $[t_0, \infty)$, then y and $D(\alpha_i, \ldots, \alpha_1)y$, $i = 1, \ldots, n-1$, are continuously differentiable functions, that is,

$$\frac{d}{dt}y(t) \quad \text{and} \quad \frac{d}{dt}\left[D(\alpha_i, \ldots, \alpha_1)y(t)\right], \qquad i = 1, \ldots, n-1,$$

exist and are continuous on $[t_0, \infty)$.

To formulate and prove our next lemma, we define the numbers $r_i(k), 1 \leq i \leq n-1$ and $k = 0, 1, \ldots, i$, by

$$r_i(0) = 1 \quad \text{and} \quad r_i(k) = \frac{1}{\alpha_{i-k+1}} r_i(k-1) + 1 \quad \text{for } k = 1, \ldots, i. \tag{2.6}$$

We also set

$$r_i := r_i(i) = 1 + \frac{1}{\alpha_1} + \frac{1}{\alpha_1\alpha_2} + \cdots + \frac{1}{\alpha_1\alpha_2\cdots\alpha_i}.$$

Lemma 2.3. *If $y \in C(\alpha_l, \dots, \alpha_1)[t_0, \infty)$ satisfies $D(\alpha_i, \dots, \alpha_1)y(t) > 0$, $i = 0, \dots, l$ and $D(\alpha_{l+1}, \dots, \alpha_1)y(t) < 0$ for $t \geq t_0$, then*

$$(t - t_0)D(\alpha_{l-k}, \dots, \alpha_1)y(t) \leq r_l(k)\big[D(\alpha_{l-k-1}, \dots, \alpha_1)y(t)\big]^{\alpha_{l-k}}, \qquad k = 0, 1, \dots, l-1, \tag{2.7$_k$}$$

for $t \geq t_0$.

Proof. Since $D(\alpha_l, \dots, \alpha_1)y(t)$ is decreasing for $t \geq t_0$, integrating on $[t_0, t]$ we obtain

$$\begin{aligned}(t - t_0)D(\alpha_l, \dots, \alpha_1)y(t) &\leq \int_{t_0}^{t} D(\alpha_l, \dots, \alpha_1)y(s)ds = \int_{t_0}^{t} \left([D(\alpha_{l-1}, \dots, \alpha_1)y(s)]^{\alpha_l}\right)' ds \\ &= \big[D(\alpha_{l-1}, \dots, \alpha_1)y(t)\big]^{\alpha_l} - \big[D(\alpha_{l-1}, \dots, \alpha_1)y(t_0)\big]^{\alpha_l} \\ &\leq \big[D(\alpha_{l-1}, \dots, \alpha_1)y(t)\big]^{\alpha_l}, \end{aligned} \tag{2.8}$$

which gives inequality (2.7$_k$) for $k = 0$. Next, since by the remark after Lemma 2.1, $D(\alpha_{l-1}, \dots, \alpha_1)y(t)$ is continuously differentiable function, we can express (2.8) explicitly as

$$\alpha_l(t - t_0)\big[D(\alpha_{l-1}, \dots, \alpha_1)y(t)\big]^{\alpha_l - 1}\big(D(\alpha_{l-1}, \dots, \alpha_1)y(t)\big)' \leq \big[D(\alpha_{l-1}, \dots, \alpha_1)y(t)\big]^{\alpha_l},$$

or, equivalently,

$$\big[(t - t_0)D(\alpha_{l-1}, \dots, \alpha_1)y(t)\big]' \leq \frac{1 + \alpha_l}{\alpha_l} D(\alpha_{l-1}, \dots, \alpha_1)y(t), \tag{2.9}$$

for $t \geq t_0$. Integrating (2.9) from t_0 to t we obtain

$$(t - t_0)D(\alpha_{l-1}, \dots, \alpha_1)y(t) \leq \frac{1 + \alpha_l}{\alpha_l}\big[D(\alpha_{l-2}, \dots, \alpha_1)y(t)\big]^{\alpha_{l-1}}, \qquad t \geq t_0, \tag{2.10}$$

which is (2.7$_k$) for $k = 1$.

Repeated application of the above procedure yields (2.7$_k$) also for $k = 2, \dots, l-1$ where $D(\alpha_j, \dots, \alpha_1)y(t)$ for $j = 0$ should be interpreted as $y(t)$. □

The following comparison lemma will play an important role in our later discussions. For the proof see Naito [19].

Lemma 2.4. *Let $l \in \{1, 3, \dots, n-1\}$ be a fixed odd number and let the differential inequality*

$$D(\alpha_n, \dots, \alpha_1)y + q(t)|y|^{\alpha_1 \cdots \alpha_n} \operatorname{sgn} y \leq 0, \qquad t \geq a > 0, \tag{2.11}$$

where $q : [a, \infty) \to (0, \infty)$ is a continuous function, have a positive solution $y(t)$ of degree l for $t \geq t_0$. Then there exists a positive solution $x(t)$ of equation (1.1) which has the same degree l.

3 Reduction to the existence of solutions of minimal degree

Define numbers R_i, $1 \leq i \leq n-1$, by

$$R_1 = 1 \quad \text{and} \quad R_i = \left(\frac{1}{r_i(i-1)}\right)^{\frac{1}{\alpha_1}} \left(\frac{1}{r_i(i-2)}\right)^{\frac{1}{\alpha_1\alpha_2}} \cdots \left(\frac{1}{r_i(1)}\right)^{\frac{1}{\alpha_1 \cdots \alpha_{i-1}}}, \qquad i = 2, \dots, n-1,$$

where $r_i(k)$, $k = 0, 1, \dots, i$, are given by (2.6).

Theorem 3.1. *Eq.* (1.1) *has a nonoscillatory solution of the Kiguradze's degree* $l, 1 \le l \le n-1$, *if and only if the differential equation*

$$D(\alpha_n, \dots, \alpha_l)z + R_l^{\beta}(t-t_0)^{(r_{l-1}-1)\beta} q(t)|z|^{\alpha_l \cdots \alpha_n} \operatorname{sgn} z = 0, \qquad t \ge t_0, \tag{3.1_l}$$

has a nonoscillatory solution of the Kiguradze's degree 1.

Proof. (Necessity.) Suppose that (1.1) has a nonoscillatory solution $x(t)$ whose Kiguradze's degree is l, $1 \le l \le n-1$. We may assume that $x(t)$ is positive and satisfies (1.2) and (1.3) on $[t_0, \infty)$. If we chain the inequalities (2.7_k), $k = 1, \dots, l-1$, together, we obtain

$$x(t) \ge R_l (t-t_0)^{r_{l-1}-1} \left[D(\alpha_{l-1}, \dots, \alpha_1)x(t) \right]^{\frac{1}{\alpha_1 \cdots \alpha_{l-1}}}, \qquad t \ge t_0. \tag{3.2}$$

Substituting this inequality into (1.1), we obtain that $x(t)$ satisfies the inequality

$$D(\alpha_n, \dots, \alpha_1)x(t) + R_l^{\alpha_1 \cdots \alpha_n}(t-t_0)^{(r_{l-1}-1)\alpha_1 \cdots \alpha_n} q(t) \left[D(\alpha_{l-1}, \dots, \alpha_1)x(t) \right]^{\alpha_l \cdots \alpha_n} \le 0.$$

Put $y(t) = D(\alpha_{l-1}, \dots, \alpha_1)x(t)$. Then the function $y(t)$ satisfies

$$D(\alpha_n, \dots, \alpha_l)y(t) + R_l^{\alpha_1 \cdots \alpha_n}(t-t_0)^{(r_{l-1}-1)\alpha_1 \cdots \alpha_n} q(t)|y(t)|^{\alpha_l \cdots \alpha_n} \operatorname{sgn} y(t) \le 0, \qquad t \ge t_0, \tag{3.3}$$

and its Kiguradze's degree is 1. By Lemma 2.4, the corresponding differential equation (3.1_l) has a positive solution $z(t)$ of the same degree 1.

(Sufficiency.) Let (3.1_l) have a nonoscillatory solution $z(t)$ of degree 1. We may assume that $z(t) > 0$ for $t \ge t_0$. Then the function

$$w(t) = (R_l / R_{l-1}) \left(\int_{t_0}^{t} \left(\int_{t_0}^{s_1} \cdots \left(\int_{t_0}^{s_{l-2}} z(s_{l-1}) ds_{l-1} \right)^{\frac{1}{\alpha_{l-1}}} \dots \, ds_2 \right)^{\frac{1}{\alpha_2}} ds_1 \right)^{\frac{1}{\alpha_1}} \tag{3.4}$$

satisfies

$$D(\alpha_{l-1}, \dots, \alpha_1)w(t) = (R_l / R_{l-1})^{\alpha_1 \cdots \alpha_{l-1}} z(t)$$

and since $z(t)$ has degree 1, the function $w(t)$ satisfies

$$D(\alpha_k, \dots, \alpha_1)w(t) > 0 \quad \text{for } k = 1, \dots, l,$$

and

$$(-1)^{n+k} D(\alpha_k, \dots, \alpha_1)w(t) < 0 \quad \text{for } k = l+1, \dots, n.$$

Hence, $w(t)$ is a function having degree l for $t \ge t_0$. Since $z(t)$ is increasing, from (3.4) we obtain

$$\begin{aligned} w(t) &\le (R_l / R_{l-1}) z(t)^{1/(\alpha_1 \cdots \alpha_{l-1})} \left(\int_{t_0}^{t} \left(\int_{t_0}^{s_1} \cdots \left(\int_{t_0}^{s_{l-2}} ds_{l-1} \right)^{\frac{1}{\alpha_{l-1}}} \dots ds_2 \right)^{\frac{1}{\alpha_2}} ds_1 \right)^{\frac{1}{\alpha_1}} \\ &= R_l (t-t_0)^{r_{l-1}-1} z(t)^{1/(\alpha_1 \cdots \alpha_{l-1})}. \end{aligned}$$

Now, as a consequence of the relation

$$r_l(k) = r_{l-1}(k-1) + \frac{1}{\alpha_{l-k+1} \cdots \alpha_l}, \qquad k = 1, \dots, l,$$

we get $r_l(k) \geq r_{l-1}(k-1), k = 1, \ldots, l$, which implies

$$(R_l/R_{l-1})^{\alpha_1 \cdots \alpha_{l-1}} \leq 1.$$

Thus,

$$D(\alpha_n, \ldots, \alpha_1)w(t) = (R_l/R_{l-1})^{\alpha_1 \cdots \alpha_{l-1}} D(\alpha_n, \ldots, \alpha_l)z(t) \leq D(\alpha_n, \ldots, \alpha_l)z(t)$$

and so for $t \geq t_0$,

$$D(\alpha_n, \ldots, \alpha_1)w(t) + q(t)w(t)^{\alpha_1 \cdots \alpha_n} \leq D(\alpha_n, \ldots, \alpha_l)z(t) + R_l^{\alpha_1 \cdots \alpha_n}(t-t_0)^{(r_{l-1}-1)\alpha_1 \cdots \alpha_n} q(t) z(t)^{\alpha_l \cdots \alpha_n}$$

showing that $w(t)$ is a solution of (2.11) for $t \geq t_0$ since $z(t)$ is a solution of (3.1_l). Finally, by Lemma 2.4, there exists a positive solution $x(t)$ of (1.1) of degree l. This completes the proof of the theorem. □

Remark 3.2. If $l = n-1$, then (3.1_l) reduces to the second-order equation

$$D(\alpha_n, \alpha_{n-1})z + R_{n-1}^{\beta}(t-t_0)^{(r_{n-2}-1)\beta} q(t)|z|^{\alpha_{n-1}\alpha_n} \operatorname{sgn} z = 0. \tag{3.1_{n-1}}$$

From Theorem 3.1 it follows that if (3.1_{n-1}) is nonoscillatory, then equation (1.1) is nonoscillatory, too. (More precisely, it has a nonoscillatory solution of the maximal degree $l = n-1$.)

However, if $l < n-1$, then equations (3.1_l) are of orders greater than 2 and it may not be an easy matter to determine whether or not (3.1_l) has a nonoscillatory solutions of degree 1.

Thus, we proceed further and associate with (1.1) a set of half-linear differential equations all of which are of the second order.

For this purpose we assume that the integrals

$$\begin{aligned}
I_1(q) &= \int_a^\infty q(t)dt, \\
I_2(q) &= \int_a^\infty \left(\int_t^\infty q(s)ds \right)^{\frac{1}{\alpha_n}} dt, \\
&\vdots \\
I_{n-l-1}(q) &= \int_a^\infty \left(\int_{s_{l+3}}^\infty \cdots \left(\int_{s_{n-1}}^\infty q(s)ds \right)^{\frac{1}{\alpha_n}} \cdots ds_{l+4} \right)^{\frac{1}{\alpha_{l+3}}} ds_{l+3}, \qquad 1 \leq l \leq n-2,
\end{aligned}$$

converge and define continuous functions $\rho_0(t), \ldots, \rho_{n-l-1}(t)$ by

$$\rho_0(t) = q(t), \qquad \rho_k(t) = \left[\int_t^\infty \rho_{k-1}(s)ds \right]^{\frac{1}{\alpha_{n-k+1}}}, \qquad k = 1, \ldots, n-l-1. \tag{3.5}$$

The following theorem is the main result of this paper.

Theorem 3.3. *Suppose that* (1.1) *has a nonoscillatory solution* $x(t)$ *which is of degree* l, $1 \leq l \leq n-1$, *for* $t \geq t_0$. *Then, the second order half-linear differential equation*

$$D(\alpha_{l+1}, \alpha_l)z + R_l^{\alpha_1 \cdots \alpha_{l+1}}(t-t_0)^{(r_{l-1}-1)\alpha_1 \cdots \alpha_{l+1}} \rho_{n-l-1}(t)|z|^{\alpha_l \alpha_{l+1}} \operatorname{sgn} z = 0, \qquad t \geq t_0, \tag{3.6_l}$$

has a nonoscillatory solution of degree 1.

Proof. Suppose that equation (1.1) has an eventually positive solution $x(t)$ which is of degree l, $1 \le l \le n-1$, for $t \ge t_0$. (If $x(t)$ is a solution which is eventually negative, the proof is similar and is omitted.)

By Theorem 3.1, there exists a positive solution $z(t)$ of the lower order differential equation (3.1_l) which is of degree 1, i.e., it satisfies for $t \ge t_0$

$$D(\alpha_l)z(t) > 0 \quad \text{and} \quad (-1)^{n+j}D(\alpha_j,\ldots,\alpha_l)z(t) < 0 \quad \text{for } j = l+1,\ldots,n. \tag{3.7}$$

Integrating (3.1_l) from t to ∞ and using (3.7), we get

$$D(\alpha_{n-1},\ldots,\alpha_l)z(t) \ge R_l^{\alpha_1\cdots\alpha_{n-1}}\left(\int_t^\infty (s-t_0)^{(r_{l-1}-1)\alpha_1\cdots\alpha_n} q(s)z(s)^{\alpha_l\cdots\alpha_n}ds\right)^{1/\alpha_n}, \qquad t \ge t_0.$$

Continuing in this fashion and using the fact that $z(t)$ and $(t-t_0)^{(r_{l-1}-1)\alpha_1\cdots\alpha_n}$ are increasing functions for $t \ge t_0$, we obtain

$$\begin{aligned}&-\left[D((\alpha_{l+1},\alpha_l)z(t)\right]^{\alpha_{l+2}}\\ &\quad\ge R_l^{\alpha_1\cdots\alpha_{l+1}}(t-t_0)^{(r_{l-1}-1)\alpha_1\cdots\alpha_{l+1}}z(t)^{\alpha_l\alpha_{l+1}\alpha_{l+2}}\left(\int_t^\infty\left(\cdots\left(\int_{s_{n-1}}^\infty q(s)ds\right)^{\frac{1}{\alpha_n}}\cdots\right)^{\frac{1}{\alpha_{l+3}}}ds_{l+2}\right),\end{aligned}$$

or, equivalently,

$$D(\alpha_{l+1},\alpha_l)z(t) + R_l^{\alpha_1\cdots\alpha_{l+1}}(t-t_0)^{(r_{l-1}-1)\alpha_1\cdots\alpha_{l+1}}\rho_{n-l-1}(t)z(t)^{\alpha_l\alpha_{l+1}} \le 0, \qquad t \ge t_0, \tag{3.8}$$

where $\rho_{n-l-1}(t)$ is defined by (3.5). Thus, by Lemma 2.4, the differential equation (3.6_l) has a positive solution of degree 1 as claimed. The proof of the theorem is complete. □

As an immediate consequence of Theorem 3.3 we get the following oscillation result.

Corollary 3.4. *If all of the second order half-linear differential equations* (3.6_l)*,* $l = 1,3,\ldots,n-1$*, are oscillatory, then all solutions of the n-th order differential equation* (1.1) *are oscillatory.*

Example 3.5. Consider the Euler-type nonlinear differential equation

$$D(\alpha_n,\ldots,\alpha_1)x + \gamma t^{-(\alpha_2\cdots\alpha_n+\alpha_3\cdots\alpha_n+\cdots+\alpha_n+1)}|x|^{\alpha_1\cdots\alpha_n}\operatorname{sgn} x = 0, \qquad t \ge 1, \tag{3.9}$$

where n is an even integer and $\alpha_1,\ldots,\alpha_n$ and γ are positive constants.

To simplify notation and formulation of our results for equation (3.9), we define the numbers q_i and Q_i, $i = 1,\ldots,n$, by

$$q_1 = 0, \qquad q_i = \alpha_i(q_{i-1}+1) \quad \text{for } i = 2,\ldots,n, \tag{3.10}$$

and

$$Q_1 = 1, \qquad Q_i = \left(\frac{1}{q_i}\right)^{\frac{1}{\alpha_i}}\left(\frac{1}{q_{i+1}}\right)^{\frac{1}{\alpha_i\alpha_{i+1}}}\cdots\left(\frac{1}{q_{n-1}}\right)^{\frac{1}{\alpha_i\cdots\alpha_{n-1}}}\left(\frac{1}{q_n}\right)^{\frac{1}{\alpha_i\cdots\alpha_n}}, \qquad i = 2,\ldots,n. \tag{3.11}$$

It is a matter of easy computation to verify that if $q(t) = \gamma t^{-q_n-1}$, $\gamma > 0$, then the functions ρ_{n-l-1} defined by (3.5) become

$$\rho_{n-l-1}(t) = \gamma^{1/(\alpha_{l+2}\cdots\alpha_n)}Q_{l+2}t^{-q_{l+1}+1}, \qquad l = 1,\ldots,n-3, \tag{3.12}$$

and the second order half-linear differential equations (3.6_l) associated with (3.9) reduce respectively to

$$\left(|z'|^{\alpha_{l+1}}\operatorname{sgn} z'\right)' + \gamma^{1/(\alpha_1\cdots\alpha_n)} R_l^{\alpha_1\cdots\alpha_{l+1}} Q_{l+2} t^{-q_{l+1}-1} |z|^{\alpha_{l+1}} \operatorname{sgn} z = 0, \qquad t \ge 1, \tag{3.13_l}$$

if $1 \le l \le n-3$, and

$$\left(|z'|^{\alpha_n}\operatorname{sgn} z'\right)' + \gamma R_{n-1}^{\alpha_1\cdots\alpha_n} t^{-q_n-1} |z|^{\alpha_n} \operatorname{sgn} z = 0, \qquad t \ge 1, \tag{3.14}$$

if $l = n-1$.

If we apply the well-known result which says that all solutions of the generalized second order Euler differential equation

$$\left(|z'|^{\alpha}\operatorname{sgn} z\right)' + \lambda t^{-\alpha-1} |z|^{\alpha} \operatorname{sgn} z = 0, \qquad t \ge 1, \tag{3.15}$$

are oscillatory if and only if

$$\lambda > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}, \tag{3.16}$$

(see, for example, [3]), then we get that for oscillation of all solutions of equation (3.7) it is sufficient that

$$\gamma^{1/(\alpha_{l+2}\cdots\alpha_n)} R_l^{\alpha_1\cdots\alpha_{l+1}} Q_{l+2} > \left(\frac{\alpha_{l+1}}{\alpha_{l+1}+1}\right)^{\alpha_{l+1}+1}, \qquad l = 1,3,\ldots,n-3, \tag{3.17_l}$$

and

$$\gamma R_{n-1}^{\alpha_1\cdots\alpha_n} > \left(\frac{\alpha_n}{\alpha_n+1}\right)^{\alpha_n+1}. \tag{3.18}$$

Example 3.6. Consider the fourth order half-linear differential equation

$$D(\alpha_4,\alpha_3,\alpha_2,\alpha_1)x + q(t)|x|^{\alpha_1\alpha_2\alpha_3\alpha_4} \operatorname{sgn} x = 0, \qquad t \ge a > 0, \tag{3.19}$$

where $\alpha_i, 1 \le i \le 4$, are positive constants and $q : [a,\infty) \to (0,\infty)$ is continuous function. Second order equations associated with (3.19) are

$$\left(|z'|^{\alpha_2}\operatorname{sgn} z'\right)' + \left(\int_t^\infty \left(\int_s^\infty q(\tau)d\tau\right)^{1/\alpha_4} ds\right)^{1/\alpha_3} |z|^{\alpha_2} \operatorname{sgn} z = 0, \qquad t \ge t_0, \tag{3.20}$$

and

$$\left(|z'|^{\alpha_4}\operatorname{sgn} z'\right)' + \left(\frac{\alpha_2\alpha_3}{1+\alpha_3+\alpha_2\alpha_3}\right)^{\alpha_2\alpha_3\alpha_4} \left(\frac{\alpha_3}{1+\alpha_3}\right)^{\alpha_3\alpha_4} (t-t_0)^{(1+\alpha_2)\alpha_3\alpha_4} q(t)|z|^{\alpha_4} \operatorname{sgn} z = 0, \quad t \ge t_0. \tag{3.21}$$

From Corollary 3.4 we know that oscillation of both equations (3.20) and (3.21) implies oscillation of all solutions of equation (3.19).

This occurs, for example, if for some $\varepsilon \in (0,1]$

$$\int_a^\infty t^{1-\varepsilon}\left(\int_t^\infty \left(\int_s^\infty q(\tau)d\tau\right)^{1/\alpha_4} ds\right)^{1/\alpha_3} dt = \infty \tag{3.22}$$

and

$$\int_a^\infty t^{(1+\alpha_2)\alpha_3\alpha_4+1-\varepsilon} q(t)dt = \infty, \tag{3.23}$$

(see [6]).

4 Reduction to the existence of solutions of maximal degree

In the last section we indicate an alternative way how to obtain the set of second-order equations (3.6_l) associated with the even order half-linear differential equation (1.1). Here, the problem of the existence of nonoscillatory solutions of an arbitrary degree l of equation(1.1) is converted into the problem of the existence of solutions of the maximal Kiguradze's degree of certain lower order half-linear differential equation.

Theorem 4.1. *If the n-th order equation (1.1) has a nonoscillatory solution of degree l, then the $(l+1)$-order differential equation*

$$D(\alpha_{l+1},\dots,\alpha_1)z(t)+\rho_{n-l-1}(t)|z(t)|^{\alpha_1\cdots\alpha_{l+1}}\operatorname{sgn} z(t)=0,\qquad t\ge t_0,\tag{4.1_l}$$

has a nonoscillatory solution of the same degree l.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1) which is of Kiguradze's degree l. We may suppose that $x(t)$ is eventually positive and satisfies (1.2) and (1.3) on $[t_0,\infty), t_0\ge a$.

If $l=n-1$, then the proof is trivial because (4.1_{n-1}) is the same as (1.1).

Let $1\le l<n-1$. Integrating (1.1) from $t(\ge t_0)$ to ∞, we get

$$D(\alpha_{n-1},\dots,\alpha_1)x(t)\ge\left(\int_t^\infty q(s)x(s)^{\alpha_1\cdots\alpha_n}ds\right)^{1/\alpha_n},\qquad t\ge t_0.$$

Continuing in this way, we finally arrive at

$$\begin{aligned}&-D(\alpha_{l+1},\dots,\alpha_1)x(t)\\&\qquad\ge\left(\int_t^\infty\left(\int_{s_{l+2}}^\infty\cdots\left(\int_{s_{n-1}}q(s)x(s)^{\alpha_1\cdots\alpha_n}ds\right)^{1/\alpha_n}\dots ds_{l+3}\right)^{1/\alpha_{l+3}}ds_{l+2}\right)^{1/\alpha_{l+2}}\end{aligned}\tag{4.2}$$

for $t\ge t_0$. Since $x(t)$ is increasing for $t\ge t_0$, from (4.2) it follows that

$$D(\alpha_{l+1},\dots,\alpha_1)x(t)+\rho_{n-l-1}(t)x(t)^{\alpha_1\cdots\alpha_n}\le 0,\qquad t\ge t_0.$$

Application of Lemma 2.4 shows that (4.1_l) has a positive solution $z(t)$ which satisfies (1.2) and (1.3) with n replaced by $l+1$. The proof of the theorem is complete. □

If we estimate $x(t)$ from below as in the proof of Theorem 3.1 and substitute it into (4.1_l), we obtain

$$D(\alpha_{l+1},\dots,\alpha_1)x(t)+R_l^{\alpha_1\cdots\alpha_{l+1}}(t-t_0)^{(r_{l-1}-1)\alpha_1\cdots\alpha_{l+1}}\rho_{n-l-1}(t)\left[D(\alpha_{l-1},\dots,\alpha_1)x(t)\right]^{\alpha_l\alpha_{l+1}}\le 0\tag{4.3}$$

for $t\ge t_0$. Let $y(t)$ be given by

$$y(t)=\left[D(\alpha_{l-1},\dots,\alpha_1)x(t)\right]^{\alpha_l}.$$

Then $y(t)$ satisfies the second order differential inequality

$$\left(|y'(t)|^{\alpha_{l+1}}\operatorname{sgn} y'(t)\right)'+R_l^{\alpha_1\cdots\alpha_{l+1}}(t-t_0)^{(r_{l-1}-1)\alpha_1\cdots\alpha_{l+1}}\rho_{n-l-1}(t)|y(t)|^{\alpha_{l+1}}\operatorname{sgn} y(t)\le 0,\qquad t\ge t_0,$$

and, by Lemma 2.4, there exists a nonoscillatory solution $z(t)$ (of degree 1) of the corresponding differential equation

$$\left(|z'(t)|^{\alpha_{l+1}}\operatorname{sgn} z'(t)\right)'+R_l^{\alpha_1\cdots\alpha_{l+1}}(t-t_0)^{(r_{l-1}-1)\alpha_1\cdots\alpha_{l+1}}\rho_{n-l-1}(t)|z(t)|^{\alpha_{l+1}}\operatorname{sgn} z(t)=0,\qquad t\ge t_0,\tag{4.4_l}$$

which is the same as (3.6_l).

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Periodic solutions of relativistic Liénard-type equations

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Abstract. In this paper, we prove that the relativistic Liénard-type equation

$$\frac{d}{dt}\left(\frac{\dot{x}\,|\dot{x}|^{p-2}}{\left(1-|\dot{x}|^{p}\right)^{\frac{p-1}{p}}}\right)+f\left(x\right)\dot{x}+g\left(x\right)=0,\qquad p>1,$$

and its special case, relativistic Van der Pol-type equation, have a periodic solution. Our results are inspired by the results obtained by Mawhin and Villari [*Nonlinear Anal.* **160**(2017), 16–24] and extend their results to this more general case.

Keywords: closed orbits, periodic solutions, limit cycles, relativistic Liénard-type equations.

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1 Introduction

In 1926, Van der Pol [16] considered the equation

$$\ddot{x}+\mu\left(x^{2}-1\right)\dot{x}+x=0,\qquad \mu\neq 0, \tag{1.1}$$

to obtain the results about relaxation oscillations which are important in physics and engineering problems. In 1928, Liénard [9] gave a more general description of relaxation oscillations for the equation

$$\ddot{x}+f\left(x\right)\dot{x}+g\left(x\right)=0, \tag{1.2}$$

where $g\left(x\right)$ is positive when $x>0$ and negative when $x<0$, $f\left(x\right)$ is negative for small values of $|x|$ and positive for large values of $|x|$. In point of fact, he takes $g(x)=x$. The more general form was first dealt with by Levinson and Smith [8]. The equations (1.1) and (1.2) are known as Van der Pol and Liénard equations, respectively. Since the appearance of Van der

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Pol and Liénard's fundamental papers, various proofs and generalizations or improvements have appeared in the literature. For example, in 1942, Levinson and Smith [8] obtained the relaxation oscillations for a more general equation

$$\ddot{x} + f(x, \dot{x})\,\dot{x} + g(x) = 0,$$

where $g(x)$ is positive when $x > 0$ and negative when $x < 0$, $f(x, \dot{x})$ is damping coefficient which for large $|x|$ is positive and for small $|\dot{x}|$ and $|x|$ is negative.

In the last ten years, the study of the existence and multiplicity of periodic solutions of second order equations where $\ddot{x}$, with $\dot{x}$ denoting the derivative of x with respect to t, is replaced by a relativistic acceleration $\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}\right)$ has been considered by many authors [2, 12, 13, 15]. To the best of our knowledge, this is the first paper using a generalized relativistic acceleration

$$\frac{d}{dt}\left(\frac{\dot{x}\,|\dot{x}|^{p-2}}{(1-|\dot{x}|^p)^{\frac{p-1}{p}}}\right)$$

to study the following problems. It should be noted that the definition of the generalized relativistic acceleration is given by the generalizations chosen in the numerator and the denominator, and this choice of the denominator will be clear in the following Section 2. It is easy to see that the inverse of the generalized curvature operator

$$\Phi_p(v) = \frac{v\,|v|^{p-2}}{\left(1-|v|^p\right)^{\frac{p-1}{p}}}, \qquad v \in (-1, 1),$$

is

$$\Phi_q^{-1}(v) = \frac{v\,|v|^{q-2}}{\left(1+|v|^q\right)^{\frac{q-1}{q}}}, \qquad v \in \mathbb{R}, \tag{1.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\left|\Phi_q^{-1}\right| < 1$. In the literature, the authors obtained some results for equations with relativistic acceleration by using various methods [1–3, 6, 10–12].

More recently, Fujimoto and Yamaoka [7] and Pérez-González et al. [13] have obtained the results about the existence and uniqueness of limit cycles of the Liénard-type differential equations of forms

$$\frac{d}{dt}(\phi(\dot{x})) + f(x)\,\phi(\dot{x}) + g(x) = 0$$

and

$$\frac{d}{dt}(\varphi(\dot{x})) + f(x)\,\psi(\dot{x}) + g(x) = 0$$

involving the curvature operators, respectively.

The aim of this paper is to obtain new results about the existence and uniqueness of limit cycles for the generalized relativistic Liénard equations of the form

$$\frac{d}{dt}\left(\frac{\dot{x}\,|\dot{x}|^{p-2}}{(1-|\dot{x}|^p)^{\frac{p-1}{p}}}\right) + f(x)\,\dot{x} + g(x) = 0, \qquad p > 1, \tag{1.4}$$

where the continuous functions f and g satisfy some conditions, inspired by Mawhin and Villari [12].

2 Relativistic duffing and Liénard-type equations

We now consider the relativistic Liénard-type equation (1.4), with $xg(x) > 0$ and $g(0) = 0$, so that $(0,0)$ is an equilibrium. Solutions of Eq. (1.4) must of course be such that $|\dot{x}(t)| < 1$ for all $t \in \mathbb{R}$, so that, instead of considering the usual phase plane $\mathbb{R}^2$, one is a priori restricted to the strip $\mathbb{R} \times (-1,1)$. A way to avoid this difficulty is to make a change of variable

$$y = \frac{\dot{x}\,|\dot{x}|^{p-2}}{\left(1-|\dot{x}|^{p}\right)^{\frac{p-1}{p}}}, \qquad p>1,\ |\dot{x}|<1,$$

which is equivalent to

$$\dot{x} = \frac{y\,|y|^{q-2}}{\left(1+|y|^{q}\right)^{\frac{q-1}{q}}}, \qquad \frac{1}{p}+\frac{1}{q}=1, \quad y\in\mathbb{R},$$

from (1.3). Then, Eq. (1.4) can be written as a pair of first order equations

$$\dot{x} = \frac{y\,|y|^{q-2}}{\left(1+|y|^{q}\right)^{\frac{q-1}{q}}}, \qquad \dot{y} = -f(x)\,\frac{y\,|y|^{q-2}}{\left(1+|y|^{q}\right)^{\frac{q-1}{q}}} - g(x). \tag{2.1}$$

On the other hand, Eq. (1.4) can be rewritten in the form below

$$\frac{d}{dt}\left[\frac{\dot{x}\,|\dot{x}|^{p-2}}{\left(1-|\dot{x}|^{p}\right)^{\frac{p-1}{p}}} + F(x)\right] + g(x) = 0,$$

where $F(x) = \int_0^x f(s)ds$. If we make the change of variable

$$y = \frac{\dot{x}\,|\dot{x}|^{p-2}}{\left(1-|\dot{x}|^{p}\right)^{\frac{p-1}{p}}} + F(x), \qquad p>1, \quad |\dot{x}|<1,$$

then we have

$$\dot{x} = \frac{(y-F(x))\,|y-F(x)|^{q-2}}{\left(1+|y-F(x)|^{q}\right)^{\frac{q-1}{q}}}, \qquad \frac{1}{p}+\frac{1}{q}=1, \quad (y-F(x))\in\mathbb{R},$$

from (1.3). Thus, Eq. (1.4) can be written as a pair of first order equations

$$\dot{x} = \frac{(y-F(x))\,|y-F(x)|^{q-2}}{\left(1+|y-F(x)|^{q}\right)^{\frac{q-1}{q}}}, \qquad \dot{y} = -g(x). \tag{2.2}$$

From this follows immediately the following regularity result.

Lemma 2.1. *If $q > 2$, $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $g : \mathbb{R} \to \mathbb{R}$ locally Lipschitzian, the Cauchy problem for Eq. (1.4) or (2.1) or (2.2) is locally uniquely solvable.*

Proof. It suffices to notice that F is of class C^1, and apply standard results [5] to system (2.2). □

Note that for $1 < q \leq 2$, the first equation of system (2.2) does not satisfy the locally Lipschitz conditions at the origin, and this case will be discussed below.

We now consider the corresponding Duffing-type equation, for which $f \equiv 0$,

$$\frac{d}{dt}\left(\frac{\dot{x}\,|\dot{x}|^{p-2}}{\left(1-|\dot{x}|^{p}\right)^{\frac{p-1}{p}}}\right)+g\left(x\right)=0, \tag{2.3}$$

and the system (2.1) reduces to

$$\dot{x}=\frac{y\,|y|^{q-2}}{\left(1+|y|^{q}\right)^{\frac{q-1}{q}}},\qquad \dot{y}=-g\left(x\right). \tag{2.4}$$

We observe that the system (2.4) has the Hamiltonian structure

$$\dot{x}=\frac{\partial H}{\partial y}\left(x,y\right),\qquad \dot{y}=-\frac{\partial H}{\partial x}\left(x,y\right),$$

where the Hamiltonian function $H\left(x,y\right)$ is given by

$$H\left(x,y\right)=\left(1+|y|^{q}\right)^{\frac{1}{q}}-1+G\left(x\right)$$

and the function $G(x)$ is the integral of $g(x)$, $G\left(x\right)=\int_0^x g\left(s\right)ds$. It is well known that the level curves of the function $H\left(x,y\right)$ are its solutions. If we consider the level curve

$$\left(1+|y|^{q}\right)^{\frac{1}{q}}-1+G\left(x\right)=C \tag{2.5}$$

in the dynamical interpretation as motion of a particle, the first term represents its kinetic energy and (2.5) expresses the law of conservation of energy as applied to the particle. Note that the constant 1 from $\left(1+|y|^{q}\right)^{\frac{1}{q}}$ is subtracted in order that, for $|y|$ small, the result $(1+|y|^{q})^{\frac{1}{q}}-1$ is close to the classical expression $\frac{y^2}{2}$.

Now, we mention a result given by Rebelo [14].

Theorem A ([14, Theorem 1])**.** *If the initial value (x_0,y_0) is not an equilibrium, that is, that $\nabla H\left(x_0,y_0\right)\neq(0,0)$, the Cauchy problem for Eq.* (2.3) *or* (2.4) *is locally uniquely solvable.*

We observe that in virtue of this result for system (2.4) Lemma 2.1 holds also for $1 < q \leq 2$ if the initial value is not the origin.

It is easy to see that the origin $(0,0)$ of our (x,y)-phase plane is a global center for the system (2.4) if and only if $G(x)\to+\infty$ as $|x|\to\infty$ as in the classical case. The time rate of change of H along a solution trajectory is given by

$$\begin{aligned}\frac{\partial H}{\partial t}\left(x,y\right)&=\frac{\partial H}{\partial x}\left(x,y\right)\frac{dx}{dt}+\frac{\partial H}{\partial y}\left(x,y\right)\frac{dy}{dt}\\&=g\left(x\right)\frac{y\,|y|^{q-2}}{\left(1+|y|^{q}\right)^{\frac{q-1}{q}}}-\frac{y\,|y|^{q-2}}{\left(1+|y|^{q}\right)^{\frac{q-1}{q}}}\left(f\left(x\right)\frac{y\,|y|^{q-2}}{\left(1+|y|^{q}\right)^{\frac{q-1}{q}}}+g\left(x\right)\right)\\&=-f\left(x\right)\left(\frac{y\,|y|^{q-2}}{\left(1+|y|^{q}\right)^{\frac{q-1}{q}}}\right)^{2}.\end{aligned}$$

Therefore, at points where $f(x)$ is positive, the trajectories of system (2.1) enter trajectories of system (2.4), while, at points where $f(x)$ is negative, the trajectories of system (2.1) exit trajectories of system (2.4). In virtue of this result, being $f(0) < 0$, the unique equilibrium $(0,0)$ for system (2.1) and system (2.4) as well, is a source. Therefore, for both systems, the Cauchy problem is uniquely solvable in future also for $1 < q \leq 2$ and this completes the result of Lemma 2.1. Moreover, the slope of the trajectories of system (2.1) is given by the following expression, where y' denotes the derivative of y with respect to x,

$$y'(x) = \frac{\dot{y}}{\dot{x}} = -f(x) - g(x)\,\frac{\left(1+|y|^q\right)^{\frac{q-1}{q}}}{y\,|y|^{q-2}}, \tag{2.6}$$

and the 0-isocline, namely the curve in which $\dot{y} = 0$, is given by

$$\frac{y\,|y|^{q-2}}{\left(1+|y|^q\right)^{\frac{q-1}{q}}} = -\frac{g(x)}{f(x)}.$$

At this point, we need to prove the existence of a winding trajectory for system (2.1) in order to apply the Poincaré–Bendixson theorem [5].

3 The relativistic Van der Pol-type equation

At first, we discuss the relativistic Van der Pol-type equation

$$\frac{d}{dt}\left(\frac{\dot{x}\,|\dot{x}|^{p-2}}{\left(1-|\dot{x}|^p\right)^{\frac{p-1}{p}}}\right) + \mu\left(x^2-1\right)\dot{x} + x = 0, \tag{3.1}$$

where $p > 1$ and $\mu \neq 0$, although interesting results, and in particular the existence of limit cycles, can be proved in a similar way for Eq. (1.4). Notice the case where $\mu < 0$ is reduced to the case where $\mu > 0$ by changing t into $-t$, so that we can assume without loss of generality that $\mu > 0$.

For this particular equation, system (2.1) becomes

$$\dot{x} = \frac{y\,|y|^{q-2}}{\left(1+|y|^q\right)^{\frac{q-1}{q}}}, \qquad \dot{y} = -\mu\left(x^2-1\right)\frac{y\,|y|^{q-2}}{\left(1+|y|^q\right)^{\frac{q-1}{q}}} - x, \tag{3.2}$$

and the 0-isocline is given by

$$\frac{y\,|y|^{q-2}}{\left(1+|y|^q\right)^{\frac{q-1}{q}}} = -\frac{x}{\mu\left(x^2-1\right)}. \tag{3.3}$$

Observe first that for $f(x) = \mu(x^2-1)$, $f(0) = -1 < 0$ and hence the origin of the phase plane is a source.

The 0-isocline in the classical Van der Pol equation is given by

$$y = -\frac{x}{\mu\left(x^2-1\right)}. \tag{3.4}$$

Of course, points of (3.3) only correspond to those x for which $-\frac{x}{\mu(x^2-1)} \in (-1,1)$, i.e., as easily shown, to the x belonging to the set

$$(-\infty,-x_2)\cup(-x_1,x_1)\cup(x_2,+\infty),$$

where

$$x_1=-\frac{1}{2\mu}+\sqrt{\frac{1}{4\mu^2}+1}\in(0,1)\,,\qquad x_2=\frac{1}{2\mu}+\sqrt{\frac{1}{4\mu^2}+1}\in(1,+\infty)\,.$$

Hence, (3.3) can be seen as 'stretching' the restriction of (3.4) to $\mathbb{R}\times(-1,1)$ to $\mathbb{R}^2$ (see Figs. 3.1 and 3.2).

We know that define $\gamma^+(S)$ as the positive semi-trajectory starting from S, and assume that $\gamma^+(S)$ moves around the origin and intersects again the y-axis in the same half-plane of S at a point $R=(0,y_R)$. Clearly, such semi-trajectory is winding if $|y_R|<|y_S|$, unwinding if $|y_R|>|y_S|$, and a cycle if $|y_R|=|y_S|$ [4]. At this point, arguing in the same way as in the classical case considered in [17], we are able to produce a winding trajectory. As the origin is a source, we can apply the Poincaré–Bendixson theorem [5] and get the existence of at least one limit cycle for (3.2).

We assume that Λ_1 is the graph of the function

$$\frac{y\,|y|^{q-2}}{\left(1+|y|^q\right)^{\frac{q-1}{q}}}=-\frac{x}{\mu\left(x^2-1\right)}$$

for $x\in(-\infty,-x_2)$. In this case, from (1.3), the function

$$y_1(x)=-\frac{x\,|x|^{p-2}}{\left(|\mu\left(x^2-1\right)|^p-|x|^p\right)^{\frac{p-1}{p}}},\ \frac{1}{p}+\frac{1}{q}=1, \tag{3.5}$$

is an increasing positive function. Similarly, we define that Λ_2 is the graph of (3.3) for $x\in(x_2,+\infty)$ and so the function $y_2(x)$ given by (3.5) is an increasing negative function. Then, we get

$$\lim_{x\to-x_2^-}y_1(x)=+\infty,\qquad \lim_{x\to x_2^+}y_2(x)=-\infty,$$
$$\lim_{x\to-\infty}y_1(x)=0,\qquad \lim_{x\to+\infty}y_2(x)=0.$$

From the assumptions in [17], we can choose a point γ in the curve Λ_1 whose abscissa x_γ is to the left of $-x_2$ and whose ordinate is larger than the values which $y_1(x)$ takes for $x<x_\gamma$. We now define the function

$$G(x,y)=-\mu\left(x^2-1\right)\frac{y\,|y|^{q-2}}{\left(1+|y|^q\right)^{\frac{q-1}{q}}}-x.$$

Since

$$\frac{d}{dy}\left(\frac{y\,|y|^{q-2}}{\left(1+|y|^q\right)^{\frac{q-1}{q}}}\right)=\frac{(q-1)\,|y|^{q-2}}{\left(1+|y|^q\right)^{\frac{2q-1}{q}}}>0$$

for $q>1$ and $y\in\mathbb{R}\backslash\{0\}$, the function $\frac{y|y|^{q-2}}{\left(1+|y|^q\right)^{\frac{q-1}{q}}}$ is an increasing function of y and so $\dot{y}=G(x,y)$ is a decreasing function of y for each fixed $x\notin[-1,1]$. The trajectory which passes through the point γ comes from 'infinity' without intersecting the x-axis before reaching the point $\gamma = (x_\gamma, y_1(x_\gamma))$ in the curve Λ_1. Since

$$y'(x) = \frac{\dot{y}}{\dot{x}} = -\mu(x^2 - 1) - x \frac{\left(1 + |y|^q\right)^{\frac{q-1}{q}}}{y\,|y|^{q-2}} \tag{3.6}$$

gives the slope of tangent to the path of (3.2) passing through the point (x, y), the trajectory does not have vertical asymptotes and, being bounded away from the x-axis, it must cross the y-axis. By an analogous argument, we can claim that the trajectory, after entering the $x > 0$ half-plane, either will cross the x-axis on the interval $(0, x_2]$, or will cross the line $x = x_2$. In the latter case, $y(x)$ will decrease after $x = x_2$. Since the inequality

$$|x| + \mu\left(x^2 - 1\right) > |x| > x_2 > 0 \quad \text{for } |x| > x_2 \tag{3.7}$$

holds, the trajectory does not have a horizontal asymptote and it must eventually cross the x-axis for $x > x_2$. From (3.6), the trajectory must meet the y-axis at some $y < 0$.

Afterwards, as a consequence of (3.7) again, the trajectory cuts the x-axis either on the $-x_2 < x < 0$ segment, or at some $x \le -x_2$. In the latter case, the trajectory may cut the curve Λ_1, but the ordinate of crossing point must be smaller than $\sup_{x \in (-\infty, x_\gamma)} y_1(x)$. Eventually, the trajectory must remain below the graph Λ_1, and so it is bounded.

Similarly, $y_2(x)$ is bounded to corresponding treatment which starts from a point $\delta \in \Lambda_2$ with abscissa $x_\delta > x_2$. Thus, we have found that starting at $t = 0$ from a point γ (or δ), the state $(x(t), y(t))$ moves for $t > 0$ along a bounded trajectory. The limit set is compact and non-empty. Since the only critical point (the origin) is repulsive, we can conclude that the limit set must be a cycle. Therefore, there exists at least one periodic solution for (3.1).

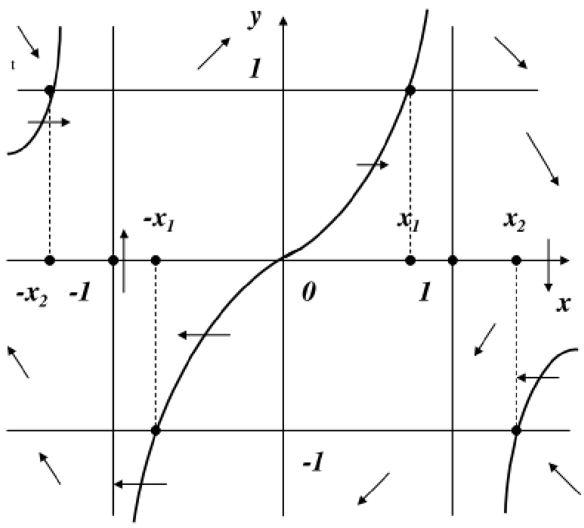

Figure 3.1: Classical Van der Pol equation for $p = 2$. Vertical asymptotes points are -1 and 1.

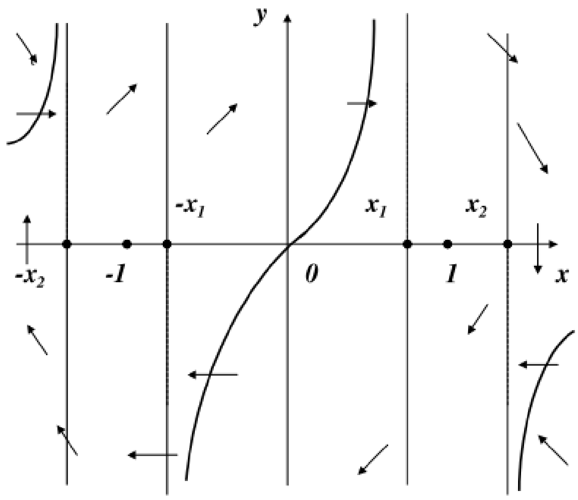

Figure 3.2: Relativistic Van der Pol-type equation for $p > 1$. Vertical asymptotes points are $\pm x_1$ and $\pm x_2$.*

As a result of the above, the following result is given.

Theorem 3.1. *For each $\mu \neq 0$, Eq. (3.1) has a least one nontrivial periodic solution.*

4 The relativistic Liénard-type equation

Following the strategy used in reference [12], we return to system (2.1) and first compare the slope of the relativistic Liénard-type system (2.6) with the slope of the classical Liénard system, namely

$$y'(x) = -f(x) - \frac{g(x)}{y}.$$

Now, we show that a direct comparison of the slopes at the same point (x, y). Since we have $|y|^q < |y|^{\frac{q}{q-1}} + |y|^{q+\frac{q}{q-1}}$ for $q > 1$ and all $y \in \mathbb{R} \setminus \{0\}$, we get

$$1 < \frac{(1+|y|^q)\,|y|^{\frac{q}{q-1}}}{|y|^q} \quad \text{and so} \quad 1 < \frac{(1+|y|^q)^{\frac{q-1}{q}}\,|y|}{|y|^{q-1}}.$$

Without loss of generality, we may take $y > 0$. The case when y is negative can similarly be dealt with. It is easy to see that while we have

$$-f(x) - \frac{g(x)}{y} > f(x) - g(x)\,\frac{(1+|y|^q)^{\frac{q-1}{q}}}{y\,|y|^{q-2}}$$

when $x > 0$, we have

$$-f(x) - \frac{g(x)}{y} < f(x) - g(x)\,\frac{(1+|y|^q)^{\frac{q-1}{q}}}{y\,|y|^{q-2}}$$

*Figures 3.1 and 3.2 are taken from the reference [12, p. 20].

when $x < 0$. Therefore, if $xy > 0$, the trajectories of system (2.1) enter the trajectories of the classical Liénard system

$$\dot{x} = y, \qquad \dot{y} = -f(x)\,y - g(x)\,, \tag{4.1}$$

while if $xy < 0$, the trajectories of system (2.1) exit the trajectories of system (4.1). So, when $xy > 0$, the trajectories of (2.1) are guided by those of (4.1). The question is then the intersection of a positive semitrajectory with the x-axis, because in this way one can prove that trajectories are clockwise and then apply the Poincaré–Bendixson theorem [5].

When $F(x)$ is bounded from below for x positive large enough and bounded from above for x negative large enough, Villari [18] has proved that the condition

$$\limsup_{x\to+\infty} \left(G(x) + F(x)\right) = +\infty \tag{4.2}$$

is necessary and sufficient in order that a positive semitrajectory starting with a nonnegative y intersects the x-axis, and that the condition

$$\limsup_{x\to-\infty} \left(G(x) - F(x)\right) = +\infty$$

is necessary and sufficient in order that a positive semitrajectory starting with a nonpositive y intersects the x-axis. The results are proved in the Liénard plane but hold as well in the phase plane.

More general situations have been considered by Villari and Zanolin in [19], that we shall adapt to the present situation. Likewise in [19], given $f : \mathbb{R} \to \mathbb{R}$ continuous, $F(x) = \int_0^x f(s)ds$, $g : \mathbb{R} \to \mathbb{R}$ continuous, we define $\Gamma_+ : \mathbb{R} \to \mathbb{R}$ by

$$\Gamma_+ = \int_0^x \left(1 + F_+(s)\right)^{-1} g(s)\, ds,$$

where $F_+(x) = \max\{0, F(x)\}$. We also define $G(x) = \int_0^x g(s)ds$.

Theorem 4.1. *Assume that the following conditions hold.*

(1) $f : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitzian, $xg(x) > 0$ for $x \neq 0$, and $f(0) < 0$.

(2) There exists $a > 0$ such that $f(x) > 0$ when $x > a$,

$$\lim_{x\to+\infty} G(x) = K < +\infty, \qquad \lim_{x\to+\infty} F(x) = +\infty.$$

(3) There exists $0 < \gamma < 4$ such that

$$\limsup_{x\to-\infty} \left(\gamma\Gamma_+(x) - F(x)\right) = +\infty.$$

Then Eq. (1.4) has at least a stable limit cycle.

Proof. Notice that Assumption 2 rules the behavior of f and g for $x > 0$ and Assumption 3 for $x < 0$. We first consider the behavior of a trajectory when $x > 0$. Let $K > 0$ be such that $G(x) < K$ for all $x \in \mathbb{R}$, according to the second condition in Assumption 2. We define $H : \mathbb{R}^2 \to \mathbb{R}$

$$H(x,y) := \left(1 + |y|^q\right)^{\frac{1}{q}} - 1 + G(x)$$

and consider the corresponding curve of equation

$$K = \left(1 + |y|^q\right)^{\frac{1}{q}} - 1 + G(x). \tag{4.3}$$

It intersects the y-axis at the point $\left(0, -\left[(K+1)^q - 1\right]^{\frac{1}{q}}\right)$. On the other hand, as $G(x) < K$ for all $x \in \mathbb{R}$ the curve with Eq. (4.3) does not intersect the x-axis. For $a > 0$ given in Assumption 2, the curve with Eq. (4.3) intersects the line $x = a$ at the point of ordinate

$$y = -\beta := -\left\{\left[(K+1) - G(a)\right]^q - 1\right\}^{\frac{1}{q}}.$$

When $G(x) \to K$, this expression tends to 0, as expected. Following an argument that appeared in [4] and [18] and a slope comparison, we observe that the negative semi-trajectory $\gamma^-(P)$ with $P = (a, -\beta)$ does not intersect the x-axis. On the other hand, as its slope is bounded, the semi-trajectory $\gamma^+(P)$ intersects the y-axis, say at point $Q = (0, \overline{y})$ with $\overline{y} < 0$.

We now consider the behavior of a trajectory when $x < 0$. For the classical Liénard system

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x),$$

we know from [19] that if Assumption 3 holds, then the positive semi-trajectory $\widehat{\gamma}^+(Q)$ starting from some point $Q = (\gamma, -\beta)$ with $\gamma \in (0, 4)$ given in Assumption 3 and $\beta > 0$ intersects the vertical isocline, and therefore the x-axis at some point $R = (\widehat{x}, 0)$. The interesting case is the one where $f(x)$ is eventually negative, which corresponds to the last condition in Assumption 2. Hence, by definition of Γ_+, $G(x)$ must dominate $F(x)$. Using a comparison argument, the positive semi-trajectory $\gamma^+(Q)$ of (2.1) must intersect the x-axis at some point $S = (x, 0)$, with $\widehat{x} < x < 0$. Now, as its slope is bounded, the semi-trajectory $\gamma^*(S)$ must intersect the y-axis at some point $(0, y)$ with $y > 0$ and, in virtue of (4.2), eventually intersects the x-axis at some point $(x, 0)$ with $x > 0$.

Therefore $\gamma(P)$ is winding. The origin being a source because of the last condition in Assumption 1, we apply the Poincaré–Bendixson theorem [5] and obtain the existence of a stable limit cycle. Like in [19], a 'dual' result holds if the conditions for $x > 0$ and $x < 0$ are inverted, whose statement is left to the reader. □

Remark 4.2. It is easy to see that if we take $p = 2$ in our results, then they reduce to that of [12].

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Study of a cyclic system of difference equations with maximum

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Abstract. In this paper we study the behaviour of the solutions of the following cyclic system of difference equations with maximum:

$$x_i(n+1) = \max\left\{A_i, \frac{x_i(n)}{x_{i+1}(n-1)}\right\}, \qquad i = 1,2,\dots,k-1,$$
$$x_k(n+1) = \max\left\{A_k, \frac{x_k(n)}{x_1(n-1)}\right\}$$

where $n = 0,1,2,\dots$, A_i, $i = 1,2,\dots,k$, are positive constants, $x_i(-1), x_i(0)$, $i = 1,2,\dots,k$, are real positive numbers. Finally for $k = 2$ under some conditions we find solutions which converge to periodic six solutions.

Keywords: difference equations with maximum, equilibrium, eventually equal to equilibrium, periodic solutions.

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1 Introduction

Max operators play an important role in the study of some problems in automatic control (see [16, 17]). This fact was one, among others, which motivated some authors to consider differences equations with maximum (see [1–7, 10–15, 20, 21, 23–37, 40–42, 45–47]).

In the beginning, majority of the papers in the topic studied special cases of difference equations in the following form:

$$y_{n+1} = \max\left\{\frac{A_0}{y_n}, \frac{A_1}{y_{n-1}}, \dots, \frac{A_k}{y_{n-k}}\right\}, \qquad n = 0,1,2,\dots,$$

where k is a natural number, whereas the coefficients A_j, $j = 0,1,\dots,k$, are real numbers (see, for example, [2, 5, 7, 12–15, 23, 45–47]).

The study of positive solutions of the following difference equation with maximum

$$x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{B}{x_{n-2}}\right\}, \qquad n = 0,1,2,\dots,$$

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conducted in [14] showed that a suitable change of variables transforms it to the difference equation with maximum of the form:

$$y_{n+1} = \max\left\{D, \frac{y_n}{y_{n-1}}\right\} \tag{1.1}$$

where $D = AB^{-1}$, which suggested the investigation of the equation. Among other things, [14] studied the periodicity of positive solutions of equation (1.1).

This also naturally suggested investigations of difference equations in the following form:

$$y_{n+1} = \max\left\{D, \frac{y_{n-k}}{y_{n-m}}\right\}, \qquad n = 0, 1, 2, \ldots,$$

where k and m are nonnegative integers (for some important results on the difference equation see [1]), which was soon after publication of [1] continued in a comprehensive study of the following difference equation

$$y_{n+1} = \max\left\{D, \frac{y^p_{n-k}}{y^q_{n-m}}\right\}, \qquad n = 0, 1, 2, \ldots,$$

and its natural generalizations, by S. Stević and his collaborators (see, for example, [10, 11, 25–31, 35–37, 40, 42]).

On the other hand, equation (1.1) suggested also studying of the corresponding close-to-symmetric systems of difference equations (some related rational ones had been previously studied for example in [18, 19]).

In [6] the authors studied the periodicity of the positive solutions of the system of difference equations with maximum which is a close-to-symmetric cousin of equation (1.1) :

$$x_{n+1} = \max\left\{A, \frac{y_n}{x_{n-1}}\right\},$$
$$x_{n+1} = \max\left\{B, \frac{x_n}{y_{n-1}}\right\},$$

where $n = 0, 1, 2, \ldots$, and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive real numbers.

Some other results on systems of difference equations with maximum can be found in [21, 24, 33–35, 37, 42]. Recall also that many difference equations and systems with maximum are connected with periodicity (see, e.g., [3–5, 12, 29, 32, 34, 41, 45–47]), a typical characteristic of positive solutions of the equations and systems. For some results on the boundedness character of difference equations and systems with maximum see [1, 3, 13, 20, 40]. The paper [1] is interesting since it also considers real solutions to a difference equations with maximum, unlike great majority of other ones.

On the third side, in [8] Iričanin and Stević suggested investigation of cyclic systems of difference equations, which later motivated some further investigations in the direction (see, for example, [9, 22, 38]).

In what follows we use the following convention (see [8]). If i and j are integers such that $i = j \pmod k$, then we will regard that $A_i = A_j$ and $x_i(n) = x_j(n)$. For example, we identify the number A_0 with A_k, and identify the sequence $x_{k+1}(n)$ with $x_1(n)$ (the convention is used in the systems which follows).

Motivated by above mentioned facts, in this paper we study the behaviour of the solutions of the following cyclic system of difference equations with maximum:

$$x_i(n+1) = \max\left\{A_i, \frac{x_i(n)}{x_{i+1}(n-1)}\right\}, \qquad i = 1,2,\dots,k, \tag{1.2}$$

where $n = 0,1,2,\dots$, the coefficients A_i, $i = 1,2,\dots,k$, are positive constants, and the initial values $x_i(-1), x_i(0)$, $i = 1,2,\dots,k$, are positive real numbers. Moreover for $k = 2$ under some conditions we find solutions which converge to periodic six solutions.

2 Study of system (1.2)

First we study the existence of equilibrium point for (1.2).

Proposition 2.1. *Consider system (1.2) where A_i, $i = 1,2,\dots,k$, are positive constants and $x_i(-1)$, $x_i(0)$, $i = 1,2,\dots,k$, are positive real numbers. Then the following statements are true:*

I. *Suppose that*

$$A_i > 1, \qquad i = 1,2,\dots,k. \tag{2.1}$$

Then (1.2) has a unique equilibrium $(x_1, x_2, \dots, x_k) = (A_1, A_2, \dots, A_k)$.

II. *Suppose that there exists an r, $r \in \{1,2,\dots,k\}$ such that*

$$(A_r - 1)(A_{r+1} - 1) < 0. \tag{2.2}$$

Then (1.2) has no equilibrium.

III. *Let*

$$0 < A_i < 1, \qquad i = 1,2,\dots,k \tag{2.3}$$

be satisfied. Then system (1.2) has a unique equilibrium $(x_1, x_2, \dots, x_k) = (1,1,\dots,1)$.

Proof. **I.** We consider the system of algebraic equations

$$x_i = \max\left\{A_i, \frac{x_i}{x_{i+1}}\right\}, \qquad i = 1,2,\dots,k. \tag{2.4}$$

We would like to point out that in (2.4) we use the following convention: if i and j are integers, then we regard that $x_i = x_j$ if $i = j \,(\operatorname{mod} k)$ (see the previous section). Since $x_i \geq A_i > 1$, $i = 1,2,\dots,k$ it is obvious that

$$x_i \neq \frac{x_i}{x_{i+1}}, \qquad i = 1,2,\dots,k.$$

From this it easily follows that system (2.4) has a unique solution

$$(x_1, x_2, \dots, x_k) = (A_1, A_2, \dots, A_k).$$

II. Suppose that there exists $r \in \{1,2,\dots,k\}$ such that inequalities (2.2) hold. Then either

$$A_r < 1, \qquad A_{r+1} > 1 \tag{2.5}$$

or

$$A_r > 1, \qquad A_{r+1} < 1 \tag{2.6}$$

are satisfied.

Suppose firstly that (2.5) hold. From (2.4) we get

$$x_r = \max\left\{A_r, \frac{x_r}{x_{r+1}}\right\}. \tag{2.7}$$

Relations (2.4) and (2.5) imply that $x_{r+1} \geq A_{r+1} > 1$. Hence we have

$$\frac{x_r}{x_{r+1}} \leq \frac{x_r}{A_{r+1}} < x_r$$

and so from (2.7) we take $x_r = A_r$. Moreover, from (2.4) we get

$$x_{r-1} = \max\left\{A_{r-1}, \frac{x_{r-1}}{x_r}\right\} = \max\left\{A_{r-1}, \frac{x_{r-1}}{A_r}\right\} \geq \frac{x_{r-1}}{A_r}$$

which is a contradiction since $0 < A_r < 1$, $r = 1, 2, \dots, k$. So (1.2) has no equilibrium.

Assume now that (2.6) is satisfied. Suppose that there exists a $j \in \{1, 2, \dots, r\}$ such that $A_j < 1$. Let $s = \max\{j : A_j < 1,\ j \in \{1, 2, \dots, r\}\}$. Then it is obvious that

$$A_s < 1, \qquad A_{s+1} > 1. \tag{2.8}$$

Then arguing as in the case where (2.5) hold, system (1.2) has no equilibrium. Assume that there exists a $j \in \{r+2, r+3, \dots, k\}$ such that $A_j > 1$. Let $v = \min\{j : A_j > 1,\ j \in \{r+2, r+3, \dots, k\}\}$. Then we get

$$A_{v-1} < 1, \qquad A_v > 1. \tag{2.9}$$

So, arguing again as above we have that (1.2) has no equilibrium.

Finally suppose that

$$A_j > 1, \quad j = 1, 2, \dots, r, \qquad A_v < 1, \quad v = r+1, r+2, \dots, k. \tag{2.10}$$

Then since from (2.10) $A_1 > 1$ we take $\frac{x_k}{x_1} \leq \frac{x_k}{A_1} < x_k$. Thus we get from (2.4)

$$x_k = \max\left\{A_k, \frac{x_k}{x_1}\right\} = A_k < 1. \tag{2.11}$$

Moreover, from (2.4), (2.10) and (2.11) it holds,

$$x_{k-1} = \max\left\{A_{k-1}, \frac{x_{k-1}}{x_k}\right\} = \max\left\{A_{k-1}, \frac{x_{k-1}}{A_k}\right\} \geq \frac{x_{k-1}}{A_k} > x_{k-1}$$

which is a contradiction and so (1.2) has no equilibrium.

III. We claim that there exists $r \in \{1, 2, \dots, k\}$ such that

$$\frac{x_r}{x_{r+1}} \geq 1. \tag{2.12}$$

Suppose on the contrary that

$$\frac{x_i}{x_{i+1}} < 1, \qquad i = 1, 2, \dots, k,$$

(recall that for $i = k$ it means $\frac{x_k}{x_1} < 1$). Then we get

$$1 = \frac{x_1}{x_2}\frac{x_2}{x_3}\cdots\frac{x_k}{x_1} < 1$$

which is not true.

Therefore there exists an r such that (2.12) holds. From (2.3), (2.7) and (2.12) we have

$$x_r = \frac{x_r}{x_{r+1}}.$$

Hence $x_{r+1} = 1$. In addition from (2.4) we take

$$1 = x_{r+1} = \max\left\{A_{r+1}, \frac{x_{r+1}}{x_{r+2}}\right\} = \max\left\{A_{r+1}, \frac{1}{x_{r+2}}\right\}.$$

Then from (2.3) it is obvious that $x_{r+2} = 1$. Working inductively we take $x_j = 1$, $j = r+1,\dots,k$. From (2.4) we get

$$1 = x_k = \max\left\{A_k, \frac{x_k}{x_1}\right\} = \max\left\{A_k, \frac{1}{x_1}\right\}$$

and so $x_1 = 1$. Then we get

$$1 = x_1 = \max\left\{A_1, \frac{1}{x_2}\right\}.$$

Then since (2.3) is satisfied it is obvious that $x_2 = 1$. Working inductively we take $x_j = 1$, $j = 1,2,\dots,r$. This completes the proof of the proposition. □

Proposition 2.2. *Suppose that* (2.1) *is satisfied. Then every solution of* (1.2) *is eventually equal to the unique equilibrium of* (1.2) $(x_1, x_2., , , , x_k) = (A_1, A_2, \dots, A_k)$.

Proof. Let $(x_1(n), x_2(n), \dots, x_k(n))$ be an arbitrary solution of (1.2). From (1.2) we get

$$x_i(n) \geq A_i, \qquad i = 1,2,\dots,k. \tag{2.13}$$

Let $s \in \{1,2,\dots,k\}$. We prove that there exists an $m_s \geq 3$ such that

$$x_s(m_s) = A_s. \tag{2.14}$$

Suppose on the contrary that for all $n \geq 3$

$$x_s(n) > A_s. \tag{2.15}$$

Then from (1.2), (2.13) and (2.15) we take for $n \geq 3$

$$x_s(n) = \max\left\{A_s, \frac{x_s(n-1)}{x_{s+1}(n-2)}\right\} = \frac{x_s(n-1)}{x_{s+1}(n-2)} \leq \frac{x_s(n-1)}{A_{s+1}}.$$

Then we take

$$x_s(3) \leq \frac{x_s(2)}{A_{s+1}},\ x_s(4) \leq \frac{x_s(2)}{A_{s+1}^2}, \dots,\ x_s(n) \leq \frac{x_s(2)}{A_{s+1}^{n-2}}.$$

Since from (2.1) $A_{s+1} > 1$ there exists an $n_0 \geq 3$ such that

$$\frac{x_s(2)}{A_{s+1}^{n-2}} < A_s, \qquad n \geq n_0$$

which implies that $x_s(n) < A_s$, $n \geq n_0$. This contradicts to (2.15) and so there exists a $m_s \geq 3$ such that (2.14) holds. From (1.2) we have

$$x_s(m_s+1) = \max\left\{A_s, \frac{x_s(m_s)}{x_{s+1}(m_s-1)}\right\}. \tag{2.16}$$

In addition relations (2.1), (2.14) imply that

$$\frac{x_s(m_s)}{x_{s+1}(m_s-1)} \leq \frac{A_s}{A_{s+1}} < A_s$$

and so from (2.16) it holds

$$x_s(m_s+1) = A_s.$$

Working inductively we can prove that

$$x_s(n) = A_s, \qquad n \geq m_s. \tag{2.17}$$

So, if $m = \max\{m_1, m_2, \ldots, m_k\}$ we have that $x_i(n) = A_i$, $i = 1, 2, \ldots, k$, for $n \geq m$. This completes the proof of the proposition. □

In the following proposition we prove that all solutions of (1.2) are unbounded if (2.2) are satisfied.

Proposition 2.3. *Consider system (1.2). Suppose that there exists an $r \in \{1, 2, \ldots, k\}$ such that (2.2) hold. Then all the solutions of system (1.2) are unbounded.*

Proof. Let $(x_1(n), x_2(n), \ldots, x_k(n))$ be an arbitrary solution of system (1.2).

Suppose firstly that there exists an $r \in \{1, 2, \ldots, k\}$ such that (2.5) is satisfied. Then since $A_{r+1} > 1$, and using the same argument in the proof of relations (2.14) and (2.17) we can prove that there exists an $n_r \geq 3$ such that

$$x_r(n) = A_r, \qquad n \geq n_r. \tag{2.18}$$

Then from (1.2) and (2.18) we obtain

$$x_{r-1}(n_r+2) = \max\left\{A_{r-1}, \frac{x_{r-1}(n_r+1)}{x_r(n_r)}\right\} \geq \frac{x_{r-1}(n_r+1)}{x_r(n_r)} = \frac{x_{r-1}(n_r+1)}{A_r},$$

and working inductively

$$x_{r-1}(n_r+3) \geq \frac{x_{r-1}(n_r+1)}{A_r^2}, \ldots, x_{r-1}(n_r+n) \geq \frac{x_{r-1}(n_r+1)}{A_r^{n-1}}.$$

Since $A_r < 1$ we have that $\lim\limits_{n\to\infty} x_{r-1}(n) = \infty$. So, the solution of (1.2) is unbounded.

Finally suppose that (2.6) hold. If there exists either an s such that (2.8) hold or a v such that (2.9) are satisfied, then arguing as in the case (2.18) we take that the solution is unbounded. Suppose that (2.10) are satisfied. Therefore since $A_1 > 1$, arguing as in (2.17) we take that there exists an n_k such that

$$x_k(n) = A_k, \qquad n \geq n_k$$

and so using the same argument as above we take

$$x_{k-1}(n_k+n) \geq \frac{x_{k-1}(n_k+1)}{A_k^{n-1}}.$$

Thus since $A_k < 1$ it holds and so $\lim_{n\to\infty} x_{k-1}(n) = \infty$. This completes the proof of the proposition. □

In the next proposition we find unbounded solutions for the system (1.2) in the case where (2.3) hold and k is an even number.

Proposition 2.4. *Consider system (1.2) where k is an even number and let condition (2.3) hold. Let $(x_1(n), x_2(n), \ldots, x_k(n))$ be a solution of (1.2). Suppose that there exists an s, $s \in \{0, 1, \ldots, \}$ such that either*

$$\begin{aligned}
&\frac{x_{2r}(s)}{x_{2r+1}(s-1)} > 1, \qquad \frac{x_{2r}(s)}{x_{2r+1}(s-1)x_{2r+1}(s)} > 1, \qquad r = 1, 2, \ldots, \frac{k-2}{2},\\
&\frac{x_{2r-1}(s)}{x_{2r}(s-1)} < A_{2r-1}, \qquad x_{2r}(s) > 1, \qquad r = 1, 2, \ldots, \frac{k}{2},\\
&\frac{x_k(s)}{x_1(s-1)} > 1, \qquad \frac{x_k(s)}{x_1(s-1)x_1(s)} > 1
\end{aligned} \tag{2.19}$$

or

$$\begin{aligned}
&\frac{x_{2r-1}(s)}{x_{2r}(s-1)} > 1, \qquad \frac{x_{2r-1}(s)}{x_{2r}(s-1)x_{2r}(s)} > 1, \qquad x_{2r-1}(s) > 1, \qquad r = 1, 2, \ldots, \frac{k}{2},\\
&\frac{x_{2r}(s)}{x_{2r+1}(s-1)} < A_{2r}, \qquad r = 1, 2, \ldots, \frac{k-2}{2},\\
&\frac{x_k(s)}{x_1(s-1)} < A_k, \qquad \frac{x_k(s)}{x_1(s-1)x_1(s)} < A_k
\end{aligned} \tag{2.20}$$

are satisfied. Then if (2.19) holds we get

$$\lim_{n\to\infty} x_{2r}(n) = \infty, \qquad x_{2r-1}(n) = A_{2r-1}, \qquad n \geq s+1, \; r = 1, 2, \ldots, \frac{k}{2} \tag{2.21}$$

and if (2.20) is satisfied we have

$$\lim_{n\to\infty} x_{2r-1}(n) = \infty, \qquad x_{2r}(n) = A_{2r}, \qquad n \geq s+1, \; r = 1, 2, \ldots, \frac{k}{2}. \tag{2.22}$$

Proof. Suppose that the conditions in (2.19) are satisfied. Then form (1.2) and (2.19) we get

$$\begin{aligned}
x_{2r-1}(s+1) &= \max\left\{A_{2r-1}, \frac{x_{2r-1}(s)}{x_{2r}(s-1)}\right\} = A_{2r-1}, \qquad r = 1, 2, \ldots, \frac{k}{2},\\
x_{2r}(s+1) &= \max\left\{A_{2r}, \frac{x_{2r}(s)}{x_{2r+1}(s-1)}\right\} = \frac{x_{2r}(s)}{x_{2r+1}(s-1)} > 1, \qquad r = 1, 2, \ldots, \frac{k-2}{2},\\
x_k(s+1) &= \max\left\{A_k, \frac{x_k(s)}{x_1(s-1)}\right\} = \frac{x_k(s)}{x_1(s-1)} > 1.
\end{aligned}$$

Moreover,

$$\begin{aligned}
x_{2r-1}(s+2) &= \max\left\{A_{2r-1}, \frac{x_{2r-1}(s+1)}{x_{2r}(s)}\right\} = \max\left\{A_{2r-1}, \frac{A_{2r-1}}{x_{2r}(s)}\right\} = A_{2r-1},\\
x_{2r}(s+2) &= \max\left\{A_{2r}, \frac{x_{2r}(s+1)}{x_{2r+1}(s)}\right\} = \max\left\{A_{2r}, \frac{x_{2r}(s)}{x_{2r+1}(s)x_{2r+1}(s-1)}\right\}\\
&= \frac{x_{2r}(s)}{x_{2r+1}(s-1)x_{2r+1}(s)} > 1,\\
x_k(s+2) &= \max\left\{A_k, \frac{x_k(s+1)}{x_1(s)}\right\} = \frac{x_k(s)}{x_1(s)x_1(s-1)} > 1.
\end{aligned}$$

In addition

$$
\begin{aligned}
x_{2r-1}(s+3) &= \max\left\{A_{2r-1}, \frac{x_{2r-1}(s+2)}{x_{2r}(s+1)}\right\} = \max\left\{A_{2r-1}, \frac{A_{2r-1}}{x_{2r}(s+1)}\right\} = A_{2r-1}, \\
x_{2r}(s+3) &= \max\left\{A_{2r}, \frac{x_{2r}(s+2)}{x_{2r+1}(s+1)}\right\} = \max\left\{A_{2r}, \frac{x_{2r}(s)}{x_{2r+1}(s)x_{2r+1}(s-1)A_{2r+1}}\right\} \\
&= \frac{x_{2r}(s)}{x_{2r+1}(s-1)x_{2r+1}(s)A_{2r+1}} > 1, \\
x_k(s+3) &= \max\left\{A_k, \frac{x_k(s+2)}{x_1(s+1)}\right\} = \frac{x_k(s)}{x_1(s)x_1(s-1)A_1} > 1.
\end{aligned}
$$

Working inductively we can prove that

$$
\begin{aligned}
x_{2r-1}(s+v) &= A_{2r-1}, \qquad v = 1,2,\dots, \quad r = 1,2,\dots,\frac{k}{2}, \\
x_{2r}(s+v) &= \frac{x_{2r}(s)}{x_{2r+1}(s-1)x_{2r+1}(s)A_{2r+1}^{v-2}}, \qquad v = 2,3,\dots, \quad r = 1,2,\dots,\frac{k-2}{2}, \\
x_k(s+v) &= \frac{x_k(s)}{x_1(s)x_1(s-1)A_1^{v-2}}.
\end{aligned}
$$

Then (2.21) is true if inequalities (2.19) hold. Similarly we can prove that if inequalities (2.20) are satisfied, then (2.22) hold. This completes the proof of the proposition. □

Now we find solutions of system (1.2) where $k = 2$ which converge to period six solutions. A related situation appears in [33]. For simplicity we set

$$x_1(n) = x_n, \qquad x_2(n) = y_n.$$

We use a product-type system of difference equations, which is solvable. There has been some considerable recent interest on solvable product-type systems of difference equations (see, for example, [39,43,44], and the related references therein).

Proposition 2.5. *Consider system*

$$x_{n+1} = \max\left\{A, \frac{x_n}{y_{n-1}}\right\}, \qquad y_{n+1} = \max\left\{B, \frac{y_n}{x_{n-1}}\right\} \tag{2.23}$$

where A, B are positive constants which satisfy

$$0 < A < 1, \qquad 0 < B < 1.$$

Let ϵ be a positive number such that

$$0 < \epsilon < \min\{1-A,\ 1-B\}. \tag{2.24}$$

Let (x_n, y_n) be a solution of (2.23) such that

$$\frac{x_0}{y_0} = \left(\frac{x_{-1}}{y_{-1}}\right)^{\lambda}, \qquad \lambda = \frac{1-\sqrt{5}}{2} \tag{2.25}$$

and

$$C_n \geq r = \max\left\{\frac{A}{1-\epsilon}, \ \frac{B}{1-\epsilon}\right\}, \tag{2.26}$$

where

$$C_n = \begin{cases} (x_0y_0)^{1/2}, & \textit{when } n = 6k, \\ (\frac{x_0y_0}{x_{-1}y_{-1}})^{1/2}, & \textit{when } n = 6k+1, \\ (x_{-1}y_{-1})^{-1/2}, & \textit{when } n = 6k+2, \\ (x_0y_0)^{-1/2}, & \textit{when } n = 6k+3, \\ (\frac{x_0y_0}{x_{-1}y_{-1}})^{-1/2}, & \textit{when } n = 6k+4, \\ (x_{-1}y_{-1})^{1/2}, & \textit{when } n = 6k+5. \end{cases}$$

Then there exists an n_0 *such that for* $n \geq n_0$ (x_n, y_n) *the form*

$$x_n = C_n \left(\frac{x_{-1}}{y_{-1}}\right)^{\frac{1}{2}\lambda^{n+1}}, \qquad y_n = C_n \left(\frac{x_{-1}}{y_{-1}}\right)^{-\frac{1}{2}\lambda^{n+1}} \tag{2.27}$$

and so (x_n, y_n) *tends to a period six solution of (2.23).*

Proof. First of all we prove that there exist x_0, x_{-1}, y_0, y_{-1} such that (2.26) is satisfied. It is obvious that $0 < r < 1$ since (2.24) holds. We choose a number θ such that

$$0 < -r + \sqrt{r} < \theta < 1 - r. \tag{2.28}$$

Let now numbers v, w be such that

$$r < r + \theta < v < (r+\theta)^{-1} < r^{-1}, \qquad r < r + \theta < w < (r+\theta)^{-1} < r^{-1}. \tag{2.29}$$

From (2.28) we get $r < (r+\theta)^2$. So,

$$r < (r+\theta)^2 < \frac{v}{w} < (r+\theta)^{-2} < r^{-1}.$$

Then if we choose x_0, x_{-1}, y_0, y_{-1}, such that the numbers

$$v = (x_0y_0)^{1/2}, \qquad w = (x_{-1}y_{-1})^{1/2}$$

satisfy inequalities (2.29), relation (2.26) is true.

We consider the system of difference equations

$$x_{n+1} = \frac{x_n}{y_{n-1}}, \qquad y_{n+1} = \frac{y_n}{x_{n-1}}, \qquad n = 0, 1, 2, \ldots \tag{2.30}$$

Let (x_n, y_n) be a solution of (2.30) which satisfies (2.25) and (2.26). Then we get

$$x_{n+4} = \frac{x_{n+3}^2 x_n}{x_{n+2}}$$

which implies that

$$\ln x_{n+4} - 2\ln x_{n+3} + \ln x_{n+2} - \ln x_n = 0.$$

By setting

$$z_n = \ln x_n \tag{2.31}$$

we get

$$z_{n+4} - 2z_{n+3} + z_{n+2} - z_n = 0. \tag{2.32}$$

The characteristic equation of (2.32) is the following

$$p^4 - 2p^3 + p^2 - 1 = (p^2 - p - 1)(p^2 - p + 1) = 0. \tag{2.33}$$

Then z_n has the form

$$z_n = d_1\mu^n + d_2\lambda^n + d_3 \cos\left(\frac{n\pi}{3}\right) + d_4 \sin\left(\frac{n\pi}{3}\right), \tag{2.34}$$

where $\mu = \frac{1+\sqrt{5}}{2}$, $\lambda = \frac{1-\sqrt{5}}{2}$, d_1, d_2, d_3, d_4 are constants.

If we set

$$w_n = \ln y_n \tag{2.35}$$

from (2.30) we get

$$\begin{aligned} w_n = z_{n+1} - z_{n+2} &= d_1(1-\mu)\mu^{n+1} + d_2(1-\lambda)\lambda^{n+1} \\ &+ d_3\left(\cos\left(\frac{(n+1)\pi}{3}\right) - \cos\left(\frac{(n+2)\pi}{3}\right)\right) \\ &+ d_4\left(\sin\left(\frac{(n+1)\pi}{3}\right) - \sin\left(\frac{(n+2)\pi}{3}\right)\right) \\ &= -d_1\mu^n - d_2\lambda^n + d_3\cos\left(\frac{n\pi}{3}\right) + d_4 \sin\left(\frac{n\pi}{3}\right). \end{aligned} \tag{2.36}$$

From (2.34) and (2.36) we get

$$\begin{aligned} z_{-1} &= d_1\mu^{-1} + d_2\lambda^{-1} + d_3\frac{1}{2} - d_4\frac{\sqrt{3}}{2}, \\ z_0 &= d_1 + d_2 + d_3, \\ w_{-1} &= -d_1\mu^{-1} - d_2\lambda^{-1} + d_3\frac{1}{2} - d_4\frac{\sqrt{3}}{2}, \\ w_0 &= -d_1 - d_2 + d_3. \end{aligned} \tag{2.37}$$

From (2.37) we have

$$\begin{aligned} d_1 &= \frac{1+\sqrt{5}}{8\sqrt{5}}\left(2(z_0 - w_0) - (1-\sqrt{5})(z_{-1} - w_{-1})\right), \\ d_2 &= \left(\frac{1}{4} - \frac{1}{4\sqrt{5}}\right)(z_0 - w_0) - \frac{1}{2\sqrt{5}}(z_{-1} - w_{-1}), \\ d_3 &= \frac{z_0 + w_0}{2}, \\ d_4 &= \frac{\sqrt{3}}{6}\left(-2(z_{-1} + w_{-1}) + z_0 + w_0\right). \end{aligned} \tag{2.38}$$

Relation (2.25) implies that

$$\begin{aligned} 2(z_0 - w_0) - (1-\sqrt{5})(z_{-1} - w_{-1}) &= 2(\ln x_0 - \ln y_0) - (1-\sqrt{5})(\ln x_{-1} - \ln y_{-1}) \\ &= 2\left(\ln\frac{x_0}{y_0} - \lambda \ln\frac{x_{-1}}{y_{-1}}\right) = 0 \end{aligned}$$

and so $d_1 = 0$.

From (2.25), (2.34), (2.36), (2.38) we get

$$z_n = \frac{1}{2}(z_{-1} - w_{-1})\lambda^{n+1} + \frac{z_0 + w_0}{2}\cos\frac{n\pi}{3} + \frac{\sqrt{3}}{6}\Big(-2(z_{-1} + w_{-1}) + z_0 + w_0\Big)\sin\frac{n\pi}{3}$$
$$w_n = -\frac{1}{2}(z_{-1} - w_{-1})\lambda^{n+1} + \frac{z_0 + w_0}{2}\cos\frac{n\pi}{3} + \frac{\sqrt{3}}{6}\Big(-2(z_{-1} + w_{-1}) + z_0 + w_0\Big)\sin\frac{n\pi}{3}.$$

By using (2.31) and (2.35) we get

$$\ln x_n = \frac{1}{2}\ln\left(\frac{x_{-1}}{y_{-1}}\right)\lambda^{n+1} + \frac{1}{2}\ln(x_0y_0)\cos\frac{n\pi}{3} + \frac{\sqrt{3}}{6}\left(-2\ln(x_{-1}y_{-1}) + \ln(x_0y_0)\right)\sin\frac{n\pi}{3},$$
$$\ln y_n = -\frac{1}{2}\ln\left(\frac{x_{-1}}{y_{-1}}\right)\lambda^{n+1} + \frac{1}{2}\ln(x_0y_0)\cos\frac{n\pi}{3} + \frac{\sqrt{3}}{6}\left(-2\ln(x_{-1}y_{-1}) + \ln(x_0y_0)\right)\sin\frac{n\pi}{3}.$$

From this and by some standard algebraic calculations we can easily prove that the relations in (2.27) are satisfied, with the constants C_n as defined in above.

Since $|\lambda| < 1$ it is obvious that

$$\lim_{n\to\infty}\left(\frac{x_{-1}}{y_{-1}}\right)^{\frac{1}{2}\lambda^{n+1}} = 1, \qquad \lim_{n\to\infty}\left(\frac{x_{-1}}{y_{-1}}\right)^{-\frac{1}{2}\lambda^{n+1}} = 1.$$

Then if ϵ is a positive number which satisfy (2.24) there exists a n_0 such that for $n \geq n_0$

$$\left(\frac{x_{-1}}{y_{-1}}\right)^{\frac{1}{2}\lambda^{n+1}} > 1 - \epsilon, \qquad \left(\frac{x_{-1}}{y_{-1}}\right)^{-\frac{1}{2}\lambda^{n+1}} > 1 - \epsilon. \tag{2.39}$$

Therefore using (2.27), (2.39) we get for $n \geq n_0$

$$\begin{aligned} x_n &\geq \max\left\{\frac{A}{1-\epsilon}, \frac{B}{1-\epsilon}\right\}(1-\epsilon) = \max\{A, B\}, \\ y_n &\geq \max\left\{\frac{A}{1-\epsilon}, \frac{B}{1-\epsilon}\right\}(1-\epsilon) = \max\{A, B\}. \end{aligned} \tag{2.40}$$

Then from (2.30) and (2.40) we have that (x_n, y_n) is a bounded solution of (2.23) where for $n \geq n_0$ satisfies system (2.30). This completes the proof of the proposition. □

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On the principal eigenvalues of the degenerate elliptic systems

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Abstract. We study some qualitative properties for the set of principal eigenvalues of a degenerate elliptic system such as strict monotonicity with respect to the domain, local isolation and monotonicity and continuity of the principal eigenvalue with respect to the weight functions. Finally, explicit lower bounds for principal eigenvalues in terms of the measure of domain are also proved.

Keywords: principal eigenvalue, monotonicity, lower bound of eigenvalues.

2020 Mathematics Subject Classification: 35J70, 35J92, 35P15.

1 Introduction

In this paper we study the following system:

$$\begin{cases} -\Delta_p u = \lambda a(x)|v|^{\beta_1-1}v & \text{in } \Omega; \\ -\Delta_q v = \mu b(x)|u|^{\beta_2-1}u & \text{in } \Omega; \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\beta_1, \beta_2 > 0$ with $\beta_1\beta_2 = (p-1)(q-1)$, $(\lambda, \mu) \in \mathbb{R}^2$, $p, q \in (1, \infty)$, Ω is a bounded domain in $\mathbb{R}^n$ with a C^2-boundary and a and b are bounded functions on Ω satisfying

$$\operatorname*{ess\,inf}_{x\in\Omega} a(x) > 0 \quad \text{and} \quad \operatorname*{ess\,inf}_{x\in\Omega} b(x) > 0. \tag{1.2}$$

The p-Laplacian operator $\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,\frac{p}{p-1}}(\Omega)$ is defined by

$$\langle -\Delta_p u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx,$$

where $W^{-1,\frac{p}{p-1}}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$.

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Consider the classical problem

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Notice that, if $f \in L^\infty(\Omega)$, then problem (1.3) admits a unique weak solution $(-\Delta_p)^{-1}(f) := u \in W_0^{1,p}(\Omega)$. In this case, there exists $\alpha \in (0,1)$ such that $u \in C^{1,\alpha}(\overline{\Omega})$ (see [12,18,24,35]).

Thus, $(-\Delta_p)^{-1} : L^\infty(\Omega) \to C_0^{1,\alpha'}(\overline{\Omega})$ is continuous and bounded for $\alpha' = \alpha$ and compact whenever $0 < \alpha' < \alpha$. Moreover, the (weak and strong) comparison principles related to the p-Laplacian operator (see [6–10,15–19,31,34]) shows that $(-\Delta_p)^{-1}$ is order preserving, that is, for all $f, g \in L^\infty(\Omega)$, $f \leq g$ in Ω implies $(-\Delta_p)^{-1} f \leq (-\Delta_p)^{-1} g$ and it is also strictly order preserving, i.e., $f \leq (\not\equiv)\ g$ and $g\ (\not\equiv) \geq 0$ in Ω imply

$$(-\Delta_p)^{-1} f < (-\Delta_p)^{-1} g \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial(-\Delta_p)^{-1} g}{\partial\nu} < \frac{\partial(-\Delta_p)^{-1} f}{\partial\nu} \quad \text{on } \partial\Omega,$$

where $\nu \equiv \nu(y_0)$ denotes the exterior unit normal to $\partial\Omega$ at $y_0 \in \partial\Omega$. More generally, we have

$$(-\Delta_p)^{-1} : W^{-1,\frac{p}{p-1}}(\Omega) \to L^p(\Omega)$$

is well defined, compact and order preserving, when $p > 2$ (see [18, Corollary 8]).

By weak maximum principle in Ω means that for any weak solution $u \in W_0^{1,p}(\Omega)$ to

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega; \\ u \geq 0 & \text{on } \partial\Omega, \end{cases}$$

with $f \geq 0$ in Ω implies that $u \geq 0$ in Ω. Besides, the strong maximum principle is said to hold in Ω if, in addition, $u > 0$ in Ω whenever $f \not\equiv 0$ in Ω. The validity of the (weak and strong) maximum principles related to the p-Laplacian operator was established in [34,36]. Later, the paper [18] generalizes such results for operators involving the p-Laplacian. More generally, in [15] the authors showed an anti-maximum principle for a class of strictly cooperative elliptic systems.

In 1994, López-Gómez and Molina-Meyer [27] made a fairly complete characterization on maximum principles for linear second order elliptic operators and, more generally, in the context of cooperative systems. More recently, in [23] the authors established the connection between maximum principle for Lane–Emden systems and their principal spectral curves. We refer to [26] for a more detailed discussion of the maximum principle for elliptic problems and cooperative systems involving linear second order elliptic operators.

We shall introduce a bit of notation. Here X stands for the space $\left[C_0^1(\overline{\Omega})\right]^2$, X_+ is given by $\{(u,v) \in X : u \geq 0 \text{ and } v \geq 0 \text{ in } \Omega\}$, and $\mathring{X}_+$ is the topological interior of X_+ in X. Then, $\mathring{X}_+$ is nonempty and given by:

$$\left\{(u,v) \in X : u, v > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial\nu}, \frac{\partial v}{\partial\nu} < 0 \text{ on } \partial\Omega\right\}.$$

Let (u,v) in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. The weak formulation of (1.1) is given by

$$\lambda \int_\Omega a(x)|v|^{\beta_1 - 1} v\Phi dx = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \Phi dx \tag{1.4}$$

and

$$\mu \int_\Omega b(x)|u|^{\beta_2 - 1} u \Psi dx = \int_\Omega |\nabla v|^{q-2} \nabla v \nabla \Psi dx$$

for any $(\Phi, \Psi) \in (C_0^1(\Omega))^2$.

We say that $(\lambda, \mu) \in \mathbb{R}_+^* \times \mathbb{R}_+^* = (0, \infty)^2$ is an eigenvalue of (1.1) if the system admits a nontrivial weak solution (φ, ψ) in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ which is called an eigenfunction corresponding to (λ, μ). We also say that (λ, μ) is a principal eigenvalue if admits a positive eigenfunction (φ, ψ). Finally, the couple (λ, μ) is said to be simple in $\mathring{X}_+$ if for any eigenfunctions $(\varphi, \psi), (\tilde{\varphi}, \tilde{\psi}) \in \mathring{X}_+$, there exists $\theta > 0$ such that $\tilde{\varphi} = \theta \varphi$ and $\tilde{\psi} = \theta \mu^{\frac{1}{\beta_2}} \psi$ in Ω.

During the past decades, the system (1.1) has been extensively studied in the case $p = q = 2$. For example, we can list the papers [4, 11, 14, 20, 28, 32], where several results on existence, nonexistence and uniqueness of nontrivial solutions have been developed when $\beta_1 \beta_2 \neq 1$. The case $\beta_1 \beta_2 = 1$ was treated in Montenegro [29]. Namely, the author proved that the set of principal eigenvalues (λ, μ) of the system (1.1) is nonempty and determines a curve in the cartesian plane which satisfies some properties as simplicity, continuity, monotonicity and local isolation. We also refer to [30] where a biparameter elliptic system was considered.

For the general case $p, q > 1$, we refer to [5] when $\beta_1 \beta_2 > (p-1)(q-1)$ and [7] when $\beta_1 \beta_2 = (p-1)(q-1)$. For instance, Cuesta and Takáč [7] showed that the set of principal eigenvalues of (1.1) is given by

$$\mathcal{C}_1(a, b, \Omega) := \left\{ (\lambda, \mu) \in (\mathbb{R}_+^*)^2 : \lambda^{\frac{1}{\sqrt{\beta_1 (p-1)}}} \mu^{\frac{1}{\sqrt{\beta_2 (q-1)}}} = \Lambda'(a, b, \Omega) \right\}$$

for some $\Lambda'(a, b, \Omega) > 0$, satisfying:

(a) (Uniqueness) $(\lambda, \mu) \in \mathcal{C}_1(a, b, \Omega)$ if and only if $(\lambda, \mu) \in \mathbb{R}_+ \times \mathbb{R}_+$ is a principal eigenvalue of the problem (1.1);

(b) (Simplicity in $\mathring{X}_+$) The principal curve $\mathcal{C}_1(a, b, \Omega)$ is simple in $\mathring{X}_+$, i.e., (λ, μ) is simple in $\mathring{X}_+$ for all $(\lambda, \mu) \in \mathcal{C}_1(a, b, \Omega)$;

(c) (Simplicity in X) Let $(\varphi, \psi) \in X$ be an eigenfunction associated to $(\lambda, \mu) \in \mathcal{C}_1(a, b, \Omega)$. So, either $(\varphi, \psi) \in \mathring{X}_+$ or $(-\varphi, -\psi) \in \mathring{X}_+$.

Let $\mathcal{R}_1(a, b, \Omega)$ be the open region in the first quadrant below $\mathcal{C}_1(a, b, \Omega)$, that is,

$$\mathcal{R}_1(a, b, \Omega) = \left\{ (\lambda, \mu) \in (\mathbb{R}_+^*)^2 : \lambda^{\frac{1}{\sqrt{\beta_1 (p-1)}}} \mu^{\frac{1}{\sqrt{\beta_2 (q-1)}}} < \Lambda'(a, b, \Omega) \right\}.$$

We say that the principal curve $\mathcal{C}_1(a, b, \Omega)$ is locally isolated above (or below) if for each $(\lambda_1, \mu_1) \in \mathcal{C}_1(a, b, \Omega)$, there is $\varepsilon = \varepsilon(\lambda_1, \mu_1) > 0$ such that the system (1.1) does not have any eigenvalue in $B_\varepsilon(\lambda_1, \mu_1) \cap \overline{\mathcal{R}_1(a, b, \Omega)}^c$ (or $B_\varepsilon(\lambda_1, \mu_1) \cap \mathcal{R}_1(a, b, \Omega)$).

Theorem 1.1. *Let $p, q \in (1, \infty)$, Ω be a bounded domain in $\mathbb{R}^n$ with a C^2-boundary, $\beta_1, \beta_2 > 0$ be such that $\beta_1 \beta_2 = (p-1)(q-1)$ and a, b, $\tilde{a}$ and $\tilde{b}$ be functions in $L^\infty(\Omega)$ satisfying (1.2) in Ω. Then, the curve $\mathcal{C}_1(a, b, \Omega)$ to the system (1.1) satisfies:*

(i) (Strict monotonicity with respect to the domain) Let D be a bounded domain in $\mathbb{R}^n$ with a C^2-boundary, such that $\overline{D} \subset \Omega$. Then, $\Lambda'(a, b, \Omega) < \Lambda'(a, b, D)$;

(ii) (Monotonicity with respect to the weights) Suppose that $a \leq \tilde{a}$ and $b \leq \tilde{b}$ in Ω. Then, $\Lambda'(a, b, \Omega) \geq \Lambda'(\tilde{a}, \tilde{b}, \Omega)$. Moreover, if $(a, b) \not\equiv (\tilde{a}, \tilde{b})$ then $\Lambda'(a, b, \Omega) > \Lambda'(\tilde{a}, \tilde{b}, \Omega)$;

(iii) (Local isolation above) The curve $\mathcal{C}_1(a, b, \Omega)$ is locally isolated above;

(iv) (Local isolation below) The system (1.1) does not admit any eigenvalues in $\mathcal{R}_1(a, b, \Omega)$. In particular, the curve $\mathcal{C}_1(a, b, \Omega)$ is locally isolated below;

(v) (Continuity of the principal eigenvalue with respect to the weight functions a and b) Let $(a_k)_{k\geq 1}$ and $(b_k)_{k\geq 1}$ be sequences of weight functions in $L^\infty(\Omega)$ which are positive in Ω. Assume that $a_k \to a$ and $b_k \to b$ uniformly in Ω. If $a, b > 0$ in $\overline{\Omega}$, then $\Lambda'(a_k, b_k, \Omega) \to \Lambda'(a, b, \Omega)$.

Note that the part (i) of Theorem 1.1 is essential for establish the Harnack inequality associated to the system (1.1). A very important application of Harnack inequality is the obtention of principal eigenvalues associated to the problems in general domains. Parts (ii) and (v) of Theorem 1.1 are important tools to furnish a min–max type characterization for principal curves associated to the problems whose solutions are not usually classical.

Now, we show an explicit lower estimate for principal eigenvalues of system (1.1) in terms of the Lebesque measure of Ω, more specifically, a counterpart of [2, Theorem 10.1] to degenerate elliptic systems. More recently, it was proved in [23] for Lane–Emden systems involving second order uniformly elliptic operators. Their proof use in a crucial way the celebrated Faber–Krahn inequality due to Faber [13] and Krahn [22]. We present now some essential ingredients:

For $1 \leq p < n$, we use the sharp Sobolev inequality for any $u \in W_0^{1,p}(\Omega)$,

$$\|u\|_{L^{p^*}(\Omega)} \leq c_{n,p} \|\nabla u\|_{L^p(\Omega)}, \tag{1.5}$$

where $p^* = \frac{np}{n-p}$ and an explicit formula of $c_{n,p}$ depending only on n and p was proved in [1, 33].

For $p = n$ and $u \in W_0^{1,p}(\Omega)$, we have

$$\|u\|_{L^\eta(\Omega)} \leq C(n) |\Omega|^{\frac{1}{\eta}} \|\nabla u\|_{L^p(\Omega)}, \tag{1.6}$$

where $C(n) > 0$, $1 \leq \eta < \infty$ and $|\cdot|$ stands for the Lebesgue measure of $\mathbb{R}^n$.

For $p > n$, there is a constant $C(n, p) > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C(n, p) |\Omega|^{-\frac{1}{p^*}} \|\nabla u\|_{L^p(\Omega)}, \tag{1.7}$$

for all $u \in W_0^{1,p}(\Omega)$.

Consider now the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u - \lambda |u|^{p-2} u = 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In [25], the author proved that the first eigenvalue $\lambda_{1,p}(\Omega)$ has the following properties, it is strictly positive, simple in any bounded connected Ω and characterized by

$$\lambda_{1,p}(\Omega) = \min_{\varphi \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla \varphi(x)|^p dx}{\int_\Omega |\varphi(x)|^p dx}.$$

By the Cheeger's constant (see [3,21]), we have

$$\lambda_{1,p}(B_1) \geq \left(\frac{n}{p}\right)^p, \tag{1.8}$$

where B_1 is the unit ball of $\mathbb{R}^n$.

Faber–Krahn inequality for the first eigenvalue of $-\boldsymbol{\Delta}_{\boldsymbol{p}}$. Let $1 < p < \infty$ and Ω be a open set in $\mathbb{R}^n$ with finite Lebesgue measure. Then,

$$\lambda_{1,p}(\Omega) \geq \lambda_{1,p}(B_1)|B_1|^{\frac{p}{n}}|\Omega|^{-\frac{p}{n}}.$$

Our main result gives an explicit lower estimate for principal eigenvalues of system (1.1) in terms of the measure of Ω and the weighted functions a and b.

Precisely, we have:

Theorem 1.2. *Let (λ, μ) be a principal eigenvalue of (1.1). Suppose $\beta_1 \geq \beta_2$, $p \leq q$ and $|\Omega| \leq 1$.*

(i) For $1 < p < n$, $q < p^*$ and

$$q - 1 \leq \beta_1 < \frac{np - n + p}{n - p},$$

there exists an explicit constant $C = C(p, q, \beta_1, \beta_2, n, a, b) > 0$ such that

$$\lambda + \mu^{\frac{p(q-1)}{q\beta_2}} \geq C\left(\frac{n}{q}\right)^{p\theta_p}|B_1|^{\theta_p\frac{p}{n}}|\Omega|^{-\theta_p\frac{p}{n}}, \tag{1.9}$$

where

$$\frac{1}{\beta_1 + 1} = \frac{\theta_p}{p} + \frac{1 - \theta_p}{p^*};$$

(ii) For $p = q = n$ and $q - 1 \leq \beta_1 < \infty$, the estimate (1.9) holds with

$$\frac{1}{\beta_1 + 1} = \frac{\theta_p}{p} + \frac{1 - \theta_p}{2(\beta_1 + 1)};$$

(iii) For $n < p$ and $q - 1 \leq \beta_1 < \infty$, there is an explicit constant $C = C(p, q, \beta_1, \beta_2, n, a, b) > 0$ such that

$$\lambda + \mu^{\frac{p(q-1)}{q\beta_2}} \geq C\left(\frac{n}{q}\right)^{\frac{\theta_q\beta_1 p}{p-1}}|B_1|^{\theta_p\frac{p}{n}}|\Omega|^{-\theta_p\frac{p}{n}}, \tag{1.10}$$

where $\theta_p = \frac{p}{\beta_1+1}$ and $\theta_q = \frac{q}{\beta_1+1}$;

(iv) For $n = p < q$ and $q - 1 \leq \beta_1 < \infty$, we have (1.10) holds, with

$$\frac{1}{\beta_1 + 1} = \frac{\theta_p}{p} + \frac{1 - \theta_p}{2(\beta_1 + 1)} \quad \text{and} \quad \theta_q = \frac{q}{\beta_1 + 1}.$$

In particular,

$$\lim_{|\Omega| \downarrow 0} \Lambda'(a, b, \Omega) = \infty.$$

Using the ideas of the proof of Theorem 1.2, we obtain the following result:

Theorem 1.3. *Let (λ, μ) be a principal eigenvalue of (1.1). Suppose $\beta_1 \leq \beta_2$, $p \leq q$ and $|\Omega| \leq 1$.*

(i) For $1 < p < n$, $q < p^*$ and

$$\frac{(p-1)(q-1)(n-p)}{np-n+p} < \beta_1 \leq p-1,$$

there exists an explicit constant $C = C(p, q, \beta_1, \beta_2, n, a, b) > 0$ such that

$$\lambda^{\frac{q(p-1)}{p\beta_1}} + \mu \geq C\left(\frac{n}{q}\right)^{rq} |B_1|^{r\frac{q}{n}} |\Omega|^{-r\frac{q}{n}}, \tag{1.11}$$

where $r := \min\left\{\theta_p, \theta_q \frac{\beta_2}{q-1}\right\}$,

$$\frac{1}{\beta_2+1} = \frac{\theta_p}{p} + \frac{1-\theta_p}{p^*}$$

and

$$\begin{cases} \frac{1}{\beta_2+1} = \frac{\theta_q}{q} + \frac{1-\theta_q}{q^*} & \text{if } 1 < q < n; \\ \frac{1}{\beta_2+1} = \frac{\theta_q}{q} + \frac{1-\theta_q}{p^*} & \text{if } q = n; \\ \theta_q = \frac{q}{\beta_2+1} & \text{if } q > n; \end{cases}$$

(ii) For $p = q = n$ and $0 < \beta_1 \leq p-1$, the estimate (1.11) holds with $r = \theta_q \frac{\beta_2}{q-1}$ and

$$\frac{1}{\beta_2+1} = \frac{\theta_q}{q} + \frac{1-\theta_q}{2(\beta_2+1)};$$

(iii) For $n < p$ and $0 < \beta_1 \leq p-1$, there is an explicit constant $C = C(p, q, \beta_1, \beta_2, n, a, b) > 0$ such that

$$\lambda^{\frac{q(p-1)}{p\beta_1}} + \mu \geq C\left(\frac{n}{q}\right)^{sq} |B_1|^{r\frac{q}{n}} |\Omega|^{-r\frac{q}{n}}, \tag{1.12}$$

where $s := \max\left\{\theta_p, \theta_q \frac{\beta_2}{q-1}\right\}$, $\theta_p = \frac{p}{\beta_2+1}$ and $\theta_q = \frac{q}{\beta_2+1}$;

(iv) For $n = p < q$ and $0 < \beta_1 \leq p-1$, we have (1.12) holds, with

$$\frac{1}{\beta_2+1} = \frac{\theta_p}{p} + \frac{1-\theta_p}{2(\beta_2+1)} \quad \text{and} \quad \theta_q = \frac{q}{\beta_2+1}.$$

In particular,

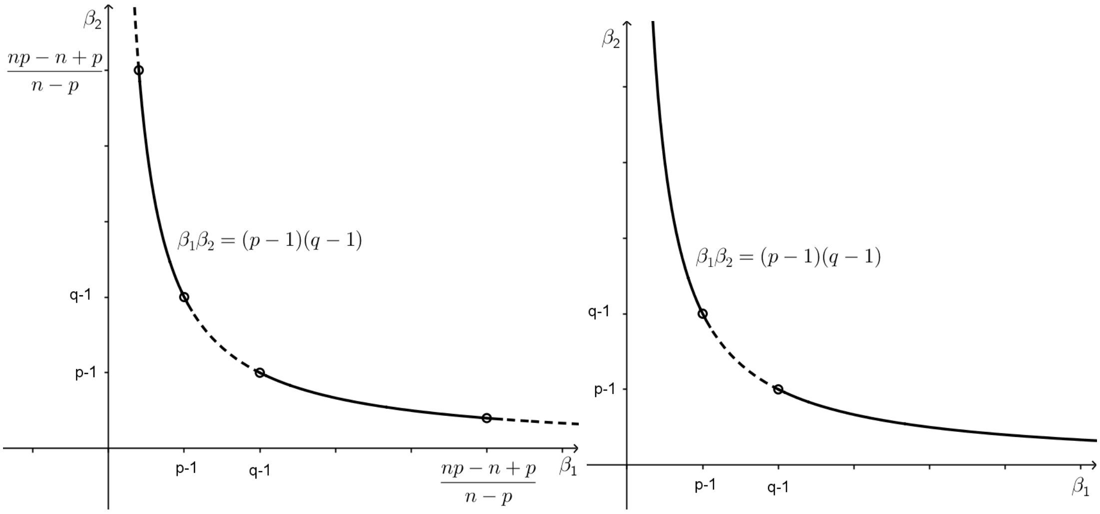
$$\lim_{|\Omega| \downarrow 0} \Lambda'(a, b, \Omega) = \infty.$$

Note that, supposing $p \leq q$, we get an explicit lower estimate for principal eigenvalues of system (1.1) for the range on β_1 and β_2,

$$\frac{(p-1)(q-1)(n-p)}{np-n+p} < \beta_1, \beta_2 \leq p-1 \quad \text{and} \quad q-1 \leq \beta_1, \beta_2 < \frac{np-n+p}{n-p}$$

for $1 < p < n$ and $0 < \beta_1, \beta_2 \leq p-1$ and $q-1 \leq \beta_1, \beta_2 < \infty$ for $p \geq n$. In particular, the result holds for all hyperbole $\beta_1\beta_2 = (p-1)(q-1)$ if $p = q \geq n$. The problem remains open in other remaining cases (see Figure 1.1). Clearly, the case $q < p$ follows similarly.

Our approach is inspired by the papers [2, 7, 23, 29]. By mean of topological arguments, strong maximum principle, Hopf's lemma and (weak and strong) comparison principles related to the p-Laplacian operator, we prove five properties of $\mathcal{C}_1(a, b, \Omega)$ which will be presented in Section 2. In Section 3, by using the Faber–Krahn inequality for the first eigenvalue of $-\Delta_p$, variational characterization of $\lambda_{1,p}(\Omega)$, Hölder, Young, interpolation and Sobolev inequalities, we show Theorem 1.2.



(a) Case $1 < p < n$ and $p \leq q < p^*$

(b) Case $p \geq n$ and $p \leq q$

Figure 1.1: Couples (β_1, β_2).

2 Proof of Theorem 1.1

In this section we provide some essential properties satisfied by the principal curve $\mathcal{C}_1(a, b, \Omega)$ which is organized into five propositions.

We first show the strict monotonicity of the principal eigenvalues with respect to the domain stated in the part (i) of Theorem 1.1. Precisely:

Proposition 2.1. *Let D and Ω be two bounded domain in $\mathbb{R}^n$ with a C^2-boundary, such that $\overline{D} \subset \Omega$ and $\mathcal{C}_1(a, b, \Omega)$ and $\mathcal{C}_1(a, b, D)$ principal curves. Then, $\Lambda'(a, b, \Omega) < \Lambda'(a, b, D)$.*

Proof. Assume by contradiction that $\Lambda'(a, b, \Omega) \geq \Lambda'(a, b, D)$. Let $(\lambda_1, \mu_1) \in \mathcal{C}_1(a, b, \Omega)$ and $(\tilde{\lambda}_1, \tilde{\mu}_1) \in \mathcal{C}_1(a, b, D)$ be such that $\frac{\lambda_1}{\mu_1} = \frac{\tilde{\lambda}_1}{\tilde{\mu}_1}$. Thus, $\lambda_1 \geq \tilde{\lambda}_1$ and $\mu_1 \geq \tilde{\mu}_1$. Let $(\varphi, \psi), (\tilde{\varphi}, \tilde{\psi})$ be positive eigenfunctions associated to the principal eigenvalues $(\lambda_1, \mu_1), (\tilde{\lambda}_1, \tilde{\mu}_1)$, respectively. Define

$$c := \min\left\{\min_{x \in \overline{D}} \varphi(x), \min_{x \in \overline{D}} \psi(x)\right\} > 0.$$

We claim that $\varphi \geq \tilde{\varphi}$ and $\psi \geq \tilde{\psi}$ in D. In fact, assume by contradiction that $\varphi < \tilde{\varphi}$ or $\psi < \tilde{\psi}$ somewhere in D. In this case, the set $\Gamma := \{\gamma > 0 : \varphi > \gamma\tilde{\varphi}$ and $\psi > \gamma^\omega \tilde{\psi}$ in $D\}$ is upper bounded, where $\omega := \frac{p-1}{\beta_1}$. In addition, the positivity of φ and ψ in $\overline{D}$ imply that Γ is nonempty. Define $0 < \overline{\gamma} := \sup \Gamma < 1$. It is clear that $\varphi \geq \overline{\gamma}\tilde{\varphi}$ and $\psi \geq \overline{\gamma}^\omega \tilde{\psi}$ in D, with $\varphi \not\equiv \overline{\gamma}\tilde{\varphi}$ and $\psi \not\equiv \overline{\gamma}^\omega \tilde{\psi}$ in D. Moreover, $\varphi \geq \overline{\gamma}\tilde{\varphi} + c$ and $\psi \geq \overline{\gamma}^\omega \tilde{\psi} + c$ on ∂D. So, we get

$$\begin{cases} -\Delta_p(\overline{\gamma}\tilde{\varphi} + c) = -\Delta_p(\overline{\gamma}\tilde{\varphi}) = \tilde{\lambda}_1 a(x)(\overline{\gamma}^\omega \tilde{\psi})^{\beta_1} \leq (\not\equiv)\ \lambda_1 a(x)\psi^{\beta_1} = -\Delta_p(\varphi) \\ -\Delta_q(\overline{\gamma}^\omega \tilde{\psi} + c) = -\Delta_q(\overline{\gamma}^\omega \tilde{\psi}) = \tilde{\mu}_1 b(x)(\overline{\gamma}\tilde{\varphi})^{\beta_2} \leq (\not\equiv)\ \mu_1 b(x)\psi^{\beta_2} = -\Delta_q(\psi) \end{cases} \quad \text{in } D.$$

Then, applying the weak comparison principle to each above equation (see [18] or [34, Lemma 3.1]), we derive $\varphi \geq \overline{\gamma}\tilde{\varphi} + c$ and $\psi \geq \overline{\gamma}^\omega \tilde{\psi} + c$ in D. Thus, $\varphi > \overline{\gamma}\tilde{\varphi}$ and $\psi > \overline{\gamma}^\omega \tilde{\psi}$ in D. So, we can find $0 < \varepsilon < 1$ such that $\varphi > (\overline{\gamma} + \varepsilon)\tilde{\varphi}$ and $\psi > (\overline{\gamma} + \varepsilon)^\omega \tilde{\psi}$ in D, contradicting the definition of $\overline{\gamma}$. Therefore, $\varphi \geq \tilde{\varphi}$ and $\psi \geq \tilde{\psi}$ in D. Note that $(\kappa\tilde{\varphi}, \kappa^\omega \tilde{\psi})$, $\kappa > 0$, are also eigenfunctions associated to $(\tilde{\lambda}_1, \tilde{\mu}_1)$. Then, $\varphi \geq \kappa\tilde{\varphi}$ and $\psi \geq \kappa^\omega \tilde{\psi}$ in D for all $\kappa > 0$; and from there we arrive at a contradiction. This concludes the desired proof. $\square$

We now show the monotonicity of principal eigenvalues with respect to the weights which corresponds to the part (ii) of Theorem 1.1.

Proposition 2.2. *Let a, b, $\tilde{a}$ and $\tilde{b}$ be functions in $L^\infty(\Omega)$ satisfying (1.2) such that $a \le \tilde{a}$ and $b \le \tilde{b}$ in Ω. Then, $\Lambda'(a,b,\Omega) \ge \Lambda'(\tilde{a},\tilde{b},\Omega)$. Moreover, if $(a,b) \not\equiv (\tilde{a},\tilde{b})$ then $\Lambda'(a,b,\Omega) > \Lambda'(\tilde{a},\tilde{b},\Omega)$.*

Proof. Assume by contradiction that $\Lambda'(a,b,\Omega) < \Lambda'(\tilde{a},\tilde{b},\Omega)$. Let $(\lambda_1(a,b),\mu_1(a,b)) \in \mathcal{C}_1(a,b,\Omega)$ and $(\lambda_1(\tilde{a},\tilde{b}),\mu_1(\tilde{a},\tilde{b})) \in \mathcal{C}_1(\tilde{a},\tilde{b},\Omega)$ be such that $\frac{\lambda_1(a,b)}{\mu_1(a,b)} = \frac{\lambda_1(\tilde{a},\tilde{b})}{\mu_1(\tilde{a},\tilde{b})}$. Thus,

$$\lambda_1(a,b) < \lambda_1(\tilde{a},\tilde{b}) \quad \text{and} \quad \mu_1(a,b) < \mu_1(\tilde{a},\tilde{b}).$$

Let (φ,ψ) and $(\tilde{\varphi},\tilde{\psi})$ be positive eigenfunctions associated to the principal eigenvalues

$$(\lambda_1(a,b),\mu_1(a,b)) \quad \text{and} \quad (\lambda_1(\tilde{a},\tilde{b}),\mu_1(\tilde{a},\tilde{b})),$$

respectively. Consider the set $\Gamma = \{\gamma > 0 : \tilde{\varphi} > \gamma\varphi \text{ and } \tilde{\psi} > \gamma^\omega\psi \text{ in } \Omega\}$, where $\omega := \frac{p-1}{\beta_1}$. Note that Γ is upper bounded, and by strong maximum principle (see [18, 34, 36]) Γ is nonempty. Define $\overline{\gamma} = \sup\Gamma > 0$. Note that, $\tilde{\varphi} \ge \overline{\gamma}\varphi$ and $\tilde{\psi} \ge \overline{\gamma}^\omega\psi$ in Ω.

Since $(-\Delta_p)^{-1}$ and $(-\Delta_q)^{-1}$ are strictly order preserving, we can find $0 < \varepsilon < 1$ such that $\tilde{\varphi} > (\overline{\gamma}+\varepsilon)\varphi$ and $\tilde{\psi} > (\overline{\gamma}+\varepsilon)^\omega\psi$ in Ω which clearly contradicts the definition of $\overline{\gamma}$. Therefore, $\Lambda'(a,b,\Omega) \ge \Lambda'(\tilde{a},\tilde{b},\Omega)$.

Finally, assume that $(a,b) \not\equiv (\tilde{a},\tilde{b})$. Arguing again by contradiction, assume that

$$\Lambda'(a,b,\Omega) = \Lambda'(\tilde{a},\tilde{b},\Omega).$$

Let (φ,ψ) and $(\tilde{\varphi},\tilde{\psi})$ be positive eigenfunctions corresponding to the principal eigenvalues $(\lambda_1(a,b),\mu_1(a,b)) = (\lambda_1(\tilde{a},\tilde{b}),\mu_1(\tilde{a},\tilde{b}))$. Proceeding similarly to the first part of the proof, we obtain $\Lambda'(a,b,\Omega) > \Lambda'(\tilde{a},\tilde{b},\Omega)$. This ends the proof. □

The two next propositions are dedicated the local isolation above and below the principal curve $\mathcal{C}_1(a,b,\Omega)$. These correspond to the parts (iii) and (iv) of Theorem 1.1, respectively. Precisely:

Proposition 2.3. *The curve $\mathcal{C}_1(a,b,\Omega)$ is locally isolated above.*

Proof. Assume by contradiction that the claim is false. Thus, there are $(\lambda_1,\mu_1) \in \mathcal{C}_1(a,b,\Omega)$ and a sequence of eigenvalues $((\lambda_k,\mu_k))_{k\ge1}$ contained in $B_{\varepsilon_k}(\lambda_1,\mu_1) \cap \overline{\mathcal{R}_1(a,b,\Omega)}^c$, where $\varepsilon_k \to 0$ with $\varepsilon_k > 0$ for all $k \in \mathbb{N}$. Let (φ_k,ψ_k) an eigenfunction associated to (λ_k,μ_k); that is, a weak solution of the system

$$\begin{cases} -\Delta_p\varphi_k = \lambda_k a(x)|\psi_k|^{\beta_1-1}\psi_k & \text{in } \Omega; \\ -\Delta_q\psi_k = \mu_k b(x)|\varphi_k|^{\beta_2-1}\varphi_k & \text{in } \Omega; \\ \varphi_k = \psi_k = 0 & \text{on } \partial\Omega, \end{cases}$$

where at least one of $-\varphi_k$ or $-\psi_k$ does not belong to $\mathring{X}_+$. Define the functions

$$u_k := \frac{\varphi_k}{\|\psi_k\|_{L^\infty(\Omega)}^{\frac{\beta_1}{p-1}}}, \quad \tilde{u}_k := \frac{\varphi_k}{\|\varphi_k\|_{L^\infty(\Omega)}}, \quad \tilde{v}_k := \frac{\psi_k}{\|\psi_k\|_{L^\infty(\Omega)}} \quad \text{and} \quad v_k := \frac{\psi_k}{\|\varphi_k\|_{L^\infty(\Omega)}^{\frac{\beta_2}{q-1}}}.$$

Then, we have $0 \le |\tilde{u}_k|, |\tilde{v}_k| \le 1$ in Ω. Therefore, the right-hand side of the following system

$$\begin{cases} -\Delta_p u_k = \lambda_k a(x)|\tilde{v}_k|^{\beta_1-1}\tilde{v}_k & \text{in } \Omega; \\ -\Delta_q v_k = \mu_k b(x)|\tilde{u}_k|^{\beta_2-1}\tilde{u}_k & \text{in } \Omega; \\ \varphi_k = \psi_k = 0 & \text{on } \partial\Omega; \end{cases} \tag{2.1}$$

is uniformly bounded in $(L^\infty(\Omega))^2$. It follows the sequences $(u_k)_{k\ge1}$ and $(v_k)_{k\ge1}$ are bounded in $C_0^{1,\alpha}(\overline{\Omega})$, by regularity and, in addition, also bounded in $L^\infty(\Omega)$; i.e., there exists a constant $C > 0$ such that $\|u_k\|_{L^\infty(\Omega)}, \|v_k\|_{L^\infty(\Omega)} \le C$ for all $k \in \mathbb{N}$. Therefore, $\|\varphi_k\|_{L^\infty(\Omega)}$ is uniformly bounded if, and only if, $\|\psi_k\|_{L^\infty(\Omega)}$ is uniformly bounded.

First, we assume that, both $\|\varphi_k\|_{L^\infty(\Omega)}$ and $\|\psi_k\|_{L^\infty(\Omega)}$ are uniformly bounded. Applying the regularity result in $C_0^{1,\alpha}(\overline{\Omega})$, we get $(\varphi_k)_{k\ge1}$ and $(\psi_k)_{k\ge1}$, are bounded in $C_0^{1,\alpha}(\overline{\Omega})$. Since Ω is bounded, by Arzelà–Ascoli Theorem, up to a subsequence, we derive the convergence

$$\varphi_k \to \varphi \quad \text{and} \quad \psi_k \to \psi \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } k \to \infty. \tag{2.2}$$

Thus, $(\varphi, \psi) \in (C_0^1(\Omega))^2$ is a weak solution of the system

$$\begin{cases} -\Delta_p \varphi = \lambda_1 a(x)|\psi|^{\beta_1-1}\psi & \text{in } \Omega; \\ -\Delta_q \psi = \mu_1 b(x)|\varphi|^{\beta_2-1}\varphi & \text{in } \Omega; \\ \varphi = \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

By simplicity in X property (c), we must have either $(\varphi, \psi) \in \mathring{X}_+$ or $(-\varphi, -\psi) \in \mathring{X}_+$. If $(\varphi, \psi) \in \mathring{X}_+$, from the convergence in (2.2), we obtain $(\varphi_k, \psi_k) \in \mathring{X}_+$ for k sufficiently large. So, by uniqueness property (a), we have $(\lambda_k, \mu_k) \in \mathcal{C}_1(a, b, \Omega)$ for k large enough, contradicting that $(\lambda_k, \mu_k) \in \overline{\mathcal{R}_1(a, b, \Omega)}^c$ for all $k \in \mathbb{N}$. Then, we must have $(-\varphi, -\psi) \in \mathring{X}_+$. We now obtain $(-\varphi_k, -\psi_k) \in \mathring{X}_+$ for k sufficiently large, by convergence in (2.2). But this contradicts our hypothesis that at least one of $-\varphi_k$ or $-\psi_k$ doesn't belong to $\mathring{X}_+$ for all $k \in \mathbb{N}$.

Now, we assume that, $\|\varphi_k\|_{L^\infty(\Omega)} \to \infty$ and $\|\psi_k\|_{L^\infty(\Omega)} \to \infty$ as $k \to \infty$. For a subsequence indicated again by $((\varphi_k, \psi_k))_{k\ge1}$, there is a function $(\tilde{\varphi}, \tilde{\psi}) \in (C_0^1(\overline{\Omega}))^2$, such that $\|\tilde{\varphi}\|_{L^\infty(\Omega)} = \|\tilde{\psi}\|_{L^\infty(\Omega)} = 1$,

$$\tilde{u}_k \to \tilde{\varphi} \quad \text{and} \quad \tilde{v}_k \to \tilde{\psi} \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } k \to \infty. \tag{2.3}$$

Moreover, there are $\tilde{\lambda}, \tilde{\mu} \in \mathbb{R}$ such that $\tilde{\lambda}^{\beta_2}\tilde{\mu}^{p-1} = 1$,

$$\|u_k\|_{L^\infty(\Omega)}^{\beta_2} \to \tilde{\mu} \quad \text{and} \quad \|v_k\|_{L^\infty(\Omega)}^{\beta_1} \to \tilde{\lambda} \quad \text{as } k \to \infty.$$

Letting $k \to \infty$ in problem (2.1), we obtain $(\tilde{\varphi}, \tilde{\psi}) \in (C_0^1(\overline{\Omega}))^2$ is a weak solution of the problem

$$\begin{cases} -\Delta_p \tilde{\varphi} = \lambda_1 \tilde{\lambda} a(x)|\tilde{\psi}|^{\beta_1-1}\tilde{\psi} & \text{in } \Omega; \\ -\Delta_q \tilde{\psi} = \mu_1 \tilde{\mu} b(x)|\tilde{\varphi}|^{\beta_2-1}\tilde{\varphi} & \text{in } \Omega; \\ \tilde{\varphi} = \tilde{\psi} = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, $(\lambda_1\tilde{\lambda}, \mu_1\tilde{\mu}) \in \mathcal{C}_1(a, b, \Omega)$. By simplicity in X property (c), we must have either $(\tilde{\varphi}, \tilde{\psi}) \in \mathring{X}_+$ or $(-\tilde{\varphi}, -\tilde{\psi}) \in \mathring{X}_+$. Again, we obtain a contradiction in an analogous way, instead of convergence in (2.2), we invoke convergence in (2.3). This ends the proof. □

Proposition 2.4. *The system* (1.1) *does not admit any eigenvalues in* $\mathcal{R}_1(a, b, \Omega)$*. In particular, the curve* $\mathcal{C}_1(a, b, \Omega)$ *is locally isolated below.*

Proof. Arguing by contradiction, assume that the system (1.1) has an eigenvalue $(\lambda, \mu) \in \mathcal{R}_1(a, b, \Omega)$. Let $(\lambda_1, \mu_1) \in \mathcal{C}_1(a, b, \Omega)$ be such that $\frac{\mu}{\lambda} = \frac{\mu_1}{\lambda_1}$. So, we have $\lambda < \lambda_1$ and $\mu < \mu_1$. Consider a positive eigenfunction (φ, ψ) corresponding to (λ_1, μ_1) and an eigenfunction (u, v) to (λ, μ). Now, we can assume that u or v is positive somewhere in Ω. Otherwise, we take $(-u, -v)$ in place of (u, v). Consider the set $\Gamma = \{\gamma > 0 : \varphi > \gamma u \text{ and } \psi > \gamma^{\omega} v \text{ in } \Omega\}$, where $\omega := \frac{p-1}{\beta_1}$. Notice that Γ is upper bounded. Moreover, by strong maximum principle, Γ is nonempty. Define the positive number $\overline{\gamma} = \sup \Gamma$. Note that, $\varphi \geq \overline{\gamma} u$ and $\psi \geq \overline{\gamma}^{\omega} v$ in Ω.

Since $\lambda < \lambda_1$, $\mu < \mu_1$ and $(-\Delta_p)^{-1}$ and $(-\Delta_q)^{-1}$ are strictly order preserving, we can find $0 < \varepsilon < 1$ such that $\varphi > (\overline{\gamma} + \varepsilon) u$ and $\psi > (\overline{\gamma} + \varepsilon)^{\omega} v$ in Ω. But this contradicts the definition of $\overline{\gamma}$. This concludes the proof. □

The last proposition establishes the continuity of the principal eigenvalue with respect to the weight functions a and b which corresponds to the part (v) of Theorem 1.1.

Proposition 2.5. *Let $(a_k)_{k \geq 1}$ and $(b_k)_{k \geq 1}$ be sequences of weight functions in $L^{\infty}(\Omega)$ which are positive in Ω. Assume that $a_k \to a$ and $b_k \to b$ uniformly in Ω. If $a, b > 0$ in $\overline{\Omega}$, then $\Lambda'(a_k, b_k, \Omega) \to \Lambda'(a, b, \Omega)$.*

Proof. Given a fixed number $r_0 > 0$, let $(\lambda_1(a, b), \mu_1(a, b)) \in \mathcal{C}_1(a, b, \Omega)$ and $(\lambda_1(a_k, b_k), \mu_1(a_k, b_k)) \in \mathcal{C}_1(a_k, b_k, \Omega)$ be such that

$$\frac{\lambda_1(a, b)}{\mu_1(a, b)} = \frac{\lambda_1(a_k, b_k)}{\mu_1(a_k, b_k)} = \frac{1}{r_0}, \quad \text{for all } k \in \mathbb{N}. \tag{2.4}$$

By definitions of $\Lambda'(a_k, b_k, \Omega)$ and $\Lambda'(a, b, \Omega)$ and equalities in (2.4), it suffices to prove only that $\lambda_1(a_k, b_k) \to \lambda_1(a, b)$ as $k \to \infty$. Assume by contradiction that there is a number $\varepsilon > 0$ such that

$$|\lambda_1(a_k, b_k) - \lambda_1(a, b)| \geq \varepsilon$$

for $k \in \mathbb{N}$. Without loss of generality, we can assume

$$\lambda_1(a_k, b_k) - \lambda_1(a, b) \geq \varepsilon\,.$$

Since a and b are positive on $\overline{\Omega}$, we can define $\delta \in \mathbb{R}$ to be such that

$$0 < \delta < \frac{\varepsilon}{\lambda_1(a, b) + \varepsilon} \min\left\{\inf_{x \in \Omega} a(x), \inf_{x \in \Omega} b(x)\right\}\,.$$

By uniform convergence of the sequences $(a_k)_{k \geq 1}$ and $(b_k)_{k \geq 1}$, up to a subsequence, we can assume without loss of generality that

$$a_k(x) \geq a(x) - \delta, \qquad b_k(x) \geq b(x) - \delta$$

for all $x \in \Omega$ and $k \in \mathbb{N}$. Let (φ_k, ψ_k) and (φ, ψ) be positive eigenfunctions associated to the principal eigenvalues

$$(\lambda_1(a_k, b_k), \mu_1(a_k, b_k)) \quad \text{and} \quad (\lambda_1(a, b), \mu_1(a, b))\,,$$

respectively. Then, by strong maximum principle, the usual set $\Gamma = \{\gamma > 0 : \varphi_k > \gamma \varphi \text{ and } \psi_k > \gamma^{\omega} \psi \text{ in } \Omega\}$ is nonempty and upper bounded, where $\omega := \frac{p-1}{\beta_1}$. Set $\overline{\gamma} := \sup \Gamma > 0$. Using the definitions of r_0, ε and δ and the above inequalities, we get

$$\begin{aligned}
-\Delta_p(\overline{\gamma}\varphi) &= \lambda_1(a,b)a(x)(\overline{\gamma}^{\omega}\psi)^{\beta_1}\\
&= (\lambda_1(a,b)+\varepsilon)(a(x)-\delta)(\overline{\gamma}^{\omega}\psi)^{\beta_1} + (-\varepsilon a(x)+\lambda_1(a,b)\delta+\varepsilon\delta)(\overline{\gamma}^{\omega}\psi)^{\beta_1}\\
&< \lambda_1(a_k,b_k)a_k(x)\psi_k^{\beta_1} = -\Delta_p(\varphi_k);
\end{aligned}$$

$$\begin{aligned}
-\Delta_q(\overline{\gamma}^{\omega}\psi) &= \mu_1(a,b)b(x)(\overline{\gamma}\varphi)^{\beta_2}\\
&= r_0(\lambda_1(a,b)+\varepsilon)(b(x)-\delta)(\overline{\gamma}\varphi)^{\beta_2} + r_0(-\varepsilon b(x)+\lambda_1(a,b)\delta+\varepsilon\delta)(\overline{\gamma}\varphi)^{\beta_2}\\
&< r_0\lambda_1(a_k,b_k)b_k(x)\varphi_k^{\beta_2} = \mu_1(a_k,b_k)b_k(x)\varphi_k^{\beta_2} = -\Delta_q(\psi_k);
\end{aligned}$$

and $\varphi_k = \overline{\gamma}\varphi = \psi_k = \overline{\gamma}^{\omega}\psi = 0$ on $\partial\Omega$. Applying the strong comparison principle to each above equation (see [7, Theorem A.1]), we derive

$$\varphi_k > \overline{\gamma}\varphi,\ \psi_k > \overline{\gamma}^{\omega}\psi \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial\varphi_k}{\partial\nu} < \frac{\partial\overline{\gamma}\varphi}{\partial\nu},\ \frac{\partial\psi_k}{\partial\nu} < \frac{\partial\overline{\gamma}^{\omega}\psi}{\partial\nu} \quad \text{on } \partial\Omega.$$

Then, $\varphi_k > (\overline{\gamma}+\varepsilon)\varphi$ and $\psi_k > (\overline{\gamma}+\varepsilon)^{\omega}\psi$ in Ω for $0<\varepsilon<1$. But this contradicts the definition of $\overline{\gamma}$, and so concluding the proof. □

3 Proof of Theorem 1.2

We first prove the case $1 < p, q < n$. Let (φ,ψ) denote a principal eigenfunction corresponding to (λ,μ). Since

$$-\Delta_p\varphi = \lambda a(x)\psi^{\beta_1}$$

in the weak sense, then applying the equality (1.4) with $\Phi = \varphi$, we obtain

$$\lambda\int_\Omega a(x)\psi^{\beta_1}\varphi dx = \int_\Omega |\nabla\varphi|^p dx.$$

Moreover, by using Hölder and Young inequalities, we get

$$\int_\Omega a(x)\psi^{\beta_1}\varphi dx \le \|a\|_{L^\infty(\Omega)}\left(\frac{1}{p}\|\varphi\|^p_{L^{\beta_1+1}(\Omega)} + \frac{p-1}{p}\|\psi\|^{p\beta_1/(p-1)}_{L^{\beta_1+1}(\Omega)}\right).$$

Consequently,

$$\lambda D_1\left(\|\varphi\|^p_{L^{\beta_1+1}(\Omega)} + \|\psi\|^{p\beta_1/(p-1)}_{L^{\beta_1+1}(\Omega)}\right) \ge \int_\Omega|\nabla\varphi|^p dx, \tag{3.1}$$

where

$$D_1 = \max\left\{\frac{1}{p}\|a\|_{L^\infty(\Omega)}, \frac{p-1}{p}\|a\|_{L^\infty(\Omega)}, \frac{1}{q}\|b\|_{L^\infty(\Omega)}, \frac{q-1}{q}\|b\|_{L^\infty(\Omega)}\right\}.$$

Similarly, it follows from

$$-\Delta_q\psi = \mu b(x)\varphi^{\beta_2}$$

in the weak sense that

$$\mu\|b\|_{L^\infty(\Omega)}|\Omega|^{\frac{\beta_1-\beta_2}{\beta_1+1}}\left(\frac{q-1}{q}\|\varphi\|^{q\beta_2/(q-1)}_{L^{\beta_1+1}(\Omega)} + \frac{1}{q}\|\psi\|^q_{L^{\beta_1+1}(\Omega)}\right) \ge \int_\Omega|\nabla\psi|^q dx.$$

Now, since $|\Omega|\le 1$ and $\frac{p(q-1)}{q\beta_2}\ge 1$, we have

$$(D_1\mu)^{\frac{p(q-1)}{q\beta_2}}D_2\left(\|\varphi\|^p_{L^{\beta_1+1}(\Omega)} + \|\psi\|^{p\beta_1/(p-1)}_{L^{\beta_1+1}(\Omega)}\right) \ge \left(\int_\Omega|\nabla\psi|^q dx\right)^{\frac{p(q-1)}{q\beta_2}}, \tag{3.2}$$

where $D_2 = 2^{\frac{p(q-1)}{q\beta_2}-1}$. Thus, adding up (3.1) and (3.2) inequalities shows that

$$\lambda + \mu^{\frac{p(q-1)}{q\beta_2}} \geq \frac{1}{D_3}\left(\frac{\int_\Omega |\nabla\varphi|^p dx + \left(\int_\Omega |\nabla\psi|^q dx\right)^{\frac{p(q-1)}{q\beta_2}}}{\|\varphi\|^p_{L^{\beta_1+1}(\Omega)} + \|\psi\|^{p\beta_1/(p-1)}_{L^{\beta_1+1}(\Omega)}}\right),$$

where $D_3 = \max\left\{D_2 D_1^{\frac{p(q-1)}{q\beta_2}}, D_1\right\}$.

On the other hand, by interpolation inequality, inequality (1.5) and variational characterization of $\lambda_{1,p}(\Omega)$, we obtain

$$\frac{\int_\Omega |\nabla\varphi|^p dx}{\|\varphi\|^p_{L^{\beta_1+1}(\Omega)}} \geq \left(c_{n,p}\right)^{(\theta_p-1)p} \lambda_{1,p}(\Omega)^{\theta_p},$$

where

$$\frac{1}{\beta_1+1} = \frac{\theta_p}{p} + \frac{1-\theta_p}{p^*}$$

and

$$\frac{\left(\int_\Omega |\nabla\psi|^q dx\right)^{\frac{p(q-1)}{q\beta_2}}}{\|\psi\|^{p\beta_1/(p-1)}_{L^{\beta_1+1}(\Omega)}} \geq \left(c_{n,q}\right)^{(\theta_q-1)\frac{p\beta_1}{p-1}} \lambda_{1,q}(\Omega)^{\theta_q\frac{p\beta_1}{(p-1)q}},$$

where

$$\frac{1}{\beta_1+1} = \frac{\theta_q}{q} + \frac{1-\theta_q}{q^*}.$$

Furthermore, by Faber-Krahn inequality for the first eigenvalue of $-\Delta_p$ and inequality (1.8), we get

$$\lambda_{1,p}(\Omega) \geq \lambda_{1,p}(B_1)|B_1|^{\frac{p}{n}}|\Omega|^{-\frac{p}{n}} \geq \left(\frac{n}{p}\right)^p |B_1|^{\frac{p}{n}}|\Omega|^{-\frac{p}{n}}.$$

Then, using that $p \leq q$, $|\Omega| \leq 1$ and $\beta_1 \geq p-1$, we obtain

$$\lambda_{1,p}(\Omega)^{\theta_p}, \lambda_{1,q}(\Omega)^{\theta_q\frac{p\beta_1}{(p-1)q}} \geq \left(\frac{n}{q}\right)^{p\theta_p} |B_1|^{\theta_p\frac{p}{n}}|\Omega|^{-\theta_p\frac{p}{n}}.$$

Therefore,

$$\lambda + \mu^{\frac{p(q-1)}{q\beta_2}} \geq C\left(\frac{n}{q}\right)^{p\theta_p} |B_1|^{\theta_p\frac{p}{n}}|\Omega|^{-\theta_p\frac{p}{n}},$$

where $C = \frac{1}{D_3}\min\left\{\left(c_{n,p}\right)^{(\theta_p-1)p}, \left(c_{n,q}\right)^{(\theta_q-1)\frac{p\beta_1}{p-1}}\right\}$.

The rest of proof is analogue, by using interpolation inequality with θ_p and θ_q appropriate and instead of inequality (1.5), we invoke inequalities (1.6) and (1.7). This concludes the proof of the theorem.

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Existence of weak solutions for quasilinear Schrödinger equations with a parameter

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Abstract. In this paper, we study the following quasilinear Schrödinger equation of the form

$$-\Delta_p u + V(x)|u|^{p-2}u - \left[\Delta_p(1+u^2)^{\alpha/2}\right]\frac{\alpha u}{2(1+u^2)^{(2-\alpha)/2}} = k(u), \qquad x \in \mathbb{R}^N,$$

where p-Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ $(1 < p \leq N)$ and $\alpha \geq 1$ is a parameter. Under some appropriate assumptions on the potential V and the nonlinear term k, using some special techniques, we establish the existence of a nontrivial solution in $C^{1,\beta}_{\mathrm{loc}}(\mathbb{R}^N)$ $(0 < \beta < 1)$, we also show that the solution is in $L^\infty(\mathbb{R}^N)$ and decays to zero at infinity when $1 < p < N$.

Keywords: quasilinear Schrödinger equation, variational method, mountain-pass theorem, p-Laplace operator.

2020 Mathematics Subject Classification: 35J62, 35J20, 35Q55.

1 Introduction

In this work, we are interested in the existence of nontrivial solution to the following quasilinear Schrödinger equation

$$-\Delta_p u + V(x)|u|^{p-2}u - \left[\Delta_p(1+u^2)^{\alpha/2}\right]\frac{\alpha u}{2(1+u^2)^{(2-\alpha)/2}} = k(u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

where p-Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ $(1 < p \leq N)$ and $\alpha \geq 1$ is a parameter. V is a positive continuous potential and $k(u)$ is a nonlinear term of subcritical type.

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Such equations arise in various branches of mathematical physics. For instance, solutions of equation (1.1), in the case $p = 2$ and $\alpha = 1$ are closed related to the existence of solitary wave solutions for quasilinear Schrödinger equations

$$iz_t = -\Delta z + W(x)z - \tilde{k}(|z|^2)z - \Delta l(|z|^2)l'(|z|^2)z, \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $\tilde{k}$, $l : \mathbb{R}^+ \to \mathbb{R}$ are real functions. The form of (1.2) has been derived as models of several physical phenomena corresponding to various types of l. For instance, the case $l(s) = s$ models the time evolution of the condensate wave function in super-fluid film [15, 16], and is called the superfluid film equation in fluid mechanics by Kurihara [15]. In the case $l(s) = (1+s)^{1/2}$, problem (1.2) models the self-channeling of a high-power ultra short laser in matter, the propagation of a high-irradiance laser in a plasma creates an optical index depending nonlinearly on the light intensity and this leads to interesting new nonlinear wave equation (see [2, 4, 8, 28]). For more physical motivations and more references dealing with applications, we refer the reader to [1, 13, 17, 25–27] and references therein.

It is well known that, via the ansatz $z(t, x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and u is a real function, (1.2) can be reduced to the following elliptic equation

$$-\Delta u + V(x)u - \left[\Delta(l(u^2))\right] l'(u^2)u = k(u), \qquad x \in \mathbb{R}^N, \tag{1.3}$$

where $V(x) = W(x) - E$ and $k(u) = \tilde{k}(u^2)u$.

If we take $l(s) = s$ in (1.3), then we obtain the superfluid film equation in plasma physics

$$-\Delta u + V(x)u - \left[\Delta(u^2)\right] u = k(u), \qquad x \in \mathbb{R}^N. \tag{1.4}$$

Clearly, when $p = 2$ and $\alpha = 2$, equation (1.1) turns into equation (1.4). Equation (1.4) has been paid much attention in the past two decades. Many existence and multiplicity results of nontrivial solutions have been established by differential methods such as constrained minimization argument, changes of variables, Nehari method, a dual approach, perturbation method, see [7, 12, 14, 20–24, 26, 29, 31] and references therein.

If we take $l(s) = (1+s)^{1/2}$ in (1.3), then we get the equation

$$-\Delta u + V(x)u - \left[\Delta(1+u^2)^{1/2}\right] \frac{u}{2(1+u^2)^{1/2}} = k(u), \qquad x \in \mathbb{R}^N, \tag{1.5}$$

which models the self-channeling of a high-power ultrashort laser in matter. Obviously, equation (1.1) turns into (1.5) for the case $p = 2$ and $\alpha = 1$.

The existence of positive solutions for (1.5) has been studied recently. In [32], by a change of variables and the Ambrosetti–Rabinowitz mountain-pass theorem, the authors proved that (1.5) has a positive solution. They assume that the potential $V \in C(\mathbb{R}^N, \mathbb{R})$ and the nonlinearity $k : \mathbb{R} \to \mathbb{R}$ is Hölder continuous and satisfy the following conditions:

(V_1) $V(x) \geq V_0 > 0$, for all $x \in \mathbb{R}^N$;

(V_2) $\lim_{|x|\to\infty} V(x) = V(\infty) < \infty$ and $V(x) \leq V(\infty)$, for all $x \in \mathbb{R}^N$;

(H_1) $k(s) = 0$ if s ≤ 0;

(H_2) $k(s) = o(s)$ as $s \to 0^+$;

(H_3) There exists $2 < \theta < 2^*$ such that $|k(s)| \leq C(1 + |s|^{\theta-1})$;

(H_4) There exists $\mu > \sqrt{6}$ such that $0 < \mu K(s) \leq sk(s)$ for all $s > 0$, where $K(s) = \int_0^s k(t)dt$.

In [5], by a dual approach, the authors studied the existence of positive solution for the following equation

$$-\Delta u + Ku - \left[\Delta(1+u^2)^{\alpha/2}\right]\frac{\alpha u}{2(1+u^2)^{(2-\alpha)/2}} = |u|^{q-1}u + |u|^{p-1}u, \qquad x \in \mathbb{R}^N, \tag{1.6}$$

where $K > 0$, $N \geq 3, \alpha \geq 1$ and $2 < q+1 < p+1 < \alpha 2^*$. Similar works can be found in [3,6,18,22] and reference therein.

However, to the best of our knowledge, in all works mentioned above, there are no existence results in the literature on the case $p \neq 2, \alpha \geq 1$ and the nonlinear term becomes general function. Motivated by the works mentioned above and [5,7,20,22,31,32], our purpose in this paper is to study the existence of nontrivial weak solutions of (1.1) under some assumptions on the potential $V(x)$ and nonlinear term $k(s)$.

Definition 1.1. We say that $u : \mathbb{R}^N \to \mathbb{R}$ is a weak solution of (1.1) if $u \in W^{1,p}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and

$$\begin{aligned}\int_{\mathbb{R}^N}\left[1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right]|\nabla u|^{p-2}\nabla u\nabla\psi dx&\\ +\frac{\alpha^p}{2}\int_{\mathbb{R}^N}\frac{[1+(\alpha-1)u^2]}{(1+u^2)^{1+(2-\alpha)p/2}}|\nabla u|^p|u|^{p-2}u\psi dx&\\ =\int_{\mathbb{R}^N}\eta(x,u)\psi dx,\ \forall\psi\in C_0^\infty(\mathbb{R}^N),&\end{aligned} \tag{1.7}$$

where $\eta(x,u) = k(u) - V(x)|u|^{p-2}u$.

In such a case, we can deduce formally that the Euler–Lagrange functional associated with the equation (1.1) is

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^N}\left[1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right]|\nabla u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \int_{\mathbb{R}^N} K(u)dx,$$

where $K(s) = \int_0^s k(t)dt$.

For (1.1), due to the appearance of the nonlocal term $\int_{\mathbb{R}^N}\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}|\nabla u|^p dx$, J may be not well defined. To overcome this difficulty, enlightened by [7,20,32], we make a change of variables as

$$v = H(u) = \int_0^u h(t)dt, \tag{1.8}$$

where $h(t) = \left[1+\frac{\alpha^p|t|^p}{2(1+t^2)^{(2-\alpha)p/2}}\right]^{1/p}$, $t \in \mathbb{R}$. Since $H(t)$ is strictly increasing on $\mathbb{R}$, the inverse function $H^{-1}(t)$ of $H(t)$ exists. Then after the change of variables, $J(u)$ can be written by

$$\mathcal{F}(v) = J(H^{-1}(v)) = \frac{1}{p}\int_{\mathbb{R}^N}|\nabla v|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|H^{-1}(v)|^p dx - \int_{\mathbb{R}^N} K(H^{-1}(v))dx. \tag{1.9}$$

According to Lemma 2.1 and our hypotheses on $V(x)$ and $k(s)$ below, it is clear that $\mathcal{F}$ is well defined in $W^{1,p}(\mathbb{R}^N)$ and $\mathcal{F} \in C^1$. The Euler–Lagrange equation associated to the functional $\mathcal{F}$ is

$$-\Delta_p v = \frac{\eta(x, H^{-1}(v))}{h(H^{-1}(v))}, \qquad x \in \mathbb{R}^N. \tag{1.10}$$

In Proposition 2.2, we will show the relationship between the solutions of (1.10) and the solutions of (1.1).

Throughout this paper, let $1 < p \leq N$, $\alpha \geq 1$. Besides, we assume that the potential $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and satisfies $(\mathrm{V}_1) - (\mathrm{V}_2)$, the nonlinearity $k(s) \in C(\mathbb{R}, \mathbb{R})$ and satisfies the following conditions:

(K_1) k is odd and $k(s) = o(|s|^{p-2}s)$ as $s \to 0$;

(K_2) There exists a constant $C > 0$ such that

$$|k(s)| \leq C(1 + |s|^{\theta-1}), \qquad \forall s \in \mathbb{R},$$

where $\alpha p < \theta < \alpha p^*$ if $1 < p < N$ and $\theta > \alpha p$ if $p = N$;

(K_3) There exists $\mu \geq \widetilde{T}(p,\alpha)p$ such that $0 < \mu K(s) \leq sk(s)$ for all $s > 0$, where $K(s) = \int_0^s k(t)dt$, $\widetilde{T}(p,\alpha) = 1 + T(p,\alpha)$ and

$$T(p,\alpha) = \sup_{t\geq 0} \frac{th'(t)}{h(t)} = \sup_{t\geq 0} \frac{\alpha^p t^p \left[1 + (\alpha-1)t^2\right]}{(1+t^2)\left[2(1+t^2)^{(2-\alpha)p/2} + \alpha^p t^p\right]} > 0. \tag{1.11}$$

Our main result is the following.

Theorem 1.2. *Let $1 < p \leq N$, $\alpha \geq 1$. Suppose (V_1)–(V_2) and (K_1)–(K_2) hold. Then (1.1) admits a nontrivial weak solution $u \in C^{1,\beta}_{\mathrm{loc}}(\mathbb{R}^N)$ $(0 < \beta < 1)$ provided that one of the following conditions is satisfied:*

(a) (K_3) *holds with $\mu > \widetilde{T}(p,\alpha)p$;*

(b) (K_3) *holds with $\mu = \widetilde{T}(p,\alpha)p = 2p$ and $p < \theta < p^*$ if $1 < p < N$ or $\theta > p$ if $p = N$ in* (K_2).

Furthermore, if $1 < p < N$, then $u \in L^\infty(\mathbb{R}^N)$ and $u(x) \to 0$ as $|x| \to \infty$.

Remark 1.3. It is not difficult to verify that $T(p,\alpha) = \alpha - 1$ if $\alpha \geq 2$ and $\alpha - 1 \leq T(p,\alpha) < 1$ if $1 \leq \alpha < 2$. If $p = 2$, then $T(p,\alpha) = T(2,\alpha)$, which equals to the $T(\alpha)$ in [5]. If $p = 2$ and $\alpha = 1$, we obtain $T(2,1) = 5 - 2\sqrt{6}$. Thus, $\mu \geq \widetilde{T}(2,1)2 = (1 + T(2,1))2 \approx 2.202$ in (K_3) is better than $\mu > 2\sqrt{6} \approx 2.449$ in (H_4). If $p = 2$ and $\alpha = 2$, we have $\widetilde{T}(2,2)2 = 4$, which coincides with that in [7]. Therefore, our conclusion in Theorem 1.2 can be viewed as an extension result in [5,7,20,32].

The organization of this paper is as follows. In Section 2, we give some properties of $H(t)$ and some preliminary results. In Section 3, we present an auxiliary problem and some related results. In Section 4, we complete the proof of Theorem 1.2.

Throughout this paper, C and C_i stand for positive constants which may take different values at different places. B_R denotes the open ball centered at the origin and radius $R > 0$, $C_0^\infty(\mathbb{R}^N)$ denotes functions infinitely differentiable with impact support in $\mathbb{R}^N$. For $1 \leq p \leq \infty$, $L^p(\mathbb{R}^N)$ denotes the usual Lebesgue space with the norms

$$\|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{1/p}, \qquad 1 \leq p < \infty;$$

$$\|u\|_\infty = \inf\left\{M > 0 : |u(x)| \leq M \text{ almost everywhere in } \mathbb{R}^N\right\}.$$

$W^{1,p}(\mathbb{R}^N)$ denotes the Sobolev spaces modelled in $L^p(\mathbb{R}^N)$ with its usual norm

$$\|u\| = \left(\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) dx\right)^{1/p}.$$

$\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and its dual X^*. The weak (strong) convergence in X is denoted by $\rightharpoonup$ $(\to)$, respectively.

2 Preliminaries

We first give some properties of the change of variables $H : \mathbb{R} \to \mathbb{R}$ defined by (1.8), which will be used frequently in the sequel of the paper.

Lemma 2.1. *For functions h, H and H^{-1}, the following properties hold:*

(1) H is odd, strictly increasing, invertible and C^2 in $\mathbb{R}$;

(2) $0 < (H^{-1})'(t) \leq 1, \forall t \in \mathbb{R}$;

(3) $|H^{-1}(t)| \leq |t|, \forall t \in \mathbb{R}$;

(4) $\lim_{t\to 0} \frac{H^{-1}(t)}{t} = 1$;

(5) $\lim_{t\to+\infty} \frac{\left(H^{-1}(t)\right)^\alpha}{t} = \begin{cases} \sqrt[p]{\frac{2}{3}}, & \alpha = 1, \\ \sqrt[p]{2}, & \alpha > 1; \end{cases}$

(6) $h(H^{-1}(t))H^{-1}(t) \leq \widetilde{T}(p,\alpha)t \leq \widetilde{T}(p,\alpha)h(H^{-1}(t))H^{-1}(t), \forall t \geq 0$;

(7) $h(H^{-1}(t))\left(H^{-1}(t)\right)^2 \leq \widetilde{T}(p,\alpha)tH^{-1}(t) \leq \widetilde{T}(p,\alpha)h(H^{-1}(t))\left(H^{-1}(t)\right)^2, \forall t \in \mathbb{R}$;

(8) $|H^{-1}(t)| \leq C|t|^{1/\alpha}$ for some $C > 0$ and $\forall t \in \mathbb{R}$;

(9) There exists $C > 0$ such that

$$|H^{-1}(t)| \geq \begin{cases} C|t|, & |t| \leq 1, \\ C|t|^{1/\alpha}, & |t| \geq 1. \end{cases}$$

Proof. By the definition of H, it is easy to verify that (1)–(4) hold.

(5) If $\alpha > 1$, since

$$h(t) = \left[1 + \frac{\alpha^p t^p}{2(1+t^2)^{(2-\alpha)p/2}}\right]^{1/p} = \left[1 + \frac{\alpha^p t^p}{2(1+t^2)^{p/2}}(1+t^2)^{(\alpha-1)p/2})\right]^{1/p}, \ t > 0,$$

one has $h(t) \sim (\frac{\alpha^p}{2} t^{p(\alpha-1)})^{1/p} = \frac{\alpha}{\sqrt[p]{2}} t^{\alpha-1}$ as $t \to +\infty$. Moreover, $H(t) = \int_0^t h(s)ds \sim \frac{1}{\sqrt[p]{2}} t^\alpha$ as $t \to +\infty$. Remember the fact $H^{-1}(t)$ is the inverse of $H(t)$, so we get $H^{-1}(t) \sim (\sqrt[p]{2}t)^{1/\alpha}$ as $t \to +\infty$, which implies $\lim_{t\to+\infty} \frac{(H^{-1}(t))^\alpha}{t} = \sqrt[p]{2}$. If $\alpha = 1$, the result is obvious since $h(t)$ is an increasing bounded function when $t > 0$.

(6) Denote $g_1(t) = h(H^{-1}(t))H^{-1}(t) - t, \ t \geq 0$. Obviously $g_1(0) = 0$. Since $\alpha \geq 1$, one has

$$\begin{aligned} g_1'(t) &= \frac{H^{-1}(t)h'(H^{-1}(t))}{h(H^{-1}(t))} \\ &= \frac{\alpha^p (H^{-1}(t))^p \left[1 + (\alpha-1)(H^{-1}(t))^2\right]}{(1+(H^{-1}(t))^2)\left[2\left(1+(H^{-1}(t))^2\right)^{(2-\alpha)p/2} + \alpha^p (H^{-1}(t))^p\right]} \geq 0, \qquad \forall t \geq 0, \end{aligned}$$

which implies

$$h(H^{-1}(t))H^{-1}(t) \geq t, \qquad \forall t \geq 0.$$

Consequently,

$$\widetilde{T}(p,\alpha)t \le \widetilde{T}(p,\alpha)h(H^{-1}(t))H^{-1}(t), \qquad \forall t \ge 0.$$

Set $g_2(t) = \widetilde{T}(p,\alpha)t - h(H^{-1}(t))H^{-1}(t),\ t \ge 0$. Clearly $g_2(0) = 0$. By virtue of $H^{-1}(t) \ge 0,\ t \ge 0$ and (1.11), we can deduce that

$$\begin{aligned} g_2'(t) &= T(p,\alpha) - \frac{H^{-1}(t)h'(H^{-1}(t))}{h(H^{-1}(t))} \\ &= T(p,\alpha) - \frac{sh'(s)}{h(s)}|_{s=H^{-1}(t)} \\ &\ge 0, \qquad \forall t \ge 0, \end{aligned}$$

which implies

$$h(H^{-1}(t))H^{-1}(t) \le \widetilde{T}(p,\alpha)t, \qquad \forall t \ge 0.$$

(7) Since $tH^{-1}(t) \ge 0,\ \forall t \in \mathbb{R}$, utilizing (6), we have

$$h(H^{-1}(t))\big(H^{-1}(t)\big)^2 \le \widetilde{T}(p,\alpha)tH^{-1}(t) \le \widetilde{T}(p,\alpha)h(H^{-1}(t))\big(H^{-1}(t)\big)^2, \qquad \forall t \in \mathbb{R}.$$

It is not difficult to verify that (8) and (9) are right from (1), (4) and (5). □

Under the hypotheses (V_1)–(V_2) and (K_1)–(K_3), we readily derive that $\mathcal{F} \in C^1(W^{1,p}(\mathbb{R}^N))$ and

$$\langle \mathcal{F}'(v), \omega \rangle = \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla \omega dx - \int_{\mathbb{R}^N} \frac{\eta(x, H^{-1}(v))}{h(H^{-1}(v))}\omega dx$$

for $v, \omega \in W^{1,p}(\mathbb{R}^N)$. Thus, the critical points of $\mathcal{F}$ correspond exactly to the weak solutions of (1.10). The following results characterize the relationship between the solutions of (1.10) and (1.1).

Proposition 2.2.

(i) If $v \in W^{1,p}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$ is a critical point of the functional $\mathcal{F}$, then $u = H^{-1}(v)$ is a weak solution of (1.1)*;*

(ii) if v is a classical solution of (1.10)*, then $u = H^{-1}(v)$ is a classical solution of* (1.1).

Proof. (i) It is easy to see that $|u|^p = |H^{-1}(v)|^p \le |v|^p$ and $|\nabla u|^p = |(H^{-1})'(v)|^p|\nabla v|^p \le |\nabla v|^p$. Hence, $u \in W^{1,p}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$. Since v is a critical point of $\mathcal{F}$, we get

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla \omega dx = \int_{\mathbb{R}^N} \frac{\eta(x, H^{-1}(v))}{h(H^{-1}(v))}\omega dx, \qquad \forall \omega \in W^{1,p}(\mathbb{R}^N). \tag{2.1}$$

Note that

$$\nabla v = H'(u)\nabla u = h(u)\nabla u = \left(1 + \frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{1/p} \nabla u. \tag{2.2}$$

For all $\psi \in C_0^\infty(\mathbb{R}^N)$, one can achieve

$$h(H^{-1}(v))\psi = h(u)\psi = \left(1 + \frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{1/p} \psi \in W^{1,p}(\mathbb{R}^N),$$

and

$$\begin{aligned}\nabla\left(h(H^{-1}(v))\psi\right) &= h'(u)\psi\nabla u + h(u)\nabla\psi \\ &= \frac{\alpha^p}{2}\left(1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{(1-p)/p}\frac{(1+(\alpha-1)u^2)}{(1+u^2)^{1+(2-\alpha)p/2}}|u|^{p-2}u\psi\nabla u \\ &\quad+\left(1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{1/p}\nabla\psi.\end{aligned} \tag{2.3}$$

Letting $\omega = h(H^{-1}(v))\psi$ in (2.1) and combining (2.2)–(2.3) enable us to deduce (1.7), which means that $u = H^{-1}(v)$ is a weak solution of (1.1).

(ii) From

$$\Delta_p v = \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(|\nabla v|^{p-2}\frac{\partial v}{\partial x_i}\right) = \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(h^{p-1}(u)|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right),$$

we deduce that

$$\begin{aligned}\Delta_p v &= h^{p-1}(u)\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right) + |\nabla u|^{p-2}\sum_{i=1}^N \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}\left(h^{p-1}(u)\right) \\ &= h^{p-1}(u)\Delta_p u + (p-1)h^{p-2}(u)h'(u)|\nabla u|^p \\ &= \left(1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{(p-1)/p}\Delta_p u \\ &\quad+\left(1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{-1/p}\frac{(p-1)\alpha^p\left(1+(\alpha-1)u^2\right)}{2(1+u^2)^{1+(2-\alpha)p/2}}|u|^{p-2}u|\nabla u|^p.\end{aligned}$$

Consequently,

$$\begin{aligned}&\left(1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{(p-1)/p}\Delta_p u \\ &\quad+\left(1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{-1/p}\frac{(p-1)\alpha^p\left(1+(\alpha-1)u^2\right)}{2(1+u^2)^{1+(2-\alpha)p/2}}|u|^{p-2}u|\nabla u|^p \\ &= -\left(1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{-1/p}\eta(x,u),\end{aligned}$$

that is,

$$\Delta_p u + \frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\Delta_p u + \frac{(p-1)\alpha^p\left(1+(\alpha-1)u^2\right)}{2(1+u^2)^{1+(2-\alpha)p/2}}|u|^{p-2}u|\nabla u|^p = -\eta(x,u). \tag{2.4}$$

Noticing that

$$\begin{aligned}&\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\Delta_p u + \frac{(p-1)\alpha^p(1+(\alpha-1)u^2)}{2(1+u^2)^{1+(2-\alpha)p/2}}|u|^{p-2}u|\nabla u|^p \\ &= \left[\Delta_p(1+u^2)^{\alpha/2}\right]\frac{\alpha u}{2(1+u^2)^{(2-\alpha)/2}}.\end{aligned}$$

This together with the (2.4) derive

$$-\Delta_p u - \left[\Delta_p(1+u^2)^{\alpha/2}\right]\frac{\alpha u}{2(1+u^2)^{(2-\alpha)/2}} = \eta(x,u).$$

The proof is finished. □

3 Auxiliary problem

To prove the main result, we employ the results [9] for the equation

$$-\Delta_p v = g(v), \qquad x \in \mathbb{R}^N. \tag{3.1}$$

The energy functional associated to (3.1) is

$$I(v) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p dx - \int_{\mathbb{R}^N} G(v)dx,$$

where $G(s) = \int_0^s g(t)dt$. Obviously, $I \in C^1(W^{1,p}(\mathbb{R}^N))$ under the assumptions on $g(s)$ below:

(G_0) g is odd and $g \in C(\mathbb{R}, \mathbb{R})$;

(G_1) $-\infty < \liminf_{s\to 0} \frac{g(s)}{|s|^{p-2}s} \le \limsup_{s\to 0} \frac{g(s)}{|s|^{p-2}s} = -\sigma < 0$ if $1 < p < N$,

$-\infty < \lim_{s\to 0} \frac{g(s)}{|s|^{N-2}s} = -\sigma < 0$ if $p = N$;

(G_2) When $1 < p < N$, $\lim_{s\to\infty} \frac{|g(s)|}{|s|^{p^*-1}} = 0$, where $p^* = \frac{Np}{N-p}$; when $p = N$, for some positive constants C and β_0, that

$$|g(s)| \le C\left[\exp\left(\beta_0|s|^{N/(N-1)}\right) - S_{N-2}(\beta_0, s)\right]$$

for all $|s| \ge R > 0$, where

$$S_{N-2}(\beta_0, s) = \sum_{k=0}^{N-2} \frac{\beta_0^k}{k!}|s|^{kN/(N-1)};$$

(G_3) There exists $\xi > 0$ such that $G(\xi) > 0$.

We recall that a solution $v(x)$ of (3.1) is said to be *a least energy solution (or ground state solution)* if and only if

$$I(v) = a, \quad \text{where } a = \inf\{I(w) : w \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} \text{ is a solution of (3.1)}\}. \tag{3.2}$$

Theorem 3.1 ([9, Theorem 1.4]). *Let $1 < p \le N$ and suppose (G_0)–(G_2) hold. Then setting*

$$\Lambda = \left\{\gamma \in C\big([0,1], W^{1,p}(\mathbb{R}^N)\big) : \ \gamma(0) = 0,\ I(\gamma(1)) < 0\right\}, \qquad b = \inf_{\gamma\in\Lambda}\max_{0\le t\le 1} I(\gamma(t)),$$

we have $\Lambda \ne \emptyset$ and $b = a$. Furthermore, for each least energy solution w of (3.1), there exists a path $\gamma \in \Lambda$ such that $w \in \gamma([0,1])$ and

$$\max_{t\in[0,1]} I(\gamma(t)) = I(w).$$

Theorem 3.2 ([9, Theorem 1.6]). *Let $1 < p \le N$ and assume that (G_0)–(G_3) are satisfied, then equation (3.1) has a least energy solution v which is positive.*

Theorem 3.3 ([9, Theorem 1.8]). *Assume that all conditions of Theorem 3.1 hold, then there exist $\lambda > 0$ and $\delta > 0$ such that $I(v) \ge \lambda\|v\|^p$ if $\|v\| \le \delta$.*

Lemma 3.4. *Assume that* (V_1)–(V_2) *and* (K_1)–(K_2) *are satisfied, then the functional* $\mathcal{F}$ *has a mountain-pass geometry.*

Proof. Let the energy functionals corresponding to the equations $-\Delta_p v = m_0(v)$ and $-\Delta_p v = m_\infty(v)$ be

$$\begin{aligned}
\mathcal{F}_0(v) &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V_0 |H^{-1}(v)|^p dx - \int_{\mathbb{R}^N} K(H^{-1}(v))dx,\\
\mathcal{F}_\infty(v) &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(\infty) |H^{-1}(v)|^p dx - \int_{\mathbb{R}^N} K(H^{-1}(v))dx,
\end{aligned}$$

respectively, where

$$\begin{aligned}
m_0(v) &= \frac{1}{h(H^{-1}(v))}\left[k(H^{-1}(v)) - V_0|H^{-1}(v)|^{p-2}H^{-1}(v)\right],\\
m_\infty(v) &= \frac{1}{h(H^{-1}(v))}\left[k(H^{-1}(v)) - V(\infty)|H^{-1}(v)|^{p-2}H^{-1}(v)\right].
\end{aligned}$$

Notice that $\mathcal{F}_0(v) \le \mathcal{F}(v) \le \mathcal{F}_\infty(v)$ for all $v \in W^{1,p}(\mathbb{R}^N)$.

Now, we claim that m_0 and m_∞ satisfy (G_0)–(G_2).

Obviously, m_0 and m_∞ satisfy (G_0).

By use of $k(s) = o(|s|^{p-2}s)$ as $s \to 0$ and Lemma 2.1 (4), we derive that

$$\lim_{s\to 0}\frac{m_0(s)}{|s|^{p-2}s} = -V_0 < 0, \qquad \lim_{s\to 0}\frac{m_\infty(s)}{|s|^{p-2}s} = -V(\infty) < 0, \quad \text{if } 1<p<N,$$

and

$$\lim_{s\to 0}\frac{m_0(s)}{|s|^{N-2}s} = -V_0 < 0, \qquad \lim_{s\to 0}\frac{m_\infty(s)}{|s|^{N-2}s} = -V(\infty) < 0, \quad \text{if } p=N.$$

Hence, m_0 and m_∞ satisfy (G_1).

Similarly to the argument in the proof of Lemma 2.1 (5), we can show that

$$\lim_{s\to\infty}\frac{|H^{-1}(s)|^{\alpha-1}}{h(H^{-1}(s))} = \begin{cases} \sqrt[p]{\frac{2}{3}}, & \alpha = 1,\\ \frac{\sqrt[p]{2}}{\alpha}, & \alpha > 1. \end{cases} \tag{3.3}$$

When $1<p<N$, it follows from (K_2) and Lemma 2.1 (2), (3), (8) that

$$\begin{aligned}
|m_0(s)| &\le \frac{1}{h(H^{-1}(s))}(C + C|H^{-1}(s)|^{\theta-1} + V_0|H^{-1}(s)|^{p-1})\\
&\le C + C\frac{|H^{-1}(s)|^{\theta-1}}{h(H^{-1}(s))} + V_0|s|^{p-1}\\
&= C + C|H^{-1}(s)|^{\theta-\alpha}\frac{|H^{-1}(s)|^{\alpha-1}}{h(H^{-1}(s))} + V_0|s|^{p-1}\\
&\le C + C|s|^{(\theta-\alpha)/\alpha}\frac{|H^{-1}(s)|^{\alpha-1}}{h(H^{-1}(s))} + V_0|s|^{p-1},
\end{aligned} \tag{3.4}$$

where $\alpha p < \theta < \alpha p^*$. Combining (3.3) and (3.4), we can deduce

$$\lim_{s\to\infty}\frac{|m_0(s)|}{|s|^{p^*-1}} = 0.$$

On the other hand, when $p = N$, applying (K_2) and Lemma 2.1 (2), (3), we conclude that

$$|m_0(s)| \leq C_1 + C_2|s|^{\theta-1}.$$

Then there exist positive constants C and β_0 such that

$$|m_0(s)| \leq C\left[\exp\left(\beta_0|s|^{N/(N-1)}\right) - S_{N-2}(\beta_0, s)\right]$$

for all $|s| \geq R > 0$, where $S_{N-2}(\beta_0, s) = \sum_{k=0}^{N-2} \frac{\beta_0^k}{k!}|s|^{kN/(N-1)}$. Therefore, m_0 satisfies (G_2). Analogously, m_∞ also satisfies (G_2).

Based upon Theorem 3.3, there exist $\lambda_1 > 0$ and $\delta_1 > 0$ such that

$$\mathcal{F}(v) \geq \mathcal{F}_0(v) \geq \lambda_1\|v\|^p \quad \text{if } \|v\| \leq \delta_1.$$

Moreover, for the functional $\mathcal{F}_\infty$, by virtue of Theorem 3.1, we obtain that there exists $e \in W^{1,p}(\mathbb{R}^N)$ with $\|e\| > \delta_1$ such that $\mathcal{F}_\infty(e) < 0$, which implies $\mathcal{F}(e) < 0$. Thus $\Gamma \neq \emptyset$, where

$$\Gamma = \left\{\gamma \in C\big([0,1], W^{1,p}(\mathbb{R}^N)\big) : \ \gamma(0) = 0, \ \mathcal{F}(\gamma(1)) < 0\right\}.$$

The proof is complete. □

Remark 3.5. By (K_3), for any given $s_0 > 0$, there exists $C > 0$ depending on s_0 such that $K(s) \geq Cs^\mu$ for all $s \geq s_0$. Particularly, we have $\lim_{s\to+\infty} K(s)/s^p = +\infty$. Thus, there exists $\xi > 0$ such that $M_0(\xi) > 0$ and $M_\infty(\xi) > 0$, where

$$M_0(s) = \int_0^s m_0(t)dt = K(H^{-1}(s)) - \frac{V_0}{p}|H^{-1}(s)|^p,$$
$$M_\infty(s) = \int_0^s m_\infty(t)dt = K(H^{-1}(s)) - \frac{V(\infty)}{p}|H^{-1}(s)|^p.$$

Hence, m_0 and m_∞ also satisfy (G_3). Taking advantage of Theorem 3.2, the equations

$$-\Delta_p v = m_0(v) \quad \text{and} \quad -\Delta_p v = m_\infty(v), \qquad x \in \mathbb{R}^N$$

have least energy solutions in $W^{1,p}(\mathbb{R}^N)$ which are positive.

4 Proof of Theorem 1.2

Since $\mathcal{F}$ has the mountain-pass geometry, we know (see [10]) that for the constant

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{F}(\gamma(t)) > 0,$$

where

$$\Gamma = \left\{\gamma \in C\big([0,1], W^{1,p}(\mathbb{R}^N)\big) : \ \gamma(0) = 0, \ \mathcal{F}(\gamma(1)) < 0\right\},$$

there exists a Cerami sequence $\{v_n\}$ for $\mathcal{F}$ at the level c, that is,

$$\mathcal{F}(v_n) \to c \quad \text{and} \quad \|\mathcal{F}'(v_n)\|(1 + \|v_n\|) \to 0 \quad \text{as } n \to \infty. \tag{4.1}$$

Lemma 4.1. *Assume that (V_1)–(V_2) and (K_1)–(K_3) are satisfied. Let $\{v_n\} \subset W^{1,p}(\mathbb{R}^N)$ be a Cerami sequence for $\mathcal{F}$ at the level $c > 0$, then $\{v_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$.*

Proof. First, we will prove that if $\{v_n\}$ satisfies

$$\int_{\mathbb{R}^N} |\nabla v_n|^p dx + \int_{\mathbb{R}^N} V(x)|H^{-1}(v_n)|^p dx \le C \tag{4.2}$$

for some constant $C > 0$, then it is bounded in $W^{1,p}(\mathbb{R}^N)$. In fact, we only need to verify that $\int_{\mathbb{R}^N} |v_n|^p dx$ is bounded. We start splitting

$$\int_{\mathbb{R}^N} |v_n|^p dx = \int_{\{x:|v_n(x)|\le 1\}} |v_n|^p dx + \int_{\{x:|v_n(x)|>1\}} |v_n|^p dx.$$

Note that $\mu \ge \widetilde{T}(p,\alpha)p \ge \alpha p$, then it follows from Lemma 2.1 (9) and Remark 3.5 that there exists $C > 0$ such that $K(H^{-1}(s)) \ge C|s|^p$ for all $|s| > 1$. Consequently,

$$\int_{\{x:|v_n(x)|>1\}} |v_n|^p dx \le C^{-1} \int_{\{x:|v_n(x)|>1\}} K(H^{-1}(v_n))dx \le C^{-1} \int_{\mathbb{R}^N} K(H^{-1}(v_n))dx. \tag{4.3}$$

Using Lemma 2.1 (9) again, we derive that

$$\begin{aligned} \int_{\{x:|v_n(x)|\le 1\}} |v_n|^p dx &\le C^{-p} \int_{\{x:|v_n(x)|\le 1\}} |H^{-1}(v_n)|^p dx \\ &\le C^{-p} V_0^{-1} \int_{\mathbb{R}^N} V(x)|H^{-1}(v_n)|^p dx. \end{aligned} \tag{4.4}$$

Combining (4.1)–(4.4), we can achieve that $\{v_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$.

Next, we will show that (4.2) holds. By (4.1), we obtain

$$\frac{1}{p}\int_{\mathbb{R}^N} |\nabla v_n|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|H^{-1}(v_n)|^p dx - \int_{\mathbb{R}^N} K(H^{-1}(v_n))dx = c + o_n(1), \tag{4.5}$$

and for all $\psi \in W^{1,p}(\mathbb{R}^N)$,

$$\begin{aligned} \langle \mathcal{F}'(v_n), \psi\rangle = &\int_{\mathbb{R}^N} |\nabla v_n|^{p-2}\nabla v_n \nabla \psi dx + \int_{\mathbb{R}^N} V(x)\frac{|H^{-1}(v_n)|^{p-2}H^{-1}(v_n)}{h(H^{-1}(v_n))}\psi dx \\ &- \int_{\mathbb{R}^N} \frac{k(H^{-1}(v_n))}{h(H^{-1}(v_n))}\psi dx. \end{aligned} \tag{4.6}$$

Denote $\psi_n = h(H^{-1}(v_n))H^{-1}(v_n)$, taking advantage of Lemma 2.1 (6), one can find $|\psi_n| \le \widetilde{T}(p,\alpha)|v_n|$ and

$$|\nabla \psi_n| = \left[1 + \frac{\alpha^p t^p[1+(\alpha-1)t^2]}{(1+t^2)\left(2(1+t^2)^{(2-\alpha)p/2} + \alpha^p t^p\right)}|_{t=|H^{-1}(v_n)|}\right] |\nabla v_n| \le \widetilde{T}(p,\alpha)|\nabla v_n|.$$

Thus, $\|\psi_n\| \le \widetilde{T}(p,\alpha)\|v_n\|$. By choosing $\psi = \psi_n$ in (4.6), we deduce that

$$\begin{aligned} &\int_{\mathbb{R}^N} \left[1 + \frac{\alpha^p t^p[1+(\alpha-1)t^2]}{(1+t^2)\left(2(1+t^2)^{(2-\alpha)p/2} + \alpha^p t^p\right)}|_{t=|H^{-1}(v_n)|}\right] |\nabla v_n|^p dx \\ &\quad + \int_{\mathbb{R}^N} V(x)|H^{-1}(v_n)|^p dx - \int_{\mathbb{R}^N} k(H^{-1}(v_n))H^{-1}(v_n)dx \\ &= \langle \mathcal{F}'(v_n), \psi_n\rangle = o_n(1). \end{aligned} \tag{4.7}$$

Combining (4.5), (4.7) and (K_3), one has

$$\begin{aligned}&\int_{\mathbb{R}^N}\left\{\frac{1}{p}-\frac{1}{\mu}\left[1+\frac{\alpha^p t^p[1+(\alpha-1)t^2]}{(1+t^2)\left(2(1+t^2)^{(2-\alpha)p/2}+\alpha^p t^p\right)}\Big|_{t=|H^{-1}(v_n)|}\right]\right\}|\nabla v_n|^p dx\\&\quad+\left(\frac{1}{p}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}V(x)|H^{-1}(v_n)|^p dx\\&\le c+o_n(1).\end{aligned}\tag{4.8}$$

If $\mu>\widetilde{T}(p,\alpha)p$ in (K_3), by virtue of (1.11), it follows that

$$\frac{\left[\mu-\widetilde{T}(p,\alpha)p\right]}{p\mu}\int_{\mathbb{R}^N}|\nabla v_n|^p dx+\frac{T(p,\alpha)}{\mu}\int_{\mathbb{R}^N}V(x)|H^{-1}(v_n)|^p dx\le c+o_n(1),$$

which implies that (4.2) holds and hence $\{v_n\}$ is bounded. If $\mu=\widetilde{T}(p,\alpha)p=2p$, applying Remark 1.3, we derive $\alpha=2$. In this case, we can apply the estimate (4.8) to derive

$$\frac{1}{2p}\int_{\mathbb{R}^N}\frac{|\nabla v_n|^p}{1+2^{p-1}|H^{-1}(v_n)|^p}dx+\frac{1}{2p}\int_{\mathbb{R}^N}V(x)|H^{-1}(v_n)|^p dx\le c+o_n(1).\tag{4.9}$$

Set $u_n=H^{-1}(v_n)$, we get that

$$|\nabla v_n|^p=\left(1+2^{p-1}|H^{-1}(v_n)|^p\right)|\nabla u_n|^p.\tag{4.10}$$

According to (V_1) and (4.9)–(4.10), it holds that

$$\frac{1}{2p}\min\{1,V_0\}\|u_n\|^p\le\frac{1}{2p}\int_{\mathbb{R}^N}|\nabla u_n|^p dx+\frac{1}{2p}\int_{\mathbb{R}^N}V(x)|u_n|^p dx\le c+o_n(1).$$

This implies $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. The conditions (K_1)–(K_2) yield that

$$K(s)\le|s|^p+C|s|^\theta.\tag{4.11}$$

Combining the condition (b) in Theorem 1.2 with (4.11), we can apply Sobolev embedding theorem to achieve that $\int_{\mathbb{R}^N}K(H^{-1}(v_n))dx=\int_{\mathbb{R}^N}K(u_n)dx$ is bounded. Thus, utilizing (4.5), we derive (4.2), which implies $\{v_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. The proof is finished. □

4.1 Existence of nontrivial critical points for $\mathcal{F}$

According to Lemma 4.1, $\{v_n\}$ is a bounded Cerami sequence in $W^{1,p}(\mathbb{R}^N)$. Since $W^{1,p}(\mathbb{R}^N)$ is a reflexive Banach space, up to a subsequence, still denoted by $\{v_n\}$, such that $v_n\rightharpoonup v$. We assert that $\mathcal{F}'(v)=0$. In fact, since $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, we only need to verify that $\langle\mathcal{F}'(v),\psi\rangle=0$ for all $\psi\in C_0^\infty(\mathbb{R}^N)$. Note that

$$\begin{aligned}&\langle\mathcal{F}'(v_n),\psi\rangle-\langle\mathcal{F}'(v),\psi\rangle\\&=\int_{\mathbb{R}^N}(|\nabla v_n|^{p-2}\nabla v_n-|\nabla v|^{p-2}\nabla v)\nabla\psi dx\\&\quad+\int_{\mathbb{R}^N}\left[\frac{|H^{-1}(v_n)|^{p-2}H^{-1}(v_n)}{h(H^{-1}(v_n))}-\frac{|H^{-1}(v)|^{p-2}H^{-1}(v)}{h(H^{-1}(v))}\right]V(x)\psi dx\\&\quad-\int_{\mathbb{R}^N}\left[\frac{k(H^{-1}(v_n))}{h(H^{-1}(v_n))}-\frac{k(H^{-1}(v))}{h(H^{-1}(v))}\right]\psi dx.\end{aligned}$$

Remember the fact that $v_n \to v$ in $L^q_{\text{loc}}(\mathbb{R}^N)$ for $q \in [1, p^*)$ if $1 < p < N$ and $q \geq 1$ if $p = N$, by virtue of the Lebesgue dominated convergence theorem and (K_1)–(K_2), we derive that for all $\psi \in C_0^\infty(\mathbb{R}^N)$,

$$\langle \mathcal{F}'(v_n), \psi \rangle - \langle \mathcal{F}'(v), \psi \rangle \to 0 \quad \text{as } n \to \infty.$$

Since $\mathcal{F}'(v_n) \to 0$ as $n \to \infty$, the desired result is obtained immediately.

Now we will prove that $v \neq 0$. Assume on the contrary that $v = 0$. The argument will be divided into the following three steps.

Step 1. We claim that $\{v_n\}$ is also a Cerami sequence for the functional $\mathcal{F}_\infty$, which defined in Lemma 3.4, at the level c.

Indeed, since $V(x) \to V(\infty)$ as $|x| \to \infty$, $v_n \to 0$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ and Lemma 2.1 (3), one can get that

$$\begin{aligned}
\mathcal{F}_\infty(v_n) - \mathcal{F}(v_n) &= \frac{1}{p} \int_{\mathbb{R}^N} \big(V(\infty) - V(x)\big) |H^{-1}(v_n)|^p dx \\
&\leq \frac{1}{p} \int_{\mathbb{R}^N} \big(V(\infty) - V(x)\big) |v_n|^p dx \to 0,
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{F}'_\infty(v_n) - \mathcal{F}'(v_n)\| &= \sup_{\|\psi\| \leq 1} |\langle \mathcal{F}'_\infty(v_n), \psi \rangle - \langle \mathcal{F}'(v_n), \psi \rangle| \\
&\leq \sup_{\|\psi\| \leq 1} \int_{\mathbb{R}^N} |v_n|^{p-1} |V(\infty) - V(x)| |\psi| dx \\
&\leq \Big(\int_{\mathbb{R}^N} |v_n|^p |V(\infty) - V(x)|^{p/(p-1)} dx \Big)^{(p-1)/p} \to 0
\end{aligned}$$

as $n \to \infty$, which implies

$$\|\mathcal{F}'_\infty(v_n)\|(1 + \|v_n\|) \leq \|\mathcal{F}'_\infty(v_n) - \mathcal{F}'(v_n)\|(1 + \|v_n\|) + \|\mathcal{F}'(v_n)\|(1 + \|v_n\|) \to 0$$

as $n \to \infty$.

Step 2. We claim that for all $R > 0$,

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |v_n|^p dx = 0 \tag{4.12}$$

cannot occur. Assume on the contrary that (4.12) occurs, that is, $\{v_n\}$ vanish, then by the Lions compactness lemma [19], we have $v_n \to 0$ in $L^q(\mathbb{R}^N)$ for any $q \in (p, p^*)$ if $1 < p < N$ and $q > p$ if $p = N$. It follows from $(K_1) - (K_2)$ that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$0 \leq k(H^{-1}(s))H^{-1}(s) \leq \varepsilon |H^{-1}(s)|^p + C_\varepsilon |H^{-1}(s)|^\theta, \qquad \forall s \in \mathbb{R}. \tag{4.13}$$

In view of (4.13) and Lemma 2.1 (3), (8), for any $v \in W^{1,p}(\mathbb{R}^N)$, one can get

$$\int_{\mathbb{R}^N} k(H^{-1}(v))H^{-1}(v) dx \leq \varepsilon \int_{\mathbb{R}^N} |v|^p dx + C_\varepsilon \int_{\mathbb{R}^N} |v|^\theta dx, \tag{4.14}$$

$$\int_{\mathbb{R}^N} k(H^{-1}(v))H^{-1}(v) dx \leq \varepsilon \int_{\mathbb{R}^N} |v|^p dx + C_\varepsilon \int_{\mathbb{R}^N} |v|^{\theta/\alpha} dx. \tag{4.15}$$

If $\mu = \widetilde{T}(p, \alpha)p = 2p$, we use inequality (4.14) , if $\mu > \widetilde{T}(p, \alpha)p$, we use inequality (4.15), we just think about the case $\mu > \widetilde{T}(p, \alpha)p$ because the other one is similar. Since $\theta/\alpha \in (p, p^*)$

if $1 < p < N$ and $\theta/\alpha > p$ if $p = N$. Combining Lemma 2.1 (6) and (4.15) enable us to deduce that for any $\varepsilon > 0$,

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{k(H^{-1}(v_n))}{h(H^{-1}(v_n))}v_n dx &\le \lim_{n\to\infty}\int_{\mathbb{R}^N}k(H^{-1}(v_n))H^{-1}(v_n)dx\\ &\le \lim_{n\to\infty}\left(\varepsilon\int_{\mathbb{R}^N}|v_n|^p dx + C_\varepsilon\int_{\mathbb{R}^N}|v_n|^{\theta/\alpha}dx\right)\\ &\le \varepsilon\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^p dx,\end{aligned}$$

which implies

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{k(H^{-1}(v_n))}{h(H^{-1}(v_n))}v_n dx = 0, \qquad \lim_{n\to\infty}\int_{\mathbb{R}^N}k(H^{-1}(v_n))H^{-1}(v_n)dx = 0. \tag{4.16}$$

Combining the first limit in (4.16) with the fact $\langle \mathcal{F}'(v_n), v_n\rangle \to 0$ as $n\to\infty$, we get

$$\int_{\mathbb{R}^N}|\nabla v_n|^p dx + \int_{\mathbb{R}^N}V(x)\frac{|H^{-1}(v_n)|^{p-2}H^{-1}(v_n)}{h(H^{-1}(v_n))}v_n dx \to 0 \tag{4.17}$$

as $n\to\infty$. Based upon (4.17) and Lemma 2.1 (7), we derive

$$\int_{\mathbb{R}^N}|\nabla v_n|^p dx + \int_{\mathbb{R}^N}V(x)|H^{-1}(v_n)|^p \to 0 \tag{4.18}$$

as $n\to\infty$. According to the second limit in (4.16) and (K_3), we deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}K(H^{-1}(v_n))dx = 0. \tag{4.19}$$

$\lim_{n\to\infty}\mathcal{F}(v_n) = 0$ is obtained immediately from (4.18) and (4.19), we get a contradiction since $\lim_{n\to\infty}\mathcal{F}(v_n) = c > 0$. Thus, $\{v_n\}$ does not vanish and there exist $\tau, R > 0$ and $\{y_n\}\subset\mathbb{R}^N$ such that

$$\lim_{n\to\infty}\int_{B_R(y_n)}|v_n|^p dx \ge \tau > 0. \tag{4.20}$$

Step 3. Set $\widetilde{v}_n(x) = v_n(x+y_n)$. Since $\{v_n\}$ is a Cerami sequence for $\mathcal{F}_\infty$, it is easy to verify that $\{\widetilde{v}_n\}$ is also a Cerami sequence for $\mathcal{F}_\infty$. Arguing as in the case of $\{v_n\}$, up to a subsequence, still denoted by $\{\widetilde{v}_n\}$, we have $\widetilde{v}_n \rightharpoonup \widetilde{v}$ with $\mathcal{F}'_\infty(\widetilde{v}) = 0$. Since $\widetilde{v}_n \to \widetilde{v}$ in $L^p(B_R)$, by (4.20), we derive that

$$\int_{B_R}|\widetilde{v}|^p dx = \lim_{n\to\infty}\int_{B_R}|\widetilde{v}_n|^p dx = \lim_{n\to\infty}\int_{B_R(y_n)}|v_n|^p dx \ge \tau > 0,$$

which implies $\widetilde{v}\ne 0$.

Make use of Lemma 2.1 (7), we get

$$|H^{-1}(\widetilde{v}_n)|^p - \frac{|H^{-1}(\widetilde{v}_n)|^{p-2}H^{-1}(\widetilde{v}_n)}{h(H^{-1}(\widetilde{v}_n))}\widetilde{v}_n \ge 0, \ \forall n\in\mathbb{N}.$$

On the other hand, in view of Lemma 2.1 (6) and (K_3), it can be deduced that

$$\frac{k(H^{-1}(\widetilde{v}_n))}{h(H^{-1}(\widetilde{v}_n))}\widetilde{v}_n - pK(H^{-1}(\widetilde{v}_n)) \ge \frac{k(H^{-1}(\widetilde{v}_n))H^{-1}(\widetilde{v}_n)}{\widetilde{T}(p,\alpha)} - pK(H^{-1}(\widetilde{v}_n)) \ge 0, \qquad \forall n\in\mathbb{N}.$$

Note that $\widetilde{v}_n$ is a Cerami sequence for $\mathcal{F}_\infty$, by Fatou's lemma, straightforward computations generate that

$$\begin{aligned}
pc &= \liminf_{n\to\infty} \left[p\mathcal{F}_\infty(\widetilde{v}_n) - \langle \mathcal{F}'_\infty(\widetilde{v}_n), \widetilde{v}_n\rangle\right] \\
&\geq \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(\infty)\left[|H^{-1}(\widetilde{v}_n)|^p - \frac{|H^{-1}(\widetilde{v}_n)|^{p-2}H^{-1}(\widetilde{v}_n)}{h(H^{-1}(\widetilde{v}_n))}\widetilde{v}_n\right] dx \\
&\quad + \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left[\frac{k(H^{-1}(\widetilde{v}_n))}{h(H^{-1}(\widetilde{v}_n))}\widetilde{v}_n - pK(H^{-1}(\widetilde{v}_n))\right] dx \\
&\geq \int_{\mathbb{R}^N} V(\infty)\left[|H^{-1}(\widetilde{v})|^p - \frac{|H^{-1}(\widetilde{v})|^{p-2}H^{-1}(\widetilde{v})}{h(H^{-1}(\widetilde{v}))}\widetilde{v}\right] dx \\
&\quad + \int_{\mathbb{R}^N} \left[\frac{k(H^{-1}(\widetilde{v}))}{h(H^{-1}(\widetilde{v}))}\widetilde{v} - pK(H^{-1}(\widetilde{v}))\right] dx \\
&= p\mathcal{F}_\infty(\widetilde{v}) - \langle \mathcal{F}'_\infty(\widetilde{v}), \widetilde{v}\rangle \\
&= p\mathcal{F}_\infty(\widetilde{v}).
\end{aligned}$$

Thus, $\widetilde{v} \neq 0$ is a critical point of $\mathcal{F}_\infty$ satisfying $\mathcal{F}_\infty(\widetilde{v}) \leq c$.

In view of *Step 3*, we derive that the least energy level a_∞ for $\mathcal{F}_\infty$ satisfies $a_\infty \leq c$. Denoting $\widehat{\omega}$ as a least energy solution of the equation $-\Delta_p v = m_\infty(v)$ (see Remark 3.5). Applying Theorem 3.1 to the functional $\mathcal{F}_\infty$, there exists a path $\gamma \in C\big([0,1], W^{1,p}(\mathbb{R}^N)\big)$ such that $\gamma(0) = 0$, $\mathcal{F}_\infty(\gamma(1)) < 0$, $\widehat{\omega} \in \gamma([0,1])$ and

$$\max_{t\in[0,1]} \mathcal{F}_\infty(\gamma(t)) = \mathcal{F}_\infty(\widehat{\omega}).$$

If $V(x) \equiv V(\infty)$, we prove the desired conclusion. So we assume that $V(x) \not\equiv V(\infty)$, we have

$$\mathcal{F}(\gamma(t)) < \mathcal{F}_\infty(\gamma(t)), \qquad \forall t \in (0,1]$$

and hence

$$c \leq \max_{t\in[0,1]} \mathcal{F}(\gamma(t)) < \max_{t\in[0,1]} \mathcal{F}_\infty(\gamma(t)) = \mathcal{F}_\infty(\widehat{\omega}) = a_\infty \leq c.$$

We get a contradiction. Therefore, v is a nontrivial critical point of $\mathcal{F}$.

4.2 L^∞-estimate and decay to zero at infinity

Let $v \in W^{1,p}(\mathbb{R}^N)$ be a nontrivial weak solution of (1.10), then for all $\omega \in W^{1,p}(\mathbb{R}^N)$, it holds that

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla \omega dx + \int_{\mathbb{R}^N} V(x)\frac{|H^{-1}(v)|^{p-2}H^{-1}(v)}{h(H^{-1}(v))}\omega dx = \int_{\mathbb{R}^N} \frac{k(H^{-1}(v))}{h(H^{-1}(v))}\omega dx. \tag{4.21}$$

Assume that $1 < p < N$. Without loss of generality, we suppose that $v \geq 0$. Otherwise, we work with the positive and negative parts of v. For each $m \geq 1$, define

$$v_m = \begin{cases} v, & \text{if } 0 \leq v \leq m, \\ m, & \text{if } v \geq m, \end{cases}$$

$$\zeta_m = v_m^{p(r-1)} v, \qquad \phi_m = v v_m^{r-1}$$

with $r > 1$ which will be given later. Choosing ζ_m as a test function in (4.21). Note that

$$k(H^{-1}(v)) \leq \frac{V_0}{2}(H^{-1}(v))^{p-1} + C(H^{-1}(v))^{\theta-1}$$

and (V_1), we can deduce

$$\begin{aligned}\int_{\mathbb{R}^N} v_m^{p(r-1)}|\nabla v|^p dx + p(r-1)\int_{\mathbb{R}^N} v_m^{p(r-1)-1} v|\nabla v|^{p-2}\nabla v_m \nabla v dx \\ \leq C\int_{\mathbb{R}^N} \frac{(H^{-1}(v))^{\theta-1}}{h(H^{-1}(v))} v v_m^{p(r-1)} dx.\end{aligned}$$

Noticing $\nabla v_m \nabla v \geq 0$ in $\mathbb{R}^N$, using Lemma 2.1 (6) and (8), one has

$$\int_{\mathbb{R}^N} v_m^{p(r-1)}|\nabla v|^p dx \leq C\int_{\mathbb{R}^N} v^{\theta/\alpha} v_m^{p(r-1)} dx = C\int_{\mathbb{R}^N} v^{\hat{\theta}-p}\phi_m^p dx, \tag{4.22}$$

where $\hat{\theta} = \theta/\alpha$. It follows from the Gagliardo–Nirenberg inequality [11] and (4.22) that

$$\begin{aligned}\Big(\int_{\mathbb{R}^N} \phi_m^{p^*} dx\Big)^{p/p^*} &\leq C_1\int_{\mathbb{R}^N} |\nabla\phi_m|^p dx \\ &\leq C_1 2^{p-1}\Big(\int_{\mathbb{R}^N} v_m^{p(r-1)}|\nabla v|^p dx + (r-1)^p\int_{\mathbb{R}^N} v^p v_m^{p(r-2)}|\nabla v_m|^p dx\Big) \\ &\leq C_1 2^{p-1} r^p\int_{\mathbb{R}^N} v_m^{p(r-1)}|\nabla v|^p dx \\ &\leq C_2 r^p\int_{\mathbb{R}^N} v^{\hat{\theta}-p}\phi_m^p dx.\end{aligned}$$

According to the Hölder inequality, one sees that

$$\Big(\int_{\mathbb{R}^N} \phi_m^{p^*} dx\Big)^{p/p^*} \leq C_2 r^p\Big(\int_{\mathbb{R}^N} v^{p^*} dx\Big)^{(\hat{\theta}-p)/p^*}\Big(\int_{\mathbb{R}^N} \phi_m^{pp^*/(p^*-\hat{\theta}+p)} dx\Big)^{(p^*-\hat{\theta}+p)/p^*}.$$

As $0 \leq \phi_m \leq v^r$, the continuity of the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ leads to

$$\Big(\int_{\mathbb{R}^N} (v v_m^{r-1})^{p^*} dx\Big)^{p/p^*} \leq C_3 r^p\|v\|^{\hat{\theta}-p}\Big(\int_{\mathbb{R}^N} v^{rpp^*/(p^*-\hat{\theta}+p)} dx\Big)^{(p^*-\hat{\theta}+p)/p^*},$$

that is,

$$\Big(\int_{\mathbb{R}^N} (v v_m^{r-1})^{p^*} dx\Big)^{p/p^*} \leq C_3 r^p\|v\|^{\hat{\theta}-p}\|v\|_{r\lambda^*}^{pr} \tag{4.23}$$

with $\lambda^* = pp^*/(p^*-\hat{\theta}+p)$ and $r = p^*/\lambda^* = 1 + (p^*-\hat{\theta})/p > 1$. By virtue of Fatou's lemma, we conclude from (4.23) that

$$\|v\|_{rp^*} \leq \big(C_3 r^p\|v\|^{\hat{\theta}-p}\big)^{1/pr}\|v\|_{r\lambda^*}$$

or

$$\|v\|_{rp^*} \leq A^{1/r} r^{1/r}\|v\|_{r\lambda^*} \tag{4.24}$$

with $A > 0$ and $A^p = C_3\|v\|^{\hat{\theta}-p}$.

We now use the classical Moser's iteration scheme to prove $v \in L^\infty(\mathbb{R}^N)$. For each $k = 0, 1, 2, \ldots$, we define $r_{k+1}\lambda^* := p^* r_k$ with $r_0 = r$. Clearly, we have $r_k = r^{k+1} \uparrow +\infty$ as $k \to \infty$. Employing the previous argument for r_1, we get from (4.24) that

$$\begin{aligned}\|v\|_{r_1 p^*} &\le A^{1/r_1} r_1^{1/r_1} \|v\|_{r_1 \lambda^*} \\ &= A^{1/r_1} r_1^{1/r_1} \|v\|_{r p^*} \\ &\le A^{1/r + 1/r_1} r^{1/r} r_1^{1/r_1} \|v\|_{p^*}.\end{aligned}$$

By iteration scheme, we have

$$\|v\|_{r_k p^*} \le A^{S_k} e^{T_k} \|v\|_{p^*} \tag{4.25}$$

with $S_k = \sum_{i=0}^k \frac{1}{r_i} = \sum_{i=0}^k \frac{1}{r^{i+1}}$ and $T_k = \sum_{i=0}^k \frac{\ln r_i}{r_i} = \sum_{i=0}^k \frac{(i+1)\ln r}{r^{i+1}}$. Recall $r = p^*/\lambda^* > 1$, we get

$$\lim_{k\to\infty} S_k = p/(p^* - \hat{\theta}), \qquad \lim_{k\to\infty} T_k = r\ln r/(r-1)^2.$$

Letting $k \to \infty$ in (4.25) and by the Sobolev embedding theorem, we can deduce that $v \in L^\infty(\mathbb{R}^N)$ and

$$\begin{aligned}\|v\|_\infty &\le A^{p/(p^*-\hat{\theta})} r^{r/(r-1)^2} \|v\|_{p^*} \\ &\le C_3^{1/(p^*-\hat{\theta})} \|v\|^{(\hat{\theta}-p)/(p^*-\hat{\theta})} r^{r/(r-1)^2} \|v\|_{p^*} \\ &\le C_4 \|v\|^{(p^*-p)/(p^*-\hat{\theta})}.\end{aligned}$$

In the case $p = N$, [30, Theorem 1] enables us to derive that v is locally bounded in $\mathbb{R}^N$. By a result in [33], we conclude that $v \in C^{1,\beta}_{\mathrm{loc}}(\mathbb{R}^N)(0 < \beta < 1)$ for $1 < p \le N$.

Next, when $1 < p < N$, we will show that $v(x) \to 0$ as $|x| \to \infty$. Since $v \in L^\infty(\mathbb{R}^N)$, it follows from (V_1), (K_2), Lemma 2.1 (8) and (4.21) that

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \nabla \psi dx \le C \int_{\mathbb{R}^N} (1 + |v|^{p-1}) \psi dx$$

for all $\psi \in C_0^\infty(\mathbb{R}^N), \psi \ge 0$. Applying [34, Theorem 1.3], one sees that for any $x \in \mathbb{R}^N$,

$$\sup_{y \in B_1(x)} v(y) \le C \|v\|_{L^p(B_2(x))}.$$

In particular, $v(x) \le C\|v\|_{L^p(B_2(x))}$. Since

$$\|v\|_{L^p(B_2(x))} \to 0 \quad \text{as } |x| \to \infty,$$

one has $v(x) \to 0$ as $|x| \to \infty$.

We conclude that $u = H^{-1}(v)$ is a nontrivial weak solution of (1.1) in $C^{1,\beta}_{\mathrm{loc}}(\mathbb{R}^N)(0 < \beta < 1)$ by Proposition 2.2. Since $|u| = |H^{-1}(v)| \le |v|$, we get that $u(x) \to 0$ as $|x| \to \infty$, which finalizes the proof of Theorem 1.2.

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p-biharmonic equation with Hardy–Sobolev exponent and without the Ambrosetti–Rabinowitz condition

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Abstract. This paper is concerned with the existence and multiplicity to p-biharmonic equation with Sobolev–Hardy term under Dirichlet boundary conditions and Navier boundary conditions, respectively. We focus on the case of the nonlinear terms without the Ambrosetti–Rabinowitz conditions. Our method is based on the variational method.

Keywords: variational methods, p-biharmonic equation, Sobolev–Hardy inequality, Fountain Theorem.

2020 Mathematics Subject Classification: 35J35, 35J62, 35J75, 35D30.

1 Introduction

We consider the following p-biharmomic equations with clamped Dirichlet boundary conditions

$$\begin{cases} \triangle_p^2 u = \frac{\mu|u|^{r-2}u}{|x|^s} + f(x,u) & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{PD}$$

and p-biharmomic equations with hinged Navier boundary conditions

$$\begin{cases} \triangle_p^2 u = \frac{\mu|u|^{r-2}u}{|x|^s} + f(x,u) & \text{in } \Omega, \\ u = \triangle u = 0 & \text{on } \partial\Omega \end{cases} \tag{PNa}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, $0 \in \Omega$, $2 < 2p < N$, $p \leq r < p^*(s) = \frac{(N-s)p}{N-2p} \leq p^*(0) := p^*$, $\mu \geq 0$.

Since Lazer and McKenna [11] provided a model for discussing the traveling waves in suspension bridges, existence and multiple of solutions for nonlinear biharmonic equations and p-biharmonic equations have been studied under the framework of nonlinear functional analysis. Bhakta [4] studied existence, multiplicity and qualitative properties of entire solutions

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of the p-biharmonic equations with Hardy term. Huang and Liu [16] obtained sign-changing solutions for p-biharmonic equations with Hardy potential. Bueno et al. [5] get multiplicity of solutions for p-biharmonic problems with with concave-convex nonlinearities. Wang and Zhao [25] studied the existence and multiplicity of solutions of p-biharmonic type equations with critical growth. On this topic, we also refer to [3,6,22,26] and references therein.

Ghoussoub and Yuan [14] obtained multiple solutions for $-\triangle_p u = \mu\frac{|u|^{r-2}u}{|x|^s} + \lambda|u|^{q-2}u$ with homogeneous Dirichlet boundary conditions in $W_0^{1,p}(\Omega)$. Perera and Zou [23] studied the multiplicity, and bifurcation results for p-Laplacian problems involving critical Hardy–Sobolev exponents in $W_0^{1,p}(\Omega)$. One of the starting points of this paper is to generalize the part results in [14,23] to the fourth-order elliptic equation.

Definition 1.1. The function u in $W_0^{2,p}(\Omega)$ is called a weak solution of Problem (PD), if

$$\int_\Omega \left[|\triangle u|^{p-2}\triangle u\triangle\phi - \frac{\mu|u|^{r-2}u\phi}{|x|^s} - f(x,u)\phi\right]dx = 0 \quad \text{for any } \phi \in W_0^{2,p}(\Omega);$$

u in $W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega)$ is said to be a weak solution of Problem (PNa), in case

$$\int_\Omega \left[|\triangle u|^{p-2}\triangle u\triangle\phi - \frac{\mu|u|^{r-2}u\phi}{|x|^s} - f(x,u)\phi\right]dx = 0, \qquad \forall\phi \in W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega).$$

Since Problem (PNa) is handled similarly to Problem (PD), we discuss the problem (PD) and only give a simple explanation for Problem (PNa).

The starting point for the variational methods of the questions (PD) and (PNa) is the following Sobolev–Hardy inequality (we refer to Lemma 2.2 in Section 2). Let $2 < 2p < N$, $r \le p^*(s)$, then

$$\left(\int_\Omega \frac{|u|^r}{|x|^s}dx\right)^{\frac{1}{r}} \lesssim \left(\int_\Omega |\triangle u|^p dx\right)^{\frac{1}{p}}, \qquad \forall u \in C_0^\infty(\Omega\setminus\{0\})$$

Therefore, we may define

$$\mu_{s,r}(\Omega) = \inf_{\substack{u\in W_0^{2,p}(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\triangle u|^p dx}{\left(\int_\Omega \frac{|u|^r}{|x|^s}dx\right)^{\frac{p}{r}}}$$

and

$$\widetilde{\mu}_{s,r}(\Omega) = \inf_{\substack{u\in W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\triangle u|^p dx}{\left(\int_\Omega \frac{|u|^r}{|x|^s}dx\right)^{\frac{p}{r}}}.$$

We replace $|u|^{q-2}u$ in [14,23] by a more general nonlinear perturbation $f(x,t)$, and we impose naturally some structural conditions on the nonlinear term $f(x,t)$, so that the associated Euler-Lagrange functional is expected to have some mountain pass geometry and compactness results. Specifically, we consider the following assumptions:

f_1) $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ is continuous and $f(x,0) = 0$ for all $x \in \overline{\Omega}$;

f_2) $\lim_{|t|\to+\infty} \frac{F(x,t)}{t^p} = +\infty$ uniformly on $x \in \overline{\Omega}$, where $F(x,t) = \int_0^t f(x,\tau)\mathrm{d}\tau$;

f_3) $\limsup_{|t|\to 0} \frac{pF(x,t)}{|t|^p} < \lambda_1$(or $\tilde{\lambda}_1$) uniformly on $x \in \overline{\Omega}$, where $\lambda_1 > 0$ is the first eigenvalue of the operator $\triangle_p^2$ in Ω with homogeneous Dirichlet boundary conditions (or homogeneous Navier boundary conditions);

(SCPI) $f(x,t)$ has subcritical polynomial growth, i.e.

$$\lim_{|t|\to+\infty|} \frac{f(x,t)}{|t|^{p^*-1}} = 0.$$

The critical point theory is based on the existence of some linking structure and deformation lemmas. To obtain such deformation results, some compactness condition of the functional is necessary. In order to get compactness, the standard approach is to apply the Ambrosetti–Rabinowitz conditions ((A–R) for short) to $f(x,t)$ and $F(x,t)$ due to Ambrosetti–Rabinowitz [1]:

(A–R) $\exists\ R_0 > 0, \theta > p$ such that $0 < \theta F(x,s) \le sf(x,s)$ for any $(|s|, x)$ in $[R_0, +\infty) \times \Omega$.

The main role of (A–R) condition is to ensure the boundedness of Palais–Smale or Cerami sequence of Euler–Lagrange functional associated to Eq. (PD) and (PNa). But (A–R) condition is a relatively restrictive eliminating many nonlinearities, for example, $f(x,t) = t \log t^2$. The absence of (A–R) condition in the second order elliptic equation goes back to Costa, Magalhães [7], Miyagaki, Souto [24], Li, Yang [19] and Liu [20], and was improved by Mugnai and Papageorgiou [21]. On this topic, we also refer to [2,8,13,17] and references therein. Inspired by [19,21], we assume the following conditions (without the (A–R) condition):

f_4) There exist $C_* \ge 0, \theta \ge 1$ such that

$$H(x,t) \le \theta H(x,\tau) + C_* \qquad \forall t,\tau \in \mathbb{R},\ 0 < |t| < |\tau|,\ \forall x \in \Omega,$$

where $H(x,t) = \frac{1}{p} t f(x,t) - F(x,t)$.

Theorem 1.2. *Assume that $f(x,t)$ satisfies (f_1)–(f_3) and* (SCPI) *condition. Then*

- *Problem* (PD) *admits at least a nontrivial weak solution $u \in W_0^{2,p}(\Omega)$;*
- *Problem* (PNa) *admits at least a nontrivial weak solution $u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$.*

For convenience, we first define the Euler–Lagrange functional I_μ as follows:

$$I_\mu(u) = \frac{1}{p}\int_\Omega |\triangle u|^p dx - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s} dx - \int_\Omega F(x,u)dx,$$

where $F(x,t) = \int_0^t f(x,\tau)d\tau$.

Additionally if we assume that $f(x,t)$ is an odd function in t, then we can prove the existence of infinitely many weak solutions to Problem (PD) and (PNa). Specifically, we can get the following results:

Theorem 1.3. *Suppose that (f_1)–(f_3) hold and*

f_5) there exist $a, b > 0$ and $q \in (p, p^*)$ such that

(SCP) $$|f(x,t)| \le a + b|t|^{q-1} \quad \text{for any } (x,t) \in \Omega \times \mathbb{R};$$

f_6) $f(x,-t) = -f(x,t)$, $\forall (x,t) \in \Omega \times \mathbb{R}$,

in addition, if $p = r$, then

- *Problem (PD) possesses a sequence of solutions* $\{u_n\} \in W_0^{2,p}(\Omega)$ *such that* $I_\mu(u_n) \to +\infty$ *provided* $0 \le \mu < \mu_{s,r}(\Omega)$*;*

- *Problem (PNa) contains a sequence of solutions* $\{u_n\} \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ *such that* $I_\mu(u_n) \to +\infty$ *in case* $0 \le \mu < \widetilde{\mu}_{s,r}(\Omega)$*.*

This paper is organized as follows: Section 2 is devoted to review some necessary mathematical knowledge about function spaces, embedding and associated functional settings. In Section 3, we gets the existence of solution to Eq. (PD) and (PNa) under $g(x,t)$ with A–R condition. In Section 4, we obtain the multiplicity of Eq. (PD) and (PNa). Section A is an appendix.

2 Functional framework

In this paper, $W_0^{2,p}(\Omega)$ and $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ are equipped with norm

$$\|u\| = \left(\int_\Omega |\triangle u|^p dx\right)^{\frac{1}{p}},$$

then $W_0^{2,p}(\Omega)$ and $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ are all Banach space.

Davies [9] extends the Rellich inequality to L^p spaces. But we only need one special case here.

Lemma 2.1 ([9, Corollary 14])**.** *For any* $p \in (1, \frac{N}{2})$ *and* $u \in C_0^\infty(\Omega \setminus \{0\})$*, the following inequality*

$$\int_\Omega |\triangle u|^p dx \ge \left(\frac{(p-1)N(N-2p)}{p^2}\right)^p \int_\Omega \frac{|u|^p}{|x|^{2p}} dx$$

is established.

Next, we will prove the corresponding Sobolev–Hardy inequality in the space $W^{2,p}(\Omega)$. Our method is derived from the proof method of Lemma 2.1 in [28] and Lemma 3.2 in [14].

Lemma 2.2 (Sobolev–Hardy inequality)**.** *Suppose that* $2 < 2p < N$*, then*

(1) If $0 < r < p^*(s)$*, there exists a constant* $C > 0$ *such that*

$$\left(\int_\Omega \frac{|u|^r}{|x|^s} dx\right)^{\frac{1}{r}} \le C \left(\int_\Omega |\triangle u|^p dx\right)^{\frac{1}{p}} \tag{2.1}$$

for any $u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$*.*

(2) If $p \le r < p^*(s)$*, then the map* $u \to \frac{u}{|x|^{\frac{s}{r}}}$ *is compact from* $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ *to* $L^r(\Omega)$*.*

Proof. (1) When $s = 0$ or $s = 2p$, (2.1) is Sobolev's inequality or Rellich's inequality, respectively. Since $p^*(s) \ge p$, we only need to consider the scenario of $0 < s < 2p$. According to

Rellich's inequality, Sobolev's inequality and Hölder's inequality, we can get

$$\begin{aligned}\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}dx &= \int_\Omega \frac{|u|^{\frac{s}{2}}}{|x|^s}|u|^{p^*(s)-\frac{s}{2}}dx \\ &\le \left(\int_\Omega \frac{|u|^p}{|x|^{2p}}dx\right)^{\frac{s}{2p}} \left(\int_\Omega |u|^{p^*}dx\right)^{\frac{2p-s}{2p}} \\ &\le \left(\frac{p^2}{(p-1)N(N-2p)}\right)^{\frac{s}{2}} \left(\int_\Omega |\triangle u|^p dx\right)^{\frac{s}{2p}} S_2 \left(\int_\Omega |\triangle u|^p dx\right)^{\frac{2p-s}{2p}\cdot\frac{p^*}{p}} \\ &= C_1 \left(\int_\Omega |\triangle u|^p dx\right)^{\frac{N-s}{N-2p}},\end{aligned}$$

where

$$C_1 = \left(\frac{p^2}{(p-1)N(N-2p)}\right)^{\frac{s}{2}} S_2, S_2 = \inf_{\substack{u\in W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\triangle u|^p dx}{(\int_\Omega |u|^{p^*}dx)^{\frac{p}{p^*}}}$$

is the corresponding optimal Sobolev constant.

(2) Let $\{u_n\}$ be a bounded sequence in $W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega)$, then there is a convergent subsequence of $\{u_n\}$ (still represented by $\{u_n\}$) such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega), \\ u_n &\to u \quad \text{strongly in } L^r(\Omega),\ \ p\le r< p^*(s).\end{aligned}$$

On the other hand,

$$\int_\Omega \frac{|u_n-u|^r}{|x|^s}dx \le C\int_{B_\delta(0)} \frac{|u_n-u|^r}{|x|^s}dx + C\|u_n-u\|^r_{L^r(\Omega)}, \quad \text{where } B_\delta(0)=B(0,\delta).$$

In the light of Hölder's inequality, we have

$$\begin{aligned}\int_\Omega \frac{|u_n-u|^r}{|x|^s}dx &\le C\left(\int_\Omega |u_n-u|^{p^*}dx\right)^{\frac{r}{p^*}} \left(\int_{B_\delta(0)} |x|^{-\frac{p^*s}{p^*-r}}dx\right)^{1-\frac{r}{p^*}} + C\|u_n-u\|^r_{L^r(\Omega)} \\ &\le C\left(\delta^{-\frac{p^*s}{p^*-r}+N}\right)^{1-\frac{r}{p^*}} + C\|u_n-u\|^r_{L^r(\Omega)}.\end{aligned}$$

Considering $p\le r<p^*(s)$ and $N-\frac{p^*s}{p-r}>0$ and let $\delta\to 0, n\to\infty$, we can get immediately inequalities

$$\int_\Omega \frac{|u_n-u|^r}{|x|^s}dx \to 0. \qquad \square$$

In order to study Eq. (PD) and (PNa), we need to discuss some properties of operator $\triangle_p^2$ on $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$.

Proposition 2.3. *For any bounded Ω in $\mathbb{R}^N$ and any p in $(1,+\infty)$, $\triangle_p^2$ satisfies the following properties:*

1) ([10]) $\triangle_p^2 : W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega) \to (W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))^*$ is a hemicontinuous operator;

2) $\triangle_p^2$ is a bounded continuous and uniformly convex coercive operator;

3) $\triangle_p^2 : W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega) \to (W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))^*$ is homeomorphic.

Proof. 2) Obviously, $\triangle_p^2$ is bounded continuously coercive. And the strict monotonicity of $\triangle_p^2$ can be derived from the following inequality [15, Lemma 5.1 and Lemma 5.2]:

Let $x,\ y \in \mathbb{R}^N$ and $\langle\cdot,\cdot\rangle$ is the usual inner product in $\mathbb{R}^N$, then

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \geq \begin{cases} C_p|x-y|^p & \text{if } p \geq 2, \\ C_p\frac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } 1 < p < 2. \end{cases} \tag{2.2}$$

3) Applying the Browder–Minty theorem, 1) and 2), we known that $\triangle_p^2$ is surjection. Similar to [12, Lemma 3.1 (iii)], it is not difficult to prove $\triangle_p^2$ is a homeomorphism. □

Remark 2.4. If $\triangle_p^2$ is an operator from $W_0^{2,p}(\Omega)$ to $(W_0^{2,p}(\Omega))^*$, Proposition 2.3 is also valid [18, Proposition 2.1].

Since $f(x,t)$ satisfies the condition (SCPI), $I_\mu(u)$ is well-posed on $W^{2,p}(\Omega)$ and is C^1, the weak solution to the problem (PD) is the critical point of $I_\mu(u)$ in $W_0^{2,p}(\Omega)$. Because the boundary condition $\triangle u|_{\partial\Omega} \equiv 0$ in Problem (PNa) is not included in natural space $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$, so Problem (PNa) must be considered in another way. Specifically, we need the regularity of the critical point to $I_\mu(u)$ in space $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ to ensure this boundary condition.

Proposition 2.5 ([26], Proposition 4.7)**.** *Suppose that $f(x,t)$ satisfies the condition* (SCPI) *and $|\mu| \leq \widetilde{\mu}_{s,r}(\Omega)$, every critical point u of I_μ satisfies $\triangle u|_{\partial\Omega} \equiv 0$ in the sense of the trace in $W_0^{1,p}(\Omega)$ $\cap W^{2,p}(\Omega)$.*

3 Proof of Theorem 1.2

In order to use Theorem A.2 to study Eq. (PD) and (PNa), we need to verify that the functionals I_μ satisfies the mountain pass geometry structure and compactness conditions.

Lemma 3.1. *Let f satisfies conditions (f_1)–(f_3) and* (SCPI)*. Then the functional I_μ satisfies mountain pass geometry:*

1. *$I_\mu(0) = 0$.*
2. *There exist positive constants ρ and η such that $I_\mu(u)|_{\partial B_\rho} \geq \eta$.*
3. *There exists e with $\|e\| > \rho$ such that $I_\mu(e) < 0$.*

Proof. 1. $I_\mu(0) = 0$ is straightforward by the condition (f_1). For 2, it follows from (f_3) and (SCPI) that there exist C_2, λ such that

$$F(x,t) \leq \frac{1}{p}(\lambda_1 - \lambda)|t|^p + C_2|t|^{p^*} \quad \text{for any } (x,t) \text{ in } \Omega\times\mathbb{R}.$$

Considering the Sobolev embedding theorem and Lemma 2.2, we obtain

$$\begin{aligned} I_\mu(u) &= \frac{1}{p}\int_\Omega |\triangle u|^p dx - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s}dx - \int_\Omega F(x,u)dx \\ &\geq \frac{1}{p}\left(1 - \frac{\lambda_1-\lambda}{\lambda_1}\right)\|u\|^p - \frac{\mu}{r}C^r\|u\|^r - \mathfrak{C}_2\|u\|^{p^*} \\ &\geq \frac{\lambda}{p\lambda_1}\|u\|^p - \frac{\mu}{r}(\mu_{s,r}(\Omega))^{-\frac{r}{p}}\|u\|^r - \mathfrak{C}_2\|u\|^{p^*}, \end{aligned}$$

where C is the constant in Lemma 2.2.

Thanks to $\lambda > 0$, $p \le r$ and $p < p^*$, we may take an enough small positive ρ and a positive constant η such that $I_\mu(u)|_{\partial B_\rho} \ge \eta$.

Next, we give the proof of 3. According to the condition (f_2), for all $M > 0$, there is $\delta > 0$ such that $F(x,t) > M|t|^p$ for all (x,t) in $\bar{\Omega} \times [-\delta,\delta]^c$.

On the other hand, considering the continuity of F, we may get

$$m := \min_{(x,t)\in\bar{\Omega}\times[-\delta,\delta]} F(x,t) \le F(x,0) = 0.$$

Therefore, we take $M > \frac{\|u\|^p}{p\|u\|^p_{L^p}} > 0$ especially, then there is an $A > 0$ such that

$$F(x,t) \ge M|t|^p - A \quad \text{for any } (x,t) \text{ in } \bar{\Omega}\times\mathbb{R}. \tag{3.1}$$

Hence,

$$\begin{aligned} I_\mu(tu) &= \frac{1}{p}\int_\Omega |\triangle tu|^p dx - \frac{\mu}{r}\int_\Omega \frac{|tu|^r}{|x|^s}dx - \int_\Omega F(x,tu)dx \\ &\le \frac{1}{p}|t|^p\int_\Omega |\triangle u|^p dx - \frac{\mu}{r}|t|^r\int_\Omega \frac{|u|^r}{|x|^s}dx - \int_\Omega (M|t|^p|u|^p - A)dx \\ &= |t|^p(\frac{1}{p}\|u\|^p - M\|u\|^p_{L^p}) - \frac{\mu}{r}|t|^r\int_\Omega \frac{|u|^r}{|x|^s}dx + A|\Omega|. \end{aligned}$$

Thence $\lim_{t\to+\infty} I_\mu(tu) = -\infty$. □

Lemma 3.2. *Assume that f satisfies (f_1)–(f_4) and* (SCPI), *then the energy functional I_μ satisfies the Cerami condition for all c in $\mathbb{R}$.*

Proof. Let $\{u_n\}_n^\infty$ be in $W_0^{2,p}(\Omega)$ such that

$$I_\mu(u_n) \to c$$

and

$$\left(1 + \|u_n\|_{W_0^{2,p}}\right)\left\|I'(u_n)\right\|_{(W_0^{2,p})^*} \to 0,$$

that is to say,

$$\frac{1}{p}\int_\Omega |\triangle u_n|^p dx - \frac{\mu}{r}\int_\Omega \frac{|u_n|^r}{|x|^s}dx - \int_\Omega F(x,u_n)dx \to c, \tag{3.2}$$

and

$$\left(1 + \|u_n\|_{W_0^{2,p}}\right)\sup_{\|\varphi\|=1}\left|\langle I'(u_n),\varphi\rangle\right| \to 0. \tag{3.3}$$

Step 1. *The sequence $\{u_n\}$ is bounded in $W_0^{2,p}(\Omega)$.*

For if not, i.e. $\|u_n\| \to +\infty$ as $n \to +\infty$. Let $v_n =: \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$ (Bounded). Hence, up to a subsequence, $v_n \rightharpoonup v$ in $W_0^{2,p}(\Omega)$. Therefore,

$$\begin{aligned} v_n \to v &\quad \text{in } L^q(\Omega),\ q < p^*, \\ v_n(x) \to v(x) &\quad a.e. \text{ in } \Omega, \\ \frac{v_n}{|x|^{\frac{s}{r}}} \to \frac{v}{|x|^{\frac{s}{r}}} &\quad \text{in } L^r(\Omega),\ r < p^*(s). \end{aligned}$$

We discuss v in two cases.

Case (i): If $v \neq 0$, then let $\Omega_{\neq} := \{x \in \Omega : v(x) \neq 0\}$.

$$|u_n(x)| = |v_n(x)|\|u_n\| \to +\infty \quad a.e. \text{ in } \Omega_{\neq}.$$

Since $I_\mu(u_n) \to c$, we get $\frac{I_\mu(u_n)}{\|u_n\|} \to 0$, i.e.

$$o(1) = \frac{1}{p} - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s\|u_n\|^p}dx - \int_{\Omega_{\neq}} \frac{F(x,u_n)}{\|u_n\|^p}dx - \int_{\Omega\setminus\Omega_{\neq}} \frac{F(x,u_n)}{\|u_n\|^p}dx. \tag{3.4}$$

In accordance to (f_2), we have

$$\frac{F(x,u_n)}{\|u_n\|^p} = \frac{F(x,u_n)}{|u_n|^p}\cdot\frac{|u_n|^p}{\|u_n\|^p} = \frac{F(x,u_n)}{|u_n|^p}|v_n|^p \to +\infty \quad a.e. \text{ in } \Omega_{\neq} \text{ as } n \to +\infty,$$

which implies $\int_{\Omega_{\neq}} \frac{F(x,u_n)}{\|u_n\|^p}dx \to +\infty$.

We claim that

$$\int_{\Omega\setminus\Omega_{\neq}} \frac{F(x,u_n)}{\|u_n\|^p}\mathrm{d}x \geqslant -\frac{K}{\|u_n\|^p}|\Omega\setminus\Omega_{\neq}| \tag{3.5}$$

for some positive constant K.

In fact, from the condition (f_2), we get $\lim_{|t|\to+\infty} F(x,t) = +\infty$ uniformly in $x \in \bar{\Omega}$, which implies

$$F(x,t) \geq -K \text{ for any } (x,t) \quad \text{in } \bar{\Omega}\times\mathbb{R}. \tag{3.6}$$

(The proof for (3.6) is similar to the process of deriving the inequality (3.1) by the condition (f_2). These details are omitted and left to the reader.)

From the inequality (3.6), we may obtain the inequality (3.5).

Since $\|u_n\| \to +\infty$, combining (3.5) and (3.6), we get

$$\begin{aligned} \frac{I_\mu(u_n)}{\|u\|^p} &= \frac{1}{p} - \frac{\mu}{r\|u\|^p}\int_\Omega \frac{|u|^r}{|x|^s}dx - \int_{\Omega_{\neq}} \frac{F(x,u_n)}{\|u_n\|^p}dx - \int_{\Omega\setminus\Omega_{\neq}} \frac{F(x,u_n)}{\|u_n\|^p}\mathrm{d}x \\ &\leq \frac{1}{p} - \int_{\Omega_{\neq}} \frac{F(x,u_n)}{\|u_n\|^p}dx - \int_{\Omega\setminus\Omega_{\neq}} \frac{F(x,u_n)}{\|u_n\|^p}\mathrm{d}x \\ &\to -\infty, \end{aligned}$$

which contradicts inequality (3.4).

Case (ii): When $v \equiv 0$. Because $t \mapsto I_\mu(tu_n)$ is continuous in $[0,1]$, thence for all n in $\mathbb{N}$ there exists t_n in $[0,1]$ such that

$$I_\mu(t_nu_n) = \max_{t\in[0,1]} I_\mu(tu_n). \tag{3.7}$$

According to the condition (SCPI), for any $R > 0$, there exists $C_3 > 0$ such that

$$F(x,t) \leq C_3|t| + \frac{|t|^{p^*}}{R^{p^*}} \text{ for all } (x,t) \quad \text{in } \Omega \times \mathbb{R}.$$

Owing to $\frac{R}{\|u_n\|}$ in $[0,1]$ for n large enough, we get

$$I_\mu(t_n u_n) = \max_{t\in[0,1]} I_\mu(tu_n) \geq I_\mu\left(R\frac{u_n}{\|u_n\|}\right) = I_\mu(Rv_n)$$

and

$$\begin{aligned} I_\mu(Rv_n) &= \frac{1}{p}\int_\Omega |\triangle Rv_n|^p dx - \frac{\mu}{r}\int_\Omega \frac{|Rv_n|^r}{|x|^s}dx - \int_\Omega F(x,Rv_n)dx \\ &\geq \frac{1}{p}R^p - \frac{\mu}{r}R^r\int_\Omega \frac{|v_n|^r}{|x|^s}dx - C_3R\int_\Omega |v_n|dx - \int_\Omega |v_n|^{p^*}dx. \end{aligned} \tag{3.8}$$

Due to $v_n \rightharpoonup v \equiv 0$ in $W_0^{2,p}(\Omega)$, then $\int_\Omega |v_n(x)|\,\mathrm{d}x \to 0$, $\int_\Omega \frac{|v_n|^r}{|x|^s}dx \to 0$ and $\int_\Omega |v_n(x)|^{p*}\,\mathrm{d}x < C(\Omega)$. Therefore, let $n \to +\infty$ in (3.8), and then let $R \to +\infty$, we have

$$I_\mu(t_n u_n) \geq I_\mu(Rv_n) \to +\infty \quad \text{as } n \to +\infty. \tag{3.9}$$

In addition, it is not difficult to infer that $0 < t_n < 1$ from $I_\mu(0) = 0$ and $I_\mu(u_n) \to c < +\infty$ as $n \to +\infty$.

Furthermore, in the light of (3.7), we have $\frac{d}{dt}(I_\mu(tu_n))|_{t=t_n} = 0$. Therefore,

$$\begin{aligned} \langle I'_\mu(t_n u_n), t_n u_n\rangle &= t_n\langle I'_\mu(t_n u_n), u_n\rangle \\ &= t_n\frac{d}{d\tau}(I_\mu(t_n u_n + \tau u_n))|_{\tau=0} \\ &= t_n\frac{d}{d\tau}(I_\mu(tu_n + \tau u_n))|_{\tau=0,t=t_n} \\ &= t_n\frac{d}{dt}(I_\mu(tu_n + \tau u_n))|_{t=t_n,\tau=0} \\ &= t_n\frac{d}{dt}(I_\mu(tu_n))|_{t=t_n} = 0. \end{aligned}$$

And considering the condition (f_4), we have

$$\begin{aligned} \frac{1}{\theta}I_\mu(t_n u_n) &= \frac{1}{\theta}\left(I_\mu(t_n u_n) - \frac{1}{p}\langle I'_\mu(t_n u_n), t_n u_n\rangle\right) \\ &= \frac{1}{\theta}\mu\left(\frac{1}{p} - \frac{1}{r}\right)|t_n|^r\int_\Omega \frac{|u_n|^r}{|x|^s}dx \\ &\quad + \frac{1}{\theta}\int_\Omega\left(\frac{1}{p}f(x,t_n u_n)\,t_n u_n - F(x,t_n u_n)\right)\mathrm{d}x \\ &= \frac{1}{\theta}\mu\left(\frac{1}{p} - \frac{1}{r}\right)|t_n|^r\int_\Omega \frac{|u_n|^r}{|x|^s}dx + \frac{1}{\theta}\int_\Omega H(x,t_n u_n)dx \\ &= \frac{1}{\theta}\mu\left(\frac{1}{p} - \frac{1}{r}\right)|t_n|^r\int_\Omega \frac{|u_n|^r}{|x|^s}dx + \frac{1}{\theta}\int_\Omega(\theta H(x,u_n) + C_*)dx \\ &= \mu\left(\frac{1}{p} - \frac{1}{r}\right)\left(\frac{|t_n|^r}{\theta} - 1\right)\int_\Omega \frac{|u_n|^r}{|x|^s}dx \\ &\quad + I_\mu(u_n) - \frac{1}{p}\langle I'_\mu(u_n), u_n\rangle + \frac{C_*}{\theta}|\Omega| \end{aligned}$$

$$
\begin{aligned}
&\leq I_\mu(u_n) - \frac{1}{p}\langle I'_\mu(u_n), u_n\rangle + \frac{C_*}{\theta}|\Omega| \\
&\to c + \frac{C_*}{\theta}|\Omega|.
\end{aligned}
$$

Thence,

$$\limsup_{n\to+\infty} I_\mu(t_n u_n) \leq \theta c + C_*|\Omega| < +\infty,$$

which is contradictive to (3.7).

Step 2. $\{u_n\}$ *admits a convergent subsequence in* $W_0^{2,p}(\Omega)$.

Since $\{u_n\}$ is bounded in the reflexive Banach space $W_0^{2,p}(\Omega)$, up to a subsequence, $u_n \rightharpoonup u$ in $W_0^{2,p}(\Omega)$. Therefore,

$$
\begin{aligned}
u_n &\to u \quad \text{in } L^q(\Omega), q < p^*, \\
u_n(x) &\to u(x) \quad a.e. \text{ in } \Omega, \\
\frac{u_n}{|x|^{\frac{s}{r}}} &\to \frac{u}{|x|^{\frac{s}{r}}} \quad \text{in } L^r(\Omega), r < p^*(s), \\
\frac{|u_n|^{r-2}u_n}{|x|^s} &\rightharpoonup \frac{|u|^{r-2}u}{|x|^s} \quad \text{weakly in } L^{r}(\Omega), r < p^*(s).
\end{aligned}
$$

According to the condition (SCPI), for every $\varepsilon > 0$, there is a $C(\varepsilon) > 0$ such that $|f(x,t)| \leq C(\varepsilon) + \varepsilon|t|^{p^*-1}$ for any (x,t) in $\Omega \times \mathbb{R}$. Therefore, we get

$$
\begin{aligned}
\left|\int_\Omega f(x,u_n)(u_n - u)\,\mathrm{d}x\right| &\leq C(\epsilon)\int_\Omega |u_n - u|\,\mathrm{d}x + \epsilon\int_\Omega |u_n - u|\,|u_n|^{p^*-1}\,\mathrm{d}x \\
&\leq C(\epsilon)\int_\Omega |u_n - u|\,\mathrm{d}x \\
&\quad + \epsilon\left(\int_\Omega |u_n|^{p^*}\,\mathrm{d}x\right)^{\frac{p^*-1}{p^*}}\left(\int_\Omega |u_n - u|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}} \\
&\leq C(\epsilon)\int_\Omega |u_n - u|\,\mathrm{d}x + \epsilon C(\Omega).
\end{aligned}
$$

In line with $u_n \rightharpoonup u$ in $W_0^{2,p}(\Omega)$, $\int_\Omega |u_n - u|\,\mathrm{d}x \to 0$, and the arbitrariness of ϵ, we may infer that

$$\int_\Omega f(x,u_n)(u_n - u)\,\mathrm{d}x \to 0.$$

On the other hand,

$$\int_\Omega \frac{|u_n|^{r-2}u_n}{|x|^s}(u_n - u)dx \to 0.$$

Hence,

$$
\begin{aligned}
0 \leftarrow &\langle I'_{u_n}, u_n - u\rangle \\
&= \int_\Omega |\triangle u_n|^{p-2}\triangle u_n(\triangle u_n - \triangle u)dx \\
&\quad - \mu\int_\Omega \frac{|u_n|^{r-2}u_n}{|x|^s}(u_n - u)dx - \int_\Omega f(x,u_n)(u_n - u)\,\mathrm{d}x \\
&= \int_\Omega |\triangle u_n|^{p-2}\triangle u_n(\triangle u_n - \triangle u)dx + o(1).
\end{aligned}
$$

Therefore,

$$\int_{\Omega} |\triangle u_n|^{p-2} \triangle u_n (\triangle u_n - \triangle u) dx \to 0,$$

which implies that $u \to u$ strongly in $W_0^{2,p}(\Omega)$, that is to say, the functional I_μ satisfies the Cerami condition for any c in $\mathbb{R}$. □

Proof of Theorem 1.2. According to Theorem A.2, Lemma 3.1 and Lemma 3.2, we know that Problem (PD) admits a nontrivial weak solution in $W_0^{2,p}(\Omega)$.

From Proposition 2.5, we obtain Lemma 3.1 and Lemma 3.2 when $W_0^{2,p}(\Omega)$ is replaced by $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$. Hence Problem (PNa) has also a nontrivial weak solution in $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$. □

4 Proof of Theorem 1.3

In this section, we apply Theorem A.3 to prove Theorem 1.3. First of all, because $W_0^{2,p}(\Omega)$ is a Banach space, we formulate Y_k and Z_k as in (A.1). The condition (f_6) means $I_\mu(-u) = -I_\mu(u)$. Since the condition (SCP) indicates the condition (SCPI), I_μ contents the Cerami condition for any c in $\mathbb{R}$ under Lemma 3.2. Here, we mimic part of the proof of Theorem 3.7 in [27] and Theorem 1.2 in [2].

In order to estimate A6) in Theorem 1.3, we need the following lemma.

Lemma 4.1.

$$\beta_k = \sup_{\substack{u \in Z_k \\ \|u\|=1}} \|u\|_{L^q} \to 0 \quad \text{as } k \to \infty$$

provided $1 \leq q < p^*$.

Proof. $Z_{k+1} = \overline{\bigoplus_{j \geq k+1} X_j} \subset \overline{\bigoplus_{j \geq k} X_j} = Z_k$ suggests $0 \leq \beta_{k+1} \leq \beta_k$, thence $\lim_{k \to +\infty} \beta_k = b \geq 0$. According to the definition of supper bound, for any $k > 0$, there exists u_k in Z_k with $\|u\|_{L^q} > \frac{\beta_k}{2}$ on $\partial B_1(0)$ in $W_0^{2,p}(\Omega)$. Since $W_0^{2,p}(\Omega)$ is a real, reflexive, and separable Banach space, we can extract a subsequence of $\{u_k\}$ (still denoted for $\{u_k\}$) such that $u_k \rightharpoonup u$ weakly in $W_0^{2,p}(\Omega)$, i.e. $\langle u_k, \varphi \rangle \to \langle u, \varphi \rangle$ for any φ in $(W_0^{2,p}(\Omega))^*$.

Since each Z_k is convex and closed, hence it is closed for the weak topology, which implies

$$u \in \bigcap_{k=1}^{+\infty} Z_k = \{0\}.$$

Therefore, according to Sobolev embedding theorem, we have

$$0 < \frac{\beta_k}{2} < u_k \to 0 \quad \text{in } L^q(\Omega) \text{ as } k \to +\infty.$$ □

Proof of Theorem 1.3. Rewrite (3.1) to the form we need here: For some $k > 0$, there exist $C_k > 0$ and $A_k > 0$ such that

$$F(x,t) \geq C_k |t|^p - A_k \text{ for every } (x,s) \quad \text{in } \overline{\Omega} \times \mathbb{R}.$$

Step 1. *For any $k \in \mathbb{N}$, there exists $\rho_k > 0$ such that*

$$a_k = \max_{\substack{u \in Y_k \\ \|u\|=\rho_k}} I_\mu(u) \leq 0.$$

In fact, all norms on Y_k are equivalent since Y_k is finite dimensional, hence there exist two positive constants $C_{k,p}$ and $\widetilde{C}_{k,p}$ such that

$$C_{k,p}^{\frac{1}{p}}\|u\|_{L^p} \leq \|u\| \leq \widetilde{C}_{k,p}^{\frac{1}{p}}\|u\|_{L^p} \quad \text{for all } u \in Y_k.$$

Therefore, for all u in Y_k, we have

$$\begin{aligned}
I_\mu(u) &= \frac{1}{p}\int_\Omega |\triangle u|^p dx - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s}dx - \int_\Omega F(x,u)dx \\
&\leq \frac{1}{p}\|u\|^p - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s}dx - C_k\|u\|_{L^p}^p + A_k|\Omega| \\
&\leq \frac{1}{p}\|u\|^p - \|u\|^p + A_k|\Omega| - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s}dx \\
&\leq \frac{1-p}{p}\|u\|^p + A_k|\Omega|.
\end{aligned}$$

Thence, we choose u in Y_k with $\|u\| = \rho_k > 0$ large enough and obtain

$$I_\mu(u) \leq 0.$$

Step 2. *There exists r_k in $(0, \rho_k)$ such that*

$$b_k = \inf_{\substack{u\in Z_k \\ \|u\|=r_k}} I_\mu(u) \to +\infty, \textit{ as } k \to \infty.$$

Indeed, (SCP) implies that there exists $C' > 0$ such that

$$|F(x,t)| \leq C'\left(1 + |t|^q\right).$$

Hence, for any u in Z_k, we get

$$\begin{aligned}
I_\mu(u) &= \frac{1}{p}\int_\Omega |\triangle u|^p dx - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s}dx - \int_\Omega F(x,u)dx \\
&\geq \frac{1}{p}\|u\|^p - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s}dx - C'\|u\|_{L^q}^q - C'|\Omega| \\
&\geq \frac{1}{p}\|u\|^p - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s}dx - C'\|\frac{u}{\|u\|}\|_{L^q}^q\|u\|^q - C'|\Omega| \\
&\geq \frac{1}{p}\|u\|^p - \frac{\mu}{r}(\mu_{s,r}(\Omega))^{-\frac{r}{p}}\|u\|^r - C'\beta_k^q\|u\|^q - C'|\Omega| \\
&= \left[\frac{1}{p}\left(1 - \frac{\mu}{\mu_{s,r}(\Omega)}\right) - C'\beta_k^q\|u\|^{q-p}\right]\|u\|^p - C'|\Omega|.
\end{aligned}$$

According to Lemma 4.1, $\lim_{k\to+\infty}\beta_k = +\infty$. Let $r_k = \left(\frac{\mu_{s,r}(\Omega)C'q\beta_k^q}{\mu_{s,r}(\Omega)-\mu}\right)^{-\frac{1}{q-p}}$, then $\lim_{k\to+\infty} r_k = +\infty$. If for $u \in Z_k$ with $\|u\| = r_k$, then we have

$$I_\mu(u) \geq \left(\frac{1}{p} - \frac{1}{q}\right)\left(1 - \frac{\mu}{\mu_{s,r}(\Omega)}\right)r_k^p - C'|\Omega| \to +\infty, \quad \text{as } k \to +\infty,$$

which yields **Step 2**. □

Remark 4.2. If $p < r$, we seem impossible to get $I_\mu(u) \to +\infty$, as $k \to +\infty$. Therefore, in a sense, $p = r$ are sharp.

Appendix A

The machinery of the critical point theory is based on the existence of a linking structure and deformation lemmas. Generally speaking, it is necessary that some compactness condition of the functional in order to derive such deformation results. We use the famous Cerami condition:

Definition A.1 (Cerami (C) condition)**.** Let X be a real Banach space with its dual space X^* and $J \in C^1(X, \mathbb{R})$. For $c \in \mathbb{R}$ we say that J satisfies the $(C)_c$ condition if for any sequence $\{x_n\} \subset X$ with $J(x_n) \to c$ and $(1 + \|x_n\|_X) \|J'(x_n)\|_{X^*} \to 0$, then the sequence $\{x_n\}$ admits a subsequence strongly convergent in X.

Theorem A.2 (Mountain Pass Theorem with Cerami condition [8])**.** *Assume that X is a real Banach space and $J \in C^1(X, \mathbb{R})$ satisfies the $(C)_c$ condition for any $c \in \mathbb{R}$, $J(0) = 0$, and, in addition,*

A1) There exist positive constants r and η such that $J(u)|_{\partial B_r} \geq \eta$;

A2) There exists an $u_0 \in X$ with $\|u_0\| > \rho$ such that $J(u_0) \leq 0$.

Then $c = \inf\limits_{\gamma\in\Gamma} \max\limits_{t\in[0,1]} J(\gamma(t)) \geq \alpha$ is a critical value of J, where

$$\Gamma = \left\{\gamma \in C^0([0,1], X) : \gamma(0) = 0, \gamma(1) = u_0\right\}.$$

Let X be a reflexive and separable Banach space, then there exist sequences $\{e_j\} \subset X$ and $\{\varphi_j\} \subset X^*$ with

A3) $\langle \varphi_i, e_i \rangle = \delta_{i,j}$, where $\delta_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j; \end{cases}$

A4) $\overline{\operatorname{span}\{e_j\}_{j=1}^\infty} = X$ and $\overline{\operatorname{span}^{w^*}\{\varphi_j\}_{j=1}^\infty} = X^*$.

Let $X_j = \mathbb{R}e_j$, then $X = \overline{\bigoplus_{j\geq 1} X_j}$. And we define

$$Y_k = \bigoplus_{j=1}^k X_j \quad \text{and} \quad Z_k = \overline{\bigoplus_{j\geq k} X_j} \tag{A.1}$$

Theorem A.3 (Fountain Theorem with Cerami condition [2])**.** *Suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the $(C)_c$ condition for all $c \in \mathbb{R}$ and $\varphi(u) = \varphi(-u)$. If for any $k \in \mathbb{N}$, there exists $\rho_k > r_k$ such that*

A5) $a_k = \max\limits_{\substack{u\in Y_k\\ \|u\|=\rho_k}} \varphi(u) \leq 0$;

A6) $b_k = \inf\limits_{\substack{u\in Z_k\\ \|u\|=r_k}} \varphi(u) \to +\infty$, as $k \to \infty$,

then φ possesses an unbounded sequence of critical values.

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Variational differential inclusions without ellipticity condition

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Abstract. The paper sets forth a new type of variational problem without any ellipticity or monotonicity condition. A prototype is a differential inclusion whose driving operator is the competing weighted (p, q)-Laplacian $-\Delta_p u + \mu\Delta_q u$ with $\mu \in \mathbb{R}$. Local and nonlocal boundary value problems fitting into this nonstandard setting are examined.

Keywords: variational problem, hemivariational inequality, lack of ellipticity, competing (p, q)-Laplacian, local and nonlocal operators.

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1 Introduction

Let X and Y be Banach spaces and let $j : X \to Y$ be a linear compact map. There are given on X a Gâteaux differentiable function $F : X \to \mathbb{R}$ with its Gâteaux differential $DF : X \to X^*$ and on Y a locally Lipschitz function $\Phi : Y \to \mathbb{R}$ whose generalized directional derivative is denoted $\Phi^0 : Y \times Y \to \mathbb{R}$. With these data we formulate the following problem in the form of a hemivariational inequality: find $u \in X$ such that

$$\langle DF(u), w\rangle + \Phi^0(ju; jw) \geq 0, \qquad \forall w \in X. \tag{1.1}$$

Problem (1.1) qualifies as a hemivariational inequality due to the presence of the term $\Phi^0(ju; jw)$. This problem is equivalent to the differential inclusion

$$-DF(u) \in j^*\partial\Phi(ju),$$

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where the notation $\partial\Phi(u)$ stands for the generalized gradient of Φ at $u \in X$ and j^* denotes the adjoint operator of j. The hemivariational inequalities provide accurate modeling of contact phenomena involving nonconvex and nonsmooth mechanical processes. For an extensive study on applications of hemivariational inequalities we cite [10,13,14],

Problem (1.1) has a variational structure, which is nonsmooth, whose associated energy functional $I : X \to \mathbb{R}$ is

$$I = F + \Phi \circ j. \tag{1.2}$$

There is a huge literature devoted to variational problems, smooth or nonsmooth, mainly employing minimax techniques based on critical point theory (see, e.g., [11], [3], [10, Chapter 3]). Since F is only Gâteaux differentiable, no available result can be applied to problem (1.1) and its corresponding energy functional I in (1.2).

The main novelty of the present work is represented by the fact that we don't assume any ellipticity condition on the leading term $DF(u)$ in (1.1). In order to highlight this essential aspect, let us consider a particular situation in (1.1) related to boundary value problems with discontinuous nonlinearities. Their study was initiated by Chang [3].

For a fixed $\mu \in \mathbb{R}$, we state the quasilinear differential inclusion

$$\begin{cases} -\Delta_p u + \mu\Delta_q u \in [\underline{f}(u), \overline{f}(u)] & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ with the boundary $\partial\Omega$. Here Δ_p and Δ_q denote the p-Laplacian and the q-Laplacian, respectively, with $1 < q < p < +\infty$, and for a function $f \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ we set

$$\underline{f}(s) = \lim_{\delta\to 0} \operatorname*{ess\,inf}_{|\tau - s|<\delta} f(\tau), \qquad \forall s \in \mathbb{R} \tag{1.4}$$

and

$$\overline{f}(s) = \lim_{\delta\to 0} \operatorname*{ess\,sup}_{|\tau - s|<\delta} f(\tau), \qquad \forall s \in \mathbb{R}. \tag{1.5}$$

If the function f is continuous, then the interval $[\underline{f}(u(x)), \overline{f}(u(x))]$ reduces to the singleton $f(u(x))$ and (1.3) becomes the quasilinear Dirichlet equation

$$\begin{cases} -\Delta_p u + \mu\Delta_q u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

An important case in problems (1.3) and (1.6) is when $\mu = 0$ with the p-Laplacian Δ_p as driving operator. Another important case is when $\mu = -1$, where the quasilinear equation is governed by the (p, q)-Laplacian $\Delta_p + \Delta_q$. We emphasize that the behavior of $-\Delta_p + \mu\Delta_q$ with $\mu > 0$ is completely different with respect to the one of $-\Delta_p + \mu\Delta_q$ with $\mu \leq 0$, the latter being an elliptic operator. In the case of $-\Delta_p + \mu\Delta_q$ with $\mu > 0$ the ellipticity is lost as can be easily seen: for $u = \lambda u_0$ with a nonzero $u_0 \in W^{1,p}_0(\Omega)$ and a number $\lambda > 0$ the expression

$$\langle -\Delta_p u + \mu\Delta u, u\rangle = \lambda^p \|\nabla u_0\|_p^p - \mu\lambda^q \|\nabla u_0\|_q^q$$

is positive for λ large and negative for λ small. Therefore the leading operator in (1.3) is a competing (p, q)-Laplacian when $\mu > 0$. This makes (1.3), thus (1.1), a nonstandard problem where a sort of hyperbolic feature is incorporated.

We further discuss a nonlocal counterpart of problem (1.3), namely

$$\begin{cases} -\Delta_p u + \mu(-\Delta)^s_q u \in [\underline{f}(u), \overline{f}(u)] & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.7}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, where $f \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ with (1.4), (1.5) as above, and a parameter $\mu \in \mathbb{R}$. Inclusion (1.7) is driven by the nonlocal operator formed with the ordinary p-Laplacian Δ_p and the (negative) s-fractional q-Laplacian $(-\Delta)^s_q$, taking $0 < s < 1$ and $1 < q < p < +\infty$, with $sq < N$. The differential operator $-\Delta_p + \mu(-\Delta)^s_q$ is the optimal fractional substitute for the (p, q)-Laplacian $-\Delta_p - \mu\Delta_q$ as noticed below in Remark 5.2. Likewise in the case of fractional p-Laplacian (see, e.g., [15]), a motivation for studying it comes from the theory of Markov processes. In this respect, we refer to [8, Example 1.2.1] describing a typical Markovian symmetric form. A brief survey of the nonlocal setting related to (1.7) can be found in Section 2. If the function f is continuous, (1.7) reduces to the equation

$$\begin{cases} -\Delta_p u + \mu(-\Delta)^s_q u = f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.8}$$

In the nonlocal problems (1.7) and (1.8) the ellipticity is preserved if $\mu \geq 0$, but not if $\mu < 0$ for which the usual methods fail to apply.

The natural notion of solution (in the weak sense) to problem (1.1) is apparent: any $u \in X$ for which inequality (1.1) holds whenever $w \in X$. Since we do not assume any ellipticity/monotonicity condition upon the principal part of (1.1) or any compactness condition of Palais–Smale type on I in (1.2) or that I be sequentially weakly lower semicontinuous (as basically is required in [1]), in order to establish the solvability of equation (1.1) we need to relax the notion of solution to fit the specific character of problem (1.1).

Definition 1.1. A function $u \in X$ is called a *generalized solution* to (1.1) if there exists a sequence $\{u_n\}_{n\geq 1} \subset X$ with the properties:

(S_1) $u_n \rightharpoonup u$ in X as $n \to \infty$;

(S_2) $\limsup_{n\to\infty} F(u_n) \leq F(u)$;

(S_3)
$$\liminf_{n\to\infty} \langle DF(u_n), v - u_n\rangle + \Phi^0(ju; jv - ju) \geq 0, \qquad \forall v \in X.$$

Remark 1.2. The idea of weakening the notion of solution to cover more general frames is frequent (see, e.g., [12, p. 183]). Different situations where the solution is a limit of (approximate) solutions are discussed in [16, 17].

Remark 1.3. Every solution to (1.1) is a generalized solution in the sense of Definition 1.1. It suffices to take the constant sequence $u_n = u$. For the converse assertion, additional assumptions should be imposed, for instance that the differential $DF : X \to X^*$ be completely sequentially continuous (i.e., $u_n \rightharpoonup u$ implies $DFu_n \to DFu$). A key role might have property (S_2) in Definition 1.1 as will be illustrated for problems (1.3), (1.6), (1.7), (1.8).

Our main result stated as Theorem 3.2 in Section 3 provides the existence of a generalized solution to problem (1.1). The approach relies on minimization of the energy functional I in (1.2) on finite dimensional subspaces of X belonging to a Galerkin basis. Denoting by $\{v_n\}_{n\geq 1} \subset X$ the resulting minimizing sequence of I, in a further step we construct through

Ekeland's variational principle (see [6,7]) applied to I and $\{v_n\}_{n\geq 1}$ a second minimizing sequence $\{u_n\}_{n\geq 1} \subset X$ of I, with finer properties, that will be shown to comply with Definition 1.1. The proof is concluded by a passing to the limit process.

The abstract result in Theorem 3.2 for problem (1.1) is applied in two different directions. First, we establish the existence of a generalized solution to the local quasilinear differential inclusion with discontinuities (1.3), in particular (1.6) (see Theorem 4.2). Second, we obtain the existence of a generalized solution to the nonlocal quasilinear inclusion (1.7), in particular (1.8) (see Theorem 5.1). In both cases, a special attention is paid to clarify when the generalized solution is a weak solution.

2 Mathematical background

Our approach on problem (1.1) relies on two fundamental tools: Galerkin basis and Ekeland's variational principle. For easy reference we recall some basic material.

A Galerkin basis of a Banach space X is a sequence $\{X_n\}_{n\geq 1}$ of vector subspaces of X for which

(i) $\dim(X_n) < \infty, \quad \forall n;$

(ii) $X_n \subset X_{n+1}, \quad \forall n;$

(iii) $$\overline{\bigcup_{n=1}^{\infty} X_n} = X.$$

If X is separable, there exists a Galerkin basis of X. For an extensive use of Galerkin bases to various existence theorems we refer to [12,16,17].

We shall apply Ekeland's Variational Principle (see [6,7]) in the following form.

Theorem 2.1. *Assume that the functional $I : X \to \mathbb{R}$ is lower semicontinuous and bounded from below on a Banach space X. If $\{v_n\}_{n\geq 1}$ is a minimizing sequence of I, then there exists a sequence $\{u_n\}_{n\geq 1}$ in X with the properties:*

(a) *$I(u_n) \leq I(v_n)$ for all n;*

(b) *$\|u_n - v_n\| \to 0$ as $n \to \infty$;*

(c) *for all $n \geq 1$, it holds*

$$I(w) > I(u_n) - \frac{1}{n}\|w - u_n\|, \qquad \forall w \in X,\ w \neq u_n.$$

Next we outline some prerequisites of nonsmooth analysis regarding the subdifferentiability of locally Lipschitz functions (for more details we recommend [4] and also [3,10]). A function $\Phi : Y \to \mathbb{R}$ on a Banach space Y is called locally Lipschitz if for every $u \in Y$ one can find a neighborhood U of u in Y and a constant $L_u > 0$ such that

$$|\Phi(v) - \Phi(w)| \leq L_u\|v - w\|, \qquad \forall v, w \in U.$$

The generalized directional derivative of a locally Lipschitz function Φ at $u \in Y$ in the direction $v \in Y$ is defined as

$$\Phi^0(u; v) := \limsup_{w\to u,\ t\to 0^+} \frac{1}{t}(\Phi(w + tv) - \Phi(w))$$

and the generalized gradient of Φ at $u \in Y$ is the set

$$\partial\Phi(u) := \left\{u^* \in Y^* : \langle u^*, v\rangle \leq \Phi^0(u;v),\ \forall v \in Y\right\}.$$

A continuous and convex function $\Phi : Y \to \mathbb{R}$ is locally Lipschitz and its generalized gradient $\partial\Phi : Y \to 2^{Y^*}$ coincides with the subdifferential of Φ in the sense of convex analysis.

We need these notions in connection with the nonsmooth problems (1.3), (1.6), (1.7), (1.8). Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ for which we set

$$g(s) = \int_0^s f(t)\,\mathrm{d}t \quad \text{for all } s \in \mathbb{R} \tag{2.1}$$

and note that $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz. Then the generalized gradient $\partial g(s)$ of g at $s \in \mathbb{R}$ is the compact interval in $\mathbb{R}$ expressed by

$$\partial g(s) = [\underline{f}(s), \overline{f}(s)], \tag{2.2}$$

where $\underline{f}(s)$ and $\overline{f}(s)$ are defined in (1.4) and (1.5), respectively.

We also address a few things about the operators in the Dirichlet problems (1.3), (1.6), (1.7), (1.8). Given $1 < q < p < +\infty$, we denote $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$ and consider the Sobolev spaces $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$ endowed with the norms $\|\nabla u\|_p$ and $\|\nabla u\|_q$, respectively, where $\|\cdot\|_r$ stands for the usual L^r-norm. The negative p-Laplacian $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is defined by

$$\langle -\Delta_p u, \varphi\rangle = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x \quad \text{for all } u, \varphi \in W_0^{1,p}(\Omega).$$

This operator is strictly monotone and continuous, so pseudomonotone. If $p = 2$ we retrieve the ordinary Laplacian operator. Similarly, we have the definition of the negative q-Laplacian $-\Delta_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$. By virtue of the embedding $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$, the differential operator $-\Delta_p + \mu\Delta_q$ driving inclusion (1.3) and equation (1.6) is well posed in $W_0^{1,p}(\Omega)$. There exists a constant $k > 0$ such that

$$\|\nabla u\|_q \leq k\|\nabla u\|_p, \qquad \forall u \in W_0^{1,p}(\Omega). \tag{2.3}$$

By a weak solution to problem (1.3) with $f \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ we mean a $u \in W_0^{1,p}(\Omega)$ for which it holds $\underline{f}(u), \overline{f}(u) \in L^{p'}(\Omega)$ and

$$\begin{aligned}\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x - \mu\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x\\ \geq \int_\Omega \min\{\underline{f}(u(x))\varphi(x), \overline{f}(u(x))\varphi(x)\}\,\mathrm{d}x \quad \text{for all } \varphi \in W_0^{1,p}(\Omega)\end{aligned} \tag{2.4}$$

or equivalently

$$\begin{aligned}\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x - \mu\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x\\ \leq \int_\Omega \max\{\underline{f}(u(x))\varphi(x), \overline{f}(u(x))\varphi(x)\}\,\mathrm{d}x \quad \text{for all } \varphi \in W_0^{1,p}(\Omega).\end{aligned} \tag{2.5}$$

The equivalence between (2.4) and (2.5) arises by replacing $\varphi \in W_0^{1,p}(\Omega)$ with $-\varphi$. For the Dirichlet equation (1.6), the ordinary notion of weak solution is recovered. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $u \in W_0^{1,p}(\Omega)$ is a weak solution to equation (1.6) provided $f(u) \in L^{p'}(\Omega)$ and

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x - \mu\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x$$
$$= \int_\Omega f(u(x))\varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in W_0^{1,p}(\Omega). \tag{2.6}$$

This follows readily from (2.4) (or (2.5)), (1.4) and (1.5).

Finally, we sketch the framework of nonlocal problems (1.7) and (1.8). The fractional Sobolev space $W^{s,q}(\Omega)$ of differentiability order $s \in (0,1)$ and summability exponent $1 < q < +\infty$ for a bounded domain $\Omega \subset \mathbb{R}^N$ with a Lipschitz continuous boundary $\partial\Omega$ is introduced as

$$W^{s,q}(\Omega) := \left\{ u \in L^q(\Omega) : \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^q}{|x-y|^{N+qs}}dxdy < \infty \right\},$$

which is a separable and reflexive Banach space endowed with the norm

$$\|u\|_{W^{s,q}(\Omega)} := \left(\|u\|_q^q + \frac{C_{N,q,s}}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^q}{|x-y|^{N+qs}}dxdy \right)^{\frac{1}{q}},$$

with a normalization constant $C_{N,q,s} > 0$. If $sq < N$, the embedding $W^{s,q}(\Omega) \hookrightarrow L^\nu(\Omega)$ is continuous for all $1 \le \nu \le q_s^*$, and compact for all $1 \le \nu < q_s^*$, with $q_s^* = Np/(N-sq)$ called the fractional critical exponent (see [5, Theorem 6.5, Corollary 7.2]). Under the conditions $0 < s < 1$, $1 < q < p < +\infty$ and $sq < N$, the embeddings $W^{1,p}(\Omega) \hookrightarrow W^{1,q}(\Omega) \hookrightarrow W^{s,q}(\Omega)$ are continuous and thus for a constant $C = C(N,s,q) \ge 1$ one has

$$\|u\|_{W^{s,q}(\Omega)} \le C\|u\|_{W^{1,q}(\Omega)}, \qquad \forall u \in W^{1,p}(\Omega). \tag{2.7}$$

(see [5, Proposition 2.2])).

The closed linear subspace

$$W_0^{s,q}(\Omega) := \{u \in W^{s,q}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

can be endowed with the equivalent norm (determined by the Gagliardo seminorm)

$$\|u\|_{W_0^{s,q}(\Omega)} := \left(\frac{C_{N,q,s}}{2}\right)^{\frac{1}{q}} [u]_{D^{s,q}(\mathbb{R}^N)} := \left(\frac{C_{N,q,s}}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^q}{|x-y|^{N+qs}}dxdy\right)^{\frac{1}{q}}$$

becoming a uniformly convex Banach space with the dual $W^{-s,q'}(\Omega)$.

The (negative) s-fractional q-Laplacian is the nonlinear operator $(-\Delta)_q^s : W_0^{s,q}(\Omega) \to W^{-s,q'}(\Omega)$ defined for all $u, v \in W_0^{s,q}(\Omega)$ by

$$\langle(-\Delta)_q^s(u), v\rangle = \frac{C_{N,q,s}}{2}\int_{R^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sq}}\,dx\,dy \tag{2.8}$$

(see [5, 15] for more insight).

Along the pattern of the corresponding local problems, $u \in W_0^{1,p}(\Omega)$ is called a weak solution to inclusion (1.7) with $0 < s < 1$, $1 < q < p < +\infty$, $sq < N$ and $f \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ provided $\underline{f}(u), \overline{f}(u) \in L^{p'}(\Omega)$ and

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x$$
$$+\mu\frac{C_{N,q,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+qs}}\,dxdy$$
$$\ge \int_\Omega \min\{\underline{f}(u(x))\varphi(x), \overline{f}(u(x))\varphi(x)\}\,\mathrm{d}x \quad \text{for all } \varphi \in W_0^{1,p}(\Omega), \tag{2.9}$$

where we set $u = \varphi = 0$ outside Ω. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $u \in W_0^{1,p}(\Omega)$ is a weak solution to the nonlocal equation (1.8) provided $f(u) \in L^{p'}(\Omega)$ and

$$\begin{aligned}
&\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla\varphi(x)\,\mathrm{d}x \\
&\qquad + \mu\frac{C_{N,q,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+qs}}\,dxdy \\
&\quad = \int_\Omega f(u(x))\varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in W_0^{1,p}(\Omega).
\end{aligned} \tag{2.10}$$

3 Existence of a generalized solution

In order to simplify the presentation, we denote by the same symbol $\|\cdot\|$ different norms that occur below. The meaning will be clear from the context. Our hypotheses on the data in problem (1.1) are as follows:

$(H1)$ *The Banach space X is separable and reflexive, and $j : X \to Y$ is a linear compact map from X to a Banach space Y.*

$(H2)$ *The function $F : X \to \mathbb{R}$ is Gâteaux differentiable, continuous, and the function $\Phi : Y \to \mathbb{R}$ is locally Lipschitz* such that

$$I = F + \Phi \circ j \text{ is bounded from below on } X \tag{3.1}$$

and I is coercive on every finite dimensional subspace of X, i.e., if X_0 is a finite dimensional subspace of X, then $I(u) \to +\infty$ as $\|u\| \to \infty$ with $u \in X_0$.

$(H3)$ *The set*

$$\{v \in X : \langle DF(v), v\rangle \le \Phi^0(jv; -jv)\}$$

is bounded in X.

The next example shows that the coercivity on every finite dimensional subspace in hypothesis $(H2)$ is a condition weaker than the coercivity on the whole space.

Example 3.1. Let X be a separable Hilbert space. Fix an orthonormal basis $\{e_m\}_{m\ge1}$ of X. Then every vector $u \in X$ can be written uniquely as $u = \sum_{m=1}^{\infty} x_m(u)e_m$, with $x_m(u) \in \mathbb{R}$, and there holds $\|u\|^2 = \sum_{m=1}^{\infty} x_m(u)^2$. The functional $J : X \to \mathbb{R}$ given by

$$J(u) = \sum_{m=1}^{\infty} \frac{1}{m^2}|x_m(u)|$$

is well defined. It is coercive on each finite dimensional subspace X_1 of X since corresponding to X_1 there is an integer m_1 such that

$$J(u) = \sum_{m=1}^{m_1} \frac{1}{m^2}|x_m(u)|, \qquad \forall u \in X_1.$$

For $u_n = ne_n$ we have $\|u_n\| = n$ and $J(u_n) = \frac{1}{n}$, so J is not coercive on X.

Now we state our existence result for problem (1.1).

Theorem 3.2. *Assume that conditions $(H1)$–$(H3)$ hold. Then problem (1.1) admits at least one generalized solution in the sense of Definition 1.1.*

Proof. We construct a special minimizing sequence $\{v_n\}_{n\geq 1} \subset X$ of the functional I in (1.2). The construction is done through a Galerkin basis $\{X_n\}_{n\geq 1}$ of X, which exists because the Banach space X is separable as known from assumption $(H1)$.

It follows from (3.1) that for every n the functional $I|_{X_n}$ obtained restricting I to X_n is bounded from below on X_n. Due to the coercivity of I on X_n as guaranteed by assumption $(H2)$, any minimizing sequence of $I|_{X_n}$ is bounded. Since $I|_{X_n}$ is also continuous and X_n is finite dimensional (note requirement (i) in the definition of Galerkin basis in Section 2), there exists $v_n \in X_n$ satisfying

$$I(v_n) = \min_{v\in X_n} I(v). \tag{3.2}$$

Then (3.2) implies

$$I(v_n + t(v - v_n)) \geq I(v_n), \qquad \forall t > 0,\ \forall v \in X_n,$$

which reads as

$$\frac{1}{t}(F(v_n + t(v - v_n)) - F(v_n)) + \frac{1}{t}(\Phi(jv_n + t(jv - jv_n)) - \Phi(jv_n)) \geq 0.$$

Passing to the limit as $t \to 0$ and then setting $v = 0$ lead to

$$\langle DF(v_n), v_n\rangle \leq \Phi^0(jv_n; -jv_n), \qquad \forall n. \tag{3.3}$$

On account of hypothesis $(H3)$, we can infer from (3.3) that the sequence $\{v_n\}$ is bounded in X. In view of the reflexivity of X (see hypothesis $(H1)$), along a relabeled subsequence we have

$$v_n \rightharpoonup u \quad \text{in } X \tag{3.4}$$

for some $u \in X$. We shall show that u is a generalized solution to (1.1).

From condition (ii) in the definition of Galerkin basis (see Section 2) and (3.2), for every n we can write

$$I(v_n) = \min_{v\in X_n} I(v) \geq \min_{v\in X_{n+1}} I(v) = I(v_{n+1}) \geq \inf_{v\in X} I(v).$$

Therefore the sequence $\{I(v_n)\}$ is nonincreasing and bounded due to (3.1). Set

$$l := \lim_{n\to\infty} I(v_n) = \inf_{n\geq 1} I(v_n).$$

We claim that

$$\lim_{n\to\infty} I(v_n) = \inf_{w\in X} I(w). \tag{3.5}$$

In order to prove (3.5), we argue by contradiction supposing that

$$l > \inf_{v\in X} I(v).$$

So, there exists $\hat{w} \in X$ such that $I(\hat{w}) < l$. By the continuity of I, there exists an open neighborhood U of $\hat{w}$ in X such that

$$I(w) < l, \qquad \forall w \in U. \tag{3.6}$$

Then through condition (iii) in the definition of Galerkin basis (see Section 2) we derive

$$U \cap \left(\bigcup_{n=1}^{\infty} X_n \right) \neq \emptyset.$$

Hence there exists $\tilde{w} \in U \cap X_{\tilde{n}}$ for some $\tilde{n}$. Recalling that $v_{\tilde{n}}$ is a minimizer of $I_{|X_{\tilde{n}}}$ (see (3.2)), from (3.6) we get the contradiction

$$\min_{v \in X_{\tilde{n}}} I(v) \leq I(\tilde{w}) < l \leq \min_{v \in X_{\tilde{n}}} I(v).$$

The obtained contradiction ensures the validity of (3.5).

Now we construct another minimizing sequence $\{u_n\}$ of I in (1.2) that will satisfy conditions (S_1)–(S_3) in Definition 1.1. To this end we notice from (3.1) that we can apply Theorem 2.1 (Ekeland's Variational Principle, see [6,7]) to the functional I in (1.2). Through this result, using the minimizing sequence $\{v_n\}_{n\geq 1}$, we can find a sequence $\{u_n\}_{n\geq 1}$ in X with the properties (a), (b), (c) in Theorem 2.1. From property (a) and (3.5) it is clear that

$$\lim_{n\to\infty} I(u_n) = \inf_{v\in X} I(v), \tag{3.7}$$

so $\{u_n\}_{n\geq 1}$ is a minimizing sequence of the functional I. Consequently, from (3.7) it turns out

$$\lim_{n\to\infty} I(u_n) \leq I(u), \tag{3.8}$$

with $u \in X$ in (3.4). By property (b) in Theorem 2.1 and (3.4) we infer that

$$u_n \rightharpoonup u \quad \text{in } X, \tag{3.9}$$

thus condition (S_1) in Definition 1.1 is verified.

Using the compactness of the map $j : X \to Y$ and the weak convergence in (3.9), we note that (3.8) amounts to saying that

$$\begin{aligned} \limsup_{n\to\infty} F(u_n) + \Phi(j(u)) &= \limsup_{n\to\infty} (F(u_n) + \Phi(j(u_n))) \\ &\leq F(u) + \Phi(j(u)). \end{aligned}$$

This proves property (S_2) in Definition 1.1.

Insert $w = u_n + t(v - u_n)$ in assertion (c) of Theorem 2.1, with $t > 0$ and an arbitrary $v \in X$, finding that

$$\frac{1}{t}(F(u_n + t(v-u_n)) - F(u_n)) + \frac{1}{t}(\Phi(ju_n + t(jv - ju_n)) - \Phi(ju_n)) \geq -\frac{1}{n}\|v - u_n\|. \tag{3.10}$$

The Gâteaux differentiability of F yields

$$\lim_{t\to 0} \frac{1}{t}(F(u_n + t(v-u_n)) - F(u_n)) = \langle DF(u_n), v - u_n \rangle, \tag{3.11}$$

while the definition of generalized directional derivative Φ^0 of Φ (see Section 2) shows

$$\limsup_{t\to 0} \frac{1}{t}(\Phi(ju_n + t(jv - ju_n)) - \Phi(ju_n)) \leq \Phi^0(ju_n; jv - ju_n). \tag{3.12}$$

Letting $t \to 0$ in (3.10), by making use of (3.11) and (3.12), we arrive at

$$\langle DF(u_n), v - u_n \rangle + \Phi^0(ju_n; jv - ju_n) \geq -\frac{1}{n}\|v - u_n\|. \tag{3.13}$$

Notice that (3.9) and the compactness of $j : X \to Y$ yield

$$ju_n \to ju \quad \text{in } Y. \tag{3.14}$$

Then the upper semicontinuity of the generalized directional derivative Φ^0 and the strong convergence in (3.14) give

$$\limsup_{n\to\infty} \Phi^0(ju_n; jv - ju_n) \leq \Phi^0(ju; jv - ju). \tag{3.15}$$

Letting $n \to \infty$ in (3.13) and taking into account (3.15) as well as the boundedness of the sequence $\{u_n\}_{n\geq 1}$ we find that

$$\begin{aligned}
&\liminf_{n\to\infty} \langle DF(u_n), v - u_n \rangle \\
&\quad = \liminf_{n\to\infty} (\langle DF(u_n), v - u_n \rangle + \Phi^0(ju_n; jv - ju_n) - \Phi^0(ju_n; jv - ju_n)) \\
&\quad \geq \liminf_{n\to\infty} (\langle DF(u_n), v - u_n \rangle + \Phi^0(ju_n; jv - ju_n)) + \liminf_{n\to\infty} (-\Phi^0(ju_n; jv - ju_n)) \\
&\quad \geq -\limsup_{n\to\infty} \Phi^0(ju_n; jv - ju_n) \geq -\Phi^0(ju; jv - ju), \qquad \forall v \in X.
\end{aligned}$$

Thus we are led to

$$\liminf_{n\to\infty} \langle DF(u_n), v - u_n \rangle + \Phi^0(ju; jv - ju) \geq 0, \qquad \forall v \in X,$$

which is just property (S_3) in Definition 1.1. Therefore $u \in X$ is a generalized solution to problem (1.1). The proof is complete. □

We illustrate the applicability of Theorem 3.2 with verifiable growth conditions.

Corollary 3.3. (i) *Assume that the Gâteaux differentiable, continuous function $F : X \to \mathbb{R}$ and the locally Lipschitz function $\Phi : Y \to \mathbb{R}$ satisfy*

$$F(v) \geq a\|v\|^r - a_0 \quad \textit{for all } v \in X, \tag{3.16}$$

with constants $a > 0$, $a_0 > 0$, $r > 0$, and

$$\Phi(w) \geq -b\|w\|^\sigma - b_0 \quad \textit{for all } w \in Y, \tag{3.17}$$

with constants $b > 0$, $b_0 > 0$ and $\sigma \in (0, r)$. Then condition $(H2)$ holds true.

(ii) *Assume that the Gâteaux differentiable, continuous function $F : X \to \mathbb{R}$, the linear compact map $j : X \to Y$ and the locally Lipschitz function $\Phi : Y \to \mathbb{R}$ satisfy*

$$\langle DF(v), v \rangle \geq \tilde{a}\|v\|^{\tilde{r}} - \tilde{a}_0 \quad \textit{for all } v \in X, \tag{3.18}$$

with constants $\tilde{a} > 0$, $\tilde{a}_0 > 0$, $\tilde{r} > 0$, and

$$\langle \xi, jv \rangle \geq -\tilde{b}\|jv\|^{\tilde{\sigma}} - \tilde{b}_0 \quad \textit{for all } v \in X \textit{ and } \xi \in \partial\Phi(jv), \tag{3.19}$$

with constants $\tilde{b} > 0$, $\tilde{b}_0 > 0$ and $\tilde{\sigma} \in (0, \tilde{r})$. Then condition $(H3)$ is fulfilled.

Proof. (i) From (3.16) and (3.17), we estimate the functional I in (1.2) from below

$$I(v) = F(v) + \Phi(jv) \geq a\|v\|^r - a_0 - b\|j\|^\sigma \|v\|^\sigma - b_0$$

for all $v \in X$. Since $r > \sigma$, we infer that (3.1) holds true. Moreover, the preceding estimate entails

$$I \text{ is coercive on } X, \text{ i.e., } I(u) \to +\infty \text{ as } \|u\| \to \infty,$$

which ensures that condition $(H2)$ is verified.

(ii) We are going to show that the set

$$X_0 := \{v \in X : \langle DF(v), v\rangle \leq \Phi^0(jv; -jv)\}$$

is bounded in X. On the basis of (3.18) and (3.19), for every $v \in X_0$ we obtain

$$\begin{aligned}\tilde{a}\|v\|^{\tilde{r}} - \tilde{a}_0 &\leq \langle DF(v), v\rangle \leq \Phi^0(jv; -jv) = \max\{\langle \xi, -jv\rangle : \xi \in \partial\Phi(jv)\} \\ &= -\min\{\langle \xi, jv\rangle : \xi \in \partial\Phi(jv)\} \leq \tilde{b}\|jv\|^{\tilde{\sigma}} + \tilde{b}_0 \leq \tilde{b}\|j\|^{\tilde{\sigma}}\|v\|^{\tilde{\sigma}} + \tilde{b}_0.\end{aligned}$$

Taking into account that $\tilde{\sigma} < \tilde{r}$, the boundedness of the set X_0 in X follows. □

Remark 3.4. Conditions (3.16), (3.17), (3.18) and (3.19) are compatible offering a large range of applicability for Theorem 3.2.

4 Local boundary value problems without ellipticity

In this section we focus on the boundary value inclusion with discontinuities (1.3), which extends the Dirichlet equation (1.6). For $1 < q < p < +\infty$ and $\mu \in \mathbb{R}$, we shall show that problem (1.3), so a fortiori (1.6), is a special case of problem (1.1) treated in Section 3. The principal point is that the leading operator $-\Delta_p + \mu\Delta_q$ exhibits a competing (p, q)-Laplacian if μ is positive, thus the ellipticity fails.

We assume to be fulfilled:

$(H)_f$ the function $f : \mathbb{R} \to \mathbb{R}$ is measurable and there exist constants $c > 0$ and $\sigma \in (1, p)$ such that

$$|f(t)| \leq c(1 + |t|^{\sigma-1}) \quad \text{for a.e. } t \in \mathbb{R}.$$

From assumption $(H)_f$ it follows that $f \in L^\infty_{\text{loc}}(\mathbb{R})$, hence the functions $\underline{f} : \mathbb{R} \to \mathbb{R}$ and $\overline{f} : \mathbb{R} \to \mathbb{R}$ introduced in (1.4) and (1.5), respectively, are well-defined.

The notion of generalized solution to problem (1.1) introduced in Definition 1.1 reads in the case of (1.3) as follows: $u \in W_0^{1,p}(\Omega)$ is a generalized solution to (1.3) if there exists a sequence $\{u_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ such that

(S_1') $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$;

(S_2')

$$\limsup_{n\to\infty} \left[\frac{1}{p}\|\nabla u_n\|_p^p - \frac{\mu}{q}\|\nabla u_n\|_q^q\right] \leq \frac{1}{p}\|\nabla u\|_p^p - \frac{\mu}{q}\|\nabla u\|_q^q; \tag{4.1}$$

(S_3') $\liminf_{n\to\infty} \langle -\Delta_p u_n + \mu\Delta_q u_n, \varphi\rangle \geq \int_\Omega \min\{\underline{f}(u(x))\varphi(x), \overline{f}(u(x))\varphi(x)\}\, dx, \qquad \forall \varphi \in W_0^{1,p}(\Omega).$

Passing from (S_3) in Definition 1.1 to formulation (S_3') is based on the Aubin–Clarke Theorem for an integral functional (see [4, Theorem 2.7.5]).

Remark 4.1. If f is continuous, then the interval $[\underline{f}(u(x)), \overline{f}(u(x))]$ reduces to the singleton $f(u(x))$ and (S_3') becomes

$(\tilde{S}_3')$ $-\Delta_p u_n + \mu\Delta_q u_n \rightharpoonup f(u)$ in $W^{-1,p'}(\Omega)$, i.e.,

$$\lim_{n\to\infty}\langle -\Delta_p u_n + \mu\Delta_q u_n, \varphi\rangle = \int_\Omega f(u(x))\varphi(x)\,\mathrm{d}x, \qquad \forall \varphi \in W_0^{1,p}(\Omega).$$

Indeed, (S_3') entails

$$\liminf_{n\to\infty}\langle -\Delta_p u_n + \mu\Delta_q u_n, \varphi\rangle \geq \int_\Omega f(u(x))\varphi(x)\,\mathrm{d}x, \qquad \forall \varphi \in W_0^{1,p}(\Omega).$$

Changing φ into $-\varphi$ produces

$$\limsup_{n\to\infty}\langle -\Delta_p u_n + \mu\Delta_q u_n, \varphi\rangle \leq \int_\Omega f(u(x))\varphi(x)\,\mathrm{d}x, \qquad \forall \varphi \in W_0^{1,p}(\Omega),$$

whence the result.

If $q = 2 < p < +\infty$, from (S_1') and the linearity of the Laplacian Δ we deduce that $(\tilde{S}_3')$ requires $-\Delta_p u_n \rightharpoonup -\mu\Delta u + f(u)$ in $W^{-1,p'}(\Omega)$.

Now we state our result on problems (1.3) and (1.6).

Theorem 4.2. *Assume that condition $(H)_f$ holds. Then, for every $\mu \in \mathbb{R}$, problem (1.3) admits at least one generalized solution. Every generalized solution is a weak solution provided $\mu \leq 0$. In particular, problem (1.6) with f continuous possesses at least a generalized solution, which is a weak solution when $\mu \leq 0$.*

Proof. Our goal is to apply Theorem 3.2 by means of Corollary 3.3. To this end we choose $X = W_0^{1,p}(\Omega)$, which is a separable and reflexive Banach space. Further, we take $Y = L^p(\Omega)$ and let $j : W_0^{1,p}(\Omega) \to L^p(\Omega)$ be the inclusion map. By Rellich–Kondrachov Theorem j is compact. Therefore assumption $(H1)$ is satisfied.

With a fixed $\mu \in \mathbb{R}$, define the functional $F : W_0^{1,p}(\Omega) \to \mathbb{R}$ as

$$F(v) = \frac{1}{p}\|\nabla v\|_p^p - \frac{\mu}{q}\|\nabla v\|_q^q \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

It is clear that $F : W_0^{1,p}(\Omega) \to \mathbb{R}$ is continuously differentiable, so Gâteaux differentiable and continuous. By (2.3), Young's inequality and $p > q$, we infer that

$$F(v) \geq \frac{1}{p}\|\nabla v\|_p^p - \frac{|\mu|k}{q}\|\nabla v\|_p^q \geq \frac{1}{2p}\|\nabla v\|_p^p - a_0 \quad \text{for all } v \in W_0^{1,p}(\Omega),$$

with a constant $a_0 > 0$. Hence condition (3.16) is verified with $r = p$.

Next we consider the function $g : \mathbb{R} \to \mathbb{R}$ in (2.1) corresponding to $f : \mathbb{R} \to \mathbb{R}$ in the right-hand side of (1.3). Thanks to assumption $(H)_f$, $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and in turn the functional $\Phi : L^p(\Omega) \to \mathbb{R}$ given by

$$\Phi(v) = -\int_\Omega g(v(x))\,\mathrm{d}x \quad \text{for all } v \in L^p(\Omega) \tag{4.2}$$

is locally Lipschitz. Precisely, Φ is Lipschitz continuous on the bounded subsets of $L^\sigma(\Omega)$ and we use the continuous embedding $L^p(\Omega) \hookrightarrow L^\sigma(\Omega)$ with $\sigma < p$.

Hypothesis $(H)_f$ implies

$$\begin{aligned}|\Phi(v)| &\le \int_\Omega |g(v(x))|\,dx \le c\|v\|_1 + \frac{c}{\sigma}\|v\|_\sigma^\sigma \le c|\Omega|^{\frac{1}{\sigma'}}\|v\|_\sigma + \frac{c}{\sigma}\|v\|_p^\sigma \\ &\le c_0(1 + \|v\|_\sigma^\sigma), \qquad \forall v \in L^p(\Omega),\end{aligned}$$

with a constant $c_0 > 0$ and $\sigma' = \sigma/(\sigma - 1)$. We derive (3.17) due to the continuous embedding $L^p(\Omega) \hookrightarrow L^\sigma(\Omega)$. By Corollary 3.3 part (i), condition $(H2)$ is fulfilled.

We note that

$$\langle DF(v), v\rangle = \|\nabla v\|_p^p - \mu\|\nabla v\|_q^q \quad \text{for all } v \in X,$$

so condition (3.18) is satisfied with $\tilde{r} = p$ because $p > q$. Pick any $v \in W_0^{1,p}(\Omega)$ and $\xi \in \partial\Phi(jv)$, with Φ in (4.2). The Aubin–Clarke Theorem (see [4, Theorem 2.7.5]) and (2.2) guarantee that $\xi \in L^{p'}(\Omega)$ and

$$-\xi(x) \in \partial g(v(x)) = [\underline{f}(v(x)), \overline{f}(v(x))] \quad \text{for a.e. } x \in \Omega. \tag{4.3}$$

Then by (4.2), $(H)_f$, (4.3) (see also (1.4), (1.5)) and the continuous embedding $L^p(\Omega) \hookrightarrow L^\sigma(\Omega)$, we infer that

$$\begin{aligned}\langle \xi, jv\rangle &= \int_\Omega \xi(x) jv(x) dx \ge -\int_\Omega |\xi(x)||jv(x)| dx \\ &\ge -\int_\Omega c(1 + |jv(x)|^{\sigma-1})|jv(x)| dx \\ &\ge -\tilde{b}\|jv\|^\sigma - \tilde{b}_0 \quad \text{for all } v \in W_0^{1,p}(\Omega) \text{ and } \xi \in \partial\Phi(jv),\end{aligned}$$

with constants $\tilde{b} > 0$ and $\tilde{b}_0 > 0$. This confirms the validity of (3.19) with $\tilde{\sigma} = \sigma$. From Corollary 3.3 part (ii), assumption $(H3)$ holds true.

We are in a position to apply Theorem 3.2, which ensures the existence of a generalized solution to problem (1.3) in the sense of Definition 1.1. Specifically, we find $u \in W_0^{1,p}(\Omega)$ and a sequence $\{u_n\}_{n\ge1} \subset W_0^{1,p}(\Omega)$ satisfying (S_1'), (S_2') and

$$\liminf_{n\to\infty} \langle -\Delta_p u_n + \mu\Delta_q u_n, \varphi\rangle + \Phi^0(u;\varphi) \ge 0, \qquad \forall \varphi \in W_0^{1,p}(\Omega), \tag{4.4}$$

with Φ in (4.2). By the Aubin–Clarke Theorem applied to Φ in (4.2), $(H)_f$ and (2.2), we find

$$\begin{aligned}\Phi^0(u;\varphi) &\le \int_\Omega \max[-\partial g(u(x))\varphi(x)]\,dx \\ &= -\int_\Omega \min\{\underline{f}(u(x))\varphi(x), \overline{f}(u(x))\varphi(x)\}\,dx, \qquad \forall \varphi \in W_0^{1,p}(\Omega).\end{aligned} \tag{4.5}$$

At this point it is enough to insert (4.5) in (4.4) to get that (S_3') holds, which proves the first part of Theorem 4.2.

Suppose that $u \in W_0^{1,p}(\Omega)$ is a generalized solution to problem (1.3) with $\mu \le 0$. We note from property (ii) in Definition 1.1 that

$$\limsup_{n\to\infty} \left[\frac{1}{p}\|\nabla u_n\|_p^p - \frac{\mu}{q}\|\nabla u_n\|_q^q\right] \le \frac{1}{p}\|\nabla u\|_p^p - \frac{\mu}{q}\|\nabla u\|_q^q; .$$

On the other hand, using the weak lower semicontinuity of the norm in conjunction with $\mu \le 0$ and (i) of Definition 1.1, it turns out

$$\begin{aligned}\limsup_{n\to\infty}\left[\frac{1}{p}\|\nabla u_n\|_p^p - \frac{\mu}{q}\|\nabla u_n\|_q^q\right] &\ge \frac{1}{p}\limsup_{n\to\infty}\|\nabla u_n\|_p^p - \frac{\mu}{q}\liminf_{n\to\infty}\|\nabla u_n\|_p^p \\ &\ge \frac{1}{p}\limsup_{n\to\infty}\|\nabla u_n\|_p^p - \frac{\mu}{q}\|\nabla u\|_q^q.\end{aligned}$$

By a simple comparison we are led to

$$\limsup_{n\to\infty}\|\nabla u_n\|_p \le \|\nabla u\|_p,$$

which implies the strong convergence $u_n \to u$ in $W_0^{1,p}(\Omega)$ because the space $W_0^{1,p}(\Omega)$ is uniformly convex (see, e.g., [2, Proposition 3.32]). On the basis of the strong convergence $u_n \to u$, we can utilize the continuity of $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ and $-\Delta_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ with $q < p$, to pass to the limit in (S_3') obtaining (2.4). This amounts to saying that u is a weak solution of (1.3). Since (2.6) is a particular case of (2.4), the proof is complete. □

5 Nonlocal boundary value problems without ellipticity

This section deals with the nonlocal boundary value problem with discontinuities (1.7) and its particular case (1.8) under the conditions $0 < s < 1$, $1 < q < p < +\infty$, $sq < N$ and $\mu \in \mathbb{R}$, thus allowing that the local operator $-\Delta_p$ and the nonlocal operator $(-\Delta)_q^s$ be competing.

The function $f : \mathbb{R} \to \mathbb{R}$ in the right-hand side of (1.7) and (1.8) is required to satisfy condition $(H)_f$ in Section 4. Subsequently, we use the notation in Section 2, in particular the associated functions $\underline{f} : \mathbb{R} \to \mathbb{R}$ and $\overline{f} : \mathbb{R} \to \mathbb{R}$ have the meaning in (1.4) and (1.5), respectively.

We rely on the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow W_0^{s,q}(\Omega)$. As in (2.7), there is a constant $C > 0$ such that

$$\|u\|_{W_0^{s,q}(\Omega)} \le C\|\nabla u\|_p, \qquad \forall u \in W_0^{1,p}(\Omega) \tag{5.1}$$

making the sum $-\Delta_p u + \mu(-\Delta)_q^s u$ well defined for $u \in W_0^{1,p}(\Omega)$ in problems (1.7) and (1.8).

In accordance with Definition 1.1, by a generalized solution to nonlocal problem (1.7) we mean any $u \in W_0^{1,p}(\Omega)$ for which one can find a sequence $\{u_n\}_{n\ge1} \subset W_0^{1,p}(\Omega)$ satisfying

(S_1'') $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$;

(S_2'')
$$\limsup_{n\to\infty}\left[\frac{1}{p}\|\nabla u_n\|_p^p + \frac{\mu}{q}\|u_n\|_{W_0^{s,q}(\Omega)}^q\right] \le \frac{1}{p}\|\nabla u\|_p^p + \frac{\mu}{q}\|u\|_{W_0^{s,q}(\Omega)}^q; \tag{5.2}$$

(S_3'')
$$\begin{aligned}&\liminf_{n\to\infty}\langle -\Delta_p(u_n) + \mu(-\Delta)_q^s(u_n), \varphi\rangle \\ &\qquad \ge \int_\Omega \min\{\underline{f}(u(x))\varphi(x), \overline{f}(u(x))\varphi(x)\}\,\mathrm{d}x, \qquad \forall \varphi \in W_0^{1,p}(\Omega).\end{aligned}$$

Here (S_3'') is obtained from (S_3) in Definition 1.1 by applying the Aubin–Clarke Theorem (see [4, Theorem 2.7.5]).

Our result on the nonlocal problems (1.7) and (1.8) is as follows.

Theorem 5.1. *Assume that condition $(H)_f$ holds. Then, for every $\mu \in \mathbb{R}$, problem (1.7) admits at least one generalized solution, which is a weak solution provided $\mu \geq 0$. In particular, this is valid for problem (1.8) with f continuous.*

Proof. In order to address Theorem 3.2 and Corollary 3.3, we choose: $X = W_0^{1,p}(\Omega)$, $Y = L^p(\Omega)$ and $j : W_0^{1,p}(\Omega) \to L^p(\Omega)$ be the inclusion map, which is compact. Consequently, assumption $(H1)$ is verified.

For a fixed $\mu \in \mathbb{R}$, we introduce the functional $F : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$F(v) = \frac{1}{p}\|\nabla u\|_p^p + \frac{\mu}{q}\|u\|^q_{W_0^{s,q}(\Omega)} \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

This is possible thanks to (5.1). Using (2.8), it is seen that F is continuously differentiable with the differential

$$\langle DF(u), v\rangle = \langle -\Delta_p(u_n) + \mu(-\Delta)_q^s(u_n), v\rangle, \qquad \forall u, v \in W_0^{1,p}(\Omega).$$

By (5.1), Young's inequality and $p > q$, we find the estimate

$$F(v) \geq \frac{1}{p}\|\nabla v\|_p^p - \frac{|\mu|}{q}\|v\|^q_{W_0^{s,q}(\Omega)} \geq \frac{1}{2p}\|\nabla v\|_p^p - a_0 \quad \text{for all } v \in W_0^{1,p}(\Omega),$$

with a constant $a_0 > 0$. Condition (3.16) is thus verified with $r = p$.

Consider the function $\Phi : L^p(\Omega) \to \mathbb{R}$ introduced in (4.2). Taking into account $(H)_f$, condition (3.19) was already checked in the proof of Theorem 4.2. Gathering (3.16) and (3.19), we are able to refer to Corollary 3.3, which yields that Theorem 3.2 can be applied. A reasoning similar to the one in the proof of Theorem 4.2 enables us to conclude that there exists a generalized solution to problem (1.7) and thus (1.8).

The last step in the proof is to show that any generalized solution of problems (1.7) and (1.8) is a weak solution provided $\mu \geq 0$. We argue on the basis of assertion (S_2'') in the definition of generalized solution. Given a generalized solution $u \in W_0^{1,p}(\Omega)$ of problem (1.7) with $\mu \geq 0$, we compare inequality (5.2) in the definition of generalized solution and the following inequality derived from weak lower semicontinuity of the norm (note $(S_1''))$

$$\begin{aligned}\limsup_{n\to\infty}\left[\frac{1}{p}\|\nabla u_n\|_p^p + \frac{\mu}{q}\|u_n\|^q_{W_0^{s,q}(\Omega)}\right] &\geq \frac{1}{p}\limsup_{n\to\infty}\|\nabla u_n\|_p^p + \frac{\mu}{q}\liminf_{n\to\infty}\|u_n\|^q_{W_0^{s,q}(\Omega)} \\ &\geq \frac{1}{p}\limsup_{n\to\infty}\|\nabla u_n\|_p^p + \frac{\mu}{q}\|u\|^q_{W_0^{s,q}(\Omega)}\end{aligned}$$

to deduce that

$$\limsup_{n\to\infty}\|\nabla u_n\|_p \leq \|\nabla u\|_p.$$

In view of the uniform convexity of the space $W_0^{1,p}(\Omega)$, property (S_1'') entitles the strong convergence $u_n \to u$ in $W_0^{1,p}(\Omega)$. From here and (S_3''), through the continuity of $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ and $(-\Delta)_q^s : W_0^{s,q}(\Omega) \to W^{-s,q'}(\Omega)$, we reach in the limit (2.9). Therefore u is a weak solution to nonlocal problem (1.7). If f is continuous, we get (2.10), which completes the proof. □

Remark 5.2. As established in [9], one always has $W_0^{s,p}(\Omega) \not\subset W_0^{s,q}(\Omega)$. For this reason we cannot replace $-\Delta_p$ by the nonlocal operator $(-\Delta)_p^s$ in problems (1.7) and (1.8).

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Antiprincipal solutions at infinity for symplectic systems on time scales

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Abstract. In this paper we introduce a new concept of antiprincipal solutions at infinity for symplectic systems on time scales. This concept complements the earlier notion of principal solutions at infinity for these systems by the second author and Šepitka (2016). We derive main properties of antiprincipal solutions at infinity, including their existence for all ranks in a given range and a construction from a certain minimal antiprincipal solution at infinity. We apply our new theory of antiprincipal solutions at infinity in the study of principal solutions, and in particular in the Reid construction of the minimal principal solution at infinity. In this work we do not assume any normality condition on the system, and we unify and extend to arbitrary time scales the theory of antiprincipal solutions at infinity of linear Hamiltonian differential systems and the theory of dominant solutions at infinity of symplectic difference systems.

Keywords: symplectic system on time scale, antiprincipal solution at infinity, principal solution at infinity, nonoscillation, linear Hamiltonian system, normality.

2020 Mathematics Subject Classification: 34N05, 34C10, 39A12, 39A21.

1 Introduction

In this paper we focus on symplectic dynamic system

$$x^\Delta = \mathcal{A}(t)x + \mathcal{B}(t)u; \quad u^\Delta = \mathcal{C}(t)x + \mathcal{D}(t)u, \quad t \in [a,\infty)_\mathbb{T}, \tag{S}$$

where $\mathbb{T}$ is a time scale, that is, $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. We assume that $\mathbb{T}$ is unbounded from above and bounded from below with $a := \min \mathbb{T}$ and $[a,\infty)_\mathbb{T} := [a,\infty) \cap \mathbb{T}$. The coefficients $\mathcal{A}(t)$, $\mathcal{B}(t)$, $\mathcal{C}(t)$, $\mathcal{D}(t)$ of system (S) are real piecewise rd-continuous $n \times n$ matrices on $[a,\infty)_\mathbb{T}$ such that the $2n \times 2n$ matrices

$$\mathcal{S}(t) := \begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) & \mathcal{D}(t) \end{pmatrix}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

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satisfy the identity

$$\mathcal{S}^T(t)\,\mathcal{J} + \mathcal{J}\mathcal{S}(t) + \mu(t)\,\mathcal{S}^T(t)\,\mathcal{J}\mathcal{S}(t) = 0, \quad t \in [a,\infty)_{\mathbb{T}},$$

where $\mu(t)$ is the graininess function of $\mathbb{T}$. Solutions of (S) are piecewise rd-continuously Δ-differentiable functions, i.e., they are continuous on $[a,\infty)_{\mathbb{T}}$ and their Δ-derivative is piecewise rd-continuous on $[a,\infty)_{\mathbb{T}}$. Basic theory of dynamic equations on time scales, including the theory of symplectic dynamic systems, are covered for example in the monographs [6,7]. Advanced topics about symplectic systems on time scales, such as the theory of Riccati matrix dynamic equations, quadratic functionals, oscillation theorems, Rayleigh principle and their applications e.g. in the optimal control theory can be found in the references [1,11,13,15–17,28–30]. Our particular interest is connected with the theory of principal and antiprincipal solutions of (S) at infinity, which was initiated by Došlý in [9] for system (S) satisfying a certain eventual normality or controllability assumption. In 2016 the second author and Šepitka provided in [22] a generalization of the concept of the principal solution at infinity to a possibly abnormal (or uncontrollable) system (S), see also [18–21].

In the present paper we continue in this investigation by introducing the corresponding theory of antiprincipal solutions of (S) at infinity in the absence of the eventual normality or controllability assumption (Definition 4.1). Note that these solutions are also called nonprincipal solutions at infinity in the context of the reference [9], or dominant solutions at infinity in the context of the references [2,10,24,25]. We present three sets of results about the antiprincipal solutions of (S) at infinity. The first set of results is devoted to their basic properties, such as the invariance with respect to the considered interval (Theorem 4.3), a characterization in terms of the limit of the associated S-matrix (Theorem 4.4), and the invariance with respect to a certain relation between conjoined bases (Theorems 4.6 and 4.7). The second set of results is devoted to the existence of antiprincipal solutions of (S) at infinity (Theorem 5.3), which requires to derive as main tools an important characterization of minimal conjoined bases of (S) on a given interval (Theorem 5.1) and a characterization of the T-matrices associated with conjoined bases of (S) (Theorem 5.2). The third set of results is devoted to applications of antiprincipal solutions of (S) at infinity, in particular in the connection with the so-called minimal antiprincipal solutions of (S) at infinity (Theorems 6.3, 6.4, and 6.6) and maximal antiprincipal solutions of (S) at infinity (Theorems 6.5 and 6.6). These are, respectively, the antiprincipal solutions of (S) at infinity with the smallest and the largest possible rank (see Section 4).

The main condition on system (S) is the assumption of its nonoscillation, i.e., every conjoined basis (X,U) of (S) is assumed to be nonoscillatory. This means that for every (X,U) there exists a point $\alpha \in [a,\infty)_{\mathbb{T}}$ such that (X,U) has no focal points in the real interval (α,∞), which is according to [14, Definition 4.1] formulated as

$$\operatorname{Ker} X(s) \subseteq \operatorname{Ker} X(t) \quad \text{for all } t,s \in [\alpha,\infty)_{\mathbb{T}} \text{ with } t \le s, \tag{1.1}$$

$$X(t)\,[X^\sigma(t)]^\dagger\,\mathcal{B}(t) \ge 0 \quad \text{for all } t \in [\alpha,\infty)_{\mathbb{T}}. \tag{1.2}$$

Condition (1.1) means that the kernel of $X(t)$ is nonincreasing on the time scale interval $[\alpha,\infty)_{\mathbb{T}}$. Hence, the point α can be chosen large enough, so that the set $\operatorname{Ker} X(t)$ is constant on $[\alpha,\infty)_{\mathbb{T}}$. Noninvertible matrix functions, such as $X(t)$ above or $S(t)$ defined in (3.1) below, then naturally occur in our theory. For this reason we utilize the Moore–Penrose pseudoinverse matrices as the principal tool for their investigation (see Remark 2.1).

The theory of antiprincipal solutions at infinity for linear Hamiltonian differential systems and the theory of dominant solutions at infinity for symplectic difference systems were developed in [20,23] and [10,24], respectively. The present work is not a mere unification of those, however. Working on arbitrary time scales we also provide a clarification of incomplete or missing arguments in several results compared with the corresponding original continuous time or discrete time statements (see the proofs of Proposition 3.18 and Theorems 3.4, 5.1, and 6.5). This paper together with [22] can be regarded as a starting point for a unified Sturmian theory for Hamiltonian and symplectic dynamic systems on time scales, whose first steps were taken in [5,9] about twenty years ago. Recent progress in the continuous and discrete Sturmian theory, where antiprincipal solutions (or dominant solutions) at infinity play a fundamental role, is documented in the papers [25–27]. We strongly believe that future development in this unified Sturmian theory will benefit from the results obtained in the presented work (see also Section 7).

The paper is organized as follows. In Section 2 we briefly recall some results from matrix analysis, which we directly use later in this paper. In Section 3 we provide basic results about symplectic systems on time scales, which form the base for the definition of an antiprincipal solution of ($\mathbb{S}$) at infinity. In Section 4 we introduce the notion of an antiprincipal solution at infinity for system ($\mathbb{S}$) and include its main properties, which are connected to the relation being contained for conjoined bases of ($\mathbb{S}$). In Section 5 we derive the existence of antiprincipal solutions at infinity for a nonoscillatory system ($\mathbb{S}$), including the existence of antiprincipal solutions at infinity with arbitrary given rank and pointing out the essential role played by the minimal antiprincipal solutions of ($\mathbb{S}$) at infinity. In Section 6 we focus on applications of the presented theory of antiprincipal solutions at infinity, in particular in the theory of principal solutions of ($\mathbb{S}$) and in the Reid construction of the minimal principal solution of ($\mathbb{S}$) at infinity. Finally, in Section 7 we comment about the results of this paper in the context of some open problems.

2 Notation and matrix analysis

In this section we introduce basic notation and recall some properties of the Moore–Penrose pseudoinverse matrices, which we will use later. For a real matrix M we denote by $\operatorname{Im} M$, $\operatorname{Ker} M$, $\operatorname{rank} M$, M^T, M^{-1}, $M^{\dagger}$ the image, kernel, rank (i.e., the dimension of the image), transpose, inverse (if M is a square invertible matrix), and the Moore–Penrose pseudoinverse of M (see its definition below), respectively. For a symmetric matrix $M \in \mathbb{R}^{n\times n}$ we write $M \geq 0$ or $M > 0$ if M is positive semidefinite or positive definite, respectively. If M_1 and M_2 are two real symmetric $n \times n$ matrices, then we write $M_1 \leq M_2$ when $M_2 - M_1 \geq 0$, respectively we write $M_1 < M_2$ when $M_2 - M_1 > 0$. The identity matrix will be denoted by I.

Furthermore, let V and W be linear subspaces in $\mathbb{R}^n$. We denote by $V \oplus W$ the direct sum of the subspaces V and W, and by $V^{\perp}$ the orthogonal complement of the subspace V in $\mathbb{R}^n$. By $\mathcal{P}_V$ we denote the orthogonal projector onto the subspace V. Then the $n \times n$ matrix $\mathcal{P}_V$ is symmetric, idempotent, and positive semidefinite.

In our approach it is essential to use the properties of the Moore–Penrose pseudoinverse. First we recall its definition via the following four properties, which will often be used in our calculations. Let M be a real $m \times n$ matrix. A real $n \times m$ matrix $M^{\dagger}$ satisfying the equalities

$$MM^{\dagger}M = M, \quad M^{\dagger}MM^{\dagger} = M^{\dagger}, \quad M^{\dagger}M = (M^{\dagger}M)^T, \quad MM^{\dagger} = (MM^{\dagger})^T \tag{2.1}$$

is called the Moore–Penrose pseudoinverse of the matrix M. We will use the following properties of the Moore–Penrose pseudoinverse, which can be found e.g. in [3,4,8] and [14, Lemma 2.1]. These properties play an essential role in our theory.

Remark 2.1. For any matrix $M \in \mathbb{R}^{m\times n}$ there exists a unique matrix $M^\dagger \in \mathbb{R}^{n\times m}$ satisfying the identities in (2.1). Moreover, the following properties hold.

(i) $(M^\dagger)^T = (M^T)^\dagger$, $(M^\dagger)^\dagger = M$, and $\operatorname{Im} M^\dagger = \operatorname{Im} M^T$, $\operatorname{Ker} M^\dagger = \operatorname{Ker} M^T$.

(ii) The matrix $MM^\dagger$ is the orthogonal projector onto $\operatorname{Im} M$, and the matrix $M^\dagger M$ is the orthogonal projector onto $\operatorname{Im} M^T$.

(iii) Let $\{M_j\}_{j=1}^\infty$ be a sequence of $m \times n$ matrices such that $M_j \to M$ for $j \to \infty$. Then the limit of $M_j^\dagger$ for $j \to \infty$ exists if and only if there exists an index $j_0 \in \mathbb{N}$ such that $\operatorname{rank} M_j = \operatorname{rank} M$ for all $j \geq j_0$. In this case $\lim_{j\to\infty} M_j^\dagger = M^\dagger$.

(iv) Let $M(t)$ be an $m \times n$ matrix function defined on the interval $[a,\infty)_\mathbb{T}$ such that $M(t) \to M$ for $t \to \infty$. Then the limit of $M^\dagger(t)$ for $t \to \infty$ exists if and only if there exists a point $t_0 \in [a,\infty)_\mathbb{T}$ such that $\operatorname{rank} M(t) = \operatorname{rank} M$ for all $t \in [t_0,\infty)_\mathbb{T}$. In this case $\lim_{t\to\infty} M^\dagger(t) = M^\dagger$.

(v) Let M_1 and M_2 be symmetric and positive semidefinite matrices such that $M_1 \leq M_2$. Then inequality $M_2^\dagger \leq M_1^\dagger$ holds if and only if $\operatorname{Im} M_1 = \operatorname{Im} M_2$, or equivalently if and only if $\operatorname{rank} M_1 = \operatorname{rank} M_2$.

(vi) If M is symmetric positive and semidefinite, then also $M^\dagger$ is symmetric and positive semidefinite. That is, if $M \geq 0$, then also $M^\dagger \geq 0$.

(vii) For any matrices M and N with suitable dimensions, the pseudoinverse of their product is given by

$$(MN)^\dagger = (\mathcal{P}_{\operatorname{Im} M^T} N)^\dagger \, (M\,\mathcal{P}_{\operatorname{Im} N})^\dagger = (M^\dagger MN)^\dagger \, (MNN^\dagger)^\dagger. \tag{2.2}$$

(viii) Let $M(t)$ be an $m \times n$ piecewise rd-continuously Δ-differentiable matrix function defined on $[a,\infty)_\mathbb{T}$ such that the kernel of $M(t)$ is constant on $[a,\infty)_\mathbb{T}$. Then the matrix function $M^\dagger(t)$ is also piecewise rd-continuously Δ-differentiable on $[a,\infty)_\mathbb{T}$ and

$$[M^\dagger(t)]^\Delta M(t) = -[M^\dagger(t)]^\sigma \, M^\Delta(t) = -[M^\sigma(t)]^\dagger \, M^\Delta(t), \quad t \in [a,\infty)_\mathbb{T}. \tag{2.3}$$

The following proposition covers a special property of orthogonal projectors, which we will use later, see the proof of Theorem 5.2 and [19, Theorem 9.2] for details.

Proposition 2.2. *Let $P_*, P, \tilde{P} \in \mathbb{R}^{n\times n}$ be arbitrary orthogonal projectors satisfying*

$$\operatorname{Im} P_* \subseteq \operatorname{Im} P, \quad \operatorname{Im} P_* \subseteq \operatorname{Im} \tilde{P}, \quad \operatorname{rank} P = \operatorname{rank} \tilde{P}.$$

Then there exists an invertible matrix $E \in \mathbb{R}^{n\times n}$ such that $EP_* = P_*$ and $\operatorname{Im} EP = \operatorname{Im} \tilde{P}$.

According to Remark 2.1(ii), the Moore–Penrose pseudoinverse can be conveniently used for the construction of the orthogonal projectors onto the image of $X^T(t)$ or onto the image of $X(t)$ of a matrix function $X : [a,\infty)_\mathbb{T} \to \mathbb{R}^{n\times n}$. In particular, the following two orthogonal projectors play important role in our theory. Define

$$P(t) := \mathcal{P}_{\operatorname{Im} X^T(t)} = X^\dagger(t)\, X(t), \quad R(t) := \mathcal{P}_{\operatorname{Im} X(t)} = X(t)\, X^\dagger(t), \quad t \in [a,\infty)_\mathbb{T}. \tag{2.4}$$

Note that from the defining properties of Moore–Penrose pseudoinverse in (2.1) we get

$$P(t)\, X^\dagger(t) = X^\dagger(t), \quad X^\dagger(t)\, R(t) = X^\dagger(t), \quad t \in [a,\infty)_\mathbb{T}. \tag{2.5}$$

Remark 2.3. We will often work with matrix functions $X : [a,\infty)_{\mathbb{T}} \to \mathbb{R}^{n\times n}$ with constant kernel on some interval $[\alpha,\infty)_{\mathbb{T}}$. In this case the associated orthogonal projector $P(t)$ defined in (2.4) is constant on $[\alpha,\infty)_{\mathbb{T}}$, since $\mathbb{R}^n = [\operatorname{Ker} X(t)]^{\perp} \oplus \operatorname{Ker} X(t)$, where the subspace $[\operatorname{Ker} X(t)]^{\perp} = \operatorname{Im} X^T(t)$ is constant on $[\alpha,\infty)_{\mathbb{T}}$. In this case we denote by P the corresponding constant orthogonal projector in (2.4), i.e., we define

$$P := P(t) \quad \text{for } t \in [\alpha,\infty)_{\mathbb{T}}, \text{ where } \operatorname{Ker} X(t) \text{ is constant.} \tag{2.6}$$

3 Results on symplectic systems on time scales

In this section we collect basic information about symplectic systems on time scales and their conjoined bases. We split this section into three subsections, separating the introductory part and two slightly more advanced (yet still preparatory) parts.

3.1 Basic preparatory results

In this subsection we recall the facts, which need to be understood for the definition of an antiprincipal solution at infinity. The results in this subsection are not new, most of them can be found in [22], where they were presented in a slightly different logical order and, in some cases, with incomplete arguments. In particular, we present full details about the monotonicity of the S-matrices for conjoined bases of (S), which yield a correct definition of the associated T-matrix. The latter matrix is the cornerstone of our investigation of antiprincipal solutions of (S) at infinity.

A solution (X,U) of (S) is a *conjoined basis*, if $X^T(t)\,U(t)$ is a symmetric matrix and $\operatorname{rank}(X^T(t),\,U^T(t))^T = n$ at some and hence at any $t \in [a,\infty)_{\mathbb{T}}$. For any two solutions (X,U) and $(\bar{X},\bar{U})$ of (S) their *Wronskian matrix* $N := X^T(t)\,\bar{U}(t) - U^T(t)\,\bar{X}(t)$ is constant on $[a,\infty)_{\mathbb{T}}$. Two conjoined bases (X,U) and $(\bar{X},\bar{U})$ are called *normalized*, if their constant Wronskian matrix N satisfies $N = I$. A conjoined basis (X,U) of (S) is called *nonoscillatory*, if there exists $\alpha \in [a,\infty)_{\mathbb{T}}$ such that (X,U) has no focal points in the real interval (α,∞), i.e., if conditions (1.1) and (1.2) hold. We say that the system (S) is *nonoscillatory* if every conjoined basis of (S) is nonoscillatory.

Let (X,U) be a conjoined basis of system (S). For simplicity we say that (X,U) has *constant kernel* on an interval $[\alpha,\infty)_{\mathbb{T}}$ if the matrix $X(t)$ has constant kernel on this interval. Similarly, we say that (X,U) has rank r on $[\alpha,\infty)_{\mathbb{T}}$, if the matrix $X(t)$ has rank r on this interval. Note that if the system (S) is nonoscillatory, then the kernel (and hence also the rank) of any of its conjoined bases is eventually constant. If (X,U) is a conjoined basis of (S) with constant kernel on some interval $[\alpha,\infty)_{\mathbb{T}}$, then it is convenient to work with the so-called S-matrix corresponding to (X,U) on $[\alpha,\infty)_{\mathbb{T}}$. It is defined by

$$S(t) := \int_{\alpha}^{t} [X^{\sigma}(s)]^{\dagger}\, \mathcal{B}(s)\, [X^{\dagger}(s)]^T\, \Delta s, \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{3.1}$$

Note that the definition of the matrix $S(t)$ is correct, since according to Remark 2.1(viii) the matrix function $X^{\dagger}$ is piecewise rd-continuously Δ-differentiable on $[\alpha,\infty)_{\mathbb{T}}$, so that $X^{\dagger}$ is continuous on $[\alpha,\infty)_{\mathbb{T}}$ and $(X^{\sigma})^{\dagger} = (X^{\dagger})^{\sigma}$ is rd-continuous on $[\alpha,\infty)_{\mathbb{T}}$. This implies that the function $(X^{\sigma})^{\dagger}\,\mathcal{B}\,(X^{\dagger})^T$ is piecewise rd-continuous (hence Δ-integrable) on $[\alpha,\infty)_{\mathbb{T}}$. Moreover, according to [14, Lemma 3.1] the matrix

$$X(t)\, [X^{\sigma}(t)]^{\dagger}\, \mathcal{B}(t) \text{ is symmetric on } [\alpha,\infty)_{\mathbb{T}}. \tag{3.2}$$

Proposition 3.1. *Let (X, U) be a conjoined basis of the system (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$. Then the corresponding S-matrix given by (3.1) is symmetric.*

Proof. Directly from the definition of $S(t)$ and using the fact that $P(t)$ is constant and hence $P(t) = P = P^{\sigma}(t)$ on $[\alpha, \infty)_{\mathbb{T}}$, we get for $t \in [\alpha, \infty)_{\mathbb{T}}$

$$\begin{aligned} S(t) &= \int_{\alpha}^{t} [X^{\sigma}(s)]^{\dagger}\, \mathcal{B}(s)\, [X^{\dagger}(s)]^{T}\, \Delta s \overset{(2.5)}{=} \int_{\alpha}^{t} P^{\sigma}(s)\, [X^{\sigma}(s)]^{\dagger}\, \mathcal{B}(s)\, [X^{\dagger}(s)]^{T}\, \Delta s \\ &= \int_{\alpha}^{t} P(s)\, [X^{\sigma}(s)]^{\dagger}\, \mathcal{B}(s)\, [X^{\dagger}(s)]^{T}\, \Delta s \overset{(2.4)}{=} \int_{\alpha}^{t} X^{\dagger}(s)\, X(s)\, [X^{\sigma}(s)]^{\dagger}\, \mathcal{B}(s)\, [X^{\dagger}(s)]^{T}\, \Delta s. \end{aligned} \tag{3.3}$$

The latter expression together with (3.2) proves the result. □

Next we present a statement about the relation between $\operatorname{Im} S(t)$ and $\operatorname{Im} P$.

Lemma 3.2. *Let (X, U) be a conjoined basis of the system (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and let the matrices P and $S(t)$ be defined by (2.6) and (3.1). Then*

$$\operatorname{Im} S(t) \subseteq \operatorname{Im} P \quad \textit{for all } t \in [\alpha, \infty)_{\mathbb{T}}. \tag{3.4}$$

Proof. Fix $t \in [\alpha, \infty)_{\mathbb{T}}$ and let $u \in \operatorname{Im} S(t)$. Then there exists $v \in \mathbb{R}^n$ such that $S(t)\, v = u$. From (3.3) we get $S(t) = PS(t)$. Then $u = PS(t)\, v$ and hence $u \in \operatorname{Im} P$. □

Remark 3.3. Let (X, U) be a conjoined basis of (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$. If we use the symmetry of $S(t)$ on $[\alpha, \infty)_{\mathbb{T}}$ and Remark 2.1(v), then the inclusion of the sets in (3.4) from the previous lemma can be equivalently written as

$$PS(t) = S(t) = S(t)\, P \quad \text{or} \quad PS^{\dagger}(t) = S^{\dagger}(t) = S^{\dagger}(t)\, P, \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{3.5}$$

The next theorem is fundamental for the definition of an antiprincipal solution of (S) at infinity and we display its proof with full details. In the theorem the so-called T-matrix corresponding to the conjoined basis (X, U) is introduced.

Theorem 3.4. *Let (X, U) be a conjoined basis of (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) and let the matrix $S(t)$ given by (3.1). Then the limit of $S^{\dagger}(t)$ as $t \to \infty$ exists. Moreover, the matrix T defined by*

$$T := \lim_{t \to \infty} S^{\dagger}(t) \tag{3.6}$$

is symmetric, positive semidefinite, i.e., $T \geq 0$, and there exists $\beta \in [\alpha, \infty)_{\mathbb{T}}$ such that

$$\operatorname{rank} T \leq \operatorname{rank} S(t) \leq \operatorname{rank} X(t) \quad \textit{for all } t \in [\beta, \infty)_{\mathbb{T}}. \tag{3.7}$$

Proof. First we show that the limit of $S^{\dagger}(t)$ exists. According to Proposition 3.1, the matrix $S(t)$ is symmetric. The constant kernel of the conjoined basis (X, U) on $[\alpha, \infty)_{\mathbb{T}}$ guarantees that $P(t) = P$ is constant on $[\alpha, \infty)_{\mathbb{T}}$. Since the conjoined basis (X, U) has no focal points in (α, ∞), we get for $t \in [\alpha, \infty)_{\mathbb{T}}$

$$S^{\Delta}(t) = [X^{\sigma}(t)]^{\dagger}\, \mathcal{B}(t)\, [X^{T}(t)]^{\dagger} \overset{(3.3)}{=} X^{\dagger}(t)\, X(t)\, [X^{\sigma}(t)]^{\dagger}\, \mathcal{B}(t)\, [X^{T}(t)]^{\dagger} \overset{(1.2)}{\geq} 0.$$

This means that matrix $S(t)$ is nondecreasing on $[\alpha, \infty)_{\mathbb{T}}$, i.e.,

$$S(t_1) \leq S(t_2) \quad \text{for all } t_1, t_2 \in [\alpha, \infty)_{\mathbb{T}} \text{ such that } t_1 < t_2. \tag{3.8}$$

Since $S(\alpha) = 0$, we get

$$S(t) \geq 0, \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{3.9}$$

This implies that $\operatorname{Im} S(t)$ is eventually constant, i.e., there exists $\beta \in [\alpha, \infty)_{\mathbb{T}}$ such that

$$\operatorname{Im} S(t_1) = \operatorname{Im} S(t_2) \quad \text{for all } t_1, t_2 \in [\beta, \infty)_{\mathbb{T}}. \tag{3.10}$$

Now we use Remark 2.1(v), where we put $M_1 := S(t_1)$ and $M_2 := S(t_2)$ for $t_1, t_2 \in [\beta, \infty)_{\mathbb{T}}$. The symmetry of $S(t)$ on $[\alpha, \infty)_{\mathbb{T}}$ and conditions (3.8), (3.9) and (3.10) on $[\beta, \infty)_{\mathbb{T}}$ imply that

$$S^{\dagger}(t_2) \leq S^{\dagger}(t_1) \quad \text{for all } t_1, t_2 \in [\beta, \infty)_{\mathbb{T}} \text{ such that } t_1 < t_2, \tag{3.11}$$

i.e., the matrix $S^{\dagger}(t)$ is nonincreasing on $[\beta, \infty)_{\mathbb{T}}$. By (3.9) and Remark 2.1(vi) we then get

$$S^{\dagger}(t) \geq 0 \quad \text{for all } t \in [\beta, \infty)_{\mathbb{T}}. \tag{3.12}$$

This implies that the limit of $S^{\dagger}(t)$ for $t \to \infty$ exists and the matrix T in (3.6) is correctly defined. Finally, matrix T is symmetric and positive semidefinite as the limit of matrices with the same properties. Condition (3.7) then follows from inclusion (3.4) and from the inclusion $\operatorname{Im} T \subseteq \operatorname{Im} S^{\dagger}(t) = \operatorname{Im} S(t)$ on $[\beta, \infty)_{\mathbb{T}}$ derived from (3.11) together with (3.12). □

Remark 3.5. The proof of the Theorem 3.4 reveals some properties of the S-matrix corresponding to a conjoined basis (X, U) of ($\mathbb{S}$) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). Namely,

(i) the matrix $S(t)$ is nondecreasing on $[\alpha, \infty)_{\mathbb{T}}$,

(ii) the set $\operatorname{Im} S(t)$ is nondecreasing on $[\alpha, \infty)_{\mathbb{T}}$ and eventually constant, i.e., there exists $\beta \in [\alpha, \infty)_{\mathbb{T}}$ such that $\operatorname{Im} S(t)$ is constant on $[\beta, \infty)_{\mathbb{T}}$,

(iii) the set $\operatorname{Ker} S(t) = [\operatorname{Im} S(t)]^{\perp}$ is nonincreasing on $[\alpha, \infty)_{\mathbb{T}}$ and eventually constant.

In the next part we define the order of abnormality of system ($\mathbb{S}$) in the same way as in [14, 22]. For any $\alpha \in [a, \infty)_{\mathbb{T}}$ denote by $\Lambda[\alpha, \infty)_{\mathbb{T}}$ the linear space of n-vector functions $u : [\alpha, \infty)_{\mathbb{T}} \to \mathbb{R}^n$ such that $\mathcal{B}(t)\, u(t) = 0$ and $u^{\Delta} = \mathcal{D}(t)\, u(t)$ on $[\alpha, \infty)_{\mathbb{T}}$. The number $d[\alpha, \infty)_{\mathbb{T}} := \dim \Lambda[\alpha, \infty)_{\mathbb{T}}$ is called the *order of abnormality* of system ($\mathbb{S}$) on the interval $[\alpha, \infty)_{\mathbb{T}}$. The limit

$$d_{\infty} := \lim_{t \to \infty} d[t, \infty)_{\mathbb{T}} \tag{3.13}$$

is then called the *maximal order of abnormality* of the system ($\mathbb{S}$). Note that this definition is correct since limit in (3.13) exists and equals to $\max\{d[t, \infty)_{\mathbb{T}},\ t \in [\alpha, \infty)_{\mathbb{T}}\}$. This can be seen from the fact that a solution $(x \equiv 0, u)$ of ($\mathbb{S}$) on $[\alpha, \infty)_{\mathbb{T}}$ is also the solution of ($\mathbb{S}$) on $[\beta, \infty)_{\mathbb{T}}$ for any $\beta \in (\alpha, \infty)_{\mathbb{T}}$. Then $\Lambda[\alpha, \infty)_{\mathbb{T}} \subseteq \Lambda[\beta, \infty)_{\mathbb{T}}$ for $\alpha, \beta \in [a, \infty)_{\mathbb{T}}$ with $\alpha < \beta$ and hence, the function $d[t, \infty)_{\mathbb{T}}$ as a function of t is nondecreasing on $[a, \infty)_{\mathbb{T}}$. Then the integer values $d[t, \infty)_{\mathbb{T}}$ and d_{∞} satisfy

$$0 \leq d[t, \infty)_{\mathbb{T}} \leq d_{\infty} \leq n, \quad t \in [a, \infty)_{\mathbb{T}}. \tag{3.14}$$

In a similar way we define the order of abnormality $d[\alpha, t]_{\mathbb{T}}$ of system ($\mathbb{S}$) on the interval $[\alpha, t]_{\mathbb{T}}$. Then, obviously, the relation $d[\alpha, \infty)_{\mathbb{T}} = \lim_{t \to \infty} d[\alpha, t]_{\mathbb{T}}$ holds.

In addition, denote by $\Lambda_0[\alpha, \infty)_{\mathbb{T}}$ the subspace of $\mathbb{R}^n$ of the initial values $u(\alpha)$ of the elements $u \in \Lambda[\alpha, \infty)_{\mathbb{T}}$. Then $\dim \Lambda_0[\alpha, \infty)_{\mathbb{T}} = \dim \Lambda[\alpha, \infty)_{\mathbb{T}} = d[t, \infty)_{\mathbb{T}}$. This auxiliary subspace will be used e.g. in Proposition 3.18 when dealing with minimal conjoined bases of ($\mathbb{S}$).

3.2 Additional preparatory results

In this subsection we recall several results, which we will use in order to derive the properties of antiprincipal solutions of ($\mathbb{S}$) at infinity. We consider a conjoined basis (X,U) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞). We also consider the associated matrix $S(t)$ defined in (3.1), for which $\operatorname{Im} S(t)$ is constant on some interval $[\beta,\infty)_{\mathbb{T}}$ with $\beta\in[\alpha,\infty)_{\mathbb{T}}$, see Remark 3.5. The following additional properties of the matrices $S(t)$ and T are proven in [22, Theorem 3.2].

Proposition 3.6. *Let (X,U) be a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and let matrices P, $R(t)$, $S(t)$ be defined in (2.6), (2.4), and (3.1). Then*

(i) $\operatorname{Im}[U(t)(I-P)]=\operatorname{Ker} R(t)$ *and hence* $R(t)\,U(t)=R(t)\,U(t)P$ *on* $[\alpha,\infty)_{\mathbb{T}}$,

(ii) $R^{\sigma}(t)\,\mathcal{B}(t)=\mathcal{B}(t)$ *and* $\mathcal{B}(t)\,R(t)=\mathcal{B}(t)$ *on* $[\alpha,\infty)_{\mathbb{T}}$.

If in addition (X,U) has no focal points in (α,∞) and if T is defined in (3.6), then

(iii) $PT=T=TP$ *and* $PT^{\dagger}=T^{\dagger}=T^{\dagger}P$.

On the intervals $[\beta,\infty)_{\mathbb{T}}$, where the subspace $\operatorname{Im} S(t)$ is constant, we can define the associated constant orthogonal projector

$$P_{S\infty}:=P_S(t),\quad t\in[\beta,\infty)_{\mathbb{T}},\quad P_S(t):=\mathcal{P}_{\operatorname{Im} S(t)}=S(t)\,S^{\dagger}(t)=S^{\dagger}(t)\,S(t). \tag{3.15}$$

From (3.5) we can see that the following inclusions

$$\operatorname{Im} S(t)\subseteq\operatorname{Im} P_{S\infty}\subseteq\operatorname{Im} P,\quad t\in[\beta,\infty)_{\mathbb{T}}, \tag{3.16}$$

hold. By using the symmetry of $S(t)$, the inclusions in (3.16) can be written as

$$P_{S\infty}\,S(t)=S(t)=S(t)\,P_{S\infty},\quad t\in[\beta,\infty)_{\mathbb{T}},\quad PP_{S\infty}=P_{S\infty}=P_{S\infty}P. \tag{3.17}$$

Finally, using the definition of Moore–Penrose pseudoinverse in (2.1) and observing the limit

$$T=\lim_{t\to\infty}S^{\dagger}(t)=\lim_{t\to\infty}S^{\dagger}(t)\,S(t)\,S^{\dagger}(t)=P_{S\infty}\left(\lim_{t\to\infty}S^{\dagger}(t)\right)=P_{S\infty}\,T,$$

we obtain the equalities

$$P_{S\infty}\,T=T=TP_{S\infty},\qquad\text{i.e.,}\qquad\operatorname{Im} T\subseteq\operatorname{Im} P_{S\infty}. \tag{3.18}$$

By the principal solution of ($\mathbb{S}$) at the point $\alpha\in[a,\infty)_{\mathbb{T}}$, denoted by $(\hat{X}^{[\alpha]},\hat{U}^{[\alpha]})$, we mean the conjoined basis of ($\mathbb{S}$) satisfying the initial conditions $\hat{X}^{[\alpha]}(\alpha)=0$ and $\hat{U}^{[\alpha]}(\alpha)=I$. The following important result provides an information about any conjoined basis of ($\mathbb{S}$) through the properties of the principal solution $(\hat{X}^{[\alpha]},\hat{U}^{[\alpha]})$. It is proven as a part of [22, Proposition 3.9].

Lemma 3.7. *Let (X,U) be a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞). Let the matrices P, $R(t)$, $S(t)$, $P_{S\infty}$ be defined by (2.6), (2.4), (3.1), (3.15). Then the principal solution $(\hat{X}^{[\alpha]},\hat{U}^{[\alpha]})$ satisfies for all $t\in[\alpha,\infty)_{\mathbb{T}}$ the following properties:*

$$\hat{X}^{[\alpha]}(t)=X(t)\,S(t)\,X^{T}(\alpha), \tag{3.19}$$

$$\operatorname{rank} S(t)=\operatorname{rank}\hat{X}^{[\alpha]}(t)=n-d[\alpha,t]_{\mathbb{T}}, \tag{3.20}$$

$$\operatorname{rank} P_{S\infty}=n-d[\alpha,\infty)_{\mathbb{T}}, \tag{3.21}$$

$$\Lambda_0[\alpha,\infty)_{\mathbb{T}}=\operatorname{Im}[X^{\dagger T}(\alpha)\,(I-P_{S\infty})]\oplus\operatorname{Im}[U(\alpha)\,(I-P)], \tag{3.22}$$

$$n-d[\alpha,\infty)_{\mathbb{T}}\le\operatorname{rank} X(t)\le n. \tag{3.23}$$

Remark 3.8. From Theorem 3.4 and (3.20) it follows that $0 \leq \operatorname{rank} T \leq n - d_\infty$ for the T-matrix associated with an arbitrary conjoined basis (X, U) of (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞).

The next two results contain additional properties of the matrices $S(t)$ and T, which are proven in [22, Propositions 5.5 and 5.6] and, with a slightly different formulation, in [22, Remark 5.7] (see also the proof of [10, Proposition 6.105] in the discrete case).

Proposition 3.9. *Let (X, U) be a conjoined basis of (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). Let $S(t)$ and T be defined in (3.1) and (3.6). If $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$, then there exists $\beta \in [\alpha, \infty)_{\mathbb{T}}$ such that*

$$S^\dagger(t) \geq T \geq 0 \quad \text{and} \quad \operatorname{rank}[S^\dagger(t) - T] = n - d_\infty \quad \text{on } [\beta, \infty)_{\mathbb{T}}. \tag{3.24}$$

Proposition 3.10. *Let (X, U) be a conjoined basis of (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞), where the point $\alpha \in [a, \infty)_{\mathbb{T}}$ is that $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Then*

$$\operatorname{Im}[P_{S\infty} - S(\beta)\, S^\dagger(t)] = \operatorname{Im} P_{S\infty} = \operatorname{Im}[P_{S\infty} - S(\beta)\, S^\dagger(t)]^T, \quad \beta, t \in [\alpha, \infty)_{\mathbb{T}},\ t \geq \beta, \tag{3.25}$$

$$\operatorname{Im}[P_{S\infty} - S(t)\, T] = \operatorname{Im} P_{S\infty} = \operatorname{Im}[P_{S\infty} - S(t)\, T]^T, \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{3.26}$$

The following relation is a useful tool for the construction of conjoined bases of (S) with certain desired properties from a conjoined basis of (S), which the same properties already has. This relation is studied in [22, Section 4] in more details. For this purpose we also recall the concept of *equivalent solutions* (X_1, U_1) and (X_2, U_2) of (S) on some interval $[\alpha, \infty)_{\mathbb{T}}$, which is defined by the property $X_1(t) = X_2(t)$ on $[\alpha, \infty)_{\mathbb{T}}$.

Definition 3.11. Let (X, U) be a conjoined basis of (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) and let the matrices P and $P_{S\infty}$ be defined by (2.6) and (3.15). Consider an orthogonal projector P_* satisfying

$$\operatorname{Im} P_{S\infty} \subseteq \operatorname{Im} P_* \subseteq \operatorname{Im} P. \tag{3.27}$$

We say that a conjoined basis (X_*, U_*) of (S) is *contained* in (X, U) on $[\alpha, \infty)_{\mathbb{T}}$ with respect to P_*, or that (X, U) *contains* (X_*, U_*) on $[\alpha, \infty)_{\mathbb{T}}$ with respect to P_*, if the solutions (X_*, U_*) and (XP_*, UP_*) are equivalent, that is, if $X_*(t) = X(t)\, P_*$ on $[\alpha, \infty)_{\mathbb{T}}$.

It should be stressed that the relation in Definition 3.11 is between a conjoined basis (X, U) of (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) and an arbitrary conjoined basis (X_*, U_*). This means that if we say that a conjoined basis (X, U) contains a conjoined basis (X_*, U_*) on $[\alpha, \infty)_{\mathbb{T}}$, then we automatically suppose that (X, U) has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). The following proposition is proven in [22, Proposition 4.2] and it shows that the conjoined basis (X_*, U_*) from Definition 3.11 inherits the properties of (X, U) on the interval $[\alpha, \infty)_{\mathbb{T}}$.

Proposition 3.12. *Let (X, U) be a conjoined basis of (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) and assume that a conjoined basis (X_*, U_*) of (S) is contained in (X, U) on $[\alpha, \infty)_{\mathbb{T}}$ with respect to an orthogonal projector P_* satisfying (3.27).*

(i) Then (X_*, U_*) has also constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). Moreover, the matrix P_* is then the associated orthogonal projector defined in (2.6) for (X_*, U_*), i.e., $P_* = \mathcal{P}_{\operatorname{Im} X_*^T(t)} = X_*^\dagger(t)\, X_*(t)$ on $[\alpha, \infty)_{\mathbb{T}}$.

(ii) If $S(t)$ and $S_*(t)$ are the S-matrices corresponding to the conjoined bases (X, U) and (X_*, U_*) on $[\alpha, \infty)_{\mathbb{T}}$, then $S(t) = S_*(t)$ on $[\alpha, \infty)_{\mathbb{T}}$.

The next proposition from [22, Theorem 5.1] guarantees the existence of a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞), which has any given rank between the numbers $n - d_\infty$ and n. We will use it later in the construction of antiprincipal solutions of ($\mathbb{S}$) with a desired rank. Note that the conjoined bases with the given rank r are constructed by the relation being contained in Definition 3.11.

Proposition 3.13. *Assume that there exists a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). Then for any integer r between $n - d_\infty$ and n there exists a conjoined basis (X, U) of ($\mathbb{S}$), which has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) too, such that* $\operatorname{rank} X(t) = r$ *on $[\alpha, \infty)_{\mathbb{T}}$.*

For the investigation of all solutions (or conjoined bases) of ($\mathbb{S}$) it is important to choose a suitable fundamental matrix of system ($\mathbb{S}$). When one conjoined basis (X, U) of ($\mathbb{S}$) is given, it turns out that it is possible to complete it to a fundamental matrix of ($\mathbb{S}$) by a specific conjoined basis $(\bar{X}, \bar{U})$. Some of the properties of this conjoined basis $(\bar{X}, \bar{U})$ were presented in [22, Proposition 3.3 and Remarks 3.4 and 3.5]. We include some additional properties based on the discrete time results in [25, Proposition 3.5] or in [10, Proposition 6.67].

Proposition 3.14. *Let (X, U) be a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$, let the matrices P and $S(t)$ defined by (2.6) and (3.1). Then there exists a conjoined basis $(\bar{X}, \bar{U})$ of ($\mathbb{S}$) such that (X, U) and $(\bar{X}, \bar{U})$ satisfy*

(i) the Wronskian $N := X^T(t)\, \bar{U}(t) - U^T(t)\, \bar{X}(t) \equiv I$ on $[a, \infty)_{\mathbb{T}}$, and

(ii) $X^\dagger(\alpha)\, \bar{X}(\alpha) = 0$.

Moreover, such a conjoined basis $(\bar{X}, \bar{U})$ then satisfies

(iii) $X^\dagger(t)\, \bar{X}(t)\, P = S(t)$ for all $t \in [\alpha, \infty)_{\mathbb{T}}$,

(iv) $\bar{X}(t)\, P = X(t)\, S(t)$ and $\bar{U}(t)\, P = U(t)\, S(t) + X^{\dagger T}(t) + U(t)\,(I - P)\, \bar{X}^T(t)\, X^{\dagger T}(t)$ for all $t \in [\alpha, \infty)_{\mathbb{T}}$ (in particular $\bar{X}(\alpha)\, P = 0$), and the solution $(\bar{X}P, \bar{U}P)$ of ($\mathbb{S}$) is uniquely determined by (X, U),

(v) $\operatorname{Ker} \bar{X}(t) = \operatorname{Im}[P - P_S(t)] = \operatorname{Im} P \cap \operatorname{Ker} S(t)$ for all $t \in [\alpha, \infty)_{\mathbb{T}}$,

(vi) the function $\bar{X}(t)$ is uniquely determined by (X, U),

(vii) $\bar{P}(t) = I - P + P_S(t)$ for all $t \in [\alpha, \infty)_{\mathbb{T}}$, where $\bar{P}(t) := \bar{X}^\dagger(t)\, \bar{X}(t)$,

(viii) $S^\dagger(t) = \bar{X}^\dagger(t)\, X(t)\, P_S(t) = \bar{X}^\dagger(t)\, X(t)\, \bar{P}(t)$ for all $t \in [\alpha, \infty)_{\mathbb{T}}$,

(ix) $\operatorname{Im} \bar{X}(\alpha) = \operatorname{Im}[I - R(\alpha)]$ and $\operatorname{Im} \bar{X}^T(\alpha) = \operatorname{Im}(I - P)$,

(x) the matrix $X(\alpha) - \bar{X}(\alpha)$ is invertible with $[X(\alpha) - \bar{X}(\alpha)]^{-1} = X^\dagger(\alpha) - \bar{X}^\dagger(\alpha)$,

(xi) $\bar{X}^\dagger(\alpha) = -(I - P)\, U^T(\alpha)$.

If in addition the conjoined basis (X, U) has no focal points in (α, ∞), then

(xii) $X(t)\, \bar{X}^T(t) \geq 0$ for all $t \in [\alpha, \infty)_{\mathbb{T}}$.

Proof. The properties (i)–(iii) and (vi) are shown in [22, Proposition 3.3], property (iv) is shown in [22, Remark 3.5]. The remaining properties (v) and (vii)–(xii) can be proven analogously to the discrete case, see the proof of [22, Proposition 6.67]. □

The following result from [22, Proposition 3.6] shows important properties of two conjoined bases of ($\mathbb{S}$), which are mutually representable in terms of symplectic fundamental matrices involving the conjoined bases from Proposition 3.14.

Proposition 3.15. *Let (X_1, U_1) and (X_2, U_2) be conjoined bases of ($\mathbb{S}$) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) and let P_1 and P_2 be the constant orthogonal projectors defined in (2.6) through the functions X_1 and X_2, respectively. Let the conjoined basis (X_2, U_2) be expressed in terms of (X_1, U_1) via the matrices M_1 and N_1, and let the conjoined basis (X_1, U_1) be expressed in terms of (X_2, U_2) via the matrices M_2 and N_2, i.e.,*

$$\begin{pmatrix} X_2(t) \\ U_2(t) \end{pmatrix} = \begin{pmatrix} X_1(t) & \bar{X}_1(t) \\ U_1(t) & \bar{U}_1(t) \end{pmatrix} \begin{pmatrix} M_1 \\ N_1 \end{pmatrix}, \quad \begin{pmatrix} X_1(t) \\ U_1(t) \end{pmatrix} = \begin{pmatrix} X_2(t) & \bar{X}_2(t) \\ U_2(t) & \bar{U}_2(t) \end{pmatrix} \begin{pmatrix} M_2 \\ N_2 \end{pmatrix}$$

on $[\alpha, \infty)_{\mathbb{T}}$, where $(\bar{X}_1, \bar{U}_1)$ and $(\bar{X}_2, \bar{U}_2)$ are the conjoined bases of ($\mathbb{S}$) satisfying the properties in Proposition 3.14 with respect to (X_1, U_1) and (X_2, U_2), respectively. If the equality $\operatorname{Im} X_1(\alpha) = \operatorname{Im} X_2(\alpha)$ *is satisfied, then*

(i) the matrices $M_1^T N_1$ and $M_2^T N_2$ are symmetric and $N_2 = -N_1^T$,

(ii) the matrices M_1 and M_2 are invertible and $M_2 = M_1^{-1}$,

(iii) the inclusions $\operatorname{Im} N_1 \subseteq \operatorname{Im} P_1$ *and* $\operatorname{Im} N_2 \subseteq P_2$ *hold.*

Moreover, the matrices M_1 and N_1 do not depend on the choice of $(\bar{X}_1, \bar{U}_1)$, and the matrices M_2 and N_2 do not depend on the choice of $(\bar{X}_2, \bar{U}_2)$.

The following properties are from [22, Remark 3.7] and they complement the results in Proposition 3.15 about the representation matrices M_i and N_i (for $i \in \{1, 2\}$).

Remark 3.16. With the notation in Proposition 3.15, let us define the matrices

$$L_1 := X_1^{\dagger}(\alpha)\, X_2(\alpha), \quad L_2 := X_2^{\dagger}(\alpha)\, X_1(\alpha).$$

Then following properties hold for $i \in \{1, 2\}$:

$$L_i L_{3-i} = P_i, \quad L_{3-i} = L_i^{\dagger}, \quad L_i = P_i M_i, \quad N_i = P_i N_i, \tag{3.28}$$

$$P_i \text{ is the projector onto } \operatorname{Im} L_i, \quad L_i^T N_i = M_i^T P_i N_i = M_i^T N_i \text{ is symmetric}, \tag{3.29}$$

$$X_{3-i}(t) = X_i(t)\, [L_i + S_i(t)\, N_i], \quad M_i + S_i(t)\, N_i \text{ is invertible}, \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{3.30}$$

$$[L_i + S_i(t)\, N_i]^{\dagger} = L_{3-i} + S_{3-i}(t)\, N_{3-i}, \quad \operatorname{Im}\, [L_i + S_i(t)\, N_i] = \operatorname{Im} P_i, \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{3.31}$$

$$S_{3-i}(t) = [L_i + S_i(t)\, N_i]^{\dagger} S_i(t)\, L_i^{\dagger T}, \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{3.32}$$

where the matrix $S_i(t)$ is defined in (3.1) via the matrix $X_i(t)$.

3.3 Minimal conjoined bases and their properties

In this subsection we focus on minimal conjoined bases of (S). A conjoined basis (X, U) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) is called *minimal* on $[\alpha, \infty)_{\mathbb{T}}$, if $\operatorname{rank} X(t) = n - d[\alpha, \infty)_{\mathbb{T}}$ holds for all $t \in [\alpha, \infty)_{\mathbb{T}}$. These special conjoined bases have the smallest possible rank according to estimate (3.23). They are used in order to derive many properties of other conjoined bases of (S). Note that if (X, U) is a minimal conjoined basis of (S) on the interval $[\alpha, \infty)_{\mathbb{T}}$, then necessarily the abnormality of (S) on $[\alpha, \infty)_{\mathbb{T}}$ is maximal, i.e., $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$ holds. This follows from (3.14) and from estimate (3.23), since the rank of $X(t)$ is constant on $[\alpha, \infty)_{\mathbb{T}}$.

The following property will be used in the proof of Theorem 5.1 and it reveals a connection between orthogonal projectors P and $P_{S\infty}$ for a minimal conjoined basis (X, U) of (S). The stated equality $P = P_{S\infty}$ actually characterizes the property of (X, U) being a minimal conjoined basis of (S) on $[\alpha, \infty)_{\mathbb{T}}$, as mentioned (without the proof) in [22, Remark 5.3.]. In order to highlight its importance we state it separately and provide the details of its proof.

Lemma 3.17. *Let (X, U) be a conjoined basis of (S) on $[\alpha, \infty)_{\mathbb{T}}$ with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞), let the matrices P and $P_{S\infty}$ defined by (2.6) and (3.15), and assume that $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Then (X, U) is a minimal conjoined basis of (S) on $[\alpha, \infty)_{\mathbb{T}}$ if and only if*

$$P = P_{S\infty}. \tag{3.33}$$

Proof. Let (X, U) be a minimal conjoined basis of (S) on $[\alpha, \infty)_{\mathbb{T}}$, so that (X, U) has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). From Lemma 3.2 it follows that

$$\operatorname{Im} S(t) \subseteq \operatorname{Im} P = \operatorname{Im} X^T(t), \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{3.34}$$

and from Remark 3.5 we know that $\operatorname{Im} S(t)$ is nondecreasing on $[\alpha, \infty)_{\mathbb{T}}$. Moreover, from equation (3.21) in Lemma 3.7 we get $\operatorname{rank} P_{S\infty} = n - d_\infty = \lim_{t\to\infty} \operatorname{rank} S(t)$. Now from the fact that (X, U) is a minimal conjoined basis we get $\operatorname{rank} X(t) = n - d_\infty = \operatorname{rank} X^T(t)$ on $[\alpha, \infty)_{\mathbb{T}}$, which together with the inclusion (3.34) shows that

$$\operatorname{Im} S(t) = \operatorname{Im} X^T(t) \text{ for } t \in (\alpha, \infty)_{\mathbb{T}}. \tag{3.35}$$

This proves (3.33), since P is the orthogonal projector onto $\operatorname{Im} X^T(t)$ on $[\alpha, \infty)_{\mathbb{T}}$ and $P_{S\infty}$ is the orthogonal projector onto $\operatorname{Im} S(t)$ on $(\alpha, \infty)_{\mathbb{T}}$. Conversely, let (3.33) hold. Then by the definition of $P(t)$ in (2.4) and by Lemma 3.7 we have

$$\operatorname{rank} X(t) \overset{(2.4)}{=} \operatorname{rank} P \overset{(3.33)}{=} \operatorname{rank} P_{S\infty} \overset{(3.21)}{=} n - d_\infty, \quad t \in [\alpha, \infty)_{\mathbb{T}}.$$

This shows that (X, U) is a minimal conjoined basis on $[\alpha, \infty)_{\mathbb{T}}$. □

In the next result we present important properties of some special conjoined bases of (S) and their S-matrices, which are based on formulas (3.20) and (3.22) in Lemma 3.7 and on the properties of the Moore–Penrose pseudoinverse in Remark 2.1. These properties hold, in particular, for minimal conjoined bases of (S). We note that the formulation is slightly more general than in [22, Proposition 5.4], which we comment in the proof.

Proposition 3.18. *The following properties of conjoined bases of* (S) *hold.*

(i) Let (X, U) *be a conjoined basis of* (S) *with constant kernel on the interval* $[\alpha, \infty)_{\mathbb{T}}$ *and with* $\operatorname{rank} X(t) = n - d_\infty$ *on* $[\alpha, \infty)_{\mathbb{T}}$*. Let* P *be the associated projector defined in* (2.6)*. Then* $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$ *and*

$$\Lambda_0[\alpha, \infty)_{\mathbb{T}} = \operatorname{Im}[U(\alpha)\,(I - P)], \quad \operatorname{Im} X(\alpha) = \big(\Lambda_0[\alpha, \infty)_{\mathbb{T}}\big)^{\perp}. \tag{3.36}$$

Consequently, the initial subspace $\operatorname{Im} X(\alpha)$ *does not depend on the choice of the conjoined basis* (X, U) *of* (S) *with constant kernel and minimal rank on* $[\alpha, \infty)_{\mathbb{T}}$*.*

(ii) Let (X_1, U_1) *and* (X_2, U_2) *be two conjoined bases of* (S) *with constant kernel on the interval* $[\alpha, \infty)_{\mathbb{T}}$ *and with* $\operatorname{rank} X_1(t) = n - d_\infty = \operatorname{rank} X_2(t)$ *on* $[\alpha, \infty)_{\mathbb{T}}$*. Let* $S_1(t)$ *and* $S_2(t)$ *be the associated matrices defined in* (3.1)*. If* $\beta \in [\alpha, \infty)_{\mathbb{T}}$ *is a point such that*

$$\operatorname{rank} S_1(t) = n - d[\alpha, \infty)_{\mathbb{T}} = \operatorname{rank} S_2(t), \quad t \in [\beta, \infty)_{\mathbb{T}},$$

then for $i \in \{1, 2\}$ *we have*

$$S_{3-i}^{\dagger}(t) = L_i^T S_i^{\dagger}(t)\, L_i + L_i^T N_i, \quad t \in [\beta, \infty)_{\mathbb{T}}, \tag{3.37}$$

where the matrices L_i *and* N_i *are from Proposition* 3.15 *and Remark* 3.16*.*

Proof. These results are proven in [22, Proposition 5.4], where it is in addition assumed that the conjoined basis (X, U) in part (i) has no focal points in the interval (α, ∞) (so that it is a minimal conjoined basis on $[\alpha, \infty)_{\mathbb{T}}$) and that the conjoined bases (X_1, U_1) and (X_2, U_2) in part (ii) have no focal points in the interval (α, ∞) (so that they are minimal conjoined bases on $[\alpha, \infty)_{\mathbb{T}}$). We emphasize that this additional assumption on no focal points of (X, U) or (X_1, U_1), (X_2, U_2) in the interval (α, ∞) is not needed for deriving the statements in (3.36) and (3.37), since the proofs actually follow only the continuous time case in [18, Theorems 5.15 and 5.17]. □

The last result of this subsection shows that for all minimal conjoined bases (X, U) of (S) on $[\alpha, \infty)_{\mathbb{T}}$ the first component of the associated conjoined basis $(\bar{X}, \bar{U})$ (that is, the matrix $\bar{X}$) is the same up to a right constant nonsingular multiple.

Lemma 3.19. *Let* (X_1, U_1) *and* (X_2, U_2) *be minimal conjoined bases of* (S) *on* $[\alpha, \infty)_{\mathbb{T}}$ *and let* $(\bar{X}_1, \bar{U}_1)$ *and* $(\bar{X}_2, \bar{U}_2)$ *be their associated conjoined bases from Proposition* 3.14*. Then there exists a constant invertible matrix* K *such that*

$$\bar{X}_2(t) = \bar{X}_1(t)\, K, \quad t \in [\alpha, \infty)_{\mathbb{T}}.$$

Proof. The proof is analogous to the proof of the continuous case in [23, Lemma 1] or to the proof of the discrete case in [24, Lemma 7.9] or [10, Lemma 6.100]. The details are therefore omitted. □

4 Antiprincipal solutions at infinity

In this section we introduce the main notion of this paper, i.e., an antiprincipal solution of (S) at infinity. This definition is based on the basic results about the matrices $S(t)$ and T in Subsection 3.1. We then derive several properties of antiprincipal solutions at infinity with the aid of Subsections 3.2 and 3.3. The results in this section are new in the time scales setting and they extend and unify their corresponding continuous and discrete time counterparts, as we emphasize by providing particular references with each statement.

Definition 4.1. A conjoined basis (X, U) of (S) is said to be an *antiprincipal solution at infinity* with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$ if

(i) the order of abnormality of (S) on the interval $[\alpha, \infty)_{\mathbb{T}}$ is maximal, i.e.,

$$d[\alpha, \infty)_{\mathbb{T}} = d_{\infty}, \tag{4.1}$$

(ii) the conjoined basis (X, U) has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞),

(iii) the matrix T defined in (3.6) corresponding to (X, U) satisfies $\operatorname{rank} T = n - d_{\infty}$.

Remark 4.2. By Theorem 3.4 we know that the limit of $S^{\dagger}(t)$ exists for all conjoined bases (X, U) of (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). Therefore the definition of an antiprincipal solution of (S) at infinity using the corresponding T-matrix is possible. Note that so far we do not know anything about the existence of limit $S(t)$ itself. In addition, according to Remark 3.8 the rank of the matrix T of an antiprincipal solution of (S) at infinity is maximal possible.

Let (X, U) be an antiprincipal solution of (S) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$. By (3.21) and (3.23) together with property (4.1) from the above definition we obtain that $n - d_{\infty} \leq \operatorname{rank} X(t) \leq n$ for all $t \in [\alpha, \infty)_{\mathbb{T}}$. Denote by r the rank of (X, U) near infinity, i.e., $r := \operatorname{rank} X(t)$ for $t \in [\alpha, \infty)_{\mathbb{T}}$. If $r = n - d_{\infty}$, then (X, U) is called a *minimal antiprincipal solution* at infinity, which we denote by $(X_{\min}, U_{\min})$. If $r = n$, then (X, U) is called a *maximal antiprincipal solution* at infinity, which we denote by $(X_{\max}, U_{\max})$. In this case the matrix $X_{\max}(t)$ is invertible for all $t \in [\alpha, \infty)_{\mathbb{T}}$. Such minimal and maximal antiprincipal solutions of (S) at infinity will be considered e.g. in Theorems 5.3, 6.4 and 6.5 or in Remark 6.7.

The next theorem shows that the definition of an antiprincipal solution does not depend on the choice of point $\alpha \in [a, \infty)_{\mathbb{T}}$ determining the interval $[\alpha, \infty)_{\mathbb{T}}$, on which we impose the conditions (i) and (ii) in Definition 4.1. For this reason the term "with respect to interval $[\alpha, \infty)_{\mathbb{T}}$" will be dropped in the terminology of antiprincipal solutions of (S) at infinity in some situations. This statement is a unification of [20, Theorem 5.5] in the continuous case and of [24, Proposition 4.4] in the discrete case, see also [10, Proposition 6.125].

Theorem 4.3. *Every antiprincipal solution of (S) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$ is also an antiprincipal solution of (S) at infinity with respect to the interval $[\beta, \infty)_{\mathbb{T}}$ for all $\beta \in (\alpha, \infty)_{\mathbb{T}}$.*

Proof. Let (X, U) be an antiprincipal solution of (S) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$ and let $\beta \in (\alpha, \infty)_{\mathbb{T}}$ be a given point. Since $d[t, \infty)_{\mathbb{T}}$ is a nondecreasing function in the argument t, we get $d[\beta, \infty)_{\mathbb{T}} \geq d[\alpha, \infty)_{\mathbb{T}} = d_{\infty}$. This implies $d[\beta, \infty)_{\mathbb{T}} = d_{\infty}$. The property

$$(X, U) \text{ has constant kernel on } [\beta, \infty)_{\mathbb{T}} \text{ and no focal point in } (\beta, \infty) \tag{4.2}$$

holds trivially since $\beta > \alpha$. In order to prove that (X, U) is an antiprincipal solution of (S) at infinity with respect to the interval $[\beta, \infty)_{\mathbb{T}}$, we need to show that the associated matrices

$$S_{\beta}(t) := \int_{\beta}^{t} [X^{\sigma}(s)]^{\dagger}\, \mathcal{B}(s)\, [X^{\dagger}(s)]^{T}\, \Delta s, \quad t \in [\beta, \infty)_{\mathbb{T}}, \quad T_{\beta} := \lim_{t \to \infty} S_{\beta}^{\dagger}(t),$$

satisfy the relation $\operatorname{rank} T_{\beta} = n - d_{\infty}$. By (4.2) and Theorem 3.4 we know that matrix T_{β}, being defined as the above limit, exists. We will show that $\operatorname{Im} T_{\beta} = \operatorname{Im} T$, which will imply the desired equality for the rank of T_{β}. Note that, with the aid of $S(t)$ defined in (3.1), the matrix $S_\beta(t)$ can be easily expressed as $S_\beta(t) = S(t) - S(\beta)$ for all $t \in [\beta,\infty)_{\mathbb{T}}$. Using (3.17) and $S(\beta) = S(\beta)\, P_{S\infty} = S(\beta)\, S^\dagger(t)\, S(t)$ on $[\beta,\infty)_{\mathbb{T}}$ we then obtain

$$S_\beta(t) = S(t) - S(\beta) = P_{S\infty}\, S(t) - S(\beta)\, S^\dagger(t)\, S(t) = [P_{S\infty} - S(\beta)\, S^\dagger(t)]\, S(t), \quad t \in [\beta,\infty)_{\mathbb{T}}.$$

Then by Remark 2.1(vii) and using (3.25) we obtain

$$S_\beta^\dagger(t) \overset{(2.2)}{=} (P_{S\infty}\, S(t))^\dagger\, ([P_{S\infty} - S(\beta)\, S^\dagger(t)]\, P_{S\infty})^\dagger = S^\dagger(t)\, [P_{S\infty} - S(\beta)\, S^\dagger(t)]^\dagger \tag{4.3}$$

for all $t \in [\beta,\infty)_{\mathbb{T}}$, see also the proof of [22, Proposition 6.4]. Moreover, by (3.25) and (3.26) in Proposition 3.10 together with (3.21) we know that

$$\operatorname{rank}[P_{S\infty} - S(\beta)\, S^\dagger(t)] = n - d_\infty = \operatorname{rank}[P_{S\infty} - S(\beta)\, T], \quad t \in [\beta,\infty)_{\mathbb{T}}.$$

Then by Remark 2.1(iv) the limit of the pseudoinverse $[P_{S\infty} - S(\beta)\, S^\dagger(t)]^\dagger$ for $t \to \infty$ exists and is equal to $[P_{S\infty} - S(\beta)\, T]^\dagger$. Therefore, we obtain that

$$T_\beta = \lim_{t\to\infty} S_\beta^\dagger(t) \overset{(4.3)}{=} \lim_{t\to\infty} S^\dagger(t)\, [P_{S\infty} - S(\beta)\, S^\dagger(t)]^\dagger = T\, [P_{S\infty} - S(\beta)\, T]^\dagger. \tag{4.4}$$

Equality (4.4) implies that $\operatorname{Im} T_\beta \subseteq \operatorname{Im} T$. On the other hand, by (3.26) in Proposition 3.10 and Remark 2.1(ii) we can express the matrix T as

$$T \overset{(3.18)}{=} T P_{S\infty} \overset{(3.26)}{=} T\, [P_{S\infty} - S(t)\, T]^\dagger\, [P_{S\infty} - S(t)\, T] \overset{(4.4)}{=} T_\beta\, [P_{S\infty} - S(t)\, T]$$

for all $t \in [\beta,\infty)_{\mathbb{T}}$. This yields that $\operatorname{Im} T \subseteq \operatorname{Im} T_\beta$, and hence we proved $\operatorname{Im} T = \operatorname{Im} T_\beta$. Consequently, $\operatorname{rank} T_\beta = \operatorname{rank} T = n - d_\infty$, which completes the proof. □

In the next theorem we show that an antiprincipal solution of (S) at infinity is characterized by the property of the existence of the limit of $S(t)$ for $t \to \infty$. It is a unification of [24, Theorem 4.3] or [10, Theorem 6.124] in the discrete case and of [20, Theorem 5.3 and Remark 5.4] in the continuous case. We will see important applications of this result in the proofs of Proposition 4.5 and of Theorems 6.4 and 6.5.

Theorem 4.4. *Let (X, U) be a conjoined basis of (S) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞), let the matrices $S(t)$ and T be given by (3.1) and (3.6), and assume that $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$. Then the following statements are equivalent.*

(i) The conjoined basis (X, U) is an antiprincipal solution of (S) at ∞.

(ii) The limit of $S(t)$ for $t \to \infty$ exists.

(iii) The condition $\lim_{t\to\infty} S(t) = T^\dagger$ holds.

Proof. We divide the proof into three steps. First we show that (i) ⇒ (ii). Let (X, U) be an antiprincipal solution at infinity. By Theorem 3.4 we know that the limit of $S^\dagger(t)$ for $t \to \infty$ exists and is equal to T. In addition,

$$\operatorname{rank} S(t) \overset{(3.20)}{=} \operatorname{rank} \hat{X}^{[\alpha]}(t) \overset{(3.20)}{=} n - d[\alpha,\infty) = n - d_\infty = \operatorname{rank} T, \tag{4.5}$$

holds for all sufficiently large $t \in [\alpha,\infty)_{\mathbb{T}}$ by Lemma 3.7 and by the assumption that (X, U) is an antiprincipal solution of (S) at infinity. Therefore, by Remark 2.1(iv) with $M(t) := S^\dagger(t)$ and $M := T$ we conclude that the limit of $[S^\dagger(t)]^\dagger = S(t)$ for $t \to \infty$ exists.

Next we prove the implication (ii) ⇒ (iii). Suppose that the limit of $S(t)$ for $t \to \infty$ exists and let us denote this limit by S_∞. From Theorem 3.4 we know that limit T of $S^\dagger(t)$ for $t \to \infty$ also exists. Moreover, by Remark 2.1(i) and (4.5) we know that

$$\operatorname{rank} S^\dagger(t) = \operatorname{rank} S(t) \overset{(4.5)}{=} n - d_\infty = \operatorname{rank} T$$

for all sufficiently large $t \in [\alpha,\infty)_{\mathbb{T}}$. Now by using Remark 2.1(iv) in which we put $M(t) := S^\dagger(t)$ and $M := T$ we conclude that the limit of $S(t)$ for $t \to \infty$ exists with

$$S_\infty = \lim_{t\to\infty} S(t) = \lim_{t\to\infty} [S^\dagger(t)]^\dagger = T^\dagger.$$

Finally, we prove the implication (iii) ⇒ (i). Suppose that $\lim_{t\to\infty} S(t) = T^\dagger$. Since by Theorem 3.4 we also know that $\lim_{t\to\infty} S^\dagger(t) = T$, it follows from Remark 2.1(iv) that there exists $\beta \in [\alpha,\infty)_{\mathbb{T}}$ such that $\operatorname{rank} S(t) = \operatorname{rank} T^\dagger$ holds for all $t \in [\beta,\infty)_{\mathbb{T}}$. Then from (3.20) in Lemma 3.7 together with the assumptions of the theorem we get $\operatorname{rank} S(t) = \operatorname{rank} S^\dagger(t) = n - d_\infty$ for all $t \in [\beta,\infty)_{\mathbb{T}}$. Therefore, considering the symmetry of T, we get $\operatorname{rank} T = \operatorname{rank} T^\dagger = n - d_\infty$, which proves that (X,U) is an antiprincipal solution of (S) at infinity. □

Our next result shows that the property of being an antiprincipal solution of (S) at infinity is preserved under the multiplication by a constant nonsingular matrix.

Proposition 4.5. *Let (X,U) be an antiprincipal solution of (S) at infinity with respect to the interval $[\alpha,\infty)_{\mathbb{T}}$. Then for every invertible $n \times n$ matrix M the solution (XM,UM) of (S) is also an antiprincipal solution at infinity with respect to the interval $[\alpha,\infty)_{\mathbb{T}}$ and the rank of (XM,UM) is the same as the rank of (X,U).*

Proof. The solution $(\tilde{X},\tilde{U}) := (XM,UM)$ is obviously a conjoined basis of (S) with the same rank as (X,U). Since $\operatorname{Ker} X(t)$ is constant on $[\alpha,\infty)_{\mathbb{T}}$, then also $\operatorname{Ker} \tilde{X}(t) = \operatorname{Ker}[X(t)\, M]$ is constant on $[\alpha,\infty)_{\mathbb{T}}$. Moreover, by (2.2) in Remark 2.1(vii) we have

$$\tilde{X}^\dagger(t) = [X(t)\, M]^\dagger = (PM)^\dagger X^\dagger(t), \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{4.6}$$

This yields that for $t \in [\alpha,\infty)_{\mathbb{T}}$ we have

$$\begin{aligned}\tilde{X}(t)\,[\tilde{X}^\sigma(t)]^\dagger \mathcal{B}(t) &\overset{(4.6)}{=} X(t)\, M (PM)^\dagger [X^\sigma(t)]^\dagger \mathcal{B}(t) = X(t)\, PM(PM)^\dagger PMM^{-1}[X^\sigma(t)]^\dagger \mathcal{B}(t) \\ &= X(t)\, PMM^{-1}[X^\sigma(t)]^\dagger \mathcal{B}(t) = X(t)\,[X^\sigma(t)]^\dagger \mathcal{B}(t) \geq 0,\end{aligned}$$

showing that the conjoined basis $(\tilde{X},\tilde{U})$ has no focal points in (α,∞). For the matrix $\tilde{S}(t)$ in (3.1) associated with $(\tilde{X},\tilde{U})$ we have

$$\tilde{S}(t) := \int_\alpha^t [\tilde{X}^\sigma(s)]^\dagger \mathcal{B}(s)\, [\tilde{X}^\dagger(s)]^T \,\Delta s \overset{(4.6)}{=} (PM)^\dagger\, S(t)\, [(PM)^\dagger]^T, \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{4.7}$$

Since the limit of $S(t)$ as $t \to \infty$ exists and is equal to $T^\dagger$ by Theorem 4.4(iii), then the limit

$$\lim_{t\to\infty} \tilde{S}(t) \overset{(4.7)}{=} (PM)^\dagger\, T^\dagger\, [(PM)^\dagger]^T$$

also exists. Therefore, by Theorem 4.4(ii) again (now applied to $(X,U) := (\tilde{X},\tilde{U})$) the conjoined basis $(\tilde{X},\tilde{U})$ is an antiprincipal solution of (S) at infinity. □

The following two theorems show that the relation "to be contained in" or "to contain" (in Definition 3.11) preserves the property of being an antiprincipal solution of ($\mathbb{S}$) at infinity. It is a unification of [20, Theorem 5.7] in the continuous case and of [24, Proposition 4.6] in the discrete case, see also [10, Proposition 6.127].

Theorem 4.6. *Let (X, U) be an antiprincipal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$. Then every conjoined basis, which is contained in (X, U) on $[\alpha, \infty)_{\mathbb{T}}$, is also an antiprincipal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$.*

Proof. From the assumptions of the theorem we directly get $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Let (X_*, U_*) be a conjoined basis of ($\mathbb{S}$), which is contained in (X, U) on $[\alpha, \infty)_{\mathbb{T}}$. The conjoined basis (X_*, U_*) has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) due to Proposition 3.12(i), because (X, U) possess the same properties. Finally according to Proposition 3.12(ii) we get that the S-matrices corresponding to (X, U) and (X_*, U_*) coincide, i.e., $S(t) = S_*(t)$ on $[\alpha, \infty)_{\mathbb{T}}$. Therefore the limit $T_* := \lim_{t\to\infty} S_*^{\dagger}(t)$ exists and equals to T. This yields that $\operatorname{rank} T_* = \operatorname{rank} T = n - d_\infty$, which proves that (X_*, U_*) is also an antiprincipal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$. □

Theorem 4.7. *Let (X, U) be an antiprincipal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$. Then every conjoined basis with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞), which contains (X, U) on $[\alpha, \infty)_{\mathbb{T}}$, is also an antiprincipal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$.*

Proof. From the assumptions of the theorem we directly get $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Denote by (X_*, U_*) the conjoined basis with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞), which contains (X, U) on the interval $[\alpha, \infty)_{\mathbb{T}}$. Then by Proposition 3.12(ii), applied to $(X, U) := (X_*, U_*)$ and $(X_*, U_*) := (X, U)$, we obtain the equality $S_*(t) = S(t)$ on $[\alpha, \infty)_{\mathbb{T}}$. This implies that for the T-matrices corresponding to (X_*, U_*) and (X, U) the equality $T_* = T$ also holds. Therefore, $\operatorname{rank} T_* = \operatorname{rank} T = n - d_\infty$ holds and (X_*, U_*) is an antiprincipal solution of ($\mathbb{S}$) at infinity with respect to $[\alpha, \infty)_{\mathbb{T}}$. □

5 Existence of antiprincipal solutions at infinity

In this section we prove the existence of antiprincipal solutions at infinity for a nonoscillatory system ($\mathbb{S}$). We show the existence of such solutions (Theorem 5.3) for any rank in the admissible range given by estimate (3.23) in Lemma 3.7. As a main tool for this construction we derive (Theorem 5.2, through Theorem 5.1) a characterization of the T-matrices associated with conjoined bases of a nonoscillatory system ($\mathbb{S}$).

Our first result describes all minimal conjoined bases of ($\mathbb{S}$) on some interval $[\alpha, \infty)_{\mathbb{T}}$. It is a generalization to arbitrary time scales of [20, Theorem 4.4 and Remark 4.5] for the continuous case and of [24, Theorem 3.4] for the discrete case, see also [10, Theorem 6.106].

Theorem 5.1. *Let (X, U) be a minimal conjoined basis of ($\mathbb{S}$) on $[\alpha, \infty)_{\mathbb{T}}$, let $P_{S\infty}$ and T defined by (3.15) and (3.6), and assume that $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Then a solution $(\tilde{X}, \tilde{U})$ is a minimal conjoined basis on $[\alpha, \infty)_{\mathbb{T}}$ if and only if there exist matrices $M, N \in \mathbb{R}^{n\times n}$ such that*

$$\tilde{X}(\alpha) = X(\alpha)\, M, \quad \tilde{U}(\alpha) = U(\alpha)\, M + X^{\dagger T}(\alpha)\, N, \tag{5.1}$$

$$M \text{ is nonsingular}, \quad M^T N = N^T M, \quad \operatorname{Im} N \subseteq \operatorname{Im} P_{S\infty}, \tag{5.2}$$

$$NM^{-1} + T \geq 0. \tag{5.3}$$

In this case the matrix $\tilde{T}$ in (3.6) *corresponding to* $(\tilde{X},\tilde{U})$ *satisfies*

$$\tilde{T} = M^T T M + M^T N, \quad \operatorname{rank} \tilde{T} = \operatorname{rank}(NM^{-1} + T). \tag{5.4}$$

Proof. Let (X,U) be the conjoined basis of (S) from the assumptions of the theorem, that is, (X,U) has constant kernel on $[\alpha,\infty)_{\mathbb{T}}$, no focal points in (α,∞), and $\operatorname{rank} X(t) = n - d_\infty$ on $[\alpha,\infty)_{\mathbb{T}}$. Assume first that $(\tilde{X},\tilde{U})$ is also a minimal conjoined basis of (S) on $[\alpha,\infty)_{\mathbb{T}}$. Then $\operatorname{rank} \tilde{X}(t) = n - d_\infty = \operatorname{rank} X(t)$ and from (3.36) in Proposition 3.18(i) we obtain

$$\operatorname{Im} \tilde{X}(\alpha) = \big(\Lambda_0[\alpha,\infty)_{\mathbb{T}}\big)^\perp = \operatorname{Im} X(\alpha). \tag{5.5}$$

Applying now Proposition 3.15, where we put $(X,U) = (X_1,U_1)$ and $(\tilde{X},\tilde{U}) = (X_2,U_2)$ on $[\alpha,\infty)_{\mathbb{T}}$, we get that there exist matrices $M,N \in \mathbb{R}^{n\times n}$ such that

- M is nonsingular by Proposition 3.15(ii),
- $M^T N = N^T M$ by Proposition 3.15(i),
- $\operatorname{Im} N \subseteq \operatorname{Im} P = \operatorname{Im} P_{S\infty}$ by Proposition 3.15(iii) and Lemma 3.17.

This which proves the properties in (5.2). Moreover, the mutual representation between $(\tilde{X},\tilde{U})$ and (X,U), which we use here, is provided by the relation

$$\begin{pmatrix} \tilde{X}(t) \\ \tilde{U}(t) \end{pmatrix} = \begin{pmatrix} X(t) & \bar{X}(t) \\ U(t) & \bar{U}(t) \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix}, \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{5.6}$$

where $(\bar{X},\bar{U})$ is the conjoined basis chosen according to Proposition 3.14. In particular,

$$X^\dagger(\alpha)\,\bar{X}(\alpha) = 0 \tag{5.7}$$

holds. Let $R(t)$ and $\tilde{R}(t)$ be the orthogonal projectors defined in (2.4), which are associated respectively with (X,U) and $(\tilde{X},\tilde{U})$. Then from (5.5) we get $R(\alpha) = \tilde{R}(\alpha)$. Now from (5.6) for $t = \alpha$ we get that $\tilde{X}(\alpha) = X(\alpha)\,M + \bar{X}(\alpha)\,N$. Multiplying this equality by $X^\dagger(\alpha)$ from the left, using (5.7), and $\tilde{R}(\alpha)\,\tilde{X}(\alpha) = \tilde{X}(\alpha)$ derived from the definition of the Moore–Penrose pseudoinverse, we get that $\tilde{X}(\alpha) = X(\alpha)\,M$. Similarly, condition (5.6) for $t = \alpha$ gives that $\tilde{U}(\alpha) = U(\alpha)\,M + \bar{U}(\alpha)\,N$. Now using the information that (X,U) and $(\bar{X},\bar{U})$ are normalized we get $\bar{U}(\alpha)\,X^T(\alpha) - U(\alpha)\,\bar{X}^T(\alpha) = I$. Multiplying this equality by $X^{\dagger T}(t)\,N$ from the right, and using $PN = P_{S\infty}N = N$ derived from the property $\operatorname{Im} N \subseteq \operatorname{Im} P = \operatorname{Im} P_{S\infty}$, we get $\bar{U}(\alpha)\,N = X^{\dagger T}(\alpha)\,N$. This together with the previous part implies that (5.1) holds. Let T and $\tilde{T}$ be, respectively, the matrices defined in (3.6) corresponding to the minimal conjoined bases (X,U) and $(\tilde{X},\tilde{U})$ on $[\alpha,\infty)_{\mathbb{T}}$. Then from Proposition 3.18(ii) and Remark 3.16 (where we put $(X_1,U_1) := (X,U)$, $T_1 := T$ and $(X_2,U_2) := (\tilde{X},\tilde{U})$, $T_2 := \tilde{T}$ and consider $t \to \infty$) we obtain

$$\tilde{T} = L_1^T T L_1 + L_1^T N \overset{(3.28)}{=} M^T P T P M + M^T N \overset{(3.29)}{=} M^T T M + M^T N, \tag{5.8}$$

This together with the existence of M^{-1} implies that

$$NM^{-1} + T = M^{T^{-1}} \tilde{T} M^{-1} \geq 0, \tag{5.9}$$

since $\tilde{T} \geq 0$ by Theorem 3.4. Therefore, condition (5.3) holds. From inequality (5.9) we then conclude that $\operatorname{rank} \tilde{T} = \operatorname{rank} NM^{-1} + T$, which together with (5.8) shows (5.4).

Conversely, assume that $(\tilde{X},\tilde{U})$ is a solution of ($\mathbb{S}$) and let $M,N\in\mathbb{R}^{n\times n}$ be such that (5.1), (5.2), and (5.3) hold. First we will show that $(\tilde{X},\tilde{U})$ is a conjoined basis of ($\mathbb{S}$), i.e., we will show that the solution $(\tilde{X},\tilde{U})$ satisfies the condition on the symmetry of $\tilde{X}^T(t)\,\tilde{U}(t)$ and the condition on $\operatorname{rank}(\tilde{X}^T(t),\tilde{U}^T(t))^T=n$ at some point $t\in[\alpha,\infty)_{\mathbb{T}}$. The symmetry of $\tilde{X}^T(t)\,\tilde{U}(t)$ can be seen by using (5.1), by the symmetry of M^TN and $X^T(t)\,U(t)$ as a property of the conjoined basis (X,U), and by the relation

$$M^TX^{\dagger}(t)\,X(t)\,N=M^TPN\overset{(5.2)}{=}M^TN,\quad t\in[a,\infty)_{\mathbb{T}}. \tag{5.10}$$

More precisely, we have

$$\tilde{X}^T(\alpha)\,\tilde{U}(\alpha)\overset{(5.1)}{=}M^TX^T(\alpha)\,[U(\alpha)\,M+X^{\dagger T}(\alpha)\,N]\overset{(5.10)}{=}M^TX^T(\alpha)\,U(\alpha)\,M+M^TN,$$

where the last matrix is symmetric for all $t\in[a,\infty)_{\mathbb{T}}$. The condition on the rank of the matrix $(\tilde{X}^T(t),\tilde{U}^T(t))^T$ is also satisfied, since it follows again from (5.1) together with the fact that $\operatorname{rank}(X^T(t),U^T(t))^T=n$ on $[\alpha,\infty)_{\mathbb{T}}$, and from the fact that the subspaces $\operatorname{Im}X(\alpha)$ and $\operatorname{Im}X^{\dagger T}(\alpha)$ are equal. Thus, $(\tilde{X},\tilde{U})$ is a conjoined basis of ($\mathbb{S}$). Next we will show that $(\tilde{X},\tilde{U})$ is a minimal conjoined basis of ($\mathbb{S}$) on $[\alpha,\infty)_{\mathbb{T}}$. This means to prove, according to the definition, that $(\tilde{X},\tilde{U})$ has constant kernel on $[\alpha,\infty)_{\mathbb{T}}$, the rank of $\tilde{X}(t)$ is equal to $n-d_\infty$ on $[\alpha,\infty)_{\mathbb{T}}$, and $(\tilde{X},\tilde{U})$ has no focal points in (α,∞). Let $S(t)$ be the matrix in (3.1) corresponding to (X,U) on $[\alpha,\infty)_{\mathbb{T}}$ and let $(\tilde{X},\tilde{U})$ be expressed in terms of (X,U) as in Proposition 3.15 (where we put $(X_1,U_1)=(X,U)$ and $(X_2,U_2)=(\tilde{X},\tilde{U})$). More precisely, $(\tilde{X},\tilde{U})$ is represented as

$$\begin{pmatrix}\tilde{X}(t)\\ \tilde{U}(t)\end{pmatrix}=\Phi(t)\begin{pmatrix}\underline{M}\\ \underline{N}\end{pmatrix},\quad \Phi(t):=\begin{pmatrix}X(t) & \bar{X}(t)\\ U(t) & \bar{U}(t)\end{pmatrix},\quad t\in[\alpha,\infty)_{\mathbb{T}}, \tag{5.11}$$

that is, $(\bar{X}_1,\bar{U}_1):=(\bar{X},\bar{U})$, $M_1:=\underline{M}$, and $N_1:=\underline{N}$ in Proposition 3.15. We will show that $\underline{M}=M$ and $\underline{N}=N$ by using the fact that the matrix $\Phi(t)$ is symplectic as a fundamental matrix of ($\mathbb{S}$). Thus, we can express its inverse as $\Phi^{-1}(t)=-\mathcal{J}\,\Phi^T(t)\,\mathcal{J}$ and evaluate it in (5.11) at $t=\alpha$ to get

$$\begin{pmatrix}\underline{M}\\ \underline{N}\end{pmatrix}=\Phi^{-1}(\alpha)\begin{pmatrix}\tilde{X}(\alpha)\\ \tilde{U}(\alpha)\end{pmatrix}=\begin{pmatrix}\bar{U}^T(\alpha) & -\bar{X}^T(\alpha)\\ -U^T(\alpha) & X^T(\alpha)\end{pmatrix}\begin{pmatrix}\tilde{X}(\alpha)\\ \tilde{U}(\alpha)\end{pmatrix}. \tag{5.12}$$

Using the fact that Wronskian of (X,U) and $(\bar{X},\bar{U})$ equals to the identity matrix and using that $(\tilde{X},\tilde{U})$ now satisfies condition (5.1), equality (5.12) implies that

$$\begin{aligned}\underline{M}&=\bar{U}^T(\alpha)\,X(\alpha)\,M-\bar{X}^T(\alpha)\,[U(\alpha)\,M+X^{\dagger T}(\alpha)\,N]\\ &=[\bar{U}^T(\alpha)\,X(\alpha)-\bar{X}^T(\alpha)\,U(\alpha)]\,M-\bar{X}^T(\alpha)\,X^{\dagger T}(\alpha)\,N=M-[X^{\dagger}(\alpha)\,\bar{X}(\alpha)]^TN\overset{(5.7)}{=}M.\end{aligned}$$

Considering now the symmetry of $X^T(t)\,U(t)$ and the third condition in assumption (5.2) in the form $P_{S\infty}N=N$ we get $PN=P_{S\infty}N=N$, since (X,U) is a minimal conjoined basis of ($\mathbb{S}$) on $[\alpha,\infty)_{\mathbb{T}}$. Therefore, from (5.12) we get

$$\begin{aligned}\underline{N}&=-U^T(\alpha)\,X(\alpha)\,M+X^T(\alpha)\,[U(\alpha)\,M+X^{\dagger T}(\alpha)\,N]\\ &=[X^T(\alpha)\,U(\alpha)-U^T(\alpha)\,X(\alpha)]\,M+X^T(\alpha)\,X^{\dagger T}(\alpha)\,N=X^{\dagger}(\alpha)\,X(\alpha)\,N=PN=N.\end{aligned}$$

From Remark 3.16 and equation (3.30) we then obtain

$$\tilde{X}(t)=X(t)\,[P_{S\infty}\,\underline{M}+S(t)\,\underline{N}]=X(t)\,[P_{S\infty}\,M+S(t)\,N],\quad t\in[\alpha,\infty)_{\mathbb{T}}. \tag{5.13}$$

Note that equation (5.13) is valid when the kernel of the first basis (X, U) is constant on $[\alpha, \infty)_{\mathbb{T}}$, which is now satisfied, and there is no requirement on the kernel of the second basis $(\tilde{X}, \tilde{U})$, analogically to discrete case, see [10, Remark 6.70(iii)]. Now we show that also $(\tilde{X}, \tilde{U})$ has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$. More precisely, we show that $\operatorname{Ker} \tilde{X}(t) = \operatorname{Ker}(P_{S\infty} M)$ on $[\alpha, \infty)_{\mathbb{T}}$ in the following two steps.

(i) We show that $\operatorname{Ker}(P_{S\infty} M) \subseteq \operatorname{Ker} \tilde{X}(t)$ on $[\alpha, \infty)_{\mathbb{T}}$. Let $u \in \operatorname{Ker} P_{S\infty} M$ be given, i.e., $P_{S\infty} Mu = 0$. Multiplying the last equality by $X(t)\,[I + S(t)\, NM^{-1}]$ from the left and using

$$NM^{-1} P_{S\infty} = M^{T-1} N^T P_{S\infty} = M^{T-1} N^T = NM^{-1}$$

derived from (5.2), we get

$$\tilde{X}(t)\, u \overset{(5.13)}{=} X(t)\, [P_{S\infty} + S(t)\, N]\, u = X(t)\, [I + S(t)\, NM^{-1}]\, P_{S\infty} Mu = 0$$

for all $t \in [\alpha, \infty)_{\mathbb{T}}$. Thus, $u \in \operatorname{Ker} \tilde{X}(t)$ on $[\alpha, \infty)_{\mathbb{T}}$.

(ii) We show that $\operatorname{Ker} \tilde{X}(t) \subseteq \operatorname{Ker}(P_{S\infty} M)$ on $[\alpha, \infty)_{\mathbb{T}}$. Let $v \in \operatorname{Ker} \tilde{X}(t)$ on $[\alpha, \infty)_{\mathbb{T}}$ and set $w := P_{S\infty} Mv$. Our aim now is to show that $w = 0$. By (5.13) we get

$$X(t)\, [w + S(t)\, NM^{-1} w] = 0, \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{5.14}$$

Multiplying (5.14) by $X^{\dagger}(t)$ from the left, using $P_{S\infty} w = w$ derived from the properties of any orthogonal projector, and from (3.17) we get

$$w = -S(t)\, NM^{-1} w, \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{5.15}$$

which implies that

$$w \in \operatorname{Im} S(t) \overset{(3.35)}{=} \operatorname{Im} X^{\dagger}(t) = \operatorname{Im} P \overset{(3.33)}{=} \operatorname{Im} P_{S\infty}, \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{5.16}$$

where we used that (X, U) is a minimal conjoined basis of ($\mathbb{S}$) on $[\alpha, \infty)_{\mathbb{T}}$. Moreover, from $w = P_{S\infty} w$ and the above derived results we obtain for $t \in [\alpha, \infty)_{\mathbb{T}}$

$$w^T S^{\dagger}(t)\, w \overset{(5.15)}{=} -w^T S^{\dagger}(t)\, S(t)\, NM^{-1} w = -w^T P_{S\infty} NM^{-1} P_{S\infty} w.$$

Considering (5.3), which can be rewritten as $-NM^{-1} \leq T$, we get

$$w^T S^{\dagger}(t)\, w = -w^T P_{S\infty} NM^{-1} P_{S\infty} w \leq w^T P_{S\infty} T P_{S\infty} w \overset{(3.18)}{=} w^T T w$$

for $t \in [\alpha, \infty)_{\mathbb{T}}$, which implies that

$$w^T [S^{\dagger}(t) - T]\, w \leq 0, \quad t \in [\alpha, \infty)_{\mathbb{T}}.$$

But according to Proposition 3.9 the inequality $S^{\dagger}(t) \geq T \geq 0$ holds for large t, so that $w \in \operatorname{Ker}[S^{\dagger}(t) - T]$ for large t. But since $\operatorname{Im} S(t) = \operatorname{Im} P_{S\infty} = \operatorname{Im} P$ for $t \in (\alpha, \infty)_{\mathbb{T}}$ and (3.18) holds, we derive by using Proposition 3.10 the equality of kernels

$$\operatorname{Ker}[S^{\dagger}(t) - T] = \operatorname{Ker}[S(t)\, S^{\dagger}(t) - S(t)\, T] = \operatorname{Ker}[P_{\infty} - S(t)\, T] \overset{(3.26)}{=} \operatorname{Ker} P_{S\infty}$$

on $(\alpha, \infty)_{\mathbb{T}}$, which implies that $w \in \operatorname{Ker} P_{S\infty}$. This together with $w \in \operatorname{Im} P_{S\infty}$ from (5.16) implies that $w = 0$. Thus, $P_{S\infty} Mv = 0$, which proves that $v \in \operatorname{Ker}(P_{S\infty} M)$.

The proof of the equality $\operatorname{Ker} \tilde{X}(t) = \operatorname{Ker}(P_{S\infty} M)$ on $[\alpha, \infty)_{\mathbb{T}}$ is now complete. It follows that

$$\operatorname{rank} \tilde{X}(t) = \operatorname{rank}(P_{S\infty} M) = \operatorname{rank} P_{S\infty} = n - d_\infty, \quad t \in [\alpha, \infty)_{\mathbb{T}}.$$

Thus, the conjoined basis $(\tilde{X}, \tilde{U})$ has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and the lowest possible rank $n - d_\infty$ on $[\alpha, \infty)_{\mathbb{T}}$. It remains to prove that $(\tilde{X}, \tilde{U})$ has no focal points in the interval (α, ∞). Since $P = P_{S\infty}$ and $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$, we have $\operatorname{Im} S(t) \equiv \operatorname{Im} P_{S\infty}$ on the interval $(\alpha, \infty)_{\mathbb{T}}$. Recall that $S(\alpha) = 0$. Since $\operatorname{Ker} \tilde{X}(t)$ is constant on $[\alpha, \infty)_{\mathbb{T}}$, the matrix

$$\tilde{S}(t) := \int_\alpha^t [\tilde{X}^\sigma(s)]^\dagger \, \mathcal{B}(s) \, [\tilde{X}^\dagger(s)]^T \, \Delta s, \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{5.17}$$

is symmetric and, by (3.32) in Remark 3.16, the formula

$$\tilde{S}(t) = [PM + S(t)\, N]^\dagger S(t)\, M^{T-1} \tilde{P}, \quad t \in [\alpha, \infty)_{\mathbb{T}},$$

holds. Then, by (3.37) in Proposition 3.18(ii), the pseudoinverse of $\tilde{S}(t)$ has the form

$$\tilde{S}^\dagger(t) = M^T S^\dagger(t)\, M + M^T N, \quad t \in (\alpha, \infty)_{\mathbb{T}}. \tag{5.18}$$

Note that if the point α is right-scattered, then formula (5.18) holds for $t \in [\sigma(\alpha), \infty)_{\mathbb{T}}$ only. Since by Proposition 3.9 the matrix function $S^\dagger(t)$ is nonincreasing on $(\alpha, \infty)_{\mathbb{T}}$, it follows from (5.18) that the matrix function $\tilde{S}^\dagger(t)$ is nonincreasing on $(\alpha, \infty)_{\mathbb{T}}$ as well and hence, by Remark 2.1(v), the the matrix function $\tilde{S}(t)$ is nondecreasing on $(\alpha, \infty)_{\mathbb{T}}$. Moreover,

$$\tilde{S}^\dagger(t) \overset{(5.18)}{=} M^T S^\dagger(t)\, M + M^T N \overset{(3.24)}{\geq} M^T T M + M^T N \overset{(5.3)}{\geq} 0, \quad t \in (\alpha, \infty)_{\mathbb{T}}.$$

Therefore, in view of Remark 2.1(vi) we also have

$$\tilde{S}(t) \geq 0, \quad t \in (\alpha, \infty)_{\mathbb{T}}. \tag{5.19}$$

From the already established monotonicity of $\tilde{S}(t)$ on $(\alpha, \infty)_{\mathbb{T}}$ it now follows that $\tilde{S}^\Delta(t) \geq 0$ on $(\alpha, \infty)_{\mathbb{T}}$. Then with the aid of Proposition 3.6(ii) (applied to $(X, U) := (\tilde{X}, \tilde{U})$) we get

$$\begin{aligned} \tilde{X}(t)\, [\tilde{X}^\sigma(t)]^\dagger \, \mathcal{B}(t) \;&=\; \tilde{X}(t)\, [\tilde{X}^\sigma(t)]^\dagger \, \mathcal{B}(t)\, \tilde{R}(t) = \tilde{X}(t)\, [\tilde{X}^\sigma(t)]^\dagger \, \mathcal{B}(t)\, [\tilde{X}^\dagger(t)]^T \tilde{X}^T(t) \\ &\overset{(5.17)}{=} \tilde{X}(t)\, \tilde{S}^\Delta(t)\, \tilde{X}^T(t) \geq 0, \quad t \in (\alpha, \infty)_{\mathbb{T}}. \end{aligned} \tag{5.20}$$

This shows that the conjoined basis $(\tilde{X}, \tilde{U})$ has no focal points in the interval (α, ∞) if the point α is right-dense, and no focal points in the interval $(\sigma(\alpha), \infty)$ if the point α is right-scattered. However, in the latter situation (that is, for $\sigma(\alpha) > \alpha$) we know by property (5.19) at $t = \sigma(\alpha)$ that $\tilde{S}^\sigma(\alpha) \geq 0$, so that in this case

$$\tilde{S}^\Delta(\alpha) = [\tilde{S}^\sigma(\alpha) - \tilde{S}(\alpha)]/\mu(\alpha) \overset{(5.17)}{=} \tilde{S}^\sigma(\alpha)/\mu(\alpha) \geq 0.$$

As in (5.20) we then conclude that $(\tilde{X}, \tilde{U})$ has no focal points in the interval $(\alpha, \sigma(\alpha)]$ when α is right-scattered. This proves that $(\tilde{X}, \tilde{U})$ has no focal points in the interval (α, ∞) and hence, it is a minimal conjoined basis of (S) on $[\alpha, \infty)_{\mathbb{T}}$. The proof is complete. □

The next theorem serves as a criterion for the classification of all T-matrices, which correspond to conjoined bases of (S) with constant kernel on some interval $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). It is a unification of the continuous and discrete cases in [20, Theorem 4.9 and Corollary 4.11] and [24, Theorem 3.5], see also [10, Theorem 6.107 and Remark 6.108].

Theorem 5.2. *Assume that* ($\mathbb{S}$) *is nonoscillatory. Then $D \in \mathbb{R}^{n\times n}$ is a T-matrix of some conjoined basis (X, U) of* ($\mathbb{S}$) *with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞) and with $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$ if and only if*

$$\text{the matrix } D \text{ is symmetric, } D \geq 0, \text{ and } \operatorname{rank} D \leq n - d_\infty. \tag{5.21}$$

Moreover, (X, U) can be chosen to be a minimal conjoined basis of ($\mathbb{S}$) *on $[\alpha,\infty)_{\mathbb{T}}$.*

Proof. First we prove that the result holds for minimal conjoined bases of ($\mathbb{S}$) on $[\alpha,\infty)_{\mathbb{T}}$. Let D be a T-matrix of a minimal conjoined basis (X, U) on an interval $[\alpha,\infty)_{\mathbb{T}}$ with $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$. Then according to Theorem 3.4 the matrix D is symmetric and $D \geq 0$. Note that by equation (3.21) in Lemma 3.7 we have $\operatorname{rank} P = \operatorname{rank} P_{S\infty} = n - d_\infty$. The inclusion in (3.18) implies that $\operatorname{rank} D \leq n - d_\infty$.

Conversely, assume that D is a symmetric matrix with $D \geq 0$ and $\operatorname{rank} D \leq n - d_\infty$. We will show through Theorem 5.1 that D is the T-matrix of some minimal conjoined basis of ($\mathbb{S}$) on some interval $[\alpha,\infty)_{\mathbb{T}}$ satisfying $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$. First, we show that there exists a minimal conjoined basis $(X_{\min}, U_{\min})$ of ($\mathbb{S}$) on $[\alpha,\infty)_{\mathbb{T}}$. Then, using this conjoined basis, we will construct another one denoted by $(\tilde{X}, \tilde{U})$ (via Theorem 5.1), such that its associated matrix $\tilde{T}$ is equal to D. Since we assume that system ($\mathbb{S}$) is nonoscillatory, then every conjoined basis of ($\mathbb{S}$) is nonoscillatory and by Proposition 3.13 (with $r := n - d_\infty$) there exists a minimal conjoined basis $(X_{\min}, U_{\min})$ of ($\mathbb{S}$) on $[\alpha,\infty)_{\mathbb{T}}$. Then by Proposition 3.18(i) condition $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$ holds. The assumption $\operatorname{rank} D \leq n - d_\infty$ guarantees that there exists an orthogonal projector Q with $\operatorname{rank} Q = n - d_\infty$ such that

$$\operatorname{Im} D \subseteq \operatorname{Im} Q. \tag{5.22}$$

Then by Lemma 3.7 condition (3.21) holds, i.e., $\operatorname{rank} P_{S\infty} = n - d_\infty = \operatorname{rank} Q$. Moreover, by Proposition 2.2 (where we put $P_* := 0$) there exists an invertible matrix $E \in \mathbb{R}^{n\times n}$ satisfying $\operatorname{Im} EP_{S\infty} = \operatorname{Im} Q$, i.e., $\operatorname{Im} P_{S\infty} = \operatorname{Im} E^{-1}Q$. In particular, the equality $P_{S\infty} E^{-1}Q = E^{-1}Q$ holds. Define the matrices $M, N \in \mathbb{R}^{n\times n}$ by

$$M := E^T, \quad N := E^{-1}D - TE^T, \tag{5.23}$$

where T is the matrix in (3.6) corresponding to $(X_{\min}, U_{\min})$. We will show that the matrices M and N satisfy conditions (5.2) and (5.3) from Theorem 5.1, i.e., the following four properties of matrices M and N hold:

(i) The matrix M is invertible. This follows from the definition of M in (5.23).

(ii) The matrix M^TN is symmetric. This follows from the symmetry of D and T and from the calculation $M^TN = E(E^{-1}D - TE^T) = D - ETE^T$.

(iii) The inclusion $\operatorname{Im} N \subseteq \operatorname{Im} P_{S\infty}$ holds, since

$$\begin{aligned} N &= E^{-1}D - TE^T \overset{(5.22)}{=} E^{-1}QD - TE^T = P_{S\infty} E^{-1}QD - TE^T \\ &\overset{(3.18)}{=} P_{S\infty}E^{-1}QD - P_{S\infty} TE^T = P_{S\infty}(E^{-1}QD - TE^T) = P_{S\infty}N. \end{aligned}$$

(iv) The matrix $NM^{-1} + T$ is positive semidefinite, since $D \geq 0$ and

$$NM^{-1} + T = (E^{-1}D - TE^T)\, E^{T-1} + T = E^{-1}DE^{T-1} \geq 0.$$

Consider now the conjoined basis $(\tilde{X}, \tilde{U})$ of (S) on $[a, \infty)_{\mathbb{T}}$ with the initial conditions at the point α given by (5.1), where matrices M and N are given in (5.23) above. Then by Theorem 5.1 the solution $(\tilde{X}, \tilde{U})$ is a minimal conjoined basis of (S) on $[\alpha, \infty)_{\mathbb{T}}$ and, moreover, its associated $\tilde{T}$ satisfies (5.4). This yields that

$$\tilde{T} \overset{(5.4)}{=} M^T T M + M^T N \overset{(5.23)}{=} E T E^T + E(E^{-1} D - T E^T) = D.$$

Therefore, we showed that the matrix D is the T-matrix of the minimal conjoined basis $(\tilde{X}, \tilde{U})$ on $[\alpha, \infty)_{\mathbb{T}}$.

The general statement of the theorem now follows from Proposition 3.12(ii). Let (X, U) be a conjoined basis of (S) with constant constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) and with $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Let (X_*, U_*) be a minimal conjoined basis on $[\alpha, \infty)_{\mathbb{T}}$, which is contained in (X, U) on $[\alpha, \infty)_{\mathbb{T}}$. Note that such a minimal conjoined basis always exists by [22, Theorem 4.3] (with the choice $P_* := P_{S\infty}$). Then by the first part of the proof the matrix $D := T_*$ in (3.6) associated with (X_*, U_*) satisfies the conditions in (5.21). But by Proposition 3.12(ii) the matrices $S_*(t)$ and $S(t)$ coincide on the interval $[\alpha, \infty)_{\mathbb{T}}$, so that $T_* = T$ and hence, the matrix T satisfies (5.21) as well. □

Next we derive the existence of antiprincipal solutions at infinity with any admissible rank r for a nonoscillatory system (S), see the continuous case in [20, Theorem 5.8] and the discrete case in [24, Theorem 4.7] or in [10, Theorem 6.128]. It can also be viewed as a counterpart of [22, Theorem 6.8] regarding the principal solutions of (S) at infinity. The most important part consists of the existence of a minimal antiprincipal solution of (S) at infinity. This property will also be used later in Section 6 in the applications of antiprincipal solutions at infinity.

Theorem 5.3. *Assume that system (S) is nonoscillatory. Then there exists a minimal antiprincipal solution of (S) at infinity. Moreover, in this case for any integer r between $n - d_\infty$ and n there exists an antiprincipal solution (X, U) of (S) at infinity with the rank of $X(t)$ equal to r for large t.*

Proof. Assume that system (S) is nonoscillatory and let $D \in \mathbb{R}^{n \times n}$ be an arbitrary symmetric and positive semidefinite matrix with $\operatorname{rank} D = n - d_\infty$. Let $\alpha \in [a, \infty)_{\mathbb{T}}$ be large enough so that $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$ holds. According to Theorem 5.2, there exists a minimal conjoined basis $(X_{\min}, U_{\min})$ on $[\alpha, \infty)_{\mathbb{T}}$ such that its corresponding matrix T is equal to D. By Definition 4.1, this conjoined basis is an antiprincipal solution of (S) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$, since $\operatorname{rank} T = \operatorname{rank} D = n - d_\infty$ due to above choice of D. In addition, since $\operatorname{rank} X_{\min}(t) = n - d_\infty$ on $[\alpha, \infty)_{\mathbb{T}}$, we get that $(X_{\min}, U_{\min})$ is a minimal antiprincipal solution of (S) at infinity, which proves the first part of the theorem. Furthermore, choose any integer r between $n - d_\infty$ and n. Then by Proposition 3.13, using the already established existence of $(X_{\min}, U_{\min})$, there exists a conjoined basis (X, U) of (S) with $\operatorname{rank} X(t) = r$ on $[\alpha, \infty)_{\mathbb{T}}$, which contains $(X_{\min}, U_{\min})$ on $[\alpha, \infty)_{\mathbb{T}}$. Then by Theorem 4.7 we know that the conjoined basis (X, U) is also an antiprincipal solution of (S), having also the desired rank r. □

Remark 5.4. On special time scales $\mathbb{T}$, which consist of disjoint closed intervals and/or isolated points, see [22, Section 7] and [28, Section 6], the converse statement in Theorem 5.3 also holds. That is, on such special time scales the existence of an antiprincipal solution at infinity implies the nonoscillation of system (S).

6 Applications of antiprincipal solutions at infinity

In this section we derive further properties of principal and antiprincipal solutions of system ($\mathbb{S}$) at infinity. First we recall the definition and basic properties of principal solutions of ($\mathbb{S}$) at infinity, which are a natural counterpart to antiprincipal solutions at infinity, when comparing the rank of their associated T-matrices.

According to [22, Definition 6.1], a conjoined basis $(\hat{X}, \hat{U})$ of ($\mathbb{S}$) is a *principal solution at infinity*, if there exists $\alpha \in [a, \infty)_{\mathbb{T}}$ such that $(\hat{X}, \hat{U})$ has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) and its associated matrix $\hat{T}$ defined in (3.6) through $\hat{X}(t)$ satisfies $\hat{T} = 0$. If $\operatorname{rank} \hat{X}(t) = n - d_\infty$ or $\operatorname{rank} \hat{X}(t) = n$ on $[\alpha, \infty)_{\mathbb{T}}$, then $(\hat{X}, \hat{U})$ is called a *minimal principal solution at infinity* or a *maximal principal solution at infinity*, respectively. According to [22, Theorem 6.6], if system ($\mathbb{S}$) is nonoscillatory, then the minimal principal solution exists and is unique up to a constant right invertible multiple. Complying with the previous notation, we will denote this (unique) minimal principal solution of ($\mathbb{S}$) at infinity by $(\hat{X}_{\min}, \hat{U}_{\min})$. The result of [22, Theorem 6.9] then shows that the minimality property of the rank of $\hat{X}_{\min}(t)$ on $[\alpha, \infty)_{\mathbb{T}}$ and the uniqueness property of $(\hat{X}_{\min}, \hat{U}_{\min})$ are in fact equivalent conditions.

The following result shows a construction of the minimal principal solution of ($\mathbb{S}$) at infinity from an arbitrary minimal conjoined basis of ($\mathbb{S}$). This construction is used in the proof of [22, Theorem 6.6] in order to establish the uniqueness of the minimal principal solution at infinity. For our future reference we present it as a separate statement.

Theorem 6.1. *Assume that system ($\mathbb{S}$) is nonoscillatory. Suppose that $\alpha \in [a, \infty)_{\mathbb{T}}$ is such that $d[\alpha, \infty) = d_\infty$ and there exists a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). Then a solution $(\hat{X}, \hat{U})$ of ($\mathbb{S}$) is a minimal principal solution at infinity if and only if*

$$\begin{pmatrix} \hat{X}(t) \\ \hat{U}(t) \end{pmatrix} = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix} - \begin{pmatrix} \bar{X}(t) \\ \bar{U}(t) \end{pmatrix} T, \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{6.1}$$

for some minimal conjoined basis (X, U) of ($\mathbb{S}$) on $[\alpha, \infty)_{\mathbb{T}}$, where $(\bar{X}, \bar{U})$ is the conjoined basis of ($\mathbb{S}$) from Proposition 3.14 associated with (X, U) and T is the matrix defined in (3.6).

Proof. Let $\alpha \in [a, \infty)_{\mathbb{T}}$ be as in the statement. If $(\hat{X}, \hat{U})$ is a minimal principal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$, then the corresponding matrix $\hat{T}$ in (3.6) satisfies $\hat{T} = 0$ and $(\hat{X}, \hat{U})$ is a minimal conjoined basis on $[\alpha, \infty)_{\mathbb{T}}$. Formula (6.1) then holds with $(X, U) := (\hat{X}, \hat{U})$. Conversely, if (X, U) is a minimal conjoined basis of ($\mathbb{S}$) on $[\alpha, \infty)_{\mathbb{T}}$ and define the solution $(\hat{X}, \hat{U})$ of ($\mathbb{S}$) by (6.1). Then in the proof of [22, Theorem 6.6] it is shown that $(\hat{X}, \hat{U})$ is a minimal conjoined basis on $[\alpha, \infty)_{\mathbb{T}}$. Moreover, by (3.37) in Proposition 3.18(ii) (with $(X_1, U_1) := (X, U)$, $(X_2, U_2) := (\hat{X}, \hat{U})$, $L_1 := X^\dagger(\alpha)\, \hat{X}(\alpha) = P$, and $N_1 := -T$) its associated matrix $\hat{S}(t)$ in (3.1) satisfies

$$\hat{S}^\dagger(t) = S^\dagger(t) - T, \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{6.2}$$

Taking the limit for $t \to \infty$ in (6.2) and using that $S^\dagger(t) \to T$ for $t \to \infty$ we obtain that $\hat{S}^\dagger(t) \to \hat{T} = 0$ for $t \to \infty$, i.e., $(\hat{X}, \hat{U})$ is a minimal principal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$. □

Remark 6.2. In [22, Theorem 6.7] it is shown that the minimal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ of ($\mathbb{S}$) at infinity can be determined from an arbitrary minimal conjoined basis (X, U) of ($\mathbb{S}$) on the interval $[\alpha, \infty)_{\mathbb{T}}$ by the initial conditions

$$\hat{X}_{\min}(\alpha) = X(\alpha), \quad \hat{U}_{\min}(\alpha) = U(\alpha) - [X^\dagger(\alpha)]^T\, T,$$

In the following considerations we will use an estimate for the maximal interval, on which the minimal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ at infinity has constant kernel and no focal points. Thus, we define the point $\hat{\alpha}_{\min} \in [\alpha, \infty)_{\mathbb{T}}$ as

$$\hat{\alpha}_{\min} := \left\{ \begin{matrix} \inf \alpha \in [a, \infty)_{\mathbb{T}},\ (\hat{X}_{\min}, \hat{U}_{\min}) \text{ has constant kernel on } [\alpha, \infty)_{\mathbb{T}} \\ \text{and no focal points in } (\alpha, \infty) \end{matrix} \right\}. \tag{6.3}$$

Moreover, by estimate (3.23) and by $\operatorname{rank} \hat{X}_{\min}(t) = n - d_\infty$ on $[\alpha, \infty)_{\mathbb{T}}$ we obtain

$$d[\hat{\alpha}_{\min}, \infty)_{\mathbb{T}} = d_\infty = d[\alpha, \infty)_{\mathbb{T}} \quad \text{for every } \alpha \in [\hat{\alpha}_{\min}, \infty)_{\mathbb{T}}. \tag{6.4}$$

In the next theorem we use minimal antiprincipal solutions of (S) at infinity for a characterization of all antiprincipal solutions of (S) at infinity through the relation being contained. It is a unification of the continuous case in [20, Theorem 5.11] and the discrete case in [24, Theorem 4.11(ii)], see also [10, Theorem 6.131(ii)].

Theorem 6.3. *Assume that system (S) is nonoscillatory, let $\hat{\alpha}_{\min} \in [a, \infty)_{\mathbb{T}}$ be defined in (6.3). Then a solution (X, U) of (S) is an antiprincipal solution at infinity if and only if (X, U) is a conjoined basis of (S), which contains some minimal antiprincipal solution of (S) at infinity on $[\alpha, \infty)_{\mathbb{T}}$ for some $\alpha \in [\hat{\alpha}_{\min}, \infty)_{\mathbb{T}}$.*

Proof. Let $(\hat{X}_{\min}, \hat{U}_{\min})$ be the minimal principal solution of (S) at infinity. Then condition (6.4) holds. Let (X, U) be an antiprincipal solution of (S) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$. Due to Theorem 4.3 we may assume that $\alpha \in [\hat{\alpha}_{\min}, \infty)_{\mathbb{T}}$. By Proposition 3.13, there exists a conjoined basis $(X_{\min}, U_{\min})$ of (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) and, moreover, with $\operatorname{rank} X_{\min} = n - d_\infty$ on $[\alpha, \infty)_{\mathbb{T}}$ such that $(X_{\min}, U_{\min})$ is contained in (X, U). From Theorem 4.7 it then follows that $(X_{\min}, U_{\min})$ is also an antiprincipal solution of (S) at infinity. Conversely, let (X, U) be a conjoined basis of (S), which contains some minimal antiprincipal solution $(X_{\min}, U_{\min})$ of (S) at infinity on $[\alpha, \infty)_{\mathbb{T}}$ for some $\alpha \in [\hat{\alpha}_{\min}, \infty)_{\mathbb{T}}$. Then by Definition 4.1 (applied to $(X_{\min}, U_{\min})$) we know that $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Therefore, by Theorem 4.6 we conclude that (X, U) is also an antiprincipal solution of (S) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$. □

The following result shows that principal solutions at finite points α for sufficiently large $\alpha \in [a, \infty)_{\mathbb{T}}$ are examples of minimal antiprincipal solutions of (S) at infinity. It is a unification of the continuous case in [18, Proposition 5.15] and the discrete case in [24, Theorem 5.10], see also [10, Theorem 6.143]. We recall from Lemma 3.7 that the principal solution of (S) at the point $\alpha \in [a, \infty)_{\mathbb{T}}$, denoted by $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$, is the solution of (S) with the initial conditions $\hat{X}^{[\alpha]}(\alpha) = 0$ and $\hat{U}^{[\alpha]}(\alpha) = I$.

Theorem 6.4. *Assume that system (S) is nonoscillatory. Let the point $\hat{\alpha}_{\min}$ be defined in (6.3). Then for every $\alpha \in [\hat{\alpha}_{\min}, \infty)_{\mathbb{T}}$ the principal solution $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$ is a minimal antiprincipal solution of (S) at infinity.*

Proof. From [22, Theorem 6.6] we know that when system (S) is nonoscillatory, then there exists the minimal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ of (S) at infinity, which we denote for simplicity by $(X, U) := (\hat{X}_{\min}, \hat{U}_{\min})$ in this proof. Consider its associated matrices P, $P_{S\infty}$, $S(t)$, and T defined in (2.6), (3.15), (3.1), and (3.6). Choose a point $\beta \in [\hat{\alpha}_{\min}, \infty)_{\mathbb{T}}$ such that $\operatorname{Im} S(t)$ is constant on $[\beta, \infty)_{\mathbb{T}}$. Then from (3.20) and (3.19) in Lemma 3.7 we get

$$\operatorname{rank} X^{[\alpha]}(t) = \operatorname{rank} S(t) = \operatorname{rank} P_{S\infty} \overset{(3.33)}{=} \operatorname{rank} P = \operatorname{rank} X(t) = n - d_\infty, \quad t \in [\beta, \infty)_{\mathbb{T}},$$

so that $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$ is a minimal conjoined basis on $[\beta,\infty)_{\mathbb{T}}$, and

$$\hat{X}^{[\alpha]}(t) = X(t)\, S(t)\, X^T(\alpha) \quad t \in [\beta,\infty)_{\mathbb{T}}.$$

By checking the four properties in (2.1) of the Moore–Penrose pseudoinverse it follows that

$$[\hat{X}^{[\alpha]}(t)]^{\dagger} = [X^{\dagger}(\alpha)]^T S^{\dagger}(t)\, X^{\dagger}(t), \quad t \in [\beta,\infty)_{\mathbb{T}}. \tag{6.5}$$

Since the image of $S(t)$ is constant on $[\beta,\infty)_{\mathbb{T}}$, hence the kernel of $S(t)$ is constant on $[\beta,\infty)_{\mathbb{T}}$, it follows by formula (2.3) in Remark 2.1(viii) that

$$[S^{\dagger}(t)]^{\Delta} S(t) = -[S^{\sigma}(t)]^{\dagger} S^{\Delta}(t), \quad t \in [\beta,\infty)_{\mathbb{T}}. \tag{6.6}$$

Multiplying this equation by the matrix $S^{\dagger}(t)$ from the right and using the definition of the constant orthogonal projector $P_{S\infty}$ in (3.15) we obtain

$$\begin{aligned} [S^{\dagger}(t)]^{\Delta} &= [S^{\dagger}(t)\, P_{S\infty}]^{\Delta} = [S^{\dagger}(t)]^{\Delta} P_{S\infty} \overset{(3.15)}{=} [S^{\dagger}(t)]^{\Delta} S(t)\, S^{\dagger}(t) \\ &\overset{(6.6)}{=} -[S^{\sigma}(t)]^{\dagger} S^{\Delta}(t)\, S^{\dagger}(t), \quad t \in [\beta,\infty)_{\mathbb{T}}. \end{aligned} \tag{6.7}$$

Consider now the matrix $\hat{S}^{[\alpha]}_{\beta}(t)$ in (3.1) associated with the principal solution $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$ on the interval $[\beta,\infty)_{\mathbb{T}}$, namely,

$$\hat{S}^{[\alpha]}_{\beta}(t) := \int_{\beta}^{t} [\hat{X}^{[\alpha]}(s)]^{\sigma\,\dagger} \mathcal{B}(s)\, [\hat{X}^{[\alpha]}(s)]^{\dagger\, T}\, \Delta s, \quad t \in [\beta,\infty)_{\mathbb{T}}. \tag{6.8}$$

Then by using (6.5), (6.7), and (6.8) we obtain for $t \in [\beta,\infty)_{\mathbb{T}}$

$$\begin{aligned} \hat{S}^{[\alpha]}_{\beta}(t) &\overset{(6.5)}{=} \int_{\beta}^{t} [X^{\dagger}(\alpha)]^T [S^{\dagger}(s)]^{\sigma} [X^{\dagger}(s)]^{\sigma} \mathcal{B}(s)\, [X^{\dagger}(s)]^T S^{\dagger}(s)\, X^{\dagger}(\alpha)\, \Delta s, \\ &= [X^{\dagger}(\alpha)]^T \left(\int_{\beta}^{t} [S^{\dagger}(s)]^{\sigma} S^{\Delta}(s)\, S^{\dagger}(s)\, \Delta s \right) X^{\dagger}(\alpha), \\ &\overset{(6.7)}{=} -[X^{\dagger}(\alpha)]^T \left(\int_{\beta}^{t} [S^{\dagger}(s)]^{\Delta}\, \Delta s \right) X^{\dagger}(\alpha) = [X^{\dagger}(\alpha)]^T\, [S^{\dagger}(\beta) - S^{\dagger}(t)]\, X^{\dagger}(\alpha). \end{aligned} \tag{6.9}$$

Now using the fact that (X, U) is the principal solution of (S) at infinity (i.e., $T = 0$), we get from (6.9) that the limit of $\hat{S}^{[\alpha]}_{\beta}(t)$ as $t \to \infty$ exists and

$$\lim_{t\to\infty} \hat{S}^{[\alpha]}_{\beta}(t) = [X^{\dagger}(\alpha)]^T\, [S^{\dagger}(\beta) - T]\, X^{\dagger}(\alpha) = [X^{\dagger}(\alpha)]^T S^{\dagger}(\beta)\, X^{\dagger}(\alpha).$$

This implies through Theorem 4.4(ii) that $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$ is an antiprincipal solution of (S) at infinity. Since we have already proved that $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$ is a minimal conjoined basis on $[\beta,\infty)_{\mathbb{T}}$, it follows that $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$ is a minimal antiprincipal solution of (S) at infinity. □

In the following result we present another example of antiprincipal solutions of (S) at infinity. It is a unification of the continuous case in [23, Proposition 1] and the discrete case in [24, Proposition 7.5], see also [10, Proposition 6.155].

Theorem 6.5. *Assume that system* (S) *is nonoscillatory and let* (X, U) *be a minimal conjoined basis of* (S) *on an interval* $[\alpha,\infty)_{\mathbb{T}}$ *satisfying* $d[\alpha,\infty)_{\mathbb{T}} = d_{\infty}$. *Then the associated conjoined basis* $(\bar{X}, \bar{U})$ *from Proposition 3.14 is a maximal antiprincipal solution of* (S) *at infinity.*

Proof. Let the conjoined bases (X, U) and $(\bar{X}, \bar{U})$ be as in the assumptions of the theorem. Let P, $S(t)$, $P_{S\infty}$ be the matrices in (2.6), (3.1), (3.15) corresponding to (X, U). Since (X, U) is a minimal conjoined basis of ($\mathbb{S}$) on $[\alpha, \infty)_{\mathbb{T}}$, we have $P = P_{S\infty}$ by Lemma 3.17. Moreover, $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$ it follows that $\operatorname{Im} S(t) \equiv \operatorname{Im} P_{S\infty}$ on $(\alpha, \infty)_{\mathbb{T}}$. Therefore, by Proposition 3.14(v) we then derive for all $t \in (\alpha, \infty)_{\mathbb{T}}$ that

$$\operatorname{Ker} \bar{X}(t) = \operatorname{Im} P \cap \operatorname{Ker} S(t) = \operatorname{Im} P_{S\infty} \cap [\operatorname{Im} S(t)]^\perp \equiv \operatorname{Im} P_{S\infty} \cap (\operatorname{Im} P_{S\infty})^\perp = \{0\}.$$

This shows that the matrix $\bar{X}(t)$ is invertible on $(\alpha, \infty)_{\mathbb{T}}$, in particular its kernel is constant on $(\alpha, \infty)_{\mathbb{T}}$. Fix any $\beta \in (\alpha, \infty)_{\mathbb{T}}$. We will show that $(\bar{X}, \bar{U})$ has no focal points in the interval (β, ∞). Recall that the matrix $S^\dagger(t)$ is nonincreasing on $[\beta, \infty)_{\mathbb{T}}$ and that, by Remark 2.1(viii) or by (6.7),

$$-[S^\sigma(t)]^\dagger S^\Delta(t)\, S^\dagger(t) = [S^\dagger(t)]^\Delta \le 0, \quad t \in [\beta, \infty)_{\mathbb{T}}. \tag{6.10}$$

Moreover, by Proposition 3.14(viii) we obtain for $t \in [\beta, \infty)_{\mathbb{T}}$ the equality

$$S^\dagger(t)\, X^\dagger(t) = \bar{X}^{-1}(t)\, X(t)\, P_{S\infty}\, X^\dagger(t) = \bar{X}^{-1}(t)\, X(t)\, P X^\dagger(t) = \bar{X}^{-1}(t)\, R(t). \tag{6.11}$$

Then by Proposition 3.6(ii) and by (6.11) with (6.10) we deduce that

$$\begin{aligned}
[\bar{X}^\sigma(t)]^{-1} \mathcal{B}(t)\, \bar{X}^{T-1}(t) &= [\bar{X}^\sigma(t)]^{-1} R^\sigma(t)\, \mathcal{B}(t)\, R(t)\, \bar{X}^{T-1}(t) \\
&\overset{(6.11)}{=} [S^\dagger(t)]^\sigma\, [X^\dagger(t)]^\sigma\, \mathcal{B}(t)\, [X^\dagger(t)]^T S^\dagger(t) = [S^\dagger(t)]^\sigma S^\Delta(t)\, S^\dagger(t) \\
&\overset{(6.10)}{=} -[S^\dagger(t)]^\Delta \ge 0, \quad t \in [\beta, \infty)_{\mathbb{T}},
\end{aligned} \tag{6.12}$$

and consequently

$$\bar{X}(t)\, [\bar{X}^\sigma(t)]^{-1} \mathcal{B}(t) = \bar{X}(t)\, [\bar{X}^\sigma(t)]^{-1} \mathcal{B}(t)\, \bar{X}^{T-1}(t)\, \bar{X}^T(t) \overset{(6.12)}{\ge} 0, \quad t \in [\beta, \infty)_{\mathbb{T}}.$$

This proves that $(\bar{X}, \bar{U})$ has no focal points in the interval (β, ∞) and hence, it is a maximal conjoined basis on $[\beta, \infty)_{\mathbb{T}}$. It remains to show that $(\bar{X}, \bar{U})$ is an antiprincipal solution of ($\mathbb{S}$) at infinity. According to (3.1), we define the associated matrix $\bar{S}(t)$ by

$$\bar{S}(t) := \int_\beta^t [\bar{X}^\sigma(s)]^{-1} \mathcal{B}(s)\, \bar{X}^{T-1}(s)\, \Delta s, \quad t \in [\beta, \infty)_{\mathbb{T}}. \tag{6.13}$$

Then by using (6.12) in (6.13) we get

$$\bar{S}(t) \overset{(6.12)}{=} -\int_\beta^t [S^\dagger(t)]^\Delta\, \Delta s = S^\dagger(\beta) - S^\dagger(t), \quad t \in [\beta, \infty)_{\mathbb{T}}. \tag{6.14}$$

This implies that the limit

$$\lim_{t \to \infty} \bar{S}(t) \overset{(6.14)}{=} \lim_{t \to \infty} [S^\dagger(\beta) - S^\dagger(t)] = S^\dagger(\beta) - T$$

exists. By Theorem 4.4(ii) (applied to $(X, U) := (\bar{X}, \bar{U})$) it then follows that the conjoined basis $(\bar{X}, \bar{U})$ is an antiprincipal solution of ($\mathbb{S}$) at infinity. The proof is complete. □

In our next result we utilize antiprincipal solutions of ($\mathbb{S}$) at infinity in the Reid construction of the minimal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ of ($\mathbb{S}$) at infinity. It is a unification of the continuous case in [23, Theorem 1] and the discrete case in [24, Theorem 7.3], see also [10, Theorem 6.153].

Theorem 6.6. *Assume that system* ($\mathbb{S}$) *is nonoscillatory. Let* (X, U) *be a minimal conjoined basis of* ($\mathbb{S}$) *on an interval* $[\alpha, \infty)_{\mathbb{T}}$ *satisfying* $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$ *and let* $\beta \in [\alpha, \infty)_{\mathbb{T}}$ *be such that the associated conjoined basis* $(\bar{X}, \bar{U})$ *from Proposition 3.14 is a maximal antiprincipal solution of* ($\mathbb{S}$) *at infinity with respect to the interval* $[\beta, \infty)_{\mathbb{T}}$*. Then for all* $\tau \in [\beta, \infty)_{\mathbb{T}}$ *the solutions* (X_τ, U_τ) *of* ($\mathbb{S}$) *given by the initial conditions*

$$X_\tau(\tau) = 0 \quad \textit{and} \quad U_\tau(\tau) = -[\bar{X}^{-1}(\tau)]^T \tag{6.15}$$

are conjoined bases of ($\mathbb{S}$) *satisfying*

$$\big(\hat{X}_{\min}(t), \hat{U}_{\min}(t)\big) = \lim_{\tau\to\infty} \big(X_\tau(t), U_\tau(t)\big), \quad t \in [a, \infty)_{\mathbb{T}}. \tag{6.16}$$

Proof. Let P, $P_{S\infty}$, $S(t)$, and T be the matrices in (2.6), (3.15), (3.1), and (3.6) associated with (X, U). Then $P = P_{S\infty}$ by Lemma 3.17 and by Proposition 3.14(viii) we get

$$S^\dagger(t) = \bar{X}^{-1}(t)\, X(t)\, P_{S\infty} = \bar{X}^{-1}(t)\, X(t)\, P = \bar{X}^{-1}(t)\, X(t), \quad t \in [\beta, \infty)_{\mathbb{T}}. \tag{6.17}$$

Fixed a point $\tau \in [\beta, \infty)_{\mathbb{T}}$. From (6.16) it follows that the solution (X_τ, U_τ) is a conjoined basis of ($\mathbb{S}$). Let us represent (X_τ, U_τ) in terms of (X, U) by using Proposition 3.15, i.e.,

$$\begin{pmatrix} X_\tau(t) \\ U_\tau(t) \end{pmatrix} = Z(t) \begin{pmatrix} M_\tau \\ N_\tau \end{pmatrix}, \quad Z(t) := \begin{pmatrix} X(t) & \bar{X}(t) \\ U(t) & \bar{U}(t) \end{pmatrix}, \quad t \in [a, \infty)_{\mathbb{T}}, \tag{6.18}$$

where the matrix $Z(t)$ is symplectic, i.e., $Z^{-1}(t) = -\mathcal{J} Z^T(t)\, \mathcal{J}$. Then the matrix $-M_\tau$ is the Wronskian of $(\bar{X}, \bar{U})$ and (X_τ, U_τ), and the matrix N_τ is the Wronskian of (X, U) and (X_τ, U_τ). Evaluating these Wronskians at the point τ we obtain

$$\begin{aligned} M_\tau &= -[\bar{X}^T(\tau)\, U_\tau(\tau) - \bar{U}^T(\tau)\, X_\tau(\tau)] \overset{(6.15)}{=} I, \\ N_\tau &= X^T(\tau)\, U_\tau(\tau) - U^T(\tau)\, X_\tau(\tau) \overset{(6.15)}{=} -X^T(\tau)\, [\bar{X}^{-1}(\tau)]^T \overset{(6.17)}{=} -[S^\dagger(\tau)]^T = -S^\dagger(\tau). \end{aligned}$$

This shows that the limits of M_τ and N_τ for $\tau \to \infty$ exist and

$$\lim_{\tau\to\infty} M_\tau = I, \quad \lim_{\tau\to\infty} N_\tau = -\lim_{\tau\to\infty} S^\dagger(\tau) = -T. \tag{6.19}$$

Therefore, the limit of (X_τ, U_τ) for $\tau \to \infty$ also exists and by (6.18) it is equal to the solution

$$\begin{pmatrix} \hat{X}(t) \\ \hat{U}(t) \end{pmatrix} := \lim_{\tau\to\infty} \begin{pmatrix} X_\tau(t) \\ U_\tau(t) \end{pmatrix} \overset{(6.18)}{=} \lim_{\tau\to\infty} Z(t) \begin{pmatrix} M_\tau \\ N_\tau \end{pmatrix} \overset{(6.19)}{=} Z(t) \begin{pmatrix} I \\ -T \end{pmatrix}, \quad t \in [a, \infty)_{\mathbb{T}}. \tag{6.20}$$

In fact, since $\operatorname{rank}(I, -T)^T = n$ and the matrix T is symmetric, the solution $(\hat{X}, \hat{U})$ defined in (6.20) is a conjoined basis of ($\mathbb{S}$). By Theorem 6.1 we then conclude that $(\hat{X}, \hat{U})$ is the minimal principal solution of ($\mathbb{S}$) at infinity, i.e., $(\hat{X}, \hat{U}) = (\hat{X}_{\min}, \hat{U}_{\min})$. $\square$

The following three comments complement the construction of $(\hat{X}_{\min}, \hat{U}_{\min})$ in Theorem 6.6.

Remark 6.7. The initial conditions in (6.15) show that the conjoined basis (X_τ, U_τ) is a constant nonsingular multiple of the principal solution $(\hat{X}^{[\tau]}, \hat{U}^{[\tau]})$ of ($\mathbb{S}$) at the point τ, namely

$$(X_\tau, U_\tau) = \big(\hat{X}^{[\tau]} M, \hat{U}^{[\tau]} M\big), \quad M := -\bar{X}^T(\tau).$$

Then, in view of Theorem 6.4 and Proposition 4.5, we may conclude that for all $\tau \in [a, \infty)_{\mathbb{T}}$ with $\tau \geq \max\{\hat{\alpha}_{\min}, \beta\}$ the conjoined bases (X_τ, U_τ) in Theorem 6.6 are minimal antiprincipal solutions of ($\mathbb{S}$) at infinity.

Remark 6.8. The limit formula for $(\hat{X}, \hat{U})$ in (6.20) shows, how this construction depend on the chosen initial conditions of (X_τ, U_τ) in (6.15). More precisely, let us consider instead of (6.15) the initial conditions $X_\tau(\tau) = 0$ and $U_\tau(\tau) = K_\tau$, where K_τ are invertible matrices for all $\tau \in [\beta, \infty)_{\mathbb{T}}$. Then the matrices M_τ and N_τ from the representation in (6.18) satisfy

$$M_\tau = -\bar{X}^T(\tau)\, K_\tau, \quad N_\tau = X^T(\tau)\, K_\tau \overset{(6.17)}{=} S^\dagger(\tau)\, \bar{X}^T(\tau)\, K_\tau,$$

where we used the fact that the matrix $\bar{X}(\tau)$ is invertible. This shows that for $t \in [a, \infty)_{\mathbb{T}}$ the limit of $(X_\tau(t), U_\tau(t))$ as $\tau \to \infty$ exists if and only the limit

$$M_\infty := \lim_{\tau \to \infty} \bar{X}^T(\tau)\, K_\tau$$

exists, and in this case the limiting solution $(\hat{X}, \hat{U})$ in (6.20) is equal to $(\hat{X}_{\min} M_\infty, \hat{U}_{\min} M_\infty)$.

Remark 6.9. The construction in Theorem 6.6 does not depend on the chosen minimal conjoined basis (X, U) on $[\alpha, \infty)_{\mathbb{T}}$. More precisely, suppose that we start with another minimal conjoined basis (X_*, U_*) of (S) on $[\alpha, \infty)_{\mathbb{T}}$ and denote by $(\bar{X}_*, \bar{U}_*)$ its associated conjoined basis from Proposition 3.14. Let us we represent (X_*, U_*) in terms of (X, U) and $(\bar{X}, \bar{U})$ as

$$\begin{pmatrix} X_*(t) \\ U_*(t) \end{pmatrix} = Z(t) \begin{pmatrix} M \\ N \end{pmatrix}, \quad t \in [a, \infty)_{\mathbb{T}},$$

where the fundamental matrix $Z(t)$ is given in (6.18) and the matrix M is invertible (see Proposition 3.15 with $(X_2, U_2) := (X_*, U_*)$, $(X_1, U_1) := (X, U)$, and $M_1 := M$). Then by using Lemma 3.19 we have $\bar{X}_*(t) = \bar{X}(t)\, M^{T-1}$ on $[\alpha, \infty)_{\mathbb{T}}$. Similarly to (6.15) we now consider for $\tau \in [\beta, \infty)_{\mathbb{T}}$ the conjoined bases $(X_{*\tau}, U_{*\tau})$ given by the initial conditions

$$X_{*\tau}(\tau) = 0 \quad \text{and} \quad U_{*\tau}(\tau) = -[\bar{X}_*^{-1}(\tau)]^T = -[\bar{X}^{-1}(\tau)]^T M.$$

Then $(X_{*\tau}, U_{*\tau}) = (X_\tau M, U_\tau M)$ on $[a, \infty)_{\mathbb{T}}$ and we derive that

$$\lim_{\tau \to \infty} \big(X_{*\tau}(t), U_{*\tau}(t)\big) = \lim_{\tau \to \infty} \big(X_\tau(t)\, M, U_\tau(t)\, M\big) = \big(\hat{X}_{\min}(t)\, M, \hat{U}_{\min}(t)\, M\big), \quad t \in [a, \infty)_{\mathbb{T}},$$

i.e., this modified construction leads to a constant nonsingular multiple of the minimal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ of (S) at infinity given in (6.16).

7 Concluding remarks

In this paper we developed the theory of antiprincipal solutions at infinity for nonoscillatory symplectic dynamic systems on time scales. The motivation for this study comes from the theory of principal solutions at infinity for these systems, and from the corresponding theory of antiprincipal or dominant solutions at infinity, which exists in the continuous or discrete time setting. Our main results include in particular a characterization of antiprincipal solutions of (S) at infinity in terms of the limit of its associated S-matrix (Theorem 4.4), a characterization of minimal conjoined bases of (S) on a given interval $[\alpha, \infty)_{\mathbb{T}}$ in terms of the initial conditions at α (Theorem 5.1), the existence of antiprincipal solutions of (S) at infinity (Theorem 5.3), and several additional properties or applications of antiprincipal solutions of (S) at infinity (presented in Theorems 6.4, 6.5, and 6.6).

Note that, unlike in the continuous or discrete cases, the existence of a nonoscillatory conjoined basis of (S) does not (so far) imply the nonoscillation of system (S) on arbitrary time scale $\mathbb{T}$, see Remark 5.4. The reason is a nonexisting pointwise definition of the multiplicity of a focal point for general time scales. We believe that this problem might be solved by using the comparative index theory, see [12] or [10, Chapter 3], in combination with the theory of principal and antiprincipal solutions of symplectic systems on time scales. This is a work in progress.

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Linear even order homogenous difference equation with delay in coefficient

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Abstract. We use many classical results known for the self-adjoint second-order linear equation and extend them for a three-term even order linear equation with a delay applied to coefficients. We derive several conditions concerning the oscillation and the existence of positive solutions. Our equation for a choice of parameter is disconjugate, and for a different choice can have positive and oscillatory solutions at the same time. However, it is still, in a sense, disconjugate if we use a weaker definition of oscillation.

Keywords: coefficient delayed equations, separately disconjugate, oscillation theory, minimal solution, difference equation.

2020 Mathematics Subject Classification: 39A06, 39A21, 39A22, 47B36, 47B39.

1 Introduction

This paper is divided into two parts. In the first part, we analyse the linear second-order homogeneous difference equation with a delay in a coefficient

$$a_{n-k}y_{n-1} + b_n y_n + a_n y_{n+1} = 0, \quad n \in \mathbb{Z}. \tag{1.1}$$

Equations with a delay in term y_{n-1} are usually considered. Nevertheless, we did not find a situation where the considered delay is in the coefficient a_n. This may be because Eq. (1.1) for $k = 1$ is often discussed together with its self-adjoint form $\triangle(p_n \triangle y_n) + q_n y_{n+1} = 0$.

Properties of this special case were discussed many times. Some necessary and sufficient conditions for the equation to be oscillatory were derived in [6, 8, 10, 19, 20, 22, 29] and for a matrix case in [7]. Properties of eventually positive solutions were observed in [28]. Minimal solutions of the special case were discussed in [14]. Recessive solutions and their connection to oscillation were discussed in [27], for a matrix case in [3], and for nonoscillatory symplectic systems in [33]. Notion of generalized zero was developed in [15] and the Sturm comparison theorem on $\mathbb{Z}$ together with the existence of a recessive solutions was discussed in [2, 5]. Many classic results about this special case can be found in [21]. Boundedness and growth of the special case were investigated in [30,31]. Generalization of the special case were considered

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for example in [24–26, 32]. If we consider a continuous case, criteria for oscillation can be found, for example, in [11], and the existence of a principal solution of a $2n$-order self-adjoint equation was recently discussed in [34]. Some ideas about how to extend the results for the fourth-order equation can be found in [9].

In Section 2, we would like to extend the results from [14], where the special case is also considered. The results from [14] were already extended in [12, 13, 17] and for the time scales in [18], but there was used the symmetrical case for $k = 1$. Arbitrary choice of $k \in \mathbb{Z}$ will lead to the generalization of some already known results.

We derive equivalent conditions for which the equation has a positive solution, and later through the deriving of a suitable version of the Sturm comparison theorem, we will get criteria of disconjugacy for Eq. (1.1). These results will be used in Section 3 as a tool, as well.

In Section 3 we analyse the linear even order homogeneous difference equation with a delay in a coefficient

$$a_{n-kH}y_n + b_{n+H}y_{n+H} + a_{n+H}y_{n+2H} = 0, \quad n \in \mathbb{Z}, \tag{1.2}$$

which is a generalization of Eq. (1.1). For $k = 0$ we get a equation discussed in [16]. We can assume that results obtained in Section 2 can be extended for Eq. (1.2) in the similar way as in [16].

We derive conditions under which Eq. (1.2) can or cannot have positive or eventually positive solutions. We also discuss a situation when Eq. (1.2) has recessive and dominant solutions. Among others, we use a combination of ideas as were established in [19, 27]. We find that Eq. (1.2) can have both positive and sign-changing solutions. A situation where an equation has oscillatory and nonoscillatory solutions at the same time was discussed for example in [1]. The same situation can appear in our equation, but we use a weaker version of oscillation to avoid this situation.

2 Second-order linear coefficient delayed equation

Let real valued sequences a_n, b_n satisfy $a_n < 0$, $b_n > 0$, for every $n \in \mathbb{Z}$. In the first part we study the equation

$$a_{n-k}y_{n-1} + b_n y_n + a_n y_{n+1} = 0, \quad k \in \mathbb{Z}. \tag{2.1}$$

If we consider a solution y_n of Eq. (2.1), then we have a solution $x_n = (-1)^n y_n$ of the equation

$$a_{n-k}x_{n-1} + d_n x_n + a_n x_{n+1} = 0,$$

where sequence $d_n < 0$ for every n. In a similar sense if we consider the equation

$$c_{n-p}x_{n-1} + b_n x_n + c_{n+l}x_{n+1} = 0,$$

where $c_n < 0$ for every n. Then we can take $a_n = c_{n+l}$ and this will result in Eq. (2.1) for $k = -l - p$.

There is a natural relation of Eq. (2.1) to the infinite matrix operator, whose truncations for

$n \leq p$, $n, p \in \mathbb{Z}$, are the matrices

$$d_{n,p} = \begin{pmatrix} b_n & a_n & 0 & \dots & 0 \\ a_{n-k+1} & b_{n+1} & a_{n+1} & \ddots & \vdots \\ 0 & a_{n-k+2} & b_{n+2} & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & a_{p-1} \\ 0 & \dots & \dots & a_{p-k} & b_p \end{pmatrix}$$

and we denote their determinants by $D_{n,p} = \det(d_{n,p})$. Note that for $k = 1$ is $d_{n,p}$ symmetrical.

For simplification of formulas, we take $D_{i+1,i} = 1$ and $D_{i+j,i} = 0$ for any $i \in \mathbb{Z}$ and $j > 1$, as well as $\prod_i^{i-1} x_i = 1$. Moreover, we will use recurrence relations

$$D_{n,p} = b_n D_{n+1,p} - a_{n-k+1} a_n D_{n+2,p}, \tag{2.2}$$
$$D_{n,p} = b_p D_{n,p-1} - a_{p-k} a_{p-1} D_{n,p-2}, \tag{2.3}$$

for $n \leq p$.

Lemma 2.1. *Let $n < p$ and real vectors $\mathbf{X} = (x_n, \dots, x_p)^T$, $\mathbf{B} = (y, 0 \dots, 0, z)^T$, then the equation*

$$d_{n,p}\mathbf{X} = \mathbf{B},$$

implies

$$x_h D_{n,p} = y D_{h+1,p} \prod_{j=n+1}^{h} (-a_{j-k}) + z D_{n,h-1} \prod_{j=h}^{p-1} (-a_j), \tag{2.4}$$

where $n \leq h \leq p$.

Proof. The proof follows from the Cramer's rule. Signs at $-a_j$ and $-a_{j-k}$ follow from comparing the sign and number of terms in a given product. □

Lemma 2.2. *Let*

$$D_{i,j} > 0, \quad \textit{for } i \leq j, \tag{2.5}$$

and let x_n^1, x_n^2 be two solutions of Eq. (2.1), which satisfy $x_m^1 = x_m^2$ for some $m \in \mathbb{Z}$. If also $x_h^1 > x_h^2$ (respectively $x_h^1 = x_h^2$) for some $h > m$, then it holds that $x_j^1 > x_j^2$ (respectively $x_j^1 = x_j^2$) for all $j > m$.

Proof. Obviously, two solutions x_n^1, x_n^2 of Eq. (2.1) have to also satisfy Lemma 2.1 where

$$y = -a_{m-k+1} x_m^1 = -a_{m-k+1} x_m^2,$$
$$z^1 = -a_{h-1} x_h^1 > -a_{h-1} x_h^2 = z^2.$$

Where for $i \in \{1, 2\}$ we have $\mathbf{X}^i = \left(x_{m+1}^i, \dots, x_{h-1}^i\right)^T$ and $\mathbf{B}^i = (y, 0 \dots, 0, z^i)^T$. Together with (2.5), we obtain from (2.4) that

$$\begin{aligned} x_j^1 D_{m+1,h-1} &= y D_{j+1,h-1} \prod_{i=m+2}^{j} (-a_{i-k}) + z^1 D_{m+1,j-1} \prod_{i=j}^{h-2} (-a_i) \\ &> y D_{j+1,h-1} \prod_{i=m+2}^{j} (-a_{i-k}) + z^2 D_{m+1,j-1} \prod_{i=j}^{h-2} (-a_i) = x_j^2 D_{m+1,h-1}, \end{aligned}$$

holds for all $n < j < h$ and thus $x_j^1 > x_j^2$. Taking $x_j^1 < x_j^2$ for some $j > h$ leads to a contradiction with $x_h^1 > x_h^2$ in the same manner. Therefore, $x_j^1 > x_j^2$ for all $j > m$. The case of $x_h^1 = x_h^2$ follows analogously. □

Similarly, we get a version of Lemma 2.2 for some $h < m$ and all $j < m$. It means that if two solutions of Eq. (2.1) are equal at two points, then they are equal everywhere.

Lemma 2.3. *Assume* (2.5), *then for any $h < p$ it holds that*

$$\frac{1}{b_h} < \frac{D_{h+1,p}}{D_{h,p}} < \frac{b_{h-1}}{a_{h-k}a_{h-1}}, \tag{2.6}$$

and the sequence $x_p = \frac{D_{h+1,p}}{D_{h,p}}$ is increasing for any h where $h < p$.

Proof. Because of (2.2) we get

$$D_{h,p} = b_h D_{h+1,p} - a_{h-k+1}a_h D_{h+2,p} < b_h D_{h+1,p},$$

which implies the left inequality of (2.6). Further, we compute

$$\begin{aligned} 0 < D_{h-1,p} &= b_{h-1}D_{h,p} - a_{h-k}a_{h-1}D_{h+1,p}, \\ a_{h-k}a_{h-1}D_{h+1,p} &< b_{h-1}D_{h,p}, \\ \frac{D_{h+1,p}}{D_{h,p}} &< \frac{b_{h-1}}{a_{h-k}a_{h-1}}, \end{aligned}$$

which implies the right inequality in (2.6).

In the second part of the proof, we will proceed by induction. First, we assume $p = h+1$ and we get

$$\begin{aligned} \frac{D_{h+1,h+2}}{D_{h,h+2}} - \frac{D_{h+1,h+1}}{D_{h,h+1}} &= \frac{D_{h,h+1}D_{h+1,h+2} - D_{h+1,h+1}D_{h,h+2}}{D_{h,h+2}D_{h,h+1}} \\ &= \frac{a_{h-k+1}a_{h-k+2}a_h a_{h+1}}{D_{h,h+2}D_{h,h+1}} > 0. \end{aligned}$$

Next, again by (2.2), we get

$$\frac{D_{h,p}}{D_{h+1,p}} - \frac{D_{h,p+1}}{D_{h+1,p+1}} = a_{h-k+1}a_h \left(\frac{D_{h+2,p+1}}{D_{h+1,p+1}} - \frac{D_{h+2,p}}{D_{h+1,p}} \right) > 0,$$

by the induction assumption, which together with (2.5) results in

$$\begin{aligned} \frac{D_{h,p}}{D_{h+1,p}} &> \frac{D_{h,p+1}}{D_{h+1,p+1}}, \\ \frac{D_{h+1,p}}{D_{h,p}} &< \frac{D_{h+1,p+1}}{D_{h,p+1}}. \end{aligned}$$

Therefore, the sequence is increasing and the proof is complete. □

Similarly, using (2.3), we get for $n < h$ that

$$\frac{1}{b_h} < \frac{D_{n,h-1}}{D_{n,h}} < \frac{b_{h+1}}{a_{h-k+1}a_h},$$

and the sequence $x_n = \frac{D_{n,h-1}}{D_{n,h}}$ is decreasing for any h which $n < h$.

Now, thanks to Lemma 2.3, we can define the sequences

$$c_n^+ = \lim_{p\to\infty} \frac{D_{n+1,p}}{D_{n,p}},$$
$$c_n^- = \lim_{p\to-\infty} \frac{D_{p,n-1}}{D_{p,n}},$$

and

$$u(j,n) = \begin{cases} 1, & j = n, \\ \prod_{h=n}^{j-1}(-a_h)c_h^-, & n < j, \\ \prod_{h=j}^{n-1}(-a_{h-k+1})c_{h+1}^+, & n > j. \end{cases}$$

Notice that by Lemma 2.3 together with $a_i < 0$ for every i, we get that $u(j,n) > 0$ for any j, n.

Definition 2.4. We say that a solution u_n of Eq. (2.1) is minimal on $[j+1,\infty)\cap\mathbb{Z}$ if any linearly independent solution v_n of Eq. (2.1) such that $u_j = v_j$ satisfies $u_k < v_k$ for every $k \geq j+1$. The minimal solution on $(-\infty, j-1]\cap\mathbb{Z}$ is defined analogously.

Lemma 2.5. *Assume (2.5), then $\alpha_n = u(j,n)$ is a positive minimal solution of Eq. (2.1) on the interval $[j+1,\infty)\cap\mathbb{Z}$ and also on the interval $(-\infty, j-1]\cap\mathbb{Z}$.*

Proof. Using Lemma 2.1 with $y = -a_{j-k+1}$ and $z = 0$ we obtain that

$$v_n(j,p) = \begin{cases} 1, & n = j, \\ \prod_{h=j}^{n-1}(-a_{h-k+1})\frac{D_{n+1,p}}{D_{j+1,p}}, & j+1 \leq n \leq p, \\ 0, & n = p+1, \end{cases}$$

is a solution on the interval $[j+1,p]\cap\mathbb{Z}$. Moreover, it holds that $u(j,n) = \lim_{p\to\infty} v_n(j,p)$ and so $\alpha_n = u(j,n)$ is a solution on the interval $[j+1,\infty)\cap\mathbb{Z}$, where $\alpha_j = u(j,j) = 1$.

Next, we assume that there is a positive solution v_n such that $v_j = \alpha_j$ and which is also linearly independent on α_n. Then we know that $v_{p+1} > v_{p+1}(j,p) = 0$ and $v_j = v_j(j,p) = 1$, for every p. Therefore, due to Lemma 2.2, we know that $v_n > v_n(j,p)$ for all p. Because $\alpha_n = \lim_{p\to\infty} v_n(j,p)$, we get that $v_n \geq \alpha_n$. But v_n is linearly independent and, again by Lemma 2.2, this inequality must hold strictly, i.e. $v_n > \alpha_n$.

Similarly, we get that $\alpha_n = u(j,n)$ is a solution on interval $(-\infty, j-1]\cap\mathbb{Z}$ using function

$$v_n(j,m) = \begin{cases} 1, & n = j, \\ \prod_{h=n}^{j-1}(-a_h)\frac{D_{m,n-1}}{D_{m,j-1}}, & m \leq n \leq j-1, \\ 0, & n = m-1. \end{cases}$$

□

Further, we will use the following notation. We define

$$u_n^+ = \begin{cases} 1, & n = 0, \\ u(0,n), & n \in \mathbb{N}, \\ u(n,0)^{-1}, & -n \in \mathbb{N}, \end{cases} \quad \text{and} \quad u_n^- = \begin{cases} 1, & n = 0, \\ u(n,0)^{-1}, & n \in \mathbb{N}, \\ u(0,n), & -n \in \mathbb{N}. \end{cases}$$

Lemma 2.6. *Assume (2.5), then $u_n^\pm$ are positive solutions of Eq. (2.1) on $\mathbb{Z}$.*

Proof. From Lemma 2.5 we know, that u_n^+ is a solution on $\mathbb{N}$. Moreover, for arbitrary $n, B \in \mathbb{N} \cup \{0\}$, $n < B$, it holds

$$u(-B,0) = u(-B,-n)u(-n,0),$$

and so

$$u_{-n}^+ = \frac{1}{u(-n,0)} = \frac{u(-B,-n)}{u(-B,0)}.$$

Using Lemma 2.5 we obtain that u_n^+ is a solution on interval $[-B+1,\infty) \cap \mathbb{Z}$. Because B is arbitrary, we have that u_n^+ is a solution on $\mathbb{Z}$. The second part involving u_n^- is done in the similar way. □

Theorem 2.7. *Condition (2.5) holds if and only if there is a positive solution of Eq. (2.1).*

Proof. The sufficiency of (2.5) comes directly from Lemma 2.6. For the second part, we assume the existence of a positive solution u_n. Then, using Lemma 2.1 for arbitrary n, $n < p$, with $y = -a_{n-k}u_{n-1}, z = -a_p u_{p+1}$, we get from (2.4) that

$$u_n D_{n,p} = -a_{n-k}u_{n-1}D_{n+1,p} - a_p u_{p+1} \prod_{j=n}^{p-1}(-a_j).$$

If we put $p = n+1$, then because $D_{n+1,n+1} = b_{n+1} > 0$ we obtain that the right-hand side is positive which implies the positivity of $D_{n,n+1} > 0$. Next, by induction we obtain that if $D_{n+1,p} > 0$, then also $D_{n,p} > 0$ through the same procedure. Therefore, the condition (2.5) is satisfied. □

We emphasize that for $k = 1$ is $d_{n,p}$ symmetrical, thus condition (2.5) gives the positive definiteness of all $d_{n,p}$. Now we recall the definitions of generalized zero and disconjugacy.

Definition 2.8. Solution y_n has a generalized zero at n_0 if $y_{n_0} = 0$ or $y_{n_0-1}y_{n_0} < 0$.

Definition 2.9. The given difference equation is disconjugate on an interval I if every nontrivial solution has at most one generalized zero on I.

Lemma 2.10. *Let Eq. (2.1) be disconjugate on interval $[a,b]$ then the boundary value problem*

$$\begin{aligned} &a_{n-k}y_{n-1} + b_n y_n + a_n y_{n+1} = 0, \\ &y_{n_1} = A, \quad y_{n_2} = B, \end{aligned}$$

where $a \le n_1 < n_2 \le b$ and $A, B \in \mathbb{R}$, has an unique solution.

Proof. General solution of Eq. (2.1) is

$$y_n = Cz_n^1 + Dz_n^2,$$

for some linearly independent z_n^1 and z_n^2. The boundary conditions result in the system

$$\begin{aligned} Cz_{n_1}^1 + Dz_{n_1}^2 &= A, \\ Cz_{n_2}^1 + Dz_{n_2}^2 &= B. \end{aligned}$$

We see that the boundary value problem has a solution whenever

$$\det \begin{pmatrix} z_{n_1}^1 & z_{n_1}^2 \\ z_{n_2}^1 & z_{n_2}^2 \end{pmatrix} \neq 0.$$

Now assume that this determinant is equal to zero. Then there would exist constants $C, D \in \mathbb{R}$ such that

$$\begin{aligned} Cz^1_{n_1} + Dz^2_{n_1} &= 0, \\ Cz^1_{n_2} + Dz^2_{n_2} &= 0. \end{aligned}$$

Thus, $y_{n_1} = y_{n_2} = 0$. This contradicts that Eq. (2.1) is disconjugate. □

Theorem 2.11. *Let Eq.* (2.1) *be disconjugate on* $\mathbb{Z}$*, then* (2.5) *holds.*

Proof. We will show that $D_{i,i+k-1} > 0$ by induction on $k \in \mathbb{N}$ for arbitrary i. Because $b_i > 0$ we have that $D_{i,i} > 0$.

Let y_n be a solution of

$$\begin{aligned} &a_{n-k}y_{n-1} + b_n y_n + a_n y_{n+1} = 0, \\ &y_{i-1} = 0, \quad y_{i+k+1} = 1, \end{aligned}$$

and assume that $D_{i,i+k-1} > 0$. By Lemma 2.10, we know that such y_n exists and it must satisfy system

$$d_{i,i+k}\mathbf{y} = \mathbf{b},$$

where $\mathbf{y} = (y_i, \dots, y_{i+k})^T$, $\mathbf{b} = (0, \dots, 0, -a_{i+k})$. Now, using Lemma 2.1 we get that

$$y_{i+k}D_{i,i+k} = -a_{i+k}D_{i,i+k-1}.$$

By disconjugacy we know that $y_{i+k} > 0$ and together with the assumption $D_{i,i+k-1} > 0$ we see that $D_{i,i+k} > 0$, as well. □

Corollary 2.12. *Let Eq.* (2.1) *be disconjugate on* $\mathbb{Z}$*, then there exists a positive solution of Eq.* (2.1)*.*

Proof. This is a direct consequence of Theorem 2.7. □

The natural question is whether the converse statement is valid as well. We will solve this problem by formulating an appropriate version of Sturm's comparison theorem. Nevertheless, it can be solved using Theorem 2.7 and Lemma 2.6 together with $u^{\pm}_n$ being minimal solutions as well. Note that we have two separate situations where $u^+_n = u^-_n$ and $u^+_n \neq u^-_n$.

Lemma 2.13. *If y_n is a nontrivial solution of Eq.* (2.1) *such that $y_{n_0} = 0$, then $y_{n_0-1}y_{n_0+1} < 0$.*

Proof. If y_n is a nontrivial solution and $y_{n_0} = 0$ for some $n_0 \in \mathbb{Z}$, then $y_{n_0-1} \neq 0 \neq y_{n_0+1}$. The rest follows from y_n being a solution of Eq. (2.1). □

Lemma 2.14. *Assume* (2.5)*. If a nontrivial solution y_n of Eq.* (2.1) *has two generalized zeros at n_1 and n_2, then any other linearly independent solution has a generalized zero in $[n_1, n_2]$.*

Proof. Without loss of generality assume that there are not other generalized zeros of y_n on (n_1, n_2). Now by contradiction, we assume that $y_n > 0$ on (n_1, n_2) and that there is a linearly independent solution z_n such that $z_n > 0$ on $[n_1, n_2]$ and $z_{n_1-1} \geq 0$, i.e. it does not have a generalized zero on $[n_1, n_2]$. We consider some n_0 from (n_1, n_2) and we can find $K \in \mathbb{R}$ such that $Kz_{n_0} = y_{n_0}$. Because $y_{n_2} \leq 0$ and it has to hold that $y_{n_1} = 0$ or $y_{n_1-1} < 0$ we can use Lemma 2.2 to get that $Kz_n > y_n$. Moreover, $u_n = Kz_n - y_n$ is also a solution of Eq. (2.1) and $u_{n_0} = 0$, $u_n > 0$ for $n \neq n_0$. Finally, $u_{n_0-1}u_{n_0+1} > 0$ gives us a contradiction with Lemma 2.13. □

Theorem 2.15. *Eq. (2.1) is disconjugate on $\mathbb{Z}$ if and only if it has a positive solution on $\mathbb{Z}$.*

Proof. We already have the first part from Corollary 2.12. Next, assume that Eq. (2.1) has a positive solution. By Theorem 2.7 we know, that (2.5) holds and so does Lemma 2.14. However, because we have a positive solution, then by Lemma 2.14, we know that there cannot be a solution with more than one generalized zero. □

3 Even order linear coefficient delayed equation

In this section we will focus on the equation

$$a_{n-kH}y_n + b_{n+H}y_{n+H} + a_{n+H}y_{n+2H} = 0, \tag{3.1}$$

for $n \in \mathbb{Z}$, with the parameters $H \in \mathbb{N}$, $k \in \mathbb{Z}$.

Lemma 3.1. *If $a_i < 0$ for every i and there is a subsequence b_{n_l} such that $b_{n_l} \le 0$ for $n_l \to \infty$ then Eq. (3.1) cannot have an eventually positive solution (i.e. a solution y_n, where $y_n > 0$ for all $n \ge N$, for some $N \in \mathbb{Z}$).*

Proof. Suppose that there exist an eventually positive solution y_n. It implies

$$a_{n_l-k\cdot H}y_{n_l} + b_{n_l+H}y_{n_l+H} + a_{n_l+H}y_{n_l+2H} < 0,$$

for $n_l \to \infty$. This is a contradiction with y_n being a solution of Eq. (3.1). □

Similar statement holds even if $n_l \to -\infty$ and $y_n > 0$ for all $n \le N$ for some $N \in \mathbb{Z}$. Because of this, we will again assume that $a_j < 0$, $b_j > 0$ for every j.

Theorem 3.2. *The following statements are true.*

1. *Let H be an even number, then Eq. (3.1) has a solution y_n if and only if it has a solution $(-1)^n y_n$.*
2. *Let H be an odd number, then Eq. (3.1) cannot have a solution $(-1)^n p_n$ where $p_n > 0$ for all $|n| \ge N$ and some $N \in \mathbb{N}$.*

Proof. For the first part, it suffices to use $z_n = (-1)^n y_n$ in Eq. (3.1) and the rest follows from H being even.

To prove the second part, we suppose that Eq. (3.1) has a solution $(-1)^n p_n$. Then we have that

$$a_{n-k\cdot H}p_n + b_{n+H}(-1)^H p_{n+H} + a_{n+H}p_{n+2H} = 0.$$

For $|n|$ sufficiently large, the terms are negative, hence the left-hand side cannot be equal zero and such a solution cannot exist. □

Corollary 3.3. *Let H be an even number, then Eq. (3.1) has at least on solution, which is not eventually positive.*

Proof. Assume that all solutions of Eq. (3.1) are eventually positive. Then there is a solution y_n, which is positive for n greater than some N. However, because H is an even number, then $(-1)^n y_n$ is also a solution of Eq. (3.1) and is not eventually positive. Thus we arrive to a contradiction. □

We obtain further generalization if we let p_n^k be real sequences and consider a linear equation

$$\sum_{k=0}^{m} p_n^k y_{n+2k} = 0. \tag{3.2}$$

Then Eq. (3.2) has a solution, which is not eventually positive.

We see that, in some cases, the studied equation cannot have a positive solution. Later we show that there is an equation that has positive and sign-changing solutions at the same time, which is a case that for $k = 0$ cannot occur. For this reason, it is more useful to focus on the situation when the equation has a positive solution. Nevertheless, we start by reminding us of the lemma, which can be found in [21].

Lemma 3.4. *Let us consider the equation*

$$\sum_{k=0}^{m} p_n^k u_{n+k} = 0, \tag{3.3}$$

where p_n^k, $k \in \{0, \dots, m\}$, are real sequences, for some $m \in \mathbb{N}$. If Eq. (3.3) has a solution u_n, then Eq. (3.3) has another solution in the form $v_n u_n$, where v_n solves the equation

$$\sum_{k=0}^{m-1} \left(\sum_{i=0}^{k} p_n^i u_{n+i} \right) \triangle v_{n+k} = 0. \tag{3.4}$$

Proof. We expand the sum $\sum_{k=0}^{m} p_n^k v_{n+k} u_{n+k}$ by Abel's summation formula and use the fact that u_n is a solution of Eq. (3.3) to obtain Eq. (3.4). □

Assume that we have a solution u_n of Eq. (3.1) and using Lemma 3.4 we obtain other solution as $v_n u_n$, where v_n solves

$$a_{n-k\cdot H} u_n \sum_{j=0}^{H-1} \triangle v_{n+j} + (a_{n-k\cdot H} u_n + b_{n+H} u_{n+H}) \sum_{j=0}^{H-1} \triangle v_{n+H+j} = 0.$$

Using the substitution $z_n = v_{n+H} - v_n$ we get using u_n being a solution of Eq. (3.1) that

$$0 = a_{n-kH} u_n z_n + (a_{n-k\cdot H} u_n + b_{n+H} u_{n+H}) z_{n+H} = a_{n-kH} u_n z_n - a_{n+H} u_{n+2H} z_{n+H}. \tag{3.5}$$

Whenever $u_n \neq 0$ for all n, then the solution of Eq. (3.5) is

$$z_n = \frac{D \prod_{j=1}^{-k-1} a_{n+jH}}{u_n u_{n+H} \prod_{j=-k}^{0} a_{n+jH}},$$

for some $D \in \mathbb{R}$. Finally, we can use the fact that $z_n = v_{n+H} - v_n$. Hence,

$$v_n = -\sum_{g=0}^{\infty} z_{n+gH}, \tag{3.6}$$

$$v_n = \sum_{g=1}^{\infty} z_{n-gH}.$$

Definition 3.5. We say that a solution u_n of Eq. (3.1) is minimal on $[\mu, \infty) \cap \mathbb{Z}$ if any linearly independent solution v_n of Eq. (3.1) with $u_\mu = v_\mu, \dots, u_{\mu+H-1} = v_{\mu+H-1}$ satisfies $v_n > u_n$, for every $n \geq \mu + H$.

Theorem 3.6. *Let Eq. (3.1) have a positive solution u_n on $\mathbb{Z}$, which is minimal on an interval $[l,\infty)$, where $l \in \mathbb{Z}$. Then for every $\mu \in \mathbb{Z}$ it holds*

$$\sum_{g=0}^{\infty} \frac{\prod_{j=g+1}^{g-k-1}(-a_{\mu+jH})}{u_{\mu+gH}u_{\mu+(g+1)H}\prod_{j=g-k}^{g}(-a_{\mu+jH})} = \infty. \tag{3.7}$$

Proof. Assume that for some $\mu \in \mathbb{Z}$ the sum in (3.7) is finite. Since u_n is a positive solution, by (3.6) we know that also

$$w_n = \begin{cases} u_n \sum_{g=0}^{\infty} \frac{\prod_{j=g+1}^{g-k-1}(-a_{n+jH})}{u_{n+gH}u_{n+(g+1)H}\prod_{j=g-k}^{g}(-a_{n+jH})}, & n \equiv \mu (\mathrm{mod}\ H), \\ u_n, & n \not\equiv \mu (\mathrm{mod}\ H), \end{cases}$$

is a positive solution.

Next, we introduce

$$w_n^* = \frac{w_n}{w_\mu}u_\mu, \qquad \text{when} \qquad n \equiv \mu (\mathrm{mod}\ H).$$

Therefore, w_n^* is also a solution where values of w_n^* and u_n are equal for H consecutive indices around μ. Because the sum in (3.7) is finite, we get

$$\liminf_{n\to\infty} \frac{w_n^*}{u_n} = \frac{u_\mu}{w_\mu} \lim_{n\to\infty} \sum_{g=0}^{\infty} \frac{\prod_{j=g+1}^{g-k-1}(-a_{n+jH})}{u_{n+gH}u_{n+(g+1)H}\prod_{j=g-k}^{g}(-a_{n+jH})} = 0.$$

It means that from some $N > l$ we have $w_N^* < u_N$ which is a contradiction with u_n being a minimal solution on $[l,\infty)$. $\square$

Through similar means as were used in [27], we can deduce the following statements. But first, we have to define a generalization of Casoratian as

$$\omega_{n,\mu} = \det \begin{pmatrix} u_{\mu+nH} & v_{\mu+nH} \\ u_{\mu+(n+1)H} & v_{\mu+(n+1)H} \end{pmatrix}.$$

Lemma 3.7. *Let u_n, v_n be two solutions of Eq. (3.1), then $\omega_{n,\mu}$ satisfies for all $\mu \in \mathbb{Z}$ the equation*

$$\omega_{n+1,\mu} = \frac{-a_{\mu+(n-k)H}}{-a_{\mu+(n+1)H}}\omega_{n,\mu}.$$

Proof. Because u_n, v_n are solutions of (3.1) we have

$$\begin{aligned} \omega_{n,\mu} &= \det \begin{pmatrix} -\frac{a_{\mu+(n+1)H}}{a_{\mu+(n-k)H}}u_{\mu+(n+2)H} & -\frac{a_{\mu+(n+1)H}}{a_{\mu+(n-k)H}}v_{\mu+(n+2)H} \\ u_{\mu+(n+1)H} & v_{\mu+(n+1)H} \end{pmatrix} \\ &= (-1)\left(-\frac{a_{\mu+(n+1)H}}{a_{\mu+(n-k)H}}\right)\omega_{n+1,\mu} = \frac{-a_{\mu+(n+1)H}}{-a_{\mu+(n-k)H}}\omega_{n+1,\mu}. \end{aligned}$$

$\square$

Hence, we can compute for some $D \in \mathbb{R}$ that

$$\omega_{n,\mu} = \frac{D}{\prod_{j=n-k}^{n}(-a_{\mu+jH})} \prod_{j=n+2}^{n-k}\left(-a_{\mu+(j-1)H}\right).$$

Note that if for some $\omega_{n,\mu}$ is $D < 0$, we get by swapping values of u_n and v_n on the set $\{\mu + jH | j \in \mathbb{Z}\}$ that u_n and v_n are still solutions of Eq. (3.1) and $D > 0$.

Theorem 3.8. *If Eq. (3.1) has two independent eventually positive solutions, then there are two independent eventually positive solutions u_n, v_n for which $\lim_{n\to\infty} \frac{u_n}{v_n} = 0$. Moreover, for arbitrary $\mu \in \mathbb{Z}$ sufficiently large*

$$\sum_{n}^{\infty} \frac{\prod_{j=n+2}^{n-k}(-a_{\mu+(j-1)H})}{u_{\mu+nH}u_{\mu+(n+1)H}\prod_{j=n-k}^{n}(-a_{\mu+jH})} = \infty, \tag{3.8}$$

$$\sum_{n}^{\infty} \frac{\prod_{j=n+2}^{n-k}(-a_{\mu+(j-1)H})}{v_{\mu+nH}v_{\mu+(n+1)H}\prod_{j=n-k}^{n}(-a_{\mu+jH})} < \infty. \tag{3.9}$$

Proof. We can expect that u_n, v_n are linearly independent, eventually positive and also that in $\omega_{n,\mu}$ is $D < 0$, for all μ. Considering μ sufficiently large we have

$$\begin{aligned} \triangle\left(\frac{u_{\mu+nH}}{v_{\mu+nH}}\right) &= \frac{u_{\mu+nH}v_{\mu+(n+1)H} - u_{\mu+(n+1)H}v_{\mu+nH}}{v_{\mu+nH}v_{\mu+(n+1)H}} \\ &= \frac{D}{v_{\mu+nH}v_{\mu+(n+1)H}\prod_{j=n-k}^{n}\left(-a_{\mu+jH}\right)} \prod_{j=n+2}^{n-k}\left(-a_{\mu+(j-1)H}\right). \end{aligned} \tag{3.10}$$

Hence, (3.10) is negative, therefore $\frac{u_{\mu+nH}}{v_{\mu+nH}}$ is strictly decreasing in n, but $\frac{u_{\mu+nH}}{v_{\mu+nH}}$ is also positive and thus bounded from below. We have that $\lim_{n\to\infty} \frac{u_{\mu+nH}}{v_{\mu+nH}} = L_\mu \geq 0$. In case that for some μ is $L_\mu > 0$, we replace u_n by $u_n - L_\mu v_n$, for $n \in \{\mu + jH | j \in \mathbb{Z}\}$. Hence, u_n will still be a solution and we get that $\lim_{n\to\infty} \frac{u_n}{v_n} = 0$.

Moreover, by summing equality (3.10) we obtain

$$D \sum_{g=k}^{n-1} \frac{1}{v_{\mu+gH}v_{\mu+(g+1)H}\prod_{j=g-k}^{g}\left(-a_{\mu+jH}\right)} \prod_{j=g+2}^{g-k}\left(-a_{\mu+(j-1)H}\right) = \frac{u_{\mu+nH}}{v_{\mu+nH}} - \frac{u_{\mu+kH}}{v_{\mu+kH}},$$

$$\xrightarrow{n\to\infty} D \sum_{g=k}^{\infty} \frac{1}{v_{\mu+gH}v_{\mu+(g+1)H}\prod_{j=g-k}^{g}\left(-a_{\mu+jH}\right)} \prod_{j=g+2}^{g-k}\left(-a_{\mu+(j-1)H}\right) = -\frac{u_{\mu+kH}}{v_{\mu+kH}},$$

which confirms the validity of (3.9). Using the unboundedness of $\frac{v_{\mu+nH}}{u_{\mu+nH}}$, we get (3.8). $\square$

Corollary 3.9. *Let for some μ be*

$$\sum_{n}^{\infty} \frac{\prod_{j=n+2}^{n-k}(-a_{\mu+(j-1)H})}{\prod_{j=n-k}^{n}(-a_{\mu+jH})} = \infty,$$

and every solution of Eq. (3.1) be eventually bounded, then Eq. (3.1) has at most one linearly independent eventually positive solution.

Proof. Suppose that Eq. (3.1) has two such solutions. Then from Theorem 3.8 there has to be a solution v_n such that $0 < v_n < M$ for n sufficiently large and some M. Moreover, for ν sufficiently large and satisfying $\nu \equiv \mu \pmod{H}$ we get from (3.9) that

$$\infty > \sum_{n}^{\infty} \frac{\prod_{j=n+2}^{n-k}(-a_{\nu+(j-1)H})}{v_{\nu+nH}v_{\nu+(n+1)H}\prod_{j=n-k}^{n}(-a_{\nu+jH})} > \frac{1}{M^2}\sum_{n}^{\infty} \frac{\prod_{j=n+2}^{n-k}(-a_{\nu+(j-1)H})}{\prod_{j=n-k}^{n}(-a_{\nu+jH})}.$$

Which is a contradiction. $\square$

As an example we consider the equation

$$-\frac{1}{2}y_n + y_{n+2} - \frac{1}{2}y_{n+4} = 0. \tag{3.11}$$

It has two solutions $u_n = K$, $v_n = Kn$ of eventually one sigh as well as two sign changing ones $(-1)^n u_n$, $(-1)^n v_n$. Moreover, it holds that

$$\sum^{\infty} \frac{\prod_{j=n+2}^{n-k}(-a_{\mu+2(j-1)})}{u_{\mu+2n}u_{\mu+2(n+1)}\prod_{j=n-k}^{n}(-a_{\mu+2j})} = \infty,$$
$$\sum^{\infty} \frac{\prod_{j=n+2}^{n-k}(-a_{\mu+2(j-1)})}{v_{\mu+2n}v_{\mu+2(n+1)}\prod_{j=n-k}^{n}(-a_{\mu+2j})} < \infty,$$

where $a_i \equiv -1/2$ and we can choose k arbitrarily. According to [16] Eq. (3.11) has a minimal solution on intervals $[2, \infty)$ and $(-\infty, -2]$.

We define the Riccati transformation through the substitution

$$s_n = \frac{b_{n+H}y_{n+H}}{(-a_{n-kH})y_n}, \quad \text{and} \quad q_n = \frac{a_n a_{n-kH}}{b_n b_{n+H}}, \tag{3.12}$$

to obtain

$$\begin{aligned} a_{n-k\cdot H}y_n + b_{n+H}y_{n+H} + a_{n+H}y_{n+2H} &= 0, \\ \frac{a_{n-k\cdot H}y_n}{b_{n+H}y_{n+H}} + 1 + \frac{a_{n+H}y_{n+2H}}{b_{n+H}y_{n+H}} &= 0, \\ -\frac{1}{s_n} + 1 - \frac{a_{n+H}a_{n-(k-1)H}}{b_{n+H}b_{n+2H}}s_{n+H} &= 0, \\ q_{n+H}s_{n+H} + \frac{1}{s_n} &= 1. \end{aligned} \tag{3.13}$$

We emphasize that $q_n > 0$ for all n.

Theorem 3.10. *Eq. (3.13) has a positive solution if and only if Eq. (3.1) has also a positive solution.*

Proof. First, if Eq. (3.1) has a positive solution y_n then via the transformation $s_n = \frac{b_{n+H}y_{n+H}}{(-a_{n-kH})y_n}$ we can see that s_n is also a positive solution of Eq. (3.13).

Second, if s_n is a positive solution of Eq. (3.13) then we can consider the initial conditions $y_N = 1, \ldots, y_{N+H-1} = 1$ for some $N \in \mathbb{Z}$ and the recurrence relation

$$y_{n+H} = \frac{(-a_{n-kH})s_n}{b_{n+H}}y_n.$$

Then, for $n \geq N$, y_n is a positive solution of Eq. (3.1). The rest of y_n is computed through the relation

$$y_n = \frac{b_{n+H}y_{n+H}}{(-a_{n-kH})s_n}.$$ □

Note that the Theorem 3.10 holds even if we consider eventually positive solutions instead of positive ones. Moreover, at this place, we can see a connection to Theorem 3.2. If H is an even number, then solutions y_n and $(-1)^n y_n$ give the same positive solution s_n of Eq. (3.13). For H being an odd number, the existence of a solution $(-1)^n y_n$ would give a solution s_n of Eq. (3.13) that is eventually negative. Nevertheless, such s_n cannot exist.

Lemma 3.11. *Let $q_n \geq p_n > 0$ and let s_n be a positive solution of*

$$q_{n+H}s_{n+H} + \frac{1}{s_n} = 1$$

on $[N, \infty)$, where $N \in \mathbb{Z}$. Then the equation

$$p_{n+H}u_{n+H} + \frac{1}{u_n} = 1,$$

has a solution u_n such that $u_n \geq s_n > 1$ on $[N, \infty)$.

Proof. If s_n is a positive solution, then also $q_{n+H}s_{n+H} > 0$, and so $\frac{1}{s_n} = 1 - q_{n+H}s_{n+H} < 1$ implies that $s_n > 1$ on $[N, \infty)$.

Now we consider initial conditions such that $u_N \geq s_N, \dots, u_{N+H-1} \geq s_{N+H-1}$ and we get that if $u_n \geq s_n$ then

$$p_{n+H}u_{n+H} = 1 - \frac{1}{u_n} = q_{n+H}s_{n+H} + \frac{1}{s_n} - \frac{1}{u_n} \geq q_{n+H}s_{n+H}.$$

Therefore, $u_{n+H} \geq \frac{q_{n+H}s_{n+H}}{p_{n+H}} \geq s_{n+H}$ and the statement of the lemma holds by induction. □

Theorem 3.12. *If q_n of (3.12) satisfy $1/(4-\varepsilon) \leq q_n$ for some $\varepsilon > 0$ and for all n sufficiently large, then Eq. (3.1) cannot have an eventually positive solution.*

Proof. If $\varepsilon \geq 4$ it would mean that $\frac{b_n b_{n+H}}{a_n a_{n-kH}} \leq (4-\varepsilon) \leq 0$, however because $a_i < 0$, $b_i > 0$ this cannot be true. Here the statement shadows Lemma 3.1.

Now we know that $\varepsilon < 4$ and assume that (3.1) has an eventually positive solution. Then there is an eventually positive solution s_n of Eq. (3.13). By Lemma 3.11 we have that the equation

$$\frac{u_{n+H}}{4-\varepsilon} + \frac{1}{u_n} = 1, \tag{3.14}$$

has a solution $u_n \geq s_n > 1$ on some $[N, \infty)$, for a sufficiently large N. If we take a positive sequence given by $x_N = 1, \dots, x_{N+H-1} = 1$, and $x_{n+H} = \frac{u_n x_n}{\sqrt{4-\varepsilon}}$, then also $u_n = \sqrt{4-\varepsilon}\frac{x_{n+H}}{x_n}$ and by substituting into (3.14) we get that x_n is a positive solution of

$$x_{n+2H} - \sqrt{4-\varepsilon}x_{n+H} + x_n = 0, \tag{3.15}$$

for $n \geq N$. This is a contradiction because Eq. (3.15) does not have an eventually positive solution. In fact Eq. (3.15) has constant coeficients and we can find all its solutions through the characteristic polynomial and de Moivre's formula. They are $\cos n\theta_k$ and $\sin n\theta_k$ where $\theta_k = \left(\arctan \frac{\varepsilon}{4-\varepsilon} + 2k\pi\right)/H$, for $k = 0, \dots, H-1$. □

Remark 3.13. We discussed eventually positive solutions, which are positive as $n \to \infty$. We can discuss the same situation if $n \to -\infty$ by taking these results and rewriting Eq. (3.1) appropriately. We emphasize that if an equation does not have and eventually positive solution, hence it even does not have a positive solution. If an equation has a positive solution, it is also an eventually positive solution.

Theorem 3.14. *If q_n of (3.12) satisfy $q_n \leq 1/4$, for all n, then Eq. (3.1) has a positive solution.*

Proof. First, let s_n be a solution of Eq. (3.13). If $s_N \geq 2$ for some N then $q_{N+H}s_{N+H} = 1 - \frac{1}{s_N} \geq 1/2$. Therefore, because $1/q_n \geq 4$, we have $s_{N+H} \geq \frac{1}{2q_{N+H}} \geq 1/2 \cdot 4 = 2$. By induction, we know that $s_n \geq 2$, for all $n \in \{N + lH | l \in [0, \infty) \cap \mathbb{Z}\}$.

Second, let again s_n be a solution of Eq. (3.13). If $0 < s_{N+H} \leq 2$ for some N then $\frac{1}{s_N} = 1 - q_{N+H}s_{N+H} \geq 1 - 1/4 \cdot 2 = 1/2$ and therefore $s_N \leq 2$. But also $1/s_N > 0$ implies that $s_N > 0$. By induction, we know that $0 < s_n \leq 2$ for all $n \in \{N + lH | l \in (-\infty, 1] \cap \mathbb{Z}\}$.

Finally, let s_n be a solution of Eq. (3.13) together with initial conditions $s_N = 2, \dots, s_{N+H-1} = 2$, for some $N \in \mathbb{Z}$. From previous two parts we have, that s_n is a positive solution of Eq. (3.13) on $\mathbb{Z}$ and by Theorem 3.10 we know that Eq. (3.1) has also a positive solution. □

Corollary 3.15. *If $b_n \geq \max\{-a_{n-H}\lambda, -4a_{n-kH}/\lambda\}$ for some $\lambda > 0$ then Eq. (3.1) has a positive solution.*

Proof. The assumption of the corollary implies that $b_n \geq -4a_{n-kH}/\lambda$ and $b_{n+H} \geq -a_n\lambda$. It follows that $b_n b_{n+H} \geq 4a_n a_{n-kH}$ and the rest is due to Theorem 3.14. □

We can connect Eq. (2.1) with Eq. (3.1) for $H = 1$ by shifting it. In the first part, the equivalence condition for Eq. (2.1) to have a positive solution was formulated. One could probably obtain similar relation by extension of the results of [16] for Eq. (3.1).

Moreover, it remains a question how this connects to q_n. By Theorem 3.12 we know that if Eq. (3.1) has a positive solution, then surely $q_n \leq 1/4$ for n sufficiently large. But we can ask whether Eq. (3.1) can have a positive solution even if $q_n > 1/4$ for some n and how Condition (2.5) connects to it.

Using again Eq. (3.11), we see that $q_n = 1/4$ and so by Theorem 3.14, we know that this equation has a positive solution.

Theorem 3.16. *If Eq. (3.1) has a solution y_n such that $y_{\mu+nH}$ is a positive sequence for some $\mu \in \mathbb{Z}$, then for every other solution $\bar{y}_n$ of Eq. (3.1), the sequence $\bar{y}_{\mu+nH}$ must have at most one generalized zero (from Definition 2.8) on $\mathbb{Z}$.*

Proof. Consider the substitution $x_p = y_{\mu+(p+1)H}$ in Eq. (3.1) and by taking $n = \mu + pH$, Eq. (3.1) changes into

$$a_{\mu+(p-k)H}y_{\mu+pH} + b_{\mu+(p+1)H}y_{\mu+(p+1)H} + a_{\mu+(p+1)H}y_{\mu+(p+2)H} = 0.$$

Now if we take $\tilde{a}_p = a_{\mu+(p+1)H}$, $\tilde{b}_p = b_{\mu+(p+1)H}$, it transforms into

$$\tilde{a}_{p-k-1}x_{p-1} + \tilde{b}_p x_p + \tilde{a}_p x_{p+1} = 0,$$

which corresponds to Eq. (2.1) and so by Theorem 2.15 we know that this equation is disconjugate. □

To further refine results obtained in Theorem 3.16, we formulate the definition of the separately nonoscillatory solution. However, let us first recall the following definition, which can be found, for example, in [4].

Definition 3.17. A nontrivial solution y_n of self-adjoint difference equation of order $2m$ has a generalized zero of order m at $n_0 + 1$ if $y_{n_0} \neq 0$, $y_{n_0+1} = \cdots = y_{n_0+m-1} = 0$, and $(-1)^m y_{n_0} y_{n_0+m} \geq 0$.

This definition corresponds to Definition 2.8 if $m = 1$. Nevertheless, for our purposes we need a combination of Definitions 2.8 and 3.17. We start by defining for some $p \in \mathbb{N}$ equivalence relation $x \sim y$ on $\mathbb{Z}$ such that $x \sim y$ if and only if $x = y + jp$ for some $j \in \mathbb{Z}$. From this equivalence we obtain equivalence classes $A_1(p), \dots, A_p(p) \subseteq \mathbb{Z}$ such that $i \in A_i(p)$. Of course $A_1(1) = \mathbb{Z}$.

Next, we define on a linearly ordered set S for $x \in S$ function

$$\rho(x) = \max\{y \in S | y < x\}.$$

Definition 3.18. Solution y_n of a given difference equation has n_0 a generalized zero on a linearly ordered set S if y_n is nontrivial on S and for $n_0 \in S$ is $y_{n_0} = 0$ or $y_{\rho(n_0)} y_{n_0} < 0$ provided that $\rho(n_0)$ exists. Solution y_n is nonoscillatory on $A_i(p) \cap I$ provided that y_n has on $A_i(p) \cap I$ only finitely many generalized zeros.

For example, recall again Eq. (3.11), which is of the fourth-order and has a solution

$$y_n = \begin{cases} 1, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Such solution has infinitely many generalized zeros with respect to both Definition 2.8 and 3.17. On the other hand, such solution does not have a generalized zero on $A_i(2)$ for both $i = 1$ (here y_n is positive) and $i = 2$ (here y_n is trivial). Another solution of Eq. (3.11) is $z_n = 1$ which does not have a generalized zero under any Definition of 2.8, 3.17 and 3.18.

Definition 3.19. Solution y_n of a given difference equation is separately i-nonoscillatory on $I(p)$ if there is a set $J \subseteq \{1, \dots, p\}$, $|J| = i$, such that y_n is nonoscillatory on $A_j(p) \cap I$ for all $j \in J$. If all solutions of the equation are separately i-nonoscillatory on $I(p)$, then this equation is called separately i-nonoscillatory on $I(p)$.

In this paper, we consider for I only $\mathbb{Z}$ or $[N, \infty)$ as well as $p = H$, because they make the most sense to us. We assume that we could get some interesting or strange results for a different choice of I and p. Moreover, with the choice of $p = 1$ and $I = [N, \infty)$, we get the usual definition of nonoscillatory solutions used for second-order linear equations through generalized zeros of Definition 2.8. Such solutions are eventually positive or negative. Hence, if a solution is separately nonoscillatory on $I(1)$, then it is also separately nonoscillatory on $I(p)$.

Corollary 3.20. *Assume there is a set $J \subseteq \{1, \dots, H\}$, $|J| = i$ such that q_n of (3.12) satisfies $q_n \leq 1/4$, for all $n \in A_j(H)$ and $j \in J$, then Eq. (3.1) is separately i-nonoscillatory on $\mathbb{Z}(H)$.*

Proof. By the proof of Theorem 3.14 we know that Eq. (3.1) has a solution which is positive on $A_j(H)$, $j \in J$. Hence, by Theorem 3.16 we know that every solution is nonoscillatory on A_j, where $j \in J$. □

Theorem 3.21. *If there is a subsequence q_{n_l} of q_n such that $q_{n_l} \geq 1$ for $n_l \to \infty$, $n_l \in A_i(H)$ and some $i \in \{1, \dots, H\}$, then Eq. (3.1) cannot have y_n a nonoscillatory solution on $A_i(h) \cap [N, \infty)$, for some $N \in \mathbb{N}$.*

Proof. Suppose that there is such a solution, then we can assume that it is positive on $I = A_i(H) \cap [N, \infty)$ for N sufficiently large. Therefore, Eq. (3.13) has a solution s_n such that $s_n > 0$

on I. Moreover, by definition $q_n > 0$ for all n and if $n \in A_i(H)$, then also $n + H \in A_i(H)$. Hence,

$$q_{n+H}s_{n+H} + \frac{1}{s_n} = 1,$$

and we have that $1/s_n < 1$ on I, thus $s_n > 1$ on I. Nevertheless, for the same reason $q_{n+H}s_{n+H} < 1$ on I and so $q_n < 1$, for all $n \geq N + H$, $n \in I$. That is a contradiction with our assumption. $\square$

In such a case, equation cannot be separately H-nonoscillatory on $I(H)$ where $I = [N, \infty)$.

Corollary 3.22. *If*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} q_{i+jH} > 1,$$

then Eq. (3.1) cannot have y_n a nonoscillatory solution on $A_i(H) \cap [N, \infty)$ for some $N \in \mathbb{N}$.

Proof. Suppose there is such a solution. Then by Theorem 3.21, $q_n < 1$ on $A_i(H) \cap [N, \infty)$, for N sufficiently large and let $m \in A_i(H) \cap [N, \infty)$ be arbitrary. Then it holds $\sum_{j=1}^{n} q_{m+jH} < n$ and also $\frac{1}{n}\sum_{j=1}^{n} q_{i+jH} < 1 + \frac{C}{n}$, for some $C \in \mathbb{R}$. Therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} q_{i+jH} \leq 1,$$

which is a contradiction. $\square$

Theorem 3.23. *If Eq. (3.1) has a solution $y_n > 0$ on $A_i(H) \cap [N, \infty)$ and $\prod_{j=1}^{n} \frac{b_{i+jH}}{(-a_{i+jH})}$ is a bounded sequence, then y_n is bounded on $A_i(H) \cap [N, \infty)$.*

Proof. Taking $z_n = \frac{y_{n+H}}{y_n}$ on $I = A_i(H) \cap [N, \infty)$, for N sufficiently large, we can see that $z_n > 0$ is a solution of the equation

$$(-a_{n-kH})\frac{1}{z_n} + (-a_{n+H})z_{n+H} = b_{n+H},$$

on I. Because all the terms are positive, it holds that $(-a_{n+H})z_{n+H} < b_{n+H}$ on I. Let $M \in I$ be arbitrary and we have

$$\frac{y_{M+nH}}{y_{M+H}} = \prod_{j=1}^{n-1} \frac{y_{M+(j+1)H}}{y_{M+jH}} = \prod_{j=1}^{n-1} z_{M+jH} < \prod_{j=1}^{n-1} \frac{b_{M+jH}}{(-a_{M+jH})}.$$

Hence, $y_{M+nH} < y_{M+H} \prod_{j=1}^{n-1} \frac{b_{M+jH}}{(-a_{M+jH})}$ for all $n \in \mathbb{N}$ is giving us the result. $\square$

Corollary 3.24. *If Eq. (3.1) has a positive solution y_n and $\prod_{j=1}^{n} \frac{b_{i+jH}}{(-a_{i+jH})}$, $\prod_{j=-n}^{1} \frac{b_{i+jH}}{(-a_{i+(j-k)H})}$ are bounded sequences for every $i \in \{1, \ldots, H\}$, then y_n is bounded on $\mathbb{Z}$.*

Proof. By Theorem 3.23 we see that y_n is bounded on all $A_i(H) \cap [N, \infty)$ for $n \to \infty$. Via the same way, we can see that

$$b_{n+H} > (-a_{n-kH})\frac{1}{z_n} = (-a_{n-kH})\frac{y_n}{y_{n+H}},$$

for every n and similarly we see that y_n is bounded even for $n \to -\infty$. $\square$

Corollary 3.25. *If $\prod_{j=1}^{n} \frac{b_{i+jH}}{(-a_{i+jH})}$ is a bounded sequence for every $i \in \{1,\ldots,H\}$ and for some $\mu \in \mathbb{Z}$ is*

$$\sum_{n}^{\infty} \frac{\prod_{j=n+2}^{n-k}(-a_{\mu+(j-1)H})}{\prod_{j=n-k}^{n}(-a_{\mu+jH})} = \infty,$$

then Eq. (3.1) has at most one linearly independent eventually positive solution.

Proof. Suppose that there are two such solutions. Then by Theorem 3.23, they are bounded as $n \to \infty$. Using the proof of Corollary 3.9, we get a contradiction. □

It is posible to extend previous ideas to other equations. As an example, we consider the equation

$$c_{n-1}a_n y_n + b_{n+1}y_{n+1} + c_{n+1}a_{n+1}y_{n+2} = 0.$$

It would result in similar but more complicated statements. However, our results can be extended even more in a similar fashion, how [23] extends the results of [27]. It should also be possible to find other criteria of separate oscillation shadowing the approach used for the case of $H = 1$, $k = 0$.

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A sharp oscillation result for second-order half-linear noncanonical delay differential equations

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Abstract. In the paper, new single-condition criteria for the oscillation of all solutions to a second-order half-linear delay differential equation in noncanonical form are obtained, relaxing a traditionally posed assumption that the delay function is nondecreasing. The oscillation constant is best possible in the sense that the strict inequality cannot be replaced by the nonstrict one without affecting the validity of the theorem. This sharp result is new even in the linear case and, to the best of our knowledge, improves all the existing results reporting in the literature so far. The advantage of our approach is the simplicity of the proof, only based on sequentially improved monotonicities of a positive solution.

Keywords: second-order differential equation, delay, half-linear, oscillation.

2020 Mathematics Subject Classification: 34C10, 34K11.

1 Introduction

Consider the second-order half-linear delay differential equation of the form

$$\Big(r(t)\left(y'(t)\right)^{\alpha}\Big)' + q(t)y^{\alpha}(\tau(t)) = 0, \qquad t \ge t_0 > 0, \tag{1.1}$$

where we assume that $\alpha > 0$ is a quotient of odd positive integers; functions r, q, and τ are continuous positive functions, $\tau(t) \le t$ and $\lim_{t\to\infty} \tau(t) = \infty$. Without further mentioning, we will assume that (1.1) is in so-called noncanonical form, i.e.,

$$\pi(t_0) := \int_{t_0}^{\infty} \frac{\mathrm{d}t}{r^{1/\alpha}(t)} < \infty.$$

By a solution of Eq. (1.1) we mean any differentiable function y which does not vanishes eventually such that $r(y')^{\alpha}$ is differentiable, satisfying (1.1) for sufficiently large t. As is customary, a solution $y(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor

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eventually negative. Otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

The oscillation theory of second-order functional differential equations has attracted a great portion of attention, which is evidenced by extensive research in the area. For a compact summary of the most recent results and appearing open problems, the reader is referred to the recent monographs the monographs by Agarwal et al. [2–5], Došlý and Řehák [12] Győri and Ladas [16], and Saker [22].

In the paper, we obtain new single-condition criteria for the oscillation of all solutions to (1.1) with unimprovable constants. This sharp result is new even in the linear case and, to the best of our knowledge, improves all the existing results reported in the literature so far. In the linear case, we also formulate analogous results for canonical equations.

The structure of the paper is the following. In Section 2, we revise the oscillatory properties of various useful equations serving as models for comparison of the obtained results. In Section 3, main results of the paper are stated, and their proofs are given in Section 4.

2 Comparison equations in the oscillation theory

Euler-type differential equations have been of utmost importance in the oscillation theory since Sturm's pioneering work in 1863. Till now, they are commonly used to examine the sharpness of general criteria derived by different methods. The optimal scenario is when the obtained criterion gives a sharp result for the Euler-type equation; or at least it is closer to it for a given set of parameters, compared to another one. Perhaps the most familiar one is the second-order linear Euler equation

$$y''(t) + \frac{q_0}{t^2} y(t) = 0 \tag{2.1}$$

which is oscillatory if and only if

$$q_0 > \frac{1}{4}. \tag{2.2}$$

In 1893, A. Kneser [17] firstly used Sturmian methods and (2.1) to show that the linear equation

$$y''(t) + q(t)y(t) = 0$$

is oscillatory if

$$\liminf_{t\to\infty} t^2 q(t) > \frac{1}{4}$$

and nonoscillatory if

$$\limsup_{t\to\infty} t^2 q(t) < \frac{1}{4}.$$

For our purposes, we consider, as a particular case of (1.1), the generalized Euler-type half-linear ordinary differential equation

$$\left(r(t)\left(y'(t)\right)^{\alpha}\right)' + \frac{q_0}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)} y^{\alpha}(t) = 0, \qquad q_0 > 0. \tag{2.3}$$

It is well-known that (2.3) is oscillatory if and only if its characteristic equation

$$c_1(m) := \alpha m^{\alpha}(1-m) = q_0$$

has no real roots what happens if

$$q_0 > \max\{c_1(m) : 0 < m < 1\} = c_1\left(\frac{\alpha}{\alpha+1}\right) = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}, \tag{2.4}$$

cf. [2, Remark 3.7.1] or [12]. If condition (2.4) fails, then (2.3) has a nonoscillatory solution $y(t) = \pi^m(t)$. As an immediate consequence of the Sturmian comparison theorem and the above result concerning (2.3), we get the following version of the classical Kneser oscillation and nonoscillation criterion for the noncanonical equation

$$\left(r(t)\left(y'(t)\right)^{\alpha}\right)' + q(t)y^{\alpha}(t) = 0. \tag{2.5}$$

Proposition 2.1. *Suppose that*

$$\liminf_{t\to\infty} r^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t) > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \tag{2.6}$$

Then (2.5) *is oscillatory. If*

$$\limsup_{t\to\infty} r^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t) < \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1},$$

then (2.5) *is nonoscillatory.*

As another important particular case of (1.1), we consider the linear Euler-type equation with proportional delay, namely,

$$\left(r(t)y'(t)\right)' + \frac{q_0}{r(t)\pi^2(t)}y(kt) = 0, \qquad 0 < k \le 1, \tag{2.7}$$

where $r(t) = t^{p+1}$, $p > 0$. By a simple change of variables

$$s = \frac{1}{\pi(t)} \quad \text{and} \quad y(t) = \frac{u(s)}{s}, \tag{2.8}$$

(2.7) can be rewritten as

$$u''(s) + \frac{q_0}{k^p s^2}u(k^p s) = 0. \tag{2.9}$$

By transforming (2.9) into a constant-coefficient-constant-delay equation, Kulenović [18] showed that (2.9) is oscillatory if and only if the associated characteristic equation

$$c_2(m) := m(1-m)k^{mp} = q_0$$

has no real root what happens if

$$q_0 > \max\{c_2(m) : 0 < m < 1\} = c_2(m_{\max}), \tag{2.10}$$

where

$$m_{\max} = \frac{-\sqrt{r^2+4}+r+2}{2r}, \qquad r = -p\ln k.$$

It is well-known that the Sturmian comparison theorem fails to extend to the more general equation

$$\left(r(t)y'(t)\right)' + q(t)y(\tau(t)) = 0 \tag{2.11}$$

due to the delayed argument. For delay differential equations, Kusano and Naito established an alternative comparison principle [19] in the sense that oscillation of the studied differential equation can be deduced from the oscillation of a simpler one. Using their result [19, Theorem 3] for (2.7), one can conclude that the equation

$$\left(t^{p+1}y'(t)\right)' + q(t)y(kt) = 0, \qquad p > 0,\ 0 < k \le 1, \tag{2.12}$$

is oscillatory if

$$\liminf_{t\to\infty} \frac{t^{1-p}q(t)}{p^2} > \max\{c_2(m) : 0 < m < 1\}.$$

As a generalized version of (2.7), we consider

$$\left(r(t)y'(t)\right)' + \frac{q_0}{r(t)\pi^2(t)}y(\tau(t)) = 0, \tag{2.13}$$

with the constant ratio $\pi(\tau(t))/\pi(t) = \lambda$. It can be verified by a direct substitution that (2.13) has a nonoscillatory solution $y(t) = \pi^m(t)$ if

$$q_0 \le \max\{c_3(m) : 0 < m < 1\},$$

where

$$c_3(m) := m(1-m)\lambda^{-m}.$$

The "only if" part here is difficult to prove because the transformation to a constant-coefficient-constant-delay form is obviously impossible. To the best of the authors' knowledge, there is no oscillation criterion for (2.11) which would be sharp for (2.13).

Finally, we consider the most general Euler-type half-linear delay differential equation

$$\left(r(t)\left(y'(t)\right)^\alpha\right)' + \frac{q_0}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)}y^\alpha(\tau(t)) = 0, \quad q_0 > 0, \quad t \ge t_0, \tag{2.14}$$

where the functions r and τ are general and such that $\pi(\tau(t))/\pi(t) = \lambda$. Note that (2.14) includes both (2.3) and (2.13) as particular cases. As previously, we find that (2.14) has a nonoscillatory solution $y(t) = \pi^m(t)$ if there is a positive root of the equation

$$c_4(m) := \alpha m^\alpha(1-m)\lambda^{-\alpha m} = q_0, \tag{2.15}$$

what happens if

$$q_0 \le \max\{c_4(m) : 0 < m < 1\}. \tag{2.16}$$

In the paper, we will show that (2.16) is not only sufficient but necessary for the existence of nonoscillatory solution of (2.14). Before that, we conclude the introductory section by revising briefly different approaches and oscillation results available for the equation (1.1). Here, it is important to stress that all below-mentioned results require that τ is a nondecreasing function.

Because of its simpler structure of nonoscillatory solutions, (1.1) has been mostly studied in canonical form and much less efforts have been undertaken for noncanonical equations. Since Trench canonical theory [24] fails to extend to half-linear equations, a common approach in the literature for investigation of such equations consists in extending known results for canonical ones, see [1, 11, 13–15, 20, 21, 23, 25]. In 2017, Džurina and Jadlovská [9] revised a variety of existing results by removing a traditionally imposed condition and obtained several one-condition oscillation criteria for (1.1).

In general, there are two main factors contributing to the oscillatory behavior of (1.1): the second-order nature of the equation and the presence of the delay; mostly treated independently by an application of one of the following methods:

1. using comparison with a second-order half-linear ordinary differential equation, directly or indirectly via generalized Riccati generalized inequality
$$u'(t) + q(t) + \alpha r^{-1/\alpha}(t) u^{(\alpha+1)/\alpha}(t) \leq 0, \tag{2.17}$$
2. using comparison with a second-order linear differential equation; by employing linearization techniques,
3. using comparison with a first-order linear delay differential equation; where the delay is essential, but the information about the second-order nature of the equation is lost.

Another method based on the weighted Hardy inequality was presented in [8]. Any of works [1,8,11,13–15,20,21,23,25], employing the methods (1) or (2) gives at best
$$q_0 > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$$
for the Euler equation (2.14) with $r(t) = t^{\alpha+1}$ and $\tau(t) = kt$, $k \in (0,1]$, which is sharp only in the ordinary case (2.3). Here, it is easy to see that the influence of the delay is completely lost. Some improvement was recently made by present authors [10] under assumption that $\pi(\tau(t))/\pi(t) \geq \lambda > 1$, which yields
$$\lambda^{q_0} q_0 > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}.$$
On the other hand, the method (3) employed in [7] requires
$$q_0^{1/\alpha} \ln \frac{1}{k} > \frac{1}{\mathrm{e}}.$$
for (2.14) with $r(t) = t^{\alpha+1}$ to be oscillatory.

The purpose of the paper is to obtain the best-possible single-condition oscillation criterion for (1.1), where both the above-mentioned factors jointly contribute. The ideas partly exploit the very recent ones from [6] for the linear equation
$$\big(r(t)y'(t)\big)' + q(t)y(\tau(t)) = 0. \tag{2.18}$$

Theorem A (See [6, Theorem 3.4])**.** *Assume that $\tau(t)$ is nondecreasing, $\tau(t) < t$,*
$$\int_{t_0}^{\infty} q(s)\pi(s)\mathrm{d}s = \infty, \tag{2.19}$$
and there exists a constant $\beta_0 > 0$ such that
$$q(t)\pi^2(t)r(t) \geq \beta_0$$
eventually. If there exists $n \in \mathbb{N}$, such that $\beta_n < 1$ for $n = 0, 1, 2, \ldots, n-1$, and
$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} q(s)\tau(s) > \frac{1-\beta_n}{e},$$
where
$$\beta_n := \frac{\beta_0 \lambda^{\beta_{n-1}}}{1-\beta_{n-1}}$$
for $n \in \mathbb{N}$ and λ satisfying
$$\frac{\pi(\tau(t))}{\pi(t)} \geq \lambda$$
eventually, then (2.18) is oscillatory.

Our newly obtained results (Theorems 3.1 and 3.4) can be regarded as a natural extension of the oscillation part of Proposition 2.1 to a half-linear delay differential equation. Their advantage over the known results is threefold: first of all, Theorem 3.1 involves the oscillation constant which is optimal for the most general Euler-type comparison equation (2.14), and hence unimprovable. Secondly, in contrast with related works [1,7,8,10,11,13–15,20,21,23,25], we relaxed the assumption that τ is nondecreasing. Thirdly, our results in a special case $\alpha = 1$ improve Theorem A in several ways:

1. we use the limit inferior of quantities $q(t)\pi^2(t)r(t)$ and $\pi(\tau(t))/\pi(t)$ in definitions of corresponding constants, which is less-restrictive to apply;
2. we show that the iteration process can be omitted in final criteria;
3. our results do not require $\tau(t) < t$ nor the monotonicity of τ, as we have already mentioned.

3 Main results

In this section, we state the main results of the paper.

Theorem 3.1. *Let*

$$\lambda_* := \liminf_{t\to\infty} \frac{\pi(\tau(t))}{\pi(t)} < \infty. \tag{3.1}$$

If

$$\liminf_{t\to\infty} r^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t) > \max\{c(m) := \alpha m^{\alpha}(1-m)\lambda_*^{-\alpha m} : 0 < m < 1\}, \tag{3.2}$$

then (1.1) is oscillatory.

Corollary 3.2. *By some computations, one has*

$$\max\{c(m) : 0 < m < 1\} = c(m_{\max}),$$

where

$$m_{\max} = \begin{cases} \dfrac{\alpha}{\alpha+1}, & \text{for } \lambda_* = 1 \\ \dfrac{-\sqrt{(\alpha r+\alpha+1)^2 - 4\alpha^2 r} + \alpha r + \alpha + 1}{2\alpha r}, & \text{for } \lambda_* \neq 1 \text{ and } r = \ln \lambda_*, \end{cases}$$

and $c(m)$ is defined by (3.2).

Remark 3.3. It is easy to verify that for $\tau(t) = t$, condition (3.2) reduces to (2.6). In view of (2.16), it is clear that condition (3.2) is sharp in the sense that the strict inequality cannot be replaced by the nonstrict one without affecting the validity of the theorem. Hence, Theorem 3.1 can be viewed as a sharp extension of Kneser oscillation criterion (2.6) to a delay half-linear equation.

For the remaining case when (3.1) is violated, we have the following result.

Theorem 3.4. *Let*

$$\lim_{t\to\infty} \frac{\pi(\tau(t))}{\pi(t)} = \infty. \tag{3.3}$$

If

$$\liminf_{t\to\infty} r^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t) > 0,$$

then (1.1) is oscillatory.

In the linear case $\alpha = 1$, it is possible to transfer the oscillation property from (1.1) to the canonical equation

$$\big(\tilde{r}(t)x'(t)\big)' + \tilde{q}(t)x(\tau(t)) = 0, \qquad t \geq t_0 > 0, \tag{3.4}$$

where $\tilde{r}$ and $\tilde{q}$ are continuous positive functions, and

$$R(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{\tilde{r}(s)} \to \infty \quad \text{as } t \to \infty.$$

Theorem 3.5. *Let*

$$\delta_* := \liminf_{t\to\infty} \frac{R(t)}{R(\tau(t))} < \infty.$$

If

$$\liminf_{t\to\infty} \tilde{r}(t)\tilde{q}(t)R(t)R(\tau(t)) > \max\{m(1-m)\delta_*^{-m} : 0 < m < 1\},$$

then (3.4) *is oscillatory.*

Theorem 3.6. *Let*

$$\lim_{t\to\infty} \frac{R(t)}{R(\tau(t))} = \infty.$$

If

$$\liminf_{t\to\infty} \tilde{r}(t)\tilde{q}(t)R(t)R(\tau(t)) > 0,$$

then (3.4) *is oscillatory.*

4 Auxiliary lemmas and proofs of main results

Let us define

$$\beta_* := \liminf_{t\to\infty} \frac{1}{\alpha} r^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t). \tag{4.1}$$

The arguments in the proofs are based on the existence of positive β_*, which is also necessary for the validity of Theorems 3.1 and 3.4. Then, for arbitrary fixed $\beta \in (0, \beta_*)$ and $\lambda \in [1, \lambda_*)$, there is a $t_1 \geq t_0$, such that

$$\frac{1}{\alpha} q(t)r^{1/\alpha}(t)\pi^{\alpha+1}(t) \geq \beta \quad \text{and} \quad \frac{\pi(\tau(t))}{\pi(t)} \geq \lambda \quad \text{on } [t_1, \infty). \tag{4.2}$$

In the sequel, we assume that all functional inequalities hold eventually, that is, they are satisfied for all t large enough.

Lemma 4.1. *Let $\beta_* > 0$. If* (1.1) *has an eventually positive solution y, then*

- (i) *y is eventually decreasing with $\lim_{t\to\infty} y(t) = 0$;*
- (ii) *y/π is eventually nondecreasing.*

Proof. (i). By [9, Theorem 1], the conclusion applies if

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)} \left(\int_{t_0}^{t} q(s)\mathrm{d}s\right)^{1/\alpha} \mathrm{d}t = \infty. \tag{4.3}$$

Indeed, by simple computations, we see that

$$\begin{aligned}\int_{t_1}^{t} \frac{1}{r^{1/\alpha}(u)} \left(\int_{t_1}^{u} q(s) \mathrm{d}s \right)^{1/\alpha} \mathrm{d}u &\geq \sqrt[\alpha]{\beta} \int_{t_1}^{t} \frac{1}{r^{1/\alpha}(u)} \left(\int_{t_1}^{u} \frac{\alpha}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)} \mathrm{d}s \right)^{1/\alpha} \mathrm{d}u \\ &= \sqrt[\alpha]{\beta} \int_{t_1}^{t} \frac{1}{r^{1/\alpha}(u)} \left(\frac{1}{\pi^{\alpha}(u)} - \frac{1}{\pi^{\alpha}(t_1)} \right)^{1/\alpha} \mathrm{d}u\end{aligned}$$

with β defined by (4.2). Since $\pi^{-\alpha}(t) \to \infty$ as $t \to \infty$, for any $\ell \in (0,1)$ and t large enough, we have $\pi^{-\alpha}(t) - \pi^{-\alpha}(t_1) \geq \ell^{\alpha}\pi^{-\alpha}(t)$ and hence

$$\int_{t_1}^{t} \frac{1}{r^{1/\alpha}(u)} \left(\int_{t_1}^{u} q(s) \mathrm{d}s \right)^{1/\alpha} \mathrm{d}u \geq \ell \sqrt[\alpha]{\beta} \int_{t_1}^{t} \frac{1}{r^{1/\alpha}(u)\pi(u)} \mathrm{d}u = \ell \sqrt[\alpha]{\beta} \ln \frac{\pi(t_1)}{\pi(t)} \to \infty \quad \text{as } t \to \infty.$$

(ii). Using the fact that $r^{1/\alpha}y'$ is nondecreasing, we obtain

$$y(t) \geq -\int_{t}^{\infty} \frac{1}{r^{1/\alpha}(s)} r^{1/\alpha}(s) y'(s) \mathrm{d}s \geq -r^{1/\alpha}(t) y'(t) \pi(t),$$

i.e.

$$\left(\frac{y}{\pi} \right)' = \frac{r^{1/\alpha} y' \pi + y}{r^{1/\alpha} \pi^2} \geq 0.$$

The proof is complete. □

Remark 4.2. Compared to the original Lemma statement used in [9, Theorem 1], we replaced the integral condition (4.3) by the requirement of positive β_*. In Theorem A, condition (2.19) was used to arrive at the same conclusion.

To improve the (i)-part of Lemma 4.1, we define a sequence $\{\beta_n\}$ by

$$\begin{aligned} \beta_0 &= \sqrt[\alpha]{\beta_*} \\ \beta_n &= \frac{\beta_0 \lambda_*^{\beta_{n-1}}}{\sqrt[\alpha]{1 - \beta_{n-1}}}, \quad n \in \mathbb{N}. \end{aligned} \tag{4.4}$$

By induction, it is easy to show that if for any $n \in \mathbb{N}$, $\beta_i < 1$, $i = 0, 1, 2, \ldots, n$, then β_{n+1} exists and

$$\beta_{n+1} = \ell_n \beta_n > \beta_n, \tag{4.5}$$

where ℓ_n is defined by

$$\begin{aligned} \ell_0 &= \frac{\lambda_*^{\beta_0}}{\sqrt[\alpha]{1 - \beta_0}} \\ \ell_{n+1} &= \lambda_*^{\beta_0(\ell_n - 1)} \sqrt[\alpha]{\frac{1 - \beta_n}{1 - \ell_n \beta_n}}, \qquad n \in \mathbb{N}_0. \end{aligned}$$

Lemma 4.3. *Let $\beta_* > 0$ and $\lambda_* < \infty$. If (1.1) has an eventually positive solution y, then for any $n \in \mathbb{N}_0$ y/π^{β_n} is eventually decreasing.*

Proof. Let y be a positive solution of (1.1) on $[t_1, \infty)$ where $t_1 \geq t_0$ is such that $y(\tau(t)) > 0$ and (4.2) holds for $t \geq t_1$. Integrating (1.1) from t_1 to t, we have

$$-r(t) \left(y'(t) \right)^{\alpha} = -r(t_1) \left(y'(t_1) \right)^{\alpha} + \int_{t_1}^{t} q(s) y^{\alpha}(\tau(s)) \mathrm{d}s. \tag{4.6}$$

By (i) of Lemma 4.1, y is decreasing and so $y(\tau(t)) \geq y(t)$ for $t \geq t_1$. Therefore,

$$-r(t)\left(y'(t)\right)^{\alpha} \geq -r(t_1)\left(y'(t_1)\right)^{\alpha} + \int_{t_1}^{t} q(s)y^{\alpha}(s)\mathrm{d}s \geq -r(t_1)\left(y'(t_1)\right)^{\alpha} + y^{\alpha}(t)\int_{t_1}^{t} q(s)\mathrm{d}s.$$

Using (4.2) in the above inequality, we get

$$\begin{aligned} -r(t)\left(y'(t)\right)^{\alpha} &\geq -r(t_1)\left(y'(t_1)\right)^{\alpha} + \beta y^{\alpha}(t)\int_{t_1}^{t} \frac{\alpha}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)}\mathrm{d}s \\ &\geq -r(t_1)\left(y'(t_1)\right)^{\alpha} + \beta\frac{y^{\alpha}(t)}{\pi^{\alpha}(t)} - \beta\frac{y^{\alpha}(t)}{\pi^{\alpha}(t_1)}. \end{aligned} \tag{4.7}$$

From (i)-part of Lemma 4.1, we have that $\lim_{t\to\infty} y(t) = 0$. Hence, there is a $t_2 \in [t_1, \infty)$ such that

$$-r(t_1)\left(y'(t_1)\right)^{\alpha} - \beta\frac{y^{\alpha}(t)}{\pi^{\alpha}(t_1)} > 0, \qquad t \geq t_2.$$

Thus,

$$-r(t)\left(y'(t)\right)^{\alpha} > \beta\frac{y^{\alpha}(t)}{\pi^{\alpha}(t)} \tag{4.8}$$

or

$$-r^{1/\alpha}(t)y'(t)\pi(t) > \sqrt[\alpha]{\beta}y(t) = \varepsilon_0\beta_0 y(t),$$

where $\varepsilon_0 = \sqrt[\alpha]{\beta}/\beta_0$ is an arbitrary constant from $(0,1)$. Therefore,

$$\left(\frac{y}{\pi^{\sqrt[\alpha]{\beta}}}\right)' = \frac{r^{1/\alpha}y'\pi^{\sqrt[\alpha]{\beta}} + \sqrt[\alpha]{\beta}\pi^{\sqrt[\alpha]{\beta}-1}y}{r^{1/\alpha}\pi^{2\sqrt[\alpha]{\beta}}} = \frac{\pi^{\sqrt[\alpha]{\beta}-1}\left(\sqrt[\alpha]{\beta}y + \pi r^{1/\alpha}y'\right)}{r^{1/\alpha}\pi^{2\sqrt[\alpha]{\beta}}} \leq 0, \qquad t \geq t_2. \tag{4.9}$$

Integrating (1.1) from t_2 to t and using that $y/\pi^{\sqrt[\alpha]{\beta}}$ is decreasing, we have

$$\begin{aligned} -r(t)\left(y'(t)\right)^{\alpha} &\geq -r(t_2)\left(y'(t_2)\right)^{\alpha} + \left(\frac{y(\tau(t))}{\pi^{\sqrt[\alpha]{\beta}}(\tau(t))}\right)^{\alpha}\int_{t_2}^{t} q(s)\pi^{\alpha\sqrt[\alpha]{\beta}}(\tau(s))\mathrm{d}s \\ &\geq -r(t_2)\left(y'(t_2)\right)^{\alpha} + \left(\frac{y(t)}{\pi^{\sqrt[\alpha]{\beta}}(t)}\right)^{\alpha}\int_{t_2}^{t} q(s)\left(\frac{\pi(\tau(s))}{\pi(s)}\right)^{\alpha\sqrt[\alpha]{\beta}}\pi^{\alpha\sqrt[\alpha]{\beta}}(s)\mathrm{d}s. \end{aligned}$$

By virtue of (4.2), we see that

$$\begin{aligned} -r(t)\left(y'(t)\right)^{\alpha} \geq &-r(t_2)\left(y'(t_2)\right)^{\alpha} + \beta\left(\frac{y(t)}{\pi^{\sqrt[\alpha]{\beta}}(t)}\right)^{\alpha}\int_{t_2}^{t}\frac{\alpha\left(\frac{\pi(\tau(s))}{\pi(s)}\right)^{\alpha\sqrt[\alpha]{\beta}}}{r^{1/\alpha}(s)\pi^{\alpha+1-\alpha\sqrt[\alpha]{\beta}}(s)}\mathrm{d}s \\ \geq &-r(t_2)\left(y'(t_2)\right)^{\alpha} \\ &+\frac{\beta}{1-\sqrt[\alpha]{\beta}}\lambda^{\alpha\sqrt[\alpha]{\beta}}\left(\frac{y(t)}{\pi^{\sqrt[\alpha]{\beta}}(t)}\right)^{\alpha}\int_{t_2}^{t}\frac{\alpha(1-\sqrt[\alpha]{\beta})}{r^{1/\alpha}(s)\pi^{\alpha+1-\alpha\sqrt[\alpha]{\beta}}(s)}\mathrm{d}s \\ = &-r(t_2)\left(y'(t_2)\right)^{\alpha} \\ &+\frac{\beta}{1-\sqrt[\alpha]{\beta}}\lambda^{\alpha\sqrt[\alpha]{\beta}}\left(\frac{y(t)}{\pi^{\sqrt[\alpha]{\beta}}(t)}\right)^{\alpha}\left(\frac{1}{\pi^{\alpha(1-\sqrt[\alpha]{\beta})}(t)} - \frac{1}{\pi^{\alpha(1-\sqrt[\alpha]{\beta})}(t_2)}\right). \end{aligned} \tag{4.10}$$

Now, we claim that $\lim_{t\to\infty} y(t)/\pi^{\sqrt[\alpha]{\beta}}(t) = 0$. It suffices to show that there is $\varepsilon > 0$ such that $y/\pi^{\sqrt[\alpha]{\beta}+\varepsilon}$ is eventually decreasing. Since $\pi(t)$ tends to zero, there is a constant

$$\ell \in \left(\frac{\sqrt[\alpha]{1-\sqrt[\alpha]{\beta}}}{\lambda^{\sqrt[\alpha]{\beta}}}, 1 \right)$$

and a $t_3 \geq t_2$ such that

$$\frac{1}{\pi^{\alpha(1-\sqrt[\alpha]{\beta})}(t)} - \frac{1}{\pi^{\alpha(1-\sqrt[\alpha]{\beta})}(t_2)} > \ell^\alpha \frac{1}{\pi^{\alpha(1-\sqrt[\alpha]{\beta})}(t)}, \qquad t \geq t_3.$$

Using the above inequality in (4.10) yields

$$-r(t)\left(y'(t)\right)^\alpha \geq \frac{\ell^\alpha \beta}{1-\sqrt[\alpha]{\beta}} \lambda^{\alpha\sqrt[\alpha]{\beta}} \left(\frac{y(t)}{\pi(t)}\right)^\alpha,$$

i.e.,

$$-r^{1/\alpha} y'(t) \geq \left(\sqrt[\alpha]{\beta}+\varepsilon\right) \frac{y(t)}{\pi(t)}, \tag{4.11}$$

where

$$\varepsilon = \sqrt[\alpha]{\beta}\left(\frac{\ell \lambda^{\sqrt[\alpha]{\beta}}}{\sqrt[\alpha]{1-\sqrt[\alpha]{\beta}}} - 1\right) > 0.$$

Thus, from (4.11),

$$\left(\frac{y}{\pi^{\sqrt[\alpha]{\beta}+\varepsilon}}\right)' \leq 0, \qquad t \geq t_3,$$

and hence the claim holds. Therefore, for $t_4 \in [t_3, \infty)$,

$$-r(t_2)\left(y'(t_2)\right)^\alpha - \frac{\beta}{1-\sqrt[\alpha]{\beta}} \lambda^{\alpha\sqrt[\alpha]{\beta}} \left(\frac{y(t)}{\pi^{\sqrt[\alpha]{\beta}}(t)}\right)^\alpha \frac{1}{\pi^{\alpha-\alpha\sqrt[\alpha]{\beta}}(t_2)} > 0, \qquad t \geq t_4.$$

Turning back to (4.10) and using the above inequality,

$$\begin{aligned} -r(t)\left(y'(t)\right)^\alpha \geq & -r(t_2)\left(y'(t_2)\right)^\alpha + \frac{\beta}{1-\sqrt[\alpha]{\beta}} \lambda^{\alpha\sqrt[\alpha]{\beta}} \left(\frac{y(t)}{\pi(t)}\right)^\alpha \\ & - \frac{\beta}{1-\sqrt[\alpha]{\beta}} \lambda^{\alpha\sqrt[\alpha]{\beta}} \left(\frac{y(t)}{\pi^{\sqrt[\alpha]{\beta}}(t)}\right)^\alpha \frac{1}{\pi^{\alpha-\alpha\sqrt[\alpha]{\beta}}(t_2)} \\ > & \frac{\beta}{1-\sqrt[\alpha]{\beta}} \lambda^{\alpha\sqrt[\alpha]{\beta}} y^\alpha, \end{aligned}$$

or

$$-r^{1/\alpha} y' \pi > \frac{\sqrt[\alpha]{\beta}}{\sqrt[\alpha]{1-\sqrt[\alpha]{\beta}}} \lambda^{\sqrt[\alpha]{\beta}} y = \varepsilon_1 \beta_1 y, \qquad t \geq t_4,$$

where

$$\varepsilon_1 = \sqrt[\alpha]{\frac{\beta(1-\sqrt[\alpha]{\beta_*})}{\beta_*(1-\sqrt[\alpha]{\beta})}} \frac{\lambda^{\sqrt[\alpha]{\beta}}}{\lambda_*^{\sqrt[\alpha]{\beta_*}}}$$

is arbitrary constant from $(0,1)$ approaching 1 if $\beta \to \beta_*$ and $\lambda \to \lambda_*$. Hence,

$$\left(\frac{y}{\pi^{\varepsilon_1\beta_1}}\right)' < 0, \qquad t \geq t_4.$$

By induction, one can show that for any $n \in \mathbb{N}_0$ and t large enough,

$$\left(\frac{y}{\pi^{\varepsilon_n\beta_n}}\right)' < 0,$$

where ε_n given by

$$\varepsilon_0 = \sqrt[\alpha]{\frac{\beta}{\beta_*}}$$

$$\varepsilon_{n+1} = \varepsilon_0 \sqrt[\alpha]{\frac{1-\beta_n}{1-\varepsilon_n\beta_n}\frac{\lambda^{\varepsilon_n\beta_n}}{\lambda_*^{\beta_n}}}, \qquad n \in \mathbb{N}_0$$

is arbitrary constant from $(0,1)$ approaching 1 if $\beta \to \beta_*$ and $\lambda \to \lambda_*$. Finally, we claim that from any $n \in \mathbb{N}_0$, $y/\pi^{\varepsilon_{n+1}\beta_{n+1}}$ decreasing implies that that y/π^{β_n} is decreasing as well. To see this, we use that from (4.5) and the fact that ε_{n+1} is arbitrarily close to 1,

$$\varepsilon_{n+1}\beta_{n+1} > \beta_n.$$

Hence, for t large enough,

$$-r^{1/\alpha}y'\pi > \varepsilon_{n+1}\beta_{n+1}y > \beta_n y$$

and so for any $n \in \mathbb{N}_0$ and t large enough,

$$\left(\frac{y}{\pi^{\beta_n}}\right)' < 0.$$

The proof is complete. □

Now, we are prepared to give straightforward proofs of the main results.

Proof of Theorem 3.1. Assume that y is an eventually positive solution of (1.1). Lemmas 4.1 and 4.3 ensure that $(y/\pi)' \geq 0$ and $(y/\pi^{\beta_n})' < 0$ for any $n \in \mathbb{N}_0$ and t large enough. This is the case when

$$\beta_n < 1 \quad \text{for any } n \in \mathbb{N}_0.$$

Hence the sequence $\{\beta_n\}$ defined by (4.4) is increasing and bounded from above, and so there exists a finite limit

$$\lim_{n\to\infty} \beta_n = m,$$

where m is the smaller positive root of the equation

$$c(m) = \liminf_{t\to\infty} r^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t). \tag{4.12}$$

However, by (3.2), equation (4.12) cannot have positive solutions. This contradiction concludes the proof. □

Proof of Theorem 3.4. Let y be a positive solution of (1.1) on $[t_1, \infty)$ where $t_1 \geq t_0$ is such that $y(\tau(t)) > 0$ for $t \geq t_1$. In view of (3.3), for any $M > 0$ there is sufficiently large t such that

$$\frac{\pi(\tau(t))}{\pi(t)} \geq M^{1/\sqrt[\alpha]{\beta}}. \tag{4.13}$$

Proceeding as in the proof of Lemma 4.3, we show that $y/\pi^{\sqrt[\alpha]{\beta}}$ is decreasing eventually, say for $t \geq t_2 \geq t_1$. Using this monotonicity in (4.6), we have

$$\begin{aligned} -r(t)\left(y'(t)\right)^{\alpha} &= -r(t_2)\left(y'(t_2)\right)^{\alpha} + \int_{t_2}^{t} q(s) y^{\alpha}(\tau(s)) \mathrm{d}s \\ &\geq -r(t_2)\left(y'(t_2)\right)^{\alpha} + M^{\alpha}\beta y^{\alpha}(t) \int_{t_2}^{t} \frac{\alpha}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)} \mathrm{d}s > M^{\alpha}\left(\frac{y(t)}{\pi(t)}\right)^{\alpha}, \end{aligned}$$

from which we deduce that y/π^M is decreasing. Since M is arbitrary, we get a contradiction with (ii)-part of Lemma 4.1 upon which y/π is nondecreasing. The proof is complete. □

Proof of Theorem 3.5. It can be directly verified that the canonical equation (3.4) is equivalent to a noncanonical equation (1.1) with $\alpha = 1$,

$$\begin{aligned} r(t) &= \tilde{r}(t) R^2(t) \\ q(t) &= \tilde{q}(t) R(t) R(\tau(t)) \end{aligned}$$

and

$$y(t) = \frac{x(t)}{R(t)}.$$

Here,

$$\pi(t) = \int_t^{\infty} \frac{\mathrm{d}s}{\tilde{r}(s) R^2(s)} = \frac{1}{R(t)}.$$

Then the conclusion immediately follows from Theorem 3.1. □

Proof of Theorem 3.6. Using the equivalent noncanonical representation of (3.4) as in the proof of Theorem 3.5, the conclusion follows from Theorem 3.4. □

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Multiple and particular solutions of a second order discrete boundary value problem with mixed periodic boundary conditions

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Abstract. In this paper, a second order discrete boundary value problem with a pair of mixed periodic boundary conditions is considered. Sufficient conditions on the existence of multiple solutions are obtained by using the critical point theory. Necessary conditions for a particular solution subject to pre-defined criteria are also investigated. Examples are given to illustrate the applications of the results as well.

Keywords: discrete boundary value problem, mixed periodic boundary conditions, variational methods, mountain pass lemma, Lagrange multiplier.

2020 Mathematics Subject Classification: 39A10, 34B15, 49K30.

1 Introduction

In this paper, we consider a boundary value problem (BVP) consisting of a second order difference equation

$$-\Delta(r(t-1)\Delta u(t-1)) = f(t, u(t)), \qquad t \in [2, N]_{\mathbb{Z}}, \tag{1.1}$$

and a pair of mixed periodic boundary conditions (BCs)

$$u(0) = u(N), \qquad r(0)\Delta u(0) = -r(N)\Delta u(N), \tag{1.2}$$

where

- $N \geq 2$ is an integer and $[a, b]_{\mathbb{Z}}$ denotes the discrete interval $\{a, \ldots, b\}$ for any integers a and b with $a \leq b$;
- Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$;
- $r(t) > 0$, $t \in [0, N]_{\mathbb{Z}}$; and

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- $f : [2, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is odd and continuous with respect to the second variable, i.e. $f(t, -x) = -f(t, x)$ and $f(t, \cdot) \in C(\mathbb{R})$, $t \in [2, N]_{\mathbb{Z}}$.

By a solution of BVP (1.1), (1.2), we mean a function $u : [0, N+1]_{\mathbb{Z}} \to \mathbb{R}$ that satisfies (1.1) and (1.2).

BVPs with various BCs have been widely studied for decades due to both theoretic importance and extensive applications in science and engineering areas. Great effort has been made to study the existence, multiplicity, and uniqueness of solutions of BVPs, see for example [4–11, 13–18] and references therein for some recent advances in this area.

Recently, Kong and Wang [15] studied the existence and multiplicity of solutions of the mixed periodic BVP

$$-\Delta^2 u(t-1) = f(u(t)), \quad t \in [2, N]_{\mathbb{Z}}, \tag{1.3}$$

$$u(0) = -u(N), \quad \Delta u(0) = \Delta u(N), \tag{1.4}$$

by using the critical point theory. In that work, the asymmetry at the boundaries of the domain caused by the mixed periodic BC (1.4) was the major obstacle in the construction of a suitable functional for applying the variational technique. As the result, a particular Banach space and an associated functional were proposed to overcome the asymmetry of the mixed periodic BC (1.4). The reader is referred to [15, Lemma 2.3] for the details. We want to point out that there was a typo in Eq. (1.1) in [15] where the domain was mistakenly written as $t \in [1, N]_{\mathbb{Z}}$, which should be replaced by $t \in [2, N]_{\mathbb{Z}}$ as seen in Eq. (1.3) above. The reason why we propose $t \in [2, N]_{\mathbb{Z}}$ will be explained in Remark 2.5 below.

Clearly, Eq. (1.1) covers Eq. (1.3) as a special case and BC (1.2) and BC (1.4) are closely related to each other. So BVP (1.1), (1.2) is parallel to BVP (1.3), (1.4) but more general. Moreover, BC (1.2) leads to an asymmetry at the boundaries as well. This obstacle must be first eliminated to construct the functional. We will use an idea similar to [15] to overcome this difficulty and further apply the variational arguments and the critical point theory to study the existence of multiple solutions of BVP (1.1), (1.2). This will be the first contribution of this paper.

Once the multiplicity of solutions is proven, it is natural to raise a new question: *Which solution is the "right" one (in the sense that some pre-defined criteria are met)?* This question is practical in applications as there is a common need to identify a particular solution following certain pre-defined criteria, among all the solutions, due to constraints or demands of particular circumstances. In this paper, a framework to derive the necessary conditions for a particular solution of BVP (1.1), (1.2) following a set of pre-defined criteria, i.e. a *target solution*, will be presented. To the best of our knowledge, this type of questions have not been considered in the literature on BVPs. Our work will fill the void and be applicable to other problems with multiple solutions. This will be the second contribution of this paper.

The remainder of this paper is organized as follows. The Banach space, the functional, and the needed lemmas are given in Section 2; criteria on the existence of multiple solutions are proven in Section 3; the necessary conditions of the target solutions are derived in Section 4; and three examples are given in Section 5 to demonstrate the applications of our results.

2 Preliminary

We first introduce a few definition and lemmas needed to prove our existence results.

Definition 2.1. Assume H is a real Banach space. We say that a functional $J \in C^1(H, \mathbb{R})$ satisfies the Palais–Smale (PS) condition if every sequence $\{u_n\} \subset H$, such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence. The sequence $\{u_n\}$ is called a PS sequence.

The following version of Clark's Theorem is taken from [19] and will play a key role in proving our existence theorem.

Lemma 2.2 ([19, Theorem 9.1]). *Let H be a real Banach space with $\mathbf{0}$ the zero of H, S^{n-1} be the $(n-1)$-dimensional unit sphere, and $J \in C^1(H, \mathbb{R})$ with J even, bounded from below and satisfying the PS condition. Suppose $J(\mathbf{0}) = 0$, and there is a set $K \subset H$ such that K is homeomorphic to S^{n-1} by an odd map, and $\sup_K J < 0$. Then J possesses at least n distinct pairs of critical points.*

In the sequel, we let H be defined by

$$H = \{u : [0, N+1]_{\mathbb{Z}} \to \mathbb{R} \mid u(0) = u(N),\ u(1) = 0,\ r(0)\Delta u(0) = -r(N)\Delta u(N)\}. \tag{2.1}$$

Remark 2.3. By (2.1), we see that any $u \in H$ must satisfy

$$u(0) = u(N), \quad u(1) = 0, \quad u(N+1) = \frac{r(0) + r(N)}{r(N)} u(N). \tag{2.2}$$

So H is isomorphic to $\mathbb{R}^{N-1}$. Then, equipped with the norm $\|u\| = \left(\sum_{t=1}^{N} u^2(t)\right)^{\frac{1}{2}}$, H is an $N-1$ dimensional Banach space. When we write the vector $u = (0, u(2), \ldots, u(N)) \in \mathbb{R}^N$, we always imply that u can be extended as a vector in H so that (2.2) holds, i.e., u can be extended to the vector

$$\left(u(N), 0, u(2), \ldots, u(N), \frac{r(0) + r(N)}{r(N)} u(N)\right).$$

Moreover, for any $u \in H$, when we write $u = (0, u(2), \ldots, u(N)) \in \mathbb{R}^N$, we mean that u have been extended in the above sense.

Let $\tilde{f} : [1, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ and $\tilde{F} : [1, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\tilde{f}(t, x) = \begin{cases} 0, & t = 1, \\ f(t, x), & t \in [2, N-1]_{\mathbb{Z}}, \\ f(N, x) + 2r(0)x, & t = N, \end{cases} \tag{2.3}$$

and

$$\tilde{F}(t, x) = \int_0^x \tilde{f}(t, s) ds, \qquad t \in [1, N]_{\mathbb{Z}}, \tag{2.4}$$

resectively. It is clear that $\tilde{f}(t, x)$ and $\tilde{F}(t, x)$ are continuous in x and $\tilde{f}(t, x)$ is odd in x if $f(t, x)$ is odd in x.

Define $J : H \to \mathbb{R}$ by

$$J(u) = -\frac{1}{2} \sum_{t=1}^{N} r(t-1)(\Delta u(t-1))^2 + \sum_{t=1}^{N} \tilde{F}(t, u(t)). \tag{2.5}$$

Lemma 2.4. *If $u \in H$ is a critical point of J, then u is a solution of BVP (1.1), (1.2).*

Proof. By (2.3)–(2.5), for any $u \in H$,

$$J(u) = -\frac{1}{2}\sum_{t=1}^{N} r(t-1)(\Delta u(t-1))^2 + \sum_{t=2}^{N}\int_0^{u(t)} f(t,s)ds + 2\int_0^{u(N)} r(0)sds.$$

Then J is continuously differentiable and its derivative $J'(u)$ at $u \in H$ is given by

$$\langle J'(u), v\rangle = -\sum_{t=1}^{N} r(t-1)\Delta u(t-1)\Delta v(t-1) + \sum_{t=2}^{N} f(t,u(t))v(t) + 2r(0)u(N)v(N) \tag{2.6}$$

for any $v \in H$.

By the summation by parts formula and (2.1),

$$\begin{aligned}\sum_{t=1}^{N} r(t-1)\Delta u(t-1)\Delta v(t-1) &= r(N)\Delta u(N)v(N) - r(0)\Delta u(0)v(0) \\ &\quad - \sum_{t=1}^{N}\Delta(r(t-1)\Delta u(t-1))v(t) \\ &= -2r(0)\Delta u(0)v(0) - \sum_{t=1}^{N}\Delta(r(t-1)\Delta u(t-1))v(t) \\ &= 2r(0)u(N)v(N) - \sum_{t=2}^{N}\Delta(r(t-1)\Delta u(t-1))v(t). \end{aligned} \tag{2.7}$$

Then by (2.6) and (2.7), we have $\langle J'(u), v\rangle = \sum_{t=2}^{N}\left[\Delta(r(t-1)\Delta u(t-1)) + f(t,u(t))\right]v(t)$. This completes the proof of the lemma. □

Remark 2.5. Below, we provide some justification why we introduce the space H and the functional J as given above and why Eq. (1.1) is defined on $[2,N]_{\mathbb{Z}}$ instead of $[1,N]_{\mathbb{Z}}$. To see this, assume Eq. (1.1) is defined on $[1,N]_{\mathbb{Z}}$, and as in the traditional way, let

$$\tilde{H} = \{u : [0, N+1]_{\mathbb{Z}} \to \mathbb{R} \mid u \text{ satisfies the BCs (1.2)}\}$$

and

$$\tilde{J}(u) = -\frac{1}{2}\sum_{t=1}^{N} r(t-1)(\Delta u(t-1))^2 + \sum_{t=1}^{N}\int_0^{u(t)} f(t,s)ds.$$

Then, if $u \in \tilde{H}$ is a critical point of $\tilde{J}(u)$, by summation by parts formula and (1.2), we have

$$\begin{aligned}\langle \tilde{J}'(u), v\rangle &= -\sum_{t=1}^{N} r(t-1)\Delta u(t-1)\Delta v(t-1) + \sum_{t=1}^{N} f(t,u(t))v(t) \\ &= -2r(N)\Delta u(N)v(N) + \sum_{t=1}^{N}[\Delta(r(t-1)\Delta u(t-1)) + f(t,u(t))]v(t)\end{aligned}$$

for any $v \in H$. Since $v \in \tilde{H}$ is arbitrary, u satisfies (1.1) at $t \in [1, N-1]_{\mathbb{Z}}$. However, u satisfies Eq. (1.1) at $t = N$ only if $\Delta u(N) = 0$. Then the BCs (1.2) now become

$$u(0) = u(N), \quad \Delta u(0) = \Delta u(N) = 0,$$

which is very restrictive and is a special case of the periodic BCs studied in the literature, for example, in [12,16]. We do not have an interest in such a simple case. In this work, in order to make u satisfy Eq. (1.1) at $t = N$ without introducing the extra assumption $\Delta u(N) = 0$, unlike the traditional way, we introduce a modification, $\tilde{f}$, of the function f, as given in (2.3), and the corresponding functional J in (2.5). In addition to the BCs, we also impose an extra condition $u(1) = 0$ in our working space H defined by (2.1). Then, as seen in Lemma 2.4, any critical point $u \in H$ of J satisfies Eq. (1.1) for all $t \in [2, N-1]_{\mathbb{Z}}$ and the BCs

$$u(0) = u(N), \quad u(1) = 0, \quad r(0)\Delta u(0) = -r(N)\Delta u(N).$$

That is, u is a solution of BVP (1.1), (1.2) with the property that $u(1) = 0$. This type of problems are new and are worthy of our studies. The above explanations also explain why we only require Eq. (1.1) to be defined on $[2, N]_{\mathbb{Z}}$. We propose Eq. (1.3) in [15] due to a similar reason.

Remark 2.6. Lemma 2.5 offers a general setting to study the BVPs with mixed periodic BCs. With the functional defined by (2.5), other variational techniques may be applied as well, see, for example, [1,3].

Next, let us consider an equivalent form of J. Let

$$A = \begin{bmatrix} r(0)+r(1) & -r(1) & 0 & \dots & 0 & -r(0) \\ -r(1) & r(1)+r(2) & -r(2) & \dots & 0 & 0 \\ 0 & -r(2) & r(2)+r(3) & \dots & 0 & 0 \\ & \cdots & & \cdots & & \\ -r(0) & 0 & 0 & \dots & -r(N-1) & r(N-1)+r(0) \end{bmatrix}_{N\times N}. \tag{2.8}$$

Then it can be verified by direct computation that for any $u \in H$,

$$J(u) = -\frac{1}{2}uAu^T + \sum_{t=1}^{N} \tilde{F}(t, u(t)), \tag{2.9}$$

where $(\cdot)^T$ denotes the transpose.

Matrix A has been studied in [16]. Some needed conclusions are summarized in the following lemma. The reader is referred to [16] for the details.

Lemma 2.7. *Let A be defined by (2.8) with $r(t) > 0$, $t \in [0, N-1]_{\mathbb{Z}}$. Then*

(a) A is positively semi-definite with $\mathrm{Rank}(A) = N - 1$.

(b) A has N nonnegative eigenvalues $0 = \lambda_0 < \lambda_1 \le \cdots \le \lambda_{N-1}$ with the associated orthonormal eigenvectors $\{\eta_0, \dots, \eta_{N-1}\}$, where $\eta_0 = (\frac{\sqrt{N}}{N}, \frac{\sqrt{N}}{N}, \dots, \frac{\sqrt{N}}{N})$.

(c) Let $\|\cdot\|$ denote the standard Euclidean norm of $\mathbb{R}^N$. For any $u \in \mathbb{R}^N$, $uAu^T \le \lambda_{N-1}\|u\|^2$; for any $u \in \mathrm{span}\{\eta_2, \dots, \eta_{N-1}\}$, $uAu^T \ge \lambda_1 \|u\|^2$.

Similary to [16, Lemma 3.1], we can prove the following lemma.

Lemma 2.8. *Assume there exists a constant $\beta > \lambda_{N-1}$ such that*

$$\lim_{x\to\infty} \frac{f(t,x)}{x} \ge \beta, \qquad t \in [2, N]_{\mathbb{Z}}. \tag{2.10}$$

Then J satisfies the PS condition.

Proof. Let $\{u_n\}_{n=1}^{\infty} \subset H$ be any sequence with $\{J(u_n)\}$ bounded and $J'(u_n) \to 0$ as $n \to \infty$. For any u_n, by (2.6), (2.5), and (2.9),

$$\begin{aligned}\langle J'(u_n), u_n\rangle &= -\sum_{t=1}^{N} r(t-1)(\Delta u_n(t-1))^2 + \sum_{t=2}^{N} f(t, u_n(t))u_n(t) + 2r(0)(u_n(N))^2 \\ &= -u_n A u_n^T + \sum_{t=2}^{N} f(t, u_n(t))u_n(t) + 2r(0)(u_n(N))^2.\end{aligned}$$

Then by Lemma 2.7,

$$\begin{aligned}\sum_{t=2}^{N} f(t, u_n(t))u_n(t) + 2r(0)(u_n(N))^2 &= \langle J'(u_n), u_n\rangle + u_n A u_n^T \\ &\leq \langle J'(u_n), u_n\rangle + \lambda_{N-1}\|u_n\|^2. \end{aligned} \tag{2.11}$$

On the other hand, by the oddity of f and (2.10), there exists constant $C > 0$ such that

$$f(t, u_n(t))u_n(t) \geq \left(\frac{\beta + \lambda_{N-1}}{2}\right)(u_n(t))^2 - C, \qquad t \in [2, N]_{\mathbb{Z}}.$$

Hence

$$\sum_{t=2}^{N} f(t, u_n(t))u_n(t) + 2r(0)(u_n(N))^2 \geq \left(\frac{\beta + \lambda_{N-1}}{2}\right)\|u_n\|^2 - NC. \tag{2.12}$$

By (2.11) and (2.12),

$$\left(\frac{\beta - \lambda_{N-1}}{2}\right)\|u_n\|^2 \leq \langle J'(u_n), u_n\rangle + NC \leq \|J'(u_n)\|\|u_n\| + NC.$$

Since $(\beta - \lambda_{N-1})/2 > 0$ and $J'(u_n) \to 0$ as $n \to \infty$, $\{u_n\}$ is bounded. Therefore, the PS condition holds. □

3 Existence of solutions

In this section, we consider the existence of multiple solutions of BVP (1.1), (1.2).

Theorem 3.1. *Let $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_{N-1}$ be the eigenvalues of A defined by (2.8) respectively. Assume that $f(t, x)$ is continuous and odd in its second variable x, and satisfies (2.10) for some $\beta > \lambda_{N-1}$. If in addition there exists a constant $\mu < \lambda_m$, $m \in [1, N-1]_{\mathbb{Z}}$, such that*

$$\lim_{x\to 0} \frac{f(t,x)}{x} \leq \mu, \quad t \in [2, N-1]_{\mathbb{Z}}, \quad \textit{and} \quad \lim_{x\to 0} \frac{f(N,x)}{x} + 2r(0) \leq \mu. \tag{3.1}$$

Then BVP (1.1), (1.2) has at least $2N - 2m$ distinct solutions.

Remark 3.2. In (3.1), when $N = 2$, we have $[2, N-1]_{\mathbb{Z}} = \emptyset$. Then, the first limit disappears.

Proof. By Lemma 2.8, J satisfies the PS condition. Since f is odd in x, by (2.3) and (2.4), $\tilde{F}(t, x)$ is even in x.

Let $\{\eta_0, \ldots, \eta_{N-1}\}$ be the orthonormal eigenvectors of A defined in Lemma 2.7, $X = \operatorname{span}\{\eta_1, \ldots, \eta_{N-1}\}$, and $Y = \operatorname{span}\{\eta_0\}$. Then it is easy to see that $\mathbb{R}^N = X \oplus Y$. By (2.1),

$H \cap Y = \mathbf{0}$, so $H = X$. For any $u \in H$, there exist $b_1, \dots, b_{N-1} \in \mathbb{R}$ such that $u = \sum_{i=1}^{N-1} b_i \eta_i$ and $\|u\|^2 = \sum_{i=1}^{N-1} b_i^2$. By (2.9) and Lemma 2.7, for any $u \in H$,

$$\begin{aligned} J(u) &= -\frac{1}{2}uAu^T + \sum_{t=1}^{N} \tilde{F}(t,u(t)) = -\frac{1}{2}\sum_{i=1}^{N-1} \lambda_i b_i^2 + \sum_{t=1}^{N} \tilde{F}(t,u(t)) \\ &\geq -\frac{1}{2}\lambda_{N-1}\sum_{i=1}^{N-1} b_i^2 + \sum_{t=1}^{N} \tilde{F}(t,u(t)) = -\frac{1}{2}\lambda_{N-1}\|u\|^2 + \sum_{t=1}^{N} \tilde{F}(t,u(t)). \end{aligned}$$

Similar to the proof of Lemma 2.8, there exists $\tilde{C} > 0$ such that

$$\sum_{t=1}^{N} \tilde{F}(t,u(t)) \geq \left(\frac{\beta + \lambda_{N-1}}{4}\right)\|u\|^2 - N\tilde{C}, \qquad u \in H.$$

Therefore, $\inf_{u \in H} J(u) > -\infty$, i.e. J is bounded below.

By (3.1), there exist $\rho > 0$ and $0 < D < \lambda_m$ such that for any $x \in [-\rho, \rho]$,

$$\int_0^x f(t,s)ds \leq \frac{D}{2}x^2, \quad t \in [2, N-1] \quad \text{and} \quad \int_0^x f(N,s)ds + r(0)x^2 \leq \frac{D}{2}x^2. \tag{3.2}$$

Let $K = \{u \in \operatorname{span}\{\eta_m, \dots, \eta_{N-1}\} \subset H \mid \|u\| = \rho\}$. It is clear that K is homeomorphic to S^{N-m-1} by an odd map $\Gamma : K \to X$ defined by $\Gamma u = \frac{u}{\rho}$. By (2.9), (2.3), (2.4), (3.2), and Lemma 2.7, for any $u \in K$,

$$\begin{aligned} J(u) &= -\frac{1}{2}uAu^T + \sum_{t=1}^{N} \tilde{F}(t,u(t)) = -\frac{1}{2}\sum_{i=m}^{N-1} \lambda_i b_i^2 + \sum_{t=1}^{N} \tilde{F}(t,u(t)) \\ &\leq -\frac{1}{2}\lambda_m \sum_{i=1}^{N-1} b_i^2 + \sum_{t=1}^{N} \tilde{F}(t,u(t)) = -\frac{1}{2}\lambda_m\|u\|^2 + \sum_{t=1}^{N} \tilde{F}(t,u(t)) \leq \frac{D-\lambda_m}{2}\rho^2 < 0. \end{aligned}$$

Therefore, $\sup_K J < 0$. By Lemma 2.2, J possesses at least $N - m$ distinct pairs of critical points. Hence BVP (1.1), (1.2) has at least $2N - 2m$ solutions by Lemma 2.4. □

The following corollary is an immediate conclusion of Theorem 3.1.

Corollary 3.3. *Assume that $f(t,x)$ is continuous and odd in its second variable x, and satisfies*

$$\liminf_{x \to \infty} \min_{t \in [2,N]_{\mathbb{Z}}} \frac{f(t,x)}{x} = \infty$$

and

$$\max\left\{\limsup_{x \to 0} \max_{t \in [2,N-1]_{\mathbb{Z}}} \frac{f(t,x)}{x}, \limsup_{x \to 0} \frac{f(N,x)}{x} + 2r(0)\right\} < \lambda_m, \tag{3.3}$$

where λ_m is the mth positive eigenvalue of A following the increasing order. Then BVP (1.1), (1.2) has at least $2N - 2m$ distinct solutions.

A note similar to Remark 3.2 applies to Eq. (3.3) in Corollary 3.3.

4 Necessary conditions of the target solution

In this section, we investigate how to identify a target solution among multiple solutions following a set of pre-defined criteria. The main idea is to find the target solution by solving an optimization problem (OP) with constraints.

Let I be a subset of $[0, N+1]_{\mathbb{Z}}$ and $u^* : I \to \mathbb{R}$ be a function defined on I. Assume the pre-defined criteria is given as a performance index, or objective function, $L : \mathbb{R}^{N+2} \to \mathbb{R}$ defined by

$$L(u) = \sum_{t \in I} (u(t) - u^*(t))^2. \tag{4.1}$$

We need to find a particular solution of BVP (1.1), (1.2) that minimizes the objective function L. In other words, BVP (1.1), (1.2) is the constraints of the OP.

We first introduce some auxiliary functions to simplify the notations. Define $G : \mathbb{R}^{N+2} \times [2, N]_{\mathbb{Z}} \to \mathbb{R}$, $B_0 : \mathbb{R}^{N+2} \to \mathbb{R}$, $B_1 : \mathbb{R}^{N+2} \to \mathbb{R}$, and $B_2 : \mathbb{R}^{N+2} \to \mathbb{R}$ by

$$\begin{aligned} G(u,t) &= r(t)u(t+1) - (r(t) + r(t-1))u(t) + r(t-1)u(t-1) + f(t, u(t)), \\ B_0(u) &= u(1), \; B_1(u) = u(0) - u(N), \text{ and} \\ B_2(u) &= r(0)u(1) - r(0)u(0) + r(N)u(N+1) - r(N)u(N). \end{aligned}$$

It is easy to verify that BVP (1.1), (1.2) is equivalent to the following system consisting of $N+2$ equations

$$G(u,t) = 0, \; t \in [2, N]_{\mathbb{Z}}, \tag{4.2}$$
$$B_0(u) = 0, \tag{4.3}$$
$$B_1(u) = 0, \tag{4.4}$$
$$B_2(u) = 0. \tag{4.5}$$

In the sequel, we use Eq. (4.2)–(4.5) as the constraints and solve the OP (4.1), (4.2)–(4.5) by the Lagrange multiplier method, see for example [2]. Clearly, $N+2$ Lagrange multipliers are needed. Let $\theta : [0, N+1] \to \mathbb{R}$ be the Lagrange multipliers and $\Phi : \mathbb{R}^{N+2} \times \mathbb{R}^{N+2} \to \mathbb{R}$ be defined by

$$\Phi(u, \theta) = \zeta L(u) + \sum_{t=2}^{N} \theta(t+1) G(u,t) + \theta(0) B_0(u) + \theta(1) B_1(u) + \theta(2) B_2(u), \tag{4.6}$$

where $\zeta > 0$ is a parameter. Then by the Lagrange multiplier method, we obtain the following necessary conditions for the target solution.

Theorem 4.1. *A target solution of BVP (1.1), (1.2) subject to L must satisfy Eq. (4.2)–(4.5) and*

$$\frac{\partial \Phi(u, \theta)}{\partial u(t)} = 0, \qquad t \in [0, N+1]_{\mathbb{Z}}.$$

Remark 4.2. The value of ζ in (4.6) does not impact the theoretic result in Theorem 4.1. However, numerical experiments reveal that the value of ζ impacts the performance of numerical optimization algorithms. This is the main reason to introduce the parameter ζ.

5 Examples

In this section, we will demonstrate the applications of our results by considering the BVP

$$-\Delta^2 u(t-1) = (u(t))^3,\ t \in [2,10]_{\mathbb{Z}}, \tag{5.1}$$
$$u(0) = u(10),\ \Delta u(0) = -\Delta u(10). \tag{5.2}$$

Let $r(t) \equiv 1$ on $[0,N]_Z$ and $f(t,x) \equiv x^3$. It is easy to verify that

$$A = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}_{10\times 10},$$

$$\lim_{x\to\infty} \frac{f(x)}{x} = \infty, \quad \text{and} \quad \lim_{x\to 0} \frac{f(x)}{x} = 0.$$

Computing the eigenvalues of A with Matlab, we have $\lambda_4 < 2 < \lambda_5$. Hence all the conditions of Corollary 3.3 are satisfied. Therefore, BVP (5.1), (5.2) has at least 10 solutions.

Next, we choose different objective functions to demonstrate the applications of Theorem 4.1.

Example 5.1. We first consider a solution of BVP (5.1), (5.2) that minimizes the objective function

$$L_1(u) = \sum_{t=4}^{6} (u(t)-1)^2.$$

Let $\zeta = 2$. By Theorem 4.1, the target solution u must satisfy the following system

$$u(t-1) - 2u(t) + u(t+1) + (u(t))^3 = 0, \quad t \in [2,10]_{\mathbb{Z}}, \tag{5.3}$$
$$u(1) = 0 \tag{5.4}$$
$$u(0) - u(10) = 0, \tag{5.5}$$
$$-u(0) + u(1) - u(10) + u(11) = 0, \tag{5.6}$$
$$\tilde{\Theta}(u,\theta,t) + \tilde{\Phi}(u,\theta,t) = 0, \quad t \in [0,11]_{\mathbb{Z}}, \tag{5.7}$$

where

$$\tilde{\Theta}(u,\theta,0) := \theta(1) - r(0)\theta(2), \tag{5.8}$$
$$\tilde{\Theta}(u,\theta,1) := \theta(0) + r(0)\theta(2) + r(1)\theta(3), \tag{5.9}$$
$$\tilde{\Theta}(u,\theta,2) := (3(u(2))^2 - (r(1)+r(2)))\theta(3) + r(2)\theta(4), \tag{5.10}$$
$$\begin{aligned}\tilde{\Theta}(u,\theta,t) := {} & r(t-1)\theta(t) + (3(u(t))^2 - (r(t-1)+r(t)))\theta(t+1) \\ & + r(t)\theta(t+2), \quad t = [3,9]_{\mathbb{Z}},\end{aligned} \tag{5.11}$$
$$\tilde{\Theta}(u,\theta,10) := -\theta(1) - r(10)\theta(2) + r(9)\theta(10) + (3(u(10))^2 - (r(9)+r(10)))\theta(11), \tag{5.12}$$
$$\tilde{\Theta}(u,\theta,11) := r(10)\theta(2) + r(10)\theta(11), \tag{5.13}$$

and

$$\tilde{\Phi}(u,\theta,t) := 4(u(t)-1), \quad t \in [4,6]_{\mathbb{Z}}, \tag{5.14}$$
$$\tilde{\Phi}(u,\theta,t) := 0, \quad \text{otherwise}. \tag{5.15}$$

Note that by (4.2)–(4.5), Eq. (5.3)–(5.6) are equivalent to BVP (5.1), (5.2); $\tilde{\Theta}$ defined by (5.8)–(5.13) are the partial derivatives of

$$\sum_{t=2}^{N} \theta(t+1)G(u,t) + \theta(0)B_0(u) + \theta(1)B_1(u) + \theta(2)B_2(u)$$

in (4.6) with respect to $u(t)$, $t \in [0,11]_{\mathbb{Z}}$; and $\tilde{\Phi}$ defined by (5.14) and (5.15) are the partial derivatives of $\zeta L(u)$ in (4.6) with respect to $u(t)$, $t \in [0,11]_{\mathbb{Z}}$.

System (5.3)–(5.7) is solved with Matlab. The graph of the numerical solution u_1 subject to L_1 is given in Figure 5.1. Clearly, the behavior of u_1 is consistent with our expectation.

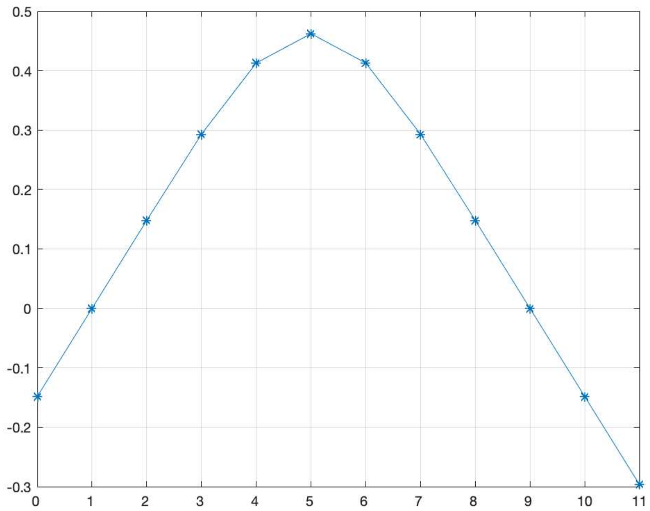

Figure 5.1: Numerical solution u_1 subject to L_1.

Example 5.2. For the comparison purpose, we also consider the solution of BVP (5.1), (5.2) that minimizes the objective function

$$L_2(u) = \sum_{t=4}^{6} (u(t)+1)^2.$$

Let $\zeta = 2$. By Theorem 4.1, the target solution must satisfy Eq. (5.3)–(5.7) with

$$\tilde{\Phi}(u,\theta,t) := 4(u(t)+1), \quad t \in [4,6]_{\mathbb{Z}},$$
$$\tilde{\Phi}(u,\theta,t) := 0, \quad \text{otherwise}.$$

The graph of the numerical solution u_2 subject to L_2 is given in Figure 5.2.

Example 5.3. In this example, we seek a solution of BVP (5.1), (5.2) that minimizes the objective function

$$L_3(u) = \sum_{t=7}^{9} (u(3) - 10)^2.$$

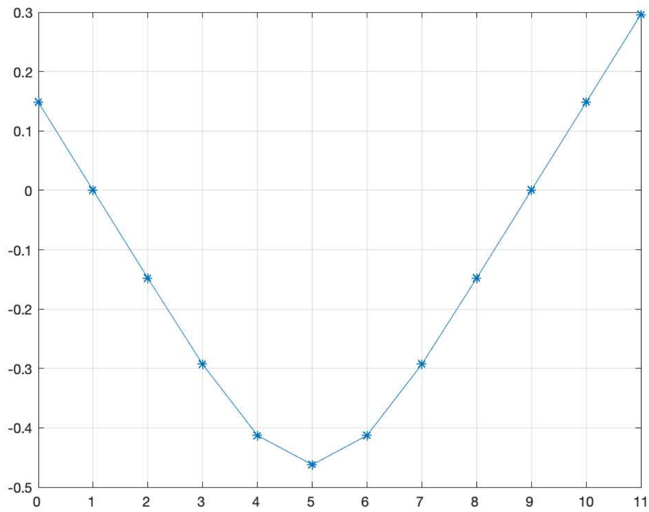

Figure 5.2: Numerical solution u_2 subject to L_2.

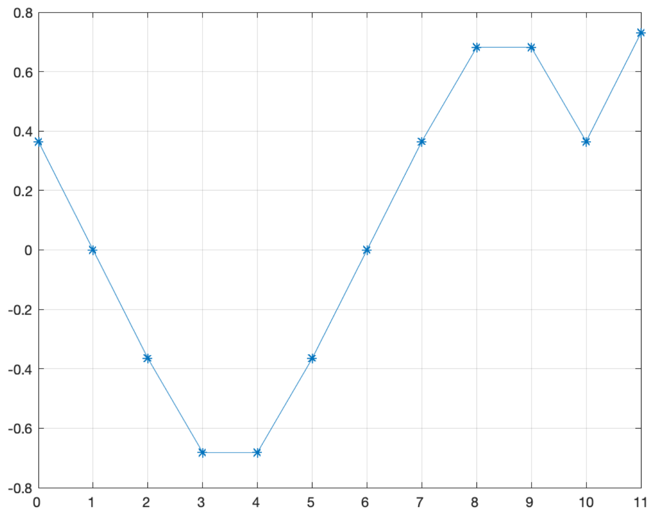

Figure 5.3: Numerical solution u_3 subject to L_3.

Let $\zeta = 1$. By Theorem 4.1, the target solution must satisfy Eq. (5.3)–(5.7) with

$$\tilde{\Phi}(u, \theta, t) := 2(u(t) - 10), \quad t \in [7, 9]_{\mathbb{Z}},$$
$$\tilde{\Phi}(u, \theta, t) := 0, \quad \text{otherwise.}$$

The graph of the numerical solution u_3 subject to L_3 is given in Figure 5.3.

Remark 5.4. Examples 5.1, 5.2, and 5.3 found three different solutions from the same BVP following different criteria. These examples demonstrated the effectiveness of Theorem 4.1. This idea can also be extended to the objective functions of other forms as well as other BVPs with multiple solutions.

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Algebraic traveling waves for the modified Korteweg–de Vries–Burgers equation

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Abstract. In this paper we characterize all traveling wave solutions of the Generalized Korteweg–de Vries–Burgers equation. In particular we recover the traveling wave solutions for the well-known Korteweg–de Vries–Burgers equation.

Keywords: traveling wave, modified Korteweg–de Vries–Burgers equation, Korteweg–de Vries–Burgers equation.

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1 Introduction and statement of the main results

Looking for traveling waves to nonlinear evolution equations has long been the major problem for mathematicians and physicists. These solutions may well describe various phenomena in physics and other fields and thus they may give more insight into the physical aspects of the problems. Many methods for obtaining traveling wave solutions have been established [4–6,19,20,25,26] with more or less success. When the degree of the nonlinearity is high most of the methods fail or can only lead to a kind of special solution and the solution procedures become very complex and do not lead to an efficient way to compute them.

In this paper we will focus on obtaining algebraic traveling wave solutions to the modified Korteweg–de Vries–Burgers equation (mKdVB) of the form

$$au_{xxx} + bu_{xx} + du^n u_x + u_t = 0 \tag{1.1}$$

where $n = 1, 2$ and a, b, d are real constants with $abd \neq 0$. When $n = 1$ is the well-known Korteweg–de Vries–Burgers equation (KdVB) that has been intensively investigated. When $n = 2$ we will call it modified Korteweg–de Vries–Burgers equation (mKdVB). These equations are widely used in fields as solid-states physics, plasma physics, fluid physics and quantum field theory (see, for instance [12,31] and the references therein). They mainly appear when seeking the asymptotic behavior of complicated systems governing physical processes in solid and fluid mechanics.

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An special attention is done to the KdVB, often considered as a combination of the Burgers equation and KdV equation since in the limit $a \to 0$, the equation reduces to the Burgers equation (named after its use by Burgers [2] for studying the turbulence in 1939), and taking the limit as $b \to 0$ we get the KdV equation (first suggested by Korteweg and de Vries [18] who used it as a nonlinear model to study the change of forms of long waves advancing in a rectangular channel).

The KdVB equation is the simplest form of the wave equation in which the nonlinear term uu_x, the dispersion u_{xxx} and the dissipation u_{xx} all occur. It arises from many physical context such as the undulant bores in a shallow water [1, 16], the flow of liquids containing gas bubbles [27], the propagation of waves in an elastic tube filled with a viscous fluid [15], weakly nonlinear plasma waves with certain dissipative effects [9, 11], the cascading down process of turbulence [7] and the atmospheric dynamics [17].

It is nonintegrable in the sense that its spectral problem is nonexistent. The existence of traveling wave solutions for the (KdVB) was obtained by the first time in [29] and after that many other papers computing the traveling wave of the KdVB appeared (see for instance [10, 13, 14, 21, 25, 28, 30]), but most of them did not obtain all the possible traveling wave solutions. However, regardless the attention done to the (KdVB), nothing is known for the existence of traveling wave solutions for the (mKdVB). This is due to the presence of high nonlinear terms. In this paper we will fill in this gap. We will use a method that will supply the already known traveling wave solution for the (KdVG) and will allows us to prove that there are no traveling wave solutions for the KdVG (i.e., equation (1.1) with $n = 2$).

As explained above, there are various approaches for constructing traveling wave solutions, but these methods become more and more useless as the degree of the nonlinear terms increase. However, in [8] the authors gave a technique to prove the existence of traveling wave solutions for general n-th order partial differential equations by showing that traveling wave solutions exist if and only if the associated n-dimensional first order ordinary differential equation has some invariant algebraic curve. In this paper we will consider only the case of 2-nd order partial differential equations.

More precisely, consider the 2-nd order partial differential equations of the form

$$\frac{\partial^2 u}{\partial x^2} = F\Big(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\Big), \tag{1.2}$$

where x and t are real variables and F is a smooth map. The traveling wave solutions of system (1.2) are particular solutions of the form $u = u(x,t) = U(x - ct)$ where $U(s)$ satisfies the boundary conditions

$$\lim_{s\to-\infty} U(s) = A \quad \text{and} \quad \lim_{s\to\infty} U(s) = B, \tag{1.3}$$

where A and B are solutions, not necessarily different, of $F(u,0,0) = 0$. Note that $U(s)$ has to be a solution, defined for all $s \in \mathbb{R}$, of the 2-nd order ordinary differential equation

$$U'' = F(U, U', -cU') = \tilde{F}(U, U'), \tag{1.4}$$

where $U(s)$ and the derivatives are taken with respect to s. The parameter c is called the *speed* of the traveling wave solution.

We say that $u(x,t) = U(x - ct)$ is an algebraic traveling wave solution if $U(s)$ is a non-constant function that satisfies (1.3) and (1.4) and there exists a polynomial p such that $p(U(s), U'(s)) = 0$.

As pointed out in [8] the term algebraic traveling wave means that the waves that we will find correspond to the algebraic curves on the phase plane and do not refer to traveling waves that approach to the constant boundary conditions (1.3) algebraically fast. The traveling wave solutions correspond to homoclinic (when $A = B$) or heteroclinic (when $A \neq B$) solutions of the associated two-dimensional system of ordinary differential equations. In many cases the critical points where this invariant manifolds start and end are hyperbolic. To motivate the definition of algebraic traveling wave solutions initiated in [8] and used in the present paper, we recall that when F is sufficiently regular, using normal form theory, in a neighborhood of these critical points, this manifold can be parameterized as $\varphi(e^{\lambda s})$ for some smooth function φ, where λ is one of the eigenvalues of the critical points.

Note that this definition of algebraic traveling wave revives the interest in the well-known and classic problem of finding invariant algebraic curves. Invariant algebraic curves are the main objects used in several subjects with special emphasis in integrability theory. The search and computation of these objects have been intensively investigated. However to determine the properties and number of them for a given planar vector field is very difficult in particular because there is no bound a priori on the degree of such curves. However in the present paper we will be able to characterize completely the algebraic traveling wave solutions of the Korteweg–de Vries–Burgers equation and of the Generalized Korteweg–de Vries–Burgers equation under some additional assumptions on the constants. We recall that for irreducible polynomials we have the following algebraic characterization of invariant algebraic curves: Given an irreducible polynomial of degree n, $g(x, y)$, we have that $g(x, y) = 0$ is an invariant algebraic curve for the system $x' = P(x, y)$, $y' = Q(x, y)$ for $P, Q \in \mathbb{C}[x, y]$, if there exists a polynomial $K = (x, y)$ of degree at most $n - 1$, called the cofactor of g such that

$$P(x,y)\frac{\partial g}{\partial x} + Q(x,y)\frac{\partial g}{\partial y} = K(x,y)g. \tag{1.5}$$

The main result that we will use is the following theorem, see [8] for its proof.

Theorem 1.1. *The partial differential equation (1.2) has an algebraic traveling wave solution if and only if the first order differential system*

$$\begin{cases} y_1' = y_2, \\ y_2' = G_c(y_1, y_2), \end{cases}$$

where

$$G_c(y_1, y_2) = \tilde{F}(y_1, y_2)$$

has an invariant algebraic curve containing the critical points $(A, 0)$and $(B, 0)$ and no other critical points between them.

The main result is, with the techniques in [8], obtain all algebraic traveling wave solutions of the (KdVB) and (mKdVB), i.e., all explicit traveling wave solutions of the equation (1.1) when $n = 1$ and when $n = 2$.

Theorem 1.2. *The following holds for system (1.1):*

(i) If $n = 1$ (KdVB), it has the algebraic traveling wave solution

$$u(x,t) = -\frac{12b^2}{25da}\Big(\frac{1}{1+\kappa_1 e^{b(x-vt)/(5a)}}\Big)^2 + \frac{6b^2}{25da} + \frac{v}{d},$$

where

$$v^2 = \frac{36b^4 - 1250da^3\kappa_2}{625a^2},$$

being κ_1, κ_2 arbitrary constants with $\kappa_1 > 0$.

(ii) *If $n = 2$ (mKdVB), it has no algebraic traveling wave solutions.*

The proof of Theorem 1.2 is given in Section 3 when $n = 1$ and in Section 4 when $n = 2$. In section 2 we have included some preliminary results that will be used to prove the results in the paper. The technique used in the paper is very powerful and has been used successfully in the papers [23, 24].

2 Preliminary results

In this section we introduce some notions and results that will be used during the proof of Theorem 1.2.

The first result based on the previous works of Seidenberg [22] was stated and proved in [3]. In the next theorem we included only the results from [3] that will be used in the paper.

Theorem 2.1. *Let $g(x, y) = 0$ be an invariant algebraic curve of a planar system with corresponding cofactor $K(x, y)$. Assume that $p = (x_0, y_0)$ is one of the critical points of the system. If $g(x_0, y_0) \neq 0$, then $K(x_0, y_0) = 0$. Moreover, assume that λ and μ are the eigenvalues of such critical point. If either $\mu \neq 0$ and λ and μ are rationally independent or $\lambda\mu < 0$, or $\mu = 0$, then either $K(x_0, y_0) = \lambda$, or $K(x_0, y_0) = \mu$, or $K(x_0, y_0) = \lambda + \mu$ (that we write as $K(x_0, y_0) \in \{\lambda, \mu, \lambda + \mu\}$).*

A polynomial $g(x, y)$ is said to be a *weight homogeneous polynomial* if there exist $s = (s_1, s_2) \in \mathbb{N}^2$ and $m \in \mathbb{N}$ such that for all $\mu \in \mathbb{R} \setminus \{0\}$,

$$g(\mu^{s_1} x, \mu^{s_2} y) = \alpha^m g(x, y),$$

where $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{N}$ the set of positive integers. We shall refer to $s = (s_1, s_2)$ to the weight of g, m the weight degree and $x = (x_1, x_2) \mapsto (\alpha^{s_1} x, \alpha^{s_2} y)$ the weight change of variables.

We first note that if there exists a solution of the form $u(x, t) = U(x - ct)$ then substituting in (1.1) and performing one integration yield

$$U'' = -\beta U' - \gamma U^{n+1} + \delta U + \theta,$$

where $\beta = b/a$, $\gamma = d/(a(n + 1))$, $\delta = c/a$ and θ is the integration constant. Therefore, we will look for the invariant algebraic curves of the system

$$\begin{aligned} x' &= y, \\ y' &= -\beta y - \gamma x^{n+1} + \delta x + \theta, \end{aligned} \tag{2.1}$$

where $x(s) = U(s)$ and $\beta, \gamma, \delta, \theta \in \mathbb{R}$ with $\beta\gamma\delta \neq 0$.

When $n = 1$, the solution of $\gamma x^2 - \delta x - \theta = 0$, that is,

$$x_{1,2} = \frac{\delta}{2\gamma} \mp \frac{\sqrt{\delta^2 + 4\gamma\theta}}{2\gamma}$$

must be real, otherwise there would be no algebraic traveling wave solutions. Therefore, $\delta^2 + 4\gamma\theta \geq 0$. Set $x = \overline{x} + x_1$, and $y = \overline{y}$. Then we rewrite system (2.1) with $n = 1$ in the variables $(\overline{x}, \overline{y})$ as

$$\begin{aligned} \overline{x}' &= \overline{y}, \\ \overline{y}' &= -\beta\overline{y} - \gamma(\overline{x} + x_1)^2 + \delta(\overline{x} + x_1) + \theta \\ &= -\beta\overline{y} - \gamma\overline{x}^2 - 2\gamma x_1\overline{x} - \gamma x_1^2 + \delta\overline{x} + \delta x_1 + \theta \\ &= -\beta\overline{y} - \gamma\overline{x}^2 + \overline{\delta}\overline{x}, \end{aligned} \tag{2.2}$$

where $\overline{\delta} = \delta - 2\gamma x_1 = \sqrt{\delta^2 + 4\gamma\theta}$.

When $n = 2$, the solution of $\gamma x^3 - \delta x - \theta = 0$ has at least one real solution, that we denote by x_1. Set $x = \overline{x} + x_1$, and $y = \overline{y}$. Then we rewrite system (2.1) with $n = 2$ in the variables $(\overline{x}, \overline{y})$ as

$$\begin{aligned} \overline{x}' &= \overline{y}, \\ \overline{y}' &= -\beta\overline{y} - \gamma(\overline{x} + x_1)^3 + \delta(\overline{x} + x_1) - \theta \\ &= -\beta\overline{y} - \gamma\overline{x}^3 - 3\gamma x_1\overline{x}^2 - 3\gamma x_1^2\overline{x} - \gamma x_1^3 + \delta\overline{x} + \delta x_1 - \theta \\ &= -\beta\overline{y} - \gamma\overline{x}^3 - \overline{\gamma}\overline{x}^2 + \overline{\delta}\overline{x}, \end{aligned} \tag{2.3}$$

where $\overline{\gamma} = 3\gamma x_1$ and $\overline{\delta} = \delta - 3\gamma x_1^2$.

3 Proof of Theorem 1.2 with $n = 1$

In this section we consider system (2.1) with $n = 1$. By the results in Section 2 this is equivalent to work with system (2.2).

Theorem 3.1. *System (2.2) has an invariant algebraic curve $g(\overline{x}, \overline{y}) = 0$ if and only if*

$$\beta = \pm\frac{5\sqrt{\overline{\delta}}}{\sqrt{6}}.$$

Moreover, if $\beta = 5\sqrt{\overline{\delta}}/\sqrt{6}$ then

$$g(\overline{x}, \overline{y}) = \frac{\overline{y}^2}{2} - \frac{\sqrt{2}}{\sqrt{3}}\frac{\sqrt{\overline{\delta}}}{\gamma}(\overline{\delta} - \gamma\overline{x})\overline{y} + \frac{\overline{x}}{3\gamma}(\overline{\delta} - \gamma\overline{x})^2,$$

and if $\beta = -5\sqrt{\overline{\delta}}/\sqrt{6}$ then

$$g(\overline{x}, \overline{y}) = \frac{\overline{y}^2}{2} + \frac{\sqrt{2}}{\sqrt{3}}\frac{\sqrt{\overline{\delta}}}{\gamma}(\overline{\delta} - \gamma\overline{x})\overline{y} + \frac{\overline{x}}{3\gamma}(\overline{\delta} - \gamma\overline{x})^2.$$

System (2.2) with $\overline{\delta} = \gamma$ is system (15) in [24]. Proceeding exactly as in the proof of Theorem 2 in [24] (with $\overline{\delta}$ instead of γ when needed) we get the proof of Theorem 3.1. So, the proof of Theorem 3.1 will be omitted.

Proof of Theorem 1.2. Consider first the case $\beta = \frac{5\sqrt{\overline{\delta}}}{\sqrt{6}}$. It follows from Theorem 3.1 that the invariant algebraic curve is

$$g(\overline{x}, \overline{y}) = \frac{\overline{y}^2}{2} - \frac{\sqrt{2}}{\sqrt{3}}\frac{\sqrt{\overline{\delta}}}{\gamma}(\overline{\delta} - \gamma)\overline{y} + \frac{\overline{x}}{3\gamma}(\overline{\delta} - \gamma\overline{x})^2.$$

The branch of $g(x,y)=0$ that contains the origin is

$$y=\frac{\sqrt{2}}{\sqrt{3}\gamma}(\overline{\delta}-\gamma\overline{x})\left(\sqrt{\overline{\delta}}-\sqrt{\overline{\delta}-\gamma\overline{x}}\right).$$

Since $\overline{x}'=\overline{y}$ we obtain

$$\overline{x}'=\frac{\sqrt{2}}{\sqrt{3}\gamma}(\overline{\delta}-\gamma\overline{x})\left(\sqrt{\overline{\delta}}-\sqrt{\overline{\delta}-\gamma\overline{x}}\right)=\frac{\sqrt{2}\overline{\delta}^{3/2}}{\sqrt{3}\gamma}\Big(1-\frac{\gamma}{\overline{\delta}}\overline{x}\Big)\Big(1-\sqrt{1-\frac{\gamma}{\overline{\delta}}\overline{x}}\Big).$$

Set $U(s)=x(s)=\overline{x}(s)+x_1$ and take $W(s)=\sqrt{1-\frac{\gamma}{\overline{\delta}}(U(s)-x_1)}$ Then

$$W'(s)=-\frac{\gamma}{\overline{\delta}}\frac{U'(s)}{2\sqrt{1-\frac{\gamma}{\overline{\delta}}(U(s)-x_1)}}=-\frac{\sqrt{\overline{\delta}}}{\sqrt{6}}W(s)(1-W(s)).$$

Its non-constant solutions that are defined for all $s\in\mathbb{R}$ are

$$W(s)=\frac{1}{1+\kappa e^{\sqrt{\overline{\delta}}s/\sqrt{6}}},\quad \kappa>0.$$

Hence,

$$U(s)=x_1+\frac{\overline{\delta}}{\gamma}\left(1-\left(\frac{1}{1+\kappa e^{\sqrt{\overline{\delta}}s/\sqrt{6}}}\right)^2\right),\quad \kappa>0.$$

This, together with the definition x_1, $\overline{\delta}$, δ, γ and β, yields the traveling wave solution in the statement of the theorem.

If we take the branch of $g(\overline{x},\overline{y})=0$ that does not contain the origin then

$$y=\frac{\sqrt{2}}{\sqrt{3}\gamma}(\overline{\delta}-\gamma\overline{x})\left(\sqrt{\overline{\delta}}+\sqrt{\overline{\delta}-\gamma\overline{x}}\right)$$

Proceeding exactly as above we get that

$$W(s)=\frac{1}{1-\kappa e^{\sqrt{\overline{\delta}}s/\sqrt{6}}},\quad \kappa>0,$$

which is not a global solution. So, in this case there are no traveling wave solutions.

Now take $\beta=-\frac{5\sqrt{\overline{\delta}}}{\sqrt{6}}$. It follows from Theorem 3.1 that the invariant algebraic curve is

$$g(\overline{x},\overline{y})=\frac{\overline{y}^2}{2}+\frac{\sqrt{2}}{\sqrt{3}}\frac{\sqrt{\overline{\delta}}}{\gamma}(\overline{\delta}-\overline{\gamma})\overline{y}+\frac{\overline{x}}{3\gamma}(\overline{\delta}-\gamma\overline{x})^2.$$

The branch of $g(,\overline{y})=0$ that contains the origin is

$$\overline{y}=-\frac{\sqrt{2}}{\sqrt{3}\gamma}(\overline{\delta}-\gamma\overline{x})\left(\sqrt{\overline{\delta}}-\sqrt{\overline{\delta}-\gamma\overline{x}}\right)$$

Since $\overline{x}'=\overline{y}$ we obtain

$$\overline{x}'=-\frac{\sqrt{2}}{\sqrt{3}\gamma}(\overline{\delta}-\gamma\overline{x})\left(\sqrt{\overline{\delta}}-\sqrt{\overline{\delta}-\gamma\overline{x}}\right)=-\frac{\sqrt{2}\overline{\delta}^{3/2}}{\sqrt{3}\gamma}\left(1-\frac{\gamma}{\overline{\delta}}\overline{x}\right)\left(1-\sqrt{1-\frac{\gamma}{\overline{\delta}}\overline{x}}\right)$$

Set $U(s) = x(s) = \overline{x}(s) + x_1$ and take $W(s) = \sqrt{1 - \frac{\gamma}{\overline{\delta}}(U(s) - x_1)}$. Then

$$W'(s) = \frac{\gamma}{\overline{\delta}} \frac{U'(s)}{2\sqrt{1 - \frac{\gamma}{\overline{\delta}}(U(s) - x_1)}} = \frac{\sqrt{\delta}}{\sqrt{6}} W(s)(1 - W(s)).$$

Its nonconstant solutions that are defined for all $s \in \mathbb{R}$ are

$$W(s) = \frac{1}{1 + \kappa e^{-\sqrt{\delta} s/\sqrt{6}}}, \quad \kappa > 0.$$

Hence

$$U(s) = x_1 + \frac{\overline{\delta}}{\gamma}\left(1 - \left(\frac{1}{1 + \kappa e^{-\sqrt{\overline{\delta}} s/\sqrt{6}}}\right)^2\right), \quad \kappa > 0.$$

This, together with the definition x_1, $\overline{\delta}$, δ, γ and β, yields the traveling wave solution in the statement of the theorem.

If we take the branch of $g(\overline{x}, \overline{y}) = 0$ that does not contain the origin then

$$y = -\frac{\sqrt{2}}{\sqrt{3}\gamma}(\overline{\delta} - \gamma x)\left(\sqrt{\overline{\delta}} + \sqrt{\overline{\delta} - \gamma \overline{x}}\right).$$

Proceeding exactly as above we get that

$$W(s) = \frac{1}{1 - \kappa e^{-\sqrt{\overline{\delta}} s/\sqrt{6}}}, \quad \kappa > 0,$$

which is not a global solution. So, in this case there are no traveling wave solutions and concludes the proof of the theorem. □

4 Proof of Theorem 1.2 with $n = 2$

In this section we consider system (2.1) with $n = 2$. By the results in Section 2 this is equivalent to work with system (2.3).

The proof of Theorem 1.2 with $n = 2$ follows directly from the following theorem that states that system (2.3) has no invariant algebraic curves.

Theorem 4.1. *System (2.3) has no invariant algebraic curve.*

Proof of Theorem 4.1. Let $g = g(\overline{x}, \overline{y}) = 0$ be an invariant algebraic curve of system (2.3) with cofactor K. We write both g and K in their power series in the variable y as

$$K(\overline{x}, \overline{y}) = \sum_{j=0}^{2} K_j(x)\overline{y}^j, \quad g = \sum_{j=0}^{\ell} g_j(\overline{x})\overline{y}^\ell,$$

for some integer ℓ and where K_j is a polynomial in $\overline{x}$ of degree j. Without loss of generality, since $g \neq 0$ we can assume that $g_\ell = g_\ell(\overline{x}) \neq 0$. Moreover, note that if system (2.3) has an invariant algebraic curve then

$$\overline{y}\frac{\partial g}{\partial \overline{x}} - \left(\beta\overline{y} + \gamma\overline{x}^3 + \overline{\gamma x}^2 - \overline{\delta}\overline{x}\right)\frac{\partial g}{\partial \overline{y}} = Kg. \tag{4.1}$$

We compute the coefficient of $\overline{y}^{2+\ell}$ in (4.1) and we get

$$g_\ell K_2 = 0, \quad \text{that is} \quad K_2 = 0$$

because $g_\ell \neq 0$. So, $K(\overline{x}) = K_0(\overline{x}) + K_1(\overline{x})\overline{y}$. Computing the coefficient of $\overline{y}^{\ell+1}$ in (4.1) we obtain

$$g_\ell'(\overline{x}) = K_1 g_\ell$$

which yields $g_\ell = \kappa e^{\int K_1(\overline{x})\,d\overline{x}}$, for $\kappa \in \mathbb{C} \setminus \{0\}$. Since g_ℓ must be a polynomial then $K_1 = 0$. This implies that $K(\overline{x}) = K_0(\overline{x})$ that we write as

$$K(\overline{x}) = K_0(\overline{x}) = \sum_{j=0}^{2} k_j \overline{x}^j, \quad k_j \in \mathbb{R}.$$

Now, equation (1.5) writes as

$$\overline{y}\frac{\partial g}{\partial \overline{x}} - (\beta\overline{y} + \gamma\overline{x}^3 + \overline{\gamma}\overline{x}^2 - \overline{\delta}\overline{x})\frac{\partial g}{\partial \overline{y}} = \sum_{j=0}^{m} k_j \overline{x}^j g.$$

We introduce the weight-change of variables of the form

$$\overline{x} = \mu^{-2}X, \quad \overline{y} = \mu^{-4}Y, \quad t = \mu^2\tau.$$

In this form, system (2.3) becomes

$$\begin{aligned} X' &= Y, \\ Y' &= -\gamma X^3 - \mu^2\beta Y - \mu^2\overline{\gamma}X^2 + \overline{\delta}\mu^4 X, \end{aligned}$$

where the prime denotes derivative in τ. Now let

$$G(X, Y) = \mu^N g(\mu^{-2}X, \mu^{-4}Y)$$

and

$$\overline{K} = \mu^2 K = \mu^2(k_0 + k_1\mu^{-2}X + \mu^{-4}X^2) = \mu^2 k_0 + k_1 X + \mu^{-2}X^2,$$

where N is the highest weight degree in the weight homogeneous components of g in the variables x and y, with weight $(2, 4)$.

We note that $G = 0$ is an invariant algebraic curve of system (2.3) with cofactor $\mu^2 K$. Indeed

$$\frac{dG}{d\tau} = \mu^N \frac{dg}{d\tau} = \mu^N \mu^2 K g = \mu^N \overline{K} G.$$

Assume that $G = \sum_{i=0}^{\ell} G_i$ where G_i is a weight homogeneous polynomial in X, Y with weight degree $\ell - i$ for $i = 0, \ldots, \ell$ and $\ell \geq N$. Obviously

$$g = G|_{\mu=1}.$$

From the definition of invariant algebraic curve we have

$$\begin{aligned} &Y\sum_{i=0}^{\ell} \mu^i \frac{\partial G_i}{\partial X} - \left(\gamma X^3 + \mu^2\beta Y + \mu^2\overline{\gamma}X^2 - \overline{\delta}\mu^4 X\right)\sum_{i=0}^{\ell} \frac{\partial G_i}{\partial Y} \\ &\quad = (\mu^2 k_0 + k_1 X + \mu^{-2}k_2 X^2)\sum_{i=0}^{\ell} \mu^i G_i. \end{aligned} \tag{4.2}$$

Computing the terms with μ^{-2} we get that $k_2 = 0$. Now the terms with μ^0 in (4.2) become

$$L[G_0] = k_1 G_0, \quad L = Y\frac{\partial}{\partial X} - \gamma X^3 \frac{\partial}{\partial Y}. \tag{4.3}$$

The characteristic equations associated with the first linear partial differential equation of system (2.3) are

$$\frac{dX}{dY} = -\gamma \frac{Y}{X^3}.$$

This system has the general solution $u = Y^2/2 + \gamma X^4/4 = \kappa$, where κ is a constant. According with the method of characteristics we make the change of variables

$$u = \frac{Y^2}{2} + \frac{\gamma}{4}X^4, \quad v = X. \tag{4.4}$$

Its inverse transformation is

$$Y = \pm\sqrt{2u - 2\gamma v^4/2}, \quad X = v. \tag{4.5}$$

In the following for simplicity we only consider the case $Y = +\sqrt{2u - \gamma v^4/2}$. Under changes (4.4) and (4.5), equation (4.3) becomes the following ordinary differential equation (for fixed u)

$$\sqrt{2u - \gamma v^4/2}\frac{d\overline{G}_0}{dv} = k_1 \overline{G}_0,$$

where $\overline{G}_0$ is G_0 written in the variables u, v. In what follows we always write $\overline{\theta}$ to denote a function $\theta = \theta(X, Y)$ written in the (u, v) variables, that is, $\overline{\theta} = \overline{\theta}(u, v)$. The above equation has the general solution

$$\overline{G}_0 = u^{\ell}\overline{F}_0(u) \exp\left(\frac{k_1}{\sqrt{2u}}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{4}, \frac{5}{4}, \frac{\gamma v^4}{4u}\right)\right),$$

where $\overline{F}_0$ is an arbitrary smooth function in the variable u and

$${}_2F_1(a, b, c, y) = \sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)}{b(b+1)\cdots(b+k-1)c(c+1)\cdots(c+k-1)} \frac{x^k}{k!} \tag{4.6}$$

is the hypergeometric function that is well defined if b, c are not negative integers. In particular, it is a polynomial if and only if a is a negative integer. Note that in this case ${}_2F_1$ is never a polynomial. Since

$$G_0(X, Y) = \overline{F}_0(u) = \overline{F}_0(Y^2/2 + \gamma X^4/4)$$

in order that $\overline{G}_0$ is a weight homogeneous polynomial of weight degree ℓ, since X and Y have weight degrees 2 and 4, respectively, we get that G_0 should be of weight degree $N = 8\ell$ and that $k_1 = 0$. Hence,

$$G_0 = a_{\ell}\left(\frac{Y^2}{2} + \gamma\frac{X^4}{4}\right)^{\ell}, \quad a_{\ell} \in \mathbb{R} \setminus \{0\}.$$

Computing the terms with μ in (4.2) using G_0 we get

$$L[G_1] = 0.$$

By the transformations in (4.4) and (4.5) and working in a similar way as we did to solve $\overline{G}_0$ we get the following ordinary differential equation

$$\sqrt{2u-\gamma v^4/2}\frac{d\overline{G}_1}{dv}=0,$$

that is $\overline{G}_1=\overline{G}_1(u)$. Since $\overline{G}_1$ is a weight homogeneous polynomial of weight degree $N-1=8\ell-1$ and u has even weight degree, we must have $\overline{G}_1=0$ and so $G_1=0$.

Computing the terms with μ^2 in (4.2) using the expression of G_0 and the fact that $G_1=0$ we get

$$\begin{aligned}
L[G_2] &= \beta a_\ell \ell Y^2\left(\frac{Y^2}{2}+\gamma\frac{X^4}{4}\right)^{\ell-1}+\overline{\gamma}a_\ell \ell X^2 Y\left(\frac{Y^2}{2}+\gamma\frac{X^4}{4}\right)^{\ell-1}+k_0 a_\ell\left(\frac{Y^2}{2}+\gamma\frac{X^4}{4}\right)^{\ell}\\
&= \beta a_\ell \ell\left(2\left(\frac{Y^2}{2}+\gamma\frac{X^4}{4}\right)-\frac{2}{3}\gamma X^4\right)\left(\frac{Y^2}{2}+\gamma\frac{X^4}{4}\right)^{\ell-1}+\overline{\gamma}a_\ell \ell X^2 Y\left(\frac{Y^2}{2}+\gamma\frac{X^4}{4}\right)^{\ell-1}\\
&\quad +k_0 a_\ell\left(\frac{Y^2}{2}+\gamma\frac{X^4}{4}\right)^{\ell}\\
&= a_\ell(2\beta\ell+k_0)\left(\frac{Y^2}{2}+\gamma\frac{X^4}{4}\right)^{\ell}-\frac{1}{2}\beta a_\ell \ell\gamma X^4\left(\frac{Y^2}{2}+\gamma\frac{X^4}{4}\right)^{\ell-1}+\overline{\gamma}a_\ell \ell X^2 Y\left(\frac{Y^2}{2}+\gamma\frac{X^4}{4}\right)^{\ell-1}.
\end{aligned}$$

By the transformations in (4.4) and (4.5) and working in a similar way to solve $\overline{G}_0$ we get the following ordinary differential equation

$$\sqrt{2u-\gamma v^4/2}\frac{d\overline{G}_2}{dv}=a_\ell(2\beta\ell+k_0)u^\ell-\frac{1}{2}\beta a_\ell \ell\gamma v^4 u^{\ell-1}+\overline{\gamma}a_\ell \ell v^2\sqrt{2u-\gamma v^4/2}u^{\ell-1}.$$

Integrating this equation with respect to v we get

$$\begin{aligned}
\overline{G}_2 &= \overline{F}_2(u)+\frac{\beta\ell u^{\ell-1}}{6}v\sqrt{2u-\gamma v^4/2}+\frac{\overline{\gamma}a_\ell\ell}{3}v^3u^{\ell-1}\\
&\quad+\frac{1}{3\sqrt{2}}u^{\ell-1/2}v(4\beta\ell+3k_0)\,{}_2F_1\left(\frac12,\frac14,\frac54,\frac{\gamma v^4}{8u}\right),
\end{aligned}$$

where $\overline{F}_2$ is a smooth function in the variable u and ${}_2F_1$ is the hypergeometric function introduced in (4.6). Here, ${}_2F_1$ is never a polynomial. Since G_2 should be a polynomial in the variable X we must have that

$$4\beta\ell+3k_0=0, \quad \text{that is} \quad k_0=-\frac{4\beta\ell}{3}.$$

Now we apply Theorem 2.1. We recall that k_0 is a constant, $k_0\neq 0$, and that in view of Theorem 2.1, g must vanish in the critical points of system (2.3), which are $(0,0)$ and $(\psi_+,0)$ and $(\psi_-,0)$ where

$$\psi_\pm=\frac{-\overline{\gamma}\pm\sqrt{\overline{\gamma}^2+4\overline{\delta}\gamma}}{2\gamma}.$$

Moreover, the critical point $(0,0)$ has the eigenvalues

$$\lambda^+=-\frac{\beta}{2}+\frac{\sqrt{\beta^2+4\overline{\delta}}}{2} \quad\text{and}\quad \lambda^-=-\frac{\beta}{2}-\frac{\sqrt{\beta^2+4\overline{\delta}}}{2},$$

the critical point $(\psi_+, 0)$ has the eigenvalues

$$\mu^+ = -\frac{\beta}{2} + \frac{\sqrt{\beta^2 + 4T_+}}{2} \quad \text{and} \quad \mu^- = -\frac{\beta}{2} - \frac{\sqrt{\beta^2 + 4T_+}}{2}$$

being

$$T_+ = \frac{\left(\overline{\gamma} - \sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\right)\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}}{2\gamma},$$

and the critical point $(\psi_-, 0)$ has the eigenvalues

$$\nu^+ = -\frac{\beta}{2} + \frac{\sqrt{\beta^2 + 4T_-}}{2} \quad \text{and} \quad \nu^- = -\frac{\beta}{2} - \frac{\sqrt{\beta^2 + 4T_-}}{2}$$

being

$$T_- = \frac{\left(-\overline{\gamma} - \sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\right)\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}}{2\gamma}$$

We consider different cases.

Case 1: $\overline{\delta}\gamma > 0$ *and* $\gamma < 0$. In this case both $(\psi_+, 0)$ and $(\psi_-, 0)$ are saddles. In view of Theorem 2.1 we must have that

$$k_0 \in \{\mu^+, \mu^-, \mu^+ + \mu^-\} = \{\mu^+, \mu^-, -\beta\} \quad \text{and} \quad k_0 \in \{\nu^+, \nu^-, \nu^+ + \nu^-\} = \{\nu^+, \nu^-, -\beta\}.$$

Note that if $k_0 = -\beta$ then

$$-\frac{4\beta\ell}{3} = -\beta, \quad \text{that is} \quad \beta\frac{3 - 4\ell}{3} = 0,$$

which is not possible because $\beta \neq 0$ and ℓ is an integer with $\ell \geq 1$. So, $k_0 \in \{\mu^+, \mu^-\}$ and $k_0 \in \{\nu^+, \nu^-\}$. The only possibility is that $\overline{\gamma} = 0$. In this case

$$-\frac{4\beta\ell}{3} = -\frac{\beta}{2} \pm \frac{\sqrt{\beta^2 - 8\overline{\delta}}}{2}$$

which yields

$$\beta = \pm\frac{3\sqrt{-\overline{\delta}}}{\sqrt{14}}.$$

Moreover the eigenvalues on $(0, 0)$ are λ^+ and λ^-. If $\beta^2 + 4\overline{\delta} < 0$ then λ^+ and λ^- would be rationally independent and in view of Theorem 2.1, then $k_0 \in \{\lambda^+, \lambda^-, \lambda^+ + \lambda^-\} = \{\lambda^+, \lambda^-, -\beta\}$. But then this would imply that

$$\sqrt{-\overline{\delta}}(i\sqrt{47} \pm (8\ell + 3)) = 0,$$

which is not possible. Hence, $\beta^2 + 4\overline{\delta} > 0$. However

$$\beta^2 + 4\overline{\delta} = \frac{47\overline{\delta}}{14} < 0$$

and so this case is not possible.

Case 2*: $\overline{\delta}\gamma > 0$ and $\gamma > 0$.* In this case $(0,0)$ is a saddle. In view of Theorem 2.1 we must have that $k_0 \in \{\lambda^+, \lambda^-, \lambda^+ + \lambda^-\} = \{\lambda^+, \lambda^-, -\beta\}$. As in Case 1 we cannot have $k_0 = -\beta$. So, imposing that $k_0 \in \{\lambda^+, \lambda^-\}$ we conclude that

$$\beta = \pm\frac{3\sqrt{\overline{\delta}}}{2\sqrt{\ell(3+4\ell)}}.$$

Moreover if $\beta^2 + 4T_+ < 0$ we would have that μ^+ and μ^- are rationally independent and so $k_0 \in \{\mu^+, \mu^-, -\beta\}$. However, $\mu^+ = \lambda^+$ (respectively $\mu^- = \lambda^-$) if and only if

$$\overline{\gamma} = \frac{3i\sqrt{\overline{\delta}\gamma}}{\sqrt{2}},$$

which is not possible. So $\beta^2 + 4T_+ > 0$. Equivalently, if $\beta^2 + 4T_- < 0$ we would have that ν^- and ν^- are rationally independent and so $k_0 \in \{\nu^+, \nu^-, -\beta\}$. However, $\nu^+ = \lambda^+$ (respectively $\nu^- = \lambda^-$) if and only if

$$\overline{\gamma} = \frac{3i\sqrt{\overline{\delta}\gamma}}{\sqrt{2}},$$

which is not possible. So $\beta^2 + 4T_- > 0$. This implies that

$$\frac{9\overline{\delta}}{2\ell(3+4\ell)} > \frac{2}{\gamma}\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\left(\overline{\gamma} + \sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\right)$$

and

$$\frac{9\overline{\delta}}{2\ell(3+4\ell)} > \frac{2}{\gamma}\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\left(-\overline{\gamma} + \sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\right)$$

or, in short,

$$\frac{9\overline{\delta}}{2\ell(3+4\ell)} > \frac{2}{\gamma}\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\left(|\overline{\gamma}| + \sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\right) = 8\overline{\delta} + \frac{2}{\gamma}\left(|\overline{\gamma}|\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma} + \overline{\gamma}^2\right),$$

being $|\overline{\gamma}|$ the absolute value of $\overline{\gamma}$. Note that this in particular implies that

$$-\frac{\overline{\delta}(64\ell^2 + 48\ell - 9)}{2\ell(3+4\ell)} > \frac{2}{\gamma}\left(|\overline{\gamma}|\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma} + \overline{\gamma}^2\right) > 0,$$

which is not possible because $\overline{\delta} > 0$ and $\ell \geq 1$. So, this case is not possible.

Case 3*: $\overline{\delta}\gamma < 0$ and $\gamma < 0$.* In this case $(0,0)$ is a saddle. In view of Theorem 2.1 we must have that $k_0 \in \{\lambda^+, \lambda^-, \lambda^+ + \lambda^-\} = \{\lambda^+, \lambda^-, -\beta\}$. As in case 1 we cannot have $k_0 = -\beta$. So, imposing that $k_0 \in \{\lambda^+, \lambda^-\}$ we conclude that

$$\beta = \pm\frac{3\sqrt{\overline{\delta}}}{2\sqrt{\ell(3+4\ell)}}.$$

Now we assume that $\overline{\gamma} \leq 0$ (otherwise we will do the argument with T_- instead of T_+). Since T_+ is a saddle we must have $k_0 \in \{\mu^+, \mu^-, \mu^+ + \mu^-\} = \{\mu^+, \mu^-, -\beta\}$. Proceeding as in Case 2, we cannot have $k_0 = -\beta$ and equating it to either μ^+ or μ^- we obtain that

$$\overline{\gamma} = \frac{3i\sqrt{\overline{\delta}\gamma}}{\sqrt{2}} = -\frac{3\sqrt{|\overline{\delta}\gamma|}}{\sqrt{2}},$$

Now proceeding as in Case 1 we have that $\mu^+ = \nu^+$ (respectively $\mu^- = \nu^-$) if and only if $\overline{\gamma} = 0$, which in this case is not possible because then $\overline{\delta} = \delta$ and $\delta\gamma \neq 0$. So, $\beta^2 + 4T_- > 0$, otherwise we would have that ν^+ and ν^- would be rationally independent and so $k_0 \in \{\nu^+, \nu^-, -\beta\}$ which we already shown that it is not possible. So, $\beta^2 + 4T_- > 0$. However, using that $\mu^+ = \lambda^+$ and $\mu^- = \lambda^-$ (that is, $T_+ = \overline{\delta}$) we get that

$$\overline{\gamma}\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma} = 2\gamma\overline{\delta} + \overline{\gamma}^2 + 4\overline{\delta}\gamma$$

and so

$$\begin{aligned}\beta^2 + 4T_- &= \frac{9\overline{\delta}}{4(\ell(3+4\ell))} - \frac{4}{2\gamma}(2\overline{\gamma}^2 + 10\overline{\delta}\gamma) = \frac{9\overline{\delta}}{4(\ell(3+4\ell))} + \frac{2}{\gamma}|\overline{\delta}\gamma| \\ &= \frac{9\overline{\delta}}{4(\ell(3+4\ell))} - 2\overline{\delta} = \frac{\overline{\delta}}{4(\ell(3+4\ell))}(9 - 24\ell - 32\ell^2) < 0,\end{aligned}$$

because $\ell \geq 1$. In short, this case is not possible.

Case 4: $\overline{\delta}\gamma < 0$ and $\gamma > 0$. We consider the case $\overline{\gamma} \geq 0$ because the case $\overline{\gamma} < 0$ is the same working with T_- instead of T_+. Since $\overline{\gamma} \geq 0$ we have that T_+ is a saddle. In view of Theorem 2.1 we must have that $k_0 \in \{\lambda^+, \lambda^-, \lambda^+ + \lambda^-\} = \{\lambda^+, \lambda^-, -\beta\}$. As in Case 1 we cannot have $k_0 = -\beta$. So, imposing that $k_0 \in \{\lambda^+, \lambda^-\}$ we conclude that

$$\beta = \pm\frac{3\sqrt{T_+}}{2\sqrt{\ell(3+4\ell)}}.$$

Now proceeding as in Case 1, it follows from Theorem 2.1 that we have either $\mu^+ = \nu^+$ (respectively $\mu^- = \nu^-$) in the case in which $\beta^2 + 4T_- < 0$ (because they will be rationally independent), or $\beta^2 + 4T_- > 0$. In the first case, proceeding as in Case 1 we must have $\overline{\gamma} \geq 0$. Assume first that $\overline{\gamma} > 0$. Then,

$$\begin{aligned}\beta^2 + 4T_- &= \frac{1}{4\ell(3+4\ell)}(9T_+ + 16\ell(3+4\ell)T_-) \\ &= \frac{1}{8\gamma\ell(3+4\ell)}\Big(\overline{\gamma}\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}(9 - 16\ell(3+4\ell)\Big) \\ &\quad - \left(\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\right)^2(9 + 16\ell(3+4\ell))) < 0,\end{aligned}$$

which is not possible. So, $\overline{\gamma} = 0$. Then

$$\beta = \pm\frac{3\sqrt{-4\overline{\delta}}}{\sqrt{2}\sqrt{\ell(3+4\ell)}}.$$

Note that

$$\beta^2 + 4\overline{\delta} = \frac{9}{2\ell(3+4\ell)}|\overline{\delta}| - 4|\overline{\delta}| = \frac{|\overline{\delta}|}{2\ell(3+4\ell)}(9 - 8\ell(3+4\ell)) < 0.$$

So, again proceeding as in Case 1 we must have $k_0 \in \{\lambda^+, \lambda^-\}$. Imposing it we conclude that $\overline{\delta} = 0$ which is not possible because $\overline{\delta} = \delta \neq 0$ whenever $\overline{\gamma} = 0$. This concludes the proof of the theorem. $\square$

5 Conclusions

In this paper we have characterized completely the algebraic traveling wave solutions of the Korteweg–de Vries–Burgers equation and of the Generalized Korteweg–de Vries–Burgers equation under some additional assumptions on the constants. The importance of this method is that can be used to completely characterize the algebraic traveling wave solutions of other well-known partial differential equations of any order provided that we are able to obtain the so-called Darboux polynomials. We emphasize that all the methods up to moment are not definite in the sense that if they do not work we cannot conclude that the system does not have traveling wave solutions, whereas in this method, if it fails, we can guarantee that there are not.

The cases of the Generalized Korteweg–de Vries–Burgers equation with $n \geq 3$ is unapproachable right now due to the fact that we are not able to compute the resulting Darboux polynomials, so these cases remain open.

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Existence of solutions for subquadratic convex or *B*-concave operator equations and applications to second order Hamiltonian systems

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Abstract. This paper investigates solutions for subquadratic convex or B-concave operator equations. First, some existence results are obtained by the index theory and the critical point theory. Then, some applications to second order Hamiltonian systems satisfying generalized periodic boundary value conditions and Sturm–Liouville boundary value conditions are pointed out. In particular, some well known theorems about periodic solutions for second order Hamiltonian systems are special cases of these results.

Keywords: subquadratic, operator equations, index theory, critical point, second order Hamiltonian systems.

2010 Mathematics Subject Classification: 34B15, 34C25, 58E05, 70H05.

1 Introduction and main results

Mawhin and Willem [9] investigated the second order Hamiltonian system

$$\begin{cases} -\ddot{x}(t) - m^2\omega^2 x(t) = \nabla_x V(t, x(t)), & \text{a.e. } t \in [0, T], \\ x(0) - x(T) = \dot{x}(0) - \dot{x}(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0, \omega = \frac{2\pi}{T}, m \in \{0, 1, 2, \dots\}, V \in C([0, T] \times \mathbb{R}^n, \mathbb{R}), \nabla_x V$ denotes the gradient of V with respect to $x, \nabla_x V \in C([0, T] \times \mathbb{R}^n, \mathbb{R}^n)$, and for each $x \in \mathbb{R}^n, V(t, x)$ is periodic in t with period T. Using the dual least action principle and the perturbation technique, the Authors, in theirs excellent book [9], proved some existence theorems of solutions for problem (1.1) with subquadratic convex or concave potential. Recently, using the reduction method, the perturbation argument and the least action principle, Tang and Wu [12] proved an abstract critical point theorem without the compactness assumptions which generalizes the results in [7]. As a main application, they successively obtained some existence theorems of problem

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(1.1) with $m = 0$ and subquadratic convex potential or $k(t)$-concave potential, which unify and generalize some earlier results in [9,13,14,16,17]. Later on, applying the abstract critical point theory established in [12], Ye [15] proved some existence theorems of problem (1.1), where $m \geq 1$ and the potential is convex and satisfies conditions which are more general than the subquadratic conditions in [9]. In this paper we reconsider in the framework of the operator equations some theorems proved in [9,12,15].

Let X be a real infinite-dimensional separable Hilbert space with inner product $(\cdot,\cdot)_X$ and the corresponding norm $\|\cdot\|_X$. Let $A : D(A) \subset X \to X$ be an unbounded linear self-adjoint operator with $\sigma(A) = \sigma_d(A)$ bounded from below. Hence, there is an orthonormal basis $\{e_j\}_{j=1}^{\infty}$ of X and $\lambda_1 \leq \lambda_2 \leq \cdots$ such that $Ae_j = \lambda_j e_j, D(A) = \{\sum_{j=1}^{\infty} c_j e_j | \sum_{j=1}^{\infty} \lambda_j^2 c_j^2 < \infty\}$. In addition, let $Z \equiv D(|A|^{\frac{1}{2}}) = \{\sum_{j=1}^{\infty} c_j e_j | \sum_{j=1}^{\infty} |\lambda_j| c_j^2 < \infty\}$ equipped with the norm $\|x\|_Z^2 = \|x\|^2 = \sum_{j=1}^{\infty}(1+|\lambda_j|)c_j^2$. For any $x = \sum_{j=1}^{\infty} c_j e_j \in Z, y = \sum_{j=1}^{\infty} d_j e_j \in Z$, we can define a bilinear form

$$a(x,y) = \sum_{j=1}^{\infty} \lambda_j c_j d_j.$$

Note that $(Ax,y)_X = a(x,y)$ if $x \in D(A), y \in Z$, this shows that $a(x,y)$ is the extension of $(Ax,y)_X$ on Z. Moreover, let $\mathcal{L}_s(X)$ be the usual space consisting of bounded symmetric operators in X. For given $B \in \mathcal{L}_s(X)$, we define

$$\nu_A(B) = \dim \ker(A-B),$$
$$i_A(B) = \sum_{\lambda<0} \nu_A(B+\lambda Id),$$

as introduced by Dong, see Definition 7.1.1 in [5] or Definition 3.1.1 and Proposition 3.1.4 in [4]. We consider the following operator equation

$$Ax - B_1 x = \nabla\Phi(x), \tag{1.2}$$

where $B_1 \in \mathcal{L}_s(X), \nu_A(B_1) \neq 0$, and Φ satisfies

(Φ_0) $\Phi \in C^1(Z,\mathbb{R})$ is weakly continuous with weakly continuous derivative, that is, $x_n \rightharpoonup x_0$ in Z implies that $\Phi(x_n) \to \Phi(x_0)$ and $\Phi'(x_n) \to \Phi'(x_0)$. Moreover, for every $x \in Z$ there exists $\nabla\Phi(x) \in X$ such that $\Phi'(x)y = (\nabla\Phi(x),y)_X$ for all $y \in Z$.

Let X_1 be a nontrivial subspace of X. For $B_1, B_2 \in \mathcal{L}_s(X)$ we write $B_1 \leq B_2$ with respect to X_1 if and only if $(B_1x,x)_X \leq (B_2x,x)_X$ for all $x \in X_1$; we write $B_1 < B_2$ w.r.t. X_1 if and only if $(B_1x,x)_X < (B_2x,x)_X$ for all $x \in X_1\backslash\{\theta\}$. If $X_1 = X$, then we just write $B_1 \leq B_2$ or $B_1 < B_2$. In addition, we write $B_1 < B_2$ properly if and only if $B_1 \leq B_2$ and $B_1 < B_2$ w.r.t. $\ker(A-B)$ for all $B \in \mathcal{L}_s(X)$.

Our main results can be stated as follows.

Theorem 1.1. *Assume that Φ satisfies (Φ_0) and*

(Φ_1) *Φ is convex in X;*

(Φ_2) *Φ and Φ' are bounded in Z;*

(Φ_3) *$\Phi(x) \to +\infty$ as $\|x\| \to \infty$ with $x \in \ker(A-B_1)$;*

(Φ_4) *there exist $c > 0$ and $B_2 \in \mathcal{L}_s(X)$ with $B_2 \geq B_1$ and $B_2 > B_1$ w.r.t. $\ker(A-B_1)$, $\nu_A(B_2) \neq 0$ and $i_A(B_2) = i_A(B_1) + \nu_A(B_1)$, such that*

$$\Phi(x) \leq \frac{1}{2}((B_2-B_1)x,x)_X + c \tag{1.3}$$

for all $x \in X$, *and*

$$\Phi(x) - \frac{1}{2}((B_2 - B_1)x, x)_X \to -\infty \tag{1.4}$$

as $\|\overline{x}\| \to \infty$, *where* $x = \widetilde{x} + \overline{x}$ *with* $\overline{x} \in \ker(A - B_2)$ *and* $\|\tilde{x}\|$ *is bounded.*

Then problem (1.2) *has a solution in* Z.

Theorem 1.2. *The conclusion of Theorem 1.1 still holds if we replace* (Φ_4) *with*

(Φ_4') *there exist* $c > 0$ *and* $B_2 \in \mathcal{L}_s(X)$ *with* $B_2 \geq B_1$ *and* $B_2 > B_1$ *w.r.t.* $\ker(A - B_1)$, $\nu_A(B_2) = 0$ *and* $i_A(B_2) = i_A(B_1) + \nu_A(B_1)$, *such that*

$$\Phi(x) \leq \frac{1}{2}((B_2 - B_1)x, x)_X + c \tag{1.5}$$

for all $x \in X$.

Theorem 1.3. *The conclusion of Theorem 1.1 still holds if we replace* (Φ_1) *and* (Φ_4) *with*

(Φ_1') Φ *is* $(B_2 - B_1)$*-concave, that is,* $-\Phi(x) + \frac{1}{2}((B_2 - B_1)x, x)_X$ *is convex in* X.

(Φ_4'') *there exists* $B_2 \in \mathcal{L}_s(X)$ *with* $B_2 \geq B_1$ *and* $B_2 > B_1$ *w.r.t.* $\ker(A - B_1), i_A(B_1) = 0, \nu_A(B_2) \neq 0$ *and* $i_A(B_2) = i_A(B_1) + \nu_A(B_1)$, *such that*

$$-\Phi(x) + \frac{1}{2}((B_2 - B_1)x, x)_X \to +\infty \tag{1.6}$$

as $\|x\| \to \infty$ *with* $x \in \ker(A - B_2)$, *respectively.*

Theorem 1.4. *The conclusion of Theorem 1.1 still holds if we replace* (Φ_1) *and* (Φ_4) *with* (Φ_1'),

(Φ_4''') *there exists* $B_2 \in \mathcal{L}_s(X)$ *with* $B_2 \geq B_1$ *and* $B_2 > B_1$ *w.r.t.* $\ker(A - B_1), i_A(B_1) = 0, \nu_A(B_2) =$ 0, *such that*

$$i_A(B_2) = i_A(B_1) + \nu_A(B_1),$$

respectively.

The paper is organized as follows. In Section 2, we first recall a critical point theorem as given in [12]. Then, following [4, 5], we recall some useful conclusions of index theory for linear self-adjoint operator equations. Finally, we quote a lemma in [3], which shows that (1.2) possesses a variational structure. In Section 3, we prove Theorems 1.1–1.4. In Section 4, we investigate their applications to second order Hamiltonian systems with generalized periodic boundary conditions and Sturm–Liouville boundary conditions. The corresponding results in [9, 12, 15] are special cases of these results.

2 Preliminaries

In order to prove our main results, we recall first two lemmas due to Tang and Wu [12].

Lemma 2.1 ([12, Theorem 1.1])**.** *Suppose that* X_1 *and* X_2 *are reflexive Banach spaces,* $I \in C^1(X_1 \times X_2, \mathbb{R})$. $I(x_1, \cdot)$ *is weakly upper semi-continuous for all* $x_1 \in X_1$ *and* $I(\cdot, x_2) : X_1 \to \mathbb{R}$ *is convex for all* $x_2 \in X_2$, *and* I' *is weakly continuous. Assume that*

$$I(\theta, x_2) \to -\infty \tag{2.1}$$

as $\|x_2\| \to +\infty$ *and, for every* $M > 0$

$$I(x_1, x_2) \to +\infty \tag{2.2}$$

as $\|x_1\| \to +\infty$ *uniformly for* $\|x_2\| \leq M$. *Then* I *has at least one critical point.*

Lemma 2.2 ([12, Lemma 5.1]). *Suppose that* H *is a real Hilbert space,* $f : H \times H \to \mathbb{R}$ *is a bilinear functional. Then* $g : H \to \mathbb{R}$ *given by*

$$g(x) = f(x, x), \qquad \forall x \in H$$

is convex if and only if

$$g(x) \geq 0, \qquad \forall x \in H.$$

Now we also recall some definitions and propositions in [4, 5].

Definition 2.3 ([5, Page 108]). For any $B \in \mathcal{L}_s(X)$, we define

$$\psi_{a,B}(x, y) = a(x, y) - (Bx, y)_X, \qquad \forall x, y \in Z.$$

For any $x, y \in Z$ if $\psi_{a,B}(x, y) = 0$ we say that x and y are $\psi_{a,B}$-orthogonal. For any two subspaces Z_1 and Z_2 of Z if $\psi_{a,B}(x, y) = 0$ for any $x \in Z_1, y \in Z_2$ we say that Z_1 and Z_2 are $\psi_{a,B}$-orthogonal.

Proposition 2.4 ([5, Proposition 7.2.1]). *For any* $B \in \mathcal{L}_s(X)$, *the space* Z *has a* $\psi_{a,B}$*-orthogonal decomposition*

$$Z = Z_a^+(B) \oplus Z_a^0(B) \oplus Z_a^-(B)$$

such that $\psi_{a,B}$ *is positive definite, null and negative definite on* $Z_a^+(B)$, $Z_a^0(B)$ *and* $Z_a^-(B)$ *respectively. Moreover,* $Z_a^0(B)$ *and* $Z_a^-(B)$ *are finitely dimensional.*

Definition 2.5 ([5, Definition 7.2.1]). For any $B \in \mathcal{L}_s(X)$, we define $\nu_a(B) = \dim Z_a^0(B)$, $i_a(B) = \dim Z_a^-(B)$.

Proposition 2.6.

(1) *For any* $B \in \mathcal{L}_s(X)$, *we have*

$$\nu_A(B) = \nu_a(B), \qquad i_A(B) = i_a(B), \qquad \ker(A - B) = Z_a^0(B).$$

([5], Proposition 7.2.2 (i))

(2) *For any* $B_1, B_2 \in \mathcal{L}_s(X)$, *if* $B_1 \leq B_2$ *with respect to* $Z_a^-(B_1)$, *then* $i_a(B_1) \leq i_a(B_2)$; *if* $B_1 \leq B_2$ *with respect to* $Z_a^-(B_1) \oplus Z_a^0(B_1)$, *then* $i_a(B_1) + \nu_a(B_1) \leq i_a(B_2) + \nu_a(B_2)$; *if* $B_1 < B_2$ *with respect to* $Z_a^0(B_1)$ *and* $B_1 \leq B_2$ *with respect to* $Z_a^-(B_1)$, *then* $i_a(B_1) + \nu_a(B_1) \leq i_a(B_2)$. ([5], Proposition 7.2.2 (ii))

(3) *For any* $B_1, B_2 \in \mathcal{L}_s(X)$, *if* $B_1(t) \leq B_2(t)$ *and* $B_1(t) < B_2(t)$ *properly, then*

$$i_a(B_2) - i_a(B_1) = \sum_{\lambda \in [0,1)} \nu_a(B_1 + \lambda(B_2 - B_1)).$$

([5], Proposition 7.2.2 (iii))

(4) *(Poincaré inequality.) For any $B \in \mathcal{L}_s(X)$, if $i_a(B) = 0$, then*

$$\psi_{a,B}(x,x) \geq 0, \qquad \forall x \in Z.$$

And the equality holds if and only if $x \in Z_a^0(B)$. ([5], Proposition 7.2.2 (v))

(5) *For any $B_1, B_2 \in \mathcal{L}_s(X)$, if $B_1 \leq B_2$ and $B_1 < B_2$ w.r.t.* $\ker(A - B_1)$ *and $i_A(B_2) = i_A(B_1) + \nu_A(B_1)$, then $Z = Z_a^-(B_1) \oplus Z_a^0(B_1) \oplus Z_a^0(B_2) \oplus Z_a^+(B_2)$, and $(-\psi_{a,B_1}(x_1,x_1))^{\frac{1}{2}} + (\psi_{a,B_2}(x_2,x_2))^{\frac{1}{2}}$ is an equivalent norm on Z for $x = x_1 + x_2$ with $x_1 \in Z_a^-(B_1)$, $x_2 \in Z_a^+(B_2)$. In particular, for any $B_1 \in \mathcal{L}_s(X)$, then $Z = Z_a^-(B_1) \oplus Z_a^0(B_1) \oplus Z_a^+(B_1)$ and $(-\psi_{a,B_1}(x_1,x_1))^{\frac{1}{2}} + (\psi_{a,B_1}(x_2,x_2))^{\frac{1}{2}}$ is also an equivalent norm on Z for $x = x_1 + x_2$ with $x_1 \in Z_a^-(B_1), x_2 \in Z_a^+(B_1)$.*

Proof. We only prove (5). Let $Z_1 = Z_a^-(B_1) \oplus Z_a^0(B_1), Z_2 = Z_a^0(B_2) \oplus Z_a^+(B_2)$. Noticing that $\psi_{a,B_1}(x,x) \geq \psi_{a,B_2}(x,x)$ for all $x \in Z$, $\psi_{a,B_1}(x,x) \leq 0$ for all $x \in Z_1$ and $\psi_{a,B_2}(x,x) \geq 0$ for all $x \in Z_2$, if $x \in Z_1 \bigcap Z_2$ then $\psi_{a,B_2}(x,x) = 0 = \psi_{a,B_1}(x,x)$, which shows that $x \in Z_a^0(B_2) \bigcap Z_a^0(B_1)$. By $B_1 \leq B_2$ and $B_1 < B_2$ w.r.t. $\ker(A - B_1)$, we have $0 = \psi_{a,B_1}(x,x) > \psi_{a,B_2}(x,x) = 0$ provided $x \in Z_a^0(B_2) \bigcap Z_a^0(B_1) \backslash \{\theta\}$. This is a contradiction, which implies that $Z_1 \bigcap Z_2 = \{\theta\}$. It remains to prove that $Z = Z_1 + Z_2$. By Proposition 2.4, we have $Z = Z_2 \oplus Z_a^-(B_2)$ and for any $x \in Z$ there exists a unique pair $(x_1, x_2) \in Z_2 \times Z_a^-(B_2)$ such that $x = x_1 + x_2$. Let $\{e_j\}_{j=1}^k$ be a basis of $Z_1, e_j = e_j^2 + e_j^-$ with $e_j^2 \in Z_2, e_j^- \in Z_a^-(B_2)$ for $j = 1, 2, \cdots, k = i_A(B_1) + \nu_A(B_1)$. By $i_A(B_2) = i_A(B_1) + \nu_A(B_1) = k$, in order to prove $\{e_j^-\}_{j=1}^k$ is a basis of $Z_a^-(B_2)$ we only need to show that $\{e_j^-\}_{j=1}^k$ is linear independent. In fact, otherwise there exist not all zero constants $c_1, \ldots, c_k$ such that $\sum_{j=1}^k c_j e_j^- = 0$. This leads to $\sum_{j=1}^k c_j e_j \in Z_1 \bigcap Z_2$, a contradiction. The linear independent shows that there exist constants $\{\alpha_j\}_{j=1}^k$ such that $x_2 = \sum_{j=1}^k \alpha_j e_j^-$. And hence $x = x_1 + x_2 = x = x_1 + \sum_{j=1}^k \alpha_j e_j^- = \sum_{j=1}^k \alpha_j e_j + \left(x_1 - \sum_{j=1}^k \alpha_j e_j^2\right)$.

Similar to the proof of Proposition 7.2.2 (iv) in [5], we can prove that $(-\psi_{a,B_1}(x_1,x_1))^{\frac{1}{2}} + (\psi_{a,B_2}(x_2,x_2))^{\frac{1}{2}}$ is an equivalent norm on Z for $x = x_1 + x_2$ with $x_1 \in Z_a^-(B_1), x_2 \in Z_a^+(B_2)$, and $(-\psi_{a,B_1}(x_1,x_1))^{\frac{1}{2}} + (\psi_{a,B_1}(x_2,x_2))^{\frac{1}{2}}$ is also an equivalent norm on Z for $x = x_1 + x_2$ with $x_1 \in Z_a^-(B_1), x_2 \in Z_a^+(B_1)$. □

Finally, let us consider the functional I defined by

$$I(x) = -\frac{1}{2}a(x,x) + \frac{1}{2}(B_1 x, x)_X + \Phi(x), \tag{2.3}$$

for every $x \in Z$. Under assumption (Φ_0), from Theorem 1.2 in [9] it is easy to verify that $I \in C^1(Z, \mathbb{R})$ is weakly upper semi-continuous on Z and I' is weakly continuous with

$$I'(x)y = -a(x,y) + (B_1 x, y)_X + \Phi'(x)y, \tag{2.4}$$

for every $x, y \in Z$.

The following important lemma is an immediate conclusion of Lemma 2.1 in [3].

Lemma 2.7. *Assume that (Φ_0) holds. Then a critical point of $I(x)$ is a solution for problem (1.2).*

3 Proofs of the Theorems

In this section, we present the proof of Theorems 1.1–1.4.

Proof of Theorem 1.1. By $\nu_A(B_1) \neq 0$, $B_1 \leq B_2$ and $B_1 < B_2$ w.r.t. $\ker(A - B_1)$ and $i_A(B_2) = i_A(B_1) + \nu_A(B_1)$, we have $Z = Z_a^-(B_1) \oplus Z_a^0(B_1) \oplus Z_a^0(B_2) \oplus Z_a^+(B_2)$ via (5) of Proposition 2.6. Set $X_1 = Z_a^-(B_1) \oplus Z_a^0(B_1), X_2 = Z_a^0(B_2) \oplus Z_a^+(B_2) = Z_a^+(B_1), x \in Z, x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. Next, we divide the proof into three steps.

Step 1. We show that $I(\cdot, x_2) : X_1 \to \mathbb{R}$ is convex for all $x_2 \in X_2$. By (Φ_1), it is obvious that $\Phi(x_1 + x_2)$ is convex in $x_1 \in X_1$. From Definition 2.3 and Proposition 2.4 we can see that for every $x_1 \in X_1$,

$$-\frac{1}{2}\psi_{a,B_1}(x_1, x_1) = -\frac{1}{2}a(x_1, x_1) + \frac{1}{2}(B_1 x_1, x_1)_X \geq 0,$$

which implies that $-\frac{1}{2}\psi_{a,B_1}(x_1, x_1)$ is convex in $x_1 \in X_1$ via Lemma 2.2. Hence, for every $x_2 \in X_2$,

$$\begin{aligned} I(x_1 + x_2) &= -\frac{1}{2}a(x_1 + x_2, x_1 + x_2) + \frac{1}{2}(B_1(x_1 + x_2), x_1 + x_2)_X + \Phi(x_1 + x_2) \\ &= -\frac{1}{2}\psi_{a,B_1}(x_1, x_1) + \Phi(x_1 + x_2) - \frac{1}{2}\psi_{a,B_1}(x_2, x_2) \end{aligned}$$

is convex in $x_1 \in X_1$.

Step 2. By contradiction, we prove that (2.2) of Lemma 2.1 holds. Assume that (2.2) of Lemma 2.1 does not hold. Then there exist $M > 0, c_0 > 0$ and two sequences $\{x_{1,n}\} \subset X_1$ and $\{x_{2,n}\} \subset X_2$ with $\|x_{1,n}\| \to +\infty$ as $n \to \infty$ and $\|x_{2,n}\| \leq M$ for all n such that

$$I(x_{1,n} + x_{2,n}) \leq c_0, \ \forall n \in \mathbf{N}. \tag{3.1}$$

For $x_1 \in X_1$, write $x_1 = x_1^- + x_1^0$, where $x_1^- \in Z_a^-(B_1)$ and $x_1^0 \in Z_a^0(B_1)$. We consider the functional $\Phi|_{Z_a^0(B_1)}$. By (Φ_0), we easily see that $\Phi|_{Z_a^0(B_1)}$ is weakly lower semi-continuous on $Z_a^0(B_1)$. Using (Φ_3), by the least action principle (see Theorem 1.1 in [9]), $\Phi|_{Z_a^0(B_1)}$ has a minimum at some $x_{1,0}^0 \in Z_a^0(B_1)$ for which

$$0 = \Phi'(x_{1,0}^0)x_1^0 = (\nabla\Phi(x_{1,0}^0), x_1^0)_X, \ \forall x_1^0 \in Z_a^0(B_1).$$

By assumption (Φ_0) and the convexity of Φ, we have

$$\begin{aligned} \Phi(x_1 + x_2) - \Phi(x_{1,0}^0) &\geq (\nabla\Phi(x_{1,0}^0), x_1^- + x_2 + x_1^0 - x_{1,0}^0)_X \\ &= (\nabla\Phi(x_{1,0}^0), x_1^- + x_2)_X, \end{aligned}$$

and then, from $\|x\|_X \leq \|x\|$ for all $x \in Z$,

$$\begin{aligned} \Phi(x_1 + x_2) &\geq \Phi(x_{1,0}^0) - \|\nabla\Phi(x_{1,0}^0)\|_X \cdot \|x_1^- + x_2\|_X \\ &\geq \Phi(x_{1,0}^0) - \|\nabla\Phi(x_{1,0}^0)\|_X \cdot (\|x_1^-\| + \|x_2\|) \\ &= c_1 - c_2 \cdot (\|x_1^-\| + \|x_2\|) \end{aligned}$$

where $c_1 = \Phi(x_{1,0}^0), c_2 = \|\nabla\Phi(x_{1,0}^0)\|_X \geq 0$. Rewrite $x_{1,n} = x_{1,n}^- + x_{1,n}^0$, where $x_{1,n}^- \in Z_a^-(B_1)$ and $x_{1,n}^0 \in Z_a^0(B_1)$. By (3.1), we have

$$\begin{aligned} c_0 &\geq I(x_{1,n} + x_{2,n}) \\ &= -\frac{1}{2}\psi_{a,B_1}(x_{1,n}^-, x_{1,n}^-) - \frac{1}{2}\psi_{a,B_1}(x_{2,n}, x_{2,n}) + \Phi(x_{1,n} + x_{2,n}) \\ &\geq -\frac{1}{2}\psi_{a,B_1}(x_{1,n}^-, x_{1,n}^-) - \frac{1}{2}\psi_{a,B_1}(x_{2,n}, x_{2,n}) + c_1 - c_2 \cdot (\|x_{1,n}^-\| + \|x_{2,n}\|). \end{aligned}$$

From (Φ_4) and (5) of Proposition 2.6, we know that $(-\psi_{a,B_1}(x_1^-, x_1^-))^{\frac{1}{2}}$ is an equivalent norm on Z for $x_1^- \in Z_a^-(B_1)$ and $(\psi_{a,B_1}(x_2, x_2))^{\frac{1}{2}}$ is an equivalent norm on Z for $x_2 \in Z_a^+(B_1)$. This means that there exist $c_3 > 0$ and $c_4 > 0$ such that

$$\begin{aligned} c_0 &\geq I(x_{1,n} + x_{2,n}) \\ &\geq \frac{c_3^2}{2}\|x_{1,n}^-\|^2 - \frac{c_4^2}{2}\|x_{2,n}\|^2 + c_1 - c_2 \cdot (\|x_{1,n}^-\| + \|x_{2,n}\|) \\ &\geq \frac{c_3^2}{2}\|x_{1,n}^-\|^2 - \frac{c_4^2 M^2}{2} + c_1 - c_2 \cdot (\|x_{1,n}^-\| + M) \end{aligned}$$

via $\|x_{2,n}\| \leq M$, which shows that $\{\|x_{1,n}^-\|\}$ is bounded. Combining this with assumption (Φ_2) and the convexity of Φ, we see that there exist $c_5 > 0$ and $c_6 = \sup_n \Phi(-x_{1,n}^- - x_{2,n})$ such that

$$\begin{aligned} c_0 &\geq I(x_{1,n} + x_{2,n}) \\ &= -\frac{1}{2}\psi_{a,B_1}(x_{1,n}^-, x_{1,n}^-) - \frac{1}{2}\psi_{a,B_1}(x_{2,n}, x_{2,n}) + \Phi(x_{1,n} + x_{2,n}) \\ &\geq \frac{(c_3 c_5)^2}{2} - \frac{c_4^2 M^2}{2} + 2\Phi\left(\frac{1}{2}x_{1,n}^0\right) - \Phi(-x_{1,n}^- - x_{2,n}) \\ &\geq \frac{(c_3 c_5)^2}{2} - \frac{c_4^2 M^2}{2} + 2\Phi\left(\frac{1}{2}x_{1,n}^0\right) - c_6. \end{aligned}$$

By (Φ_3), we know that $\{\|x_{1,n}^0\|\}$ is also bounded. This contradicts the fact that $\|x_{1,n}^-\| + \|x_{1,n}^0\| \geq \|x_{1,n}\| \to +\infty$ as $n \to \infty$. Therefore (2.2) of Lemma 2.1 holds.

Step 3. We check that (2.1) of Lemma 2.1 holds. If not, there exist a constant c_7 and a sequence $\{x_{2,n}\}$ in X_2 such that $\|x_{2,n}\| \to +\infty$ as $n \to \infty$ and

$$I(x_{2,n}) \geq c_7 \tag{3.2}$$

for all n. For $x_2 \in X_2$, write $x_2 = x_2^0 + x_2^+$, where $x_2^0 \in Z_a^0(B_2)$ and $x_2^+ \in Z_a^+(B_2)$. Notice that $\nu_M^s(B_2) \neq 0$ and $X_2 = Z_a^0(B_2) \oplus Z_a^+(B_2)$. Let $x_{2,n} = x_{2,n}^0 + x_{2,n}^+, x_{2,n}^0 \in Z_a^0(B_2), x_{2,n}^+ \in Z_a^+(B_2)$. Then by (1.3) of (Φ_4), (3.2), Definition 2.3 and Proposition 2.4, we have

$$\begin{aligned} c_7 &\leq I(x_{2,n}) \\ &\leq -\frac{1}{2}a(x_{2,n}^0 + x_{2,n}^+, x_{2,n}^0 + x_{2,n}^+) + \frac{1}{2}(B_2(x_{2,n}^0 + x_{2,n}^+), x_{2,n}^0 + x_{2,n}^+)_X + c \\ &= -\frac{1}{2}\psi_{a,B_2}(x_{2,n}^+, x_{2,n}^+) + c \end{aligned}$$

which implies that $\{x_{2,n}^+\}$ is bounded since $(-\psi_{a,B_1}(x_1, x_1))^{\frac{1}{2}} + (\psi_{a,B_2}(x_2, x_2))^{\frac{1}{2}}$ is an equivalent norm on Z for $x = x_1 + x_2$ with $x_1 \in Z_a^-(B_1)$ and $x_2 \in Z_a^+(B_2)$, where $x_1 = \theta$. Since $\|x_{2,n}\| \leq \|x_{2,n}^0\| + \|x_{2,n}^+\|$, we have $\|x_{2,n}^0\| \to \infty$ as $n \to +\infty$. By $x_{2,n} \in X_2 = Z_a^0(B_2) \oplus Z_a^+(B_2)$, we have $\psi_{a,B_2}(x_{2,n}, x_{2,n}) \geq 0$ for all n via Proposition 2.4. From $\|x_{2,n}^0\| \to \infty$ as $n \to +\infty$ we have

$$I(x_{2,n}) \leq \Phi(x_{2,n}) - \frac{1}{2}((B_2 - B_1)x_{2,n}, x_{2,n})_X \to -\infty$$

via (1.4) of (Φ_4), which contradicts (3.2). Hence (2.1) of Lemma 2.1 holds.

By Lemma 2.1, I has at least one critical point. Hence problem (1.2) has at least one solution in Z via Lemma 2.7. The proof is complete. □

Proof of Theorem 1.2. By $\nu_A(B_1) \neq 0$, $B_1 \leq B_2$ and $B_1 < B_2$ w.r.t. $\ker(A - B_1)$ and $i_A(B_2) = i_A(B_1) + \nu_A(B_1)$, we have $Z = Z_a^-(B_1) \oplus Z_a^0(B_1) \oplus Z_a^0(B_2) \oplus Z_a^+(B_2)$ via (5) of Proposition 2.6. Note that $\nu_A(B_2) = 0$, we have $Z_a^0(B_2) = \{\theta\}$, which implies that $Z = Z_a^-(B_1) \oplus Z_a^0(B_1) \oplus Z_a^+(B_2)$ and $Z_a^+(B_2) = Z_a^+(B_1)$. Set $X_1 = Z_a^-(B_1) \oplus Z_a^0(B_1), X_2 = Z_a^+(B_2) = Z_a^+(B_1), x \in Z, x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$.

Let us follow the proof of Theorem 1.1 until (3.2). For $x_{2,n} \in Z_a^+(B_2) = Z_a^+(B_1)$, by (1.5) of (Φ_4'), (3.2), Definition 2.3 and Proposition 2.4, we have

$$\begin{aligned} c_7 &\leq I(x_{2,n}) \\ &\leq -\frac{1}{2}a(x_{2,n}, x_{2,n}) + \frac{1}{2}(B_2 x_{2,n}, x_{2,n})_X + c \\ &= -\frac{1}{2}\psi_{a,B_2}(x_{2,n}, x_{2,n}) + c. \end{aligned}$$

Since $(-\psi_{a,B_1}(x_1, x_1))^{\frac{1}{2}} + (\psi_{a,B_2}(x_2, x_2))^{\frac{1}{2}}$ is an equivalent norm on Z for $x = x_1 + x_2$ with $x_1 \in Z_a^-(B_1)$ and $x_2 \in Z_a^+(B_2)$, where $x_1 = \theta$, we have $\psi_{a,B_2}(x_{2,n}, x_{2,n}) \to +\infty$ via $\|x_{2,n}\| \to \infty$ as $n \to +\infty$. Thus, we have

$$I(x_{2,n}) \leq -\frac{1}{2}\psi_{a,B_2}(x_{2,n}, x_{2,n}) + c \to -\infty$$

as $n \to +\infty$, which contradicts (3.2). Hence (2.1) of Lemma 2.1 holds.

By Lemma 2.1, I has at least one critical point. Hence problem (1.2) has at least one solution in Z via Lemma 2.7. The proof is complete. □

Proof of Theorem 1.3. We apply Lemma 2.1. Consider the functional I_1 defined by

$$I_1(x) = -I(x) = \frac{1}{2}a(x, x) - \frac{1}{2}(B_1 x, x)_X - \Phi(x), \tag{3.3}$$

for every $x \in Z$. Under assumption (Φ_0), it is easy to verify that $I_1 \in C^1(Z, \mathbb{R})$ and I_1' is weakly continuous.

Note that $i_A(B_1) = 0$, we have $Z_a^-(B_1) = \{\theta\}$. By $\nu_A(B_1) \neq 0$, $B_1 \leq B_2$ and $B_1 < B_2$ w.r.t. $\ker(A - B_1)$ and $i_A(B_2) = i_A(B_1) + \nu_A(B_1)$, we have $Z = Z_a^0(B_1) \oplus Z_a^0(B_2) \oplus Z_a^+(B_2)$ via (5) of Proposition 2.6. Set $X_1 = Z_a^0(B_2) \oplus Z_a^+(B_2) = Z_a^+(B_1), X_2 = Z_a^0(B_1), x \in Z, x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. From Definition 2.3 and Proposition 2.4, we have

$$I_1(x) = I_1(x_1 + x_2) = \frac{1}{2}a(x_1, x_1) - \frac{1}{2}(B_1 x_1, x_1)_X - \Phi(x_1 + x_2),$$

for every $x \in Z$. Thus, $I_1(x_1, \cdot)$ is weakly upper semi-continuous for all $x_1 \in X_1$ via $\Phi \in C^1(Z, \mathbb{R})$ is weakly continuous.

Next, we still divide the proof into three steps.

Step 1. We show that $I_1(\cdot, x_2) : X_1 \to \mathbb{R}$ is convex for all $x_2 \in X_2$. By (Φ_1'), it is obvious that $-\Phi(x_1 + x_2) + \frac{1}{2}((B_2 - B_1)(x_1 + x_2), x_1 + x_2)_X$ is convex in $x_1 \in X_1$. From Definition 2.3 and Proposition 2.4 we know that for every $x_1 \in X_1$,

$$\frac{1}{2}\psi_{a,B_2}(x_1, x_1) = \frac{1}{2}a(x_1, x_1) - \frac{1}{2}(B_2 x_1, x_1)_X \geq 0,$$

which shows that $\frac{1}{2}\psi_{a,B_2}(x_1, x_1)$ is convex in $x_1 \in X_1$ via Lemma 2.2. Hence, for every $x_2 \in X_2$,

$$\begin{aligned} I_1(x_1 + x_2) &= \frac{1}{2}a(x_1 + x_2, x_1 + x_2) - \frac{1}{2}(B_1(x_1 + x_2), x_1 + x_2)_X - \Phi(x_1 + x_2) \\ &= \frac{1}{2}\psi_{a,B_2}(x_1, x_1) - \Phi(x_1 + x_2) + \frac{1}{2}((B_2 - B_1)(x_1 + x_2), x_1 + x_2)_X + \frac{1}{2}\psi_{a,B_2}(x_2, x_2) \end{aligned}$$

is convex in $x_1 \in X_1$.

Step 2. By contradiction, we verify that (2.2) of Lemma 2.1 holds. If (2.2) of Lemma 2.1 does not hold, there exist $M > 0, c_8 > 0$ and two sequences $\{x_{1,n}\} \subset X_1$ and $\{x_{2,n}\} \subset X_2$ with $\|x_{1,n}\| \to +\infty$ as $n \to \infty$ and $\|x_{2,n}\| \le M$ for all n such that

$$I_1(x_{1,n} + x_{2,n}) \le c_8, \qquad \forall n \in \mathbf{N}. \tag{3.4}$$

For $x_1 \in X_1$, write $x_1 = x_1^0 + x_1^+$, where $x_1^0 \in Z_a^0(B_2)$ and $x_1^+ \in Z_a^+(B_2)$. Let us consider the functional

$$\varphi(x) = -\Phi(x) + \frac{1}{2}((B_2 - B_1)x, x)_X$$

for all $x \in X$. By (Φ_0) and (Φ_1'), we easily see that $\varphi \in C^1(Z, \mathbb{R})$ and φ is weakly lower semi-continuous on $Z_a^0(B_2)$. Using (1.6) of (Φ_4''), by the least action principle (see Theorem 1.1 in [9]), φ has a minimum at some $x_{1,0}^0 \in Z_a^0(B_2)$ for which

$$0 = \varphi'(x_{1,0}^0)x_1^0 = -(\nabla\Phi(x_{1,0}^0), x_1^0)_X + ((B_2 - B_1)x_{1,0}^0, x_1^0)_X, \qquad \forall x_1^0 \in Z_a^0(B_2).$$

By $\varphi \in C^1(Z, \mathbb{R})$ and the $(B_2 - B_1)$-concavity of Φ, we have

$$\begin{aligned}
&\varphi(x_1 + x_2) - \varphi(x_{1,0}^0) \\
&\quad \ge -(\nabla\Phi(x_{1,0}^0), x_1^+ + x_2 + x_1^0 - x_{1,0}^0)_X + ((B_2 - B_1)x_{1,0}^0, x_1^+ + x_2 + x_1^0 - x_{1,0}^0)_X \\
&\quad = -(\nabla\Phi(x_{1,0}^0), x_1^+ + x_2)_X + ((B_2 - B_1)x_{1,0}^0, x_1^+ + x_2)_X,
\end{aligned}$$

and then, from $\|x\|_X \le \|x\|$ for all $x \in Z$,

$$\begin{aligned}
\varphi(x_1 + x_2) &\ge \varphi(x_{1,0}^0) - (\|\nabla\Phi(x_{1,0}^0)\|_X + \|(B_2 - B_1)x_{1,0}^0\|_X) \cdot \|x_1^+ + x_2\|_X \\
&\ge \varphi(x_{1,0}^0) - (\|\nabla\Phi(x_{1,0}^0)\|_X + \|(B_2 - B_1)x_{1,0}^0\|_X) \cdot (\|x_1^+\| + \|x_2\|) \\
&= c_9 - c_{10} \cdot (\|x_1^+\| + \|x_2\|)
\end{aligned}$$

where $c_9 = \varphi(x_{1,0}^0), c_{10} = \|\nabla\Phi(x_{1,0}^0)\|_X + \|(B_2 - B_1)x_{1,0}^0\|_X \ge 0$. Rewrite $x_{1,n} = x_{1,n}^+ + x_{1,n}^0$, where $x_{1,n}^+ \in Z_a^+(B_2)$ and $x_{1,n}^0 \in Z_a^0(B_2)$. By (3.4), we have

$$\begin{aligned}
c_8 \ge I_1(x_{1,n} + x_{2,n}) &= \frac{1}{2}\psi_{a,B_2}(x_{1,n} + x_{2,n}, x_{1,n} + x_{2,n}) \\
&\quad + \frac{1}{2}((B_2 - B_1)(x_{1,n} + x_{2,n}), x_{1,n} + x_{2,n})_X - \Phi(x_{1,n} + x_{2,n}) \\
&= \frac{1}{2}\psi_{a,B_2}(x_{1,n}^+, x_{1,n}^+) + \frac{1}{2}\psi_{a,B_2}(x_{2,n}, x_{2,n}) + \varphi(x_{1,n} + x_{2,n}) \\
&\ge \frac{1}{2}\psi_{a,B_2}(x_{1,n}^+, x_{1,n}^+) + \frac{1}{2}\psi_{a,B_2}(x_{2,n}, x_{2,n}) + c_9 - c_{10} \cdot (\|x_{1,n}^+\| + \|x_{2,n}\|).
\end{aligned}$$

From (Φ_4'') and (5) of Proposition 2.6, we know that $(\psi_{a,B_2}(x, x))^{\frac{1}{2}}$ is an equivalent norm on Z for $x \in Z_a^+(B_2)$. Noticing that $-\psi_{a,B_2}(x, x) > 0$ for all $x \in Z_a^-(B_2)\backslash\{\theta\}$, so $(-\psi_{a,B_2}(x, x))^{\frac{1}{2}}$ is a norm on $Z_a^-(B_2)$, which is equivalent to $\|\cdot\|_Z = \|\cdot\|$ because of the finiteness of the subspace $Z_a^-(B_2)$. This means that there exist $c_{11} > 0$ and $c_{12} > 0$ such that

$$\begin{aligned}
c_8 &\ge I_1(x_{1,n} + x_{2,n}) \\
&\ge \frac{c_{11}^2}{2}\|x_{1,n}^+\|^2 - \frac{c_{12}^2}{2}\|x_{2,n}\|^2 + c_9 - c_{10} \cdot (\|x_{1,n}^+\| + \|x_{2,n}\|) \\
&\ge \frac{c_{11}^2}{2}\|x_{1,n}^+\|^2 - \frac{c_{12}^2 M^2}{2} + c_9 - c_{10} \cdot (\|x_{1,n}^+\| + M)
\end{aligned}$$

via $\|x_{2,n}\| \leq M$, which shows that $\{\|x_{1,n}^{+}\|\}$ is bounded. Combining this with assumption (Φ_2) and the $(B_2 - B_1)$-concavity of Φ, we see that there exist $c_{13} > 0$ and $c_{14} = \sup_n \varphi(-x_{1,n}^{+} - x_{2,n})$ such that

$$\begin{aligned} c_8 &\geq I_1(x_{1,n} + x_{2,n}) \\ &= \frac{1}{2}\psi_{a,B_2}(x_{1,n}^{+}, x_{1,n}^{+}) + \frac{1}{2}\psi_{a,B_2}(x_{2,n}, x_{2,n}) + \varphi(x_{1,n} + x_{2,n}) \\ &\geq \frac{(c_{11}c_{13})^2}{2} - \frac{c_{12}^2 M^2}{2} + 2\varphi\left(\frac{1}{2}x_{1,n}^{0}\right) - \varphi(-x_{1,n}^{+} - x_{2,n}) \\ &\geq \frac{(c_{11}c_{13})^2}{2} - \frac{c_{12}^2 M^2}{2} + 2\varphi\left(\frac{1}{2}x_{1,n}^{0}\right) - c_{14}. \end{aligned}$$

By (1.6) of (Φ_4''), we know that $\{\|x_{1,n}^{0}\|\}$ is also bounded. This contradicts the fact that $\|x_{1,n}^{+}\| + \|x_{1,n}^{0}\| \geq \|x_{1,n}\| \to +\infty$ as $n \to \infty$. Therefore (2.2) of Lemma 2.1 holds.

Step 3. By $X_2 = Z_a^0(B_1)$, we have $I_1(x_2) = -\Phi(x_2)$ for all $x_2 \in X_2$. Thus, (2.1) of Lemma 2.1 holds via (Φ_3).

By Lemma 2.1, I_1 has at least one critical point. Hence problem (1.2) has at least one solution in Z via Lemma 2.7. The proof is complete. □

Proof of Theorem 1.4. we still consider the functional I_1 defined by (3.3). Under assumption (Φ_0), it is easy to verify that $I_1 \in C^1(Z, \mathbb{R})$ and I_1' is weakly continuous.

By $\nu_A(B_1) \neq 0$, $B_1 \leq B_2$ and $B_1 < B_2$ w.r.t. $\ker(A - B_1)$ and $i_A(B_2) = i_A(B_1) + \nu_A(B_1)$, we have $Z = Z_a^-(B_1) \oplus Z_a^0(B_1) \oplus Z_a^0(B_2) \oplus Z_a^+(B_2)$ via (5) of Proposition 2.6. Note that $i_A(B_1) = 0$ and $\nu_A(B_2) = 0$, we have $Z_a^-(B_1) = Z_a^0(B_2) = \{\theta\}$, which implies that $Z = Z_a^0(B_1) \oplus Z_a^+(B_2)$, $Z_a^-(B_2) = Z_a^0(B_1)$ and $Z_a^+(B_2) = Z_a^+(B_1)$. Set $X_1 = Z_a^+(B_2) = Z_a^+(B_1), X_2 = Z_a^0(B_1), x \in Z, x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$.

From the proof of Theorem 1.3, it is not difficult to see that we only need to verify the validity of (2.2) in Lemma 2.1. If (2.2) of Lemma 2.1 does not hold, there exist $M > 0, c_{15} > 0$ and two sequences $\{x_{1,n}\} \subset X_1$ and $\{x_{2,n}\} \subset X_2$ with $\|x_{1,n}\| \to +\infty$ as $n \to \infty$ and $\|x_{2,n}\| \leq M$ for all n such that

$$I_1(x_{1,n} + x_{2,n}) \leq c_{15}, \qquad \forall n \in \mathbf{N}. \tag{3.5}$$

We consider the functional

$$\varphi(x) = -\Phi(x) + \frac{1}{2}((B_2 - B_1)x, x)_X$$

for all $x \in X$. By (Φ_0) and (Φ_1'), we easily see that $\varphi \in C^1(Z, \mathbb{R})$. From the $(B_2 - B_1)$-concavity of Φ, we have

$$\begin{aligned} \varphi(x_1 + x_2) - \varphi(\theta) &\geq -(\nabla\Phi(\theta), x_1 + x_2)_X + ((B_2 - B_1)\theta, x_1 + x_2)_X \\ &= -(\nabla\Phi(\theta), x_1 + x_2)_X, \end{aligned}$$

and then, from $\|x\|_X \leq \|x\|$ for all $x \in Z$,

$$\begin{aligned} \varphi(x_1 + x_2) &\geq \varphi(\theta) - \|\nabla\Phi(\theta)\|_X \cdot \|x_1 + x_2\|_X \\ &\geq \varphi(\theta) - \|\nabla\Phi(\theta)\|_X(\|x_1\| + \|x_2\|). \end{aligned}$$

From (Φ_4''') and (5) of Proposition 2.6, we know that $(\psi_{a,B_2}(x,x))^{\frac{1}{2}}$ is an equivalent norm on Z for $x \in Z_a^+(B_2)$. Noticing that $-\psi_{a,B_2}(x,x) > 0$ for all $x \in Z_a^-(B_2)\backslash\{\theta\}$, so $(-\psi_{a,B_2}(x,x))^{\frac{1}{2}}$ is a norm on $Z_a^-(B_2)$, which is equivalent to $\|\cdot\|_Z = \|\cdot\|$ because of the finiteness of the subspace $Z_a^-(B_2)$. Combining (3.5), we know that there exist $c_{16} > 0$ and $c_{17} > 0$ such that

$$\begin{aligned} c_{15} &\geq I_1(x_{1,n} + x_{2,n}) \\ &= \frac{1}{2}\psi_{a,B_2}(x_{1,n}, x_{1,n}) + \frac{1}{2}\psi_{a,B_2}(x_{2,n}, x_{2,n}) + \varphi(x_{1,n} + x_{2,n}) \\ &\geq \frac{c_{16}^2}{2}\|x_{1,n}\|^2 - \frac{c_{17}^2 M^2}{2} + \varphi(\theta) - \|\nabla\Phi(\theta)\|_X(\|x_{1,n}\| + M), \end{aligned}$$

which shows that $\{\|x_{1,n}\|\}$ is bounded. This contradicts the fact that $\|x_{1,n}\| \to +\infty$ as $n \to \infty$. Therefore (2.2) of Lemma 2.1 holds. The proof is complete. □

4 Applications to the second order Hamiltonian systems

In this section, we consider the applications of the main results to the second order Hamiltonian systems satisfying two boundary value conditions including generalized periodic boundary value conditions and Sturm-Liouville boundary value conditions. For more details about Hamiltonian systems, we refer to the excellent books [6,8,9,11] and the papers [1,2,10].

4.1 Second order Hamiltonian systems satisfying generalized periodic boundary value conditions

As a first example, we consider a generalized periodic boundary value problem

$$-\ddot{x} - \bar{B}_1(t)x = \nabla_x V(t,x) \quad \text{a.e. } t \in [0,1], \tag{4.1}$$

$$x(1) = Mx(0), \qquad \dot{x}(1) = N\dot{x}(0), \tag{4.2}$$

where $\bar{B}_1(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbb{R}^n)) = \{B(t) = (b_{jk})_{n\times n} | b_{jk}(t) = b_{kj}(t),\ t \in [0,1],\ b_{jk}(t) \in L^\infty([0,1])\}$, $M, N \in GL(n) = \{A = (a_{jk})_{n\times n}\ |a_{jk} \in \mathbb{R}$ and $\det(A) \neq 0\}$, and $MN^T = I_n$, where I_n is the unit matrix of order n, and $\nabla_x V(t,x)$ denotes the gradient of $V(t,x)$ for $x \in \mathbb{R}^n$. We suppose that $V : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following condition:

(H_0) $V(t,x)$ is measurable in t for every $x \in \mathbb{R}^n$ and continuously differentiable in x for a.e. $t \in [0,1]$.

Moreover, there exist $a(\cdot) \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b(t) \in L^1([0,1], \mathbb{R}^+)$ such that

$$|V(t,x)| \leq a(|x|)b(t) \quad \text{and} \quad |\nabla_x V(t,x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0,1]$, where $\mathbb{R}^+ = [0,+\infty)$.

Let $X = L^2([0,1], \mathbb{R}^n)$. Define $A_1 : D(A_1) \to X$ by $(A_1x)(t) = -\ddot{x}(t)$ where $D(A_1) = \{x \in H^2([0,1], \mathbb{R}^n) | x$ satisfies (4.2)$\}$. Set $(B_1x)(t) = \bar{B}_1(t)x(t)$ with $D(B_1) = X$. From Corollary 1.21 in [3], we know that A_1 is self-adjoint in X and $\sigma(A_1) = \sigma_d(A_1) \subset [0,+\infty)$. Define $i_M(\bar{B}_1) = i_{A_1}(B_1), \nu_M(\bar{B}_1) = \nu_{A_1}(B_1)$, that is, $\nu_M(\bar{B}_1)$ is the dimension of the solution subspace of (4.1)–(4.2) with $V(t,x) \equiv 0$ and $i_M(\bar{B}_1) = \sum_{\lambda<0} \nu_M(\bar{B}_1 + \lambda I_n)$.

Assume that $\nu_M(\bar{B}_1) \neq 0$. Meanwhile, set $Z_1 = \{x \in H^1([0,1], \mathbb{R}^n) | x(1) = Mx(0)\}$. Then, from Corollary 1.21 in [3] again, we have $Z_1 = D(|A_1|^{\frac{1}{2}})$.

Remark 4.1 ([5, Remark 7.1.3], [4, Example 2.4.3])**.** Let $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$ be the eigenvalues of a constant $n \times n$ symmetric matrix B. Then

$$i_{I_n}(B) = {}^{\#}\{k : \alpha_k > 0\} + 2\sum_{k=1}^{n} {}^{\#}\{j \in \mathbf{N} : 4(j\pi)^2 < \alpha_k\}, \tag{4.3}$$

$$\nu_{I_n}(B) = {}^{\#}\{k : \alpha_k = 0\} + 2\sum_{k=1}^{n} {}^{\#}\{j \in \mathbf{N} : 4(j\pi)^2 = \alpha_k\}, \tag{4.4}$$

$$i_{-I_n}(B) = 2\sum_{k=1}^{n} {}^{\#}\{j \in \mathbf{N} : ((2j-1)\pi)^2 < \alpha_k\},$$

$$\nu_{-I_n}(B) = 2\sum_{k=1}^{n} {}^{\#}\{j \in \mathbf{N} : ((2j-1)\pi)^2 = \alpha_k\},$$

where ${}^{\#}E$ denotes the number of elements in a set E. For $\eta \in \mathbb{R}\backslash\{\pm 1, 0\}$ with $\lambda_0 = \arccos \frac{2}{\eta^{-1}+\eta}$, we have

$$i_{\eta I_n}(B) = \sum_{k=1}^{n} {}^{\#}\{j \in \mathbf{N} : (2j\pi + \lambda_0)^2 < \alpha_k\} + \sum_{k=1}^{n} {}^{\#}\{j \in \mathbf{N} : (2\pi - \lambda_0 + 2j\pi)^2 < \alpha_k\},$$

$$\nu_{\eta I_n}(B) = \sum_{k=1}^{n} {}^{\#}\{j \in \mathbf{N} : (2j\pi + \lambda_0)^2 = \alpha_k\} + \sum_{k=1}^{n} {}^{\#}\{j \in \mathbf{N} : (2\pi - \lambda_0 + 2j\pi)^2 = \alpha_k\}.$$

In particular, formulae (4.3) and (4.4) were given first by Mawhin and Willem in [9].

In addition, set

$$\Phi(x) = \int_0^1 V(t,x)dt, \qquad \forall x \in Z_1.$$

Then, $\Phi \in C^1(Z_1, \mathbb{R})$ is weakly continuous with weakly continuous derivative and for every $x \in Z_1$,

$$\Phi'(x)y = \int_0^1 (\nabla_x V(t,x), y)dt, \qquad \forall y \in Z_1$$

because of (H_0). Hence, (Φ_0) holds. Moreover, for each $x \in Z_1$, we can write the norm

$$\|x\|^2 = \int_0^1 [|\dot{x}(t)|^2 + |x(t)|^2]dt.$$

Let $\|\cdot\|_\infty$ be the norm of $C([0,1], \mathbb{R}^n)$. Then, there is a constant $\delta_0 > 0$ such that

$$|x| \le \|x\|_\infty \le \delta_0 \|x\| \tag{4.5}$$

for any $x \in Z_1$. By (4.5) and (H_0), we can verify that (Φ_2) holds.

For any $\bar{B}_1(t), \bar{B}_2(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbb{R}^n))$, we write $\bar{B}_1 \le \bar{B}_2$ if $\bar{B}_1(t) \le \bar{B}_2(t)$ for a.e. $t \in [0,1]$ and define $\bar{B}_1 < \bar{B}_2$ if $\bar{B}_1 \le \bar{B}_2$ and $\bar{B}_1(t) < \bar{B}_2(t)$ on a subset of $(0,1)$ with positive measure.

Now, the following four results hold.

Theorem 4.2. *Assume that $V(t,x)$ satisfies (H_0) and*

(H_1) *$V(t,x)$ is convex in x for a.e. $t \in [0,1]$;*

(H_2) *$\int_0^1 V(t,x)dt$ as $\|x\| \to \infty$ with $x \in \ker(A_1 - \bar{B}_1)$;*

(H$_3$) *there exist* $\gamma(t) \in L^1([0,1],\mathbb{R}^+)$ *and* $\bar{B}_2(t) \in L^\infty([0,1],\mathcal{L}_s(\mathbb{R}^n))$ *with* $\bar{B}_2 > \bar{B}_1$, $\nu_M(\bar{B}_2) \neq 0$ *and* $i_M(\bar{B}_2) = i_M(\bar{B}_1) + \nu_M(\bar{B}_1)$, *such that*

$$V(t,x) \leq \frac{1}{2}((\bar{B}_2(t) - \bar{B}_1(t))x, x) + \gamma(t) \tag{4.6}$$

for all $x \in \mathbb{R}^n$ *and a.e.* $t \in [0,1]$, *and*

$$\text{meas}\left\{t \in [0,1] \middle| V(t,x) - \frac{1}{2}((\bar{B}_2(t) - \bar{B}_1(t))x, x) \to -\infty \text{ as } \|\bar{x}\| \to \infty\right\} > 0, \tag{4.7}$$

where $x = \tilde{x} + \bar{x} \in Z_1$ *with* $\bar{x} \in \ker(A_1 - \bar{B}_2)$ *and* $\|\tilde{x}\|$ *is bounded.*

Then problem (4.1)–(4.2) *has a solution in* Z_1.

Proof. Clearly, (H$_0$) implies that (Φ_0) and (Φ_2) hold, (H$_1$) implies that (Φ_1) holds, and (H$_2$) implies that (Φ_3). We need only to show that (Φ_4) follows from (H$_3$). First, since $\bar{B}_2 > \bar{B}_1$, then exists $E_0 \subset [0,1]$ with meas$E_0 > 0$ such that $\bar{B}_2(t) > \bar{B}_1(t)$ for all $t \in E_0$ and $\bar{B}_2(t) \geq \bar{B}_1(t)$ for all $t \in [0,1]\backslash E_0$. Hence

$$\begin{aligned}((\bar{B}_2 - \bar{B}_1)x, x)_X &= \int_0^1 ((\bar{B}_2(t) - \bar{B}_1(t))x(t), x(t))dt \\ &\geq \int_{E_0} ((\bar{B}_2(t) - \bar{B}_1(t))x(t), x(t))dt > 0\end{aligned}$$

for all $x \in \ker(A_1 - \bar{B}_1)$, because $x(t) \in \ker(A_1 - \bar{B}_1)$ only has finite zeros. This implies that $\bar{B}_2 \geq \bar{B}_1$ and $\bar{B}_2 > \bar{B}_1$ w.r.t. $\ker(A_1 - \bar{B}_1)$. Next, by (4.6), we have

$$\begin{aligned}\Phi(x) = \int_0^1 V(t,x)dt &\leq \frac{1}{2}\int_0^1 ((\bar{B}_2(t) - \bar{B}_1(t))x(t), x(t))dt + \int_0^1 \gamma(t)dt \\ &= \frac{1}{2}((\bar{B}_2 - \bar{B}_1)x, x)_X + c\end{aligned}$$

for all $x \in X$, where $c = \int_0^1 \gamma(t)dt$, which shows that (1.3) of (Φ_4) holds. Finally, set $E_1 = \{t \in [0,1] \mid V(t,x) - \frac{1}{2}((\bar{B}_2(t) - \bar{B}_1(t))x, x) \to -\infty \text{ as } \|\bar{x}\| \to \infty\}$, where $x = \tilde{x} + \bar{x} \in Z_1$ with $\bar{x} \in \ker(A_1 - \bar{B}_2)$ and $\|\tilde{x}\|$ is bounded. Thus, by (4.7) and meas$E_1 > 0$, we have

$$\begin{aligned}&\Phi(x) - \frac{1}{2}((\bar{B}_2 - \bar{B}_1)x, x)_X \\ &= \int_0^1 [V(t,x) - \frac{1}{2}((\bar{B}_2(t) - \bar{B}_1(t))x, x)]dt \\ &\leq \int_{E_1} [V(t,x) - \frac{1}{2}((\bar{B}_2(t) - \bar{B}_1(t))x, x)]dt + \int_0^1 \gamma(t)dt \to -\infty\end{aligned}$$

as $\|\bar{x}\| \to \infty$ with $x = \tilde{x} + \bar{x}$, $\bar{x} \in \ker(A_1 - \bar{B}_2)$ and $\|\tilde{x}\|$ is bounded, which implies that (1.4) of (Φ_4) holds. Now, we can apply Theorem 1.1 to conclude that the system (4.1) – (4.2) has a solution in Z_1. □

Remark 4.3. In particular, set $\bar{B}_1(t) \equiv m^2(2\pi)^2, \bar{B}_2(t) = (m+1)^2(2\pi)^2, m \in \{0,1,2,\dots\}$ and $M = I_n$. Then, $Z_1 = \{x \in H^1([0,1],\mathbb{R}^n) | x(1) = x(0)\}$, $\sigma(A_1) = \{(2m\pi)^2 | m \in \mathbf{N}\}$ and $\ker(A_1 - \bar{B}_1) = \{a\cos(2mt\pi) + b\sin(2mt\pi) | a, b \in \mathbb{R}^n\}$. Hence, the following problem

$$-\ddot{x}(t) - m^2(2k\pi)^2 x(t) = \nabla_x V(t, x(t)), \qquad x(0) - x(1) = \dot{x}(0) - \dot{x}(1) = 0$$

has a solution via Theorems 4.2. In addition, for the interval $[0, T]$ considered in second order Hamiltonian systems satisfying periodic boundary value conditions, if $T = 1$, in Theorem 3.1 ($m = 0$) of [12] and Theorem 1.1 ($m \neq 0$) of [15], assume that $V(t, x)$ satisfies (H_0), (H_1), (H_2), and

($\mathrm{H}_{3,1}$) there exists $\gamma(t) \in L^1([0,1], \mathbb{R}^+)$ such that

$$V(t,x) \leq \frac{2m+1}{2}(2\pi)^2|x|^2 + \gamma(t) \tag{4.8}$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0,1]$, and

$$\operatorname{meas}\left\{t \in [0,1] \middle| V(t,x) - \frac{2m+1}{2}(2\pi)^2|x|^2 \to -\infty \text{ as } |x| \to \infty\right\} > 0, \tag{4.9}$$

then the conclusion of Theorem 4.2 is also true. In fact, set $\bar{B}_2(t) = (m+1)^2(2\pi)^2$, $x = \tilde{x} + \bar{x} \in Z_1$ with $\bar{x} \in \ker(A_1 - (m+1)^2(2\pi)^2)$. If $\|\bar{x}\| \to \infty$ and $\|\tilde{x}\|$ is bounded, we can obtain that $|x| \to \infty$ via the proof of Theorem 1.1 in [15]. From (4.9), we know that (4.7) holds. Noticing that $i_{I_n}((m+1)^2(2\pi)^2) = \nu_{I_n}(m^2(2\pi)^2) + i_{I_n}(m^2(2\pi)^2)$, we have ($\mathrm{H}_3$) holds via ($\mathrm{H}_{3,1}$). So Theorem 4.2 generalizes in Theorem 3.1 ($m = 0$) of [12] and Theorem 1.1 ($m \neq 0$) of [15]. By the remarks in [12] and [15] we can see that Theorem 4.2 also generalizes the corresponding theorems in [9] as $T = 1$.

Theorem 4.4. *The conclusion of Theorem 4.2 still holds if we replace* (H_3) *with*

(H_3') *there exist $\alpha(t) \in L^\infty([0,1], \mathbb{R}^+)$, $\gamma(t) \in L^1([0,1], \mathbb{R}^+)$ and $\bar{B}_3(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbb{R}^n))$ with $\bar{B}_3 > \bar{B}_1$, $\nu_M(\bar{B}_3) \neq 0$ and $i_M(\bar{B}_3) = i_M(\bar{B}_1) + \nu_M(\bar{B}_1)$, such that $\alpha(t)I_n \leq \bar{B}_3(t) - \bar{B}_1(t)$ for a.e. $t \in [0,1]$ with*

$$\operatorname{meas}\left\{t \in [0,1] \middle| 0 < \alpha(t)I_n < \bar{B}_3(t) - \bar{B}_1(t)\right\} > 0, \tag{4.10}$$

and

$$V(t,x) \leq \frac{1}{2}\alpha(t)|x|^2 + \gamma(t) \tag{4.11}$$

for a.e. $t \in [0,1]$ and for all $x \in \mathbb{R}^n$.

Proof. Similarly to the proof of Theorem 4.2, We need only to show that (Φ_4') follows from (H_3'). Set $\bar{B}_2(t) = \bar{B}_1(t) + \alpha(t)I_n$, we have $\bar{B}_2(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbb{R}^n))$ via $\alpha(t) \in L^\infty([0,1], \mathbb{R}^+)$ and $\bar{B}_2(t) \geq \bar{B}_1(t)$. By (4.10), we have $\bar{B}_2 \geq \bar{B}_1$ and $\bar{B}_2 > \bar{B}_1$ w.r.t. $\ker(A_1 - \bar{B}_1)$ and $\bar{B}_3 \geq \bar{B}_2$ and $\bar{B}_3 > \bar{B}_2$ w.r.t. $\ker(A_1 - \bar{B}_2)$ via the similar proof in Theorem 4.2. From (2) of Proposition 2.6, we can find that

$$i_M(\bar{B}_1) + \nu_M(\bar{B}_1) = i_M(\bar{B}_3) \geq i_M(\bar{B}_2) + \nu_M(\bar{B}_2) \geq i_M(\bar{B}_2) \geq i_M(\bar{B}_1) + \nu_M(\bar{B}_1),$$

which implies that $i_M(\bar{B}_2) = i_M(\bar{B}_1) + \nu_M(\bar{B}_1)$ and $\nu_M(\bar{B}_2) = 0$. Again by (4.11), we have

$$\begin{aligned}\Phi(x) = \int_0^1 V(t,x)dt &\leq \frac{1}{2}\int_0^1 ((\bar{B}_2(t) - \bar{B}_1(t))x(t), x(t))dt + \int_0^1 \gamma(t)dt \\ &= \frac{1}{2}((\bar{B}_2 - \bar{B}_1)x, x)_X + c\end{aligned}$$

for all $x \in X$, where $c = \int_0^1 \gamma(t)dt$. This shows that (Φ_4') holds. Next, we can apply Theorem 1.2 to conclude that the system (4.1)–(4.2) has a solution in Z_1. □

Remark 4.5. In particular, set $\bar{B}_1(t) \equiv m^2(2\pi)^2, \bar{B}_2(t) = (m+1)^2(2\pi)^2, m \in \{0,1,2,\dots\}$ and $M = I_n$. Then, the following problem

$$-\ddot{x}(t) - m^2(2k\pi)^2 x(t) = \nabla_x V(t,x(t)), \qquad x(0)-x(1) = \dot{x}(0)-\dot{x}(1) = 0$$

has a solution via Theorems 4.4. In addition, as $T=1$, in Theorem 3.3 ($m=0$) of [12] and Theorem 1.10 ($m \neq 0$) of [15], assume that $V(t,x)$ satisfies (H_0), (H_1), (H_2), and

$(\mathrm{H}'_{3,1})$ there exist $\gamma(t), \alpha(t) \in L^1([0,1],\mathbb{R}^+)$ with $\int_0^1 \alpha(t)dt < \frac{12(2m+1)}{(m+1)^2}$, such that (4.11) holds.

Then the conclusion of Theorem 4.4 is also true.

Obviously, $\alpha(t) \in L^\infty([0,1],\mathbb{R}^+) \subset L^1([0,1],\mathbb{R}^+)$. But, for $\alpha(t) \in L^\infty([0,1],\mathbb{R}^+)$, we have $\int_0^1 \alpha(t)dt < \frac{12(2m+1)}{(m+1)^2} \not\Rightarrow 0 < \alpha(t) < (2m+1)(2\pi)^2$ and $0 < \alpha(t) < (2m+1)(2\pi)^2 \not\Rightarrow \int_0^1 \alpha(t)dt < \frac{12(2m+1)}{(m+1)^2}$. Indeed, if

$$\alpha(t) = \begin{cases} (2m+1)(2\pi)^2, & x \in [0, \frac{1}{(2m+1)(2\pi)^2}], \\ 0, & x \in (\frac{1}{(2m+1)(2\pi)^2}, 1], \end{cases}$$

then $\int_0^1 \alpha(t)dt = 1 \le \frac{12(2m+1)}{(m+1)^2}$ as $m \le 22$ and $\alpha(t) \ge (2m+1)(2\pi)^2$ for $x \in [0, \frac{1}{(2m+1)(2\pi)^2}]$; if $\frac{12(2m+1)}{(m+1)^2} < \alpha(t) < (2m+1)(2\pi)^2$, then $\int_0^1 \alpha(t)dt > \frac{12(2m+1)}{(m+1)^2}$. So Theorem 4.4 is a new result and, in some sence, it represent a development of Theorem 3.3 ($m=0$) of [12] and Theorem 1.10 ($m \neq 0$) of [15].

Theorem 4.6. *The conclusion of Theorem 4.2 still holds if we replace* (H_1) *and* (H_3) *with*

(H'_1) $V(t,\cdot)$ *is* $(\bar{B}_2(t) - \bar{B}_1(t))$*-concave, that is,* $-V(t,x) + \frac{1}{2}((\bar{B}_2(t) - \bar{B}_1(t))x, x)$ *is convex in* x *for a.e.* $t \in [0,1]$.

(H''_3) *there exists* $\bar{B}_2(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbb{R}^n))$ *with* $\bar{B}_2 > \bar{B}_1$, $i_M(\bar{B}_1) = 0$, $\nu_M(\bar{B}_2) \neq 0$ *and* $i_M(\bar{B}_2) = i_M(\bar{B}_1) + \nu_M(\bar{B}_1)$, *such that*

$$\int_0^1 \left(-V(t,x) + \frac{1}{2}((\bar{B}_2(t) - \bar{B}_1(t))x, x) \right) dt \to +\infty \tag{4.12}$$

as $\|x\| \to \infty$ *with* $x \in \ker(A_1 - \bar{B}_2)$,

respectively.

The proof Theorem 4.6 is similar to that of Theorem 4.2. Here we omit it.

Remark 4.7. In particular, set $\bar{B}_1(t) \equiv 0, \bar{B}_2(t) = (2\pi)^2$ and $M = I_n$. Then, the following problem

$$-\ddot{x}(t) = \nabla_x V(t,x(t)), \qquad x(0)-x(1) = \dot{x}(0)-\dot{x}(1) = 0$$

has a solution via Theorems 4.6. In addition, as $T=1$, then Theorem 4.6 reduces to Theorem 5.2 in [12].

Theorem 4.8. *The conclusion of Theorem 4.2 still holds if we replace* (H_1) *and* (H_3) *with*

(H''_1) $V(t,\cdot)$ *is* $\beta(t)$*-concave, that is,* $-V(t,x) + \frac{1}{2}\beta(t)|x|^2$ *is convex in* x *for a.e.* $t \in [0,1]$.

(H$_3'''$) *there exist* $\beta(t) \in L^\infty([0,1],\mathbb{R}^+)$ *and* $\bar{B}_3(t) \in L^\infty([0,1],\mathcal{L}_s(\mathbb{R}^n))$ *with* $\bar{B}_3 > \bar{B}_1$, $i_M(\bar{B}_1) = 0$, $\nu_M(\bar{B}_3) \neq 0$ *and* $i_M(\bar{B}_3) = i_M(\bar{B}_1) + \nu_M(\bar{B}_1)$, *such that* $\beta(t) \leq \bar{B}_3(t) - \bar{B}_1(t)$ *for a.e.* $t \in [0,1]$ *with*

$$\operatorname{meas}\left\{t \in [0,1] \;\middle|\; 0 < \beta(t) < \bar{B}_3(t) - \bar{B}_1(t)\right\} > 0, \tag{4.13}$$

respectively.

The proof Theorem 4.8 is similar to that of Theorem 4.4. Here we omit it.

Remark 4.9. In particular, set $\bar{B}_1(t) \equiv 0$, $\bar{B}_2(t) = (2\pi)^2$ and $M = I_n$. Then, the following problem

$$-\ddot{x}(t) = \nabla_x V(t,x(t)), \qquad x(0) - x(1) = \dot{x}(0) - \dot{x}(1) = 0$$

has a solution via Theorems 4.8. Moreover, as $T = 1$, then Theorem 4.8 reduces to Theorem 5.1 of [12] as $k(t) \in L^\infty([0,1],\mathbb{R}^+)$.

In addition, as $T = 1$, in Theorem 1.4 of [12], assume that $V(t,x)$ satisfies (H$_0$), (H$_2$), and

(H$_{1,1}''$) there exist $\beta(t) \in L^1([0,1],\mathbb{R}^+)$ with $\int_0^1 \beta(t)dt < 12$, such that $V(t,\cdot)$ is $\beta(t)$-concave.

Then the conclusion of Theorem 4.8 is also true.

Obviously, $\beta(t) \in L^\infty([0,1],\mathbb{R}^+) \subset L^1([0,1],\mathbb{R}^+)$. But, for $\beta(t) \in L^\infty([0,1],\mathbb{R}^+)$, we have $\int_0^1 \beta(t)dt < 12 \not\Rightarrow 0 < \beta(t) < (2\pi)^2$ and $0 < \beta(t) < (2\pi)^2 \not\Rightarrow \int_0^1 \beta(t)dt < 12$. Indeed, if

$$\beta(t) = \begin{cases} (2\pi)^2, & x \in [0, \frac{1}{(2\pi)^2}], \\ 0, & x \in (\frac{1}{(2\pi)^2}, 1], \end{cases}$$

then $\int_0^1 \beta(t)dt = 1$ and $\beta(t) \geq (2\pi)^2$ for $x \in [0, \frac{1}{(2\pi)^2}]$; if $12 < \beta(t) < (2\pi)^2$, then $\int_0^1 \beta(t)dt > 12$. So Theorem 4.8 is a new result and, in some sence, it represent a development of Theorem 1.4 of [12]. By the remarks in [12] we can see that Theorem 4.8 also generalizes the corresponding theorems in [9,14,16,17] as $T = 1$.

4.2 Second order Hamiltonian systems satisfying Sturm–Liouville boundary value conditions

As a second example, we consider Sturm–Liouville boundary value problem

$$-\ddot{x} - \tilde{B}_1(t)x = \nabla_x V(t,x), \tag{4.14}$$

$$x(0)\cos\alpha - \dot{x}(0)\sin\alpha = 0, \tag{4.15}$$

$$x(1)\cos\beta - \dot{x}(1)\sin\beta = 0, \tag{4.16}$$

where $\tilde{B}_1 \in L^\infty([0,1],\mathcal{L}_s(\mathbb{R}^n))$, $\nabla_x V(t,x)$ denotes the gradient of $V(t,x)$ for $x \in \mathbb{R}^n$ and $0 \leq \alpha < \pi, 0 < \beta \leq \pi$. We suppose that $V : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ satisfies (H$_0$).

Let $X = L^2([0,1],\mathbb{R}^n)$. Define $A_2 : D(A_2) \to X$ by $(A_2x)(t) = -\ddot{x}(t)$ with $D(A_2) = \{x \in H^2([0,1],\mathbb{R}^n) | x$ satisfies (4.15) and (4.16)$\}$. Set $(B_1x)(t) = \tilde{B}_1(t)x(t)$ with $D(B_1) = X$. From Proposition 1.17 in [3], we know that A_2 is self-adjoint in X and $\sigma(A_2) = \sigma_d(A_2)$ is bounded from below. Define $i_{\alpha,\beta}(\tilde{B}_1) = i_{A_1}(B_1), \nu_{\alpha,\beta}(\tilde{B}_1) = \nu_{A_1}(B_1)$, that is, $\nu_{\alpha,\beta}(\tilde{B}_1)$ is the dimension of the solution subspace of (4.14)–(4.16) with $V(t,x) \equiv 0$.

Assume that $\nu_{\alpha,\beta}(\tilde{B}_1) \neq 0$. Meanwhile, set

$$Z_2 = \begin{cases} \{x \in H^1([0,1],\mathbb{R}^n) | x(1) = 0\}, & \alpha = 0, \beta \in (0,\pi); \\ \{x \in H^1([0,1],\mathbb{R}^n) | x(0) = 0\}, & \alpha \in (0,\pi), \beta = \pi; \\ \{x \in H^1([0,1],\mathbb{R}^n) | x(1) = x(0) = 0\}, & \alpha = 0, \beta = \pi; \\ H^1([0,1],\mathbb{R}^n), & \alpha, \beta \in (0,\pi). \end{cases}$$

Then, from Proposition 1.17 in [3] again, we have $Z_2 = D(|A_1|^{\frac{1}{2}})$. Moreover, set

$$\Phi(x) = \int_0^1 V(t,x)dt, \qquad \forall x \in Z_2.$$

Then, $\Phi \in C^1(Z_2, \mathbb{R})$ is weakly continuous with weakly continuous derivative and for every $x \in Z_2$,

$$\Phi'(x)y = \int_0^1 (\nabla_x V(t,x), y)dt, \qquad \forall y \in Z_2$$

because of (H_0). Hence, (Φ_0) holds. Further, for each $x \in Z_2$, we can write the norm

$$\|x\|^2 = \int_0^1 [|\dot{x}(t)|^2 + |x(t)|^2]dt.$$

By (4.5) and (H_0), we can verify that (Φ_2) holds. Then, the following four results hold. Since their proofs are similar to Theorems 4.2–4.8, and we omit them here.

Theorem 4.10. *Assume that $V(t,x)$ satisfies (H_0), (H_1), (H_2) and (H_3) with $\bar{B}_1, \bar{B}_2$ and A_1 replaced with $\tilde{B}_1, \tilde{B}_2$ and A_2 respectively, then problem (4.14)–(4.16) has a solution in Z_2.*

Theorem 4.11. *The conclusion of Theorem 4.10 still holds if we replace (H_3) and $\bar{B}_3$ with (H_3') and $\tilde{B}_3$, respectively.*

Theorem 4.12. *The conclusion of Theorem 4.10 still holds if we replace (H_1) and (H_3) with (H_1') and (H_3''), respectively.*

Theorem 4.13. *The conclusion of Theorem 4.10 still holds if we replace $\bar{B}_3$, (H_1) and (H_3) with $\tilde{B}_3$, (H_1'') and (H_3'''), respectively.*

Remark 4.14. In particular, set $\tilde{B}_1(t) \equiv \pi^2 I_n$ and $\alpha = 0, \beta = \pi$. Then, $Z_2 = H_0^1$, $\sigma(A_2) = \{k^2\pi^2 | k \in \mathbf{N}\backslash\{0\}\}$ and $\ker(A_2 - \tilde{B}_1) = \{a \sin t\pi \mid a \in \mathbb{R}^n\}$. Hence, the following problem

$$-\ddot{x}(t) = \nabla_x V(t, x(t)), \qquad x(0) = x(1) = 0$$

has a solution via Theorems 4.10–4.13 respectively, where $\tilde{B}_2(t) \equiv 4\pi^2 I_n = \tilde{B}_3(t)$.

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Infinitely many solutions for a quasilinear Schrödinger equation with Hardy potentials

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Abstract. In this article, we study the following quasilinear Schrödinger equation

$$-\Delta u - \mu \frac{u}{|x|^2} + V(x)u - (\Delta(u^2))u = f(x,u), \qquad x \in \mathbb{R}^N,$$

where $V(x)$ is a given positive potential and the nonlinearity $f(x,u)$ is allowed to be sign-changing. Under some suitable assumptions, we obtain the existence of infinitely many nontrivial solutions by a change of variable and Symmetric Mountain Pass Theorem.

Keywords: quasilinear Schrödinger equation, Hardy potential, Mountain Pass Theorem.

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1 Introduction and main results

In this paper, we consider the following equation

$$-\Delta u - \mu \frac{u}{|x|^2} + V(x)u - (\Delta(u^2))u = f(x,u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $0 \leq \mu < \bar{\mu} := \frac{(N-2)^2}{4}$, $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ is a given potential and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

For problem (1.1), if $\mu = 0$, $f(x,u) = f(u)$, then (1.1) becomes

$$-\Delta u + V(x)u - (\Delta(u^2))u = f(u), \qquad x \in \mathbb{R}^N. \tag{1.2}$$

Recently, the existence of solutions for (1.2) has drawn much attention, see for example [5,7, 19,21,22,25]. Particularly, it was established the existence of both one-sign and nodal ground states of soliton type solutions in [21] by Nehari method. Furthermore, using a constrained minimization argument, the existence of a positive ground state solution has been proved in [25]. Later, by using a change of variables, [19] and [7] studied the existence of solutions in

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different working spaces with different classes of nonlinearities. For more results we can refer to [18,20,23,33,34].

Moreover, if we take $\mu \equiv 0$ in (1.1), we have

$$-\Delta u + V(x)u - (\Delta(u^2))u = f(x,u), \qquad x \in \mathbb{R}^N. \tag{1.3}$$

In [39], Zhang and Tang proved there are infinitely many solutions of quasilinear Schrödinger equation with sign-changing potential by Mountain Pass Theorem. When $f(x,u) = \frac{|u|^{2^*(s)-2}u}{|x|^s}$, where $0 \le s < 2$ and $2^*(s) = \frac{2(N-s)}{N-2}$ is the critical Sobolev–Hardy exponents, the problem (1.3) was studied in [10,12]. If $f(x,u) = \lambda|u|^{q-2}u + \frac{|u|^{p-2}u}{|x|^s}$, the authors in [40] have proved the existence of solutions by using a change of variable.

Recently, great attention has been attracted to the study of the following problem

$$-\Delta u - \mu\frac{u}{|x|^2} + V(x)u = f(x,u), \qquad x \in \mathbb{R}^N. \tag{1.4}$$

This class of quasilinear equations are often referred as modified nonlinear Schrödinger equations, whose solutions are related to the existence of standing wave solutions. For example, by use of variational method, Kang and Deng in [13] proved the existence of solutions for $V(x) = 0$ and $f(x,u) = \frac{|u|^{2^*(s)-2}u}{|x|^s} + K(x)|u|^{r-2}u$. Using the similar method, Li in [14] proved the existence of nontrivial solutions for $V(x) = 0$ and $f(x,u) = \frac{|u|^{2^*(s)-2}u}{|x|^s} + K(x)|u|^{r-2}u + \lambda u$. In [4], Cao and Zhou studied the problem (1.4) with $V(x) \equiv 1$ and general subcritical nonlinearity $f(x,u)$, they obtained the existence and multiplicity of positive solutions in some different conditions, their method relies upon the proof of Tarantello in [30]. Under certain conditions, using Ekeland's variational principle, Chen and Peng in [6] obtained the existence of a positive solution with $V(x) \equiv 1$ and nonlinearity $\lambda(f(x,u) + h(x))$. For more results about (1.4), we can refer to [9,11,29] and the references therein.

As regards other relevant papers, we mention here [8,15–17,27,28,31,35,38]. Motivated by facts mentioned above, in this paper, we study the existence of infinitely many solutions for problem (1.1) by Mountain Pass Theorem. Before giving the main result of this paper, we give the assumptions of the potential $V(x)$ and the nonlinear term $f(x,u)$ as follows, firstly

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) = V_0 > 0$;

(V_2) for any $L > 0$, there exists a constant $\vartheta > 0$ such that

$$\lim_{|y|\to\infty} \operatorname{meas}\{x \in \mathbb{R}^N : |x-y| \le \vartheta, V(x) \le L\} = 0;$$

(F_0) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exist constants $c_1, c_2 > 0$ and $4 < p < 22^*$ such that

$$|f(x,u)| \le c_1|u| + c_2|u|^{p-1}, \qquad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R};$$

(F_1) $\lim_{|u|\to\infty} \frac{F(x,u)}{u^4} = \infty$ uniformly in x, and there exists $a_0 \ge 0$ such that $F(x,u) \ge 0$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$ and $|u| \ge a_0$, where $F(x,u) = \int_0^u f(x,s)ds$;

(F_2) $\tilde{F}(x,u) := \frac{1}{4}f(x,u)u - F(x,u) \ge 0$ and there exist $c_0 > 0$ and $\sigma \in \big(\max\{1, \frac{2N}{N+2}\}, 2\big)$ such that

$$|F(x,u)|^\sigma \le c_0|u|^{2\sigma}\tilde{F}(x,u)$$

for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$ with u large enough;

(F_3) $f(x,u) = -f(x,-u)$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.

Now, we are ready to state the main result of this paper.

Theorem 1.1. *Assume that (V_1)–(V_2), (F_0)–(F_3) are satisfied, then problem (1.1) has infinitely many nontrivial solutions $\{u_n\}$ such that $\|u_n\| \to \infty$ and $I(u_n) \to \infty$(I will be defined later).*

Remark 1.2 (see [30]). It follows from (F_1) and (F_2) that

$$\tilde{F}(x,u) \geq \frac{1}{c_0}\left(\frac{|F(x,u)|}{|u|^2}\right)^{\sigma} \to \infty, \tag{1.5}$$

uniformly in x as $|u| \to \infty$.

This paper is organized as follows. In Section 2, we will introduce the variational setting for the problem and some preliminary results. In Section 3, we give the proof of main result.

Notations. In what follows we will adopt the following notations

- $C, C_i, i = 1,2,3,\ldots$ denote possibly different positive constants which may change from line to line;
- For $1 \leq p < \infty$, $L^p(\mathbb{R}^N)$ denotes the usual Lebesgue spaces with norms
$$\|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{1/p}, \qquad 1 \leq p < \infty;$$
- $H^1(\mathbb{R}^N)$ denotes the Sobolev spaces modeled in $L^2(\mathbb{R}^N)$ with norm
$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 dx\right)^{1/2}.$$
- B_R denotes the open ball centered at the origin and radius $R > 0$.

2 Variational setting and preliminary results

Before establishing the variational setting for problem (1.1), we give our working space firstly. Under the assumption (V_1) we define

$$E := \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty\right\},$$

then E is a Hilbert space equipped with the inner product and norm

$$(u,v) = \int_{\mathbb{R}^N} \left(\nabla u \nabla v - \mu \frac{uv}{|x|^2} + V(x)uv\right)dx, \qquad \|u\| = (u,u)^{1/2}.$$

In view of (V_1) and for $u \in E$, the following norm

$$\|u\|_E = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx\right)^{\frac{1}{2}}$$

is equivalent to the classic one in $H^1(\mathbb{R}^N)$.

Now, let us recall the Hardy inequality, which is the main tool and allows us to deal with Hardy-type potentials.

Lemma 2.1 (see [1]). *Assume that* $1 < p < N$ *and* $u \in W^{1,p}(\mathbb{R}^N)$*, then*

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \leq \left(\frac{p}{(N-p)} \right)^p \int_{\mathbb{R}^N} |\nabla u|^p dx.$$

Thus, by Lemma 2.1, $\|u\|$ is well defined. In fact

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx \\ &\quad \geq \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{4}{(N-2)^2} |\nabla u|^2 \right) dx \\ &\quad = \left(1 - \mu \frac{4}{(N-2)^2} \right) \int_{\mathbb{R}^N} |\nabla u|^2 dx \\ &\quad > \left(1 - \frac{(N-2)^2}{4} \frac{4}{(N-2)^2} \right) \int_{\mathbb{R}^N} |\nabla u|^2 dx \\ &\quad = 0. \end{aligned} \tag{2.1}$$

Lemma 2.2. *Assume that* $0 \leq \mu < \bar{\mu} = \frac{(N-2)^2}{4}$*, then there exist* $C_1, C_2 > 0$ *such that*

$$C_1 \|u\|_E^2 \leq \|u\|^2 \leq C_2 \|u\|_E^2,$$

for any $u \in H^1(\mathbb{R}^N)$*.*

Proof. For $\mu \geq 0$, we have

$$\begin{aligned} \|u\|^2 &= \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} + V(x) u^2 \right) dx \\ &\leq \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x) u^2) dx \\ &= \|u\|_E^2. \end{aligned} \tag{2.2}$$

On the other hand, since $0 \leq \mu < \bar{\mu} = \frac{(N-2)^2}{4}$, we can get

$$1 \geq 1 - \frac{4\mu}{(N-2)^2} > 0.$$

Then, we have

$$\begin{aligned} \|u\|^2 &= \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} + V(x) u^2 \right) dx \\ &\geq \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \frac{4\mu}{(N-2)^2} |\nabla u|^2 + V(x) u^2 \right) dx \\ &\geq \left(1 - \frac{4\mu}{(N-2)^2} \right) \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x) u^2) dx \\ &= \left(1 - \frac{4\mu}{(N-2)^2} \right) \|u\|_E^2. \end{aligned} \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$C_1 \|u\|_E^2 \leq \|u\|^2 \leq C_2 \|u\|_E^2. \qquad \square$$

As we all known, under the assumption (V_1), the embedding $E \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for $r \in [2, 2^*]$ and $E \hookrightarrow L^r_{loc}(\mathbb{R}^N)$ is compact for $[2, 2^*)$ i.e. there is a constant $d_r > 0$ such that

$$\|u\|_s \leq d_r \|u\|_E, \qquad \forall u \in E,\ r \in [2, 2^*].$$

From this, by Lemma 2.2, there is $C_3 > 0$ such that

$$\|u\|_r \leq d_r \|u\|_E \leq C_3 \|u\|, \qquad \forall u \in E,\ r \in [2, 2^*].$$

Furthermore, under the assumptions (V_1) and (V_2), we have the following compactness lemma due to [3] (see also [2, 41]).

Lemma 2.3. *Assume that (V_1) and (V_2) hold, the embedding $E \hookrightarrow L^r(\mathbb{R}^N)$ is compact for $2 \leq r < 2^*$.*

In order to solve problem (1.1), we define the energy functional $I : E \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (1 + 2|u|^2)|\nabla u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} \frac{\mu}{|x|^2} u^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x) u^2 dx - \int_{\mathbb{R}^N} F(x, u) dx.$$

It is well known that I is not well defined in E. To overcome this difficulty, we make the change of variables by $v = h^{-1}(u)$, where h is defined by

$$h'(t) = \frac{1}{\sqrt{1 + 2|h(t)|^2}} \quad \text{on } [0, \infty),$$

and

$$h(-t) = -h(t) \quad \text{on } (-\infty, 0].$$

Therefore, after the change of variables, from $I(u)$ we obtain the following functional

$$\begin{aligned} J(v) &:= I(h(v)) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} \frac{\mu}{|x|^2} h^2(v) dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x) h^2(v) dx - \int_{\mathbb{R}^N} F(x, h(v)) dx. \end{aligned} \tag{2.4}$$

It is easy to check that J is well defined on E. Under our hypotheses, $J \in C^1(E, \mathbb{R})$ and

$$\begin{aligned} \langle J'(v), \phi \rangle = &\int_{\mathbb{R}^N} \nabla v \nabla \phi dx - \int_{\mathbb{R}^N} \frac{\mu}{|x|^2} h(v) h'(v) \phi dx \\ &+ \int_{\mathbb{R}^N} V(x) h(v) h'(v) \phi dx - \int_{\mathbb{R}^N} f(x, h(v)) h'(v) \phi dx. \end{aligned} \tag{2.5}$$

for all $\phi \in E$.

Moreover, the critical points of J are the weak solutions of the following equation

$$-\Delta v = \frac{1}{\sqrt{1 + 2|h(v)|^2}} \left(f(x, h(v)) - V(x) h(v) + \frac{\mu}{|x|^2} h(v) \right) \quad \text{in } \mathbb{R}^N. \tag{2.6}$$

We also observe that if v is a critical point of the functional J, then $u = h(v)$ is a critical point of the functional I, i.e. $u = h(v)$ is a solution of (1.1).

Now, let us recall some properties of the change of variables $h : \mathbb{R} \to \mathbb{R}$.

Lemma 2.4. (see [24]) *The function $h(t)$ and its derivative satisfy the following properties*

(h_1) *h is uniquely defined, C^∞ and invertible;*

(h_2) *$|h'(t)| \leq 1$ for all $t \in \mathbb{R}$;*

(h_3) *$|h(t)| \leq |t|$ for all $t \in \mathbb{R}$;*

(h_4) *$\frac{h(t)}{t} \to 1$ as $t \to 0$;*

(h_5) *$\frac{h(t)}{\sqrt{t}} \to 2^{\frac{1}{4}}$ as $t \to \infty$;*

(h_6) *$\frac{h(t)}{2} \leq th'(t) \leq h(t)$ for all $t > 0$;*

(h_7) *$\frac{h^2(t)}{2} \leq th(t)h'(t) \leq h^2(t)$ for all $t \in \mathbb{R}$;*

(h_8) *$|h(t)| \leq 2^{\frac{1}{4}}|t|^{\frac{1}{2}}$ for all $t \in \mathbb{R}$;*

(h_9) *there exists a positive constant C such that*

$$|h(t)| \geq \begin{cases} C|t|, & |t| \leq 1 \\ C|t|^{\frac{1}{2}}, & |t| \geq 1; \end{cases}$$

(h_{10}) *for each $\alpha > 0$, there exists a positive constant $C(\alpha)$ such that*

$$|h(\alpha t)|^2 \leq C(\alpha)|h(t)|^2;$$

(h_{11}) *$|h(t)h'(t)| \leq \frac{1}{\sqrt{2}}$.*

For convenience of our proof, we give the following basic and important definition.

Definition 2.5 (see [36])**.** Assume that $J \in C^1(E, \mathbb{R})$, sequence $\{u_n\} \subset E$ is called $(C)_c$ sequence if

$$J(v_n) \to c \quad \text{and} \quad (1 + \|v_n\|)J'(v_n) \to 0.$$

If any $(C)_c$ sequence has a convergent subsequence, we say J satisfies Cerami condition at level c.

Lemma 2.6. *Assume that (V_1), (V_2), (F_0)–(F_2) hold, then any $(C)_c$-sequence of J is bounded in E for each $c \in \mathbb{R}$.*

Proof. Let $\{v_n\} \subset E$ be a $(C)_c$-sequence of J, we have

$$J(v_n) \to c, \qquad (1 + \|v_n\|)J'(v_n) \to 0 \quad \text{as } n \to \infty. \tag{2.7}$$

Then, there is a constant $C_4 > 0$ such that

$$J(v_n) - \frac{1}{4}\langle J'(v_n), v_n\rangle \leq C_4. \tag{2.8}$$

Let

$$\|v_n\|_h^2 := \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 - \frac{\mu}{|x|^2}h^2(v_n) + V(x)h^2(v_n)\right)dx.$$

First, we prove that there exists $C_5 > 0$ such that

$$\|v_n\|_h^2 \leq C_5. \tag{2.9}$$

On the contrary, we suppose that

$$\|v_n\|_h^2 \to \infty.$$

Taking $\tilde{h}(v_n) = \frac{h(v_n)}{\|v_n\|_h}$, then $\|\tilde{h}(v_n)\| \leq 1$. Passing to a subsequence, we assume that

$$\tilde{h}(v_n) \rightharpoonup \nu \quad \text{in } E,$$

$$\tilde{h}(v_n) \to \nu \quad \text{in } L^r(\mathbb{R}^N),\ 2 \leq r < 2^*,$$

and

$$\tilde{h}(v_n) \to \nu \quad \text{a.e. on } \mathbb{R}^N.$$

From (2.4) and (2.7), we have

$$\lim_{|n|\to\infty} \int_{\mathbb{R}^N} \frac{|F(x,h(v_n))|}{\|v_n\|_h^2} dx = \frac{1}{2}. \tag{2.10}$$

On the other hand, set $\xi_n = \frac{h(v_n)}{h'(v_n)}$, then there exists $C_6 > 0$ such that $\|\xi_n\| \leq C_6\|v_n\|$. Since $\{v_n\}$ is a $(C)_c$ sequence of J, by (2.8) we have

$$\begin{aligned}
C_6 &\geq J(v_n) - \frac{1}{4}\langle J'(v_n), \xi_n\rangle \\
&= \frac{1}{4}\int_{\mathbb{R}^N} \left(|\nabla h(v_n)|^2 - \frac{\mu}{|x|^2}h^2(v_n) + V(x)h^2(v_n)\right)dx \\
&\quad + \int_{\mathbb{R}^N} \left(\frac{1}{4}f(x,h(v_n))h(v_n) - F(x,h(v_n))\right)dx \\
&= \frac{1}{4}\|h(v_n)\|^2 + \int_{\mathbb{R}^N} \tilde{F}(x,h(v_n))dx \\
&\geq \int_{\mathbb{R}^N} \tilde{F}(x,h(v_n))dx.
\end{aligned} \tag{2.11}$$

Take $l(a) = \inf\{\tilde{F}(x,h(v_n)) \mid x \in \mathbb{R}^N,\ |h(v_n)| \geq a\}$, for $a \geq 0$. By (1.5), we have $l(a) \to \infty$ as $a \to \infty$. For $0 \leq b_1 < b_2$, let

$$B_n(b_1,b_2) = \{x \in \mathbb{R}^N : b_1 \leq |h(v_n(x))| < b_2\}.$$

Combining with (2.11) that

$$\begin{aligned}
C_6 &\geq \int_{B_n(0,a)} \tilde{F}(x,h(v_n))dx + \int_{B_n(a,\infty)} \tilde{F}(x,h(v_n))dx \\
&\geq \int_{B_n(0,a)} \tilde{F}(x,h(v_n))dx + l(a)\text{meas}\{B_n(a,\infty)\},
\end{aligned}$$

from this we get $\text{meas}\{B_n(a,\infty)\} \to 0$ as $a \to \infty$ uniformly in n. Hence, for $r \in [2,2^*)$ and

(h_{11}), there exist $C, C_7 > 0$ such that

$$\begin{aligned}
\int_{B_n(a,\infty)} \tilde{h}^r(v_n)dx &\leq \left(\int_{B_n(a,\infty)} \tilde{h}^{22^*}(v_n)dx\right)^{\frac{r}{22^*}} \left(\text{meas}\{B_n(a,\infty)\}\right)^{\frac{22^*-r}{22^*}} \\
&= \frac{1}{\|v_n\|_h^r}\left(\int_{B_n(a,\infty)} h^{22^*}(v_n)dx\right)^{\frac{r}{22^*}} \left(\text{meas}\{B_n(a,\infty)\}\right)^{\frac{22^*-r}{22^*}} \\
&\leq \frac{C}{\|v_n\|_h^r}\left(\int_{B_n(a,\infty)} |\nabla h^2(v_n)|^2dx\right)^{\frac{r}{4}} \left(\text{meas}\{B_n(a,\infty)\}\right)^{\frac{22^*-r}{22^*}} \\
&\leq \frac{C_6}{\|v_n\|_h^r}\left(\int_{B_n(a,\infty)} |\nabla v_n|^2dx\right)^{\frac{r}{4}} \left(\text{meas}\{B_n(a,\infty)\}\right)^{\frac{22^*-r}{22^*}} \\
&\leq C_7\|v_n\|_h^{-r/2}\left(\text{meas}\{B_n(a,\infty)\}\right)^{\frac{22^*-r}{22^*}} \to 0,
\end{aligned} \tag{2.12}$$

as $a \to \infty$ uniformly in n.

If $\nu = 0$, then $\tilde{h}(v_n) \to 0$ in $L^r(\mathbb{R}^N)$, $2 \leq r < 2^*$. For any $0 < \varepsilon < \frac{1}{8}$, there exist a_1, L large enough, such that

$$\begin{aligned}
\int_{B_n(0,a_1)} \frac{|F(x,h(v_n))|}{|h(v_n)|^2}|\tilde{h}(v_n)|^2dx &\leq \int_{B_n(0,a_1)} \frac{c_1|h(v_n)|^2 + c_2|h(v_n)|^p}{|h(v_n)|^2}|\tilde{h}(v_n)|^2dx \\
&\leq (c_1 + c_2a_1^{p-2}) \int_{B_n(0,r_1)} |\tilde{h}(v_n)|^2dx \\
&\leq (c_1 + c_2a_1^{p-2}) \int_{\mathbb{R}^N} |\tilde{h}(v_n)|^2dx \\
&< \varepsilon,
\end{aligned} \tag{2.13}$$

for $n > L$. Set $\tau' = \frac{\sigma}{\sigma-1}$, since $\sigma \in \left(\max\{1, \frac{2N}{N+2}\}, 2\right)$, then $2\tau' \in (2, 22^*)$. Thus, by (F_2) and (2.12) we have

$$\begin{aligned}
&\int_{B_n(a_1,\infty)} \frac{|F(x,h(v_n))|}{|h(v_n)|^2}|\tilde{h}(v_n)|^2dx \\
&\leq \left(\int_{B_n(a_1,\infty)} \left(\frac{|F(x,h(v_n))|}{|h(v_n)|^2}\right)^{\sigma}dx\right)^{\frac{1}{\sigma}} \left(\int_{B_n(a_1,\infty)} |\tilde{h}(v_n)|^{2\tau'}dx\right)^{\frac{1}{\tau'}} \\
&\leq c_0^{\frac{1}{\sigma}}\left(\int_{B_n(a_1,\infty)} \tilde{F}(x,h(v_n))dx\right)^{\frac{1}{\sigma}} \left(\int_{B_n(a_1,\infty)} |\tilde{h}(v_n)|^{2\tau'}dx\right)^{\frac{1}{\tau'}} \\
&\leq C_8\left(\int_{B_n(a_1,\infty)} |\tilde{h}(v_n)|^{2\tau'}dx\right)^{\frac{1}{\tau'}} \\
&< \varepsilon.
\end{aligned} \tag{2.14}$$

From (2.13) and (2.14), we can get

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{|F(x,h(v_n))|}{\|v_n\|_h^2}dx &= \int_{B_n(0,a_1)} \frac{|F(x,h(v_n))|}{|h(v_n)|^2}|\tilde{h}(v_n)|^2dx + \int_{B_n(a_1,\infty)} \frac{|F(x,h(v_n))|}{|h(v_n)|^2}|\tilde{h}(v_n)|^2dx \\
&< 2\varepsilon < \frac{1}{4},
\end{aligned}$$

for $n > L$, which contradicts (2.10).

If $\nu \neq 0$, then meas$\{B\} > 0$, where $B = \{x \in \mathbb{R}^N : \nu \neq 0\}$. For $x \in B$, we have $|h(v_n)| \to \infty$ as $n \to \infty$. Hence $B \subset B_n(a_0, \infty)$ for $n \in \mathbb{N}$ large enough, where a_0 is given in (F_1). By (F_1), we have

$$\frac{F(x, h(v_n))}{|h(v_n)|^4} \to \infty \quad \text{as } n \to \infty.$$

Using Fatou's Lemma, then

$$\int_B \frac{F(x, h(v_n))}{|h(v_n)|^4} dx \to \infty \quad \text{as } n \to \infty. \tag{2.15}$$

We see from (2.7) and (2.15)

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c + o(1)}{\|v_n\|_h^2} = \lim_{n\to\infty} \frac{J(v_n)}{\|v_n\|_h^2} \\
&= \lim_{n\to\infty} \frac{1}{\|v_n\|_h^2} \left(\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v_n|^2 - \frac{\mu}{|x|^2} h^2(v_n) + V(x) h^2(v_n)) dx - \int_{\mathbb{R}^N} F(x, h(v_n)) dx \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{2} - \int_{B_n(0,a_0)} \frac{F(x, h(v_n))}{|h(v_n)|^2} |\tilde{h}(v_n)|^2 dx - \int_{B_n(a_0,\infty)} \frac{F(x, h(v_n))}{|h(v_n)|^2} |\tilde{h}(v_n)|^2 dx \right) \\
&\leq \frac{1}{2} + \limsup_{n\to\infty} \left((c_1 + c_2 a_0^{p-2}) \int_{\mathbb{R}^N} |\tilde{h}(v_n)|^2 dx - \int_{B_n(a_0,\infty)} \frac{F(x, h(v_n))}{|h(v_n)|^2} |\tilde{h}(v_n)|^2 dx \right) \\
&\leq C_9 - \liminf_{n\to\infty} \int_B \frac{F(x, h(v_n))}{|h(v_n)|^4} |h(v_n) \tilde{h}(v_n)|^2 dx \\
&= -\infty,
\end{aligned}$$

which is a contradiction. Hence, (2.9) holds.

In order to prove that $\{v_n\}$ is bounded, we only need to show that there is $C_{10} > 0$ such that

$$\|v_n\|_h^2 \geq C_{10} \|v_n\|^2. \tag{2.16}$$

Arguing indirectly, for a subsequence, we assume $\frac{\|v_n\|_h^2}{\|v_n\|^2} \to 0$, where $v_n \neq 0$ (if not, the result is obvious). Take $\xi_{n,1} = \frac{v_n}{\|v_n\|}$, $\eta_{n,1} = \frac{h^2(v_n)}{\|v_n\|^2}$, then

$$\int_{\mathbb{R}^N} \left(|\nabla \xi_{n,1}|^2 - \frac{\mu}{|x|^2} \eta_{n,1}(x) + V(x) \eta_{n,1}(x) \right) dx \to 0. \tag{2.17}$$

It follows from (h_3) that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(|\nabla \xi_{n,1}|^2 - \frac{\mu}{|x|^2} \eta_{n,1}(x) + V(x) \eta_{n,1}(x) \right) dx \\
&\quad = \int_{\mathbb{R}^N} \left(|\nabla \xi_{n,1}|^2 - \frac{\mu}{|x|^2} \frac{h^2(v_n)}{\|v_n\|^2} + V(x) \eta_{n,1}(x) \right) dx \\
&\quad \geq \int_{\mathbb{R}^N} \left(|\nabla \xi_{n,1}|^2 - \frac{\mu}{|x|^2} \frac{v_n^2}{\|v_n\|^2} + V(x) \eta_{n,1}(x) \right) dx \\
&\quad = \int_{\mathbb{R}^N} \left(|\nabla \xi_{n,1}|^2 - \frac{\mu}{|x|^2} \xi_{n,1}^2 + V(x) \eta_{n,1}(x) \right) dx \\
&\quad \geq 0.
\end{aligned} \tag{2.18}$$

Combining (2.1), (2.17) and (2.18), we

$$\int_{\mathbb{R}^N}\left(|\nabla\xi_{n,1}|^2-\frac{\mu}{|x|^2}\xi_{n,1}^2+V(x)\eta_{n,1}(x)\right)dx\to 0.$$

Hence

$$\int_{\mathbb{R}^N}\left(|\nabla\xi_{n,1}|^2-\frac{\mu}{|x|^2}\xi_{n,1}^2\right)dx\to 0,\qquad \int_{\mathbb{R}^N}V(x)\eta_{n,1}(x)dx\to 0\quad\text{and}\quad \int_{\mathbb{R}^N}V(x)\xi_{n,1}^2dx\to 1.$$

Similar to the idea of [37], let $B_n=\{x\in\mathbb{R}^N:|v_n(x)|\geq C_{11}\}$, where $C_{11}>0$ is independent of n. We suppose that for $\varepsilon>0$, $\text{meas}\{B_n\}<\varepsilon$. If not, there exists $\varepsilon'>0$ and $\{v_{n_i}\}\subset\{v_n\}$ such that

$$\text{meas}\{x\in\mathbb{R}^N:|v_{n_i}(x)|\geq i\}\geq\varepsilon',$$

where $i>0$ is a integer. Set $B_{n_i}=\{x\in\mathbb{R}^N:|v_{n_i}(x)|\geq i\}$. From (2.1), (h_3) and (h_9) we have

$$\begin{aligned}\|v_{n_i}\|_h^2&=\int_{\mathbb{R}^N}\left(|\nabla v_{n_i}|^2-\frac{\mu}{|x|^2}h^2(v_{n_i})+V(x)h^2(v_{n_i})\right)dx\\&\geq\int_{\mathbb{R}^N}\left(|\nabla v_{n_i}|^2-\frac{\mu}{|x|^2}v_{n_i}^2+V(x)h^2(v_{n_i})\right)dx\\&>\int_{\mathbb{R}^N}V(x)h^2(v_{n_i})dx\\&>Ci\varepsilon'\to\infty.\end{aligned}$$

as $i\to\infty$, which is a contradiction. For constants $C_{12},C_{13}>0$, it follows $|v_n(x)|\leq C_{12}$, (h_9) and (h_{10}) that

$$\frac{C}{C_{12}^2}v_n^2\leq h^2\left(\frac{1}{C_{12}}v_n\right)\leq C_{13}h^2(v_n).$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_n}V(x)\xi_{n,1}^2dx&\leq C_{14}\int_{\mathbb{R}^N\setminus B_n}V(x)\frac{h^2(v_n)}{\|v_n\|}dx\\&\leq C_{14}\int_{\mathbb{R}^N}V(x)\eta_{n,1}(x)dx\to 0,\end{aligned}\tag{2.19}$$

where $C_{14}>0$ is a constant. For another, by absolute continuity of integral, there exists $\varepsilon>0$ such that

$$\int_{B'}V(x)\xi_{n,1}^2dx\leq\frac{1}{2},\tag{2.20}$$

where $B'\subset\mathbb{R}^N$ and $\text{meas}\{B'\}<\varepsilon$. By (2.19) and (2.20), we have

$$\int_{\mathbb{R}^N}V(x)\xi_{n,1}^2dx=\int_{\mathbb{R}^N\setminus B_n}V(x)\xi_{n,1}^2dx+\int_{B_n}V(x)\xi_{n,1}^2dx\leq\frac{1}{2}+o(1).$$

We can get a contradiction. Hence (2.16) holds. Combining (2.9) with (2.16), we complete the proof of this lemma. □

Lemma 2.7. *Assume that (V_1), (V_2), (F_0)–(F_2) hold, then J satisfies $(C)_c$-condition.*

Proof. Lemma 2.6 implies that $\{v_n\}$ is bounded in E. For a subsequence, we can assume that $v_n \rightharpoonup v$ in E. From Lemma 2.3, $v_n \to v$ in $L^r(\mathbb{R}^N)$ for all $2 \leq r < 2^*$ and $v_n \to v$ a.e. on $\mathbb{R}^N$. First, we claim that there exists $C_{15} > 0$ such that

$$\int_{\mathbb{R}^N} \left(|\nabla(v_n - v)|^2 + \left(V(x) - \frac{\mu}{|x|^2} \right) (h(v_n)h'(v_n) - h(v)h'(v)) \right) (v_n - v) dx \geq C_{15} \|v_n - v\|_E^2 \tag{2.21}$$

Indeed, we may assume $v_n \neq v$ (otherwise the conclusion is trivial). Set

$$\xi_{n,2} = \frac{v_n - v}{\|v_n - v\|} \quad \text{and} \quad \eta_{n,2} = \frac{h(v_n)h'(v_n) - h(v)h'(v)}{v_n - v},$$

we argue by contradiction and assume that

$$\int_{\mathbb{R}^N} \left(|\nabla \xi_{n,2}|^2 - \frac{\mu}{|x|^2} \eta_{n,2}(x) \xi_{n,2}^2 + V(x) \eta_{n,2}(x) \xi_{n,2}^2 \right) dx \to 0. \tag{2.22}$$

Since

$$\frac{d}{dt}(h(t)h'(t)) = h(t)h''(t) + (h'(t))^2 = \frac{1}{(1 + 2h^2(t))^2} > 0,$$

$h(t)h'(t)$ is strictly increasing and for each $C_{16} > 0$, there is $\delta_1 > 0$ such that

$$\frac{d}{dt}(h(t)h'(t)) \geq \delta_1,$$

at $|t| \leq C_{16}$. From this, we see that $\eta_{n,2}(x)$ is positive. On the other hand, for $v_n > v$, there exists $\theta \in (v, v_n)$ such that

$$\eta_{n,2} = \frac{h(v_n)h'(v_n) - h(v)h'(v)}{v_n - v} = \frac{d}{dt}(h(\theta)h'(\theta)) = \frac{1}{(1 + 2h^2(\theta))^2} \leq 1.$$

Similarly, we can prove the case $v_n < v$.

Hence,

$$\eta_{n,2}(x) \leq 1 \quad \text{for all } v_n \neq v. \tag{2.23}$$

It follows from (2.1), (2.22) and (2.23) that

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^N} \left(|\nabla \xi_{n,2}|^2 - \frac{\mu}{|x|^2} \xi_{n,2}^2 + V(x) \eta_{n,2}(x) \xi_{n,2}^2 \right) dx \\ &\leq \int_{\mathbb{R}^N} \left(|\nabla \xi_{n,2}|^2 - \frac{\mu}{|x|^2} \eta_{n,2}(x) \xi_{n,2}^2 + V(x) \eta_{n,2}(x) \xi_{n,2}^2 \right) dx \\ &\to 0. \end{aligned}$$

Then, we have

$$\int_{\mathbb{R}^N} \left(|\nabla \xi_{n,2}|^2 - \frac{\mu}{|x|^2} \xi_{n,2}^2 \right) dx \to 0, \qquad \int_{\mathbb{R}^N} V(x) \eta_{n,2}(x) \xi_{n,2}^2 dx \to 0,$$

and

$$\int_{\mathbb{R}^N} V(x) \xi_{n,2}^2 dx \to 1.$$

By a similar fashion as (2.19) and (2.20), we can conclude a contradiction.

On the other hand, by (h_2), (h_3), (h_8), (h_{11}), (F_0) and (1.5), there is $C_{17} > 0$ such that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} (f(x,h(v_n))h'(v_n) - f(x,h(v))h'(v))(v_n - v)dx \right| \\
&\quad \le \int_{\mathbb{R}^N} C_{17}(|v_n| + |v_n|^{\frac{p}{2}-1} + |v| + |v|^{\frac{p}{2}-1})|v_n - v|dx \\
&\quad \le C_{17}(\|v_n\|_2 + \|v\|_2)\|v_n - v\|_2 + \left(\|v_n\|_{\frac{p}{2}}^{\frac{p-2}{2}} + \|v\|_{\frac{p}{2}}^{\frac{p-2}{2}} \right) \|v_n - v\|_{\frac{p}{2}} \\
&\quad = o(1).
\end{aligned} \tag{2.24}$$

Therefore, by (2.21) and (2.24), we have

$$\begin{aligned}
o(1) &= \langle J'(v_n) - J'(v), v_n - v \rangle \\
&= \int_{\mathbb{R}^N} \left(|\nabla(v_n - v)|^2 + (V(x) - \frac{\mu}{|x|^2})(h(v_n)h'(v_n) - h(v)h'(v))(v_n - v) \right) dx \\
&\quad - \int_{\mathbb{R}^N} (f(x,h(v_n))h'(v_n) - f(x,h(v))h'(v))(v_n - v)dx \\
&\ge C_{15}\|v_n - v\| + o(1).
\end{aligned}$$

This implies that $\|v_n - v\| \to 0$ as $n \to \infty$. Thus, the proof is complete. □

To prove our main result in this paper, we need the following lemma.

Lemma 2.8 (Symmetric Mountain Pass Theorem [26])**.** *Let X be an infinite dimensional Banach space, $X = Y \bigoplus Z$, where Y is finite dimensional. If $\Psi \in C^1(X, \mathbb{R})$ satisfies $(C)_c$-condition for all $c > 0$, and*

- (I_1) *$\Psi(0) = 0$, $\Psi(-u) = u$ for all $u \in X$;*
- (I_2) *there exist constants ρ, $\alpha > 0$ such that $\Psi \,|_{\partial B_\rho \cap Z} \ge \alpha$;*
- (I_3) *for any finite dimensional subspace $\tilde{X} \subset X$, there is $R = R(\tilde{X}) > 0$ such that $\Psi(u) \le 0$ on $\tilde{X} \setminus B_R$;*

then Ψ possesses an unbounded sequence of critical values.

3 Proof of Theorem 1.1

Let $\{e_i\}$ is a total orthonormal basis of E and define $X_i = \mathbb{R}e_i$, then $E = \bigoplus_{i=1}^{\infty} X_i$. Let

$$Y_j = \bigoplus_{i=1}^{i} X_i, \qquad Z_j = \overline{\bigoplus_{j+1}^{\infty} X_i}, \qquad j \in \mathbb{Z},$$

then $E = Y_j \bigoplus Z_j$ and Y_j is finite-dimensional. Similar to Lemma 3.8 in [36], we have the following lemma.

Lemma 3.1 ([36])**.** *Under assumptions (V_1) and (V_2), for $2 \le r < 2^*$,*

$$\beta_j(r) := \sup_{u \in Z_j, \|v\|=1} \|v\|_r \to 0, \qquad j \to \infty.$$

Before going further, we need to show that there exists $C_{18} > 0$ such that

$$\int_{\mathbb{R}^N} \left(|\nabla v|^2 - \frac{\mu}{|x|^2} h^2(v) + V(x) h^2(v) \right) dx \geq C_{18} \|v\|^2, \qquad \forall v \in S_\rho, \tag{3.1}$$

where $S_\rho = \{v \in E : \|v\| = \rho\}$. Indeed, by a similar argument as (2.16), we can get this conclusion. Moreover, by Lemma 3.1, we can choose an integer $\kappa \geq 1$ such that

$$\|v\|_2^2 \leq \frac{C_{18}}{4c_1} \|v\|, \qquad \|v\|_{\frac{p}{2}}^{\frac{p}{2}} \leq \frac{C_{18}}{4c_2} \|v\|^{\frac{p}{2}}, \qquad \forall v \in Z_\kappa. \tag{3.2}$$

Lemma 3.2. *Assume that (V_1), (V_2) and (F_0) hold, then there exist constants ρ, $\alpha > 0$ such that $J|_{S_\rho \cap Z_\kappa} \geq \alpha$.*

Proof. For any $v \in Z_\kappa$ with $\|v\| = \rho < 1$, by (h_3), (h_8), (3.1) and (3.2), we have

$$\begin{aligned}
J(v) &= \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v|^2 - \frac{\mu}{|x|^2} h^2(v) + V(x) h^2(v) \right) dx - \int_{\mathbb{R}^N} F(x, h(v)) dx \\
&\geq \frac{C_{18}}{2} \|v\|^2 - \int_{\mathbb{R}^N} (c_1 |h(v)|^2 + c_2 |h(v)|^p) dx \\
&\geq \frac{C_{18}}{2} \|v\|^2 - \int_{\mathbb{R}^N} (c_1 |v|^2 + c_2 |v|^{\frac{p}{2}}) dx \\
&\geq \frac{C_{18}}{2} \|v\|^2 - \frac{C_{18}}{4} \|v\|^2 - \frac{C_{18}}{4} \|v\|^{\frac{p}{2}} \\
&= \frac{C_{18}}{4} \|v\|^2 \left(1 - \|v\|^{\frac{p-4}{2}} \right) \\
&> 0.
\end{aligned}$$

since $p \in (4, 22^*)$. This completes the proof. ☐

Lemma 3.3. *Assume that (V_1), (V_2), (F_0) and (F_1) hold, for any finite dimensional subspace $\tilde{E} \subset E$, there is $R = R(\tilde{E}) > 0$ such that*

$$J(v) \leq 0, \qquad \forall v \in \tilde{E} \setminus B_R.$$

Proof. For any finite dimensional subspace $\tilde{E} \subset E$, there is a positive integral number κ such that $\tilde{E} \subset Y_\kappa$. Suppose to the contrary that there is a sequence $\{v_n\} \subset \tilde{E}$ such that $\|v_n\| \to \infty$ and $J(v_n) > 0$. Hence

$$\frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 - \frac{\mu}{|x|^2} h^2(v_n) + V(x) h^2(v_n) \right) dx > \int_{\mathbb{R}^N} F(x, h(v_n)) dx. \tag{3.3}$$

Jointly with (h_3), we have

$$\frac{\int_{\mathbb{R}^N} F(x, h(v_n)) dx}{\|v_n\|^2} < \frac{1}{2}. \tag{3.4}$$

Set $\eta_n = \frac{v_n}{\|v_n\|}$. Then up to a subsequence, we can assume that

$$\begin{aligned}
&\eta_n \rightharpoonup \eta \quad \text{in } E, \\
&\eta_n \to \eta \quad \text{in } L^r(\mathbb{R}^N) \quad \text{for } 2 \leq r < 2^*
\end{aligned}$$

and

$$\eta_n \to \eta \quad \text{a.e. on } \mathbb{R}^N.$$

Set $A_1 = \{x \in \mathbb{R}^N : \eta(x) \neq 0\}$ and $A_2 = \{x \in \mathbb{R}^N : \eta(x) = 0\}$. If $\text{meas}\{A_1\} > 0$, then by (F_1), (h_5) and Fatou's Lemma, we have

$$\int_{A_1} \frac{F(x, h(v_n))}{\|v_n\|^2} dx = \int_{A_1} \frac{F(x, h(v_n))}{h^4(v_n)} \frac{h^4(v_n)}{v_n^2} \eta_n^2 dx \to \infty.$$

By (F_0) and (F_1), there exists $C_{19} > 0$ such that

$$F(x,t) \geq -C_{19} t^2, \qquad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Hence

$$\int_{A_2} \frac{F(x, h(v_n))}{\|v_n\|^2} dx \geq -C_{19} \int_{A_2} \frac{h^2(v_n)}{\|v_n\|^2} dx \geq -C_{19} \int_{A_2} \eta_n^2 dx.$$

Since $\eta_n \to \eta$ in $L^2(\mathbb{R}^N)$, it is clear that

$$\liminf_{n \to \infty} \int_{A_2} \frac{F(x, h(v_n))}{\|v_n\|^2} dx = 0.$$

Consequently,

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{F(x, h(v_n))}{\|v_n\|^2} dx = \infty.$$

By (3.4) we obtain $\frac{1}{2} > \infty$, a contradiction. This shows $\text{meas}\{A_1\} = 0$ i.e. $\eta(x) = 0$ a.e. on $\mathbb{R}^N$. By the equivalency of all norms in $\tilde{E}$, there exists $C > 0$ such that

$$\|v\|_2^2 \geq C\|v\|^2, \qquad \forall v \in \tilde{E}.$$

Hence

$$0 = \lim_{n \to \infty} \|\eta_n\|_2^2 \geq C \lim_{n \to +\infty} \|\eta_n\|^2 = C,$$

a contradiction. This completes the proof. □

Now, we prove our main result.

Proof of Theorem 1.1. Let $\Psi = J$, $X = E$, $Y = Y_\kappa$ and $Z = Z_\kappa$. Obviously, $J(0) = 0$ and (F_3) implies that J is even. By Lemma 2.7, 3.2 and Lemma 3.3, all conditions of Lemma 2.5 are satisfied. Thus, problem (2.6) has infinitely many nontrivial solutions sequence $\{v_n\}$ such that $J(v_n) \to \infty$ as $n \to \infty$. Namely, problem (1.1) also has infinitely many solutions sequence $\{u_n\}$ such that $I(u_n) \to \infty$ as $n \to \infty$. □

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Nonlinear resonant problems with an indefinite potential and concave boundary condition

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Abstract. We consider a nonlinear elliptic problem driven by the p-Laplacian plus and indefinite potential term. The reaction is $(p-1)$-linear and resonant and the boundary term is concave. The problem is nonparametric. Using variational tools, together with truncation and perturbation techniques and critical groups, we show that the problem has at least three nontrivial smooth solutions.

Keywords: resonant reaction, concave boundary term, critical group, nonlinear regularity, multiple solutions.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we deal with the following nonlinear boundary value problem

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = f(z,u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} = \beta(z)|u|^{q-2}u & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

with $1 < q < p$.

In this problem, Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(|Du|^{p-2}Du\right) \quad \text{for all } u \in W^{1,p}(\Omega),\ 1 < p < N.$$

This problem has three special features which make its study interesting. The first feature is that the potential coefficient $\xi \in L^\infty(\Omega)$ is indefinite (that is, sign changing) and so the left hand side of the problem is noncoercive. The second feature is that the forcing term $f(z,x)$ which is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \mapsto f(z,x)$ is continuous) asymptotically as $x \to \pm\infty$ is resonant with respect to

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the principal eigenvalue of the differential operator $u \mapsto -\Delta_p u + \xi(z)|u|^{p-2}u$ with Neumann boundary condition. So, the problem is resonant and as it is well-known such problems are more difficult to deal with. The third feature is that combined with the resonant reaction, we have a concave boundary term (since $\beta(z) \geq 0$ for all $z \in \partial\Omega$ and $1 < q < p$). Therefore problem (1.1) is a variant of the classical *concave-convex* problem, in which the convex ($(p-1)$-superlinear) term is replaced by a resonant ($(p-1)$-linear) term and the *concave* contribution comes from the boundary condition. Problems with such *competition* phenomena, were studied recently by Abreu–Madeira [1], Hu–Papageorgiou [6], Papageorgiou–Rădulescu [9], Papageorgiou–Scapellato [12] and Sabina de Lis–Segura de Leon [14]. All these works deal with parametric problems. The presence of a parameter in the problem, makes the analysis easier, since by varying and restricting the parameter, we are able to satisfy the relevant geometry in order to apply the minimax theorems of critical point theory. In our problem there is no parameter. In addition, in all the aforementioned works the reaction is $(p-1)$-superlinear and so do not cover the resonant case treated here.

In the boundary condition, $\frac{\partial u}{\partial n_p}$ denotes the conormal derivative of $u \in W^{1,p}(\Omega)$. It is interpreted using the nonlinear Green's identity (see [11, p. 35]) and if $u \in W^{1,p}(\Omega) \cap C^{0,1}(\overline{\Omega})$, then

$$\frac{\partial u}{\partial n_p} = |Du|^{p-2}(Du, n)_{\mathbb{R}^N} = |Du|^{p-2}\frac{\partial u}{\partial n},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

Using variational tools based on the critical point theory together with critical groups (Morse theory), we show that problem (1.1) has at least three nontrivial smooth solutions.

2 Mathematical background – hypotheses

The study of problem (1.1), uses the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesgue spaces $L^\tau(\partial\Omega)$, $1 \leq \tau < \infty$.

By $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$. We have

$$\|u\| = \left[\|u\|_p^p + \|Du\|_p^p\right]^{\frac{1}{p}} \quad \text{for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is ordered using the positive (order) cone

$$C_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \ \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \ \text{ for all } z \in \overline{\Omega}\right\}.$$

Also by $\sigma(\cdot)$ we denote the $(N-1)$-dimensional Hausdorff (surface) measure on $\partial\Omega$. Using this measure, we can define the boundary Lebesgue spaces $L^\tau(\partial\Omega)$, $1 \leq \tau < \infty$. By $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ we denote the *trace map*. This map is linear, compact and $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. So, the trace map defines boundary values for all Sobolev functions. In the sequel, we drop the use of the trace map $\gamma_0(\cdot)$ and all restrictions of Sobolev functions on $\partial\Omega$, are interpreted in the sense of traces.

Let $\langle\cdot,\cdot\rangle$ denote the duality brackets for the pair $(W^{1,p}(\Omega), W^{1,p}(\Omega)^*)$ and consider the map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ to be the nonlinear operator defined by

$$\langle A(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W^{1,p}(\Omega).$$

From Gasiński–Papageorgiou [5] (p. 279), we have that this map is:

- monotone, continuous (hence maximal monotone too) and maps bounded sets to bounded sets;
- it is of type $(S)_+$, that is,

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } \limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leq 0$$
$$\text{imply that}$$
$$u_n \to u \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty.$$

Let $\xi \in L^\infty(\Omega)$ and consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \widehat{\lambda}|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an *eigenvalue*, if (2.1) admits a nontrivial solution $\widehat{u} \in W^{1,p}(\Omega)$ known as an eigenfunction corresponding to the eigenvalue $\widehat{\lambda}$.

Problem (2.1) was studied by Fragnelli–Mugnai–Papageorgiou [3] (Robin problem) and Mugnai–Papageorgiou [8] (Neumann problem), who proved that there is a smallest eigenvalue $\widehat{\lambda}_1 \in \mathbb{R}$ with the following properties:

(a) $\widehat{\lambda}_1$ is isolated, that is, if $\widehat{\sigma}(p)$ denotes the spectrum of (2.1), then we can find $\epsilon > 0$ small such that $(\widehat{\lambda}_1, \widehat{\lambda}_1 + \epsilon) \cap \widehat{\sigma}(p) = \varnothing$.

(b) $\widehat{\lambda}_1$ is simple, that is, if $\widehat{u}, \widehat{v} \in W^{1,p}(\Omega)$ are eigenfunctions corresponding to $\widehat{\lambda}_1$, then $\widehat{u} = \vartheta\widehat{v}$ for some $\vartheta \in \mathbb{R} \setminus \{0\}$.

(c) If $\gamma(u) = \|Du\|_p^p + \int_\Omega \xi(z)|u|^p\, dz$ for all $u \in W^{1,p}(\Omega)$, then

$$\widehat{\lambda}_1 = \inf\left[\frac{\gamma(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega),\ u \neq 0\right]. \tag{2.2}$$

In (2.2) the infimum is realized on the corresponding one dimensional eigenspace (see (b)). Then, it follows that the elements of this eigenspace have fixed sign. By $\widehat{u}_1 \in W^{1,p}(\Omega)$ we denote the positive, L^p-normalized (that is, $\|\widehat{u}_1\|_p = 1$) eigenfunction corresponding to $\widehat{\lambda}_1$. The nonlinear regularity theory of Lieberman [7] and the nonlinear maximum principle (see, for example, Gasiński–Papageorgiou [4], p. 738), imply that $\widehat{u}_1 \in \mathrm{int}C_+$. We mention that $\widehat{\lambda}_1$ is the only eigenvalue with eigenfunctions of constant sign. All other eigenvalues have nodal (that is, sign changing) eigenfunctions. Note that using the Ljusternik–Schnirelmann minimax scheme, we can generate a whole strictly increasing sequence $\{\widehat{\lambda}_k\}_{k\geq 1}$ of eigenvalues such that $\widehat{\lambda}_k \to +\infty$ as $k \to +\infty$. We do not know if this sequence exhausts $\widehat{\sigma}(p)$.

Let X be a Banach space and $\varphi \in C^1(X, \mathbb{R})$, $c \in \mathbb{R}$. We introduce the following two sets

$$\varphi^c = \{u \in X : \varphi(u) \leq c\},$$
$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \quad \text{(the critical set of } \varphi).$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0$. By $H_k(Y_1, Y_2)$ we denote the k^{th}-relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. If $u \in K_\varphi$ is isolated and $c = \varphi(u)$, then the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

with U being a neighborhood of u such that $\varphi^c \cap U \cap K_\varphi = \{u\}$. The excision property of singular homology, implies that the above definition is independent of the isolating neighborhood.

Finally, we fix some basic notation. Given $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then, for $u \in W^{1,p}(\Omega)$, we define $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We have

$$u^\pm \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

If $u, v \in W^{1,p}(\Omega)$ and $u \leq v$, then

$$[u, v] = \left\{ h \in W^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega \right\}.$$

Our hypotheses on the data of problem (1.1) are the following:

H_0: $\xi \in L^\infty(\Omega)$, $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$ and $\beta(z) > 0$ for all $z \in \partial\Omega$.

H_1: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \leq a(z)[1 + |x|^{p-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$;

(ii) if $F(z, x) = \int_0^x f(z, s)\, ds$, then $\lim_{x \to \pm\infty} \dfrac{pF(z, x)}{|x|^p} \leq \widehat{\lambda}_1(p)$ uniformly for a.a. $z \in \Omega$;

(iii) there exists $\tau \in (q, p)$ such that

$$0 < \gamma_0 \leq \liminf_{x \to \pm\infty} \frac{f(z, x)x - pF(z, x)}{|x|^\tau} \text{ uniformly for a.a. } z \in \Omega;$$

(iv) there exist $\delta_0 > 0$, $\widehat{c} > \|\xi^-\|_\infty$ and $\mu \in [q, p)$ such that

$$\widehat{c}|x|^p \leq F(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta_0$$

and

$$\mu F(z, x) - f(z, x)x \geq 0 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta_0.$$

Remarks 2.1. Hypotheses H_1(i),(ii), imply that the reaction $f(z, \cdot)$ is $(p-1)$-linear as $x \to \pm\infty$ and the problem is resonant with respect to $\widehat{\lambda}_1(p)$. Note that the resonance condition (hypothesis H_1(ii)) is formulated in terms of the primitive $F(z, x)$ which is more general.

3 Solutions of constant sign

In this section, we prove the existence of two nontrivial smooth solutions of constant sign (one positive and the other negative).

To this end, let $\eta > \|\xi\|_\infty$ and consider the following two C^1-functionals $\varphi_\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_\Omega [\xi(z) + \eta]|u|^p\,dz - \frac{1}{q}\int_{\partial\Omega} \beta(z)(u^+)^q\,d\sigma - \int_\Omega \left[F(z,u^+) + \frac{\eta}{p}(u^+)^p\right]dz,$$
$$\varphi_-(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_\Omega [\xi(z) + \eta]|u|^p dz + \frac{1}{q}\int_{\partial\Omega} \beta(z)(u^-)^q d\sigma - \int_\Omega \left[F(z,-u^-) - \frac{\eta}{p}(u^-)^p\right]dz,$$

for all $u \in W^{1,p}(\Omega)$.

We show that these functionals are coercive.

Proposition 3.1. *If hypotheses* H_0, H_1 *hold, then the functionals* $\varphi_\pm(\cdot)$ *are both coercive.*

Proof. We do the proof for $\varphi_+(\cdot)$, the proof for $\varphi_-(\cdot)$ being similar.

We argue by contradiction. So, suppose that $\varphi_+(\cdot)$ is not coercive. Then we can find a sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that

$$\|u_n\| \to \infty \quad \text{and} \quad \varphi(u_n) \leq M_1 \quad \text{for some } M_1 > 0, \text{ all } n \in \mathbb{N}. \tag{3.1}$$

Then we have

$$\begin{aligned} M_1 &\geq \varphi_+(u_n) \\ &= \frac{1}{p}\left[\|Du_n^+\|_p^p + \int_\Omega \xi(z)(u_n^+)^p\,dz\right] + \frac{1}{p}\left[\|Du_n^-\|_p^p + \int_\Omega [\xi(z)+\eta](u_n^-)^p\,dz\right] - \\ &\quad - \frac{1}{q}\int_{\partial\Omega} \beta(z)(u_n^+)^q\,d\sigma - \int_\Omega F(z,u_n^+)\,dz \\ &\geq \frac{1}{p}\left[\|Du_n^+\|_p^p + \int_\Omega \xi(z)(u_n^+)^p\,dz\right] - \frac{1}{q}\int_{\partial\Omega} \beta(z)(u_n^+)^q\,d\sigma - \int_\Omega F(z,u_n^+)\,dz \\ &\quad (\text{since } \eta > \|\xi\|_\infty). \end{aligned} \tag{3.2}$$

We will use (3.2) to show that $\{u_n^+\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. We proceed indirectly. So, suppose that at least for a subsequence, we have

$$\|u_n^+\| \to \infty \quad \text{as } n \to \infty. \tag{3.3}$$

Let $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. We have $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^p(\Omega) \text{ and in } L^p(\partial\Omega); \quad y \geq 0. \tag{3.4}$$

From (3.2) we have

$$\frac{1}{p}\left[\|Dy_n\|_p^p + \int_\Omega \xi(z)y_n^p\,dz\right] - \frac{1}{q\|u_n^+\|^{p-q}}\int_{\partial\Omega} \beta(z)y_n^q\,d\sigma - \int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^p}\,dz \leq \frac{M_1}{\|u_n^+\|^p}, \tag{3.5}$$

for all $n \in \mathbb{N}$

Hypothesis H_1(i) implies that

$$\begin{aligned} &|F(z,x)| \leq c_1[1 + |x|^p] \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_1 > 0, \\ \Rightarrow\quad &\left\{\frac{F(\cdot,u_n^+(\cdot))}{\|u_n^+\|^p}\right\}_{n\geq 1} \subseteq L^1(\Omega) \quad \text{is uniformly integrable.} \end{aligned}$$

Then, by the Dunford–Pettis theorem (see Papageorgiou–Winkert [13], Theorem 4.1.18, p. 289), we have that $\{\frac{F(\cdot,u_n^+(\cdot))}{\|u_n^+\|^p}\}_{n\geq 1} \subseteq L^1(\Omega)$ is relatively weakly compact. Then, by the Eberlein–Smulian theorem and by passing to a subsequence if necessary, we have

$$\frac{F(\cdot,u_n^+(\cdot))}{\|u_n^+\|^p} \xrightarrow{w} \frac{1}{p}\vartheta(\cdot)y(\cdot)^p \quad \text{in } L^1(\Omega) \text{ as } n\to\infty \tag{3.6}$$

with $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leq \widehat{\lambda}_1(p)$ for a.a. $z\in\Omega$
(see hypothesis H_1(ii) and Aizicovici–Papageorgiou–Staicu [2], proof of Proposition 16).

We return to (3.5), pass to the limit as $n\to\infty$ and use (3.4), (3.3), (3.6) and the fact that $q<p$, to obtain

$$\|Dy\|_p^p + \int_\Omega \xi(z)y^p\,\mathrm{d}z \leq \int_\Omega \vartheta(z)y^p\,\mathrm{d}z. \tag{3.7}$$

First suppose that $\vartheta \not\equiv \widehat{\lambda}_1(p)$ (see (3.6)). Then from (3.7) and Mugnai–Papageorgiou [8] (Lemma 4.11), we have

$$\begin{aligned} & c_2\|y\|^p \leq 0 \quad \text{for some } c_2>0,\\ \Rightarrow\quad & y=0. \end{aligned} \tag{3.8}$$

Then from (3.5), (3.7), (3.8), (3.4) and (3.6), we obtain

$$\begin{aligned} & \|Du_n\|_p \to 0,\\ \Rightarrow\quad & y_n\to 0 \quad \text{in } W^{1,p}(\Omega), \end{aligned}$$

a contradiction since $\|y_n\|=1$ for all $n\in\mathbb{N}$.

Next we suppose that $\vartheta(z)=\widehat{\lambda}_1(p)$ for a.a. $z\in\Omega$. Then from (3.7) and (2.2), we have that

$$y=\mu\widehat{u}_1(p) \quad \text{with } \mu\geq 0 \text{ (recall that } y\geq 0).$$

If $\mu=0$, then $y=0$ and as above, we show that

$$y_n\to 0 \quad \text{in } W^{1,p}(\Omega),$$

a contradiction to the fact that $\|y_n\|=1$ for all $n\in\mathbb{N}$. So, suppose $\mu>0$. Then $y\in \operatorname{int}C_+$. This implies that

$$u_n^+(z)\to+\infty \quad \text{for a.a. } z\in\Omega. \tag{3.9}$$

From (3.2) we have

$$M_1 \geq \frac{1}{p}\int_\Omega \left[\widehat{\lambda}_1(p)(u_n^+)^p - pF(z,u_n^+)\right]\mathrm{d}z - \frac{1}{q}\int_{\partial\Omega}\beta(z)(u_n^+)^q\,\mathrm{d}\sigma$$

(see (3.7), (2.2) and recall that $\eta>\|\eta\|_\infty$),

$$\Rightarrow\quad \frac{M_1}{\|u_n^+\|^\tau} \geq \frac{1}{p}\int_\Omega \frac{\left[\widehat{\lambda}_1(p)(u_n^+)^p - pF(z,u_n^+)\right]}{(u_n^+)^p}y_n^p\,\mathrm{d}z - \frac{1}{q\|u_n^+\|^{p-q}}\int_{\partial\Omega}\beta(z)y_n^q\,\mathrm{d}\sigma, \tag{3.10}$$

for all $n\in\mathbb{N}$.

On $\mathring{\mathbb{R}}_+=(0,\infty)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{F(z,x)}{x^p}\right] = \frac{f(z,x)x^p - px^{p-1}F(z,x)}{x^{2p}} = \frac{f(z,x)x-pF(z,x)}{x^{p+1}}.$$

On account of hypothesis H_1(iii), we can find $\gamma_1 \in (0, \gamma_0)$ and $M_2 > 0$ such that

$$\begin{aligned}
& \frac{f(z,x)x - pF(z,x)}{x^{p+1}} \geq \frac{\gamma_1}{x^{p+1-\tau}} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M_2, \\
\Rightarrow \quad & \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{F(z,x)}{x^p}\right] \geq \frac{\gamma_1}{x^{p+1-\tau}} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M_2, \\
\Rightarrow \quad & \frac{F(z,v)}{v^p} - \frac{F(z,x)}{x^p} \geq \frac{\gamma_1}{p-\tau}\left[\frac{1}{x^{p-\tau}} - \frac{1}{v^{p-\tau}}\right] \quad \text{for a.a. } z \in \Omega, \text{ all } v \geq x \geq M_2.
\end{aligned}$$

Passing to the limit as $v \to \infty$ and since $\frac{F(z,v)}{v^p} \to \frac{1}{p}\widehat{\lambda}_1(p)$ as $v \to +\infty$, we obtain

$$\begin{aligned}
& \frac{\widehat{\lambda}_1(p)}{p} - \frac{F(z,x)}{x^p} \geq \frac{\gamma_1}{p-\tau} \cdot \frac{1}{x^{p-\tau}} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M_2, \\
\Rightarrow \quad & \frac{\widehat{\lambda}_1(p)x^p - pF(z,x)}{x^\tau} \geq \frac{p\gamma_1}{p-\tau} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M_2, \\
\Rightarrow \quad & \liminf_{x \to +\infty} \frac{\widehat{\lambda}_1(p)x^p - pF(z,x)}{x^\tau} \geq \frac{p\gamma_1}{p-\tau} > 0 \quad \text{uniformly for a.a. } z \in \Omega. \qquad (3.11)
\end{aligned}$$

Returning to (3.10), passing to the limit as $n \to \infty$ and using (3.9), (3.11) and Fatou's lemma, we obtain

$$0 \geq \liminf_{n \to \infty} \int_\Omega \frac{\left[\widehat{\lambda}_1(p)(u_n^+)^p - pF(z,u_n^+)\right]}{(u_n^+)^p} y_n^p \,\mathrm{d}z > 0$$

(recall that $q < p$ and see (3.3)),

a contradiction. We infer that

$$\{u_n^+\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \quad \text{is bounded.} \qquad (3.12)$$

From (3.1) and (3.12), we have

$$\begin{aligned}
& \frac{1}{p}\left[\|Du_n^-\|_p^p + \int_\Omega [\xi(z) + \eta](u_n^-)^p \,\mathrm{d}z\right] \leq M_3 \quad \text{for some } M_3 > 0, \text{ all } n \in \mathbb{N}, \\
\Rightarrow \quad & c_3\|u_n^-\|^p \leq M_3 \quad \text{for some } c_3 > 0, \text{ all } n \in \mathbb{N}, \\
\Rightarrow \quad & \{u_n^-\} \subseteq W^{1,p}(\Omega) \quad \text{is bounded.} \qquad (3.13)
\end{aligned}$$

From (3.12) and (3.13) it follows that

$$\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \quad \text{is bounded,}$$

which contradicts (3.1). This proves that $\varphi_+(\cdot)$ is coercive.

In a similar fashion we show that $\varphi_-(\cdot)$ is coercive too. □

Now we are ready to produce the two constant sign solutions.

Proposition 3.2. *If hypotheses* H_0, H_1 *hold, then problem* (1.1) *has at least two constant sign smooth solutions*

$$u_0 \in \mathrm{int}C_+ \quad \textit{and} \quad v_0 \in -\mathrm{int}C_+.$$

Proof. From Proposition 3.1 we know that $\varphi_+(\cdot)$ is coercive. Also by the Sobolev embedding theorem and the compactness of the trace map, we see that $\varphi_+(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\varphi_+(u_0) = \min\left[\varphi_+(u) : u \in W^{1,p}(\Omega)\right]. \tag{3.14}$$

Since $\widehat{u}_1(p) \in \operatorname{int} C_+$, we can choose $t \in (0,1)$ small such that

$$0 < t\widehat{u}_1(p)(z) \le \delta_0 \quad \text{for all } z \in \overline{\Omega},$$

with $\delta_0 > 0$ as in hypothesis H_1(iv). We have

$$0 \le F(z, t\widehat{u}_1(p)(z)) \quad \text{for a.a. } z \in \Omega. \tag{3.15}$$

Then we have

$$\varphi_+(t\widehat{u}_1(p)) \le \frac{t^p}{p}\widehat{\lambda}_1(p) - \frac{t^q}{q}\int_{\partial\Omega}\beta(z)\widehat{u}_1(p)^q\,\mathrm{d}\sigma \quad \text{(see (3.15))}.$$

Since $q < p$, choosing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned} & \varphi_+(t\widehat{u}_1(p)) < 0, \\ \Rightarrow\quad & \varphi_+(u_0) < 0 = \varphi_+(0) \quad \text{(see (3.14))}, \\ \Rightarrow\quad & u_0 \ne 0. \end{aligned}$$

From (3.14), we have

$$\begin{aligned} & \varphi'_+(u_0) = 0, \\ \Rightarrow\quad & \langle A(u_0), h\rangle + \int_\Omega [\xi(z)+\eta]|u_0|^{p-2}u_0 h\,\mathrm{d}z \\ & \quad = \int_{\partial\Omega}\beta(z)(u_0^+)^{q-1}h\,\mathrm{d}\sigma + \int_\Omega [f(z,u_0^+) + \eta(u_0^+)^{p-1}]h\,\mathrm{d}z \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.16}$$

In (3.16) we choose $h = -u_n^- \in W^{1,p}(\Omega)$ and obtain

$$\begin{aligned} & \|Du_0^-\|_p^p + \int_\Omega [\xi(z)+\eta](u_0^-)^p\,\mathrm{d}z = 0, \\ \Rightarrow\quad & c_4\|u_0^-\|^p \le 0 \quad \text{for some } c_4 > 0 \text{ (since } \eta > \|\xi\|_\infty), \\ \Rightarrow\quad & u_0 \ge 0,\ u_0 \ne 0. \end{aligned}$$

Then from (3.16) we have

$$\begin{cases} -\Delta_p u_0(z) + \xi(z)u_0(z)^{p-1} = f(z, u_0(z)) & \text{for a.a. } z \in \Omega, \\ \dfrac{\partial u_0}{\partial n_p} = \beta(z)u_0^{q-1} & \text{on } \partial\Omega. \end{cases} \tag{3.17}$$

From (3.17) and Proposition 2.10 of Papageorgiou–Rădulescu [10] (see also Theorem 4.1 of Winkert [15]), we have that $u_0 \in L^\infty(\Omega)$. Then Theorem 2 of Lieberman [7], implies that $u_0 \in C_+ \setminus \{0\}$.

Let $\rho = \|u_0\|_\infty$. We can find $\widehat{\xi}_\rho > 0$ such that $f(z,x)x + \widehat{\xi}_\rho|x|^p \geq 0$ for a.a. $z \in \Omega$, all $|x| \leq \rho$. Then from (3.17) we have

$$\begin{aligned} & -\Delta_p u_0(z) + \left[\xi(z) + \widehat{\xi}_\rho\right] u_0(z)^{p-1} \geq 0 \quad \text{for a.a. } z \in \Omega \text{ (see hypothesis H}_1\text{(iv))}, \\ \Rightarrow \quad & \Delta_p u_0(z) \leq \left[\|\xi\|_\infty + \widehat{\xi}_\rho\right] u_0(z)^{p-1} \quad \text{for a.a. } z \in \Omega, \\ \Rightarrow \quad & u_0 \in \operatorname{int} C_+ \quad \text{(by the nonlinear maximum principle; see [4, p. 738]).} \end{aligned}$$

Similarly working this time with the functional $\varphi_-(\cdot)$, we obtain a negative solution $v_0 \in -\operatorname{int} C_+$ for problem (1.1). ☐

It is easy to check that

$$K_{\varphi_+} \subseteq \operatorname{int} C_+ \cup \{0\} \quad \text{and} \quad K_{\varphi_-} \subseteq (-\operatorname{int} C_+) \cup \{0\}.$$

So, we may assume that

$$K_{\varphi_+} = \{0, u_0\} \quad \text{and} \quad K_{\varphi_-} = \{0, v_0\}, \tag{3.18}$$

or otherwise we already have a third nontrivial smooth solution which in fact has fixed sign. So, we are done. In the next section we produce a third nontrivial smooth solution for problem (1.1).

4 Three nontrivial solutions

Starting from (3.18), we introduce the following truncation-perturbation of $f(z,\cdot)$ (as before $\eta > \|\xi\|_\infty$):

$$\widehat{f}(z,x) = \begin{cases} f(z,v_0(z)) + \eta|v_0(z)|^{p-2}v_0(z) & \text{if } x < v_0(z), \\ f(z,x) + \eta|x|^{p-2}x & \text{if } v_0(z) \leq x \leq u_0(z), \\ f(z,u_0(z)) + \eta u_0(z)^{p-1} & \text{if } u_0(z) < x. \end{cases} \tag{4.1}$$

We also consider the positive and negative truncations of $\widehat{f}(z,x)$, namely the functions

$$\widehat{f}_\pm(z,x) = \widehat{f}(z, \pm x^\pm). \tag{4.2}$$

It is clear that $\widehat{f}$ and $\widehat{f}_\pm$ are all three Carathéodory functions. We see that

$$\widehat{F}(z,x) = \int_0^x \widehat{f}(z,s)\,\mathrm{d}s \quad \text{and} \quad \widehat{F}_\pm(z,x) = \int_0^x \widehat{f}_\pm(z,s)\,\mathrm{d}s.$$

We also introduce similar truncations of the boundary term:

$$\widehat{g}(z,x) = \begin{cases} \beta(z)|v_0(z)|^{q-2}v_0(z) & \text{if } x < v_0(z), \\ \beta(z)|x|^{q-2}x & \text{if } v_0(z) \leq x \leq u_0(z), \\ \beta(z)u_0(z)^{q-1} & \text{if } u_0(z) < x, \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathbb{R}. \tag{4.3}$$

We also consider the positive and negative truncations of $g(z,\cdot)$, namely the functions

$$\widehat{g}_\pm(z,x) = \widehat{g}(z, \pm x^\pm). \tag{4.4}$$

Evidently $\widehat{g}$ and $\widehat{g}_\pm$ are all three Carathéodory functions on $\partial\Omega \times \mathbb{R}$. We set

$$\widehat{G}(z,x) = \int_0^x \widehat{g}(z,s)\,\mathrm{d}s \quad \text{and} \quad \widehat{G}_\pm(z,x) = \int_0^x \widehat{g}_\pm(z,s)\,\mathrm{d}s.$$

We introduce the C^1-functionals $\widehat{\psi}, \widehat{\psi}_\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned}
\widehat{\psi}(u) &= \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_\Omega [\xi(z)+\eta]|u|^p\,\mathrm{d}z - \int_\Omega \widehat{F}(z,u)\,\mathrm{d}z - \int_{\partial\Omega} \widehat{G}(z,u)\,\mathrm{d}\sigma,\\
\widehat{\psi}_\pm(u) &= \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_\Omega [\xi(z)+\eta]|u|^p\,\mathrm{d}z - \int_\Omega \widehat{F}_\pm(z,u)\,\mathrm{d}z - \int_{\partial\Omega} \widehat{G}_\pm(z,u)\,\mathrm{d}\sigma,
\end{aligned}$$

for all $u \in W^{1,p}(\Omega)$.

Finally, let $\varphi : W^{1,p}(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_\Omega \xi(z)|u|^p\,\mathrm{d}z - \int_\Omega F(z,u)\,\mathrm{d}z - \frac{1}{q}\int_{\partial\Omega} \beta(z)|u|^q\,\mathrm{d}\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

We have that $\varphi \in C^1(W^{1,p}(\Omega))$. Also

$$K_\varphi = \text{set of solutions of problem (1.1)}, \tag{4.5}$$

while from (4.3), (4.4) and the nonlinear regularity theory [7], we have

$$K_{\widehat{\psi}} \subseteq [v_0,u_0] \cap C^1(\overline{\Omega}), \quad K_{\widehat{\psi}_+} \subseteq [0,u_0] \cap C_+, \quad K_{\widehat{\psi}_-} \subseteq [v_0,0] \cap C_+. \tag{4.6}$$

Note that

$$\varphi|_{[v_0,u_0]} = \widehat{\psi}|_{[v_0,u_0]} \quad \text{and} \quad \varphi'|_{[v_0,u_0]} = \widehat{\psi}'|_{[v_0,u_0]}, \tag{4.7}$$

$$\varphi|_{[0,u_0]} = \varphi_+|_{[0,u_0]} = \widehat{\psi}_+|_{[0,u_0]} \quad \text{and} \quad \varphi'|_{[0,u_0]} = \varphi'_+|_{[0,u_0]} = \widehat{\psi}'_+|_{[0,u_0]}, \tag{4.8}$$

$$\varphi|_{[v_0,0]} = \varphi_-|_{[v_0,0]} = \widehat{\psi}_-|_{[v_0,0]} \quad \text{and} \quad \varphi'|_{[v_0,0]} = \varphi'_-|_{[v_0,0]} = \widehat{\psi}'_-|_{[v_0,0]}. \tag{4.9}$$

From (4.5) we see that we may assume that K_φ is finite or otherwise we already have an infinity of nontrivial smooth solutions for problem (1.1) and so we are done. Combining this fact with (4.6) and (4.7), we see that $K_{\widehat{\psi}}$ is finite too. Moreover, from (3.18), (4.6), (4.8), (4.9) we infer that

$$K_{\widehat{\psi}} \subseteq [v_0,u_0] \cap C^1(\overline{\Omega}) \text{ is finite}, \quad K_{\widehat{\psi}_+} = \{0,u_0\}, \quad K_{\widehat{\psi}_-} = \{0,v_0\}. \tag{4.10}$$

These observations permit the consideration of the critical groups of φ and $\widehat{\psi}$ at $u = 0$ and for these groups we have the following result.

Proposition 4.1. *If hypotheses* H_0, H_1 *hold, then* $C_k(\varphi,0) = C_k(\widehat{\psi},0)$ *for all* $k \in \mathbb{N}_0$.

Proof. Recall that we assume that K_φ is finite. We consider the homotopy $\widehat{h}(t,u)$ defined by

$$\widehat{h}(t,u) = t\widehat{\psi}(u) + (1-t)\varphi(u) \quad \text{for all } (t,u) \in [0,1] \times W^{1,p}(\Omega).$$

Suppose we could find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that

$$t_n \to t \in [0,1], \quad u_n \to 0 \text{ in } W^{1,p}(\Omega), \quad \widehat{h}'_u(t_n,u_n) = 0 \text{ for all } n \in \mathbb{N}. \tag{4.11}$$

From the equation in (4.11), we have

$$\begin{cases} -\Delta_p u_n(z) + [\xi(z) + t_n\eta]|u_n(z)|^{p-2}u_n(z) & \\ = t_n\widehat{f}(z,u_n(z)) + (1-t_n)f(z,u_n(z)) & \text{for a.a. } z \in \Omega, \\ \dfrac{\partial u_n}{\partial n_p} = t_n\widehat{g}(z,u_n) + (1-t_n)\beta(z)|u_n|^{q-2}u_n & \text{on } \partial\Omega. \end{cases} \tag{4.12}$$

From (4.12) and Proposition 2.10 of Papageorgiou-Rădulescu [10], we can find $c_5 > 0$ such that

$$\|u_n\|_\infty \le c_5 \quad \text{for all } n \in \mathbb{N}.$$

Then from Theorem 2 of Lieberman [7], we see that there exist $\alpha_0 \in (0,1)$ and $c_6 > 0$ such that

$$u_n \in C^{1,\alpha_0}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha_0}(\overline{\Omega})} \le c_6 \quad \text{for all } n \in \mathbb{N}. \tag{4.13}$$

From (4.13), the compact embedding of $C^{1,\alpha_0}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and (4.11) we infer that

$$u_n \to 0 \quad \text{in } C^1(\overline{\Omega}) \text{ as } n \to \infty. \tag{4.14}$$

Then, on account of (4.14), we can find $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} & u_n \in [v_0, u_0], \quad \text{for all } n \ge n_0, \\ \Rightarrow \quad & \{u_n\}_{n\ge n_0} \subseteq K_\varphi \quad \text{(see (4.7) and (4.10))}, \end{aligned}$$

which contradicts our assumption that K_φ is finite. Therefore (4.11) can not occur and then the homotopy invariance property of critical groups (see Papageorgiou–Rădulescu–Repovš [11, Theorem 6.3.6, p. 505]), implies that

$$C_k(\varphi,0) = C_k(\widehat{\psi},0) \quad \text{for all } k \in \mathbb{N}_0. \qquad \square$$

Next we compute the critical groups of φ at $u = 0$.

Proposition 4.2. *If hypotheses* H_0, H_1 *hold, then* $C_k(\varphi,0) = 0$ *for all* $k \in \mathbb{N}_0$.

Proof. On account of hypotheses H_1(i),(iv), we have

$$F(z,x) \ge -c_7|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \tag{4.15}$$

with $c_7 > 0$ and $r > p$. Then, using (4.15), for every $u \in W^{1,p}(\Omega)$ and every $t > 0$, we have

$$\varphi(tu) \le t^p c_8\|u\|^p + t^r c_9\|u\|^r - t^q \int_{\partial\Omega} \beta(z)|u|^q \,d\sigma \quad \text{for some } c_8, c_9 > 0.$$

Note that $\int_{\partial\Omega} \beta(z)|u|^q \,d\sigma > 0$. Therefore since $q < p < r$, we can find $t^* = t^*(u) \in (0,1)$ such that

$$\varphi(tu) < 0 \quad \text{for all } t \in (0,t^*). \tag{4.16}$$

Let $u \in W^{1,p}(\Omega)$ with $0 < \|u\| \le 1$, $\varphi(u) = 0$ and $\vartheta \in (\mu,p)$. We have

$$\begin{aligned} \frac{d}{dt}\varphi(tu)\bigg|_{t=1} &= \langle \varphi'(u), u\rangle \quad \text{(by the chain rule)} \\ &= \langle A(u),u\rangle + \int_\Omega \xi(z)|u|^p\,dz - \int_\Omega f(z,u)u\,dz - \int_{\partial\Omega}\beta(z)|u|^q\,d\sigma \\ &= \left[1-\frac{\vartheta}{p}\right]\|Du\|_p^p + \left[1-\frac{\vartheta}{p}\right]\int_\Omega \xi(z)|u|^p\,dz + \left[\frac{\vartheta}{q}-1\right]\int_{\partial\Omega}\beta(z)|u|^q\,d\sigma \\ &\quad + (\vartheta-\mu)\int_\Omega F(z,u)\,dz + \int_\Omega [\mu F(z,u) - f(z,u)u]\,dz \\ &\quad \text{(since } \varphi(u) = 0). \end{aligned} \tag{4.17}$$

By hypothesis H_1(iv), we have that

$$F(z,x) \geq \hat{c}|x|^p \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta_0. \tag{4.18}$$

Combining (4.18) with hypothesis H_1(i) we have that

$$F(z,x) \geq \hat{c}|x|^p - c_{10}|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \tag{4.19}$$

for some $c_{10} > 0$.

In addition, hypotheses H_1(i),(iv) imply that

$$\mu F(z,x) - f(z,x)x \geq -c_{11}|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_{11} > 0. \tag{4.20}$$

We return to (4.17) and use (4.18), (4.19), (4.20) and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu)\bigg|_{t=1} \geq c_{12}\|Du\|_p^p + \left[\hat{c} - \|\xi^-\|_\infty\right] \|u\|_p^p - c_{13}\|u\|^r$$

for some $c_{12}, c_{13} > 0$ (recall that $q < \mu < \vartheta$).

But by hypothesis H_1(iv) we have that $\hat{c} > \|\xi^-\|_\infty$. So, from the above inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu)\bigg|_{t=1} \geq c_{14}\|u\|^p - c_{13}\|u\|^r \quad \text{for some } c_{14} > 0.$$

Since $p < r$, we can find $\rho \in (0,1)$ small such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu)\bigg|_{t=1} > 0 \quad \text{for all } u \in W^{1,p}(\Omega) \text{ with } < \|u\| \leq \rho,\ \varphi(u) = 0. \tag{4.21}$$

Consider a $u \in W^{1,p}(\Omega)$ as in (4.21), namely that

$$0 < \|u\| < \rho \quad \text{and} \quad \varphi(u) = 0.$$

We show that

$$\varphi(tu) \leq 0 \quad \text{for all } t \in [0,1]. \tag{4.22}$$

Suppose that (4.22) is not true. Then we can find $t_0 \in (0,1)$ such that $\varphi(t_0u) > 0$. Since $\varphi(u) = 0$ and $\varphi(\cdot)$ is continuous, by Bolzano's theorem, we can find $\hat{t} \in (t_0, 1]$ such that $\varphi(\hat{t}u) = 0$. We set

$$t^* = \min\left\{\hat{t} \in (t_0, 1] : \varphi(tu) = 0\right\} > t_0 > 0.$$

We have

$$\varphi(tu) > 0 \quad \text{for all } t \in [t_0, t^*). \tag{4.23}$$

If $v = t^*u$, then $0 < \|v\| \leq \rho$ and $\varphi(v) = 0$. So, from (4.21) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu)\bigg|_{t=1} > 0. \tag{4.24}$$

On the other hand

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu)\bigg|_{t=1} = t^*\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu)\bigg|_{t=t^*} = t^* \lim_{t\to(t^*)^-} \frac{\varphi(tu)}{t-t^*} \leq 0 \quad \text{(see (4.23))}. \tag{4.25}$$

Comparing (4.24) and (4.25), we have a contradiction. This proves (4.22).

Recall that K_φ is finite. So, we can always choose $\rho \in (0,1)$ small so that $K_\varphi \cap \overline{B_\rho} = \{0\}$ (recall that $B_\rho = \{u \in W^{1,p}(\Omega) : \|u\| < \rho\}$). Consider the continuous deformation $h_0 : [0,1] \times (\varphi^0 \cap \overline{B_\rho}) \to \varphi^0 \cap \overline{B_\rho}$ defined by

$$h_0(t,u) = (1-t)u \quad \text{for all } (t,u) \in [0,1] \times (\varphi^0 \cap \overline{B_\rho}).$$

On account of (4.22) this deformation is well-defined and shows that $\varphi^0 \cap \overline{B_\rho}$ is contractible in itself.

Let $u \in \overline{B_\rho}$ with $\varphi(u) > 0$. We claim that there is a unique $t(u) \in (0,1)$ such that

$$\varphi(t(u)u) = 0. \tag{4.26}$$

The existence of such $t(u) \in (0,1)$ follows from (4.16) and Bolzano's theorem. For the uniqueness, suppose we could find $0 < t_1 < t_2 < 1$ such that

$$\varphi(t_1 u) = \varphi(t_2 u) = 0. \tag{4.27}$$

Consider the function

$$\eta(t) = \varphi(t t_2 u) \quad \text{for all } t \in [0,1].$$

From (4.27) and (4.22), it follows that that $t = \frac{t_1}{t_2} \in (0,1)$ is a maximizer of $\eta(\cdot)$. Therefore we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\eta(t)\bigg|_{t=\frac{t_1}{t_2}} = 0,\\ \Rightarrow\quad &\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t t_1 u)\bigg|_{t=1} = 0, \end{aligned}$$

which contradicts (4.21). So, $t(u) \in (0,1)$ satisfying (4.26) is unique. Therefore we have

$$\varphi(tu) < 0 \text{ for } t \in (0, t(u)) \quad \text{and} \quad \varphi(tu) > 0 \ \text{ if } t \in (t(u), 1]. \tag{4.28}$$

Then we introduce the function $\lambda : \overline{B_\rho} \setminus \{0\} \to [0,1]$ defined by

$$\lambda(u) = \begin{cases} 1 & \text{if } u \in \overline{B_\rho} \setminus \{0\},\ \varphi(u) \le 0,\\ t(u) & \text{if } u \in \overline{B_\rho} \setminus \{0\},\ \varphi(u) > 0. \end{cases}$$

It is easy to see that $\lambda(\cdot)$ is continuous. So, if we consider the map $k : \overline{B_\rho} \setminus \{0\} \to (\varphi^0 \cap \overline{B_\rho}) \setminus \{0\}$ defined by

$$k(u) = \begin{cases} u & \text{if } u \in \overline{B_\rho} \setminus \{0\},\ \varphi(u) \le 0,\\ \lambda(u)u & \text{if } u \in \overline{B_\rho} \setminus \{0\},\ \varphi(u) > 0, \end{cases}$$

then $k(\cdot)$ is continuous and $k|_{(\varphi^0 \cap \overline{B_\rho}) \setminus \{0\}} = \text{identity}$. It follows that $(\varphi^0 \cap \overline{B_\rho}) \setminus \{0\}$ is a retract of $\overline{B_\rho} \setminus \{0\}$, which is contractible. Therefore $(\varphi^0 \cap \overline{B_\rho}) \setminus \{0\}$ is contractible and so we have

$$\begin{aligned} &H_k(\varphi^0 \cap \overline{B_\rho}, (\varphi^0 \cap \overline{B_\rho}) \setminus \{0\}) = 0 \quad \text{for all } k \in \mathbb{N}_0 \text{ (see [11], p. 469)},\\ \Rightarrow\quad &C_k(\varphi, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \end{aligned}$$

$\square$

Corollary 4.3. *If hypotheses* H_0, H_1 *hold, then* $C_k(\widehat{\psi}, 0) = 0$ *for all* $k \in \mathbb{N}_0$.

Now we are ready for the multiplicity theorem. It is interesting to point out that the solutions we produce are ordered.

Theorem 4.4. *If hypotheses* H_0, H_1 *hold, then problem* (1.1) *has at least three nontrivial smooth solutions*

$$u_0 \in \mathrm{int}C_+, \quad v_0 \in -\mathrm{int}C_+ \quad \textit{and} \quad y_0 \in C^1(\overline{\Omega}),\ v_0 \le y_0 \le u_0.$$

Proof. From Proposition 3.2 we already have two nontrivial constant sign solutions

$$u_0 \in \mathrm{int}C_+ \quad \text{and} \quad v_0 \in -\mathrm{int}C_+.$$

<u>Claim:</u> $u_0 \in \mathrm{int}C_+$ *and* $v_0 \in -\mathrm{int}C_+$ *are local minimizers of* $\widehat{\psi}(\cdot)$.

From (4.1), (4.2), (4.3) and (4.4), we see that $\widehat{\psi}_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$\widehat{\psi}_+(\widetilde{u}_0) = \min\left[\widehat{\psi}_+(u) : u \in W^{1,p}(\Omega.)\right] \tag{4.29}$$

Let $u \in \mathrm{int}C_+$. Since $u_0 \in \mathrm{int}C_+$, we can find $t \in (0,1)$ small such that

$$0 \le tu \le \min\{u_0, \delta_0\}$$

(see Papageorgiou–Rădulescu–Repovš [11], Proposition 4.1.22, p. 274). Then, since $\mu < p$, we have

$$\begin{aligned} & \widehat{\psi}_+(tu) < 0 \quad \text{for } t \in (0,1) \text{ small},\\ \Rightarrow\quad & \widehat{\psi}_+(\widetilde{u}_0) < 0 = \widehat{\psi}_+(0) \quad (\text{see (4.29)}),\\ \Rightarrow\quad & \widetilde{u}_0 \ne 0,\\ \Rightarrow\quad & \widetilde{u}_0 = u_0 \quad (\text{see (4.10) and (4.29)}). \end{aligned}$$

Note that $\widehat{\psi}\big|_{C_+} = \widehat{\psi}_+\big|_{C_+}$. Since $u_0 \in \mathrm{int}C_+$, it follows that

$$\begin{aligned} & u_0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \widehat{\psi}(\cdot),\\ \Rightarrow\quad & u_0 \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \widehat{\psi}(\cdot)\\ & (\text{see Papageorgiou–Rădulescu [10, Proposition 2.12]}). \end{aligned}$$

Similarly for $v_0 \in -\operatorname{int} C_+$ using this time the functional $\psi_-(\cdot)$.
This proves the claim.

Without any loss of generality we may assume that

$$\widehat{\psi}(v_0) \le \widehat{\psi}(u_0).$$

From (4.10), the Claim and Theorem 5.7.6, p. 449, of Papageorgiou–Rădulescu–Repovš [11], we know that we can find $\rho \in (0,1)$ small such that

$$\widehat{\psi}(v_0) \le \widehat{\psi}(u_0) < \inf\left[\widehat{\psi}(u) : \|u - u_0\| = \rho\right] = \widehat{m}_\rho, \quad \|v_0 - u_0\| > \rho. \tag{4.30}$$

Since $\widehat{\psi}(\cdot)$ is coercive, from Proposition 5.1.15, p. 369, of Papageorgiou–Rădulescu–Repovš [11], we have that

$$\widehat{\psi}(\cdot) \text{ satisfies the Palais–Smale condition.} \tag{4.31}$$

Then (4.30) and (4.31) permit the use of the mountain pass theorem. So, we can find $y_0 \in W^{1,p}(\Omega)$ such that

$$y_0 \in K_{\widehat{\psi}} \subseteq [v_0, u_0] \cap C^1(\overline{\Omega}) \text{ (see (4.10))} \quad \text{and} \quad \widehat{m}_\rho \leq \widehat{\psi}(y_0) \text{ (see (4.30))}. \tag{4.32}$$

From (4.30) and (4.32), we have that

$$y_0 \neq u_0 \quad \text{and} \quad y_0 \neq v_0.$$

Moreover, since y_0 is a critical point of $\widehat{\psi}$ of mountain pass type, from Corollary 6.6.9, p. 533, of Papageorgiou–Rădulescu–Repovš [11], we have

$$C_1(\widehat{\psi}, y_0) \neq 0. \tag{4.33}$$

On the other hand, from Corollary 4.3, we have

$$C_k(\widehat{\psi}, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{4.34}$$

Comparing (4.33) and (4.34) we infer that $y_0 \neq 0$. Therefore $y_0 \in C^1(\overline{\Omega})$ is the third nontrivial solution of (1.1) and $v_0(z) \leq y_0(z) \leq u_0(z)$ for all $z \in \overline{\Omega}$. □

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On discreteness of spectrum of a second order differential operator

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Abstract. A new form of a necessary and sufficient conditions for the discreteness of the spectrum of singular operator $-\frac{1}{\rho(x)}(p(x)u')'$, $-\infty \le a \le x \le b \le +\infty$ is obtained. A simpler proof of the necessity is obtained.

Keywords: discreteness of spectrum, second order differential operator.

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1 Introduction

Let $I = (a, b)$ where $-\infty \le a < b \le +\infty$. The differential operator

$$\mathcal{L}u = \frac{1}{\rho}\left(-(pu')' + qu\right), \qquad x \in I = (a, b), \tag{1.1}$$

was the first to be studied from the point of view of the properties of its spectrum, in particular, the discreteness of the spectrum. Recall that the spectrum of an operator A acting in a Hilbert space H is discrete if it consists only of eigenvalues of finite multiplicity [2]. Operator (1.1) is studied in the space $L_2(I, \rho)$ of functions that are square integrable on I with positive weight ρ.

In the case $(a, b) = (-\infty, \infty)$, and $\rho = 1$ the operator $\mathcal{L}u = -u'' + qu$ has discrete spectrum, if [3] $\lim_{x\to\infty} q(x) = +\infty$. It is a sufficient condition. A. M. Molchanov obtained [11] the following necessary and sufficient condition: for any $\delta > 0$

$$\lim_{x\to\infty} \int_x^{x+\delta} q(x)dx = +\infty. \tag{1.2}$$

Note that Molchanov studied an operator in the n-dimensional space R^n. Here we consider only the case when $q = 0$. In this case for the operator*

$$\mathcal{L}u(x) := -\frac{1}{\rho(x)}(p(x)u')', \qquad x \in I = (a, b), \tag{1.3}$$

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*sign := means *equal by definition*

a necessary and sufficient condition is obtained by I. Kac and M. G. Krein [6]. However, the result in [6] is formulated in such a way that equivalence with the form proposed below (Theorem 2.4) is not obvious (see section 7). Note also that the method in [6] pursued other goals, and is more complicated. We use some method (see Lemma 5.2) close to the Glazman splitting method [5]. The essential point here is a simpler proof of necessity (Lemma 4.1). As test functions, sections $G(x,s)$ of the Green function were chosen, where $s \to a$ or $s \to b$. This simplifies the proof of necessity (see below two-sided estimates (4.5) and (4.6)).

In this regard, we have to note the result of M. Sh. Birman [1, p. 148], [5, p. 93] for an even-order equation on semiaxis $[0,\infty)$. For the operator $\mathcal{L}_0 u = -(1/\rho)u''$ this condition has the following form

$$\lim_{s\to\infty} s \int_s^\infty \rho(x)\,dx = 0. \tag{1.4}$$

It is assumed that $\int_0^\infty \rho(x)\,dx < \infty$. If $\int_0^1 \rho(x)\,dx = \infty$, the condition

$$\lim_{s\to 0} s \int_s^1 \rho(x)\,dx = 0 \tag{1.5}$$

together with (1.4) guarantees [10] discreteness of spectrum of $-(1/\rho)u''$. The result of presented article was announced in [9] for a more general functional differential operator of the form

$$\mathcal{L}u(x) := -\frac{1}{\rho(x)}(p(x)u')' + \int_a^b u(s)r(x,ds), \qquad x \in I = (a,b).$$

For simplicity, we omit the integral term here.

2 Assumptions. Conditions of discreteness

For the operator (1.3) assume that the functions $p(x)$ and $\rho(x)$ are measurable and positive almost everywhere on a finite or infinite interval $I := (a,b)$, $-\infty \le a < b \le \infty$. Assume that $1/p$ and ρ are locally on I integrable, that is, for any s_1, s_2, $a < s_1 < s_2 < b$

$$\int_{s_1}^{s_2} \frac{dx}{p(x)} < \infty, \qquad \int_{s_1}^{s_2} \rho(x)\,dx < \infty.$$

Definition 2.1. If for some $s \in I = (a,b)$

$$\int_a^s \rho(x)\,dx = \infty, \qquad \int_a^s \frac{dx}{p(x)} < \infty \tag{2.1}$$

then $\mathcal{L}$ has singularity at the point $x = a$ by $\rho(x)$. If for some $s \in I = (a,b)$

$$\int_a^s \frac{dx}{p(x)} = \infty, \qquad \int_a^s \rho(x)\,dx < \infty \tag{2.2}$$

say that $\mathcal{L}$ has singularity at the point $x = a$ by $p(x)$. Similarly, we can define the singularity at the right end of the interval.

Only one type of singularity at each end of the interval is allowed. It is clear that the singularity at the right end of the interval can be considered similarly to the left end. Moreover, the singularity at the right end can be reduced to the singularity at the left end by the change of variable $x = -x'$. Therefore, one could consider the singularity only at the left end of the interval. Assuming that

$$\int_s^b \frac{dx}{p(x)} < \infty \quad \text{and} \quad \int_s^b \rho(x)\,dx < \infty \qquad (a < s < b) \tag{2.3}$$

and letting

$$\Phi_1(s) := \int_a^s \frac{dx}{p(x)} \int_s^b \rho(x)\,dx, \qquad \Phi_2(s) := \int_a^s \rho(x)\,dx \int_s^b \frac{dx}{p(x)}$$

we have the following theorem.

Theorem 2.2. *For the spectrum of operator* (1.3) *to be discrete, it is necessary and sufficient that at least one of relations*

$$\lim_{s\to a} \Phi_1(s) = 0 \quad \text{or} \quad \lim_{s\to a} \Phi_2(s) = 0$$

be true.

Remark 2.3. If there is a singularity, then one of the integrals $\Phi_1(s)$ or $\Phi_2(s)$ does not exist. Therefore, only one type of singularity is allowed.

However, it is more convenient to represent Theorem 2.2 in a simpler form (Theorem 2.4 below). For this, we consider *both types of singularities* at different ends of the interval simultaneously. The essence of the content of Theorem 2.2 will not change. So, we assume that

$$\int_s^b \rho(x)\,dx < \infty, \qquad \int_a^s \frac{dx}{p(x)} < \infty, \qquad a < s < b \tag{2.4}$$

but

$$\int_a^s \rho(x)\,dx = \infty, \qquad \int_s^b \frac{dx}{p(x)} = \infty. \tag{2.5}$$

Let

$$\Phi(s) := \int_a^s \frac{dx}{p(x)} \int_s^b \rho(x)\,dx. \tag{2.6}$$

Theorem 2.2 takes the following form.

Theorem 2.4. *For discreteness of the spectrum of the operator* (1.3)*, it is necessary and sufficient that*

$$\lim_{s\to a} \Phi(s) = \lim_{s\to b} \Phi(s) = 0.$$

Proof. It follows from Lemma 5.3 and Section 3. □

To simplify the notation, assume that $a = 0$ and $b = l \leq \infty$ (l is the *length of a string*). We use also the boundary condition

$$u(0) = 0. \tag{2.7}$$

Condition (2.7) is not essential for the study of discreteness. It affects the estimate of the first eigenvalue (lower boundary of the spectrum).

3 Variational method

We use the following form of the variational method [8]. In the space $L_2(I,\rho)$ of square integrable functions the scalar product is defined by $(f,g) := \int_I f(x)g(x)\rho(x)\,dx$. Here $I = (a,b) = (0,l)$, $l \le \infty$. The bilinear form

$$[u,v] := \int_0^l p(x)u'(x)v'(x)\,dx \tag{3.1}$$

serves as a scalar product in Hilbert space W of all locally absolutely continuous on $[0,l)$ functions satisfying the boundary condition (2.7). Let $T\colon W \to L_2(I,\rho)$ be defined by the equality $Tu(x) = u(x)$. Note that $T(W)$ is dense in $L_2(I,\rho)$. The equation in variational form

$$[u,v] = (f,Tv) \qquad (\forall v \in W), \tag{3.2}$$

$f \in L_2(I,\rho)$ with respect to u has unique solution $u = T^*f$. Equation (3.2) is equivalent to equation $\mathcal{L}u = f$, where $\mathcal{L} := (T^*)^{-1}$.

If form $[u,v]$ is defined by (3.1), operator $\mathcal{L}$ can be represented by (1.3) under boundary conditions $u(0) = 0$, $pu'\big|_{x=l} = 0$. Thus, eigenvalue problem

$$\mathcal{L}u = \lambda Tu$$

has the representation

$$-\frac{1}{\rho}(pu')' = \lambda u, \qquad u(0) = 0, \qquad pu'\big|_{x=l} = 0. \tag{3.3}$$

Discreteness of spectrum of operator $\mathcal{L}$ is equivalent to compactness of the operator T. If T is compact, the eigenvalue problem (3.3) has a system eigenfunctions u_n that forms an orthogonal basis in the space W. The system Tu_n forms an orthogonal basis in $L_2(I,\rho)$.

4 Auxiliary inequalities

Let $u \in W$ and

$$A_u := \int_I \frac{|u(s)u'(s)|}{\omega(s)}\,ds,$$

where the positive parameter function ω will be defined below. By the Cauchy inequality

$$A_u^2 \le \int_I \frac{u(s)^2}{\omega(s)^2}\frac{ds}{p(s)} \cdot \int_I p(s)u'(s)^2\,ds = B_u \cdot [u,u], \tag{4.1}$$

where $B_u := \int_I \frac{u(s)^2}{\omega(s)^2}\frac{ds}{p(s)}$. Hence and since $u(0) = 0$

$$B_u = 2\int_I \frac{ds}{\omega(s)^2 p(s)} \int_0^s u(x)u'(x)\,dx = 2\int_I u(x)u'(x)dx \int_x^l \frac{ds}{\omega(s)^2 p(s)}.$$

Let the function ω be chosen so that

$$\int_x^l \frac{ds}{\omega(s)^2 p(s)} = \frac{1}{\omega(x)} - \frac{1}{\omega(l)} \le \frac{1}{\omega(x)}. \tag{4.2}$$

Then $B_u \le 2\int_I \frac{|u(x)u'(x)|}{\omega(x)}\,dx = 2A_u$. From here and (4.1) $A_u^2 \le 2A_u[u,u]$ and

$$A_u \le 2[u,u]. \tag{4.3}$$

From (4.2) we obtain $-\frac{1}{\omega^2 p} = -\frac{1}{\omega^2}\omega'$ and

$$\omega(s) = \int_0^s \frac{dx}{p(x)}. \tag{4.4}$$

Lemma 4.1. *Let* $0 < c < l$, $0 < d < l$. *The following inequalities hold:*

$$\sup_{s\in[0,c]}\left(\Phi(s) - \int_0^s \frac{dx}{p(x)}\int_c^l \rho\,dx\right) \le \sup_{\|u\|\le 1}(Tu,Tu)_{[0,c]} \le 4\sup_{s\in[0,c]}\Phi(s), \tag{4.5}$$

$$\sup_{s\in[d,l)}\Phi(s) \le \sup_{\|u\|\le 1}(Tu,Tu)_{[d,l]} \le \Phi(d) + 4\sup_{s\in[d,l)}\Phi(s). \tag{4.6}$$

Proof. The left inequality of (4.5). Let $s \in (0,c]$, $\omega := \int_0^s \frac{dx}{p(x)}$ and

$$u(x) := \begin{cases} \dfrac{1}{\sqrt{\omega}}\displaystyle\int_0^x \frac{dt}{p(t)}, & \text{if } 0 \le x \le s,\\ \sqrt{\omega}, & \text{if } s < x < l. \end{cases}$$

Then $[u,u] = \int_0^s p(x)(u')^2 = \frac{1}{\omega}\int_0^s p(x)\frac{dx}{p(x)^2} = 1$,

$$(Tu,Tu)_{[0,c]} \ge \int_s^c u^2\rho\,dx = \omega\int_s^c \rho\,dx = \Phi(s) - \int_0^s \frac{dx}{p(x)}\int_c^l \rho\,dx.$$

The left inequality of (4.6). Let $s \in [d,l)$, ω and u be defined by the same equalities. Then $[u,u] = 1$,

$$(Tu,Tu)_{[d,l)} \ge \int_s^l u^2\rho\,dx = \omega\int_s^l \rho\,dx = \Phi(s).$$

The right inequality of (4.5). Let $\|u\| \le 1$. By virtue of (4.3) and (4.4)

$$\begin{aligned}\int_0^c (u(x))^2\rho(x)\,dx &= \int_0^c \left(2\int_0^x u(s)u'(s)\,ds\right)\rho(x)\,dx\\ &= 2\int_0^c \frac{u(s)u'(s)}{\omega(s)}\left(\omega(s)\int_s^c \rho(x)\,dx\right)ds\\ &\le 2\sup_{0<s<c}\Phi(s)\int_0^c \frac{u(s)u'(s)}{\omega(s)}\,ds \le 2\sup_{0<s<c}\Phi(s)A_u \le 4\sup_{0<s<c}\Phi(s).\end{aligned}$$

The right inequality of (4.6). Let $\|u\| \le 1$. We have

$$\int_d^l (u(x))^2\rho(x)\,dx = \int_d^l \rho(x)\left((u(d))^2 + 2\int_d^x u(s)u'(s)\,ds\right)dx.$$

Since

$$(u(d))^2 = \left(\int_0^d u'(s)\,ds\right)^2 \le \int_0^d p(s)(u'(s))^2\,ds\int_0^d \frac{ds}{p(s)} \le \int_0^d \frac{ds}{p(s)}$$

we have

$$(u(d))^2\int_d^l \rho\,dx \le \Phi(d).$$

For the second term, in view (4.3)

$$\int_d^l \left(2\int_d^x u(s)u'(s)\,ds\right)\rho(x)\,dx = 2\int_d^l \frac{u(s)u'(s)}{\omega(s)}\left(\omega(s)\int_s^l \rho(x)\,dx\right)ds$$
$$\le 2\sup_{d<s<l}\Phi(s)\int_0^d \frac{u(s)u'(s)}{\omega(s)}\,ds \le 2\sup_{d<s<l}\Phi(s)A_u \le 4\sup_{d<s<l}\Phi(s). \quad \square$$

5 Boundedness and compactness

The boundedness of operator T and its action from space W to space $L_2(I,\rho)$ are necessary for further investigation of the spectrum. The compactness of operator T, as mentioned in Section 3, is equivalent to the discreteness of the spectrum of operator (1.3).

5.1 Boundedness

Since

$$(Tu,Tu) = \int_0^l u^2\rho\,dx = 2\int_0^l \rho(x)\,dx\int_0^x u(s)u'(s)\,ds = 2\int_0^l \frac{u(s)u'(s)}{\omega(s)}\,\omega(s)\int_s^l \rho(x)\,dx\,ds,$$

by virtue of (4.3) and (2.6)

$$(Tu,Tu) \le 4[u,u]\sup_{s\in(0,l)}\omega(s)\int_s^l \rho(x)\,dx = 4[u,u]\sup_{s\in(0,l)}\Phi(s). \tag{5.1}$$

So, the boundedness of function $\Phi(s)$ guarantees the boundedness of operator T. It seems this is necessary condition. Let λ_0 be the lower boundary of spectrum of $\mathcal{L}$. It satisfies the representation

$$(\lambda_0)^{-1} = \sup_{u\neq 0}\frac{(Tu,Tu)}{[u,u]}.$$

From (5.1) we have the estimate

$$(\lambda_0)^{-1} \le 4\sup\Phi(s).$$

5.2 Compactness

- Let $(Tu,Tu)_\Delta := \int_\Delta u^2\rho\,dx$. Below we will use $\Delta = [0,c]$ and $\Delta = [d,l)$.

  Below we use the following compactness criterion [4, p. 268], [7, p. 318].

Theorem 5.1 (I. Gelfand)**.** *For the relative compactness of the set A in a Banach space E, it is necessary and sufficient that for any sequence f_n of linear functionals converging on each element of a Banach space E, the convergence is uniform on the set A.*

The following statement is closed to the localization principle [5].

Lemma 5.2. *The condition*

$$\lim_{c\to 0}\sup_{\|u\|\le 1}(Tu,Tu)_{[0,c]} = 0 \bigwedge \lim_{d\to l}\sup_{\|u\|\le 1}(Tu,Tu)_{[d,l)} = 0 \tag{5.2}$$

is a necessary and sufficient condition for compactness of T.

Proof. Necessity. Suppose $\exists\sigma > 0$, $\exists c_n \to 0$, $\exists u_n$ such that $\|u_n\| = 1$ and

$$(Tu_n, Tu_n)_{\Delta_n} > \sigma,$$

where $\Delta_n := [0, c_n]$. Let $f_n = \chi_{\Delta_n} \frac{1}{\|Tu_n\|_{\Delta_n}} Tu_n$ (χ_{Δ_n} is the characteristic function of the set Δ_n). Since

$$(f_n, z)^2 \le \frac{1}{\|Tu_n\|^2_{\Delta_n}} \int_0^{c_n} u_n^2 \rho\, dx \int_0^{c_n} z^2 \rho\, dx = \int_0^{c_n} z^2 \rho\, dx \to 0$$

(f_n, z) converges for any $z \in L_2(I, \rho)$. But the following contradicts Theorem 5.1:

$$(f_n, Tu_n) = \frac{1}{\|Tu_n\|_{\Delta_n}} \int_0^{c_n} u_n^2 \rho\, dx = \sqrt{\int_0^{c_n} u_n^2 \rho\, dx} > \sqrt{\sigma}.$$

The necessity of the second condition in (5.2) is proved in exactly the same way.

Sufficiency. Let $f_n \in L_2(I, \rho)$ be a sequence such that $(f_n, z) \to 0$ for any $z \in L_2(I, \rho)$. We have to show that $f_n(Tu) = (f_n, Tu) \to 0$ uniformly on $[u, u] \le 1$. First,

$$\left(\int_0^c f_n(x) u(x) \rho(x)\, dx\right)^2 \le \int_0^c f_n(x)^2 \rho(x)\, dx \int_0^c u(x)^2 \rho(x)\, dx \le C \int_0^c u(x)^2 \rho(x)\, dx.$$

From here and by virtue of (5.2)

$$\lim_{c \to 0} \int_0^c f_n(x) u(x) \rho(x)\, dx = 0$$

uniformly on the set $\{(u, n)\colon [u, u] \le 1,\ n = 1, 2, \ldots\}$. Similarly,

$$\lim_{d \to l} \int_d^l f_n(x) u(x) \rho(x)\, dx = 0$$

uniformly on the set $\{(u, n)\colon [u, u] \le 1,\ n = 1, 2, \ldots\}$.

Therefore, it suffices to establish for any $\alpha, \beta \in (0, l)$ uniform on $[u, u] \le 1$ convergence of the sequence $\int_\alpha^\beta f_n(x) u(x) \rho(x)\, dx$. We have

$$\int_\alpha^\beta f_n(x) u(x) \rho(x)\, dx = \int_\alpha^\beta f_n(x) \left(u(\alpha) + \int_\alpha^x u'(s)\, ds\right) \rho(x)\, dx.$$

The first term converges uniformly since $\int_\alpha^\beta f_n(x) \rho(x)\, dx$ converges and

$$(u(\alpha))^2 = \left(\int_0^\alpha u'(x)\, dx\right)^2 \le \int_0^\alpha p(x)(u'(x))^2\, dx \int_0^\alpha \frac{dx}{p(x)} \le [u, u] \int_0^\alpha \frac{dx}{p(x)}.$$

Let us estimate the second term:

$$\begin{aligned}\left(\int_\alpha^\beta f_n(x) \left(\int_\alpha^x u'(s)\, ds\right) \rho(x)\, dx\right)^2 &= \left(\int_\alpha^\beta u'(s)\, ds \int_s^\beta f_n(x) \rho(x)\, dx\right)^2 \\ &\le \int_\alpha^\beta p(s)(u'(s))^2\, ds \int_\alpha^\beta (\varphi_n(s))^2\, ds \le \int_\alpha^l (\varphi_n(s))^2\, ds,\end{aligned}$$

where $\varphi_n(s) = (p(s))^{-1/2} \int_s^\beta f_n(x) \rho(x)\, dx$. Note, that $\varphi_n(s) = (f_n, z_s)$ where $z_s(x) = 0$, if $x \notin [s, l]$, and $z_s(x) = (p(s))^{-1/2}$, if $x \in [s, l]$. Thus $\varphi_n(s) = (f_n, z_s) \to 0$ for all $s \in I$.

Since

$$(\varphi_n(s))^2 \le \frac{1}{p(s)} \int_s^\beta \rho(x)\, dx \int_s^\beta (f_n(x))^2 \rho(x)\, dx \le \|f_n\|^2 \frac{1}{p(s)} \int_s^\beta \rho(x)\, dx$$

by virtue of the Lebesgue theorem $\int_\alpha^\beta (\varphi_n(s))^2\, ds \to 0$. $\square$

Lemma 5.3. *The condition* $\lim_{s\to 0}\Phi(s) = 0$ *and* $\lim_{s\to l}\Phi(s) = 0$ *is a necessary and sufficient condition for compactness of the operator* T.

Proof. It follows from Lemma 5.2 and from inequalities (4.5) and (4.6). For example, consider in detail the proof of the necessity of condition $\lim_{s\to 0}\Phi(s) = 0$. The compactness of operator T implies (5.2). Suppose $\lim_{s\to 0}\Phi(s) = 0$ is not true. Then there are $\varepsilon > 0$ and $s_n \to 0$ such that $\Phi(s_n) \geq \varepsilon$. Let $c > 0$. For some $s_n < c$

$$\Phi(s_n) - \int_0^{s_n} \frac{dx}{p(x)} \int_c^l \rho(x)dx \geq \varepsilon/2.$$

From (4.5) we have $\sup_{\|u\|\leq 1}(Tu, Tu)_{[0,c]} \geq \varepsilon/2$. Since c is arbitrary, this contradicts (5.2).

The other three statements are proved similarly. □

6 Example. Laguerre polynomials

Consider equation $xy'' + (1-x)y' + ny = 0$ generating the Laguerre polynomials. Multiplying by e^{-x}, we get

$$(xe^{-x}y')' + ne^{-x}y = 0.$$

In this case $p(x) = xe^{-x}$, $\rho(x) = e^{-x}$. Let's verify the discreteness conditions for the interval $(0,\infty)$. At the point $x = 0$ it is

$$\int_s^1 \frac{dx}{p(x)} \int_0^s \rho(x)\,dx \to 0$$

when $s \to 0$. It is so since $\int_0^s e^{-x}\,dx = O(s)$ and $\int_s^1 \frac{e^x}{x}\,dx \sim \int_s^1 \frac{dx}{x} = -\ln s$.

At the $x = \infty$ we have to check

$$\int_1^s \frac{dx}{p(x)} \int_s^\infty \rho(x)\,dx \to 0,$$

when $s \to \infty$, that is $\int_1^s \frac{e^x}{x}\,dx \cdot e^{-s} \to 0$. For arbitrary $\varepsilon > 0$ take $A > 0$ such that $1/A < \varepsilon/2$. Then

$$\int_1^s \frac{e^x}{x}\,dx \cdot e^{-s} \leq \int_1^A \frac{e^x}{x}\,dx \cdot e^{-s} + \varepsilon/2.$$

7 Criterion formulation in the article by Krein and Kac

Article [6] discusses equation

$$y'' + \lambda\rho y = 0, \qquad 0 \leq x < L,$$

in which the generalized density is considered to be the derivative dM/dx, $L \leq +\infty$. L is considered the length of the string, and M is its mass.

Spectrum discreteness criterion: for the spectrum of the string to be discrete, it is necessary and sufficient that in case $L = \infty$ condition

$$\lim_{x\to\infty} x(M(\infty) - M(x)) = 0$$

is fulfilled, and in case $M(L) = \infty$ the dual condition

$$\lim_{x\to L} M(x)(L-x).$$

In the first case, it is assumed that $M(L) < \infty$, and in the second $L < \infty$.

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Delayed linear difference equations: the method of $\mathcal{Z}$-transform

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Abstract. A system of nonhomogeneous linear difference equations with linear parts given by non-commutative matrices is studied. Representation of its solution is derived by means of newly defined delayed perturbation of discrete matrix exponential using the $\mathcal{Z}$-transform. We discard the invertibility condition of matrix of non delayed term used in recent works related to the representation of solutions for delayed linear difference systems.

Keywords: $\mathcal{Z}$-transform, difference equations.

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1 Introduction

Throughout the paper we denote:

- Θ and I the $d \times d$ zero and identity matrix, respectively;
- $\mathbb{Z}_a^b := \{a, a+1, \ldots, b\}$ for $a, b \in \mathbb{Z} \cup \{\pm\infty\}$, $a \leq b$;
- An empty sum $\sum_{i=a}^{b} z(i) = 0$ and an empty product $\prod_{i=a}^{b} z(i) = 1$ for integers $a < b$, where $z(i)$ is a given function which does not have to be defined for each $i \in \mathbb{Z}_b^a$ in this case;
- $\Delta x(k) = x(k+1) - x(k)$ is the forward difference operator;

In the present paper we consider the following discrete systems with delay,

$$x(k+1) = Ax(k) + Bx(k-m) + f(k), \qquad k \geq 0, \tag{1.1}$$

where $m \geq 1$ is a fixed integer, $k \in \mathbb{Z}_0^{\infty}$, A, B are constant $d \times d$ matrices, $x : \mathbb{Z}_{-m}^{\infty} \to \mathbb{R}^d$ is an unknown solution, C is a constant $d \times d$ matrix and $f : \mathbb{Z}_0^{\infty} \to \mathbb{R}^d$ is a function.

Let $\varphi : \mathbb{Z}_{-m}^{0} \to \mathbb{R}^d$ be a function. We consider an initial value problem

$$x(k) = \varphi(k), \qquad k \in \mathbb{Z}_{-m}^{0}. \tag{1.2}$$

We recall that the initial problem (1.1), (1.2) has a unique solution in $\mathbb{Z}_{-m}^{\infty}$.

In 2006, J. Diblik and D. Ya. Khusainov published two papers [2,3] on a matrix representation of solutions of linear discrete systems with a single delay using so called delayed discrete matrix exponential. In [8,9] the concept of discrete matrix delayed exponential is extended to two matrices with a representation derived of solutions to systems with two delayed linear terms. Along these lines, [21] presents rather general results giving a representation of solutions to discrete systems with multiple delayed terms assuming that matrices of these terms pairwise permute, while the paper by the author [15] treats the case of non-permutable matrices. The results of these papers are widely used. These basic results of these papers are widely used to deal with control theory, iterative learning control and stability analysis for time-delay equations; see for example, [1,4,5,7,11–14,16,18–20,22,23] and references therein.

In the paper [6] is an open problem formulated - to prove that the case of non-permutable matrice can be treated with the method of $\mathcal{Z}$-transform. This paper gives positive answer to this problem in the case of two matrices. Representation of solutions is derived by means of newly defined delayed perturbation of matrix exponential using the $\mathcal{Z}$-transform where the existence of inverse of the matrix A is not assumed (the assumption of regularity of matrix A plays important role in [15]).

The $\mathcal{Z}$-transform is a mathematical device similar to a generating function which provides an alternate method for solving linear difference equations as well as certain summation equations. The $\mathcal{Z}$-transform is important in the analysis and design of digital control systems. Note that in [21] the $\mathcal{Z}$-transform is applied to the following multiple delayed linear discrete systems with permutable matrices:

$$x(k+1) = x(k) + \sum_{j=1}^{m} B_j x(k-m_j) + f(k), \qquad k \geq 0$$
$$x(k) = \varphi(k), \qquad k \in \mathbb{Z}_{-m}^{0},$$

where $B_1, \ldots, B_m$ are pairwise permutable matrices.

Motivated by [21] we apply the $\mathcal{Z}$-transform to study the problem (1.1), (1.2) assuming that the linear parts A, B in (1.1) are given by pairwise nonpermutable matrices. This does not allow to change the order when multiplying matrices and problem becomes much more difficult.

2 Delayed perturbation of discrete matrix exponential

The main tool in our study is the $\mathcal{Z}$-transform defined as

$$\mathcal{Z}\{f(k)\}(z) = \sum_{k=0}^{\infty} \frac{f(k)}{z^k}$$

for $z \in \mathbb{R}$ and an exponentially bounded function $f : \mathbb{Z}_0^{\infty} \to \mathbb{R}^d$ such that $\|f(k)\| \leq c_1 c_2^k$ for all $k \in \mathbb{Z}_0^{\infty}$ and some constants $c_1, c_2 \in \mathbb{R}^+$. Note that if f is exponentially bounded, then $\mathcal{Z}\{f(k)\}(z)$ exists for all z sufficiently large. The $\mathcal{Z}$-transform is considered componentwisely. σ is the Heaviside step function defined as

$$\sigma(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

The next lemma gathers up some features of the $\mathcal{Z}$-transform.

Lemma 2.1 ([10]). *The following equalities are true for sufficiently large $z \in \mathbb{R}$ and exponentially bounded functions f, g:*

1. $\mathcal{Z}\left\{af(k)+bg(k)\right\} = a\mathcal{Z}\left\{f(k)\right\} + b\mathcal{Z}\left\{g(k)\right\},\ a,b \in \mathbb{R}$;
2. $\mathcal{Z}^{-1}\left\{z^{-l}\right\}(k) = \delta(l,k)$ *for* $l \in \mathbb{Z}_0^{\infty}$, *where* δ *is the Kronecker delta,*
$$\delta(l,k) = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}$$
3. $\mathcal{Z}^{-1}\left\{F(z)\,G(z)\right\}(k) = (f * g)(k)$. *Here the convolution operator* $*$ *is given by*
$$(f*g)(k) = \sum_{j=0}^{k} f(j)\,g(k-j);$$

The next lemma is a corollary of the latter one.

Lemma 2.2. *The following identities are true for sufficiently large $z \in \mathbb{R}$:*

$$\mathcal{Z}^{-1}\left\{\left((zI-A)^{-1}B\right)^{j}(zI-A)^{-1}\right\}(k) = Q(k-1;j), \tag{2.1}$$

$$\mathcal{Z}^{-1}\left\{\frac{1}{z^{mj+\gamma}}\left((zI-A)^{-1}B\right)^{j}(zI-A)^{-1}\right\}(k) = Q(k-mj-\gamma-1;j), \tag{2.2}$$

where

$$Q(k;0) = A^{k}\sigma(k), \qquad Q(k;j) = \sum_{l=j}^{k} A^{k-l}BQ(l-1;j-1)\,\sigma(k-j).$$

Proof. To prove the formula (2.1) we recall the following identity

$$(I-C)^{j}\sum_{k=0}^{\infty}\begin{pmatrix} k+j-1 \\ j-1 \end{pmatrix} C^{k} = I, \qquad \|C\| < 1.$$

Using this formula, we have

$$(zI-A)^{-j} = \frac{1}{z^{j}}\sum_{k=0}^{\infty}\begin{pmatrix} k+j-1 \\ j-1 \end{pmatrix}\frac{1}{z^{k}}A^{k}.$$

We use the mathematical induction. For $j = 0$, we have

$$\begin{aligned}\mathcal{Z}^{-1}\left\{(zI-A)^{-1}\right\} &= \mathcal{Z}^{-1}\left\{\frac{1}{z^{1}}\right\} * \mathcal{Z}^{-1}\left\{\sum_{l=0}^{\infty}\frac{1}{z^{l}}A^{l}\right\} \\ &= \left(\delta(1,\cdot) * A^{\cdot}\right)(k) = \sum_{l=0}^{k}\delta(1,l)\,A^{k-l} = A^{k-1}\sigma(k-1) = Q(k-1;0).\end{aligned} \tag{2.3}$$

For $j = 1$, we have

$$\begin{aligned}\mathcal{Z}^{-1}&\left\{(zI-A)^{-1}B(zI-A)^{-1}\right\}(k) = \mathcal{Z}^{-1}\left\{(zI-A)^{-1}B\right\} * \mathcal{Z}^{-1}\left\{(zI-A)^{-1}\right\}(k) \\ &= \left\{A^{\cdot-1}\sigma(\cdot-1)\,B * Q(\cdot-1;0)\right\}(k) = \sum_{j=0}^{k} A^{k-j-1}\sigma(k-j-1)\,BA^{j-1}\sigma(j-1) \\ &= \sum_{j=1}^{k-1} A^{k-j-1}BA^{j-1}\sigma(k-2) =: Q(k-1;1).\end{aligned}$$

For $j=2$, we get

$$\begin{aligned}
&\mathcal{Z}^{-1}\left\{(zI-A)^{-2}B^{2}(zI-A)^{-1}\right\}(k)\\
&=\mathcal{Z}^{-1}\left\{(zI-A)^{-1}B\right\}*\mathcal{Z}^{-1}\left\{(zI-A)^{-1}B(zI-A)^{-1}\right\}(k)\\
&=\left\{A^{\cdot-1}\sigma(\cdot-1)B*Q(\cdot-1;1)\sigma(\cdot-2)\right\}(k)\\
&=\sum_{j=0}^{k}A^{k-j-1}\sigma(k-j-1)BQ(j-1;1)\sigma(j-2)\\
&=\sum_{j=2}^{k-1}A^{k-j-1}\sigma(k-j-1)BQ(j-1;1)\sigma(j-2)\\
&=\sum_{j=2}^{k-1}A^{k-j-1}BQ(j-1;1)\sigma(k-3)=:Q(k-1;2).
\end{aligned}$$

Now, suppose that it holds for $j=n$. Then convolution property yields

$$\begin{aligned}
&\mathcal{Z}^{-1}\left\{(zI-A)^{-(n+1)}B^{n+1}(zI-A)^{-1}\right\}(k)\\
&=\left(\mathcal{Z}^{-1}\left\{(zI-A)^{-1}B\right\}*\mathcal{Z}^{-1}\left\{(zI-A)^{-n}B^{n}(zI-A)^{-1}\right\}\right)(k)\\
&=\left\{A^{\cdot-1}\sigma(\cdot-1)B*Q(\cdot-1,n)\right\}(k)=\sum_{j=0}^{k}A^{k-j-1}\sigma(k-j-1)BQ(j-1;n)\sigma(j-n-1)\\
&=\sum_{j=n+1}^{k-1}A^{k-j-1}\sigma(k-j-1)BQ(j-1;n):=Q(k-1;n+1).
\end{aligned}$$

what was to be proved.

The identity (2.2) is obvious:

$$\begin{aligned}
&\mathcal{Z}^{-1}\left\{\frac{1}{z^{mj+\gamma}}\left((zI-A)^{-1}B\right)^{j}(zI-A)^{-1}\right\}(k)\\
&=\mathcal{Z}^{-1}\left\{\frac{1}{z^{mj+\gamma}}\right\}*\mathcal{Z}^{-1}\left\{\left((zI-A)^{-1}B\right)^{j}(zI-A)^{-1}\right\}\\
&=\left(\delta(mj+\gamma,\cdot)Q(\cdot-1;j)\sigma(\cdot-j-1)\right)\\
&=\sum_{s=0}^{k}\delta(mj+\gamma,s)Q(k-s-1;j)\sigma(k-s-j-1)\\
&=Q(k-mj-\gamma-1;j).
\end{aligned}$$

□

Lemma 2.3. *Let $m\geq 1$, A,B be a constant $d\times d$ matrices, $\varphi:\mathbb{Z}_{-m}^{0}\to\mathbb{R}^{d}$ be given function. Assume that $f:\mathbb{Z}_{0}^{\infty}\to\mathbb{R}^{d}$ is exponentially bounded. Then the solution of Cauchy problem (1.1), (1.2) is exponentially bounded.*

For given matrices A,B and delay m, we define delayed perturbation of discrete matrix exponential $X_{m}^{A,B}(k)$ by the following definition.

Definition 2.4. Let $m\geq 1$, A,B be a constant $d\times d$ matrices. Delayed perturbation of discrete matrix exponential is defined as

$$X_{m}^{A,B}(k)=\sum_{j=0}^{\left\lfloor\frac{k+m}{m+1}\right\rfloor}Q(k+m-mj;j):\mathbb{Z}_{0}^{\infty}\to\mathbb{R}^{d\times d},$$

where

$$Q(k;j) = \begin{cases} 0, & j \in \mathbb{Z}_{-\infty}^{-1}, \\ A^k \sigma(k), & j = 0, \\ \sum_{l=j}^{k} A^{k-l} B Q(l-1; j-1) \sigma(k-j), & j \in \mathbb{Z}_1^{\infty}. \end{cases} \tag{2.4}$$

Remark 2.5. It should be stressed out that $Q(k;j)$ was used in [17] to define delayed perturbation of Mittag-Leffler functions. Using the definition (2.4) of $Q(k;j)$ one may show that

	$j=0$	$j=1$	$j=2$	$j=3$	$\cdots$	$j=p,$
$Q(0,j)$	I	Θ	Θ			
$Q(1,j)$	A	B	Θ	Θ	$\cdots$	$\Theta,$
$Q(2,j)$	A^2	$AB+BA$	B^2	Θ	$\cdots$	$\Theta,$
$Q(3,j)$	A^3	$A(AB+BA)+BA^2$	$AB^2+B(AB+BA)$	B^3	$\cdots$	$\Theta,$
					$\cdots$	
$\cdots$	$\cdots$	$\cdots$	$\cdots$		$\cdots$	$\Theta,$
$Q(p,j)$	A^p	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$B^p.$

From the above table, it is easily seen that, in the case of commutativity $AB = BA$, we have $Q(k;j) := \binom{k}{j} A^{k-j} B^j \sigma(k-j), \quad k, j \in \mathbb{Z}_0^{\infty}$.

3 Representation of a solution

Below using the $\mathcal{Z}$-transform we prove the main result of the paper on the representation of solution of the problem (1.1), (1.2) in terms of the delayed perturbation of discrete matrix exponential.

Theorem 3.1. *Let $m \geq 1$, A, B be a constant $d \times d$ matrices, $\varphi : \mathbb{Z}_{-m}^{0} \to \mathbb{R}^d$ be given function. Assume that $f : \mathbb{Z}_0^{\infty} \to \mathbb{R}^d$ is exponentially bounded. The solution $x(k)$ of the Cauchy problem* (1.1), (1.2) *has the following form*

$$x(k) = X_m^{A,B}(k-m)\varphi(0) + \sum_{i=-m}^{-1} X_m^{A,B}(k-1-2m-i) B\varphi(i) + \sum_{i=1}^{k} X_m^{A,B}(k-m-i) f(i-1),$$

for $k \in \mathbb{Z}_{-m}^{\infty}$.

Proof. We recall that existence of $\mathcal{Z}$-transform of $f(k)$ and $x(k)$ is guaranteed by Lemma 2.3. Thus we may apply the $\mathcal{Z}$-transform to the equation (1.1) to get

$$\sum_{k=0}^{\infty} \frac{x(k+1)}{z^k} = A \sum_{k=0}^{\infty} \frac{x(k)}{z^k} + B \sum_{k=0}^{\infty} \frac{x(k-m)}{z^k} + \sum_{k=0}^{\infty} \frac{f(k)}{z^k},$$

$$z(X(z) - \varphi(0)) = AX(z) + \frac{B}{z^m}\left(X(z) + \sum_{k=-m}^{-1} \frac{\varphi(k)}{z^k}\right) + F(z),$$

$$\left(zI - A - \frac{B}{z^m}\right) X(z) = z\varphi(0) + \frac{B}{z^m} \sum_{k=-m}^{-1} \frac{\varphi(k)}{z^k} + F(z)$$

$$\begin{aligned} X(z) &= z\left(zI - A - \frac{B}{z^m}\right)^{-1} \varphi(0) + \left(zI - A - \frac{B}{z^m}\right)^{-1} \sum_{k=-m}^{-1} B \frac{\varphi(k)}{z^{k+m}} \\ &\quad + \left(zI - A - \frac{B}{z^m}\right)^{-1} F(z). \end{aligned} \tag{3.1}$$

On the other hand, for sufficiently large $z \in \mathbb{R}$ so that $\left\| (zI - A)^{-1} \frac{B}{z^m} \right\| < 1$

$$\begin{aligned} \left(zI - A - \frac{B}{z^m} \right)^{-1} &= \left(I - (zI - A)^{-1} \frac{B}{z^m} \right)^{-1} (zI - A)^{-1} \\ &= \sum_{j=0}^{\infty} \frac{1}{z^{mj}} \left((zI - A)^{-1} B \right)^j (zI - A)^{-1}. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2) it follows that

$$\begin{aligned} X(z) = & \sum_{j=0}^{\infty} \frac{z}{z^{mj}} \left((zI - A)^{-1} B \right)^j (zI - A)^{-1} \varphi(0) \\ & + \sum_{j=0}^{\infty} \frac{1}{z^{mj}} \left((zI - A)^{-1} B \right)^j (zI - A)^{-1} \sum_{k=-m}^{-1} B \frac{\varphi(k)}{z^{k+m}} \\ & + \sum_{j=0}^{\infty} \frac{1}{z^{mj}} \left((zI - A)^{-1} B \right)^j (zI - A)^{-1} F(z), \end{aligned}$$

for sufficiently large z. Taking the inverse $\mathcal{Z}$-transform, we have

$$x(k) = A_0(k) + \sum_{i=-m}^{-1} A_i(k) + A_f(k),$$

where

$$\begin{aligned} A_0(k) &= \mathcal{Z}^{-1} \left\{ \sum_{j=0}^{\infty} \frac{1}{z^{mj}} \left((zI - A)^{-1} B \right)^j \frac{1}{z^{-1}} (zI - A)^{-1} \varphi(0) \right\} (k), \\ A_i(k) &= \mathcal{Z}^{-1} \left\{ \sum_{j=0}^{\infty} \frac{1}{z^{mj}} \left((zI - A)^{-1} B \right)^j \frac{1}{z^{i+m}} (zI - A)^{-1} B\varphi(i) \right\} (k), \quad i \in \mathbb{Z}_{-m}^{-1}, \\ A_f(k) &= \mathcal{Z}^{-1} \left\{ \sum_{j=0}^{\infty} \frac{1}{z^{mj}} \left((zI - A)^{-1} B \right)^j (zI - A)^{-1} F(z) \right\} (k). \end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned} x(k) = & \sum_{j=0}^{\left\lfloor \frac{k}{m+1} \right\rfloor} Q(k - jm; j) \varphi(0) + \sum_{i=-m}^{-1} \sum_{j=0}^{\left\lfloor \frac{k-i}{m+1} + 1 \right\rfloor} Q(k - jm - i - m - 1; j) B\varphi(i) \\ & + \sum_{l=1}^{k} \sum_{j=0}^{\left\lfloor \frac{k-l}{m+1} \right\rfloor} Q(k - l - jm; j) f(l - 1). \end{aligned}$$

□

Lemma 3.2. *Matrix $Q(k; j)$ has the following properties*

(i) $Q(k+1; j) = AQ(k; j) + BQ(k; j-1)$, $k, j \in \mathbb{Z}_0^{\infty}$.

(ii) If $AB = BA$, then we have

$$Q(k; j) := \binom{k}{j} A^{k-j} B^j \sigma(k - j), \qquad k, j \in \mathbb{Z}_0^{\infty}.$$

Proof. (i) follows directly from the definition (2.4) of $Q(k;j)$. To show (ii) we use the definition of $Q(k;j)$:

$$Q(k,0) = A^k \sigma(k), \qquad Q(k;j) = \sum_{l=j}^{k} A^{k-l} B Q(l-1; j-1) \sigma(k-j), \qquad j \geq 1.$$

For $j = 0, 1$, we have

$$Q(k,0) = A^k \sigma(k), \qquad Q(k,1) = \sum_{l=1}^{k} A^{k-l} B A^{l-1} = k A^{k-1} B = \binom{k}{1} A^{k-1} B.$$

Assume that it is true for $j = n$, and let us prove it for $j = n+1$:

$$\begin{aligned} Q(k; n+1) &= \sum_{l=n+1}^{k} A^{k-l} B Q(l-1; n) \sigma(k-n-1) \\ &= \sum_{l=n+1}^{k} A^{k-l} B \binom{l-1}{n} A^{l-1-n} B^n \sigma(k-n-1) \sigma(l-n-1) \\ &= A^{k-n-1} B^{n+1} \sum_{l=n+1}^{k} \binom{l-1}{n} \sigma(k-n-1) \\ &= \binom{k}{n+1} A^{k-n-1} B^{n+1} \sigma(k-n-1). \end{aligned}$$

□

Lemma 3.3. *We have the following special cases:*

(i) If $A = I$, then $X_m^{A,B}(k) = e_m^{Bk}$;

(ii) If $B = \Theta$, then $X_m^{A,\Theta}(k) = A^{k+m}$.

Proof. It follows

$$Q(k - jm; j) = \binom{k - jm}{j} A^{k-jm-j} B^j$$

(i) It follows that

$$X_m^{I,B}(k) = \sum_{j=0}^{\left\lfloor \frac{k+m}{m+1} \right\rfloor} Q(k + m - mj; j) = \sum_{j=0}^{\left\lfloor \frac{k+m}{m+1} \right\rfloor} \binom{k + m - jm}{j} B^j = e_m^{Bk}.$$

(ii) $B = \Theta$:

$$X_m^{A,\Theta}(k) = \sum_{j=0}^{\left\lfloor \frac{k+m}{m+1} \right\rfloor} Q(k + m - mj; j) = \sum_{j=0}^{\left\lfloor \frac{k+m}{m+1} \right\rfloor} \binom{k + m - jm}{j} A^{k+m-jm-j} B^j = A^{k+m}.$$

□

Lemma 3.4 ([21]). *Let $l \in \mathbb{Z}_0^{\infty}$. $k \in \mathbb{Z}_{(l-1)(m+1)+1}^{l(m+1)}$ if and only if*

$$l = \left\lfloor \frac{k-1}{m+1} \right\rfloor + 1 = \left\lfloor \frac{k+m}{m+1} \right\rfloor.$$

Proof. Indeed, for this l,

$$(l-1)(m+1)+1=\left\lfloor \frac{k-1}{m+1}\right\rfloor (m+1)+1\leq k$$

and

$$l(m+1)=\left\lfloor \frac{k+m}{m+1}\right\rfloor (m+1)=\left\lceil \frac{k}{m+1}\right\rceil (m+1)\geq k.$$

On the other hand, if $k\in \mathbb{Z}_{(l-1)(m+1)+1}^{l(m+1)}$ for some $l\in \mathbb{Z}_0^{\infty}$, then $l\leq \frac{k+m}{m+1}$ and $\frac{k}{m+1}\leq l$. Hence, $l\leq \left\lfloor \frac{k+m}{m+1}\right\rfloor$ and $\left\lceil \frac{k}{m+1}\right\rceil \leq l$, i.e. $l=\left\lfloor \frac{k+m}{m+1}\right\rfloor$. □

Using this lemma, one can easily show that

$$X_m^{A,B}(k)=\begin{cases}\Theta, & k\in \mathbb{Z}_{-\infty}^{-m-1},\\ A^{k+m}+\sum\limits_{j=1}^{l}Q(k+m-mj;j), & k\in \mathbb{Z}_{(l-1)(m+1)+1}^{l(m+1)},\ l\in \mathbb{Z}_0^{\infty}.\end{cases}$$

Lemma 3.5. *$X_m^{A,B}(k)$ is a solution of*

$$X_m^{A,B}(k+1)=AX_m^{A,B}(k)+BX_m^{A,B}(k-m),$$
$$X_m^{A,B}(k)=A^{k+m},\qquad k\in \mathbb{Z}_{-m}^{0},\ \ X_m^{A,B}(k)=\Theta,\ \ k\in \mathbb{Z}_{-\infty}^{-m-1}.$$

Proof. By Lemma 3.2, we have

$$\begin{aligned}X_m^{A,B}(k+1)&=\sum_{j=0}^{\left\lfloor \frac{k+1+m}{m+1}\right\rfloor}Q(k+1+m-mj;j)\\&=\sum_{j=0}^{\left\lfloor \frac{k+m}{m+1}\right\rfloor}AQ(k+m-mj;j)+\sum_{j=1}^{\left\lfloor \frac{k+1+m}{m+1}\right\rfloor}BQ(k+m-mj;j-1)\\&=AX_m^{A,B}(k)+B\sum_{j=0}^{\left\lfloor \frac{k+m}{m+1}\right\rfloor}Q(k-mj;j)\\&=AX_m^{A,B}(k)+BX_m^{A,B}(k-m).\end{aligned}$$ □

It should be stressed out that the assumption on the exponential boundedness of the function f can be omitted.

Theorem 3.6. *The solution of initial value problem* (1.1), (1.2) *can be written in the following form*

$$\begin{aligned}x(k)&=X_m^{A,B}(k-m)\varphi(0)+\sum_{i=-m}^{-1}X_m^{A,B}(k-1-2m-i)B\varphi(i)\\&\quad+\sum_{i=1}^{k}X_m^{A,B}(k-m-i)f(i-1),\qquad k\in \mathbb{Z}_0^{\infty}.\end{aligned}\tag{3.3}$$

Proof. If $k\in \mathbb{Z}_0^{m-1}$, then $k-m\in \mathbb{Z}_{-m}^{1}$ and

$$X_m^{A,B}(k-1-2m-i)=\begin{cases}\Theta, & i\in \mathbb{Z}_{k-m}^{-1},\ \ (k-1-2m-i\leq -m-1)\\ A^{k}, & i\in \mathbb{Z}_{-m}^{k-m-1}\ \ (-m\leq k-1-2m-i\leq 0).\end{cases}$$

Thus (3.3) gives

$$x(k) = A^k \varphi(0) + \sum_{i=-m}^{k-m-1} A^{k-1-m-i} B\varphi(i) + \sum_{i=1}^{k} A^{k-i} f(i-1)$$

and

$$\begin{aligned} x(k+1) &= A^{k+1}\varphi(0) + \sum_{i=-m}^{k-m} A^{k-m-i} B\varphi(i) + \sum_{i=1}^{k+1} A^{k+1-i} f(i-1) \\ &= A\left(A^k \varphi(0) + \sum_{i=-m}^{k-m-1} A^{k-1-m-i} B\varphi(i) + \sum_{i=1}^{k} A^{k-i} f(i-1)\right) + B\varphi(k-m) + f(k) \\ &= Ax(k) + B\varphi(k-m) + f(k). \end{aligned}$$

For $k \in \mathbb{Z}_m^{\infty}$:

$$\begin{aligned} x(k+1) &= X_m^{A,B}(k+1-m)\varphi(0) + \sum_{i=-m}^{-1} X_m^{A,B}(k-2m-i) B\varphi(i) \\ &\quad + \sum_{i=1}^{k+1} X_m^{A,B}(k+1-m-i) f(i-1) \\ &= AX_m^{A,B}(k-m)\varphi(0) + BX_m^{A,B}(k-2m)\varphi(0) \\ &\quad + A\sum_{i=-m}^{-1} X_m^{A,B}(k-1-2m-i) B\varphi(i) + B\sum_{i=-m}^{-1} X_m^{A,B}(k-1-3m-i) B\varphi(i) \\ &\quad + A\sum_{i=1}^{k} X_m^{A,B}(k-m-i) f(i-1) + B\sum_{i=1}^{k} X_m^{A,B}(k-2m-i) f(i-1) \\ &\quad + X_m^{A,B}(-m) f(k) \\ &= Ax(k) + Bx(k-m) + f(k). \end{aligned}$$

For $k \in \mathbb{Z}_{-m}^{-1}$:

$$\begin{aligned} x(k) &= X_m^{A,B}(k-m)\varphi(0) + \sum_{i=-m}^{-1} X_m^{A,B}(k-1-2m-i) B\varphi(i) \\ &\quad + \sum_{i=1}^{k} X_m^{A,B}(k-m-i) f(i-1). \end{aligned} \qquad \square$$

4 Conclusion

The paper solves a problem of representation of solution for discrete linear delay system using the delayed perturbation of discrete matrix exponential. In [2,3] discrete delayed matrix exponential is suggested to express solutions of delayed equations with first-order differences: $x(k+1) = Ax(k) + Bx(k-m) + f(k)$. These results are obtained under the commutativity of A and B, and under the condition $\det A \neq 0$. Commutativity condition was omitted in [15]. In this paper we drop the condition of existence of a matrix A^{-1}. The result has been obtained by defining the new delayed perturbation of discrete matrix exponential and employing the $\mathcal{Z}$-transform.

One possible direction in which to generalise the results of this paper is by looking at higher-order linear delay difference equations. It would be interesting to see how the theorems proved above can be extended to these cases. Another direction in which we would like to extend is to consider the classical, fractional and discrete linear systems containing multiple delays.

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Periodic orbits for periodic eco-epidemiological systems with infected prey

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Abstract. We address the existence of periodic orbits for periodic eco-epidemiological system with disease in the prey for two distinct families of models. For the first one, we use Mawhin's continuation theorem in a wide general system that includes some models discussed in the literature, and for the second family we obtain a sharp result using a recent strategy that relies on the uniqueness of periodic orbits in the disease-free space.

Keywords: eco-epidemiological model, existence, periodic.

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1 Introduction

Eco-epidemiological models are ecological models that include infected compartments. In many situations, these models describe more accurately the real ecological system than models where the disease is not taken into account.

There is already a large number of works concerning eco-epidemiological models. To mention just a few recent works, we refer [4] where a mathematical study on disease persistence and extinction is carried out; [2] where the authors study the global stability of a delayed eco-epidemiological model with Holling-type III functional response, and [11] where an eco-epidemiological model with harvesting is considered.

One of the main concerns when studying eco-epidemiological models is to determine conditions under which one can predict if the disease persists or dies out. In mathematical epidemiology, these conditions are usually given in terms of the so called basic reproduction number $\mathcal{R}_0$, defined in [5] for autonomous systems as the spectral radius of the next generation matrix.

In [3], $\mathcal{R}_0$ was introduced for the periodic models, and later on, in [16], the definition of $\mathcal{R}_0$ was adapted to the study of periodic patchy models. In the recent article [6] the theory in [16] was used in the study of persistence of the predator in general periodic predator-prey models.

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When persistence is guaranteed, the obtention of conditions that assure the existence of periodic orbits for periodic eco-epidemiological models is an important issue in the deepening of the description of these models since these orbits correspond to situations where possibly there is some equilibrium in the described ecological system, reflected in the fact that the behaviour of the theoretical model is the same over the years. In [13] it was proved that there is an endemic periodic orbit for the periodic version of the model considered in [18] when the infected prey is permanent and some additional conditions are fulfilled, partially giving a positive answer to a conjecture in this last paper.

The models in [18] and [13] assume that there is no predation on uninfected preys. In spite of that, this assumption is not suitable for the description of many eco-epidemiological models. The main purpose of this paper is to present some results on the existence of an endemic periodic orbit for periodic eco-epidemiological systems with disease in the prey that generalize the systems in [18] and [13] by including in the model a general function corresponding to the predation of uninfected preys. Two slightly distinct families of models will be considered separately, one of them in section 2 and the other is section 4. The proof of the main result in section 2 relies on Mawhin's continuation theorem. Following the approach in [13], we begin by locating the components of possible periodic orbits for the one parameter family of systems that arise in Mawhin's result, allowing us to check that the conditions of that theorem are fulfilled. Although the main steps in our proof correspond to the ones in [13], several additional nontrivial arguments are needed in our case. Additionally, there is also a substantial difference between our approach and the one in [13, 18]. In fact, we take as a departure point some prescribed behaviour of the uninfected subsystem, corresponding to the dynamics of preys and predators in the absence of disease: we will assume in this work that we have global asymptotic stability of solutions of some special perturbations of the bidimensional predator-prey system (the system obtained by letting $I = 0$ in the first and third equations in (1.1)). Thus, when applying our results to particular situations, one must verify that the underlying uninfected subsystem satisfies our assumptions. On the other hand, our approach allows us to construct an eco-epidemiological model from a previously studied predator-prey model (the uninfected subsystem) that satisfies our assumptions. This approach has the advantage of highlighting the link between the dynamics of the eco-epidemiological model and the dynamics of the predator-prey model used in its construction. For the family of systems in section 4, we were able to obtain a sharp result using a recent strategy available in the literature instead of Mawhin's continuation theorem.

Considering what was said, as a generalization of the model studied in [13], a periodic version of the general non-autonomous model introduced in [18], we consider the following periodic eco-epidemiological model:

$$\begin{cases} S' = \Lambda(t) - \mu(t)S - a(t)f(S,I,P)P - \beta(t)SI, \\ I' = \beta(t)SI - \eta(t)g(S,I,P)I - c(t)I, \\ P' = h(t,P) + \gamma(t)a(t)f(S,I,P)P + \theta(t)\eta(t)g(S,I,P)I, \end{cases} \tag{1.1}$$

where S, I and P correspond, respectively, to the susceptible prey, infected prey and predator. In our model $h(t,P)$ correspond to the vital dynamics of predators in the absence of this prey.

In this work we consider two different scenarios: in the first one we will take

$$h(t,P) = (r(t) - b(t)P)P. \tag{1.2}$$

When $r(t) > 0$ for all t, we obtain a model with linear vital dynamics of susceptible prey in the absence of predators and disease and with logistic vital dynamics of predators in the absence of the considered prey. This model generalizes [18]. When $r(t) < 0$ for all t, we obtain a model with a classical vital dynamics of the predators as in the family of Lotka–Volterra models considered in [6]. In the second scenario we consider a linear vital dynamics for predators by taking

$$h(t, P) = \Upsilon(t) - \zeta(t)P. \tag{1.3}$$

This model has no periodic solutions on the axis, allowing us to use a different set of arguments to establish the existence of an endemic periodic orbit. Note that, when h is given by (1.2), there is space in our model for the possibility that predators survive in the absence of this prey. In fact, when $r(t)$ is nonnegative, predator have a logistic behaviour. A possible biological interpretation is that predators in this ecosystem possess different sources of food and, in the absence of the prey in this model, the behaviour of the predator population is logistic. When $r(t)$ is nonpositive we obtain a usual behaviour for predators in the absence of preys. When h is given by (1.3) predators always survive in the absence of the prey considered in the model and we also interpret this fact as in the corresponding situation for the first scenario.

In the first scenario, for technical reasons, we have to make the restriction $g(S, I, P) = P$, while in the second scenario we let g be a general function that satisfies some natural assumptions.

In the first situation, $r(t)$ and $b(t)$ are parameters related to the vital dynamics of the predator population that include the intra-specific competition between predators. This vital dynamics is assumed to follow a logistic law when $r(t) > 0$ for all $t \geqslant 0$ and that is similar to the vital dynamics of predator in a family of Lotka–Volterra models considered in [6] when $r(t) < 0$ for all $t \geqslant 0$. In both scenarios $\Lambda(t)$ is the recruitment rate of the prey population, $\mu(t)$ is the natural death rate of the prey population, $a(t)$ is the predation rate of susceptible prey, $\beta(t)$ is the incidence rate, $\eta(t)$ is the predation rate of infected prey, $c(t)$ is the death rate in the infective class ($c(t) \geqslant \mu(t)$), $\gamma(t)$ is the rate of converting susceptible prey into predator (biomass transfer), $\theta(t)$ is the rate of converting infected prey into predator. It is assumed that only susceptible preys S are capable of reproducing, i.e, the infected prey is removed by death (including natural and disease-related death) or by predation before having the possibility of reproducing.

2 Eco-epidemiological models with classical or logistic vital dynamics for predators

In this section we let $g(S, I, P) = P$ and $h(t, P) = (r(t) - b(t)P)P$, obtaining a model that generalizes the model in [13] by considering a function that corresponds to the predation of uninfected preys:

$$\begin{cases} S' = \Lambda(t) - \mu(t)S - a(t)f(S, I, P)P - \beta(t)SI, \\ I' = \beta(t)SI - \eta(t)PI - c(t)I, \\ P' = (r(t) - b(t)P)P + \gamma(t)a(t)f(S, I, P)P + \theta(t)\eta(t)PI. \end{cases} \tag{2.1}$$

Given an ω-periodic function f, we will use throughout the paper the notations $f^\ell = \inf_{t\in(0,\omega]} f(t)$, $f^u = \sup_{t\in(0,\omega]} f(t)$ and $\bar{f} = \frac{1}{\omega}\int_0^\omega f(s)\,ds$. We will assume the following structural hypothesis concerning the parameter functions and the function f appearing in our model:

S1) The real valued functions Λ, μ, a, β, η, c, γ, θ and b are periodic with period ω, nonnegative and continuous; the real valued function r is periodic with period ω and continuous and can be nonnegative or nonpositive;

S2) Function f is nonnegative and C^1;

S3) Function $x \mapsto f(x,y,z)$ is nondecreasing;

S4) Functions $z \mapsto f(x,y,z)$ and $y \mapsto f(x,y,z)$ are nonincreasing;

S5) For all (x,y,z) we have

$$f(x,y,z) + z\frac{\partial f}{\partial z}(x,y,z) > 0, \quad \overline{\eta} + \overline{a}\frac{\partial f}{\partial y}(x,y,z) > 0 \quad \text{and} \quad \overline{\theta\eta} + \overline{\gamma a}\frac{\partial f}{\partial y}(x,y,z) > 0;$$

S6) $\bar{\Lambda} > 0$, $\bar{\mu} > 0$ and $\bar{b} > 0$;

S7) There is $\alpha \geqslant 1$ and $K > 0$ such that $f(x,0,0) \leqslant Kx^\alpha$.

Note that our functional response must depend on I to be able to include functional response functions with saturation, that must depend on the total population of preys (see [1, 14]). Our setting includes several of the most common functional responses for the functional response function f, including, among others, $f(S,I,P) = kS$ (Holling-type I), $f(S,I,P) = kS/(1+m(S+I))$ (Holling-type II), $f(S,I,P) = kS^\alpha/(1+m(S+I)^\alpha)$ (Holling-type III), $f(S,I,P) = kS/(a+b(S+I)+c(S+I)^2)$ (Holling-type IV), $f(S,I,P) = kS/(a+b(S+I)+cP)$ (Beddington–De Angelis), $f(S,I,P) = kS/(a+b(S+I)+cP+d(S+I)P)$ (Crowley–Martin). Also note that conditions S3), S4) are natural from a biological perspective and naturally are satisfied by the usual functional responses considered in the literature. Conditions S5) and S7) are satisfied by most of the usual functional response functions.

To formulate our next assumptions we need to consider two auxiliary equations and one auxiliary system. First, for each $\lambda \in (0,1]$, we need to consider the following equations:

$$x' = \lambda(\Lambda(t) - \mu(t)x) \tag{2.2}$$

and

$$z' = \lambda(r(t) - b(t)z)z. \tag{2.3}$$

Note that, if we identify x with the susceptible prey population, equation (2.2) gives the behaviour of the susceptible preys in the absence of infected preys and predator and identifying z with the predator population, equation (2.3) gives the behaviour of the predator in the absence of preys.

Equation (2.2) is a linear equation that was considered in countless papers on epidemiological models and equation (2.3) was already studied in [8]. These equations have a well known behaviour, given in the following lemmas:

Lemma 2.1. *For each $\lambda \in (0,1]$ there is a unique ω-periodic solution of equation (2.2), $x^*_\lambda(t)$, that is globally asymptotically stable in $\mathbb{R}^+$.*

Lemma 2.2. *If the function r is nonnegative, for each $\lambda \in (0,1]$ there is a unique ω-periodic solution of equation (2.3), $z^*_\lambda(t)$, that is globally asymptotically stable in $\mathbb{R}^+$. If the function r is nonpositive for each $\lambda \in (0,1]$ the zero solution of equation (2.3), that we still denote by $z^*_\lambda(t)$, is globally asymptotically stable in $\mathbb{R}^+_0$.*

For each $\lambda \in (0,1]$, we also need to consider the next family of systems, which corresponds to behaviour of the preys and predators in the absence of infected preys (system (1.1) with $I = 0$, $S = x$ and $P = z$):

$$\begin{cases} x' = \lambda(\Lambda(t) - \mu(t)x - a(t)f(x, \varepsilon_3, z)z - \varepsilon_1 x), \\ z' = \lambda(\gamma(t)a(t)f(x, \varepsilon_4, z) + r(t) - b(t)z + \varepsilon_2)z. \end{cases} \tag{2.4}$$

We now make our last structural assumption on system (1.1):

S9) For each $\lambda \in (0,1]$ and each $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \geqslant 0$ sufficiently small, system (2.4) has a unique ω-periodic solution, $(x^*_{\lambda,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(t), z^*_{\lambda,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(t))$, with

$$x^*_{\lambda,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(t) > 0 \quad \text{and} \quad z^*_{\lambda,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(t) > 0,$$

that is globally asymptotically stable in the set

$$\{(x,z) \in (\mathbb{R}_0^+)^2 : x \geqslant 0 \ \wedge \ z > 0\}.$$

We assume that $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \mapsto (x^*_{\lambda,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(t),\ z^*_{\lambda,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(t))$ is continuous.

Denoting $x^*_\lambda = x^*_{\lambda,0,0,0,0}$ and $z^*_\lambda = z^*_{\lambda,0,0,0,0}$, we introduce the numbers

$$\overline{\mathcal{R}}_0 = \frac{\bar{\beta}\bar{\Lambda}/\bar{\mu}}{\bar{c} + \bar{\eta}\bar{r}/\bar{b}}, \quad \mathcal{R}_0^\lambda = \frac{\overline{\beta x^*_\lambda}}{\bar{c} + \overline{\eta z^*_\lambda}} \quad \text{and} \quad \widetilde{\mathcal{R}}_0 = \inf_{\lambda \in (0,1]} \mathcal{R}_0^\lambda \tag{2.5}$$

Before presenting our main result we have to consider the averaged system corresponding to (2.1):

$$\begin{cases} S' = \overline{\Lambda} - \overline{\mu}S - \overline{a}f(S,I,P)P - \overline{\beta}SI, \\ I' = \overline{\beta}SI - \overline{\eta}PI - \overline{c}I, \\ P' = (\overline{r} - \overline{b}P)P + \overline{\gamma a}f(S,I,P)P + \overline{\theta\eta}PI. \end{cases} \tag{2.6}$$

The number $\overline{\mathcal{R}}_0$ is the basic reproductive number of (2.6) when $f \equiv 0$ (see [13, 18]). We now present our main result.

Theorem 2.3. *If $\widetilde{R}_0 > 1$, conditions S1) to S9) hold and there is a unique equilibrium of the averaged system (2.6) in $(\mathbb{R}^+)^3$, the interior of the first octant, then system (1.1) possesses an endemic periodic orbit of period ω.*

Our proof relies on an application of Mawhin's continuation theorem. We will proceed in several steps. Firstly, in subsection 2.1, we consider a one parameter family of systems and obtain uniform bounds for the components of any periodic solution of these systems. Next, in subsection 2.2 we make a suitable change of variables in our family of systems to establish the setting where we will apply Mawhin's continuation Theorem. Finally, in subsection 2.3, we use Mawhin's continuation Theorem to obtain our result.

2.1 Uniform persistence for the periodic orbits of a one parameter family of systems.

In this section, to obtain uniform bounds for the components of any periodic solution of the family of systems that we can obtain multiplying the right hand side of (1.1) by $\lambda \in (0,1]$, we need to consider the auxiliary system

$$\begin{cases} S_\lambda' = \lambda(\Lambda(t) - \mu(t)S_\lambda - a(t)f(S_\lambda, I_\lambda, P_\lambda)P_\lambda - \beta(t)S_\lambda I_\lambda), \\ I_\lambda' = \lambda(\beta(t)S_\lambda I_\lambda - \eta(t)P_\lambda I_\lambda - c(t)I_\lambda), \\ P_\lambda' = \lambda(\gamma(t)a(t)f(S_\lambda, I_\lambda, P_\lambda)P_\lambda + \theta(t)\eta(t)P_\lambda I_\lambda + r(t)P_\lambda - b(t)P_\lambda^2). \end{cases} \tag{2.7}$$

We will consider separately each of the several components of any periodic orbit.

Lemma 2.4. *Let $x_\lambda^*(t)$ be the unique solution of (2.2). There is $L_1 > 0$ such that, for any $\lambda \in (0,1]$ and any periodic solution $(S_\lambda(t), I_\lambda(t), P_\lambda(t))$ of (2.7) with initial conditions $S_\lambda(t_0) = S_0 > 0$, $I_\lambda(t_0) = I_0 > 0$ and $P_\lambda(t_0) = P_0 > 0$, we have $S_\lambda(t) + I_\lambda(t) \leqslant x_\lambda^*(t) \leqslant \Lambda^u/\mu^\ell$ and $S_\lambda \geqslant L_1$, for all $t \in \mathbb{R}$.*

Proof. Let $(S_\lambda(t), I_\lambda(t), P_\lambda(t))$ be some periodic solution of (2.7) with initial conditions $S_\lambda(t_0) = S_0 > 0$, $I_\lambda(t_0) = I_0 > 0$ and $P_\lambda(t_0) = P_0 > 0$. Since $c(t) \geqslant \mu(t)$, we have, by the first and second equations of (2.7),

$$(S_\lambda + I_\lambda)' \leqslant \lambda\Lambda(t) - \lambda\mu(t)S_\lambda - \lambda c(t)I_\lambda \leqslant \lambda\Lambda(t) - \lambda\mu(t)(S_\lambda + I_\lambda).$$

Since, by Lemma 2.1, equation (2.2) has a unique periodic orbit, $x_\lambda^*(t)$, that is globally asymptotically stable, we conclude that $S_\lambda(t) + I_\lambda(t) \leqslant x_\lambda^*(t)$ for all $t \in \mathbb{R}$. Comparing equation (2.2) with equation $x' = \lambda\Lambda^u - \lambda\mu^\ell x$, we conclude that $x_\lambda^*(t) \leqslant \Lambda^u/\mu^\ell$.

Using conditions S3) and S4), by the third equation of (2.7), we have

$$P_\lambda' \leqslant \lambda(r(t) + \gamma(t)a(t)f(x_\lambda^*(t), 0, 0) + \theta(t)\eta(t)x_\lambda^*(t) - b(t)P_\lambda)P_\lambda \leqslant (\Theta^u - b^\ell P_\lambda)P_\lambda,$$

where function Θ is given by

$$\Theta(t) = \max_{t\in[0,\omega]}\{r(t), 0\} + \gamma(t)a(t)f(x_\lambda^*(t), 0, 0) + \theta(t)\eta(t)x_\lambda^*(t).$$

Thus, comparing with equation (2.3) and using Lemma 2.2, we get $P_\lambda(t) \leqslant P_\lambda^*(t) \leqslant \Theta^u/b^\ell$. Using the bound obtained above, since $-\beta(t)S_\lambda(t) \geqslant -\beta(t)x_\lambda^*(t)$, we have, by conditions S3), S4) and S7),

$$\begin{aligned} S_\lambda' &= \lambda\Lambda(t) - \lambda\mu(t)S_\lambda - \lambda a(t)f(S_\lambda, I_\lambda, P_\lambda)P_\lambda - \lambda\beta(t)S_\lambda I_\lambda \\ &\geqslant \lambda\Lambda^\ell - \left(\lambda\mu^u + \lambda a^u \frac{f(S_\lambda, 0, 0)}{S_\lambda}\frac{\Theta^u}{b^\ell} + \lambda\beta^u(x_\lambda^*)^u\right)S_\lambda \\ &\geqslant \lambda\Lambda^\ell - \left(\lambda\mu^u + \lambda a^u K((x_\lambda^*)^u)^{\alpha-1}\Theta^u/b^\ell + \lambda\beta^u(x_\lambda^*)^u\right)S_\lambda. \end{aligned}$$

According to computations above we have $x_\lambda^*(t) \leqslant \Lambda^u/\mu^\ell$ and thus

$$S_\lambda(t) \geqslant \frac{\lambda\Lambda^\ell}{\lambda\mu^u + \lambda a^u K(\Lambda^u/\mu^\ell)^{\alpha-1}\Theta^u/b^\ell + \lambda\beta^u\Lambda^u/\mu^\ell} =: L_1.$$ □

Lemma 2.5. *Let $z_\lambda^*(t)$ be the unique solution of (2.3). There is $L_2 > 0$ such that, for any $\lambda \in (0,1]$ and any periodic solution $(S_\lambda(t), I_\lambda(t), P_\lambda(t))$ of (2.7) with initial conditions $S_\lambda(t_0) = S_0 > 0$, $I_\lambda(t_0) = I_0 > 0$ and $P_\lambda(t_0) = P_0 > 0$, we have $z_\lambda^*(t) \leqslant P_\lambda(t) \leqslant L_2$, for all $t \in \mathbb{R}$.*

Proof. Let $\lambda \in (0,1]$ and $(S_\lambda(t), I_\lambda(t), P_\lambda(t))$ be any periodic solution of (2.7) with initial conditions $S_\lambda(t_0) = S_0 > 0$, $I_\lambda(t_0) = I_0 > 0$ and $P_\lambda(t_0) = P_0 > 0$. We have

$$P'_\lambda = \lambda P_\lambda(\gamma(t)a(t)f(S_\lambda, I_\lambda, P_\lambda) + \theta(t)\eta(t)I_\lambda + r(t) - b(t)P_\lambda) \geqslant (\lambda r(t) - \lambda b(t)P_\lambda)P_\lambda.$$

Comparing the previous inequality with equation (2.3) and using Lemma 2.2, we get $P_\lambda(t) \geqslant z^*_\lambda(t)$. Using the computations in proof of the previous lemma, we have $P_\lambda(t) \leqslant L_1$ and we take $L_2 = L_1$. □

Lemma 2.6. *Let $\widetilde{\mathcal{R}}_0 > 1$. There are $L_3, L_4 > 0$ such that, for any $\lambda \in (0,1]$ and any periodic solution $(S_\lambda(t), I_\lambda(t), P_\lambda(t))$ of (2.7) with initial conditions $S_\lambda(t_0) = S_0 > 0$, $I_\lambda(t_0) = I_0 > 0$ and $P_\lambda(t_0) = P_0 > 0$, we have $L_3 \leqslant I_\lambda(t) \leqslant L_4$, for all $t \in \mathbb{R}$.*

Proof. We will first prove that there is $\varepsilon_1 > 0$ such that, for any $\lambda \in (0,1]$, we have

$$\limsup_{t\to+\infty} I_\lambda(t) \geqslant \varepsilon_1. \tag{2.8}$$

By contradiction, assume that (2.8) does not hold. Then, for any $\varepsilon > 0$, there must be $\lambda > 0$ such that $I_\lambda(t) < \varepsilon$ for all $t \in \mathbb{R}$. We have

$$\begin{cases} S'_\lambda \leqslant \lambda\Lambda(t) - \lambda\mu(t)S_\lambda - \lambda a(t)f(S_\lambda, \varepsilon, P_\lambda)P_\lambda, \\ P'_\lambda \leqslant \lambda(\gamma(t)a(t)f(S_\lambda, 0, P_\lambda) + r(t) - b(t)P_\lambda + \lambda\varepsilon\theta^u\eta^u)P_\lambda \end{cases}$$

and

$$\begin{cases} S'_\lambda \geqslant \lambda\Lambda(t) - \lambda\mu(t)S_\lambda - \lambda a(t)f(S_\lambda, 0, P_\lambda)P_\lambda - \varepsilon\lambda\beta^u S_\lambda, \\ P'_\lambda \geqslant \lambda(\gamma(t)a(t)f(S_\lambda, \varepsilon, P_\lambda) + r(t) - b(t)P_\lambda)P_\lambda. \end{cases}$$

By condition S9), we conclude that

$$x^*_{\lambda,\varepsilon\lambda\beta^u,0,0,\varepsilon}(t) \leqslant S_\lambda(t) \leqslant x^*_{\lambda,0,\varepsilon\lambda\theta^u\eta^u,\varepsilon,0}(t)$$

and

$$z^*_{\lambda,\varepsilon\lambda\beta^u,0,0,\varepsilon}(t) \leqslant P_\lambda(t) \leqslant z^*_{\lambda,0,\varepsilon\lambda\theta^u\eta^u,\varepsilon,0}(t).$$

Thus, using condition S9), we have

$$\begin{aligned} I'_\lambda &= \lambda(\beta(t)S_\lambda - \eta(t)P_\lambda - c(t))I_\lambda \\ &\geqslant (\lambda\beta(t)x^*_{\lambda,\varepsilon\lambda\beta^u,0,0,\varepsilon}(t) - \lambda\eta(t)z^*_{\lambda,0,\varepsilon\lambda\theta^u\eta^u,\varepsilon,0}(t) - \lambda c(t))I_\lambda \\ &\geqslant (\lambda\beta(t)x^*_\lambda(t) - \lambda\eta(t)z^*_\lambda(t) - \lambda c(t) - \varphi(\varepsilon))I_\lambda, \end{aligned} \tag{2.9}$$

where φ is a nonnegative function such that $\varphi(\varepsilon) \to 0$ as $\varepsilon \to 0$ (notice that, by continuity, we can assume that φ is independent of λ and, by periodicity of the parameter functions, it is independent of t).

Integrating in $[0,\omega]$ and using (S9)), we get

$$\begin{aligned} 0 &= \frac{1}{\omega}(\ln I_\lambda(\omega) - \ln I_\lambda(0)) = \frac{1}{\omega}\int_0^\omega I'_\lambda(s)/I_\lambda(s)\,ds \\ &\geqslant \lambda\left(\overline{\beta x^*_\lambda} - \bar{c} - \overline{\eta z^*_\lambda}\right) + \varphi(\varepsilon) = \lambda(\bar{c} + \overline{\eta z^*_\lambda})(\mathcal{R}^\lambda_0 - 1) + \varphi(\varepsilon) \end{aligned}$$

and since

$$\mathcal{R}^\lambda_0 \geqslant \inf_{\ell\in(0,1]} \mathcal{R}^\ell_0 = \widetilde{\mathcal{R}}_0 > 1,$$

we have a contradiction. We conclude that (2.8) holds. Next we will prove that there is $\varepsilon_2 > 0$ such that, for any $\lambda \in (0,1]$, we have

$$\liminf_{t\to+\infty} I_\lambda(t) \geqslant \varepsilon_2. \tag{2.10}$$

Assuming by contradiction that (2.10) does not hold, we conclude that there is a sequence $(\lambda_n, I_{\lambda_n}(s_n), I_{\lambda_n}(t_n)) \subset (0,1] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$ such that $s_n < t_n$, $t_n - s_n \leqslant \omega$,

$$I_{\lambda_n}(s_n) = 1/n, \quad I_{\lambda_n}(t_n) = \varepsilon_2/2 \quad \text{and} \quad I_{\lambda_n}(t) \in (1/n, \varepsilon_2/2), \text{ for all } t \in (s_n, t_n).$$

Since $\lambda_n \leqslant 1$, by Lemma 2.4 we have

$$I'_{\lambda_n} = (\lambda_n\beta(t)S_{\lambda_n} - \lambda_n\eta(t)P_{\lambda_n} - \lambda_n c(t))I_{\lambda_n} \leqslant \beta^u\Lambda^u I_{\lambda_n}/\mu^\ell$$

and thus

$$\ln(\varepsilon_2 n/2) = \ln(I_{\lambda_n}(t_n)/I_{\lambda_n}(s_n)) = \int_{s_n}^{t_n} I'_{\lambda_n}(s)/I_{\lambda_n}(s)\, ds \leqslant \beta^u\Lambda^u\omega/\mu^\ell,$$

which is a contradiction since the sequence $(\ln(\varepsilon_2 n/2))_{n\in\mathbb{N}}$ goes to $+\infty$ as $n \to +\infty$, and thus is not bounded.

We conclude that there is $\varepsilon_2 > 0$ such that (2.10) holds. Letting $L_3 = \varepsilon_2$, we obtain $I_\lambda(t) \geqslant L_3$ for all $\lambda \in (0,1]$.

Since $I_\lambda(t) \leqslant S_\lambda(t) + I_\lambda(t)$, by Lemma 2.4, we can take $L_4 = L_2$ and the result is established.

□

2.2 Setting where Mawhin's continuation theorem will be applied.

To apply Mawhin's continuation theorem to our model we make the change of variables: $S(t) = e^{u_1(t)}$, $I(t) = e^{u_2(t)}$ and $P(t) = e^{u_3(t)}$. With this change of variables, system (1.1) becomes

$$\begin{cases} u_1' = \Lambda(t)e^{-u_1} - a(t)f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3-u_1} - \beta(t)e^{u_2} - \mu(t), \\ u_2' = \beta(t)e^{u_1} - \eta(t)e^{u_3} - c(t), \\ u_3' = \gamma(t)a(t)f(e^{u_1}, e^{u_2}, e^{u_3}) + \theta(t)\eta(t)e^{u_2} - b(t)e^{u_3} + r(t). \end{cases} \tag{2.11}$$

Note that, if $(u_1^*(t), u_2^*(t), u_3^*(t))$ is an ω-periodic solution of (2.11) then $(e^{u_1(t)}, e^{u_2(t)}, e^{u_3(t)})$ is an ω-periodic solution of system (1.1).

To define the operators in Mawhin's theorem (see appendix A), we need to consider the Banach spaces $(X, \|\cdot\|)$ and $(Z, \|\cdot\|)$ where X and Z are the space of ω-periodic continuous functions $u : \mathbb{R} \to \mathbb{R}^3$:

$$X = Z = \{u = (u_1, u_2, u_3) \in C(\mathbb{R}, \mathbb{R}^3) : u(t) = u(t+\omega)\}$$

and

$$\|u\| = \max_{t\in[0,\omega]} |u_1(t)| + \max_{t\in[0,\omega]} |u_2(t)| + \max_{t\in[0,\omega]} |u_3(t)|.$$

Next, we consider the linear map $\mathcal{L} : X \cap C^1(\mathbb{R}, \mathbb{R}^3) \to Z$ given by

$$\mathcal{L}u(t) = \frac{du(t)}{dt} \tag{2.12}$$

and the map $\mathcal{N} : X \to Z$ defined by

$$\mathcal{N}u(t) = \begin{bmatrix} \Lambda(t)e^{-u_1(t)} - a(t)f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3(t)-u_1(t)} - \beta(t)e^{u_2(t)} - \mu(t) \\ \beta(t)e^{u_1(t)} - \eta(t)e^{u_3(t)} - c(t) \\ \gamma(t)a(t)f(e^{u_1}, e^{u_2}, e^{u_3}) + \theta(t)\eta(t)e^{u_2(t)} - b(t)e^{u_3(t)} + r(t) \end{bmatrix}. \tag{2.13}$$

In the following lemma we show that the linear map in (2.12) is a Fredholm mapping of index zero

Lemma 2.7. *The linear map $\mathcal{L}$ in (2.12) is a Fredholm mapping of index zero.*

Proof. We have

$$\begin{aligned} \ker \mathcal{L} &= \left\{ (u_1, u_2, u_3) \in X \cap C^1(\mathbb{R}, \mathbb{R}^3) : \frac{du_i(t)}{dt} = 0, \quad i = 1, 2, 3 \right\} \\ &= \left\{ (u_1, u_2, u_3) \in X \cap C^1(\mathbb{R}, \mathbb{R}^3) : u_i \text{ is constant}, \quad i = 1, 2, 3 \right\} \end{aligned}$$

and thus $\ker \mathcal{L}$ can be identified with $\mathbb{R}^3$. Therefore $\dim \ker \mathcal{L} = 3$. On the other hand

$$\begin{aligned} \operatorname{Im} \mathcal{L} &= \left\{ (z_1, z_2, z_3) \in Z : \ \exists\, u \in X \cap C^1(\mathbb{R}, \mathbb{R}^3) : \frac{du_i(t)}{dt} = z_i(t), \ i = 1, 2, 3 \right\} \\ &= \left\{ (z_1, z_2, z_3) \in Z : \int_0^\omega z_i(s)\, ds = 0, \ i = 1, 2, 3 \right\}. \end{aligned}$$

and any $z \in Z$ can be written as $z = \tilde{z} + \alpha$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ and $\tilde{z} \in \operatorname{Im} \mathcal{L}$. Thus the complementary space of $\operatorname{Im} \mathcal{L}$ consists of the constant functions. Thus, the complementary space has dimension 3 and therefore $\operatorname{codim} \operatorname{Im} \mathcal{L} = 3$.

Given any sequence (z_n) in $\operatorname{Im} \mathcal{L}$ such that

$$z_n = ((z_1)_n, (z_2)_n, (z_3)_n) \to z = (z_1, z_2, z_3),$$

we have, for $i = 1, 2, 3$ (note that $z \in Z$ since Z is a Banach space and thus it is integrable in $[0, \omega]$ since it is continuous in that interval),

$$\int_0^\omega z_i(s)\, ds = \int_0^\omega \lim_{n \to +\infty} (z_i)_n(s)\, ds = \lim_{n \to +\infty} \int_0^\omega (z_i)_n(s)\, ds = 0.$$

Thus, $z \in \operatorname{Im} \mathcal{L}$ and we conclude that $\operatorname{Im} \mathcal{L}$ is closed in Z. Thus $\mathcal{L}$ is a Fredholm mapping of index zero. □

Consider the projectors $P : X \to X$ and $Q : Z \to Z$ given by

$$Pu(t) = \frac{1}{\omega} \int_0^\omega u(s) ds \quad \text{and} \quad Qz(t) = \frac{1}{\omega} \int_0^\omega z(s) ds.$$

Note that $\operatorname{Im} P = \ker \mathcal{L}$ and that $\ker Q = \operatorname{Im}(I - Q) = \operatorname{Im} \mathcal{L}$.

Consider the generalized inverse of $\mathcal{L}$, $\mathcal{K} : \operatorname{Im} \mathcal{L} \to D \cap \ker P$, given by

$$\mathcal{K}z(t) = \int_0^t z(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^r z(s)\, ds\, dr$$

the operator $Q\mathcal{N} : X \to Z$ given by

$$Q\mathcal{N}u(t) = \begin{bmatrix} \frac{1}{\omega}\int_0^\omega \Lambda(s)e^{-u_1(s)} - a(s)f(e^{u_1(s)}, e^{u_2(s)}, e^{u_3(s)})e^{u_3(s)} - \beta(s)e^{u_2(s)}\,ds - \overline{\mu} \\ \frac{1}{\omega}\int_0^\omega \beta(s)e^{u_1(s)} - \eta(s)e^{u_3(s)}\,ds - \overline{c} \\ \frac{1}{\omega}\int_0^\omega \gamma(s)a(s)f(e^{u_1(s)}, e^{u_2(s)}, e^{u_3(s)})e^{u_3(s)} + \theta(s)\eta(s)e^{u_2(s)} - b(s)e^{u_3(s)}\,ds + \overline{r} \end{bmatrix}$$

and the mapping $\mathcal{K}(I-Q)\mathcal{N} : X \to D \cap \ker P$ given by

$$\mathcal{K}(I-Q)\mathcal{N}u(t) = B_1(t) - B_2(t) - B_3(t),$$

where

$$B_1(t) = \begin{bmatrix} \int_0^t \Lambda(s)e^{-u_1(s)} - a(s)f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3(s)} - \beta(s)e^{u_2(s)} - \mu(s)\,ds \\ \int_0^t \beta(s)e^{u_1(s)} - \eta(s)e^{u_3(s)} - c(s)\,ds \\ \int_0^t \gamma(s)a(s)f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3(s)} + \theta(s)\eta(s)e^{u_2(s)} - b(s)e^{u_3(s)}dt + r(s)\,ds \end{bmatrix},$$

$$B_2(t) = \begin{bmatrix} \frac{1}{\omega}\int_0^\omega\int_0^r \Lambda(s)e^{-u_1(s)} - a(s)f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3(s)} - \beta(s)e^{u_2(s)} - \mu(s)\,ds\,dr \\ \frac{1}{\omega}\int_0^\omega\int_0^r \beta(s)e^{u_1(s)} - \eta(s)e^{u_3(s)} - c(s)\,ds\,dr \\ \frac{1}{\omega}\int_0^\omega\int_0^r \gamma(s)a(s)f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3(s)} + \theta(s)\eta(s)e^{u_2(s)} - b(s)e^{u_3(s)} + r(s)\,ds\,dr \end{bmatrix},$$

and

$$B_3(t) = \left(\frac{t}{\omega} - \frac{1}{2}\right)\begin{bmatrix} \int_0^\omega \Lambda(s)e^{-u_1(s)} - a(s)f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3(s)} - \beta(s)e^{u_2(s)} - \mu(s)\,ds \\ \int_0^\omega \beta(s)e^{u_1(s)} - \eta(s)e^{u_3(s)} - c(s)\,ds \\ \int_0^\omega \gamma(s)a(s)f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3(s)} + \theta(s)\eta(s)e^{u_2(s)} - b(s)e^{u_3(s)} + r(s)\,ds \end{bmatrix}.$$

The next lemma shows that $\mathcal{N}$ is $\mathcal{L}$-compact in the closure of any open bounded subset of its domain.

Lemma 2.8. *The map $\mathcal{N}$ is $\mathcal{L}$-compact in the closure of any open bounded set $U \subseteq X$.*

Proof. Let $U \subseteq X$ be an open bounded set and $\overline{U}$ its closure in X. Then, there is $M > 0$ such that, for any $u = (u_1, u_2, u_3) \in \overline{U}$, we have that $|u_i(t)| \leqslant M$, $i = 1, 2, 3$. Letting $Q\mathcal{N}u = ((Q\mathcal{N})_1 u, (Q\mathcal{N})_2 u, (Q\mathcal{N})_3 u)$, we have

$$\begin{aligned} |(Q\mathcal{N})_1 u(t)| &\leqslant \mathrm{e}^M\left(\bar{\Lambda} + \bar{a}f(e^M, 0, 0) + \bar{\beta}\right) + \bar{\mu}, \\ |(Q\mathcal{N})_2 u(t)| &\leqslant e^M(\bar{\beta} + \bar{\eta}) + \bar{c}, \\ |(Q\mathcal{N})_3 u(t)| &\leqslant \mathrm{e}^M\left(\overline{\gamma a}f(e^M, 0, 0) + \overline{\theta\eta} + \bar{b}\right) + \bar{r} \end{aligned}$$

and we conclude that $Q\mathcal{N}(\overline{U})$ is bounded.

Let now

$$\mathcal{K}(I-Q)\mathcal{N}u = ((\mathcal{K}(I-Q)\mathcal{N})_1 u, (\mathcal{K}(I-Q)\mathcal{N})_2 u, (\mathcal{K}(I-Q)\mathcal{N})_3 u).$$

Let $B \subset X$ be a bounded set. Note that the boundedness of B implies that there is M such that $|u_i| < M$, for all $i = 1,2,3$, and all $u = (u_1, u_2, u_3) \in B$. It is immediate that $\{\mathcal{K}(I-Q)\mathcal{N}u : u \in B\}$ is pointwise bounded. Given $u = (u_1, u_2, u_3)_{n\in\mathbb{N}} \in B$ we have

$$\begin{aligned}
&(K(I-Q)\mathcal{N})_1 u(t) - (\mathcal{K}(I-Q)\mathcal{N})_1 u(v) \\
&\quad = \int_v^t \Lambda(s)e^{-u_1(s)} - a(s)f(e^{u_1(s)}, e^{u_2(s)}, e^{u_3(s)})e^{u_2(s)} - \beta(s)e^{u_2(s)} - \mu(s)\, ds \\
&\qquad - \frac{t-v}{\omega}\int_0^\omega \Lambda(s)e^{-u_1(s)} - a(s)f(e^{u_1(s)}, e^{u_2(s)}, e^{u_3(s)})e^{u_2(s)} - \beta(s)e^{u_2(s)} - \mu(s)\, ds \\
&\quad \leqslant 2(t-v)\left[e^M(\Lambda^u + a^u f(e^M,0,0) + \beta^u e^M) + \mu^M\right],
\end{aligned} \tag{2.14}$$

and similarly

$$(\mathcal{K}(I-Q)\mathcal{N})_2 u(t) - (\mathcal{K}(I-Q)\mathcal{N})_2 u(v) \leqslant 2(t-v)\left[e^M(\beta^u + \eta^u) + c^u\right] \tag{2.15}$$

and

$$\begin{aligned}
&(\mathcal{K}(I-Q)\mathcal{N})_3 u(t) - (\mathcal{K}(I-Q)\mathcal{N})_3 u(v)) \\
&\qquad \leqslant 2(t-v)\left[(\gamma^u a^u f(e^M,0,0) + \theta^u\eta^u + b^u)e^M + r^u\right].
\end{aligned} \tag{2.16}$$

By (2.14), (2.15) and (2.16), we conclude that $\{\mathcal{K}(I-Q)\mathcal{N}u : u \in B\}$ is equicontinuous. Therefore, by the Ascoli–Arzelà theorem, $\mathcal{K}(I-Q)\mathcal{N}(B)$ is relatively compact. Thus the operator $\mathcal{K}(I-Q)\mathcal{N}$ is compact.

We conclude that $\mathcal{N}$ is $\mathcal{L}$-compact in the closure of any bounded set contained in X. □

2.3 Application of Mawhin's continuation theorem.

In this section we will construct the set where, applying Mahwin's continuation theorem, we will find the periodic orbit in the statement of our result.

Consider the system of algebraic equations:

$$\begin{cases}
\overline{\Lambda}e^{-u_1} - \overline{a}f(e^{u_1}, e^{u_2}, e^{u_3})e^{u_3-u_1} - \overline{\beta}e^{u_2} - \overline{\mu} = 0, \\
\overline{\beta}e^{u_1} - \overline{\eta}e^{u_3} - \overline{c} = 0, \\
\overline{\gamma a}f(e^{u_1}, e^{u_2}, e^{u_3}) + \overline{\theta\eta}e^{u_2} - \overline{b}e^{u_3} + \overline{r} = 0.
\end{cases} \tag{2.17}$$

Note that, by hypothesis, the system above has a unique solution on the interior of the first octant. Denote this solution by $p^*(t) = (p_1^*, p_2^*, p_3^*)$. Note also that, by the second equation, we get

$$\overline{\eta}e^{u_3} = \overline{\beta}e^{u_1} - c. \tag{2.18}$$

By Lemmas 2.4, 2.5 and 2.6, there is a constant $M_0 > 0$ such that $\|u_\lambda(t)\| < M_0$, for any $t \in [0,\omega]$ and any periodic solution $u_\lambda(t)$ of (2.7). Let

$$U = \{(u_1, u_2, u_3) \in X : \|(u_1, u_2, u_3)\| < M_0 + \|p^*\|\}. \tag{2.19}$$

Conditions M1. and M2. in Mawhin's continuation theorem (see Appendix A) are fulfilled in the set U defined in (2.19).

Using the notation $v = (\mathrm{e}^{p_1^*}, \mathrm{e}^{p_2^*}, \mathrm{e}^{p_3^*})$, the Jacobian matrix of the vector field corresponding to (2.17) computed in (p_1^*, p_2^*, p_3^*) is

$$J = \begin{bmatrix}
-\overline{a}\frac{\partial f}{\partial S}(v)\,\mathrm{e}^{p_3^*} - \overline{\beta}\,\mathrm{e}^{p_2^*} - \overline{\mu} & -\overline{\beta}\,\mathrm{e}^{p_2^*} - \overline{a}\frac{\partial f}{\partial I}(v)\,\mathrm{e}^{p_3^*+p_2^*-p_1^*} & -\overline{a}\frac{\partial f}{\partial P}(v)\,\mathrm{e}^{2p_3^*-p_1^*} - \overline{a}f(v)\,\mathrm{e}^{p_3^*-p_1^*} \\
\overline{\beta}\,\mathrm{e}^{p_1^*} & 0 & -\overline{\eta}\,\mathrm{e}^{p_3^*} \\
\overline{\gamma a}\frac{\partial f}{\partial S}(v)\,\mathrm{e}^{p_1^*} & \overline{\theta\eta}\,\mathrm{e}^{p_2^*} + \overline{\gamma a}\frac{\partial f}{\partial I}(v)\,\mathrm{e}^{p_2^*} & \overline{\gamma a}\frac{\partial f}{\partial P}(v)\,\mathrm{e}^{p_3^*} - \overline{b}\,\mathrm{e}^{p_3^*}
\end{bmatrix}.$$

Thus

$$
\begin{aligned}
&\det J(p_1^*,p_2^*,p_3^*)\\
&= -\overline{\beta}\,\mathrm{e}^{p_1^*}\left(-\overline{\beta}\,\mathrm{e}^{p_2^*}\left(\overline{\gamma a}\frac{\partial f}{\partial P}(v)\,\mathrm{e}^{p_3^*}-\overline{b}\,\mathrm{e}^{p_3^*}\right)+\left(\overline{a}\frac{\partial f}{\partial P}(v)\,\mathrm{e}^{2p_3^*-p_1^*}+\overline{a}f(v)\,\mathrm{e}^{p_3^*-p_1^*}\right)\overline{\theta\eta}\,\mathrm{e}^{p_2^*}\right)\\
&\quad -\overline{\beta}\,\mathrm{e}^{p_1^*}\left(-\overline{a}\frac{\partial f}{\partial I}(v)\,\mathrm{e}^{p_2^*+p_3^*-p_1^*}\left(\overline{\gamma a}\frac{\partial f}{\partial P}(v)\,\mathrm{e}^{p_3^*}-\overline{b}\,\mathrm{e}^{p_3^*}\right)+\left(\overline{a}\frac{\partial f}{\partial P}(v)\,\mathrm{e}^{2p_3^*-p_1^*}+\overline{a}f(v)\,\mathrm{e}^{p_3^*-p_1^*}\right)\overline{\gamma a}\frac{\partial f}{\partial I}(v)\,\mathrm{e}^{p_2^*}\right)\\
&\quad +\overline{\eta}\,\mathrm{e}^{p_3^*}\left(\left(-\overline{a}\frac{\partial f}{\partial S}(v)\,\mathrm{e}^{p_3^*}-\overline{\beta}\,\mathrm{e}^{p_2^*}-\overline{\mu}\right)\overline{\theta\eta}\,\mathrm{e}^{p_2^*}+\overline{\beta}\,\mathrm{e}^{p_2^*}\,\overline{\gamma a}\frac{\partial f}{\partial S}(v)\,\mathrm{e}^{p_1^*}\right)\\
&\quad +\overline{\eta}\,\mathrm{e}^{p_3^*}\left(\left(-\overline{a}\frac{\partial f}{\partial S}(v)\,\mathrm{e}^{p_3^*}-\overline{\beta}\,\mathrm{e}^{p_2^*}-\overline{\mu}\right)\overline{\gamma a}\frac{\partial f}{\partial I}(v)\,\mathrm{e}^{p_2^*}+\overline{a}\frac{\partial f}{\partial I}(v)\,\mathrm{e}^{p_2^*+p_3^*-p_1^*}\,\overline{\gamma a}\frac{\partial f}{\partial S}(v)\,\mathrm{e}^{p_1^*}\right)\\
&= -\overline{\beta}\,\mathrm{e}^{p_1^*}\left(-\left(\overline{\beta}+\overline{a}\frac{\partial f}{\partial I}(v)\,\mathrm{e}^{p_3^*-p_1^*}\right)\mathrm{e}^{p_2^*}\left(\overline{\gamma a}\frac{\partial f}{\partial P}(v)\,\mathrm{e}^{p_3^*}-\overline{b}\,\mathrm{e}^{p_3^*}\right)\right.\\
&\quad \left.+\left(\overline{a}\frac{\partial f}{\partial P}(v)\,\mathrm{e}^{2p_3^*-p_1^*}+\overline{a}f(v)\,\mathrm{e}^{p_3^*-p_1^*}\right)\left(\overline{\theta\eta}+\overline{\gamma a}\frac{\partial f}{\partial I}(v)\right)\mathrm{e}^{p_2^*}\right)\\
&\quad +\overline{\eta}\,\mathrm{e}^{p_3^*}\left(\left(-\overline{a}\frac{\partial f}{\partial S}(v)\,\mathrm{e}^{p_3^*}-\overline{\beta}\,\mathrm{e}^{p_2^*}-\overline{\mu}\right)\left(\overline{\theta\eta}+\overline{\gamma a}\frac{\partial f}{\partial I}(v)\right)\mathrm{e}^{p_2^*}\right.\\
&\quad \left.+\left(\overline{\beta}\,\mathrm{e}^{p_2^*}+\overline{a}\frac{\partial f}{\partial I}(v)\,\mathrm{e}^{p_2^*+p_3^*-p_1^*}\right)\overline{\gamma a}\frac{\partial f}{\partial S}(v)\,\mathrm{e}^{p_1^*}\right).
\end{aligned}
$$

Taking into account S5) and (2.18), we have

$$
\begin{aligned}
\det J(p_1^*,p_2^*,p_3^*) = &-\overline{\beta}\,\mathrm{e}^{p_1^*}\left(-\frac{\overline{\beta}}{\overline{\eta}}\left(\overline{\eta}+\overline{a}\frac{\partial f}{\partial I}(v)-\frac{\overline{ac}}{\overline{\beta}}\frac{\partial f}{\partial I}(v)\,\mathrm{e}^{-p_1^*}\right)\mathrm{e}^{p_2^*}\left(\overline{\gamma a}\frac{\partial f}{\partial P}(v)\,\mathrm{e}^{p_3^*}-\overline{b}\,\mathrm{e}^{p_3^*}\right)\right.\\
&\left.+\overline{a}\,\mathrm{e}^{p_3^*-p_1^*}\left(\frac{\partial f}{\partial P}(v)\,\mathrm{e}^{p_3^*}+f(v)\right)\left(\overline{\theta\eta}+\overline{\gamma a}\frac{\partial f}{\partial I}(v)\right)\mathrm{e}^{p_2^*}\right)\\
&+\overline{\eta}\,\mathrm{e}^{p_3^*}\left(\left(-\overline{a}\frac{\partial f}{\partial S}(v)\,\mathrm{e}^{p_3^*}-\overline{\beta}\,\mathrm{e}^{p_2^*}-\overline{\mu}\right)\left(\overline{\theta\eta}+\overline{\gamma a}\frac{\partial f}{\partial I}(v)\right)\mathrm{e}^{p_2^*}\right.\\
&\left.+\frac{\overline{\beta}}{\overline{\eta}}\left(\overline{\eta}+\overline{a}\frac{\partial f}{\partial I}(v)-\frac{\overline{ac}}{\overline{\beta}}\frac{\partial f}{\partial I}(v)\,\mathrm{e}^{-p_1^*}\right)\mathrm{e}^{p_2^*}\,\overline{\gamma a}\frac{\partial f}{\partial S}(v)\,\mathrm{e}^{p_1^*}\right)<0.
\end{aligned}
$$

Let $\mathcal{I}:\mathrm{Im}Q\to\ker\mathcal{L}$ be an isomorphism. Thus

$$\deg(\mathcal{I}Q\mathcal{N},U\cap\ker\mathcal{L},0)=\det J(p_1^*,p_2^*,p_3^*)\neq 0 \tag{2.20}$$

and condition M3) in Mawhin's continuation theorem (see appendix A) holds. Taking into account Lemma 2.6, the proof of Theorem 2.3 is completed.

3 Examples.

In this section we present some examples to illustrate the main result in the previous section.

3.1 A model with Holling-type I functional response.

Letting $f(S,I,P)=S$ (Holling-type I functional response) in system (2.1), we obtain the model:

$$
\begin{cases}
S' = \Lambda(t)-\mu(t)S-a(t)SP+\beta(t)SI,\\
I' = \beta(t)SI-\eta(t)PI-c(t)I,\\
P' = (r(t)-b(t)P)P+\gamma(t)a(t)SP+\theta\eta(t)PI.
\end{cases}\tag{3.1}
$$

Since $f(S,I,P)=S$, conditions S2) to S5) are trivially satisfied and S7) is satisfied with $K=\alpha=1$. We obtain the following corollary.

Corollary 3.1. *Assume that that conditions S1), S6) and S9) hold. If $\widetilde{R}_0 > 1$, $\overline{\beta}\,\overline{b} - \overline{\gamma\,a\eta} > 0$ and*

$$\overline{\mathcal{R}}_0 > 1 + \overline{a\eta}\,\frac{\overline{\gamma}\,\overline{\Lambda}}{\overline{\mu}(\overline{r}\,\overline{\eta} + \overline{b}\,\overline{c})} + a\frac{\overline{\beta}\,\overline{r} + \overline{\gamma a}\,\overline{c}}{\overline{\mu}(\overline{b}\,\overline{\beta} - \overline{\gamma}\,\overline{a\eta})} \tag{3.2}$$

then system (3.1) possesses an endemic periodic orbit of period ω.

Proof. Consider the system of algebraic equations

$$\begin{cases} \overline{\Lambda}e^{-u_1} - \overline{a}e^{u_3} - \overline{\beta}e^{u_2} - \overline{\mu} = 0, \\ \overline{\beta}e^{u_1} - \overline{\eta}e^{u_3} - \overline{c} = 0, \\ \overline{\gamma a}e^{u_1} + \overline{\theta\eta}e^{u_2} - \overline{b}e^{u_3} + \overline{r} = 0. \end{cases} \tag{3.3}$$

By the second and third equations we get

$$\mathrm{e}^{u_1} = \frac{\overline{\eta}\,\mathrm{e}^{u_3} + \overline{c}}{\overline{\beta}} \quad \text{and} \quad \mathrm{e}^{u_2} = \frac{\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta}}{\overline{\beta}\,\overline{\theta\eta}}\,\mathrm{e}^{u_3} - \frac{\overline{\beta}\,\overline{r} + \overline{\gamma a}\,\overline{c}}{\overline{\beta}\,\overline{\theta\eta}}.$$

Notice that by hypothesis $\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta} > 0$ and the right hand side of the second equation is positive as long as $\mathrm{e}^{u_3} > (\overline{\beta}\,\overline{r} + \overline{c}\,\overline{\gamma a})/(\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta})$. Using the first equation we get

$$\frac{\overline{\beta}\,\overline{\Lambda}}{\overline{\eta}\,\mathrm{e}^{u_3} + \overline{c}} - \left(\overline{a} + \frac{\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta}}{\overline{\theta\,\eta}}\right)\mathrm{e}^{u_3} + \frac{\overline{\beta}\,\overline{r} + \overline{\gamma a}\,\overline{c}}{\overline{\theta\,\eta}} - \overline{\mu} = 0.$$

Taking into account that we must have $\mathrm{e}^{u_3} > (\overline{\beta}\,\overline{r} + \overline{c}\,\overline{\gamma a})/(\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta})$, we consider the function $F : [(\overline{\beta}\,\overline{r} + \overline{c}\,\overline{\gamma a})/(\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta}), +\infty[\to \mathbb{R}$ given by

$$F(x) = \frac{\overline{\beta}\,\overline{\Lambda}}{\overline{\eta}x + \overline{c}} - \left(\overline{a} + \frac{\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta}}{\overline{\theta\,\eta}}\right)x + \frac{\overline{\beta}\,\overline{r} + \overline{\gamma a}\,\overline{c}}{\overline{\theta\,\eta}} - \overline{\mu}.$$

It is immediate that F is decreasing and that, by the hypothesis in our corollary, we have

$$F\left(\frac{\overline{\beta}\,\overline{r} + \overline{c}\,\overline{\gamma a}}{\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta}}\right) = \overline{\mu}\left(\overline{\mathcal{R}}_0 - 1 - \frac{\overline{a\eta}\,\overline{\gamma}\,\overline{\Lambda}}{\overline{\mu}(\overline{r}\,\overline{\eta} + \overline{b}\,\overline{c})} - a\frac{\overline{\beta}\,\overline{r} + \overline{\gamma a}\,\overline{c}}{\overline{\mu}(\overline{b}\,\overline{\beta} - \overline{\gamma}\,\overline{a\eta})}\right) > 0$$

and $\lim_{x\to+\infty} F(x) = -\infty$. We conclude that there is $x_0 \in [(\overline{\beta}\,\overline{r} + \overline{c}\,\overline{\gamma a})/(\overline{\beta}\,\overline{b} - \overline{\gamma}\,\overline{a\eta}), +\infty[$ such that $F(x_0) = 0$. This implies that there is a unique solution of (3.3). The result follows now from Theorem 2.3. □

We now assume that the real valued functions Λ, μ, r, b, γ and a are constant and positive. Model (3.1) becomes

$$\begin{cases} S' = \Lambda - \mu S - aSP + \beta(t)SI, \\ I' = \beta(t)SI - \eta(t)PI - c(t)I, \\ P' = (r - bP)P + \gamma aSP + \theta\eta(t)PI. \end{cases} \tag{3.4}$$

We have the following corollary.

Corollary 3.2. *Assume that that conditions S1) and S6) hold. If $\widetilde{R}_0 > 1$, $b\overline{\beta} - \gamma a\overline{\eta} > 0$, $\Lambda < \mu^2/a$ and*

$$\overline{\mathcal{R}}_0 > 1 + \frac{a}{\mu}\left(\frac{\overline{\eta}\gamma\,\Lambda}{r\,\overline{\eta} + b\,\overline{c}} + \frac{\overline{\beta}\,r + \gamma a\,\overline{c}}{b\,\overline{\beta} - \gamma\,a\overline{\eta}}\right)$$

then system (3.4) possesses an endemic periodic orbit of period ω.

Proof. We begin by noticing that system (2.4) becomes in our context

$$\begin{cases} x' = \lambda(\Lambda - \mu x - axz - \varepsilon_1 x), \\ z' = \lambda(r - bz + \gamma ax + \varepsilon_2)z. \end{cases} \tag{3.5}$$

System (3.5) has two equilibriums: $E_1 = (\Lambda/(\mu + \varepsilon_1), 0)$ and

$$E_2 = \left(\frac{\sqrt{V^2 + 4\Lambda\gamma a^2/b} - V}{2\gamma a^2/b}, \frac{\sqrt{V^2 + 4\Lambda\gamma a^2/b} - V}{2\gamma a^2/b} + r + \varepsilon_2 \right),$$

where $V = \mu + \varepsilon_1 + a(r + \varepsilon_2)/b$. It is easy to check that E_2 is locally attractive and that E_1 is a saddle point whose stable manifold coincides with the x-axis. If $0 < \alpha < (r + \varepsilon_2)/b$ then, in the line $z = \alpha$ the flow points upward. Additionally, if $\Lambda < \mu(\mu + \varepsilon_1)/a$, in the line $x = \mu/a$ the flow points to the left and the x-coordinate of E_1 is less than μ/a. Thus the region $R = \{(x, z) \in \mathbb{R}^2 : 0 \leqslant x \leqslant \mu/a \wedge z \geqslant \alpha\}$ is positively invariant. Since the divergence of the vector field is given by $-\mu - \varepsilon_1 + \varepsilon_2 - (a + 2b)z + \gamma ax$, we conclude that it is null on the line $z = \frac{-\mu - \varepsilon_1 + \varepsilon_2}{a + 2b} + \frac{\gamma a}{a + 2b}x$. Thus the divergence of the vector field doesn't change sign on the region R and this forbids the existence of a periodic orbit on R. There is also no periodic orbit on $(\mathbb{R}_0^+)^2 \setminus R$ since there is no additional equilibrium in $(\mathbb{R}_0^+)^2$. Since E_2 is locally asymptotically stable, there is no homoclinic orbit connecting E_2 to itself. Therefore, the ω-limit of any orbit in $(\mathbb{R}^2)^+$ must be the equilibrium point E_2 and the global asymptotic stability of (3.5) for sufficiently small $\varepsilon_1, \varepsilon_2 > 0$ follows. We conclude that condition S9) holds. $\square$

To do some simulation, we consider the following particular set of parameters: $\Lambda = 0.1$; $\mu = 0.6$; $\beta(t) = 20(1 + 0.9\cos(2\pi t))$; $\eta(t) = 0.7(1 + 0.7\cos(\pi + 2\pi t))$; $c(t) = 0.1$; $r = 0.2$; $b = 0.3$; $\theta = 10$, $\gamma(t) = 0.1$ and $a = 3$. We obtain the model

$$\begin{cases} S' = 0.1 - 0.6S - 20(1 + 0.9\cos(2\pi t))SI - 3SP, \\ I' = 20(1 + 0.9\cos(2\pi t))SI - 0.7(1 + 0.7\cos(\pi + 2\pi t))PI - 0.1I, \\ P' = (0.2 - 0.3P)P + 7(1 + 0.7\cos(\pi + 2\pi t))PI + 0.3SP. \end{cases} \tag{3.6}$$

Notice that, for our model, $\Lambda = 0.1 > 0.012 = \mu^2/a$, $b\overline{\beta} - \gamma a\overline{\eta} = 3.99 > 0$, $\overline{R}_0 \approx 5.88 > 1 + 1.86$ and $\widetilde{R}_0 \approx 24.8 > 1$, and thus the conditions in Corollary 3.1 are fulfilled. Considering the initial condition $(S_0, I_0, P_0) = (0.03567, 0.02047, 0.88021)$ we obtain the periodic orbit in Figure 3.1. Although

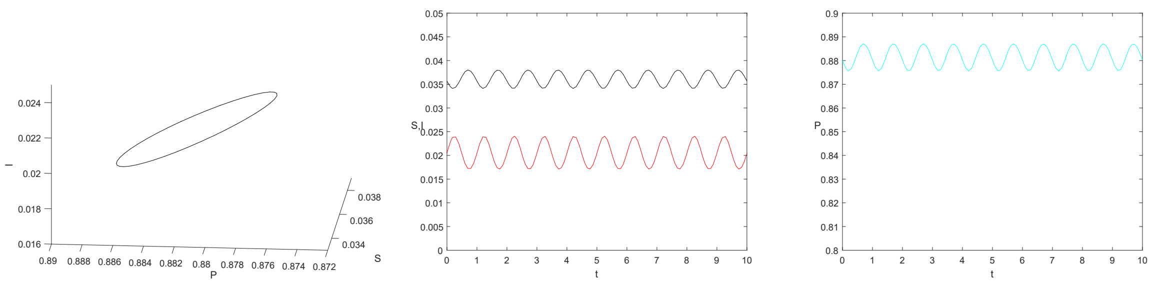

Figure 3.1: Periodic orbit for model (3.6)

our theoretical result doesn't imply the attractivity of the periodic solution, the simulations carried out suggest that this is the case.

3.2 A model with no predation on susceptible preys.

Letting $f \equiv 0$ in system (1.1), and still assuming that the real valued functions Λ, μ, β, η, c, γ, r, θ and b are periodic with period ω, nonnegative, continuous and also that $\bar{\Lambda} > 0$, $\bar{\mu} > 0$, $\bar{r} > 0$ and $\bar{b} > 0$, we obtain the periodic model considered in [13,18]:

$$\begin{cases} S' = \Lambda(t) - \mu(t)S - \beta(t)SI, \\ I' = \beta(t)SI - \eta(t)PI - c(t)I, \\ P' = (r(t) - b(t)P)P + \theta(t)\eta(t)PI. \end{cases} \tag{3.7}$$

In [18], the authors refer that the assumption that predator mainly eats the infected prey (that is modelled by assuming that no predation on uninfected preys occur) is in accordance with the fact that the infected individuals are less active and can be caught more easily, or that infection modifies the behavior of the preys in such a way that they start living in parts of the habitat which are accessible to the predator. Some examples available in the literature are also provided in [18]: as an example of a situation where infected individuals can be caught more easily, the authors cite [10], where it is showed that wolf attacks on moose on Isle Royale in Lake Superior are more successful if the moose are heavily infected with a lungworm; as an example of a situation where the behavior of the prey individuals is modified, favoring predation, the authors cite [7].

Note that conditions S2) to S5) and S7) are trivially satisfied since $f \equiv 0$. Also note that system (2.4) becomes in this context

$$\begin{cases} x' = \lambda(\Lambda(t) - \mu(t)x - \varepsilon_1 x), \\ z' = \lambda(r(t) - b(t)z + \varepsilon_2)z. \end{cases} \tag{3.8}$$

and, by Lemmas 1 to 4 in [18] we conclude that condition S9) holds in this setting. Note also that condition (3.2) becomes $\overline{\mathcal{R}}_0 > 1$ and condition $\bar{b}\,\bar{\beta} - \bar{\gamma}\,\overline{a\eta} \leqslant 0$ is trivially satisfied since we can take $\gamma = 0$ or $a = 0$. We obtain the following corollary that recovers the result in [13]:

Corollary 3.3. *If $\widetilde{\mathcal{R}}_0 > 1$ and $\overline{\mathcal{R}}_0 > 1$ hold, then system (3.7) possesses an endemic periodic orbit of period ω.*

4 Eco-epidemiological models with linear vital dynamics for predators

In this section we let $h(t,P) = \Upsilon(t) - \zeta(t)P$, obtaining the following model:

$$\begin{cases} S' = \Lambda(t) - \mu(t)S - a(t)f(S,I,P)P - \beta(t)SI, \\ I' = \beta(t)SI - \eta(t)g(S,I,P)I - c(t)I, \\ P' = \Upsilon(t) - \zeta(t)P + \gamma(t)a(t)f(S,I,P)P + \theta(t)\eta(t)g(S,I,P)I. \end{cases} \tag{4.1}$$

To establish the existence of an endemic periodic orbit for system (4.1) we assume the following natural conditions:

R1) The real valued functions Λ, μ, a, β, η, c, Υ, ζ, γ and θ are periodic with period ω, nonnegative and continuous;

R2) Functions $y \mapsto f(x,y,z)$ and $z \mapsto f(x,y,z)$ is nonincreasing; function $x \mapsto f(x,y,z)$ is nondecreasing;

R3) Functions $x \mapsto g(x,y,z)$, $y \mapsto g(x,y,z)$ are nonincreasing; function $z \mapsto g(x,y,z)$ is nondecreasing;

R4) Function f is C^1;

R5) $\bar{\Lambda} > 0$, $\bar{\mu} > 0$, $\bar{\Upsilon} > 0$ and $\bar{\zeta} > 0$.

Note that our setting includes several of the most common functional responses for both functions f and g: $f(S,I,P) = kS$ and $g(S,I,P) = kP$ (Holling-type I), $f(S,I,P) = kS/(1+m(S+I))$ and $g(S,I,P) = kP/(1+m(S+I))$ (Holling-type II), $f(S,I,P) = kS^\alpha/(1+m(S+I)^\alpha)$ and $g(S,I,P) = kP^\alpha/(1+m(S+I)^\alpha)$ (Holling-type III), $f(S,I,P) = kS/(a+b(S+I)+c(S+I)^2)$ and $g(S,I,P) = kP/(a+b(S+I)+c(S+I)^2)$ (Holling-type IV), $f(S,I,P) = kS/(a+b(S+I)+cP)$ and $g(S,I,P) =$

$kP/(a+b(S+I)+cP)$ (Beddington–De Angelis), $f(S,I,P) = kS/(a+b(S+I)+cP+d(S+I)P)$ and $g(S,I,P) = kP/(a+b(S+I)+cP+d(S+I)P)$ (Crowley–Martin). Also note that conditions S3), S4) are natural from a biological perspective and naturally are satisfied by the usual functional responses considered in the literature. Conditions S5) and S7) are satisfied by most of the usual functional response functions.

We also need to consider the following auxiliary system that corresponds to perturbations of the disease-free system for (4.1):

$$\begin{cases} x' = \Lambda(t) - \mu(t)x - a(t)f(x,\varepsilon_3,z)z - \varepsilon_1 x, \\ z' = \Upsilon(t) - \zeta(t)z + \gamma(t)a(t)f(x,\varepsilon_4,z)z + \varepsilon_2 z. \end{cases} \tag{4.2}$$

We now make our last structural assumption on system (4.1):

R5) For each $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \geqslant 0$ sufficiently small, system (4.2) has a unique ω-periodic solution

$$(x^*_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(t), z^*_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(t)),$$

with

$$x^*_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(t) > 0 \quad \text{and} \quad z^*_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(t) > 0,$$

that is globally asymptotically stable in the set

$$\{(x,z) \in (\mathbb{R}^+_0)^2 : x \geqslant 0 \ \wedge \ z \geqslant 0\}.$$

We assume that $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \mapsto (x^*_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(t),\, z^*_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(t))$ is continuous.

To obtain the basic reproductive number for our model we consider the ordering (I,S,P) instead of (S,I,P), so that the infected compartment becomes the first one and the uninfected compartments became the last ones. Our new notation corresponds to the one in [12]. With this ordering, the functions $\mathcal{F}$, $\mathcal{V}^+$ and $\mathcal{V}^-$ in [12] become respectively

$$\mathcal{F}(t,(I,S,P)) = (\beta(t)SI, 0, 0),$$

$$\mathcal{V}^+(t,(I,S,P)) = (0,0,\Upsilon(t) + \gamma(t)a(t)f(S,I,P)P + \theta(t)\eta(t)g(S,I,P)I)$$

and

$$\mathcal{V}^-(t,(I,S,P)) = (\eta(t)g(S,I,P)I + c(t)I, \mu(t)S + a(t)f(S,I,P)P + \beta(t)SI, \zeta(t)P).$$

Having identified $\mathcal{F}$ and $\mathcal{V}$ we can compute the matrices $F(t)$ and $V(t)$ in [12] that in our context reduce to one dimensional matrices (that we identify with real numbers). In fact, we have

$$F(t) = \frac{\partial}{\partial I}(\beta(t)SI)|_{(x^*(t),0,z^*(t))} = \beta(t)x^*(t)$$

and

$$V(t) = \frac{\partial}{\partial I}(\eta(t)g(S,P,I)I + c(t)I)|_{(x^*(t),0,z^*(t))} = \eta(t)g(x^*(t),0,z^*(t)) + c(t).$$

The evolution operator $W(s,t,\lambda)$ associated with the linear ω-periodic parametric system $w' = (-V(t) + F(t)/\lambda)w$ is easily seen to be given by

$$W(s,t,\lambda) = \mathrm{e}^{-\int_s^t \beta(r)x^*(r)/\lambda - c(r) - \eta(r)g(x^*(r),0,z^*(r))\,dr}$$

and thus

$$W(\omega,0,\lambda) = 1 \quad \Leftrightarrow \quad \overline{\beta x^*}/\lambda - \bar{c} - \overline{\eta g(x^*,0,z^*)} = 0 \quad \Leftrightarrow \quad \lambda = \frac{\overline{\beta x^*}}{\bar{c} + \overline{\eta g(x^*,0,z^*)}}.$$

Define

$$\mathcal{R}_0 = \frac{\overline{\beta x^*}}{\bar{c} + \overline{\eta g(x^*,0,z^*)}}. \tag{4.3}$$

Note that our system satisfies conditions (A_1) to (A_7) in [6].

Theorem 4.1. *Assume conditions R1) to R5). If $\mathcal{R}_0 > 1$, then model (4.1) has an endemic periodic orbit in* $(\mathbb{R}_0^+)^3$.

The proof of our theorem adapts to our situation the strategy in [6,12]. It will be developed in two steps: using a result derived in [12], we obtain persistence of the infective prey in subsection 4.1 and then, using a Poincaré map, we establish the existence of a periodic orbit in subsection 4.2.

4.1 Uniform persistence

The first step in the proof of Theorem 4.1 is to establish the persistence of all the compartments in our model. To do so we will use Theorem 2 in [12]. Note first that, as long as $\alpha_3 \max\{\theta, \gamma\} < \alpha_2 < \alpha_1$, we have

$$\begin{aligned}\langle (S', I', P'), (\alpha_1, \alpha_2, \alpha_3) \rangle &= \alpha_1 \left(\Lambda(t) - \mu(t)S - a(t)f(S, I, P)P - \beta(t)SI\right) \\ &\quad + \alpha_2 \left(\beta(t)SI - \eta(t)g(S, I, P)I - c(t)I\right) \\ &\quad + \alpha_3 \left(\Upsilon(t) - \zeta(t)P + \gamma(t)a(t)f(S, I, P)P + \theta(t)\eta(t)g(S, I, P)I\right) \\ &< \alpha_1\Lambda^u + \alpha_3\Upsilon^u - \min\{\mu^\ell + c^\ell + \zeta^\ell\}(\alpha_1 S + \alpha_2 I + \alpha_3 P).\end{aligned} \tag{4.4}$$

Thus, defining

$$K = \frac{\alpha_1\Lambda^u + \alpha_3\Upsilon^u}{\min\{\mu^\ell + c^\ell + \zeta^\ell\}},$$

we conclude $\langle (S', I', P'), (\alpha_1, \alpha_2, \alpha_3) \rangle < 0$ when $\alpha_1 S + \alpha_2 I + \alpha_3 P < K$ and that the set

$$\mathcal{K} = \{(S, I, P) \in (\mathbb{R}_0^+)^3 : \alpha_1 S + \alpha_2 I + \alpha_3 P \leq K\} \tag{4.5}$$

is forward invariant for the flow of system (4.1). Additionally, letting $W = \alpha_1 S + \alpha_2 I + \alpha_3 P$, $t_0 \geqslant 0$ and $W_0 = \alpha_1 S(t_0) + \alpha_2 I(t_0) + \alpha_3 P(t_0)$, by (4.4) we have for $t \geqslant t_0$

$$W(t) < K - (K - W_0)\,\mathrm{e}^{-\min\{\mu^\ell + c^\ell + \zeta^\ell\}(t - t_0)}$$

and thus $\limsup_{t\to+\infty} W(t) < K$. We conclude that $\mathcal{K}$ is an absorbing set for the flow. Thus the set $\mathcal{K}$ satisfies assumption (A_8) in [6].

Let now $(S(t), I(t), P(t))$ be a solution of (4.1) such that $I(t) \leqslant \varepsilon$, for $t \geqslant 0$. Since, by the first and third equations in (4.1), we have

$$\begin{cases} S' \geqslant \Lambda(t) - \mu(t)S - a(t)f(S, 0, P)P - \beta^u S\varepsilon, \\ P' \geqslant \Upsilon(t) - \zeta(t)P + \gamma(t)a(t)f(S, \varepsilon, P)P \end{cases}$$

and

$$\begin{cases} S' \leqslant \Lambda(t) - \mu(t)S - a(t)f(S, \varepsilon, P)P, \\ P' \leqslant \Upsilon(t) - \zeta(t)P + \gamma(t)a(t)f(S, 0, P)P + \theta^u\eta^u P\varepsilon, \end{cases}$$

condition R5), allows us to conclude that for sufficiently large $t > 0$ we have $S(t) \geqslant x^*_{\beta^u\varepsilon,0,0,\varepsilon}(t) \geqslant x^*(t) - \sigma_1(\varepsilon)$ and $P(t) \leqslant z^*_{0,\theta^u\eta^u\varepsilon,\varepsilon,0}(t) \leqslant z^*(t) + \sigma_2(\varepsilon)$ with $\sigma_1(\varepsilon), \sigma_2(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, taking into account R2) and R3), if $I(t) \leqslant \varepsilon$ we have

$$\begin{aligned} I' &= \beta(t)SI - \eta(t)g(S, I, P)I - c(t)I \\ &\geqslant \Big(\beta(t)x^*(t) - \beta^\ell\sigma_1(\varepsilon) - \eta(t)g(x^*(t) - \sigma_1(\varepsilon), 0, z^*(t) + \sigma_2(\varepsilon)) - c(t)\Big)\, I \\ &\geqslant \left(F(t)/\lambda(\varepsilon) - V(t)\right) I \end{aligned}$$

where $\lambda :]0, \varepsilon^*[\to \mathbb{R}$, well-defined when we take $\varepsilon^* > 0$ sufficiently small, is given by

$$\lambda(\varepsilon) = \max_{t \in]0,\varepsilon[} \frac{\beta(t)x^*(t)}{\beta(t)x^*(t) - \beta^\ell\sigma_1(\varepsilon) + \eta(t)g(x^*(t), 0, z^*(t)) - \eta(t)g(x^*(t) - \sigma_1(\varepsilon), 0, z^*(t) + \sigma_2(\varepsilon))}$$

and we can immediately see that $\lambda(\varepsilon) \to 1$ as $\varepsilon \to 0$.

By Theorem 2 in [12], we conclude that the infective prey is uniformly strong persistent in system (4.1). The uniform strong persistence of the susceptible prey and the predator, in our situation, is a consequence of the uniform strong persistence of the infectives. In fact, given $\delta > 0$, if $\limsup_{t\to+\infty} S(t) < \delta$ for some solution $(S(t), I(t), P(t))$ then $I' \leqslant (\beta^u\delta - c^\ell)I$. Thus, if we had a solution such that $\delta < c^\ell/\beta^u$ it would follow that $I(t) \to 0$, contradicting the uniform persistence of I. Therefore S is uniformly weak persistent. By Theorem 1.3.3 in [17], we conclude that S must be uniformly strong persistent. Finally, the uniform strong persistence of P is a consequence of the bound $P' \geqslant \Upsilon^\ell - \zeta^u P$.

4.2 Existence of a periodic orbit

Next, to establish the existence of a positive periodic orbit for (4.1) we use the following result.

Theorem 4.2 ([17, Theorem 1.3.6]). *Let $\tau : X \to X$ be a continuous map with $\tau(X_0) \subset X_0$ that is point dissipative, compact and uniform persistent with respect to $(X_0, \partial X_0)$. Then there exists a global attractor A_0 for S in X_0 that attracts strongly bounded sets in X_0 and S has a coexistence state $x_0 \in A_0$.*

To apply this result to our model we let $X = (\mathbb{R}_0^+)^3$, $X_0 = \mathcal{K}$ and $S = \tau$, where $\tau : (\mathbb{R}_0^+)^3 \to (\mathbb{R}_0^+)^3$ ia a time-ω map associated to our system and given by $\tau(S_0, I_0, P_0) = (S(\omega), I(\omega), P(\omega))$, where $(S(t), I(t), P(t))$ is the solution of (4.1) such that $(S(0), I(0), P(0)) = (S_0, I_0, P_0)$.

Since the bounded set $\mathcal{K}$ is an absorbing set for the flow of (4.1), we conclude that τ is point dissipative. It is immediate that τ is compact and, by the discussion in subsection 4.1, we conclude that τ is uniformly persistent with respect to $(\mathcal{K}, \partial\mathcal{K})$. Therefore, Theorem 4.2 allows us to conclude that τ has a coexistence state in $\mathcal{K}$. This coexistence state is a periodic orbit of our system contained in $\mathcal{K}$. This established our result.

A Mawhin's continuation theorem

In this appendix we state Mawhin's continuation theorem [9, Part IV]. Let X and Z be Banach spaces.

Definition A.1. A linear map $\mathcal{L} : D \subseteq X \to Z$ is called a Fredholm mapping of index zero if

1. $\dim\ker\mathcal{L} = \operatorname{codim}\operatorname{Im}\mathcal{L} \leqslant \infty$;
2. $\operatorname{Im}\mathcal{L}$ is closed in Z.

Given a Fredholm mapping of index zero $\mathcal{L} : D \subseteq X \to Z$ it is well known that there are continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that:

1. $\operatorname{Im} P = \ker\mathcal{L}$;
2. $\ker Q = \operatorname{Im}\mathcal{L} = \operatorname{Im}(I - Q)$;
3. $X = \ker\mathcal{L} \oplus \ker P$;
4. $Z = \operatorname{Im}\mathcal{L} \oplus \operatorname{Im} Q$.

It follows that $\mathcal{L}|_{D\bigcap\ker P} : (I - P)X \to \operatorname{Im}\mathcal{L}$ is invertible. We denote the inverse of that map by $\mathcal{K}$.

Definition A.2. A continuous mapping $\mathcal{N} : X \to Z$ is called $\mathcal{L}$-compact on $\overline{U} \subset X$, where U is an open bounded set, if

1. $QN(\overline{U})$ is bounded;
2. $\mathcal{K}(I - Q)\mathcal{N} : \overline{U} \to X$ is compact.

Note that, since $\operatorname{Im} Q$ is isomorphic to $\ker\mathcal{L}$, there is an isomophism $\mathcal{I} : \operatorname{Im}Q \to \ker\mathcal{L}$. We are now prepared to state the Mawhin's continuation theorem.

Theorem A.3 (Mawhin's continuation theorem)**.** *Let X and Z be Banach spaces and let $U \subset X$ be an open set. Assume that $\mathcal{L} : D \subseteq X \to Z$ is a Fredholm mapping of index zero and let $\mathcal{N} : X \to Z$ be $\mathcal{L}$-compact on $\overline{U}$. Additionally, assume that*

M1) for each $\lambda \in (0,1)$ and $x \in \partial U \cap D$ we have $\mathcal{L}x \neq \lambda \mathcal{N}x$;

M2) for each $x \in \partial U \cap \ker \mathcal{L}$ we have $Q\mathcal{N}x \neq 0$;

M3) $\deg(\mathcal{I}Q\mathcal{N}, U \cap \ker \mathcal{L}, 0) \neq 0$.

Then the operator equation $\mathcal{L}x = \mathcal{N}x$ has at least one solution in $D \cap \overline{U}$.

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On integrability and cyclicity of cubic systems

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Abstract. In this paper we study the integrability of a few families of the complex cubic system. We have obtained necessary and sufficient conditions for existence of a local analytic first integral. Sufficiency of the obtained conditions was proven using different methods: time-reversibility, Darboux integrability and others. Using the obtained results on integrability of complex cubic system, we have obtained results for corresponding real cubic systems. Then the study of bifurcation of limit cycles from each component of the center variety of real system was performed.

Keywords: two dimensional systems, cubic systems, integrability, cyclicity, limit cycles.

2020 Mathematics Subject Classification: 34C05, 37C10.

1 Introduction

One of the main problems of qualitative theory is the problem of integrability. The integrability is not often seen phenomena, but never the least less important. A first integral determines the phase portrait of the plane system and for higher dimensional systems first integral can be used to reduce the dimension of the system, hence the importance. This problem can be linked to another problem of qualitative theory, the problem of distinguishing between a center or a focus. The so-called center problem goes back to Dulac [19], who published in year 1908 a paper on integrability of real quadratic ones. The integrability problem for quadratic system is resolved by Dulac, Kapteyn and others, see [19, 30–32, 39, 48, 50, 51]. Since the publication of Dulac's work, a lot of studies have been made on higher degrees systems, real and complex systems. The integrability conditions for some cubic systems were presented in [4, 14, 17, 18, 22, 36–38, 43, 47] and for results on higher degree systems see [5, 6, 8, 23, 24, 45].

When the systems that contain a center are known, there appears the question: "What is the bound of the number of limit cycles that can bifurcate from the center under small perturbation of parameters of the system?" This is a part of the 16th Hilbert's problem, one of the twenty-three problems introduced by David Hilbert in 1900. It is stated as: "What is the maximum number of limit cycles of system $\dot{x} = P_n, \quad \dot{y} = Q_n$, where P_n and Q_n are polynomial of degree n or less? What are possible relative positions of the limit cycles?"

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In attempt to solve this open Hilbert's problem, the cyclicity problem became one of the main problems in the qualitative theory of differential equations (survey by J. Li, [34]).

The beginning of the study of cyclicity problem goes back to Bautin, who introduced the concept of cyclicity [3]. In the seminar paper of Bautin it was proven that the minimal bound on the number of limit cycles of quadratic system is 3. Since then a lot of studies were made on this problem. For quadratic systems it was believed for some time that there are only 3 limit cycles that can bifurcate, but some examples of quadratic systems with 4 limit cycles were constructed [7,49]. Due to the faulty proof of Dulac on the fixed number of limit cycles of fixed polynomial system, see [19], was his statement a big uncertainty for some time. But one step closer to reviling the correctness of it were Chicone and Shafer [9] in year 1983, where it was proven that a fixed quadratic system has a finite number of limit cycles in any bounded region. The result was extended to the whole phase plane by Bamón [2] and Romanovski [42]. Dulac's Theorem for an arbitrary polynomial system was then proven by Ecalle [20] and Il'yaschenko [27]. Even though a lot of studies on this problem is done, the question on the uniform bound on the number of limit cycles in polynomial systems of fixed degree remains unknown. For more results on cyclicity see [25,26,28,33,44,46,52–55,57].

In this paper we present results of integrability of a complex family of cubic polynomial systems of the following form

$$\begin{aligned} \dot{x} &= x - a_{10}x^2 - a_{20}x^3 - a_{11}x^2y - a_{02}xy^2 - a_{-13}y^3, \\ \dot{y} &= -y + b_{01}y^2 + b_{3,-1}x^3 + b_{20}x^2y + b_{11}xy^2 + b_{02}y^3. \end{aligned} \tag{1.1}$$

The computations for the general family (1.1) were complicated, hence we studied four different subfamilies of it. We explore integrability of the systems (1.1) where

$$\begin{aligned} &1)\ a_{-13} = b_{3,-1} = 1, \qquad 2)\ a_{-13} = b_{3,-1} = 0, \qquad 3)\ a_{-13} = 1, \quad b_{3,-1} = 0, \\ &4)\ a_{-13} = 0, \quad b_{3,-1} = 1. \end{aligned} \tag{1.2}$$

By choosing these specific subfamilies we enable determination of general conditions for integrability of complex systems of the form (1.1). In our case it is only necessary to study three of four cases, since the involution $a_{ij} \leftrightarrow b_{ji}$ transforms case 3) into case 4). As it will be shown in Section 3, obtained conditions for these subsystems can be transformed to more general system, where a_{-13} and $b_{3,-1}$ are arbitrary. The approach is describe into details in the same section.

The main result of this paper is presented here.

Theorem 1.1. *The system (1.1) is integrable if and only if one of the following conditions holds:*

1. $a_{11} = a_{-13} = a_{02} = b_{11} = b_{02} = 0,$
2. $a_{11} = a_{-13} = a_{02} = b_{11} = b_{3,-1} = b_{20} = 0,$
3. $a_{11} = a_{20} = b_{11} = b_{3,-1} = b_{20} = 0,$
4. $a_{11} - b_{11} = a_{-13} = b_{3,-1} = a_{20} + b_{20} = a_{02} + b_{02} = 0,$
5. $a_{11} - b_{11} = a_{20}^2 a_{-13} - b_{02}^2 b_{3,-1} = a_{02}b_{02}b_{3,-1} - a_{20}b_{20}a_{-13} = a_{02}a_{20} - b_{20}b_{02} = a_{02}^2 b_{3,-1} - b_{20}^2 a_{-13} = a_{10}^2 b_{02} - a_{20}b_{01}^2 = a_{10}^2 a_{-13}b_{20} - a_{02}b_{01}^2 b_{3,-1} = a_{10}^2 a_{20}a_{-13} - b_{02}b_{01}^2 b_{3,-1} = a_{10}^2 a_{02} - b_{01}^2 b_{20} = a_{10}^4 a_{-13} - b_{01}^4 b_{3,-1} = 0,$

6. $a_{11} = a_{10} = b_{01} = b_{11} = 3a_{-13}b_{3,-1} + 4b_{20}b_{02} = a_{20} + 3b_{20} = 3a_{02} + b_{02} = 0$,
7. $a_{11} - b_{11} = a_{10} = b_{01} = a_{02} - 3b_{02} = 3a_{20} - b_{20} = 0$.

Using obtained components of center variety of complex system (1.1), we have computed the center variety of the general real system which complexification is complex systems (1.1), Theorem 4.1. In Section 4 we have researched the cyclicity of each real component.

2 Preliminaries

Let us study the system

$$\begin{aligned} \dot{u} &= au + bv + f_1(u,v), \\ \dot{v} &= cu + dv + f_2(u,v). \end{aligned} \tag{2.1}$$

The behavior of the nondegenerate singular point at the origin of two-dimensional systems (2.1) is the same as for the linearized system of (2.1), that is the system

$$\dot{u} = au + bv, \quad \dot{v} = cu + dv,$$

except in the case of center. In the case of two purely imaginary eigenvalues of the linearized system the singularity can be either a focus or a center. In that case some additional study needs to be done.

The important theorem, which is the link between the center-focus problem and the integrability problem, studied in this paper, is the Poincaré–Lyapunov Theorem [35,40].

It states the following:

Theorem 2.1. *The system*

$$\begin{aligned} \dot{u} &= \lambda u - v + \tilde{P}(u,v) = \lambda u - v + \sum_{j+k=2}^{n} A_{jk} u^j v^k, \\ \dot{v} &= u + \lambda v + \tilde{Q}(u,v) = u + \lambda v + \sum_{j+k=2}^{n} B_{jk} u^j v^k. \end{aligned} \tag{2.2}$$

on $\mathbb{R}^2$ has a center in the origin if and only if it there exists the a formal first integral of the form $\psi(u,v) = u^2 + v^2 + \cdots$

By transformation $x = u + iv$ the real system can be transformed to

$$\dot{x} = ix + P\left(\frac{(x+\bar{x})}{2}, \frac{(x-\bar{x})}{2i}\right) + iQ\left(\frac{(x+\bar{x})}{2}, \frac{(x-\bar{x})}{2i}\right) = i(x + X_1(x,\bar{x})).$$

The complex system obtained after (complex) time transformation $idt = d\tau$ is

$$\dot{x} = \lambda x + i\left(x - \sum_{p+q=2}^{n} a_{pq} x^{p+1} \bar{x}^q\right). \tag{2.3}$$

The system (2.3) for $\lambda = 0$, with $\bar{x} \to y$, $\bar{a}_{pq} \to b_{qp}$ and after time rescaling is written as

$$\begin{aligned} \dot{x} &= x - \sum_{p+q=2}^{n} a_{pq} x^{p+1} y^q = P_1(x,y), \\ \dot{y} &= -y + \sum_{p+q=2}^{n} b_{qp} x^q y^q = Q_1(x,y), \end{aligned} \tag{2.4}$$

where $P_1(x, y)$ and $Q_1(x, y)$ are polynomials of degree at most n.

The system (2.4) is *locally analytically integrable* if and only if it admits a formal first integral in the form

$$\psi(x, y) = xy + \sum_{l+m\geq 3} v_{l-1,m-1} x^l y^m. \tag{2.5}$$

Since the first integral is constant on any solution, it is obvious that it needs to satisfy $\mathcal{X}\psi(x, y) = \frac{\partial \psi}{\partial x} P_1 + \frac{\partial \psi}{\partial y} Q_1 \equiv 0$.

The construction of the first integral in the form (2.5) yields a series for which $\mathcal{X}\psi(x, y)$ reduces to

$$\mathcal{X}\psi(x, y) = \frac{\partial \psi}{\partial x} P_1 + \frac{\partial \psi}{\partial y} Q_1 := \sum_{k_1+k_2\geq 2} g_{k_1,k_2} x^{k_1} y^{k_2} \ldots \tag{2.6}$$

The coefficients g_{k_1,k_2} of series (2.6) can be obtain with some computations from

$$\begin{aligned} \mathcal{X}\psi(x, y) = &\left(y + \sum_{l+m\geq 3} l v_{l-1,m-1} x^{l-1} y^m \right) \left(x - \sum_{p+q=2}^{n} a_{pq} x^{p+1} y^q \right) \\ &+ \left(x + \sum_{l+m\geq 3} m v_{l-1,m-1} x^l y^{m-1} \right) \left(-y + \sum_{p+q=2}^{n} b_{qp} x^q y^q \right) \end{aligned} \tag{2.7}$$

and are of the form

$$g_{k_1,k_2} = (k_1 - k_2) v_{k_1,k_2} - \sum_{\substack{s_1+s_2=0,\\ s_1,s_2\geq -1}}^{k_1+k_2-1} ((s_1 + 1) a_{k_1-s_1,k_2-s_2} - (s_2 + 1) b_{k_1-s_1,k_2-s_2}) v_{s_1,s_2}. \tag{2.8}$$

In order for the series $\psi(x, y)$ to be a first integral each coefficient g_{k_1,k_2} must be equal to zero. By step-by-step construction of series (2.5), we see that for $k_1 \neq k_2$ the coefficients $v_{l,m}$ can be chosen so that $g_{k_1,k_2} = 0$. But when $k_1 = k_2 = i$ this is not the case and g_{k_1,k_2} depends on previous $v_{l,m}$. The polynomial of coefficients of the system (2.4) appearing in (2.6),

$$g_{i,i} = \sum_{\substack{s_1+s_2=0,\\ s_1,s_2\geq -1}}^{2k-1} ((s_1 + 1) a_{k-s_1,k-s_2} - (s_2 + 1) b_{k-s_1,k-s_2}) v_{s_1,s_2},$$

is called *i-th focus quantity* and the ideal $\mathcal{B} = \langle g_{1,1}, g_{2,2}, \ldots \rangle$ is called the *Bautin ideal*. The ideal generated by the first k focus quantities is denoted by $\mathcal{B}_k$. The variety of the ideal $\mathcal{B}$, $\mathbf{V}(\mathcal{B})$, is called *the center variety*.

The ideals $\mathcal{B}_1, \mathcal{B}_2, \ldots$ form the ascending chain of ideals,
$\mathcal{B}_1 \subseteq \ldots \subseteq \mathcal{B}_{k-1} \subseteq \mathcal{B}_k \subseteq \ldots$, and by the Hilbert Basis Theorem, this chain stabilizes at some k.

Hence in order to obtain subfamilies of the system (1.1) which are locally integrable it is necessary to compute irreducible decomposition of $\mathbf{V}(\mathcal{B}_k)$, where k is the number for which the ascending chain of $\mathcal{B}_k$ stabilizes. For obtained conditions it remains to be shown that these conditions are sufficient, i.e. find the first integral of the form (2.5). For more detailed on this see [1, 44].

From obtained center variety of any polynomial family one can produce, using different approaches, a bound for the cyclicity of the system. An efficient computational technique which we used in this paper and which allows estimation of the generic cyclicity of a family of centers was described in the paper by Christopher [10].

Before the formulation of the theorem presented in [10], let us explain some notations and give some additional definitions.

Denote with $(\lambda,(A,B))$ the coefficient string $(\lambda, A_{20},\ldots,B_{0n})$ and with $E((\lambda,(A,B)))$ the space of parameters of the family (2.2). For the family (2.3) the coefficient string is $(\lambda,a) = (\lambda, a_{p_1q_1},\ldots,a_{p_lq_l})$, where l is the number of coefficients of the system (2.3) and $E((\lambda,a))$ is the space of parameters. By $g^{\mathbb{R}}_{kk}$ the polynomial obtained by substitution of coefficients b_{ji} with $\overline{a}_{ij}$ in the polynomial g_{kk} is denoted and let $\mathcal{B}^{\mathbb{R}}_k$ be the ideal $\mathcal{B}^{\mathbb{R}}_k = \langle g^{\mathbb{R}}_{11}, g^{\mathbb{R}}_{22},\ldots,g^{\mathbb{R}}_{kk}\rangle$.

Since the parameters of the system (2.2) and of the system (2.3) are connected, the definition is given for the complex system (2.3).

Definition 2.2. For parameters (λ,a), let $n((\lambda,a),\epsilon)$ denote the number of limit cycles of the corresponding system (2.3) that lie wholly within an ϵ-neighborhood of the origin. The singularity at the origin for the system (2.3) with fixed coefficients $(\lambda^*,a^*)\in E((\lambda,a))$ has *cyclicity c with respect to the space $E((\lambda,a))$*, if there exist positive constant δ_0 and ϵ_0 such that for every pair ϵ and δ satisfying $0<\epsilon<\epsilon_0$ and $0<\delta<\delta_0$,

$$\max\{n((\lambda,a),\epsilon) : |(\lambda,a)-(\lambda^*,a^*)|<\delta\} = c.$$

The approach for the estimation of the number of limit cycles of our system was based on the following theorem by C. Christopher [10]:

Theorem 2.3. *Suppose that s is a point on the center variety and that* $\operatorname{rank} J_p(\mathcal{B}^{\mathbb{R}}_k) = k$. *Then s lies on a component of the center variety of codimension at least k and there are bifurcations of* (2.3) *which produce k limit cycles locally from the center corresponding to the parameter value s.*

If furthermore, we know that s lies on a component of the center variety of codimension k, then s is smooth point of the variety, and the cyclicity of the center for the parameter value s is exactly $k-1$.

In the latter case, $k-1$ is also the cyclicity of generic point on this component of the center variety.

3 Results on integrability

Before presenting the main results on integrability we recall some important methods used approaching the problem of integrability.

The so-called Darboux method is based on Darboux factors and using them we can sometimes construct the Darboux integrals, more on this can be found in [11, 12, 44].

Definition 3.1. A nonconstant polynomial $f(x,y)\in\mathbb{C}[x,y]$ is called a *Darboux factor* of system (2.4) if there exists a polynomial $K(x,y)\in\mathbb{C}[x,y]$ such that

$$\mathcal{X}f = \frac{\partial f}{\partial x}P_1 + \frac{\partial f}{\partial y}Q_1 = Kf. \tag{3.1}$$

The polynomial $K(x,y)$ is called a *cofactor* of $f(x,y)$ and it has degree at most n.

If sufficient number of Darboux factors are found, then so-called *Darboux first integral* can be constructed.

Let $f_1,\ldots,f_s$ be Darboux factors such that $\alpha_j\in\mathbb{C}$ for $1\le j\le s$. A first integral of system (2.4) of the form

$$H = f_1^{\alpha_1}\cdots f_s^{\alpha_s}$$

is called a *Darboux first integral* of system (2.4).

For two specific systems of the form (2.4), *Hamiltonian system* and *time-reversible system*, it is known that the singularity of the origin is a center, see [44].

We recall that: System (2.4) is a *Hamiltonian system* if there is a function $H : \mathbb{C}^2 \to \mathbb{C}$ called Hamiltonian, such that $P_1 = -H_y$ and $Q_1 = H_x$.

Clearly, the Hamiltonian is a first integral of the system.

The definition of time-reversibility of the system is the following.

Definition 3.2. The system $\frac{dz}{dt} = F(z)$, where $z = (x, y) \in \mathbb{C}^2$, is time-reversible if there exists a transformation $T(x, y) = (\gamma x, \gamma^{-1} y)$, for $\gamma \in \mathbb{C} \setminus \{0\}$, such that

$$\frac{d(Tz)}{dt} = F(Tz).$$

In the proofs of the following theorems the results of [29] on time-reversibility of the cubic systems will be important.

Next we present the results on integrability of system (1.1).

Theorem 3.3. *System (1.1) with $a_{-13} = b_{3,-1} = 1$ is integrable if and only if one of the following conditions holds:*

1. $a_{11} - b_{11} = b_{01} = a_{10} = a_{02} - 3b_{02} = 3a_{20} - b_{20} = 0,$
2. $a_{11} - b_{11} = a_{20} + b_{02} = a_{02} + b_{20} = a_{10}^2 + b_{01}^2 = 0,$
3. $a_{11} - b_{11} = a_{20} - b_{02} = a_{02} - b_{20} = a_{10} - b_{01} = 0,$
4. $a_{11} - b_{11} = a_{20} - b_{02} = a_{02} - b_{20} = a_{10} + b_{01} = 0,$
5. $a_{11} = b_{11} = a_{10} = b_{01} = a_{20} + 3b_{20} = 3a_{02} + b_{02} = 4b_{20}b_{02} + 3 = 0.$

Proof. The computation of necessary conditions
With the computer algebra system Mathematica we were able to compute first nine non-zero focus quantities using algorithm presented in [44]. Due to the large size of the focus quantities, we present here only two

$$\begin{aligned}
g_{11} &= a_{01}a_{10} + a_{11} - b_{01}b_{10} - b_{11};\\
g_{22} &= (24a_{01}^2a_{10}^2 + 24a_{01}a_{10}a_{11} + 6a_{01}^2a_{20} + 3a_{02}a_{20} + 2a_{10}a_{-12}a_{20}\\
&\quad - 18a_{01}a_{10}^2b_{01} - 18a_{10}a_{11}b_{01} - 3a_{01}a_{20}b_{01} - 27a_{01}^2a_{10}b_{10}\\
&\quad + 3a_{02}a_{10}b_{10} - 27a_{01}a_{11}b_{10} + 2a_{10}^2a_{-12}b_{10} + 5a_{-12}a_{20}b_{10}\\
&\quad + 21a_{11}b_{01}b_{10} + 18a_{10}b_{01}^2b_{10} + 3a_{10}b_{02}b_{10} + 3a_{10}a_{-12}b_{10}^2\\
&\quad + 27a_{01}b_{01}b_{10}^2 - 24b_{01}^2b_{10}^2 - 6b_{02}b_{10}^2 - 2a_{-12}b_{10}^3 - 21a_{01}a_{10}b_{11}\\
&\quad + 18a_{10}b_{01}b_{11} + 27a_{01}b_{10}b_{11} - 24b_{01}b_{10}b_{11} + 2a_{10}a_{-12}b_{20}\\
&\quad - 3a_{01}b_{01}b_{20} - 3b_{02}b_{20} - 3a_{-12}b_{10}b_{20} + 2a_{01}^3b_{2,-1} + 3a_{01}a_{02}b_{2,-1}\\
&\quad - 4a_{11}a_{-12}b_{2,-1} + 2a_{10}a_{-13}b_{2,-1} - 3a_{01}^2b_{01}b_{2,-1} - 2a_{02}b_{01}b_{2,-1}\\
&\quad - 2a_{01}b_{01}^2b_{2,-1} - 5a_{01}b_{02}b_{2,-1} - 2b_{01}b_{02}b_{2,-1} - a_{-13}b_{10}b_{2,-1}\\
&\quad + 4a_{-12}b_{11}b_{2,-1} + a_{01}a_{-12}b_{3,-1} - 2a_{-12}b_{01}b_{3,-1})/3.
\end{aligned}$$

To obtain the necessary conditions for system to be integrable, the irreducible decomposition of integrability variety, $\mathbf{V}(B_9)$ needs to computed. The irreducible decomposition was computed using Singular [15] routine `minAssGTZ` [16].

Since the computation of irreducible decomposition is difficult, in many cases it is necessary to work in modular arithmetics instead of over the field of rational numbers. Since the obtained ideals have rational coefficients, the rational reconstruction needs to be done. For more informations on rational reconstruction algorithm see [53]. Working with modular arithmetics sometimes produces wrong conditions or do not produces all conditions, some can be lost. For this reason additional few steps need to be done.

The approach which can be used to check the conditions was suggested in [41].

The irreducible decomposition was computed over four different characteristics; 7919, 32003, 100109 and 104729. The approach described in [41] was not done completely, but in many cases computations are difficult even for more capable computers. But with high probability the list of conditions of Theorem 3.3 is complete.

The existence of the analytic first integral

Now we prove that under each of the conditions of Theorem 3.3 the system has a first integral.

Case 1. The system under conditions $a_{11} - b_{11} = b_{01} = a_{10} = a_{02} - 3b_{02} = 3a_{20} - b_{20} = 0$ is

$$\begin{aligned}\dot{x} &= x - a_{20}x^3 - b_{11}x^2y - a_{02}xy^2 - y^3,\\ \dot{y} &= -y + x^3 + b_{11}xy^2 + 3a_{20}x^2y + \frac{a_{02}}{3}y^3.\end{aligned}$$

It is a Hamiltonian system. The first integral is $\psi(x,y) = xy - \frac{x^4}{4} - \frac{y^4}{4} - a_{20}x^3y - \frac{b_{11}}{2}x^2y^2 - \frac{a_{02}}{3}xy^3$.

Case 2. Conditions $a_{11} - b_{11} = a_{20} + b_{02} = a_{02} + b_{20} = a_{10}^2 + b_{01}^2 = 0$ satisfy the conditions for time-reversible cubic system written in [44], hence the system is time-reversible.

Case 3 and *Case 4.* systems are of form

$$\begin{aligned}\dot{x} &= x - a_{10}x^2 - a_{20}x^3 - a_{11}x^2y - a_{02}xy^2 - y^3,\\ \dot{y} &= -y \pm a_{10} + x^3 + a_{11}xy^2 + a_{02}x^2y + a_{20}y^3.\end{aligned}$$

The system, the same as in Case 2, is time-reversible, since it satisfies the conditions for time-reversible cubic system.

Case 5. The conditions $a_{11} = b_{11} = a_{10} = b_{01} = a_{20} + 3b_{20} = 3a_{02} + b_{02} = 4b_{20}b_{02} + 3 = 0$ yield the system

$$\begin{aligned}\dot{x} &= x - \frac{9}{4b_{02}}x^3 + \frac{b_{02}}{3}xy^2 - y^3,\\ \dot{y} &= x^3 - y - \frac{3}{4b_{02}}x^2y + b_{02}y^3.\end{aligned}$$

We obtain three Darboux factors of this system, one of degree four,

$$\begin{aligned}l_1(x,y) = 1 &- \frac{3}{2b_{02}}x^2 + \frac{b_{02}^2}{9}x^4 - \frac{9}{4b_{02}^2}xy - \frac{4b_{02}^2}{9}xy + \frac{2b_{02}}{3}x^3y - \frac{2b_{02}}{3}y^2 + \frac{3}{2}x^2y^2\\ &+ \frac{3}{2b_{02}}xy^3 + \frac{9}{16b_{02}^2}y^4,\end{aligned}$$

and two of degree six,

$$\begin{aligned} l_2(x,y) = 1 &- \frac{9}{2b_{02}}x^2 + \frac{81}{16b_{02}^2}x^4 - \frac{b_{02}}{3}x^6 + 2b_{02}x^3y - \frac{3}{2}x^5y - 2b_{02}y^2 + \frac{3}{2}x^2y^2 \\ &- \frac{9}{4b_{02}}x^4y^2 + \frac{9}{2b_{02}}xy^3 - \frac{9}{8b_{02}^2}x^3y^3 - \frac{2b_{02}^2}{9}x^3y^3 + b_{02}^2y^4 - b_{02}x^2y^4 \\ &- \frac{3}{2}xy^5 + -\frac{3}{4b_{02}}y^6 \end{aligned}$$

and

$$\begin{aligned} l_3(x,y) = 1 &- \frac{9x^2}{4b_{02}} - \frac{216b_{02}^2xy}{81+16b_{02}^4} - b_{02}y^2 + \frac{54b_{02}^2x^4}{81+16b_{02}^4} + \frac{9}{2}x^2y^2 + \frac{54b_{02}^2y^4}{81+16b_{02}^4} \\ &+ \frac{b_{02}(243+16b_{02}^4)x^3y}{81+16b_{02}^4} + \frac{27(27+16b_{02}^4)xy^3}{4b_{02}(81+16b_{02}^4)} - \frac{8b_{02}^5x^6}{3(81+16b_{02}^4)} \\ &- \frac{24b_{02}^4x^5y}{81+16b_{02}^4} - \frac{90b_{02}^3x^4y^2}{81+16b_{02}^4} - \frac{180b_{02}^2x^3y^3}{81+16b_{02}^4} - \frac{405b_{02}x^2y^4}{2(81+16b_{02}^4)} \\ &- \frac{243xy^5}{2(81+16b_{02}^4)} - \frac{243y^6}{8b_{02}(81+16b_{02}^4)}. \end{aligned}$$

Two of these three Darboux factors construct the first integral

$$\psi(x,y) = C(l_1^3 l_2 - l_1^3 l_2^{-1}) = xy + \ldots,$$

where $C = \frac{6b_{02}^2}{81+16b_{02}^4}$ and $81 + 16b_{02}^4 \neq 0$.

In case $81 + 16b_{02}^4 = 0$, the first integral is of form

$$\psi(x,y) = \frac{1}{4}(4 - 4(-1)^{\frac{3}{4}}x^2 + ix^4 - 4(-1)^{\frac{1}{4}}x^3y + 4(-1)^{\frac{1}{4}}y^2 + 6x^2y^2 + 4(-1)^{\frac{3}{4}}xy^3 - iy^4). \quad \square$$

Theorem 3.4. *The system* (1.1) *with* $a_{-13} = b_{3,-1} = 0$ *is integrable if and only if one of the following conditions holds:*

1. $a_{11} = b_{11} = b_{20} = a_{20} = 0$,
2. $a_{11} = b_{11} = b_{20} = a_{02} = 0$,
3. $a_{11} = b_{11} = b_{02} = a_{02} = 0$,
4. $a_{11} - b_{11} = a_{02}a_{20} - b_{20}b_{02} = a_{20}b_{01}^2 - a_{10}^2b_{02} = a_{10}^2a_{02} - b_{01}^2b_{20} = 0$,
5. $a_{11} - b_{11} = a_{20} + b_{20} = a_{02} + b_{02} = 0$.

Proof. The computation of necessary conditions The computation of irreducible decomposition of variety of ideal $\mathcal{B}_9$ with additional conditions $a_{-13} = b_{3,-1} = 0$, was not too extensive and difficult, hence it was done over the field of rational numbers. This way conditions of Theorem 3.4 were obtained.

The existence of the analytic first integral

The system (1.1) with $a_{-13} = b_{3,-1} = 0$ is Lotka–Volterra system, which was studied in [18].

Case 1. The system under conditions $a_{11} = b_{11} = b_{20} = a_{20} = 0$ is equivalent to the system of Case 4 of Theorem 1.4 in [18].

Case 2. Conditions $a_{11} = b_{11} = b_{20} = a_{02} = 0$ yield the Case 3 of Theorem 1.4 in [18].

Case 3. Conditions $a_{11} = b_{11} = b_{02} = a_{02} = 0$ yield the system that is equivalent to the system of Case 5 of Theorem 1.4 in [18].

Case 4. The Case 4 is Case 2 of Theorem 1.4 in [18].

Case 5. Conditions $a_{11} - b_{11} = a_{20} + b_{20} = a_{02} + b_{02} = 0$ are conditions of Case 1 of Theorem 1.4 in [18]. □

Theorem 3.5. *The system* (1.1) *with* $a_{-13} = 1$ *and* $b_{3,-1} = 0$ *is integrable if and only if one of the following conditions holds:*

1. $a_{11} - b_{11} = b_{20} = a_{20} = a_{10} = 0,$
2. $a_{11} = b_{11} = b_{20} = a_{20} = 0,$
3. $a_{11} - b_{11} = a_{10} = b_{01} = 3a_{20} - b_{20} = a_{02} - 3b_{02} = 0,$
4. $a_{11} = b_{11} = a_{20} + 3b_{20} = b_{01} = b_{02} = a_{02} = a_{10} = 0.$

Proof. The computation of necessary conditions

The conditions were obtained similar as in case of Theorem 3.3.

The existence of the analytic first integral

Case 1. The corresponding system for conditions $a_{11} - b_{11} = b_{20} = a_{20} = a_{10} = 0$ is

$$\begin{aligned} \dot{x} &= x - a_{11}x^2y - a_{02}xy^2 - y^3, \\ \dot{y} &= -y + b_{01}y^2 + a_{11}xy^2 + b_{02}y^3. \end{aligned}$$

This system is time-reversible, hence integrable.

Case 2. In this case system is of the form

$$\begin{aligned} \dot{x} &= x - a_{10}x^2 - a_{02}xy^2 - y^3, \\ \dot{y} &= -y + b_{01}y^2 + b_{02}y^3. \end{aligned}$$

Darboux factors found for this system are

$$l_1(x,y) = y,\, l_{2,3}(x,y) = \frac{1}{2}\left(2 - b_{01}y \pm \sqrt{b_{01}^2 + 4b_{02}y}\right),$$

but using them we were not able to construct Darboux first integral or Darboux integrating factor. For this reason we looked for a first integral of the form $\psi(x,y) = \sum_{k=1}^{\infty} f_k(x)y^k$. The function $f_k(x)$ is defined by recursive differential equation

$$\begin{aligned} (k-2)b_{02}f_{k-2}(x) + (k-1)b_{01}f_{k-1}(x) - kf_k(x) - f'_{k-3}(x) + \\ - a_{02}xf'_{k-2}(x) + x(1 - a_{10}x)f'_k(x) = 0. \quad (3.2) \end{aligned}$$

Using induction we show that for every odd number, $k = 2n - 1$, is $f_{2n-1}(x) = \frac{p_n(x)}{(-1+a_{10}x)^{2n-1}}$ and for every even number, $k = 2n$, is $f_{2n}(x) = \frac{p_n(x)}{(-1+a_{10}x)^{2n}}$.

Proving first the assumption for odd numbers.

For $k=1$: $f_1(x)=\frac{-x}{(-1+a_{10}x)}$. Let us assume that the assumption holds for all $l<2n-1$. We need to show that it holds for $2n-1$. Using assumptions in (3.2) for every $l<2n-1$ we obtain differential equation

$$\frac{p_n(x)}{x(-1+a_{10}x)^{2n-1}}=\frac{(2n-1)}{x(-1+a_{10}x)}f_{2n-1}(x)+f'_{2n-1}(x),$$

which has solution

$$\begin{aligned}f_{2n-1}(x)&=\frac{x^{2n-1}}{(-1)^{2n-1}(-1+a_{10}x)^{2n-1}}\int\frac{(-1)^{2n-1}p_n(x)(-1+a_{10}x)^{2n-1}}{x^{2n}(-1+a_{10}x)^{2n-1}}dx\\&=\frac{x^{2n-1}}{(-1+a_{10}x)^{2n-1}}\int\frac{p_n(x)}{x^{2n}}dx=\frac{x^{2n-1}}{(-1+a_{10}x)^{2n-1}}\frac{p_n(x)}{x^{2n-1}}=\frac{p_n(x)}{(-1+a_{10}x)^{2n-1}}.\end{aligned}$$

In the same way this can be proven for even numbers k.

For $k=2$: $f_2(x)=\frac{b_{01}x}{(-1+a_{10}x)^2}$ and

$$\frac{p_n(x)}{x(-1+a_{10}x)^{2n}}=\frac{2n}{x(-1+a_{10}x)}f_{2n}(x)+f'_{2n}(x)$$

needs to hold. Solving this differential equation we obtain $f_{2n}(x)=\frac{p_n(x)}{(-1+a_{10}x)^{2n}}$, as needed.

Case 3. The system corresponding to conditions $a_{11}-b_{11}=a_{10}=b_{01}=3a_{20}-b_{20}=a_{02}-3b_{02}=0$ is

$$\begin{aligned}\dot{x}&=x-b_{11}x^2y-\frac{b_{20}}{3}x^3-3b_{02}xy^2-y^3,\\\dot{y}&=-y+b_{11}xy^2+b_{20}x^2y+b_{02}y^3.\end{aligned}$$

This is Hamiltonian system and the first integral is

$$\psi(x,y)=xy-\frac{b_{20}}{3}x^3y-\frac{b_{11}}{2}x^2y^2-b_{02}xy^3-\frac{y^4}{4}.$$

Case 4. The system in this case is

$$\begin{aligned}\dot{x}&=x-a_{20}x^3-y^3,\\\dot{y}&=y(-1+\frac{a_{20}}{3}x^2).\end{aligned}$$

Darboux factors of this system are $l_1(x,y)=y$, $l_2(x,y)=x-\frac{y^3}{4}$ and two Darboux factors of degree six,

$$l_3(x,y)=\frac{1}{9}(9-18a_{20}x^2+9a_{20}x^4+18a_{20}xy^3-2a_{20}^2x^3y^3-3a_{20}y^6)$$

and

$$l_4(x,y)=\frac{1}{6}(6-6a_{20}x^2+6a_{20}xy^3-a_{20}y^6).$$

Using three of four Darboux factors we obtain first integral

$$\psi(x,y)=l_1l_2l_3^{-\frac{1}{3}}=xy+\cdots$$

□

Studying integrability of the systems of higher degrees is difficult, mostly because of computation of irreducible decomposition. Due to these problem we splitted the research of the system (1.1) to four cases, as explained before in Section 1. The fact is that by the involution of parameters $a_{ij} \leftrightarrow b_{ji}$ we can transforms case 3) of (1.2), where additional conditions are $a_{-13} = 1$ and $b_{3,-1} = 0$, into case 4), where $a_{-13} = 0$ and $b_{3,-1} = 1$. Hence only three of four cases needed to be studied. In theorems 3.3, 3.4 and 3.5 the obtained results are presented and in the proofs all procedures of obtaining these conditions are explained into details.

By fixing some coefficients and splitting the study of the system (1.1), the general conditions of integrability of this system were not obtained. But as it will be explained here the general conditions of integrability of the system (1.1) can be computed using conditions of Theorems 3.3, 3.4 and 3.5.

The main theory behind obtaining the general results is the elimination theory. More on this theory can be read in [13, Chapter 3] or [44, Chapter 1.3]. Before explaining the whole procedure for obtaining the general conditions, some important facts on the elimination theory need to be given.

Definition 3.6. Let $I = \langle f_1, \dots, f_m \rangle$ be ideal in $k[x_1, \dots, x_n]$ (with the implicit ordering of the variables $x_1 > x_2 > \dots > x_n$) and fix $l \in \{0, 1, \dots, n-1\}$. The *$l$-th elimination ideal of I is the ideal $I_l = I \cap k[x_{l+1}, x_{l+2}, \dots, x_n]$. Any point $(a_{l+1}, \dots, a_n) \in \mathbf{V}(I_l)$ is called partial solution of the system $\{f = 0; f \in I\}$.*

Geometrically, the elimination is the projection of $\mathbf{V}(I) \subset k^n$ on the lower dimensional subspace k^{n-l}.

The method for computing the elimination ideal I_l is provided in the following theorem.

Theorem 3.7. *Fix the lexicographic term order on the ring $k[x_1, \dots, x_n]$ with $x_1 > x_2 > \dots > x_n$ and let G be a Gröbner basis for an ideal I of $k[x_1, \dots, x_n]$ with respect to this order. Then for every l, $0 \le l \le n-1$, the set $G_l := G \cap k[x_{l+1}, \dots, x_n]$ is a Gröbner basis for the l-th elimination ideal I_l.*

The procedure of obtaining the general results is based on the following observations.

Taking the variables

$$x_1 \to ax, \quad y_1 \to by$$

changes the system (1.1) into the system

$$\begin{aligned}
\dot{x_1} &= x_1 - \alpha_{10}x_1^2 - \alpha_{20}x_1^3 - \alpha_{01}x_1y_1 - \alpha_{11}x_1^2y_1 - \alpha_{-12}y_1^2 - \alpha_{02}x_1y_1^2 - \alpha_{-13}y_1^3,\\
\dot{y_1} &= -y_1 + \beta 2, -1x_1^2 + \beta 3, -1x_1^3 + \beta_{10}x_1y_1 + \beta_{02}x_1^2y_1 + \beta_{01}y_1^2 + \beta_{11}x_1y_1^2 + \beta_{02}y_1^3,
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
\alpha_{10} &= \frac{a_{10}}{a}, & \beta_{2,-1} &= \frac{bb_{21}}{a^2},\\
\alpha_{20} &= \frac{a_{20}}{a^2}, & \beta_{3,-1} &= \frac{bb_{3,-1}}{a^3},\\
\alpha_{01} &= \frac{a_{01}}{b}, & \beta_{10} &= \frac{b_{10}}{a},\\
\alpha_{11} &= \frac{a_{11}}{ab}, & \beta_{20} &= \frac{b_{20}}{a^2},\\
\alpha_{-12} &= \frac{aa_{-12}}{b^2}, & \beta_{01} &= \frac{b_{01}}{b},
\end{aligned}$$

$$\alpha_{02} = \frac{a_{02}}{b^2}, \qquad \beta_{11} = \frac{b_{11}}{ab},$$
$$\alpha_{-13} = \frac{aa_{-13}}{b^3}, \qquad \beta_{02} = \frac{b_{02}}{b^2}.$$

The focus quantities of both systems, (1.1) and (3.3), are different only by the constant factor. This constant factor does not make a difference for the center variety, hence the irreducible decomposition of both varieties generates the same conditions.

As it is seen from the system (3.3), each nonzero coefficient can be rescaled so that obtained coefficient is equal to 1. Similar, coefficients can be set equal to zero.

Hence by splitting our studies as presented in Section 1, the general results were not lost. These can be obtained with the approach described below.

For the case 1), where $a_{-13} = b_{3,-1} = 1$, the coefficients α_{-13} and $\beta_{3,-1}$ need to fulfil $\alpha_{-13} = ab^{-3}$ and $\beta_{3,-1} = a^{-3}b$, with additional restrictions $a \neq 0$ and $b \neq 0$. These additional restrictions can be written in the term of polynomial as $1 - wa$, respectively $1 - vb$. The other conditions of Theorem 3.3 change regarding $a_{i,j} = \alpha_{i,j}a^{-i}b^j$ and $b_{i,j} = \beta_{i,j}a^{-i}b^j$, where $i, j \in \{-1, \ldots, 3\}$. This way ideals $I_1, \ldots, I_5 \in \mathbb{C}[w, v, a, b, A, B]$, where $A = \{a_{10}, a_{20}, a_{11}, a_{02}, a_{-13}\}$ and $B = \{b_{01}, b_{02}, b_{11}, b_{20}, b_{3,-1}\}$ are formed,

$$\begin{aligned}
I_1 = \langle &1 - wa, 1 - vb, ab(a_{11} - b_{11}), bb_{01}, aa_{10}, b^2(a_{02} - 3b_{02}), a^2(3a_{20} - b_{20}),\\
&- a + b^3a_{-13}, -b + a^3b_{3,-1}\rangle\\
I_2 = \langle &1 - wa, 1 - vb, ab(a_{11} - b_{11}), a^2a_{20} + b^2b_{02}, b^2a_{02} + a^2b_{20}, a^2a_{10}^2 + b^2b_{01}^2,\\
&- a + b^3a_{-13}, -b + a^3b_{3,-1}\rangle\\
I_3 = \langle &1 - wa, 1 - vb, ab(a_{11} - b_{11}), a^2a_{20} - b^2b_{02}, b^2a_{02} - a^2b_{20}, aa_{10} - bb_{01},\\
&- a + b^3a_{-13}, -b + a^3b_{3,-1}\rangle\\
I_4 = \langle &1 - wa, 1 - vb, ab(a_{11} - b_{11}), a^2a_{20} - b^2b_{02}, b^2a_{02} - a^2b_{20}, aa_{10} + bb_{01},\\
&- a + b^3a_{-13}, -b + a^3b_{3,-1}\rangle\\
I_5 = \langle &1 - wa, 1 - vb, aba_{11}, abb_{11}, aa_{10}, bb_{01}, a^2(a_{20} + 3b_{20}), b^2(3a_{02} + b_{02},\\
&3 + 4a^2b^2b_{02}b_{20}, -a + b^3a_{-13}, -b + a^3b_{3,-1}\rangle.
\end{aligned}$$

Similar we obtain ideals $I_6, \ldots, I_{10}$ from conditions of Theorem 3.4. Ideals $I_{11}, \ldots, I_{14}$ were gained from Theorem 3.5 and $I_{15}, \ldots, I_{18}$ by involution of coefficients in conditions of Theorem 3.5.

From the obtained ideals $I_1, \ldots, I_{18}$ we eliminate, using Singular routine `eliminate`, variables w, v, a and b. The elimination ideals are

$$J_1' = I_1 \cap \mathbb{C}[a_{10}, a_{20}, a_{11}, a_{02}, a_{-13}, b_{01}, b_{02}, b_{11}, b_{20}, b_{3,-1}], \ldots, J_{18}'.$$

Then we compute irreducible decomposition (Singular routine `minAssGTZ`) of each obtained eliminated ideal, gaining ideals $J_1, \ldots, J_{18}$:

$$\begin{aligned}
J_1 = \langle &b_{01}, a_{11} - b_{11}, 3a_{20} - b_{20}, a_{02} - 3b_{02}, a_{10}\rangle,\\
J_2 = \langle &a_{11} - b_{11}, a_{20}^2a_{-13} - b_{02}^2b_{3,-1}, a_{02}b_{02}b_{3,-1} - a_{20}a_{-13}b_{20}, a_{02}a_{20} - b_{20}b_{02},\\
&a_{02}^2b_{3,-1} - a_{-13}b_{20}^2, a_{10}^2b_{02} - a_{20}b_{01}^2, a_{10}^2a_{-13}b_{20} - a_{02}b_{01}^2b_{3,-1},\\
&a_{10}^2a_{20}a_{-13} - b_{01}^2b_{02}b_{3,-1}, a_{10}^2a_{02} - b_{01}^2b_{20}, a_{10}^4a_{-13} - b_{01}^4b_{3,-1}\rangle,
\end{aligned}$$

$$\begin{aligned}
J_3 = \langle & a_{11} - b_{11}, a_{20}^2 a_{-13} - b_{02}^2 b_{3,-1}, a_{02} b_{02} b_{3,-1} - a_{20} a_{-13} b_{20}, a_{02} a_{20} - b_{20} b_{02}, \\
& a_{02}^2 b_{3,-1} - a_{-13} b_{20}^2, a_{10}^2 b_{02} - a_{20} b_{01}^2, a_{10}^2 a_{-13} b_{20} - a_{02} b_{01}^2 b_{3,-1}, \\
& a_{10}^2 a_{20} a_{-13} - b_{01}^2 b_{02} b_{3,-1}, a_{10}^2 a_{02} - b_{01}^2 b_{20}, a_{10}^4 a_{-13} - b_{01}^4 b_{3,-1} \rangle, \\
J_4 = \langle & a_{11} - b_{11}, a_{20}^2 a_{-13} - b_{02}^2 b_{3,-1}, a_{02} b_{02} b_{3,-1} - a_{20} a_{-13} b_{20}, a_{02} a_{20} - b_{20} b_{02}, \\
& a_{02}^2 b_{3,-1} - a_{-13} b_{20}^2, a_{10}^2 b_{02} - a_{20} b_{01}^2, a_{10}^2 a_{-13} b_{20} - a_{02} b_{01}^2 b_{3,-1}, \\
& a_{10}^2 a_{20} a_{-13} - b_{01}^2 b_{02} b_{3,-1}, a_{10}^2 a_{02} - b_{01}^2 b_{20}, a_{10}^4 a_{-13} - b_{01}^4 b_{3,-1} \rangle, \\
J_5 = \langle & b_{11}, b_{01}, a_{11}, 3a_{-13} b_{3,-1} + 4 b_{20} b_{02}, a_{20} + 3 b_{20}, 3 a_{02} + b_{02}, a_{10} \rangle, \ldots
\end{aligned}$$

By computing intersection of obtained ideals J_i, $J = \cap_{i=1}^{18} J_i$ (Singular routine `intersect`) and then using Singular routine `minAssGTZ` to compute irreducible decomposition of $\mathbf{V}(J)$, we obtain list of conditions from Theorem 1.1. For more details on this approach see [21].

4 Cyclicity of components of the center variety

In this section we will presented results connected to cyclicity of the specific family of real cubic system.

The researched real system was obtained from the complex system (1.1) by setting

$$\begin{aligned}
&a_{10} = A_{10} + iB_{10}, \quad b_{01} = A_{10} - iB_{10}, \quad a_{20} = A_{20} + iB_{20}, \quad b_{02} = A_{20} - iB_{20}, \\
&a_{02} = A_{02} + iB_{02}, \quad b_{20} = A_{02} - iB_{02}, \quad a_{11} = A_{11} + iB_{11}, \quad b_{11} = A_{11} - iB_{11}, \\
&\qquad\qquad a_{-13} = A_{-13} + iB_{-13}, \quad b_{3,-1} = A_{-13} - iB_{-13}.
\end{aligned} \tag{4.1}$$

In the same way, by setting (4.1), the real center variety was obtained from the center variety presented in Theorem 1.1. The studied real system is of the form

$$\begin{aligned}
\dot{x} = i(x &- (A_{10}x^2 + A_{20}x^3 + A_{11}x^2\bar{x} + A_{02}x\bar{x}^2 + A_{-13}\bar{x}^3) \\
&- i(B_{10}x^2 + B_{20}x^3 + B_{11}x^2\bar{x} + B_{02}x\bar{x}^2 + B_{-13}\bar{x}^3)).
\end{aligned} \tag{4.2}$$

Theorem 4.1. *The center variety in $\mathbb{R}^{10}$ of the real system (4.2) consists of the following irreducible components:*

1) $3B_{20} + B_{02} = B_{11} = B_{10} = 3A_{20} - A_{02} = A_{10} = 0$,

2) $B_{20} - 3B_{02} = B_{11} = B_{10} = A_{20} + 3A_{02} = A_{11} = A_{10} = A_{-13}^2 + B_{-13}^2 - 4A_{02}^2 - 4B_{02}^2 = 0$,

3) $B_{11} = A_{02}B_{20} + A_{20}B_{02} = A_{02}^2 B_{-13} - 2A_{-13}A_{02}B_{02} - B_{-13}B_{02}^2 = A_{20}A_{02}B_{-13} - 2A_{-13}A_{20}B_{02} + B_{-13}B_{20}B_{02} = A_{20}^2 B_{-13} + 2A_{-13}A_{20}B_{20} - B_{-13}B_{20}^2 = 2A_{10}A_{02}B_{10} + A_{10}^2 B_{02} - B_{10}^2 B_{02} = 2A_{10}A_{20}B_{10} - A_{10}^2 B_{20} + B_{10}^2 B_{20} = A_{10}^2 A_{02} B_{-13} - A_{02} B_{10}^2 B_{-13} - 2A_{10}^2 A_{-13} B_{02} + 2A_{-13} B_{10}^2 B_{02} + 2A_{10} B_{10} B_{-13} B_{02} = A_{10}^2 A_{20} B_{-13} - A_{20} B_{10}^2 B_{-13} + 2A_{10}^2 A_{-13} B_{20} - 2A_{-13} B_{10}^2 B_{20} - 2A_{10} B_{10} B_{-13} B_{20} = 2A_{02} B_{10}^3 B_{-13} + 4A_{10}^2 A_{-13} B_{10} B_{02} - 4A_{-13} B_{10}^3 B_{02} + A_{10}^3 B_{-13} B_{02} - 5A_{10} B_{10}^2 B_{-13} B_{02} = 2A_{20} B_{10}^3 B_{-13} - 4A_{10}^2 A_{-13} B_{10} B_{20} + 4A_{-13} B_{10}^3 B_{20} - A_{10}^3 B_{-13} B_{20} + 5A_{10} B_{10}^2 B_{-13} B_{20} = 4A_{10}^3 A_{-13} B_{10} - 4A_{10} A_{-13} B_{10}^3 + A_{10}^4 B_{-13} - 6A_{10}^2 B_{10}^2 B_{-13} + B_{10}^4 B_{-13} = 0$,

4) $B_{20} - B_{02} = B_{-13} = B_{11} = A_{02} + A_{20} = A_{-13} = 0$,

5) $B_{02} = B_{-13} = B_{11} = A_{02} = A_{11} = A_{-13} = 0$.

The dimension of these components is 5, 3, 5, 5, 4, respectively.

Proof. The center variety of the real system (4.2) was obtain from complex variety of Theorem 1.1. The change of coefficients in the way as written at (4.1) and then by elimination of complex unit i from obtained ideals produced the conditions of Theorem 4.1. The conditions 4), 5), 6), 7) of Theorem 1.1 yield conditions 4), 3), 2), 1) of this theorem and the condition 5) was obtain from 2). The other obtained conditions are subvarieties of 3), 4) and 5).

As we can see from 1), 4) and 5) the number of parameters in these components is equal to 5, 5 and 6. Hence the dimension is 5, 5 and 4, since the number of all parameters is 10 and 5 (respectively 5 and 4) parameters are free.

The dimension of remaining components is not obvious as in three cases before. By the Theorem 2 of [13, Chapter 3.3] the upper bound of dimension can be determine from obtained rational parametrization. For the case 2) the parametrization is

$$\begin{aligned}
B_{10} = A_{10} = B_{11} = A_{11} = f_0 = 0, \quad & A_{02} = f_1(u_1, u_2, u_3)/g_2(u_3),\\
B_{02} = f_2(u_1, u_2, u_3)/g_2(u_3), \quad & A_{20} = f_3(u_1, u_2, u_3)/g_2(u_3),\\
B_{20} = f_4(u_1, u_2, u_3)/g_2(u_3), \quad & B_{-13} = f_5(u_1, u_2, u_3)/(g_1(u_2)g_2(u_3)),\\
A_{-13} = f_6(u_1, u_2, u_3)/(g_1(u_2)g_2(u_3)), &
\end{aligned}$$

where

$$\begin{gathered}
f_1(u_1,u_2,u_3) = u_1(1-u_3^2), \quad f_2(u_1,u_2,u_3) = 2u_1u_3,\\
f_3(u_1,u_2,u_3) = -3u_1(1-u_3^2), \quad f_4(u_1,u_2,u_3) = 6u_1u_3,\\
f_5(u_1,u_2,u_3) = -u_1(u_2+u_3)(-1+u_2u_3),\\
f_6(u_1,u_2,u_3) = \frac{1}{2}u_1(-1-u_2-u_3+u_2u_3)(-1+u_2+u_3+u_2u_3),\\
g_1(u_2) = 1+u_2^2, \quad g_2(u_3) = 1+u_3^2
\end{gathered}$$

and the components dimension is less or equal three, since these functions depends on three variables, u_1, u_2 and u_3. To know if the dimension is exactly three, Jacobian of the functions $f_0(u_1,u_2,u_3), \ldots, f_6(u_1,u_2,u_3)$ needs to be computed. The Jacobian in some arbitrary point, $u_1 = 1, u_2 = 4, u_3 = 2$, is three, hence the dimension is equal to three.

In the same way we obtain the dimension for component 3). The parametrization is

$$\begin{gathered}
B_{11} = f_0 = 0, \quad B_{10} = f_1(u_1,u_2,u_3,u_4,u_5) = u_1,\\
A_{10} = f_2(u_1,u_2,u_3,u_4,u_5) = u_2, \quad A_{20} = f_3(u_1,u_2,u_3,u_4,u_5) = u_3,\\
B_{20} = f_4(u_1,u_2,u_3,u_4,u_5)/(g_3(u_1,u_2)g_4(u_1,u_2)),\\
A_{02} = f_5(u_1,u_2,u_3,u_4,u_5) = u_4, \quad B_{02} = f_6(u_1,u_2,u_3,u_4,u_5)/(g_3(u_1,u_2)g_4(u_1,u_2)),\\
A_{-13} = f_7(u_1,u_2,u_3,u_4,u_5)/((f_1(u_1)f_2(u_2)g_3(u_1,u_2)g_4(u_1,u_2))),
\end{gathered}$$

where

$$\begin{gathered}
f_4(u_1,u_2,u_3,u_4,u_5) = -2u_1u_2u_3, \ f_6(u_1,u_2,u_3,u_4,u_5) = u_4,\\
f_7(u_1,u_2,u_3,u_4,u_5) = -2u_1u_2u_4, \ g_3(u_1,u_2) = u_1-u_2,\\
g_4(u_1,u_2) = u_1+u_2.
\end{gathered}$$

The dimension of this component is less or equal five and the Jacobian of $f_1(u_1,u_2,u_3,u_4,u_5)$, $\ldots, f_7(u_1,u_2,u_3,u_4,u_5)$ in random point $u_1 = 4, u_2 = 6, u_3 = 2, u_4 = 1, u_5 = 2$ is five, hence the dimension of this component of center variety is five. □

Theorem 4.2. *Let us define polynomials* $F_1 = A_{20}^2B_{-13} + 2A_{-13}A_{20}B_{20} - B_{-13}B_{20}^2$, $F_2 = (A_{02}B_{-13} - B_{02}A_{-13})(B_{02}B_{-13} + A_{02}A_{-13})$, $F_3 = 3A_{02}^2 + 2A_{02}A_{10}^2 - 8A_{02}A_{20} + 2A_{10}^2A_{20} - 3A_{20}^2 - 2A_{02}B_{10}^2 - 2A_{20}B_{10}^2$, $F_4 = (A_{10}^2 + B_{10}^2)(A_{02} + B_{02})(A_{02} - B_{02})$ *and* $F_5 = A_{10}^2B_{20} - B_{10}^2B_{20} - 2A_{10}A_{20}B_{10}$.

There are bifurcations of the system (4.2) which produce 3 limit cycles locally from the center corresponding to the parameter value p_1, where p_1 is a component 1) with $F_1(p_1) \neq 0$ of $\mathbb{R}^{10}$. The cyclicity of a generic point p_2 of component 2) with $F_2(p_2) \neq 0$ and of point p_4 with $F_4(p_4) \neq 0$ is 5. For the component 3) with $F_3(p_3) \neq 0$ the cyclicity is 4 and 6 for the component 5) with $F_5(p_5) \neq 0$.

Proof. Component 1) We choose an arbitrary point $p = (A_{10}, B_{10}, A_{20}, B_{20}, A_{02}, B_{02}, A_{11}, B_{11}, A_{-13}, B_{-13})$ of this component, $(0,0,1,1,3,-3,2,0,1,1)$, the rank of the Jacobian of the focus quantities, rank $J_p^{(k)} = 3$, is equal to three. By Theorem 2.3 the cyclicity of a generic point of this component is three.

Component 2) For the random point $p = (0,0,-3,3,1,1,0,0,-\sqrt{7},1)$ the rank of the Jacobian is five, rank $J_p^{(k)} = 5$, hence five limit cycles can bifurcate for these systems.

Component 3) The rank of Jacobian of the focus quantities at the point p, where $p = (2,1,\frac{3}{4},1,-\frac{3}{4},1,1,0,\frac{7}{24},1)$ of the component 3) is equal to four, rank $J_p^{(4)} = 4$.

Component 4) For the point $p = (1,1,-2,3,2,3,1,0,0,0)$ of the component 4) there can bifurcate up to five limit cycles, since the rank of Jacobian at the point p is five, rank $J_p^{(5)} = 5$.

Component 5) The cyclicity of the component 5) is six, since the rank of Jacobian of the focus quantities at the point $p = (2,3,1,1,0,0,0,0,0,0)$ of this component is six, rank $J_p^{(k)} = 6$. □

5 Conclusions

The main results in this paper are on integrability and cyclicity of cubic system. The computation of necessary conditions for system of the form (1.1) were difficult. It was impossible to compute over the field of rational numbers. To overcome the difficulties we have splitted our system into four subsystems, solved the integrability problem and from the integrability conditions for these subsystems we have reconstructed integrability variety of general system (1.1). From the results on integrability of complex cubic system, where seven conditions were obtained, see Theorem 1.1, we have obtained the conditions of associated real cubic systems. Results are presented in Thereom 4.1. For each of five obtained components of integrability variety of a real systems we studied the number of limit cycles that can bifurcate from it. It was shown that maximum limit cycles that can bifurcate from system (4.2) under some specific conditions is six. This number is, in comparison to result from Żołądek [56], where he proven that there are up to eleven limit cycles appearing, small.

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Ground state solutions for nonlinearly coupled systems of Choquard type with lower critical exponent

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Abstract. In this paper, we study the existence of ground state solutions for the following nonlinearly coupled systems of Choquard type with lower critical exponent by variational methods

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u + p|u|^{p-2}u|v|^q, & \text{in } \mathbb{R}^N, \\ -\Delta v + V(x)v = (I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}-1}v + q|v|^{q-2}v|u|^p, & \text{in } \mathbb{R}^N. \end{cases}$$

Where $N \geq 3$, $\alpha \in (0, N)$, I_α is the Riesz potential, $p, q \in \left(1, \sqrt{\frac{N}{N-2}}\right)$ and $Np + (N+2)q < 2N+4$, $\frac{N+\alpha}{N}$ is the lower critical exponent in the sense of Hardy–Littlewood–Sobolev inequality and $V \in C(\mathbb{R}^N, (0, \infty))$ is a bounded potential function. As far as we have known, little research has been done on this type of coupled systems up to now. Our research is a promotion and supplement to previous research.

Keywords: nonlinearly coupled systems, lower critical exponent, Choquard type equation, ground state solutions, variational methods.

2020 Mathematics Subject Classification: 35J10, 35J60, 35J65.

1 Introduction and main result

We are interested in the following nonlinearly coupled systems of Choquard type with lower critical exponent

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u + p|u|^{p-2}u|v|^q, & \text{in } \mathbb{R}^N, \\ -\Delta v + V(x)v = (I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}-1}v + q|v|^{q-2}v|u|^p, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.1}$$

Where the dimension $N \geq 3$ of $\mathbb{R}^N$ is given and function $I_\alpha : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is a Riesz potential of order $\alpha \in (0, N)$ defined for each $x \in \mathbb{R}^N \setminus \{0\}$,

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha|x|^{N-\alpha}},$$

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Γ denotes the classical Gamma function, $*$ represents the convolution product on $\mathbb{R}^N$, $p,q \in (1, \sqrt{\frac{N}{N-2}})$ and $Np + (N+2)q < 2N+4$, $V \in C(\mathbb{R}^N, (0,\infty))$ is a bounded potential function. More precisely, we make the following assumptions on V,

(V_1) $V_0 := \inf\limits_{x\in\mathbb{R}^N} V(x) > 0$;

(V_2) $V(x) < \lim\limits_{|y|\to\infty} V(y) = V_\infty < \infty$.

For the following Choquard equation

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

when $N=3$, $\alpha=2$, $p=2$ and V is a positive constant, this equation appears in several physical contexts, such as standing waves for the Hartree equation, the description of the quantum physics of a polaron at rest by S. I. Pekar in [13] and the modeling of an electron trapped in its own hole in 1976 in the work of Choquard, as a certain approximation to Hartree–Fock theory of one–component plasma (see [4]). In some particular cases, this equation is also known as the Schrödinger–Newton equation, which was introduced by R. Penrose [14] in his discussion on the selfgravitational collapse of a quantum mechanical wave function. The existence and uniqueness of positive solutions for equation (1.2) with $N=3$, $V(x)\equiv 1$, $\alpha=2$ and $p=2$ was firstly obtained by E. H. Lieb in [4]. Later, P. L. Lions [6,7] got the existence and multiplicity results of normalized solution on the same topic. Since then, the existence and qualitative properties of solutions for equation (1.2) have been widely studied by variational methods in the recent decades. For related topics, we refer the reader to the recent survey paper [12].

To study equation (1.2) variationally, the well-known Hardy–Littlewood–Sobolev inequality is the starting point. Particularly, V. Moroz and J. Van Schaftingen [9] established the existence, qualitative properties and decay estimates of ground state solutions for the autonomous case of equation (1.2) with $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ and $V(x)\equiv 1$. In view of the Pohožaev identity [9–11], Choquard equation (1.2) with V is a positive constant has no nontrivial smooth H^1 solution when either $p \le \frac{N+\alpha}{N}$ or $p \ge \frac{N+\alpha}{N-2}$. Usually, $\frac{N+\alpha}{N}$ is called the lower critical exponent and $\frac{N+\alpha}{N-2}$ is the upper critical exponent for Choquard equation in the sense of Hardy–Littlewood–Sobolev inequality. The upper critical exponent plays a similar role as the Sobolev critical exponent in the local semilinear equations. C. O. Alves, S. Gao, M. Squassina and M. Yang[1] established the existence of ground states for a type of critical Choquard equation with constant coefficients and also studied the existence and multiplicity of semi–classical solutions and characterized the concentration behavior by variational methods. G. Li and C. Tang [8] obtained a positive ground state solution for Choquard equation with upper critical exponent when the nonlinear perturbation satisfies the general subcritical growth conditions. The lower critical exponent seems to be a new feature for Choquard equation, which is related to a new phenomenon of "bubbling at infinity" (for more details see [10]).

J. Van Schaftingen and J. Xia [15] studied the ground state solutions of the following Choquard equation with lower critical exponent and coercive potential V,

$$-\Delta u + V(x)u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u, \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Later, J. Van Schaftingen and J. Xia [16] also obtained a ground state solution for the following Choquard equation with lower critical exponent and a local nonlinear perturbation

$$-\Delta u + u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u + f(x,u), \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

For the autonomous case $f(x,u) = f(u)$ satisfies some superlinear assumptions, the existence and symmetry of ground state for equation (1.4) were also got. Furthermore, they derived a ground state solution of equation (1.4) for the nonautonomous case $f(x,u) = K(x)|u|^{q-2}u$ with $q \in (2, 2+\frac{4}{N})$ and $K \in L^\infty(\mathbb{R}^N)$ satisfying $\inf_{x\in\mathbb{R}^N} K(x) = K_\infty = \lim_{|x|\to\infty} K(x) > 0$.

As we mentioned above, all the results in the literature are concerned with a single equation. More recently, P. Chen and X. Liu [2] obtained the existence of ground state solutions for the following linearly coupled systems of Choquard type with subcritical exponent $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$,

$$\begin{cases} -\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u + \lambda v, & \text{in } \mathbb{R}^N, \\ -\Delta v + v = (I_\alpha * |v|^p)|v|^{p-2}v + \lambda u, & \text{in } \mathbb{R}^N. \end{cases}$$

Later, M. Yang, J. de Albuquerque, E. Silva and M. Silva [19] obtained the existence of positive ground state solutions for the following linearly coupled systems of Choquard type

$$\begin{cases} -\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u + \lambda v, & \text{in } \mathbb{R}^N, \\ -\Delta v + v = (I_\alpha * |v|^q)|v|^{q-2}v + \lambda u, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.5}$$

when the exponents satisfy one of case 1, case 2 and case 3, and also obtained that there is no nontrivial solution for system (1.5) in case 4, where

case 1, $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ and $q = \frac{N+\alpha}{N-2}$,

case 2, $p = \frac{N+\alpha}{N}$ and $\frac{N+\alpha}{N} < q < \frac{N+\alpha}{N-2}$,

case 3, $p = \frac{N+\alpha}{N}$ and $q = \frac{N+\alpha}{N-2}$,

case 4, $p, q \le \frac{N+\alpha}{N}$ or $p, q \ge \frac{N+\alpha}{N-2}$.

Motivated by [2, 15, 16, 19], in this paper, we will study the existence of ground state solutions for system (1.1). Our main result reads as followed.

Theorem 1.1. *Let $N \ge 3$, $\alpha \in (0, N)$, $p, q \in \left(1, \sqrt{\frac{N}{N-2}}\right)$, $Np + (N+2)q < 2N+4$ and V satisfies $(V_1), (V_2)$, then system (1.1) admits at least one ground state solution.*

Remark 1.2. The assumption $Np + (N+2)q < 2N+4$ is mainly used to get the energy estimate of c_0 in Lemma 2.6. In particular, $p, q \in \left(1, \frac{N+2}{N+1}\right)$ satisfy our assumptions on p, q.

The method used to prove Theorem 1.1 is as follows. Firstly, we establish the variational framework for system (1.1). Let $H^1(\mathbb{R}^N)$ denote the normal Sobolev space equipped with the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) dx \right)^{\frac{1}{2}}.$$

Define $X = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ equipped with norm

$$\|(u,v)\| = (\|u\|^2 + \|v\|^2)^{\frac{1}{2}}.$$

Similar to $H^1(\mathbb{R}^N)$, X is a Hilbert space and satisfies

$$X \hookrightarrow L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N), \qquad p \in [2, 2^*], \quad \text{where } 2^* = \frac{2N}{N-2}.$$

By Hardy–Littlewood–Sobolev inequality and Sobolev embedding theorem, the energy functional associated to system (1.1)

$$
\begin{aligned}
J_V(u,v) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)|u|^2)dx + \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + V(x)|v|^2)dx \\
&\quad - \frac{N}{2(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}dx - \frac{N}{2(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1}dx \\
&\quad - \int_{\mathbb{R}^N}|u|^p|v|^q dx
\end{aligned}
$$

is $C^1(X,\mathbb{R})$ and

$$
\begin{aligned}
\langle J_V'(u,v),(\phi,\varphi)\rangle &= \int_{\mathbb{R}^N}(\nabla u\nabla\phi + V(x)u\phi)dx + \int_{\mathbb{R}^N}(\nabla v\nabla\varphi + V(x)v\varphi)dx \\
&\quad - \int_{\mathbb{R}^N}(I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u\phi)dx - \int_{\mathbb{R}^N}(I_\alpha * |v|^{\frac{\alpha}{N}+1}|v|^{\frac{\alpha}{N}-1}v\varphi)dx \\
&\quad - p\int_{\mathbb{R}^N}|v|^q|u|^{p-2}u\phi dx - q\int_{\mathbb{R}^N}|u|^p|v|^{q-2}v\varphi dx, \quad \text{for } (\phi,\varphi)\in X.
\end{aligned}
$$

Thus, any critical point of J_V is a weak solution of system (1.1). As usual, a nontrivial solution $(u,v)\in X$ of system (1.1) is called a ground state solution if

$$
J_V(u,v) = c_g^V := \inf\{J_V(u,v) : (u,v)\in X\setminus\{(0,0)\} \quad \text{and} \quad J_V'(u,v) = 0\}.
$$

Secondly, in the process of finding ground state solutions for system (1.1), the following limiting problem plays a significant role

$$
\begin{cases}
-\Delta u + V_\infty u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u + p|u|^{p-2}u|v|^q, & \text{in } \mathbb{R}^N, \\
-\Delta v + V_\infty v = (I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}-1}v + q|v|^{q-2}v|u|^p, & \text{in } \mathbb{R}^N.
\end{cases} \tag{1.6}
$$

Compared with the autonomous system (1.6), the potential V in system (1.1) breaks down the invariance under translations in $\mathbb{R}^N$, then we cannot use the translation-invariant concentration–compactness argument. The strategy to prove Theorem 1.1 is a comparison of the energy of the functional J_V with the functional J_{V_∞} associated to system (1.6). On the one hand, we construct a Palais–Smale sequence $\{(u_n,v_n)\}$ of J_{V_∞} at the level c_0 defined in (2.4), that is, a sequence $\{(u_n,v_n)\}$ in X such that $J_{V_\infty}(u_n,v_n)\to c_0$ and $J_{V_\infty}'(u_n,v_n)\to 0$ as $n\to\infty$. On the other hand, we prove that up to translations the sequence $\{(u_n,v_n)\}$ converges to a nontrivial solution (u,v) of system (1.6). Then, in the same way we obtain a $(\mathrm{PS})_{c_V}$ sequence of J_V. Furthermore, by the equivalent characterization of c_0, we can show that $c_V < c_0$ under the assumptions on the potential V. Based on $c_V < c_0$, the compactness maintains and a ground state solution for system (1.1) is obtained.

The rest of the paper is organized as follows. We give some preliminaries in Section 2. We obtain a ground state solution for system (1.6) in Section 3. Theorem 1.1 is proved in Section 4.

2 Preliminary

In this section, we first provide some preliminary results.

The following well-known Hardy–Littlewood–Sobolev inequality will be frequently used in this paper.

Lemma 2.1 (Hardy–Littlewood–Sobolev inequality, [5])**.** *Let $p, q > 1$, $\alpha \in (0, N)$, $1 \le r < s < \infty$ and $s \in (1, \frac{N}{\alpha})$ such that*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\alpha}{N}, \qquad \frac{1}{r} - \frac{1}{s} = \frac{\alpha}{N}.$$

(i) Let $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$, we have

$$\Big| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N-\alpha}} dxdy \Big| \le C(N, \alpha, p) \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)}.$$

(ii) For any $f \in L^r(\mathbb{R}^N)$, $I_\alpha * f \in L^s(\mathbb{R}^N)$ and

$$\|I_\alpha * f\|_{L^s(\mathbb{R}^N)} \le C(N, \alpha, r) \|f\|_{L^r(\mathbb{R}^N)}.$$

By Hardy–Littlewood–Sobolev inequality mentioned above and the classical Sobolev embedding theorem, we obtain

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1}) |u|^{\frac{\alpha}{N}+1} dx \le C(N, \alpha) \Big(\int_{\mathbb{R}^N} |u|^2 dx \Big)^{\frac{\alpha}{N}+1}. \tag{2.1}$$

This inequality can be restated as the following minimization problem

$$S = \inf \Big\{ \int_{\mathbb{R}^N} |u|^2 dx : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1}) |u|^{\frac{\alpha}{N}+1} dx = 1 \Big\}.$$

By Theorem 4.3 in [5], the infimum S is achieved by a function $u \in H^1(\mathbb{R}^N)$ if and only if

$$u(x) = A \left(\frac{\varepsilon}{\varepsilon^2 + |x-a|^2} \right)^{\frac{N}{2}}, \qquad x \in \mathbb{R}^N, \tag{2.2}$$

for some given constants $A \in \mathbb{R}$, and $a \in \mathbb{R}^N$, $\varepsilon \in (0, \infty)$. The form of the minimizers in (2.2) suggests that a loss of compactness in equation (1.3) with V is a positive constant may occur by both of translations and dilations.

First, we recall that pointwise convergence of a bounded sequence implies weak convergence.

Lemma 2.2 ([18, Proposition 5.4.7])**.** *Let $N \ge 3$, $q \in (1, \infty)$ and $\{u_n\}$ be a bounded sequence in $L^q(\mathbb{R}^N)$. If $u_n(x) \to u(x)$ almost everywhere in $\mathbb{R}^N$ as $n \to \infty$, then $u_n \rightharpoonup u$ weakly in $L^q(\mathbb{R}^N)$.*

Similarly as in [3], we can get the following lemma.

Lemma 2.3. *Assume that $\{u_n\} \subset H^1(\mathbb{R}^N)$ is a sequence satisfying that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, then for any $\varphi \in H^1(\mathbb{R}^N)$,*

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{\alpha}{N}+1}) |u_n|^{\frac{\alpha}{N}-1} u_n \varphi dx = \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1}) |u|^{\frac{\alpha}{N}-1} u \varphi dx.$$

Proof. For the reader's convenience, we give a complete proof here. Up to a subsequence, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. By Sobolev's embedding theorem, $\{u_n\}$ is bounded in $L^2(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$, the sequence $\{|u_n|^{\frac{N+\alpha}{N}}\}$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Then by Lemma 2.2

$$|u_n|^{\frac{\alpha}{N}+1} \rightharpoonup |u|^{\frac{\alpha}{N}+1}, \quad \text{in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N).$$

$$|u_n|^{\frac{\alpha}{N}-1}u_n\varphi \to |u|^{\frac{\alpha}{N}-1}u\varphi, \quad \text{in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N), \text{ for any } \varphi \in H^1(\mathbb{R}^N).$$

By Lemma 2.1, the Riesz potential defines a linear continuous map from $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ to $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$. We know that,

$$I_\alpha * (|u_n|^{\frac{\alpha}{N}-1}u_n\varphi) \to I_\alpha * (|u|^{\frac{\alpha}{N}-1}u\varphi), \quad \text{in } L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N).$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}-1}u_n\varphi dx - \int_{\mathbb{R}^N}(I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u\varphi dx \\
&= \int_{\mathbb{R}^N}|u_n|^{\frac{\alpha}{N}+1}(I_\alpha * (|u_n|^{\frac{\alpha}{N}-1}u_n\varphi)dx - \int_{\mathbb{R}^N}|u|^{\frac{\alpha}{N}+1}(I_\alpha * (|u|^{\frac{\alpha}{N}-1}u\varphi)dx \\
&= \int_{\mathbb{R}^N}|u_n|^{\frac{\alpha}{N}+1}(I_\alpha * (|u_n|^{\frac{\alpha}{N}-1}u_n\varphi) - I_\alpha * (|u|^{\frac{\alpha}{N}-1}u\varphi))dx \\
&\quad + \int_{\mathbb{R}^N}(|u_n|^{\frac{\alpha}{N}+1} - |u|^{\frac{\alpha}{N}+1})(I_\alpha * (|u|^{\frac{\alpha}{N}-1}u\varphi))dx \\
&\to 0, \quad \text{as } n \to \infty.
\end{aligned} \tag{2.3}$$

The proof is complete. □

Lemma 2.4 ([17, Lemma 1.21]). *Let $r_0 > 0$ and $s \in [2, 2^*)$. If $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and*

$$\sup_{y\in\mathbb{R}^N}\int_{B(y,r_0)}|u_n|^s \to 0, \quad \text{as } n \to \infty,$$

then $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for $t \in (2, 2^*)$.

Lemma 2.5. *The functional J_{V_∞} satisfies the following properties:*

(1) there exists $\rho > 0$ such that $\inf_{(u,v)\in X,\ \|(u,v)\|=\rho} J_{V_\infty}(u,v) > 0$;

(2) for any $(u,v) \in X\backslash\{(0,0)\}$, it holds $\lim_{t\to\infty} J_{V_\infty}(tu,tv) = -\infty$.

Proof. (1) By (2.1) and the classical Sobolev inequality, we can deduce that

$$\begin{aligned}
J_{V_\infty}(u,v) &\geq \frac{1}{2}\min\{1, V_\infty\}(\|u\|^2 + \|v\|^2) - \frac{N}{2(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}dx \\
&\quad - \frac{N}{2(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1}dx - \int_{\mathbb{R}^N}|u|^p|v|^q dx \\
&\geq \frac{1}{2}\min\{1, V_\infty\}\|(u,v)\|^2 - C_1(\|u\|^{\frac{2\alpha}{N}+2} + ||v||^{\frac{2\alpha}{N}+2}) - \int_{\mathbb{R}^N}(|u|^{2p} + |v|^{2q})dx \\
&\geq \frac{1}{2}\min\{1, V_\infty\}\|(u,v)\|^2 - C_1\|(u,v)\|^{\frac{2\alpha}{N}+2} - C_2\|(u,v)\|^{2p} - C_3\|(u,v)\|^{2q},
\end{aligned}$$

where C_1, C_2 are positive constants. Since $p, q > 1$ and $\alpha > 0$, we have that

$$\inf_{(u,v)\in X,\ \|(u,v)\|=\rho} J_{V_\infty}(u,v) > 0,$$

provided that $\rho > 0$ is sufficiently small.

(2) For any $(u,v) \in X\backslash\{(0,0)\}$, we have

$$\begin{aligned}
J_{V_\infty}(tu,tv) &\le \frac{t^2}{2}\max\{1,V_\infty\}(\|u\|^2+\|v\|^2) - \frac{Nt^{\frac{2\alpha}{N}+2}}{2(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}dx \\
&\quad - \frac{Nt^{\frac{2\alpha}{N}+2}}{2(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1}dx - t^{p+q}\int_{\mathbb{R}^N}|u|^p|v|^q dx \\
&\le \frac{t^2}{2}\max\{1,V_\infty\}\|(u,v)\|^2 - \frac{Nt^{\frac{2\alpha}{N}+2}}{2(N+\alpha)}\Big(\int_{\mathbb{R}^N}(I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}dx \\
&\quad + \int_{\mathbb{R}^N}(I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1}dx\Big).
\end{aligned}$$

Then the conclusion (2) follows. □

By the classical Mountain Pass theorem [17], we have a minimax description at the energy level c_0 defined by

$$c_0 = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_{V_\infty}(\gamma(t)), \tag{2.4}$$

where

$$\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = (0,0), J_{V_\infty}(\gamma(1)) < 0\}.$$

Lemma 2.6. *Let $N \ge 3$, $\alpha \in (0,N)$, $p,q \in \big(1,\sqrt{\frac{N}{N-2}}\big)$ and $Np+(N+2)q < 2N+4$, then $c_0 < c_* := \frac{\alpha}{2(N+\alpha)}(V_\infty S)^{\frac{N}{\alpha}+1}$.*

Proof. We first show that $c_0 \le c_1$, where

$$c_1 = \inf_{(u,v)\in X\backslash\{(0,0)\}}\max_{t\ge 0} J_{V_\infty}(tu,tv).$$

Indeed, for any $(u,v) \in X\backslash\{(0,0)\}$, by Lemma 2.5 (2), there exists $t_{u,v} > 0$ such that

$$J_{V_\infty}(t_{u,v}u, t_{u,v}v) < 0.$$

Hence, by (2.4), we have

$$c_0 \le \max_{\tau\in[0,1]} J_{V_\infty}(\tau t_{u,v}u, \tau t_{u,v}v) \le \max_{t\ge 0} J_{V_\infty}(tu,tv). \tag{2.5}$$

It leads to $c_0 \le c_1$.

By the representation formula (2.2) for the optimal functions of Hardy–Littlewood–Sobolev inequality, for each $\varepsilon > 0$, we set

$$U(x) = A(1+|x|^2)^{-\frac{N}{2}}, \qquad x \in \mathbb{R}^N,$$

$U_\varepsilon(x) = \varepsilon^{\frac{N}{2}}U(\varepsilon x)$ and $V_\varepsilon(x) = \varepsilon^{\frac{N+\beta}{2}}U(\varepsilon x)$, where $\beta \in \big(\frac{N(p+q-2)}{2-q}, \frac{4-N(p+q-2)}{q}\big)$. For each $\varepsilon > 0$ the function U_ε satisfies

$$\int_{\mathbb{R}^N}|U_\varepsilon|^2dx = S \quad \text{and} \quad \int_{\mathbb{R}^N}(I_\alpha * |U_\varepsilon|^{\frac{\alpha}{N}+1})|U_\varepsilon|^{\frac{\alpha}{N}+1}dx = 1.$$

Through direct computations, we have that

$$\int_{\mathbb{R}^N}|V_\varepsilon|^2dx = \varepsilon^\beta\int_{\mathbb{R}^N}|U|^2dx, \qquad \int_{\mathbb{R}^N}(I_\alpha * |V_\varepsilon|^{\frac{\alpha}{N}+1})|V_\varepsilon|^{\frac{\alpha}{N}+1}dx = \varepsilon^{\frac{\beta(N+\alpha)}{N}},$$

$$\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 dx = \varepsilon^2 \int_{\mathbb{R}^N} |\nabla U|^2 dx, \qquad \int_{\mathbb{R}^N} |\nabla V_\varepsilon|^2 dx = \varepsilon^{\beta+2} \int_{\mathbb{R}^N} |\nabla U|^2 dx.$$

For every $\varepsilon > 0$, we now consider the function $\xi_\varepsilon : [0,\infty) \to \mathbb{R}$ defined by

$$\xi_\varepsilon(t) := J_{V_\infty}(tU_\varepsilon, tV_\varepsilon) = g(t) + h_\varepsilon(t) + f_\varepsilon(t), \qquad t \in [0,\infty),$$

where the functions $g, h_\varepsilon, f_\varepsilon : [0,\infty) \to \mathbb{R}$ are defined by

$$\begin{aligned}
g(t) &= \frac{1}{2} V_\infty S t^2 - \frac{N}{2(N+\alpha)} t^{\frac{2\alpha}{N}+2}, \\
h_\varepsilon(t) &= \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla V_\varepsilon|^2 dx + \frac{t^2}{2} V_\infty \int_{\mathbb{R}^N} |V_\varepsilon|^2 dx - \frac{N t^{\frac{2(N+\alpha)}{N}}}{2(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |V_\varepsilon|^{\frac{\alpha}{N}+1}) |V_\varepsilon|^{\frac{\alpha}{N}+1} dx, \\
f_\varepsilon(t) &= \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 dx - t^{p+q} \int_{\mathbb{R}^N} |U_\varepsilon|^p |V_\varepsilon|^q dx.
\end{aligned}$$

Since $\xi_\varepsilon(t) > 0$ whenever $t > 0$ is small enough, $\lim_{t\to 0} \xi_\varepsilon(t) = 0$ and $\lim_{t\to\infty} \xi_\varepsilon(t) = -\infty$, for each $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\xi_\varepsilon(t_\varepsilon) = \max_{t\geq 0} \xi_\varepsilon(t).$$

By the definition of the function g, we have

$$c_1 \leq \max_{t\geq 0} \xi_\varepsilon(t) = \xi_\varepsilon(t_\varepsilon) = g(t_\varepsilon) + h_\varepsilon(t_\varepsilon) + f_\varepsilon(t_\varepsilon) \leq g(t_*) + h_\varepsilon(t_\varepsilon) + f_\varepsilon(t_\varepsilon), \tag{2.6}$$

where $t_* = (V_\infty S)^{\frac{N}{2\alpha}}$ satisfies that

$$g(t_*) = \max_{t\geq 0} g(t) = \frac{\alpha}{2(N+\alpha)} V_\infty^{\frac{N}{\alpha}+1} S^{\frac{N}{\alpha}+1} = c_*.$$

Since $\xi'_\varepsilon(t_\varepsilon) = 0$, we have

$$\begin{aligned}
&\varepsilon^2 \int_{\mathbb{R}^N} |\nabla U|^2 dx + \varepsilon^{\beta+2} \int_{\mathbb{R}^N} |\nabla U|^2 dx + \varepsilon^\beta \int_{\mathbb{R}^N} |U|^2 dx + V_\infty S \\
&\quad = t_\varepsilon^{\frac{2\alpha}{N}} + t_\varepsilon^{\frac{2\alpha}{N}} \int_{\mathbb{R}^N} (I_\alpha * |V_\varepsilon|^{\frac{\alpha}{N}+1}) |V_\varepsilon|^{\frac{\alpha}{N}+1} dx + (p+q) t_\varepsilon^{p+q-2} \int_{\mathbb{R}^N} |U_\varepsilon|^p |V_\varepsilon|^q dx \\
&\quad \geq t_\varepsilon^{\frac{2\alpha}{N}}.
\end{aligned} \tag{2.7}$$

Hence, we have $\limsup_{\varepsilon\to 0} t_\varepsilon^{\frac{2\alpha}{N}} \leq V_\infty S$, which is equivalent to $\limsup_{\varepsilon\to 0} t_\varepsilon \leq V_\infty^{\frac{N}{2\alpha}} S^{\frac{N}{2\alpha}}$. Notice that

$$\begin{aligned}
&t_\varepsilon^{\frac{2\alpha}{N}} \int_{\mathbb{R}^N} (I_\alpha * |V_\varepsilon|^{\frac{\alpha}{N}+1}) |V_\varepsilon|^{\frac{\alpha}{N}+1} dx + (p+q) t_\varepsilon^{p+q-2} \int_{\mathbb{R}^N} |U_\varepsilon|^p |V_\varepsilon|^q dx \\
&\quad = \varepsilon^{\frac{\beta(N+\alpha)}{N}} t_\varepsilon^{\frac{2\alpha}{N}} + (p+q) \varepsilon^{\frac{N(p+q-2)+\beta q}{2}} t_\varepsilon^{p+q-2} \int_{\mathbb{R}^N} |U|^{p+q} dx,
\end{aligned}$$

we can obtain that

$$\lim_{\varepsilon\to 0} \left(t_\varepsilon^{\frac{2\alpha}{N}} \int_{\mathbb{R}^N} (I_\alpha * |V_\varepsilon|^{\frac{\alpha}{N}+1}) |V_\varepsilon|^{\frac{\alpha}{N}+1} dx + (p+q) t_\varepsilon^{p+q-2} \int_{\mathbb{R}^N} |U_\varepsilon|^p |V_\varepsilon|^q dx \right) = 0. \tag{2.8}$$

Then (2.7) and (2.8) imply $\liminf_{\varepsilon\to 0} t_\varepsilon^{\frac{2\alpha}{N}} \geq V_\infty S$. Therefore, $\lim_{\varepsilon\to 0} t_\varepsilon^{\frac{2\alpha}{N}} = V_\infty S$. It leads to $\lim_{\varepsilon\to 0} t_\varepsilon = t_*$.

We now observe that

$$f_\varepsilon(t_\varepsilon)+h_\varepsilon(t_\varepsilon)\le \frac{1}{2}\varepsilon^{\beta+2}t_\varepsilon^2\int_{\mathbb{R}^N}|\nabla U|^2dx+\frac{1}{2}\varepsilon^2t_\varepsilon^2\int_{\mathbb{R}^N}|\nabla U|^2dx$$
$$+\frac{1}{2}\varepsilon^{\beta}t_\varepsilon^2V_\infty\int_{\mathbb{R}^N}|U|^2dx-\varepsilon^{\frac{N(p+q-2)+\beta q}{2}}t_\varepsilon^{p+q-2}\int_{\mathbb{R}^N}|U|^{p+q}dx.$$

Since $p,q\in\left(1,\sqrt{\frac{N}{N-2}}\right)$, $Np+(N+2)q<2N+4$ and $\beta\in\left(\frac{N(p+q-2)}{2-q},\frac{4-N(p+q-2)}{q}\right)$, through direct computations, we can get that $\frac{N(p+q-2)+\beta q}{2}<\min\{\beta,2\}$. Thus

$$f_\varepsilon(t_\varepsilon)+h_\varepsilon(t_\varepsilon)<0,\quad \text{when } \varepsilon>0 \text{ is small enough.}$$

Then it follows from (2.6) that $c_1<c_*$ and thus $c_0<c_*$ in view of (2.5). □

3 Existence of ground state solutions for the limiting problem (1.6)

In this section, we will prove that the limiting problem (1.6) admits at least one ground state solution.

Before giving a complete proof, we state the following lemmas, which will be frequently used in the sequel proofs. Set

$$\|(u,v)\|_{V_\infty}=\left(\int_{\mathbb{R}^N}(|\nabla u|^2+V_\infty u^2)dx+\int_{\mathbb{R}^N}(|\nabla v|^2+V_\infty v^2)dx\right)^{\frac{1}{2}}.$$

Define

$$c_g^{V_\infty}:=\inf\{J_{V_\infty}(u,v):(u,v)\in X\setminus\{(0,0)\}\quad\text{and}\quad J'_{V_\infty}(u,v)=0\}.$$

Lemma 3.1. *If $\{(u_n,v_n)\}$ is a sequence in X such that*

$$\liminf_{n\to\infty}\|(u_n,v_n)\|_{V_\infty}>0,\quad \textit{and}\quad \lim_{n\to\infty}\langle\Phi'(u_n,v_n),(u_n,v_n)\rangle=0,$$

where the functional $\Phi: X\to\mathbb{R}$ is defined by

$$\Phi(u,v)=\frac{1}{2}\|(u,v)\|_{V_\infty}^2-\frac{N}{2(N+\alpha)}\left(\int_{\mathbb{R}^N}(I_\alpha*|u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}dx+\int_{\mathbb{R}^N}(I_\alpha*|v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1}dx\right),$$

then $\liminf_{n\to\infty}\Phi(u_n,v_n)\ge c_*$.

Proof. From $\lim_{n\to\infty}\langle\Phi'(u_n,v_n),(u_n,v_n)\rangle=0$, we observe that

$$\|(u_n,v_n)\|_{V_\infty}^2=\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1}dx+\int_{\mathbb{R}^N}(I_\alpha*|v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1}dx+o_n(1).$$

By the assumption $\liminf_{n\to\infty}\|(u_n,v_n)\|_{V_\infty}>0$ and (2.1), we can deduce that

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}(|u_n|^2+|v_n|^2)dx>0.$$

It follows from the definition of S that

$$\begin{aligned}&\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1}dx+\int_{\mathbb{R}^N}(I_\alpha*|v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1}dx+o_n(1)\\
&\quad\ge\int_{\mathbb{R}^N}(V_\infty|u_n|^2+V_\infty|v_n|^2)dx\\
&\quad\ge V_\infty S\left[\left(\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1}dx\right)^{\frac{N}{N+\alpha}}+\left(\int_{\mathbb{R}^N}(I_\alpha*|v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1}dx\right)^{\frac{N}{N+\alpha}}\right]\\
&\quad\ge V_\infty S\left(\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1}dx+\int_{\mathbb{R}^N}(I_\alpha*|v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1}dx\right)^{\frac{N}{N+\alpha}},\end{aligned}$$

which leads to

$$\begin{aligned}
&\liminf_{n\to\infty} \|(u_n, v_n)\|^2_{V_\infty} \\
&\quad = \liminf_{n\to\infty} \left(\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1} dx + \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1} dx \right) \\
&\quad \geq (V_\infty S)^{\frac{N}{\alpha}+1}.
\end{aligned} \tag{3.1}$$

Therefore,

$$\begin{aligned}
\Phi(u_n, v_n) &= \Phi(u_n, v_n) - \frac{N}{2(N+\alpha)} \langle \Phi'(u_n, v_n), (u_n, v_n) \rangle + o_n(1) \\
&= \frac{\alpha}{2(N+\alpha)} \|u_n, v_n\|^2_{V_\infty} + o_n(1).
\end{aligned} \tag{3.2}$$

Then combine (3.1) with (3.2),

$$\liminf_{n\to\infty} \Phi(u_n, v_n) = \liminf_{n\to\infty} \frac{\alpha}{2(N+\alpha)} \|u_n, v_n\|^2_{V_\infty} \geq \frac{\alpha}{2(N+\alpha)} (V_\infty S)^{1+\frac{N}{\alpha}} = c_*.$$

The proof is complete. □

Lemma 3.2. *Let $\{(u_n, v_n)\}$ be a bounded $(PS)_c$ sequence with $c \in (0, c_*)$ for functional J_{V_∞}, then up to a subsequence and translations, the sequence $\{(u_n, v_n)\}$ converges weakly to some $(u, v) \in X \backslash \{(0, 0)\}$ such that*

$$J'_{V_\infty}(u, v) = 0 \quad \textit{and} \quad J_{V_\infty}(u, v) \in (0, c].$$

Proof. First we show that

$$\limsup_{n\to\infty} \frac{1}{2} \int_{\mathbb{R}^N} (|u_n|^{2p} + |v_n|^{2q}) dx > 0. \tag{3.3}$$

Otherwise, up to a subsequence, we have

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^p |v_n|^q dx \leq \limsup_{n\to\infty} \int_{\mathbb{R}^N} (|u_n|^{2p} + |v_n|^{2q}) dx = 0. \tag{3.4}$$

Since $\lim\limits_{n\to\infty} \langle J'_{V_\infty}(u_n, v_n), (u_n, v_n) \rangle = 0$, we have

$$\|(u_n, v_n)\|^2_{V_\infty} = \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1} dx + \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1} dx + o_n(1).$$

While, $J_{V_\infty}(u_n, v_n) \to c > 0,\ n \to \infty$, together with (3.4) and (2.1), imply that

$$\liminf_{n\to\infty} \|(u_n, v_n)\|_{V_\infty} > 0.$$

Then we deduce from Lemma 3.1 that

$$\begin{aligned}
c &= \liminf_{n\to\infty} J_{V_\infty}(u_n, v_n) \\
&= \liminf_{n\to\infty} \Phi(u_n, v_n) - \limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^p |v_n|^q dx \\
&= \liminf_{n\to\infty} \Phi(u_n, v_n) \\
&\geq c_*,
\end{aligned}$$

which contradicts with the fact $c \in (0, c_*)$. Thus (3.3) holds. It implies that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2p} dx > 0, \quad \text{or} \quad \limsup_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^{2q} dx > 0.$$

By the Lions inequality (Lemma 1.21 in [17]),

$$\int_{\mathbb{R}^N} |u_n|^s dx \le C \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2 \right) dx \left(\sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^s dx \right)^{1-\frac{2}{s}},$$

$$\int_{\mathbb{R}^N} |v_n|^s dx \le C \left(\int_{\mathbb{R}^N} |\nabla v_n|^2 + |v_n|^2 \right) dx \left(\sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |v_n|^s dx \right)^{1-\frac{2}{s}},$$

where $s \in (2, 2^*)$. Then there exists sequences of points $\{y_n\} \in \mathbb{R}^N$ such that

$$\limsup_{n\to\infty} \int_{B_1(y_n)} |u_n|^{2p} dx > 0, \quad \text{or} \quad \limsup_{n\to\infty} \int_{B_1(y_n)} |v_n|^{2q} dx > 0.$$

Thus we have

$$\limsup_{n\to\infty} \int_{B_1(y_n)} (|u_n|^{2p} + |v_n|^{2q}) dx > 0. \tag{3.5}$$

Define $\widetilde{u}_n := u_n(\cdot + y_n)$, $\widetilde{v}_n := v_n(\cdot + y_n)$. Since the functional J_{V_∞} is invariant under translations, the sequence $\{(\widetilde{u}_n, \widetilde{v}_n)\} \subset X$ is also a bounded $(\mathrm{PS})_c$ sequence of J_{V_∞}. Then by (3.5) there exists some $(u, v) \in X \backslash \{(0, 0)\}$ such that

$$\begin{aligned}
&(\widetilde{u}_n, \widetilde{v}_n) \rightharpoonup (u, v) \ \text{ in } X.\\
&\widetilde{u}_n \rightharpoonup u,\ \widetilde{v}_n \rightharpoonup v \ \text{ in } H^1(\mathbb{R}^N),\\
&\widetilde{u}_n \to u,\ \widetilde{v}_n \to v \ \text{ in } L^r_{loc}(\mathbb{R}^N),\ r \in [1, 2^*),\\
&\widetilde{u}_n(x) \to u(x),\ \widetilde{v}_n(x) \to v(x),\ \text{a.e. } x \in \mathbb{R}^N.
\end{aligned}$$

Since $1 < p, q < \sqrt{\frac{N}{N-2}}$ implies that $2 < 2p, 2q, 2pq < 2^*$, we have

$$\int_{\mathbb{R}^N} |\widetilde{v}_n|^{2q} |\widetilde{u}_n|^{2(p-1)} dx \le \left(\int_{\mathbb{R}^N} |\widetilde{v}_n|^{2pq} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |\widetilde{u}_n|^{2p} dx \right)^{\frac{p-1}{p}} < \infty.$$

That is to say $\{|\widetilde{v}_n|^q |\widetilde{u}_n|^{p-2} \widetilde{u}_n\}$ is bounded in $L^2(\mathbb{R}^N)$. Then by Proposition 5.4.7 in [18],

$$|\widetilde{v}_n|^q |\widetilde{u}_n|^{p-2} \widetilde{u}_n \rightharpoonup |\widetilde{v}|^q |\widetilde{u}|^{p-2} \widetilde{u}, \quad \text{in } L^2(\mathbb{R}^N).$$

Since $\phi \in H^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\widetilde{v}_n|^q |\widetilde{u}_n|^{p-2} \widetilde{u}_n \phi dx \to \int_{\mathbb{R}^N} |\widetilde{v}|^q |\widetilde{u}|^{p-2} \widetilde{u} \phi dx, \qquad n \to \infty. \tag{3.6}$$

Similarly, we can also get

$$\int_{\mathbb{R}^N} |\widetilde{u}_n|^p |\widetilde{v}_n|^{q-2} \widetilde{v}_n \varphi dx \to \int_{\mathbb{R}^N} |\widetilde{u}|^p |\widetilde{v}|^{q-2} \widetilde{v} \varphi dx, \qquad n \to \infty. \tag{3.7}$$

We now claim that $J'_{V_\infty}(u,v)=0$. For any $(\phi,\varphi)\in X$, by Lemma 2.3, (3.6) and (3.7), we have

$$\begin{aligned}
&\langle J'_{V_\infty}(u_n,v_n),(\phi(x-y_n),\varphi(x-y_n))\rangle\\
&\quad=\int_{\mathbb{R}^N}(\nabla u_n\cdot\nabla\phi(x-y_n)+V_\infty u_n\phi(x-y_n)+\nabla v_n\cdot\nabla\varphi(x-y_n)+V_\infty v_n\varphi(x-y_n))dx\\
&\qquad-\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}-1}u_n\phi(x-y_n)dx-\int_{\mathbb{R}^N}(I_\alpha*|v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}-1}v_n\varphi(x-y_n)dx\\
&\qquad-\int_{\mathbb{R}^N}|v_n|^q|u_n|^{p-2}u_n\phi(x-y_n)dx-\int_{\mathbb{R}^N}|u_n|^p|v_n|^{q-2}v_n\varphi(x-y_n)dx\\
&\quad=\int_{\mathbb{R}^N}(\nabla\widetilde{u}_n\cdot\nabla\phi+V_\infty\widetilde{u}_n\phi+\nabla\widetilde{v}_n\cdot\nabla\varphi+V_\infty\widetilde{v}_n\varphi)dx\\
&\qquad-\int_{\mathbb{R}^N}(I_\alpha*|\widetilde{u}_n|^{\frac{\alpha}{N}+1})|\widetilde{u}_n|^{\frac{\alpha}{N}-1}\widetilde{u}_n\phi dx-\int_{\mathbb{R}^N}(I_\alpha*|\widetilde{v}_n|^{\frac{\alpha}{N}+1})|\widetilde{v}_n|^{\frac{\alpha}{N}-1}\widetilde{v}_n\varphi dx\\
&\qquad-\int_{\mathbb{R}^N}|\widetilde{v}_n|^q|\widetilde{u}_n|^{p-2}\widetilde{u}_n\phi dx-\int_{\mathbb{R}^N}|\widetilde{u}_n|^p|\widetilde{v}_n|^{q-2}\widetilde{v}_n\varphi dx\\
&\quad=\langle J'_{V_\infty}(u,v),(\phi,\varphi)\rangle+o_n(1).
\end{aligned}$$

Thus $J'_{V_\infty}(u,v)=0$.

By the Fatou lemma,

$$\begin{aligned}
J_{V_\infty}(u,v)&=J_{V_\infty}(u,v)-\frac{1}{2}\langle J'_{V_\infty}(u,v),(u,v)\rangle\\
&=\frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}((I_\alpha*|u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}+(I_\alpha*|v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1})dx\\
&\quad+\Big(\frac{p+q}{2}-1\Big)\int_{\mathbb{R}^N}|u|^p|v|^q dx\\
&\le\liminf_{n\to\infty}\Big(\frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}((I_\alpha*|\widetilde{u}_n|^{\frac{\alpha}{N}+1})|\widetilde{u}_n|^{\frac{\alpha}{N}+1}+(I_\alpha*|\widetilde{v}_n|^{\frac{\alpha}{N}+1})|\widetilde{v}_n|^{\frac{\alpha}{N}+1})dx\\
&\quad+\Big(\frac{p+q}{2}-1\Big)\int_{\mathbb{R}^N}|\widetilde{u}_n|^p|\widetilde{v}_n|^q dx\Big)\\
&=\liminf_{n\to\infty}\Big(J_{V_\infty}(\widetilde{u}_n,\widetilde{v}_n)-\frac{1}{2}\langle J'_{V_\infty}(\widetilde{u}_n,v_n),(\widetilde{u}_n,\widetilde{v}_n)\rangle\Big)\\
&=c.
\end{aligned}$$

Thus $J_{V_\infty}(u,v)\le c$.

We finally conclude that

$$\begin{aligned}
J_{V_\infty}(u,v)&=J_{V_\infty}(u,v)-\frac{1}{2}\langle J'_{V_\infty}(u,v),(u,v)\rangle\\
&=\frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}((I_\alpha*|u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}+(I_\alpha*|v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1})dx\\
&\quad+\Big(\frac{p+q}{2}-1\Big)\int_{\mathbb{R}^N}|u|^p|v|^q dx\\
&>0.
\end{aligned}\tag{3.8}$$

Therefore, the lemma follows. □

By Lemma 2.5 and Mountain Pass theorem, there exists a Palais–Smale sequence $\{(u_n,v_n)\}$ of J_{V_∞} at the energy level c_0. It then follows lemma 2.6 that $c_0\in(0,c_*)$. The sequence $\{(u_n,v_n)\}$

is bounded in X. In fact, by taking $\mu \in \big(2, \min\big\{\frac{2(N+\alpha)}{N}, p+q\big\}\big]$, we can get

$$\begin{aligned}
c_0 + o_n(1) &= J_{V_\infty}(u_n, v_n) - \frac{1}{\mu}\langle J'_{V_\infty}(u_n, v_n), (u_n, v_n)\rangle \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|(u_n, v_n)\|^2_{V_\infty} \\
&\quad + \left(\frac{1}{\mu} - \frac{N}{2(N+\alpha)}\right)\left(\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1}dx + \int_{\mathbb{R}^N}(I_\alpha * |v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1}dx\right) \\
&\quad + \left(\frac{p+q}{\mu} - 1\right)\int_{\mathbb{R}^N}|u_n|^p|v_n|^q dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|(u_n, v_n)\|^2_{V_\infty}.
\end{aligned}$$

Thus $\{(u_n, v_n)\}$ is bounded in X. Up to a subsequence if necessary, there exists $(u, v) \in X$ such that

$$(u_n, v_n) \rightharpoonup (u, v) \quad \text{in } X, \qquad (u_n(x), v_n(x)) \to (u(x), v(x)) \quad \text{a.e. in } \mathbb{R}^N.$$

Then Lemma 3.2 infers that (u, v) is a nontrivial critical point of functional J_{V_∞} and $J_{V_\infty}(u, v) \in (0, c_0]$.

Let$\{(z_n, w_n)\}$ be a sequence of nontrivial critical points of J_{V_∞} such that

$$\lim_{n\to\infty} J_{V_\infty}(z_n, w_n) = c_g^{V_\infty}.$$

It is easy to see that $c_g^{V_\infty} \leq c_0 < c_*$. By using the same arguments as above, we can get that $\{(z_n, w_n)\}$ is bounded in X. In view of $\langle J'_{V_\infty}(z_n, w_n), (z_n, w_n)\rangle = 0$, it follows that $\{\|(z_n, w_n)\|\}$ has a positive lower bound, which together with (3.8) implies that $c_g^{V_\infty} > 0$. Therefore, $\{(z_n, w_n)\}$ is a $(\mathrm{PS})_{c_g^{V_\infty}}$ sequence of J_{V_∞} with $c_g^{V_\infty} \in (0, c_0]$. It follows from Lemma 3.2 that up to a sequence of $\{(z_n, w_n)\}$ and translations,

$$(z_n, w_n) \rightharpoonup (z, w) \neq 0 \quad \text{in } X, \quad \text{as } n \to \infty, \qquad J'_{V_\infty}(z, w) = 0 \quad \text{and} \quad J_{V_\infty}(z, w) \in (0, c_g^{V_\infty}].$$

Furthermore, by the definition of $c_g^{V_\infty}$, we conclude that $J_{V_\infty}(z, w) = c_g^{V_\infty}$. Hence, (z, w) is a ground state solution of system (1.6). □

4 Proof of Theorem 1.1

Lemma 4.1. *For any solution $(u, v) \in X\backslash\{(0, 0)\}$ of system (1.6), the function $J_{V_\infty}(tu, tv)$, $t \geq 0$ achieves its unique strict global maximum at $t = 1$, that is to say*

$$J_{V_\infty}(u, v) = \max_{t\geq 0} J_{V_\infty}(tu, tv) > J_{V_\infty}(tu, tv), \quad \textit{for } t \geq 0 \textit{ and } t \neq 1.$$

Proof. Let $(u, v) \in X\backslash\{(0, 0)\}$ be a solution of system (1.6), for every $t \geq 0$, we have

$$\begin{aligned}
J_{V_\infty}(tu, tv) &= \frac{t^2}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V_\infty|u|^2 + |\nabla v|^2 + V_\infty|v|^2)dx \\
&\quad - \frac{N}{2(N+\alpha)}t^{\frac{2\alpha}{N}+2}\int_{\mathbb{R}^N}((I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1} + (I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1})dx \\
&\quad - t^{p+q}\int_{\mathbb{R}^N}|u|^p|v|^q dx \\
&= \frac{A}{2}t^2 - \frac{BN}{2(N+\alpha)}t^{\frac{2\alpha}{N}+2} - Ct^{p+q},
\end{aligned} \tag{4.1}$$

where

$$A := \int_{\mathbb{R}^N} (|\nabla u|^2 + V_\infty |u|^2 + |\nabla v|^2 + V_\infty |v|^2) dx;$$

$$B := \int_{\mathbb{R}^N} ((I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1} + (I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1}) dx;$$

$$C := \int_{\mathbb{R}^N} |u|^p |v|^q dx.$$

By (4.1), it is easy to get that $J_{V_\infty}(tu, tv) \in C^1([0,\infty), \mathbb{R})$ and $\lim_{t\to\infty} J_{V_\infty}(tu, tv) = -\infty$. Thus $J_{V_\infty}(tu, tv)$ can achieve its global maximum. Since $0 = \langle J'_{V_\infty}(u, v), (u, v)\rangle = A - B - (p + q)C$, by a direct calculation, we can get that $t = 1$ is the only point such that $\frac{dJ_{V_\infty}(tu,tv)}{dt} = 0$. Then $J_{V_\infty}(tu, tv)$ achieves the unique strict global maximum at $t = 1$. □

Lemma 4.2. *Assume $(V_1), (V_2)$ hold, then there exists a $(PS)_{c_V}$ sequence for J_V with $0 < c_V < c_g^{V_\infty}$.*

Proof. Firstly, we claim that there exists $(u^0, v^0) \in X$ such that $J_V(u^0, v^0) < 0$. Indeed, for any $(u, v) \in X\backslash\{(0, 0)\}$, we have $J_V(u, v) < J_{V_\infty}(u, v)$. In view of (4.1), by taking $u^0 = tu, v^0 = tv$ with t large enough, where (u, v) is a ground state solution of system (1.6). Then we get that $J_V(u^0, v^0) < J_{V_\infty}(u^0, v^0) < 0$.

Similar to Lemma 2.5, we see that the functional J_V also enjoys the Mountain Pass geometry. Then we have a minimax description at c_V. We show that

$$c_V := \inf_{\gamma\in \Upsilon} \max_{t\in[0,1]} J_V(\gamma(t)) > \max\{J_V(0, 0), J_V(u^0, v^0)\},$$

where

$$\Upsilon = \{\gamma \in C([0, 1], X) : \gamma(0) = (0, 0), \gamma(1) = (u^0, v^0)\}.$$

In fact, (V_1), (V_2) and (2.1) imply that

$$\begin{aligned} J_V(u, v) &\geq \frac{1}{2} \min\{1, V(x)\}(\|u\|^2 + \|v\|^2) - C_1 \left(\|u\|^{\frac{2(N+\alpha)}{N}} + \|v\|^{\frac{2(N+\alpha)}{N}}\right) - \frac{1}{2}\int_{\mathbb{R}^N} (|u|^{2p} + |v|^{2q}) dx \\ &\geq \frac{1}{2} \min\{1, V(x)\}\|(u, v)\|^2 - C_1\|(u, v)\|^{\frac{2(N+\alpha)}{N}} - C_2\|(u, v)\|^{2p} - C_3\|(u, v)\|^{2q}, \end{aligned}$$

where C_1, C_2 are positive constants. Since $p, q \in \left(1, \sqrt{\frac{N}{N-2}}\right)$, J_V has a strict local minimum at 0 and then $c_V > 0$.

Next, we show that $c_V < c_g^{V_\infty}$. Let (u, v) be the ground state solution of system (1.6) mentioned above. From the proof of Lemma 4.1 and by using (V_2), we see that

$$c_g^{V_\infty} = J_{V_\infty}(u, v) = \max_{t\geq 0} J_{V_\infty}(tu, tv) > \max_{t\geq 0} J_V(tu, tv) \geq c_V.$$

The proof is complete. □

Proof of Theorem 1.1. The proof is divided into four steps.

Step 1. Let $\{(u_n,v_n)\}$ be a $(\mathrm{PS})_{c_V}$ sequence of functional J_V with $0<c_V<c_g^{V_\infty}$. Then take $\mu\in\big(2,\min\big\{\frac{2(N+\alpha)}{N},p+q\big\}\big]$, we have

$$\begin{aligned} c_V+o_n(1) &= J_V(u_n,v_n)-\frac{1}{\mu}\langle J_V'(u_n,v_n),(u_n,v_n)\rangle\\ &=\Big(\frac12-\frac1\mu\Big)\int_{\mathbb{R}^N}(|\nabla u_n|^2+V(x)|u_n|^2+|\nabla v_n|^2+V(x)|v_n|^2)dx\\ &\quad+\Big(\frac1\mu-\frac{N}{2(N+\alpha)}\Big)\int_{\mathbb{R}^N}((I_\alpha*|u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1}+(I_\alpha*|v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1})dx\\ &\quad+\Big(\frac{p+q}{\mu}-1\Big)\int_{\mathbb{R}^N}|u_n|^p|v_n|^q dx\\ &\geq\Big(\frac12-\frac1\mu\Big)\int_{\mathbb{R}^N}(|\nabla u_n|^2+V(x)|u_n|^2+|\nabla v_n|^2+V(x)|v_n|^2)dx. \end{aligned}$$

Thus (V_1) and (V_2) imply that $\{(u_n,v_n)\}$ is bounded in X. Therefore, there exists $(u,v)\in X$ such that up to a subsequence if necessary,

$$(u_n,v_n)\rightharpoonup(u,v)\quad\text{weakly in } X,\quad (u_n(x),v_n(x))\to(u(x),v(x)),\ \text{for almost every } x\in\mathbb{R}^N.$$

By a similar argument as in the proof of Lemma 3.2, we see that there exists $\{y_n\}\subset\mathbb{R}^N$ such that

$$\limsup_{n\to\infty}\int_{B_1(y_n)}(|u_n|^{2p}+|v_n|^{2q})dx>0. \tag{4.2}$$

Step 2. We can claim that $\{y_n\}$ is bounded in $\mathbb{R}^N$. In fact, suppose that for a subsequence still denoted by $\{y_n\}$ such that

$$\lim_{n\to\infty}|y_n|\to\infty, \tag{4.3}$$

we define $z_n(\cdot)=u_n(\cdot+y_n)$, $w_n(\cdot)=v_n(\cdot+y_n)$, then $\{(z_n,w_n)\}$ is bounded in X, and by (4.2) $(z_n,w_n)\rightharpoonup(z,w)\neq(0,0)$. In the following, we will show that $J'_{V_\infty}(z,w)=0$ and $J_{V_\infty}(z,w)\leq c_V$, which contradict that $c_V<c_g^{V_\infty}$. Hence $\{y_n\}$ is bounded.

In order to prove $J'_{V_\infty}(z,w)=0$, by (4.3), (V_1), (V_2) and Hölder inequality, for any $(\phi,\varphi)\in X$, we have

$$\begin{aligned} &\Big|\int_{\mathbb{R}^N}(V(x+y_n)-V_\infty)(z_n(x)\phi(x)+w_n(x)\varphi(x))dx\Big|\\ &\quad\leq\Big|\int_{B_{|y_n|/2}}(V(x+y_n)-V_\infty)(z_n(x)\phi(x)+w_n(x)\varphi(x))dx\Big|\\ &\qquad+\Big|\int_{\mathbb{R}^N\setminus B_{|y_n|/2}}(V(x+y_n)-V_\infty)(z_n(x)\phi(x)+w_n(x)\varphi(x))dx\Big|\\ &\quad\leq\sup_{B_{|y_n|/2}}|V(x+y_n)-V_\infty|(|z_n|_{L^2(\mathbb{R}^N)}|\phi|_{L^2(\mathbb{R}^N)}+|w_n|_{L^2(\mathbb{R}^N)}|\varphi|_{L^2(\mathbb{R}^N)})\\ &\qquad+C\Big(|z_n|_{L^2(\mathbb{R}^N)}\Big(\int_{\mathbb{R}^N\setminus B_{|y_n|/2}}|\phi|^2dx\Big)^{\frac12}+|w_n|_{L^2(\mathbb{R}^N)}\Big(\int_{\mathbb{R}^N\setminus B_{|y_n|/2}}|\varphi|^2dx\Big)^{\frac12}\Big)\\ &\quad=o_n(1). \end{aligned} \tag{4.4}$$

Thus Lemma 2.3 and (4.4) imply that

$$
\begin{aligned}
&\langle J_V'(u_n,v_n),(\phi(x-y_n),\varphi(x-y_n))\rangle\\
&\quad=\int_{\mathbb{R}^N}(\nabla u_n(x)\nabla\phi(x-y_n)+V(x)u_n(x)\phi(x-y_n))dx\\
&\qquad-\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}-1}u_n\phi(x-y_n)dx\\
&\qquad+\int_{\mathbb{R}^N}(\nabla v_n(x)\nabla\varphi(x-y_n)+V(x)v_n(x)\varphi(x-y_n))dx\\
&\qquad-\int_{\mathbb{R}^N}(I_\alpha*|v_n|^{\frac{\alpha}{N}+1}|v_n|^{\frac{\alpha}{N}-1}v_n\varphi(x-y_n))dx\\
&\qquad-p\int_{\mathbb{R}^N}|v_n|^q|u_n|^{p-2}u_n\phi(x-y_n)dx-q\int_{\mathbb{R}^N}|u_n|^p|v_n|^{q-2}v_n\varphi(x-y_n)dx\\
&\quad=\int_{\mathbb{R}^N}(\nabla z_n(x)\nabla\phi(x)+V(x+y_n)z_n(x)\phi(x))dx\\
&\qquad+\int_{\mathbb{R}^N}(\nabla w_n(x)\nabla\varphi(x)+V(x+y_n)w_n(x)\varphi(x))dx\\
&\qquad-\int_{\mathbb{R}^N}(I_\alpha*|z_n|^{\frac{\alpha}{N}+1})|z_n|^{\frac{\alpha}{N}-1}z_n\phi dx-\int_{\mathbb{R}^N}(I_\alpha*|w_n|^{\frac{\alpha}{N}+1})|w_n|^{\frac{\alpha}{N}-1}w_n\varphi dx\\
&\qquad-p\int_{\mathbb{R}^N}|w_n|^q|z_n|^{p-2}z_n\phi dx-q\int_{\mathbb{R}^N}|z_n|^p|w_n|^{q-2}w_n\varphi dx\\
&\quad=\langle J_{V_\infty}'(z_n,w_n),(\phi,\varphi)\rangle+\int_{\mathbb{R}^N}(V(x+y_n)-V_\infty)(z_n(x)\phi(x)+w_n(x)\varphi(x))dx\\
&\quad=\langle J_{V_\infty}'(z,w),(\phi,\varphi)\rangle+o_n(1).
\end{aligned}
\tag{4.5}
$$

Then from (4.5) we deduce that $J_{V_\infty}'(z,w)=0$.

To prove $J_{V_\infty}(z,w)\le c_V$, by the Fatou lemma, we have

$$
\begin{aligned}
J_{V_\infty}(z,w)&=J_{V_\infty}(z,w)-\frac{1}{2}\langle J_{V_\infty}'(z,w),(z,w)\rangle\\
&=\frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}((I_\alpha*|z|^{\frac{\alpha}{N}+1})|z|^{\frac{\alpha}{N}+1}+(I_\alpha*|w|^{\frac{\alpha}{N}+1})|w|^{\frac{\alpha}{N}+1})dx\\
&\quad+\Big(\frac{p+q}{2}-1\Big)\int_{\mathbb{R}^N}|z|^p|w|^q dx\\
&\le\liminf_{n\to\infty}\Big[\frac{\alpha}{2(N+\alpha)}\Big(\int_{\mathbb{R}^N}(I_\alpha*|z_n|^{\frac{\alpha}{N}+1})|z_n|^{\frac{\alpha}{N}+1}dx+\\
&\quad\int_{\mathbb{R}^N}(I_\alpha*|w_n|^{\frac{\alpha}{N}+1})|w_n|^{\frac{\alpha}{N}+1}dx\Big)+\Big(\frac{p+q}{2}-1\Big)\int_{\mathbb{R}^N}|z_n|^p|w_n|^q dx\Big]\\
&=\liminf_{n\to\infty}\Big[\frac{\alpha}{2(N+\alpha)}\Big(\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1}dx\\
&\quad+\int_{\mathbb{R}^N}(I_\alpha*|v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1}dx\Big)+\Big(\frac{p+q}{2}-1\Big)\int_{\mathbb{R}^N}|u_n|^p|v_n|^q dx\Big]\\
&=\liminf_{n\to\infty}(J_V(u_n,v_n)-\frac{1}{2}\langle J_V'(u_n,v_n),(u_n,v_n)\rangle)=c_V.
\end{aligned}
\tag{4.6}
$$

Therefore, $J_{V_\infty}(z,w)\le c_V$.

Step 3. We show that (u,v) obtained in step 1 is a nontrivial solution of (1.1) and $J_V(u,v)\in(0,c_V]$. By the classical Sobolev embedding theorem, (4.2) and step 2, we have $(u,v)\ne(0,0)$. In view of Lemma 2.3, Lemma 3.2, (V_1) and (V_2), we can show that (u,v) is a critical point of

J_V. Similarly to the proof of (4.6), we have $J_V(u,v) \le c_V$. Direct calculation gives that

$$\begin{aligned}
J_V(u,v) &= J_V(u,v) - \frac{1}{2}\langle J'_V(u,v),(u,v)\rangle \\
&= \frac{\alpha}{2(N+\alpha)}\Big(\int_{\mathbb{R}^N}(I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}dx + \int_{\mathbb{R}^N}(I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1}dx\Big) \\
&\quad + \Big(\frac{p+q}{2}-1\Big)\int_{\mathbb{R}^N}|u|^p|v|^q dx \\
&> 0.
\end{aligned}$$

Thus $0 < J_V(u,v) \le c_V < c_g^{V_\infty}$.

Step 4. We show that there exists a ground state solution of system (1.1). By Step 3 and the definition of c_g^V, we see that $c_g^V < c_g^{V_\infty}$. Let $\{(z_n,w_n)\}$ be a sequence of nontrivial critical points of J_V satisfying $J_V(z_n,w_n) \to c_g^V$ as $n \to \infty$. By using the same arguments as in Step 1, we can show that $\{(z_n,w_n)\}$ is bounded in X. In view of $\langle J'_V(z_n,w_n),(z_n,w_n)\rangle = 0$, it follows that $\{\|(z_n,w_n)\|_X\}$ has a positive lower bound. By similar arguments as step 1, we can show that $c_g^V > 0$. Therefore, $\{(z_n,w_n)\}$ is a $(\mathrm{PS})_{c_g^V}$ sequence of functional J_V with $0 < c_g^V < c_g^{V_\infty}$. Repeating Step 1–Step 3, we obtain some $(z,w) \in X\setminus\{(0,0)\}$ such that $J'_V(z,w) = 0$ and $J_V(z,w) \le c_g^V$. Thus (z,w) is a ground state solution of system (1.1). The proof of Theorem 1.1 is complete. □

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An invariant set bifurcation theory for nonautonomous nonlinear evolution equations

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Abstract. In this paper we establish an invariant set bifurcation theory for the nonautonomous dynamical system $(\varphi_\lambda, \theta)_{X,\mathcal{H}}$ generated by the evolution equation

$$u_t + Au = \lambda u + p(t,u), \quad p \in \mathcal{H} = \mathcal{H}[f(\cdot,u)] \tag{0.1}$$

on a Hilbert space X, where A is a sectorial operator, λ is the bifurcation parameter, $f(\cdot,u) : \mathbb{R} \to X$ is translation compact, $f(t,0) \equiv 0$ and $\mathcal{H}[f]$ is the hull of $f(\cdot,u)$. Denote by $\varphi_\lambda := \varphi_\lambda(t,p)u$ the cocycle semiflow generated by the system. Under some other assumptions on f, we show that as the parameter λ crosses an eigenvalue $\lambda_0 \in \mathbb{R}$ of A, the system bifurcates from 0 to a nonautonomous invariant set $B_\lambda(\cdot)$ on one-sided neighborhood of λ_0. Moreover,

$$\lim_{\lambda\to\lambda_0} H_{X^\alpha}\left(B_\lambda(p), 0\right) = 0, \quad p \in P,$$

where $H_{X^\alpha}(\cdot,\cdot)$ denotes the Hausdorff semidistance in X^α (here X^α ($\alpha \geq 0$) defined below is the fractional power spaces associated with A).

Our result is based on the pullback attractor bifurcation on the local central invariant manifolds $\mathcal{M}^\lambda_{\mathrm{loc}}(\cdot)$.

Keywords: stability of pullback attractors, local invariant manifolds, nonautonomous invariant set bifurcations.

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1 Introduction

Invariant set bifurcation theory of autonomous dynamical systems has been extremely well developed [1, 6, 16, 17, 19, 23–27, 30–32]. A relatively simpler but important case is that of bifurcations from equilibria, including bifurcation to multiple equilibria (static bifurcation) and to periodic solutions (Hopf bifurcation) (see among others, [6, 27]). Ma and Wang [23] and Sanjurjio [31] developed a local attractor bifurcation theory. Roughly speaking, if the trivial equilibrium e of an autonomous system changes from an attractor to a repeller on the

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local center manifold of the equilibrium when the bifurcation parameter λ crosses a critical value λ_0, then the system bifurcates a compact invariant set K which is an attractor of the system restricted to the center manifold. Chow and Hale [6] started to discuss stability and bifurcation phenomena associated with more general invariant sets, e.g. periodic orbits. Using Conley index theory, Rybakowski [30] and Li and Wang [19] developed global bifurcation theorems to discuss bifurcation phenomena of nonlinear autonomous evolution equations.

The study of invariant set bifurcation for nonautonomous dynamical system has also received a lot of attention. Langa et al. [18] presented a collection of examples to illustrate bifurcation phenomena in nonautonomous ordinary differential equations. Carvalho et al. [4] studied the structure of the pullback attractor for a nonautonomous version of the Chafee–Infante equation, and investigated the bifurcations that this attractor undergoes as bifurcation parameter varies. In [28], Rasmussen introduced various concepts of bifurcation and transition for nonautonomous systems, corresponding to different time domains. And several examples were presented to illustrate these definitions.

The main aim of the paper is to develop a counterpart for the classical autonomous invariant set bifurcation patterns of Ma and Wang [23] and Sanjurjio [31] in the context of nonautonomous invariant set bifurcation. Unlike autonomous dynamical systems for which forward dynamics is studied, pullback dynamics is much more natural than the more familiar forward dynamics for nonautonomous dynamical systems. But this makes it very difficult to extend the invariant set bifurcation theory of autonomous systems to nonautonomous systems when pullback dynamics is considered. Our approach in the paper is to treat a nonautonomous system as a cocycle semiflow over a suitable base space. One of the advantage of a cocycle semiflow approach is that the synchronizing solutions or the other synchronizing behaviors with the nonautonomous driving force can be studied [12,14,15]. Moreover, in the framework of a cocycle semiflow, the base spaces are compact in many important cases. For example, if the nonlinearity f of (0.1) is periodic (resp. quasiperiodic, almost periodic, local almost periodic) in the time variable t, then the base space $\mathcal{H}$ is compact. Based on the compactness of the base spaces, we can establish the equivalence between pullback attraction of cocycle semiflow and forward attraction of the associated autonomous semiflow. This device makes the dynamics of such a nonautonomous system appear like those of an autonomous system.

Without the compactness assumption on the base spaces, the upper semicontinuity of global pullback attractors for nonautonomous systems was obtained in Caraballo and Langa [2]. However, compact forward invariant sets of the perturbed systems are required to guarantee the existence of perturbed pullback attractors. In the paper, we suppose that the base spaces of cocycle semiflows considered are compact. As a result, after introducing the notion of (local) pullback attractors (see Definition 2.6), we can establish a general result on the stability of local pullback attractors as the perturbation parameter is varied. Based on this result, a local pullback attractor bifurcation theory can be developed. This can be regarded as a generalization of autonomous attractor bifurcation theory in [23] for nonautonomous cases. Finally, we study the bifurcation of invariant sets for the cocycle semiflow φ_λ generated by the nonautonomous nonlinear evolution equation (0.1). We first construct a local central invariant manifold $\mathcal{M}^\lambda_{\rm loc}(\cdot)$ for φ_λ as λ near λ_0. Under further assumptions on f to ensure that 0 is a pullback attractor for φ_{λ_0}, we then restrict φ_λ to $\mathcal{M}^\lambda_{\rm loc}(\cdot)$ and obtain a pullback attractor bifurcation on $\mathcal{M}^\lambda_{\rm loc}(\cdot)$ as λ crosses λ_0. It leads to an invariant set bifurcation for φ_λ. It is worth mentioning that if 0 is not an attractor but a repeller for φ_{λ_0}, our result still holds. Denote by

$B_\lambda(\cdot)$ the bifurcated invariant set. We further know that

$$\lim_{\lambda\to\lambda_0} H_{X^\alpha}(B_\lambda(p),0)=0,\quad p\in P.$$

This paper is organized as follows. In Section 2, we present respectively some basic facts in autonomous and nonautonomous dynamical systems which will be required in the rest of the work. Section 3 deals with the stability of pullback attractors as bifurcation parameter varies. In Section 4, we establish an invariant set bifurcation theory for (0.1). We illustrate the main results with an example in Section 5. Finally, Section 6 contains the proofs of two propositions presented earlier in the paper.

2 Preliminaries

In this section we introduce some basic definitions and notions [7,8].

Let X be a complete metric space with metric $d(\cdot,\cdot)$. Given $M\subset X$, we denote $\overline{M}$ and $\operatorname{int} M$ the closure and interior of any subset M of X, respectively. A set $U\subset X$ is called a neighborhood of $M\subset X$, if $\overline{M}\subset \operatorname{int} U$. For any $\rho>0$, denote by

$$\mathrm{B}_X(M,\rho):=\{x\in X:\ d(x,M)<\rho\}$$

the ρ-neighborhood of M in X, where $d(x,M)=\inf_{y\in M} d(x,y)$.

The Hausdorff semidistance in X is defined as

$$H_X(M,N)=\sup_{x\in M} d(x,N),\qquad \forall\, M,N\subset X.$$

2.1 Semiflows and attractors

Let $\mathbb{R}^+=[0,\infty)$. A continuous mapping $S:\mathbb{R}^+\times X\to X$ is called a semiflow on X, if it satisfies

i) $S(0,x)=x$ for all $x\in X$; and

ii) $S(t+s,x)=S(t,S(s,x))$ for all $x\in X$ and $t,s\in\mathbb{R}^+$.

Let S be a given semiflow on X. As usual, we will rewrite $S(t,x)$ as $S(t)x$.

A set $B\subset X$ is called **invariant** (resp. **positively invariant**) under S if $S(t)B=B$ (resp. $S(t)B\subset B$) for all $t\geq 0$.

Let B and C be subsets of X. We say that B **attracts** C under S, if

$$\lim_{t\to\infty} H_X(S(t)C,B)=0.$$

Definition 2.1. A compact subset $\mathcal{A}\subset X$ is called an **attractor** for S, if it is invariant under S and attracts one of neighborhood of itself.

It is well known that if U is a compact positively invariant set of S, then the omega-limit set $\omega(U):=\bigcap_{T\geq 0}\overline{\bigcup_{t\geq T} S(t)U}$ is an attractor of S. The definition of the attraction basin of the attractor and other properties can be found in [10,22,30].

2.2 Cocycle semiflows and pullback attractors

A nonautonomous system consists of a "base flow" and a "cocycle semiflow" that is in some sense driven by the base flow.

A **base flow** $\{\theta_t\}_{t\in\mathbb{R}} := \{\theta(t)\}_{t\in\mathbb{R}}$ is a flow on a metric space P such that $\theta_t P = P$ for all $t \in \mathbb{R}$.

Definition 2.2. A **cocycle semiflow** φ on the phase space X over θ is a continuous mapping $\varphi : \mathbb{R}^+ \times P \times X \to X$ satisfying

- $\varphi(0, p, x) = x$,
- $\varphi(t+s, p, x) = \varphi(t, \theta_s p, \varphi(s, p, x))$ (cocycle property).

Remark 2.3. If we replace $\mathbb{R}^+$ by $\mathbb{R}$ in the above definition, then φ is called a cocycle flow on X.

We usually denote $\varphi(t, p)x := \varphi(t, p, x)$. Then $\{\varphi(t, p)\}_{t\geq 0,\, p\in P}$ can be viewed as a family of continuous mappings on X.

For convenience in statement, a family of subsets $\{B_p\}_{p\in P}$ of X is called a **nonautonomous set** in X. Let $B(\cdot) = \{B_p\}_{p\in P}$ be a nonautonomous set. For convenience, we will rewrite B_p as $B(p)$, called the p-**section** of $B(\cdot)$. We also denote $\mathbb{B}$ the union of the sets $B(p) \times \{p\}$ ($p \in P$), i.e.,

$$\mathbb{B} = \bigcup_{p\in P} B(p) \times \{p\}.$$

Note that $\mathbb{B}$ is a subset of $X \times P$.

A nonautonomous set $B(\cdot)$ is said to be closed (resp. open, compact), if each section $B(p)$ is closed (resp. open, compact) in X. A nonautonomous set $U(\cdot)$ is called a **neighborhood** of $B(\cdot)$, if $\overline{B}(p) \subset \operatorname{int} U(p)$ for each $p \in P$.

A nonautonomous set $B(\cdot)$ is said to be **invariant** (resp. **forward invariant**) under φ if for $t \geq 0$,

$$\varphi(t, p)B(p) = B(\theta_t p), \qquad p \in P.$$
$$(\text{resp. } \varphi(t, p)B(p) \subset B(\theta_t p), \qquad p \in P.)$$

Let $B(\cdot)$ and $C(\cdot)$ be two nonautonomous subsets of X. We say that $B(\cdot)$ **pullback attracts** $C(\cdot)$ under φ if for any $p \in P$,

$$\lim_{t\to\infty} H_X(\varphi(t, \theta_{-t}p)C(\theta_{-t}p),\, B(p)) = 0.$$

Let φ be a given cocycle semiflow on X with driving system θ on base space P. The (autonomous) semiflow $\Phi := \{\Phi(t)\}_{t\geq 0}$ on $Y := P \times X$, given by

$$\Phi(t)(p, x) = (\theta_t p, \phi(t, p)x), \quad t \geq 0,$$

is called the **skew product semiflow** associated to φ.

The following fundamental result studies the relationship between the pullback attraction of φ and attraction of Φ. The proof is given in the Appendixes.

Proposition 2.4. *Let $(\varphi, \theta)_{X,P}$ be a nonautonomous system, and let Φ be the skew-product flow associated to φ. Let $K(\cdot)$ and $B(\cdot)$ be two nonautonomous sets. Suppose P and $K_P := \overline{\bigcup_{p\in P} K(p)} \subset X$ are both compact. Then $K(\cdot)$ pullback attracts $B(\cdot)$ through φ if and only if $\mathbb{K} := \bigcup_{p\in P} K(p) \times \{p\}$ attracts $\mathbb{B} := \bigcup_{p\in P} B(p) \times \{p\}$ through Φ.*

Remark 2.5. The special case that $K(\cdot)$ is a global pullback attractor was considered in Theorem 15.7 and Theorem 15.8 of [5].

Definition 2.6. Let $(\varphi, \theta)_{X,P}$ be a nonautonomous system. A nonautonomous set $A(\cdot)$ is called a **(local) pullback attractor** for φ if it is compact, invariant and pullback attracts a neighborhood $U(\cdot)$ of itself.

The local pullback attractor defined here, very similar to the notion of a past attractor in Rasmussen [29], can be seen as a nature generalization of the local attractor from the autonomous theory. Similar to the case of autonomous systems, if $U(\cdot)$ is a compact forward invariant set of φ, then the omega-limit set $\omega(U)(\cdot)$ defined as

$$\omega(U)(\omega) = \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \varphi(t, \theta_{-t}\omega)U(\theta_{-t}\omega)}, \quad \omega \in \Omega$$

is a pullback attractor of φ. For instance, consider the following simple system on $X = \mathbb{R}$:

$$x'(t) = -3x + p(t)x^3, \quad p \in \mathcal{H}[h], \tag{2.1}$$

where $h(t) = 2 + \sin t$ and $\mathcal{H}[h]$ is its hull which is the closure for the uniform convergence topology of the set of t-translates of h. The translation map $\theta_t : \mathcal{H} \to \mathcal{H}$ given by $\theta_t p(s) = p(t+s)$ defines a flow on $\mathcal{H}$. Then the unique solution of (2.1) define a cocycle flow on X given by $\varphi(t, p)x_0 = x(t, 0; p; x_0)$. Since

$$\frac{1}{2}\frac{d}{dt}x^2 = -3x^2 + p(t)x^4 \leq -3(x^2 - x^4) < 0$$

provided that $|x| \leq 1/2$. Therefore $[-1/2, 1/2]$ is a forward invariant set of φ and it is pullback attracted by the pullback attractor 0. It is worth noting that 0 is only a local pullback attractor. Indeed,

$$\frac{1}{2}\frac{d}{dt}x^2 = -3x^2 + p(t)x^4 \geq -3x^2 + x^4 > 0$$

provided that $|x| \geq 2$. Thus 0 is only a local pullback attractor of φ.

In general, it is difficult to define the attraction basin of a pullback attractor. Fortunately, under the assumptions of Proposition 2.4, we can define the pullback attraction basin of a pullback attractor $A(\cdot)$. Specifically, we have

Definition 2.7. Let $(\varphi, \theta)_{X,P}$ be a nonautonomous system, and let Φ be the skew-product flow associated to φ. Suppose P is compact. Let $A(\cdot)$ be a pullback attractor of φ such that $A_P := \overline{\bigcup_{p\in P} A(p)}$ is compact. Then the **pullback attraction basin** of $A(\cdot)$ can be given by

$$B(A)(\cdot) = \{x(\cdot) : \mathbb{A} \text{ attracts } \mathbb{x} \text{ under } \Phi\},$$

where $x(\cdot)$ is any singleton nonautonomous set in X and $\mathbb{x} = \bigcup_{p\in P}\{p\} \times x(p)$.

3 Stability of pullback attractors

We now establish a result on the stability of pullback attractors under a small perturbation. In fact, we prove a continuity result with respect to the Hausdorff semi-distance.

Let X be a Banach space with norm $\|\cdot\|$, and let A be a sectorial operator on X. Pick a number $a > 0$ such that

$$\operatorname{Re}\sigma(A + aI) > 0.$$

Denote $\Lambda = A + aI$. For each $\alpha \geq 0$, define the fractional power space as $X^\alpha = D(\Lambda^\alpha)$, which is equipped with the norm $\|\cdot\|_\alpha$ defined by

$$\|x\|_\alpha = \|\Lambda^\alpha x\|, \quad x \in X^\alpha.$$

Note that the definition of X^α is independent of the choice of the number a. If A has compact resolvent, the inclusion $X^{\alpha'} \hookrightarrow X^\alpha$ is compact for $\alpha' > \alpha \geq 0$.

Let φ_{λ_0} ($\lambda_0 \in \mathbb{R}$) be a given cocycle semiflow on X with driving system θ on base space P. For $\delta > 0$, denote $I_{\lambda_0}(\delta) := (\lambda_0 - \delta, \lambda_0 + \delta)$. Assume that φ_λ, $\lambda \in I_{\lambda_0}(\delta)$ is a small perturbation of the given flow φ_{λ_0} based on P. Let us make the following assumptions:

(H1): The base space P is compact.

(H2): For every $T > 0$ and compact subset B of X, we have

$$\lim_{\lambda \to \lambda_0} \|\varphi_\lambda(t,p)x - \varphi_{\lambda_0}(t,p)x\|_\alpha = 0, \tag{3.1}$$

uniformly with respect to $(t,x) \in [0,T] \times B$ and $p \in P$.

Under the assumptions **(H1), (H2)**, we can get a result on the stability of pullback attractors.

Theorem 3.1. *Let $A_{\lambda_0}(\cdot) := \{A_{\lambda_0}(p)\}_{p \in P}$ be an attractor of the cocycle semiflow φ_{λ_0} which pullback attracts a neighborhood $U(\cdot)$ of itself. Let*

$$\mathbb{U} := \bigcup_{p \in P} U(p) \times \{p\} \quad \text{and} \quad \mathbb{A}_{\lambda_0} := \overline{\bigcup_{p \in P} A_{\lambda_0}(p) \times \{p\}}.$$

Assume $\mathbb{U}$ is a compact neighborhood of $\mathbb{A}_{\lambda_0}$ in $Y = X \times P$, then under the assumptions **(H1), (H2)**, *the following statements hold.*

(a) *There exists a small $\delta > 0$ such that for each $\lambda \in I_{\lambda_0}(\delta)$, φ_λ has a pullback attractor $A_\lambda(\cdot)$ such that*

$$\lim_{\lambda \to \lambda_0} H_X(A_\lambda(p), \bigcup_{p \in \mathcal{H}} A_{\lambda_0}(p)) = 0. \tag{3.2}$$

(b) *In addition, if $U(\cdot)$ is forward invariant, then*

$$\lim_{\lambda \to \lambda_0} H_X(A_\lambda(p), A_{\lambda_0}(p)) = 0. \tag{3.3}$$

Proof. (a) By the compactness of $\mathbb{U}$, we know that $A_{\lambda_0 P} := \overline{\bigcup_{p \in P} A_{\lambda_0}(p)}$ is compact. Since $A_{\lambda_0}(\cdot)$ pullback attracts $U(\cdot)$ and P is compact, by Proposition 2.4, $\mathbb{A}_{\lambda_0}$ attracts $\mathbb{U}$ through Φ_{λ_0}. Since $\mathbb{U}$ is a neighborhood of $\mathbb{A}_{\lambda_0}$, one knows that $\mathbb{A}_{\lambda_0}$ is an attractor of Φ_{λ_0}. By the assumption **(H2)**, for any compact set $B \subset X$, we have that

$$\lim_{\lambda \to \lambda_0} H_Y(\Phi_\lambda(t)(x,p), \Phi_{\lambda_0}(t)(x,p)) = \lim_{\lambda \to \lambda_0} \|\varphi_\lambda(t,p)x - \varphi_{\lambda_0}(t,p)x\|_\alpha = 0 \tag{3.4}$$

uniformly with respect to $t \in [0,T]$ and $(x,p) \in B \times P$. Then by the stability of the autonomous attractors [21, Theorem 4.1], there exists a $\delta > 0$ (independent of $p \in P$) such that for each $\lambda \in I_{\lambda_0}(\delta) := (\lambda_0 - \delta, \lambda_0 + \delta)$, Φ_λ has an attractor $\mathbb{A}_\lambda$ contained in $\mathbb{U}$. Moreover,

$$\lim_{\lambda \to \lambda_0} H_Y(\mathbb{A}_\lambda, \mathbb{A}_{\lambda_0}) = 0. \tag{3.5}$$

Write $\mathbb{A}_\lambda$ as $\bigcup_{p\in P} A_\lambda(p) \times \{p\}$, $\lambda \in I_{\lambda_0}(\delta)$. Using Proposition 2.4 again, we have that $A_\lambda(\cdot)$ pullback attracts $U(\cdot)$ through φ_λ, i.e., $A_\lambda(\cdot)$ is a pullback attractor of φ_λ. (3.2) is a direct consequence of (3.5).

To complete the proof of $(\mathbb{b})$, we shall prove (3.3) by contradiction. Thus, let us assume that there exist $\sigma > 0$ and a sequence $\lambda_j \to \lambda_0$, as $j \to \infty$, $x_j \in A_{\lambda_j}(p)$ such that

$$d_X(x_j, x) > \sigma, \qquad \text{for all } x \in A_{\lambda_0}(p). \tag{3.6}$$

Note that

$$x_j = \varphi_{\lambda_j}(n, \theta_{-n}p)x_j^n, \qquad \text{for some } x_j^n \in A_{\lambda_j}(\theta_{-n}p).$$

Similar to the argument in $(\mathbb{a})$, we can assume that $A_{\lambda_j}(p) \subset U(p)$, thus $x_j \in U(p)$. By the compactness of $U(p)$, there exists a subsequence of x_j (still denoted by x_j) which converges to some $x_0 \in U(p)$. Now, for each fixed n we have $x_j^n \in U(\theta_{-n}p)$ so that there is a further subsequence of x_j^n (still denoted by x_j^n) which converges to some $x_0^n \in U(\theta_{-n}p)$. On the other hand, for any given $\nu > 0$, we can use the assumption **(H2)** and the continuity of $\varphi(n, \theta_{-n}p)$ to show that for j large enough,

$$\begin{aligned}
&3d(\varphi_{\lambda_j}(n, \theta_{-t}p)x_j^n, \varphi_{\lambda_0}(n, \theta_{-t}p)x_0^n) \\
&\quad \le d(\varphi_{\lambda_j}(n, \theta_{-t}p)x_j^n, \varphi_{\lambda_0}(n, \theta_{-t}p)x_j^n) + d(\varphi_{\lambda_0}(n, \theta_{-t}p)x_j^n, \varphi_{\lambda_0}(n, \theta_{-t}p)x_0^n) \\
&\quad \le \nu + \nu.
\end{aligned}$$

Then, for each fixed $n \in \mathbb{N}$,

$$x_0 = \lim_{j\to\infty} x_j = \lim_{j\to\infty} \varphi_{\lambda_j}(n, \theta_{-n}p)x_j^n = \varphi_{\lambda_0}(n, \theta_{-n}p)x_0^n.$$

Since $U(p)$ is forward invariant, we have

$$x_0 \in \bigcap_{n\in\mathbb{N}} \varphi_{\lambda_0}(n, \theta_{-n}p)U(\theta_{-n}p) = A_{\lambda_0}(p),$$

which contradicts (3.6). The proof is complete. □

The main contribution of Theorem 3.1 is the existence of pullback attractor $A_\lambda(\cdot)$ for φ_λ as λ near λ_0, while the argument of the upper semicontinuity of pullback attractors is an adaptation of that of [2].

The conditions of the following results may be easier to be verified in applications.

Corollary 3.2. *Let $A_{\lambda_0}(\cdot) := \{A_{\lambda_0}(p)\}_{p\in P}$ be an attractor of the cocycle semiflow φ_{λ_0} and $U \subset X$ be a compact forward invariant neighborhood of $A_{\lambda_0}(\cdot)$. Then under the assumptions* **(H1), (H2)***, there exists a small $\delta > 0$ such that for each $\lambda \in I_{\lambda_0}(\delta)$, φ_λ has a pullback attractor $A_\lambda(\cdot)$ satisfying*

$$\lim_{\lambda\to\lambda_0} H_X(A_\lambda(p), A_{\lambda_0}(p)) = 0.$$

4 Invariant set bifurcation for nonautonomous nonlinear evolution equations

Based on the general result of the stability of pullback attractors, in the section we can establish some results on invariant set bifurcation for nonautonomous dynamical systems.

4.1 Problem and mathematical setting

From now on, we assume X is a Hilbert space with inner product $(\cdot,\cdot)$. We will consider and study invariant set bifurcation of the evolution equation

$$u_t + Au = \lambda u + f(t,u) \tag{4.1}$$

on X, where $\lambda \in \mathbb{R}$ is a bifurcation parameter, the nonlinearity $f : \mathbb{R} \times X^\alpha \to X$ is bounded continuous mapping satisfying

(F1)

$$f(t,u) = o(\|u\|_\alpha), \quad \text{as } \|u\|_\alpha \to 0 \tag{4.2}$$

uniformly on $t \in \mathbb{R}$. Moreover, there is $\beta > 0$ such that

$$((f(t,u),u) \le -\beta \cdot \kappa(u) \tag{4.3}$$

for $t \in \mathbb{R}$ and $u \in X^\alpha$, where $\kappa : X \to \mathbb{R}^+$ is a nonnegative function satisfying that $\kappa(u) = 0$ if and only if $u = 0$.

Denote $k(\rho)$ the Lipschitz constant of $f(t,\cdot)$ in $\overline{\mathrm{B}}_{X^\alpha}(\rho)$. Then by (4.2),

$$\lim_{\rho\to 0} k(\rho) = 0$$

and

$$\|f(t,u_1) - f(t,u_2)\| \le k(\rho)\|u_1 - u_2\|_\alpha, \qquad \forall\, u_1,u_2 \in \overline{\mathrm{B}}_{X^\alpha}(\rho). \tag{4.4}$$

Denote $C_b(\mathbb{R},X)$ the set of bounded continuous functions from $\mathbb{R}$ to X. Equip $C_b(\mathbb{R},X)$ with either the uniform convergence topology generated by the metric

$$r(h_1,h_2) = \sup_{t\in\mathbb{R}} \|h_1(t) - h_2(t)\|,$$

or the compact-open topology generated by the metric

$$r(h_1,h_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\max_{t\in[-n,n]} \|h_1(t) - h_2(t)\|}{1 + \max_{t\in[-n,n]} \|h_1(t) - h_2(t)\|}.$$

Let $f(\cdot,u) \in C_b(\mathbb{R},X)$ be the function in (4.1). Define the hull of $f(\cdot,u)$ as follows

$$\mathcal{H} := \mathcal{H}[f(\cdot,u)] = \overline{\{f(\tau+\cdot,u) :\ \tau \in \mathbb{R}\}}^{\,C_b(\mathbb{R},X)}.$$

In application, $f(\cdot,u)$ is often taken as a periodic function, quasiperiodic function, almost periodic function, local almost periodic function [7,20] or uniformly almost automorphic function [33]. In this case, the hull $\mathcal{H}$ is a **compact** metric space. Accordingly, the translation group θ on $\mathcal{H}$ is given by

$$\theta_\tau p(\cdot,u) = p(\tau+\cdot,u), \qquad t \in \mathbb{R},\ p \in \mathcal{H}.$$

Instead of (3.2), we will consider the more general cocycle system in X^α (where $\alpha \in [0,1)$):

$$u_t + Au = \lambda u + p(t,u), \qquad p \in \mathcal{H}. \tag{4.5}$$

Proposition 4.1 ([11]). *Let A and p be given as above. Assume that p is locally Hölder continuous in t. Then for each $u_0 \in X^\alpha$, there is a $T > t_0$ such that (4.5) has a unique solution $u(t) = u_\lambda(t, t_0; u_0, p)$ on $[t_0, T)$ satisfying*

$$u(t) = e^{-A(t-t_0)}x_0 + \int_{t_0}^{t} e^{-A(t-s)}[\lambda u(s) + p(s, u(s))]ds, \quad t \in [t_0, T). \tag{4.6}$$

For convenience, from now on we always assume that the unique solution (4.6) is globally defined. Define

$$\varphi_\lambda(t, p)u := u_\lambda(t, 0; u, p), \qquad u \in X^\alpha, \ p \in \mathcal{H}.$$

Then φ_λ is a **cocycle semiflow** on X^α driven by the base flow θ on $\mathcal{H}$.

Remark 4.2. Note that for each $p \in \mathcal{H}$, $u(t)$ is a p-solution of φ_λ on an interval J *if and only if* it solves the equation (4.5) on J.

4.2 Local invariant manifolds

Let $\lambda_0 \in \mathbb{R}$ be an isolated eigenvalue of A. Suppose that

(F2) there is a $\eta > 0$ such that the spectrum

$$\sigma(A) \cap \{z \in \mathbb{C} : \lambda_0 - \eta < \operatorname{Re} z < \lambda_0 + \eta\} = \lambda_0.$$

Denote $A_\lambda := A - \lambda$. Then for $\lambda \in I_{\lambda_0}(\eta/4) := (\lambda_0 - \eta/4, \lambda_0 + \eta/4)$, the spectrum $\sigma(A_\lambda)$ has a decomposition $\sigma(A_\lambda) = \sigma_c \cup \sigma_+ \cup \sigma_-$, where

$$\sigma_c = \{\lambda_0 - \lambda\}, \quad \sigma_+ = \sigma(A_\lambda) \cap \{\operatorname{Re}\lambda > 0\} \quad \text{and} \quad \sigma_- = \sigma(A_\lambda) \cap \{\operatorname{Re}\lambda < 0\}.$$

Accordingly, the space X has a direct sum decomposition: $X = X_c \oplus X_+ \oplus X_-$. Denote $X_\pm = X_+ \oplus X_-$ and

$$X_i^\alpha := X_i \cap X^\alpha, \qquad i = c, +, -, \pm.$$

Note that X_c^α is finite dimensional.

Under the assumptions on A and f, we can construct a local invariant manifold for φ_λ, $\lambda \in I_{\lambda_0}(\eta/8)$.

Proposition 4.3. *Suppose the assumptions* **(F1), (F2)** *hold. Then there exists $\varrho > 0$ such that the cocycle semiflow φ_λ, $\lambda \in I_{\lambda_0}(\eta/8)$ has a local invariant manifold $\mathcal{M}^\lambda_{\text{loc}}(\cdot) := \{\mathcal{M}^\lambda_{\text{loc}}(p)\}_{p\in\mathcal{H}}$ in X^α which is represented as*

$$\mathcal{M}^\lambda_{\text{loc}}(p) = \{y + \xi^\lambda_p(y) : y \in \overline{B}_{X_c^\alpha}(\varrho)\},$$

where $\xi^\cdot_p(\cdot) : I_{\lambda_0}(\eta/8) \times \overline{B}_{X_c^\alpha}(\varrho) \to X^\alpha_\pm$ is a Lipschitz continuous mapping satisfying that

$$\xi^\lambda_p(0) = 0 \text{ and } \|\xi^\lambda_p(y) - \xi^\lambda_p(z)\|_\alpha \le L_1\|y - z\|_\alpha \tag{4.7}$$

and

$$\|\xi^{\lambda_1}_p(y) - \xi^{\lambda_2}_p(y)\|_\alpha \le L_2|\lambda_1 - \lambda_2|, \tag{4.8}$$

where $L_1 > 0$ is independent of $p \in P$ and $\lambda \in I_{\lambda_0}(\eta/8)$, and $L_2 > 0$ is independent of $p \in P$ and $y \in \overline{B}_{X_c^\alpha}(\varrho)$.

The proof of the above proposition is given in the Appendixes.

4.3 Invariant set bifurcation

Firstly, let us restrict the equation (4.5) on the invariant manifold $\mathcal{M}^{\lambda}_{loc}(\cdot)$, $\lambda \in I_{\lambda_0}(\eta/8)$. Specifically, we study the finite dimensional equation

$$y_t + (\lambda_0 - \lambda)y = p(t, y + \xi^{\lambda}_{\theta_t p}(y)), \quad y \in \overline{\mathrm{B}}_{X^\alpha_c}(\varrho),\, p \in \mathcal{H}. \tag{4.9}$$

Denote ϕ_λ, $\lambda \in I_{\lambda_0}(\eta/8)$ the cocycle flow on $\overline{\mathrm{B}}_{X^\alpha_c}(\varrho)$ with driving system θ on the base space $\mathcal{H}$ generated by (4.9).

We first say that the condition **(H2)** (in Section 3) holds for the cocycle flow ϕ_λ, $\lambda \in I_{\lambda_0}(\eta/8)$. Specifically, we have the following result.

Lemma 4.4. *For every $T > 0$, we have*

$$\lim_{\lambda\to\lambda_0} \|\phi_\lambda(t,p)y - \phi_{\lambda_0}(t,p)y\|_\alpha = 0, \tag{4.10}$$

uniformly with respect to $(t, y) \in [0, T] \times \overline{\mathrm{B}}_{X^\alpha_c}(\varrho)$ and $p \in P$.

Proof. For $\lambda \in I_{\lambda_0}(\eta/8)$, denote $y_\lambda(t) := \phi_\lambda(t,p)y$ and $v(t) = y_\lambda(t) - y_{\lambda_0}(t)$, then v satisfies

$$v_t + (\lambda_0 - \lambda)y_\lambda = p(t, y_\lambda + \xi^{\lambda}_{\theta_t p}(y_\lambda)) - p(t, y_{\lambda_0} + \xi^{\lambda_0}_{\theta_t p}(y_{\lambda_0})). \tag{4.11}$$

Note that $\|y_\lambda\| \le \rho$ and

$$\begin{aligned}
&\|p(t, y_\lambda + \xi^{\lambda}_{\theta_t p}(y_\lambda)) - p(t, y_{\lambda_0} + \xi^{\lambda_0}_{\theta_t p}(y_{\lambda_0}))\| \\
&\quad\le k(\rho)\big((L_1+1)\|v\|_\alpha + L_2|\lambda - \lambda_0|\big) \\
&\quad\le C'\left(\|v\|^2 + (\lambda-\lambda_0)^2\right) \text{ for some constant } C',
\end{aligned} \tag{4.12}$$

where $\rho > 0$ is the bound of $u \in \mathcal{M}^{\lambda}_{loc}(\cdot)$, which is independent of λ by (4.6). Taking the inner product of the equation (4.11) with v and using (4.12) to obtain that there is a constant $C > 0$ being independent of λ such that

$$\frac{d}{dt}\|v\|^2 \le C\left(\|v\|^2 + (\lambda-\lambda_0)^2\right).$$

Applying the classical Gronwall lemma to get that

$$\|v(t)\|^2 \le \left(e^{Ct} - 1\right)(\lambda-\lambda_0)^2, \qquad \square$$

Lemma 4.5. *Under the assumptions* **(F1), (F2)**, *$y = 0$ is locally asymptotically stable for ϕ_{λ_0}. Therefore 0 is a pullback attractor of ϕ_{λ_0}.*

Proof. Since X^α_c is finite dimensional, all the norms on X^α_c are equivalent. Hence for convenience, we equip X^α_c the norm $\|\cdot\|$ of X in the following argument.

Note that ϕ_{λ_0} is generated by the equation

$$y_t = p(t, y + \xi^{\lambda_0}_{\theta_t p}(y)), \quad y \in \overline{\mathrm{B}}_{X^\alpha_c}(\varrho),\, p \in \mathcal{H}. \tag{4.13}$$

Taking the inner product of the equation (4.13) with $y + \xi^{\lambda_0}_{\theta_t p}(y)$ in X, using the fact that $(y, \xi^{\lambda_0}_{\theta_t p}(y)) = 0$ and the assumption **(F1)**, it yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|y\|^2 &= \left(p\left(t, y + \xi^{\lambda_0}_{\theta_t p}(y)\right), y + \xi^{\lambda_0}_{\theta_t p}(y)\right) \\
&\le -\beta\cdot\kappa\left(y + \xi^{\lambda_0}_{\theta_t p}(y)\right).
\end{aligned} \tag{4.14}$$

It is clear that $\kappa\left(y+\xi_{\theta_t p}^{\lambda_0}(y)\right)=0$ if and only if $y=0$. Therefore $\lim_{t\to\infty}\|y\|=0$. The proof is complete. $\square$

Henceforth we will suppose that

(F3) The hull $\mathcal{H}$ is a compact metric space.

We then obtain a pullback attractor bifurcation theory for ϕ_λ as λ crosses λ_0.

Theorem 4.6. *Under the assumptions* **(F1), (F2)** *and* **(F3)**, *the cocycle semiflow ϕ_λ bifurcates from $(0,\lambda_0)$ a pullback attractor $A_\lambda(\cdot)$ for $\lambda>\lambda_0$, and*

$$\lim_{\lambda\to\lambda_0^+} H_{X_c^\alpha}(A_\lambda(p),\{0\})=0. \tag{4.15}$$

Proof. Recall from Lemma 4.5 that 0 is a pullback attractor for ϕ_{λ_0} and it pullback attracts $\overline{\mathrm{B}}_{X_c^\alpha}(\varrho)$ for sufficiently small $\varrho>0$. The bounded set $\overline{\mathrm{B}}_{X_c^\alpha}(\varrho)\subset X_c^\alpha$ is compact due to X_c^α being finite dimensional. Moreover, $\overline{\mathrm{B}}_{X_c^\alpha}(\varrho)$ is forward invariant under ϕ_{λ_0}. Then by Theorem 3.1, there is a $\eta'\in(0,\eta/8)$ such that for each $\lambda\in I_{\lambda_0}(\eta')$, the cocycle semiflow ϕ_λ has a pullback attractor $A_\lambda(\cdot)$ and (4.15) holds.

In the following, we prove that $0\notin A_\lambda(\cdot)$ for $\lambda\in I_{\lambda_0}^+(\eta'):=(\lambda_0,\lambda_0+\eta')$, which completes the proof.

Let $\lambda\in I_{\lambda_0}^+(\eta')$ be fixed, and let $w(t)=y(-t)$. Then $w(t)$ satisfies

$$w_t-(\lambda_0-\lambda)w=-p(-t,w+\xi_{\theta_{-t}p}^{\lambda}(w)). \tag{4.16}$$

Taking the inner product of the equation (4.16) with w in X^α, we have

$$\frac{1}{2}\frac{d}{dt}\|w\|^2-(\lambda_0-\lambda)\|w\|^2=-(p(t,w+\xi_{\theta_{-t}p}^{\lambda}(w)),w). \tag{4.17}$$

Since

$$\|p(t,u)\|\le k(\|u\|_\alpha)\|u\|_\alpha \quad\text{and}\quad \|\xi_{\theta_{-t}p}^{\lambda}(w)\|_\alpha\le L_1\|w\|_\alpha,$$

we have

$$\begin{aligned}
\|p(-t,w+\xi_{\theta_{-t}p}^{\lambda}(w))\| &\le k(\|w+\xi_{\theta_{-t}p}^{\lambda}(w)\|_\alpha)\|w+\xi_{\theta_{-t}p}^{\lambda}(w)\|_\alpha\\
&\le k(\|w+\xi_{\theta_{-t}p}^{\lambda}(w)\|_\alpha)\left(\|w\|_\alpha+L_1\|w\|_\alpha\right)\\
&\le k(\|w+\xi_{\theta_{-t}p}^{\lambda}(w)\|_\alpha)\cdot(1+L_1)\|w\|_\alpha\\
&\le [(1+L_1)ck(\|w+\xi_{\theta_{-t}p}^{\lambda}(w)\|_\alpha)]\cdot\|w\|\\
&\le \frac{1}{2}(\lambda-\lambda_0)\|w\|,\ \text{for sufficiently small } \|w\|_\alpha.
\end{aligned} \tag{4.18}$$

We get from (4.17) and (4.18) that

$$\frac{d}{dt}\|w\|^2\le-(\lambda-\lambda_0)\|w\|^2$$

for sufficiently small $\|w\|_\alpha$, which shows for fixed $\lambda\in I_{\lambda_0}^+(\eta')$, 0 locally asymptotically stable for the cocycle flow generated by the equation (4.16). In other words, 0 is a repeller of ϕ_λ when $\lambda\in I_{\lambda_0}^+(\eta')$ and repels a neighborhood of 0 in X_c^α. This implies that $0\notin A_\lambda(\cdot)$, $\lambda\in I_{\lambda_0}^+(\eta')$. The proof is complete. $\square$

We are now in position to give and prove the main result of this paper.

Theorem 4.7. *Under the assumptions* **(F1), (F2)** *and* **(F3)**, *the cocycle semiflow φ_λ bifurcates from $(0, \lambda_0)$ an invariant compact set $B_\lambda(\cdot)$ for $\lambda > \lambda_0$, and for each $p \in P$,*

$$\lim_{\lambda \to \lambda_0^+} H_X(B_\lambda(p), \{0\}) = 0. \tag{4.19}$$

Proof. Let $A_\lambda(\cdot)$ be the bifurcated attractor obtained in Theorem 4.6. Define $B_\lambda(\cdot)$ by

$$B_\lambda(p) = \{y + \xi_p^\lambda(y) : y \in A_\lambda(p)\}, \quad p \in \mathcal{H}. \tag{4.20}$$

We know from Theorem 4.6 that $0 \notin B_\lambda(\cdot)$ and $B_\lambda(\cdot) \subset \mathcal{M}_{loc}^\lambda(\cdot)$. Based on the compactness of $A_\lambda(p)$ and the continuity of $\xi_p^\lambda(y)$ in y, we can directly derive the compactness of $B_\lambda(p)$. So $B_\lambda(\cdot)$ is compact.

We claim that $B_\lambda(\cdot)$ is invariant under φ_λ. Indeed, let $p \in P$ and $y + \xi_p^\lambda(y) \in B_\lambda(p)$. Since $\phi_\lambda(t,p)y \in A_\lambda(\theta_t p)$, $t \geq 0$, by the invariance of $\mathcal{M}_{loc}^\lambda(\cdot)$, we have

$$\varphi_\lambda(t,p)(y + \xi_p^\lambda(y)) = \phi_\lambda(t,p)y + \xi_{\theta_t p}^\lambda(\phi_\lambda(t,p)y) \in B_\lambda(\theta_t p),$$

which shows

$$\varphi_\lambda(t,p)B_\lambda(p) \subset B_\lambda(\theta_t p), \quad t \geq 0.$$

On the other hand, for any $y + \xi_{\theta_t p}^\lambda(y) \in B_\lambda(\theta_t p)$, $t \geq 0$. Using the invariance of $A_\lambda(\cdot)$, there is a $y' \in A_\lambda(p)$ such that $y = \phi_\lambda(t,p)y'$. Then

$$\begin{aligned} y + \xi_{\theta_t p}^\lambda(y) &= \phi_\lambda(t,p)y' + \xi_{\theta_t p}^\lambda(\phi_\lambda(t,p)y') \\ &= \varphi_\lambda(t,p)(y' + \xi_{\theta_t p}^\lambda(y')) \in \varphi(t,p)B_\lambda(p), \end{aligned}$$

which shows

$$B_\lambda(\theta_t p) \subset \varphi_\lambda(t,p)B_\lambda(p), \quad t \geq 0.$$

Therefore $B_\lambda(\cdot)$ is invariant under φ_λ.

Finally, (4.19) is an immediately consequence of (4.15) and (4.7). □

We now give a result which parallels Theorem 4.7.

Corollary 4.8. *Let the assumptions* **(F1), (F2), (F3)** *hold, but replace* (4.3) *by the assumption that*

$$(f(t,u), u) \geq \beta \cdot \kappa(u).$$

Then the cocycle semiflow φ_λ bifurcates from $(0, \lambda_0)$ an invariant compact set $B_\lambda(\cdot)$ for $\lambda < \lambda_0$, and for each $p \in P$,

$$\lim_{\lambda \to \lambda_0^-} H_X(B_\lambda(p), \{0\}) = 0. \tag{4.21}$$

Proof. Let $\lambda \in I_{\lambda_0}(\eta/8)$. Consider the following equation

$$z_t - (\lambda_0 - \lambda)z = -p(-t, z + \xi_{\theta_{-t}p}^\lambda(z)), \quad z \in \overline{\mathrm{B}}_{X_c^\alpha}(\varrho),\ p \in \mathcal{H}. \tag{4.22}$$

Denote by ϕ_λ^- be the cocycle flow generated by (4.22). Then ϕ_λ^- be the inverse flow of ϕ_λ.

Repeating the argument of Lemma 4.4, Lemma 4.5 and Theorem 4.6 (replacing ϕ_λ by ϕ_λ^-) to show ϕ_λ^- bifurcates from $(0, \lambda_0)$ a pullback attractor $R_\lambda(\cdot)$ for $\lambda < \lambda_0$, and

$$\lim_{\lambda \to \lambda_0^-} H_{X_c^\alpha}(R_\lambda(p), \{0\}) = 0.$$

It is clear that $R_\lambda(\cdot)$ is also an invariant set of ϕ_λ. Define a set $B_\lambda(\cdot)$ by

$$B_\lambda(p) = \{y + \xi_p^\lambda(y) : y \in R_\lambda(p)\}, \quad p \in \mathcal{H}.$$

Similar to Theorem 4.7, we can show $B_\lambda(\cdot)$ is an invariant set of φ_λ and (4.21) holds. □

5 An example

Consider the nonautonomous system

$$\begin{cases} u_t - \Delta u = \lambda u \pm h(t)u^3, & t > 0,\ x \in \Omega; \\ u = 0, & t > 0,\ x \in \partial\Omega, \end{cases} \tag{5.1}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary, h is a function such that $h(t) \geq \delta > 0$ for some $\delta > 0$.

Denote by A the operator $-\Delta$ associated with the homogeneous Dirichlet boundary condition. Then A is a sectorial operator on $X = L^2(\Omega)$ with compact resolvent, and $\mathcal{D}(A) = H^2(\Omega) \bigcap H_0^1(\Omega)$. Note that A has eigenvalues $0 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots$. Denote $V = H_0^1(\Omega)$. By $(\cdot,\cdot)$ and $|\cdot|$ we denote the usual inner product and norm on H, respectively. The inner product and norm on V, denoted by $((\cdot,\cdot))$ and $\|\cdot\|$, respectively, are defined as

$$((u,v)) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x, \quad \|u\| = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{1/2}$$

for $u, v \in V$.

The system (5.1) can be written into an abstract equation on X:

$$u_t + Au = \lambda u \pm h(t)u^3.$$

Define the hull $\mathcal{H} := \mathcal{H}[h(\cdot)u^3]$. By the assumption on h, it is clear that

$$(p(t,u),u) \geq \delta \int_\Omega u^4 dx, \quad p \in \mathcal{H}.$$

Consider the cocycle system:

$$u_t + Au = \lambda u \pm p(t,u), \qquad p \in \mathcal{H}. \tag{5.2}$$

Denote $\varphi_\lambda^\pm := \varphi_\lambda^\pm(t,p)u$ the cocycle semiflow on $H_0^1(\Omega)$ driven by the base flow (translation group) θ on $\mathcal{H}$.

Since all the hypotheses in the main theorem above are fulfilled, we obtain some interesting results concerning the dynamics of the perturbed system. In particular,

Theorem 5.1. *Suppose $\mathcal{H}$ is compact. Then the cocycle semiflow φ_λ^- (resp. φ_λ^+) bifurcates from $(0,\mu_k)$, $k = 1,2,\cdots$ an invariant compact set $B_\lambda^-(\cdot)$ for $\lambda > \lambda_0$ (resp. $B_\lambda^+(\cdot)$ for $\lambda < \lambda_0$) and for each $p \in P$,*

$$\lim_{\lambda \to \lambda_0^+} H_X(B_\lambda^-(p), \{0\}) = 0.$$

$$\Big(\textit{resp.} \ \lim_{\lambda \to \lambda_0^-} H_X(B_\lambda^+(p), \{0\}) = 0.\Big)$$

6 Appendixes

6.1 Relationship between the pullback attraction of φ and the attraction of Φ

Proof of Proposition 2.4. Necessity: By the compactness of P, one finds that

$$\begin{aligned}\lim_{t\to\infty} H_Y\left(\Phi(t)\mathbb{B}, P\times K_P\right) &= \lim_{t\to\infty} H_X\left(\varphi(t,p)B(p), K_P\right)\\ &\le \lim_{t\to\infty}\sup_{p\in P} H_X\left(\varphi(t,p)B(p), K_P\right)\\ &= \lim_{t\to\infty}\sup_{p\in P} H_X\left(\varphi(t,\theta_{-t}p)B(\theta_{-t}p), K_P\right)\\ &= 0.\end{aligned}$$

This means the compact set $P\times K_P$ attracts $\mathbb{B}$ through Φ. Therefore the omega-limit set $\omega(\mathbb{B})$ of $\mathbb{B}$ exists and attracts $\mathbb{B}$.

In the following, we prove $\omega(\mathbb{B})\subset\mathbb{K}$, which completes the necessity. For this purpose, define a nonautonomous set $\tilde{B}(\cdot)$ as follows

$$\tilde{B}(p) := \overline{\bigcup_{s\ge 0}\varphi(s,\theta_{-s}p)B(\theta_{-s}p)},\quad p\in P.$$

It is clear that $B(\cdot)\subset\tilde{B}(\cdot)$. We first say $\tilde{B}(\cdot)$ is forward invariant. Indeed, for any $t\ge 0$ and $p\in P$,

$$\begin{aligned}\varphi(t,p)\tilde{B}(p) &= \varphi(t,p)\overline{\bigcup_{s\ge 0}\varphi(s,\theta_{-s}p)B(\theta_{-s}p)}\\ &\subset \overline{\bigcup_{s\ge 0}\varphi(t,p)\circ\varphi(s,\theta_{-s}p)B(\theta_{-s}p)}\\ &= \overline{\bigcup_{s\ge 0}\varphi(t+s,\theta_{-(t+s)}\circ\theta_t p)B(\theta_{-(t+s)}\circ\theta_t p)}\\ &\subset \overline{\bigcup_{s\ge 0}\varphi(s,\theta_{-s}\circ\theta_t p)B(\theta_{-s}\circ\theta_t p)} = \tilde{B}(\theta_t p).\end{aligned} \tag{6.1}$$

So $\tilde{B}(\cdot)$ is forward invariant, which implies the omega-limit set $\omega(\tilde{B})(\cdot)$ of $\tilde{B}(\cdot)$ is the maximal invariant set in $\tilde{B}(\cdot)$. Furthermore, for any $p\in P$,

$$\begin{aligned}\omega(\tilde{B})(p) &= \bigcap_{\tau\ge 0}\overline{\bigcup_{t\ge\tau}\varphi(t,\theta_{-t}p)\tilde{B}(\theta_{-t}p)}\\ &= \bigcap_{\tau\ge 0}\overline{\bigcup_{t\ge\tau}\varphi(t,\theta_{-t}p)\overline{\bigcup_{s\ge 0}\varphi(s,\theta_{-(s+t)}p)B(\theta_{-(s+t)}p)}}\\ &= \bigcap_{\tau\ge 0}\overline{\bigcup_{t\ge\tau}\varphi(t,\theta_{-t}p)\bigcup_{s\ge 0}\varphi(s,\theta_{-(s+t)}p)B(\theta_{-(s+t)}p)}\\ &= \bigcap_{\tau\ge 0}\overline{\bigcup_{t\ge\tau}\bigcup_{s\ge 0}\varphi(t,\theta_{-t}p)\circ\varphi(s,\theta_{-(s+t)}p)B(\theta_{-(s+t)}p)}\\ &= \bigcap_{\tau\ge 0}\overline{\bigcup_{t\ge\tau}\bigcup_{s\ge 0}\varphi(t+s,\theta_{-(s+t)}p)B(\theta_{-(s+t)}p)}\\ &= \bigcap_{\tau\ge 0}\overline{\bigcup_{t\ge\tau}\varphi(t,\theta_{-t}p)B(\theta_{-t}p)}\\ &= \omega(B)(p),\end{aligned}$$

where the third "=" holds since for each fixed $t \geq 0$ and $p \in P$, $\varphi(t, \theta_{-t}p)$ is a continuous map on X. It follows that $\omega(B)(\cdot)$ is the maximal forward invariant set in $\tilde{B}(\cdot)$. Therefore $\mathbb{C} := \bigcup_{p\in P}\left(\{p\} \times \omega(B)(p)\right)$ is the maximal invariant set in $\tilde{\mathbb{B}} := \bigcup_{p\in P}\left(\{p\} \times \tilde{B}(p)\right)$. By the forward invariance of $\tilde{B}(\cdot)$,

$$
\begin{aligned}
\varphi(t)\tilde{\mathbb{B}} &= \varphi(t) \bigcup_{p\in P} \left(\{p\} \times \tilde{B}(p)\right) \\
&\subset \bigcup_{p\in P} \varphi(t)\left(\{p\} \times \tilde{B}(p)\right) \\
&= \bigcup_{p\in P} \left(\{\theta_t p\} \times \varphi(t,p)\tilde{B}(p)\right) \\
&\subset \text{(by (6.1))} \subset \bigcup_{p\in P} \left(\{\theta_t p\} \times \tilde{B}(\theta_t p)\right) \\
&= \tilde{\mathbb{B}}, \quad t \geq 0,
\end{aligned}
$$

i.e. $\tilde{\mathbb{B}}$ is positively invariant under φ. Then $\omega(\tilde{\mathbb{B}})$ is the maximal invariant set in $\tilde{\mathbb{B}}$. Recall that $\mathbb{C}$ is also the maximal invariant set in $\tilde{\mathbb{B}}$, we have

$$\omega(\mathbb{B}) \subset \omega(\tilde{\mathbb{B}}) = \mathbb{C}.$$

Finally, by the assumption that $K(\cdot)$ attracts $B(\cdot)$, one knows that $\omega(B)(\cdot) \subset K(\cdot)$, and thus $\mathbb{C} \subset \mathbb{K}$, which shows

$$\omega(\mathbb{B}) \subset \mathbb{K}.$$

Sufficiency: In a very similar way as above, we can prove the sufficiency.

By the compactness of P,

$$
\begin{aligned}
\lim_{t\to\infty} H_X\left(\varphi(t, \theta_{-t}p)B(\theta_{-t}p), K_P)\right] &\leq \lim_{t\to\infty} \sup_{p\in P} H_X\left(\varphi(t,p)B(p), K_P\right) \\
&= \lim_{t\to\infty} \sup_{p\in P} H_Y\left(\Phi(t)\mathbb{B}, P \times K_P\right) \\
&= \lim_{t\to\infty} H_Y\left(\Phi(t)\mathbb{B}, P \times K_P\right) \\
&= 0,
\end{aligned}
$$

which implies $\omega(B)(\cdot)$ exists and pullback attracts $B(\cdot)$.

To complete the proof, it suffices to show $\omega(B)(\cdot) \subset K(\cdot)$. We first define a set

$$\hat{\mathbb{B}} = \overline{\bigcup_{s\geq 0} \Phi(s)\mathbb{B}}.$$

Then $\hat{\mathbb{B}}$ is positively invariant and

$$\omega(\hat{\mathbb{B}}) = \omega(\mathbb{B}).$$

This implies that $\Omega(\mathbb{B})$ is the maximal invariant set in $\hat{\mathbb{B}}$. Write $\omega(\mathbb{B}) := \bigcup_{p\in P}\{p\} \times C(p)$, then $C(\cdot)$ is the maximal invariant set in $\hat{B}(\cdot)$, where $\hat{B}(\cdot)$ is the set defined by $\hat{\mathbb{B}} := \bigcup_{p\in P}\{p\} \times \hat{B}(p)$. By the positive invariance of $\hat{\mathbb{B}}$, one also knows that $\hat{B}(\cdot)$ is forward invariant. This implies $\Omega_{\hat{B}}(\cdot)$ is the maximal invariant set in $\hat{B}(\cdot)$. We then have that $\omega(B)(\cdot) \subset \omega(\hat{B})(\cdot) = C(\cdot)$. We learn from the condition $\omega(\mathbb{B}) \subset \mathbb{K}$ that $C(\cdot) \subset K(\cdot)$. In summary, $\omega(B)(\cdot) \subset K(\cdot)$, which completes the sufficiency. $\square$

6.2 Construction of local invariant manifold

Let $M > 0$. For $\mu \geq 0$, define a Banach space as

$$\mathscr{X}_\mu = \left\{ u \in C(\mathbb{R}; X^\alpha) : \sup_{t\in\mathbb{R}} e^{-\mu|t|}\|x(t)\|_\alpha < M \right\},$$

which is equipped with the norm $\|\cdot\|_{\mathscr{X}_\mu}$,

$$\|x\|_{\mathscr{X}_\mu} = \sup_{t\in\mathbb{R}} e^{-\mu|t|}\|x(t)\|_\alpha, \qquad \forall\, x \in \mathscr{X}_\mu.$$

Let $A^\lambda = A - \lambda$. Write $\sigma(A_\lambda) = \sigma_- \cup \sigma_c \cup \sigma_+$, where

$$\sigma_c = \{\lambda_0 - \lambda\},$$

$$\sigma_- = \sigma(A_\lambda) \cap \{\operatorname{Re}\lambda < 0\}, \quad \sigma_+ = \sigma(A_\lambda) \cap \{\operatorname{Re}\lambda > 0\}.$$

According to the spectral decomposition, the space X has a direct sum decomposition: $X = X_- \oplus X_c \oplus X_+$. Denote $X_\pm := X_- \bigcup X_+$. Note that each X_i, $i = -,+,\pm,c$ is independent of λ. Let

$$\Pi_i : X \to X_i, \qquad i = -,+,\pm,c$$

be the projection from X to X_i. Denote $A_i^\lambda = A^\lambda|_{X_i}$. By the assumption **(F2)**, we deduce that if $\lambda \in (\lambda_0 - \eta/4, \lambda_0 + \eta/4)$ then for $\alpha \in [0,1)$,

$$\|A^\alpha e^{-A_-^\lambda t}\| \leq e^{\frac{3\eta}{4}t}, \quad \|e^{-A_-^\lambda t}\| \leq e^{-\frac{3\eta}{4}t}, \qquad t \leq 0, \tag{6.2}$$

$$\|A^\alpha e^{-A_+^\lambda t}\Pi_+ A^{-\alpha}\| \leq e^{-\frac{3\eta}{4}t}, \quad \|A^\alpha e^{-A_+^\lambda t}\| \leq t^{-\alpha}e^{-\frac{3\eta}{4}t}, \qquad t > 0, \tag{6.3}$$

$$\|A^\alpha e^{-A_c^\lambda t}\| \leq e^{\frac{\eta}{4}|t|}, \quad \|e^{-A_c^\lambda t}\| \leq e^{\frac{\eta}{4}|t|}, \qquad t \in \mathbb{R}. \tag{6.4}$$

Proof of Proposition 4.3. The proof of the existence result for a local invariant manifold is an adaptation of the corresponding result in Chicone and Latushkin [9]. Here we give the details for completeness and the reader's convenience. The main aim of the proof is to show the Lipschitz continuity of $\xi_p^\lambda(y)$ in λ and y, respectively.

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\chi(z) = \begin{cases} 1, & |z| \leq 1/2; \\ 0, & |z| \geq 1. \end{cases}$$

For $\rho > 0$, one can then define a smooth function such that

$$p_\rho(t,u) = \chi\left(\frac{\|u\|_\alpha}{\rho}\right) p(t,u).$$

Select suitable χ such that

$$\|p_\rho(t,u) - p_\rho(t,v)\| \leq k(\rho)\|u - v\|, \tag{6.5}$$

where $k(\rho)$ is the local Lipschitz constant of f given in (4.4). Instead of (4.5), we consider the truncated system

$$u_t + Au = \lambda u + p_\rho(t,u), \qquad p \in \mathcal{H}. \tag{6.6}$$

Suppose that ρ is so small that

$$M_\rho := k(\rho) \int_0^\infty \left(2 + \tau^{-\alpha}\right) e^{-\frac{\eta}{4}\tau} d\tau < 1. \tag{6.7}$$

Let $u \in \mathscr{X}_{\eta/2}$. By simple computations, we know that u is the solution of (6.6) if and only if it solves the integral equation

$$\begin{aligned} u(t) = e^{-A_c^\lambda t}\Pi_c u(0) &+ \int_0^t e^{-A_c^\lambda (t-\tau)}\Pi_c p_\rho(\tau, u(\tau)) d\tau \\ &+ \int_{-\infty}^t e^{-A_+^\lambda (t-\tau)}\Pi_+ p_\rho(\tau, u(\tau)) d\tau \\ &- \int_t^\infty e^{-A_-^\lambda (t-\tau)}\Pi_- p_\rho(\tau, u(\tau)) d\tau. \end{aligned} \tag{6.8}$$

Take a $\tilde{\varrho} > 0$ small enough so that

$$\tilde{\varrho} < \left(1 - M_\rho\right) M. \tag{6.9}$$

Let $p \in \mathcal{H}$ and $\lambda \in I_{\lambda_0}(\eta/8)$ be fixed. For each $y \in \overline{\mathrm{B}}_{X_c^\alpha}(\tilde{\varrho})$, one can use the righthand side of equation (6.8) to define a contraction mapping $\mathcal{T} := \mathcal{T}_y$ on $\mathscr{X}_{\eta/2}$ as follows:

$$\begin{aligned} \mathcal{T}u(t) = e^{-A_c^\lambda t} y &+ \int_0^t e^{-A_c^\lambda (t-\tau)}\Pi_c p_\rho(\tau, u(\tau)) d\tau \\ &+ \int_{-\infty}^t e^{-A_+^\lambda (t-\tau)}\Pi_+ p_\rho(\tau, u(\tau)) d\tau \\ &- \int_t^\infty e^{-A_-^\lambda (t-\tau)}\Pi_- p_\rho(\tau, u(\tau)) d\tau. \end{aligned}$$

We first verify that $\mathcal{T}$ maps $\mathscr{X}_{\eta/2}$ into itself.

For notational convenience, we write

$$0 \wedge t = \min\{0, t\}, \quad 0 \vee t = \max\{0, t\}, \quad \text{for } t \in \mathbb{R}.$$

Let $u \in \mathscr{X}_{\eta/2}$. By (6.2)–(6.4) and (6.5) we have

$$\begin{aligned} \|\mathcal{T}u(t)\|_\alpha \le e^{\frac{\eta}{4}|t|}\|y\|_\alpha &+ \int_{0\wedge t}^{0\vee t} e^{\frac{\eta}{4}|t-\tau|} k(\rho)\|u(\tau)\|_\alpha d\tau \\ &+ \int_{-\infty}^t (t-\tau)^{-\alpha} e^{-\frac{3\eta}{4}(t-\tau)} k(\rho)\|u(\tau)\|_\alpha d\tau \\ &+ \int_t^\infty e^{\frac{3\eta}{4}(t-\tau)} k(\rho)\|u(\tau)\|_\alpha d\tau. \end{aligned} \tag{6.10}$$

It is trivial to verify that

$$e^{-\frac{\eta}{2}|t|} \int_{0\wedge t}^{0\vee t} e^{\frac{\eta}{4}|t-\tau|} k(\rho)\|u(\tau)\|_\alpha d\tau = \int_{0\wedge t}^{0\vee t} e^{-\frac{\eta}{4}|t-\tau|} \left[e^{-\frac{\eta}{2}|\tau|} k(\rho)\|u(\tau)\|_\alpha\right] d\tau. \tag{6.11}$$

Observing that

$$e^{-\frac{\eta}{2}|t|} = e^{-\frac{\eta}{2}|(t-\tau)+\tau|} \le e^{\frac{\eta}{2}|t-\tau|} e^{-\frac{\eta}{2}|\tau|},$$

by (6.9), (6.10) and (6.11) we find that

$$
\begin{aligned}
e^{-\frac{\eta}{2}|t|}\|\mathcal{T}x(t)\|_\alpha \leq e^{-\frac{\eta}{4}|t|}\|y\|_\alpha &+ \int_{0\wedge t}^{0\vee t} e^{-\frac{\eta}{4}|t-\tau|}\left[e^{-\frac{\eta}{2}|\tau|}k(\rho)\|u(\tau)\|_\alpha\right]d\tau \\
&+ \int_{-\infty}^{t}(t-\tau)^{-\alpha}e^{\frac{\eta}{2}|t-\tau|}e^{-\frac{3\eta}{4}(t-\tau)}\left[e^{-\frac{\eta}{2}|\tau|}k(\rho)\|u(\tau)\|_\alpha\right]d\tau \\
&+ \int_{t}^{\infty} e^{\frac{\eta}{2}|t-\tau|}e^{\frac{3\eta}{4}(t-\tau)}\left[e^{-\frac{\eta}{2}|\tau|}k(\rho)\|u(\tau)\|_\alpha\right]d\tau \\
= e^{-\frac{\eta}{4}|t|}\|y\|_\alpha &+ \int_{0\wedge t}^{0\vee t} e^{-\frac{\eta}{4}|t-\tau|}\left[e^{-\frac{\eta}{2}|\tau|}k(\rho)\|u(\tau)\|_\alpha\right]d\tau \\
&+ \int_{-\infty}^{t}(t-\tau)^{-\alpha}e^{-\frac{\eta}{4}(t-\tau)}\left[e^{-\frac{\eta}{2}|\tau|}k(\rho)\|u(\tau)\|_\alpha\right]d\tau \\
&+ \int_{t}^{\infty} e^{\frac{\eta}{4}(t-\tau)}\left[e^{-\frac{\eta}{2}|\tau|}k(\rho)\|u(\tau)\|_\alpha\right]d\tau \\
\leq \|y\|_\alpha &+ M_\rho\|u\|_{\mathscr{X}_{\eta/2}} < M, \quad \forall\, t\in\mathbb{R}.
\end{aligned}
\tag{6.12}
$$

Hence $\mathcal{T}u \in \mathscr{X}_{\eta/2}$.

Next, we check that $\mathcal{T}$ is contractive. Indeed, in a quite similar fashion as above, it can be shown that for any $u, u' \in \mathscr{X}_{\eta/2}$,

$$
\begin{aligned}
e^{-\frac{\eta}{2}|t|}\|\mathcal{T}u(t)-\mathcal{T}u'(t)\|_\alpha &\leq k(\rho)\int_{0\wedge t}^{0\vee t} e^{-\frac{\eta}{4}|t-\tau|}\left(e^{-\frac{\eta}{2}|\tau|}\|u(\tau)-u'(\tau)\|_\alpha\right)d\tau \\
&+ k(\rho)\int_{-\infty}^{t}(t-\tau)^{-\alpha}e^{-\frac{\eta}{4}(t-\tau)}\left(e^{-\frac{\eta}{2}|\tau|}\|u(\tau)-u'(\tau)\|_\alpha\right)d\tau \\
&+ k(\rho)\int_{t}^{\infty} e^{\frac{\eta}{4}(t-\tau)}\left(e^{-\frac{\eta}{2}|\tau|}\|u(\tau)-u'(\tau)\|_\alpha\right)d\tau \\
&\leq \left(k(\rho)\int_0^\infty \left(2+\tau^{-\alpha}\right)e^{-\frac{\eta}{4}\tau}d\tau\right)\|u-u'\|_{\mathscr{U}_{\eta/2}} \\
&:= M_\rho\|u-u'\|_{\mathscr{X}_{\eta/2}}, \qquad \forall\, t\in\mathbb{R}.
\end{aligned}
\tag{6.13}
$$

Thus

$$
\|\mathcal{T}u-\mathcal{T}u'\|_{\mathscr{X}_{\eta/2}} \leq M_\rho\|u-u'\|_{\mathscr{X}_{\eta/2}}.
$$

The conditon (6.7) then asserts that $\mathcal{T}$ is contractive.

Thanks to the Banach fixed-point theorem, $\mathcal{T}$ has a unique fixed point $\gamma^y_{p,\lambda} \in \mathscr{X}_{\eta/2}$ which is precisely a full solution of (4.5) with $\Pi_c\gamma^y_{p,\lambda}(0) = y$ and solves the integral equation

$$
\begin{aligned}
\gamma^y_{p,\lambda}(t) = e^{-A_c^\lambda t}y &+ \int_0^t e^{-A_c^\lambda(t-\tau)}\Pi_c p_\rho(\tau,\gamma^y_{p,\lambda}(\tau))d\tau \\
&+ \int_{-\infty}^{t} e^{-A_+^\lambda(t-\tau)}\Pi_+ p_\rho(\tau,\gamma^y_{p,\lambda}(\tau))d\tau \\
&- \int_t^\infty e^{-A_-^\lambda(t-\tau)}\Pi_- p_\rho(\tau,\gamma^y_{p,\lambda}(\tau))d\tau.
\end{aligned}
\tag{6.14}
$$

For $y, z \in \overline{\mathrm{B}}_{X_c^\alpha}(\tilde{\varrho})$ and $t \in \mathbb{R}$, similarly to (6.13), by (6.14) we have

$$\begin{aligned}
& e^{-\frac{\eta}{2}|t|}\|\gamma^y_{p,\lambda}(t) - \gamma^z_{p,\lambda}(t)\|_\alpha \\
& \quad \le e^{-\frac{\eta}{4}|t|}\|y-z\|_\alpha + k(\rho)\int_{0\wedge t}^{0\vee t} e^{-\frac{\eta}{4}|t-\tau|}\big(e^{-\frac{\eta}{2}|\tau|}\|\gamma^y_{p,\lambda}(\tau) - \gamma^z_{p,\lambda}(\tau)\|_\alpha\big)d\tau \\
& \qquad + k(\rho)\int_{-\infty}^{t}(t-\tau)^{-\alpha}e^{-\frac{\eta}{4}(t-\tau)}\big(e^{-\frac{\eta}{2}|\tau|}\|\gamma^y_{p,\lambda}(\tau) - \gamma^z_{p,\lambda}(\tau)\|_\alpha\big)d\tau \\
& \qquad + k(\rho)\int_{t}^{\infty} e^{\frac{\eta}{4}(t-\tau)}\big(e^{-\frac{\eta}{2}|\tau|}\|\gamma^y_{p,\lambda}(\tau) - \gamma^z_{p,\lambda}(\tau)\|_\alpha\big)d\tau \\
& \quad \le \|y-z\|_\alpha + M_\rho\|\gamma^y_{p,\lambda} - \gamma^z_{p,\lambda}\|_{\mathscr{X}_{\eta/2}}.
\end{aligned}$$

Hence

$$\|\gamma^y_{p,\lambda} - \gamma^z_{p,\lambda}\|_{\mathscr{X}_{\eta/2}} \le \frac{M}{1-M_\rho}\|y-z\|_\alpha,$$

which implies that

$$\|\gamma^y_{p,\lambda}(0) - \gamma^z_{p,\lambda}(0)\|_\alpha \le \frac{M}{1-M_\rho}\|y-z\|_\alpha. \tag{6.15}$$

For each $p \in \mathcal{H}$ and $\lambda \in I_{\lambda_0}(\eta/8)$, define a mapping from X_c^α to X_{us}^α as

$$\begin{aligned}
\xi^\lambda_p(y) := & \int_{-\infty}^{0} e^{A^\lambda_+\tau}\Pi_+ p_\rho(\tau, \gamma^y_{p,\lambda}(\tau))d\tau \\
& - \int_0^\infty e^{A^\lambda_-\tau}\Pi_- p_\rho(\tau, \gamma^y_{p,\lambda}(\tau))d\tau, \quad y \in \overline{\mathrm{B}}_{X_c^\alpha}(\tilde{\varrho}).
\end{aligned} \tag{6.16}$$

Setting $t = 0$ in (6.14) leads to

$$\gamma^y_{p,\lambda}(0) = y + \xi^\lambda_p(y), \qquad y \in \overline{\mathrm{B}}_{X_c^\alpha}(\tilde{\varrho}). \tag{6.17}$$

We conclude from (6.15), (6.16) and (6.17) that $\xi^\lambda_p(\cdot) : \overline{\mathrm{B}}_{X_c^\alpha}(\tilde{\varrho}) \to X_{us}^\alpha$ is a Lipschitz continuous mapping uniformly on p and λ. More specifically, let

$$L_1 := \frac{M}{1-M_\rho} + 1.$$

Then for each $y, z \in \overline{\mathrm{B}}_{X_c^\alpha}(\tilde{\varrho})$,

$$\begin{aligned}
\|\xi^\lambda_p(y) - \xi^\lambda_p(z)\|_\alpha & \le \|\gamma^y_{p,\lambda}(0) - \gamma^z_{p,\lambda}(0)\|_\alpha + \|y-z\|_\alpha \\
& \le L_1\|y-z\|_\alpha.
\end{aligned}$$

Since $\gamma^y_{p,\lambda} \equiv 0$ is a full solution of (6.6), we have $\xi^\lambda_p(0) \equiv 0$, and thus

$$\lim_{\|y\|_\alpha \to 0}\|\xi^\lambda_p(y)\|_\alpha = 0$$

uniformly on $p \in \mathcal{H}$ and $\lambda \in I_{\lambda_0}(\eta/8)$.

Take a sufficiently small $\varrho > 0$ such that

$$\|y + \xi^\lambda_p(y)\| \le \frac{\rho}{2}, \quad y \in \overline{\mathrm{B}}_{X_c^\alpha}(\varrho).$$

Define for each $p \in \mathcal{H}$ the p-section as

$$\mathcal{M}^\lambda_{\mathrm{loc}}(p) = \{y + \xi^\lambda_p(y) : y \in \overline{\mathrm{B}}_{X_c^\alpha}(\varrho)\}.$$

By the definition of p_ρ, $\mathcal{M}^\lambda_{\mathrm{loc}}(\cdot) := \{\mathcal{M}^\lambda_{\mathrm{loc}}(p)\}_{p\in\mathcal{H}}$ is a local invariant manifold of the cocycle semiflow φ_λ, $\lambda \in I_{\lambda_0}(\eta/8)$ generated by (4.5). And for each $p \in \mathcal{H}$, the section $\mathcal{M}^\lambda_{loc}(p)$ is homeomorphic to $\overline{\mathrm{B}}_{X^\alpha_c}(\varrho)$.

In the last part, we show $\xi^\cdot_p(y) : I_{\lambda_0}(\eta/8) \to X^\alpha_{us}$ is Lipschitz uniformly on $y \in \overline{\mathrm{B}}_{X^\alpha_c}(\varrho)$ and $p \in P$. Indeed, for $\lambda_1, \lambda_2 \in I_{\lambda_0}(\eta/8)$ with $\lambda_1 \le \lambda_2$, we have for $t \in \mathbb{R}$ that

$$\begin{aligned}\|e^{-A_c^{\lambda_1}t} - e^{-A_c^{\lambda_2}t}\| &\le \|e^{-A_c^{\lambda_1}t}\| \cdot \left|1 - e^{-(\lambda_2-\lambda_1)t}\right| \\ &\le e^{\frac{\eta}{4}|t|} \cdot \left|1 - e^{-(\lambda_2-\lambda_1)t}\right|.\end{aligned}$$

Then for $t \in \mathbb{R}$,

$$\begin{aligned}&e^{-\frac{\eta}{2}|t|}\int_0^t \|e^{-A_c^{\lambda_1}(t-\tau)}p(\tau,\gamma^y_{p,\lambda_1}(\tau)) - e^{-A_c^{\lambda_2}(t-\tau)}p_\rho(\tau,\gamma^y_{p,\lambda_2}(\tau))\|d\tau \\ &\quad\le \int_0^t e^{-\frac{\eta}{4}|t-\tau|}k_1(\rho)\left(e^{-\frac{\eta}{2}|\tau|}\|\gamma^y_{p,\lambda_1}(\tau) - \gamma^y_{p,\lambda_2}(\tau)\|_\alpha\right)d\tau \\ &\qquad + \int_0^t e^{-\frac{\eta}{4}|t-\tau|}\cdot k_1(\rho)\left|1 - e^{-(\lambda_2-\lambda_1)(t-\tau)}\right|\cdot\left(e^{-\frac{\eta}{2}|\tau|}\|\gamma^y_{p,\lambda_2}(\tau)\|_\alpha\right)d\tau \\ &\quad\le k(\rho)\int_0^t e^{-\frac{\eta}{4}|t-\tau|}\left(e^{-\frac{\eta}{2}|\tau|}\|\gamma^y_{p,\lambda_1}(\tau) - \gamma^y_{p,\lambda_2}(\tau)\|_\alpha\right)d\tau \\ &\qquad + k(\rho)M\int_0^t e^{-\frac{\eta}{4}|t-\tau|}\left|1 - e^{-(\lambda_2-\lambda_1)(t-\tau)}\right|d\tau.\end{aligned} \tag{6.18}$$

We can apply very similar arguments to get that

$$\begin{aligned}&e^{-\frac{\eta}{2}|t|}\int_{-\infty}^t \|e^{-A_s^{\lambda_1}(t-\tau)}p_\rho(\tau,\gamma^y_{p,\lambda_1}(\tau)) - e^{-A_s^{\lambda_2}(t-\tau)}p_\rho(\tau,\gamma^y_{p,\lambda_2}(\tau))\|d\tau \\ &\quad\le k(\rho)\int_{-\infty}^t (t-\tau)^\alpha e^{-\frac{\eta}{4}(t-\tau)}\left(e^{-\frac{\eta}{2}|\tau|}\|\gamma^y_{p,\lambda_1}(\tau) - \gamma^y_{p,\lambda_2}(\tau)\|_\alpha\right)d\tau \\ &\qquad + k(\rho)M\int_{-\infty}^t (t-\tau)^\alpha e^{-\frac{\eta}{4}(t-\tau)}\left|1 - e^{-(\lambda_2-\lambda_1)(t-\tau)}\right|d\tau\end{aligned} \tag{6.19}$$

and

$$\begin{aligned}&e^{-\frac{\eta}{2}|t|}\int_t^\infty \|e^{-A_u^{\lambda_1}(t-\tau)}p(\tau,\gamma^y_{p,\lambda_1}(\tau)) - e^{-A_u^{\lambda_2}(t-\tau)}p(\tau,\gamma^y_{p,\lambda_2}(\tau))\|d\tau \\ &\quad\le k(\rho)\int_t^\infty e^{\frac{\eta}{4}(t-\tau)}\left(e^{-\frac{\eta}{2}|\tau|}\|\gamma^y_{p,\lambda_1}(\tau) - \gamma^y_{p,\lambda_2}(\tau)\|_\alpha\right)d\tau \\ &\qquad + k(\rho)M\int_t^\infty e^{\frac{\eta}{4}(t-\tau)}\left|1 - e^{-(\lambda_2-\lambda_1)(t-\tau)}\right|d\tau.\end{aligned} \tag{6.20}$$

By (6.18), (6.19) and (6.20), we derive that

$$\begin{aligned}&e^{-\frac{\eta}{2}|t|}\|\gamma^y_{p,\lambda_1}(t) - \gamma^y_{p,\lambda_2}(t)\|_\alpha \\ &\quad\le e^{-\frac{\eta}{2}|t|}\int_0^t \|e^{-A_c^{\lambda_1}(t-\tau)}p(\tau,\gamma^y_{p,\lambda_1}(\tau)) - e^{-A_c^{\lambda_2}(t-\tau)}p(\tau,\gamma^y_{p,\lambda_2}(\tau))\|d\tau \\ &\qquad + e^{-\frac{\eta}{2}|t|}\int_{-\infty}^t \|e^{-A_s^{\lambda_1}(t-\tau)}p(\tau,\gamma^y_{p,\lambda_1}(\tau)) - e^{-A_s^{\lambda_2}(t-\tau)}p(\tau,\gamma^y_{p,\lambda_2}(\tau))\|d\tau \\ &\qquad + e^{-\frac{\eta}{2}|t|}\int_t^\infty \|e^{-A_u^{\lambda_1}(t-\tau)}p(\tau,\gamma^y_{p,\lambda_1}(\tau)) - e^{-A_u^{\lambda_2}(t-\tau)}p(\tau,\gamma^y_{p,\lambda_2}(\tau))\|d\tau \\ &\quad\le k(\rho)\int_0^\infty (2+t^{-\alpha})e^{-\frac{\eta}{4}t}dt\cdot\sup_{t\in\mathbb{R}} e^{-\frac{\eta}{2}|t|}\|\gamma^y_{p,\lambda_1}(t) - \gamma^y_{p,\lambda_2}(t)\|_\alpha \\ &\qquad + k(\rho)M\int_0^\infty (2+t^{-\alpha})e^{-\frac{\eta}{4}t}\left(e^{(\lambda_2-\lambda_1)t} - 1\right)dt.\end{aligned} \tag{6.21}$$

It follows that

$$\begin{aligned}
\|\xi_p^{\lambda_1}(y) - \xi_p^{\lambda_2}(y)\|_\alpha &= \|u_{\lambda_1}(0) - u_{\lambda_2}(0)\|_\alpha \\
&\le \sup_{t\in\mathbb{R}} e^{-\frac{\eta}{2}|t|} \|\gamma^y_{p,\lambda_1}(t) - \gamma^y_{p,\lambda_2}(t)\|_\alpha \\
&\le \frac{k_1(\rho)M}{1-M_\rho} \int_0^\infty (2+t^{-\alpha}) e^{-\frac{\eta}{4}t} \left(e^{(\lambda_2-\lambda_1)t} - 1\right) dt \\
&\le \frac{k_1(\rho)M}{1-M_\rho} \int_0^\infty (2+t^{-\alpha}) t\, e^{-[\frac{\eta}{4}-(\lambda_2-\lambda_1)]t} dt \cdot |\lambda_1 - \lambda_2|,
\end{aligned}$$

where the differential mean value is applied to $e^{(\lambda_2-\lambda_1)t} - 1$ to get the last term. It is clear that the integral

$$\int_0^\infty (2+t^{-\alpha}) t\, e^{-[\frac{\eta}{4}-(\lambda_2-\lambda_1)]t} dt = \int_0^\infty (2t + t^{1-\alpha})\, e^{-[\frac{\eta}{4}-(\lambda_2-\lambda_1)]t} dt$$

converges. Therefore

$$\xi_p^{\lambda_1}(y) - \xi_p^{\lambda_2}(y)\| \le L_2|\lambda_1 - \lambda_2|,$$

where

$$L_2 := \frac{k_1(\rho)M}{1-M_\rho} \int_0^\infty (2t + t^{1-\alpha})\, e^{-[\frac{\eta}{4}-(\lambda_2-\lambda_1)]t} dt,$$

and thus $\xi_p^{\cdot}(y)$ is Lipschitz continuous on $I_{\lambda_0}(\eta/8)$ uniformly on $p \in P$ and $y \in \overline{\mathrm{B}}_{X_c^\alpha}(\varrho)$. $\square$

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Admissibility and general dichotomies for evolution families

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Abstract. For an arbitrary noninvertible evolution family on the half-line and for $\rho\colon [0,\infty) \to [0,\infty)$ in a large class of rate functions, we consider the notion of a ρ-dichotomy with respect to a family of norms and characterize it in terms of two admissibility conditions. In particular, our results are applicable to exponential as well as polynomial dichotomies with respect to a family of norms. As a nontrivial application of our work, we establish the robustness of general nonuniform dichotomies.

Keywords: admissibility, dichotomies with growth rates, robustness.

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1 Introduction

Among many methods used to study the asymptotic behavior of nonautonomous systems, one of the most famous is the so-called admissibility method. This line of research in the context of differential equations has a long history that goes back to the pioneering work of Perron [26]. The main idea of Perron's work was to characterize the asymptotic properties of the linear differential equation

$$\dot{x}(t) = A(t)x(t), \qquad t \in \mathbb{J},$$

in terms of the (unique) solvability in $O(\mathbb{J}, X)$ of the equation

$$\dot{x}(t) = A(t)x(t) + f(t), \qquad t \in \mathbb{J},$$

for each test function $f \in I(\mathbb{J}, X)$, where $\mathbb{J} \in \{[0,\infty), \mathbb{R}\}$. Here X is a Banach space, while $I(\mathbb{J}, X)$ – the input-space and $O(\mathbb{J}, X)$ – the output space are suitably constructed function spaces. The milestones of this theory were grounded in the sixtieth in the remarkable works of Massera and Schäffer [18–20] and respectively in the seventies in the outstanding monographs of Coppel [10] and Daleckĭi and Kreĭn [11].

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Since then various authors obtained valuable contributions to this line of the research. For the results dealing with characterizations of *uniform* exponential behavior in terms of appropriate admissibility properties, we refer to the works of Huy [15], Latushkin, Randolph and Schnaubelt [16], Van Minh, Räbiger and Schnaubelt [22], Van Minh and Huy [23], Preda, Pogan and Preda [28, 29] as well as Sasu and Sasu [31–35]. For contributions dealing with various concepts of nonuniform exponential behavior, we refer to [4, 5, 17, 21, 27, 30, 36] and references therein. For a detailed description of this line of the research, we refer to [6].

We point out that all the above works deal with *exponential* behavior. Although this type of behavior has a somewhat privileged role due to its connections with the hyperbolic smooth dynamics, it is certainly not the only type of behavior that appears in the qualitative study of nonautonomous differential equations. To the best of our knowledge, the study of dichotomies with not necessarily exponential rates of expansion and contraction was initiated by Muldowney [24] and Naulin and Pinto [25]. More recently, in the context of nonuniform asymptotic behavior such dichotomies have been studied by Barreira and Valls [1,3] and Bento and Silva [8,9]. A special emphasis was devoted to the so-called polynomial dichotomies [2,7]. A complete characterization of polynomial dichotomies in terms of admissibility for evolution families was obtained by Dragičević [12] (see also [13] for related results in the case of discrete time) by building on the work of Hai [14], who considered polynomial stability.

The main objective of the present paper is to obtain similar results to that in [12] but for a much wider class of dichotomies. More precisely, for a large class of rate functions $\rho\colon [0,\infty) \to [0,\infty)$, we introduce the notion of a ρ-dichotomy with respect to a family of norms. We then obtain a complete characterization of this concept in terms of appropriate admissibility conditions. We point out that our results are new even in the particular case of *uniform* ρ-dichotomies. Indeed, although the proofs use the somewhat standard techniques, the major task accomplished in the present paper was to formulate appropriate admissibility conditions for the general dichotomies we study. In addition, the obtained results are new even for the class of polynomial dichotomies since in comparison to [12], we do not require that our evolution family exhibits polynomial bounded growth property. Consequently, we need to impose two admissibility conditions (rather than just one as in [12]) to characterize polynomial dichotomies. We stress that in the present paper we also use different admissibility spaces from those in [12].

The paper is organized as follows. In Section 2 we introduce the class of dichotomies we study as well as input and output spaces we are going to use. In Section 3, we show that the existence of ρ-dichotomies yields two types of admissibility properties. Then, in Section 4 we obtain a converse result by showing that those admissibility properties imply the existence of a ρ-dichotomy. Finally, in Section 5 we apply those results to establish the robustness of ρ-dichotomies.

2 Preliminaries

2.1 Generalized dichotomies

Let $X = (X, \|\cdot\|)$ be an arbitrary Banach space and let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on X. A family $\mathcal{T} = \{T(t,s)\}_{t\geq s\geq 0}$ of operators in $\mathcal{B}(X)$ is said to be an *evolution family* on X if the following properties hold:

- $T(t,t) = \mathrm{Id}$, for $t \geq 0$;

- $T(t,s)T(s,\tau) = T(t,\tau)$, for $t \geq s \geq \tau \geq 0$;
- for all $s \geq 0$ and $x \in X$ the mapping $t \mapsto T(t,s)x$ is continuous on $[s,\infty)$ and the mapping $t \mapsto T(s,t)x$ is continuous on $[0,s]$.

In this paper we always assume that $\mathcal{T} = \{T(t,s)\}_{t\geq s\geq 0}$ is an evolution family on X and let $\rho\colon [0,\infty) \to [0,\infty)$ be a strictly increasing function of class C^1 such that

$$\rho(0) = 0 \quad \text{and} \quad \lim_{t\to\infty} \rho(t) = \infty.$$

In particular, observe that ρ is bijective. Furthermore, assume that $\{\|\cdot\|_t\}_{t\geq 0}$ is a family of norms on X such that:

- there exist $C > 0$ and $\varepsilon \geq 0$ with
$$\|x\| \leq \|x\|_t \leq Ce^{\varepsilon\rho(t)}\|x\|, \quad \text{for } x \in X \text{ and } t \geq 0; \tag{2.1}$$
- the mapping $t \mapsto \|x\|_t$ is continuous for each $x \in X$.

We say that the evolution family $\mathcal{T}$ admits a *ρ-dichotomy* with respect to the family of norms $\|\cdot\|_t$, $t \geq 0$, if there exists a family $\{P(t)\}_{t\geq 0}$ of projections on X satisfying

$$T(t,s)P(s) = P(t)T(t,s), \quad \text{for } t \geq s \geq 0, \tag{2.2}$$

such that

$$T(t,s)|_{\operatorname{Ker} P(s)}\colon \operatorname{Ker} P(s) \to \operatorname{Ker} P(t) \text{ is invertible for all } t \geq s \geq 0, \tag{2.3}$$

and there exist constants $\lambda, D > 0$ such that:

- for $x \in X$ and $t \geq s \geq 0$,
$$\|T(t,s)P(s)x\|_t \leq De^{-\lambda(\rho(t)-\rho(s))}\|x\|_s; \tag{2.4}$$
- for $x \in X$ and $0 \leq t \leq s$,
$$\|T(t,s)(\mathrm{Id} - P(s))x\|_t \leq De^{-\lambda(\rho(s)-\rho(t))}\|x\|_s, \tag{2.5}$$
where
$$T(t,s) := \left(T(s,t)|_{\operatorname{Ker} P(t)}\right)^{-1}\colon \operatorname{Ker} P(s) \to \operatorname{Ker} P(t),$$
for $0 \leq t \leq s$.

In the following we recall the concept of *ρ-nonuniform exponential dichotomy* for evolution families (see [1,3]) and establish its connection with the notion of ρ-dichotomy with respect to a family of norms. An evolution family $\mathcal{T}$ is said to admit a *ρ-nonuniform exponential dichotomy* if there exists a family $\{P(t)\}_{t\geq 0}$ of projections on X satisfying (2.2) and (2.3), and there exist constants $\lambda, D > 0$ and $\varepsilon \geq 0$ such that

$$\|T(t,s)P(s)\| \leq De^{-\lambda(\rho(t)-\rho(s))+\varepsilon\rho(s)}, \quad \text{for } t \geq s \geq 0, \tag{2.6}$$

and

$$\|T(t,s)(\mathrm{Id} - P(s))\| \leq De^{-\lambda(\rho(s)-\rho(t))+\varepsilon\rho(s)}, \quad \text{for } 0 \leq t \leq s. \tag{2.7}$$

The concept of ρ-nonuniform exponential dichotomy includes as a special case the usual *exponential behavior* when $\rho(t) = t$. Also, for $\rho(t) = \ln(t+1)$ we obtain the concept of *nonuniform polynomial dichotomy* introduced independently by Barreira and Valls [2] and Bento and Silva [7], and more general for $\rho(t) = \int_0^t \mu(t)dt$, where $\mu : [0,\infty) \to (0,\infty)$ is a continuous function such that $\lim_{t\to\infty} \int_0^t \mu(t)dt = \infty$, we obtain the nonuniform version of the *generalized dichotomy* in the sense of Muldowney [24].

Proposition 2.1. *The following statements are equivalent:*

1. *$\mathcal{T}$ admits a ρ-nonuniform exponential dichotomy;*

2. *$\mathcal{T}$ admits a ρ-dichotomy with respect to a family of norms $\|\cdot\|_t$, $t \geq 0$ such that $t \mapsto \|x\|_t$ is continuous for each $x \in X$.*

Proof. Assume that $\mathcal{T}$ admits a ρ-nonuniform exponential dichotomy. For $t \geq 0$ and $x \in X$, set

$$\|x\|_t = \sup_{\tau \geq t} e^{\lambda(\rho(\tau)-\rho(t))}\|T(\tau,t)P(t)x\| + \sup_{\tau\in[0,t]} e^{\lambda(\rho(t)-\rho(\tau))}\|T(\tau,t)(\mathrm{Id}-P(t))x\|.$$

A simple computation shows that (2.1) holds for $C = 2D$. Moreover, by repeating the arguments in the proof of [6, Proposition 5.6], one can show that $t \mapsto \|x\|_t$ is continuous for each $x \in X$. Furthermore, for $t \geq s \geq 0$ and $x \in X$ we have

$$\begin{aligned}
\|T(t,s)P(s)x\|_t &= \sup_{\tau\geq t} e^{\lambda(\rho(\tau)-\rho(t))}\|T(\tau,s)P(s)x\| \\
&= \sup_{\tau\geq t} e^{-\lambda(\rho(t)-\rho(s))}e^{\lambda(\rho(\tau)-\rho(s))}\|T(\tau,s)P(s)x\| \\
&\leq e^{-\lambda(\rho(t)-\rho(s))} \sup_{\tau\geq s} e^{\lambda(\rho(\tau)-\rho(s))}\|T(\tau,s)P(s)x\| \\
&\leq e^{-\lambda(\rho(t)-\rho(s))}\|x\|_s,
\end{aligned}$$

and thus (2.4) holds. Similarly, one can prove (2.5). Hence, the evolution family $\mathcal{T}$ admits a ρ-dichotomy with respect to the family of norms $\|\cdot\|_t$, $t \geq 0$, defined above.

Conversely, assume that $\mathcal{T}$ admits a ρ-dichotomy with respect to a family of norms $\|\cdot\|_t$ on X satisfying (2.1) for some $C > 0$ and $\varepsilon \geq 0$. For $t \geq s \geq 0$ and $x \in X$ we have

$$\begin{aligned}
\|T(t,s)P(s)x\| &\leq \|T(t,s)P(s)x\|_t \\
&\leq De^{-\lambda(\rho(t)-\rho(s))}\|x\|_s \\
&\leq DCe^{\varepsilon\rho(s)}e^{-\lambda(\rho(t)-\rho(s))}\|x\|,
\end{aligned}$$

and thus (2.6) holds. Similarly, one can show (2.7). Therefore, the evolution family $\mathcal{T}$ admits a ρ-nonuniform exponential dichotomy. □

2.2 Admissible spaces

Let Y_1 be the space of all Bochner measurable functions $x\colon [0,\infty) \to X$ such that

$$\|x\|_1 := \int_0^\infty \|x(t)\|_t \, dt < \infty,$$

identifying functions that are equal Lebesque-almost everywhere. It is easy to show that $(Y_1, \|\cdot\|_1)$ is a Banach space (see [4, Theorem 1]). Moreover, consider the space Y_∞ of all continuous functions $x\colon [0,\infty) \to X$ such that

$$\|x\|_\infty := \sup_{t\geq 0} \|x(t)\|_t < \infty.$$

One can easily prove that $(Y_\infty, \|\cdot\|_\infty)$ is a Banach space. For a closed subspace $Z \subset X$, Y_∞^Z is the space of all $x \in Y_\infty$ such that $x(0) \in Z$. Obviously, Y_∞^Z is a closed subspace of Y_∞, therefore it is also a Banach space.

We consider another Banach function space $(Y'_\infty, \|\cdot\|'_\infty)$, which consists of all Bochner measurable functions $x\colon [0,\infty) \to X$ such that

$$\|x\|'_\infty := \operatorname*{ess\,sup}_{t\geq 0} \|x(t)\|_t < \infty,$$

where ess sup is taken with respect to the Lebesgue measure on $[0,\infty)$.

3 From dichotomy to admissibility

In this section we show that the existence of a ρ-dichotomy with respect to a family of norms for an evolution family $\mathcal{T} = \{T(t,s)\}_{t\geq s\geq 0}$ yields the admissibility of the pairs $\left(Y_\infty^Z, Y_1\right)$, $\left(Y_\infty^Z, Y'_\infty\right)$ for a certain closed subspace $Z \subset X$.

Proposition 3.1. *Assume that the evolution family $\mathcal{T}$ admits a ρ-dichotomy with respect to a family of norms $\|\cdot\|_t$, $t \geq 0$, and set $Z = \operatorname{Ker} P(0)$. Then, for each $y \in Y_1$ there exists a unique $x \in Y_\infty^Z$ such that*

$$x(t) = T(t,s)x(s) + \int_s^t T(t,\tau)y(\tau)\,d\tau, \quad \textit{for } t \geq s \geq 0. \tag{3.1}$$

Proof. Take an arbitrary $y \in Y_1$. For $t \geq 0$, set

$$x(t) = \int_0^t T(t,s)P(s)y(s)\,ds - \int_t^\infty T(t,s)(\mathrm{Id} - P(s))y(s)\,ds.$$

It follows from (2.4) and (2.5) that

$$\begin{aligned}
\|x(t)\|_t &\leq \int_0^t \|T(t,s)P(s)y(s)\|_t\,ds + \int_t^\infty \|T(t,s)(\mathrm{Id} - P(s))y(s)\|_t\,ds \\
&\leq D\int_0^t e^{-\lambda(\rho(t)-\rho(s))}\|y(s)\|_s\,ds + D\int_t^\infty e^{-\lambda(\rho(s)-\rho(t))}\|y(s)\|_s\,ds \\
&\leq D\int_0^t \|y(s)\|_s\,ds + D\int_t^\infty \|y(s)\|_s\,ds = D\,\|y\|_1,
\end{aligned}$$

for every $t \geq 0$, and thus $x \in Y_\infty$. On the other hand, it is easy to check that $x(0) \in Z$. Therefore, $x \in Y_\infty^Z$. Moreover, for $t \geq s \geq 0$ we have

$$\begin{aligned}
x(t) - T(t,s)x(s) &= \int_0^t T(t,\tau)P(\tau)y(\tau)\,d\tau - T(t,s)\int_0^s T(s,\tau)P(\tau)y(\tau)\,d\tau \\
&\quad - \int_t^\infty T(t,\tau)(\mathrm{Id} - P(\tau))y(\tau)\,d\tau \\
&\quad + T(t,s)\int_s^\infty T(s,\tau)(\mathrm{Id} - P(\tau))y(\tau)\,d\tau \\
&= \int_s^t T(t,\tau)P(\tau)y(\tau)\,d\tau + \int_s^t T(t,\tau)(\mathrm{Id} - P(\tau))y(\tau)\,d\tau \\
&= \int_s^t T(t,\tau)y(\tau)\,d\tau,
\end{aligned}$$

and therefore we conclude that (3.1) holds. In order to establish the uniqueness, it is sufficient to consider the case when $y = 0$. Let $x \in Y_\infty^Z$ such that

$$x(t) = T(t,s)x(s), \quad \text{for } t \geq s \geq 0.$$

Then, from (2.5) we have

$$\begin{aligned}
\|x(0)\|_0 &= \|(\mathrm{Id} - P(0))x(0)\|_0 = \|T(0,t)(\mathrm{Id} - P(t))x(t)\|_0 \\
&\leq De^{-\lambda\rho(t)}\|x(t)\|_t \\
&\leq De^{-\lambda\rho(t)}\|x\|_\infty,
\end{aligned}$$

for every $t \geq 0$. Passing to the limit when $t \to \infty$, we conclude that $x(0) = 0$, which implies that $x = 0$. □

Proposition 3.2. *Assume that the evolution family $\mathcal{T}$ admits a ρ-dichotomy with respect to a family of norms $\|\cdot\|_t$, $t \geq 0$, and set $Z = \operatorname{Ker} P(0)$. Then, for each $y \in Y'_\infty$ there exists a unique $x \in Y_\infty^Z$ such that*

$$x(t) = T(t,s)x(s) + \int_s^t \rho'(\tau)T(t,\tau)y(\tau)\,d\tau, \quad \text{for } t \geq s \geq 0. \tag{3.2}$$

Proof. Take $y \in Y'_\infty$. For $t \geq 0$, set

$$x(t) = \int_0^t \rho'(s)T(t,s)P(s)y(s)\,ds - \int_t^\infty \rho'(s)T(t,s)(\mathrm{Id} - P(s))y(s)\,ds.$$

It follows from (2.4) and (2.5) that

$$\begin{aligned}
\|x(t)\|_t &\leq \int_0^t \rho'(s)\|T(t,s)P(s)y(s)\|_t\,ds + \int_t^\infty \rho'(s)\|T(t,s)(\mathrm{Id} - P(s))y(s)\|_t\,ds \\
&\leq D\int_0^t \rho'(s)e^{-\lambda(\rho(t)-\rho(s))}\|y(s)\|_s\,ds + D\int_t^\infty \rho'(s)e^{-\lambda(\rho(s)-\rho(t))}\|y(s)\|_s\,ds \\
&\leq D\|y\|'_\infty\left(\int_0^t \rho'(s)e^{-\lambda(\rho(t)-\rho(s))}\,ds + \int_t^\infty \rho'(s)e^{-\lambda(\rho(s)-\rho(t))}\,ds\right) \\
&\leq \frac{2D}{\lambda}\|y\|'_\infty, \quad \text{for every } t \geq 0.
\end{aligned}$$

Since $x(0) \in Z$, we conclude that $x \in Y_\infty^Z$. A simple computation shows that (3.2) holds. The uniqueness part can be established as in the proof of Proposition 3.1. □

4 From admissibility to dichotomy

The aim of this section is to prove that the admissibility of the pairs (Y_∞^Z, Y_1), (Y_∞^Z, Y'_∞) for a closed subspace $Z \subset X$ yields the existence of a ρ-dichotomy with respect to the family of norms $\{\|\cdot\|_t\}_{t\geq 0}$. More precisely, our goal is to establish the following result.

Theorem 4.1. *Assume that there exists a closed subspace $Z \subset X$ such that:*

(i) for each $y \in Y_1$ there exists a unique $x \in Y_\infty^Z$ satisfying (3.1);

(ii) for each $y \in Y'_\infty$ there exists a unique $x \in Y_\infty^Z$ satisfying (3.2).

Then, the evolution family $\mathcal{T}$ admits a ρ-dichotomy with respect to the family of norms $\|\cdot\|_t$, $t \geq 0$.

Proof. Let

$$T_Z : \mathcal{D}(T_Z) \subset Y_\infty^Z \to Y_1, \quad T_Z x = y,$$

where

$$\mathcal{D}(T_Z) = \left\{ x \in Y_\infty^Z : \text{ there exists } y \in Y_1 \text{ satisfying (3.1)} \right\}.$$

Furthermore, let

$$T'_Z : \mathcal{D}(T'_Z) \subset Y_\infty^Z \to Y'_\infty, \quad T'_Z x = y,$$

where

$$\mathcal{D}(T'_Z) = \left\{ x \in Y_\infty^Z : \text{ there exists } y \in Y'_\infty \text{ satisfying (3.2)} \right\}.$$

Lemma 4.2. *The operators $T_Z \colon \mathcal{D}(T_Z) \to Y_1$, $T'_Z \colon \mathcal{D}(T'_Z) \to Y'_\infty$ are well-defined, linear and closed.*

Proof of the lemma. Assume that $x \in Y_\infty^Z$ and $y_1, y_2 \in Y_1$ such that

$$x(t) = T(t,\tau)x(\tau) + \int_\tau^t T(t,s)y_i(s)\, ds,$$

for $t \geq \tau \geq 0$ and $i \in \{1,2\}$. Hence,

$$\int_\tau^t T(t,s)(y_1(s) - y_2(s))\, ds = 0, \quad \text{for } t > \tau \geq 0.$$

Dividing by $t - \tau$ and letting $t - \tau \to 0$, it follows from the Lebesgue differentiation theorem that

$$y_1(t) = y_2(t) \quad \text{for almost every } t \geq 0.$$

We conclude that $y_1 = y_2$ in Y_1. Thus, T_Z is well-defined and, by definition it is linear.

We now show that T_Z is closed. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(T_Z)$ converging to $x \in Y_\infty^Z$ such that $y_n = T_Z x_n$ converges to $y \in Y_1$. Then, for $t \geq \tau \geq 0$ we have that

$$x(t) - T(t,\tau)x(\tau) = \lim_{n\to\infty}(x_n(t) - T(t,\tau)x_n(\tau)) = \lim_{n\to\infty}\int_\tau^t T(t,s)y_n(s)\, ds.$$

On the other hand, we have

$$\begin{aligned}\left\| \int_\tau^t T(t,s)y_n(s)\, ds - \int_\tau^t T(t,s)y(s)\, ds \right\| &\leq M \int_\tau^t \|y_n(s) - y(s)\|\, ds \\ &\leq M \int_\tau^t \|y_n(s) - y(s)\|_s\, ds \\ &\leq M\|y_n - y\|_1,\end{aligned}$$

where $M = M(t,\tau) = \sup\{\|T(t,s)\| : s \in [\tau,t]\}$ is finite by the Banach–Steinhaus theorem. Since $y_n \to y$ in Y_1, we conclude that

$$\lim_{n\to\infty}\int_\tau^t T(t,s)y_n(s)\, ds = \int_\tau^t T(t,s)y(s)\, ds,$$

and therefore (3.1) holds. We conclude that $x \in \mathcal{D}(T_Z)$ and $T_Z x = y$. Therefore, T_Z is a closed linear operator. Similarly, one can show that T'_Z is well-defined, linear and closed. $\square$

By the assumption in Theorem 4.1, the linear operators T_Z, T'_Z are bijective, and by previous lemma and the Closed Graph Theorem they have bounded inverse $G_Z\colon Y_1 \to Y_\infty^Z$ and $G'_Z\colon Y'_\infty \to Y_\infty^Z$, respectively.

For $\tau \geq 0$, set

$$S(\tau) = \left\{ v \in X : \sup_{t \geq \tau} \|T(t,\tau)v\|_t < \infty \right\} \quad \text{and} \quad U(\tau) = T(\tau, 0)Z.$$

Clearly, $S(\tau)$ and $U(\tau)$ are subspaces of X for each $\tau \geq 0$.

Lemma 4.3. *For $\tau \geq 0$, we have that*

$$X = S(\tau) \oplus U(\tau). \tag{4.1}$$

Proof of the lemma. Let $\tau \geq 0$ and take $v \in X$. Set

$$g(s) = \chi_{[\tau,\tau+1]}(s) T(s,\tau)v, \qquad s \geq 0.$$

Clearly, $g \in Y_1$. Since T_Z is invertible, there exists $h \in \mathcal{D}(T_Z) \subset Y_\infty^Z$ such that $T_Z h = g$. It follows from (3.1) that

$$h(t) = T(t,\tau)(h(\tau) + v) \quad \text{for } t \geq \tau + 1.$$

Since $h \in Y_\infty$, we conclude that $h(\tau) + v \in S(\tau)$. Similarly, it follows from (3.1) that

$$h(\tau) = T(\tau, 0)h(0).$$

Since $h(0) \in Z$, we have that $h(\tau) \in U(\tau)$ and thus

$$v = (h(\tau) + v) + (-h(\tau)) \in S(\tau) + U(\tau).$$

We have proved that $X = S(\tau) + U(\tau)$.

Take now $v \in S(\tau) \cap U(\tau)$. Then, there exists $z \in Z$ such that $v = T(\tau, 0)z$. We consider a function $h\colon [0, \infty) \to X$, defined by

$$h(t) = T(t, 0)z \quad \text{for } t \geq 0.$$

Clearly, $h \in Y_\infty^Z$. Since $h(t) = T(t,s)h(s)$ for all $t \geq s \geq 0$, it follows that $T_Z h = 0$ and thus $h = 0$. We conclude that $v = h(\tau) = 0$, and hence $S(\tau) \cap U(\tau) = \{0\}$. This completes the proof of the lemma. $\square$

Let $P(\tau)\colon X \to S(\tau)$ and $Q(\tau)\colon X \to U(\tau)$ be the projections associated with the decomposition (4.1), with $P(\tau) + Q(\tau) = \text{Id}$. Observe that (2.2) holds. Indeed, observe that

$$T(t,\tau)S(\tau) \subset S(t) \quad \text{and} \quad T(t,\tau)U(\tau) \subset U(t), \quad \text{for } t \geq \tau \geq 0.$$

Hence, we have that for every $x \in X$ and $t \geq \tau \geq 0$,

$$P(t)T(t,\tau)x = P(t)T(t,\tau)P(\tau)x + P(t)T(t,\tau)Q(\tau)x = T(t,\tau)P(\tau)x.$$

We conclude that (2.2) holds.

Lemma 4.4. *For $t \geq \tau \geq 0$, the restriction $T(t,\tau)|_{U(\tau)}\colon U(\tau) \to U(t)$ is invertible.*

Proof of the lemma. Let $t \geq \tau \geq 0$ and take $x \in U(t)$. Then, there exists $z \in Z$ such that $x = T(t,0)z$. Since $T(\tau,0)z \in U(\tau)$ and $x = T(t,\tau)T(\tau,0)z$, we conclude that $T(t,\tau)|_{U(\tau)}$ is surjective.

Let now $x \in U(\tau)$ such that $T(t,\tau)x = 0$. Take $z \in Z$ such that $x = T(\tau,0)z$. We define $u\colon [0,\infty) \to X$ by $u(s) = T(s,0)z$, $s \geq 0$. Since $u(s) = 0$ for $s \geq t$, we have that $u \in Y_\infty^Z$ and $T_Z u = 0$. Consequently, $u = 0$ and $x = u(\tau) = 0$. This proves that $T(t,\tau)|_{U(\tau)}$ is also injective. The proof of the lemma is completed. □

Lemma 4.5. *There exists $M > 0$ such that*

$$\|P(\tau)v\|_\tau \leq M\|v\|_\tau, \quad \textit{for all } v \in X \textit{ and } \tau \geq 0. \tag{4.2}$$

Proof of the lemma. Take $v \in X$ and $\tau \geq 0$. Moreover, given $h > 0$, we define a function $g_h\colon [0,\infty) \to X$ by

$$g_h(t) = \frac{1}{h}\chi_{[\tau,\tau+h]}(t)T(t,\tau)v.$$

Clearly, $g_h \in Y_1$ and thus there exists $x_h \in \mathcal{D}(T_Z)$ such that $T_Z x_h = g_h$. We have

$$\|P(\tau)v\|_\tau = \|x_h(\tau) + v\|_\tau \leq \|x_h(\tau)\|_\tau + \|v\|_\tau \leq \|G_Z g_h\|_\infty + \|v\|_\tau.$$

Moreover,

$$\|G_Z g_h\|_\infty \leq \|G_Z\| \cdot \|g_h\|_1 = \|G_Z\| \, \frac{1}{h}\int_\tau^{\tau+h} \|T(t,\tau)v\|_t \, dt.$$

Letting $h \to 0$, we obtain

$$\|P(\tau)v\|_\tau \leq (1 + \|G_Z\|)\, \|v\|_\tau,$$

and we conclude that (4.2) holds for $M = 1 + \|G_Z\|$. □

Lemma 4.6. *There exist constants $\lambda, D > 0$ such that*

$$\|T(t,\tau)v\|_t \leq De^{-\lambda(\rho(t)-\rho(\tau))}\|v\|_\tau, \quad \textit{for } t \geq \tau \geq 0 \textit{ and } v \in S(\tau). \tag{4.3}$$

Proof of the lemma. Fix $\tau \geq 0$ and let $v \in S(\tau)$. We consider the function

$$u\colon [0,\infty) \to X, \quad u(t) = \chi_{[\tau,\infty)}(t)T(t,\tau)v.$$

Moreover, for any fixed $h > 0$, we define two functions $\varphi_h\colon [0,\infty) \to \mathbb{R}$ and $g_h\colon [0,\infty) \to X$ by

$$\varphi_h(t) = \begin{cases} 0, & 0 \leq t \leq \tau, \\ \frac{1}{h}(t-\tau), & \tau \leq t \leq \tau + h, \\ 1, & t \geq \tau + h, \end{cases}$$

and

$$g_h(t) = \frac{1}{h}\,\chi_{[\tau,\tau+h]}(t)\, T(t,\tau)v, \qquad t \geq 0.$$

It is easy to show that $g_h \in Y_1$, $\varphi_h u \in \mathcal{D}(T_Z)$ and $T_Z(\varphi_h u) = g_h$. We have

$$\begin{aligned}\sup_{t\geq\tau+h} \|u(t)\|_t = \sup_{t\geq\tau+h} \|\varphi_h(t)u(t)\|_t &\leq \|\varphi_h u\|_\infty = \|G_Z g_h\|_\infty \\ &\leq \|G_Z\| \cdot \|g_h\|_1 \\ &= \|G_Z\|\frac{1}{h}\int_\tau^{\tau+h} \|u(s)\|_s \, ds.\end{aligned}$$

Hence, letting $h \to 0$ we obtain the inequality

$$\|u(t)\|_t \le \|G_Z\| \cdot \|v\|_\tau, \quad \text{for every } t \ge \tau.$$

Thus,

$$\|T(t,\tau)v\|_t \le \|G_Z\| \cdot \|v\|_\tau, \quad \text{for every } t \ge \tau. \tag{4.4}$$

Let us take $t \ge \tau$ and $v \in S(\tau)$ such that $T(t,\tau)v \ne 0$, thus $T(s,\tau)v \ne 0$ for all $s \in [\tau, t]$. Let us consider $x, y\colon [0,\infty) \to X$ defined by

$$y(s) = \chi_{[\tau,t]}(s)\, \frac{T(s,\tau)v}{\|T(s,\tau)v\|_s}, \qquad s \ge 0,$$

and

$$x(s) = \begin{cases} 0, & 0 \le s \le \tau, \\ \int_\tau^s \rho'(r) \frac{T(s,\tau)v}{\|T(r,\tau)v\|_r}\, dr, & \tau < s \le t, \\ \int_\tau^t \rho'(r) \frac{T(s,\tau)v}{\|T(r,\tau)v\|_r}\, dr, & s > t. \end{cases}$$

Note that $y \in Y'_\infty$ and $\|y\|'_\infty = 1$. Furthermore, since $v \in S(\tau)$ we get that

$$\|x(s)\|_s \le \int_\tau^t \frac{\rho'(r)}{\|T(r,\tau)v\|_r}\, dr\, \|T(s,\tau)v\|_s \le a_{t,\tau,v} \sup_{r \ge \tau} \|T(r,\tau)v\|_r < \infty,$$

for all $s \ge \tau$, where

$$a_{t,\tau,v} = \int_\tau^t \frac{\rho'(r)}{\|T(r,\tau)v\|_r}\, dr < \infty,$$

and thus $x \in Y_\infty^Z$. It is straightforward to show that $T'_Z x = y$. Consequently,

$$\|x\|_\infty = \|G'_Z y\|_\infty \le \|G'_Z\| \cdot \|y\|'_\infty = \|G'_Z\|.$$

Therefore,

$$\|G'_Z\| \ge \|x\|_\infty \ge \|x(t)\|_t = \|T(t,\tau)v\|_t \int_\tau^t \frac{\rho'(r)}{\|T(r,\tau)v\|_r}\, dr. \tag{4.5}$$

From (4.4) it follows that

$$\frac{1}{\|T(r,\tau)v\|_r} \ge \frac{1}{\|G_Z\| \cdot \|v\|_\tau}, \quad \text{for all } r \in [\tau, t],$$

and thus, from (4.5) we get

$$\|G'_Z\| \cdot \|G_Z\| \cdot \|v\|_\tau \ge \|T(t,\tau)v\|_t\, (\rho(t) - \rho(\tau)), \text{ for } t \ge \tau \text{ and } v \in S(\tau).$$

Consequently,

$$(t-\tau) \left\| T\left(\rho^{-1}(t), \rho^{-1}(\tau)\right) v \right\|_{\rho^{-1}(t)} \le \|G'_Z\| \cdot \|G_Z\| \cdot \|v\|_{\rho^{-1}(\tau)},$$

for $t \ge \tau$ and $v \in S\left(\rho^{-1}(\tau)\right)$. Let $N_0 \in \mathbb{N}^*$ such that $N_0 > e\|G'_Z\| \cdot \|G_Z\|$, and let $t \ge \tau + N_0$. Then,

$$\begin{aligned} N_0 \left\| T\left(\rho^{-1}(t), \rho^{-1}(\tau)\right) v \right\|_{\rho^{-1}(t)} &\le (t-\tau) \left\| T\left(\rho^{-1}(t), \rho^{-1}(\tau)\right) v \right\|_{\rho^{-1}(t)} \\ &\le \|G'_Z\| \cdot \|G_Z\| \cdot \|v\|_{\rho^{-1}(\tau)}, \end{aligned}$$

which implies that there exists $N_0 \in \mathbb{N}^*$ such that

$$\|T(\rho^{-1}(t), \rho^{-1}(\tau))v\|_{\rho^{-1}(t)} \leq \frac{1}{e}\|v\|_{\rho^{-1}(\tau)}, \tag{4.6}$$

for $t \geq \tau$ with $t - \tau \geq N_0$ and $v \in S\left(\rho^{-1}(\tau)\right)$. Take an arbitrary $t \geq \tau$ with $t - \tau \geq N_0$ and write $t - \tau$ in the form

$$t - \tau = kN_0 + r,\ k = k(t,\tau) \in \mathbb{N}^* \quad \text{and} \quad r = r(t,\tau) \in [0, N_0).$$

Observing that

$$\begin{aligned} &T\left(\rho^{-1}(t), \rho^{-1}(\tau)\right) \\ &\quad = T\left(\rho^{-1}(t), \rho^{-1}(\tau + kN_0)\right)\prod_{j=0}^{k-1} T\left(\rho^{-1}(\tau + (k-j)N_0), \rho^{-1}(\tau + (k-j-1)N_0)\right), \end{aligned}$$

it follows from (4.4) and (4.6) that

$$\begin{aligned} \|T\left(\rho^{-1}(t), \rho^{-1}(\tau)\right) v\|_{\rho^{-1}(t)} &\leq \|G_Z\|\, e^{-k}\, \|v\|_{\rho^{-1}(\tau)} \\ &\leq e\,\|G_Z\|\, e^{-\frac{1}{N_0}(t-\tau)}\, \|v\|_{\rho^{-1}(\tau)}, \end{aligned}$$

and thus (4.3) holds with $\lambda = 1/N_0$ and $D = e\,\|G_Z\|$. The proof of the lemma is completed. □

Lemma 4.7. *There exist $\lambda, D > 0$ such that*

$$\|T(t,\tau)v\|_t \leq De^{-\lambda(\rho(\tau)-\rho(t))}\|v\|_\tau, \text{ for } 0 \leq t \leq \tau \text{ and } v \in U(\tau). \tag{4.7}$$

Proof of the lemma. Take $\tau > 0$ and $z \in Z$. We define a function $u\colon [0,\infty) \to X$ by

$$u(t) = T(t,0)z, \quad \text{for } t \geq 0.$$

For sufficiently small $h > 0$, we define $\psi_h\colon [0,\infty) \to \mathbb{R}$,

$$\psi_h(t) = \begin{cases} 1, & 0 \leq t \leq \tau - h, \\ -\frac{t-\tau}{h}, & \tau - h \leq t \leq \tau, \\ 0, & t \geq \tau. \end{cases}$$

Finally, we consider

$$g_h\colon [0,\infty) \to X, \qquad g_h = -\frac{1}{h}\chi_{[\tau-h,\tau]}\, u.$$

It is easy to check that $g_h \in Y_1$, $\psi_h u \in \mathcal{D}(T_Z)$ and $T_Z(\psi_h u) = g_h$. Hence,

$$\begin{aligned} \sup_{t\in[0,\tau-h]} \|u(t)\|_t = \sup_{t\in[0,\tau-h]} \|\psi_h(t)u(t)\|_t &\leq \|\psi_h u\|_\infty = \|G_Z g_h\|_\infty \\ &\leq \|G_Z\| \cdot \|g_h\|_1 \\ &= \|G_Z\| \cdot \frac{1}{h}\int_{\tau-h}^{\tau} \|u(s)\|_s\, ds. \end{aligned}$$

Letting $h \to 0$, we get

$$\|u(t)\|_t \leq \|G_Z\| \cdot \|u(\tau)\|_\tau, \quad \text{for } 0 \leq t \leq \tau,$$

which implies

$$\|T(t,0)z\|_t \le \|G_Z\| \cdot \|T(\tau,0)z\|_\tau, \quad \text{for } z \in Z \text{ and } 0 \le t \le \tau. \tag{4.8}$$

Take now $z \in Z \setminus \{0\}$ and $0 \le t \le \tau$. We define $x, y \colon [0,\infty) \to X$ by

$$y(s) = \begin{cases} -\frac{T(s,0)z}{\|T(s,0)z\|_s}, & 0 \le s \le \tau, \\ 0, & s > \tau, \end{cases}$$

and

$$x(s) = \begin{cases} \int_s^\tau \rho'(r)\frac{T(s,0)z}{\|T(r,0)z\|_r}\, dr, & 0 \le s \le \tau, \\ 0, & s > \tau. \end{cases}$$

Observe that $y \in Y'_\infty$ and $\|y\|'_\infty = 1$. Moreover, $x \in Y^Z_\infty$ and it is easy to check that $T'_Z x = y$. Hence,

$$\|x\|_\infty = \|G'_Z y\|_\infty \le \|G'_Z\|.$$

Consequently, for each $0 \le s \le \tau$ we have

$$\|G'_Z\| \ge \|T(s,0)z\|_s \int_s^\tau \rho'(r)\frac{1}{\|T(r,0)z\|_r}\, dr.$$

Letting $\tau \to \infty$, we conclude that

$$\|G'_Z\| \ge \|T(s,0)z\|_s \int_s^\infty \rho'(r)\frac{1}{\|T(r,0)z\|_r}\, dr \quad \text{for } s \ge 0 \text{ and } z \in Z \setminus \{0\}. \tag{4.9}$$

Take now $0 \le t \le \tau$ and $z \in Z \setminus \{0\}$. It follows from (4.8) and (4.9) that

$$\begin{aligned} \frac{1}{\|T(\rho^{-1}(t),0)z\|_{\rho^{-1}(t)}} &\ge \frac{1}{\|G'_Z\|}\int_{\rho^{-1}(t)}^\infty \rho'(r)\frac{1}{\|T(r,0)z\|_r}\, dr \\ &\ge \frac{1}{\|G'_Z\|}\int_{\rho^{-1}(t)}^{\rho^{-1}(\tau)} \rho'(r)\frac{1}{\|T(r,0)z\|_r}\, dr \\ &\ge \frac{1}{\|G'_Z\|}\int_{\rho^{-1}(t)}^{\rho^{-1}(\tau)} \rho'(r)\frac{1}{\|G_Z\| \cdot \|T(\rho^{-1}(\tau),0)z\|_{\rho^{-1}(\tau)}}\, dr \\ &= \frac{\tau - t}{\|G'_Z\| \cdot \|G_Z\|} \cdot \frac{1}{\|T(\rho^{-1}(\tau),0)z\|_{\rho^{-1}(\tau)}} \end{aligned}$$

and thus

$$(\tau - t)\|T(\rho^{-1}(t),0)z\|_{\rho^{-1}(t)} \le \|G_Z\| \cdot \|G'_Z\| \cdot \|T(\rho^{-1}(\tau),0)z\|_{\rho^{-1}(\tau)}.$$

We conclude that there exists $N_0 \in \mathbb{N}^*$ such that

$$\|T(\rho^{-1}(t),0)z\|_{\rho^{-1}(t)} \le \frac{1}{e}\|T(\rho^{-1}(\tau),0)z\|_{\rho^{-1}(\tau)},$$

for $z \in Z$ and $0 \le t \le \tau$ such that $\tau - t \ge N_0$. Hence,

$$\|T(\rho^{-1}(t),\rho^{-1}(\tau))v\|_{\rho^{-1}(t)} \le \frac{1}{e}\|v\|_{\rho^{-1}(\tau)},$$

for $v \in U(\rho^{-1}(\tau))$ and $0 \le t \le \tau$ such that $\tau - t \ge N_0$. By arguing as in the proof of Lemma 4.6, we find that there exist $\lambda, D > 0$ such that

$$\|T(\rho^{-1}(t),\rho^{-1}(\tau))v\|_{\rho^{-1}(t)} \le De^{-\lambda(\tau - t)}\|v\|_{\rho^{-1}(\tau)},$$

for $v \in U(\rho^{-1}(\tau))$ and $0 \le t \le \tau$, which readily implies the conclusion of the lemma. □

In order to complete the proof of the theorem, it is sufficient to observe that (4.2), (4.3) and (4.7) imply that (2.4) and (2.5) hold. □

Remark 4.8. It is worth observing that in order to deduce the existence of a ρ-dichotomy we imposed two admissibility conditions. In the following two examples we will illustrate that this was necessary.

Example 4.9. We consider an evolution family $\mathcal{T} = \{T(t,s)\}_{t \geq s \geq 0}$ given by

$$T(t,s) = \mathrm{Id}, \qquad t \geq s \geq 0.$$

Furthermore, take $Z = \{0\}$ and let $\|\cdot\|_t = \|\cdot\|$ for $t \geq 0$. Then for each $y \in Y_1$, the unique $x \in Y_Z$ satisfying (3.1) is given by

$$x(t) = \int_0^t T(t,s)y(s)\,ds = \int_0^t y(s)\,ds, \qquad t \geq 0.$$

Thus, the first assumption of Theorem 4.1 is fulfilled. On the other hand, $\mathcal{T}$ obviously doesn't admit a ρ-dichotomy with respect to the family of norms $\|\cdot\|_t$, $t \geq 0$.

The following example is a simple modification of [12, Example 1].

Example 4.10. Let $X = \mathbb{R}$ with the standard Euclidean norm $|\cdot|$. Furthermore, let $\|\cdot\|_t = |\cdot|$ for $t \geq 0$ and take $Z = \{0\}$. Furthermore, let $\rho(t) = \ln(1+t)$ for $t \geq 0$. We consider the sequence $(A_n)_{n \in \mathbb{N}}$ of operators on X (which can of course be identified with numbers) given by

$$A_n = \begin{cases} n & \text{if } n = 2^l \text{ for some } l \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, for $t \geq s \geq 0$ we define

$$T(t,s) = \begin{cases} A_{\lfloor t \rfloor - 1} \cdots A_{\lfloor s \rfloor}, & \lfloor t \rfloor \geq \lfloor s \rfloor + 1, \\ 1, & \lfloor t \rfloor = \lfloor s \rfloor. \end{cases}$$

Then, $\mathcal{T} = \{T(t,s)\}_{t \geq s \geq 0}$ is an evolution family. By arguing as in [12, Example 1], it is easy to check that the second assumption of Theorem 4.1 is satisfied and $\mathcal{T}$ doesn't admit a ρ-dichotomy with respect to the family of norms $\|\cdot\|_t$, $t \geq 0$.

5 Robustness of generalized dichotomies

In this section we apply our main results to prove that the concept of ρ-dichotomy with respect to a family $\{\|\cdot\|_t\}_{t \geq 0}$ of norms on X persist under sufficiently small linear perturbations. As a consequence, we establish the robustness property of ρ-nonuniform exponential dichotomy.

Theorem 5.1. *Assume that the evolution family $\{T(t,s)\}_{t \geq s \geq 0}$ admits a ρ-dichotomy with respect to a family $\{\|\cdot\|_t\}_{t \geq 0}$ of norms on X satisfying*

$$\|x\| \leq \|x\|_t \leq Ce^{\varepsilon\rho(t)}\|x\|, \quad \textit{for } x \in X \textit{ and } t \geq 0,$$

for some $C > 0$ and $\varepsilon \geq 0$, such that the mapping $t \mapsto \|x\|_t$ is continuous for each $x \in X$. If $B : [0, \infty) \to \mathcal{B}(X)$ is a strongly continuous operator-valued function such that

$$\|B(t)\| \leq \delta e^{-(\varepsilon+a)\rho(t)}\rho'(t), \qquad t \geq 0, \tag{5.1}$$

for some $a > 0$ and sufficiently small $\delta > 0$, then the perturbed evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ satisfying

$$U(t,s) = T(t,s) + \int_s^t T(t,\tau)B(\tau)U(\tau,s)\,d\tau, \qquad t \geq s \geq 0, \tag{5.2}$$

admits a ρ-dichotomy with respect to the family of norms $\|\cdot\|_t$, $t \geq 0$.

Proof. Since $\{T(t,s)\}_{t\geq s\geq 0}$ admits a ρ-dichotomy with respect to the family of norms $\|\cdot\|_t$, $t \geq 0$, it follows from Proposition 3.1 and Proposition 3.2 that there exists a closed subspace $Z \subset X$ such that the operators

$$T_Z\colon \mathcal{D}(T_Z) \subset Y_\infty^Z \to Y_1 \quad \text{and} \quad T'_Z\colon \mathcal{D}(T'_Z) \subset Y_\infty^Z \to Y'_\infty,$$

defined in the proof of Theorem 4.1, are invertible and closed. We consider the graph norms:

$$\|x\|_{T_Z} := \|x\|_\infty + \|T_Z x\|_1, \qquad x \in \mathcal{D}(T_Z),$$

and

$$\|x\|_{T'_Z} := \|x\|_\infty + \|T'_Z x\|'_\infty, \qquad x \in \mathcal{D}(T'_Z).$$

Since T_Z, T'_Z are closed, it follows that $(\mathcal{D}(T_Z), \|\cdot\|_{T_Z})$, $\left(\mathcal{D}(T'_Z), \|\cdot\|_{T'_Z}\right)$ are Banach spaces. Furthermore,

$$T_Z : (\mathcal{D}(T_Z), \|\cdot\|_{T_Z}) \to (Y_1, \|\cdot\|_1)$$

and

$$T'_Z : \left(\mathcal{D}(T'_Z), \|\cdot\|_{T'_Z}\right) \to (Y'_\infty, \|\cdot\|'_\infty)$$

are bounded linear operators, denoted simply by T_Z and T'_Z, respectively.

We consider the linear operators $D : \mathcal{D}(T_Z) \to Y_1$, $D' : \mathcal{D}(T'_Z) \to Y'_\infty$ defined by

$$(Dx)(t) = B(t)x(t) \text{ and } (D'x)(t) = \frac{1}{\rho'(t)}B(t)x(t), \text{ for } t \geq 0.$$

One can easy check that these operators are well-defined. Furthermore, for each $x \in \mathcal{D}(T_Z)$ we have

$$\begin{aligned}
\|Dx\|_1 &= \int_0^\infty \|B(t)x(t)\|_t\,dt \\
&\leq C\int_0^\infty e^{\varepsilon\rho(t)}\|B(t)x(t)\|\,dt \\
&\leq \delta C\int_0^\infty e^{-a\rho(t)}\,\rho'(t)\,\|x(t)\|\,dt \\
&\leq \frac{\delta C}{a}\,\|x\|_\infty,
\end{aligned}$$

and thus

$$\|Dx\|_1 \leq \frac{\delta C}{a}\,\|x\|_{T_Z}, \quad x \in \mathcal{D}(T_Z). \tag{5.3}$$

On the other hand, for $x \in \mathcal{D}(T'_Z)$ we get

$$\begin{aligned}
\|(D'x)(t)\|_t &= \frac{1}{\rho'(t)}\|B(t)x(t)\|_t \\
&\leq \frac{1}{\rho'(t)}Ce^{\varepsilon\rho(t)}\|B(t)x(t)\| \\
&\leq \delta\,Ce^{-a\rho(t)}\|x(t)\| \\
&\leq \delta\,C\,\|x\|_{T'_Z},
\end{aligned}$$

for all $t \geq 0$, hence

$$\|D'x\|'_\infty \leq \delta\, C\, \|x\|_{T'_Z}, \qquad x \in \mathcal{D}(T'_Z). \tag{5.4}$$

We define now the linear operators

$$U_Z : \mathcal{D}(U_Z) \to Y_1, \qquad U_Z x = y,$$

where $\mathcal{D}(U_Z)$ is the set of all functions $x \in Y^Z_\infty$ such that there exists $y \in Y_1$ satisfying

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\tau)y(\tau)\, d\tau, \quad \text{for } t \geq s \geq 0,$$

and respectively,

$$U'_Z : \mathcal{D}(U'_Z) \to Y'_\infty, \qquad U'_Z x = y,$$

where $\mathcal{D}(U'_Z)$ is the set of all functions $x \in Y^Z_\infty$ such that there exists $y \in Y'_\infty$ satisfying

$$x(t) = U(t,s)x(s) + \int_s^t \rho'(\tau)U(t,\tau)y(\tau)\, d\tau, \quad \text{for } t \geq s \geq 0.$$

Lemma 5.2. *We have:*

$$\mathcal{D}(T_Z) = \mathcal{D}(U_Z) \quad \textit{and} \quad T_Z = U_Z + D, \tag{5.5}$$

and respectively,

$$\mathcal{D}(T'_Z) = \mathcal{D}(U'_Z) \quad \textit{and} \quad T'_Z = U'_Z + D'. \tag{5.6}$$

Proof of the lemma. Take $x \in \mathcal{D}(U_Z)$, that is $x \in Y^Z_\infty$ such that there exists $y \in Y_1$ with $U_Z x = y$. Then, for $t \geq s \geq 0$ we have

$$\begin{aligned}
x(t) &= U(t,s)x(s) + \int_s^t U(t,\tau)y(\tau)\, d\tau \\
&= T(t,s)x(s) + \int_s^t T(t,\tau)B(\tau)U(\tau,s)x(s)\, d\tau + \int_s^t T(t,\tau)y(\tau)\, d\tau \\
&\quad + \int_s^t \int_\tau^t T(t,r)B(r)U(r,\tau)y(\tau)\, dr\, d\tau \\
&= T(t,s)x(s) + \int_s^t T(t,r)y(r)\, dr + \int_s^t T(t,r)B(r)U(r,s)x(s)\, dr \\
&\quad + \int_s^t \int_s^r T(t,r)B(r)U(r,\tau)y(\tau)\, d\tau\, dr \\
&= T(t,s)x(s) + \int_s^t T(t,r)\left(y(r) + B(r)x(r)\right)\, dr,
\end{aligned}$$

thus $x \in \mathcal{D}(T_Z)$ and

$$T_Z x = y + Dx = (U_Z + D)x.$$

Reversing the arguments, we conclude that (5.5) holds. Similarly, one can prove (5.6). ☐

Now, we continue the proof of the theorem. From (5.5) and (5.3) we have

$$\|(U_Z - T_Z)x\|_1 = \|Dx\|_1 \leq \frac{\delta C}{a}\, \|x\|_{T_Z}, \quad \text{for all } x \in \mathcal{D}(T_Z) = \mathcal{D}(U_Z),$$

which implies that $U_Z : \mathcal{D}(U_Z) \to Y_1$ is bounded. Since T_Z is invertible, we obtain that U_Z is also invertible for sufficiently small $\delta > 0$. Similarly, one can show that U'_Z is invertible for sufficiently small $\delta > 0$. By Theorem 4.1 we conclude that the perturbed evolution family $\{U(t,s)\}_{t \geq s \geq 0}$ admits a ρ-dichotomy with respect to the family of norms $\|\cdot\|_t$, $t \geq 0$. ☐

From Proposition 2.1 and Theorem 5.1 we are able now to establish the robustness property of ρ-nonuniform exponential dichotomy.

Corollary 5.3. *Assume that $\mathcal{T} = \{T(t,s)\}_{t\geq s\geq 0}$ admits a ρ-nonuniform exponential dichotomy. If $B : [0,\infty) \to \mathcal{B}(X)$ is a strongly continuous operator-valued function satisfying (5.1) for some $a > 0$ and sufficiently small $\delta > 0$, then the perturbed evolution family satisfying (5.2) admits also a ρ-nonuniform exponential dichotomy.*

Remark 5.4. We stress that the robustness of ρ-nonuniform exponential dichotomies was established in [3, Theorem 1] using different techniques. However, we point out that we establish robustness under a wider class of perturbations than those considered in [3, Theorem 1]. On the other hand, we consider a smaller class of rate functions ρ.

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Existence, uniqueness, and global asymptotic stability of an equilibrium in a multiple unbounded distributed delay network

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Abstract. By employing the notion of M-matrices and Banach's contraction mapping principle, we provide complete characterisation of the existence and uniqueness of an equilibrium of a Cohen–Grossberg–Hopfield-type neural network endowed with multiple unbounded distributed time delays. Invoking similar arguments, and by constructing a suitable Lyapunov functional, we establish sufficient conditions for the global asymptotic stability of the equilibrium, independent of time delays.

Keywords: Cohen–Grossberg–Hopfield-type neural networks, unbounded distributed time delays, Banach's contraction mapping principle, M-matrix, equilibrium, Lyapunov functional, global asymptotic stability.

2020 Mathematics Subject Classification: 34K05, 34K06, 34K20, 34K99.

1 Introduction

The principal objective of this article is to put on a firm mathematical foundation the existence, uniqueness, and global asymptotic stability of an equilibrium of a Cohen–Grossberg–Hopfield-type neural network [9,20,21] motif endowed with multiple distributed time delays. The neural network model studied in this article falls within the class of so-called static neural network models with S-type distributed delays [31, 32]. We characterise, in a rigorous manner, the delay-independent global asymptotic stability of the unique equilibrium using only the notion of M-matrices [3, 12] and the technique of Lyapunov functionals. Let us begin by recalling that the idea of an artificial neural network equipped with signal transmission time delays was first studied by Marcus et al. [27], and since then, the research area has blossomed. Marcus et al. [27] studied a certain class of Hopfield and Cohen–Grossberg [9, 20, 21] artificial neural networks, and demonstrated that the introduction of discrete signal transmission time delays in the neuronal responses induced sustained oscillations and chaos in the emergent network dynamics. In the electronic implementation of analog artificial neural networks,

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signal transmission time delays are a consequence of the finite switching speed of individual amplifiers (neurons) in the network [8,27]. It is well-known that time delays abound in biological neuronal networks [10,25,27] and in electronic artificial neural networks [27]. Discrete time delays are a good first approximation in mathematical models of simple neural network circuits comprised of only a small number of units or neurons [8,34]. Such neural network circuits are characterised by a compact network structure, with negligible spatial extent effects. However, the undisputed biophysical reality is that biological neuronal networks are characterised by an intricate spatial structure of parallel neural pathways in the form of axons (or bundles of axons) of varying thicknesses and lengths. As these neural pathways are known to conduct signals between various neurons, it is self-evident that a biophysically reasonable mathematical modelling paradigm for neuronal networks is one that incorporates signal transmission time delays in which the time delays are distributed rather than discrete. Artificial neural networks incorporating discrete time delays have been widely studied in the literature [2,18,26,30,33]. The problem of neuronal networks endowed with distributed time delays has received some attention in the literature in recent times (see [5,8,11,29,34] and references therein). Nonetheless, the dynamics of artificial neuronal networks endowed with distributed time delays remain largely poorly understood today. In this article, much of our analysis is inspired by the work of Zhang et al. [34] and Chen [8], who studied a special class of Cohen–Grossberg–Hopfield artificial neural networks endowed with distributed time delays, and whose work in turn was a further development of the results of [13] and [14] who had previously established global asymptotic stability results for a class of additive neural networks without any time delays. Extending the results of Gopalsamy et al. [16] and Hofbauer et al. [19], Campbell [4] established delay independent local and global asymptotic stability results for a certain class of additive neural networks endowed with multiple discrete time delays using technical machinery from matrix theory and the method Lyapunov functionals. Wang et al. [32] studied the asymptotic robust stability of the static neural network model endowed with so-called S-type *finitely* distributed time delays, by employing the framework of Lebesgue–Stieltjes integrals. Oliveira [31] studied the global asymptotic stability of a general class of retarded functional differential equations using ideas from matrix theory and Lebesgue–Stieltjes integration, and avoided employing the well-known technique of Lyapunov functionals. Of particular interest, Oliveira [31] studied the existence and the global asymptotic stability of an equilibrium point in the case of two neural network models with *finitely* distributed time delays without using the technique of Lyapunov functionals, namely, the Cohen–Grossberg and the static models.

Our work in this article draws much of its technical motivation from [4,6,16,19,31]. In particular, we consider the *infinitely* distributed time-delayed Hopfield-type network [6,20,21] of n artificial neurons described by the system

$$x_k'(t) = -x_k(t) + g_k\left(\sum_{j=1}^{n} a_{kj}\int_0^\infty x_j(t-u)f_{kj}(u)du\right), \qquad k = 1,\ldots,n\,, \tag{1.1}$$

where $a_{kj} \in \mathbb{R}$, $k, j = 1,\ldots,n$, and the nonlinearity g_k is responsible for modulating the activity of the k^{th} neuron. It is clear that the system (1.1) is a generalisation of the static neural network model [31, equation (3), page 82] with multiple general infinitely distributed time delays, and devoid of any external input signals. Construction of a phase space for infinitely distributed time delay systems such as (1.1) is a little nuanced and technically delicate. Let $\rho > 0$ be a fixed real number. An appropriate (see [6], and references contained therein) phase space for systems with infinite time delays, such as (1.1), is the Banach space $\mathcal{C}_n := C_{0,\rho}\left((-\infty,0],\mathbb{R}^n\right)$

comprising of all continuous $\mathbb{R}^n$-valued functions $\psi(\theta)$ such that the function $e^{\rho\theta}\psi(\theta)$, $\theta \in (-\infty, 0]$, is bounded, uniformly continuous, and satisfies ([23, page 102], [6])

$$\lim_{\theta\to-\infty} e^{\rho\theta}\psi(\theta) = 0\,. \tag{1.2}$$

Furthermore, the Banach space $\mathcal{C}_n$ is equipped with the weighted sup-norm ([23, page 102], [6])

$$\|\psi\|_{\infty,\rho} := \sup_{\theta\in(-\infty,0]} e^{\rho\theta}|\psi(\theta)|\,. \tag{1.3}$$

We assume the following hypotheses on the nonlinearity g_k [4].

(H1) $g_k \in C^2(\mathbb{R})$, $g_k'(u) > 0$, $\sup_{u\in\mathbb{R}} g_k'(u) = g_k'(0) = 1$;

(H2) $g_k(0) = 0$, $\lim_{u\to\pm\infty} g_k(u) = \pm 1$.

Without loss of generality, we adopt throughout this article the specific g_k given by the hyperbolic tangent function

$$g_k(x) = \tanh(\gamma x), \quad \gamma > 0\,. \tag{1.4}$$

We assume that the time delay kernels $f_{kj} : [0,\infty) \mapsto [0,\infty)$, for $k, j = 1,\dots,n$, are continuous functions satisfying the constraints

$$\int_0^\infty f_{kj}(s)ds = 1\,, \qquad \int_0^\infty s f_{kj}(s)ds < \infty\,, \quad \text{and} \quad f_{kj} = f_{jk}\,, \quad \forall k,j = 1,\dots,n\,. \tag{1.5}$$

The usual initial conditions associated with (1.1) are given by [8,34]

$$x_k(\theta) = \phi_k(\theta)\,, \quad \theta \in (-\infty, 0]\,, \; k = 1,\dots,n\,, \tag{1.6}$$

where the ϕ_k are bounded continuous functions on $(-\infty, 0]$. The linearisation of (1.1) about its trivial equilibrium is given by

$$x_k'(t) = -x_k(t) + \sum_{j=1}^n \ell_{kj} \int_{-\infty}^t x_j(s) f_{kj}(t-s)ds\,, \; k = 1,\dots,n\,, \tag{1.7}$$

where $\ell_{kj} := g_k'(0)a_{kj} = a_{kj} \in \mathbb{R}$, $k, j = 1,\dots,n$, are constants. With respect to (1.7), let $\mathbb{R}^n \ni x \mapsto (x_1,\dots,x_n)^\mathsf{T}$ and denote the interconnection matrix by $A := (\ell_{kj}) = (a_{kj})$, $k, j = 1,\dots,n$.

The goal of the present article is to characterise the existence and uniqueness of an equilibrium of (1.1) on one hand, and the global asymptotic stability of this equilibrium on the other. We do so by appealing to the well-known Banach's contraction mapping principle and by constructing an appropriate Lyapunov functional, and employing arguments from the theory of M-matrices [3,12].

2 Existence and uniqueness of the equilibrium

In this section, we establish sufficient conditions for the existence and uniqueness of an equilibrium point of the system (1.1). The approach adopted here hinges on Banach's contraction mapping theorem, and is largely motivated by the inspirational work of [16] and [4].

Theorem 2.1. *If*

$$\beta := \max_{1\le j\le n}\left(\sum_{k=1}^{n}|a_{kj}|\right) < 1\,, \tag{2.1}$$

then the system of algebraic equations

$$x_k = g_k\left(\sum_{j=1}^{n} a_{kj}x_j\right)\ ,\ k=1,\dots,n \tag{2.2}$$

admits a unique solution.

Proof. For calculational convenience, let $v_k := x_k$, $k = 1,\dots,n$, so that (2.2) becomes

$$v_k = g_k\left(\sum_{j=1}^{n} a_{kj}v_j\right) := G_k(v_1,\dots,v_n)\ ,\ k=1,\dots,n\,. \tag{2.3}$$

Our goal is to establish the existence of fixed points of the map $\mathbf{G} : \mathbb{R}^n \mapsto \mathbb{R}^n$ defined by $\mathbf{G} := (G_1(\mathbf{v}),\dots,G_n(\mathbf{v}))$, with $\mathbf{v} := (v_1,\dots,v_n)$. From the hypotheses (H1) and (H2), we have that

$$-1 \le g_k\left(\sum_{j=1}^{n} a_{kj}v_j\right) \le 1\,,\quad k=1,\dots,n\,. \tag{2.4}$$

This observation implies that the set D defined by

$$D := \{(x_1,\dots,x_n)\in\mathbb{R}^n \mid -1\le x_k\le 1,\ k=1,\dots,n\} \tag{2.5}$$

is invariant with respect to the mapping $\mathbf{G}$ [4,16]. In what follows, we establish that $\mathbf{G}$ is a contraction mapping on D. By Banach's contraction mapping principle, it will follow that $\mathbf{G}$ has a unique fixed point. First, let $\mathbf{v} := (v_1,\dots,v_n)$ and $\mathbf{u} := (u_1,\dots,u_n)$. We begin by noting from (2.3) that

$$\begin{aligned}
\|\mathbf{G}(\mathbf{v})-\mathbf{G}(\mathbf{u})\| &= \sum_{k=1}^{n}|G_k(\mathbf{v})-G_k(\mathbf{u})| \\
&= \sum_{k=1}^{n}\left|g_k\left(\sum_{j=1}^{n}a_{kj}v_j\right)-g_k\left(\sum_{j=1}^{n}a_{kj}u_j\right)\right| \\
&\le \sum_{k=1}^{n}|g_k'(\theta_k)|\sum_{j=1}^{n}|a_{kj}|\,|v_j-u_j| \\
&= \sum_{k=1}^{n}c_k\sum_{j=1}^{n}|a_{kj}|\,|v_j-u_j| \\
&= \sum_{j=1}^{n}\left(\sum_{k=1}^{n}c_k\,|a_{kj}|\right)|v_j-u_j| \\
&\le \beta\sum_{j=1}^{n}|v_j-u_j| \\
&= \beta\,\|\mathbf{v}-\mathbf{u}\|\ ,
\end{aligned} \tag{2.6}$$

where

$$\sum_{j=1}^{n}a_{kj}u_j \le \theta_k \le \sum_{j=1}^{n}a_{kj}v_j\ ,\quad k=1,\dots,n\,,\quad c_k := g_k'(\theta_k)\in(0,1]\,,$$

and

$$\beta := \max_{1\le j\le n}\left(\sum_{k=1}^{n} c_k \left|a_{kj}\right|\right) = \max_{1\le j\le n}\left(\sum_{k=1}^{n} \left|a_{kj}\right|\right) < 1 \tag{2.7}$$

by hypothesis. Without loss of generality, and by recourse to hypothesis (H1), we have here set $c_k = 1$, $\forall k = 1,\ldots,n$. Consequently, $\mathbf{G}$ is a contraction on D, and by Banach's contraction mapping principle, it has a unique fixed point, say $\mathbf{v}^* := (v_1^*,\ldots,v_n^*)$, such that

$$v_k^* = g_k\left(\sum_{j=1}^{n} a_{kj} v_j^*\right), \quad k = 1,\ldots,n\,.$$

Thus, (1.1) has a unique equilibrium point. This completes the proof. □

3 Global asymptotic stability of the equilibrium

We now establish the global asymptotic stability of the equilibrium $\mathbf{x}^* := (x_1^*,\ldots,x_n^*)$ of (1.1) by recourse to the theory of M-matrices, and by constructing an appropriate Lyapunov functional. Let $y_k(t) := x_k(t) - x_k^*$, $k = 1,\ldots,n$. From the hypothesis (H1) and Lagrange's Mean Value Theorem, there exists

$$\vartheta_k \in \left(\sum_{j=1}^{n} a_{kj}x_j^*\,,\ \sum_{j=1}^{n} a_{kj}\int_0^\infty y_j(t-u)f_{kj}(u)du + \sum_{j=1}^{n} a_{kj}x_j^*\right), \quad k = 1,\ldots,n\,, \tag{3.1}$$

such that

$$g_k\left(\sum_{j=1}^{n} a_{kj}\int_0^\infty y_j(t-u)f_{kj}(u)du + \sum_{j=1}^{n} a_{kj}x_j^*\right) - g_k\left(\sum_{j=1}^{n} a_{kj}x_j^*\right)$$
$$= g_k'(\vartheta_k)\sum_{j=1}^{n} a_{kj}\int_0^\infty y_j(t-u)f_{kj}(u)du\,, \tag{3.2}$$

for $k = 1,\ldots,n$. It is important to stress the fact that ϑ_k identified in (3.1) is not a constant – it depends on the solution y_j, $j = 1,\ldots,n$, and the time t. By virtue of the coordinate translation $y_k(t) := x_k(t) - x_k^*$, $k = 1,\ldots,n$, and (3.2), the system (1.1) transforms to

$$y_k'(t) = -y_k(t) + g_k\left(\sum_{j=1}^{n} a_{kj}\int_0^\infty \left(y_j(t-u) + x_j^*\right) f_{kj}(u)du\right) - g_k\left(\sum_{j=1}^{n} a_{kj}x_j^*\right),$$
$$k = 1,\ldots,n, \tag{3.3}$$

which subsequently leads to the linearisation

$$y_k'(t) = -y_k(t) + c_k\sum_{j=1}^{n} a_{kj}\int_0^\infty y_j(t-u)f_{kj}(u)du, \quad k = 1,\ldots,n\,, \tag{3.4}$$

where $c_k := g_k'(\vartheta_k) \in (0,1]$, $\forall k = 1,\ldots,n$, by the hypothesis (H1). We note that c_k depends on t, and this observation has some consequential ramifications as will be shown in the analysis to come. Now, borrowing some of the notation of [4], let $A := (a_{kj})$, $|A| := (|a_{kj}|)$, $K := -I + A$, and $\widehat{K} := -I + |A|$, where I is the $n\times n$ identity matrix. Sufficient conditions for the local asymptotic stability of the equilibrium $\mathbf{x}^* := (x_1^*,\ldots,x_n^*)$ of (1.1) can be established in a manner analogous to that presented in [4, Theorem 2.6 and Corollary 2.7, page 6], and are given in [6]. To prepare the groundwork for the analysis to follow, we note that the off-diagonal entries of the matrix $-\widehat{K}$ are less than or equal to zero, which means that it is a Z-matrix. The matrix $-\widehat{K}$ is expressible in the form

$$\begin{aligned} -\widehat{K} &:= \begin{pmatrix} 1-|a_{11}| & -|a_{12}| & -|a_{13}| & \cdots & -|a_{1n}| \\ -|a_{21}| & 1-|a_{22}| & -|a_{23}| & \cdots & -|a_{2n}| \\ -|a_{31}| & -|a_{32}| & 1-|a_{33}| & \cdots & -|a_{3n}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -|a_{n1}| & -|a_{n2}| & -|a_{n3}| & \cdots & 1-|a_{nn}| \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} - \begin{pmatrix} |a_{11}| & |a_{12}| & |a_{13}| & \cdots & |a_{1n}| \\ |a_{21}| & |a_{22}| & |a_{23}| & \cdots & |a_{2n}| \\ |a_{31}| & |a_{32}| & |a_{33}| & \cdots & |a_{3n}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |a_{n1}| & |a_{n2}| & |a_{n3}| & \cdots & |a_{nn}| \end{pmatrix} \\ &:= sI - B\,, \end{aligned} \tag{3.5}$$

where B is the non-negative matrix given by

$$B := \begin{pmatrix} |a_{11}| & |a_{12}| & |a_{13}| & \cdots & |a_{1n}| \\ |a_{21}| & |a_{22}| & |a_{23}| & \cdots & |a_{2n}| \\ |a_{31}| & |a_{32}| & |a_{33}| & \cdots & |a_{3n}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |a_{n1}| & |a_{n2}| & |a_{n3}| & \cdots & |a_{nn}| \end{pmatrix}, \tag{3.6}$$

$s := 1 > 0$, and I is the $n \times n$ identity matrix. The following lemma will be instrumental in the proof of our main result in the present Section.

Lemma 3.1. *If $-\widehat{K}$ is a Z-matrix and $\rho(B) < 1$, then $-\widehat{K}$ is a non-singular M-matrix.*

Proof. That $-\widehat{K}$ is a Z-matrix is trivial. Suppose that $\rho(B) < 1$. Since $-\widehat{K} = I - B$, the result follows [12, page 129, Theorem 5.1.1.]. □

As an example to amplify the implication of Lemma 3.1, consider $n = 2$ populations of artificial neurons, with $a_{11} = a_{22} = 0$, $a_{12} = 2$, and $a_{21} = 1$. Then, we have that

$$\begin{aligned} A = (a_{kj}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \\ \implies -\widehat{K} &= \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} := I - B\,, \end{aligned} \tag{3.7}$$

where $\rho(B) = \sqrt{2} > 1$. Hence, the matrix $-\widehat{K}$ in this example is not a non-singular M-matrix for the simple reason that it does not satisfy at least one of the hypotheses stipulated in Lemma 3.1. In the view of Lemma 3.1, we arrive at our main result in the present Section.

Theorem 3.2. *If $-\widehat{K}$ is a non-singular M-matrix, then the system (1.1) has a unique globally asymptotically stable equilibrium.*

Proof. Assume that $-\widehat{K}$ is a non-singular M-matrix. That is, assume that $-\widehat{K} = I - B$ is a Z-matrix and that $\rho(B) < 1$ [12, page 129, Theorem 5.1.1.]. It is well-known that if the spectral radius of a matrix is less than 1, then the matrix has a norm which is less than 1 [22, page 347, Lemma 5.6.10]. Since $B = (|a_{ij}|)$, $i, j = 1, \dots, n$, the maximum column sum matrix norm of B is given by [22]

$$|||B|||_1 = \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}| \right) < 1 \,, \tag{3.8}$$

which is identical to the hypothesis of Theorem 2.1.

Now, since $-\widehat{K} := I - |A|$ is a non-singular M-matrix from Lemma 3.1, it follows [3, 12] that $\exists\, \xi_j > 0$, $j = 1, \dots, n$, such that

$$-\xi_j + \sum_{k=1}^{n} |a_{kj}| \xi_k < 0 \,, \quad j = 1, \dots, n \,. \tag{3.9}$$

Consider the Lyapunov functional $V(t) = V(y)(t)$ defined by [4, 8, 32, 34]

$$V(y)(t) := \sum_{k=1}^{n} \xi_k \left\{ |y_k(t)| + \sum_{j=1}^{n} |a_{kj}| \int_0^\infty f_{kj}(s) \left(\int_{t-s}^{t} |y_j(\tau)| d\tau \right) ds \right\} \,. \tag{3.10}$$

Computing the upper Dini derivative of (3.10) along the solutions of the nonlinear system (3.3) yields

$$\begin{aligned}
D^+V(t) &= \sum_{k=1}^{n} \xi_k \left\{ \operatorname{sgn}(y_k(t)) y_k'(t) + \sum_{j=1}^{n} |a_{kj}| \int_0^\infty f_{kj}(s) \left(|y_j(t)| - |y_j(t-s)| \right) ds \right\} \\
&= \sum_{k=1}^{n} \xi_k \left\{ -\operatorname{sgn}(y_k(t)) y_k(t) + \operatorname{sgn}(y_k(t)) g_k \left(\sum_{j=1}^{n} a_{kj} \int_0^\infty \left(y_j(t-u) + x_j^* \right) f_{kj}(u) du \right) \right. \\
&\quad \left. -\operatorname{sgn}(y_k(t)) g_k \left(\sum_{j=1}^{n} a_{kj} x_j^* \right) + \sum_{j=1}^{n} |a_{kj}| \int_0^\infty f_{kj}(s) \left(|y_j(t)| - |y_j(t-s)| \right) ds \right\} \\
&\le \sum_{k=1}^{n} \xi_k \left\{ -|y_k(t)| + \left| g_k \left(\sum_{j=1}^{n} a_{kj} \int_0^\infty \left(y_j(t-u) + x_j^* \right) f_{kj}(u) du \right) - g_k \left(\sum_{j=1}^{n} a_{kj} x_j^* \right) \right| \right. \\
&\quad \left. + \sum_{j=1}^{n} |a_{kj}| \int_0^\infty |y_j(t)| f_{kj}(s) ds - \sum_{j=1}^{n} |a_{kj}| \int_0^\infty |y_j(t-s)| f_{kj}(s) ds \right\} \\
&\le \sum_{k=1}^{n} \xi_k \left\{ -|y_k(t)| + \sum_{j=1}^{n} |a_{kj}| \int_0^\infty |y_j(t-u)| f_{kj}(u) du + \sum_{j=1}^{n} |a_{kj}| |y_j(t)| \right. \\
&\quad \left. - \sum_{j=1}^{n} |a_{kj}| \int_0^\infty |y_j(t-u)| f_{kj}(u) du \right\} \\
&= \sum_{k=1}^{n} \xi_k \left\{ -|y_k(t)| + \sum_{j=1}^{n} |a_{kj}| |y_j(t)| \right\} \\
&= \sum_{k=1}^{n} \left(-\xi_k |y_k(t)| \right) + \sum_{k=1}^{n} \sum_{j=1}^{n} \left(|a_{jk}| \xi_j |y_k(t)| \right) \\
&= \sum_{k=1}^{n} \left(-\xi_k + \sum_{j=1}^{n} |a_{jk}| \xi_j \right) |y_k(t)| \le \mu \sum_{k=1}^{n} |y_k(t)| < 0 \,,
\end{aligned}$$

where, by virtue of the condition (3.9),

$$\mu := \max_{1\le k\le n}\left\{-\xi_k + \sum_{j=1}^{n}|a_{jk}|\xi_j\right\} < 0\,. \tag{3.11}$$

Hence, the trivial equilibrium of (3.3) is globally asymptotically stable [24, corollary 5.2, page 30]. Therefore, the equilibrium $\mathbf{x}^* := (x_1^*, \ldots, x_n^*)$ of (1.1) is globally asymptotically stable (see [1,4,17] and [15, pages 4-5]). This completes the proof. $\square$

4 A numerical example

We give a numerical example to illustrate an application of Theorem 3.2. Consider $n = 2$ populations of artificial neurons, with $a_{11} = a_{22} = \frac{1}{2}$, $a_{12} = \frac{1}{16}$, and $a_{21} = 1$. Thus, we have that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{16} \\ 1 & \frac{1}{2} \end{pmatrix}$$
$$\Rightarrow -\widehat{K} = I - |A| = \begin{pmatrix} \frac{1}{2} & -\frac{1}{16} \\ -1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & \frac{1}{16} \\ 1 & \frac{1}{2} \end{pmatrix} := I - B\,, \tag{4.1}$$

with $\rho(B) = \frac{3}{4} < 1$. That $-\widehat{K}$ is a Z-matrix is trivial. This observation, in conjunction with the fact that $\rho(B) < 1$, implies that $-\widehat{K}$ is a non-singular M-matrix by Lemma 3.1. For the specified interconnection matrix A, the system (1.1) condenses to

$$\begin{cases} x_1'(t) = -x_1(t) + g_1\left(\displaystyle\sum_{j=1}^{2} a_{1j}\int_0^\infty x_j(t-u)f_{1j}(u)du\right)\,, \\ x_2'(t) = -x_2(t) + g_2\left(\displaystyle\sum_{j=1}^{2} a_{2j}\int_0^\infty x_j(t-u)f_{2j}(u)du\right)\,, \end{cases} \tag{4.2}$$

with the initial conditions given in (1.6) for $n = 2$. Since $-\widehat{K}$ is a non-singular M-matrix, we are guaranteed by Theorem 3.2 that the system (4.2) admits a unique globally asymptotically stable equilibrium. For the sake of completeness, we establish the existence and uniqueness of an equilibrium of (4.2). Now, since $\rho(B) < 1$, it follows that there exists a matrix norm such that $|||A||| < 1$ [22, page 347, Lemma 5.6.10]. To characterise such a norm, we proceed in the manner adumbrated below. Let $J := P^{-1}AP = \mathrm{diag}\left(\frac{1}{4}, \frac{3}{4}\right)$ be the Jordan form of A, with $P := \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ 1 & 1 \end{pmatrix}$, and let $D := I$ be the 2×2 identity matrix. Note that the matrix A has eigenvalues $\lambda_1 := \frac{1}{4}$ and $\lambda_2 := \frac{3}{4}$. The two columns of P are the eigenvectors of A. The eigenspace for $\lambda_1 = \frac{1}{4}$ is spanned by $\mathbf{u} := \begin{pmatrix} -\frac{1}{4} \\ 1 \end{pmatrix}$ whilst that for $\lambda_2 = \frac{3}{4}$ is spanned by $\mathbf{v} := \begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix}$. Now, define a norm by $|||A||| := |||D^{-1}P^{-1}APD|||_p = |||P^{-1}AP|||_p$, where $|||\cdot|||_p$ denotes the induced p-norm. In other words,

$$|||A||| := \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p}\,, \tag{4.3}$$

where the two norms $\|\cdot\|_p$ on the right hand side denote the usual p-norm for vectors. When $p = 1$, $|||A|||$ is identical to the maximum column sum of the entrywise absolute value of

A. For the matrix P in this example, we have that $P^{-1}AP = \operatorname{diag}\left(\frac{1}{4}, \frac{3}{4}\right)$, and consequently, $|||A||| = \max_{1\le j\le 2}\left(\sum_{i=1}^{2}|a_{ij}|\right) = \frac{3}{4} < 1$; this last inequality matches the hypothesis (2.1) of Theorem 2.1. Hence, the existence and uniqueness of an equilibrium of the system (4.2) is guaranteed.

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About existence and regularity of positive solutions for a quasilinear Schrödinger equation with singular nonlinearity

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Abstract. We establish the existence of positive solutions for the singular quasilinear Schrödinger equation

$$\begin{cases} -\Delta u - \Delta(u^2)u = h(x)u^{-\gamma} + f(x,u) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $1 < \gamma$, $h \in L^1(\Omega)$ and $h > 0$ almost everywhere in Ω. The function f may change sign on Ω. By using the variational method and some analysis techniques, the necessary and sufficient condition for the existence of a solution is obtained.

Keywords: strong singularity, variational methods, regularity, fibering methods, indefinite nonlinearity.

2020 Mathematics Subject Classification: 35J62, 35J20, 55J75.

1 Introduction

In this paper we study the existence of solution for the following quasilinear Schrödinger equation

$$\begin{cases} -\Delta u - \Delta(u^2)u = h(x)u^{-\gamma} + f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $1 < \gamma$, $h \in L^1(\Omega)$, $h > 0$ almost everywhere (a.e.) in Ω and $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function. We assume that the function f satisfies one of the following conditions:

$(f)_1$ $f(x,s) = b(x)s^p$, where $p \in (0,1), b \in L^\infty(\Omega)$ and $b^+ = \max\{b, 0\} \not\equiv 0$.

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$(f)_2$ $f(x,s) = -b(x)s^{22^*-1}$, where $b \in L^\infty(\Omega)$ and $b \geq 0$ a.e. in Ω.

We say that a function $u \in H_0^1(\Omega)$ is a weak solution (solution, for short) of (P) if $u > 0$ a.e. in Ω, and, for every $\varphi \in H_0^1(\Omega)$,

$$hu^{-\gamma}\varphi \in L^1(\Omega) \tag{1.1}$$

and

$$\int_\Omega [(1+2u^2)\nabla u \nabla \varphi + 2u|\nabla u|^2\varphi] = \int_\Omega h(x)u^{-\gamma}\varphi + \int_\Omega f(x,u)\varphi.$$

Consider the following quasilinear Schrödinger equation

$$-\Delta u - \Delta(u^2)u = g(x,u) \quad \text{in } \Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. When $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, recently, there appeared some works dealing with (1.2), see for example [1,17,18] and its references. In these works the nonlinearity is non-singular, and so the authors were able to combine the dual approach of [4] with classic results of variational methods to prove their main results.

When g is singular, problems of type (1.2) was studied by Do Ó–Moameni [6], Liu–Liu–Zhao [16], Wang [26] and Dos Santos–Figueiredo–Severo [24]. In [6] the authors studied the problem

$$\begin{cases} -\Delta u - \frac{1}{2}\Delta(u^2)u = \lambda|u|^2u - u - u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a ball in $\mathbb{R}^N$ centered at the origin, $0 < \gamma < 1$ and $N \geq 2$. They showed that problem (1.3) has a radially symmetric solution $u \in H_0^1(\Omega)$ for $\lambda \in I$, where I is an open interval.

Liu–Liu–Zhao in [16] considered the problem

$$\begin{cases} -\Delta_s u - \frac{s}{2^{s-1}}\Delta(u^2)u = h(x)u^{-\gamma} + \lambda u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $N \geq 3$, Δ_s is the s-Laplacian operator, $2 < 2s < p+1 < \infty$, $0 < \gamma$ and $h \geq 0$ is a nontrivial measurable function satisfying the following condition: there exist a function $\phi_0 \geq 0$ in $C_0^1(\overline{\Omega})$ and $q > N$ such that $h\phi_0^{-\gamma} \in L^q(\Omega)$. The authors used sub-supersolution method, truncation arguments and variational methods to prove the existence of a $\lambda_* > 0$ such that problem (1.4) has at least two solutions for $\lambda \in (0, \lambda_*)$.

Wang in [26], by using minimax methods and some analysis techniques, showed the existence and uniqueness of solutions to the problem

$$\begin{cases} -\Delta u - \Delta(u^2)u = h(x)u^{-\gamma} - u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $N \geq 3$, $\gamma \in (0,1)$, $p \in [2, 22^*]$, $h \in L^{\frac{22^*}{22^*-1+\gamma}}(\Omega)$ and $h > 0$ a.e. in Ω.

In [24], Dos Santos–Figueiredo–Severo studied the problem

$$\begin{cases} -\Delta u - \Delta(u^2)u = h(x)u^{-\gamma} + z(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $N \geq 3$, h is a nonnegative function, $\gamma > 0$ is a constant and the nonlinearity $z : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies some conditions. By using sub-supersolution method, truncation arguments and the Mountain Pass Theorem they showed the existence of two solutions. We would like to emphasize that for the authors to use the sub-supersolution method, the following assumption was very important: there exist $\phi_0 \in C_0^1(\overline{\Omega})$, $\phi_0 \geq 0$, and $q > N$ such that $h\phi_0^{-\gamma} \in L^q(\Omega)$. Furthermore, we note that our assumption on the function h is different (see (1.7) below), because it does not guarantee that $hv_0^{-\gamma} \in L^q(\Omega)$ for some $q > N$.

Singular elliptic problems has been studied extensively in recent years, see [5,7,11–14,21–23,25] and the references therein. Especially, Sun in [25] considered the problem

$$\begin{cases} -\Delta u = h(x)u^{-\gamma} + b(x)u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $b \in L^\infty(\Omega)$ is a nonnegative function, $0 < p < 1 < \gamma$, $h \in L^1(\Omega)$ and $h > 0$ a.e. in Ω. By using variational methods the author showed that the existence of $H_0^1(\Omega)$–solutions of (1.6) is related to a compatibility hypothesis between on the couple $(h(x), \gamma)$. More precisely, problem (1.6) has a solution in $H_0^1(\Omega)$ if and only if there exists $v_0 \in H_0^1(\Omega)$ such that

$$\int_\Omega h(x)|v_0|^{1-\gamma} < \infty. \tag{1.7}$$

Motivated by above results, our main purpose in this paper is to investigate the existence of $H_0^1(\Omega)$-solutions for problem (P). We shall show that the compatibility condition (1.7) on the couple $(h(x), \gamma)$ is also optimal for the existence of weak solutions to problem (P). Under additional assumption on the function h we show that the solutions of (P) belong to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, and as a consequence we obtain uniqueness of solution.

Before giving our main results, we need an additional assumption. The function $d(x) = d(x, \partial\Omega)$ denotes the distance from a point $x \in \overline{\Omega}$ to the boundary $\partial\Omega$, where $\overline{\Omega} = \Omega \cup \partial\Omega$ is the closure of $\Omega \subset \mathbb{R}^N$.

We introduce the following assumption:

(bh) $b \geq 0$ a.e. in Ω and there exist constants $c > 0$ and $\beta \in (0,1)$ such that

$$h(x) \leq c d^{\gamma-\beta}(x), \qquad \forall x \in \Omega. \tag{1.8}$$

Our first result is the following.

Theorem 1.1. *If $(f)_1$ holds, then:*

a) problem (P) admits a solution $u \in H_0^1(\Omega)$ if and only if there exists a function $v_0 \in H_0^1(\Omega)$ satisfying (1.7);

b) *under the additional assumption* (bh) *the solution* u *obtained in* a) *belongs to* $C^{1,\alpha}(\overline{\Omega})$ *for some* $\alpha \in (0,1)$. *In particular, problem* (P) *has a unique solution in* $H_0^1(\Omega)$.

It is worth pointing out that there are some differences between problems (P) and (1.5). We give one in the following example.

Example 1.2. Let Ω_0 be an open set with $\overline{\Omega}_0 \subset \Omega$ and $\beta, p \in (0,1)$. Then the functions $h(x) = d^{\gamma-\beta}(x), x \in \Omega$ and $f(x,s) = (2\chi_{\overline{\Omega}_0}(x) - 1)s^p, (x,s) \in \Omega \times \mathbb{R}$ satisfy (1.7) and $(f)_1$, respectively (see Remark 3.3). Here we denote by $\chi_{\overline{\Omega}_0}$ the characteristic function of $\overline{\Omega}_0$. We claim that the functions h and f do not satisfy the assumption (h_1) in [24]. To see this let $y \in \partial\Omega$ and $k, s > 0$. Since $\lim_{x\to y} -kh(x) = \lim_{x\to y} -kd^{\gamma-\beta}(x) = 0$, we can find $\epsilon > 0$ such that

$$f(x,s) = -s^p < -kh(x) \quad \text{for every } x \in \{x \in \Omega : |x - y| < \epsilon\} \setminus \overline{\Omega}_0.$$

This proves the claim.

Regularity results for singular elliptic equations have been studied in Giacomoni–Schindler–Takáč [8], Giacomoni–Saoudi [9] and Marino–Winkert [19] in the particular context of weak singularity, that is $\gamma \in (0,1)$. Specifically, in [8] the $C^{1,\alpha}(\overline{\Omega})$ regularity is proved. In the present paper, we consider the opposite situation where $\gamma > 1$ (namely, strong singularity) and give conditions on h which guarantee the $C^{1,\alpha}(\overline{\Omega})$ regularity of weak solutions of (P). We observe that due to the difference between the types of singularities, and also due to the structure of problem (P_A) below, the regularity result of [8] can not be applied to prove Theorem 1.1-*b*).

Now we state our second result.

Theorem 1.3. *Suppose* $(f)_2$ *holds. Then problem* (P) *admits a unique solution* $u \in H_0^1(\Omega)$ *if and only if there exists a function* $v_0 \in H_0^1(\Omega)$ *satisfying* (1.7).

To prove the existence of a solution for problem (P), we use the method of changing variables developed in Colin–Jeanjean [4]. With this approach, the energy functional associated to the new problem has nonhomogeneous terms (see problem (P_A)) and some difficulties arise. For example, the techniques used by the works mentioned above do not apply directly here. In order to deal with these difficulties, we make a careful analysis of the fiber maps associated to the energy functional associated to the new problem and we will approach it in a new way.

We emphasize that Theorem 1.1 extends the main result of Sun [25] (see Theorem 1.2 in [25]), in the sense that we consider the operator $Lu = -\Delta u - \Delta(u^2)u$ instead of the Laplacian operator and the potential b may change sign on Ω. As far as we know, the regularity of solution (and consequently the uniqueness) obtained in Theorem 1.1-*b*) is new. Also, Theorem 1.3 extends Theorem 1.1 of Wang [26] in the sense that we consider the case $\gamma > 1$.

The paper is organized as follows. In the next section we reformulate problem (P) into a new one which finds its natural setting in the Sobolev space $H_0^1(\Omega)$ and we prove some important lemmas. In section 3, we give the proof of Theorem 1.1. In section 4, we prove Theorem 1.3 and in the Appendix we prove some properties of the positive solutions of problem $-\Delta u - \Delta(u^2)u = h(x)u^{-\gamma} + \lambda b(x)u^p$ in Ω, where the parameter $\lambda \geq 0$ varies.

Notation. Throughout the paper we make use of the following notation:

- c, C denote positive constants, which may vary from line to line.
- $H_0^1(\Omega)$ denotes the Sobolev space equipped with the norm $\|u\| = \left(\int_\Omega |\nabla u|^2 dx\right)^2$.

- $L^s(\Omega)$, $1 \le s \le \infty$, denotes the Lebesgue space with the norms $\|u\|_s = \left(\int_\Omega |\nabla u|^s dx\right)^{1/s}$, for $1 \le p < \infty$, $\|u\|_\infty = \inf\{C > 0 : |u(x)| \le C \text{ a.s. in } \Omega\}$.
- For $0 < \alpha \le 1$, $C^{1,\alpha}(\overline{\Omega})$ denotes the space of Hölder functions with exponent α. The norm of $C^{1,\alpha}(\overline{\Omega})$ is denoted by $|\cdot|_{1,\alpha}$.
- We denote by ϕ_1 the L^∞-normalized (that is, $|\phi_1|_\infty = 1$) positive eigenfunction of $(-\Delta, H_0^1(\Omega))$.
- If B is a measurable set in $\mathbb{R}^N$, we denote by χ_B the characteristic function of B.

2 Reformulation of the problem and preliminaries

The natural energy functional corresponding to the problem (P) is the following:

$$J(u) = \frac{1}{2}\int_\Omega (1+2u^2)|\nabla u|^2 + \frac{1}{\gamma-1}\int_\Omega h(x)|u|^{1-\gamma} - \int_\Omega F(x,u), \qquad u \in D(J),$$

where

$$D(J) = \left\{ u \in H_0^1(\Omega) : \int_\Omega h(x)|u|^{1-\gamma} < \infty \right\}$$

and $F(x,s) = \int_0^s f(x,t)dt$.

However, this functional is not well defined, because $\int_\Omega u^2|\nabla u|^2 dx$ is not finite for all $u \in H_0^1(\Omega)$, hence it is difficult to apply variational methods directly. In order to overcome this difficulty, we use the following change of variables introduced by [4], namely, $v := g^{-1}(u)$, where g is defined by

$$\begin{cases} g'(t) = \frac{1}{(1+2|g(t)|^2)^{\frac{1}{2}}} & \text{in } [0,\infty), \\ g(t) = -g(-t) & \text{in } (-\infty, 0]. \end{cases}$$

We list some properties of g, whose proofs can be found in Liu [15].

Lemma 2.1. *The function g satisfies the following properties:*

(1) *g is uniquely defined, C^∞ and invertible;*

(2) $g(0) = 0$*;*

(3) $0 < g'(t) \le 1$ *for all* $t \in \mathbb{R}$*;*

(4) $\frac{1}{2}g(t) \le tg'(t) \le g(t)$ *for all* $t > 0$*;*

(5) $|g(t)| \le |t|$ *for all* $t \in \mathbb{R}$*;*

(6) $|g(t)| \le 2^{1/4}|t|^{1/2}$ *for all* $t \in \mathbb{R}$*;*

(7) $(g(t))^2 - g(t)g'(t)t \ge 0$ *for all* $t \in \mathbb{R}$*;*

(8) *There exists a positive constant C such that $|g(t)| \ge C|t|$ for $|t| \le 1$ and $|g(t)| \ge C|t|^{1/2}$ for $|t| \ge 1$;*

(9) $g''(t) < 0$ *when* $t > 0$ *and* $g''(t) > 0$ *when* $t < 0$*;*

(10) *the functions $(g(t))^{1-\gamma}$ and $(g(t))^{-\gamma}$ are decreasing for all $t > 0$;*

(11) *the function* $(g(t))^p t^{-1}$ *is decreasing for all* $t > 0$;

(12) $|g(t)g'(t)| < 1/\sqrt{2}$ *for all* $t \in \mathbb{R}$.

Proof. We only prove (10) and (11). From $g(t), g'(t) > 0$ for $t > 0$ and $\gamma > 1$, we obtain

$$\left[(g(t))^{1-\gamma}\right]' = (1-\gamma)(g(t))^{-\gamma}g'(t) < 0, \qquad \forall t > 0$$

and

$$\left[(g(t))^{-\gamma}\right]' = -\gamma(g(t))^{-\gamma-1}g'(t) < 0, \qquad \forall t > 0,$$

which imply that $(g(t))^{1-\gamma}$ and $(g(t))^{-\gamma}$ are decreasing for all $t > 0$. Therefore, (10) has been proved.

(11) Using the fact that $p < 1$ and (4) we find

$$\begin{aligned}\left[(g(t))^p t^{-1}\right]' &= p(g(t))^{p-1}g'(t)t^{-1} - (g(t))^p t^{-2}\\ &= p(g(t))^{p-1}(g'(t)t)t^{-2} - (g(t))^p t^{-2}\\ &< t^{-2}\left[(g(t))^{p-1}g(t) - (g(t))^p\right]\\ &= 0,\end{aligned}$$

for all $t > 0$. Hence the function $(g(t))^p t^{-1}$ is decreasing for all $t > 0$. The lemma is proved. □

After a change of variable $v = g^{-1}(u)$, we define an associated problem

$$\begin{cases} -\Delta v = \left[h(x)(g(v))^{-\gamma} + f(x, g(v))\right] g'(v) & \text{in } \Omega,\\ v > 0 & \text{in } \Omega,\\ v(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_A}$$

We say that a function $v \in H_0^1(\Omega)$ is a weak solution (solution, for short) of (P_A) if $v > 0$ a.e. in Ω, and, for every $\varphi \in H_0^1(\Omega)$,

$$h(x)(g(v))^{-\gamma}g'(v)\varphi \in L^1(\Omega)$$

and

$$\int_\Omega \nabla v \nabla \varphi = \int_\Omega h(x)(g(v))^{-\gamma}g'(v)\varphi + \int_\Omega f(x, g(v))g'(v)\varphi.$$

It is easy to see that problem (P_A) is equivalent to our problem (P), which takes $u = g(v)$ as its solutions.

The energy functional associated with problem (P_A) is defined as

$$\Phi(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{\gamma - 1}\int_\Omega h(x)|g(v)|^{1-\gamma} - \int_\Omega F(x, g(v)), \qquad v \in D(\Phi),$$

if $D(\Phi) \neq \emptyset$, where

$$D(\Phi) = \left\{v \in H_0^1(\Omega) : \int_\Omega h(x)|g(v)|^{1-\gamma} < \infty\right\}$$

and $F(x, s) = \int_0^s f(x, t)dt$.

We shall justify that Φ is well defined by showing that $D(\Phi) \neq \emptyset$. We first remark that if v_0 satisfies (1.7), then $|v_0|$ satisfies (1.7), too. Hence, without loss of generality we can assume that $v_0 > 0$ a.e. in Ω.

We have the following lemma.

Lemma 2.2. *Let v be Lebesgue measurable and suppose that $v > 0$ a.e. in Ω. The following statements are equivalent:*

(a) $\int_\Omega h(x)|v|^{1-\gamma} < \infty$;

(b) $\int_\Omega h(x)(g(v))^{-\gamma}g'(v)v < \infty$;

(c) $\int_\Omega h(x)(g(v))^{1-\gamma} < \infty$.

In particular, if condition (1.7) holds, then $D(\Phi) \neq \emptyset$.

Proof. $(a) \Rightarrow (b)$: First, we decompose Ω as $\Omega = A_1 \cup A_2$, where

$$A_1 = \{x \in \Omega : |v(x)| \leq 1\} \text{ and } A_2 = \{x \in \Omega : |v(x)| > 1\}.$$

It is easy to see that

$$h(x)(g(v))^{-\gamma}g'(v)v = h(x)(g(v))^{-\gamma}g'(v)v\chi_{A_1} + h(x)(g(v))^{-\gamma}g'(v)v\chi_{A_2},$$

thus

$$\int_\Omega h(x)(g(v))^{-\gamma}g'(v)v < \infty$$

if and only if

$$h(x)(g(v))^{-\gamma}g'(v)v\chi_{A_1} \in L^1(\Omega) \quad \text{and} \quad h(x)(g(v))^{-\gamma}g'(v)v\chi_{A_2} \in L^1(\Omega). \tag{2.1}$$

Let us show that (2.1) holds, and consequently that $\int_\Omega h(x)(g(v))^{-\gamma}g'(v)v < \infty$. Indeed, by Lemma 2.1 (4), (8) we have

$$\begin{aligned}|h(x)(g(v(x)))^{-\gamma}g'(v(x))v(x)| &\leq h(x)(g(v(x)))^{1-\gamma}\\ &\leq C^{1-\gamma}h(x)v^{1-\gamma}(x), \qquad \forall x \in A_1\end{aligned}$$

and

$$\begin{aligned}|h(x)(g(v(x)))^{-\gamma}g'(v(x))v(x)| &\leq h(x)(g(v(x)))^{1-\gamma}\\ &\leq C^{1-\gamma}h(x)v^{(1-\gamma)/2}(x)\\ &\leq C^{1-\gamma}h(x), \qquad \forall x \in A_2,\end{aligned}$$

which shows (2.1), because $h|v|^{1-\gamma}, h \in L^1(\Omega)$.

$(b) \Rightarrow (c)$: By Lemma 2.1 (4) we obtain

$$\int_\Omega h(x)(g(v))^{1-\gamma} = \int_\Omega h(x)(g(v))^{-\gamma}g(v) \leq 2\int_\Omega h(x)(g(v))^{-\gamma}g'(v)v < \infty.$$

$(c) \Rightarrow (a)$: From Lemma 2.1 (5) we find

$$\int_\Omega h(x)|v|^{1-\gamma} \leq \int_\Omega h(x)(g(v))^{1-\gamma} < \infty.$$

The proof of the lemma is completed. □

From now on we will assume (1.7) and as a consequence, by Lemma 2.2 we obtain $D(J) \neq \emptyset$ and $D(\Phi) \neq \emptyset$. Moreover $D(J) = D(\Phi)$.

The fact that we are looking for positive solutions leads us to introduce the sets

$$V_+ = \left\{ v \in H_0^1(\Omega) \setminus \{0\} : v \geq 0 \right\}$$

and

$$D_+(J) = \{ v \in V_+ : v \in D(J) \}.$$

For each $v \in D_+(J)$ we define the fiber map $\phi_v : (0, \infty) \to \mathbb{R}$ by

$$\phi_v(t) := \Phi(tv) = \frac{t^2}{2} \int_\Omega |\nabla v|^2 + \frac{1}{\gamma - 1} \int_\Omega h(x)(g(tv))^{1-\gamma} - \int_\Omega F(x, g(tv)).$$

In what follows, we will study the main properties of the fiber maps.

Lemma 2.3. *If $v \in D_+(J)$, then $\phi_v \in C^1((0, \infty), \mathbb{R})$.*

Proof. It is clear that $\tilde{\Gamma} \in C^1((0, \infty), \mathbb{R})$, where

$$\tilde{\Gamma}(t) = \frac{t^2}{2} \int_\Omega |\nabla v|^2 - \int_\Omega F(x, g(tv)).$$

Therefore, it is sufficient to show that $\Gamma \in C^1((0, \infty), \mathbb{R})$, where Γ is defined by

$$\Gamma(t) = \int_\Omega h(x)(g(tv))^{1-\gamma}.$$

Let $t > 0$. For every $s > 0$, by the Mean Value Theorem there exists a measurable function $\theta = \theta(s, x) \in (0, 1)$ such that $t + \theta(s, x)s \to t$ as $s \to 0$ and

$$\Gamma(t + s) - \Gamma(t) = (1 - \gamma) \int_\Omega h(x)(g((t + \theta s)v))^{-\gamma} g'((t + \theta s)v) sv.$$

Since, by Lemma 2.1(9), (10), the function $g^{-\gamma} g'$ is decreasing on $(0, \infty)$ it follows that

$$(g((t + \theta s)v))^{-\gamma} g'((t + \theta s)v) \leq (g(tv))^{-\gamma} g'(tv) \quad \text{a.e. in } \Omega.$$

Furthermore, as a consequence of Lemma 2.2 we have $h(g(tv))^{-\gamma} g'(tv)v \in L^1(\Omega)$. Hence, applying the Lebesgue's dominated convergence theorem we obtain

$$\Gamma'(t) = \lim_{s \to 0} \frac{\Gamma(t + s) - \Gamma(t)}{s} = (1 - \gamma) \int_\Omega h(x)(g(tv))^{-\gamma} g'(tv)v,$$

that is, Γ is differentiable at t. Finally, using Lemma 2.2 and the Lebesgue's dominated convergence theorem we deduce that the function $\Gamma' : (0, \infty) \longrightarrow \mathbb{R}$ defined by

$$\Gamma'(t) = (1 - \gamma) \int_\Omega h(x)(g(tv))^{-\gamma} g'(tv)v,$$

is continuous, namely, $\Gamma \in C^1((0, \infty), \mathbb{R})$. The proof is complete. □

Our next result deals with the existence of global minima of ϕ_v, for every $v \in D_+(J)$.

Lemma 2.4. *If $v \in D_+(J)$, then there exists a $t(v) > 0$ such that*

$$\phi_v(t(v)) = \inf_{t > 0} \phi_v(t).$$

Proof. We only give here the proof for the case in which $(f)_1$ holds. The case that $(f)_2$ holds is similar.

First, we claim that

$$\lim_{t\to 0}\phi_v(t)=\infty \quad \text{and} \quad \lim_{t\to\infty}\phi_v(t)=\infty. \tag{2.2}$$

In fact, by Lemma 2.1 (5) we have

$$\int_\Omega h(x)(g(tv))^{1-\gamma}dx \geq t^{1-\gamma}\int_\Omega h(x)|v|^{1-\gamma}$$

and

$$t^{p+1}\int_\Omega |b(x)||v|^{p+1} \geq \left|\int_\Omega b(x)(g(tv))^{p+1}\right| \geq 0,$$

whence

$$\lim_{t\to 0}\int_\Omega h(x)(g(tv))^{1-\gamma}dx=\infty \quad \text{and} \quad \lim_{t\to 0}\int_\Omega b(x)(g(tv))^{p+1}=0.$$

Since $\gamma>1$, we deduce from this that $\lim_{t\to 0}\phi_v(t)=\infty$. Moreover, one has

$$\lim_{t\to\infty}\phi_v(t)\geq \lim_{t\to\infty}t^2\left[\|v\|^2-t^{p-2}\frac{\|b\|_\infty}{p+1}\int_\Omega |v|^{p+1}dx\right]=\infty,$$

that is, $\lim_{t\to\infty}\phi_v(t)=\infty$.

Finally, from the continuity of ϕ_v and (2.2) we deduce that there exists a $t(v)>0$ such that $\phi_v(t(v))=\inf_{t>0}\phi_v(t)$. This concludes the proof of the lemma. □

The following pictures give the possible graphs of the fiber maps.

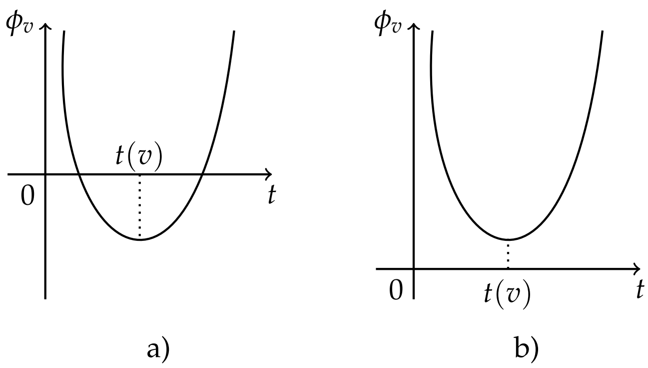

Figure 2.1: Possible graphs of the fiber maps.

Motivated by [25], we define the following constraint sets

$$\mathcal{N}_1=\left\{v\in V_+ : \|v\|^2-\int_\Omega f(x,g(v))g'(v)v \geq \int_\Omega h(x)(g(v))^{-\gamma}g'(v)v\right\}$$

and

$$\mathcal{N}_2=\left\{v\in V_+ : \|v\|^2-\int_\Omega f(x,g(v))g'(v)v = \int_\Omega h(x)(g(v))^{-\gamma}g'(v)v\right\}.$$

Observe that if v is a solution of (P_A) then $v\in\mathcal{N}_2$ and $\mathcal{N}_2\subset\mathcal{N}_1$.

It should be noted that for $\gamma>1$, $\mathcal{N}_2$ is not closed as usual (certainly not weakly closed).

We prove that every function in $D_+(J)$ may be projected on the set $\mathcal{N}_2$. In particular, $\mathcal{N}_1\neq\emptyset$.

Lemma 2.5. *For any $v \in D_+(J)$ we have $t(v)v \in \mathcal{N}_2$.*

Proof. From Lemma 2.4 we infer that $t(v)$ is a global minimum of ϕ_v and hence, by Lemma 2.3 one has $\phi_v'(t(v)) = 0$. Thus

$$\begin{aligned}0 &= t(v)\phi_v'(t(v)) \\ &= \|t(v)v\|^2 - \int_\Omega h(x)(g(t(v)v))^{-\gamma}g'(t(v)v)(t(v)v) - \int_\Omega f(x,t(v)v)g'(t(v)v)(t(v)v) = 0,\end{aligned}$$

namely, $t(v)v \in \mathcal{N}_2 \subset \mathcal{N}_1$. The proof is complete. □

We end this section with the following lemmas, which will be used to prove the regularity of the solutions.

Lemma 2.6. *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Let $u \in L^1_{loc}(\Omega)$ and assume that, for some $k \geq 0$, u satisfies, in the sense of distributions,*

$$\begin{cases} -\Delta u + ku \geq 0 & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega. \end{cases}$$

Then either $u \equiv 0$, or there exists $\epsilon > 0$ such that $u(x) \geq \epsilon d(x, \partial\Omega)$, $x \in \Omega$.

Proof. See Brezis–Nirenberg [3, Theorem 3]. □

Lemma 2.7. *Let $a \in L^1(\Omega)$ and suppose that there exist constants $\delta \in (0,1)$ and $C > 0$ such that $|a(x)| \leq C\phi_1^{-\delta}(x)$, for a.e. $x \in \Omega$. Then, the problem*

$$\begin{cases} -\Delta u = a & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $u \in H_0^1(\Omega)$. Furthermore, there exist constants $\alpha \in (0,1)$ and $M > 0$ depending only on C, α, Ω such that $u \in C^{1,\alpha}(\overline{\Omega})$ and $|u|_{1,\alpha} < M$.

Proof. See Hai [11, Lemma 2.1, Remark 2.2]. □

Remark 2.8. For future use we recall that there exist constants $l_1, l_2 > 0$ such that

$$l_1 d(x, \partial\Omega) \leq \phi_1(x) \leq l_2 d(x, \partial\Omega), \qquad x \in \Omega,$$

where ϕ_1 is the first eigenfunction of $(-\Delta, H_0^1(\Omega))$.

Lemma 2.9. *Let $\psi_j : \Omega \times (0,\infty) \longrightarrow [0,\infty)$, $j = 1,2$ are measurable functions such that*

$$\psi_1(x,s) \leq \psi_2(x,s) \quad \text{for all } (x,s) \in \Omega \times (0,\infty),$$

and for each $x \in \Omega$, the function $s \longmapsto \psi_1(x,s)s^{-1}$ is decreasing on $(0,\infty)$. Furthermore let $u, v \in H^1(\Omega)$, with $u \in L^\infty(\Omega), u > 0, v > 0$ on Ω are such that

$$-\Delta u \leq \psi_1(x,u) \quad \text{and} \quad -\Delta v \geq \psi_2(x,v) \quad \text{on } \Omega.$$

If $u \leq v$ on $\partial\Omega$ and $\psi_1(\cdot,u)$ (or $\psi_2(\cdot,u)$) belongs to $L^1(\Omega)$, then $u \leq v$ on Ω.

Proof. See Mohammed [20, Theorem 4.1]. □

3 Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. First, we shall show the existence of a global minimum of Φ on $\mathcal{N}_1$. For this purpose, we need the following lemma.

Lemma 3.1. *The set $\mathcal{N}_1$ is not empty and the functional Φ is coercive on $\mathcal{N}_1$.*

Proof. Since (1.7) holds, Lemmas 2.2 and 2.5 imply $\mathcal{N}_1 \neq \emptyset$. We now show that Φ is coercive on $\mathcal{N}_1$. Indeed, for every $v \in \mathcal{N}_1$,

$$\begin{aligned}\Phi(v) &= \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{\gamma-1}\int_\Omega h(x)(g(v))^{1-\gamma} - \frac{1}{p+1}\int_\Omega b(x)(g(v))^{p+1} \\ &\geq \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{\|b\|_\infty}{p+1}\int_\Omega (g(v))^{p+1},\end{aligned}$$

and from Lemma 2.1 (5) and Sobolev embedding we obtain

$$\Phi(v) \geq \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{\|b\|_\infty}{p+1}\int_\Omega |v|^{p+1} \geq \frac{\|v\|^2}{2} - C\frac{\|v\|^{p+1}}{p+1},$$

for some constant $C > 0$. Since $p \in (0,1)$ one infers that Φ is coercive on $\mathcal{N}_1$. □

As an immediate consequence of Lemma 3.1, we can deduce that

$$J_1 = \inf_{v\in\mathcal{N}_1} \Phi(v) \quad \text{and} \quad J_2 = \inf_{v\in\mathcal{N}_2} \Phi(v)$$

are well defined with $J_1, J_2 \in \mathbb{R}$ and $J_2 \geq J_1$.

We now prove that the infimum of Φ on $\mathcal{N}_1$ is attained.

Lemma 3.2. *There exists $v \in \mathcal{N}_2$ such that $J_1 = \Phi(v) = J_2$.*

Proof. Let $\{v_n\} \subset \mathcal{N}_1$ be a minimizing sequence for Φ. From Lemma 3.1 the sequence $\{v_n\} \subset \mathcal{N}_1$ is bounded and then, up to subsequences, there exists $v \in H_0^1(\Omega)$ such that

$$\begin{cases} v_n \rightharpoonup v & \text{in } H_0^1(\Omega), \\ v_n \longrightarrow v & \text{in } L^s(\Omega) \text{ for all } s \in (0, 2^*), \\ v_n \longrightarrow v & \text{a.e. in } \Omega. \end{cases}$$

Since $v_n > 0$ a.e. in Ω, we have $v \geq 0$ a.e. in Ω, that is, $v \in V_+$. From the definition of $\mathcal{N}_1$ and Lemma 2.1 (3), (4), (5) it follows that for some constant C one has

$$\begin{aligned}\frac{1}{2}\int_\Omega h(x)(g(v_n))^{1-\gamma} &\leq \int_\Omega h(x)(g(v_n))^{-\gamma} g'(v_n)v_n \\ &\leq \|v_n\|^2 - \int_\Omega b(x)(g(v_n))^p g'(v_n)v_n \\ &\leq \|v_n\|^2 + \int_\Omega |b(x)||v_n|^{p+1} \\ &\leq \|v_n\|^2 + c||v_n||^{p+1} \\ &\leq C.\end{aligned}$$

Therefore, using Fatou's lemma we get $\int_\Omega \theta(x) \le C < \infty$, where

$$\theta(x) = \begin{cases} h(x)(g(v(x)))^{1-\gamma}, & \text{if } v(x) \neq 0 \\ \infty, & \text{if } v(x) = 0. \end{cases}$$

Since $g(0) = 0$ (by Lemma 2.1 (2)) and $\int_\Omega \theta(x) < \infty$, it follows that $v > 0$ a.e. in Ω. Thus, using Fatou's lemma again, we obtain

$$0 < \int_\Omega h(x)(g(v))^{-\gamma} g'(v) v \le C$$

and this jointly with Lemma 2.2 imply that $v \in D_+(J)$. As a consequence, Lemmas 2.4 and 2.5 apply yielding a global minimum $t(v) > 0$ such that $\phi_v(t(v)) = \inf_{t>0} \phi_v(t)$ and $t(v)v \in \mathcal{N}_2$. Furthermore, we have

$$\begin{aligned}
J_1 &= \lim_{n\to\infty} \Phi(v_n) = \liminf_{n\to\infty} \Phi(v_n) \\
&= \liminf_{n\to\infty} \left[\frac{1}{2} \int_\Omega |\nabla v_n|^2 + \frac{1}{\gamma - 1} \int_\Omega h(x) g(v_n)^{1-\gamma} - \frac{1}{p+1} \int_\Omega b(x)(g(v_n))^{p+1} \right] \\
&\ge \liminf_{n\to\infty} \left[\frac{1}{2} \int_\Omega |\nabla v_n|^2 \right] + \liminf_{n\to\infty} \left[\frac{1}{\gamma-1} \int_\Omega h(x)(g(v_n))^{1-\gamma} \right] - \frac{1}{p+1} \int_\Omega b(x)(g(v))^{p+1} \\
&\ge \frac{1}{2} \int_\Omega |\nabla v|^2 + \frac{1}{\gamma-1} \int_\Omega h(x)(g(v))^{1-\gamma} - \frac{1}{p+1} \int_\Omega b(x)(g(v))^{p+1} = \phi_v(1) \\
&\ge \phi_v(t(v)) = \Phi(t(v)v) \\
&\ge J_2 \\
&\ge J_1.
\end{aligned}$$

Hence

$$J_1 = \phi_v(1) = \Phi(v) = J_2,$$

that is, $\phi_v(1) = \phi_v(t(v)) = \inf_{t>0} \phi_v(t)$. This implies $\phi_v'(1) = 0$ and consequently $v \in \mathcal{N}_2 \subset \mathcal{N}_1$. $\square$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. a) Necessity. Suppose that $u \in H_0^1(\Omega)$ is a solution of (P), by taking $\varphi = u$ in (1.1), we have

$$\int_\Omega h(x)|u|^{1-\gamma} < \infty.$$

Sufficiency. Let v be the global minimum obtained in Lemma 3.2. We will prove that v is a solution of (P_A). Let $\varphi \in H_0^1(\Omega)$, $\varphi \ge 0$. Applying Lemma 2.1 (10) we find

$$\int_\Omega h(x)(g(v+\epsilon\varphi))^{1-\gamma} \le \int_\Omega h(x)(g(v))^{1-\gamma} < \infty \qquad \forall \epsilon > 0,$$

namely, $v + \epsilon\varphi \in D_+(J)$ for every $\epsilon > 0$. Then, from Lemmas 2.4 and 2.5 there exists a $t(\epsilon) > 0$ such that $\phi_{v+\epsilon\varphi}(t(\epsilon)) = \inf_{t>0} \phi_{v+\epsilon\varphi}(t)$ and $t(\epsilon)(v+\epsilon\varphi) \in \mathcal{N}_2$. Therefore

$$\Phi(v+\epsilon\varphi) = \phi_{v+\epsilon\varphi}(1) \ge \phi_{v+\epsilon\varphi}(t(\epsilon)) = \Phi(t(\epsilon)(v+\epsilon\varphi)) \ge J_2 = \Phi(v),$$

that is,

$$\int_\Omega \frac{h(x)(g(v+\epsilon\varphi))^{1-\gamma} - h(x)(g(v))^{1-\gamma}}{1-\gamma} \leq \frac{\|v+\epsilon\varphi\|^2 - \|v\|^2}{2} - \int_\Omega \frac{b(x)(g(v+\epsilon\varphi))^{p+1} - b(x)(g(v))^{p+1}}{p+1}.$$

Thus, dividing both sides of the above inequality by $\epsilon > 0$, passing to the limit inferior as $\epsilon \longrightarrow 0$ and using Fatou's Lemma, we have

$$\begin{aligned}\int_\Omega h(x)(g(v))^{-\gamma}g'(v)\varphi &= \int_\Omega \liminf \frac{h(x)(g(v+\epsilon\varphi))^{1-\gamma} - h(x)(g(v))^{1-\gamma}}{1-\gamma} \\ &\leq \int_\Omega \nabla v \nabla \varphi - \int_\Omega b(x)(g(v))^p g'(v)\varphi. \end{aligned} \tag{3.1}$$

Finally, we can use an argument inspired by Graham–Eagle [10] to show that v is a solution of (P_A). Since $v \in \mathcal{N}_2$, one has

$$\|v\|^2 - \int_\Omega b(x)(g(v))^p g'(v)v - \int_\Omega h(x)(g(v))^{-\gamma}g'(v)v = 0.$$

For arbitrary $\varphi \in H_0^1(\Omega)$ and $\epsilon > 0$, set $\Psi = (v+\epsilon\varphi)^+$ and

$$\Omega_1^\epsilon = \{x \in \Omega : b(x) < 0 \text{ and } v(x) + \epsilon\varphi(x) < 0\}.$$

Then, inserting Ψ into (3.1) and using $v \in \mathcal{N}_2$, we obtain that

$$\begin{aligned} 0 &\leq \int_\Omega \nabla v \nabla \Psi - \int_\Omega b(x)(g(v))^p g'(v)\Psi - \int_\Omega h(x)(g(v))^{-\gamma}g'(v)\Psi \\ &= \int_{[v+\epsilon\varphi \geq 0]} \nabla v \nabla (v+\epsilon\varphi) - b(x)(g(v))^p g'(v)(v+\epsilon\varphi) - h(x)(g(v))^{-\gamma}g'(v)(v+\epsilon\varphi) \\ &= \left(\int_\Omega - \int_{[v+\epsilon\varphi<0]}\right) \nabla v \nabla (v+\epsilon\varphi) - b(x)(g(v))^p g'(v)(v+\epsilon\varphi) - h(x)(g(v))^{-\gamma}g'(v)(v+\epsilon\varphi) \\ &= \|v\|^2 - \int_\Omega b(x)(g(v))^p g'(v)v - \int_\Omega h(x)(g(v))^{-\gamma}g'(v)v \\ &\quad + \epsilon\left[\int_\Omega \nabla v \nabla \varphi - b(x)(g(v))^p g'(v)\varphi - h(x)(g(v))^{-\gamma}g'(v)\varphi\right] \\ &\quad - \int_{[v+\epsilon\varphi<0]} \nabla v \nabla (v+\epsilon\varphi) - b(x)(g(v))^p g'(v)(v+\epsilon\varphi) - h(x)(g(v))^{-\gamma}g'(v)(v+\epsilon\varphi) \\ &\leq \epsilon\left[\int_\Omega \nabla v \nabla \varphi - b(x)(g(v))^p g'(v)\varphi - h(x)(g(u))^{-\gamma}g'(v)\varphi\right] \\ &\quad - \epsilon\int_{[v+\epsilon\varphi<0]} \nabla v \nabla \varphi + \epsilon\int_{\Omega_1^\epsilon} b(x)(g(v))^p g'(v)\varphi. \end{aligned}$$

Since the measure of the domains of integration $[v+\epsilon\varphi < 0]$ and Ω_1^ϵ tends to zero as $\epsilon \to 0$, we then divide the above expression by $\epsilon > 0$ to obtain

$$0 \leq \int_\Omega \nabla v \nabla \varphi - b(x)(g(v))^p g'(v)\varphi - h(x)(g(v))^{-\gamma}g'(v)\varphi,$$

as $\epsilon \to 0$. Replacing φ by $-\varphi$ we conclude:

$$\int_\Omega \nabla v \nabla \varphi - b(x)(g(v))^p g'(v)\varphi - h(x)(g(v))^{-\gamma}g'(v)\varphi = 0, \qquad \forall \varphi \in H_0^1(\Omega),$$

and therefore v is a solution of (P_A). This means that $u = g(v)$ is a solution of problem (P). We complete the proof of a).

b) Suppose that v is a solution of (P_A). We will show that $v \in C^{1,\alpha}(\overline{\Omega})$ and hence, as $g \in C^\infty$ we get $u = g(v) \in C^{1,\alpha}(\overline{\Omega})$. Since $v \not\equiv 0$ satisfies in the sense of distributions

$$\begin{cases} -\Delta v \geq 0 & \text{in } \Omega, \\ v \geq 0 & \text{in } \Omega, \end{cases}$$

we can apply Lemma 2.6 yielding a $\epsilon > 0$ such that

$$v(x) \geq \epsilon d(x, \partial\Omega), \qquad x \in \Omega,$$
$$\epsilon d(x, \partial\Omega) < 1, \qquad x \in \Omega. \tag{3.2}$$

Then, by (1.8) and Lemma 2.1 (3), (8), (10) there exist constants $c, C > 0$ and $\beta \in (0,1)$ such that

$$\begin{aligned} |h(x)(g(v))^{-\gamma} g'(v)| &\leq h(x)(g(\epsilon d(x,\partial\Omega)))^{-\gamma} \leq h(x)C(\epsilon d(x,\partial\Omega))^{-\gamma} \\ &\leq C c d^{\gamma-\beta}(x,\partial\Omega) d^{-\gamma}(x,\partial\Omega) \\ &= C d^{-\beta}(x,\partial\Omega) \\ &\leq C\phi_1^{-\beta}(x) \end{aligned} \tag{3.3}$$

for every $x \in \Omega$, and hence $h(g(v))^{-\gamma} g'(v) \in L^1(\Omega)$. Thus, by Lemma 2.7 there exists a solution $\Psi_1 \in C^{1,\alpha_1}(\overline{\Omega})$, for some $\alpha_1 \in (0,1)$, of the problem

$$\begin{cases} -\Delta w = h(x)(g(v))^{-\gamma} g'(v) & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Next, we prove that the problem

$$\begin{cases} -\Delta w = b(x)(g(v))^{p} g'(v) & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

has a unique solution $\Psi_2 \in C^{1,\alpha_2}(\overline{\Omega})$, for some $\alpha_2 \in (0,1)$.

Let $\delta := 1 - p \in (0,1)$. From (3.2) and Lemma 2.1 (8), (12) we have

$$|b(x)g^p(v(x))g'(v(x))| \leq \|b\|_\infty g^{-\delta}(v(x))(g(v(x))g'(v(x))) \leq C\phi_1^{-\delta}(x),$$

that is,

$$|b(x)g^p(v(x))g'(v(x))| \leq C\phi_1^{-\delta}(x),$$

for every $x \in \Omega$ and some constant $C > 0$. Therefore, by Lemma 2.7 problem (3.4) has a unique solution $\Psi_2 \in C^{1,\alpha_2}(\overline{\Omega})$, for some $\alpha_2 \in (0,1)$.

We claim that $v = \Psi_1 + \Psi_2$. Indeed, using the fact that Ψ_1, Ψ_2 and v are solutions, we find

$$\int_\Omega \nabla v \nabla \varphi = \int_\Omega \left[h(x)(g(v))^{-\gamma} g'(v) + b(x)(g(v))^p g'(v) \right] \varphi = \int_\Omega \nabla(\Psi_1 + \Psi_2)\nabla\varphi,$$

for every $\varphi \in H_0^1(\Omega)$. Therefore, $v = \Psi_1 + \Psi_2$, and then $v \in C^{1,\alpha}(\overline{\Omega})$, where $\alpha := \min\{\alpha_1, \alpha_2\} \in (0,1)$. Thus, the claim follows, and consequently $u = g(v) \in C^{1,\alpha}(\overline{\Omega})$ showing the regularity of the solutions of (P).

Finally, we show the uniqueness of solution to (P). For this purpose, we show the uniqueness of solution to (P_A). Let v_1 and v_2 be two solutions of (P_A). We will prove that $v_1 \le v_2$ in Ω. First, let us set

$$j(x,s) := h(x)(g(s))^{-\gamma}g'(s) + b(x)(g(s))^p g'(s).$$

Fix $x \in \Omega$. According to Lemma 2.1 (9), (10), (11), the function $s \longmapsto j(x,s)s^{-1}$ is decreasing on $(0,\infty)$. Moreover, from (3.3) one has

$$0 \le j(x,v_i) \le C\phi_1^{-\beta}(x) + b(x)(g(v_i(x)))^p g'(v_i(x)), \qquad x \in \Omega,$$

hence $j(x,v_i) \in L^1(\Omega)$ for $i = 1,2$. Thus, we can use Lemma 2.9 with $\psi_i = j$ $(i = 1,2)$, $u = v_1$ and $v = v_2$ to get $v_1 \le v_2$ in Ω. Similarly we get $v_2 \le v_1$ in Ω, thus $v_1 = v_2$. This concludes the proof of the theorem.

Remark 3.3. If (1.8) holds, then problem (P) has a solution. Indeed, choose $v_0 = \phi_1 \in H_0^1(\Omega)$. From Remark 2.8 and (1.8) we have $h|\phi_1|^{1-\gamma} \le cl_1^{\beta-\gamma}|\phi_1|^{1-\beta} \in L^1(\Omega)$. Theorem 1.1 a) then guarantees the existence of a solution of (P).

4 Proof of Theorem 1.3

In this section, we assume $(f)_2$, that is, $f(x,s) = -b(x)s^{22^*-1}$ with $0 \le b \in L^\infty(\Omega)$ and $b \not\equiv 0$. Since the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact, the proof of Lemma 3.2 can not be applied directly here. In order to overcome this difficulty, we use the Brezis–Lieb Theorem (see [2]).

Now, we have

$$\Phi(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{\gamma - 1}\int_\Omega h(x)(g(v))^{1-\gamma} + \frac{1}{22^*}\int_\Omega b(x)(g(v))^{22^*},$$

for $v \in D(J)$. From (1.7) and Lemmas 2.2 and 2.5, one has $\mathcal{N}_1 \ne \emptyset$.

We will show the following.

Lemma 4.1. *The functional Φ is coercive on $\mathcal{N}_1$*

Proof. For every $v \in \mathcal{N}_1$, we have $\Phi(v) \ge \frac{1}{2}\|v\|^2$ and hence, Φ is coercive on $\mathcal{N}_1$. □

As an immediate consequence of Lemma 4.1, we can deduce that

$$J_1 = \inf_{v \in \mathcal{N}_1} \Phi(v) \quad \text{and} \quad J_2 = \inf_{v \in \mathcal{N}_2} \Phi(v)$$

are well defined with $J_1, J_2 \in \mathbb{R}$ and $J_2 \ge J_1$.

Next, we prove the following lemma.

Lemma 4.2. *There exists $v \in \mathcal{N}_2$ such that $J_1 = \Phi(v) = J_2$.*

Proof. Let $\{v_n\} \subset \mathcal{N}_1$ be a minimizing sequence for Φ. From Lemma 4.1 the sequence $\{v_n\} \subset \mathcal{N}_1$ is bounded in $H_0^1(\Omega)$, so in $L^{2^*}(\Omega)$ too, and then, up to subsequences, there exists $v \in H_0^1(\Omega)$ such that

$$\begin{cases} v_n \rightharpoonup v & \text{in } H_0^1(\Omega), \\ v_n \longrightarrow v & \text{in } L^s(\Omega) \text{ for all } s \in (0, 2^*), \\ v_n \longrightarrow v & \text{a.s. in } \Omega. \end{cases}$$

As a consequence, by Lemma 2.1 (6), there exists a constant $C > 0$ such that

$$\int_\Omega b(x)(g(v_n))^{22^*} = \int_\Omega \left[b^{\frac{1}{2^*}}\right]^{2^*} \left[(g(v_n))^2\right]^{2^*} \leq \|b\|_\infty K_0^{22^*} \int_\Omega |v_n|^{2^*} \leq C.$$

Moreover, $b(x)(g(v_n))^{22^*} \longrightarrow b(x)(g(v))^{22^*}$ a.s. in Ω. Hence, by virtue of the Brezis–Lieb Theorem (see [2]) it follows that

$$\begin{aligned} \int_\Omega b(x)(g(v_n))^{22^*} &= \int_\Omega b(x)(g(v))^{22^*} + \int_\Omega b(x)|(g(v_n))^{22^*} - (g(v))^{22^*}| + o(1) \\ &\geq \int_\Omega b(x)(g(v))^{22^*} + o(1). \end{aligned} \tag{4.1}$$

We can repeat the arguments used in Lemma 3.2 to prove the following.

- $v > 0$ a.e. in Ω and $\displaystyle\int_\Omega h(x)(g(v))^{-\gamma} g'(v)v < \infty$;
- there exists $t(v) > 0$ such that $t(v)v \in \mathcal{N}_2$.

Then, by (4.1) and the Fatou's lemma we find

$$\begin{aligned} J_1 &= \lim \Phi(v_n) \\ &= \liminf \left[\frac{1}{2}\int_\Omega |\nabla v_n|^2 + \frac{1}{\gamma - 1}\int_\Omega h(x)(g(v_n))^{1-\gamma} + \frac{1}{22^*}\int_\Omega b(x)(g(v_n))^{22^*}\right] \\ &\geq \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{\gamma - 1}\int_\Omega h(x)(g(v))^{1-\gamma} + \frac{1}{22^*}\int_\Omega b(x)(g(v))^{22^*} \\ &= \phi_v(1) \\ &\geq \phi_v(t(v)) = \Phi(t(v)v) \geq J_2 \geq J_1. \end{aligned}$$

Hence

$$J_1 = \phi_v(1) = \Phi(v) = J_2,$$

that is, $\phi_v(1) = \phi_v(t(v)) = \inf_{t>0} \phi_v(t)$. This implies $\phi_v'(1) = 0$ and consequently $v \in \mathcal{N}_2 \subset \mathcal{N}_1$. This ends the proof. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Necessity. Repeating the argument used to prove the corresponding claim in Theorem 1.1 *a*), the result follows.

Sufficiency. Let v be the global minimum obtained in Lemma 4.2. We will prove that v is a solution of (P_A). Let $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$ and $\epsilon > 0$. We can repeat the arguments used in Theorem 1.1 *a*) to prove the following.

- $h(\cdot)(g(v + \epsilon\varphi))^{1-\gamma} \in L^1(\Omega)$;

- there exists a $t(\epsilon) > 0$ such that $\phi_{v+\epsilon\varphi}(t(\epsilon)) = \inf_{t>0} \phi_{v+\epsilon\varphi}(t)$ and $t(\epsilon)(v+\epsilon\varphi) \in \mathcal{N}_2$;
- $\int_\Omega h(x)(g(v))^{-\gamma} g'(v)\varphi \le \int_\Omega \nabla v \nabla \varphi + \int_\Omega b(x)(g(v))^{22^*-1} g'(v)\varphi$.

From this information, as in Theorem 1.1 *a*), we can apply an argument inspired by Graham-Eagle [10] to get

$$\begin{aligned}
0 \le \|v\|^2 &+ \int_\Omega b(x)(g(v))^{22^*-1} g'(v)v - \int_\Omega h(x)(g(v))^{-\gamma} g'(v)v \\
&+ \epsilon \left[\int_\Omega \nabla v \nabla \varphi + b(x)(g(v))^{22^*-1} g'(v)\varphi - h(x)(g(v))^{-\gamma} g'(v)\varphi \right] \\
&- \int_{[v+\epsilon\varphi<0]} \nabla v \nabla (v+\epsilon\varphi) + b(x)(g(v))^{22^*-1} g'(v)(v+\epsilon\varphi) - h(x)(g(v))^{-\gamma} g'(v)(v+\epsilon\varphi) \\
\le \epsilon &\left[\int_\Omega \nabla v \nabla \varphi + b(x)(g(v))^{22^*-1} g'(v)\varphi - h(x)(g(v))^{-\gamma} g'(v)\varphi \right] \\
&- \epsilon \int_{[v+\epsilon\varphi<0]} \nabla v \nabla \varphi + b(x)(g(v))^{22^*-1} g'(v)\varphi,
\end{aligned}$$

for every $\varphi \in H_0^1(\Omega)$.

Since the measure of the domain of integration $[v+\epsilon\varphi<0]$ tends to zero as $\epsilon \to 0$, we then divide the above expression by $\epsilon > 0$ to obtain

$$0 \le \int_\Omega \nabla v \nabla \varphi - b(x)(g(v))^p g'(v)\varphi - h(x)(g(v))^{-\gamma} g'(v)\varphi,$$

as $\epsilon \to 0$. Replacing φ by $-\varphi$ we conclude:

$$\int_\Omega \nabla v \nabla \varphi - b(x)(g(v))^p g'(v)\varphi - h(x)(g(v))^{-\gamma} g'(v)\varphi = 0, \qquad \forall \varphi \in H_0^1(\Omega),$$

and therefore v is a solution of (P_A). This means that $u = g(v)$ is a solution of problem (P).

Finally, we show the uniqueness of solution to (P). For this purpose, we show the uniqueness of solution to (P_A). Let v_1 and v_2 be two solutions of (P_A). We will prove that $v_1 = v_2$ in Ω. First, let us set

$$j(x,t) = -b(x)(g(t))^{22^*-1} g'(t) + h(x)(g(t))^{-\gamma} g'(t),$$

for $x \in \Omega$ and $t > 0$. Note that $j(.,t)$ is decreasing by virtue of Lemma 2.1 (9), (10). Thus,

$$\|v_1 - v_2\|^2 = \int_\Omega (j(x,v_1) - j(x,v_2))(v_1 - v_2) < 0,$$

which yields $v_1 = v_2$. Hence, problem (P_A) has a unique solution. The proof of the theorem is complete.

Appendix A

Consider the problem

$$\begin{cases} -\Delta u - \Delta(u^2)u = h(x)u^{-\gamma} + \lambda b(x)u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $\lambda \geq 0, 0 < p < 1, 0 \leq b \in L^\infty(\Omega)$ and $b \not\equiv 0$.

This appendix is devoted to the study of some properties of the solutions of (P_λ). From now on we assume (1.7) holds. Therefore, by Theorem 1.1 problem (P_λ) has a solution, which we denote by u_λ.

The main result of this appendix is stated next.

Theorem A.1. *The following properties are valid:*

a) $u_\lambda \geq u_0$ *in* Ω *for every* $\lambda > 0$.

b) $u_\lambda \longrightarrow u_0$ *in* $H_0^1(\Omega)$ *as* $\lambda \longrightarrow 0$.

In order to prove Theorem A.1, we consider the problem

$$\begin{cases} -\Delta v = h(x)(g(v))^{-\gamma}g'(v) + \lambda b(x)(g(v))^p g'(v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{D_λ}$$

and denote by $v_\lambda = g^{-1}(u_\lambda)$ the solution obtained in the proof of Theorem 1.1.

Let Φ_λ the energy functional associated to (D_λ). For each $\lambda \geq 0$, let us set

$$\mathcal{N}_\lambda = \left\{ v \in V_+ : ||v||^2 - \int_\Omega \lambda b(g(v))^p g'(v)v \geq \int_\Omega h(x)(g(v))^{-\gamma}g'(v)v \right\}.$$

We can now state the key lemma for proving Theorem A.1.

Lemma A.2. *The following properties hold true:*

a) $v_\lambda \geq v_0$ *in* Ω.

b) $v_\lambda \longrightarrow v_0$ *in* $H_0^1(\Omega)$ *as* $\lambda \longrightarrow 0$.

c) $\lim\limits_{\lambda \to 0} \Phi_\lambda(v_\lambda) = \Phi_0(v_0) > 0$.

d) If (1.8) *holds, then the function* $[0, \infty) \ni \lambda \longmapsto \Phi_\lambda(v_\lambda)$ *is continuous and decreasing.*

Proof. *a)* Using the fact that v_0 and v_λ are solutions of (D_0) and (D_λ), respectively, and Lemma 2.1 (9), (10) we have

$$\begin{aligned} -\|(v_\lambda - v_0)^-\|^2 &= \int_\Omega ((g(v_\lambda))^{-\gamma}g'(v_\lambda) - (g(v_0))^{-\gamma}g'(v_0) + \lambda b(x)(g(v_\lambda))^p g'(v_\lambda))(v_\lambda - v_0)^- \\ &\geq \int_\Omega ((g(v_\lambda))^{-\gamma}g'(v_\lambda) - (g(v_0))^{-\gamma}g'(v_0))(v_\lambda - v_0)^- \\ &= \int_{\{v_\lambda < v_0\}} ((g(v_\lambda))^{-\gamma}g'(v_\lambda) - (g(v_0))^{-\gamma}g'(v_0))(v_\lambda - v_0)^- \geq 0. \end{aligned}$$

As a consequence one has $\|(v_\lambda - v_0)^-\| = 0$, which implies $v_\lambda \geq v_0$ in Ω.

b) Let $\{\lambda_n\} \subset (0, \infty)$ be a sequence such that $\lambda_n \to 0$ and denote by $v_{\lambda_n} = v_n$. We claim that $\{v_n\}$ is bounded in $H_0^1(\Omega)$. Indeed, since $\{v_n\} \subset \mathcal{N}_{\lambda_n}$ it follows that

$$\|v_n\|^2 = \int_\Omega h(x)(g(v_n))^{-\gamma}g'(v_n)v_n + \lambda_n \int_\Omega b(g(v_n))^p g'(v_n)v_n.$$

Thus, from Lemma 2.1 (4), (5), (10) and $v_n \geq v_0$ in Ω we get

$$\begin{aligned}\|v_n\|^2 &\leq \int_\Omega h(x)(g(v_n))^{1-\gamma} + \lambda_n \int_\Omega b(x)(g(v_n))^{p+1} \\ &\leq \int_\Omega h(x)(g(v_0))^{1-\gamma} + \lambda_n \int_\Omega b(x)|v_n|^{p+1} \\ &\leq \int_\Omega h(x)(g(v_0))^{1-\gamma} + \lambda_n C\|v_n\|^{p+1}\end{aligned}$$

and hence $\{v_n\}$ is bounded in $H_0^1(\Omega)$, because $0 < p < 1$.

Therefore, there exists $\psi \in H_0^1(\Omega), \psi \geq 0$ such that, up to a subsequence, we have

$$\begin{cases} v_n \rightharpoonup \psi & \text{in } H_0^1(\Omega), \\ v_n \to \psi & \text{in } L^s(\Omega) \text{ for all } s \in (0, 2^*), \\ v_n \to \psi & \text{a.s. in } \Omega. \end{cases}$$

As in the proof of Lemma 3.2, we derive that $\psi > 0$ in Ω. This implies that

$$h(x)(g(v_n))^{-\gamma} g'(v_n)(v_n - \psi) \to 0 \quad \text{a.s. in } \Omega,$$

and by virtue of Lemma 2.1 (4), (9), (10) and $v_n \geq v_0$ in Ω one finds

$$\begin{aligned}|h(x)(g(v_n))^{-\gamma} g'(v_n)(v_n - \psi)| &\leq h(x)(g(v_n))^{1-\gamma} + h(x)(g(v_n))^{-\gamma} g'(v_n)\psi \\ &\leq h(x)(g(v_0))^{1-\gamma} + h(x)(g(v_0))^{-\gamma} g'(v_0)\psi,\end{aligned}$$

where

$$h(x)(g(v_0))^{1-\gamma} + h(x)(g(v_0))^{-\gamma} g'(v_0)\psi \in L^1(\Omega),$$

because v_0 is a solution of (D_0). Hence, by the Lebesgue's dominated convergence theorem we get

$$\int_\Omega h(x)(g(v_n))^{-\gamma} g'(v_n)(v_n - \psi) \longrightarrow 0. \tag{A.1}$$

As a consequence of (A.1) we have

$$\begin{aligned}\lim_{n\to\infty}(v_n, v_n - \psi) &= \lim_{n\to\infty} \int_\Omega \nabla v_n \nabla (v_n - \psi) = \\ &= \lim_{n\to\infty}\left[\int_\Omega h(x)(g(v_n))^{-\gamma} g'(v_n)(v_n - \psi) + \lambda_n \int_\Omega b(x)(g(v_n))^p g'(v_n)(v_n - \psi)\right] \\ &= 0,\end{aligned}$$

and since $v_n \rightharpoonup \psi$, it follows that

$$\lim_{n\to\infty}\|v_n - \psi\|^2 = \lim_{n\to\infty}(v_n, v_n - \psi) - \lim_{n\to\infty}(\psi, v_n - \psi) = 0,$$

namely, $v_n \longrightarrow \psi$ in $H_0^1(\Omega)$ as $n \to \infty$.

To end the proof of b), it is sufficient to show that $\psi = v_0$. Indeed, because v_n is a solution of (D_{λ_n}) one has

$$\int_\Omega \nabla v_n \nabla \varphi = \int_\Omega h(x)(g(v_n))^{-\gamma} g'(v_n)\varphi + \lambda_n \int_\Omega b(x)(g(v_n))^p g'(v_n)\varphi, \tag{A.2}$$

for all $\varphi \in H_0^1(\Omega)$. Moreover, from $v_0 \leq v_n$ in Ω and Lemma 2.1 (9), (10) we find

$$h(x)(g(v_n))^{-\gamma}g'(v_n)\varphi \longrightarrow h(x)(g(\psi))^{-\gamma}g'(\psi)\varphi \quad \text{a.s. in } \Omega,$$

and

$$|h(x)(g(v_n))^{-\gamma}g'(v_n)\varphi| \leq h(x)(g(v_0))^{-\gamma}g'(v_0)\varphi.$$

Therefore, letting $n \to \infty$ in (A.2), and by using Lebesgue's dominated convergence theorem we obtain

$$\int_\Omega \nabla\psi\nabla\varphi = \int_\Omega h(x)(g(\psi))^{-\gamma}g'(\psi)\varphi,$$

for every $\varphi \in H_0^1(\Omega)$. This means that ψ is a solution of (D_0), and by uniqueness of solutions of (D_0) we deduce that $\psi = v_0$. This ends the proof of $b)$.

$c)$ From $a)$ and $b)$ it follows that $v_\lambda \geq v_0$ for all $\lambda > 0$ and $v_\lambda \longrightarrow v_0$ in $H_0^1(\Omega)$ as $\lambda \to 0$. Thus, reasoning as in $b)$, and by using Lebesgue's dominated convergence theorem we get $\lim_{\lambda\to 0}\Phi_\lambda(v_\lambda) = \Phi_0(v_0)$.

$d)$ We can argue as in $b)$ to show that the function is continuous. In order to prove that it is decreasing, let $0 \leq \lambda < \mu$. Then,

$$\Phi_\lambda(v_\lambda) > \Phi_\mu(v_\lambda) \geq \Phi_\mu(t_\mu(v_\lambda)v_\lambda) \geq \Phi_\mu(v_\mu),$$

that is, the function $[0,\infty) \ni \lambda \longmapsto \Phi_\lambda(v_\lambda)$ is decreasing. We complete the proof of the lemma. □

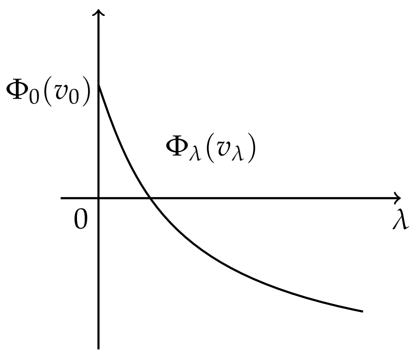

Figure A.1: Graph of function $[0,\infty) \ni \lambda \longmapsto \Phi_\lambda(v_\lambda)$.

We are now in a position to prove Theorem A.1.

Proof of Theorem A.1. $a)$ Let $u_\lambda = g(v_\lambda)$ and $u_0 = g(v_0)$. By Lemma A.2 $a)$ we have $v_\lambda \geq v_0$ in Ω, for every $\lambda \geq 0$. So, by virtue of Lemma 2.1 (3) we find

$$u_\lambda = g(v_\lambda) \geq g(v_0) = u_0 \text{ in } \Omega.$$

This finishes the proof of $a)$.

$b)$ We first observe that $\nabla u_\lambda = g'(v_\lambda)\nabla v_\lambda$, for each $\lambda \geq 0$. Then, as a consequence of the inequality $(x+y)^2 \leq 2(x^2+y^2)$, for $x, y \geq 0$, and Lemma 2.1(3) we get

$$\begin{aligned}\int_\Omega |\nabla u_\lambda - \nabla u_0|^2 &= \int_\Omega |g'(v_\lambda)\nabla v_\lambda - g'(v_0)\nabla v_0|^2 \\ &\leq \int_\Omega (g'(v_\lambda)|\nabla v_\lambda - \nabla v_0| + |g'(v_\lambda) - g'(v_0)||\nabla v_0|)^2 \\ &\leq 2\int_\Omega (g'(v_\lambda))^2 |\nabla v_\lambda - \nabla v_0|^2 + 2\int_\Omega |g'(v_\lambda) - g'(v_0)|^2 |\nabla v_0|^2 \\ &\leq 2\int_\Omega |\nabla v_\lambda - \nabla v_0|^2 + 2\int_\Omega |g'(v_\lambda) - g'(v_0)|^2 |\nabla v_0|^2.\end{aligned}$$

Hence, it is sufficient to prove that

$$\int_\Omega |\nabla v_\lambda - \nabla v_0|^2 \longrightarrow 0 \quad \text{and} \quad \int_\Omega |g'(v_\lambda) - g'(v_0)|^2 |\nabla v_0|^2 \longrightarrow 0, \quad \text{as } \lambda \to 0.$$

We already know (see Lemma A.2 *b*)) that $\int_\Omega |\nabla v_\lambda - \nabla v_0|^2 \longrightarrow 0$ as $\lambda \to 0$. Moreover, as $g'(t) \leq 1$ for every $t \geq 0$, we can apply Lebesgue's dominated convergence to infer that

$$\int_\Omega |g'(v_\lambda) - g'(v_0)|^2 |\nabla v_0|^2 \longrightarrow 0 \quad \text{as } \lambda \to 0.$$

This completes the proof of Theorem A.1.

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Existence and uniqueness of positive solutions for Kirchhoff type beam equations

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Abstract. This paper is concerned with the existence and uniqueness of positive solutions for the fourth order Kirchhoff type problem

$$\begin{cases} u''''(x) - \left(a + b\int_0^1 (u'(x))^2 dx\right) u''(x) = \lambda f(u(x)), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

where $a > 0, b \geq 0$ are constants, $\lambda \in \mathbb{R}$ is a parameter. For the case $f(u) \equiv u$, we use an argument based on the linear eigenvalue problems of fourth order equations and their properties to show that there exists a unique positive solution for all $\lambda > \lambda_{1,a}$, here $\lambda_{1,a}$ is the first eigenvalue of the above problem with $b = 0$; for the case f is sublinear, we prove that there exists a unique positive solution for all $\lambda > 0$ and no positive solution for $\lambda < 0$ by using bifurcation method.

Keywords: fourth order boundary value problem, Kirchhoff type beam equation, global bifurcation, positive solution, uniqueness.

2020 Mathematics Subject Classification: 34B10, 34B18.

1 Introduction

Consider the following nonlinear fourth order Kirchhoff type problem

$$\begin{cases} u''''(x) - \left(a + b\int_0^1 (u'(x))^2 dx\right) u''(x) = \lambda f(u(x)), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1.1}$$

where $a > 0, b \geq 0$ are constants, $\lambda \in \mathbb{R}$ is a parameter, $f : \mathbb{R} \to \mathbb{R}$ is continuous. Due to the presence of the integral term $(b\int_0^1 (u'(x))^2 dx)u''(x)$, the equation is not a pointwise identity and therefore is a nonlocal integro-differential problem.

Problem (1.1) describes the bending equilibrium of an extensible beam of length 1 which is simply supported at two endpoints $x = 0$ and $x = 1$. The right side term $\lambda f(u)$ in equation

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stands for a force exerted on the beam by the foundation. In fact, (1.1) is related to the stationary problem associated with

$$\frac{\partial^2 u}{\partial t^2}+\frac{EI}{\rho A}\frac{\partial^4 u}{\partial x^4}-\left(\frac{H}{\rho}+\frac{E}{2\rho L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

which was proposed by Woinowsky-Krieger [29] as a model for the deflection of an extensible beam of length L with hinged ends. In (1.2), $u=u(x,t)$ is the lateral displacement at the space coordinate x and the time t; the letters H, E, ρ, I and A denote, respectively, the tension in the rest position, the Young elasticity modulus, the density, the cross-sectional moment of inertia and the cross-sectional area. The nonlinear term in the brackets is a correction to the classical Euler-Bernoulli equation

$$\frac{\partial^2 u}{\partial t^2}+\frac{EI}{\rho A}\frac{\partial^4 u}{\partial x^4}=0,$$

which does not consider the changes of the tension induced by the variation of the length during the deflection. This kind of correction was proposed by Kirchhoff [9] to generalize D'Alembert's equation with clamped ends. For this reason (1.1) is often called a Kirchhoff type beam equation. Other problems involving fourth-order equations of Kirchhoff type can be found in [7,19].

In the study of problem (1.1) and its generalizations, the nonlocal term under the integral sign causes some mathematical difficulties which make the study of the problem particularly interesting. The existence and multiplicity of solutions for (1.1) and its multi-dimensional case have been studied by several authors, see [13,15–18,27,28,30] and the references there in. Meanwhile, numerical methods of (1.1) have been developed in [3,4,20,21,23,25,26,32].

In [15–17], by using variational methods, Ma considered existence and multiplicity of solutions for (1.1) with $\lambda\equiv 1$ under different nonlinear boundary conditions. In [18], based on the fixed point theorems in cones of ordered Banach spaces, Ma studied existence and multiplicity of positive solutions results for (1.1) with right side term $f(x,u,u')$ in equation.

For multi-dimensional case of (1.1) with $\lambda\equiv 1$, Wang et al. studied the existence and multiplicity of nontrivial solutions by using the mountain pass theorem and the truncation method in [27,28]; for a kind of problem similar to (1.1) in $\mathbb{R}^3$, Xu and Chen [30] established the existence and multiplicity of negative energy solutions based on the genus properties in critical point theory, and very rencently, Mao and Wang [13] studied the existence of nontrivial mountain-pass type of solutions via the Mountain Pass lemma.

It is worth noticing that, in the above mentioned research work, the uniqueness of solutions for the problem has not been discussed. As far as the author knows, there are very few results on the uniqueness of solutions for problem (1.1). In [3], when the right side term $\lambda f(u(x))=g(x)$ is nonpositive, Dang and Luan proved that problem (1.1) has a unique solution by reducing the problem to a nonlinear equation and proposed an iterative method for finding the solution. Very recently, by using contraction mapping principle, Dang and Nguyen [4] obtained a uniqueness result for (1.1) in multi-dimensional case with the right side term $\lambda f(u(x))=g(x,u)$ is bounded. To the best of our knowledge, apart from the two works mentioned above, there is no other result on the uniqueness of solutions for nonlocal integro-differential problem (1.1).

Motivated by the above described works, the object of this paper is to study the existence and uniqueness of positive solutions for (1.1), and our main tool is bifurcation method. It should be emphasized that, global bifurcation phenomena for fourth order problem (1.1) with

$b = 0$ have been investigated in [10, 14, 24], and [1, 5, 8, 11, 12] studied second order Kirchhoff type problem by using the bifurcation theory, but as far as we know, the bifurcation phenomena for fourth order Kirchhoff problem (1.1) has not been discussed.

Concretely, in the present paper we are concerned with problem (1.1) under the two cases: $f(u) \equiv u$ or f is sublinear. For $f(u) \equiv u$, (1.1) can be seen as a nonlinear eigenvalue problem, we use an argument based on the linear eigenvalue problems of fourth order equations and their properties to show that there exists a unique positive solution for all $\lambda > \lambda_{1,a}$, where $\lambda_{1,a}$ is the first eigenvalue of (1.1) with $b = 0$; for the case f is sublinear, such as $f(u) = c_1 u^p + c_2 u^q$ ($c_1, c_2 \geq 0$, $0 < p, q < 1$, see Remark 4.1), we prove that there exists a unique positive solution for all $\lambda > 0$ and no positive solution for $\lambda < 0$ by using bifurcation method.

The rest of the paper is arranged as follows: In Section 2, as preliminaries, we first construct the operator equation corresponding to (1.1). In Section 3, we deal with the case $f(u) \equiv u$ based on the linear eigenvalue problem of fourth order equations and their properties. Finally, for the case f is sublinear, we discuss the existence and uniqueness of positive solutions for (1.1) by using bifurcation method in Section 4.

2 Preliminaries

Let $P := \{u \in C[0,1] : u(x) \geq 0, \forall\ x \in [0,1]\}$ be the positive cone in $C[0,1]$ and let $U := P \cup (-P)$. A solution to problem (1.1) is a function $u \in C^4[0,1]$ which satisfies the equation and boundary conditions, and moreover, if $u \in C^4[0,1] \cap P$ we call u a positive solution.

Proposition 2.1. *For each $g \in C[0,1]$, there exists a solution $u \in C^4[0,1]$ to the problem*

$$\begin{cases} u''''(x) - (a + b\int_0^1 (u'(x))^2 dx) u''(x) = g(x), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{2.1}$$

and if $g \in U$, then u is unique. Moreover, the operator $T : U \to U$ defined by

$$T(g) := u$$

is positive and compact.

Proof. First, when $g \equiv 0$, we prove that (2.1) has only a unique solution $u \equiv 0$. Assume that u is a solution of (2.1) with $g \equiv 0$, set $w = -u''$, then by (2.1) we have

$$\begin{cases} -w''(x) + (a + b\int_0^1 (u'(x))^2 dx) w(x) = 0, & x \in (0,1), \\ w(0) = w(1) = 0, \end{cases} \tag{2.2}$$

$$\begin{cases} -u''(x) = w(x), & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{2.3}$$

We claim that the solution of (2.2) is $w \equiv 0$. In fact, suppose on the contrary that $w \not\equiv 0$ is a solution of (2.2), and without loss of generality, $w(\tau) = \max\{w(x) | x \in [0,1]\} > 0$ for some $\tau \in (0,1)$, then we have $w''(\tau) \leq 0$, which contradicts with $w''(\tau) = (a + b\int_0^1 (u'(x))^2 dx) w(\tau) > 0$. Substituting $w \equiv 0$ in (2.3), $u \equiv 0$ is an immediate conclusion.

Next, we prove the existence and uniqueness of solutions for (2.1) with $g \neq 0$. For any constant $R \geq 0$, let u_R stands for the unique solution of the linear fourth order problem

$$\begin{cases} u''''(x) - (a + bR) u''(x) = g(x), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{2.4}$$

then

$$u_R(x) = \int_0^1 \int_0^1 G_1(x,t) G_{2,R}(t,s) g(s) ds dt, \qquad x \in [0,1], \tag{2.5}$$

$$u_R''(x) = -\int_0^1 G_{2,R}(x,t) g(t) dt, \qquad x \in [0,1], \tag{2.6}$$

here

$$G_1(x,t) = \begin{cases} t(1-x), & 0 \le t \le x \le 1, \\ x(1-t), & 0 \le x \le t \le 1, \end{cases} \tag{2.7}$$

and

$$G_{2,R}(t,s) = \begin{cases} \frac{\sinh(\sqrt{a+bR}t)\sinh(\sqrt{a+bR}(1-s))}{\sqrt{a+bR}\sinh\sqrt{a+bR}}, & 0 \le t \le s \le 1, \\ \frac{\sinh(\sqrt{a+bR}s)\sinh(\sqrt{a+bR}(1-t))}{\sqrt{a+bR}\sinh\sqrt{a+bR}}, & 0 \le s \le t \le 1, \end{cases} \tag{2.8}$$

are Green functions of

$$\begin{cases} -u''(x) = 0, & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{2.9}$$

and

$$\begin{cases} -w''(t) + (a+bR)w(t) = 0, & t \in (0,1), \\ w(0) = w(1) = 0, \end{cases} \tag{2.10}$$

respectively. Since $0 \le G_1(x,t) \le G_1(x,x)$ and $0 \le G_{2,R}(t,s) \le G_{2,R}(t,t) \le \frac{(\sinh\frac{\sqrt{a}}{2})^2}{\sqrt{a}\sinh\sqrt{a}}$, then by (2.5)–(2.8) we have that there exist two positive constants C_1 and C_1 such that

$$\|u_R\|_\infty \le C_1 \|g\|_\infty, \qquad \|u_R''\|_\infty \le C_2 \|g\|_\infty. \tag{2.11}$$

Multiplying the equation in (2.4) by u_R and integrating it over $[0,1]$, based on boundary conditions and integration by parts we obtain

$$\int_0^1 (u_R'(x))^2 dx = \frac{\int_0^1 g(x) u_R(x) dx - \int_0^1 (u_R''(x))^2 dx}{a+bR}. \tag{2.12}$$

Now to get a solution of (2.1), we only need to find R such that

$$R = y(R) := \frac{\int_0^1 g(x) u_R(x) dx - \int_0^1 (u_R''(x))^2 dx}{a+bR} = \int_0^1 (u_R'(x))^2 dx, \tag{2.13}$$

that is, find a fixed point of $R = y(R)$. Obviously, $y(0) > 0$. On the other hand, by (2.11) we have

$$|y(R)| = \frac{\left| \int_0^1 g(x) u_R(x) dx - \int_0^1 (u_R''(x))^2 dx \right|}{a+bR} \le \frac{C_1 \|g\|_\infty^2 + C_2^2 \|g\|_\infty^2}{a} \le C. \tag{2.14}$$

This concludes the existence of fixed point for $R = y(R)$ which yields a solution u of (2.1) in $C^4[0,1]$.

Now, we show that if $g \in U$, the solution of (2.1) is unique. Without loss of generality, we assume on the contrary that for some $g \in P$, there exist two solutions $u \ne v$. By (2.5) and (2.6), we have

$$u \ge 0, \quad u'' \le 0; \qquad v \ge 0, \quad v'' \le 0. \tag{2.15}$$

Since u and v satisfy the equation in (2.1), we have

$$u''''(x) - v''''(x) - \left[a + b\int_0^1 (u'(x))^2 dx\right](u''(x) - v''(x)) \\ - b\left[\int_0^1 (u'(x))^2 dx - \int_0^1 (v'(x))^2 dx\right] v''(x) = 0. \quad (2.16)$$

Set $w = -(u'' - v'')$. If $\int_0^1 (u'(x))^2 dx = \int_0^1 (v'(x))^2 dx$, then (2.2) holds for $w = -(u'' - v'')$ and consequently we can obtain $u \equiv v$ arguing as above. If we assume that $\int_0^1 (u'(x))^2 dx > \int_0^1 (v'(x))^2 dx$, then by (2.16) and (2.15) we have

$$u''''(x) - v''''(x) - \left[a + b\int_0^1 (u'(x))^2 dx\right](u''(x) - v''(x)) \le 0, \quad (2.17)$$

that is

$$-w''(x) + \left[a + b\int_0^1 (u'(x))^2 dx\right] w(x) \le 0. \quad (2.18)$$

We claim that (2.18) implies $w \le 0$. In fact, suppose on the contrary that $w(\tau) = \max\{w(x) | x \in [0,1]\} > 0$ for some $\tau \in (0,1)$, then $w''(\tau) \le 0$. This contradicts with (2.18) with $x = \tau$. On the other hand, based on boundary conditions and integration by parts, from the assumption

$$\begin{aligned} \int_0^1 (u'(x))^2 dx - \int_0^1 (v'(x))^2 dx &= \int_0^1 [u'(x) + v'(x)][u'(x) - v'(x)] dx \\ &= -\int_0^1 (u(x) + v(x))(u''(x) - v''(x)) dx \\ &= \int_0^1 (u(x) + v(x)) w(x) dx > 0. \end{aligned} \quad (2.19)$$

Since (2.15) guarantees that $u(x) + v(x) \ge 0$, then (2.19) contradicts with $w \le 0$. The uniqueness of solutions for (2.1) is proved.

At the end, let $T : U \to C[0,1]$ be the operator defined by $Tg = u$, where u is the solution of (2.1). By (2.5) and the positiveness of Green functions $G_1(x,t)$, $G_{2,R}(t,s)$ in (2.7) and (2.8), we conclude that T is a positive operator, that is $T : U \to U$. Now, we show that T is compact. Without loss of generality, let $B \subseteq P$ be any bounded set. Combining (2.5) with (2.11) we can see that TB is a bounded set in P; On the other hand, (2.6) with (2.11) imply that TB is bounded in $C^2[0,1]$ and then we can deduce that TB is equicontinuous. Consequently, by Arzelà–Ascoli theorem we have that $T : P \to P$ is a completely continuous operator. Therefore $T : U \to U$ is a compact operator and the proof is completed. □

Remark 2.2. When $g(x)$ is nonpositive, Dang and Luan [3] proved that problem (2.1) has a unique solution by reducing the problem to a nonlinear equation. Compared with [3], our proof in 2.1 is more concise.

3 Nonlinear eigenvalue problem

In this section, we study (1.1) with $f(u) \equiv u$, that is the nonlinear eigenvalue problem

$$\begin{cases} u''''(x) - (a + b\int_0^1 (u'(x))^2 dx) u''(x) = \lambda u(x), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \quad (3.1)$$

The solutions of (3.1) are closely related to the following linear eigenvalue problem:

$$\begin{cases} u''''(x) - Au''(x) = \lambda u(x), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{3.2}$$

In [6], Del Pino and Manásevich proposed that: a pair of constants (λ, A) such that (3.2) possesses a nontrivial solution will be called an eigenvalue pair, and the corresponding nontrivial solution will be called an eigenfunction. Furthermore, they proved that the eigenvalue pair (λ, A) of (3.2) must satisfy

$$\frac{\lambda}{(k\pi)^4} - \frac{A}{(k\pi)^2} = 1, \quad \text{for some } k \in \mathbb{N},$$

and the corresponding eigenfunction is $\varphi_k = c \sin k\pi x (c \neq 0$ is an arbitrary constant).

Now, given a positive constant A, we use $\lambda_{1,A}$ to denote the principal eigenvalue of problem (3.2), then we have the following results:

Lemma 3.1. *(i) If A_1, A_2 are positive constants such that $A_1 < A_2$, then $\lambda_{1,A_1} < \lambda_{1,A_2}$.*
(ii) Let B, C be two fixed positive constants. Consider the map

$$\lambda_1(\mu) := \lambda_{1,B+\mu C}, \qquad \mu \geq 0,$$

then $\lambda_1(\cdot)$ is a continuous and strictly increasing function and

$$\lambda_1(0) = \lambda_{1,B}, \qquad \lim_{\mu \to +\infty} \lambda_1(\mu) = +\infty.$$

Proof. By [6], we know that the principal eigenvalue $\lambda_{1,A}$ of (3.2) satisfy

$$\frac{\lambda_{1,A}}{\pi^4} - \frac{A}{\pi^2} = 1, \tag{3.3}$$

and the corresponding first eigenfunction is $\varphi_1(x) = c \sin \pi x$, where $c \neq 0$ is an arbitrary constant. According to (3.3), $\lambda_{1,A} = (1 + \frac{A}{\pi^2})\pi^4$, then (i) and (ii) are immediate consequences. □

By using Lemma 3.1, we prove the following results on the nonlinear eigenvalue problem (3.1).

Theorem 3.2. *Problem (3.1) has a positive solution u_λ if and only if $\lambda \in (\lambda_{1,a}, +\infty)$, moreover, the solution u_λ is unique and satisfying*

$$\lim_{\lambda \to \lambda_{1,a}} \|u_\lambda\|_\infty = 0, \qquad \lim_{\lambda \to +\infty} \|u_\lambda\|_\infty = +\infty. \tag{3.4}$$

Proof. Assume that u is a positive solution of (3.1), then $\int_0^1 (u'(x))^2 dx > 0$, consequently by Lemma 3.1 (i) we have

$$\lambda = \lambda_{1,a+b\int_0^1 (u'(x))^2 dx} > \lambda_{1,a}.$$

To any $\lambda \in (\lambda_{1,a}, +\infty)$, by Lemma 3.1 (ii), there exists a unique $t_0(\lambda) > 0$ such that

$$\lambda_{1,a+bt_0} = \lambda,$$

moreover,

$$\lim_{\lambda \to \lambda_{1,a}} t_0(\lambda) = 0, \qquad \lim_{\lambda \to +\infty} t_0(\lambda) = +\infty. \tag{3.5}$$

For the fixed t_0, take appropriate principal eigenfunction $\varphi_1(x) = c\sin \pi x (c > 0)$ of (3.2) associated to $\lambda_{1,a+bt_0}$ such that

$$\int_0^1 (\varphi_1'(x))^2 dx = t_0. \tag{3.6}$$

Then it is easy to see that $u_\lambda = \varphi_1 > 0$ is a positive solution of (3.1).

To prove the uniqueness, we assume that there exist two positive solutions $u \neq v$, since

$$\lambda = \lambda_{1,a+b\int_0^1 (u'(x))^2 dx} = \lambda_{1,a+b\int_0^1 (v'(x))^2 dx'}$$

then Lemma 3.1 (ii) guarantees that $\int_0^1 (u'(x))^2 dx = \int_0^1 (v'(x))^2 dx$ and u is proportional to v, which implies that $u = v$.

Finally, we prove (3.4). Since the unique positive solution of (3.1) is $u_\lambda = \varphi_1(x) = c_\lambda \sin \pi x$, where c_λ is a positive constant depending on λ, then by (3.6) and (3.5), we have

$$\lim_{\lambda\to\lambda_{1,a}} \int_0^1 (u_\lambda'(x))^2 dx = \lim_{\lambda\to\lambda_{1,a}} \int_0^1 [(c_\lambda \sin \pi x)']^2 dx = \lim_{\lambda\to\lambda_{1,a}} \frac{1}{2} c_\lambda^2 \pi^2 \to 0, \tag{3.7}$$

and similarly

$$\lim_{\lambda\to+\infty} \int_0^1 (u_\lambda'(x))^2 dx = \lim_{\lambda\to+\infty} \frac{1}{2} c_\lambda^2 \pi^2 \to \infty, \tag{3.8}$$

that is

$$\lim_{\lambda\to\lambda_{1,a}} c_\lambda \to 0, \qquad \lim_{\lambda\to+\infty} c_\lambda \to +\infty, \tag{3.9}$$

then (3.4) is an immediate consequence. □

4 The sublinear case

In this section, we study (1.1) when the nonlinear term f is sublinear which means that f satisfying:

(H1) $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(s) > 0$ for all $s > 0$, $f(0) = 0$ and $f_0 := \lim\limits_{s\to 0+} \frac{f(s)}{s} = +\infty$;

(H2) $f_\infty := \lim\limits_{s\to+\infty} \frac{f(s)}{s} = 0$.

The main tool we will use in this section is global bifurcation theory.

We first state some notation. Let $X := \{u \in C^2[0,1] : u(0) = u(1) = u''(0) = u''(1) = 0\}$ with the norm $\|u\|_X = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}$. $B_\rho := \{u \in X : \|u\|_X < \rho\}$. For any $u \in X$, denote $u^+ = \max\{u, 0\}$. Define the operator $F : \mathbb{R} \times X \mapsto X$ by

$$F(\lambda, u)(x) := T(\lambda f(u^+(x))), \tag{4.1}$$

where T is the operator defined in Proposition 2.1. Obviously, if u is a nonnegative solution of (1.1), then u satisfies

$$u = F(\lambda, u). \tag{4.2}$$

On the other hand, if u is a solution of (4.2), we show that u must be a nonnegative solution of (1.1). In fact, by (H1) we have $f(u^+) \geq 0$ for any $u \in C[0,1]$. Then the positiveness of the operator T yields that the solution of (4.2) must be nonnegative or nonpositive according to $\lambda \geq 0$ or $\lambda \leq 0$. If we assume that the latter happens, that is, $u(x) \leq 0, \forall x \in [0,1]$, then

$f(u^+) \equiv 0$ and consequently (4.2) implies that $u \equiv 0$. From the above discussion, u is a nonnegative solution of (1.1) if and only if (4.2) holds.

Since the map from X into $U := P \cup (-P)$ defined by $u \mapsto \lambda f(u^+)$ is continuous, and $C^4[0,1] \cap X$ is compactly imbedded in X, then by Proposition 2.1, the operator $F : \mathbb{R} \times X \mapsto X$ as in (4.1) is completely continuous. In order to prove the main result of this section, we need the following lemmas.

Lemma 4.1. *For any fixed $\lambda < 0$, there exists a number $\rho > 0$ such that*

$$\deg(I - F(\lambda, \cdot), B_\rho(0), 0) = 1.$$

Proof. First, we claim that there exists $\delta > 0$ such that

$$u \neq tF(\lambda, u) = tT(\lambda f(u^+)) \quad \text{for all } u \in \overline{B_\delta},\ u \neq 0 \quad \text{and } t \in [0,1].$$

Suppose on the contrary that there exists a sequence $\{u_n\}$ in $X \setminus 0$ with $\|u_n\|_X \longrightarrow 0$ and $\{t_n\}$ in $[0,1]$ such that

$$u_n = t_n F(\lambda, u_n) = t_n T(\lambda f(u_n^+)),$$

that is

$$u_n''''(x) - \left(a + b\int_0^1 (u_n'(x))^2 dx\right) u_n''(x) = t_n \lambda f(u_n^+(x)) \leq 0, \qquad x \in (0,1). \tag{4.3}$$

Set $w_n = -u_n''$, then by (4.3) we can get an inequality for w_n similar to (2.18), which can deduce that $w_n \leq 0$. Consequently, $-u_n'' = w_n \leq 0$ and $u_n(0) = u_n(1) = 0$ guarantee that $u_n \leq 0$, which implies $f(u_n^+) \equiv 0$ according to (H1). Then by Proposition 2.1, (4.3) has only a unique solution $u_n \equiv 0$, a contradiction with $u_n \in X \setminus 0$.

Take $\rho \in (0, \delta]$, according to the homotopy invariance of topological degree and the normalization property, we have

$$\deg(I - F(\lambda, \cdot), B_\rho(0), 0) = \deg(I, B_\rho(0), 0) = 1.$$

□

Lemma 4.2. *For any fixed $\lambda > 0$, there exists a number $\rho > 0$ such that*

$$\deg(I - F(\lambda, \cdot), B_\rho(0), 0) = 0.$$

Proof. First, take a $\psi \in X, \psi > 0$, we claim that there exists $\delta > 0$ such that

$$u \neq T(\lambda f(u^+) + t\psi) \quad \text{for all } u \in \overline{B_\delta},\ u \neq 0 \quad \text{and } t \in [0,1].$$

Suppose on the contrary that there exists a sequence $\{u_n\}$ in $X \setminus 0$ with $\|u_n\|_X \longrightarrow 0$ and $\{t_n\}$ in $[0,1]$ such that

$$u_n = T(\lambda f(u_n^+) + t_n\psi),$$

that is

$$u_n''''(x) - \left(a + b\int_0^1 (u_n'(x))^2 dx\right) u_n''(x) = \lambda f(u_n^+(x)) + t_n\psi(x), \qquad x \in (0,1), \tag{4.4}$$

Since $t_n\psi(x) > 0, \forall x \in (0,1)$, from the similar argument in Lemma 4.1 we have that $u_n(x) > 0, \forall x \in (0,1)$.

On the other hand, $\|u_n\|_X \longrightarrow 0$ implies that

$$\int_0^1 (u_n'(x))^2 dx \le C$$

for some positive constant C. Hence, according to Lemma 3.1 we have that

$$\lambda_{1,a+b\int_0^1 (u_n'(x))^2 dx} \le \lambda_{1,a+bC} =: \Lambda.$$

Fix this value of Λ, since $\|u_n\|_\infty \longrightarrow 0$, then according to (H1), for n large we have that $\lambda f(u_n^+(x)) > \Lambda u_n(x), \forall x \in (0,1)$. Combining this with $u_n''(x) \le 0, \forall x \in [0,1]$ we can conclude that for any $x \in (0,1)$ the following inequality holds

$$\begin{aligned} u_n''''(x) - (a+bC)u_n''(x) \ge u_n''''(x) - \left(a + b\int_0^1 (u_n'(x))^2 dx\right) u_n''(x) \\ = \lambda f(u_n^+(x)) + t_n\psi(x) > \Lambda u_n(x), \end{aligned}$$

which implies that $\lambda_{1,a+bC} > \Lambda$, a contradiction.

Take $\rho \in (0,\delta]$, since the equation $u - T(\lambda f(u) + \psi) = 0$ has no solution in $B_\rho(0)$, then according to the homotopy invariance of topological degree we have

$$\deg(I - F(\lambda,\cdot), B_\rho(0), 0) = \deg(I - T(\lambda f(\cdot) + \psi), B_\rho(0), 0) = 0. \qquad \square$$

Now, we are ready to consider the bifurcation of positive solutions of (1.1) from the line of trivial solutions $\{(\lambda, 0) \in \mathbb{R} \times X : \lambda \in \mathbb{R}\}$.

Theorem 4.3. *Assume that (H1) and (H2) hold. Then from* $(0,0)$ *there emanate an unbounded continuum* $\mathcal{C}_0$ *of positive solutions of* (4.2) *in* $\mathbb{R} \times X$.

Proof. First, we show that $(0,0)$ is a bifurcation point from the line of trivial solutions $\{(\lambda,0) \in \mathbb{R} \times X : \lambda \in \mathbb{R}\}$ for the equation (1.1). In fact, this can be obtained following from a simple modification of the global bifurcation theorem of Rabinowitz [22, Theorem 1.3], and the similar arguments has been used in [2, Proposition 3.5] Suppose on the contrary that $(0,0)$ is not a bifurcation point for (4.2), then there is a neighborhood of $(0,0)$ containing no nontrivial solutions of (4.2). In particular there exists a small $\epsilon > 0$ such that there are no solutions of (4.2) on $[-\epsilon,\epsilon] \times \partial B_\epsilon(0)$. Then $\deg(I - F(\lambda,\cdot), B_\epsilon(0), 0)$ is well defined for $\lambda \in [-\epsilon,\epsilon]$ and, by the homotopy invariance property of degree we have

$$\deg(I - F(\lambda,\cdot), B_\epsilon(0), 0) \equiv \text{constant}, \qquad \forall \lambda \in [-\epsilon,\epsilon],$$

which is a contradict with Lemma 4.1 and 4.2.

Then according to Rabinowitz's global bifurcation theorem, an continuum $\mathcal{C}_0$ of positive solutions of (4.2) emanates from $(0,0)$, and either

(i) $\mathcal{C}_0$ is unbounded in $\mathbb{R} \times X$, or

(ii) $\mathcal{C}_0 \cap [(\mathbb{R} \setminus 0) \times \{0\}] \neq \emptyset$.

To prove the unboundedness of $\mathcal{C}_0$, we only need to show that the case (ii) cannot occur, that is: $\mathcal{C}_0$ can not meet $(\lambda, 0)$ for any $\lambda \neq 0$. It is easy to see that for $\lambda < 0$ problem (1.1) does not possess a positive solution. For the case $\lambda > 0$, we assume on the contrary that there exist some $\lambda_0 > 0$ and a sequence of parameters $\{\lambda_n\}$ and corresponding positive solutions

$\{u_n\}$ of (1.1) such that $\lambda_n \longrightarrow \lambda_0$ and $\|u_n\|_X \longrightarrow 0$. Since $\|u_n\|_\infty \longrightarrow 0$, then by (H1), for fixed $\varepsilon \in (0, \lambda_0)$ there exists $n_0 \in \mathbb{N}$ such that when $n > n_0$ we have

$$u_n''''(x) - \left(a + b\int_0^1 (u_n'(x))^2 dx\right) u_n''(x)$$
$$= \lambda_n f(u_n(x)) \geq (\lambda_0 - \varepsilon) f(u_n(x)) > \Lambda u_n(x), \qquad \forall x \in (0,1),$$

where Λ is defined as in Lemma 4.2. Now, we can get a contradiction in a similar way that in the proof of Lemma 4.2. $\square$

The main result of this section is following:

Theorem 4.4. *Assume that (H1) and (H2) hold, then* (1.1) *has a positive solution if and only if $\lambda > 0$. In addition, if f is monotone increasing and there exists $\alpha \in (0,1)$ such that*

$$f(\tau s) \geq \tau^\alpha f(s) \tag{4.5}$$

for any $\tau \in (0,1)$ and $s > 0$, then the positive solution of (1.1) *is unique.*

Proof. By Theorem 4.3, there exists an unbounded continuum $\mathcal{C}_0 \in \mathbb{R} \times X$ of positive solutions of (1.1). We will show that $\|u\|_X$ is bounded for any fixed $\lambda > 0$, that is, $\mathcal{C}_0$ can not blow up at finite $\lambda \in (0, +\infty)$. To do this, we first prove $\|u\|_\infty$ is bounded for any fixed $\lambda > 0$. Assume on the contrary that there exist $\lambda_0 > 0$ and a sequence of parameters $\{\lambda_n\}$ and corresponding positive solutions $\{u_n\}$ of (1.1) such that $\lambda_n \longrightarrow \lambda_0$, $\|u_n\|_\infty \longrightarrow \infty$. Since

$$u_n''''(x) - \left(a + b\int_0^1 (u_n'(x))^2 dx\right) u_n''(x) = \lambda_n f(u_n), \tag{4.6}$$

divide (4.6) by $\|u_n\|_\infty$ and set $v_n = \frac{u_n}{\|u_n\|_\infty}$, then we get

$$v_n''''(x) - \left(a + b\int_0^1 (u_n'(x))^2 dx\right) v_n''(x) = \lambda_n \frac{f(u_n(x))}{\|u_n\|_\infty}. \tag{4.7}$$

Multiplying (4.7) by v_n and integrating it over $[0,1]$, based on boundary conditions and integration by parts we obtain

$$\int_0^1 (v_n'(x))^2 dx = \frac{\int_0^1 \lambda_n \frac{f(u_n(x))}{\|u_n\|_\infty} v_n(x) dx - \int_0^1 (v_n''(x))^2 dx}{a + b\int_0^1 (u_n'(x))^2 dx}. \tag{4.8}$$

Since $\|v_n\|_\infty \equiv 1$, $\{\lambda_n\}$ is bounded and (H2) guarantees that $\frac{f(u_n(x))}{\|u_n\|_\infty} \longrightarrow 0$ as $n \longrightarrow \infty$, then (4.8) implies

$$0 \leq \int_0^1 (v_n'(x))^2 dx \leq \frac{\int_0^1 \lambda_n \frac{f(u_n(x))}{\|u_n\|_\infty} v_n(x) dx}{a} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

that is $\|v_n'\|_\infty \longrightarrow 0$. By the boundary conditions $v_n(0) = v_n(1) = 0$, there exist $\xi_n \in (0,1)$ such that $v_n(x) = \int_{\xi_n}^x v_n'(t) dt$, $\forall x \in [0,1]$. Combining this with $\|v_n'\|_\infty \longrightarrow 0$ we can conclude that $\|v_n\|_\infty \longrightarrow 0$. This contradicts $\|v_n\|_\infty \equiv 1$, and then we get the boundedness of $\|u\|_\infty$. Next, we show that the boundedness of $\|u\|_\infty$ can deduce the boundedness of $\|u'\|_\infty$ and $\|u''\|_\infty$. Since

$$u''''(x) - \left(a + b\int_0^1 (u'(x))^2 dx\right) u''(x) = \lambda f(u(x)), \tag{4.9}$$

multiplying (4.9) by u and integrating it over $[0,1]$, similarly we can obtain

$$\int_0^1 (u'(x))^2 dx = \frac{\int_0^1 \lambda f(u(x))u(x)dx - \int_0^1 (u''(x))^2 dx}{a + b\int_0^1 (u'(x))^2 dx} \le \frac{\int_0^1 \lambda f(u(x))u(x)dx}{a}. \tag{4.10}$$

(4.10) implies that $\|u'\|_\infty$ is bounded, and consequently, $\|u''\|_\infty$ is bounded too. According to the definition of $\|u\|_X$, the above conclusion means that $\|u\|_X$ is bounded for any fixed $\lambda > 0$. Combining this with the unboundedness of $\mathcal{C}_0$, we conclude that $\sup\{\lambda |\ (\lambda, u) \in \mathcal{C}_0\} = \infty$, then for any $\lambda > 0$ there exists a positive solution for (1.1).

Now, we prove that if f is monotone increasing and satisfies (4.5), then (1.1) has only a unique positive solution. Assume that there exist two positive solutions $u \neq v$ corresponding to some fixed $\lambda > 0$. If $\int_0^1 (u'(x))^2 dx = \int_0^1 (v'(x))^2 dx = R > 0$, consider the problem

$$\begin{cases} \omega''''(x) - \left(a + b\int_0^1 (u'(x))^2 dx\right)\omega''(x) = \lambda f(\omega(x)), & x \in (0,1), \\ \omega(0) = \omega(1) = \omega''(0) = \omega''(1) = 0, \end{cases} \tag{4.11}$$

and its corresponding integral operator $H : P \to P$ given by

$$H(\omega) = T(\lambda f(\omega)) = \lambda \int_0^1 \int_0^1 G_1(x,t) G_{2,R}(t,s) f(\omega(s)) ds dt.$$

By the monotonicity of f and (4.5), H is an increasing α-concave operator according to [31, Definition 2.3], then by [31, Theorem 2.1, Remark 2.1], the operator equation $H(\omega) = \omega$ has a unique solution, which is also the unique positive solution of (4.11). That is, $u = v$.

If we assume that $\int_0^1 (u'(x))^2 dx > \int_0^1 (v'(x))^2 dx$, since $v'' \le 0$, we have

$$\begin{aligned} v''''(x) &- \left(a + b\int_0^1 (u'(x))^2 dx\right) v''(x) \\ &\ge v''''(x) - \left(a + b\int_0^1 (v'(x))^2 dx\right) v''(x) = \lambda f(v(x)), \end{aligned} \tag{4.12}$$

which means that v is actually an upper solution of (4.11). Constructing an iterative sequence $v_{n+1} = Hv_n, n = 0,1,2,\ldots$, where $v_0 = v$, then (4.12) and the monotonicity of f guarantee that $\{v_n\}$ is decreasing. Moreover, by [31, Theorem 2.1, Remark 2.1], $\{v_n\}$ must converge to the unique solution u of (4.11), and consequently we have

$$0 \le u(x) \le v(x), \qquad \forall x \in [0,1]. \tag{4.13}$$

On the other hand, based on boundary conditions and integration by parts, from the assumption $\int_0^1 (u'(x))^2 dx > \int_0^1 (v'(x))^2 dx$ we have that

$$\begin{aligned} \int_0^1 (u'(x))^2 dx - \int_0^1 (v'(x))^2 dx &= \int_0^1 [u'(x) + v'(x)][u'(x) - v'(x)] dx \\ &= -\int_0^1 (u(x) - v(x))(u''(x) + v''(x)) dx > 0, \end{aligned} \tag{4.14}$$

since $-(u''(x) + v''(x)) \ge 0$ following from (2.15), then (4.14) contradicts with (4.13). This concludes the proof. □

Remark 4.5. If c_1, c_2 are nonnegative constants satisfying $c_1^2 + c_2^2 \neq 0$, $0 < p, q < 1$, then it is easy to check that the function

$$f(u) = c_1 u^p + c_2 u^q$$

is increasing and satisfies (H1),(H2) and (4.5). Consequently, Theorem 4.4 guarantees that the problem

$$\begin{cases} u''''(x) - (a + b\int_0^1 (u'(x))^2 dx) u''(x) = \lambda(c_1 u^p(x) + c_2 u^q(x)), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

has a positive solution if and only if $\lambda > 0$, moreover, the positive solution is unique.

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On the solvability of the periodically forced relativistic pendulum equation on time scales

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Abstract. We study some properties of the range of the relativistic pendulum operator $\mathcal{P}$, that is, the set of possible continuous T-periodic forcing terms p for which the equation $\mathcal{P}x = p$ admits a T-periodic solution over a T-periodic time scale $\mathbb{T}$. Writing $p(t) = p_0(t) + \overline{p}$, we prove the existence of a nonempty compact interval $\mathcal{I}(p_0)$, depending continuously on p_0, such that the problem has a solution if and only if $\overline{p} \in \mathcal{I}(p_0)$ and at least two different solutions when $\overline{p}$ is an interior point. Furthermore, we give sufficient conditions for nondegeneracy; specifically, we prove that if T is small then $\mathcal{I}(p_0)$ is a neighbourhood of 0 for arbitrary p_0. The results in the present paper improve the smallness condition obtained in previous works for the continuous case $\mathbb{T} = \mathbb{R}$.

Keywords: relativistic pendulum, periodic solutions, time scales, degenerate equations.

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1 Introduction

The T-periodic problem for the forced relativistic pendulum equation on time scales reads

$$\mathcal{P}x(t) := (\varphi(x^{\Delta}(t)))^{\Delta} + ax^{\Delta}(t) + b\sin x(t) = p_0(t) + s, \quad t \in \mathbb{T}, \tag{1.1}$$

where $a, b > 0$ and s are real numbers, $\mathbb{T}$ is an arbitrary T-periodic nonempty closed subset of $\mathbb{R}$ for some $T > 0$, $\varphi : (-c, c) \to \mathbb{R}$ is the relativistic operator

$$\varphi(x) := \frac{x}{\sqrt{1 - \frac{x^2}{c^2}}}$$

with $c > 0$ and p_0 is continuous and T-periodic in $\mathbb{T}$, with zero average. In this work, we are concerned with the set of all possible values of s such that (1.1) admits a T-periodic solution.

The time scales theory was introduced in 1988, in the PhD thesis of Stefan Hilger [12], as an attempt to unify discrete and continuous calculus. The time scale $\mathbb{R}$ corresponds to

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the continuous case and, hence, yields results for ordinary differential equations. If the time scale is $\mathbb{Z}$, then the results apply to standard difference equations. However, the generality of the set $\mathbb{T}$ produces many different situations in which the time scales formalism is useful in several applications. For example, in the study of hybrid discrete-continuous dynamical systems, see [6].

In the past decades, periodic problems involving the relativistic forced pendulum differential equation for the continuous case $\mathbb{T} = \mathbb{R}$ were studied by many authors, see [3,4,8,14,18,19]. In particular, the works [3, 19] are concerned with the so-called *solvability set*, that is, the set $\mathcal{I}(p_0)$ of values of s for which (1.1) has at least one T-periodic solution. We remark that problem (1.1) is 2π-periodic and, consequently, if x is a T-periodic solution then $x + 2k\pi$ is also a T-periodic solution for all $k \in \mathbb{Z}$. For this reason, the multiplicity results for (1.1) usually refer to the existence of *geometrically distinct* T-periodic solutions, i.e. solutions not differing by a multiple of 2π.

For the standard pendulum equation with $a = 0$, the solvability set was analyzed in the pioneering work [9], where it was proved that $\mathcal{I}(p_0) \subset [-b, b]$ is a nonempty compact interval containing 0. Moreover, $\mathcal{I}(p_0)$ depends continuously on p_0. These results were partially extended to the relativistic case in [8]; however, the method of proof in both works is variational and, consequently, cannot be applied to the case $a > 0$. This latter situation was studied in [11] for the standard pendulum and in [19] for the relativistic case. An interesting question, stated already in [9] is whether or not the equation may be degenerate, namely: is there any p_0 such that $\mathcal{I}(p_0)$ reduces to a single point? Many works are devoted to this problem and, for the classical pendulum, nondegeneracy has been proved for an open and dense subset of $\tilde{C}_T$, the space of zero-average T-periodic continuous functions. However, the question for arbitrary p_0 remains unsolved. For a survey on the pendulum equation and open problems see for example [15].

The purpose of this work is to extend the results in [3] and [19] to the context of time scales. To this end, we prove in the first place that the set $\mathcal{I}(p_0)$ is a nonempty compact interval depending continuously on p_0. The method of proof is inspired in a simple idea introduced in [11] for the standard pendulum equation, which basically employs the Schauder Theorem and the method of upper and lower solutions. Moreover, by a Leray–Schauder degree argument it shall be proved that if s is an interior point of $\mathcal{I}(p_0)$, then the problem admits at least two geometrically distinct periodic solutions.

Furthermore, sufficient conditions shall be given in order to guarantee that $0 \in \mathcal{I}(p_0)$. We recall that, when $a \neq 0$, this is not trivial even in the continuous case $\mathbb{T} = \mathbb{R}$. For the classical pendulum equation, there exist well known examples with $0 \notin \mathcal{I}(p_0)$ for arbitrary values of T; for the relativistic case, it was proved in [3] that, if $cT < \sqrt{3}\pi$, then $0 \in \mathcal{I}(p_0)^\circ$. In a very recent paper (see [10]), this bound was improved in terms of a, b and $\|p_0\|_{L^1}$, yielding the uniform condition $cT \leq 2\pi$. It is worth noticing that, however, the problem is still open for large values of T. As we shall see, a slight improvement of the previous bound can be deduced from the results in the present paper. Specifically, we shall prove the existence of T^* with $cT^* > \pi$ such that if $T \leq T^*$ then $0 \in \mathcal{I}(p_0)$ and it is an interior point when the inequality is strict. An inferior bound for T^* can be characterized as a zero of a real function; for the continuous case $\mathbb{T} = \mathbb{R}$, it is easily shown that the bound obtained in [4] is improved; furthermore, it is numerically seen that $cT^* > 6.318$, thus improving also the bound in [10]. We remark that the computation is independent of p_0: in other words, if $T < T^*$, then the range of the operator $\mathcal{P}$ contains a set of the form $\tilde{C}_T + [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$.

We highlight that our paper is devoted to equations on time scales that involve a φ-

laplacian of relativistic type, for which the literature is scarce. For example, in [17], the existence of heteroclinic solutions for a family of equations on time scales that includes the unforced relativistic pendulum is proved. However, to our knowledge there are no papers concerned with periodic solutions and, more precisely, the solvability set for equations with a singular φ-laplacian on time scales.

This work is organized as follows. In Section 2, we establish the notation, terminology and preliminary results which will be used throughout the paper. In Section 3 we prove that the set $\mathcal{I}(p_0)$ is a nonempty compact interval depending continuously on p_0, and that two geometrically distinct T-periodic solutions exist when s is an interior point. Finally, Section 4 is devoted to find sufficient conditions in order to guarantee that $0 \in \mathcal{I}(p_0)$ and improve the condition obtained in [3] for the continuous case.

2 Notation and preliminaries

For the reader's convenience, let us firstly recall some basic definitions concerning time-scales that shall be used in this work. For a more detailed exposition, see e.g. [6,7].

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$, with the induced topology. Throughout this work, we shall assume that $\mathbb{T}$ is T-periodic for some fixed $T > 0$, namely that $\mathbb{T} + T = \mathbb{T}$. For $a, b \in \mathbb{T}$ with $a \leq b$, we shall denote $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$.

The *forward jump* operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf \{s \in \mathbb{T} : s > t\}.$$

A point $t \in \mathbb{T}$ is called *right scattered* if $\sigma(t) > t$, and *right dense* otherwise. A function u is *delta differentiable* at $t \in \mathbb{T}$ if there exists a number (denoted by $u^\Delta(t)$) with the property that given any $\epsilon > 0$ there is a neighbourhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|(u(\sigma(t)) - u(s)) - u^\Delta(t)\,(\sigma(t) - s)\right| \leq \epsilon\,|\sigma(t) - s|$$

for all $s \in U$. Thus, we call $u^\Delta(t)$ the *delta derivative* of u at t. Moreover, we say that u is *delta differentiable* on $\mathbb{T}$ provided that $u^\Delta(t)$ exists for all $t \in \mathbb{T}$. Note that for $\mathbb{T} = \mathbb{R}$, we have $u^\Delta = u'$, the usual derivative, and for $\mathbb{T} = \mathbb{Z}$ we have that $u^\Delta(t) = \Delta u(t) = u(t+1) - u(t)$.

A function $U : \mathbb{T} \to \mathbb{R}$ is called a Δ-antiderivative of $u : \mathbb{T} \to \mathbb{R}$ provided $U^\Delta(t) = u(t)$ holds for all $t \in \mathbb{T}$. It is not difficult to prove that every continuous u has a Δ-antiderivative, which is unique up to a constant term. Thus, the Δ-integral from t_0 to t of u is well defined by

$$\int_{t_0}^{t} u(s)\Delta s = U(t) - U(t_0) \quad \text{for all } t \in \mathbb{T}.$$

Let $C_T = C_T(\mathbb{T}, \mathbb{R})$ be the Banach space of all continuous T-periodic real functions on $\mathbb{T}$ endowed with the uniform norm

$$\|x\|_\infty = \sup_{\mathbb{T}} |x(t)| = \sup_{[0,T]_{\mathbb{T}}} |x(t)|$$

and let $\tilde{C_T}$ be the subspace of those elements of C_T having zero average. By $C_T^1 = C_T^1(\mathbb{T}, \mathbb{R})$ we shall denote the Banach space of all continuous T-periodic functions on $\mathbb{T}$ that are Δ-differentiable functions with continuous Δ-derivatives, endowed with the standard norm

$$\|x\|_1 = \sup_{[0,T]_{\mathbb{T}}} |x(t)| + \sup_{[0,T]_{\mathbb{T}}} |x^\Delta(t)|.$$

Equation (1.1) can be written as

$$(\varphi(x^{\Delta}(t)))^{\Delta} = f(t, x(t), x^{\Delta}(t)), \quad t \in \mathbb{T}, \tag{2.1}$$

where $f : \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the continuous function given by $f(t,u,v) := p_0(t) + s - au - b\sin(u)$. A function $x \in C^1_T$ is said to be a solution of (2.1) if $\varphi(x^{\Delta}) \in C^1_T$ and verifies $(\varphi(x^{\Delta}(t)))^{\Delta} = f(t, x(t), x^{\Delta}(t))$ for all $t \in \mathbb{T}$. We remark that necessarily $\|x\|_\infty < c$.

For $x \in C_T$, the average, the maximum value and the minimum value of x shall be denoted respectively by $\overline{x}$, $x_{\max}$ and $x_{\min}$, namely

$$\overline{x} := \frac{1}{T}\int_0^T x(t)\Delta t, \qquad x_{\max} := \max_{t\in[0,T]_{\mathbb{T}}} x(t) \qquad x_{\min} := \min_{t\in[0,T]_{\mathbb{T}}} x(t).$$

2.1 Upper and lower solutions and degree

Let us define T-periodic lower and upper solutions for problem (2.1) as follows.

Definition 2.1. A lower T-periodic solution α (resp. upper solution β) of (2.1) is a function $\alpha \in C^1_T$ with $\|\alpha^{\Delta}\|_\infty < c$ such that $\varphi(\alpha^{\Delta})$ is continuously Δ-differentiable and

$$\left(\varphi\left(\alpha^{\Delta}(t)\right)\right)^{\Delta} \geq f(t, \alpha(t), \alpha^{\Delta}(t)) \quad \left(resp.\ \left(\varphi\left(\beta^{\Delta}(t)\right)\right)^{\Delta} \leq f(t, \beta(t), \beta^{\Delta}(t))\right) \tag{2.2}$$

for all $t \in \mathbb{T}$. Such lower (upper) solution is called strict if the inequality (2.2) is strict for all $t \in \mathbb{T}$.

It is worth recalling the problem of finding T-periodic solutions of (2.1) over the closure of the set

$$\Omega_{\alpha,\beta} := \{x \in C^1_T : \alpha(t) \leq x(t) \leq \beta(t) \text{ for all } t\}$$

can be reduced to a fixed point equation $x = M_f(x)$, where $M_f : \overline{\Omega}_{\alpha,\beta} \to C^1_T$ is a compact operator that can be defined according to the nonlinear version of the continuation method (see e.g. [16]), namely

$$M_f(x) := \overline{x} + \overline{N_f x} + K(N_f x - \overline{N_f x}),$$

where N_f is the Nemitskii operator associated to f and $K : \tilde{C}_T \to \tilde{C}_T$ is the (nonlinear) compact operator given by $K\xi = x$, with $x \in C^1_T$ the unique solution of the problem $(\varphi(x^{\Delta}(t)))^{\Delta} = \xi(t)$ with zero average. We recall, for the reader's convenience, that the definition of K based upon the existence, easy to prove, of a (unique) completely continuous map $\phi : C_T \to \mathbb{R}$ satisfying $\int_0^T \varphi^{-1}(h + \phi(h))\Delta t = 0$ for all $h \in C_T$. For the purposes of the present paper, we shall only need the following result, which is an adaptation of Theorem 3.7 in [1]:

Theorem 2.2. *Suppose that (2.1) has a T-periodic lower solution α and an upper solution β such that $\alpha(t) \leq \beta(t)$ for all $t \in \mathbb{T}$. Then problem (1.1) has at least one T-periodic solution x with $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in \mathbb{T}$. If furthermore α and β are strict, then $\deg_{LS}(I - M_f, \Omega_{\alpha,\beta}(0), 0) = 1$, where $\deg_{LS}$ stands for the Leray–Schauder degree.*

3 The solvability set $\mathcal{I}(p_0)$

In this section, we shall prove that the solution set $\mathcal{I}(p_0)$ is a nonempty compact set; furthermore, employing the method of upper and lower solutions it shall be verified that $\mathcal{I}(p_0)$ is an interval depending continuously on p_0. Finally, the excision property of the degree will be employed to verify that if s is an interior point of $\mathcal{I}(p_0)$, then the problem has at least 2 geometrically different T-periodic solutions.

Theorem 3.1. *Assume that $p_0 \in C_T$ has zero average. Then, there exist numbers $d(p_0)$ and $D(p_0)$, with $-b \le d(p_0) \le D(p_0) \le b$, such that (1.1) has at least one T-periodic solution if and only if $s \in [d(p_0), D(p_0)]$. Moreover, the functions $d, D : \tilde{C}_T \to \mathbb{R}$ are continuous.*

Proof. For the reader's convenience, we shall proceed in several steps.

Step 1 (An associated integro-differential problem). Observe that if $x \in C^1_T$ is a solution of (1.1), then, Δ-integration over $[0, T]_{\mathbb{T}}$ yields $s = \frac{b}{T}\int_0^T \sin(x(t))\Delta t$. Therefore, it proves convenient to consider the integro-differential Dirichlet problem

$$\begin{cases} (\varphi(x^\Delta(t)))^\Delta + ax^\Delta(t) + b\sin x(t) = p_0(t) + s(x), & t \in (0, T)_{\mathbb{T}} \\ x(0) = x(T), \end{cases} \tag{3.1}$$

with $s(x) := \frac{b}{T}\int_0^T \sin(x(t))\Delta t$. By Schauder's fixed point theorem, it is straightforward to prove that for each $r \in \mathbb{R}$ there exists at least one solution $x \in C([0, T]_{\mathbb{T}})$ of (3.1) such that $x(0) = x(T) = r$.

Step 2 ($\mathcal{I}(p_0)$ is is nonempty and bounded). Let x be a solution of (3.1) such that $x(0) = x(T) = r$, then integration over $[0, T]_{\mathbb{T}}$ yields

$$\varphi(x^\Delta(T)) - \varphi(x^\Delta(0)) + b\int_0^T \sin x(t)\Delta t = Ts(x),$$

and hence $\varphi(x^\Delta(T)) = \varphi(x^\Delta(0))$. It follows that x may be extended in a T-periodic fashion to a solution of (1.1) with $s = s(x)$. In other words,

$$\mathcal{I}(p_0) = \{s(x) : x \text{ is a solution of (3.1) for some } r \in [0, 2\pi]\} \neq \emptyset.$$

Moreover, it is clear from definition that $|s(x)| \le b$, so $\mathcal{I}(p_0) \subset [-b, b]$.

Step 3 ($\mathcal{I}(p_0)$ is connected). Assume that $s_1, s_2 \in \mathcal{I}(p_0)$ are such that $s_1 < s_2$, and let x_1 and x_2 be T-periodic solutions of (1.1) for s_1 and s_2, respectively. Then for any $s \in (s_1, s_2)$ it is verified that x_1 and x_2 are strict upper and a lower solutions of (1.1), respectively. Replacing x_1 by $x_1 + 2k\pi$, with k the first integer such that $x_2 < x_1 + 2k\pi$ and applying Theorem 2.2 with $\alpha = x_2$ and $\beta = x_1 + 2k\pi$, we conclude that problem (1.1) has at least one T-periodic solution, whence $s \in \mathcal{I}(p_0)$.

Step 4 ($\mathcal{I}(p_0)$ is closed). Let $\{s_n\} \subset \mathcal{I}(p_0)$ converge to some s, and let $x_n \in C^1_T$ be a solution of (1.1) for s_n. Without loss of generality, we may assume that $x_n(0) \in [0, 2\pi]$. Because $\|x_n^\Delta\|_\infty < c$, by Arzelà–Ascoli theorem there exists a subsequence (still denoted $\{x_n\}$) that converges uniformly to some x. Furthermore, from (1.1) we deduce the existence of a constant C independent of n such that $|(\varphi(x_n^\Delta(t)))^\Delta| \le C$ for all t. We claim that $\varphi(x_n^\Delta)$ is also uniformly bounded, that is, $\|x_n^\Delta\|_\infty$ is bounded away from c. Indeed, otherwise passing to a subsequence we may suppose for example that $\varphi(x_n^\Delta)_{\max} \to +\infty$. Because $\varphi(x_n^\Delta(t_1)) - \varphi(x_n^\Delta(t_0)) \le C(t_1 - t_0)$ for all $t_1 > t_0$, we deduce from periodicity that $\varphi(x_n^\Delta)_{\max} - \varphi(x_n^\Delta)_{\min} \le CT$ and, consequently, $\varphi(x_n^\Delta)_{\min} \to +\infty$. This implies that $(x_n^\Delta)_{\min} \to c$, which contradicts the fact that x_n^Δ has zero average. Using Arzelà-Ascoli again, we may assume that $\varphi(x_n^\Delta)$ converges uniformly to some function v and, from the identity $x_n(t) = x_n(0) + \int_0^t x_n^\Delta(\xi)\Delta\xi$ we deduce that $x \in C^1_T$ and $x^\Delta = \varphi^{-1}(v)$. Now integrate the equation for each n and take limit for $n \to \infty$ to obtain

$$\varphi(x^\Delta(t)) = \varphi(x^\Delta(0)) + \int_0^t [s + p_0(\xi) - b\sin(x(\xi))]\Delta\xi - a[x(t) - x(0)].$$

In turn, this implies that x is a solution of (3.1) with $s(x) = s$; hence, $\mathcal{I}(p_0)$ is closed and the proof is complete.

Step 5 (continuous dependence on p_0). Let $\{p_0^n\}_{n\in\mathbb{N}} \subset \tilde{C_T}$ be a sequence that converges to some p_0. We shall prove that $D(p_0^n) \to D(p_0)$; the proof for d is analogous. Similarly to Step 4, it is seen that if a subsequence of $\{D(p_0^n)\}$ converges to some D, then the problem for p_0 with $s = D$ admits a solution and, consequently, $D \le D(p_0)$. Thus, it suffices to prove that $\liminf_{n\to\infty} D(p_0^n) \ge D(p_0)$. Indeed, otherwise, passing to a subsequence we may suppose that $D(p_0^n) \to D < D(p_0)$. Fix $\eta > 0$ such that $D + \eta < D(p_0)$ and let x be a T-periodic solution of (1.1) for $s = D(p_0)$. Take n large enough such that

$$p_0(t) + D(p_0) > p_0^n(t) + D + \eta > p_0^n(t) + D(p_0^n) \qquad \forall\, t \in [0, T]_{\mathbb{T}}$$

and let x_n be a T-periodic solution of (1.1) for p_0^n and $s = D(p_0^n)$. The previous inequalities imply that x and x_n are respectively a lower and an upper solution of the problem for p_0^n and $s = D + \eta$ and, without loss of generality, we may assume that $x < x_n$. Thus, (1.1) has a T-periodic solution for p_0^n and $s = D + \eta > D(p_0^n)$, a contradiction. □

The following theorem establishes the existence of at least two geometrically different T-periodic solutions to problem (1.1) when s is an interior point.

Theorem 3.2. *Assume that $p_0 \in C_T$ has zero average. If $s \in (d(p_0), D(p_0))$, then the problem (1.1) has at least two geometrically different T-periodic solutions.*

Proof. For $s \in (d(p_0), D(p_0))$, let $s_1 := d(p_0) < s < D(p_0) := s_2$ and let x_1, x_2 be as in Step 3 of the previous proof. Then x_1 and x_2 are strict upper and lower solutions for s, respectively. Due to the 2π-periodicity of (1.1), we may assume that $x_2 < x_1$ and $x_2 + 2\pi \not\le x_1$ and, consequently, Ω_{x_2,x_1} and $\Omega_{x_2+2\pi,x_1+2\pi}$ are disjoint open subsets of $\Omega_{x_2,x_1+2\pi}$. From Theorem 2.2 and the excision property of the Leray–Schauder degree, we deduce the existence of three different solutions $y_1, y_2, y_3 \in C_T^1$ such that

$$\begin{aligned} x_2(t) &< y_1(t) < x_1(t), \\ x_2(t) + 2\pi &< y_2(t) < x_1(t) + 2\pi \\ x_2(t) &< y_3(t) < x_1(t) + 2\pi \end{aligned}$$

for all $t \in \mathbb{T}$. If $y_2 = y_1 + 2\pi$, then $y_3 \ne y_1, y_1 + 2\pi$ and the conclusion follows. □

4 Sufficient conditions for $0 \in \mathcal{I}(p_0)$

In this section, we shall obtain conditions guaranteeing that 0 belongs to the solvability set. Even in the continuous case, this is not clear when $a \ne 0$ since, as it is well known, counter-examples exist for the classical pendulum equation for arbitrary periods. In the relativistic case, however, it was proved that $0 \in \mathcal{I}(p_0)$ when T is sufficiently small and counter-examples for large values of T are not yet known. Here, as mentioned in the introduction, we shall improve the bounds for T obtained in previous works for $\mathbb{T} = \mathbb{R}$. The results shall be expressed in terms of $k(\mathbb{T})$, the optimal constant of the inequality

$$\|x - \overline{x}\|_\infty \le k\|x^\Delta\|_\infty, \qquad x \in C_T^1.$$

For instance, for arbitrary $\mathbb{T}$ it is readily seen that $k(\mathbb{T}) \leq \frac{T}{2}$, because x^Δ has zero average and hence, due to periodicity,

$$x_{\max} - x_{\min} \leq \int_{t_{\min}}^{t_{\max}} [x^\Delta(t)]^+ \Delta t \leq \int_0^T [x^\Delta(t)]^+ \Delta t = \frac{1}{2} \int_0^T |x^\Delta(t)| \Delta t.$$

We recall that, in the continuous case, the (optimal) Sobolev inequality $\|x - \overline{x}\|_\infty \leq \sqrt{\frac{T}{12}} \|x'\|_2$ implies that $k(\mathbb{R}) \leq \frac{T}{2\sqrt{3}}$.

The main result of this section reads as follows.

Theorem 4.1. *Assume that $ck(\mathbb{T}) < \pi$ and define the function*

$$\psi(\delta) := 2\delta \cos(\delta) + (cT - 2\delta) \cos(ck(\mathbb{T})).$$

If $\psi(\delta) \geq 0$ for some $\delta \in (0, \frac{\pi}{2})$, then $0 \in \mathcal{I}(p_0)$. Furthermore, if the previous inequality is strict, then $0 \in \mathcal{I}(p_0)^\circ$.

Before proceeding to the proof, it is worth to recall that, from Theorem 4.1 and Example 5.3 in [2], in order to prove the existence of T-periodic solutions for $s = 0$ it suffices to verify that the equation

$$\left(\frac{x^\Delta(t)}{\sqrt{1 - \frac{x^\Delta(t)^2}{c^2}}} \right)^\Delta = \lambda [p_0(t) - a x^\Delta(t) - b \sin x(t)] \tag{4.1}$$

has no T-periodic solutions with average $\pm\frac{\pi}{2}$. For example, if $x \in C^1_T$ is a solution of (4.1) such that $\overline{x} = \frac{\pi}{2}$, then it follows from the definition of $k(\mathbb{T})$ that, for all $t \in \mathbb{T}$,

$$\left| x(t) - \frac{\pi}{2} \right| \leq ck(\mathbb{T}).$$

In particular, if $ck(\mathbb{T}) \leq \frac{\pi}{2}$, then $x(t) \in [0, \pi]$ for all $t \in \mathbb{T}$ and, upon integration of equation (4.1), we deduce:

$$0 = b \int_0^T \sin(x(t)) \Delta t > 0.$$

The same contradiction is obtained also if $\overline{x} = -\frac{\pi}{2}$. For example, the condition $cT \leq \pi$ is sufficient for arbitrary $\mathbb{T}$ and, in the continuous case, the condition $cT \leq \sqrt{3}\pi$ is retrieved. However, the previous bound $ck(\mathbb{T}) \leq \frac{\pi}{2}$ can be improved, as we shall see in the following proof.

Proof of Theorem 4.1. From the preceding discussion, it may be assumed that $\frac{\pi}{2} < ck(\mathbb{T}) < \pi$. Suppose that x is a solution of (4.1) such that $\overline{x} = \frac{\pi}{2}$, then

$$x(t) \in \left[\frac{\pi}{2} - ck(\mathbb{T}), \frac{\pi}{2} + ck(\mathbb{T}) \right] \subset \left(-\frac{\pi}{2}, \frac{3\pi}{2} \right)$$

for all $t \in \mathbb{T}$ and hence

$$\sin x(t) \geq -\sin(A) > -1, \quad \text{where } A = ck(\mathbb{T}) - \frac{\pi}{2}.$$

Fix $\delta \in (0, \frac{\pi}{2})$ and consider the set

$$C_\delta = \left\{ t \in [0, T]_\mathbb{T} : \left| x(t) - \frac{\pi}{2} \right| \leq \delta \right\}.$$

Then

$$
\begin{aligned}
0 &= \int_0^T \sin(x(t))\Delta t \geq \int_{C_\delta} (\sin(x(t)) + \sin(A))\Delta t - T\sin(A) \\
&\geq \left[\sin(\frac{\pi}{2} - \delta) + \sin(A)\right] \mathfrak{m}(C_\delta) - T\sin(A) \\
&= \cos(\delta)\mathfrak{m}(C_\delta) - [T - \mathfrak{m}(C_\delta)]\sin A,
\end{aligned}
\tag{4.2}
$$

where $\mathfrak{m}(C_\delta)$ is the measure of the set C_δ associated to the Δ-integral, namely $\mathfrak{m}(C_\delta) = \int_{C_\delta} \Delta t$. Clearly, a contradiction is obtained when the latter term of (4.2) is positive.

Moreover, notice that if $x(t_0) \leq \frac{\pi}{2}$ and $t_1 > t_0$ is such that $x(t_1) \geq \frac{\pi}{2} + \delta$, then

$$
\delta \leq x(t_1) - x(t_0) = \int_{t_0}^{t_1} x^\Delta(s)\Delta s < c(t_1 - t_0).
$$

In the same way, if $t_0 < t_1$ are such that $x(t_0) \geq \frac{\pi}{2}$ and $x(t_1) \leq \frac{\pi}{2} - \delta$, then $c(t_1 - t_0) > \delta$. Thus, by periodicity, we deduce that $\mathfrak{m}(C_\delta) > \frac{2\delta}{c}$. The same conclusions are obtained if $\overline{x} = -\frac{\pi}{2}$; hence, a sufficient condition for the existence of at least one T-periodic solution is that, for some $\delta \in (0, \frac{\pi}{2})$,

$$
\cos(\delta)\frac{2\delta}{c} \geq \left(T - \frac{2\delta}{c}\right)\sin A
$$

or, equivalently, that $\psi(\delta) \geq 0$. Note, furthermore, that if the inequality is strict, then a contradiction is still obtained as in (4.2) if we add a small parameter s to the function p_0 in (4.1). $\square$

Remark 4.2. It is seen that ψ reaches its maximum at the unique $\delta^* \in (0, \frac{\pi}{2})$ such that

$$
\cos(\delta^*) - \delta^* \sin(\delta^*) = \cos\left(ck(\mathbb{T})\right). \tag{4.3}
$$

Thus, replacing (4.3) in ψ, a somewhat explicit condition on T reads:

$$
2(\delta^*)^2 \sin(\delta^*) + cT\cos\left(ck(\mathbb{T})\right) \geq 0.
$$

An immediate corollary is the following:

Corollary 4.3. *There exists a constant T^* with $cT^* > \pi$ such that $0 \in \mathcal{I}(p_0)$ for all $p_0 \in \tilde{C}_T$ if $T \leq T^*$ and it is an interior point if $T < T^*$. For the particular case $\mathbb{T} = \mathbb{R}$, it is verified that $cT^* > \sqrt{3}\pi$.*

Proof. For arbitrary $\mathbb{T}$, we know already that $k(\mathbb{T}) \leq \frac{T}{2}$, then a sufficient condition when $cT \in (\pi, 2\pi)$ is the existence of $\delta \in (0, \frac{\pi}{2})$ such that $\Psi(\delta, T) \geq 0$, where

$$
\Psi(\delta, T) := 2\delta\cos(\delta) + (cT - 2\delta)\cos\left(\frac{cT}{2}\right).
$$

The result now follows trivially from the fact that $\Psi(\delta, \frac{\pi}{c}) = 2\delta\cos(\delta)$. The proof is similar for $\mathbb{T} = \mathbb{R}$, now taking

$$
\Psi_{cont}(\delta, T) := 2\delta\cos(\delta) + (cT - 2\delta)\cos\left(\frac{cT}{2\sqrt{3}}\right).
$$

$\square$

Remark 4.4. A more quantitative version of the previous corollary follows from the fact that the function Ψ is strictly decreasing with respect to T when $cT \in (\pi, 2\pi)$ and arbitrary $\delta \in (0, \frac{\pi}{2})$. In particular, observe that if $\Psi(\delta, \hat{T}) \geq 0$ for some $\hat{T} \in (\frac{\pi}{c}, \frac{2\pi}{c})$ and some $\delta \in (0, \frac{\pi}{2})$, then $\Psi(\delta, T) > 0$ for $T \in (\frac{\pi}{c}, \hat{T})$. Thus, a lower bound for T^* is given by the unique value of $T \in (\frac{\pi}{c}, \frac{2\pi}{c})$ such that

$$\max_{\delta \in [0, \frac{\pi}{2}]} \Psi(\delta, T) = 0.$$

Analogous conclusions are obtained when $\mathbb{T} = \mathbb{R}$ using Ψ_{cont} instead of Ψ.

4.1 Numerical examples and final remarks

As shown in Corollary 4.3, the bound thus obtained always improves the simpler one $ck(\mathbb{T}) \leq \frac{\pi}{2}$ and, in particular, it guarantees that if the latter inequality is satisfied then 0 is in fact an interior point of $\mathcal{I}(p_0)$. In the continuous case, an easy numerical computation gives the sufficient condition $cT \leq 6.318$, slightly better than the bound $cT \leq 2\pi$ deduced from [10] (see Figure 1). For arbitrary $\mathbb{T}$, numerical experiments show that $0 \in \mathcal{I}(p_0)^\circ$ for $cT \leq 4.19$, as shown in Figure 2.

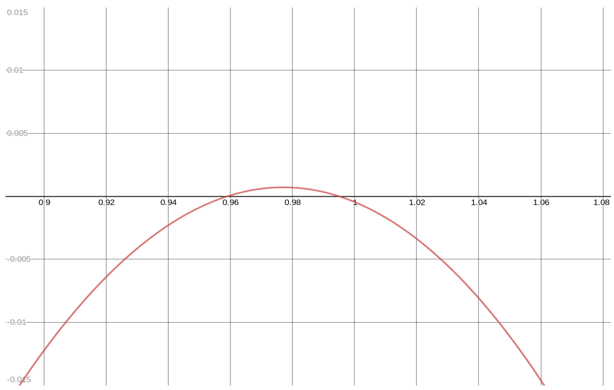

Figure 4.1: Graph of ψ for $\mathbb{T} = \mathbb{R}$ with $cT = 6.318$

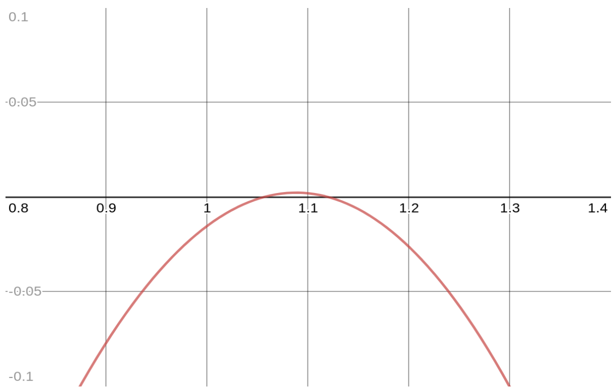

Figure 4.2: Graph of ψ for $k(\mathbb{T}) = \frac{T}{2}$ and $cT = 4.19$

Remark 4.5. An estimation of the constant $k(\mathbb{T})$ could be obtained analogously to the continuous case as shown for example in [13]. Let $\{e_n\}_{n \in \mathbb{Z}} \subset C_T$ be an orthonormal basis of $L^2(0, T)_{\mathbb{T}}$

with $e_0 \equiv \frac{1}{\sqrt{T}}$ and E_n be a primitive of e_n such that $\overline{E}_n = 0$. Writing $x^\Delta = \sum_{n\neq 0} a_n e_n$, it follows that

$$\|x - \overline{x}\|_\infty = \left|\sum_{n\neq 0} a_n E_n\right| \leq \|x^\Delta\|_{L^2} \sqrt{\sum_{n\neq 0} \|E_n\|_\infty^2} \leq \|x^\Delta\|_\infty \sqrt{T \sum_{n\neq 0} \|E_n\|_\infty^2}.$$

When $\mathbb{T} = \mathbb{R}$, taking the usual Fourier basis one has that $\|E_n\|_\infty = \frac{\sqrt{T}}{2\pi n}$ and the value $k(\mathbb{R}) \leq \frac{T}{2\sqrt{3}}$ is obtained from the well known equality $\sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Remark 4.6. As mentioned in the introduction, Theorem 4.1 allows to compute an inferior bound for the length of the solvability interval which does not depend on p_0, provided that T is small enough. In some obvious cases, inferior bounds are obtained for arbitrary T: for example, if $\|p_0\|_\infty < b$ then $[-\varepsilon, \varepsilon] \subset \mathcal{I}(p_0)$ for $\varepsilon = b - \|p_0\|_\infty$. This is readily verified taking $\alpha = \frac{\pi}{2}$ and $\beta = \frac{3\pi}{2}$ as lower and upper solutions.

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Influence of variable coefficients on global existence of solutions of semilinear heat equations with nonlinear boundary conditions

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Abstract. We consider semilinear parabolic equations with nonlinear boundary conditions. We give conditions which guarantee global existence of solutions as well as blow-up in finite time of all solutions with nontrivial initial data. The results depend on the behavior of variable coefficients as $t \to \infty$.

Keywords: semilinear parabolic equation, nonlinear boundary condition, finite time blow-up.

2020 Mathematics Subject Classification: 35B44, 35K58, 35K61.

1 Introduction

We investigate the global solvability and blow-up in finite time for semilinear heat equation

$$u_t = \Delta u + \alpha(t) f(u) \quad \text{for } x \in \Omega, \ t > 0, \tag{1.1}$$

with nonlinear boundary condition

$$\frac{\partial u(x,t)}{\partial \nu} = \beta(t) g(u) \quad \text{for } x \in \partial\Omega, \ t > 0, \tag{1.2}$$

and initial datum

$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ for $n \geq 1$ with smooth boundary $\partial\Omega$, ν is the unit exterior normal vector on the boundary $\partial\Omega$. Here $f(u)$ and $g(u)$ are nonnegative continuous functions for $u \geq 0$, $\alpha(t)$ and $\beta(t)$ are nonnegative continuous functions for $t \geq 0$, $u_0(x) \in C^1(\overline{\Omega})$, $u_0(x) \geq 0$ in $\overline{\Omega}$ and satisfies boundary condition (1.2) as $t = 0$. We will consider nonnegative classical solutions of (1.1)–(1.3).

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Blow-up problem for parabolic equations with reaction term in general form were considered in many papers (see, for example, [1,2,8,9,14,21,27] and the references therein). For the global existence and blow-up of solutions for linear parabolic equations with $\beta(t) \equiv 1$ in (1.2), we refer to previous studies [16,17,22,24–26]. In particular, Walter [24] proved that if $g(s)$ and $g'(s)$ are continuous, positive and increasing for large s, a necessary and sufficient condition for global existence is

$$\int^{+\infty} \frac{ds}{g(s)g'(s)} = +\infty.$$

Some papers are devoted to blow-up phenomena in parabolic problems with time-dependent coefficients (see, for example, [4–6,18–20,28]). So, it follows from results of Payne and Philippin [20] blow-up of all nontrivial solutions for (1.1)–(1.3) with $\beta(t) \equiv 0$ under the conditions (2.15) and

$$f(s) \geq z(s) > 0, \qquad s > 0,$$

where z satisfies

$$\int_a^{+\infty} \frac{ds}{z(s)} < +\infty \quad \text{for any } a > 0$$

and Jensen's inequality

$$\frac{1}{|\Omega|}\int_\Omega z(u)\,dx \geq z\left(\frac{1}{|\Omega|}\int_\Omega u\,dx\right). \tag{1.4}$$

In (1.4), $|\Omega|$ is the volume of Ω.

The aim of our paper is study the influence of variable coefficients $\alpha(t)$ and $\beta(t)$ on the global existence and blow-up of classical solutions of (1.1)–(1.3).

This paper is organized as follows. Finite time blow-up of all nontrivial solutions is proved in Section 2. In Section 3, we present the global existence of solutions for small initial data.

2 Finite time blow-up

In this section, we give conditions for blow-up in finite time of all nontrivial solutions of (1.1)–(1.3).

Before giving our main results, we state a comparison principle which has been proved in [7,23] for more general problems. Let $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$, $\Gamma_T = S_T \cup \overline{\Omega} \times \{0\}$, $T > 0$.

Theorem 2.1. *Let $v(x,t), w(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ satisfy the inequalities:*

$$v_t - \Delta v - \alpha(t)f(v) < w_t - \Delta w - \alpha(t)f(w) \quad \text{in } Q_T,$$
$$\frac{\partial v(x,t)}{\partial \nu} - \beta(t)g(v) < \frac{\partial w(x,t)}{\partial \nu} - \beta(t)g(w) \quad \text{on } S_T,$$
$$v(x,0) < w(x,0) \quad \text{in } \overline{\Omega}.$$

Then

$$v(x,t) < w(x,t) \quad \text{in } Q_T.$$

The first our blow-up result is the following.

Theorem 2.2. *Let $g(s)$ be a nondecreasing positive function for $s > 0$ such that*

$$\int^{+\infty} \frac{ds}{g(s)} < +\infty \tag{2.1}$$

and

$$\int_0^{+\infty} \beta(t)\,dt = +\infty. \tag{2.2}$$

Then any nontrivial nonnegative solution of (1.1)–(1.3) *blows up in finite time.*

Proof. We suppose that $u(x,t)$ is a nontrivial nonnegative solution which exists in Q_T for any positive T. Then for some $T > 0$ there exists $(\overline{x},\overline{t}) \in Q_T$ such that $u(\overline{x},\overline{t}) > 0$. Since $u_t - \Delta u = \alpha(t)f(u) \geq 0$, by strong maximum principle $u(x,t) > 0$ in $Q_T \setminus \overline{Q_{\overline{t}}}$. Let $u(x_\star,t_\star) = 0$ in some point $(x_\star,t_\star) \in S_T \setminus \overline{S_{\overline{t}}}$. According to Theorem 3.6 of [11] it yields $\partial u(x_\star,t_\star)/\partial\nu < 0$, which contradicts the boundary condition (1.2). Thus, $u(x,t) > 0$ in $Q_T \cup S_T \setminus \overline{Q_{\overline{t}}}$. Then there exists $t_0 > \overline{t}$ such that $\beta(t_0) > 0$ and

$$\min_{\overline{\Omega}} u(x,t_0) > 2\sigma, \tag{2.3}$$

where σ is a positive constant.

Let $G_N(x,y;t-\tau)$ denote the Green's function for the heat equation given by

$$u_t - \Delta u = 0 \quad \text{for } x \in \Omega,\ t > 0$$

with homogeneous Neumann boundary condition. We note that the Green's function has the following properties (see, for example, [12,13]:

$$\begin{aligned}
&G_N(x,y;t-\tau) \geq 0, && x,y \in \Omega,\ 0 \leq \tau < t, && (2.4)\\
&\int_\Omega G_N(x,y;t-\tau)\,dy = 1, && x \in \Omega,\ 0 \leq \tau < t, && (2.5)\\
&G_N(x,y;t-\tau) \geq c_1, && x,\,y \in \overline{\Omega},\ t-\tau \geq \varepsilon, && (2.6)\\
&|G_N(x,y;t-\tau) - 1/|\Omega|| \leq c_2 \exp[-c_3(t-\tau)], && x,\,y \in \overline{\Omega},\ t-\tau \geq \varepsilon, &&\\
&\int_{\partial\Omega} G_N(x,y;t-\tau)\,dS_y \leq \frac{c_4}{\sqrt{t-\tau}}, && x \in \overline{\Omega},\ 0 < t-\tau \leq \varepsilon, &&
\end{aligned}$$

for some small $\varepsilon > 0$. Here by c_i $(i \in \mathbb{N})$ we denote positive constants.

Now we introduce conditions on several auxiliary comparison functions. We suppose that $h(s) \in C^1((0,+\infty)) \cap C([0,+\infty))$, $h(s) > 0$ for $s > 0$, $h'(s) \geq 0$ for $s > 0$, $g(s) \geq h(s)$ and

$$\int^{+\infty} \frac{ds}{h(s)} < +\infty.$$

Let $\xi(t)$ be a positive continuous function for $t \geq t_0$ such that

$$\int_{t_0}^{+\infty} \xi(t)\,dt < \frac{\sigma}{2} \tag{2.7}$$

and $\gamma(t)$ be a positive continuous function for $t \geq t_0$ such that $\gamma(t_0) = \beta(t_0)h(2\sigma)$ and

$$\int_{t_0}^{t} \gamma(\tau) \int_{\partial\Omega} G_N(x,y;t-\tau)\,dS_y\,d\tau < \frac{\sigma}{2} \quad \text{for } x \in \overline{\Omega},\ t \geq t_0. \tag{2.8}$$

We consider the following problem

$$\begin{cases} v_t = \Delta v - \xi(t) \ \text{ for } \ x \in \Omega, \ \ t > t_0,\\ \dfrac{\partial v(x,t)}{\partial\nu} = \beta(t)h(v) - \gamma(t) \ \text{ for } \ x \in \partial\Omega, \ \ t > t_0,\\ v(x,t_0) = 2\sigma \ \text{ for } \ x \in \Omega. \end{cases} \tag{2.9}$$

To find lower bound for $v(x,t)$ we represent (2.9) in equivalent form

$$\begin{aligned} v(x,t) = 2\sigma \int_\Omega G_N(x,y;t)\,dy - \int_{t_0}^t \int_\Omega G_N(x,y;t-\tau)\xi(\tau)\,dy\,d\tau \\ + \int_{t_0}^t \int_{\partial\Omega} G_N(x,y;t-\tau)\left(\beta(\tau)h(v) - \gamma(\tau)\right)\,dS_y\,d\tau. \end{aligned} \tag{2.10}$$

Using (2.7), (2.8) and the properties of the Green's function (2.4), (2.5), we obtain from (2.10)

$$v(x,t) \geq 2\sigma - \int_{t_0}^t \xi(\tau)\,d\tau - \int_{t_0}^t \gamma(\tau) \int_{\partial\Omega} G_N(x,y;t-\tau)dS_y\,d\tau > \sigma. \tag{2.11}$$

As in [22] we put

$$m(t) = \int_\Omega \int_{v(x,t)}^{+\infty} \frac{ds}{h(s)}\,dx.$$

We observe that $m(t)$ is well defined and positive for $t \geq t_0$. Since $v(x,t)$ is the solution of (2.9), we get

$$\begin{aligned} m'(t) &= -\int_\Omega \frac{v_t}{h(v)}\,dx = -\int_\Omega \frac{\Delta v}{h(v)}\,dx + \xi(t)\int_\Omega \frac{dx}{h(v)} \\ &= -\int_\Omega \operatorname{div}\left(\frac{\nabla v}{h(v)}\right)\,dx - \int_\Omega \frac{h'(v)\|\nabla v\|^2}{h^2(v)}\,dx + \xi(t)\int_\Omega \frac{dx}{h(v)}. \end{aligned}$$

Applying the inequality $h'(v) \geq 0$, Gauss theorem, the boundary condition in (2.9) and (2.11), we obtain for $t \geq t_0$

$$m'(t) \leq -\int_{\partial\Omega} \frac{1}{h(v)}\frac{\partial v}{\partial \nu}\,dS + \xi(t)\frac{|\Omega|}{h(\sigma)} \leq -|\partial\Omega|\beta(t) + \frac{|\Omega|\xi(t) + |\partial\Omega|\gamma(t)}{h(\sigma)}. \tag{2.12}$$

Due to (2.2), (2.6)–(2.8) $m(t)$ is negative for large values of t. Hence $v(x,t)$ blows up in finite time T_0. Applying Theorem 2.1 to $v(x,t)$ and $u(x,t)$ in $Q_T \setminus \overline{Q_{t_0}}$ for any $T \in (t_0, T_0)$, we prove the theorem. □

Remark 2.3. If $u_0(x)$ is positive in $\overline{\Omega}$ we can obtain an upper bound for blow-up time of the solution. We put $t_0 = 0$ and $v(x,0) = u_0(x) - \varepsilon$ in (2.9) for $\varepsilon \in (0, \min_{\overline{\Omega}} u_0(x))$. Integrating (2.12) over $[0,T]$, we have

$$m(t) \leq m(0) - |\partial\Omega| \int_0^T \beta(t)\,dt + \int_0^T \frac{|\Omega|\xi(t) + |\partial\Omega|\gamma(t)}{h(\sigma)}\,dt.$$

Since $m(t) > 0$ and ε, $\xi(t)$, $\gamma(t)$ are arbitrary we conclude that the solution of (1.1)–(1.3) blows up in finite time T_b, where $T_b \leq T$ and

$$\int_\Omega \int_{u_0(x)}^{+\infty} \frac{ds}{h(s)}\,dx = |\partial\Omega| \int_0^T \beta(t)\,dt.$$

Remark 2.4. We note that (1.1)–(1.3) with $u_0(x) \equiv 0$ may have trivial and blow-up solutions under the assumptions of Theorem 2.2. Indeed, let the conditions of Theorem 2.2 hold, $\alpha(t) \equiv 0$, $\beta(t) \equiv 1$ and $g(u) = u^p$, $u \in [0,\gamma]$ for some $\gamma > 0$ and $0 < p < 1$. As it was proved in [3], problem (1.1)–(1.3) has trivial and positive for $t > 0$ solutions and last one blows up in finite time by Theorem 2.2.

To prove next blow-up result for (1.1)–(1.3) we need a comparison principle with unstrict inequality in the boundary condition.

Theorem 2.5. *Let $\delta > 0$ and $v(x,t), w(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ satisfy the inequalities:*

$$v_t - \Delta v - \alpha(t) f(v) + \delta < w_t - \Delta w - \alpha(t) f(w) \quad \text{in } Q_T,$$
$$\frac{\partial v(x,t)}{\partial \nu} \le \frac{\partial w(x,t)}{\partial \nu} \quad \text{on } S_T,$$
$$v(x,0) < w(x,0) \quad \text{in } \overline{\Omega}.$$

Then

$$v(x,t) \le w(x,t) \quad \text{in } Q_T.$$

Proof. Let τ be any positive constant such that $\tau < T$ and a positive function $\gamma(x) \in C^2(\overline{\Omega})$ satisfy the following inequality

$$\frac{\partial \gamma(x)}{\partial \nu} > 0 \quad \text{on } \partial\Omega.$$

For positive ε we introduce

$$w_\varepsilon(x,t) = w(x,t) + \varepsilon \gamma(x). \tag{2.13}$$

Obviously,

$$v(x,0) < w_\varepsilon(x,0) \quad \text{in } \overline{\Omega}, \qquad \frac{\partial v(x,t)}{\partial \nu} < \frac{\partial w_\varepsilon(x,t)}{\partial \nu} \quad \text{on } S_\tau.$$

Moreover,

$$v_t - \Delta v - \alpha(t) f(v) < w_{\varepsilon t} - \Delta w_\varepsilon - \alpha(t) f(w_\varepsilon) \quad \text{in } Q_\tau,$$

if we take ε so small that

$$\delta > \varepsilon \Delta \gamma + \alpha(t)[f(w + \varepsilon\gamma) - f(w)] \quad \text{in } Q_\tau.$$

Applying Theorem 2.1 with $\beta(t) \equiv 0$, we obtain

$$v(x,t) < w_\varepsilon(x,t) \quad \text{in } Q_\tau.$$

Passing to the limit as $\varepsilon \to 0$ and $\tau \to T$, we prove the theorem. □

Theorem 2.6. *Let $f(s) > 0$ for $s > 0$,*

$$\int^{+\infty} \frac{ds}{f(s)} < +\infty \tag{2.14}$$

and

$$\int_0^{+\infty} \alpha(t)\, dt = +\infty. \tag{2.15}$$

Then any nontrivial nonnegative solution of (1.1)–(1.3) *blows up in finite time.*

Proof. We suppose that $u(x,t)$ is a nontrivial nonnegative solution which exists in Q_T for any positive T. In Theorem 2.2 we proved (2.3). Let $\xi(t)$ be a positive continuous function for $t \ge t_0$ such that

$$\max_{[\sigma, 2\sigma]} f(s) \int_{t_0}^{+\infty} \xi(t)\, dt < \sigma. \tag{2.16}$$

We consider the following auxiliary problem

$$\begin{cases} v'(t) = \alpha(t) f(v) - \xi(t) f(v), & t > t_0, \\ v(t_0) = 2\sigma. \end{cases} \tag{2.17}$$

We prove at first that

$$v(t) > \sigma \quad \text{for } t \geq t_0. \tag{2.18}$$

Suppose there exist t_1 and t_2 such that

$$t_2 > t_1 \geq t_0, \qquad v(t_1) = 2\sigma, \qquad v(t_2) = \sigma,$$

and

$$v(t) > \sigma \quad \text{for } t \in [t_0, t_2) \quad \text{and} \quad v(t) \leq 2\sigma \quad \text{for } t \in [t_1, t_2].$$

Integrating the equation in (2.17) over $[t_1, t_2]$, we have due to (2.16)

$$v(t_2) \geq -\max_{[\sigma, 2\sigma]} f(s) \int_{t_1}^{t_2} \xi(t)\, dt + v(t_1) > \sigma.$$

A contradiction proves (2.18).

From (2.17) we obtain

$$\int_{2\sigma}^{v(t)} \frac{ds}{f(s)} = \int_{t_0}^{t} [\alpha(\tau) - \xi(\tau)]\, d\tau. \tag{2.19}$$

By (2.14)–(2.16) the left side of (2.19) is finite and the right side of (2.19) tends to infinity as $t \to \infty$. Hence the solution of (2.17) blows up in finite time T_0. Applying Theorem 2.5 to $v(t)$ and $u(x,t)$ in $Q_T \setminus \overline{Q_{t_0}}$ for any $T \in (t_0, T_0)$, we prove the theorem. □

Remark 2.7. If $u_0(x)$ is positive in $\overline{\Omega}$ we can obtain an upper bound for blow-up time of the solution. Taking $t_0 = 0$, we conclude from (2.19) that the solution of (1.1)–(1.3) blows up in finite time T_b, where $T_b \leq T$ and

$$\int_{\min_{\overline{\Omega}} u_0(x)}^{+\infty} \frac{ds}{f(s)} = \int_0^T \alpha(t)\, dt.$$

Remark 2.8. Theorem 2.6 does not hold if $f(s)$ is not positive for $s > 0$. To show this we suppose that $f(u_1) = 0$ for some $u_1 > 0$, $\beta(t) \equiv 0$, $u_0(x) = u_1$. Then problem (1.1)–(1.3) has the solution $u(x,t) = u_1$.

Remark 2.9. We note that (2.14) is necessary condition for blow-up of solutions of (1.1)–(1.3) with $\beta(t) \equiv 0$. Let $f(s) > 0$ for $s > 0$ and

$$\int^{+\infty} \frac{ds}{f(s)} = +\infty.$$

Then any solution of (1.1)–(1.3) is global. Indeed, let $u(x,t)$ be a nontrivial solution of (1.1)–(1.3). Then there exist $t_0 \geq 0$ and $x \in \Omega$ such that $u(x, t_0) > 0$.

We consider the following problem

$$\begin{cases} v'(t) = (\alpha(t) + \xi(t)) f(v), \ t > t_0, \\ v(t_0) > \max_{\overline{\Omega}} u(x, t_0) > 0, \end{cases} \tag{2.20}$$

where $\xi(t)$ is some positive continuous function for $t \geq t_0$. Obviously, $v(t)$ is global solution of (2.20). Applying Theorem 2.5 to $u(x,t)$ and $v(t)$ in $Q_T \setminus \overline{Q_{t_0}}$ for any $T > t_0$, we prove the theorem.

Remark 2.10. Problem (1.1)–(1.3) with $u_0(x) \equiv 0$ may have trivial and blow-up solutions under the assumptions of Theorem 2.6. Indeed, let the conditions of Theorem 2.6 hold, $\beta(t) \equiv 0$, $f(s)$ be a nondecreasing Hölder continuous function on $[0, \epsilon]$ for some $\epsilon > 0$ and

$$\int_0^{\epsilon} \frac{ds}{f(s)} < +\infty.$$

As it was proved in [15], problem (1.1)–(1.3) has trivial and positive for $t > 0$ solutions and last one blows up in finite time by Theorem 2.6.

3 Global existence

To formulate global existence result for problem (1.1)–(1.3) we suppose:

$$f(s) \text{ is a nonnegative locally Hölder continuous function for } s \geq 0, \tag{3.1}$$

$$\text{there exists } p > 0 \text{ such that } f(s) \text{ is a positive nondecreasing function for } s \in (0, p), \tag{3.2}$$

$$\int_0 \frac{ds}{f(s)} = +\infty, \qquad \lim_{s \to 0} \frac{g(s)}{s} = 0, \tag{3.3}$$

$$\int_0^{+\infty} (\alpha(t) + \beta(t))\, dt < +\infty \tag{3.4}$$

and there exist positive constants γ, t_0 and K such that $\gamma > t_0$ and

$$\int_{t-t_0}^{t} \frac{\beta(\tau) d\tau}{\sqrt{t - \tau}} \leq K \quad \text{for } t \geq \gamma. \tag{3.5}$$

Theorem 3.1. *Let (3.1)–(3.5) hold. Then problem (1.1)–(1.3) has bounded global solution for small initial datum.*

Proof. It is well known that problem (1.1)–(1.3) has a local nonnegative classical solution $u(x, t)$. Let $y(x, t)$ be a solution of the following problem

$$\begin{cases} y_t = \Delta y, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial y(x,t)}{\partial \nu} = \xi(t) + \beta(t), \ x \in \partial\Omega, \ t > 0, \\ y(x, 0) = 1, \ x \in \Omega, \end{cases} \tag{3.6}$$

where $\xi(t)$ is a positive continuous function that satisfies (3.4), (3.5) with $\beta(t) = \xi(t)$. According to Lemma 3.3 of [10] there exists a positive constant Y such that

$$1 \leq y(x, t) \leq Y, \qquad x \in \Omega, \ t > 0.$$

Due to (3.2), (3.3) for any $a \in (0, p)$, there exist $\varepsilon(a)$ and a positive continuous function $\eta(t)$ such that

$$0 < \varepsilon(a) < \frac{a}{Y}, \qquad \int_0^{\infty} \eta(t)\, dt < \infty \quad \text{and} \quad \int_{\varepsilon Y}^{a} \frac{ds}{f(s)} > Y \int_0^{\infty} (\alpha(t) + \eta(t))\, dt$$

for any $\varepsilon \in (0, \varepsilon(a))$. Now for any $T > 0$ we construct a positive supersolution of (1.1)–(1.3) in Q_T in such a form that

$$\overline{u}(x, t) = \varepsilon z(t) y(x, t),$$

where function $z(t)$ is defined in the following way

$$\int_{\varepsilon Y}^{\varepsilon Y z(t)} \frac{ds}{f(s)} = Y \int_0^t (\alpha(\tau) + \eta(\tau))\, d\tau.$$

It is easy to see that $\varepsilon Y z(t) < a$ and $z(t)$ is the solution of the following Cauchy problem

$$z'(t) - \frac{1}{\varepsilon}(\alpha(t) + \eta(t)) f(\varepsilon Y z(t)) = 0, \qquad z(0) = 1.$$

After simple computations it follows that

$$\begin{aligned} \overline{u}_t - \Delta \overline{u} - \alpha(t) f(\overline{u}) &= \varepsilon z' y + \varepsilon z y_t - \varepsilon z \Delta y - \alpha(t) f(\varepsilon z y) \\ &\geq \alpha(t)(f(\varepsilon Y z(t)) - f(\varepsilon z y)) + \eta(t) f(\varepsilon Y z(t)) > 0, \qquad x \in \Omega,\ t > 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \overline{u}(x,t)}{\partial \nu} - \beta(t) g(\overline{u}) &= \varepsilon z(t)(\xi(t) + \beta(t)) - \beta(t) g(\varepsilon z(t) y(x,t)) \\ &> \varepsilon z(t) \beta(t) \left[1 - \frac{g(\varepsilon z(t) y(x,t))}{\varepsilon z(t) y(x,t)} y(x,t) \right] \geq 0 \end{aligned}$$

for small values of a. Thus, by Theorem 2.1 there exists bounded global solution of (1.1)–(1.3) for any initial datum satisfying the inequality

$$u_0(x) < \varepsilon. \qquad \square$$

Remark 3.2. We suppose that $g(s)$ is a nondecreasing positive function for $s > 0$, $f(s) > 0$ for $s > 0$ and (2.1), (2.14) hold. Then by Theorem 2.2 and Theorem 2.6 (3.4) is necessary for global existence of solutions of (1.1)–(1.3).

Let for any $a > 0$ $g(s) > \delta(a) > 0$ if $s > a$. Then arguing in the same way as in the proof of Lemma 3.3 of [10] it is easy to show that (3.5) is necessary for the existence of nontrivial bounded global solutions of (1.1)–(1.3).

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Traveling front of polyhedral shape for a nonlocal delayed diffusion equation

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Abstract. This paper is concerned with the existence and stability of traveling fronts with convex polyhedral shape for nonlocal delay diffusion equations. By using the existence and stability results of V-form fronts and pyramidal traveling fronts, we first show that there exists a traveling front $V(x, y, z)$ with polyhedral shape of nonlocal delay diffusion equation associated with $z = h(x, y)$. Moreover, the asymptotic stability and other qualitative properties of such traveling front $V(x, y, z)$ are also established.

Keywords: traveling front, polyhedral shape, reaction-diffusion equation, nonlocal delayed.

2020 Mathematics Subject Classification: 34K30, 35C07, 35K57, 35B35.

1 Introduction

Recently, the study on the nonplanar traveling fronts of reaction-diffusion equations/systems has attracted an increasing attention and many types of nonplanar traveling fronts have been observed. See [5, 6, 9, 10, 14, 15, 28, 31] for V-shaped traveling fronts, see [9, 10, 23, 32] for cylindrically symmetric traveling fronts; see [4, 11, 16, 21, 22, 34] for pyramidal shaped traveling fronts and see [17–19, 23–27, 33] for other related works on multidimensional traveling fronts. It is well known that time delay and nonlocality play very important roles in the study of the population dynamics in biological and epidemiological models. Traveling fronts of reaction-diffusion equations with time delay in one or multidimensional spaces have been extensively studied, see [7, 8, 12, 20, 29, 30, 35]. Nevertheless, a very little attention has been paid to the study of nonplanar traveling fronts for reaction-diffusion equation with delay. As far as we know, Bao and Huang [1] proved that there exists two-dimensional V-shaped traveling fronts of bistable reaction-diffusion equation with delay, also see [3] for the existence of pyramidal traveling fronts. In [2], the author and Bao have established the existence of N-dimensional pyramidal traveling fronts of nonlocal delayed diffusion equation for $N \geq 3$ and see [13] for asymptotic stability of such pyramidal traveling fronts in the three-dimensional whole space.

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Motivated by [19, 23], in the current paper, we consider the existence, uniqueness and stability of three-dimensional traveling fronts with convex polyhedral shape for the following nonlocal delayed diffusion equation

$$\frac{\partial u}{\partial t}(x,y,z,t) = D\Delta u(x,y,z,t) - du(x,y,z,t) + \int_{\mathbb{R}} b\left(u(x,y,z_1,t-\tau)\right) f(z-z_1)dz_1, \tag{1.1}$$

where $D > 0$ and $d > 0$ denote the diffusion rate and death rate of the adult population, respectively, $\tau \geq 0$ is the maturation time for the species, $b(\cdot)$ is related to the birth function. The convolution in space term represents the nonlocal interaction in one direction and the kernel function $f(\cdot) \in C^\infty(\mathbb{R},\mathbb{R})$ satisfies

$$f(x) \geq 0, \int_{\mathbb{R}} f(y)dy = 1 \quad \text{and} \quad \int_{\mathbb{R}} e^{\lambda y} f(y)dy < +\infty \quad \text{for some } \lambda \geq 0. \tag{1.2}$$

Assume that

(A1) $b(\cdot) \in C^1(\mathbb{R},\mathbb{R})$ and there exists a constant $K > 0$ such that $b(0) = dK - b(K) = 0$;

(A2) $b'(u) \geq 0$ for $u \in [0,K]$ and $d > C\max\{b'(0), b'(K)\}$ for some constant $C > 1$;

(A3) there exits $u^* \in (0,K)$ such that $du^* - b(u^*) = 0$, $b'(u^*) > d$ and $du - b(u) \neq 0$ for $u \in (0,u^*) \cup (u^*,K)$.

By assumption (A1), (1.1) has at least two spatially homogeneous equilibria 0 and K and (1.1) is of nonlocal bistable structure if $b(u)$ satisfies (A1)–(A3). It is known from [12] that, under the assumption (A1)–(A3), there exists a unique solution pair (c,U) of (1.1) satisfying

$$DU''(\xi) - dU(\xi) - cU'(\xi) + \int_{\mathbb{R}} b(U(\xi - c\tau - y))f(y)dy = 0$$

and

$$U(-\infty) = 0, \quad U(+\infty) = K,$$

where $U(\cdot)$ is the monotone increasing wave profile and $c \in \mathbb{R}$ is the speed. Moreover, following from Wang et al. [30], if (A1)–(A3) hold, there exist positive constants β_1 and C_1 such that

$$\max\left\{U(-\xi), |U(\xi)-1|, |U'(\pm\xi)|, |U''(\pm\xi)|\right\} \leq C_1 e^{-\beta_1 \xi}, \quad \forall\, \xi \geq 0.$$

Define

$$[0,K]_C := \left\{\phi \in C(\mathbb{R}^3 \times [-\tau,0],\mathbb{R}) : 0 \leq \phi(x,y,z,r) \leq K, r \in [-\tau,0]\right\}.$$

Due to the effect of nonlocality in (1.1), the solution travel towards z direction. Set $z_1 = z + st$ and $u(x,y,z,t) = w(x,y,z_1,t)$. For simplicity, we still denote $w(x,y,z_1,t)$ by $w(x,y,z,t)$. Substituting w into (1.1), we have

$$\begin{cases} \frac{\partial w}{\partial t} = D\Delta w - s\frac{\partial w}{\partial z} - dw + \int_{\mathbb{R}} b(w(x,y,z-s\tau-z_1,t-\tau))f(z_1)dz_1, \\ w(x,y,z,r) = \phi(x,y,z,r), \quad (x,y,z) \in \mathbb{R}^3,\ r \in [-\tau,0]. \end{cases} \tag{1.3}$$

Let $w(x,y,z,t;\phi)$ be the solution of (1.3) with $w(x,y,z,r) = \phi(x,y,z,r) \in [0,K]_C$. Hereafter, we always assume $s > c > 0$. The objective of this paper is to seek for the solution $V(x,y,z) \in [0,K]_C$ of

$$\mathcal{L}[V] := -D\Delta V + s\frac{\partial V}{\partial z} + dV - \int_{\mathbb{R}} b\left(V(\mathbf{x}', z - s\tau - z_1)\right) f(z_1)dz_1 = 0 \quad \text{in } \mathbb{R}^3. \tag{1.4}$$

Let

$$m_* = \frac{\sqrt{s^2 - c^2}}{c}.$$

Given $n \geq 3$, assume that $\{\theta_j\}_{1\leq j\leq n}$ satisfies

$$0 < \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi \quad \text{and} \quad \max_{1\leq j\leq n} (\theta_{j+1} - \theta_j) < \pi,$$

where $\theta_{n+1} = \theta_1 + 2\pi$. Given s_j with

$$\min_{1\leq j\leq n} s_j \geq 0 \quad \text{for } 1 \leq j \leq n.$$

Then

$$\mu_j := \frac{1}{\sqrt{1+m_*^2}} \begin{pmatrix} m_* \cos\theta_j \\ m_* \sin\theta_j \\ -1 \end{pmatrix}$$

is the unit normal vector of a surface $\{z = m_*(x\cos\theta_j + y\sin\theta_j)\}$. Putting

$$\begin{aligned} &h_j(x,y) := m_*(x\cos\theta_j + y\sin\theta_j - s_j), \\ &h(x,y) := \max_{1\leq j\leq n} h_j(x,y) = m_* \max_{1\leq j\leq n} (x\cos\theta_j + y\sin\theta_j - s_j). \end{aligned} \tag{1.5}$$

Then $\{(x,y,z) \in \mathbb{R}^3 | -z \geq h(x,y)\}$ is a convex polyhedron. If $(s_1,...,s_n) = (0,0,...,0)$, the polyhedron becomes a pyramid in $\mathbb{R}^3$.

Define

$$\Theta := \max_{2\leq j\leq n-1} \frac{s_j \sin(\theta_{j+1} - \theta_{j-1}) - s_{j-1}\sin(\theta_{j+1} - \theta_j) - s_{j+1}\sin(\theta_j - \theta_{j-1})}{\sin(\theta_{j+1} - \theta_j) + \sin(\theta_j - \theta_{j-1}) - \sin(\theta_{j+1} - \theta_{j-1})}. \tag{1.6}$$

For $j = 1,2,\ldots,n$, define

$$\Omega_j := \{(x,y) \in \mathbb{R}^2 \mid h(x,y) = h_j(x,y),\, h(x,y) \geq m_*\Theta\}.$$

We note that $\Omega_j \neq \emptyset$ for all $1 \leq j \leq n$. Here $\Omega_1, \ldots, \Omega_n$ are located counterclockwise.

Set

$$S_j = \{(x,y,z) \in \mathbb{R}^3 \mid -z = h_j(x,y),\, (x,y) \in \Omega_j\}, \quad j = 1,\ldots,n.$$

Let

$$\Gamma_j = \{(x,y,z) \in \mathbb{R}^3 \mid -z = h_j(x,y) = h_{j+1}(x,y) \geq m_*\Theta\}, \quad j = 1,\ldots,n$$

be a part of an edge of a polyhedron $\{(x,y,z) \in \mathbb{R}^3 \mid -z \geq h(x,y)\}$. If $(s_1,\ldots,s_n) = (0,\ldots,0)$ and $\Theta = 0$, Γ_j and $\cup_{j=1}^n \Gamma_j$ stand for an edge and the set of all edges of a pyramid, respectively.

For each $\gamma > 0$, we define

$$D(\gamma) := \{(x,y,z) \in \mathbb{R}^3 \mid \operatorname{dist}((x,y,z), \cup_{j=1}^n \Gamma_j) > \gamma\}.$$

Define

$$v^-(x,y,z) = U\left(\frac{c}{s}(z + h(x,y))\right) = \max_{1\leqslant j\leqslant n} U\left(\frac{c}{s}(z + h_j(x,y))\right).$$

Theorem 1.1. *Let $s > c > 0$ and $h(x,y)$ be given by (1.5). Under the assumption (A1)–(A3), there exists a solution $V(x,y,z)$ of (1.4) such that*

$$\lim_{\gamma\to\infty} \sup_{(x,y,z)\in D(\gamma)} \left| V(x,y,z) - U\left(\frac{c}{s}(z+h(x,y))\right)\right| = 0, \tag{1.7}$$

$$0 < U\left(\frac{c}{s}(z+h(x,y))\right) < V(x,y,z) < K \quad \text{for all } (x,y,z)\in\mathbb{R}^3,$$

$$\lim_{R\to\infty} \sup_{|z+h(x,y)|>R} |V_z(x,y,z)| = 0,$$

$$\inf_{\delta\le V(x,y,z)\le K-\delta} V_z(x,y,z) > 0 \quad \text{for } \delta > 0 \text{ small}$$

and

$$\lim_{R\to\infty} \sup_{|\mathbf{x}|>R} \left| V(x,y,z) - \max_{1\le j\le n} E_j(x - X_j(\rho), y - Y_j(\rho), z + m_*\rho)\right| = 0,$$

where E_j is the two-dimensional V-shaped traveling front defined by (2.4) in Section 2 and $\rho \in (\Theta,\infty)$.

Theorem 1.2. *Let $V(x,y,z)$ be given by Theorem 1.1, $\widehat{s} = \max_{1\le j\le n} s_j > 0$, $\widetilde{V}$ is the pyramidal traveling front given in Theorem 2.3, $X_j(-\widehat{s})$, $Y_j(-\widehat{s})$ and $X_j(\rho), Y_j(\rho)$ satisfy $h(X_j(-\widehat{s}), Y_j(-\widehat{s})) = -m_*\widehat{s}$ and $h(X_j(\rho), Y_j(\rho)) = m_*\rho$ for $\rho \in (\Theta,\infty)$, respectively. If the initial value $\phi \in C(\mathbb{R}^3 \times [-\tau, 0], \mathbb{R})$ satisfies $\phi(x,y,z,r) \ge v^-(x,y,z)$ and*

$$\begin{aligned}&\max_{1\le j\le n} \widetilde{V}(x - X_j(-\widehat{s}), y - Y_j(-\widehat{s}), z - m_*\widehat{s})\\ \le&\phi(x,y,z,r) \le \min_{1\le j\le n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + m_*\rho),\end{aligned}$$

then the solution $w(x,y,z,t;\phi)$ of (1.3) satisfies

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^3} |w(x,y,z,t;\phi) - V(x,y,z)| = 0.$$

Note that the set $\{(x,y,z)\in\mathbb{R}^3 | -z \ge h(x,y)\}$ is a convex polyhedron for given $h(x,y)$ in (1.5). Then $V(x,y,z)$ given in Theorem 1.1 is called traveling front with convex polyhedral shape associated with $z = h(x,y)$. Since the polyhedron becomes a pyramid in $\mathbb{R}^3$ if $(s_1,\ldots,s_n) = (0,0,\ldots,0)$, then traveling front with convex polyhedral shape $V(x,y,z)$ becomes the pyramidal shape traveling front when $s_j = 0\,(j = 1,2,\ldots,n)$. Theorem 1.2 implies that such traveling front $V(x,y,z)$ is also asymptotically stable and uniquely determined by (1.4) and (1.7).

The rest of this paper is organized as follows. In Section 2, we state some preliminaries on the V-form traveling fronts and pyramidal traveling fronts. We study the existence and asymptotic stability of traveling fronts with convex polyhedral shape in Section 3.

2 Preliminary

In this section, we recall some results established in [2] and [13] including comparison principle, the existence and stability of V-form fronts and pyramidal traveling front in two dimensional space and three dimensional space, respectively.

Let $X = BUC(\mathbb{R}^3,\mathbb{R})$ be the Banach norm of all bounded and uniformly continuous functions from $\mathbb{R}^3$ to $\mathbb{R}$ with the usual supremum $|\cdot|_X$, and $X^+ = \{\phi \in X : \phi(x,y,z) \ge$

$0, \forall (x,y,z) \in \mathbb{R}^3\}$. Let $\phi \in [-\delta_0, K+\delta_0]_{\mathcal{C}} = \{\phi \in \mathcal{C} : \phi(x,y,z,s) \in [-\delta_0, K+\delta_0], s \in [-\tau, 0], (x,y,z) \in \mathbb{R}^3\}$ for some $\delta_0 > 0$.

Then, from [2, Theorem 2.1], we have the following existence and comparison theorem.

Theorem 2.1. *Assume that* (A1)–(A3) *hold. Then for any* $\phi \in [-\delta_0, K+\delta_0]_{\mathcal{C}}$, (1.3) *has a unique mild solution* $w(x,y,z,t;\phi)$ *on* $[0,\infty)$ *with* $-\delta_0 \le w(x,y,z,t;\phi) \le K+\delta_0$ *for* $(x,y,z,t) \in \mathbb{R}^3 \times [-\tau, \infty)$*, and* $w(x,y,z,t;\phi)$ *is a classical solution of* (1.3) *for* $(x,y,z,t) \in \mathbb{R}^3 \times [\tau,\infty)$*. Moreover, suppose that* $w^+(x,y,z,t)$ *and* $w^-(x,y,z,t)$ *are supersolution and subsolution of* (1.3) *on* $\mathbb{R}^3 \times \mathbb{R}^+$*, respectively, and satisfy* $-\delta_0 \le w^{\pm}(x,y,z,t) \le K+\delta_0$ *for* $t \in [-\tau,\infty)$ *and* $(x,y,z) \in \mathbb{R}^N$*, and* $w^-(x,y,z,s) \le w^+(x,y,z,s)$ *for any* $(x,y,z) \in \mathbb{R}^3$ *and* $s \in [-\tau, 0]$*. Then there holds* $w^+(x,y,z,t) \ge w^-(x,y,z,t)$ *for* $(x,y,z) \in \mathbb{R}^3, t \ge 0$*.*

Next, we state the existence and stability of V-form front of nonlocal delayed diffusion equation in two-dimensional space, see [2,13].

Let $\widehat{w}(\xi,\eta,t;\widehat{\phi})$ be the solution of

$$\begin{cases} \frac{\partial \widehat{w}}{\partial t} - D(\widehat{w}_{\xi\xi} + \widehat{w}_{\eta\eta}) + s\widehat{w}_\eta - d\widehat{w} + \int_{\mathbb{R}} b(\widehat{w}(\xi, \eta - s\tau - \eta_1, t-\tau)) f(\eta_1) d\eta_1 = 0, \\ \widehat{w}(\xi,\eta,r) = \widehat{\phi}(\xi,\eta,r), \quad (\xi,\eta) \in \mathbb{R}^2,\ r \in [-\tau, 0]. \end{cases} \tag{2.1}$$

Theorem 2.2. (See [2, Corollary 3.1]) *For any* $s > c$*, there exists a solution* $\widehat{V}(\xi,\eta)$ *satisfying*

$$-\widehat{V}_{\xi\xi} - \widehat{V}_{\eta\eta} + s\widehat{V}_\eta + d\widehat{V} - \int_{\mathbb{R}} b(\widehat{V}(\xi, \eta - s\tau - \eta_1)) f(\eta_1) d\eta_1 = 0 \tag{2.2}$$

for any $(\xi,\eta) \in \mathbb{R}^2$*. Moreover, there hold*

$$\widehat{V}(\xi,\eta) > U\left(\frac{c}{s}(\eta + m^*|\xi|)\right) \quad \text{for } (\xi,\eta) \in \mathbb{R}^2$$

and

$$\lim_{R\to\infty} \sup_{\xi^2+\eta^2 > R^2} \left| \widehat{V}(\xi,\eta) - U\left(\frac{c}{s}(\eta + m^*|\xi|)\right) \right| = 0.$$

One also has

$$\inf_{\delta \le \widehat{V}(\xi,\eta) \le K-\delta} \widehat{V}_\eta(\xi,\eta) > 0 \quad \text{for any } \delta \in (0, \delta^*]$$

and

$$\widehat{V}(\xi+\xi_0,\eta) \le \widehat{V}(\xi,\eta+\eta_0) \quad \forall (\xi,\eta) \in \mathbb{R}^2, \xi_0,\eta_0 \in \mathbb{R} \quad \text{with } \eta_0 \ge m_*|\xi_0|.$$

The solution $\widehat{w}(\xi,\eta,t;\phi)$ *of* (2.1) *satisfies*

$$\lim_{t\to\infty} \|\widehat{w}(\xi,\eta,t;\phi) - \widehat{V}(\xi,\eta)\|_{L^\infty(\mathbb{R}^2)} = 0$$

for any initial value $\phi(\xi,\eta,r) \in [0,K]_{\mathcal{C}}$ *satisfying* $\widehat{\phi}(\xi,\eta,r) \ge v^-(\xi,\eta)$ *and*

$$\lim_{\gamma\to\infty} \sup_{(\xi,\eta)\in D(\gamma), r \in [-\tau,0]} |\widehat{\phi}(\xi,\eta,r) - v^-(\xi,\eta)| = 0.$$

Set

$$\begin{aligned} p_j(x,y) &:= m_*(x\cos\theta_j + y\sin\theta_j), \\ p(x,y) &= \max_{1\le j\le n} h_j(x,y) = m_* \max_{1\le j\le n}(x\cos\theta_j + y\sin\theta_j) \end{aligned} \tag{2.3}$$

and

$$k_j := \cos\left(\frac{\theta_{j+1}-\theta_j}{2}\right) > 0, \quad \phi_j := \frac{\theta_{j+1}+\theta_j}{2}, \quad 1 \le j \le n.$$

Define

$$E_j(x,y,z) := \widehat{V}\left(x\sin\phi_j - y\cos\phi_j, \frac{z - m_* k_j(x\sin\phi_j + y\cos\phi_j)}{\sqrt{m_*^2 k_j^2 + 1}}\right). \tag{2.4}$$

It is easy to check that every $E_j(x,y,z)$ is a V-shaped traveling front with speed $\frac{s}{\sqrt{1+m_*^2k_j^2}} > 0$ for any $1 \le j \le n$, that is, $E_j(x,y,z)$ satisfies (2.2) in Theorem 2.2. By [2, Theorem 1.1] and [13, Theorem 1.2], we have the following existence and stability of pyramidal traveling front $\widetilde{V}(x,y,z)$ associated with a pyramid $z = p(x,y)$.

Theorem 2.3. *Assume that (A1)–(A3) hold true. Let $s > c > 0$ and $p(x,y)$ be given by (2.3). Then there exists a solution $\widetilde{V}(x,y,z)$ of (1.4) with*

$$\lim_{\gamma\to\infty} \sup_{(x,y,z)\in D(\gamma)} \left|\widetilde{V}(x,y,z) - U\left(\frac{c}{s}(z+p(x,y))\right)\right| = 0,$$

$$U\left(\frac{c}{s}(z+p(x,y))\right) < \widetilde{V}(x,y,z) < K \quad \text{for all } (x,y,z) \in \mathbb{R}^3,$$

$$\frac{\partial \widetilde{V}}{\partial z}(x,y,z) > 0 \quad \text{for all } (x,y,z) \in \mathbb{R}^3,$$

$$\lim_{R\to\infty} \sup_{|z+p(x,y)|\ge R} |\widetilde{V}_z(x,y,z)| = 0 \quad \text{and} \quad \inf_{\delta \le \widetilde{V}(\xi,\eta) \le K-\delta} \widetilde{V}_\eta(x,y,z) > 0 \quad \text{for any } \delta \in (0,\delta^*].$$

Suppose that the initial value $\phi(x,y,z,r) \in C(\mathbb{R}^3 \times [-\tau,0],\mathbb{R})$ satisfies $\phi(x,y,z,r) \ge v^-(x,y,z)$ and

$$\lim_{\gamma\to+\infty} \sup_{(x,y,z)\in D(\gamma), r\in[-\tau,0]} |\phi(x,y,z,r) - \widetilde{V}(x,y,z)| = 0,$$

then the solution $w(x,y,z,t;\phi)$ of (1.3) satisfies

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^3} |w(x,y,z,t;\phi) - \widetilde{V}(x,y,z)| = 0.$$

Furthermore, by [13], we have the following useful lemmas.

Lemma 2.4. *Let $\widetilde{V}(x,y,z)$ be as in Theorem 2.3 associated with pyramid $z = p(x,y)$. Then one has*

$$\lim_{R\to\infty} \sup_{|\mathbf{x}|\ge R} |\widetilde{V}(x,y,z) - \max_{1\le j\le n} E_j(x,y,z)| = 0,$$

$$\max_{1\le j\le n} E_j(x,y,z) < \widetilde{V}(x,y,z) \quad \text{for all } (x,y,z) \in \mathbb{R}^3$$

and

$$\lim_{\gamma\to\infty} \sup_{\mathbf{x}\in D(\gamma), t\in[0,T]} \left|\max_{1\le j\le n} E_j(x,y,z) - U\left(\frac{c}{s}(z+p(x,y))\right)\right| = 0.$$

Lemma 2.5 (See [13, Lemma 3.1]). *There exist a positive constant ρ sufficiently large and a positive constant β small enough such that for any $0 < \delta < \frac{\delta^*}{2}e^{-\beta\tau}$, the function*

$$w^+(x,y,z,t) := \widetilde{V}(x,y,z+\rho\delta(1-e^{-\beta t})) + \delta e^{-\beta t}$$

is a supersolution of (1.3) and the function

$$w^-(x,y,z,t) := \widetilde{V}(x,y,z-\rho\delta(1-e^{-\beta t})) - \delta e^{-\beta t}$$

is a subsolution of (1.3) for any $(x,y,z) \in \mathbb{R}^3$ and $t \ge 0$, where $\widetilde{V}(x,y,z)$ be as in Theorem 2.3.

3 Traveling front with polyhedral shape

In this section, we study the existence and asymptotic stability of traveling fronts with convex polyhedral shape of (1.1) and prove Theorems 1.1–1.2.

We first recall that $\{(x,y,z) \in \mathbb{R}^3 | -z \geq h(x,y)\}$ is a convex polyhedron. For any $\zeta \in \mathbb{R}$ and $1 \leq j \leq n$, let $(X_j(\zeta), Y_j(\zeta))$ be defined by

$$h_j(X_j(\zeta), Y_j(\zeta)) = h_{j+1}(X_j(\zeta), Y_j(\zeta)) = m_*\zeta.$$

Direct computations give

$$\begin{pmatrix} X_j(\zeta) \\ Y_j(\zeta) \end{pmatrix} = \frac{1}{\sin(\theta_{j+1} - \theta_j)} \begin{pmatrix} (\zeta + s_j)\sin\theta_{j+1} - (\zeta + s_{j+1})\sin\theta_j \\ -(\zeta + s_j)\cos\theta_{j+1} + (\zeta + s_{j+1})\cos\theta_j \end{pmatrix}.$$

As point in [23], a set $\{(x,y) \in \mathbb{R}^2 | h(x,y) \leq \zeta\}$ is either an empty set or a nonempty convex closed set in $\mathbb{R}^2$. By [23, Lemma 3.1], the set $\{(x,y) \in \mathbb{R}^2 | h(x,y) \leq m_*\rho\}$ is a convex $n-$polygon in the $x-y$ plane with vertices $\{(X_j(\rho), Y_j(\rho))\}_{1\leq j\leq n}$ for any fixed number $\rho \in (0, +\infty)$.

Proof of Theorem 1.1. Since $h(X_j(\rho), Y_j(\rho)) = m_*\rho$ for all $1 \leq j \leq n$, then we obtain

$$h(x,y) \leq m_*\rho + p(x - X_j(\rho), y - Y_j(\rho))$$

for all $(x,y) \in \mathbb{R}^N$, $1 \leq j \leq n$, where $h(x,y)$ and $p(x,y)$ are defined in (1.5) and (2.3), respectively. Set

$$v^-(x,y,z) = U\left(\frac{c}{s}(z + h(x,y))\right) = \max_{1\leq j\leq n} U\left(\frac{c}{s}(z + h_j(x,y))\right).$$

Note that the function $v^-(x,y,z)$ is a subsolution of (1.4) and the pyramidal traveling front $\widetilde{V}(x,y,z)$ defined in Theorem 2.3 is a solution of (1.4). Thus, we have

$$v^-(x,y,z) < \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + m_*\rho)$$

for all $(x,y,z) \in \mathbb{R}^3$ and $1 \leq j \leq n$. This shows that

$$\min_{1\leq j\leq n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + m_*\rho)$$

is a supersolution of (1.4) for all $(x,y,z) \in \mathbb{R}^3$. Define

$$V(x,y,z) := \lim_{t\to\infty} w(x,y,z,t;v^-), \quad \forall (x,y,z) \in \mathbb{R}^3.$$

Then the function $V(x,y,z) \in C^2(\mathbb{R}^3)$ is a solution of (1.4). As a result of the comparison principle (see Theorem 2.1), we have

$$v^-(x,y,z) < V(x,y,z) \leq \min_{1\leq j\leq n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + m_*\rho) \tag{3.1}$$

for all $(x,y,z) \in \mathbb{R}^3$. On the other hand, since

$$\max\{h_j(x,y), h_{j+1}(x,y)\} \leq h(x,y) \quad \text{in } \mathbb{R}^2, \forall 1 \leq j \leq n,$$

then

$$U\left(\frac{c}{s}(z+\max\{h_j(x,y),h_{j+1}(x,y)\})\right) \le v^-(x,y,z), \quad (x,y,z)\in\mathbb{R}^3,\ \forall 1\le j\le n.$$

We consider the left-hand side and the right hand side as an initial value of (1.3), respectively. Then Theorem 2.1 yields that

$$w\left(x,y,z,t;U\left(\frac{c}{s}(z+\max\{h_j(x,y),h_{j+1}(x,y)\})\right)\right) \le w(x,y,z,t;v^-(x,y,z)) \tag{3.2}$$

for all $1\le j\le n$. Note that

$$h_j(x,y) = p_j(x-X_j(\rho),y-Y_j(\rho)) + m_*\rho.$$

Recall that E_j $(1\le j\le n)$ is defined by (2.3). Let $t\to\infty$ in (3.2), by Lemma 2.4, we obtain

$$E_j(x-X_j(\rho),y-Y_j(\rho),z+m_*\rho) \le V(x,y,z), \quad (x,y,z)\in\mathbb{R}^3.$$

This together with (3.1), there is

$$\max_{1\le j\le n} E_j(x-X_j(\rho),y-Y_j(\rho),z+m_*\rho) \le V(x,y,z) \le \min_{1\le j\le n} \widetilde{V}(x-X_j(\rho),y-Y_j(\rho),z+m_*\rho)$$

for all $(x,y,z)\in\mathbb{R}^3$. By Theorem 2.2 and 2.3, we then have

$$\lim_{R\to\infty}\sup_{|\mathbf{x}|>R}\left|V(x,y,z) - \max_{1\le j\le n} E_j(x-X_j(\rho),y-Y_j(\rho),z+m_*\rho)\right| = 0 \tag{3.3}$$

and

$$0 < U\left(\frac{c}{s}(z+h(x,y))\right) < V(x,y,z) < K \quad \text{for all } (x,y,z)\in\mathbb{R}^3.$$

We use the Schauder interior estimate to the following equation

$$\left(\frac{\partial}{\partial t} - D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) + s\frac{\partial}{\partial z}\right)(V-E_j)$$
$$= -d(V-E_j) + \int_{\mathbb{R}} b(V(x,y,z-s\tau-z_1))f(z_1)dz - \int_{\mathbb{R}} b(E_j(x,y,z-s\tau-z_1))f(z_1)dz.$$

Then by Theorems 2.2–2.3 and (3.3), we obtain

$$\inf_{\delta\le V(x,y,z)\le K-\delta} V_z(x,y,z) > 0 \quad \text{for } \delta>0 \text{ small}.$$

Note that $|z+h(x,y)|\to\infty$ implies $\operatorname{dist}((x,y,z),\Gamma_j)\to\infty$ for $1\le j\le n$. Then we have

$$\lim_{\gamma\to\infty}\sup_{(x,y,z)\in D(\gamma)}\left|V(x,y,z) - U\left(\frac{c}{s}(z+h(x,y))\right)\right| = 0.$$

By the interpolation $\|\cdot\|_{C^1} \le 2\sqrt{\|\cdot\|_{C^0}\|\cdot\|_{C^2}}$ and the fact

$$\lim_{R\to\infty}\sup_{|z+h(x,y)|\ge R}\left|U_z\left(\frac{c}{s}(z+h(x,y))\right)\right| = 0,$$

we get

$$\lim_{R\to\infty}\sup_{|z+h(x,y)|>R}|V_z(x,y,z)| = 0.$$

This completes the proof. □

In the following, we show that the traveling front $V(x,y,z)$ with convex polyhedral shape is asymptotically stable.

Proof of Theorem 1.2. Set

$$\widehat{s} := \max_{1\le j\le n} s_j \ge 0.$$

Then there holds

$$-m_*\widehat{s} + p(x - X_j(-\widehat{s}), y - Y_j(-\widehat{s})) \le h(x,y) \quad \text{for } 1 \le j \le n. \tag{3.4}$$

It then follows that

$$U\left(\frac{c}{s}(z - m_*\widehat{s} + p(x - X_j(-\widehat{s}), y - Y_j(-\widehat{s})))\right) \le U\left(\frac{c}{s}(z + h(x,y))\right) \quad \text{for } 1 \le j \le n. \tag{3.5}$$

Consider the left-hand side and the right-hand side of (3.5) as initial values of (1.3) and let $t \to \infty$, we obtain

$$\widetilde{V}(x - X_j(-\widehat{s}), y - Y_j(-\widehat{s}), z - m_*\widehat{s}) \le V(x,y,z) \quad \text{for } (x,y,z) \in \mathbb{R}^3,\ 1 \le j \le n. \tag{3.6}$$

Together with (3.5) and (3.6), we have

$$\begin{aligned}\max_{1\le j\le n} &\widetilde{V}(x - X_j(-\widehat{s}), y - Y_j(-\widehat{s}), z - m_*\widehat{s}) \\ &\le V(x,y,z) \le \min_{1\le j\le n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + m_*\rho)\end{aligned} \tag{3.7}$$

for all $(x,y,z) \in \mathbb{R}^3$.

For all $(x,y,z) \in \mathbb{R}^3$, set

$$V^*(x,y,z) := \lim_{t\to\infty} w\left(x,y,z,t; \min_{1\le j\le n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + m_*\rho)\right).$$

Then the comparison principle gives that

$$V(x,y,z) \le V^*(x,y,z) \le \min_{1\le j\le n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + m_*\rho).$$

By (3.3) and Theorem 2.3, we have

$$\lim_{R\to\infty} \sup_{|\mathbf{x}|\ge R} |V^*(x,y,z) - V(x,y,z)| = 0.$$

It then follows the similar way in [23] that, there holds $V^*(x,y,z) \equiv V(x,y,z)$. This implies

$$\lim_{t\to\infty} \left\| w\left(x,y,z,t; \min_{1\le j\le n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + m_*\rho)\right) - V(x,y,z) \right\|_{L^\infty(\mathbb{R}^3)} = 0.$$

Using the similar process to $\max_{1\le j\le n} \widetilde{V}(x - X_j(-\widehat{s}), y - Y_j(-\widehat{s}), z - m_*\widehat{s})$, we also have

$$\lim_{t\to\infty} \left\| w\left(x,y,z,t; \max_{1\le j\le n} \widetilde{V}(x - X_j(-\widehat{s}), y - Y_j(-\widehat{s}), z - m_*\widehat{s})\right) - V(x,y,z) \right\|_{L^\infty(\mathbb{R}^3)} = 0.$$

Note that for any fixed $(x,y,z) \in \mathbb{R}^3$ and $t > 0$, $w(x,y,z,t;\cdot)$ is continuous mapping in X. By the continuity of $w(x,y,z,t;\cdot)$ and Theorems 2.1–2.3, we obtain

$$\lim_{t\to\infty} \|w(x,y,z,t;\phi) - V(x,y,z)\|_{L^\infty} = 0.$$

The proof is completed. $\square$

Furthermore, $V(x,y,z)$ also enjoys the following properties, which can be proved by the similar ways as that in [23, Lemma 3.3–3.5] and we omit them here.

Lemma 3.1. *Let $V(x,y,z)$ be as in Theorem 1.1. Then there holds*

(i) Let $h(x,y)$ be defined in (1.5), $\overline{h}(x,y)=\max_{1\le j\le n} h_j(x,y)=m_* \max_{1\le j\le n}(x\cos\theta_j+y\sin\theta_j-\overline{s}_j)$ with $\min_{1\le j\le n}\overline{s}_j\ge 0$. Define $\overline{V}(x,y,z)$ be the traveling front of polyhedral-shape associated with $\overline{h}(x,y)$. If $\overline{h}(x,y)\ge h(x,y)$ for any $(x,y)\in\mathbb{R}^2$, then $\overline{V}(x,y,z)\ge V(x,y,z)$ for all $(x,y,z)\in\mathbb{R}^3$.

(ii) One has $\frac{\partial V}{\partial \nu}(x,y,z)>0$ in $\mathbb{R}^3$ for

$$\nu=\frac{1}{\sqrt{1+t_1^2+t_2^2}}\begin{pmatrix} t_1\\ t_2\\ 1\end{pmatrix}\quad \textit{with } \sqrt{t_1^2+t_2^2}\le\frac{1}{m_*}.$$

(iii) If $h(x,y)=h(|x|,|y|)$, then there holds $V(x,y,z)=V(|x|,|y|,z)$ for all $(x,y,z)\in\mathbb{R}^3$ and

$$\begin{aligned} &V_x(x,y,z)>0 \quad \textit{for } (x,y,z)\in(0,\infty)\times\mathbb{R}^2,\\ &V_x(0,y,z)=0 \quad \textit{for } (y,z)\mathbb{R}^2,\\ &V_y(x,y,z)>0 \quad \textit{for } (x,y,z)\in\mathbb{R}\times(0,\infty)\times\mathbb{R},\\ &V_y(x,0,z)=0 \quad \textit{for } (x,z)\in\mathbb{R}^2.\end{aligned}$$

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Dynamics of a Leslie–Gower predator–prey system with cross-diffusion

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Abstract. A Leslie–Gower predator–prey system with cross-diffusion subject to Neumann boundary conditions is considered. The global existence and boundedness of solutions are shown. Some sufficient conditions ensuring the existence of nonconstant solutions are obtained by means of the Leray–Schauder degree theory. The local and global stability of the positive constant steady-state solution are investigated via eigenvalue analysis and Lyapunov procedure. Based on center manifold reduction and normal form theory, Hopf bifurcation direction and the stability of bifurcating time-periodic solutions are investigated and a normal form of Bogdanov–Takens bifurcation is determined as well.

Keywords: cross-diffusion, predator–prey system, global existence, stability, Hopf bifurcation, Bogdanov–Takens bifurcation.

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1 Introduction

In ecological systems, the interaction of predator and prey has abundant dynamical features although the investigations on predator-prey models has improved and lasted for several decades, which are based on the pioneering works of Lotka and Volterra [34]. Moreover, more realistic models are proposed in view of laboratory experiments and observations. Leslie and Gower [17] first proposed the following predator–prey model

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = u(a_1 - u - c_1 v), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = v\left(b_1 - \dfrac{d_1 v}{u}\right), \end{cases} \tag{1.1}$$

where $u(t)$ and $v(t)$ represent the densities of prey and predators at time t, respectively; the parameters a_1, b_1 c_1 and d_1 are positive constants; the term $d_1 v/u$ is called the Leslie–Gower

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terms, which measures the loss in the predator population due to rarity of its favorite food. System (1.1) is regarded as a prototypical predator-prey system in the ecological studies. But the interaction terms in (1.1) are unbounded, which is not reasonable in the real world. By using Holling type II functional response [13] in both prey and predator interaction terms, a Leslie–Gower predator–prey system with saturated functional responses is obtained and takes the form (see [4]):

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = u(a_1 - b_1 u - \dfrac{c_1 v}{u + k_1}), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = v\left(a_2 - \dfrac{c_2 v}{u + k_2}\right). \end{cases} \tag{1.2}$$

The model (1.2) is based on the biological fact that if the predator v is more capable of switching from its favorite food (the prey u) to other food options, then it has better ability to survive when the prey population is low; here a_1 and a_2 are the growth rates per capita of prey u and predator v, respectively; b_1 measures the strength of intraspecific competition among individuals of species u, and it is related to the carrying capacity of the prey; c_1 is the maximum value of the per capita reduction rate of u due to v, and c_2 is the maximum growth per capita of v due to predation of u; k_1 and k_2 measure the extent to which environment provides protection to prey u and predator v, respectively.

Non-monotonic responses appear at the microbial level; when the nutrient concentration reaches at a high level an inhibitory effect of the specific growth rate can occur [3,6]. This may frequently be noticed when micro-organisms are used for waste decomposition or for water purification. Andrews [3] suggested a response function $p(u) = \frac{mu}{k_1+k_2u+u^2}$, known as Monod–Haldane response function, to model such an inhibitory effect at high concentrations. In particular, Sokol and Howell [31] derived a simplified Monod–Haldane type $p(u) = \frac{mu}{k_1+u^2}$. A Leslie-Gower predator-prey system with a Monod–Haldane functional response takes the form:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = u\left(a_1 - b_1 u - \dfrac{mv}{k_1 + u^2}\right), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = v\left(a_2 - \dfrac{dv}{k_2 + u^2}\right). \end{cases} \tag{1.3}$$

In mathematical ecology, population may be distributed non-homogeneously, and the predators and preys naturally develop strategies for survival. Thus, we may introduce diffusive structure, which can be illustrated as different concentration levels of predators and preys causing different movements. Diffusion means the movement of individuals from a higher to a lower concentration region, while cross diffusion implies the population fluxes of one species owing to the presence of the other species. In this paper, our concern is the following system with cross-diffusion rates

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + u\left(a - u - \dfrac{v}{1+u^2}\right) & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = \Delta[(d_2 + \beta u)v] + v\left(b - \dfrac{v}{1+u^2}\right) & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = \varphi(x) \geq 0, \quad v(x,0) = \psi(x) \geq 0, & \text{in } \Omega, \end{cases} \tag{1.4}$$

whose corresponding ordinary differential equations (ODEs) is (1.3) with all the parameters b_1, m, k_1 and k_2 equal to 1. Here Δ denotes the Laplacian operator on $\mathbb{R}^N$ ($N \geq 1$), Ω is a connected bounded open domain in $\mathbb{R}^N$, with a smooth boundary $\partial\Omega$, $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega$. The homogeneous Neumann boundary condition means that the two species have zero flux across the boundary $\partial\Omega$. The diffusion terms d_j, $j = 1, 2$ stand for natural dispersive force of movement of an individual, while β describes the mutual interferences between individuals and is usually referred as the cross-diffusion pressure measuring the situation that the prey keeps away from the predator; a and b are the growth rates per capita of prey u and predator v. The parameters a, b, d_1 and d_2 are positive constants and β is non-negative constant.

In some cases, the quantity v is not influenced by any cross diffusion in the sense that the coefficient β in the second equation of (1.4) vanishes, that is, we ignore the population migration of predators due to the presence of preys. In this situation, Li et al. [20] considered the following reaction-diffusion system in the one-dimensional space domain $\Omega = (0, \pi)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + u\left(a - u - \dfrac{v}{1+u^2}\right) & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = d\Delta v + v\left(b - \dfrac{v}{1+u^2}\right) & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{1.5}$$

where d is the relative diffusion rate of predator v when the diffusion rate of prey is rescaled to 1. Li et al. [20] studied the Hopf bifurcation and steady-state bifurcation by taking d as the bifurcation parameter and described both the global structure of the steady-state bifurcation from simple eigenvalues and the local structure of the steady-state bifurcation from double eigenvalues by using space decomposition and the implicit function theorem.

The presence of the cross-diffusion term causes more abundant dynamic behaviors. For example, the effect of cross diffusion on dynamics of predator-prey models has been studied in [5, 7, 22, 24, 30, 35, 37, 43, 44, 47]. The relevant discussion is a bit difficult and requires more techniques than for models without cross-diffusion. In [5, 24, 43, 44], the researchers mainly obtained the non-existence and existence of non-constant positive steady-states (patterns) and showed cross diffusion can create non-constant steady states. Gambino et al. [7] analyzed the linear stability of the positive equilibrium of a competitive Lotka–Volterra system, and showed the cross-diffusion is the key mechanism for the formation of spatial patterns through Turing bifurcation. Liu et al. [22] not only obtained the global existence result of solutions under an appropriate parameter condition, but also gave explicit parameter ranges of the existence of non-constant positive steady-states.

For system (1.4), we first discuss the influence of the cross-diffusion coefficient β on the global existence of the solution. As far as global existence is concerned, many researchers have some relevant works, for example, [22, 26, 33, 41]. Wu et al. [41] and Tao [33] analyzed the predator-prey model with prey-taxis and discussed the effect of the prey-taxis term on the global existence of solutions of the system. Mu et al. [26] studied the global existence of classical solutions to a parabolic-parabolic chemotaxis system, but there are strict restrictions on functions in the system. Liu et al. [22] investigated the global existence of solutions of a parabolic-elliptic two-species competition model with cross diffusion.

Next, for a predator-prey system, what we are interested in is whether the various species can exist and takes the form of non-constant time-independent positive solutions. In [5, 8, 24, 25, 43, 44], the authors have established the existence of stationary patterns in some predator-prey models in the presence of self-diffusion and cross-diffusion. Our results are a little different from theirs. We not only prove the existence of non-constant solution of system (1.4) when the cross-diffusion β is sufficiently large, but also we find infinitely many intervals of $d_1 > 0$ near zero such that (1.4) admits at least one nonconstant solution if d_1 belongs to such intervals. Moreover, researchers have paid more attention to Hopf bifurcation and steady state bifurcation (cf. [9,10,15,18,19,36,42,46]), and investigated some predator-prey models without cross diffusion term. Only a few works [23, 45] have concentrated on the Bogdanov–Takens bifurcation phenomena of diffusive predator-prey systems with delay effect. In this paper, we study the Bogdanov–Takens bifurcation by regarding the cross-diffusion term β as one of bifurcation parameters.

The organization of the remaining part of the paper is as follows. In Section 2 we prove the global existence and boundedness results of solutions to (1.4) and in Section 3 we obtain a priori bounds of nonnegative steady state solutions. In Section 4 we deal with the non-existence of non-constant positive steady states for sufficient large diffusion coefficient and consider the existence of non-constant positive steady states for a small range of diffusion coefficient and sufficient large cross-diffusion coefficient by using the Leray–Schauder degree theory. Section 5 is devoted to the local and global stability of homogeneous steady states. Center manifold reduction and normal form theory are employed in Section 6 not only to discuss the existence of Hopf bifurcation but also to determine the Hopf bifurcation direction and the stability of bifurcating time-periodic solutions. In Section 7 we observe that system (1.4) exhibits Bogdanov–Takens bifurcation phenomena. Finally in Section 8, some conclusions are presented and numerical simulations are carried out to illustrate some previous theoretical results.

For convenience, we introduce the following notations. Let $H^k(\Omega)$ $(k \geq 0)$ be the Sobolev space of the L^2-functions f defined on Ω whose derivatives $f^{(n)}$ $(n = 1, \dots, k)$ also belong to $L^2(\Omega)$. Denote the spaces $\mathbb{X} = \{\phi \in H^2(\Omega) | \frac{\partial \phi}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega\}$ and $\mathbb{Y} = L^2(\Omega)$. For a space Z, we also define the complexification of Z to be $Z_{\mathbb{C}} \triangleq Z \oplus \mathrm{i}Z = \{x_1 + \mathrm{i}x_2 | x_1, x_2 \in Z\}$. Define an inner product on the complex-valued Hilbert space $\mathbb{Y}^2_{\mathbb{C}}$ by

$$\langle u, v\rangle = \int_\Omega \overline{u}(s)^T v(s) \mathrm{d}s \qquad \text{for } u, v \in \mathbb{Y}^2_{\mathbb{C}}. \tag{1.6}$$

2 Global existence and boundedness

In this section, we employ the method in [40] to obtain the global existence and boundedness of solutions of model (1.4). We need to establish some priori estimates. It is clear that the local existence of solutions to (1.4) was established by Amann [1]. This result can be summarized as follows.

Lemma 2.1. *For each fixed $p > N$, assume that the initial data $(\varphi, \psi) \in (W^{1,p}(\Omega))^2$ satisfies $\varphi \geq 0$ and $\psi \geq 0$, then there exists a positive constant $T_{\max}$ (the maximal existence time) such that (φ, ψ) determines a unique nonnegative classical solution $(u(x,t), v(x,t))$ of system (1.4) satisfying $(u, v) \in (C([0, T_{\max}), W^{1,p}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})))^2$ and*

$$0 \leq u(x,t) \leq c \triangleq \max\left\{\max_{\bar{\Omega}} \varphi(x), a\right\}, \quad v(x,t) \geq 0 \tag{2.1}$$

for all $(x,t) \in \bar{\Omega} \times [0, T_{\max})$.

Proof. (i) The local existence of the solution to (1.4) follows from [1]. Denote by $T_{\max}$ the maximal existence time of the solution. Next, we shall prove (2.1). On account of (1.4), we know that $v(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta[(d_2+\beta u)v] + v\left(b - \dfrac{v}{1+u^2}\right) & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ v(x,0) = \psi(x) \geq 0 & \text{in } \Omega. \end{cases} \tag{2.2}$$

Clearly, $\underline{v} \equiv 0$ is a sub-solution to problem (2.2). Hence, we can apply the maximum principle for parabolic equations to obtain that $v(x,t) \geq 0$. Similarly, we can obtain $u(x,t) \geq 0$. Also from (1.4) and $v \geq 0$, we obtain that

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\Delta u + u\left(a - u - \dfrac{v}{1+u^2}\right) \leq u(a-u) & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0) = \varphi(x) \geq 0 & \text{in } \Omega. \end{cases}$$

Then from comparison principle of parabolic equations, it is easy to verify $u(x,t) \leq c$, where c is given in (2.1). This completes the proof of Lemma 2.1. □

The above lemma means that, in the space $W^{1,p}(\Omega)$ each pair of the initial values φ and ψ can determine a unique nonnegative classical solution $(u(x,t), v(x,t))$, which is twice continuously differentiable with respect to $x \in \overline{\Omega}$ and continuously differentiable with respect to $t \in [0, T_{\max})$. Moreover, $u(\cdot,t)$, $v(\cdot,t) \in W^{1,p}(\Omega)$ can be regarded as a continuous mapping with respect to $t \in [0, T_{\max})$.

According to Amann's results [2], we need to establish the L^∞ bound of (u,v) in order to show its global existence. Based on Lemma 2.1, it is enough to establish the L^∞ bound of $v(x,t)$. Firstly, we shall show that the solution $v(x,t)$ is bounded in $L^1(\Omega)$. In the proof, we need to use the following elementary inequality [39].

Lemma 2.2. *Assume that $z(t) \geq 0$ satisfy*

$$\begin{cases} z'(t) \leq -a_1 z^r(t) + a_2 z(t) + a_3, & t > 0, \\ z(0) = z_0, \end{cases}$$

where $a_1, a_2, a_3 > 0$ and $r > 1$. Then there exist constants $c_1(a_1,a_2,a_3,r)$ and $c_2(z_0)$ such that

$$z(t) \leq \max\{c_1(a_1,a_2,a_3,r), c_2(z_0)\}.$$

Lemma 2.3. *There exists a constant $C_0 > 0$ such that the second component of the solution of (1.4) satisfies the following estimate*

$$\int_\Omega v(x,t)\mathrm{d}x \leq C_0 \quad \text{for all } t \in (0, T_{\max}). \tag{2.3}$$

Proof. Let

$$U(t) = \int_\Omega u(x,t)\mathrm{d}x, \quad V(t) = \int_\Omega v(x,t)\mathrm{d}x.$$

Then we have

$$\dot{U}(t) + \dot{V}(t) = \int_\Omega (au+bv)\mathrm{d}x - \int_\Omega \left(u^2 + \frac{uv}{1+u^2} + \frac{v^2}{1+u^2}\right)\mathrm{d}x.$$

In view of Lemma 2.1, we know that $0 \le u(x,t) \le c$ for all $(x,t) \in \bar{\Omega} \times [0, T_{\max})$ and hence that

$$u^2 + \frac{uv}{1+u^2} + \frac{v^2}{1+u^2} \ge \frac{(u+v)^2}{2(1+u^2)} \ge \frac{(u+v)^2}{2(1+c^2)},$$

which, together with the Hölder inequality, implies that

$$\begin{aligned}\dot{U}(t) + \dot{V}(t) &\le \int_\Omega r(u+v)\mathrm{d}x - \int_\Omega \frac{(u+v)^2}{2(1+c^2)}\mathrm{d}x \\ &\le r\int_\Omega (u+v)\mathrm{d}x - \frac{1}{2(1+c^2)|\Omega|}\left[\int_\Omega (u+v)\mathrm{d}x\right]^2 \\ &= r[U(t)+V(t)] - \frac{[U(t)+V(t)]^2}{2(1+c^2)|\Omega|}\end{aligned}$$

with $r = \max\{a, b\}$. It follows from Lemma 2.2 that there exists a positive constant M such that $U(t) + V(t) \le M$ for all $t \in (0, T_{\max})$, and hence that there exists a positive constant C_0 such that (2.3) holds. The proof is completed. □

Secondly, we will establish L^p estimates for $v(x,t)$ by using a weight function $\phi(u)$ similar to that in [32,38,41]. We now present some basic inequalities which will be used in the sequel (see [14,27]). In several places we shall need the following Poincaré's inequality:

$$\|u\|_{1,p} \le C_4(\|\nabla u\|_p + \|u\|_q) \quad \text{for all } u \in W^{1,p}(\Omega)$$

with arbitrary $p > 1$ and $q > 0$. Also, an essential role will be played by Gagliardo-Nirenberg interpolation inequality

$$\|u\|_p \le C_3 \|u\|_{1,q}^{\eta} \|u\|_m^{1-\eta} \quad \text{for all } u \in W^{1,p}(\Omega),$$

which holds for all $1 \le p, q \le \infty$ satisfying $p(n-q) < nq$ and all $m \in (0, p)$ with

$$\eta = \frac{\frac{n}{m} - \frac{n}{p}}{\frac{n}{m} + 1 - \frac{n}{q}} \in (0,1).$$

Lemma 2.4. *Let $(u(x,t), v(x,t))$ be a solution of (1.4), then for every $p \in [2, \infty)$, there exists a positive constant $E > 0$ such that*

$$\|v(x,t)\|_p \le E \quad \text{for } t \in (0, T_{\max})$$

if

$$\beta \in \left[0, \frac{d_1 d_2}{2\sqrt{2}(d_1+d_2)pc}\right]. \tag{2.4}$$

Proof. Let

$$\alpha = \frac{d_1 d_2 (p-1)}{4(d_1+d_2)^2 pc^2}, \tag{2.5}$$

and consider a weight function

$$\phi(u(x,t)) = e^{\alpha u^2(x,t)} \quad \text{when } 0 \le u(x,t) \le c. \tag{2.6}$$

Denote $\phi(u(x,t))$ by $\phi(u)$, then we have

$$1 \le \phi(u) = e^{\alpha u^2} \le e^{\alpha c^2} = h \quad \text{and} \quad ,1 \le \phi'(u) = 2\alpha u e^{\alpha u^2} \le 2\alpha c e^{\alpha c^2}, \qquad 0 \le u \le c. \tag{2.7}$$

It follows from system (1.4) that

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p\phi(u)\mathrm{d}x =& \int_\Omega v^{p-1}\phi(u)\frac{\partial v}{\partial t}\mathrm{d}x + \frac{1}{p}\int_\Omega v^p\phi'(u)\frac{\partial u}{\partial t}\mathrm{d}x \\ =& \int_\Omega v^{p-1}\phi(u)[\Delta(d_2+\beta u)v]\mathrm{d}x + \int_\Omega v^p\phi(u)\left(b - \frac{v}{1+u^2}\right)\mathrm{d}x \\ &+ \frac{1}{p}\int_\Omega v^p\phi'(u)\left[d_1\Delta u + u\left(a - u - \frac{v}{1+u^2}\right)\right]\mathrm{d}x \\ \le& -(p-1)\int_\Omega v^{p-2}(d_2+\beta u)\phi(u)|\nabla v|^2\mathrm{d}x \\ &-(p-1)\beta\int_\Omega v^{p-1}\phi(u)\nabla u\cdot\nabla v\mathrm{d}x - \int_\Omega v^{p-1}\phi'(u)(d_2+\beta u)\nabla u\cdot\nabla v\mathrm{d}x \\ &-\beta\int_\Omega v^p\phi'(u)|\nabla u|^2\mathrm{d}x + b\int_\Omega v^p\phi(u)\mathrm{d}x - d_1\int_\Omega v^{p-1}\phi'(u)\nabla u\cdot\nabla v\mathrm{d}x \\ &-\frac{d_1}{p}\int_\Omega v^p\phi''(u)|\nabla u|^2\mathrm{d}x + \frac{ac}{p}\int_\Omega v^p\phi'(u)\mathrm{d}x,\end{aligned}$$

which implies that

$$\begin{aligned}&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p\phi(u)\mathrm{d}x + (p-1)d_2\int_\Omega v^{p-2}\phi(u)|\nabla v|^2\mathrm{d}x + \frac{d_1}{p}\int_\Omega v^p\phi''(u)|\nabla u|^2\mathrm{d}x \\ &\le -\int_\Omega (d_2+\beta u)v^{p-1}\phi'(u)\nabla u\cdot\nabla v\mathrm{d}x - \beta(p-1)\int_\Omega v^{p-1}\phi(u)\nabla u\cdot\nabla v\mathrm{d}x \\ &\quad - d_1\int_\Omega v^{p-1}\phi'(u)\nabla u\cdot\nabla v\mathrm{d}x + b\int_\Omega v^p\phi(u)\mathrm{d}x + \frac{ac}{p}\int_\Omega v^p\phi'(u)\mathrm{d}x.\end{aligned} \tag{2.8}$$

In virtue of (2.7), we know that $\phi'(u), \phi(u) > 0$. Combining with $v(x,t) \ge 0$, it is easy to see that

$$\begin{aligned}&-(d_1+d_2)\int_\Omega v^{p-1}\phi'(u)\nabla u\cdot\nabla v\mathrm{d}x \\ &\le \int_\Omega \frac{\sqrt{\phi(u)d_2(p-1)}v^{\frac{p-2}{2}}|\nabla v|}{\sqrt{2}}\cdot\frac{\sqrt{2}(d_1+d_2)v^{\frac{p}{2}}\phi'(u)|\nabla u|}{\sqrt{\phi(u)d_2(p-1)}}\mathrm{d}x.\end{aligned}$$

Furthermore, using Young's inequality, we obtain

$$\begin{aligned}&-(d_1+d_2)\int_\Omega v^{p-1}\phi'(u)\nabla u\cdot\nabla v\mathrm{d}x \\ &\le \frac{d_2(p-1)}{4}\int_\Omega v^{p-2}\phi(u)|\nabla v|^2\mathrm{d}x + \frac{(d_1+d_2)^2}{d_2(p-1)}\int_\Omega v^p\frac{\phi'^2(u)}{\phi(u)}|\nabla u|^2\mathrm{d}x.\end{aligned} \tag{2.9}$$

Similar to the above, we obtain

$$\begin{aligned}&-\beta(p-1)\int_\Omega v^{p-1}\phi(u)\nabla u\cdot\nabla v\mathrm{d}x \\ &\le \frac{d_2(p-1)}{4}\int_\Omega v^{p-2}\phi(u)|\nabla v|^2\mathrm{d}x + \frac{\beta^2(p-1)}{d_2}\int_\Omega v^p\phi(u)|\nabla u|^2\mathrm{d}x.\end{aligned} \tag{2.10}$$

Together with $0 \le u \le c$, we similarly have

$$\begin{aligned}&-\beta\int_\Omega uv^{p-1}\phi'(u)\nabla u\cdot\nabla v\mathrm{d}x \\ &\le \frac{d_2(p-1)}{4}\int_\Omega v^{p-2}\phi(u)|\nabla v|^2\mathrm{d}x + \frac{\beta^2(p-1)}{d_2}\int_\Omega u^2v^p\frac{\phi'^2(u)}{\phi(u)}|\nabla u|^2\mathrm{d}x \\ &\le \frac{d_2(p-1)}{4}\int_\Omega v^{p-2}\phi(u)|\nabla v|^2\mathrm{d}x + \frac{\beta^2(p-1)c^2}{d_2}\int_\Omega v^p\frac{\phi'^2(u)}{\phi(u)}|\nabla u|^2\mathrm{d}x.\end{aligned} \tag{2.11}$$

Substituting (2.9), (2.10) and (2.11) into (2.8), we have

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p\phi(u)\mathrm{d}x+\frac{d_2(p-1)}{4}\int_\Omega v^{p-2}\phi(u)|\nabla v|^2\mathrm{d}x+\frac{d_1}{p}\int_\Omega v^p\phi''(u)|\nabla u|^2\mathrm{d}x\\
&\quad\le\left[\frac{(d_1+d_2)^2}{d_2(p-1)}+\frac{\beta^2c^2(p-1)}{d_2}\right]\int_\Omega v^p\frac{\phi'^2(u)}{\phi(u)}|\nabla u|^2\mathrm{d}x\\
&\qquad+\frac{\beta^2(p-1)}{d_2}\int_\Omega v^p\phi(u)|\nabla u|^2\mathrm{d}x+b\int_\Omega v^p\phi(u)\mathrm{d}x+\frac{ac}{p}\int_\Omega v^p\phi'(u)\mathrm{d}x.
\end{aligned}\tag{2.12}$$

Clearly,

$$\frac{\phi'^2(u)}{\phi(u)}=4\alpha^2u^2\phi(u)\quad\text{and}\quad\phi''(u)=(2\alpha+4\alpha^2u^2)\phi(u).$$

By a direct calculation, we obtain

$$\begin{aligned}
&\frac{2a_2(u)}{a_1(u)}\le\frac{4(d_1+d_2)^2c^2p}{d_1d_2(p-1)}\alpha=1,\\
&\frac{4a_3(u)}{a_1(u)}\le\frac{2\beta^2(p-1)p}{d_1d_2\alpha}=\frac{8\beta^2c^2p^2(d_1+d_2)^2}{d_1^2d_2^2}\le1,\\
&\frac{4a_4(u)}{a_1(u)}\le\frac{4c^2\beta^2p(p-1)}{d_1d_2}=\frac{4p(p-1)}{d_1d_2}\cdot\frac{d_1^2d_2^2}{8p^2(d_1+d_2)^2}=\frac{d_1d_2(p-1)}{2p(d_1+d_2)^2}<1,
\end{aligned}\tag{2.13}$$

for $0\le u\le c$, where β and α satisfy (2.4) and (2.5) respectively, and

$$\begin{aligned}
a_1(u)&=\frac{d_1}{p}\phi''(u),\\
a_2(u)&=\frac{(d_1+d_2)^2}{d_2(p-1)}\cdot\frac{\phi'^2(u)}{\phi(u)},\\
a_3(u)&=\frac{\beta^2(p-1)}{d_2}\phi(u),\\
a_4(u)&=\frac{\beta^2c^2(p-1)}{d_2}\cdot\frac{\phi'^2(u)}{\phi(u)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\beta^2(p-1)}{d_2}\int_\Omega v^p\phi(u)|\nabla u|^2\mathrm{d}x+\frac{(d_1+d_2)^2}{d_2(p-1)}\int_\Omega v^p\frac{\phi'^2(u)}{\phi(u)}|\nabla u|^2\mathrm{d}x\\
&\quad+\frac{\beta^2c^2(p-1)}{d_2}\int_\Omega v^p\frac{\phi'(u)^2}{\phi'(u)}|\nabla u|^2\mathrm{d}x\le\frac{d_1}{p}\int_\Omega v^p\phi''(u)|\nabla v|^2\mathrm{d}x.
\end{aligned}\tag{2.14}$$

It follows from (2.14) that (2.12) is simplified to be

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p\phi(u)\mathrm{d}x+\frac{d_2(p-1)}{4}\int_\Omega v^{p-2}\phi(u)|\nabla v|^2\le C_1\int_\Omega v^p\phi(u)\mathrm{d}x,\tag{2.15}$$

where $C_1=(bp+2\alpha ac^2)/p$. By the Gagliardo–Nirenberg and Poincaré's inequality and (2.7) and (2.3), we have

$$\begin{aligned}\int_\Omega v^p\phi(u)\mathrm{d}x &\le h\int_\Omega v^p\mathrm{d}x = h\|v^{\frac{p}{2}}\|_2^2 \le C_2\|v^{\frac{p}{2}}\|_{1,2}^{2\eta}\|v^{\frac{p}{2}}\|_{\frac{2}{p}}^{2(1-\eta)} \\ &\le hC_3\left(C_4\left(\frac{2}{p}\right)\right)^{2\eta}\left(\|\nabla v^{\frac{p}{2}}\|_2+\|v^{\frac{p}{2}}\|\right)^{2\eta}\|v^{\frac{p}{2}}\|_{\frac{2}{p}}^{2(1-\eta)} \\ &= hC_3\left(C_4\left(\frac{2}{p}\right)\right)^{2\eta}\left(\|\nabla v^{\frac{p}{2}}\|_2+\|v\|_1^{\frac{p}{2}}\right)^{2\eta}\|v\|_1^{p(1-\eta)} \\ &\le C_5\left(\|\nabla v^{\frac{p}{2}}\|_2^2+1\right)^{\eta},\end{aligned} \tag{2.16}$$

where

$$\eta = \frac{pn-n}{2-n+pn} \in (0,1).$$

Now from (2.7) and (2.16), we have

$$\begin{aligned}\int_\Omega v^{p-2}\phi(u)|\nabla v|^2\mathrm{d}x &\ge \int_\Omega v^{p-2}|\nabla v|^2\mathrm{d}x = \frac{4}{p^2}\int_\Omega|\nabla v^{\frac{p}{2}}|^2\mathrm{d}x \\ &\ge \frac{4}{p^2C_5^{\frac{1}{\eta}}}\left(\int_\Omega v^p\phi(u)\mathrm{d}x\right)^{\frac{1}{\eta}} - \frac{4}{p^2}.\end{aligned} \tag{2.17}$$

Hence from (2.15) and (2.17) we obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p\phi(u)\mathrm{d}x \le -\frac{d_2(p-1)}{p^2C_5^{\frac{1}{\eta}}}\left(\int_\Omega v^p\phi(u)\mathrm{d}x\right)^{\frac{1}{\eta}} + C_1\int_\Omega v^p\phi(u)\mathrm{d}x + \frac{d_2(p-1)}{p^2}$$

for all $t\in(0,T_{\max})$, where $\frac{1}{\eta}>1$. By using Lemma 2.2 and (2.7), we conclude that there exists $E>0$ such that

$$\|v(\cdot,t)\|_p \le \left(\int_\Omega v^p\phi(u)\mathrm{d}x\right)^{\frac{1}{p}} \le E \quad \text{for } t\in(0,T_{\max}),$$

which is the desired result. □

Finally, we establish the L^∞ bound of $v(x,t)$ using Lemma 2.4.

Lemma 2.5. *If β satisfies (2.4) and let $(u(x,t),v(x,t))$ be a solution of (1.4). Then there exists a positive constant A such that*

$$\|v(\cdot,t)\|_\infty \le A \quad \text{for } t\in(0,T_{\max}).$$

Proof. Define

$$f(u,v) = u\left(a-u-\frac{v}{1+u^2}\right), \qquad g(u,v)=\beta v\Delta u + v\left(b-\frac{v}{1+u^2}\right)$$

for $(u,v)\in(C([0,T_{\max}),W^{1,p}(\Omega)\cap C^{2,1}(\bar\Omega\times(0,T_{\max})))^2$. It follows from Lemmas 2.4 and 2.1 that there exists a positive constant A_1 such that

$$\|f\|_{L^p(\Omega)} \le A_1 < +\infty \quad \text{for all } t\in(0,T_{\max}). \tag{2.18}$$

In virtue of (2.18) and the first equation of system (1.4) and the L^p-estimate for parabolic equations, we obtain

$$\|u(\cdot,t)\|_{W_p^2(\Omega)} \le A_1 \quad \text{for all } t\in(0,T_{\max}). \tag{2.19}$$

This, together with the Sobolev embedding theorem (see [16]), yields

$$\|\nabla u(\cdot,t)\|_{L^\infty(\Omega)} \le A_1 \quad \text{for all } t \in (0, T_{\max}). \tag{2.20}$$

We now turn to the second equation of (1.4), which can be rewritten as the non-divergence form:

$$\frac{\partial v}{\partial t} = (d_2 + \beta u)\Delta v + 2\beta \nabla u \cdot \nabla v + g(u,v). \tag{2.21}$$

In virtue of Lemmas 2.4 and 2.1 and (2.19), we have

$$\|g(u,v)\|_{L^p(\Omega)} \le A_1 \quad \text{and} \quad \|d_2 + \beta u\|_{L^\infty(\Omega)} \le A_1 \quad \text{for all } t \in (0, T_{\max}). \tag{2.22}$$

Using (2.21), (2.20) and (2.22) and the L^p-estimate for parabolic equations, we have

$$\|v(\cdot,t)\|_{W_p^2(\Omega)} \le A_1 \quad \text{for all } t \in (0, T_{\max}).$$

Again, taking p to be sufficiently large and combing with the Sobolev embedding theorem (see [16]), we have

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le A \quad \text{for all } t \in (0, T_{\max}).$$

Hence, this proof is completed. □

Obviously, from Lemmas 2.1 and 2.5 and [2], we conclude that $T_{\max} = \infty$ and $\|v(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty \le M(\varphi,\psi)$ for all $t \in [0,\infty)$, where $M(\varphi,\psi)$ depends on the initial value (φ,ψ). Notice that in the proof of Lemma 2.1, for any positive constant ε_0, there exists $t_1 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty} \le a + \varepsilon_0 \quad \text{for all } t \in (t_1, \infty). \tag{2.23}$$

Hence we can replace c by $a + \varepsilon_0$ for $t \in (t_1,\infty)$. Similarly in Lemma 2.3, C_0 can be chosen to be independent of (φ,ψ). So $\int_\Omega u(x,t) \le C_0$ for $t \in (t_2,\infty)$ with $t_2 > t_1$. Again in the proof of Lemmas 2.4 and 2.5, we can also replace c by $a+\varepsilon_0$ and then we can find $t_0 > t_2$ such that

$$\|v(\cdot,t)\|_p \le E \quad \text{for all } t \in (t_0,\infty)$$

and

$$\|v(\cdot,t)\|_\infty \le A \quad \text{for all } t \in (t_0,\infty) \tag{2.24}$$

if

$$\beta \in \left[0, \frac{d_1 d_2}{2\sqrt{2}(d_1+d_2)pa}\right], \tag{2.25}$$

where E and A are independent of (φ,ψ). In view of (2.23) and (2.24), there exists a constant M_1 such that

$$\|v(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty \le M_1 \quad \text{for all } t \in (t_0,\infty),$$

where M_1 is independent of (φ,ψ). Therefore, we have the following theorem.

Theorem 2.6. *Suppose that $p > N$ and β satisfies (2.4), then every initial value $(\varphi,\psi) \in (W^{1,p}(\Omega))^2$ satisfying $\varphi(x) \ge 0$ and $\psi(x) \ge 0$ for all $x \in \Omega$, determines a unique global classical solution $(u(x,t), v(x,t))$ of system (1.4), which satisfies $(u,v) \in (C([0,\infty); W^{1,p}(\Omega)) \cap C^{2,1}(\Omega \times [0,\infty)))^2$. Moreover, (u,v) is uniformly bounded in $\Omega \times (0,\infty)$, that is, there exists a constant $M(\varphi,\psi) > 0$, depending on the initial (φ,ψ), such that $\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty \le M$ for all $t \in [0,\infty)$. Furthermore, if β satisfies (2.25), then there exist two positive constants M_1, independent of (φ,ψ), and $t_0 > 0$, such that $\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty \le M_1$ for all $t \in (t_0,\infty)$.*

3 A priori estimates

Steady-state solutions of (1.4) satisfy the following system:

$$\begin{cases} d_1\Delta u + u\left(a - u - \dfrac{v}{1+u^2}\right) = 0 & \text{in } \Omega, \\ \Delta[(d_2+\beta u)v] + v\left(b - \dfrac{v}{1+u^2}\right) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \\ u(x,0) = \varphi(x) \geq 0, \quad v(x,0) = \psi(x) \geq 0, & \text{in } \Omega. \end{cases} \tag{3.1}$$

It is easy to see that system (1.4) has a positive constant steady-state solution $\mathbf{e} = (u^*, v^*)^T$ if and only if $a > b$, where $u^* = \theta \triangleq a - b$, $v^* = b(1+\theta^2)$.

Next, we study the asymptotic behavior of positive solutions of (3.1) as d_1 is small or β is sufficiently large. For the first step of the asymptotic analysis, we derive a priori positive upper and lower bounds for positive solutions to (3.1).

Lemma 3.1. *Suppose that (u,v) is a solution of (3.1) and $a \neq b$, then there exists a positive constant $\check{C}$ such that (u,v) satisfies*

$$\check{C} \leq u(x) \leq a, \qquad \frac{d_2 b}{d_2+\beta} \leq v(x) \leq \kappa \triangleq \frac{b}{d_2}(d_2+\beta a)(1+a^2)$$

for all $x \in \bar{\Omega}$.

Proof. Let $x_0 \in \bar{\Omega}$ be a maximum point of u, i.e., $u(x_0) = \max_{x\in\bar{\Omega}} u(x)$. Then by using the maximum principle [24] to the first equation of (3.1), one has $a - u(x_0) - \frac{v(x_0)}{1+u^2(x_0)} \geq 0$ and hence $u \leq a$.

By setting $w = (d_2+\beta u)v$, we can reduce the second equation of (3.1) with the boundary condition to

$$\begin{cases} \Delta w + v\left(b - \dfrac{v}{1+u^2}\right) = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Let $x_1 \in \bar{\Omega}$ be a maximum point of w, i.e., $w(x_1) = \max_{x\in\bar{\Omega}} w(x)$. Applying the maximum principle [24] to (3.2), we get $v(x_1) \leq b(1+u^2(x_1)) = b(1+a^2)$. Note that $0 \leq u(x_1) \leq a$ and $v(x_1) \leq b(1+a^2)$, then we have $\max_{\bar{\Omega}} w(x) = w(x_1) \leq b(d_2+\beta a)(1+a^2)$, which in turn implies that

$$\max_{\bar{\Omega}} v(x) \leq \frac{1}{d_2}\max_{\bar{\Omega}} w(x) = \frac{1}{d_2}[d_2+\beta u(x_1)]v(x_1) \leq \frac{b}{d_2}(d_2+\beta a)(1+a^2) = \kappa.$$

To obtain the lower bound for v, we define $w(y_0) = \min_{\bar{\Omega}} w(x)$. Similarly, applying the maximum principle [24] to (3.2) yields $v(y_0) \geq b(1+u^2(y_0))$. According to the definition of w, we obtain

$$\min_{\bar{\Omega}} v \geq \frac{\min_{\bar{\Omega}} w}{d_2+\beta\max_{\bar{\Omega}} u} = \frac{w(y_0)}{d_2+\beta u(x_0)} = \frac{d_2+\beta u(y_0)}{d_2+\beta u(x_0)}v(y_0) \geq \frac{d_2 b}{d_2+\beta a}.$$

Now, denote $u(y_1) = \min_{\bar{\Omega}} u(x)$ for some $y_1 \in \bar{\Omega}$. It follows from the maximum principle [24] that

$$u(y_1) \geq a - \frac{v(y_1)}{1 + u^2(y_1)}. \tag{3.3}$$

Note that

$$\frac{v(y_1)}{1 + u^2(y_1)} \leq v(y_1) \leq \max_{\bar{\Omega}} v(x) \leq \kappa. \tag{3.4}$$

This, together with (3.3) and (3.4), implies that

$$u(y_1) \geq a - \frac{v(y_1)}{1 + u^2(y_1)} \geq a - \kappa.$$

If $a > \kappa$ then $u(x) \geq a - \kappa$ for all $x \in \overline{\Omega}$ and hence the proof of Lemma 3.1 is completed.

In what follows, we shall show that $u(x) \geq \check{C}$ in the case where $a \leq \kappa$. Let

$$c_1(x) = a - u(x) - \frac{v(x)}{1 + u^2(x)},$$

then

$$|c_1(x)| \leq 2a + \kappa.$$

From Harnack's inequality (see [21]), there exists a positive constant C^* such that

$$\max_{\bar{\Omega}} u(x) \leq C^* \min_{\bar{\Omega}} u(x).$$

Hence, it remains to prove that there is a positive constant ε such that $\max_{\bar{\Omega}} u(x) > \varepsilon$. Suppose this is not true, then there exists a sequence $\{(d_{1n}, d_{2n}, \beta_n)\}_{n=1}^{\infty}$ such that the corresponding positive solutions (u_n, v_n) of problem (3.1) with $(d_1, d_2, \beta) = (d_{1n}, d_{2n}, \beta_n)$ satisfy $\max_{\bar{\Omega}} u_n \to 0$ as $n \to \infty$.

From the Sobolev embedding theorem and elliptic estimates, there exists a subsequence of $\{(u_n, v_n)^T\}_{n=1}^{\infty}$, which we still denote by $\{(u_n, v_n)\}_{n=1}^{\infty}$, such that $u_n \to u_\infty$ and $v_n \to v_\infty$ in $C^2(\overline{\Omega})$ as $n \to \infty$. From the assumption, we have $u_\infty \equiv 0$ and (u_∞, v_∞) satisfies (3.1). Then the second equation of (3.1) implies

$$-d_2 \Delta v_\infty = v_\infty (b - v_\infty) \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega.$$

By the property of solutions of the logistic equation and $\min v_n \geq d_2 b/(d_2 + \beta)$, we have $v_\infty = b$. Denote by $\tilde{u}_n = u_n / \|u_n\|_{L^\infty}$ the L^∞ normalization of u_n. Then by dividing the first equation of (3.1) by $\|u_n\|_{L^\infty}$, we know that $\{\tilde{u}_n\}$ forms a sequence of positive solutions of

$$-d_1 \Delta \tilde{u}_n = \tilde{u}_n \left(a - u_n - \frac{v_n}{1 + u_n^2} \right) \quad \text{in } \Omega, \qquad \frac{\partial \tilde{u}_n}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega. \tag{3.5}$$

Note that $\|\tilde{u}_n\|_{L^\infty} = 1$ for $n \in \mathbb{N}$, then it follows from the elliptic regularity theory and the Sobolev embedding theorem that there exists a nonnegative function $\tilde{u}_\infty \in C^1(\bar{\Omega})$ such that $\lim_{n\to\infty} \tilde{u}_n = \tilde{u}_\infty$ in $C^1(\bar{\Omega})$. This, combining with $\|\tilde{u}_\infty\|_{L^\infty} = 1$, yields that $\tilde{u}_\infty > 0$. On the other hand, by integrating the first equation in (3.5) over Ω, we observe that

$$\int_\Omega \tilde{u}_n \left(a - u_n - \frac{v_n}{1 + u_n^2} \right) \mathrm{d}x = 0.$$

Let $n \to \infty$, and note that $\tilde{u}_\infty > 0$, $u_\infty = 0$ and $v_\infty = b$, then we have $a = b$, which contradicts our assumption. Therefore, we complete the proof of Lemma 3.1. $\square$

4 Existence/nonexistence of nonconstant solutions

Throughout the remaining part of this paper, we always assume that $a > b$.

Lemma 4.1. *Every sequence $\{(u_n, v_n)\}_{n=1}^{\infty}$ of positive solutions of (3.1) with $a > b$ and $d_1 = d_{1n} \to \infty$ as $n \to \infty$ satisfies*

$$\|u_n - u^*\|_{L^\infty} + \|v_n - v^*\|_{L^\infty} \to 0 \quad \text{as } n \to \infty,$$

where $\mathbf{e} = (u^*, v^*)$ is the unique positive constant solution.

Proof. For fixed a, b, β and Ω, Lemma 3.1 and standard regularity arguments tell that $\{(u_n, v_n)\}_{n=1}^{\infty}$ has a convergent subsequence, which we still denote by $\{(u_n, v_n)\}_{n=1}^{\infty}$. According to the argument by Lou and Ni [24], we can obtain a positive constant K, which is independent of n, such that

$$\|u_n - \bar{u}_n\|_{L^\infty} \le \frac{K}{d_{1n}} \quad \text{with } \bar{u}_n = \frac{1}{|\bar{\Omega}|} \int_\Omega u_n \mathrm{d}x$$

for $n \in \mathbb{N}$. Together with Lemma 3.1, we can find a constant $\bar{u} \in [0, a]$ such that $\lim_{n\to\infty} u_n = \bar{u}$ uniformly in $\bar{\Omega}$. Lemma 3.1 and the standard L^p-estimate for elliptic equations mean that both $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are uniformly bounded in $W^{2,p}(\Omega)$. Thus, the usual compactness argument implies

$$\lim_{n\to\infty} u_n = \bar{u} \quad \text{in } C^1(\bar{\Omega}), \tag{4.1}$$

passing to subsequence. We can similarly get a nonnegative function $\bar{v}$ such that

$$\lim_{n\to\infty} v_n = \bar{v} \quad \text{in } C^1(\bar{\Omega}), \tag{4.2}$$

passing to a subsequence. By setting $n \to \infty$ in the weak form of the second equation of (3.1) and using the elliptic regularity theory, we know that $\bar{v}$ satisfies

$$(d_2 + \beta\bar{u})\Delta\bar{v} = \bar{v}\left(b - \frac{\bar{v}}{1+\bar{u}^2}\right) \quad \text{in } C^1(\bar{\Omega}), \qquad \frac{\partial \bar{v}}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\bar{\Omega}.$$

Since $\bar{u} \in [0, a]$ is constant, the well-known property of the logistic equation implies that $\bar{v}$ is also constant and satisfies

$$\bar{v} = 0 \quad \text{or} \quad b - \frac{\bar{v}}{1+\bar{u}^2} = 0. \tag{4.3}$$

Integrating the first equation of (3.1) yields

$$\int_\Omega u_n \left(a - u_n - b - \frac{v_n}{1+u_n^2}\right) \mathrm{d}x = 0, \qquad n \in \mathbb{N}. \tag{4.4}$$

By (4.1) and (4.2), letting $n \to \infty$ in (4.4) implies

$$\bar{u} = 0 \quad \text{or} \quad a - \bar{u} - \frac{\bar{v}}{1+\bar{u}^2} = 0$$

because $\bar{u}$ and $\bar{v}$ are constants. Suppose for contradiction that $a - \bar{u} - \frac{\bar{v}}{1+\bar{u}^2} \neq 0$. Hence (4.1) and (4.2) imply $a - u_n - \frac{v_n}{1+u_n^2} \neq 0$ in Ω for sufficiently large $n \in \mathbb{N}$. Together with $u_n > 0$ in Ω, we obtain

$$\int_\Omega u_n \left(a - u_n - \frac{v_n}{1+u_n^2}\right) \mathrm{d}x \neq 0$$

for sufficiently large $n \in \mathbb{N}$. However, this contradicts (4.4). Then we obtain $a - \bar{u} - \frac{\bar{v}}{1+\bar{u}^2} = 0$. Using a similar argument, we have $b - \frac{\bar{v}}{1+\bar{u}^2} = 0$. Therefore, $(\bar{u}, \bar{v}) = (u^*, v^*)$. $\square$

Theorem 4.2. *For any fixed $(d_2, \beta, a, b, \Omega)$ satisfying $a > b$, there exists a large positive constant D such that (3.1) with $d_1 \geq D$ has no nonconstant solutions.*

Proof. Assume that (u, v) is a non-negative solution of (3.1) and denote

$$\bar{u} = \frac{1}{|\Omega|}\int_\Omega u\mathrm{d}x, \qquad \bar{v} = \frac{1}{|\Omega|}\int_\Omega v\mathrm{d}x.$$

Then, multiplying $u - \bar{u}$ the first equation in (3.1) by and integrating over Ω yield

$$\begin{aligned}
d_1 \int_\Omega |\nabla u|^2 \mathrm{d}x &= \int_\Omega u(a - u - \frac{v}{1+u^2})(u - \bar{u})\mathrm{d}x \\
&= \int_\Omega \left(a - u - \bar{u} - \frac{v}{1+u^2} + \frac{\bar{u}\bar{v}(u+\bar{u})}{(1+u^2)(1+\bar{u}^2)}\right)(u-\bar{u})^2 \mathrm{d}x \\
&\quad - \int_\Omega \frac{\bar{u}(u-\bar{u})(v-\bar{v})}{1+u^2}\mathrm{d}x \\
&\leq \int_\Omega (a + 2a^2\kappa)(u-\bar{u})^2\mathrm{d}x + \frac{\bar{u}}{2}\left[\int_\Omega \frac{(u-\bar{u})^2}{1+u^2}\mathrm{d}x + \int_\Omega \frac{(v-\bar{v})^2}{1+u^2}\mathrm{d}x\right] \\
&\leq \left(\frac{3a}{2} + 2a^2\kappa\right)\int_\Omega (u-\bar{u})^2\mathrm{d}x + \frac{a}{2}\int_\Omega (v-\bar{v})^2\mathrm{d}x,
\end{aligned} \tag{4.5}$$

where the last inequality comes from Lemma 3.1. Recall the Poincaré–Wirtinger inequality $\lambda_1\|U - \bar{U}\|_{L^2}^2 \leq \|\nabla U\|_{L^2}^2$ for any $U \in H^1(\Omega)$, where λ_1 is the least positive eigenvalue of $-\Delta$ with homogeneous Neumann boundary condition on $\partial\Omega$. Then it follows from (4.5) that

$$\left[1 - \frac{a}{\lambda_1 d_1}\left(\frac{3}{2} + 2a\kappa\right)\right]\|\nabla u\|_{L^2}^2 \leq \frac{a}{2\lambda_1 d_1}\|\nabla v\|_{L^2}^2. \tag{4.6}$$

Similarly, multiplying by $v - \bar{v}$ the second equation of (3.1) and integrating the resulting expression lead us to

$$\begin{aligned}
\int_\Omega (d_2 + \beta u)|\nabla v|^2\mathrm{d}x &= \int_\Omega v\left(b - \frac{v}{1+u^2}\right)(v-\bar{v})\mathrm{d}x - \beta\int_\Omega v\nabla u\cdot\nabla v\mathrm{d}x \\
&= \int_\Omega \left(b - \frac{v}{1+u^2} - \frac{\bar{v}}{1+u^2}\right)(v-\bar{v})^2\mathrm{d}x \\
&\quad + \int_\Omega \frac{\bar{v}^2(u-\bar{u})(v-\bar{v})(u+\bar{u})}{(1+u^2)(1+\bar{u}^2)}\mathrm{d}x - \beta\int_\Omega v\nabla u\cdot\nabla v\mathrm{d}x.
\end{aligned}$$

By Lemma 3.1 and Young's inequality, for any $\varepsilon > 0$, one can find a positive constant K such that

$$\begin{aligned}
\int_\Omega (d_2 + \beta u)|\nabla v|^2\mathrm{d}x &\leq \int_\Omega \left(b - \frac{v}{1+u^2} - \frac{\bar{v}}{1+u^2}\right)(v-\bar{v})^2\mathrm{d}x \\
&\quad + 2a\kappa^2\left[\int_\Omega \frac{K}{\varepsilon}(u-\bar{u})^2\mathrm{d}x + \int_\Omega \varepsilon(v-\bar{v})^2\mathrm{d}x\right] \\
&\quad + \beta\kappa\left[\int_\Omega \frac{K}{\varepsilon}|\nabla u|^2\mathrm{d}x + \int_\Omega \varepsilon|\nabla v|^2\mathrm{d}x\right],
\end{aligned} \tag{4.7}$$

where κ is the positive number given in Lemma 3.1. Then the Poincaré–Wirtinger inequality implies

$$\begin{aligned}
&\left[1 - \varepsilon\kappa\left(\frac{2a\kappa}{d_2\lambda_1} + \frac{\beta}{d_2}\right)\right]\|\nabla v\|_{L^2}^2 \\
&\leq \frac{1}{d_2}\int_\Omega \left(b - \frac{v}{1+u^2} - \frac{\bar{v}}{1+u^2}\right)(v-\bar{v})^2\mathrm{d}x + \frac{\kappa K}{\varepsilon}\left(\frac{2a\kappa}{d_2\lambda_1} + \frac{\beta}{d_2}\right)\|\nabla u\|_{L^2}^2.
\end{aligned} \tag{4.8}$$

Note that

$$b - \frac{v}{1+u^2} - \frac{\bar{v}}{1+u^2} = b - \frac{\bar{v}}{1+\bar{u}^2} - \left(\frac{\bar{v}}{1+u^2} - \frac{\bar{v}}{1+\bar{u}^2}\right) - \frac{v}{1+u^2},$$

then it follows from Lemma 4.1 that

$$b - \frac{v}{1+u^2} - \frac{\bar{v}}{1+u^2} < \varepsilon \quad \text{if } d_1 > 0 \text{ is sufficiently large.} \tag{4.9}$$

Thus, when $d_1 > 0$ is large, (4.6) and (4.8) enable us to find a positive constant K_1 such that

$$\|\nabla u\|_{L^2}^2 \leq \frac{K_1}{d_1}\|\nabla u\|_{L^2}^2,$$

which implies that u is a constant if d_1 is large enough. Combining with (4.8) and (4.9), we deduce that (u, v) is a constant solution if $d_1 > 0$ is sufficiently large. Then the proof of Theorem 4.2 is completed. $\square$

Remark 4.3. The conclusion of Theorem 4.2 is still valid in the case where $\beta = 0$, that is, for any fixed (d_2, a, b, Ω) with $a > b$, there exists a large positive constant D such that (3.1) with $\beta = 0$ and $d_1 \geq D$ has no nonconstant solutions.

Recall that $-\Delta$ under Neumann boundary condition has eigenvalues $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$ with $\lim_{n\to\infty} \lambda_n = +\infty$. Let $\mathbb{S}_i$ be the eigenspace associated with λ_i with multiplicity n_i. Let ϕ_{ij}, $1 \leq j \leq n_i$, be the normalized eigenfunctions corresponding to λ_i. Then the set $\{\phi_{ij} | \, i \geq 0, 1 \leq j \leq n_i\}$ forms a complete orthonormal basis of the Lebesgue space $L^2(\overline{\Omega})$ of integrable functions defined on Ω, $\phi_0(x) > 0$ for all $x \in \Omega$. Let $\mathbb{X}_{ij} = \{c\phi_{ij} | \, c \in \mathbb{R}^2\}$, and $\{\phi_{ij} | \, 1 \leq j \leq \dim \mathbb{S}_i\}$ be an orthonormal basis of $\mathbb{S}_i$. For $i \geq 0$, it can be observed that

$$\mathbb{X} = \bigoplus_{i=1}^{\infty} \mathbb{X}_i \quad \text{and} \quad \mathbb{X}_i = \bigoplus_{j=1}^{\dim \mathbb{S}_i} \mathbb{X}_{ij}. \tag{4.10}$$

Next, we study the linearization of (3.1) at (u^*, v^*), where $\mathbf{e} = (u^*, v^*)$ is the unique positive constant solution of (1.4). Let $\Phi(U) = (d_1 u, d_2 v + \beta uv)^T$ and

$$G(U) = \begin{bmatrix} u\left(a - u - \dfrac{v}{1+u^2}\right) \\ v\left(b - \dfrac{v}{1+u^2}\right) \end{bmatrix}$$

for $U = (u, v)^T$. Then (3.1) can be rewritten as

$$\begin{cases} -\Delta\Phi(U) = G(U) & \text{in } \Omega, \\ \dfrac{\partial U}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.11}$$

Define

$$\mathbb{X}^+ = \{U \in \mathbb{X} \mid u > 0, \, v > 0 \text{ on } \bar{\Omega}\}$$

and

$$\mathbb{B} = \left\{U \in \mathbb{X} \,\middle|\, \frac{1}{C} < u < C, \, \frac{1}{C} < v < C\right\},$$

where C is a positive constant whose existence is guaranteed by Lemma 3.1. Note that the derivative $\Phi_U(U)$ of $\Phi(U)$ with respect to U is a positive operator for all non-negative U, then

$\Phi_U^{-1}(U)$ exists and is a positive operator as well. Hence, U is a positive solution to (4.11) if and only if

$$F(U) \triangleq U - (I-\Delta)^{-1}\left\{\Phi_U^{-1}(U)[G(U) + \nabla U \Phi_{UU}(U)\nabla U^T] + U\right\} = 0 \quad \text{in } \mathbb{X}^+,$$

where $(I-\Delta)^{-1}$ is the inverse of $I-\Delta$ in X. As $F(\cdot)$ is a compact perturbation of the identity operator, the Leray–Schauder degree $\deg(F(\cdot), 0, \mathbb{B})$ is well-defined if $F(U) \neq 0$ on $\partial\mathbb{B}$. Note that

$$D_U F(\mathbf{e}) = I - (I-\Delta)^{-1}\{\Phi_U^{-1}(\mathbf{e})G_U(\mathbf{e}) + I\},$$

and that the index of F at $\mathbf{e}$ is defined as $\operatorname{index}(F(\cdot), \mathbf{e}) = (-1)^\gamma$ provided that $D_U F(\mathbf{e})$ is invertible, where $\gamma = \sum m_\mu$ and m_μ is the multiplicity of any negative eigenvalue μ of $D_U F(\mathbf{e})$; see [28] for more details.

We now consider the eigenvalues of $D_U F(\mathbf{e})$. First, for every integer $i \geq 0$ and $1 \leq j \leq \dim\mathbb{S}_i$, $\mathbb{X}_{ij}$ is invariant under $D_U F(\mathbf{e})$, and μ is an eigenvalue of $D_U F(\mathbf{e})$ on $\mathbb{X}_{ij}$ if and only if it is an eigenvalue of the matrix

$$I - \frac{1}{1+\lambda_i}\left[\Phi_U^{-1}(\mathbf{e})G_U(\mathbf{e}) + I\right] = \frac{1}{1+\lambda_i}\left[\lambda_i I - \Phi_U^{-1}(\mathbf{e})G_U(\mathbf{e})\right].$$

Thus, $D_U F(\mathbf{e})$ is invertible if and only if, for all $i \geq 0$, the above matrix is nonsingular. To calculate γ, we first define

$$H(\lambda) = \det\{\lambda I - \Phi_U^{-1}(\mathbf{e})G_U(\mathbf{e})\}. \tag{4.12}$$

If $H(\lambda_i) \neq 0$, then for each $1 \leq j \leq \dim\mathbb{S}_i$, the number of negative eigenvalues of $D_U F(\mathbf{e})$ on $\mathbb{X}_{ij}$ is odd if and only if $H(\lambda_i) < 0$. In conclusion, we have the following lemma (see [29]), which gives the explicit formula of calculating the index.

Lemma 4.4. *If $a > b$ and $H(\lambda_i) \neq 0$ for all $i \geq 0$, then*

$$\operatorname{index}(F(\cdot), \mathbf{e}) = (-1)^\gamma \quad \textit{with} \quad \gamma = \sum_{i \geq 0, H(\lambda_i) < 0} n_i(\lambda_i),$$

where $n_i(\lambda_i)$ is the algebraic multiplicity of λ_i.

To facilitate our computation of $\operatorname{index}(F(\cdot), \mathbf{e})$, we will consider the sign of $H(\lambda_i)$. Notice that our aim is to investigate the effect of the cross-diffusion coefficient β and diffusion coefficient d_1 on the existence of stationary patterns. Then we will concentrate on the dependence of $H(\lambda_i)$ on β and d_1. Note that

$$\lambda I - \Phi_U^{-1}(\mathbf{e})G_U(\mathbf{e}) = \begin{bmatrix} \lambda - \frac{2\theta^2 b}{d_1(1+\theta^2)} + \frac{\theta}{d_1} & \frac{\theta}{d_1(1+\theta^2)} \\ -\frac{\beta\theta b(1+\theta^2)}{d_1(d_2+\beta\theta)} + \frac{2\beta\theta^2 b^2}{d_1(d_2+\beta\theta)} - \frac{2b\theta^2}{d_2+\beta\theta} & \lambda - \frac{\beta\theta b}{d_1(d_2+\beta\theta)} + \frac{b}{d_2+\beta\theta} \end{bmatrix}.$$

Then, we have

$$H(\lambda) = \lambda^2 - \left[\frac{\beta\theta b}{d_1(d_2+\beta\theta)} + \frac{2\theta^2 b}{d_1(1+\theta^2)} - \frac{\theta}{d_1} - \frac{b}{d_2+\beta\theta}\right]\lambda + \frac{b\theta}{d_1(d_2+\beta\theta)}. \tag{4.13}$$

If

$$\Lambda(\beta, d_1) \triangleq \left[\frac{\beta\theta b}{d_1(d_2+\beta\theta)} + \frac{2\theta^2 b}{d_1(1+\theta^2)} - \frac{\theta}{d_1} - \frac{b}{d_2+\beta\theta}\right]^2 - \frac{4b\theta}{d_1(d_2+\beta\theta)} > 0,$$

then $H(\lambda) = 0$ has two roots $\lambda = \lambda^+(\beta, d_1)$ and $\lambda = \lambda^-(\beta, d_1)$, where

$$\lambda^{\pm}(\beta, d_1) = \frac{1}{2}\left[\frac{\beta\theta b}{d_1(d_2+\beta\theta)} + \frac{2\theta^2 b}{d_1(1+\theta^2)} - \frac{\theta}{d_1} - \frac{b}{d_2+\beta\theta} \pm \sqrt{\Lambda(\beta, d_1)}\right].$$

We first consider the dependence of $H(\lambda)$ on β. When β is large enough, we have $\Lambda > 0$ and the two roots of $H(\lambda)$ satisfy

$$\lim_{\beta\to\infty} \lambda^-(\beta, d_1) = 0$$

and

$$\lim_{\beta\to\infty} \lambda^+(\beta, d_1) = \frac{(1+3\theta^2)b - \theta(1+\theta^2)}{d_1(1+\theta^2)} \triangleq \bar{\lambda} \quad \text{if } b > \frac{(1+\theta^2)\theta}{1+3\theta^2}. \tag{4.14}$$

Thus, we have the following existence result about the non-constant steady state solution:

Theorem 4.5. *Assume that $a > b$, $b > \frac{(1+\theta^2)\theta}{1+3\theta^2}$ and $\bar{\lambda} \in (\lambda_n, \lambda_{n+1})$ for some $n \geq 1$ and $\sum_{i=1}^n n_i(\lambda_i)$ is odd, then there exists a positive number β^* such that system (3.1) with $\beta \geq \beta^*$ has at least one non-constant positive solution.*

Proof. In virtue of (4.14) and $\bar{\lambda} \in (\lambda_n, \lambda_{n+1})$, there exists a positive constant β^* such that, if $\beta \geq \beta^*$ then

$$0 < \lambda^-(\beta, d_1) < \lambda_1 \quad \text{and} \quad \lambda^+(\beta, d_1) \in (\lambda_n, \lambda_{n+1}). \tag{4.15}$$

We argue by contradiction. Assume that system (3.1) with $\beta \geq \beta^*$ has no non-constant positive solutions. For $s \in [0,1]$, define

$$\Psi(s, U) = ((sd_1 + (1-s)d_1^*)u, (d_2 + s\beta u)v)^T,$$

where d_1^* is a positive constant such that $d_1^* \geq D$ and $\frac{2\theta^2 b}{d_1^*(1+\theta^2)} - \frac{\theta}{d_1^*} - \frac{b}{d_2} < 0$. Obviously, $\Psi(1, \cdot) = \Phi(\cdot)$. Consider the following system

$$\begin{cases} -\Delta\Psi(s, U) = G(U) & \text{in } \Omega, \\ \dfrac{\partial U}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.16}$$

Then U is a positive non-constant solution of (3.1) if and only if it is a solution to (4.16) with $s = 1$. It is obvious that $\mathbf{e}$ is the unique constant positive solution of (4.16) for all $0 \leq s \leq 1$. U is a positive solution of (4.16) if and only if

$$\mathcal{F}(s, U) \triangleq U - (I-\Delta)^{-1}\left\{\Psi_U^{-1}(s, U)[G(U) + \nabla U \Psi_{s,UU}(U)\nabla U^T] + U\right\} = 0 \quad \text{in } \mathbb{X}^+.$$

It is obvious that $\mathcal{F}(1, U) = F(U)$. Remark 4.3 says that $\mathcal{F}(0, U) = 0$ has only one positive solution $\mathbf{e}$ in $\mathbb{X}^+$. By a direct computation, we have

$$D_U\mathcal{F}(s, \mathbf{e}) = I - (I-\Delta)^{-1}\{\Psi_U^{-1}(s, \mathbf{e})G_U(\mathbf{e}) + I\}.$$

In particular,

$$\begin{aligned} D_U\mathcal{F}(0, \mathbf{e}) &= I - (I-\Delta)^{-1}\{\Psi_U^{-1}(0, \mathbf{e})G_U(\mathbf{e}) + I\}, \\ D_U\mathcal{F}(1, \mathbf{e}) &= I - (I-\Delta)^{-1}\{\Phi_U^{-1}(\mathbf{e})G_U(\mathbf{e}) + I\} = D_U F(\mathbf{e}), \end{aligned}$$

where $\Psi_U(0, \cdot) = \mathrm{diag}(d_1^*, d_2)$. From the previous analysis, we know that the key point is to determine the sign of

$$\mathcal{H}(s, \lambda) = \det\{\lambda I - \Psi_U^{-1}(s, \mathbf{e})G_U(\mathbf{e})\}. \tag{4.17}$$

By direction calculation, we have $\mathcal{H}(0,\lambda) = \lambda^2 - (\frac{2\theta^2 b}{d_1^*(1+\theta^2)} - \frac{\theta}{d_1^*} - \frac{b}{d_2})\lambda + \frac{b\theta}{d_1^* d_2}$ and hence

$$\mathcal{H}(0,\lambda_i) > 0 \quad \text{for all } i \geq 0.$$

Clearly, $\mathcal{H}(1,\lambda) = H(\lambda)$. Therefore, in view of (4.14) and (4.15), we can get

$$\begin{cases} H(\lambda_0) = H(0) > 0, & \\ H(\lambda_i) < 0 & \text{when } 1 \leq i \leq n, \\ H(\lambda_i) > 0 & \text{when } i \geq n+1. \end{cases}$$

Therefore, zero is not an eigenvalue of $\lambda_i I - \Phi_U^{-1}(\mathbf{e})G_U(\mathbf{e})$ for all $i \geq 0$, and

$$\sum_{i\geq 1, H(\lambda_i)<0} \dim \mathbb{S}_i = \sum_{i=1}^{n} n_i(\lambda_i), \quad \text{which is odd.}$$

Thanks to Lemma 4.4, we have

$$\begin{aligned} &\operatorname{index}(\mathcal{F}(1,\cdot),\mathbf{e}) = (-1)^\gamma = (-1)^{\sum_{i=1}^{n} n_i(\lambda_i)} = -1, \\ &\operatorname{index}(\mathcal{F}(0,\cdot),\mathbf{e}) = (-1)^\gamma = (-1)^0 = 1. \end{aligned}$$

Now, by Lemma 3.1, we know that every positive solution of system (3.1) lies in $\mathbb{B}$ and $\mathcal{F}(t,\cdot) \neq 0$ on $\partial\mathbb{B}$. So $\deg(\mathcal{F}(s,\cdot),\mathbb{B},0)$ is well defined. By the homotopy invariance of topological degree, we have

$$\deg(\mathcal{F}(1,\cdot),\mathbb{B},0) = \deg(F(0,\cdot),\mathbb{B},0). \tag{4.18}$$

On the other hand, from our assumption, both equations $\mathcal{F}(1,\mathbf{e}) = 0$ and $\mathcal{F}(0,\mathbf{e}) = 0$ have only one positive solution $\mathbf{e}$ in $\mathbb{B}$, then we have

$$\begin{aligned} &\deg(\mathcal{F}(1,\cdot),\mathbb{B},0) = \operatorname{index}(\mathcal{F}(1,\cdot),\mathbf{e}) = -1, \\ &\deg(\mathcal{F}(0,\cdot),\mathbb{B},0) = \operatorname{index}(\mathcal{F}(0,\cdot),\mathbf{e}) = 1, \end{aligned}$$

which is a contradiction with (4.18). So the proof is completed. □

Next we consider the dependence of $H(\lambda)$ on d_1. From the previous analysis, it follows that the roots of $H(\lambda) = 0$ are all negative if $\frac{2\theta b}{1+\theta^2} + \frac{\beta b}{d_2+\beta\theta} - 1 < 0$ and $\Lambda(\beta,d_1) > 0$. So, in this case, we can't obtain the existence of non-constant positive solutions by using the method of degree theory.

We begin with the case $\frac{2\theta b}{1+\theta^2} + \frac{\beta b}{d_2+\beta\theta} - 1 > 0$. By straightforward computations, one can get $\Lambda(\beta,d_1) > 0$ and the two roots of $H(\lambda) = 0$ satisfy $0 < \lambda^-(\beta,d_1) < \lambda^+(\beta,d_1)$ if $d_1 \in (0,d^*)$ and $\frac{2\theta b}{1+\theta^2} + \frac{\beta b}{d_2+\beta\theta} > 1$, where

$$d^* = \left\{ \frac{1}{b}\sqrt{\theta b(d_2+\beta\theta)} \left(-1 + \sqrt{\left(\frac{\beta b}{d_2+\beta\theta} + \frac{2\theta b}{1+\theta^2} \right)} \right) \right\}^2.$$

Furthermore, one can verify that $\lambda^-(\beta,d_1)$ is monotone increasing and $\lambda^+(\beta,d_1)$ is monotone decreasing with respect to $d_1 \in (0,d^*)$. Moreover, $\lambda^+(\beta,d_1)$ and $\lambda^-(\beta,d_1)$ satisfy

$$\begin{aligned} &\lim_{d_1\to 0} \lambda^-(\beta,d_1) = \frac{b}{(d_2+\beta\theta)} \left(-1 + \frac{2\theta b}{1+\theta^2} + \frac{\beta b}{d_2+\beta\theta} \right)^{-1} \triangleq \eta, \\ &\lim_{d_1\to 0} \lambda^+(\beta,d_1) = +\infty, \\ &\lim_{d_1\to d^*} \lambda^-(\beta,d_1) = \lim_{d_1\to d^*} \lambda^+(\beta,d_1) = \frac{b}{(d_2+\beta\theta)} \left(-1 + \sqrt{\frac{2\theta b}{1+\theta^2} + \frac{\beta b}{d_2+\beta\theta}} \right)^{-1}. \end{aligned}$$

In order to state the structure of nonconstant solutions, we introduce the following two natural numbers j_0 and k_0 by

$$j_0 \triangleq \min\left\{ j \in \mathbb{N} \,\middle|\, \frac{b}{(d_2+\beta\theta)}\left(-1+\frac{2\theta b}{1+\theta^2}+\frac{\beta b}{d_2+\beta\theta}\right)^{-1} < \lambda_j \right\},$$

$$k_0 \triangleq \min\left\{ k \in \mathbb{N} \,\middle|\, \frac{b}{(d_2+\beta\theta)}\left(-1+\sqrt{\frac{2\theta b}{1+\theta^2}+\frac{\beta b}{d_2+\beta\theta}}\right)^{-1} \le \lambda_k \right\} (\ge j_0).$$

Since $\lambda^+(\beta, d_1)$ is monotone decreasing with respect to d_1 and $\lim_{d_1\to 0}\lambda^+(\beta, d_1) = +\infty$, there are positive numbers

$$d_{1k} = \sup\{d_1 > 0 |\, \lambda^+(\beta, d_1) > \lambda_k\} \quad \text{for } k = k_0, k_0+1, \ldots. \tag{4.19}$$

Solving $\lambda^+(\beta, d_1) = \lambda_k$ for d_1, we get the solution d_{1k} ($k = k_0, k_0+1, \ldots$) with

$$d_{1k} = \left(-1+\frac{2\theta b}{1+\theta^2}+\frac{\beta b}{d_2+\beta\theta}-\frac{b\theta}{\lambda_k(d_2+\beta\theta)}\right)\left(\lambda_k+\frac{b}{d_2+\beta\theta}\right)^{-1}.$$

Therefore, the sequence $\{d_{1k}\}_{k=k_0}^{\infty}$ defined by (4.19) satisfies

$$0 \leftarrow \cdots < d_{1k} < \cdots < d_{1k_0+1} < d_{1k_0} < d^* \triangleq d_{1k_0-1}. \tag{4.20}$$

If $k_0 > j_0$, we define

$$\tilde{d}_{1j} \triangleq \inf\{d_1 > 0 |\, \lambda^-(\beta, d_1) > \lambda_j\} \quad \text{for } j = j_0, j_0+1, \cdots, k_0-1.$$

Similarly, it follows from $\lambda^-(\beta, d_1) = \lambda_j$ that $\tilde{d}_{1j} = d_{1j}$. Hence the monotone increasing property of $\lambda^-(\beta, d_1)$ for $d_1 \in (0, d^*)$ induces the monotone increasing property of $\{\tilde{d}_{1j}\}_{j=j_0}^{k_0-1}$ as

$$(\tilde{d}_{j_0-1} \triangleq) 0 < \tilde{d}_{j_0} < \tilde{d}_{j_0+1} < \cdots < \tilde{d}_{k_0-1} < d^*.$$

Therefore, we have the following conclusions:

Theorem 4.6. *Assume that $a > b$ and $\frac{2\theta b}{1+\theta^2}+\frac{\beta b}{d_2+\beta\theta} > 1$, then the following* (i) *and* (ii) *hold true:*

(i) In case where $k_0 > j_0$, there exists at least one nonconstant solution of (3.1) *provided that $d_1 \in (\tilde{d}_{1j}, \tilde{d}_{1j+1}) \cap (d_{1k+1}, d_{1k})$ and $\sum_{i=j+1}^{k} n_i(\lambda_i)$ is odd or $d_1 \in (d_{1k+1}, d_{1k}) \cap (\tilde{d}_{k_0-1}, d^*)$ and $\sum_{i=k_0}^{k} n_i(\lambda_i)$ is odd.*

(ii) In case where $k_0 = j_0$, there exists at least one nonconstant solution of (3.1) *provided that $d_1 \in (d_{1k+1}, d_{1k})$ and $\sum_{i=k_0}^{k} n_i(\lambda_i)$ is odd.*

Proof. In the case where $\frac{2\theta b}{1+\theta^2}+\frac{\beta b}{d_2+\beta\theta} > 1$, suppose for contradiction that there is no nonconstant solution of (3.1). According to Lemma 3.1, we know every positive solution of system (3.1) lies in $\mathbb{B}$ and $F(U) \neq 0$ on $\partial\mathbb{B}$. Then the homotopy invariance of topological degree implies

$$\deg(F(\cdot), \mathbb{B}, 0) \quad \text{is constant for all } d_1 > 0. \tag{4.21}$$

In view of Theorem 4.2, we recall that if $d_1 \ge D$, then $F(U) = 0$ has a unique solution $\mathbf{e}$ in $\mathbb{X}^+$. Therefore, we know that

$$\deg(F(\cdot), \mathbb{B}, 0) = \operatorname{index}(F(\cdot), \mathbf{e}) \quad \text{for } d_1 \ge D.$$

It is easy to verify that $\lambda^+(\beta, d_1)$ is monotone decreasing with respect to d_1 and satisfies $\lim_{d_1\to\infty} \lambda^+(\beta, d_1) = 0$. Together with $H(\lambda_0) > 0$ and $\lambda^-(\beta, d_1) < \lambda^+(\beta, d_1)$, we obtain $H(\lambda_i) > 0$ for all $i \geq 0$ when d_1 is sufficiently large. It follows from Lemma 4.4 that if $d_1 > 0$ is large enough,

$$\deg(F(\cdot), \mathbb{B}, 0) = \operatorname{index}(F(\cdot), \mathbf{e}) = (-1)^\gamma = (-1)^0 = 1. \tag{4.22}$$

On the other hand, if $d_1 \in (\tilde{d}_{1j}, \tilde{d}_{1j+1}) \cap (d_{k+1}, d_k)$, then (4.19) and (4.20) imply that $\lambda_j < \lambda^-(\beta, d_1) < \lambda_{j+1}$ and $\lambda^+(\beta, d_1) > \lambda_k$. Hence, if $k_0 > j_0$, we can get

$$\begin{cases} H(\lambda_0) = H(0) > 0, & \\ H(\lambda_i) < 0 & \text{when } j+1 \leq i \leq k, \\ H(\lambda_i) > 0 & \text{when } i \geq k. \end{cases}$$

By Lemma 4.4, we have

$$\operatorname{index}(F(\cdot), \mathbf{e}) = (-1)^\gamma = (-1)^{\sum_{i=j+1}^k n_i(\lambda_i)}.$$

If $\sum_{i=j+1}^k n_i(\lambda_i)$ is odd , then

$$\deg(F(\cdot), B, 0) = \operatorname{index}(F(\cdot), \mathbf{e}) = (-1)^\gamma = (-1)^{\sum_{i=j+1}^k n_i(\lambda_i)} = -1,$$

which is a contradiction with (4.22). Consequently, by the contradiction argument, we obtain at least one nonconstant solution if $d_1 \in (\tilde{d}_{1j}, \tilde{d}_{1j+1}) \cap (d_{k+1}, d_k)$ and $\sum_{i=j+1}^k n_i(\lambda_i)$ is odd. Similarly, we have $\lambda(k_0 - 1) < \lambda^-(\beta, d_1) < \lambda^-(\beta, d^*) \leq \lambda(k_0)$ and $\lambda^+(\beta, d_1) > \lambda(k)$ if $d_1 \in (\tilde{d}_{k_0-1}, d^*) \cap (d_{k+1}, d_k)$. Therefore,

$$\begin{cases} H(\lambda_0) = H(0) > 0, & \\ H(\lambda_i) < 0, & \text{when } k_0 \leq i \leq k, \\ H(\lambda_i) > 0, & \text{when } i \geq k \end{cases}$$

if $k_0 > j_0$. Through similar calculations, we can get a contradiction with (4.22) if $\sum_{i=k_0}^k n_i(\lambda_i)$ is odd. So the proof for the statement (i) is completed. The proof for statement (ii) can be carried out by a similar manner. □

Remark 4.7. In particular, assume that $j_0 = k_0 = 1$, namely,

$$\frac{b}{(d_2 + \beta\theta)} \left(-1 + \sqrt{\frac{2\theta b}{1+\theta^2} + \frac{\beta b}{d_2 + \beta\theta}} \right)^{-1} \leq \lambda_1.$$

If $a > b$ and

$$\frac{2\theta b}{1+\theta^2} + \frac{\beta b}{d_2 + \beta\theta} > 1,$$

then there exists a sequence $\{d_{1k}\}_{j=0}^\infty$ such that $0 \leftarrow \cdots < d_{1k} < \cdots < d_{12} < d_{11}$ and (3.1) admits at least one nonconstant solution if $d_1 \in (d_{1k+1}, d_{1k})$ and $\sum_{i=1}^k n_i(\lambda_i)$ is odd.

5 Stability of the positive constant solution

In this section, we firstly analyze the stability of the positive constant steady-state solution by eigenvalue analysis. And then, we will investigate the global stability of the positive constant steady-state solution. To investigate the local dynamical behavior of system (1.4) near the positive constant solution **e**, we need to consider the linearized operator $\mathcal{L}_{\alpha_1,\beta}$ of (1.4) with respect to (u,v) at (u^*,v^*). Note that

$$\mathcal{L}_{\alpha_1,\beta} = \begin{bmatrix} d_1\Delta + \alpha_1 & -\alpha_2 \\ \beta b(1+\theta^2)\Delta + \alpha_3 & (d_2+\beta\theta)\Delta - b \end{bmatrix},$$

where

$$\theta = a-b, \quad \alpha_1 = -\theta + \frac{2b\theta^2}{1+\theta^2}, \quad \alpha_2 = \frac{\theta}{1+\theta^2}, \quad \alpha_3 = 2b^2\theta.$$

The characteristic equation is $\mathcal{L}_{\alpha_1,\beta}(\tilde{\phi}_1,\tilde{\phi}_2) = \sigma(\tilde{\phi}_1,\tilde{\phi}_2)$. Let $\tilde{\phi}_1 = \sum_{0\le i\le\infty} a_i\varphi_i$, $\tilde{\phi}_2 = \sum_{0\le i\le\infty} b_i\varphi_i$. Notice that $\{\varphi_i\}_{i=0}^{\infty}$ is a complete orthogonal base of $\mathbb{X}$. Substituting them into the characteristic equation yields

$$\sum_{0\le i\le\infty} \mathcal{M}(\sigma,\alpha_1,\beta,\lambda_i)(a_i,b_i)^T\varphi_i = 0,$$

where

$$\mathcal{M}(\sigma,\alpha_1,\beta,\lambda_i) = \begin{bmatrix} -d_1\lambda_i + \alpha_1 - \sigma & -\alpha_2 \\ -\beta b(1+\theta^2)\lambda_i + \alpha_3 & -(d_2+\beta\theta)\lambda_i - b - \sigma \end{bmatrix}.$$

To investigate the stability of the positive steady-state solution, it suffices to study the characteristic equation $\det\mathcal{M}(\sigma,\alpha_1,\beta,\lambda_i) = 0$, that is,

$$\sigma^2 - T_i(\alpha_1,\beta)\sigma + D_i(\alpha_1,\beta) = 0, \qquad i = 0,1,2,\ldots, \tag{5.1}$$

where

$$\begin{aligned} T_i(\alpha_1,\beta) &= -(d_1+d_2+\beta\theta)\lambda_i + \alpha_1 - b, \\ D_i(\alpha_1,\beta) &= d_1(d_2+\beta\theta)\lambda_i^2 + [bd_1 - (d_2+\beta\theta)\alpha_1 - b\beta(1+\theta^2)\alpha_2]\lambda_i + b\theta. \end{aligned}$$

It is easy to know that two solutions of equation (5.1) have negative real parts if $T_i(\alpha_1,\beta) < 0$ and $D_i(\alpha_1,\beta) > 0$ for all $i \ge 0$. Thus, we have the following results.

Lemma 5.1. *If $a > b$, then all eigenvalues of $\mathcal{L}_{\alpha_1,\beta}$ have negative real parts, or equivalently, the homogenous steady-state $\mathbf{e} = (\theta, b(1+\theta^2))$ is locally asymptotically stable, provided that one of the following conditions is satisfied:*

(i) either $\alpha_1 < -b$ or $-b < \alpha_1 < 0$ and $\beta < \frac{bd_1-d_2\alpha_1}{(\alpha_1+b)\theta}$ or $0 < \alpha_1 < \min\{b, \frac{bd_1}{d_2}\}$ and $\beta \le \frac{bd_1-d_2\alpha_1}{(\alpha_1+b)\theta}$;

(ii) $0 < \alpha_1 < \min\{b, \frac{bd_1}{d_2}\}$ and $\beta > \frac{bd_1-d_2\alpha_1}{(\alpha_1+b)\theta}$ and $[(bd_1-d_2\alpha_1)-(\alpha_1+b)\beta\theta]^2 < 4d_1(d_2+\beta\theta)b\theta$;

(iii) $d_1 < d_2$ and $\frac{d_1 b}{d_2} < \alpha_1 < b$ and $[(bd_1-d_2\alpha_1)-(\alpha_1+b)\beta\theta]^2 < 4d_1(d_2+\beta\theta)b\theta$.

Now, we consider the global stability of **e**.

Lemma 5.2. *If $a > b$, $2b(a+c) < 1$ and $\beta \le \frac{2}{c}\sqrt{\frac{d_1d_2\theta}{b(1+\theta^2)}}$, then **e** is globally asymptotically stable.*

Proof. We discuss the global stability of $\mathbf{e}$ by Lyapunov method. Define

$$L(u(x,t),v(x,t)) = \int_\Omega \int_{u^*}^{u} \frac{\xi - u^*}{\xi} \mathrm{d}\xi \mathrm{d}x + \int_\Omega \int_{v^*}^{v} \frac{\eta - u^*}{\eta} \mathrm{d}\eta \mathrm{d}x.$$

Then

$$\begin{aligned} L'(t) &= \int_\Omega \frac{u-u^*}{u}\frac{\partial u}{\partial t}\mathrm{d}x + \int_\Omega \frac{v-v^*}{v}\frac{\partial v}{\partial t}\mathrm{d}x \\ &= d_1 \int_\Omega \frac{u-u^*}{u}\Delta u \mathrm{d}x + \int_\Omega \frac{v-v^*}{v}[\Delta(d_2+\beta u)v]\mathrm{d}x \\ &\quad + \int_\Omega (u-u^*)\left(a-u-\frac{v}{1+u^2}\right)\mathrm{d}x + \int_\Omega (v-v^*)\left(b-\frac{v}{1+u^2}\right)\mathrm{d}x \\ &\triangleq I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= -d_1 \int_\Omega \frac{u^*}{u^2}|\nabla u|^2 \mathrm{d}x - \int_\Omega (d_2+\beta u)\frac{v^*}{v^2}|\nabla v|^2\mathrm{d}x - \beta\int_\Omega \frac{v^*}{v}|\nabla u||\nabla v|\mathrm{d}x \\ I_2 &= -\int_\Omega (u-u^*)\left(u-u^*+\frac{v}{1+u^2}-\frac{v^*}{1+(u^*)^2}\right)\mathrm{d}x - \int_\Omega (v-v^*)\left(\frac{v}{1+u^2}-\frac{v^*}{1+(u^*)^2}\right)\mathrm{d}x. \end{aligned}$$

It is easy to see that $4d_1(d_2+\beta u)u^* \geq \beta^2 u^2 v^*$ when $\beta \leq \frac{2}{c}\sqrt{\frac{d_1 d_2 \theta}{b(1+\theta^2)}}$. Hence, we have $I_1 \leq 0$. Further computation gives

$$\begin{aligned} I_2 = &-\int_\Omega \left[(u-u^*)^2 - (u-u^*)\left(\frac{v}{1+u^2}-\frac{v^*}{1+u^2}+\frac{v^*}{1+u^2}-\frac{v^*}{1+(u^*)^2}\right)\right]\mathrm{d}x \\ &-\int_\Omega (v-v^*)\left(\frac{v}{1+u^2}-\frac{v^*}{1+u^2}+\frac{v^*}{1+u^2}-\frac{v^*}{1+(u^*)^2}\right)\mathrm{d}x \\ = &\int_\Omega \left(-1+\frac{2v^*(u+u^*)}{(1+(u^*)^2)(1+u^2)}\right)(u-u^*)^2\mathrm{d}x - \int_\Omega \frac{1}{1+u^2}(v-v^*)^2\mathrm{d}x \\ &+\int_\Omega \left(\frac{2v^*(u+u^*)}{(1+(u^*)^2)(1+u^2)}-\frac{1}{1+u^2}\right)(u-u^*)(v-v^*)\mathrm{d}x \\ = &\int_\Omega \left(-1+\frac{2b(u+u^*)}{1+u^2}\right)(u-u^*)^2\mathrm{d}x - \int_\Omega \frac{1}{1+u^2}(v-v^*)^2\mathrm{d}x \\ &+\int_\Omega \left(\frac{2b(u+u^*)-1}{1+u^2}\right)(u-u^*)(v-v^*)\mathrm{d}x. \end{aligned}$$

Clearly,

$$\frac{2b(u+u^*)}{1+u^2} < 1 \quad \text{and} \quad \left(1-\frac{2b(u+u^*)}{1+u^2}\right)\frac{4}{1+u^2} > \left(\frac{2b(u+u^*)-1}{1+u^2}\right)^2$$

if $2b(a+c) < 1$. Therefore, we have $I_2 < 0$. It follows from the above arguments that if the conditions of Lemma 5.2 are satisfied, then $L'(t) < 0$ along all trajectories in the first quadrant except (u^*, v^*). Therefore $\mathbf{e} = (u^*, v^*)$ is globally asymptotically stable. □

6 Hopf bifurcation

This section is devoted to the Hopf bifurcation at the nontrivial steady-state solution $\mathbf{e} = (u^*, v^*)^T$ of (1.4) with $a > b$. To be more precise, as a pair of simple complex conjugate eigenvalues of the linearization around $\mathbf{e} = (u^*, v^*)^T$ cross the imaginary axis of the complex plane, the nontrivial steady-state solution $\mathbf{e} = (u^*, v^*)^T$ of (1.4) loses stability and a branch of small-amplitude limit cycles emerges from $\mathbf{e} = (u^*, v^*)^T$. Throughout this section, we always assume that

(H1) $a > b$, λ_i is a simple eigenvalues of the linear operator $-\Delta$ subject to the homogeneous boundary condition $\frac{\partial}{\partial \mathbf{n}} u$ on $\partial\Omega$, φ_i is the eigenvector associated with λ_i satisfying $\int_\Omega \varphi_i^2(x)\mathrm{d}x = 1$.

In what follows, by choosing the cross-diffusion coefficient β as the bifurcation parameter, we shall analyze the occurrence of Hopf bifurcation, the Hopf bifurcation direction and the stability of bifurcating time-periodic solutions. It follows from [11, 12] that system (1.4) with $a > b$ undergoes Hopf bifurcation near $\beta = \beta_i$ at the nontrivial steady-state solution $\mathbf{e} = (u^*, v^*)^T$, where $\beta_i \in (0, \infty)$ satisfies

$$T_i(\alpha_1, \beta_i) = 0, \qquad \frac{\partial}{\partial \beta} T_i(\alpha_1, \beta_i) \neq 0, \qquad D_i(\alpha_1, \beta_i) > 0,$$

and

$$T_j(\alpha_1, \beta_i) \neq 0, \qquad D_j(\alpha_1, \beta_i) \neq 0 \quad \text{for all } i \neq j.$$

Note that $T_i(\alpha_1, \beta)$ is monotone decreasing with respect to β, then it is easy to see that $T_i(\alpha_1, \cdot)$ has exactly one zero

$$\beta_i \triangleq \frac{\alpha_1 - b - (d_1 + d_2)\lambda_i}{\theta\lambda_i},$$

which is positive when $\alpha_1 > b + (d_1 + d_2)\lambda_i$. Obviously, $T_j(\alpha_1, \beta_i) \neq 0$ for $j \neq i$. Moreover, $D_i(\alpha_1, \beta_i) = -\alpha_1^2 + 2\alpha_1 d_1 \lambda_i + b^2 + b(d_1 + d_2)\lambda_i + b\theta - d_1^2\lambda_i^2$. Hence, it is easy to see that $D_i(\alpha_1, \beta_i) > 0$ if $b\theta > d_2\lambda_i^2 + b(d_2 - d_1)\lambda_i$ and $\alpha_1 < d_1\lambda_i + \sqrt{b(d_1 + d_2)\lambda_i + b\theta + b^2}$. Next, we only need to verify $D_j(\alpha_1, \beta_i) \neq 0$ for all $j \neq i$. Obviously,

$$D_j(\alpha_1, \beta_i) = -\frac{\lambda_j}{\lambda_i}\alpha_1^2 + \left(\frac{d_1\lambda_j^2}{\lambda_i} + d_1\lambda_j\right)\alpha_1 - \frac{bd_1\lambda_j^2}{\lambda_i} - d_1^2\lambda_j^2 + 2bd_1\lambda_j + bd_2\lambda_j + \frac{b^2\lambda_j}{\lambda_i} + b\theta.$$

Therefore, we have $D_j(\alpha_1, \beta_i) < 0$ for all $j \neq i$ if $\Re < 0$ and $D_j(\alpha_1, \beta_i) \neq 0$ for all $j \neq i$ if $\Re > 0$ and $\alpha_1 \neq \alpha_1^{\pm}$, where

$$\Re = \frac{d_1^2\lambda_j^4}{\lambda_i^2} + d_1^2\lambda_j^2 + \frac{4\lambda_j}{\lambda_i}\left[-\frac{bd_1\lambda_j^2}{\lambda_i} - \frac{1}{2}d_1^2\lambda_j^2 + (2d_1 + d_2)b\lambda_j + \frac{b^2\lambda_j}{\lambda_i} + b\theta\right]$$

and

$$\alpha_1^{\pm} = \frac{d_1(\lambda_j + \lambda_i)\lambda_j \pm \lambda_i\sqrt{\Re}}{2\lambda_j}.$$

Therefore, we shall consider Hopf bifurcation under the following assumptions:

(H2) $b\theta > d_2\lambda_i^2 + b(d_2 - d_1)\lambda_i$ and $b + (d_1 + d_2)\lambda_i < \alpha_1 < d_1\lambda_i + \sqrt{b(d_1 + d_2)\lambda_i + b\theta + b^2}$;

(H3) Either $\Re < 0$ or $\Re > 0$ and $\alpha_1 \neq \alpha_1^{\pm}$.

For convenience, we call a Hopf bifurcation forward if there exist periodic solutions when parameter value $\beta > \beta_i$; and backward if $\beta < \beta_i$. Under assumptions (H1), (H2) and (H3),

$\mathcal{L}_{\alpha_1,\beta_i}$ has exactly one pair of purely imaginary eigenvalues $\pm \mathrm{i}\omega_i$ with associated eigenvectors q_i and $\bar{q}_i$, where $\omega_i = \sqrt{D_i(\alpha_1,\beta_i)}$, $q_i = \rho_i\varphi_i$, and the nonzero vector $\rho_i \in \mathbb{C}^2$ satisfies $\mathcal{M}(\mathrm{i}\omega_i, \alpha_1, \beta_i, \lambda_i)\rho_i = 0$. It follows that $\rho_i = (\alpha_2, -d_1\lambda_i + \alpha_1 - i\omega_i)^T$. Moreover, there exist a neighborhood $N_1(\beta_i) \times N_2(\mathrm{i}\omega_i)$ of $(\beta_i, \mathrm{i}\omega_i)$ in $\mathbb{R}_+ \times \mathbb{C}$ and a continuously differentiable function σ: $N_1(\beta_i) \to N_2(\mathrm{i}\omega_i)$ such that $\sigma(\beta_i) = \pm\mathrm{i}\omega_i$ and that the only eigenvalue of $\mathcal{L}_{\alpha_1,\beta}$ in $N_2(\mathrm{i}\omega_i)$ is $\sigma(\beta)$. Moreover, as β varies such that $T_i(\alpha_1,\beta)$ decreases and passes through 0, $\sigma(\beta)$ varies from a complex number with a positive real part to a purely imaginary number and then to a complex number with a negative real part. This implies that a codimension one Hopf bifurcation for (1.4) occurs at $\beta = \beta_i$. Namely, in every neighborhood of $(U,\beta) = (\mathbf{e},\beta_i)$ there is a unique branch of time-periodic spatially non-homogeneous solutions $U_\beta(t,x)$, which tends to $\mathbf{e}$ as $\beta \to \beta_i$. The period T_β of $U_\beta(t,x)$ satisfies that $T_\beta \to 2\pi/\omega_i$ as $\beta \to \beta_i$.

Under assumptions (H1), (H2) and (H3), $-\mathrm{i}\omega_i$ is also an eigenvalue of $\mathcal{L}^*_{\alpha_1,\beta_i}$ with an associated eigenvector $p_i = \rho_i^*\varphi_i$, where $\rho_i^* \in \mathbb{C}^2 \setminus \{0\}$ satisfies

$$\mathcal{M}^T(-\mathrm{i}\omega_i, \alpha_1, \beta_i, \lambda_i)\rho_i^* = 0$$

and $\bar{\rho}_i^* \cdot \rho_i = 1$ and $\rho_i^* \cdot \rho_i = 0$. Then, we have $\rho_i^* = \left(\frac{-b-(d_2+\beta_i)\lambda_i+\mathrm{i}\omega_i}{2i\alpha_2\omega_i}, \frac{1}{2i\omega_i}\right)^T$. Next, we consider the bifurcation direction and stability of the bifurcating periodic solutions at $\beta = \beta_i$ according to [11,12]. Denote by $\mathfrak{G}^2 = (\mathfrak{G}_1^2, \mathfrak{G}_2^2)^T$, and $\mathfrak{G}^3 = (\mathfrak{G}_1^3, \mathfrak{G}_2^3)^T$ the second- and third-order Fréchet derivatives of $\Delta\Phi(U) + G(U)$ with respect to U at $\mathbf{e} = (u^*, v^*)$, respectively. A straightforward computation yields

$$\begin{aligned}
\mathfrak{G}_1^2(\xi,\zeta) &= 2\left(-1 + \frac{3b\theta(1-\theta^2)}{(1+\theta^2)^2}\right)\xi_1\zeta_1 + \frac{\theta^2-1}{(1+\theta^2)^2}(\xi_1\zeta_2 + \xi_2\zeta_1),\\
\mathfrak{G}_2^2(\xi,\zeta) &= \Delta[\beta(\xi_1\zeta_2 + \xi_2\zeta_1)] - \frac{2}{1+\theta^2}\xi_2\zeta_2 + \frac{4b\theta}{1+\theta^2}(\xi_1\zeta_2 + \xi_2\zeta_1) + \frac{2b^2(1-3\theta^2)}{1+\theta^2}\xi_1\zeta_1\\
\mathfrak{G}_1^3(\xi,\zeta,\varsigma) &= \frac{6b(1-6\theta^2+\theta^4)}{(1+\theta^2)^3}\xi_1\zeta_1\varsigma_1 + \frac{2\theta(3-\theta^2)}{(1+\theta^2)^3}(\xi_1\zeta_1\varsigma_2 + \xi_2\zeta_1\varsigma_1 + \xi_1\zeta_2\varsigma_1),\\
\mathfrak{G}_2^3(\xi,\zeta,\varsigma) &= \frac{4b(1-3\theta^2)}{(1+\theta^2)^2}(\xi_1\zeta_1\varsigma_2 + \xi_1\zeta_2\varsigma_1 + \xi_2\zeta_1\varsigma_1)\\
&\quad + \frac{4\theta}{(1+\theta^2)^2}(\xi_2\zeta_2\varsigma_1 + \xi_1\zeta_2\varsigma_2 + \xi_2\zeta_1\varsigma_2) + \frac{24b^2\theta(\theta^2-1)}{(1+\theta^2)^2}\xi_1\zeta_1\varsigma_1
\end{aligned}$$

for all $\xi = (\xi_1,\xi_2)^T$, $(\zeta_1,\zeta_2)^T$ and $\varsigma = (\varsigma_1,\varsigma_2)^T \in \mathbb{X}$. It is well known that the following quantity determines the direction and stability of bifurcating periodic orbits $U_\beta(t,x)$ (see [11, 12])

$$\Upsilon_i = \frac{\mathrm{i}}{2\omega_i}\left(\mathbf{g}_{11}\mathbf{g}_{20} - 2|\mathbf{g}_{11}|^2 - \frac{|\mathbf{g}_{02}|^2}{3}\right) + \frac{\mathbf{g}_{21}}{2}, \tag{6.1}$$

where

$$\begin{aligned}
\mathbf{g}_{20} &= \langle p_i, \mathfrak{G}^2(q_i,q_i)\rangle,\\
\mathbf{g}_{11} &= \langle p_i, \mathfrak{G}^2(q_i,\bar{q}_i)\rangle,\\
\mathbf{g}_{02} &= \langle p_i, \mathfrak{G}^2(\bar{q}_i,\bar{q}_i)\rangle,\\
\mathbf{g}_{21} &= \langle p_i, \mathfrak{G}^3(q_i,q_i,\bar{q}_i)\rangle + 2\langle p_i, \mathfrak{G}^2(W_{11},q_i)\rangle + \langle p_i, \mathfrak{G}^2(W_{20},\bar{q}_i)\rangle,
\end{aligned}$$

and

$$\begin{aligned}
W_{20} &= [2i\omega_i - \mathcal{L}_{\alpha_1,\beta_i}]^{-1}\left[\mathfrak{G}^2(q_i,q_i) - \langle p_i, \mathfrak{G}^2(q_i,q_i)\rangle q_i - \langle \bar{p}_i, \mathfrak{G}^2(q_i,q_i)\rangle \bar{q}_i\right],\\
W_{11} &= -[\mathcal{L}_{\alpha_1,\beta_i}]^{-1}\left[\mathfrak{G}^2(q_i,\bar{q}_i) - \langle p_i, \mathfrak{G}^2(q_i,\bar{q}_i)\rangle q_i - \langle \bar{p}_i, \mathfrak{G}^2(q_i,\bar{q}_i)\rangle \bar{q}_i\right].
\end{aligned}$$

Therefore, we obtain the following result.

Theorem 6.1. *In addition to assumptions (H1), (H2) and (H3), a Hopf bifurcation for* (1.4) *occurs at* $\beta = \beta_i$ *if* $a > b$. *Namely, when* $a > b$, *in a neighborhood of* $(U, \beta) = (\mathbf{e}, \beta_i)$ *there is a branch of periodic solutions* $U_\beta(x,t)$ *satisfying* $U_\beta(x,t) \to \mathbf{e}$ *as* $\beta \to \beta_i$. *The period* T_β *of* $U_\beta(x,t)$ *satisfies that* $T_\beta \to 2\pi/\omega_*$ *as* $\beta \to \beta_i$. *Moreover, the bifurcation is backward (respectively, forward) if* $\mathrm{Re}(\Upsilon_i) < 0$ $(respectively, > 0)$.

Obviously, in Theorem 6.1, if λ_i is not the principal eigenvalue of the linear operator $-\Delta$ subject to the homogeneous boundary condition $\frac{\partial}{\partial \mathbf{n}} u = 0$ on $\partial\Omega$, then the Hopf bifurcating periodic solutions $U_\beta(x,t)$ is spatially nonhomogeneous and unstable. However, if λ_i is the principal eigenvalue $\lambda_0 = 0$, then the associated eigenvector φ_0 can be a positive constant function on Ω. In this case, assumption (H1) is obviously satisfied and $\alpha_1 - b$ is sufficiently close to zero. Hence, we can regard b as a bifurcation parameter. Obviously, we have $T_0(\alpha_1,\beta) = 0$, $T_j(\alpha_1,\beta) < 0$ and $D_0(\alpha_1,\beta) = b\theta > 0$, $D_j(\alpha_1,\beta) > 0$ for all $j \in \mathbb{N}$ if $b = b_* \triangleq \frac{\theta(1+\theta^2)}{\theta^2-1}$ and one of the following conditions is satisfied

(A1) $\theta > 1$ and $d_1 - d_2 - 2\beta\theta \geq 0$;

(A2) $\theta > 1$, $d_1 > d_2$ and $d_1 - d_2 - 2\beta\theta < 0$ and $b(d_1 - d_2 - 2\beta\theta)^2 < 4d_1(d_2 + \beta\theta)\theta$;

(A3) $\theta > 1$ and $d_1 < d_2$ and $b(d_1 - d_2 - 2\beta\theta)^2 < 4d_1(d_2 + \beta\theta)\theta$.

It is easy to evaluate $\sigma(b)$ at $b = b_*$ to get $\mathrm{Re}\sigma'(b_*) = \frac{\theta^2-1}{2(1+\theta^2)} > 0$. Thus, it remains to calculate the direction of Hopf bifurcation and the stability of bifurcating periodic orbits bifurcating from $(U,b) = (\mathbf{e}, b^*)$. In virtue of (6.1), we have

$$\mathrm{Re}(\Upsilon_0) = \frac{3\theta^3(2 - 3\theta^2 + 6\theta^4 - \theta^6)}{2(1+\theta^2)^4(\theta^2-1)}.$$

Corollary 6.2. *Under one of conditions (A1)-(A3), if* $a > b$ *then in every neighborhood of* $(U,b) = (\mathbf{e}, b_*)$ *there is a branch of spatially homogeneous periodic solutions* $U_b(x,t)$ *satisfying* $U_b(x,t) \to \mathbf{e}$ *as* $b \to b_*$ *and the Hopf bifurcation is forward (respectively, backward) and the bifurcation periodic solutions are orbitally asymptotically stable (respectively, unstable) if* $2 - 3\theta^2 + 6\theta^4 - \theta^6 < 0$ *(respectively,* > 0*), where* $b_* \triangleq \frac{\theta(1+\theta^2)}{\theta^2-1}$.

7 Bogdanov–Takens bifurcation

Apart from the occurrence of Hopf bifurcation discussed so far, codimension 2 bifurcation such as Bogdanov–Takens bifurcation is also possible in system (1.4). In order to discuss codimension 2 bifurcation, in addition to taking β as a bifurcation parameter, we need another parameter. It is easy to see that $T_i(\alpha_1,\beta)$ depends on (β, α_1 θ, b) and $D_i(\alpha_1,\beta)$ on (β, b, α_1, α_2 θ). More precisely, α_1 depends on θ and b, α_2 on θ. For convenience, we choose α_1 and β as bifurcation parameters. In this section, we investigate the Bogdanov–Takens bifurcation at the nontrivial steady-state solution $\mathbf{e} = (u^*, v^*)^T$ of (1.4) under the condition (H1) and the following assumption

(H4) $T_i(\alpha_1,\beta) = 0$, $D_i(\alpha_1,\beta) = 0$, $T_j(\alpha_1,\beta) \neq 0$, $D_j(\alpha_1,\beta) \neq 0$ for $j \neq i$.

That is, the Bogdanov–Takens bifurcation is a bifurcation in a two-parameter family of system (1.4) at which $\mathbf{e} = (u^*, v^*)^T$ has a zero eigenvalue of geometric multiplicity one and algebraic multiplicity two. Assume that $\alpha_1 > b + (d_1 + d_2)\lambda_i$ and $b\theta > d_2\lambda_i^2 + b(d_2 - d_1)\lambda_i$, then the only choice of (α_1, β) satisfying assumption (H4) is (α_1^*, β^*), where

$$\alpha_1^* = d_1\lambda_i + \sqrt{b(d_1 + d_2)\lambda_i + b\theta + b^2}, \qquad \beta^* = \frac{\alpha_1 - b - (d_1 + d_2)\lambda_i}{\theta\lambda_i}.$$

Clearly, if $j \neq i$, we have $T_j(\alpha_1^*, \beta^*) \neq 0$. Furthermore,

$$D_j(\alpha_1^*, \beta^*) = \left(1 - \frac{\lambda_j}{\lambda_i}\right)\left(bd_1\lambda_j + b\theta - d_1\lambda_j\sqrt{b(d_1 + d_2)\lambda_i + b\theta + b^2}\right)$$

Hence, it is easy to see that $D_j(\alpha_1^*, \beta^*) \neq 0$ for all $j \neq i$ if $\sqrt{b(d_1 + d_2)\lambda_i + b\theta + b^2} \neq b + \frac{b\theta}{d_1\lambda_j}$. Therefore, we have the following result:

Theorem 7.1. *Under the assumption (H1), if $a > b$ and $\sqrt{b(d_1 + d_2)\lambda_i + b\theta + b^2} \neq b + \frac{b\theta}{d_1\lambda_j}$ and $b\theta > d_2\lambda_i^2 + b(d_2 - d_1)\lambda_i$, then near $(\alpha_1, \beta) = (\alpha_1^*, \beta^*)$ system (1.4) has a Bogdanov–Takens singularity at the positive constant steady-state solution $\mathbf{e} = (u^*, v^*)^T$.*

Under assumptions (H1) and (H4), $\mathcal{L}_{\alpha_1^*,\beta^*}$ has exactly a zero eigenvalue of geometric multiplicity one and algebraic multiplicity two. Let $\mathbb{P}$ be the subspace of $\mathcal{L}_{\alpha_1^*,\beta^*}$ associated with zero eigenvalues. Let $\Phi = (\phi_1, \phi_2) = (\mathbf{c}_1\varphi_i, \mathbf{c}_2\varphi_i)$ be a basis for $\mathbb{P}$, and $\Psi = (\psi_1, \psi_2)^T = (\mathbf{d}_1\varphi_i, \mathbf{d}_2\varphi_i)^T$ be the basis for the dual space $\mathbb{P}^*$ in $\mathbb{X}$, such that $\langle \psi_j, \phi_s \rangle = \delta_{js}$, where δ_{js} is the Kronecker delta. Obviously,

$$\begin{aligned} \mathcal{M}(0, \alpha_1^*, \beta^*, \lambda_i)\mathbf{c}_1 = 0 \qquad &\text{and} \qquad \mathcal{M}(0, \alpha_1^*, \beta^*, \lambda_i)\mathbf{c}_2 = \mathbf{c}_1, \\ \mathcal{M}^T(0, \alpha_1^*, \beta^*, \lambda_i)\mathbf{d}_2 = 0 \qquad &\text{and} \qquad \mathcal{M}^T(0, \alpha_1^*, \beta^*, \lambda_i)\mathbf{d}_2 = \mathbf{d}_1, \end{aligned}$$

where

$$\mathbf{c}_1 = \left(1, \frac{-d_1\lambda_i + \alpha_1^*}{\alpha_2}\right)^T, \qquad \mathbf{c}_2 = \left(0, -\frac{1}{\alpha_2}\right)^T,$$

and

$$\mathbf{d}_1 = (1, 0)^T, \qquad \mathbf{d}_2 = ((d_2 + \beta^*\theta)\lambda_i + b, -\alpha_2)^T.$$

Thus, $\langle \Psi, \Phi \rangle = \mathrm{Id}_2$ and $\dot{\Phi} = B\Phi$, where

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We adopt the framework of [11], we rewrite system (1.4) as

$$\frac{dU}{dt} = \mathcal{L}(\alpha_1, \beta, U)U + F(\alpha_1, \beta, U), \tag{7.1}$$

where

$$F(\alpha_1, \beta, U) = \begin{pmatrix} u\left(a - u - \dfrac{v}{1 + u^2}\right) - d_1\Delta u - \alpha_1 u - \alpha_2 v \\ v\left(b - \dfrac{v}{1 + u^2}\right) - \beta b(1 + \theta^2)\Delta u - \alpha_3 u - (d_2 + \beta\theta)\Delta v + bv \end{pmatrix}.$$

We decompose $\mathbb{X} = \mathbb{X}^c + \mathbb{X}^s$, with $\mathbb{X}^c \triangleq \{z\Phi | z \in \mathbb{R}^2\}$, $\mathbb{X}^s \triangleq \{U \in \mathbb{X} | \langle \Psi, U \rangle = 0\}$. For any $U = (u, v)^T \in \mathbb{X}$, there exist $z \in \mathbb{R}$ and $y = (y_1, y_2) \in \mathbb{X}^s$ such that $U = \mathbf{e} + \Phi z + y$. Then system (7.1) is reduced to the following system in (z, y) coordinates:

$$\begin{cases} \frac{dz}{dt} = Bz + \langle \Psi, F(\mathbf{e} + \Phi z + y, \alpha_1, \beta) \rangle, \\ \frac{dy}{dt} = \mathcal{L}(\alpha_1, \beta)y + H(z_1, z_2, y), \end{cases} \tag{7.2}$$

where

$$H(z_1, z_2, y) = F(\mathbf{e} + \Phi z + y, \alpha_1, \beta) - \langle \Psi, F(\mathbf{e} + \Phi z + y, \alpha_1, \beta)\rangle.$$

According to [11], the normal form of Bogdanov–Takens bifurcation under conditions (H1) and (H4) is given by

$$\begin{cases} \dfrac{dz_1}{dt} = z_2, \\ \dfrac{dz_2}{dt} = C_{10}(\alpha_1, \beta)z_1 + C_{01}(\alpha_1, \beta)z_2 + C_{20}(\alpha_1^*, \beta^*)z_1^2 + C_{11}(\alpha_1^*, \beta^*)z_1 z_2, \end{cases} \tag{7.3}$$

where

$$\begin{aligned} C_{10}(\alpha_1, \beta) &= \langle \psi_2, \mathcal{L}^1_{\alpha_1}\phi_1\alpha_1 + \mathcal{L}^1_\beta \phi_1 \beta\rangle \\ &= [(d_2 + \beta^*\theta)\lambda_i + b](\alpha_1 - \alpha_1^*) + [b(1+\theta^2)\lambda_i\alpha_2 + \theta(\alpha_1^* - d_1\lambda_i)\lambda_i](\beta - \beta^*), \\ C_{01}(\alpha_1, \beta) &= \langle \psi_2, \mathcal{L}^1_{\alpha_1}\phi_2\alpha_1 + \mathcal{L}^1_\beta\phi_2\beta\rangle + \langle \psi_1, \mathcal{L}^1_{\alpha_1}\phi_1\alpha_1 + \mathcal{L}^1_\beta(\phi_1)\beta\rangle = \alpha_1 - \alpha_1^* - \theta\lambda_i(\beta - \beta^*), \\ C_{20}(\alpha_1^*, \beta^*) &= \langle \psi_2, \mathfrak{G}^2(\phi_1, \phi_1)\rangle, \\ C_{11}(\alpha_1^*, \beta^*) &= \langle \psi_1, \mathfrak{G}^2(\phi_1, \phi_1)\rangle + \langle \psi_2, \mathfrak{G}^2(\phi_1, \phi_2)\rangle. \end{aligned}$$

For convenience, we denote

$$\tilde{G} = \begin{bmatrix} (d_2 + \beta^*\theta)\lambda_i + b & b(1+\theta^2)\alpha_2\lambda_i + \theta(\alpha_1^* - d_1\lambda_i)\lambda_i \\ 1 & -\theta\lambda_i \end{bmatrix}.$$

It is easy to see that $\det\tilde{G} > 0$. Therefore, we have the following conclusion.

Theorem 7.2. *Under assumptions (H1) and (H4), if $a > b$ and $C_{20}C_{11} \neq 0$, then system (7.1) undergoes a Bogdanov–Takens bifurcation. More precisely, if $C_{20}C_{11} < 0$, then, in the (C_{10}, C_{01}) bifurcation diagram, the Hopf bifurcation curve Γ_1 and the homoclinic bifurcation curve Γ_2 lie in the region $\mathfrak{W}$. Both the homoclinic loop and the periodic orbit are unstable, where*

$$\begin{aligned} \mathfrak{W} &= \{(C_{10}, C_{01}) \,|\, C_{10} > 0 \text{ and } C_{01} < 0\} \\ \Gamma_1 &= \left\{ (\alpha_1, \beta) \,\middle|\, C_{01}(\alpha_1, \beta) = \frac{C_{11}}{C_{20}}C_{10}(\alpha_1, \beta) + h.o.t,\ C_{10}(\alpha_1, \beta) > 0 \right\} \\ \Gamma_2 &= \left\{ (\alpha_1, \beta) \,\middle|\, C_{01}(\alpha_1, \beta) = \delta\sqrt{C_{10}(\alpha_1, \beta)}C_{10}(\alpha_1, \beta) + h.o.t,\ C_{10}(\alpha_1, \beta) > 0 \right\} \end{aligned}$$

and δ is a continuous and differentiable function satisfying $\delta(0) = \frac{6C_{11}}{7C_{20}}$.

8 Conclusions and numerical simulations

In this paper, we have shown that all solutions of system (1.4) exist globally and are uniformly bounded if β satisfies (2.4). But we don't know whether the solution of system (1.4) can blow up in a finite time or exists globally if (2.4) does not hold. This is a problem filled with challenge. Next, this paper presents the existence of the non-constant positive steady states of system (1.4). In view of Theorems 4.5 and 4.6, we see that system (1.4) has a non-constant positive steady state if either the diffusion coefficient d_1 is small or the cross-diffusion coefficient β is large. This implies the predator and prey species may coexist in the interacting habit nonuniformly if the predator disperses quickly from a high density of prey to a low density one, or the prey move slowly from a higher to a lower concentration region. Our theoretical analysis shows that the cross-diffusion phenomenon has the potential to play an important role in the coexistence information. From the biological point of view, our analysis gives a theoretical support for studying coexistence phenomena of reaction-diffusion systems with cross-diffusion.

Sections 6 and 7 show that system (1.4) is capable of producing much more abundant dynamics than the corresponding ODEs. For example, system (1.4) may have multiple bifurcation under certain conditions, and both Hopf bifurcation and homoclinic bifurcation are possible. According to Section 7, we know that the ODEs associated with (1.4) (i.e., with $d_1 = d_2 = \beta = 0$) cannot show Bogdanov–Takens singularity, but system (1.4) can show Bogdanov–Takens singularity (see Theorem 7.1); this indicates that diffusion plays a fundamental role in producing a rich dynamics and even Bogdanov–Takens bifurcation phenomena. Meanwhile, the existence and properties of the spatially nonhomogeneous Hopf bifurcation of system (1.4) (i.e., $\lambda_i \neq \lambda_0$) are established in Theorem 6.1, and Corollary 6.2 is devoted to spatially homogeneous Hopf bifurcation of system (1.4) (i.e., $\lambda_i = \lambda_0$).

Finally, we present some numerical simulations to support and supplement our analytic results given in the previous sections. For the spatially homogeneous model (1.4), it follows from Corollary 6.2 that $\mathbf{e}$ is locally asymptotically stable if $a > b$, $b < \frac{\theta(1+\theta^2)}{\theta^2-1}$ and one of conditions (A1)–(A3) is satisfied, and is unstable if $\theta > 1$ and $b > \frac{\theta(1+\theta^2)}{\theta^2-1}$. In addition, when b passes through $\frac{\theta(1+\theta^2)}{\theta^2-1}$ from the left of $\frac{\theta(1+\theta^2)}{\theta^2-1}$, $\mathbf{e}$ will lose its stability and a family of periodic solutions bifurcate from the interior equilibrium $\mathbf{e}$. It also follows from Corollary 6.2 that the direction of Hopf bifurcation is forward and the bifurcating periodic solutions are asymptotically stable if $2 - 3\theta^2 + 6\theta^4 - \theta^6 < 0$. For system (1.4) with $\Omega = (0, 2\pi)$ and initial values $u(x,t) = \cos^2 \frac{x}{\pi}$ and $v(x,t) = \cos^2 \frac{x}{\pi}$, if we fix $\theta = 2.4$, then the critical point $\frac{\theta(1+\theta^2)}{\theta^2-1} = 3.4084$ and $2 - 3\theta^2 + 6\theta^4 - \theta^6 = -7.3174 < 0$. Next, we can choose the following three sets of parameter values, which satisfy conditions (A1)–(A3) respectively:

(P1) $d_1 = 3$, $d_2 = 1$, $\beta = 0.3$;

(P2) $d_1 = 3$, $d_2 = 1$, $\beta = 1$;

(P3) $d_1 = 3$, $d_2 = 5$, $\beta = 0.5$.

Obviously, if the values of a and b are fixed, the mathematical phenomena described by the above three sets of parameters are quite similar. Without loss of generality, we illustrate our analytical results by numerical simulations only under the condition (P1). If $a = 5.55$ and $b = 3.15 < 3.4084$, then the positive constant solution $\mathbf{e}$ is locally asymptotically stable (see Figure 8.1). Choose $a = 5.81$, $b = 3.41 > 3.4084$, then we see that a limit cycle arises out of Hopf bifurcation around $\mathbf{e}$ (see Figure 8.2). Lemma 5.2 tells us the positive constant solution $\mathbf{e}$ is globally asymptotically stable if

$$a > b,\, 2b(a+c) < 1 \quad \text{and} \quad \beta \leq \frac{2}{c}\sqrt{\frac{d_1 d_2 \theta}{b(1+\theta^2)}}.$$

Here, for system (1.4) with $\Omega = (0, 2\pi)$, we choose $d_1 = d_2 = 1$, $a = 1.5$, $b = 0.15$ $\beta = 2.38$, and initial values $u(x,t) = \cos^2 \frac{x}{\pi}$, $v(x,t) = \cos^2 \frac{x}{\pi}$, then $c = a = 1.5$, $\beta < \frac{2}{c}\sqrt{\frac{d_1 d_2 \theta}{b(1+\theta^2)}} = 2.3809$ and $2b(a+c) = 0.9$. Thus, as depicted in Figure 8.3, the positive constant solution $\mathbf{e}$ is globally asymptotically stable. Nevertheless, we do not know whether the conclusion of Lemma 5.2 holds true if the value b does not satisfy $2b(a+c) < 1$. Therefore, we just find a sufficient condition ensuring the global asymptotical stability of the positive steady-state solution $\mathbf{e}$. However, when $a > b$, we can get the critical value b_* of the parameter b by fixing the value of θ. It follows from Corollary 6.2 that the positive steady-state solution $\mathbf{e}$ is locally asymptotically stable if $b < b_*$ and will lose its stability when b passes b_* from the left of b_*.

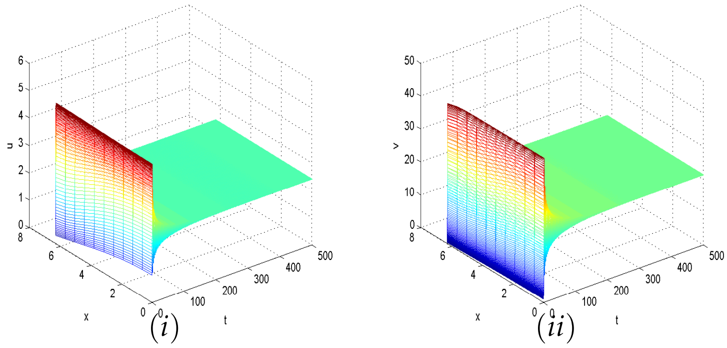

Figure 8.1: The solutions of model (1.4) tends to a positive steady state with parameters $b = 3.15 < 3.4084$ and $a = 5.55$.

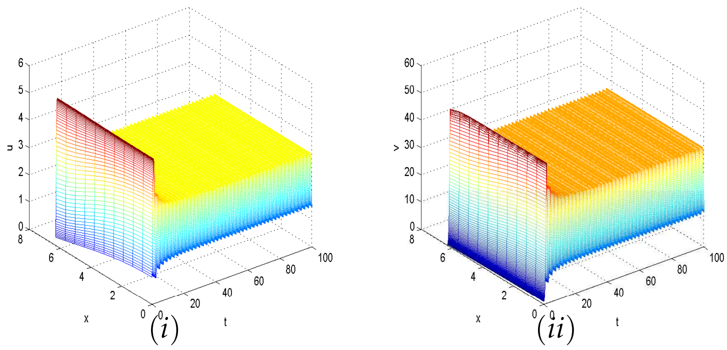

Figure 8.2: The solutions of model (1.4) with $b = 3.41 > 3.4084$ and $a = 5.81$ tends to a positive spatially homogeneous time-periodic orbit.

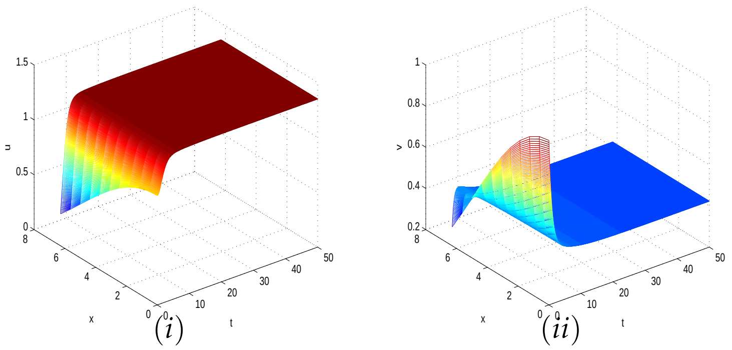

Figure 8.3: The positive steady-state solution $\mathbf{e} = (1.35, 0.4234)$ of model (1.4) with parameters $a = 1.5$ $b = 0.15$ and $\beta = 2.38$ is globally asymptotically stable
.

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Uniqueness and nonuniqueness of fronts for degenerate diffusion-convection reaction equations

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Abstract. We consider a scalar parabolic equation in one spatial dimension. The equation is constituted by a convective term, a reaction term with one or two equilibria, and a positive diffusivity which can however vanish. We prove the existence and several properties of traveling-wave solutions to such an equation. In particular, we provide a sharp estimate for the minimal speed of the profiles and improve previous results about the regularity of wavefronts. Moreover, we show the existence of an infinite number of semi-wavefronts with the same speed.

Keywords: degenerate and doubly degenerate diffusivity, diffusion-convection-reaction equations, traveling-wave solutions, sharp profiles, semi-wavefronts.

2020 Mathematics Subject Classification: 35K65, 35C07, 34B40, 35K57.

1 Introduction

We study the existence and qualitative properties of traveling-wave solutions to the scalar diffusion-convection-reaction equation

$$\rho_t + f(\rho)_x = (D(\rho)\rho_x)_x + g(\rho), \qquad t \geq 0,\ x \in \mathbb{R}. \tag{1.1}$$

Here $\rho = \rho(t,x)$ is the unknown variable and takes values in the interval $[0,1]$. The convective term f satisfies the condition

(f) $f \in C^1[0,1]$, $f(0) = 0$.

The requirement $f(0) = 0$ is not a real assumption, since f is defined up to an additive constant; we denote $h(\rho) = \dot{f}(\rho)$, where with a dot we intend the derivative with respect to the variable ρ (or φ later on). About the diffusivity D and the reaction term g we consider two different scenarios, where the assumptions are made on the pair D, g; we assume either

(D1) $D \in C^1[0,1]$, $D > 0$ in $(0,1)$ and $D(1) = 0$,

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(g0) $g \in C^0[0,1]$, $g > 0$ in $(0,1]$, $g(0) = 0$,

or else

(D0) $D \in C^1[0,1]$, $D > 0$ in $(0,1)$ and $D(0) = 0$,

(g01) $g \in C^0[0,1]$, $g > 0$ in $(0,1)$, $g(0) = g(1) = 0$.

In the above notation, the numbers suggest where it is *mandatory* that the corresponding function vanishes. Notice that (D1) leaves open the possibility for D to vanish or not at 0, and (D0) for D at 1. We refer to Figure 1.1 for a graphical illustration of these assumptions. Notice that the product Dg always vanishes at both 0 and 1 under both set of assumptions.

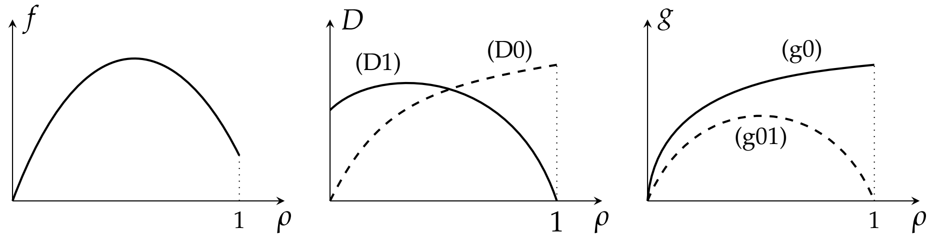

Figure 1.1: Typical plots of the functions f, D and g. In the plots of D and g, solid or dashed lines depict pairs of functions D and g that are considered together in the following. The possibility that D vanishes at the other extremum is left open.

We also require the following condition on the product of D and g:

$$\limsup_{\varphi \to 0^+} \frac{D(\varphi)g(\varphi)}{\varphi} < +\infty, \tag{1.2}$$

which is equivalent to $D(\varphi)g(\varphi) \le L\varphi$, for some $L > 0$ and φ in a right neighborhood of 0.

In (1.1), the notation $\rho = \rho(t,x)$ suggests a density; this is indeed the case. Recently, the modeling of collective movements has attracted the interest of several mathematicians [9,10,22]. This paper is partly motivated by such a research stream and carries on the analysis of a scalar parabolic model begun in [5–7]. Indeed, if $f(\rho) = \rho v(\rho)$, where the velocity v is an assigned function, then equation (1.1) can be understood as a simplified model for a crowd walking with velocity v along a straight path with side entries for other pedestrians, which are modeled by g; here ρ is understood as the crowd normalized density. Assumption (g01), for instance, means that pedestrians do not enter if the road is empty ($g(0) = 0$, modeling an aggregative behavior) or if it is fully occupied ($g(1) = 0$, because of lack of space). If the diffusivity is small, then the diffusion term accounts for some "chaotic" behavior, which is common in crowds movements. In this framework, D may degenerate at the extrema of the interval where it is defined [2, 4, 20]; for more details we refer to [6]. The assumption (g0) is better motivated by population dynamics. In this case g is a growth term which, for instance, increases with the population density ρ. We refer to [19] for analogous modelings in biology. Anyhow, apart from the above possible applications, equation (1.1) is a quite general diffusion-convection-reaction equation that deserves to be fully understood.

A *traveling-wave solution* is, roughly speaking, a solution to (1.1) of the form $\rho(t,x) = \varphi(x-ct)$, for some profile $\varphi = \varphi(\xi)$ and constant wave speed c, see [11] for general information. In this case the profile must satisfy, in some sense, the equation

$$\left(D(\varphi)\varphi'\right)' + (c - h(\varphi))\,\varphi' + g(\varphi) = 0, \tag{1.3}$$

where $'$ denotes the derivative with respect to ξ. We consider in this paper non-constant, monotone profiles, and focus on the case they are decreasing. As a consequence, we aim at determining solutions to (1.3) whose values at $\pm\infty$ are the zeroes of the function g and then satisfy either

$$\varphi(-\infty) = 1, \qquad \varphi(+\infty) = 0, \tag{1.4}$$

or simply

$$\varphi(+\infty) = 0, \tag{1.5}$$

according to we make assumption (g01) or (g0). The former profiles are called *wavefronts*, the latter are *semi-wavefronts*; precise definitions are provided in Definition 2.1. Notice that in both cases the equilibria may be reached for a finite value of the variable ξ as a consequence of the degeneracy of D at those points. These solutions represent single-shape smooth transitions between the two constant densities 0 and 1. The interest of wavefronts lies in the fact that they are viscous approximations of shock waves to the inviscid version of equation (1.1), i.e., when $D = 0$. Semi-wavefronts lack of this motivation but are nevertheless meaningful for applications [6]; moreover, wavefronts connecting “nonstandard” end states can be constructed by pasting semi-wavefronts [7]. At last, we point out that assumption (1.2) is usual in this framework, when looking for decreasing profiles, see e.g. [1].

If $D(\rho) \geq 0$, the existence of solutions to the initial-value problem for (1.1) is more or less classical [24]; however, the *fine structure* of traveling waves reveals a variety of different patterns. We refer to [15,16], respectively, for the cases where D is non degenerate, i.e., $D > 0$, and for the degenerate case, where D can vanish at either 0 or 1. The main results of those papers is that there is a critical threshold c^*, depending on both f and the product Dg, such that traveling waves satisfying (1.4) exist if and only if $c \geq c^*$. The smoothness of the profiles depend on f, D and c but not on g. In both papers the source term satisfies (g01); see [5,6] for the case when g has only one zero.

The case when D changes sign, which is not studied in this paper, also has strong motivations: we quote [13,21] for biological models and [7] for applications to collective movements. Several results about traveling waves have been obtained in [7,8,12–14].

In this paper we study semi-wavefronts and wavefronts for (1.1), thus completing the analysis of [5, 6]. We prove that in both cases there is a threshold c^* such that profiles only exists for $c \geq c^*$; we also study their regularity and strict monotonicity, namely whether they are *classical* (i.e., C^1) or *sharp* (and then reach an equilibrium at a finite ξ in a no more than continuous way). We strongly rely on [15, 16] and exploit some recent results obtained in [18]. Several examples are scattered throughout the paper to show that our assumptions are necessary in most cases.

This research has some important novelties. First, we give a refined estimate for c^*, which allows to better understand the meaning of this threshold. Second, we improve a result obtained in [16] about the appearance of wavefronts with a sharp profile. Third, in the case of semi-wavefronts, we show that for any speed $c \geq c^*$ there exists a family of profiles with speed c. This phenomenon does not show up in [5,6].

The main tool to investigate (1.3) is the analysis of singular first-order problems as

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{D(\varphi)g(\varphi)}{z(\varphi)}, & \varphi \in (0,1), \\ z(\varphi) < 0, & \varphi \in (0,1), \\ z(0) = 0. \end{cases} \tag{1.6}$$

Problem (1.6) is deduced by problem (1.3)–(1.5) by the singular change of variables $z(\varphi) := D(\varphi)\varphi'$, where the right-hand side is understood to be computed at $\varphi^{-1}(\varphi)$, see e.g. [6, 15]. Notice that φ^{-1} exists by the assumption of monotony of φ.

On the other hand, the analysis of problem (1.6) is fully exploited in the forthcoming paper [3], which deals with the case in which D changes sign. In that paper we show that there still exist wavefronts joining 1 with 0, which travel across the region where D is negative; they are constructed by pasting two semi-wavefronts obtained in the current paper. Similar results in the case $g = 0$ are proved in [7].

Here is an account of the paper. In Section 2 we provide some basic definitions and state our main results. The analysis of problem (1.6) and of other related singular problems occupies Sections 3 to 8. Then, in Sections 9 and 10 we exploit such results to construct semi-wavefronts and wavefronts, respectively; there, we prove our main results.

2 Main results

We give some definitions on traveling waves and their profiles. Let $I \subseteq \mathbb{R}$ be an open interval.

Definition 2.1. Assume $f, D, g \in C[0,1]$. Consider a function $\varphi \in C(I)$ with values in $[0,1]$, which is differentiable a.e. and such that $D(\varphi)\varphi' \in L^1_{\mathrm{loc}}(I)$; let c be a real constant. The function $\rho(x,t) := \varphi(x - ct)$, for (x,t) with $x - ct \in I$, is a *traveling-wave* solution of equation (1.1) with wave speed c and wave profile φ if, for every $\psi \in C_0^\infty(I)$,

$$\int_I \left(D\left(\varphi(\xi)\right)\varphi'(\xi) - f\left(\varphi(\xi)\right) + c\varphi(\xi)\right)\psi'(\xi) - g\left(\varphi(\xi)\right)\psi(\xi)\,d\xi = 0. \tag{2.1}$$

Definition 2.1 can be made more precise. Below, *monotonic* means that $\varphi(\xi_1) \leq \varphi(\xi_2)$ (or $\varphi(\xi_1) \geq \varphi(\xi_2)$) for every $\xi_1 < \xi_2$ in the domain of φ; in *(iii)* we assume $g(0) = g(1) = 0$, while in *(iv)* we only require that g vanishes at the point which is specified by the semi-wavefront. A traveling-wave solution is

(i) *global* if $I = \mathbb{R}$ and *strict* if $I \neq \mathbb{R}$ and φ is not extendible to $\mathbb{R}$;

(ii) *classical* if φ is differentiable, $D(\varphi)\varphi'$ is absolutely continuous and (1.3) holds a.e.; *sharp at* ℓ if there exists $\xi_\ell \in I$ such that $\varphi(\xi_\ell) = \ell$, with φ classical in $I \setminus \{\xi_\ell\}$ and not differentiable at ξ_ℓ;

(iii) *a wavefront* if it is global, with a monotonic, non-constant profile φ satisfying either (1.4) or the converse condition;

(iv) *a semi-wavefront to* 1 (or *to* 0) if $I = (a, \infty)$ for $a \in \mathbb{R}$, the profile φ is monotonic, non-constant and $\varphi(\xi) \to 1$ (respectively, $\varphi(\xi) \to 0$) as $\xi \to \infty$; *a semi-wavefront from* 1 (or *from* 0) if $I = (-\infty, b)$ for $b \in \mathbb{R}$, the profile φ is monotonic, non-constant and $\varphi(\xi) \to 1$ (respectively, $\varphi(\xi) \to 0$) as $\xi \to -\infty$.

In *(iv)* we say that φ connects $\varphi(a^+)$ (1 or 0) with 1 or 0 (resp., with $\varphi(b^-)$).

The smoothness of a profile depends on the degeneracy of D, see [11]. More precisely, assume (f), and either (D1), (g0) or (D0), (g01); let ρ be any traveling-wave solution of (1.1) with profile φ defined in I and speed c. Then φ is classical in each interval $J \subset I$ where $D(\varphi(\xi)) > 0$ for $\xi \in J$, and $\varphi \in C^2(J)$. Profiles are determined up to a space shift.

Our first main result concerns *semi-wavefronts*.

Theorem 2.2. *Assume* (f), (D1), (g0) *and* (1.2). *Then, there exists* $c^* \in \mathbb{R}$, *which satisfies*

$$\max\left\{\sup_{\varphi\in(0,1]}\frac{f(\varphi)}{\varphi}, h(0)+2\sqrt{\liminf_{\varphi\to 0^+}\frac{D(\varphi)g(\varphi)}{\varphi}}\right\} \\ \le c^* \le 2\sqrt{\sup_{\varphi\in(0,1]}\frac{D(\varphi)g(\varphi)}{\varphi}} + \sup_{\varphi\in(0,1]}\frac{f(\varphi)}{\varphi}, \quad (2.2)$$

such that (1.1) *has strict semi-wavefronts to* 0, *connecting* 1 *to* 0, *if and only if* $c \ge c^*$.

Moreover, if φ *is the profile of one of such semi-wavefronts, then it holds that*

$$\varphi'(\xi) < 0 \ \text{ for any } \ 0 < \varphi(\xi) < 1. \quad (2.3)$$

For a fixed $c > c^*$, the profiles of Theorem 2.2 are *not* unique. This lack of uniqueness is not due only to the action of space shifts but, more intimately, to the non-uniqueness of solutions to problem (1.6) that is proved in Proposition 5.1 below. Roughly speaking, these profiles depend on a parameter b ranging in the interval $[\beta(c), 0]$, for a suitable threshold $\beta(c) \le 0$. As a conclusion, the family of profiles can be precisely written as

$$\varphi_b = \varphi_b(\xi), \qquad \text{for } b \in [\beta(c), 0]. \quad (2.4)$$

Moreover, $\beta(c) < 0$ if $c > c^*$ and $\beta(c) \to -\infty$ as $c \to +\infty$. The threshold $\beta(c)$ essentially corresponds to the minimum value that the quantity $D(\varphi_b)\varphi_b'$ achieves when φ_b reaches 1, for $b \in [\beta(c), 0]$. This loss of uniqueness is a novelty if we compare Theorem 2.2 with analogous results in [5, 6]. In particular, in [6, Theorem 2.7] the assumptions on the functions D and g are *reversed*: both of them are positive in $(0,1)$ with $D(0) = 0 < g(0)$, $D(1) > 0 = g(1)$; in [5, Theorem 2.3] D and g are still positive in $(0,1)$ but the vanishing conditions are $D(1) = 0 = g(1)$. In both cases the profiles exist for every $c \in \mathbb{R}$ and are unique. The different results are due to the nature of the equilibria of the dynamical systems of (1.3).

The estimates (2.2) deserve some comments. The left estimate improves analogous bounds (see [18] for a comprehensive list) by including the term $\sup_{\varphi\in(0,1]} f(\varphi)/\varphi \ge h(0)$ on the left-hand side. This improvement looks more significative if we also assume $(\dot{D g})(0) = 0$, as we do in the Theorem 2.3. In this case (2.2) reduces to

$$\sup_{\varphi\in(0,1]}\frac{f(\varphi)}{\varphi} \le c^* \le 2\sqrt{\sup_{\varphi\in(0,1]}\frac{D(\varphi)g(\varphi)}{\varphi}} + \sup_{\varphi\in(0,1]}\frac{f(\varphi)}{\varphi}. \quad (2.5)$$

which can be written with obvious notation as

$$c_{con} \le c^* \le c_{dr} + c_{con},$$

where the indexes label velocities related to the convection or diffusion-reaction components. In (2.5) the *same* term, accounting for the dependence on f, occurs in *both* the lower and upper bound. This symmetry, which shows the shift of the critical threshold as a consequence of the convective term f, occurs in *none* of the previous estimates.

The meaning of c_{dr} is known since [1]; we comment on c_{con}. In the diffusion-convection case (i.e., when $g = 0$), there exist profiles connecting $\ell \in (0, 1]$ to 0 if and only if

$$s_\ell(\varphi) := \frac{f(\ell)}{\ell}\varphi > f(\varphi), \quad \text{for } \varphi \in (0, \ell), \tag{2.6}$$

see [11, Theorem 9.1]. The quantity c_{con} then represents the *maximal speed* that can be reached by the profiles connecting ℓ to 0, for $\ell \in (0, 1]$. Condition (2.6) is also necessary and sufficient in the purely hyperbolic case (i.e., when also $D = 0$) in order that the equation $u_t + f(u)_x = 0$ admits a shock wave of speed $f(\ell)/\ell$ with ℓ as left state and 0 as right state. This is not surprising since the viscous profiles approximate the shock wave and converge to it in the vanishing viscosity limit. Indeed, condition (2.6) does not depend on D.

The presence of the positive reaction term g satisfying (g01) (if (g0) holds we only have semi-wavefronts, but the same bounds still hold) does not allow profile speeds to be less than c_{con}: assuming that z satisfies (1.6), by the positivity of both D and g we deduce

$$c \geq \sup_{\varphi \in (0,1]} \left(\frac{f(\varphi)}{\varphi} - \frac{z(\varphi)}{\varphi} \right) \geq c_{con}. \tag{2.7}$$

Then, c_{con} now becomes a bound for the *minimal speed* of the profiles. The bound (2.7) is strict (i.e., there is a gap between c_{con} and c^*) if $(\dot{Dg})(0) > 0$; this occurs for instance if $D(0) > 0$ and $\dot{g}(0) > 0$ and follows by integrating $(1.6)_1$ from 0 to φ and (2.2), see Remark 5.6. If $f = 0$, then the corresponding strict bound $c^* > 0$ occurs for any positive and continuous D and g: if $c^* = 0$ then z should be an increasing function by (3.11), a contradiction.

In some cases, semi-wavefronts are sharp at 0. We refer to Corollary 9.4 for a detailed account of the behavior of the profiles when they reach the equilibrium.

We now present our result on *wavefronts*; we assume that D and g satisfiy (D0) and (g01). The goal is to extend results contained in [16, Theorems 2.1 and 6.1] regarding the existence and, more importantly, the regularity of wavefronts of Equation (1.1). In particular, the next theorem has the merit to derive the classification of wavefronts under (D0), merely, without additional assumptions (which were instead required in [16, Theorems 2.1 and 6.1]). Notice that in the following result we require that D vanishes at 0; this assumption leads to improve not only the left-hand bound (2.2) on c^* by (2.5), but also the right-hand bound, by means of a recent integral estimate provided in [18].

Theorem 2.3. *Assume* (f), (D0) *and* (g01) *and* (1.2)*. Then there exists c^*, satisfying*

$$\sup_{\varphi \in (0,1]} \frac{f(\varphi)}{\varphi} \leq c^* \leq \sup_{\varphi \in (0,1]} \frac{f(\varphi)}{\varphi} + 2\sqrt{\sup_{\varphi \in (0,1]} \frac{1}{\varphi} \int_0^\varphi \frac{D(\sigma)g(\sigma)}{\sigma}\, d\sigma}, \tag{2.8}$$

such that Equation (1.1) *admits a (unique up to space shifts) wavefront, whose wave profile φ satisfies* (1.4)*, if and only if $c \geq c^*$. Moreover, we have $\varphi'(\xi) < 0$, for $0 < \varphi(\xi) < 1$, and*

(i) if $c > c^*$, then φ is classical at 0*;*

(ii) if $c = c^*$ and $c^* > h(0)$, then φ is sharp at 0 *and if it reaches* 0 *at $\xi_0 \in \mathbb{R}$ then*

$$\lim_{\xi \to \xi_0^-} \varphi'(\xi) = \begin{cases} \frac{h(0)-c^*}{\dot{D}(0)} < 0 & \text{if } \dot{D}(0) > 0, \\ -\infty & \text{if } \dot{D}(0) = 0. \end{cases}$$

As in analogous cases [6], Theorem 2.3 provides no information about the smoothness of the profiles when $c = c^* = h(0)$. We show in Remark 10.1 that in such a case profiles may be either sharp or classical.

3 Singular first-order problems

Here we begin the analysis of problem (1.6). First, we consider, for $c \in \mathbb{R}$, the problem

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, & \varphi \in (0,1), \\ z(\varphi) < 0, & \varphi \in (0,1), \end{cases} \tag{3.1}$$

where we assume

$$q \in C^0[0,1] \quad \text{and} \quad q > 0 \text{ in } (0,1). \tag{3.2}$$

We point out that the differential equation $(3.1)_1$ generalizes $(1.6)_1$ since the assumptions on q are a bit less strict than the ones on Dg, under (D1)–(g0) or (D1)–(g01). In the following lemma we prove that a solution of (3.1) can be extended continuously up to the boundary.

Lemma 3.1. *Assume* (3.2). *If $z \in C^1(0,1)$ is a solution of* (3.1), *then it can be extended continuously to the interval* $[0,1]$.

Proof. Since $q/z < 0$ in $(0,1)$, then for any $0 < \varphi < \varphi_1 < 1$ the function

$$\varphi \to \int_{\varphi}^{\varphi_1} \frac{q(\sigma)}{z(\sigma)}\, d\sigma$$

is strictly increasing. Hence, we can pass to the limit as $\varphi \to 0^+$ in the expression

$$z(\varphi) = z(\varphi_1) - \int_{\varphi}^{\varphi_1} (h(\sigma) - c)\, d\sigma + \int_{\varphi}^{\varphi_1} \frac{q(\sigma)}{z(\sigma)}\, d\sigma, \tag{3.3}$$

which is obtained by integrating $(3.1)_1$ in (φ, φ_1). Then $z(0^+)$ exists and necessarily lies in $[-\infty, 0]$ because of $(3.1)_2$. If $z(0^+) = -\infty$, then by passing to the limit for $\varphi \to 0^+$ in (3.3) we find a contradiction, since the last integral converges as $\varphi \to 0^+$. Hence, $z(0^+) \in (-\infty, 0]$.

For $z(1^-)$ the proof is even simpler: by integrating $(3.1)_1$ in (φ_2, φ), for $0 < \varphi_2 < \varphi < 1$, we obtain (3.3) with φ_2 replacing φ_1. As before, we deduce that $z(1^-)$ exists. Also, since the last integral in (3.3) is now positive, we get $z(\varphi) > z(\varphi_2) + \int_{\varphi_2}^{\varphi} (h(\sigma) - c)\, d\sigma$, for any $\varphi \in (\varphi_2, 1)$. This directly rules out the alternative $z(1^-) = -\infty$ and concludes the proof. $\square$

We now summarize [6, Lemmas 4.1 and 4.3] in a version for our purposes, by also exploiting Lemma 3.1. These tools were obtained in [6] under stricter assumptions on q, but it is easy to verify that they also apply to the current case, in virtue of (3.2). For $\mu < 0$ and $\sigma \in (0,1]$ or $\sigma \in [0,1)$, they deal with the systems

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, \ \varphi < \sigma, \\ z(\sigma) = \mu, \end{cases} \qquad \begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, \ \varphi > \sigma, \\ z(\sigma) = \mu. \end{cases} \tag{3.4}$$

A function $\eta \in C^1(\sigma_1, \sigma_2)$, for $0 \le \sigma_1 < \sigma_2 \le 1$, is an upper-solution of $(3.1)_1$ in (σ_1, σ_2) if

$$\dot{\eta}(\varphi) \ge h(\varphi) - c - \frac{q(\varphi)}{\eta(\varphi)} \quad \text{for any} \quad \sigma_1 < \varphi < \sigma_2. \tag{3.5}$$

The upper-solution η is said strict if the inequality in (3.5) is strict. A function $\omega \in C^1(\sigma_1, \sigma_2)$ is a (strict) lower-solution of $(3.1)_1$ in (σ_1, σ_2) if the (strict) inequality in (3.5) is reversed.

Lemma 3.2. *Assume* (3.2) *and consider equation* $(3.1)_1$*; the following results hold.*

1. *Set* $\mu < 0$. *Then,*
   (a) *let* $\sigma \in (0, 1]$*; then problem* $(3.4)_1$ *admits a unique solution* $z \in C^0[0, \sigma] \cap C^1(0, \sigma)$*;*
   (b) *let* $\sigma \in [0, 1)$*; then problem* $(3.4)_2$ *admits a unique solution* $z \in C^0[\sigma, \delta] \cap C^1(\sigma, \delta)$*, for some maximal* $\sigma < \delta \leq 1$. *Moreover, either* $\delta = 1$ *or* $z(\delta) = 0$.

2. *Set* $0 \leq \sigma_1 < \sigma_2 \leq 1$*; let* z *be a solution of* (3.1) *in* (σ_1, σ_2). *It holds that:*
   (a) *if* η *is a strict upper-solution of* $(3.1)_1$ *in* (σ_1, σ_2)*, then*
      (i) *if* $\eta(\sigma_2) \leq z(\sigma_2) < 0$*, then* $\eta < z$ *in* (σ_1, σ_2)*;*
      (ii) *if* $0 > \eta(\sigma_1) \geq z(\sigma_1)$ *then* $\eta > z$ *in* (σ_1, σ_2)*; moreover, if* η *is defined in* $[0, 1]$*, then* z *must be defined in* $[\sigma_1, 1]$ *and* $\eta > z$ *in* $(\sigma_1, 1)$*;*
   (b) *if* ω *is a strict lower-solution of* $(3.1)_1$ *in* (σ_1, σ_2)*, then*
      (i) *if* $0 > \omega(\sigma_2) \geq z(\sigma_2)$*, then* $\omega > z$ *in* (σ_1, σ_2)*; moreover, if* ω *is defined in* $[0, 1]$*, then* z *must be defined in* $[0, \sigma_2]$ *and* $\omega > z$ *in* $(0, \sigma_2)$*;*
      (ii) *if* $\omega(\sigma_1) \leq z(\sigma_1) < 0$ *then* $\omega < z$ *in* (σ_1, σ_2).

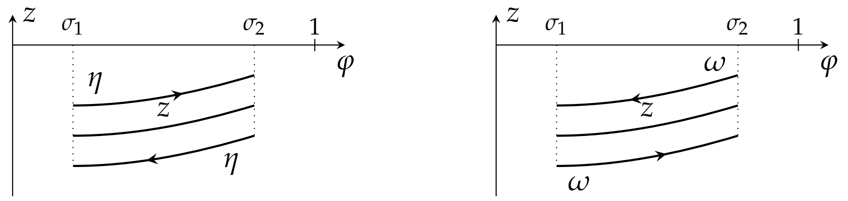

Figure 3.1: An illustration of Lemma 3.2 *(2)*. Left: supersolutions η; right: subsolutions ω.

In the context of equations as $(3.1)_1$, proper limit arguments are often needed.

Lemma 3.3. *Assume* (3.2). *Let* $\{c_n\}_n$ *be a sequence of real numbers and* $c \in \mathbb{R}$ *such that* $c_n \to c$ *as* $n \to \infty$. *Let* $z_n \in C^0[0, 1] \cap C^1(0, 1)$ *satisfy* (3.1) *corresponding to* c_n. *If* $\{z_n\}_n$ *is increasing and there exists* $v \in C^0[0, 1]$ *such that*

$$z_n(\varphi) \leq v(\varphi) < 0 \quad \text{for any} \quad n \in \mathbb{N} \quad \text{and} \quad \varphi \in (0, 1), \tag{3.6}$$

then z_n *converges (uniformly on* $[0, 1]$*) to a solution* $\bar{z} \in C^0[0, 1] \cap C^1(0, 1)$ *of* (3.1).

The same conclusion holds if $\{z_n\}_n$ *is decreasing and there exists* $w \in C^0[0, 1]$ *such that*

$$z_n(\varphi) \geq w(\varphi) \quad \text{for any} \quad n \in \mathbb{N} \quad \text{and} \quad \varphi \in (0, 1).$$

Proof. Take first $\{z_n\}_n$ increasing. From (3.6), we can define $\bar{z} = \bar{z}(\varphi)$ as

$$\lim_{n \to \infty} z_n(\varphi) =: \bar{z}(\varphi), \qquad \varphi \in (0, 1).$$

It is obvious that $z_1 \leq \bar{z} \leq v < 0$ in $(0, 1)$. By integrating $(3.1)_1$, we have

$$z_n(\varphi) - z_n(\varphi_0) = \int_{\varphi_0}^{\varphi} \left\{ h(\sigma) - c_n + \frac{q(\sigma)}{-z_n(\sigma)} \right\} d\sigma \quad \text{for any} \quad \varphi_0, \varphi \in (0, 1).$$

Since, for every $\sigma \in (0,1)$, the sequence $\{q(\sigma)/(-z_n(\sigma))\}_n$ is increasing, then the Monotone Convergence Theorem implies that

$$\bar{z}(\varphi) - \bar{z}(\varphi_0) = \int_{\varphi_0}^{\varphi} \left\{ h(\sigma) - c - \frac{q(\sigma)}{\bar{z}(\sigma)} \right\} d\sigma \quad \text{for any } \varphi_0, \varphi \in (0,1),$$

where all the involved quantities are finite. This tells us that $\bar{z}$ is absolutely continuous in every compact interval $[a,b] \subset (0,1)$. By differentiating, we then obtain that $\bar{z} \in C^1(0,1)$ satisfies (3.1). From Lemma 3.1, we also have that $\bar{z} \in C^0[0,1]$. To conclude that z_n converges to $\bar{z}$ uniformly on $[0,1]$, it only remains to prove that

$$\bar{z}(0^+) = \lim_{n\to\infty} z_n(0) \quad \text{and} \quad \bar{z}(1^-) = \lim_{n\to\infty} z_n(1). \tag{3.7}$$

Indeed, if (3.7) holds, then $\{z_n\}_n$ turns out to be a monotone sequence of continuous functions converging pointwise to $\bar{z} \in C^0[0,1]$ on a compact set. Then, by *Dini's monotone convergence theorem* (see [23, Theorem 7.13]), z_n must converge uniformly to $\bar{z}$ on $[0,1]$. We prove only $(3.7)_1$ since $(3.7)_2$ follows as well. If $z_n(0) \to 0$, as $n \to \infty$, then $\bar{z}(0^+) = 0$, because $z_n \le \bar{z} < 0$ in $(0,1)$. Hence $(3.7)_1$ is verified. If instead $z_n(0) \to \mu < 0$, we argue as follows. Consider $\delta \in \mathbb{R}$ such that $c_n > \delta$, for any $n \in \mathbb{N}$, and let $\eta = \eta(\varphi)$ satisfy

$$\begin{cases} \dot{\eta}(\varphi) = h(\varphi) - \delta - \frac{q(\varphi)}{\eta(\varphi)}, & \varphi > 0, \\ \eta(0) = \mu. \end{cases} \tag{3.8}$$

By Lemma 3.2 *(1.b)* such an η exists, in its maximal-existence interval $[0,\sigma)$, for some $\sigma \in (0,1]$. Moreover, we have

$$\dot{\eta}(\varphi) > h(\varphi) - c_n - \frac{q(\varphi)}{\eta(\varphi)}, \qquad \varphi \in (0,\sigma).$$

Hence, in $(0,\sigma)$, η is a strict upper-solution of $(3.1)_1$ with $c = c_n$ and $z_n(0) \le \eta(0) < 0$. Thus, Lemma 3.2 *(2.a.ii)* implies that $z_n \le \eta$ in $(0,\sigma)$. By passing to the pointwise limit, for $n \to \infty$, it is clear that $\bar{z} \le \eta$ in $(0,\sigma)$. Since $\bar{z}, \eta$ are continuous up to $\varphi = 0$, then $\bar{z}(0^+) \le \mu$. On the other hand we have $\bar{z}(0^+) \ge \mu$ because $z_n \le \bar{z}$ in $(0,1)$ and $z_n, \bar{z} \in C^0[0,1]$. Then $\bar{z}(0^+) = \mu$ and this concludes the proof of $(3.7)_1$.

Consider $\{z_n\}_n$ decreasing. By adapting the arguments used in the first part of this proof, we can show that z_n converges pointwise in $(0,1)$ to $\bar{z} \in C^0[0,1] \cap C^1(0,1)$ satisfying (3.1). As before we need (3.7) to conclude. To this end, we again observe that similarly to the case of $\{z_n\}_n$ increasing, we have (3.7) if both $z_n(0) \to \mu < 0$ and $z_n(1) \to \nu < 0$. Instead, the proofs of either $(3.7)_1$ when $z_n(0) \to 0$ and $(3.7)_2$ when $z_n(0) \to 0$ are now more subtle. We provide them both. First, since $z_n < 0$ in $(0,1)$, observe that requiring that $z_n(0) \to 0$ (or $z_n(1) \to 0$) corresponds to have $z_n(0) = 0$ (or $z_n(1) = 0$), for every $n \in \mathbb{N}$.

Take $z_n(0) = 0$, for $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and for $\varphi \in (0,1)$, let $\sigma_\varphi \in (0,\varphi)$ be defined by

$$\dot{z}_n(\sigma_\varphi) = \frac{z_n(\varphi)}{\varphi}.$$

Take $\delta_1 \in \mathbb{R}$ such that $\delta_1 > c_n$, for each $n \in \mathbb{N}$. By using $(3.1)_1$ and the fact that $q/z_n < 0$ in $(0,1)$, we deduce, for any $\varphi \in (0,1)$,

$$\frac{z_n(\varphi)}{\varphi} = \dot{z}_n(\sigma_\varphi) > h(\sigma_\varphi) - c_n > \inf_{\varphi\in(0,1)} h(\varphi) - \delta_1 =: C < 0. \tag{3.9}$$

The sign of C is due to $c_n \geq h(0)$, for $n \in \mathbb{N}$; otherwise, it would not be possible to have z_n satisfying (3.1) and $z_n(0) = 0$. Inequality (3.9) implies that $z_n(\varphi) > C\varphi$ for $\varphi \in (0,1)$. Hence, letting $n \to \infty$, this leads to $\bar{z}(\varphi) \geq C\varphi$, for $\varphi \in (0,1)$. Passing to the limit as $\varphi \to 0^+$ gives $\bar{z}(0^+) \geq 0$, which in turn implies that $\bar{z}(0^+) = 0$. Thus, $(3.7)_1$ is verified.

Lastly, let $z_n(1) = 0$, for any $n \in \mathbb{N}$. Fix $\varepsilon > 0$ and consider $\eta_2 = \eta_2(\varphi)$ such that

$$\begin{cases} \dot{\eta}_2(\varphi) = h(\varphi) - \delta - \frac{q(\varphi)}{\eta_2(\varphi)}, & \varphi > 0, \\ \eta_2(1) = -\varepsilon < 0, \end{cases} \tag{3.10}$$

where $\delta \in \mathbb{R}$ is such that $\delta < c_n$, for any $n \in \mathbb{N}$. Such an η_2 exists and is defined and continuous in $[0,1]$, because of Lemma 3.2 *(1.a)* and Lemma 3.1. Take an arbitrary $n \in \mathbb{N}$. From $0 = z_n(1) > \eta_2(1)$, it follows that $\eta_2 < z_n$ in $[\sigma_n, 1]$, for some $\sigma_n > 0$, with $z_n(\sigma_n) < 0$. Thus, since

$$\dot{\eta}_2(\varphi) > h(\varphi) - c_n - \frac{q(\varphi)}{\eta_2(\varphi)}, \qquad \varphi \in (0,1),$$

then η_2 is a strict upper-solution of $(3.1)_1$ with $c = c_n$ in $(0, \sigma_n)$ and $\eta_2(\sigma_n) < z_n(\sigma_n) < 0$. An application of Lemma 3.2 *(2.a.i)* implies that $\eta_2 < z_n$ in $(0, \sigma_n)$. Thus, $z_n > \eta_2$ in $(0,1)$, for any $n \in \mathbb{N}$. By passing to the pointwise limit, as $n \to \infty$, we then have $\bar{z}(\varphi) \geq \eta_2(\varphi)$, for $\varphi \in (0,1)$. By the continuity of both $\bar{z}$ and η_2 at $\varphi = 1$, we obtain $0 \geq \bar{z}(1^-) \geq -\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we deduce that necessarily $\bar{z}(1^-) = 0$. □

Because of Lemmas 3.1 and 3.3, in the following we always mean solutions z to problem (3.1), and analogous ones, in the class $C[0,1] \cap C^1(0,1)$, *without any further mention*.

Motivated by Lemma 3.1, in the next sections we focus the following problem, where the boundary condition is given on the *left* extremum of the interval of definition:

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, & \varphi \in (0,1), \\ z(\varphi) < 0, & \varphi \in (0,1), \\ z(0) = 0. \end{cases} \tag{3.11}$$

Problem (3.11) is exploited for *semi-wavefronts*. The value of $z(1)$ is not prescribed; from $(3.11)_2$, we have $z(1) \leq 0$. The extremal case $z(1) = 0$ is needed in the study of *wavefronts*:

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, & \varphi \in (0,1), \\ z(\varphi) < 0, & \varphi \in (0,1), \\ z(0) = z(1) = 0. \end{cases} \tag{3.12}$$

4 The singular problem with two boundary conditions

Problems (3.11) and (3.12) have solutions only when c is larger than a critical threshold c^*. In this section we first give a new estimate to c^* under mild conditions on q; then, we obtain a result of existence and uniqueness of solutions to (3.12) if $c \geq c^*$. Recalling (D1), (g0) and (1.2) and (D0)–(g01), throughout the next sections we need to strengthen the assumptions (3.2) of Section 3; for commodity we gather them all here below. We assume

(q) $q \in C^0[0,1]$, $q > 0$ in $(0,1)$, $q(0) = q(1) = 0$ and $\limsup_{\varphi \to 0^+} \dfrac{q(\varphi)}{\varphi} < +\infty$.

We improve, as in [18, Theorem 3.1], a well-known result [1,11,15]. If q is differentiable at 0, in [18, Theorem 3.1] it is proved that Problem (3.11) has a solution if

$$c > \sup_{\varphi\in(0,1]} \frac{f(\varphi)}{\varphi} + 2\sqrt{\sup_{\varphi\in(0,1]} \frac{1}{\varphi}\int_0^{\varphi} \frac{q(\sigma)}{\sigma}\,d\sigma}. \tag{4.1}$$

The last assumption in (q) is weaker than the differentiability of q at 0 and our result below is less stronger than the one in [18]. It is an open problem whether the existence of solutions to Problem (3.12) under (4.1) can be achieved by only assuming $\limsup_{\varphi\to 0^+} q(\varphi)/\varphi < +\infty$.

Lemma 4.1. *Assume* (q). *Then Problem* (3.12) *admits a solution if*

$$c > \sup_{\varphi\in(0,1]} \frac{f(\varphi)}{\varphi} + 2\sqrt{\sup_{\varphi\in(0,1]} \frac{q(\varphi)}{\varphi}}. \tag{4.2}$$

Proof. We follow [18, Theorem 3.1]. By (4.2) we see that there exists $K > 0$, $\varepsilon > 0$ so that

$$K^2 + \left(\sup_{\varphi\in(0,1]} \frac{f(\varphi)}{\varphi} - c\right) K + \sup_{\varphi\in(0,1]} \frac{q(\varphi)}{\varphi} < -\varepsilon K < 0 \quad \text{for} \quad \varphi \in (0,1].$$

For every $\tau > 0$, we get, for any $\varphi > \tau$,

$$\frac{1}{\varphi-\tau}\int_\tau^{\varphi} \frac{q(s)}{s}\,ds = \frac{q(s_{\varphi,\tau})}{s_{\varphi,\tau}} \le \sup_{\varphi\in(0,1]} \frac{q(\varphi)}{\varphi},$$

where $s_{\varphi,\tau} \in (\tau,\varphi)$ is given by the Mean Value Theorem. As a consequence, for any $\tau > 0$,

$$K^2 + \left(\sup_{\varphi\in(0,1]} \frac{f(\varphi)}{\varphi} + \varepsilon - c\right) K + \frac{1}{\varphi-\tau}\int_\tau^{\varphi} \frac{q(s)}{s}\,ds < 0 \quad \text{for every} \quad \varphi \in (\tau,1].$$

A continuity argument in [18] implies that there exists $\overline{\tau}$ such that for any $\tau < \overline{\tau}$ we have

$$\frac{f(\varphi)-f(\tau)}{\varphi-\tau} \le \frac{f(\varphi)}{\varphi} + \varepsilon \le \sup_{\varphi\in(0,1]} \frac{f(\varphi)}{\varphi} + \varepsilon, \qquad \varphi \in (\tau,1],$$

and thus, for such values of τ, it must hold

$$K^2 + \left(\frac{f(\varphi)-f(\tau)}{\varphi-\tau} - c\right) K + \frac{1}{\varphi-\tau}\int_\tau^{\varphi} \frac{q(s)}{s}\,ds < 0 \quad \text{for every} \quad \varphi \in (\tau,1].$$

This implies that the function $\eta_\tau = \eta_\tau(\varphi)$, defined for $\varphi \in [\tau,1]$ by

$$\eta_\tau(\varphi) := -K\tau + \int_\tau^{\varphi} \left\{ h(\sigma) - c - \frac{q(\sigma)}{-K\sigma} \right\} d\sigma,$$

is an upper-solution of $(3.11)_1$ such that $\eta_\tau(\varphi) < -K\varphi$, for $\varphi \in (\tau,1]$, and $\eta_\tau(\tau) = -K\tau < 0$. Arguments based on Lemma 3.2 *(2.a.ii)* imply that it results defined in $[\tau,1]$ a function z_τ which solves $(3.4)_2$ with $\mu = -K\tau$; we extend continuously z_τ to $[0,\tau]$ by $z_\tau(\varphi) = -K\varphi$, for $\varphi \in [0,\tau]$. This gives a family $\{z_\tau\}_{\tau>0}$ of decreasing functions as $\tau \to 0^+$ (in the sense that $z_{\tau_1} \le z_{\tau_2}$ in $[0,1]$ for $0 < \tau_1 < \tau_2$). After some manipulations of the differential equation in $(3.4)_2$, based on the sign of q/z_τ and on $\eta_\tau(\varphi) < -K\varphi$, for $\varphi \in (\tau,1]$, we deduce that

$$f(\varphi) - c\varphi \le z_\tau(\varphi) \le -K\varphi, \qquad \varphi \in [0,1].$$

Hence, applying Lemma 3.3 in each interval $(a,b) \subset [0,1]$ we finally deduce that $\bar{z}$, the limit of z_τ for $\tau \to 0^+$, solves $(3.11)_1$, $\bar{z} < 0$ in $(0,1)$ and $\bar{z}(0) = 0$. Hence, $\bar{z}$ is a solution of (3.11). Finally, as observed in [18], an application of [17, Lemma 2.1] implies the conclusion. □

We now give a result about solutions to (3.12); see Figure 5.1 on the left.

Proposition 4.2. *Assume* (q). *Then, there exists* c^* *satisfying*

$$h(0) + 2\sqrt{\liminf_{\varphi\to 0^+} \frac{q(\varphi)}{\varphi}} \le c^* \le 2\sqrt{\sup_{\varphi\in(0,1]} \frac{q(\varphi)}{\varphi}} + \sup_{\varphi\in(0,1]} \frac{f(\varphi)}{\varphi}, \tag{4.3}$$

such that there exists a unique z *satisfying* (3.12) *if and only if* $c \ge c^*$.

Proof. The result, apart from the refined estimate (4.3) is proved in [17, Proposition 1]. Estimate (4.3) follows from Lemma 4.1 and $\sup_{\varphi\in(0,1]} f(\varphi)/\varphi \le \max_{\varphi\in[0,1]} h(\varphi)$. □

5 The singular problem with left boundary condition

Now we face problem (3.11). We always assume (q) and refer to the threshold c^* introduced in Proposition 4.2; we denote by z^* the corresponding unique solution to (3.12). See Figure 5.1 on the left for an illustration of Proposition 5.1.

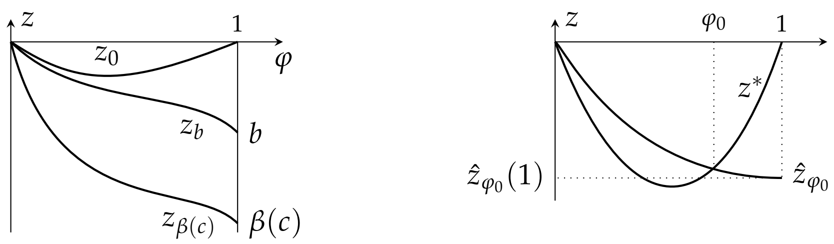

Figure 5.1: Left: an illustration of Propositions 4.2 and 5.1, for fixed $c > c^*$. Solutions to (3.11) are labelled according to their right-hand limit: z_0 occurs in the former proposition, z_b in the latter. Right: the functions $\hat{z}_{\varphi_0}$ and z^* in *Step (i)* of Proposition 5.1.

Proposition 5.1. *Assume* (q). *For every* $c > c^*$, *there exists* $\beta = \beta(c) < 0$ *satisfying*

$$\beta \ge f(1) - c, \tag{5.1}$$

such that problem (3.11) *with the additional condition* $z(1) = b < 0$ *admits a unique solution* z *if and only if* $b \ge \beta$.

In the above proposition, the threshold case $c = c^*$ is a bit more technical; we shall prove in Proposition 6.3 that $\beta(c^*) = 0$ under some further assumptions.

Proof of Proposition 5.1. For any $c > c^*$, we define the set $\mathcal{A}_c$ as

$$\mathcal{A}_c := \{b < 0 : (3.11) \text{ admits a solution with } z(1) = b\}.$$

We show that $\mathcal{A}_c = [\beta, 0)$, for some $\beta = \beta(c) < 0$, by dividing the proof into four steps.

Step (i): $\mathcal{A}_c \neq \varnothing$. We claim that there exists $\hat{z}$ which satisfies (3.11) and $\hat{z}(1) < 0$. Take $\varphi_0 \in (0,1)$ and consider the following problem, see Figure 5.1 on the right,

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, \\ z(\varphi_0) = z^*(\varphi_0). \end{cases} \tag{5.2}$$

Lemma 3.2 *(1)* implies the existence of a solution $\hat{z}_{\varphi_0}$ of (5.2) defined in its maximal-existence interval $(0, \delta)$, for some $\varphi_0 < \delta \le 1$. Since $\hat{z}_{\varphi_0}$ satisfies $(5.2)_1$ and $c > c^*$, then

$$\dot{\hat{z}}_{\varphi_0}(\varphi) = h(\varphi) - c^* - \frac{q(\varphi)}{\hat{z}_{\varphi_0}(\varphi)} + (c^* - c) < h(\varphi) - c^* - \frac{q(\varphi)}{\hat{z}_{\varphi_0}(\varphi)}, \qquad \varphi \in (0, \delta).$$

This implies that $\hat{z}_{\varphi_0}$ is a strict lower-solution of $(3.11)_1$ with $c = c^*$. From Lemma 3.2 *(2.b)*, this and $\hat{z}_{\varphi_0}(\varphi_0) = z^*(\varphi_0) < 0$ imply that

$$z^* < \hat{z}_{\varphi_0} \text{ in } (0, \varphi_0) \quad \text{and} \quad \hat{z}_{\varphi_0} < z^* \text{ in } (\varphi_0, \delta). \tag{5.3}$$

Since $z^* < \hat{z}_{\varphi_0} < 0$ in $(0, \varphi_0)$, we get $\hat{z}_{\varphi_0}(0^+) = 0$. Since $\hat{z}_{\varphi_0} < z^*$ in (φ_0, δ), we obtain that $\hat{z}_{\varphi_0}(\delta^-) \le z^*(\delta^-)$. Thus $\delta = 1$, otherwise $\hat{z}_{\varphi_0}(\delta) < 0$, in contradiction with the fact that $(0, \delta)$ is the maximal-existence interval of $\hat{z}_{\varphi_0}$.

From Lemma 3.1, $\hat{z}_{\varphi_0}(1) \in \mathbb{R}$. It remains to prove that $\hat{z}_{\varphi_0}(1) < 0$. From what we observed above, it follows that $z^* > \hat{z}_{\varphi_0}$ in $(\varphi_0, 1)$. Hence, for any $\varphi \in (\varphi_0, 1)$, we have

$$\dot{z^*}(\varphi) - \dot{\hat{z}}_{\varphi_0}(\varphi) = c - c^* + \frac{q(\varphi)}{z^*(\varphi)\hat{z}_{\varphi_0}(\varphi)}\left(z^* - \hat{z}_{\varphi_0}\right)(\varphi) > \frac{q(\varphi)}{z^*(\varphi)\hat{z}_{\varphi_0}(\varphi)}\left(z^* - \hat{z}_{\varphi_0}\right)(\varphi) > 0.$$

This implies that $(z^* - \hat{z}_{\varphi_0})$ is strictly increasing in $(\varphi_0, 1)$ and hence

$$-\hat{z}_{\varphi_0}(1) = z^*(1) - \hat{z}_{\varphi_0}(1) > z^*(\varphi_0) - \hat{z}_{\varphi_0}(\varphi_0) = 0,$$

which means $\hat{z}_{\varphi_0}(1) < 0$. Thus, $\hat{z}_{\varphi_0}(1) \in \mathcal{A}_c$.

Step (ii): if $b \in \mathcal{A}_c$ then $[b, 0) \subset \mathcal{A}_c$. Suppose that there exists $b \in \mathcal{A}_c$ and let z_b be the solution of (3.11) and $z_b(1) = b$. Take $b < b_1 < 0$. For Lemma 3.2 *(1.a)* there exists z_{b_1} defined in $(0, 1)$ satisfying $(3.11)_1$ and $z_{b_1}(1) = b_1 < 0$.

We claim that $z_b < z_{b_1}$ in $(0, 1)$. If not, then $z_b(\varphi_0) = z_{b_1}(\varphi_0) =: y_0 < 0$, for some $\varphi_0 \in (0, 1)$. Without loss of generality we can assume $z_b < z_{b_1}$ in $(\varphi_0, 1]$. We denote by $f_c(\varphi, y) = h(\varphi) - c - q(\varphi)/y$ the right-hand side of the differential equation in (3.11); the function f_c is continuous in $[0, 1] \times (-\infty, 0)$ and locally Lipschitz-continuous in y. Hence, z_b and z_{b_1} are two different solutions of

$$\begin{cases} y' = f_c(\varphi, y), & \varphi \in (\varphi_0, 1), \\ y(\varphi_0) = y_0, \end{cases}$$

which contradicts the uniqueness of the Cauchy problem. Thus, $z_b < z_{b_1} < 0$ in $(0, 1)$. Since z_b satisfies $(3.11)_3$ then $z_{b_1}(0^+) = 0$ and hence $b_1 \in \mathcal{A}_c$.

Step (iii): $\inf \mathcal{A}_c \in \mathbb{R}$. Suppose that z satisfies Equation $(3.11)_1$. As already observed, this implies $\dot{z}(\varphi) > h(\varphi) - c$, $\varphi \in (0, 1)$. Thus, for any $\varphi \in (0, 1)$,

$$z(\varphi) = z(\varphi) - z(0) \ge \int_0^{\varphi} h(\sigma) - c \, d\sigma = f(\varphi) - c\varphi. \tag{5.4}$$

This implies that $z(1) \ge f(1) - c$. Define $\beta = \beta(c)$ by

$$\beta := \inf \mathcal{A}_c.$$

Thus, $\beta \ge f(1) - c > -\infty$, which also proves (5.1).

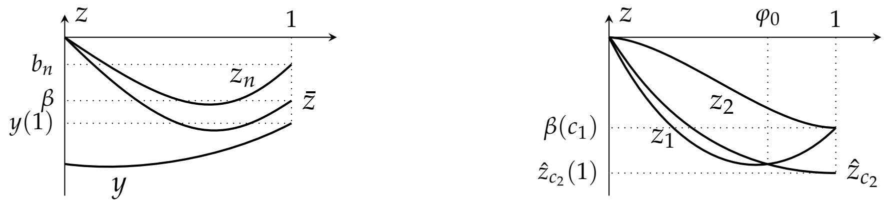

Figure 5.2: Left: the functions z_n, y and $\bar{z}$ in *Step (iv)* of Proposition 5.1. Right: the functions z_1, z_2 and $\hat{z}_{c_2}$ in the proof of *(i)* of Corollary 5.3.

Step (iv): $\beta \in \mathcal{A}_c$. Let $\{b_n\}_n \subset \mathcal{A}_c$ be a strictly decreasing sequence such that $b_n \to \beta^+$. Since $b_n \in \mathcal{A}_c$, each b_n is associated with a solution z_n of (3.11) and $z_n(1) = b_n$. From the uniqueness of the solution of Cauchy problem for $(3.11)_1$, the sequence z_n is decreasing.

For any given $\delta < \beta$, let y be defined by

$$\begin{cases} \dot{y}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{y(\varphi)}, & \varphi < 1 \\ y(1) = \delta < \beta. \end{cases}$$

Such a y exists and is defined in $[0,1]$ from Lemma 3.2 *(1.a)*. Also, $b_n > \delta$, for any $n \in \mathbb{N}$. Thus, for any $n \in \mathbb{N}$, $z_n \geq y$ in $[0,1]$. Lemma 3.3 implies that there exists $\bar{z}$ satisfying (3.1) such that $z_n \to \bar{z}$ uniformly in $[0,1]$ (see Figure 5.2 on the left). In particular, we deduce that $\bar{z}(0) = 0$ and $\bar{z}(1) = \beta$. Hence, we conclude that $\beta \in \mathcal{A}_c$.

Putting together *Steps (i)–(iv)*, we conclude that $\mathcal{A}_c = [\beta, 0)$. □

The monotonicity of solutions of (3.11) now follows. We omit the proof since it is quite standard, once that Lemma 3.2 *(2)* is given. (See [6, Lemma 5.1].)

Corollary 5.2 (Monotonicity of solutions)**.** *Assume* (q). *Let $c_2 > c_1 \geq c^*$ and assume that z_1 and z_2 satisfy* (3.11) *with $c = c_1$ and $c = c_2$, respectively. Then, if $z_1(1) \leq z_2(1)$ it occurs that $z_1 < z_2$ in $(0,1)$.*

A monotony property of $\beta(c)$ now follows.

Corollary 5.3. *Under* (q) *we have:*

(i) $\beta(c_2) < \beta(c_1)$ *for every $c_2 > c_1 > c^*$;*

(ii) $\beta(c) \to -\infty$ *as $c \to +\infty$.*

Proof. To prove *(i)*, let z_1 be a solution of (3.11) corresponding to $c = c_1$ and such that $z_1(1) = b_1 \in \mathcal{A}_{c_1}$. As a consequence of Lemma 3.2 *(1.a)*, the problem

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c_2 - \frac{q(\varphi)}{z(\varphi)}, & \varphi \in (0,1), \\ z(1) = b_1 < 0, \end{cases}$$

admits a (unique) solution z_2 defined in $[0,1]$. From the monotonicity of solutions given by Corollary 5.2, we have $z_1 < z_2 < 0$ in $(0,1)$. Since $z_1(0) = 0$, then we have $z_2(0) = 0$. Thus, $\mathcal{A}_{c_1} \subseteq \mathcal{A}_{c_2}$ and hence $\beta(c_1) \geq \beta(c_2)$. To prove $\beta(c_1) > \beta(c_2)$ we argue as follows.

For any $\varphi_0 \in (0,1)$ we can repeat the same arguments as in *Step (i)* of Proposition 5.1, by replacing c with c_2 and z^* with z_1 in (5.2). Thus, the problem

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c_2 - \frac{q(\varphi)}{z(\varphi)}, & \varphi \in (0,1), \\ z(\varphi_0) = z_1(\varphi_0) < 0, \end{cases}$$

admits a unique solution $\hat{z}_{c_2}$ defined in $[0,1]$, because necessarily any solution of the last problem must be bounded from above by z_2, see Figure 5.2 on the right. Moreover, by applying Lemma 3.2 *(2.b.ii)*, $\hat{z}_{c_2} < z_1$ in $(\varphi_0, 1)$, which implies that $\hat{z}_{c_2}(1) < z_1(1)$, since

$$\dot{\hat{z}}_{c_2}(\varphi) - \dot{z_1}(\varphi) = c_1 - c_2 + \frac{q(\varphi)}{z_1(\varphi)\hat{z}_{c_2}(\varphi)} \left(\hat{z}_{c_2}(\varphi) - z_1(\varphi)\right) < 0 \quad \text{for any} \quad \varphi \in (\varphi_0, 1).$$

Since $\beta(c_2) \le \hat{z}_{c_2}(1) < z_1(1) = b_1$ then we proved *(i)* since b_1 is arbitrary in $\mathcal{A}_{c_1}$.

Finally, we prove *(ii)*. For $c > c^*$, let z_c be the solution of (3.11) such that $z_c(1) = \beta(c)$. For any fixed $c_1 > c^*$, we have $z_c < z_{c_1}$ in $(0,1)$, if $c > c_1$. Thus, for any $c > c_1$,

$$\dot{z}_c(\varphi) = h(\varphi) - c + \frac{q(\varphi)}{-z_c(\varphi)} < h(\varphi) - c + \frac{q(\varphi)}{-z_{c_1}(\varphi)}, \qquad \varphi \in (0,1).$$

In particular, since $z_{c_1} < 0$ in $(0,1]$, then, for any $0 < \delta < 1$, there exists $M > 0$ such that $q(\varphi)/(-z_{c_1}(\varphi)) \le M$ for any $\varphi \in (\delta, 1]$. Thus, for any $\varphi \in (\delta, 1)$,

$$z_c(\varphi) \le z_c(\delta) + f(\varphi) - f(\delta) + (M - c)\,(\varphi - \delta) < f(\varphi) - f(\delta) + (M - c)\,(\varphi - \delta),$$

which implies $\beta(c) = z_c(1) \le f(1) - f(\delta) + (M - c)(1 - \delta)$. This proves *(ii)*. □

We now collect some consequences of (5.4) and Lemma 4.1, concerning a sharper estimate to c^*. To the best of our knowledge these estimates are new and we provide some comments.

Corollary 5.4. *Assume* (q). *It holds that*

$$c^* \ge \max\left\{ \sup_{\varphi \in (0,1]} \frac{f(\varphi)}{\varphi}, h(0) + 2\sqrt{\liminf_{\varphi \to 0^+} \frac{q(\varphi)}{\varphi}} \right\}. \tag{5.5}$$

Proof. Formula (5.4) in *Step (iii)* implies that $f(\varphi) < c\varphi$, for $\varphi \in (0,1)$. Thus, $f(\varphi) \le c^*\varphi$, for $\varphi \in (0,1)$. This implies $c^* \ge \sup_{\varphi \in (0,1]} \frac{f(\varphi)}{\varphi}$, which, together with (4.3) implies (5.5). □

Remark 5.5. Lemma 4.1 and Corollary 5.4 imply that, under (q), the threshold c^* verifies (2.2). Moreover, make the assumption $\dot{q}(0) = 0$, which is valid if $q = Dg$ under (D1), with $D(0) = 0$, (g0) or under (D0) and (g01). In this case, the estimates in (2.8) hold true. Indeed, the assumptions on q are covered by [18, Theorem 3.1] and hence it follows that

$$c^* \le \sup_{\varphi \in (0,1]} \frac{f(\varphi)}{\varphi} + 2\sqrt{\sup_{\varphi \in (0,1]} \frac{1}{\varphi} \int_0^\varphi \frac{q(\sigma)}{\sigma}\, d\sigma}.$$

The bound from above in (2.8) is then proved. The bound from below in (2.8) is instead due directly to (5.5), because of $\dot{q}(0) = 0$.

Remark 5.6. We can now make precise the statement following formula (2.7) about the gap between c_{con} and c^*. If c_{con} is obtained at some $\varphi \in (0,1]$, then the sup in the right-hand side of (2.7) is strictly larger than c_{con} because $z < 0$ in $(0,1)$. Then $c^* > c_{con}$. Otherwise, if $\sup_{\varphi \in (0,1]} f(\varphi)(\varphi) = h(0)$, then $c_{con} = h(0)$ and by (5.5) we still deduce $c^* > c_{con}$.

6 Further existence and non-existence results

Propositions 4.2 and 5.1 completely treat the existence of solutions of (3.12) and (3.11), respectively, in the cases $c \geq c^*$ and $c > c^*$. In this section, we investigate the remaining cases and show that such propositions are somehow optimal.

We now deal with the following problem, where $c \in \mathbb{R}$ but, differently from (3.11), the boundary condition is imposed on the *right* extremum of the interval of definition:

$$\begin{cases} \dot{\zeta}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{\zeta(\varphi)}, & \varphi \in (0,1), \\ \zeta(\varphi) < 0, & \varphi \in (0,1), \\ \zeta(1) = 0. \end{cases} \tag{6.1}$$

The differential equation in (3.11) and (6.1) is the same; it inherits the properties of the dynamical system underlying (1.3). For slightly more regular functions g, the dynamical system has a center or a node at $(0,0)$ and a saddle at $(1,0)$. The corresponding results, Proposition 5.1 and Lemma 6.1, differ as in Lemma 3.2 *(1)*.

Moreover, while in problem (3.11) the threshold c^* discriminated the existence of solutions, for problem (6.1) solutions will be proved to exist *for every* $c \in \mathbb{R}$; instead, the threshold c^* enters into the problem to discriminate whether solutions reach 0 or not (see Figure 6.1). A related behavior was pointed out in [6, Theorem 2.6]. On the contrary, the monotonicity properties stated in Corollary 5.2 and in Lemma 6.1 are the same.

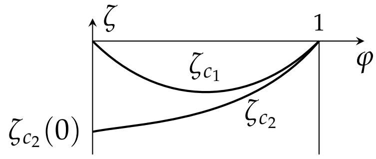

Figure 6.1: An illustration of Lemma 6.1. Here, $c_1 \geq c^*$ while $c_2 < c^*$ and $\zeta_{c_2}(0) < 0$.

Lemma 6.1. *Assume* (q). *For any $c \in \mathbb{R}$, Problem* (6.1) *admits a unique solution ζ_c. If $c \geq c^*$ then $\zeta_c(0) = 0$ and if $c < c^*$ then $\zeta_c(0) < 0$. Moreover, we have:*

(i) if $c_2 > c_1$ then $\zeta_{c_2} > \zeta_{c_1}$ in $(0,1)$;

(ii) it holds that $z^*(\varphi) = \lim_{c \to c^*} \zeta_c(\varphi)$ for any $\varphi \in [0,1]$.

Proof. The existence and uniqueness was proved in [6, Theorem 2.6], while the monotonicity as stated in *(i)* was given in [6, Lemma 5.1]. It remains to prove *(ii)*. We show that

$$\lim_{\delta \to 0^+} \zeta_{c^*-\delta}(\varphi) = \lim_{\delta \to 0^+} \zeta_{c^*+\delta}(\varphi) = z^*(\varphi) \quad \text{for } \varphi \in [0,1].$$

For any $\varphi \in [0,1]$, by *(i)* we have

$$\zeta_{c^*-\delta_2}(\varphi) < \zeta_{c^*-\delta_1}(\varphi) < z^*(\varphi) < \zeta_{c^*+\delta_1}(\varphi) < \zeta_{c^*+\delta_2}(\varphi) \text{ for any } 0 < \delta_1 < \delta_2. \tag{6.2}$$

Lemma 3.3 and (6.2) imply that there exist two functions $\overline{w}, \underline{w} \in C^0[0,1] \cap C^1(0,1)$ so that $\overline{w}(\varphi) = \lim_{\delta \to 0^+} \zeta_{c^*+\delta}(\varphi)$ and $\underline{w}(\varphi) = \lim_{\delta \to 0^+} \zeta_{c^*-\delta}(\varphi)$, $\varphi \in [0,1]$, and that both $\underline{w}$ and $\overline{w}$ satisfy (3.1) with $c = c^*$. Since $\underline{w}(1) = \overline{w}(1) = 0$, both of them then solve (6.1). By the uniqueness of solutions of (6.1) it follows that $\underline{w} = \overline{w} = z^*$. □

Remark 6.2. Note that, because of the uniqueness stated in Lemma 6.1, it follows that, for any $c \geq c^*$, the solution z given by Proposition 4.2 corresponds to ζ_c of Lemma 6.1. Moreover, for $c < c^*$ fixed, there exists a bound from below for $\zeta_c(0) < 0$. We have

$$\zeta_c(0) \geq -1 - A_c, \qquad \text{for } A_c := \max\{\max_{\varphi\in[0,1]} h(\varphi) - c, 0\} + \max_{\varphi\in[0,1]} q(\varphi) > 0.$$

Indeed, the function $\eta(\varphi) := A_c\,(\varphi - 1) - 1$, for $\varphi \in [0,1]$, is a strict upper-solution of $(6.1)_1$. Therefore, if $\zeta_c(\varphi_0) \leq \eta(\varphi_0)$, for some $\varphi_0 \in (0,1)$, then $\zeta_c < \eta$ in $(\varphi_0, 1)$ by Lemma 3.2 *(2.a.ii)*, which is in contradiction with $\zeta_c(1) = 0 > \eta(1)$. Thus, $\zeta_c(0) \geq \eta(0) = -A_c - 1$. Notice that, for $c \geq \max h$, $A_c = \max q$ does not depend on c, while $A_c \to \infty$, as $c \to -\infty$.

We now show that $\beta(c^*) = 0$ under some additional conditions. First, we assume (also for future reference) that $\dot{q}(0)$ exists:

$$\dot{q}(0) = \lim_{\varphi\to 0^+} \frac{q(\varphi)}{\varphi} \in [0, \infty). \tag{6.3}$$

Proposition 6.3. *Assume* (q), (6.3) *and also*

$$\int_0 \frac{q(\sigma)}{\sigma^2}\, d\sigma < +\infty \quad \text{and} \quad c^* > h(0). \tag{6.4}$$

Then Problem (3.11) *with $c = c^*$ admits a unique solution z, which satisfies $z(1) = 0$.*

Notice that $(6.4)_1$ above strengthens the last condition in (q) and is satisfied if $\dot{q}(\varphi) = O(\varphi^\alpha)$ for $\varphi \to 0^+$, for some $\alpha > 0$; in any case it implies $\dot{q}(0) = 0$ by (6.3).

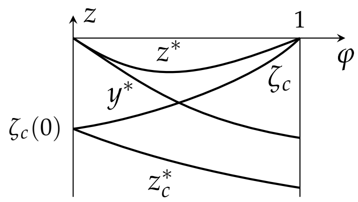

Figure 6.2: The functions z^*, ζ_c, y^* and z_c^*, for $c < c^*$.

Proof of Proposition 6.3. Suppose, by contradiction, that there exists y^* which solves (3.11) with $c = c^*$ and $y^*(1) < 0$; observe that

$$z^* > y^* \quad \text{in } (0,1]. \tag{6.5}$$

We show that y^* is an upper bound for the family of functions $\{z_c^*\}_{c<c^*}$ defined as follows, see Figure 6.2. For any $c < c^*$, let ζ_c be the solution of (6.1), given in Lemma 6.1. Consider the initial-value problem

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c^* - \frac{q(\varphi)}{z(\varphi)}, & \varphi \in (0,1), \\ z(0) = \zeta_c(0) < 0. \end{cases} \tag{6.6}$$

By Lemma 3.2 *(1.b)*, problem (6.6) admits a unique solution z_c^* in $[0,\delta]$ for some $\delta \leq 1$. Moreover, since $z_c^*(0) < 0$ and z_c^* satisfies (6.6), then $z_c^* < y^*$ in $[0,\delta)$. Thus, if $\delta < 1$ then we have $-\infty < z_c^*(\delta) < y^*(\delta) < 0$; again by Lemma 3.2 *(1.b)* we deduce $\delta = 1$. Then

$$y^* > z_c^* \quad \text{in } [0,1). \tag{6.7}$$

By both Lemma 6.1 *(ii)* and (6.7) we now find a contradiction, which implies that such a y^* cannot exist. For this, for any $c < c^*$, define η_c by

$$\eta_c(\varphi) = \zeta_c(\varphi) - z_c^*(\varphi), \qquad \varphi \in [0,1].$$

Since z_c^* is a strict lower-solution of $(3.11)_1$, then Lemma 3.2 *(2.b.ii)* implies $\eta_c > 0$ in $(0,1)$. We claim that, for any fixed $\varphi_0 \in (0,1]$, $\eta_c(\varphi_0)$ is uniformly bounded from below for c close to c^*. Indeed, for any $0 < \delta < (z^* - y^*)(\varphi_0)$, we clearly have, by (6.7) and (6.5),

$$\eta_c(\varphi_0) > \zeta_c(\varphi_0) - y^*(\varphi_0) = (\zeta_c - z^*)(\varphi_0) + (z^* - y^*)(\varphi_0) > (\zeta_c - z^*)(\varphi_0) + \delta.$$

Thus, in virtue of Lemma 6.1 *(ii)*, for any c sufficiently close to c^*, we have

$$\eta_c(\varphi_0) \geq \frac{\delta}{2} > 0, \tag{6.8}$$

which proves our claim. On the other hand, define $k = k(\varphi) > 0$ by

$$k(\varphi) := \frac{q(\varphi)}{(z^* y^*)(\varphi)}, \qquad \varphi \in (0,1).$$

From assumptions (6.3) and $(6.4)_2$ we deduce $\dot{z}^*(0) = h(0) - c^* < 0$ because of [6, Proposition 5.2]. Also, by (6.5) we deduce that $y^* z^* > z^{*2}$ in $(0,1]$. Thus,

$$k(\varphi) < \frac{q(\varphi)}{\varphi^2}\left(\frac{\varphi}{z^*(\varphi)}\right)^2 = \frac{q(\varphi)}{\varphi^2}\left\{\frac{1}{(c^* - h(0))^2} + o(1)\right\} \quad \text{for } \varphi \to 0^+.$$

This leads to

$$\int_0^{\varphi_0} k(\sigma)\, d\sigma =: M < +\infty$$

by means of $(6.4)_1$. Since ζ_c and z_c^* satisfy $(3.11)_1$ with $c < c^*$ and $c = c^*$, respectively, and since $\zeta_c z_c^* > z^* y^*$ by the monotonicity stated in Lemma 6.1 and (6.7), then

$$\dot{\eta}_c(\varphi) = c^* - c - \frac{q(\varphi)}{\zeta_c(\varphi) z_c^*(\varphi)}\left(z_c^*(\varphi) - \zeta_c(\varphi)\right) < c^* - c + k(\varphi)\eta_c(\varphi),$$

for $\varphi \in (0,1)$. After some straightforward manipulations, this gives

$$\frac{d}{d\varphi}\left(\eta_c(\varphi) e^{-\int_0^\varphi k(\sigma)\, d\sigma}\right) \leq (c^* - c)\, e^{-\int_0^\varphi k(\sigma)\, d\sigma}, \qquad \varphi \in (0,1).$$

By integrating in $(0, \varphi_0)$ (where φ_0 is the point for which (6.8) holds) we obtain

$$0 < \eta_c(\varphi_0) \leq (c^* - c)\, e^{\int_0^{\varphi_0} k(\sigma)\, d\sigma} \int_0^{\varphi_0} e^{-\int_0^\sigma k(s)\, ds}\, d\sigma \leq (c^* - c)\, e^M \varphi_0, \tag{6.9}$$

since $e^{-\int_0^\sigma k(s)\, ds} \leq 1$, for any $0 < \sigma < \varphi_0$, because of $k > 0$. Since M does not depend on c, from (6.9), we conclude that $\eta_c(\varphi_0) \to 0$, for $c \to c^*$. This contradicts (6.8). □

We notice that if $q = Dg$, with $D \in C^1[0,1]$, then $(6.4)_1$ follows if we have both $D(0) = 0$ and there exists $L \geq 0$ such that $g(\varphi) \leq L\varphi^\alpha$ for any φ in a right neighborhood of 0 and some $\alpha > 0$. The next remark deals with $(6.4)_2$.

Remark 6.4. First, from (4.3), we have $c^* \geq \sup_{\varphi\in(0,1]} \frac{f(\varphi)}{\varphi} \geq h(0)$. We show that the case $c^* = h(0)$ can indeed occur and then $(6.4)_2$ is a real assumption. Set, for $\varphi \in (0,1)$,

$$q(\varphi) = \varphi^3\,(1-\varphi)\,, \qquad h(\varphi) = 3\varphi\,(\varphi-1)\,, \tag{6.10}$$

and $z(\varphi) = \varphi^2\,(\varphi-1)$. Direct computations show that z satisfies (3.11) with $c = 0 = h(0)$. Hence, $c^* = h(0)$, because of $c^* \geq h(0)$.

Second, in the spirit of [16, Theorems 1.2 and 1.3], which concerns a similar case, we claim that $(6.4)_2$ occurs if there exists $\delta > 0$ such that $h(\varphi) \geq h(0)$ for all $\varphi \in [0,\delta]$. Indeed, if z is a solution of (3.11) with $c = c^*$, then from $(3.11)_1$ we have $\dot z(\varphi) > h(\varphi) - c^* \geq h(0) - c^*$, for $\varphi \in (0,\delta)$. This implies $h(0) - c^* \leq \inf_{\varphi\in(0,\delta)} \dot z(\varphi) < 0$, because of $(3.11)_2$ and $(3.11)_3$, which proves our claim.

Lastly, we show by a counter-example that the conclusion of Proposition 6.3 fails when $(6.4)_1$ holds but $(6.4)_2$ *does not*. Consider, for $\varphi \in [0,1]$, $q(\varphi) = \varphi^4\,(1-\varphi)$ and $y^*(\varphi) = -\varphi^2$. Clearly, $y^* < 0$ in $(0,1)$ and $y^*(0) = 0$. Furthermore, we have

$$\dot y^*(\varphi) + \frac{q(\varphi)}{y^*(\varphi)} = -2\varphi - \varphi^2\,(1-\varphi)\,, \qquad \varphi \in (0,1).$$

This implies that y^* satisfies $(3.11)_1$ with $h(\varphi) = -2\varphi - \varphi^2\,(1-\varphi)$ and $c = 0$. As a consequence, by $c^* \geq h(0) = 0$, we deduce $c^* = h(0) = 0$. Thus, we proved that there exists q satisfying $(6.4)_1$ such that (3.11) with $c = c^* = h(0)$ admits a solution $y^* \neq z^*$.

Proposition 6.5. *Assume* (q). *For no $c < c^*$ problem* (3.11) *admits solutions.*

Proof. Take $c < c^*$ and assume by contradiction that problem (3.11) has a solution z. If $\zeta = \zeta_c$ is the solution to (6.1) given by Lemma 6.1, then $\zeta(0) < 0$, by Proposition 4.2. Then $\zeta(\varphi_0) = z(\varphi_0) =: y_0 < 0$, for some $\varphi_0 \in (0,1)$; see Figure 6.3. This contradicts the uniqueness of the Cauchy problem associated to $(6.1)_1$. The proof is concluded.

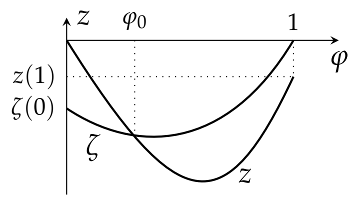

Figure 6.3: The functions z and ζ.

□

7 The behavior of z near 1

In this section and in the next one we investigate the behavior of the solutions z to (3.11) at 1 and 0. We now deal with the former case. We suppose that, analogously to (6.3),

$$\dot q(1) \in (-\infty, 0]. \tag{7.1}$$

Proposition 7.1. *Assume* (q) *and* (7.1)*; consider* $c \geq c^*$ *and let* z *be a solution of* (3.11)*. Then,* $\dot{z}(1)$ *exists and it holds that*

(i) if $z(1) \in [\beta, 0)$*, then* $\dot{z}(1) = h(1) - c$*;*

(ii) if $z(1) = 0$*, then*

$$\dot{z}(1) = \begin{cases} \frac{1}{2}\left[h(1) - c + \sqrt{(h(1)-c)^2 - 4\dot{q}(1)}\right] & \text{if } \dot{q}(1) < 0, \\ \max\{0, h(1) - c\} & \text{if } \dot{q}(1) = 0. \end{cases} \tag{7.2}$$

Proof. In case *(i)*, we only need to take the limit for $\varphi \to 1^-$ in $(3.11)_1$.

In case *(ii)*, the proof of the existence of $\dot{z}(1)$ is analogous to the proof of [16, Lemma 2.1], even if in that paper there is the further assumption $\dot{q}(1) = 0$. In the current case, $\dot{z}(1)$ must coincide with one of the roots of the equation $\gamma^2 - (h(1) - c)\,\gamma + \dot{q}(1) = 0$, which are

$$r_\pm := \frac{h(1) - c \pm \sqrt{(h(1)-c)^2 - 4\dot{q}(1)}}{2}.$$

A direct check shows that the right-hand side of (7.2) corresponds exactly to r_+. Thus, if we prove that $\dot{z}(1) = r_+$ then we conclude the proof.

If $\dot{q}(1) < 0$, the fact that $r_- < 0$ implies necessarily that $\mu = r_+$, because of $\dot{z}(1) \geq 0$.

Let $\dot{q}(1) = 0$. Since we do not yet know whether $\dot{z}$ is continuous at 1 (see Remark 9.2), we argue as follows. For any $\varphi \in (0,1)$, by the Mean Value Theorem there exists $\sigma_\varphi \in (\varphi, 1)$ satisfying $\dot{z}(\sigma_\varphi) = \frac{z(\varphi)}{\varphi - 1}$. By the definition of $\dot{z}(1)$ it then follows that

$$\lim_{\varphi \to 1^-} \dot{z}(\sigma_\varphi) = \dot{z}(1) \quad \text{and} \quad \lim_{\varphi \to 1^-} \frac{z(\sigma_\varphi)}{\sigma_\varphi - 1} = \dot{z}(1). \tag{7.3}$$

From $(3.11)_1$, the sign conditions in $(3.2)_2$ and $(3.11)_2$ imply that

$$\dot{z}(\sigma_\varphi) > h(\sigma_\varphi) - c, \qquad \varphi \in (0,1). \tag{7.4}$$

By (7.3), passing to the limit as $\varphi \to 1^-$ gives $\dot{z}(1) \geq h(1) - c$, because of the continuity of h at 1. Moreover, since $\dot{z}(1) \geq 0$ it holds that $\dot{z}(1) \geq \max\{0, h(1) - c\} = r_+$. This concludes the proof, since it necessarily follows that $\dot{z}(1) = r_+$ also in this case. □

Remark 7.2. We prove in Remark 9.2 that $z \in C^1(0,1]$ under the assumptions of Proposition 7.1. We now show that (7.1) is necessary for the existence of $\dot{z}(1)$. We define

$$q(\varphi) = \varphi^3\,(1-\varphi)\left[(\sin(\log(1-\varphi)) + 2)^2 + 2\cos(\log(1-\varphi)) + \frac{1}{2}\sin(2\log(1-\varphi))\right],$$

for $\varphi \in [0,1]$. The function q satisfies (q), while $\dot{q}(1)$ does not exist. Direct computations show that the function $z = z(\varphi)$ defined by $z(\varphi) = -(2 + \sin(\log(1-\varphi)))\,(1-\varphi)\,\varphi^2$ satisfies (3.11) with $c = 0$ and $h(\varphi) = \varphi(\varphi - 1)\,[\cos(\log(1-\varphi)) + 3\sin(\log(1-\varphi)) + 6]$. It is easy to verify that $\dot{z}(1)$ does not exist.

8 The behavior of z near 0

For $\varphi_0 \in (0,1)$ we consider the problem, see Figure 8.1 on the left,

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, & \varphi \in (0,1), \\ z(\varphi_0) = z^*(\varphi_0). \end{cases} \tag{8.1}$$

Lemma 8.1. *Assume* (q). *Fix $c > c^*$. For every $\varphi_0 \in (0,1)$ there is a unique solution $\hat{z}_{\varphi_0} \in C[0,1] \cap C^1(0,1)$ to problem* (8.1). *We have $\hat{z}_{\varphi_0}(0) = 0$, and also*

$$\hat{z}_{\varphi_0} < z^* \quad \text{in } (\varphi_0, 1] \quad \text{and} \quad \hat{z}_{\varphi_0} \geq z_\beta \quad \text{in } (0,1], \tag{8.2}$$

where z_β is the solution to (3.11) *with $z_\beta(1) = \beta$. If $0 < \varphi_1 < \varphi_0$ then $\hat{z}_{\varphi_1} < \hat{z}_{\varphi_0}$ in $(0,1]$.*

Proof. The existence and uniqueness of solutions is proved by *Step (i)* in the proof of Proposition 5.1. Inequality $(8.2)_1$ follows from the arguments contained in *Step (i)* of the proof of Proposition 5.1, while $(8.2)_2$ is obvious.

If $0 < \varphi_1 < \varphi_0$ then $\hat{z}_{\varphi_1}(\varphi_0) < \hat{z}_{\varphi_0}(\varphi_0)$, because $\hat{z}_{\varphi_1} < z^*$ in $(\varphi_1, 1]$ and $\varphi_0 \in (\varphi_1, 1]$. The monotony follows by the uniqueness of solutions to the Cauchy problem associated to $(3.11)_1$. The regularity of $\hat{z}_{\varphi_0}$ follows from both $(8.1)_1$ and Lemma 3.1; directly from $(8.2)_2$, we deduce $\hat{z}_{\varphi_0}(0) = 0$. □

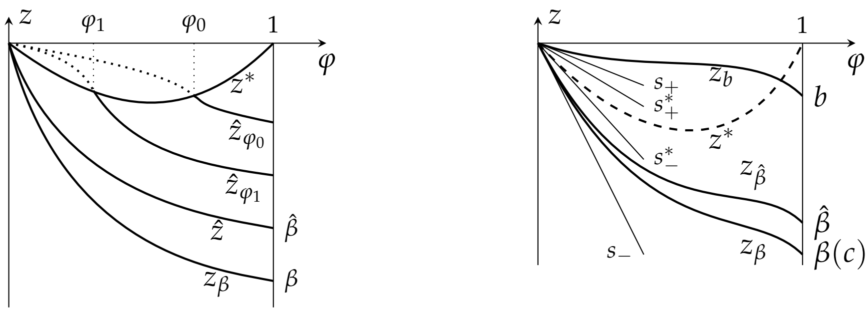

Figure 8.1: Left: the functions $\hat{z}_{\varphi_0}$, $\hat{z}$ and z_β in Lemma 8.1. Right: an illustration of Proposition 8.2 for fixed $c > c^*$. Solutions are labelled according to their right-hand limit; $s_\pm$ denote the slope of the tangent of z at 0. The dashed curve is the plot of z^*.

For every $c > c^*$, by the monotonicity of $\{\hat{z}_{\varphi_0}\}_{\varphi_0}$ and $(8.2)_2$, Lemma 3.3 implies that there exists $\hat{z} \in C^0[0,1] \cap C^1(0,1)$ which solves (3.1) such that

$$\hat{z}(\varphi) = \lim_{\varphi_0 \to 0^+} \hat{z}_{\varphi_0}(\varphi), \qquad \varphi \in [0,1]. \tag{8.3}$$

Such a $\hat{z}$ satisfies $z_\beta \leq \hat{z} \leq z^*$ in $(0,1)$ by (8.2) and then (3.11). Define $\hat{\beta} \in [\beta, 0)$ by

$$\hat{\beta} := \hat{z}(1). \tag{8.4}$$

In the following result we assume again (6.3). We shall prove in Remark 8.3 that such a condition is necessary for the existence of $\dot{z}(0)$. From (4.3) and (6.3) we deduce $(h(0) - c)^2 - 4\dot{q}(0) \geq 0$ for any $c \geq c^*$; we can then denote

$$s_\pm(c) := \frac{h(0) - c}{2} \pm \frac{\sqrt{(h(0) - c)^2 - 4\dot{q}(0)}}{2}, \quad \text{for } c \geq c^*.$$

The next proposition generalizes [6, Proposition 5.2] to the case of a more generic q, and, more deeply, to the case $z(1) < 0$. It is worth noting that this latter case reveals the behavior detected by (8.6), and shown in Figure 8.1 on the right, which was not contained in [6].

Proposition 8.2. *Assume* (q) *and* (6.3). *If $c \geq c^*$ and z is a solution of* (3.11), *then, $\dot z(0)$ exists. Moreover, it holds that*

$$\dot z(0) = \begin{cases} s_+(c) & \text{if } c > c^* \text{ and } z(1) > \hat\beta, \\ s_-(c^*) & \text{if } c = c^*, \end{cases} \tag{8.5}$$

and, if $c^* > h(0)$,

$$\dot z(0) = s_-(c) \quad \text{if } c > c^* \ \text{ and } \ z(1) \in [\beta, \hat\beta]. \tag{8.6}$$

Proof. Arguing as in the proof of [6, Proposition 5.2], we deduce that $\dot z(0)$ exists for $c \geq c^*$ and is one of the root of the equation $\gamma^2 - (h(0) - c)\,\gamma + \dot q(0) = 0$. Then $\dot z(0) \in \{s_-(c), s_+(c)\}$ for every $c \geq c^*$. Straightforward computations give

$$s_-(c) < s_-(c^*) \leq s_+(c^*) \leq s_+(c) \leq 0 \quad \text{for any } \ c > c^* \tag{8.7}$$

and $h(0) - c \leq s_-(c)$, for any $c \geq c^*$. We denote $s^*_\pm := s_\pm(c^*)$.

Take $c > c^*$. Let $\hat z_{\varphi_0}$ and $\hat z$ be defined as in the beginning of Section 8, see Figure 8.1 on the left. If $z(1) > \hat\beta$ then $z(1) > \hat z_{\varphi_1}(1)$, for some $\varphi_1 \in (0, 1)$, because of (8.3). Thus, $z > \hat z_{\varphi_1}$ in $(0, 1]$. We observed in (5.3) that $\hat z_{\varphi_1} > z^*$ in $(0, \varphi_1)$. Thus, $z > z^*$ in $(0, \varphi_1)$ and hence $\dot z(0) \geq \dot z^*(0)$. Since $s_-(c) < s^*_- \leq 0$ by (8.7), we deduce $\dot z(0) = s_+(c)$. This proves $(8.5)_1$.

Now, we prove $(8.5)_2$. If $z = z^*$, then $(8.5)_2$ was obtained in [6, Proposition 5.2] under some specific assumptions on q. Since the relevant ones were (3.2) and (6.3), we deduce that $(8.5)_2$ occurs also in the current case. If $z = y^*$ is a solution of (3.11), different from z^* (such a y^* can exist, as we proved in Remark 6.4, since (6.4) does not necessarily follow), then $y^* < z^*$ in $(0, 1]$ by Proposition 4.2. Since $\dot y^*(0) \in \{s^*_-, s^*_+\}$ and $\dot z^*(0) = s^*_-$ then we have $\dot y^*(0) = s^*_-$. Hence, $(8.5)_2$ holds.

It remains to prove (8.6) under the additional condition $h(0) - c^* < 0$. By $\beta \leq z(1) \leq \hat\beta$ we have $z \leq \hat z$ and hence $z < z^*$, which implies $\dot z(0) \leq \dot z^*(0)$. Since, under the additional condition $h(0) - c^* < 0$, we have $s^*_- < s^*_+$ and since we proved that $\dot z^*(0) = s^*_-$, we conclude that $\dot z(0) = s_-(c)$, which is (8.6). This concludes the proof. □

Remark 8.3. Now, we prove that (6.3) is necessary for the existence of $\dot z(0)$. For $\varphi \in [0, 1]$ define $q(\varphi) = \varphi(1 - \varphi)^4\,(2 + \sin(\log\varphi))\,(3 - \cos(\log\varphi) - \sin(\log\varphi))$. The function q satisfies (q), while $\dot q(0)$ does not exist, since $\liminf_{\varphi\to 0^+} q(\varphi)/\varphi < \limsup_{\varphi\to 0^+} q(\varphi)/\varphi$. Direct computations show that the function $z(\varphi) = -\,(2 + \sin(\log\varphi))\,(1 - \varphi)^2\,\varphi$ solves (3.11) with $c = 0$ and $h(\varphi) = 2\,(2 + \sin(\log\varphi))\,(1 - \varphi)\,\varphi - 5\,(1 - \varphi)^2$. Clearly, $\dot z(0)$ does not exists.

We now show that, under the assumptions of Proposition 6.3, the threshold $\hat\beta(c)$ defined in (8.4) and occurring in Proposition 8.2 coincides with the threshold $\beta(c)$ introduced in Proposition 5.1. It is an open problem whether the two thresholds differ without assuming (6.3) and (6.4).

Proposition 8.4. *Assume* (q), (6.3), (6.4) *and $c > c^*$. Then $\beta(c) = \hat\beta(c)$.*

Proof. Consider $\varepsilon > 0$ and let z_ε be the solution of

$$\begin{cases} \dot z_\varepsilon(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z_\varepsilon(\varphi)}, & \varphi > 0, \\ z_\varepsilon(0) = -\varepsilon < 0. \end{cases}$$

Lemma 3.2 *(1.b)* implies that z_ε exists and it is defined in its maximal-existence interval $[0, \delta]$, for some $0 < \delta \leq 1$. By the uniqueness of solutions of the Cauchy problem associated to $(3.1)_1$, we have necessarily $z_\varepsilon < z_\beta$ in $[0, \delta]$, where z_β was defined in the statement of Lemma 8.1. Since $z_\beta(\delta) < 0$ then $\delta = 1$.

We claim that z_ε converges for $\varepsilon \to 0^+$ to both $\hat{z}$ and z_β, where $\hat{z}$ is defined in (8.3), see Figure 8.1 on the left. From the uniqueness of the limit, it follows that $\hat{z}$ and z_β coincides and hence that $\beta = \hat{\beta}$. To prove the claim, consider

$$\eta_\varepsilon(\varphi) := \hat{z}(\varphi) - z_\varepsilon(\varphi), \qquad \varphi \in [0,1].$$

Since $\hat{z} \geq z_\beta > z_\varepsilon$ in $[0,1]$, then $\eta_\varepsilon > 0$ in $[0,1]$. Moreover, $\eta_\varepsilon(0) = \varepsilon$. We have

$$\dot{\eta}_\varepsilon(\varphi) = \frac{q(\varphi)}{z_\varepsilon(\varphi)\hat{z}(\varphi)} \eta_\varepsilon(\varphi), \qquad \varphi \in (0,1).$$

Thus,

$$\frac{\dot{\eta}_\varepsilon(\varphi)}{\eta_\varepsilon(\varphi)} = \frac{q(\varphi)}{z_\varepsilon(\varphi)\hat{z}(\varphi)}, \qquad \varphi \in (0,1)$$

and hence, for any $0 < \tau < \varphi$,

$$\log\left(\eta_\varepsilon(\varphi)\right) - \log\left(\eta_\varepsilon(\tau)\right) = \int_\tau^\varphi \frac{q(s)}{z_\varepsilon(s)\hat{z}(s)}\, ds \leq \int_\tau^1 \frac{q(s)}{z_\beta(s)\hat{z}(s)}\, ds. \tag{8.8}$$

Notice, from $(6.4)_2$ it follows that we can apply (8.6) with $\dot{q}(0) = 0$ (because of (6.3)) and obtain $z_\beta(s)\hat{z}(s) = (h(0) - c)^2 s^2 + o(s^2)$, as $s \to 0^+$. Hence, from $(6.4)_1$,

$$\sup_{\tau > 0} \int_\tau^1 \frac{q(s)}{z_\beta(s)\hat{z}(s)}\, ds =: C < +\infty.$$

From (8.8), by taking the limit as $\tau \to 0^+$ we deduce $\eta_\varepsilon(\varphi) \leq \varepsilon e^C$, $\varphi \in [0,1)$, and then

$$\lim_{\varepsilon \to 0^+} z_\varepsilon(\varphi) = \hat{z}(\varphi), \qquad \varphi \in [0,1). \tag{8.9}$$

We now apply Lemma 3.3 to deduce that z_ε converges (uniformly on $[0,1]$) to a solution $\bar{z}$ of $(3.1)_1$ in $(0,1)$ such that $\bar{z} < 0$ in $(0,1)$ and $\bar{z}(0) = 0$. Since $z_\varepsilon < z_\beta$ and z_β lies below every solution of (3.1), by the very definition of z_β, we conclude that $\bar{z}$ coincides with z_β, that is $\lim_{\varepsilon \to 0^+} z_\varepsilon(\varphi) = z_\beta(\varphi)$, $\varphi \in [0,1]$. From this formula and (8.9) we clearly have $z_\beta = \hat{z}$. □

9 Strongly non-unique strict semi-wavefronts

We now apply the previous results to study semi-wavefronts of Equation (1.1) when D and g satisfy (D1), (g0) and (1.2); in particular, we prove Theorem 2.2 and Corollary 9.4. Indeed, all the results obtained in Sections 4–8 apply when we set

$$q := Dg, \tag{9.1}$$

since such q fulfills (q). Throughout this section, by c^* we always intend the threshold given by Proposition 4.2 for q as in (9.1), for which it holds (2.2), as observed in Remark 5.5.

Lemma 9.1. *Assume* (D1), (g0) *and* (1.2)*. Consider $c \geq c^*$ and let z be the solution of* (3.12) *when* (9.1) *occurs. Then, it holds that*

$$\lim_{\varphi\to 1^-} \frac{D(\varphi)}{z(\varphi)} = \begin{cases} \frac{h(1)-c-\sqrt{(h(1)-c)^2-4\dot{D}(1)g(1)}}{2g(1)} & \text{if } \dot{D}(1) < 0, \\ \min\left\{0, \frac{h(1)-c}{g(1)}\right\} & \text{if } \dot{D}(1) = 0. \end{cases} \tag{9.2}$$

Proof. First, observe that Proposition 7.1 applies to the current case.

If either $\dot{D}(1) < 0$ or $\dot{D}(1) = 0$ and $c < h(1)$, then $\dot{z}(1) > 0$, by (7.2), because $\dot{q}(1) = \dot{D}(1)g(1)$. As a consequence, we have

$$\lim_{\varphi\to 1^-} \frac{D(\varphi)}{z(\varphi)} = \lim_{\varphi\to 1^-} \frac{\frac{D(\varphi)}{\varphi-1}}{\frac{z(\varphi)}{\varphi-1}} = \frac{\dot{D}(1)}{\dot{z}(1)},$$

which, together with (7.2), implies both $(9.2)_1$ and the first half of $(9.2)_2$.

If $\dot{D}(1) = 0$ and $c \geq h(1)$, we need a refined argument based on strict upper- and lower-solutions of $(3.11)_1$. We split the proof in two subcases.

(i) Assume first $\dot{D}(1) = 0$ and $c > h(1)$. Fix $\varepsilon > 0$ and define $\omega = \omega(\varphi)$ by

$$\omega(\varphi) := -\frac{g(1)}{c - h(1) + \varepsilon g(1)} D(\varphi), \qquad \text{for } \varphi \in (0,1). \tag{9.3}$$

First, we observe that $\omega < 0$ in $(0,1)$. Moreover, we get

$$\dot{\omega}(\varphi) = -\frac{g(1)}{c - h(1) + \varepsilon g(1)} \dot{D}(\varphi),$$

which in turn implies $\dot{\omega}(1) = 0$, since $\dot{D}(1) = 0$. Now, if we compute the right-hand side of $(3.11)_1$ applied to ω, we obtain

$$h(\varphi) - c - \frac{D(\varphi)g(\varphi)}{\omega(\varphi)} = h(\varphi) - c + \frac{g(\varphi)\left[c - h(1) + \varepsilon g(1)\right]}{g(1)}, \qquad \text{for } \varphi \in (0,1),$$

which tends to $\varepsilon g(1) > 0$ as $\varphi \to 1^-$. Hence, there exists $\sigma \in (0,1)$ such that

$$\dot{\omega}(\varphi) < h(\varphi) - c - \frac{D(\varphi)g(\varphi)}{\omega(\varphi)}, \qquad \varphi \in [\sigma, 1), \tag{9.4}$$

that is, ω is a (strict) lower-solution of $(3.11)_1$ in $[\sigma, 1)$.

Since $\dot{z}(1) = 0$, we can take a sequence $\{\varphi_n\}_n \subset (\sigma, 1)$, with $\varphi_n \to 1$ as $n \to \infty$, such that $\dot{z}(\varphi_n) \to 0$ as follows. Let $\{\sigma_n\}_n \subset (\sigma, 1)$ be such that $\sigma_n \to 1$. For any $n \in \mathbb{N}$, the Mean Value Theorem implies that there exists $\varphi_n \in (\sigma_n, 1)$ for which it holds $\dot{z}(\varphi_n) = \frac{z(\sigma_n)}{\sigma_n - 1}$. Since the sequence in the right-hand side of this last identity tends to $\dot{z}(1) = 0$, as $n \to \infty$, we obtained the desired $\{\varphi_n\}_n$. With this in mind, from $(3.11)_1$, we obtain

$$\lim_{n\to\infty} \frac{D(\varphi_n)g(\varphi_n)}{z(\varphi_n)} = h(1) - c, \tag{9.5}$$

and then

$$\lim_{n\to\infty} \frac{\omega(\varphi_n)}{z(\varphi_n)} = \frac{c - h(1)}{c - h(1) + \varepsilon g(1)} = 1 - \frac{\varepsilon g(1)}{c - h(1) + \varepsilon g(1)} < 1.$$

Hence, there exists $\overline{n}$ such that $\omega(\varphi_n) > z(\varphi_n)$ for $n \geq \overline{n}$. Without loss of generality we assume that $\overline{n} = 1$. We claim that

$$\omega(\varphi) > z(\varphi), \quad \text{for } \varphi \in (\varphi_1, 1). \tag{9.6}$$

We reason by contradiction, see Figure 9.1. Suppose that there exists $\tilde{\varphi} \in (\varphi_1, 1)$ such that $\omega(\tilde{\varphi}) \leq z(\tilde{\varphi})$. There exists $n \in \mathbb{N}$ for which $\tilde{\varphi} \in (\varphi_n, \varphi_{n+1})$. Since $\omega(\varphi_n) > z(\varphi_n)$ and $\omega(\varphi_{n+1}) > z(\varphi_{n+1})$, the existence of such a $\tilde{\varphi}$ implies that the function $(\omega - z)$ in $(\varphi_n, \varphi_{n+1})$ admits a non-positive minimum at $\tilde{\varphi}_2 \in (\varphi_n, \varphi_{n+1})$, that is $\dot{\omega}(\tilde{\varphi}_2) = \dot{z}(\tilde{\varphi}_2)$ and $\omega(\tilde{\varphi}_2) \leq z(\tilde{\varphi}_2)$. Thus, from $(3.11)_1$ and (9.4) we have that

$$h(\tilde{\varphi}_2) - c - \frac{(Dg)(\tilde{\varphi}_2)}{z(\tilde{\varphi}_2)} = \dot{z}(\tilde{\varphi}_2) = \dot{\omega}(\tilde{\varphi}_2) < h(\tilde{\varphi}_2) - c - \frac{(Dg)(\tilde{\varphi}_2)}{\omega(\tilde{\varphi}_2)},$$

which in turn implies $1/z(\tilde{\varphi}_2) > 1/\omega(\tilde{\varphi}_2)$ because of $(Dg)(\tilde{\varphi}_2) > 0$. Hence, $z(\tilde{\varphi}_2) < \omega(\tilde{\varphi}_2)$ which contradicts the existence of $\tilde{\varphi}_2$. Then (9.6) is proved. At last, we have

$$\frac{D(\varphi)}{z(\varphi)} > \frac{D(\varphi)}{\omega(\varphi)} = -\frac{c - h(1)}{g(1)} - \varepsilon, \qquad \varphi \in (\varphi_1, 1). \tag{9.7}$$

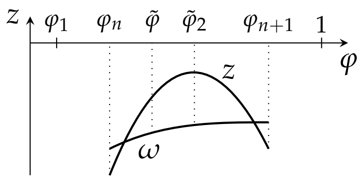

Figure 9.1: A detail of the plots of functions ω and z in case *(i)*.

Analogously, for $\varepsilon > 0$ small enough to satisfy $c > h(1) + \varepsilon g(1)$, we define $\eta = \eta(\varphi)$ by

$$\eta(\varphi) := -\frac{g(1)}{c - h(1) - \varepsilon g(1)} D(\varphi), \qquad \varphi \in (0, 1).$$

By arguing as above when we considered ω in (9.3), we deduce that η is a (strict) upper-solution of $(3.11)_1$ in $[\sigma_2, 1)$ for some $\sigma_2 \in (0, 1)$. Proceeding as we did to obtain (9.7), we now get $\eta(\varphi) < z(\varphi)$ for $\varphi \in (\varphi_1, 1)$, for some $\varphi_1 > \sigma_2$. Thus,

$$\frac{D(\varphi)}{z(\varphi)} < \frac{D(\varphi)}{\eta(\varphi)} = -\frac{c - h(1)}{g(1)} + \varepsilon, \qquad \varphi \in (\varphi_1, 1). \tag{9.8}$$

Finally, putting together (9.7) and (9.8), since $\varepsilon > 0$ is arbitrary, we deduce

$$\lim_{\varphi \to 1^-} \frac{D(\varphi)}{z(\varphi)} = \frac{h(1) - c}{g(1)}. \tag{9.9}$$

Thus, we proved $(9.2)_2$ with $c > h(1)$.

(ii) Now, we consider the case $\dot{D}(1) = 0$ and $c = h(1)$. Fix $\varepsilon > 0$. Set

$$\omega(\varphi) := -\frac{D(\varphi)}{\varepsilon}, \qquad \varphi \in (0, 1), \tag{9.10}$$

which coincides with (9.3) in the current case. By proceeding exactly as in the case *(ii)*, we obtain (9.3) for ω defined as in (9.10), namely $0 > \omega(\varphi) > z(\varphi)$, for $\varphi \in (\varphi_1, 1)$, for some $\varphi_1 \in (0,1)$. This implies, as in (9.7),

$$0 > \frac{D(\varphi)}{z(\varphi)} > \frac{D(\varphi)}{\omega(\varphi)} = -\varepsilon, \qquad \varphi \in (\varphi_1, 1). \tag{9.11}$$

Then (9.11) implies $D(\varphi)/z(\varphi) \to 0^-$ as $\varphi \to 1^-$, which is $(9.2)_2$ in the case $c = h(1)$. □

Remark 9.2. Let $c \geq c^*$ and z be any solution of (3.11). We infer that $z \in C^1(0,1]$. In fact, if $z(1) = b < 0$, in the proof of case *(i)* of Proposition 7.1 we already checked that this is true, since $\lim_{\varphi \to 1^-} \dot z(\varphi) = \dot z(1)$. If $z(1) = 0$, from (9.2) it follows that the right-hand side of $(3.11)_1$ still has a finite limit, as $\varphi \to 1^-$. As observed, this means that $z \in C^1(0,1]$.

We now prove Theorem 2.2.

Proof of Theorem 2.2. We first prove that there exists a semi-wavefront to 0 of (1.1) *if* $c \geq c^*$. For $q = Dg$, consider one of the solutions $z = z(\varphi)$ of (3.11), provided by Propositions 4.2 and 5.1. Consider the Cauchy problem

$$\begin{cases} \varphi' = \frac{z(\varphi)}{D(\varphi)}, \\ \varphi(0) = \frac{1}{2}. \end{cases} \tag{9.12}$$

The right-hand side of $(9.12)_1$ is of class C^1 in a neighborhood of $\frac{1}{2}$, and then there exists a unique solution φ in its maximal-existence interval (a, ξ_0), for $-\infty \leq a < \xi_0 \leq \infty$. Since $z(\varphi)/D(\varphi) < 0$ for $\varphi \in (0,1)$, we deduce that φ is decreasing and then $\lim_{\xi \to a^+} \varphi(\xi) = 1$, $\lim_{\xi \to \xi_0^-} \varphi(\xi) = 0$. By $(9.12)_1$, the profile φ satisfies (1.3) in (a, ξ_0). We show that, if $\xi_0 \in \mathbb{R}$, we can extend φ and obtain a solution of (1.3), in the sense of Definition 2.1, defined in the half-line $(a, +\infty)$.

Assume $\xi_0 \in \mathbb{R}$ and set $\varphi(\xi) = 0$, for any $\xi \geq \xi_0$. The new function (which without any ambiguity we still call φ) is clearly of class $C^0(a, +\infty) \cap C^2\left((a, +\infty) \setminus \{\xi_0\}\right)$ and is a classical solution of (1.3) in $(a, +\infty) \setminus \{\xi_0\}$. Moreover, observe that, as a consequence of both the fact that z satisfies $(3.11)_3$, and $(9.12)_1$, we have

$$\lim_{\xi \to \xi_0^-} D\left(\varphi(\xi)\right) \varphi'(\xi) = 0. \tag{9.13}$$

This implies that $D(\varphi)\varphi' \in L^1_{\mathrm{loc}}(a, +\infty)$.

To show that φ is a solution of (1.3) according to Definition 2.1, it remains to prove (2.1). For this purpose, consider $\psi \in C_0^\infty(a, +\infty)$, and let $a < \xi_1 < \xi_2 < \infty$ be such that $\psi(\xi) = 0$, for any $\xi \geq \xi_2$ or $\xi \leq \xi_1$. Our goal is then to prove the following:

$$\int_{\xi_1}^{\xi_2} \left(D\left(\varphi\right)\varphi' - f\left(\varphi\right) + c\varphi\right)\psi' - g(\varphi)\psi \, d\xi = 0. \tag{9.14}$$

Identity (9.14) is obvious if $\xi_2 < \xi_0$, since φ solves (1.3) in (a, ξ_0). Assume $\xi_2 \geq \xi_0$. In the interval (ξ_0, ξ_2) we have $\varphi = 0$, and since $g(0) = f(0) = 0$ we deduce

$$\int_{\xi_0}^{\xi_2} \left(D\left(\varphi\right)\varphi' - f\left(\varphi\right) + c\varphi\right)\psi' - g(\varphi)\psi \, d\xi = 0. \tag{9.15}$$

In the interval (ξ_1, ξ_0) we have, by (9.13),

$$\begin{aligned}
&\int_{\xi_1}^{\xi_0} \left(D\left(\varphi\right)\varphi' - f\left(\varphi\right) + c\varphi\right)\psi' - g(\varphi)\psi \, d\xi \\
&\quad = \lim_{\varepsilon\to 0^+} \int_{\xi_1}^{\xi_0-\varepsilon} \left(D\left(\varphi\right)\varphi' - f\left(\varphi\right) + c\varphi\right)\psi' - g(\varphi)\psi \, d\xi \\
&\quad = \lim_{\varepsilon\to 0^+} \left(\left(D\left(\varphi\right)\varphi' - f(\varphi) + c\varphi\right)\psi\right)\left(\xi_0 - \varepsilon\right) = 0.
\end{aligned} \tag{9.16}$$

Thus, identities (9.15) and (9.16) imply (9.14).

At last, we claim that $a \in \mathbb{R}$, i.e., that φ is *strict*. For this, it is sufficient to prove

$$\lim_{\xi\to a^+} \varphi'(\xi) < 0. \tag{9.17}$$

We stress that the case $\lim_{\xi\to a^+} \varphi'(\xi) \to -\infty$, for short $\varphi'(a^+) = -\infty$, is included in (9.17). To prove (9.17), we notice that, from (9.12),

$$\lim_{\xi\to a^+} \varphi'(\xi) = \lim_{\varphi\to 1^-} \frac{z(\varphi)}{D(\varphi)}.$$

Thus, (9.17) easily follows from either a direct check, in the case $z(1) < 0$, or the application of Lemma 9.1, in the case $z(1) = 0$. This concludes the first part of the proof.

Conversely, we prove that if there exists a semi-wavefront φ to 0 defined in $(a, +\infty)$, then $c \geq c^*$. Let $\bar{b}$ be defined by

$$\bar{b} := \sup\left\{\xi > a : \varphi(\xi) > 0\right\} \in (a, +\infty]. \tag{9.18}$$

We have $0 < \varphi < 1$ in $\left(a, \bar{b}\right)$ and so φ is a classical solution of (1.3) in $\left(a, \bar{b}\right)$. We claim that

$$\lim_{\xi\to \bar{b}^-} D\left(\varphi(\xi)\right)\varphi'(\xi) = 0. \tag{9.19}$$

Suppose $\bar{b} \in \mathbb{R}$. Take $\xi_1 > a$ and $\xi_2 > \bar{b}$. By choosing, in Definition 2.1, $\psi \in C_0^\infty(a, +\infty)$ with support in (ξ_1, ξ_2) such that $\psi(\bar{b}) \neq 0$, (2.1) reads as (passing to the limit in the integral as in (9.16))

$$\begin{aligned}
0 &= \int_{\xi_1}^{\xi_2} \left(D\left(\varphi\right)\varphi' + c\varphi - f\left(\varphi\right)\right)\psi' - g\left(\varphi\right)\psi \, d\xi \\
&= \int_{\xi_1}^{\bar{b}} \left(D\left(\varphi\right)\varphi' + c\varphi - f\left(\varphi\right)\right)\psi' - g\left(\varphi\right)\psi \, d\xi = \left(D(\varphi)\varphi'\right)\left(\bar{b}^-\right)\psi(\bar{b}).
\end{aligned}$$

Then we got (9.19) in this case. If $\bar{b} = +\infty$, by integrating (1.3) in $[\eta, \xi] \subset (\bar{a}, +\infty)$, we have

$$\begin{aligned}
&D\left(\varphi(\xi)\right)\varphi'(\xi) \\
&\quad = D\left(\varphi(\eta)\right)\varphi'(\eta) - c\left(\varphi(\xi) - \varphi(\eta)\right) + \left(f(\varphi(\xi)) - f(\varphi(\eta))\right) - \int_\eta^\xi g\left(\varphi(\sigma)\right) d\sigma.
\end{aligned} \tag{9.20}$$

Since the function

$$\xi \mapsto \int_\eta^\xi g(\varphi(\sigma))\, d\sigma$$

is increasing (because $g > 0$ in $(0,1)$), then $\lim_{\xi\to\infty} D(\varphi(\xi))\varphi'(\xi) = \ell$ for some $\ell \in [-\infty, 0]$. If $\ell < 0$, then, $\varphi'(\xi)$ tends either to some negative value or to $-\infty$ as $\xi \to +\infty$. In both cases, this contradicts the boundedness of φ, and so (9.19) is proved.

We show now (2.3). Suppose by contradiction that (2.3) does not occur, there exists $\xi_0 \in (a, \bar{b})$, with $0 < \varphi(\xi_0) < 1$, such that $\varphi'(\xi_0) = 0$. Then (1.3) implies $\varphi''(\xi_0) = -g(\varphi(\xi_0))/D(\varphi(\xi_0)) < 0$ and hence ξ_0 is a local maximum point of φ. It is plain to see that, in turn, this implies that there exists $a < \xi_1 < \xi_0$ which is a local minimum point of φ. From what we said about ξ_0, we necessarily have $\varphi(\xi_1) = \varphi'(\xi_1) = 0$.

Take $\xi \in (\xi_1, \bar{b})$. Integrating (1.3) in $[\xi_1, \xi]$ gives (9.20) with ξ_1 replacing η. By passing to the limit for $\xi \to \bar{b}^-$, from (9.19) we obtain the contradiction $0 < 0$. This proves (2.3).

From (2.3), we can define the function $z = z(\varphi)$, for $\varphi \in (0,1)$, by

$$z(\varphi) := D(\varphi)\varphi'(\xi(\varphi)), \tag{9.21}$$

where $\xi = \xi(\varphi)$ is the inverse function of φ. Again by (2.3), it follows also that $z < 0$ in $(0,1)$. From (9.19), we clearly have $z(0^+) = 0$; furthermore, a direct computation shows that z solves equation $(1.6)_1$. Thus, z solves problem (1.6), which is (3.11) with $q = Dg$. At last, Proposition 6.5 implies $c \geq c^*$. □

Remark 9.3. The proof of Theorem 2.2 provides a formula for $\varphi'(a^+)$. If $z(1) < 0$, then $\varphi'(a^+) = -\infty$. If $z(1) = 0$, Lemma 9.1 leads to

$$\lim_{\xi\to a^+} \varphi'(\xi) = \begin{cases} \frac{2g(1)}{h(1)-c-\sqrt{(h(1)-c)^2-4\dot{D}(1)g(1)}} & \text{if } \dot{D}(1) < 0, \\ \frac{g(1)}{h(1)-c} & \text{if } \dot{D}(1) = 0 \text{ and } c > h(1), \\ -\infty & \text{if } \dot{D}(1) = 0 \text{ and } c \leq h(1). \end{cases} \tag{9.22}$$

We now investigate the qualitative properties of the profiles when they reach the equilibrium 0. The classification is complete, apart from some cases corresponding to $c^* = h(0)$, when further assumptions are needed, see Remark 10.1. Below the existence of the $\lim_{\xi\to a^+} D(\varphi(\xi))\varphi'(\xi)$ is a consequence of the definition (9.21) and Lemma 3.1.

Corollary 9.4. *Under the assumptions of Theorem 2.2, let $c \geq c^*$ and φ be a strict semi-wavefront to 0 of (1.1), connecting 1 to 0, defined in its maximal-existence interval $(a, +\infty)$. Then, for $c > c^*$, there exists $\hat{\beta}(c) \in [\beta(c), 0]$ such that the following results hold.*

(i) $D(0) > 0$ implies that φ is classical and strictly decreasing.

(ii) $D(0) = 0$, $c > c^*$ and

$$\lim_{\xi\to a^+} D(\varphi(\xi))\varphi'(\xi) > \hat{\beta}(c), \tag{9.23}$$

imply that φ is classical; moreover, φ reaches 0 at some $\xi_0 > a$ if

$$c > h(0) + \limsup_{\varphi\to 0^+} \frac{g(\varphi)}{\varphi}. \tag{9.24}$$

(iii) $D(0) = 0$, $c^* > h(0)$ and

$$\textit{either } c = c^* \textit{ or } \lim_{\xi\to a^+} D(\varphi(\xi))\varphi'(\xi) \leq \hat{\beta}(c) \tag{9.25}$$

imply that φ is sharp at 0 *(reached at some $\xi_0 > a$) with*

$$\lim_{\xi \to \xi_0^-} \varphi'(\xi) = \begin{cases} \frac{h(0)-c}{\dot{D}(0)} < 0 & \text{if } \dot{D}(0) > 0, \\ -\infty & \text{if } \dot{D}(0) = 0. \end{cases} \tag{9.26}$$

Notice that β is related to the existence of the semi-wavefronts while $\hat{\beta}$ deals with their smoothness (see Figure 9.2). The two thresholds coincide under the assumptions of Proposition 8.4.

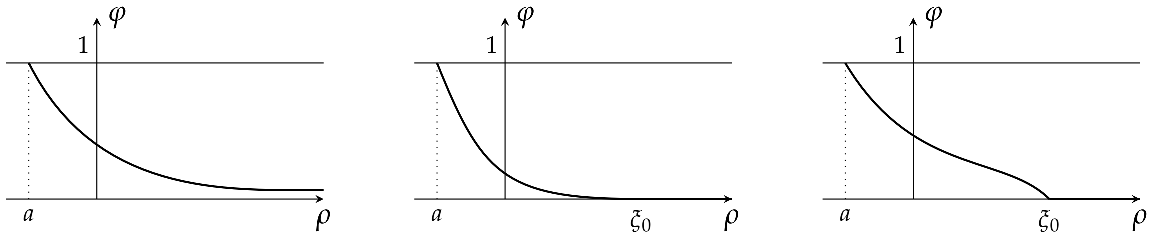

Figure 9.2: Examples of profiles occurring in Corollary 9.4. From the left to the right, they depict, respectively, what stated in Parts *(i)*, *(ii)* and *(iii)*.

Proof of Corollary 9.4. Define $\xi_0 := \sup\{\xi > a : \varphi(\xi) > 0\} \in (a, +\infty]$. We assume without loss of generality that $a < 0 < \xi_0$ and $\varphi(0) = 1/2$. Let z be the function defined in (9.21). Notice, $1 = D(\varphi)\varphi'/z(\varphi)$ if $\varphi \in (0,1)$. Thus, for any $\xi > 0$, it follows that

$$\xi = \int_0^\xi \frac{D(\varphi(s))}{z(\varphi(s))} \varphi'(s) ds = \int_{1/2}^{\varphi(\xi)} \frac{D(\sigma)}{z(\sigma)} d\sigma = \int_{\varphi(\xi)}^{1/2} \frac{D(\sigma)}{-z(\sigma)} d\sigma.$$

Therefore, $\xi_0 \in \mathbb{R}$ if and only if it holds that

$$\int_0^{1/2} \frac{D(\sigma)}{-z(\sigma)} d\sigma := \lim_{\varphi \to 0^+} \int_\varphi^{1/2} \frac{D(\sigma)}{-z(\sigma)} d\sigma < +\infty. \tag{9.27}$$

For $c > c^*$, let $\hat{\beta}(c)$ be given by (8.4).

We prove *(i)*. By Proposition 8.2 we know that $\dot{z}(0)$ exists and it is finite; since $D(0) > 0$ we deduce that (9.27) does not hold. Then, $\xi_0 = +\infty$ and so φ is strictly decreasing. This, and the fact that φ is of class C^2 when $\varphi \in (0,1)$, imply $\varphi \in C^2(a, +\infty)$, hence φ is classical. Part *(i)* is hence showed.

Assume $D(0) = 0$. In this case, Formula (6.3) holds with $\dot{q}(0) = 0$ and $\dot{z}(0)$ exists by Proposition 8.2.

We show *(ii)*. Since (9.23) holds then (8.5) reads as $\dot{z}(0) = 0$. We treat separately the cases $\dot{D}(0) > 0$ or $\dot{D}(0) = 0$. Suppose that $\dot{D}(0) > 0$. Therefore,

$$\lim_{\xi \to \xi_0^-} \varphi'(\xi) = \frac{\dot{z}(0)}{\dot{D}(0)} = 0 \tag{9.28}$$

and hence φ (not necessarily strictly monotone) is classical. Suppose then $D(0) = \dot{D}(0) = 0$. Fix $\varepsilon > 0$ and define $\eta(\varphi) := -\varepsilon D(\varphi)$, $\varphi \in (0,1)$. We have

$$\dot{\eta}(\varphi) - h(\varphi) + c + \frac{D(\varphi)g(\varphi)}{\eta(\varphi)} \to -h(0) + c > 0, \quad \text{as} \quad \varphi \to 0^+.$$

Therefore η is a strict upper-solution of $(1.6)_1$ in $(0,\delta]$, for some $\delta > 0$. Also, since $\dot{z}(0) = 0$, there exists a sequence $\{\varphi_n\}_n$, with $\delta \geq \varphi_n \to 0^+$, such that $\dot{z}(\varphi_n) \to 0$. From $(1.6)_1$, this implies that

$$\lim_{n\to\infty} \frac{\varepsilon D(\varphi_n)}{-z(\varphi_n)} = \varepsilon \lim_{n\to\infty} \frac{\dot{z}(\varphi_n) + c - h(\varphi_n)}{g(\varphi_n)} = \infty.$$

Hence, $-\eta(\delta_1) = \varepsilon D(\delta_1) > -z(\delta_1)$, for some $0 < \delta_1 \leq \delta$ small enough. An application of Lemma 3.2 *(2.a.i)* then gives

$$z(\varphi) > -\varepsilon D(\varphi), \qquad \varphi \in (0, \delta_1]. \tag{9.29}$$

This clearly implies that

$$0 > \frac{z(\varphi)}{D(\varphi)} > -\varepsilon, \qquad \varphi \in (0, \delta_1].$$

Since $\varepsilon > 0$ is arbitrary, then we have $\varphi'(\xi) \to 0$ for $\xi \to \xi_0^-$ and hence φ is classical, that is we showed the first part of *(ii)*. Define $\eta(\varphi) := -\varphi D(\varphi)$. We have, for any $\varphi \in (0,1)$,

$$\dot{\eta}(\varphi) - h(\varphi) + c + \frac{D(\varphi)g(\varphi)}{\eta(\varphi)} = -\dot{D}(\varphi)\varphi - D(\varphi) - h(\varphi) + c - \frac{g(\varphi)}{\varphi}.$$

Thus, by means of (9.24), we get

$$\liminf_{\varphi\to 0^+} \left[\dot{\eta}(\varphi) - h(\varphi) + c + \frac{D(\varphi)g(\varphi)}{\eta(\varphi)}\right] = c - h(0) - \limsup_{\varphi\to 0^+} \frac{g(\varphi)}{\varphi} > 0.$$

Therefore, η is a strict upper-solution of $(1.6)_1$ in $(0,\delta]$, for some $\delta > 0$. Furthermore, taking the same sequence $\varphi_n \to 0^+$ as above such that $\dot{z}(\varphi_n) \to 0$, as $n \to \infty$, then we have

$$\liminf_{n\to\infty} \frac{D(\varphi_n)\varphi_n}{-z(\varphi_n)} = \liminf_{n\to\infty} \frac{\dot{z}(\varphi_n) + c - h(\varphi_n)}{g(\varphi_n)/\varphi_n} = \frac{c - h(0)}{\limsup_{n\to\infty} g(\varphi_n)/\varphi_n} > 1,$$

since (9.24) holds. Thus, as in (9.29), we deduce that $D(\varphi)\varphi > -z(\varphi)$ in $(0,\delta]$, after choosing $0 < \delta \leq 1/2$ small enough. Hence,

$$\int_0^{1/2} \frac{D(\sigma)}{-z(\sigma)}\, d\sigma > \int_0^{\delta} \frac{d\sigma}{\sigma} = +\infty,$$

which concludes the proof of *(ii)*, by means of (9.27).

We show *(iii)*. By (8.5), (8.6), $c^* > h(0)$ and (9.25) we obtain $\dot{z}(0) = h(0) - c < 0$. Then,

$$\frac{D(\sigma)}{-z(\sigma)} = \frac{\dot{D}(0) + o(1)}{c - h(0) + o(1)} \quad \text{as } \sigma \to 0^+,$$

and consequently (9.27) is verified. Thus, $\xi_0 \in \mathbb{R}$. Furthermore, from (9.21),

$$\lim_{\xi\to\xi_0^-} \varphi'(\xi) = \lim_{\varphi\to 0^+} \frac{z(\varphi)/\varphi}{D(\varphi)/\varphi} = \frac{h(0) - c}{\dot{D}(0)} \in [-\infty, 0),$$

which implies that φ is sharp at 0 and that (9.26) holds. □

10 New regularity classification of wavefronts

In this section we prove Theorem 2.3. Analogously to Section 9, but now thanks to assumptions (D0)–(g01), we apply results of Sections 4–8 to the case $q = Dg$.

Proof of Theorem 2.3. We first show that wavefronts are allowed if and only if $c \geq c^*$ for c^* satisfying (2.8); the proof is mostly contained in the proof of Theorem 2.2. Then, we prove *(i)* and *(ii)*, by exploiting some of the arguments in the proof of Corollary 9.4.

Set $q = Dg$. Clearly, q satisfies (q), with in particular $\dot{q}(0) = 0$. By Proposition 4.2, Problem (3.12) admits a unique solution z if and only if $c \geq c^*$ where for c^* it holds (4.3). As observed in Remark 5.5, since (D0) and (g01) hold true, in this case c^* satisfies (2.8).

To the solution z there is associated the solution $\varphi = \varphi(\xi)$ of the problem

$$\begin{cases} \varphi' = \frac{z(\varphi)}{D(\varphi)}, \\ \varphi(0) = \frac{1}{2}. \end{cases} \tag{10.1}$$

Such a φ exists and satisfies $(10.1)_1$ in some maximal interval (ξ_1, ξ_0), so that

$$\lim_{\xi \to \xi_1^+} \varphi(\xi) = 1 \quad \text{and} \quad \lim_{\xi \to \xi_0^-} \varphi(\xi) = 0.$$

Also, φ satisfies (1.3) in (ξ_1, ξ_0). As discussed in the proof of Theorem 2.2, if $\xi_0 \in \mathbb{R}$, then φ can be extended continuously to a solution of (1.3) in $(\xi_0, +\infty)$, by setting $\varphi(\xi) = 0$, for $\xi \geq \xi_0$. Since $g(1) = 0$, it also holds that if $\xi_1 \in \mathbb{R}$ then we can extend φ to a solution of (1.3) in $(-\infty, \xi_1)$, by setting $\varphi(\xi) = 1$ for $\xi \leq \xi_1$. Thus, we can always consider φ satisfying weakly (1.3) in $\mathbb{R}$; moreover φ solves $(10.1)_1$ in (ξ_1, ξ_0) with

$$\xi_1 = \inf\{\xi \in \mathbb{R} : \varphi(\xi) < 1\} \in [-\infty, 0), \qquad \xi_0 = \sup\{\xi \in \mathbb{R} : \varphi(\xi) > 0\} \in (0, +\infty],$$

and it is constant in $\mathbb{R} \setminus (\xi_1, \xi_0)$. Thus, we showed that if $c \geq c^*$ then there exists a wavefront φ whose profile satisfies (1.4).

By reasoning as in the proof of Theorem 2.2, also the converse implication holds. Indeed, if φ is a profile of a wavefront satisfying (1.4), then the function z defined by $z(\varphi) := D(\varphi)\varphi'\left(\varphi^{-1}(\varphi)\right)$, $0 < \varphi < 1$, is a solution of (3.12). Thus, $c \geq c^*$.

We prove *(i)*. Assume $c > c^*$. From (8.5) in Proposition 8.2, we have $\dot{z}(0) = 0$. Hence, if $\dot{D}(0) \neq 0$ then it holds

$$\lim_{\xi \to \xi_0^-} \varphi'(\xi) = \lim_{\varphi \to 0^+} \frac{z(\varphi)}{D(\varphi)} = 0. \tag{10.2}$$

If $\dot{D}(0) = 0$, then we argue as in the proof of Corollary 9.4, see (9.29), to show that, for any $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that $z(\varphi) > -\varepsilon D(\varphi)$, $\varphi \in (0, \delta]$. Hence,

$$\lim_{\xi \to \xi_0^-} \varphi'(\xi) = \lim_{\varphi \to 0^+} \frac{z(\varphi)}{D(\varphi)} \geq -\varepsilon.$$

Since $\varphi' < 0$ in (ξ_1, ξ_0) and ε is arbitrarily small, it follows again (10.2).

We prove now *(ii)*. By $(8.5)_2$, from $c = c^* > h(0)$ we have $\dot{z}(0) = h(0) - c^* < 0$. Then,

$$\frac{D(\sigma)}{-z(\sigma)} = \frac{\dot{D}(0) + o(1)}{c - h(0) + o(1)} \quad \text{as } \sigma \to 0^+,$$

and consequently (9.27) is verified. Thus, $\xi_0 \in \mathbb{R}$. Furthermore, from (9.21),

$$\lim_{\xi \to \xi_0^-} \varphi'(\xi) = \lim_{\varphi \to 0^+} \frac{z(\varphi)/\varphi}{D(\varphi)/\varphi} = \frac{h(0) - c^*}{\dot{D}(0)} \in [-\infty, 0),$$

and thus the conclusions hold. □

Remark 10.1 (Case $c = c^* = h(0)$)**.** Part *(i)* and *(ii)* of Theorem 2.3 do not cover the case $c = c^* = h(0)$. The following discussion shows that, to classify the behavior in that case, further assumptions are needed. More precisely, either a classical and a sharp wavefront can indeed occur under (D0) and (g01). Take q and h as in (6.10) in Remark 6.4. There, we proved that in this case it holds $c^* = h(0) = 0$. Consider

$$\begin{cases} D_1(\varphi) = \varphi^2, \\ g_1(\varphi) = \varphi(1-\varphi), \end{cases} \qquad \begin{cases} D_2(\varphi) = \varphi, \\ g_2(\varphi) = \varphi^2(1-\varphi). \end{cases}$$

Clearly, D_1 and g_1 satisfy (D0) and (g01) and so D_2 and g_2. Also, since $D_1 g_1 = q = D_2 g_2$, then $c_1^* = c_2^* = h(0) = 0$, where c_1^* and c_2^* are the thresholds given by Proposition 4.2 associated with $D_1 g_1$ and $D_2 g_2$, respectively. Define, for $\xi \in \mathbb{R}$,

$$\varphi_1(\xi) := \begin{cases} 1 - \frac{e^\xi}{2}, & \xi < \log(2), \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \varphi_2(\xi) := \frac{1}{1+e^\xi}.$$

Direct computations show that φ_1 and φ_2 are two wave profiles defining two wavefronts, both of them associated with $c = h(0)$. Plainly, φ_1 is sharp at $\xi = \log(2)$ while φ_2 is classical.

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Existence and Ulam type stability for nonlinear Riemann–Liouville fractional differential equations with constant delay

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. A nonlinear Riemann–Liouville fractional differential equation with constant delay is studied. Initially, some existence results are proved. Three Ulam type stability concepts are defined and studied. Several sufficient conditions are obtained. Some of the obtained results are illustrated on fractional biological models.

Keywords: Riemann–Liouville fractional derivative, constant delay, initial value problem, existence, Ulam type stability.

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1 Introduction

Ulam type stability concept is quite significant in realistic problems in many applications in numerical analysis, optimization, biology and economics etc. This type of stability guarantees that there is a close exact solution. Recently, several authors extended and discussed Ulam type stability to fractional differential equations. The Ulam type stability is well studied recently for Caputo fractional differential equations. For example, about Caputo fractional differential equations we refer [8, 12], for Caputo fractional differential equations with impulses [14], about Caputo fractional differential equations with delays see, for example, [2, 13], for ψ-Hilfer fractional derivative and a constant delay [11]. Note that in the case of

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the Riemann–Louisville (RL) fractional derivative only the case without any delays is studied (see, for example, [3,7,13,16]).

In addition, many real world processes and phenomena are characterized by the influence of past values of the state variable on the recent one and this leads to the inclusion of delays in the models. The analysis of RL delay fractional differential equations is rather complex (analytical solution computation, controllability analysis, etc.) and a very small class of equations could be solved in explicit form. It requires theoretical proofs of methods guarantee existence of enough close function to the unknown solution. One of these types of method is Ulam type stability. According to our knowledge this type of stability is not studied for nonlinear RL fractional differential equations with delays.

The main goal of the paper is to study scalar nonlinear RL fractional differential equations with a constant delay, to obtain some sufficient conditions for uniqueness and existence and to study Ulam type stability. The present paper is organized as follows. In Section 2, some notations and results about fractional calculus are given. In Section 3, an existence result, based on the Banach contraction principle, for the studied problem is presented. In Section 4, we prove three types of Ulam–Hyers stability results for the given RL fractional differential equation with a constant delay. Finally, in the last section, we illustrate the application of some of th obtained results on two fractional biological models: fractional generalization of Lasota–Ważewska model and fractional generalization of the logistic equation with a biological delay depending on the mechanistic details of the model.

2 Preliminary notes on fractional derivatives and equations

In this section, we introduce notations, definitions, and preliminary facts which are used throughout the paper. Let $T: \ 0 < T < \infty$, $\bar{J} = [0, T]$, $J = (0, T]$, $\tau > 0$ be given (the delay).

There exists a natural number N such that $N\tau < T \leq (N+1)\tau$ holds, i.e. $J = \cup_{k=0}^{N-1}(k\tau, (k+1)\tau] \cup (N\tau, T]$. To be easier for the notation without lose of generalization we could assume that $T = (N+1)\tau$ and then $J = \cup_{k=0}^{N}(k\tau, (k+1)\tau]$.

By $C(J, \mathbb{R})$ we denote the set of all continuous function with the norm $\|x\| = \sup\{|x(t)| : t \in J\}$.

By C_0 we denote the set of all functions $x \in C([-\tau, 0], \mathbb{R})$ with the norm $\|x\|_0 = \sup\{|x(t)| : t \in [-\tau, 0]\}$.

We consider the weighted space of functions $C_\gamma(J) = \{y \in J \to \mathbb{R} : \ t^\gamma y(t) \in C(J, \mathbb{R})\}$ with the norm $\|y\|_{C_\gamma} = \sup_{t \in J} |t^\gamma y(t)|$.

Note $C(J, \mathbb{R})$, $C_\gamma(J)$ are Banach spaces.

In this paper we will use the following definitions for fractional derivatives and integrals:

- *Riemann–Liouville fractional integral* of order $q \in (0, 1)$ [9]

$$ {}_0I_t^q m(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{m(s)}{(t-s)^{1-q}} ds, \qquad t \in \bar{J}, $$

  where $\Gamma(\cdot)$ is the Gamma function.

- *Riemann–Liouville fractional derivative* of order $q \in (0, 1)$ [9]

$$ {}_0^{RL}D_t^q m(t) = \frac{d}{dt}\big({}_0I_t^{1-q} m(t)\big) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q}\, m(s) ds, \qquad t \in \bar{J}. $$

We will give fractional integrals and RL fractional derivatives of some elementary functions which will be used later:

Proposition 2.1 ([5]). *The following equalities are true:*

$$
\begin{aligned}
{}^{RL}_{t_0}D^q_t(t-t_0)^\beta &= \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-q)}(t-t_0)^{\beta-q},\\
{}_{t_0}I^q_t(t-t_0)^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(1+\beta+q)}(t-t_0)^{\beta+q},\\
{}_{t_0}I^{1-q}_t(t-t_0)^{q-1} &= \Gamma(q),\\
{}^{RL}_{t_0}D^q_t(t-t_0)^{q-1} &= 0.
\end{aligned}
$$

The definitions of the initial condition for fractional differential equations with RL derivatives are based on the following result:

Proposition 2.2 ([9, Lemma 3.2]). *Let $q \in (0,1)$, $t_0, T \geq 0:\ t_0 < T \leq \infty$ and $m \in L_1^{loc}([t_0,T],\mathbb{R})$.*

(a) If there exists a.e. a limit $\lim_{t\to t_0+}[(t-t_0)^{q-1}m(t)] = c$, then there also exists a limit

$$
{}_{t_0}I^{1-q}_t m(t)|_{t=t_0} := \lim_{t\to t_0+} {}_{t_0}I^{1-q}_t m(t) = c\Gamma(q).
$$

(b) If there exists a.e. a limit ${}_{t_0}I^{1-q}_t m(t)|_{t=t_0} = b$ and if the limit $\lim_{t\to t_0+}[(t-t_0)^{1-q}m(t)]$ exists, then

$$
\lim_{t\to t_0+}[(t-t_0)^{1-q}m(t)] = \frac{b}{\Gamma(q)}.
$$

In the case of a scalar linear RL fractional differential equation we have the following result:

Proposition 2.3 ([9, Example 4.1]). *The solution of the Cauchy type problem*

$$
{}^{RL}_{a}D^q_t x(t) = \lambda x(t) + f(t), \qquad {}_aI^{1-q}_t x(t)|_{t=a} = b
$$

has the following form [9, formula (4.1.14)]

$$
x(t) = \frac{b}{(t-a)^{1-q}}E_{q,q}(\lambda(t-a)^q) + \int_a^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)f(s)ds \tag{2.1}
$$

where $E_{p,q}(z) = \sum_{j=0}^\infty \frac{z^j}{\Gamma(jp+q)}$ is the Mittag-Leffler function with two parameters (see, for example, [9]).

Proposition 2.4. *The inequality*

$$
\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)ds \leq \frac{1}{|a|}(E_q(|a|t^q)-1) \tag{2.2}
$$

holds.

Proof.

$$\begin{aligned}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)ds &= \int_0^t s^{q-1}E_{q,q}(as^q)ds = \int_0^t s^{q-1}\sum_{n=0}^{\infty}\frac{(as^q)^n}{\Gamma((n+1)q)}ds\\ &= \sum_{n=0}^{\infty}\frac{a^n\int_0^t s^{(n+1)q-1}ds}{\Gamma((n+1)q)} = \Big|\sum_{n=0}^{\infty}\frac{a^n t^{(n+1)q}}{(n+1)q\Gamma((n+1)q)}\Big| \le \sum_{n=1}^{\infty}\frac{|a^{n-1}|t^{nq}}{\Gamma(nq+1)}\\ &\le \frac{1}{|a|}\sum_{n=0}^{\infty}\frac{(|a|t^q)^n}{\Gamma(nq+1)} - \frac{1}{|a|} = \frac{1}{|a|}(E_q(|a|t^q)-1).\end{aligned}$$

□

From Proposition 2.3 and Proposition 2.2 (a) we obtain the following result for the weighted form of the initial condition:

Proposition 2.5. *The solution of the Cauchy type problem*

$${}_a^{RL}D_t^q x(t) = \lambda x(t) + f(t), \qquad \lim_{t\to a+}\Big((t-a)^{1-q}x(t)\Big) = C$$

has the following form

$$x(t) = \frac{C\,\Gamma(q)}{(t-a)^{1-q}}E_{q,q}(\lambda(t-a)^q) + \int_a^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)f(s)ds. \tag{2.3}$$

Proposition 2.6 ([9])**.** *For $q\in(0,1)$ the following properties*

$$0 \le E_{q,q}(-\lambda t^q) \le \frac{1}{\Gamma(q)}, \qquad t\ge 0,\ \lambda\ge 0,$$

$$\lim_{t\to 0+} E_{q,q}(-\lambda t^q) = E_{q,q}(0) = \frac{1}{\Gamma(q)}$$

hold.

Proposition 2.7 ([17, Corollary 2])**.** *Let $a(t)$ be a nondecreasing function on J, $g(t)$ be a nonegative, nondecreasing continuous function on J, and*

$$u(t) \le a(t) + g(t)\int_0^t (t-s)^{\beta-1}u(s)ds, \qquad t\in J.$$

Then $u(t) \le a(t)E_\beta(g(t)\Gamma(\beta)t^\beta)$, $t\in J$.

3 Statement of the problem

Consider the initial value problem (IVP) for a nonlinear system of fractional differential equations with constant delay and $q\in(0,1)$

$$\begin{aligned}&{}_0^{RL}D_t^q x(t) = ax(t) + bx(t-\tau) + f(t,x(t),x(t-\tau)) \quad \text{for } t\in J,\\ &x(t) = g(t) \quad \text{for } t\in[-\tau,0],\\ &\lim_{t\to 0+}\Big(t^{1-q}x(t)\Big) = g(0)\end{aligned} \tag{3.1}$$

where ${}_0^{RL}D_t^q$ denotes the RL fractional derivative, $a,b\in\mathbb{R}$ are constants, $\tau>0$ is the constant delay, the functions $f: J\times\mathbb{R}\to\mathbb{R}\to\mathbb{R}$, $g\in C_0$.

First, we will consider the partial linear case of (3.1) without a delay, i.e.

$$\begin{aligned} &{}_0^{RL}D_t^q x(t) = ax(t) + \sigma(t, x(t)) \quad \text{for } t \in J, \\ &\lim_{t\to 0+} \left(t^{1-q}x(t)\right) = x_0. \end{aligned} \tag{3.2}$$

where $a \in \mathbb{R}$ is a constant, $\sigma \in C(J, \mathbb{R})$, $q \in (0, 1)$.

Lemma 3.1 ([9])**.** *The linear initial value problem* (3.2) *has the following integral representation for a solution*

$$x(t) = t^{q-1}x_0\Gamma(q)E_{q,q}(at^q) + \int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)\sigma(s, x(s))ds, \qquad t \in J.$$

We consider also the integral presentation (see [1]) of a special case of (3.1), i.e. we will consider the non-homogeneous scalar linear Riemann–Liouville fractional differential equations with a constant delay

$${}_0^{RL}D_t^q x(t) = Ax(t) + Bx(t-\tau) + \sigma(t) \quad \text{for } t \in J, \tag{3.3}$$

with the initial conditions

$$x(t) = g(t), \qquad t \in [-\tau, 0], \tag{3.4}$$

$$\lim_{t\to 0+} \left(t^{1-q}x(t)\right) = g(0) \tag{3.5}$$

where $\sigma \in C(\mathbb{R}_+, \mathbb{R})$, $g \in C([-\tau, 0], \mathbb{R})$, A, B are real constants.

Lemma 3.2 ([1])**.** *The solution of the IVP* (3.3)*,* (3.4)*,* (3.5) *is given by*

$$x(t) = \begin{cases} g(t) & t \in (-\tau, 0] \\ g(0)\Gamma(q)E_{q,q}(At^q)t^{q-1} + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\big(Bg(s-\tau) + \sigma(s)\big)ds & t \in (0, \tau] \\ g(0)\Gamma(q)E_{q,q}(At^q)t^{q-1} + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\sigma(s)ds & \\ \quad +B\sum_{i=0}^{n-1}\int_{i\tau}^{(i+1)\tau}(t-s)^{q-1}E_{q,q}(A(t-s)^q)x(s-\tau)ds & \\ \quad +B\int_{n\tau}^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)x(s-\tau)ds & \\ \qquad\qquad \text{for } t \in (n\tau, (n+1)\tau], \quad n = 1, 2, \dots, N. & \end{cases}$$

We will consider the assumptions:

(A1) The function $f \in C(\bar{J} \times \mathbb{R}^2, \mathbb{R})$ and there exist constants $K, L > 0$ such that

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \le K|u_1 - u_2| + L|v_1 - v_2|, \quad t \in \bar{J}, \ \ u_1, u_2, v_1, v_2 \in \mathbb{R}.$$

(A2) The function $g \in C_0$.

4 Existence and integral presentation of the solution

Now we will study the existence of the solution of (3.1) and its presentation, based on Lemma 3.2. We will use the Banach contraction principle.

Lemma 4.1. *Let the assumptions (A1), (A2) be satisfied and $q \in [0.5, 1)$.*
Then the operator $\Omega : C_{1-q}(J) \to C_{1-q}(J)$ where

$$\Omega(y(t)) = \begin{cases} g(0)\Gamma(q)E_{q,q}(at^q)t^{q-1} \\ \quad + \int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)\Big(bg(s-\tau) + f(s,y(s),g(s-\tau))\Big)ds, \quad t \in (0,\tau] \\ g(0)\Gamma(q)E_{q,q}(at^q)t^{q-1} \\ \quad + \int_0^\tau (t-s)^{q-1}E_{q,q}(a(t-s)^q)\Big(bg(s-\tau) + f(s,y(s),g(s-\tau))\Big)ds \\ \quad + \int_\tau^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)\Big(by(s-\tau) + f(s,y(s),y(s-\tau))\Big)ds, \quad t \in (\tau, T]. \end{cases} \tag{4.1}$$

Proof. Let $y \in C_{1-q}(J)$. We will prove the inclusion

$$\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)\Big(by(s-\tau) + f(s,y(s),y(s-\tau))\Big)ds \in C_{1-q}(J) \quad \text{for } t \in J.$$

Let $t \in (0,\tau]$ then according to assumption (A1) and Proposition 2.1 with $\beta = q-1$ we get

$$\begin{aligned} &\left| t^{1-q} \int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)\Big(bg(s-\tau) + f(s,y(s),g(s-\tau))\Big)ds \right| \\ &\leq Kt^{1-q}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)|y(s)|ds \\ &\quad + Lt^{1-q}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)|g(s-\tau)|ds \\ &\quad + t^{1-q}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)|f(s,0,0)|ds \\ &\quad + |b|\|g\|_0 t^{1-q}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)ds \\ &\leq KSt^{1-q}\int_0^t (t-s)^{q-1}s^{q-1}|s^{1-q}y(s)|ds + \Big(L\|g\|_0 + C\Big)\frac{tS}{q} \\ &\leq \frac{KSt^q\Gamma^2(q)}{\Gamma(2q)}\|y\|_{C_{1-q}} + \Big((L+|b|)\|g\|_0 + C\Big)\frac{St}{q}. \end{aligned} \tag{4.2}$$

where $C = \sup_{t\in J}|f(t,0,0)|$, $S = \sup_{t\in J} E_{q,q}(at^q)$.

Let $t > \tau$. Then according to assumption (A1), equality $\int_\tau^t \frac{(s-\tau)^{q-1}}{(t-s)^{1-q}}ds = \frac{\Gamma^2(q)}{\Gamma(2q)}(t-\tau)^{2q-1}$ (see Proposition 2.1 with $\beta = q-1$, $t_0 = \tau$) and $t^{1-q}(t-\tau)^{2q-1} \leq t^q$ it follows

$$\begin{aligned} &\left| t^{1-q}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)\Big(by(s-\tau) + f(s,y(s),y(s-\tau))\Big)ds \right| \\ &\leq t^{1-q}S\int_0^t (t-s)^{q-1}(|b|+L)|y(s-\tau)|ds + t^{1-q}KS\int_0^t (t-s)^{q-1}|y(s)|ds + \frac{t}{q}SC \\ &\leq t^{1-q}S\int_0^\tau (t-s)^{q-1}(|b|+L)|g(s-\tau)|ds + t^{1-q}S\int_\tau^t (t-s)^{q-1}(b+L)|y(s-\tau)|ds \\ &\quad + \frac{t}{q}SC + SK\|y(s)\|_{C_{1-q}}\frac{t^q\Gamma^2(q)}{\Gamma(2q)} \leq \frac{t-(t-\tau)^q t^{1-q}}{q}S(|b|+L)\|g\|_0 \\ &\quad + S(|b|+L)\|y(s)\|_{C_{1-q}}\frac{t^{1-q}(t-\tau)^{2q-1}\Gamma^2(q)}{\Gamma(2q)} + \frac{t}{q}SC + SK\|y(s)\|_{C_{1-q}}\frac{t^q\Gamma^2(q)}{\Gamma(2q)} \\ &\leq \frac{t}{q}S((b+L)\|g\|_0 + C) + S(|b|+L+K)\|y(s)\|_{C_{1-q}}\frac{t^q\Gamma^2(q)}{\Gamma(2q)}, \quad t > \tau. \end{aligned} \tag{4.3}$$

From inequalities (4.2) and (4.3) it follows

$$\begin{aligned}&\left|t^{1-q}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)\Big(by(s-\tau)+f(s,y(s),y(s-\tau))\Big)ds\right|\\&\quad\leq \frac{T}{q}S((|b|+L)\|g\|_0+C)+S(|b|+L+K)\|y(s)\|_{C_{1-q}}\frac{T^q\Gamma^2(q)}{\Gamma(2q)}.\end{aligned} \tag{4.4}$$

Therefore, the integrals exist and $\Omega y(t)\in C_{1-q}(J)$. □

Remark 4.2. Note the restriction $q\in[0.5,1)$ is necessary to prove the inequality (4.3) and it is deeply connected with the presence of the delay.

Lemma 4.3. *Suppose (A1) and (A2) hold and $q\in[0.5,1)$.*

(i) If the function $y\in C_{1-q}(J)$ is a solution of IVP (3.1) then it is a fixed-point of the operator Ω defined by (4.1).

(ii) If the function $y\in C_{1-q}(J)$ is a fixed-point of the operator Ω with $y(t)=g(t),\ t\in[-\tau,0]$ then it is a solution of IVP (3.1).

Proof. (i) Let the function $y\in C_{1-q}(J)$ be a solution of IVP (3.1). We will use an induction to prove the function y is a fixed point of the operator Ω.

Let $t\in(0,\tau]$. Then y satisfies the initial value problem (3.2) with $\sigma(t,x)=bg(t-\tau)+f(t,x,g(t-\tau))$ and $x_0=g(0)$. From Lemma 3.1 it follows $\Omega(y(t))=y(t),\ t\in(0,\tau]$.

Let $t\in(\tau,2\tau]$. Then y satisfies the initial value problem (3.2) with $\sigma(t,x)=by(t-\tau)+f(t,x,y(t-\tau))$ and $x_0=g(0)$. From Lemma 3.1 it follows $\Omega(y(t))=y(t),\ t\in(\tau,2\tau]$.

By induction it follows the solution y is a fixed point of the operator Ω.

(ii) Let $y\in C_{1-q}(J)$ be a fixed-point of the operator Ω with $y(t)=g(t),\ t\in[-\tau,0]$.

Then from Lemma 4.1, inequalities (4.2), (4.3) and Proposition 2.6 we obtain that

$$\lim_{t\to 0+}\Big(t^{1-q}\Omega(y(t))\Big)=g(0). \tag{4.5}$$

Therefore, the function y solves the IVP (3.1). □

Remark 4.4. If the conditions of Lemma 4.3 are satisfied, and $y\in C_{1-q}(J)$ is a fixed-point of the operator Ω, then we can spread the definition of y over the entire interval $[-\tau,T]$ by $y(t)=g(t),\ t\in[-\tau,0]$ and then y is a solution of IVP (3.1).

Theorem 4.5 (Existence result)**.** *Let $q\in[0.5,1)$ and the assumption (A1) and (A2) be satisfied and the inequality*

$$\alpha=(K+L+|b|)\frac{T^q\Gamma^2(q)}{\Gamma(2q)}\sup_{t\in J}E_{q,q}(at^q)<1 \tag{4.6}$$

holds.

Then the initial value problem (3.1) *has a unique solution* $y \in C_{1-q}(J)$ *satisfying the integral presentation*

$$y(t) = \begin{cases} g(t), \quad t \in [-\tau, 0] \\ g(0)\Gamma(q)E_{q,q}(at^q)t^{q-1} + b\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)g(s-\tau)\Big)ds \\ \qquad + \int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)f(s, y(s), g(s-\tau))ds, \quad t \in (0, \tau] \\ g(0)\Gamma(q)E_{q,q}(at^q)t^{q-1} + \int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)f(s, y(s), y(s-\tau))ds \\ \quad +b\sum_{i=0}^{n-1}\int_{i\tau}^{(i+1)\tau}(t-s)^{q-1}E_{q,q}(a(t-s)^q)y(s-\tau)ds \\ \quad +b\int_{n\tau}^{t}(t-s)^{q-1}E_{q,q}(a(t-s)^q)y(s-\tau)ds \\ \qquad\qquad\qquad for \quad t \in (n\tau, (n+1)\tau], n = 1, 2, \dots, N \end{cases} \tag{4.7}$$

Proof. According to Lemma 4.1 the operator $\Omega : C_{1-q}(J) \to C_{1-q}(J)$.

We will prove the operator Ω has an unique fixed point in $C_{1-q}(J)$.

Let $y, y^* \in C_{1-q}(J)$ and $t \in (0, \tau]$. Then we obtain

$$\begin{aligned} &|t^{1-q}[\Omega(y(t)) - \Omega(y^*(t))]| \\ &\quad \le t^{1-q}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)|f(s, y(s), g(s-\tau)) - f(s, y^*(s), g(s-\tau))|ds \\ &\quad \le t^{1-q}KS\int_0^t (t-s)^{q-1}s^{q-1}|s^{1-q}(y(s) - y^*(s))|ds \\ &\quad \le t^{1-q}KS\|y - y^*\|_{C_{1-q}}\int_0^t (t-s)^{q-1}s^{q-1}ds \\ &\quad \le \frac{KST^q\Gamma^2(q)}{\Gamma(2q)}\|y - y^*\|_{C_{1-q}} \le \alpha\|y - y^*\|_{C_{1-q}}. \end{aligned} \tag{4.8}$$

Let $y, y^* \in C_{1-q}(J)$ and $t > \tau$. Then we obtain

$$\begin{aligned} &\Big|t^{1-q}[\Omega(y(t)) - \Omega(y^*(t)]\Big| \\ &\quad \le t^{1-q}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)|f(s, y(s), y(s-\tau)) - f(s, y^*(s), y^*(s-\tau))|ds \\ &\qquad + t^{1-q}|b|\int_\tau^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)|y(s-\tau) - y^*(s-\tau)|ds \\ &\quad \le Kt^{1-q}S\int_0^t (t-s)^{q-1}|y(s) - y^*(s)|ds \\ &\qquad + t^{1-q}(L + |b|)S\int_\tau^t (t-s)^{q-1}(s-\tau)^{q-1}|(s-\tau)^{1-q}(y(s-\tau) - y^*(s-\tau))|ds \\ &\quad \le KS\|y - y^*\|_{C_{1-q}}\frac{t^q\Gamma^2(q)}{\Gamma(2q)} + (L + |b|)S\|y - y^*\|_{C_{1-q}}\frac{t^{1-q}(t-\tau)^{2q-1}\Gamma^2(q)}{\Gamma(2q)} \\ &\quad \le (K + L + |b|)\frac{St^q\Gamma^2(q)}{\Gamma(2q)}\|y - y^*\|_{C_{1-q}} \le \alpha\|y - y^*\|_{C_{1-q}}. \end{aligned} \tag{4.9}$$

Therefore

$$\|\Omega(y) - \Omega(y^*)\|_{C_{1-q}} \le \alpha\|y - y^*\|_{C_{1-q}}. \tag{4.10}$$

According to Lemma 4.3 it follows the claim of Theorem 4.5. □

Corollary 4.6. *Let the assumptions (A1), (A2) are satisfied with* $a \le 0$, $q \in [0.5, 1)$ *and*

$$(K + L + |b|)T^q\Gamma(q) < \Gamma(2q). \tag{4.11}$$

Then the initial value problem (3.1) *has a unique solution* $y \in C_{1-q}(J)$.

The proof follows from Theorem 4.5 and Proposition 2.6.

In the case of an equation without a delay we obtain the following corollary.

Corollary 4.7. *Let $\tau = 0$, $a = b = 0$, $q \in (0,1)$, the function $f \in C(J \times \mathbb{R}, \mathbb{R})$ and there exists a constant $K > 0$ such that $|f(t,u_1) - f(t,u_2)| \le K|u_1 - u_2|$, $t \in J$, $u_1, u_2 \in \mathbb{R}$ and*

$$KT^q\Gamma(q) < \Gamma(2q). \tag{4.12}$$

Then the reduced initial value problem

$${}_0^{RL}D_t^q x(t) = f(t, x(t)),\ t \in J, \qquad \lim_{t\to 0+}\left(t^{1-q}x(t)\right) = x_0 \tag{4.13}$$

has a unique solution $y \in C_{1-q}(J)$.

Note in the case without a delay it follows from the proof of Theorem 4.5 that we do not need the restriction $q \in [0.5, 1)$.

Remark 4.8. Note the result of Corollary 4.7 coincides the result of [3, Theorem 3.4] with $L = 0$.

5 Ulam type stability

Let $\varepsilon > 0$ and $\Phi \in C(\bar{J}, [0,\infty))$ be non-decreasing and such that for any $t \in \bar{J}$ the inequality $\int_0^t (t-s)^{q-1}\Phi(s)ds < \infty$ holds.

Definition 5.1 ([10])**.** The equation (3.1) is Ulam–Hyers stable if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C_{1-q}(J)$ of the inequalities

$$\begin{aligned}
&\left|{}_0^{RL}D_t^q y(t) - ay(t) - by(t-\tau) - f(t, y(t), y(t-\tau))\right| \le \varepsilon \quad \text{for } t \in J,\\
&y(t) = g(t) \quad \text{for } t \in [-\tau, 0],\\
&\lim_{t\to 0+}\left(t^{1-q}x(t)\right) = g(0)
\end{aligned} \tag{5.1}$$

there exists a solution $x \in C_{1-q}(J)$ of the problem (3.1) with

$$|x(t) - y(t)| \le \varepsilon\, c_f \quad \text{for } t \in J. \tag{5.2}$$

Definition 5.2 ([10])**.** The problem (3.1) is Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C_{1-q}(J)$ of the inequality

$$\begin{aligned}
&\left|{}_0^{C}D_t^q y(t) - ay(t) - by(t-\tau) - f(t, y(t), y(t-\tau))\right| \le \varepsilon\Phi(t) \quad \text{for } t \in J,\\
&y(t) = g(t) \quad \text{for } t \in [-\tau, 0],\\
&\lim_{t\to 0+}\left(t^{1-q}x(t)\right) = g(0)
\end{aligned} \tag{5.3}$$

there exists a solution $x \in C_{1-q}(J)$ of the problem (3.1) with

$$|y(t) - x(t)| \le \varepsilon\, c_f\Phi(t), \qquad t \in J. \tag{5.4}$$

Definition 5.3 ([10])**.** The problem (3.1) is generalized Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_f > 0$ such that for each solution $y \in C_{1-q}(J)$ of the inequality

$$\begin{aligned}
&\Big|\, {}_0^C D_t^q y(t) - a y(t) - b y(t-\tau) - f(t, y(t), y(t-\tau))\Big| \le \Phi(t) \quad \text{for } t \in J,\\
&y(t) = g(t) \quad \text{for } t \in [-\tau, 0],\\
&\lim_{t\to 0+} \Big(t^{1-q} x(t)\Big) = g(0)
\end{aligned} \tag{5.5}$$

there exists a solution $x \in C_{1-q}(J)$ of the problem (3.1) with

$$|y(t) - x(t)| \le \; c_f \Phi(t), \qquad t \in J. \tag{5.6}$$

Remark 5.4. If the function $f \in C(J \times \mathbb{R}^2, \mathbb{R})$ then the function $y \in C_{1-q}(J)$ is a solution of the inequality (5.1) iff there exist a function $G \in C_{1-q}(J)$ which depends on y such that

(i) $\|G(t)\| \le \varepsilon$;

(ii) ${}_0^C D_t^q y(t) = a y(t) + b y(t-\tau) + f(t, y(t), y(t-\tau)) + G(t)$ for $t \in J$ with initial conditions $y(t) = g(t),\ t \in [-\tau, 0],\ \lim_{t\to 0+} \big(t^{1-q} x(t)\big) = g(0)$.

Remark 5.5. If the function $f \in C(\bar{J} \times \mathbb{R}^2, \mathbb{R})$ then the function $y \in C_{1-q}(J)$ is a solution of the inequality (5.5) iff there exist a function $G \in C_{1-q}(J)$ which depends on y such that

(i) $|G(t)| \le \Phi(t)$ for $t \in J$;

(ii) ${}_0^C D_t^q y(t) = a y(t) + b y(t-\tau) + f(t, y(t), y(t-\tau)) + G(t)$ for $t \in J$ with initial conditions $y(t) = g(t),\ t \in [-\tau, 0],\ \lim_{t\to 0+} \big(t^{1-q} x(t)\big) = g(0)$.

Note we have a similar remark for inequality (5.3).

Based on Remark 5.4 and Definition 5.1 we get the following result.

Lemma 5.6. *Let the conditions of Theorem 4.5 be satisfied. If $y \in C_{1-q}(J)$ is a solution of inequalities (5.1) then it satisfies the following integral-algebraic inequalities*

$$\begin{aligned}
&\Big| y(t) - g(0)\Gamma(q) E_{q,q}(a t^q) t^{q-1} \\
&\qquad - \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) \Big(b g(s-\tau) + f(s, y(s), g(s-\tau))\Big) ds \Big| \\
&\quad \le \frac{\varepsilon}{|a|}\big(E_q(|a| t^q) - 1\big), \qquad t \in (0, \tau] \\
&\Big| y(t) - g(0)\Gamma(q) E_{q,q}(A t^q) t^{q-1} \\
&\qquad - \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) \Big(b y(s-\tau) + f(s, y(s), y(s-\tau))\Big) ds \Big| \\
&\quad \le \frac{\varepsilon}{|a|}\big(E_q(|a| t^q) - 1\big), \qquad t \in (\tau, T].
\end{aligned} \tag{5.7}$$

Proof. Let $y \in C_{1-q}(J)$ be a solution of inequalities (5.1). According to Remark 5.5 it satisfies the IVP

$$\begin{aligned}
&{}_0^C D_t^q y(t) = ay(t) + by(t-\tau) + f(t, y(t), y(t-\tau)) + G(t) \quad \text{for } t \in J, \\
&y(t) = g(t) \quad \text{for } t \in [-\tau, 0], \\
&\lim_{t \to 0+} \left(t^{1-q} x(t) \right) = g(0).
\end{aligned} \tag{5.8}$$

Then according to Lemma 4.3 (i) $y(t)$ is a fixed-point of the operator Ω defined by (4.1), where $f(t,x,y)$ is replaced by $f(t,x,y) + G(t)$.

Let $t \in (0, \tau]$. Apply the inequalities $|G(t)| \le \varepsilon$ and (2.2) and obtain

$$\left| y(t) - g(0)\Gamma(q) E_{q,q}(at^q) t^{q-1} - \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) \Big(bg(s-\tau) + f(s, y(s), g(s-\tau)) \Big) ds \right|$$

$$= \left| \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) G(s) ds \right| \le \frac{\varepsilon}{|a|} (E_q(|a|t^q) - 1).$$

The proof for $t \in (\tau, T]$ is similar and we omit it. □

Now we will study Ulam type stability of problem (3.1).

Theorem 5.7 (Stability results)**.** *Assume the conditions of Theorem 4.5 are satisfied.*

(i) Suppose for any $\varepsilon > 0$ the inequality (5.1) has at least one solution. Then problem (3.1) is Ulam–Hyers stable.

(ii) Suppose there exists a nondecreasing function $\Phi \in C(\bar{J}, [0, \infty))$ such that for any $t \in \bar{J}$ the inequality $\int_0^t (t-s)^{q-1} \Phi(s) ds \le \Lambda_\Phi \Phi(t)$ holds where $\Lambda_\Phi > 0$ is a constant and for any $\varepsilon > 0$ the inequality (5.3) has at least one solution. Then problem (3.1) is Ulam–Hyers–Rassias stable with respect to Φ.

(iii) If there exists a nondecreasing function $\Phi \in C(\bar{J}, [0, \infty))$ such that for any $t \in \bar{J}$ the inequality $\int_0^t (t-s)^{q-1} \Phi(s) ds \le \Lambda_\Phi \Phi(t)$ holds, $\Lambda_\Phi > 0$ is a constant, and inequality (5.5) has at least one solution then problem (3.1) is generalized Ulam–Hyers–Rassias stable with respect to Φ.

Proof. According to Theorem 4.5 the problem (3.1) has an unique solution $x \in C_{1-q}(J)$ for which the integral presentation (4.7) holds.

(i) Let $\varepsilon > 0$ be an arbitrary number and $y \in C_{1-q}(J)$ be a solution of the inequality (5.1). Therefore, the integral inequalities (5.7) hold.

Denote

$$Q = (K + L + |b|) C E_q(\Gamma(q) K C \tau^q) \frac{\tau^q}{q}, \qquad C = \max_{t \in J} E_{q,q}(at^q),$$

and

$$M_{k+1} = M(1+Q)^k, \qquad k = 0, 1, 2, \dots, N, \qquad M = \frac{1}{|a|}(E_q(|a|T^q) - 1).$$

Let $t \in (0, \tau]$ be an arbitrary fixed point. According to Lemma 5.6, Theorem 4.5, inequality (5.7) and equality (4.7) we obtain

$$\begin{aligned}
&|x(t) - y(t)| \\
&\quad \le \left| \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) \Big(f(s, x(s), g(s-\tau)) - f(s, y(s), g(s-\tau)) \Big) ds \right| \\
&\qquad + \left| \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) G(s) ds \right| \\
&\quad \le \frac{\varepsilon}{|a|} (E_q(|a| t^q) - 1) + KC \int_0^t (t-s)^{q-1} |x(s) - y(s)| ds \\
&\quad \le \varepsilon M + KC \int_0^t (t-s)^{q-1} |x(s) - y(s)| ds.
\end{aligned} \tag{5.9}$$

According to Proposition 2.7 we get

$$|x(t) - y(t)| \le \varepsilon M E_q(\Gamma(q) K C t^q), \qquad t \in [0, \tau], \tag{5.10}$$

and

$$|x(t) - y(t)| \le \varepsilon M E_q(\Gamma(q) K C \tau^q) = \varepsilon M_1, \qquad t \in [0, \tau], \tag{5.11}$$

Let $t \in (\tau, 2\tau]$ be an arbitrary fixed point. According to Lemma 5.6, Theorem 4.5, inequalities (5.7), (5.11) and (2.2) we obtain

$$\begin{aligned}
&|x(t) - y(t)| \\
&\quad \le \left| \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) \Big(f(s, x(s), x(s-\tau)) - f(s, y(s), y(s-\tau)) \Big) ds \right| \\
&\qquad + \left| b \int_\tau^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) \Big(x(s-\tau) - y(s-\tau) \Big) ds \right| \\
&\qquad + \left| \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) G(s) ds \right| \\
&\quad \le \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) \Big(K|x(s) - y(s)| + L|x(s-\tau) - y(s-\tau)| \Big) ds \\
&\qquad + \varepsilon |b| M_1 C \int_\tau^t (t-s)^{q-1} ds + \varepsilon M \\
&\quad \le KC \int_\tau^t (t-s)^{q-1} |x(s) - y(s)| ds + \varepsilon M E_q(\Gamma(q) K C \tau^q)(L + |b|) C \frac{(t-\tau)^q}{q} + \varepsilon M \\
&\qquad + \varepsilon M E_q(\Gamma(q) K C \tau^q) K C \frac{\tau^q}{q} \\
&\quad \le \varepsilon M (1 + Q) + KC \int_\tau^t (t-s)^{q-1} |x(s) - y(s)| ds.
\end{aligned} \tag{5.12}$$

According to Proposition 2.7 we get

$$|x(t) - y(t)| \le \varepsilon M (1 + Q) E_q(\Gamma(q) K C t^q) = \varepsilon M_2 E_q(\Gamma(q) K C (t-\tau)^q), \qquad t \in (\tau, 2\tau]. \tag{5.13}$$

and

$$|x(t) - y(t)| \le \varepsilon M_2 E_q(\Gamma(q) K C \tau^q), \qquad t \in (\tau, 2\tau]. \tag{5.14}$$

Let $t \in (2\tau, 3\tau]$ be an arbitrary fixed point. According to Lemma 5.6, Theorem 4.5, inequalities (5.9), (5.11) and (5.14) we obtain

$$\begin{aligned}
&|x(t) - y(t)| \\
&\quad \leq \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) \Big(K|x(s) - y(s)| + L|x(s-\tau) - y(s-\tau)| \Big) ds \\
&\qquad + |b| \int_\tau^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) |x(s-\tau) - y(s-\tau)| ds \\
&\qquad + \left| \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) G(s) ds \right| \\
&\quad \leq KC \int_0^t (t-s)^{q-1} |x(s) - y(s)| ds + (L + |b|) C \int_\tau^{2\tau} (t-s)^{q-1} |x(s-\tau) - y(s-\tau)| ds \\
&\qquad + (L + |b|) C \int_{2\tau}^t (t-s)^{q-1} |x(s-\tau) - y(s-\tau)| ds + \varepsilon M \\
&\quad \leq KC \int_{2\tau}^t (t-s)^{q-1} |x(s) - y(s)| ds + \varepsilon M (1+Q)^2.
\end{aligned} \tag{5.15}$$

According to Proposition 2.7 we get

$$|x(t) - y(t)| \leq \varepsilon M (1+Q)^2 E_q(\Gamma(q) K C t^q) = \varepsilon M_3 E_q(\Gamma(q) K C (t - 2\tau)^q), \qquad t \in (2\tau, 3\tau] \tag{5.16}$$

and

$$|x(t) - y(t)| \leq \varepsilon M_3 E_q(\Gamma(q) K C \tau^q), \qquad t \in (2\tau, 3\tau]. \tag{5.17}$$

Continuing the induction process we prove

$$|x(t) - y(t)| \leq \varepsilon M_{k+1} E_q(\Gamma(q) K C \tau^q), \qquad t \in (k\tau, (k+1)\tau],\ k = 0, 1, 2, \ldots, N. \tag{5.18}$$

Inequality (5.18) proves the claim (i) with $c_f = M(1+Q)^N E_q(\Gamma(q) K C \tau^q)$.

(iii) Let $y \in C_{1-q}(J)$ be a solution of the inequality (5.5) with the function $\Phi(t)$ defined in the condition (iii) of Theorem 5.7.

Denote

$$Q = (K + L + |b|) C \Lambda_\Phi E_q(\Gamma(q) K C \tau^q), \qquad C = \max_{t \in J} E_{q,q}(a t^q),$$

and

$$M_{k+1} = M(1+Q)^k, \qquad k = 0, 1, 2, \ldots, N, \qquad M = C \Lambda_\Phi.$$

Similar to the case (i) we use an induction to prove the inequality

$$|x(t) - y(t)| \leq M_{k+1} E_q(\Gamma(q) K C \tau^q) \Phi(t), \qquad t \in (k\tau, (k+1)\tau],\ k = 0, 1, 2, \ldots, N. \tag{5.19}$$

Therefore, the problem (3.1) is generalized Ulam–Hyers–Rassias stable with respect to Φ with $c_f = C \Lambda_\Phi (1+Q)^n E_q(\Gamma(q) K C \tau^q)$.

(ii) The proof is similar to the one in (i). □

6 Applications to some biological models

In this section we will apply the obtained results to some biological models and their fractional generalizations.

Model 1. The investigation of blood cell dynamics is connected with formulating and studyig mathematical methods and models, numerical results, schemes to estimate parameters and prognosticate optimal treatments to particular diseases. In order to describe the survival of red blood cells in animals, Ważewska-Czyżewska and Lasota proposed in [15] the following delayed equation $x'(t) = -\gamma x(t) + \beta e^{-\alpha x(t-\tau)}$ where $x(t)$ represents the number of red blood cells at time t, $\gamma > 0$ is the death probability for a red blood cell, a and β are positive constants related to the production of red blood cells per unit time and τ is the time delay between the production of immature red blood cells and their maturation for release in circulating blood stream. The well known Lasota–Ważewska model was extended and generalized by many authors. Now we will consider one fractional generalization.

Consider the following fractional generalization of the mentioned above model:

$$ {}_0^{RL}D_t^q x(t) = \beta e^{-\alpha x(t-\tau)} - \gamma x(t), \qquad t \in J \tag{6.1}$$

with the initial conditions (3.4), (3.5), where $q \in [0.5, 1)$, x is the number of red blood cells, $\beta > 0$ is the demand for oxygen, $\tau > 0$ is the time required for erythrocytes to attain maturity, $\gamma > 0$ is the cell destruction rate.

In this case $a = -\gamma$, $b = 0$, $f(t, x, y) = \beta e^{-\alpha y}$ and $|f(t, x, y) - f(t, u, v)| = |\beta e^{-\alpha y} - \beta e^{-\alpha v}| \leq \beta\alpha|y - v|$, i.e. the condition (A1) is satisfied with $K = 0$, $L = \beta\alpha$.

If $\beta\alpha T\Gamma(q) < \Gamma(2q)$ then according to Theorem 4.5 the initial value problem (6.1), (3.4), (3.5) has an unique solution $x \in C_{1-q}(J)$ satisfying the integral presentation

$$x(t) = g(0)\Gamma(q)E_{q,q}(-\gamma t^q)t^{q-1} + \beta \int_0^t (t-s)^{q-1}E_{q,q}(-\gamma(t-s)^q)e^{-\alpha x(s-\tau)}ds, \quad t \in (0, T]. \tag{6.2}$$

Consider the partial case of $q = 0.8$, $\beta = 0.05$, $\alpha = 0.9$,, $\gamma = 0.01$, and $\tau = 2$, $T = 12$, $g(t) = t^2$. Then the inequality $0.9(0.05)\Gamma(0.8)12 < \Gamma(1.6)$ holds and therefore the model (6.1) has a solution satisfying the integral presentation

$$x(t) = 0.05 \int_0^t (t-s)^{-0.2}E_{0.8,0.8}(-0.5(t-s)^{0.8})e^{-0.9x(s-2)}ds, \qquad t \in (0, 12].$$

Consider the function $y(t) = t^2$. Then ${}_0^{RL}D_t^{0.8} t^2 = \frac{\Gamma(3)}{\Gamma(2.2)}t^{1.2}$ and the inequality

$$\left| \frac{\Gamma(3)}{\Gamma(2.2)}t^{1.2} - 0.05e^{-0.9(t-2)^2} + 0.01t^2 \right| \leq 3.5t + 0.0015 \quad \text{for } t \in [0, 12], \tag{6.3}$$

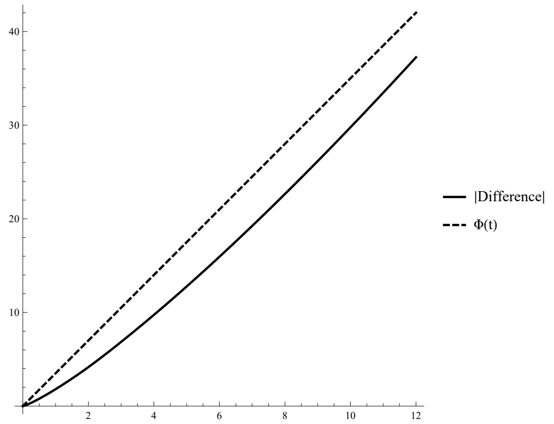
holds (see Figure 6.1a).

Consider $\Phi(t) = 3.5t + 0.0015$. Then $\int_0^t (t-s)^{-0.2}(3.5s + 0.0015)ds \leq \Lambda_\Phi(3.5t + 0.0015)$ with $\Lambda_\Phi = 5.5$ (see Figure 6.1b).

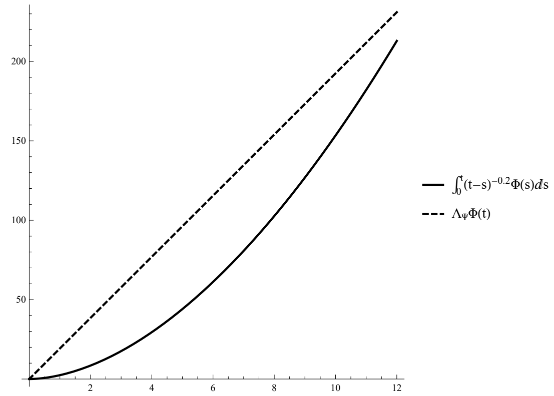
According to Theorem 5.7(iii) the solution $x(t)$ of (6.1) satisfies

$$\left| x(t) - t^2 \right| \leq 39.9618t + 0.0199809, \qquad t \in [0, 12]$$

where $c_f = C\Lambda_\Phi(1+Q)^4 E_q(0) = 5.5(1.2475)^4 = 13.3206$, $C = \max_{t\in[0,12]} E_{0.8,0.8}(-0.01t^{0.8}) = 1$, $Q = (K + L + |b|)C\Lambda_\Phi E_q(\Gamma(q)KC\tau^q = 0.2475$.



(a) Graph of the difference $\left|\frac{\Gamma(3)}{\Gamma(2.2)}t^{1.2} - 0.05e^{-0.9(t-2)^2} + 0.01t^2\right|$ and the function $\Phi(t) = 3.5t + 0.0015$ on $[0, 12]$.



(b) Graph of the fractional integral of the function $\Phi(t) = 3.5t + 0.0015$ and the function $\Lambda_\Phi\Phi(t)$ on $[0, 12]$.

Figure 6.1: Model 1.

Model 2. Consider the logistic equation where the effect of a biological delay depends on the mechanistic details of the model. For example, suppose that a period of time τ elapses between egg laying and hatching. Let us consider the case of and constant harvesting rate $\mu > 0$ ([6]): $N'(t) = BN(t-\tau)e^{-\delta\tau}(1 - \frac{N(t)}{A}) - \mu N(t)$, where δ is egg mortality and $e^{-\delta\tau}$ could be the birth rate (given as a fraction). Hatchlings were produced by parents alive τ time ago, but complete for sites with individuals alive at their dispersal and recruitment. It has a zero equilibrium.

Now, consider the fractional generalization of the model with $q \in (0, 1)$:

$$ {}_0^{RL}D_t^q N(t) = BN(t-\tau)e^{-\delta\tau}\Big(1 - \frac{N(t)}{A}\Big) - \mu N(t) \tag{6.4} $$

with the initial conditions

$$ N(t) = g(t), \qquad t \in [-\tau, 0], \tag{6.5} $$

$$ \lim_{t \to 0+} \Big(t^{1-q}N(t)\Big) = g(0) \tag{6.6} $$

Note (6.4) has a zero equilibrium.

In this case $f(t, x, y) = -\frac{B}{A}e^{\delta\tau}xy$ and $a = -\mu$, $b = Be^{-\delta\tau}$. Then $|f(t, x, y) - f(t, u, v)| = \frac{B}{A}e^{-\delta\tau}|xy - uv| \leq \frac{B}{A}e^{-\delta\tau}|x(y-u) + U(x-v)| \leq K|x-v| + L|y-u|$ with $K = \frac{B}{A}e^{-\delta\tau}\max v$ and $L = \frac{B}{A}e^{-\delta\tau}\max x$. We will consider the case $N \leq W$, $W \in (0, A]$. Therefore, $K = L = \frac{B}{A}We^{-\delta\tau}$. According to Theorem 4.5 if $\alpha = \big(\frac{2}{A}W + 1\big)Be^{-\delta\tau}\frac{T\Gamma(q)}{\Gamma(2q)} < 1$ then (6.4) has a solution in C_{1-q} satisfying the integral presentation

$$ N(t) = \begin{cases} g(0)\Gamma(q)E_{q,q}(-\mu t^q)t^{q-1} \\ \quad + \int_0^t (t-s)^{q-1}E_{q,q}(-\mu(t-s)^q)\Big(Be^{\delta\tau}g(s-\tau) - \frac{B}{A}e^{-\delta\tau}N(s)N(s-\tau)\Big)ds, \quad t \in (0, \tau] \\ g(0)\Gamma(q)E_{q,q}(-\mu t^q)t^{q-1} - \frac{B}{A}e^{-\delta\tau}\int_0^t (t-s)^{q-1}E_{q,q}(-\mu(t-s)^q)N(s)N(s-\tau)ds \\ \quad + Be^{-\delta\tau}\int_0^t (t-s)^{q-1}E_{q,q}(-\mu(t-s)^q)N(s-\tau)ds, \quad t \in (n\tau, (n+1)\tau],\ n = 1, 2, \ldots \end{cases} $$

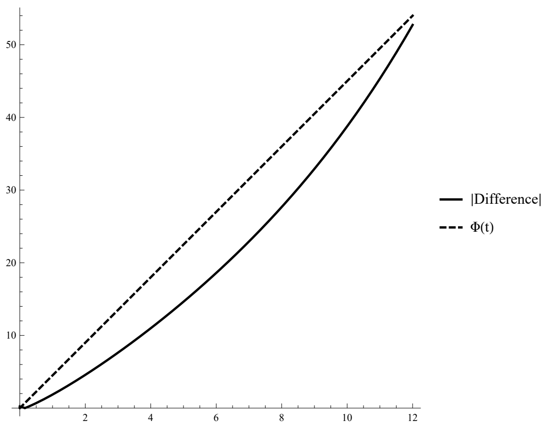
Consider the partial case $B = 0.07$, $\delta = 0.1$, $T = 6$, $q = 0.8$, $\tau = 2$, $\beta = 2$, $A = 100$, $W = 30$, $\mu = 0.1$ and $N \leq 1$. Then $\alpha \approx 0.716881 < 1$ and $K = L = \frac{0.07}{100}30e^{-0.02} = 0.0205842$, $a = -0.1$, $b = 0.07e^{-0.2}$. According to Theorem 4.5 the equation (6.4) has a solution.

Consider the function $y(t) = t^2$. Because ${}_0^{RL}D_t^{0.8}t^2 = \frac{\Gamma(3)}{\Gamma(1.5)}t^{1.2}$ the inequality

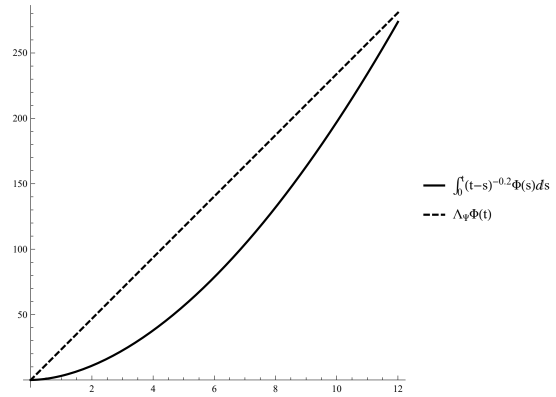
$$\left|\frac{\Gamma(3)}{\Gamma(3-0.8)}t^{2-0.8} - 0.07(t-2)^2e^{-0.2}\Big(1-\frac{t^2}{100}\Big) + 0.1t^2\right| \le 4.5t, \qquad t \in [0,12].$$

holds (see Figure 6.2a).

Consider $\Phi(t) = 4.5t$. The inequality $\int_0^t (t-s)^{0.8-1} 4.5s ds \le \Lambda_\Phi(4.5t)$ holds with $\Lambda_\Phi = 5.2$ (see Figure 6.2b).



(a) Graph of the difference $|\frac{\Gamma(3)}{\Gamma(3-0.8)}t^{2-0.8} - 0.07(t-2)^2e^{-0.2}(1-\frac{t^2}{100}) + 0.1t^2|$ and the function $\Phi(t) = 4.5t$ on $[0,12]$.



(b) Graph of the fractional integral of the function $\Phi(t) = 4.5t$ and the function $\Lambda_\Phi\Phi(t)$ on $[0,12]$.

Figure 6.2: Model 2.

According to Theorem 5.7(iii) the solution $N(t)$ of (6.4) satisfies

$$\left|N(t) - t^2\right| \le 37.5892t, \qquad t \in [0,12]$$

since $c_f = 5.2(1+Q)^4 1.04604 = 5.5(1.2475)^4 = 8.35315$, $C = \max_{t\in[0,12]} E_{0.8,0.8}(-0.1t^{0.8}) = 1$, $Q = (0.0205842 + 0.0205842 + 0.07e^{-0.2})5.2E_{0.8}(\Gamma(0.8)0.0205842\ 2^{0.8}) = 0.535671$.

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Topological entropy for impulsive differential equations

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. A positive topological entropy is examined for impulsive differential equations via the associated Poincaré translation operators on compact subsets of Euclidean spaces and, in particular, on tori. We will show the conditions under which the impulsive mapping has the forcing property in the sense that its positive topological entropy implies the same for its composition with the Poincaré translation operator along the trajectories of given systems. It allows us to speak about chaos for impulsive differential equations under consideration. In particular, on tori, there are practically no implicit restrictions for such a forcing property. Moreover, the asymptotic Nielsen number (which is in difference to topological entropy a homotopy invariant) can be used there effectively for the lower estimate of topological entropy. Several illustrative examples are supplied.

Keywords: topological entropy, impulsive differential equation, Poincaré's operator, asymptotic Nielsen number, Lefschetz number, Carathéodory periodic solution.

2020 Mathematics Subject Classification: Primary 34B37, 34C28, 37B40; Secondary 34C40, 37D45.

1 Introduction

The main aim of the present paper is to establish a positive topological entropy for impulsive differential equations via the associated Poincaré translation operators along their trajectories. We will present, under natural assumptions, the relationship for the topological entropies of given impulsive maps and their compositions with the Poincaré operators, from which a positive topological entropy of the composition, determining chaos for the impulsive differential equations, is implied by the one of the impulsive map. On tori, the Ivanov theorem (see

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[13, 17]), using effectively the asymptotic Nielsen number (which is in difference to topological entropy a homotopy invariant), is applied for the lower estimate of topological entropy. Moreover, this application can be expressed on tori in terms of the Lefschetz numbers which are significantly easier for calculations.

Although various sorts of chaos have been already investigated for impulsive differential equations (see e.g. [1, 5, 6, 18, 24], and the references therein), as far as we know, a topological entropy has been examined, with only a few exceptions like [3], exclusively for non-impulsive differential equations and dynamical systems (see e.g. [11, 14, 22, 25, 27], and the references therein). That is why we would like, besides other things, to eliminate here this handicap.

For this goal, we will firstly recall Bowen's definition of a topological entropy [7], jointly with its basic properties. We will also recall the Ivanov theorem [13] and its consequences on tori. For the systems of ordinary differential equations on $\mathbb{R}^n$ and $\mathbb{R}^n/\mathbb{Z}^n$, we will define the associated Poincaré translation operators along the trajectories and point out the relationship between Carathéodory periodic solutions and periodic points of the Poincaré operators. Before a separate formulation of the main theorems about a positive topological entropy for impulsive differential equations on Euclidean spaces and tori, we will deduce mentioned crucial relationship for topological entropies of impulsive maps and their compositions with the Poincaré operators. The obtained results will be illustrated by simple examples and commented by concluding remarks.

2 Preliminaries

Although the topological entropy, which is a central notion of our paper, was defined by Bowen [7] (cf. also [2, p. 188], [23, pp. 369–370]) for uniformly continuous maps, we will restrict ourselves (from the practical reasons) to a subclass of continuous maps on compact metric spaces. For more details about the topological entropy, see e.g. [19].

Definition 2.1. Let (X, d) be a compact metric space and $f\colon X \to X$ be a continuous map. A set $S \subset X$ is called *(n, ε)-separated for f*, for a positive integer n and $\varepsilon > 0$, if for every pair of distinct points $x, y \in S$, $x \neq y$, there is at least one k with $0 \leq k < n$ such that $d\left(f^k(x), f^k(y)\right) > \varepsilon$. Then, denoting the *number of different orbits of length n* by

$$r(n, \varepsilon, f) := \max\left\{\#S\colon S \subset X \text{ is an } (n, \varepsilon)\text{-separated set for } f\right\},$$

where $\#S$ stands for the cardinality (i.e. the number of elements) of S, the *topological entropy $h(f)$ of f* is defined as

$$h(f) := \lim_{\varepsilon \to 0}\left[\limsup_{n \to \infty} \frac{1}{n} \log(r(n, \varepsilon, f))\right].$$

It will be convenient to recall the following properties of topological entropy. The first lemma justifies Definition 2.1 in the sense that the metric d in the notation of $h(f)$ can be omitted.

Lemma 2.2 (cf. e.g. [19, Proposition 3.1.2], [26, Corollary 7.5.2])**.** *If X is a compact metrisable space and d' is any metric on X, then $h(f) = h_{d'}(f)$ holds for any continuous map $f\colon X \to X$, where $h_{d'}(f)$ denotes the topological entropy of f on X calculated with any specific metric d'.*

Lemma 2.3 (cf. e.g. [2, Lemma 4.1.10], [23, Theorem IX.1.3])**.** *Let f be a continuous map on X. Assume $X = X_1 \cup \cdots \cup X_k$ is a decomposition into disjoint closed invariant subsets which are a* *positive distance apart. Then*

$$h(f) = \max_{j=1,\dots,k} h\left(f|_{X_j}\right).$$

Lemma 2.4 (cf. e.g. [2, Lemma 4.1.5], [23, Theorem IX.1.4]). *Let f be a continuous map on a compact metric space X. Let $\Omega \subset X$ be the nonwandering points of f, i.e. the points $p \in \Omega$ such that, for every neighbourhood U of p, there is an integer $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. Then the entropy $h(f)$ of f equals the entropy of f restricted to its nonwandering set Ω, namely $h(f) = h(f|_\Omega)$.*

Lemma 2.5 (cf. e.g. [23, Theorem IX.1.5]). *Let f be a continuous map on a compact metric space X for which the nonwandering set Ω consists of a finite number of periodic orbits. Then the topological entropy $h(f)$ of f is zero, $h(f) = 0$. In particular, the same is true, provided $\bigcap_{j=0}^{\infty} f^j(X)$ is finite (see e.g. [2, p. 194]).*

Before formulating the following lemma, let us recall that a map $s\colon X \to Y$ is *uniformly finite to one* if $s^{-1}(y)$ has a finite number of points for each $y \in Y$, and there is a bound on the number of elements in $s^{-1}(y)$ which is independent of $y \in Y$.

Lemma 2.6 (cf. e.g. [23, Theorem IX.1.8]). *Assume that $f\colon X \to X$ and $g\colon Y \to Y$ are continuous maps, where (X, d) and (Y, d') are compact metric spaces with metrics d and d', respectively. Assume $s\colon X \to Y$ is a semi-conjugacy from f to g, i.e. (i) s is continuous, (ii) s is "onto", (iii) $s \circ f = g \circ s$, that is uniformly finite to one. Then $h(f) = h(g)$.*

If X is a compact polyhedron, then we can apply in the form of proposition the following Jiang's slight generalization (see [17]) of the Ivanov theorem [13], for the lower estimate of the topological entropy. For the definition and properties of the Nielsen number, which is unlike to topological entropy a homotopy invariant, see e.g. [9, 15].

Proposition 2.7. *Suppose X is a compact polyhedron and, in particular (for our needs), the torus $X = \mathbb{R}^n/\mathbb{Z}^n$. Let $f\colon X \to X$ be a continuous map. Then for any continuous map $g\colon X \to X$ homotopic to f (i.e. $g \sim f$), the topological entropy $h(g)$ satisfies $h(g) \geq \log N^\infty(f)$, where*

$$N^\infty(f) := \max\left\{1, \limsup_{m\to\infty} \left(N(f^m)\right)^{\frac{1}{m}}\right\}$$

is the asymptotic Nielsen number of f and $N(f^m)$ is the standard Nielsen number of the m-th iterate of f. Thus, if $N^\infty(f) > 1$, then

$$h(g) \geq \limsup_{m\to\infty} \frac{1}{m} \log N(f^m) > 0$$

holds for any $g \sim f$.

Remark 2.8. For the torus $X = \mathbb{R}^n/\mathbb{Z}^n$, we have still (see [8])

$$N(f) = |\lambda(f)|,$$

where $\lambda(f)$ denotes the *Lefschetz number* of f (for its definition and properties, see e.g. [9]), by which the inequality

$$h(g) \geq \log N^\infty(f) \tag{2.1}$$

can be rewritten into

$$h(g) \geq \log \max\left\{1, \limsup_{m\to\infty} |\lambda(f^m)|^{\frac{1}{m}}\right\}, \tag{2.2}$$

which is significantly easier for verification.

Hence, if

$$\limsup_{m\to\infty} |\lambda(f^m)|^{\frac{1}{m}} > 1,$$

then

$$h(g) \geq \limsup_{m\to\infty} \frac{1}{m} \log |\lambda(f^m)| > 0$$

holds for any $g \sim f$.

If, in particular, $f\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is an endomorphism defined by an integer matrix A, whose eigenvalues are $\lambda_1, \ldots, \lambda_n$, then (see e.g. [16, Example, p. 192])

$$N^\infty(f) = \begin{cases} 1, & \text{if } \lambda(f) = 0 \\ \prod_{|\lambda_k|>1} |\lambda_k|\,, & \text{otherwise,} \end{cases} \tag{2.3}$$

and $\lambda(f) = \det(\mathcal{I} - A) = \Pi_{k=1}^n (1 - \lambda_k)$, where $\lambda(f)$ stands for the Lefschetz number of f.

Now, consider the vector differential equation

$$x' = F(t, x), \tag{2.4}$$

where $F\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is the *Carathéodory mapping* such that $F(t,x) \equiv F(t+\omega, x)$, for some given $\omega > 0$, i.e.

(i) $F(\cdot, x)\colon [0,\omega] \to \mathbb{R}^n$ is measurable, for every $x \in \mathbb{R}^n$,

(ii) $F(t,\cdot)\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous, for almost all (a.a.) $t \in [0,\omega]$.

Let, furthermore (2.4) satisfy a *uniqueness condition* and all solutions of (2.4) entirely exist on the whole line $(-\infty, \infty)$.

By a *(Carathéodory) solution* $x(\cdot)$ of (2.4), we understand a locally absolutely continuous function, i.e. $x \in AC_{loc}(\mathbb{R}, \mathbb{R}^n)$, which satisfies (2.4) for a.a. $t \in \mathbb{R}$.

We can associate to (2.4) the *Poincaré translation operator* $T_\omega\colon \mathbb{R}^n \to \mathbb{R}^n$ along its trajectories as follows:

$$T_\omega(x_0) := \{x(\omega)\colon x(\cdot) \text{ is a solution of (2.4) such that } x(0) = x_0\}\,. \tag{2.5}$$

It is well known (see e.g. [20, Chapter 1.1] that T_ω is a homeomorphism such that $T_\omega^k = T_{k\omega}$, for every $k \in \mathbb{N}$.

Assuming still that

$$F(t, \ldots, x_j, \ldots) \equiv F(t, \ldots, x_j + 1, \ldots), \quad j = 1, \ldots, n, \tag{2.6}$$

where $x = (x_1, \ldots, x_n)$, we can also consider (2.4) on the torus $\mathbb{R}^n/\mathbb{Z}^n$, which can be endowed with the metric

$$\hat{d}(x,y) := \min\{d_{Eucl}(a,b)\colon a \in [x], b \in [y]\}\,,$$

for all $x, y \in \mathbb{R}^n/\mathbb{Z}^n$, where $d_{Eucl}(a,b) := \sqrt{\sum_{j=1}^n (a_j - b_j)^2}$, for all $a, b \in \mathbb{R}^n$.

The associated *Poincaré translation operator* $\hat{T}_\omega\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ along the trajectories of (2.4), considered on $\mathbb{R}^n/\mathbb{Z}^n$, takes the form $\hat{T}_\omega := \tau \circ T_\omega$, where T_ω was defined in (2.5), and $\tau\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$, $x \to [x] := \{y \in \mathbb{R}^n\colon (y - x) \in \mathbb{Z}^n\}$ is the natural (canonical) projection. It is well known (see e.g. [10, Chapter XVII]) that $\hat{T}_\omega$ is also a homeomorphism such that $\hat{T}_\omega^k = \hat{T}_{k\omega}$, for every $k \in \mathbb{N}$. In particular, for $n = 1$, $\hat{T}_\omega$ is an orientation-preserving homeomorphism.

One can easily detect the one-to-one correspondence between the $k\omega$-periodic solutions of (2.4), i.e. $x(t) \equiv x(t + k\omega)$ but $x(t) \not\equiv x(t + j\omega)$ for $j < k$, and k-periodic points of T_ω, i.e $x_0 = T_\omega^k(x_0)$ but $x_0 \neq T_\omega^j(x_0)$ for $j < k$, where $x_0 = x(0)$ and j, k are positive integers.

The same correspondence holds between $k\omega$-periodic solutions $\hat{x}(\cdot) := \tau \circ x(\cdot)$ of (2.4), considered on $\mathbb{R}^n/\mathbb{Z}^n$, and k-periodic points $\hat{x}_0 = \tau \circ x_0$ of $\hat{T}_\omega := \tau \circ T_\omega$, where $\hat{x}_0 = \hat{x}(0)$.

The impulsive differential equations, i.e. the differential equations (2.4) with impulses at $t = t_j := j\omega$, $j \in \mathbb{Z}$, will be considered separately on the spaces $\mathbb{R}^n$ and $\mathbb{R}^n/\mathbb{Z}^n$. Their solutions will be also understood in the same Carathéodory sense, i.e. $x \in AC[t_j, t_{j+1}]$, $j \in \mathbb{Z}$.

3 Topological entropy for impulsive differential equations on $\mathbb{R}^n$

Consider the vector impulsive differential equation

$$\begin{cases} x' = F(t,x), \ t \neq t_j := j\omega, \ \text{for some given } \omega > 0, \\ x(t_j^+) = I(x(t_j^-)), \ j \in \mathbb{Z}, \end{cases} \tag{3.1}$$

where $F\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is the Carathéodory mapping such that $F(t,x) \equiv F(t+\omega, x)$, equation (2.4) satisfies a uniqueness condition and a global existence of all its solutions on $(-\infty, \infty)$. Let, furthermore, $I\colon \mathbb{R}^n \to \mathbb{R}^n$ be a compact continuous impulsive mapping such that $K_0 := \overline{I(\mathbb{R}^n)}$ and $I(K_0) = K_0$.

Proposition 3.1. *Let $T_\omega\colon \mathbb{R}^n \to \mathbb{R}^n$ be the associated Poincaré translation operator along the trajectories of* (2.4)*, defined in* (2.5)*, such that $K_1 := T_\omega(K_0)$ and $K_0 \subset K_1$. Then the equality*

$$h\left(I|_{K_1} \circ T_\omega|_{K_0}\right) = h\left(I|_{K_0}\right) \tag{3.2}$$

holds for the topological entropies h of the maps $I|_{K_1} \circ T_\omega|_{K_0}\colon K_0 \to K_0$ and $I|_{K_0}\colon K_0 \to K_0$.

Proof. We have the diagram

$$\begin{tikzcd} K_0 \arrow[r, "T_\omega"] \arrow[d, "T_\omega"'] & K_1 \arrow[r, "I"] & K_0 \arrow[d, "T_\omega"] \\ K_1 \arrow[r, "I"] & K_0 \arrow[r, "T_\omega"] \arrow[u, "T_\omega"] & K_1, \end{tikzcd}$$

where $K_0, K_1 \subset \mathbb{R}^n$ are compact subsets, and $T_\omega|_{K_0}\colon K_0 \to K_1$ is (i) continuous, (ii) "onto" and uniformly finite to one, (iii) $T_\omega|_{K_0} \circ \left(I|_{K_1} \circ T_\omega|_{K_0}\right) = \left(T_\omega|_{K_0} \circ I|_{K_1}\right) \circ T_\omega|_{K_0}$, i.e. it is a semi-conjugacy.

Thus, applying Lemma 2.6, we obtain that

$$h\left(I|_{K_1} \circ T_\omega|_{K_0}\right) = h\left(T_\omega|_{K_0} \circ I|_{K_1}\right).$$

Endowing K_0, K_1 with the respective metrics d, d', where

$$\begin{aligned} d(x,y) &:= d_{\mathrm{Eucl}}(x,y), \ \text{for all } x, y \in K_0, \\ d'(x,y) &:= d_{\mathrm{Eucl}}\left(T_\omega(x), T_\omega(y)\right), \ \text{for all } x, y \in K_0, \\ d'(x',y') &:= d_{\mathrm{Eucl}}\left(x', y'\right), \ \text{for all } x' \ (= T_\omega(x)), \ y' \ (= T_\omega(y)) \in K_1, \end{aligned}$$

we can write in this notation that

$$h\left(T_\omega|_{K_0} \circ I|_{K_1}\right) = h_{d'}\left(I|_{K_1}\right), \qquad \text{resp.} \qquad h\left(I|_{K_1} \circ T_\omega|_{K_0}\right) = h_{d'}\left(I|_{K_1}\right), \tag{3.3}$$

where the lower index d' denotes the respective metric.

We can also write that

$$h_{d'}\left(I|_{K_1}\right) = h_{d'}\left(I|_{T_\omega(K_0)}\right) = h_{d'}\left(I|_{K_0}\right). \tag{3.4}$$

Furthermore, since the topological entropy of given continuous maps on compact metric spaces does not depend, according to Lemma 2.2, on the used metrics, we get still that

$$h_{d'}\left(I|_{K_0}\right) = h\left(I|_{K_0}\right). \tag{3.5}$$

Summing up the relations (3.3)–(3.5), we arrive at (3.2), as claimed. □

Remark 3.2. It can be readily seen from (3.2) that a positive topological entropy holds for $I|_{K_1} \circ T_\omega|_{K_0}$, when $h(I|_{K_0}) > 0$ and $K_0 \subset K_1$, which is a rather implicit condition. Since $K_1 \setminus K_0$ is the wandering set for I, condition (3.2) is in a certain sense sharp (cf. also (3.3)). On the other hand, if K_0 contains only a finite number of periodic orbits for $I|_{K_0}$, then according to Lemma 2.5, $h(I|_{K_0}) = 0$, by which also $h(I|_{K_1} \circ T_\omega|_{K_0}) = 0$.

Corollary 3.3. *Consider the scalar impulsive differential equation, i.e.* (3.1) *for* $n = 1$. *If* $[a, b] \subset [T_\omega(a), T_\omega(b)]$ *holds for the Poincaré translation operator* T_ω *along the trajectories of* (2.4), *defined in* (2.5), *where* $[a, b] = I([a, b])$, *then condition* (3.2) *takes the form*

$$h\left(I|_{[T_\omega(a),T_\omega(b)]} \circ T_\omega|_{[a,b]}\right) = h\left(I|_{[a,b]}\right). \tag{3.6}$$

Proof. Since $T_\omega \colon \mathbb{R} \to \mathbb{R}$ must be, under a uniqueness condition, strictly increasing, we have that $K_1 = [T_\omega(a), T_\omega(b)]$, where $K_0 = [a, b]$. In this notation, $K_0 \subset K_1$, and condition (3.2) takes the form (3.6). □

Definition 3.4. We say that the vector impulsive differential equation (3.1) exhibits *chaos in the sense of a positive topological entropy h* if $h(I|_{K_1} \circ T_\omega|_{K_0}) > 0$ holds for the composition of the associated Poincaré translation operator T_ω along the trajectories of (2.4), defined in (2.5), with the compact impulsive mapping $I \colon \mathbb{R}^n \to \mathbb{R}^n$, where $K_0 := \overline{I(\mathbb{R}^n)}$ and $K_1 := T_\omega(K_0)$.

Theorem 3.5. *The vector impulsive differential equation* (3.1) *exhibits, under the above assumptions, chaos in the sense of Definition* 3.4, *if* $I(K_0) = K_0$ *and* $K_0 \subset K_1$, *where* $K_0 := \overline{I(\mathbb{R}^n)}$ *and* $K_1 := T_\omega(K_0)$, *jointly with* $h(I|_{K_0}) > 0$.

Proof. The proof follows directly from the inequality (3.2) in Proposition 3.1. □

Corollary 3.6. *The scalar* ($n = 1$) *impulsive differential equation* (3.1) *exhibits, under the above assumptions, chaos in the sense of Definition* 3.4, *provided* $h(I|_{[a,b]}) > 0$ *holds, jointly with* $\overline{I(\mathbb{R})} = I([a, b]) = [a, b] \subset [T_\omega(a), T_\omega(b)]$.

Proof. The proof follows directly from the equality (3.6) in Corollary 3.3, where $K_0 = [a, b]$ and $K_1 = [T_\omega(a), T_\omega(b)]$. □

The following simple illustrative examples demonstrate an application of Corollary 3.6 to scalar ($n = 1$) linear and semi-linear impulsive differential equations.

Example 3.7. Consider the linear impulsive equation

$$\begin{cases} x' = p(t)x + q(t), & t \neq t_j := j\omega, \quad \text{for some given } \omega > 0, \\ x(t_j^+) = I(x(t_j^-)), & j \in \mathbb{Z}, \end{cases} \tag{3.7}$$

where $p, q\colon \mathbb{R} \to \mathbb{R}$ are measurable functions such that $p(t) \equiv p(t+\omega)$, $q(t) \equiv q(t+\omega)$, and the compact (continuous) impulsive function $I\colon \mathbb{R} \to \mathbb{R}$ satisfies $\overline{I(\mathbb{R})} = [a,b]$ and $I([a,b]) = [a,b]$.

Since the general solution of $x' = p(t)x + q(t)$ reads

$$x(t) = x(0)\,\mathrm{e}^{\int_0^t p(s)\,\mathrm{d}s} + \int_0^t \mathrm{e}^{\int_s^t p(r)\,\mathrm{d}r} q(s)\,\mathrm{d}s,$$

the required inclusion $[a,b] \subset [T_\omega(a), T_\omega(b)]$ in Corollary 3.6 takes the form

$$a \geq a\,\mathrm{e}^{\int_0^\omega p(t)\,\mathrm{d}t} + \int_0^\omega \mathrm{e}^{\int_s^\omega p(r)\,\mathrm{d}r} q(s)\,\mathrm{d}s,$$

$$b \leq b\,\mathrm{e}^{\int_0^\omega p(t)\,\mathrm{d}t} + \int_0^\omega \mathrm{e}^{\int_s^\omega p(r)\,\mathrm{d}r} q(s)\,\mathrm{d}s.$$

Specially, for $a = 0$, $b = 1$:

$$0 \geq \int_0^\omega \mathrm{e}^{\int_s^\omega p(r)\,\mathrm{d}r} q(s)\,\mathrm{d}s, \qquad 1 \leq \mathrm{e}^{\int_0^\omega p(t)\,\mathrm{d}t} + \int_0^\omega \mathrm{e}^{\int_s^\omega p(r)\,\mathrm{d}r} q(s)\,\mathrm{d}s.$$

In order to satisfy the first inequality, we can assume that $q(t) \leq 0$, for a.a. $t \in [0,\omega]$. The second inequality can be then more restrictively rewritten into

$$\mathrm{e}^{\int_0^\omega p(t)\,\mathrm{d}t} \geq 1 + \left| \int_0^\omega \mathrm{e}^{\int_0^\omega p(r)\,\mathrm{d}r} q(s)\,\mathrm{d}s \right|.$$

Denoting $P := \left|\int_0^\omega p(t)\,\mathrm{d}t\right|$ and $Q := \left|\int_0^\omega q(t)\,\mathrm{d}t\right|$, we can rewrite it finally as

$$\begin{gathered} \mathrm{e}^P(1-Q) \geq 1, \text{ resp. } Q \leq \frac{\mathrm{e}^P - 1}{\mathrm{e}^P}, \\ \text{jointly with } q(t) \leq 0, \text{ for a.a. } t \in [0,\omega]. \end{gathered} \tag{3.8}$$

Specially, for $p(t) \equiv p > 0$, we can require that

$$Q \leq \frac{\mathrm{e}^{p\omega} - 1}{\mathrm{e}^{p\omega}} \quad \text{and} \quad q(t) \leq 0,$$

for a.a. $t \in [0,\omega]$, or $-p\,\mathrm{e}^{-p\omega} \leq q(t) \leq 0$, for a.a. $t \in [0,\omega]$.

Thus, the linear impulsive equation (3.7) exhibits chaos in the sense of Definition 3.4, provided (3.8) holds jointly with $h\big(I|_{[0,1]}\big) > 0$.

The last inequality is satisfied, for instance, for the 1-periodically extended tent map $I(x) \equiv I(x+1)$, where

$$I(x) := \begin{cases} 2x, & \text{for } x \in \left[0, \tfrac{1}{2}\right], \\ 2(1-x), & \text{for } x \in \left[\tfrac{1}{2}, 1\right], \end{cases}$$

because $I(\mathbb{R}) = I([0,1]) = [0,1]$ and (cf. (3.6))

$$h\left(I|_{[T_\omega(0), T_\omega(1)]} \circ T_\omega|_{[0,1]}\right) = h\left(I|_{[0,1]}\right) = \log 2.$$

For the last inequality, see e.g. [19, Corollary 15.2.14].

Example 3.8. Consider the semi-linear impulsive equation

$$\begin{cases} x' = p(t,x)x + q(t,x), & t \neq t_j := j\omega, \quad \text{for some given } \omega > 0, \\ x(t_j^+) = I(x(t_j^-)), & j \in \mathbb{Z}, \end{cases} \tag{3.9}$$

where $p, q\colon \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions such that $p(t,x) \equiv p(t+\omega, x)$, $q(t,x) \equiv q(t+\omega, x)$, and the compact (continuous) impulsive function $I\colon \mathbb{R} \to \mathbb{R}$ satisfies $\overline{I(\mathbb{R})} = [a,b]$ and $I([a,b]) = [a,b]$.

Since the solutions $x_0(\cdot), x_1(\cdot)$ of $x' = p(t,x)x + q(t,x)$ such that $x_0(0) = 0$, $x_1(0) = 1$ can be implicitly expressed as

$$\begin{aligned} x_0(t) &= \int_0^t \mathrm{e}^{\int_s^t p(r,x_0(r))\,\mathrm{d}r} q(s, x_0(s))\,\mathrm{d}s, \\ x_1(t) &= \mathrm{e}^{\int_0^t p(s,x_1(s))\,\mathrm{d}s} + \int_0^t \mathrm{e}^{\int_s^t p(r,x_1(r))\,\mathrm{d}r} q(s, x_1(s))\,\mathrm{d}s, \end{aligned}$$

one can proceed in a similar way as in Example 3.7.

Hence, the required inclusion $[0,1] \subset [T_\omega(0), T_\omega(1)]$ (for $a = 0, b = 1$) in Corollary 3.6 takes this time the form

$$\begin{aligned} 0 &\geq \int_0^\omega \mathrm{e}^{\int_s^\omega p(r,x_0(r))\,\mathrm{d}r} q(s, x_0(s))\,\mathrm{d}s, \\ 1 &\leq \mathrm{e}^{\int_0^\omega p(t,x_1(t))\,\mathrm{d}t} + \int_0^\omega \mathrm{e}^{\int_s^\omega p(r,x_1(r))\,\mathrm{d}r} q(s, x_1(s))\,\mathrm{d}s. \end{aligned}$$

In order to satisfy the first inequality, we can assume that $q(t,x) \leq 0$, for a.a. $t \in [0,\omega]$ and all $x \in \mathbb{R}$. The second inequality can be then more restrictively rewritten into

$$\mathrm{e}^{\int_0^\omega p(t,x_1(t))\,\mathrm{d}t} \geq 1 + \left| \int_0^\omega \mathrm{e}^{\int_s^\omega p(r,x_1(r))\,\mathrm{d}r} q(s, x_1(s))\,\mathrm{d}s \right|.$$

Assuming still the existence of real constants p_0, p_1, q_1 such that

$$0 < p_0 \leq p(t,x) \leq p_1 \quad \text{and} \quad |q(t,x)| \leq q_1, \quad \text{for a.a. } t \in [0,\omega] \text{ and all } x \in \mathbb{R},$$

we still require that

$$q_1 \leq \frac{\mathrm{e}^{p_0\omega} - 1}{\omega\,\mathrm{e}^{p_1\omega}},$$

i.e. jointly with $q(t,x) \leq 0$,

$$-\frac{\mathrm{e}^{p_0\omega} - 1}{\omega\,\mathrm{e}^{p_1\omega}} \leq q(t,x) \leq 0, \quad \text{for a.a. } t \in [0,\omega] \text{ and all } x \in \mathbb{R}, \tag{3.10}$$

where $0 < p_0 \leq p(t,x)$, for a.a. $t \in [0,\omega]$ and all $x \in \mathbb{R}$.

Thus, the semi-linear impulsive equation (3.9) exhibits chaos in the sense of Definition 3.4, provided (3.10) holds jointly with $h\big(I|_{[0,1]}\big) > 0$. This inequality can be satisfied like in Example 3.7, for instance, for the 1-periodically extended tent map.

Now, we would like to apply Theorem 3.5 to the nonlinear vector impulsive differential equation (3.1).

Example 3.9. Consider (3.1), where F and I are as above, and assume that

$$\begin{cases} f_j(t,\dots,x_j,\dots) > 0 & \text{holds for all } x_j \geq b_j,\ j=1,\dots,n, \\ f_j(t,\dots,x_j,\dots) < 0 & \text{holds for all } x_j \leq a_j,\ j=1,\dots,n, \end{cases} \tag{3.11}$$

uniformly for a.a. $t \in [0,\omega]$ and all the remaining components of $x = (x_1,\dots,x_n)$, where $F(t,x) = (f_1(t,x),\dots,f_n(t,x))^T$ and $\left(\overline{I(\mathbb{R}^n)} = \right) K_0 := [a_1,b_1] \times \dots \times [a_n,b_n]$, $I(K_0) = K_0$.

Since, in view of (3.11), the inequalities $x_j(\omega,a_j) \leq a_j$ and $x_j(\omega,b_j) \geq b_j$, $j=1,\dots,n$, hold for all the components of the solutions $x(\cdot,a)$ and $x(\cdot,b)$ such that $x(0,a)=a$ and $x(0,b)=b$, where $a=(a_1,\dots,a_n)$, $b=(b_1,\dots,b_n)$, the particular inclusion $K_0 \subset K_1$ is satisfied, where $K_0 := [a_1,b_1] \times \dots \times [a_n,b_n]$ and $K_1 := T_\omega(K_0)$.

Thus, the vector impulsive equation (3.1) exhibits, according to Theorem 3.5, chaos in the sense of Definition 3.4, provided (3.11) holds jointly with $h\big(I|_{K_0}\big) > 0$, where $K_0 := [a_1,b_1] \times \dots \times [a_n,b_n]$. This inequality can be satisfied, for instance when $K_0 := [0,1]^n$ (i.e. for $[a_j,b_j] = [0,1]$, $j=1,\dots,n$), for the Cartesian product $\mathbf{I}$ of 1-periodically extended tent maps, because $\mathbf{I}([0,1]^n) = [0,1]^n$ and (see e.g. [26])

$$h\left(\mathbf{I}|_{K_1} \circ T_\omega|_{[0,1]^n}\right) = h\left(\mathbf{I}|_{[0,1]^n}\right) = n \log 2.$$

Remark 3.10. Observe that condition (3.11) imposed on the equations (3.7) and (3.9) takes the simple forms $p(t)+q(t) > 0$, $q(t) < 0$, for a.a. $t \in [0,\omega]$, resp. $p(t,1)+q(t,1) > 0$, $q(t,0) < 0$, for a.a. $t \in [0,\omega]$.

4 Topological entropy for impulsive differential equations on $\mathbb{R}^n/\mathbb{Z}^n$

Consider (3.1) and assume additionally that (2.6) holds jointly with

$$I(\dots,x_j,\dots) \equiv I(\dots,x_j+1,\dots) \pmod 1, \quad j=1,\dots,n, \tag{4.1}$$

where $x=(x_1,\dots,x_n)$.

Because of the commutative diagram

$$\begin{array}{ccccc} \mathbb{R}^n & \xrightarrow{T_\omega} & \mathbb{R}^n & \xrightarrow{I} & \mathbb{R}^n \\ \tau\downarrow & & \downarrow\tau & & \downarrow\tau \\ \mathbb{R}^n/\mathbb{Z}^n & \xrightarrow{\hat{T}_\omega} & \mathbb{R}^n/\mathbb{Z}^n & \xrightarrow{\hat{I}} & \mathbb{R}^n/\mathbb{Z}^n, \end{array}$$

where τ is the natural (canonical) projection, $\hat{T}_\omega := \tau \circ T_\omega \colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$, where $T_\omega \colon \mathbb{R}^n \to \mathbb{R}^n$ is the Poincaré translation operator along the trajectories of (2.4), defined in (2.5), and $\hat{I} = \tau \circ I : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$, where $I \colon \mathbb{R}^n \to \mathbb{R}^n$ is the impulsive mapping in (3.1), we can advantageously consider (3.1) on the torus $\mathbb{R}^n/\mathbb{Z}^n$, in the metric

$$\hat{d} \colon \mathbb{R}^n/\mathbb{Z}^n \times \mathbb{R}^n/\mathbb{Z}^n \to \left[0, \frac{\sqrt{n}}{2}\right],$$

where $\hat{d}(x,y) := \min\{d_{Eucl}(a,b) \colon a \in [x], b \in [y]\}$.

Since $\hat{T}_\omega$ is well known (see e.g. [10, Chapter XVII]) to be a homeomorphism and, in particular for $n = 1$, even an orientation-preserving homeomorphism, the composition

$$\widehat{I \circ T_\omega} := \hat{I} \circ \hat{T}_\omega \colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$$

is continuous in $\left(\mathbb{R}^n/\mathbb{Z}^n, \hat{d}\right)$.

We can therefore give the following analogy of Proposition 3.1 on $\mathbb{R}^n/\mathbb{Z}^n$.

Proposition 4.1. *The equality*

$$h\left(\widehat{I \circ T_\omega}\right) = h\left(\hat{I}\right) \tag{4.2}$$

holds, under the above assumptions and $\hat{I}\left(\mathbb{R}^n/\mathbb{Z}^n\right) = \mathbb{R}^n/\mathbb{Z}^n$, *for the topological entropies* h *of the maps* $\widehat{I \circ T_\omega} \colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ *and* $\hat{I} \colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ *in* $\left(\mathbb{R}^n/\mathbb{Z}^n, \hat{d}\right)$.

Proof. We can proceed analogously, but (since $\mathbb{R}^n/\mathbb{Z}^n$ is compact and $\hat{I}$ is "onto") in a simpler way, as in the proof of Proposition 3.1.

We have the diagram

$$\begin{tikzcd}
\mathbb{R}^n/\mathbb{Z}^n \arrow[r, "\hat{T}_\omega"] \arrow[d, "\hat{T}_\omega"'] & \mathbb{R}^n/\mathbb{Z}^n \arrow[r, "\hat{I}"] & \mathbb{R}^n/\mathbb{Z}^n \arrow[d, "\hat{T}_\omega"] \\
\mathbb{R}^n/\mathbb{Z}^n \arrow[r, "\hat{I}"] & \mathbb{R}^n/\mathbb{Z}^n \arrow[r, "\hat{T}_\omega"] \arrow[u, "\hat{T}_\omega"] & \mathbb{R}^n/\mathbb{Z}^n,
\end{tikzcd}$$

where $\hat{T}_\omega \colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is a homeomorphism and "onto".

Thus, according to Lemma 2.6, we obtain that

$$h\left(\hat{I} \circ \hat{T}_\omega\right) = h\left(\hat{T}_\omega \circ \hat{I}\right).$$

Endowing $\mathbb{R}^n/\mathbb{Z}^n$ with the new metric $\hat{d}'$, where

$$\hat{d}'(x, y) := \hat{d}\left(\hat{T}_\omega(x), \hat{T}_\omega(y)\right), \quad \text{for all} \quad x, y \in \mathbb{R}^n/\mathbb{Z}^n,$$

we have that

$$h\left(\widehat{I \circ T_\omega}\right) = h\left(\widehat{T_\omega \circ I}\right) = h_{\hat{d}'}(\hat{I}),$$

where the lower index $\hat{d}'$ denotes the respective metric. Furthermore, we get still, according to Lemma 2.2,

$$h_{\hat{d}'}(\hat{I}) = h(\hat{I}),$$

and, after all, that

$$h\left(\widehat{I \circ T_\omega}\right) = h(\hat{I}),$$

i.e. (4.2), as claimed. □

Definition 4.2. We say that the vector impulsive differential equation (3.1) exhibits on $\mathbb{R}^n/\mathbb{Z}^n$ (cf. also (2.6), (4.1)) *chaos in the sense of a positive topological entropy* h if $h\left(\widehat{I \circ T_\omega}\right) > 0$ holds for the map $\widehat{I \circ T_\omega} \colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ in $(\mathbb{R}^n/\mathbb{Z}^n, \hat{d})$, defined above.

Theorem 4.3. *The vector impulsive differential equation* (3.1) *exhibits on* $\mathbb{R}^n/\mathbb{Z}^n$, *under the above assumptions and additionally* (2.6), (4.1), *jointly with* $\hat{I}(\mathbb{R}^n/\mathbb{Z}^n) = \mathbb{R}^n/\mathbb{Z}^n$, *chaos in the sense of Definition* 4.2, *provided* $h(\hat{I}) > 0$ *holds for the impulsive mapping* $\hat{I} \colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ *in the metric* $\hat{d}$.

Proof. The proof follows directly from the equality (4.2) in Proposition 4.1. □

The following corollary can help us to calculate effectively the topological entropy $h(\hat{I})$, and to ensure chaos for (3.1) on $\mathbb{R}^n/\mathbb{Z}^n$ (cf. [6, Theorem 5.2]).

Corollary 4.4. *Let $\hat{I}\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ be defined by an integer matrix A, whose eigenvalues are $\lambda_1, \dots, \lambda_n$. Then*

$$h(\hat{I}) = \sum_{|\lambda_k|>1} \log|\lambda_k|$$

holds for the topological entropy of $\hat{I}$, provided $\prod_{k=1}^n (1-\lambda_k) \neq 0$. Therefore, if

$$\sum_{|\lambda_k|>1} \log|\lambda_k| > 0 \quad \text{and} \quad \prod_{k=1}^{n} (1-\lambda_k) \neq 0,$$

then (3.1) exhibits on $\mathbb{R}^n/\mathbb{Z}^n$ under (2.6) chaos in the sense of Definition 4.2.

Proof. The first assertion is well known (see e.g. [26, p. 203] and cf. the preliminaries in Section 2). The second part is, on this basis, an immediate consequence of Theorem 4.3. □

Example 4.5. As an illustrative example of the application of Corollary 4.4, let us consider (3.1) on $\mathbb{R}^2/\mathbb{Z}^2$ (i.e. for $n = 2$), when assuming (2.6). Let $\hat{I}\colon \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ be defined by the integer matrix A, whose real eigenvalues are one, say λ_1, of modulus $|\lambda_1| > 1$ and the other, say λ_2, with $|\lambda_2| < 1$. For instance, A can take the form,

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix},$$

because $\lambda_1 = 1 + \sqrt{2}$, $\lambda_2 = 1 - \sqrt{2}$, and so $(1-\lambda_1)(1-\lambda_2) = -2$, and $|\lambda_1| = |1+\sqrt{2}| > 1$, $|\lambda_2| = |1-\sqrt{2}| < 1$.

Then $h(\hat{I}) = \log|\lambda_1|) = \log(1+\sqrt{2}) > 0$, and (3.1) exhibits on $\mathbb{R}^2/\mathbb{Z}^2$, according to Corollary 4.4, chaos in the sense of Definition 4.2.

Observe that since $\lambda(\hat{I}) = (1-\lambda_1)(1-\lambda_2) \neq 0$ holds for the Lefschetz number, we obtain according to (2.3) that $N^\infty(\hat{I}) = |\lambda_1| = 1 + \sqrt{2}$, and subsequently (see (2.1)) $h(\hat{I}) \geq \log|\lambda_1| > 0$, with the same conclusion for (3.1).

Theorem 4.3 can be modified by means of Proposition 2.7 as follows.

Theorem 4.6. *Consider, under the above assumptions and (2.6), (4.1), jointly with $\hat{I}(\mathbb{R}^n/\mathbb{Z}^n) = \mathbb{R}^n/\mathbb{Z}^n$, the vector impulsive differential equation (3.1) on $\mathbb{R}^n/\mathbb{Z}^n$. Assume that the impulsive mapping $\hat{I}\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is homotopic to a continuous map $f\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ such that $N^\infty(f) > 1$, i.e. (see (2.2))*

$$\limsup_{m\to\infty} |\lambda(f^m)|^{\frac{1}{m}} > 1,$$

where $\lambda(f^m)$ stands for the Lefschetz number of the m-th iterate of f.

Then $h(\hat{I}) \geq \limsup_{m\to\infty} \frac{1}{m} \log N(f^m) > 0$ holds, where $N(f^m)$ denotes the Nielsen number of the m-th iterate of f, and subsequently equation (3.1) exhibits on $\mathbb{R}^n/\mathbb{Z}^n$ chaos in the sense of Definition 4.2.

Proof. The proof follows directly from Theorem 4.3, on the basis of Proposition 2.7 and Remark 2.8. □

Example 4.7. Consider the scalar ($n = 1$) impulsive differential equation (3.1) on $\mathbb{R}/\mathbb{Z}$, when assuming (2.6). Let $\hat{I}\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the doubling impulsive mapping, where

$$\hat{I} := \begin{cases} 2x, & \text{for } x \in \left[0, \frac{1}{2}\right], \\ 2x - 1, & \text{for } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Since one can easily check that (see e.g. [6])

$$N(\hat{I}^k) = \left|\lambda(\hat{I}^k)\right| = \left|1 - 2^k\right|, \quad k \in \mathbb{N},$$

holds for the Nielsen and Lefschetz numbers, we obtain that

$$N^\infty(\hat{I}) = \limsup_{m\to\infty} |\lambda(\hat{I}^m)|^{\frac{1}{m}} = \limsup_{m\to\infty} |1 - 2^m|^{\frac{1}{m}} > 1.$$

Thus, applying Theorem 4.6, $h(\hat{I}) > 0$ holds, and (3.1) exhibits on $\mathbb{R}/\mathbb{Z}$ chaos in the sense of Definition 4.2.

According to Corollary 4.4, we have $h(\hat{I}) = \log 2$, and the same conclusion.

Remark 4.8. Observe that if $I\colon [0,1] \to [0,1]$ is the standard tent map defined in Example 3.7, resp. its 1-periodic extension, then $\hat{I} := \tau \circ I\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ takes the same form as I. Thus, $h(\hat{I}) = \log 2$, which is sufficient for the application of Theorem 4.3. On the other hand,

$$N(\hat{I}^k) = \left|\lambda(\hat{I}^k)\right| = 1, \quad k \in \mathbb{N},$$

holds this time, which excludes the application of Theorem 4.6.

Example 4.9. Consider the scalar linear impulsive equation (3.7) with $p(t) \equiv 0$, i.e.

$$\begin{cases} x' = q(t), \quad t \neq t_j := j\omega, \quad \text{for some given } \omega > 0, \\ x(t_j^+) = I(x(t_j^-)), \quad j \in \mathbb{Z}, \end{cases} \tag{4.3}$$

where $q\colon \mathbb{R} \to \mathbb{R}$ is a measurable function such that $q(t) \equiv q(t+\omega)$ and $\frac{1}{\omega}\int_0^\omega q(t)\,\mathrm{d}t = 0$.

(i) One can easily check that (4.3) exhibits, according to Theorem 3.5, chaos in the sense of Definition 3.4, provided the continuous impulsive function $I\colon \mathbb{R} \to \mathbb{R}$ is compact, $I(K_0) = K_0$ and such that $h(I|_{K_0}) > 0$, where $K_0 := \overline{I(\mathbb{R})}$.

(ii) Furthermore, (4.3) exhibits on $\mathbb{R}/\mathbb{Z}$, according to Theorem 4.3, chaos in the sense of Definition 4.2, provided the continuous impulsive function $I\colon \mathbb{R} \to \mathbb{R}$ satisfies $I(x) \equiv I(x+1)(\operatorname{mod} 1)$, $\hat{I}(\mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z}$, and $h(\hat{I}) > 0$, where $\hat{I} := \tau \circ I\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$.

(iii) At last, (4.3) exhibits on $\mathbb{R}/\mathbb{Z}$, according to Theorem 4.6, chaos in the sense of Definition 4.2, provided the continuous impulsive function $I\colon \mathbb{R} \to \mathbb{R}$ satisfies $I(x) \equiv I(x+1)(\operatorname{mod} 1)$, $\hat{I}(\mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z}$, and $\hat{I}$ is homotopic to $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ (i.e. $\hat{I} \sim f$) such that

$$\limsup_{m\to\infty} |\lambda(f^m)|^{\frac{1}{m}} > 1,$$

where $\lambda(f^m)$ stands for the Lefschetz number of the m-th iterate of f.

Remark 4.10. One can easily check that since for the 1-periodically extended tent map $I\colon \mathbb{R} \to [0,1]$, defined in Example 3.7, $h(I|_{[0,1]}) = h(\hat{I}) = \log 2\,(> 0)$ and $\limsup_{m\to\infty} \left|\lambda(\hat{I}^m)\right|^{\frac{1}{m}} = 1$, Theorem 3.5 and 4.3 apply in (i),(ii), while Theorem 4.6 does not apply in (iii). On the other hand, since for the doubling map $I := 2x\colon \mathbb{R} \to \mathbb{R}$, we have $I(\mathbb{R}) = \mathbb{R}$, $h(\hat{I}) = \log 2$ and $\limsup_{m\to\infty} \left|\lambda(\hat{I}^m)\right|^{\frac{1}{m}} = \limsup_{m\to\infty} |1 - 2^m|^{\frac{1}{m}} > 1$, Theorems 4.3 and 4.6 apply in (ii), (iii), while Theorem 3.5 does not apply in (i).

5 Concluding remarks

It is well known that (see e.g. the main theorem in [21]), for continuous maps on compact intervals, a positive topological entropy is equivalent with Devaney's chaos on a closed invariant subset, i.e. (i) topological transitivity, (ii) density of periodic points, (iii) sensitive dependence on initial conditions. Moreover, transitivity implies period six (see e.g. [12]), and subsequently (in view of the celebrated Sharkovsky cycle coexistence theorem, cf. e.g. [2, Theorem 2.1.1]) the coexistence of $2k$-periodic points, for every $k \in \mathbb{N}$. Reversely, the existence of a periodic point with period $k \neq 2^n$, $n \in \mathbb{N} \cup \{0\}$, implies according to the theorem of Boven and Franks (see e.g. [2, Theorem 4.4.20]), a positive topological entropy, and subsequently Devaney's chaos on a closed invariant subset. The same, except the information about period six, but "only" with period $k \neq 2^n$, $n \in \mathbb{N} \cup \{0\}$, is true for continuous maps on a circle, provided they possess a fixed point (see e.g. [2]).

Thus, many results for scalar ($n = 1$) impulsive differential equations about Devaney's chaos and the coexistence of periodic solutions with various periods, including those of the type $k \neq 2^n$, $n \in \mathbb{N} \cup \{0\}$, can be also interpreted in terms of a positive topological entropy.

In higher ($n > 1$) dimensions, the situation is more delicate. Nevertheless, the coexistence of infinitely many periodic solutions is also there, in view of Lemma 2.5, a necessary condition for a positive topological entropy.

Under the assumptions of Corollary 4.4, we are able to prove like in [4, Theorem 4.3] the coexistence of $k\omega$-periodic $(\mathrm{mod}\, 1)$ solutions of (3.1), for infinitely many $k \in \mathbb{N}$, including those for $k \neq 2^n$, $n \in \mathbb{N} \cup \{0\}$.

In this light, at least the results about topological entropy for impulsive differential equations, obtained in higher dimensions, seem to be original.

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Nonlocal boundary value problems with BV-type data

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Only a fool can celebrate the years of upcoming death (George Bernard Shaw)

However, for you we make an exception: *Happy birthday, dear Jeff!*

Abstract. In this paper we present some existence and uniqueness results for solutions of second order boundary value problems, which are functions of bounded variation along with their derivatives. To this end, we apply fixed point theorems to an equivalent nonlinear perturbed Hammerstein integral equation. Here we consider non- standard boundary conditions like coupled boundary conditions, uncoupled boundary conditions, or integral-type boundary conditions. We also prove an abstract result concerning the spectral radii of some general classes of operators which applies to all boundary value problems mentioned above. The abstract results are throughout illustrated by a large number of examples.

Keywords: boundary value problem, bounded variation, spectral radius.

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1 Nonlocal boundary value problems

It is well known that nonlinear boundary value problems (BVPs) are closely related to Hammerstein integral equations, while nonlinear initial value problems (IVPs) are closely related to Volterra–Hammerstein integral equations. Since a linear Volterra operator has often spectral radius zero, solutions of IVPs are usually much easier to obtain than solutions of BVPs.

During the last decades, so-called nonlocal BVPs have found growing attention, mainly in view of their generality and applicability. In a very general formulation, a second-order

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nonlinear equation with nonlocal boundary conditions has the form [7]

$$\begin{aligned} &x''(t) + p(t)x'(t) + q(t)x(t) + r(t)g(t, x(t)) = 0 \qquad (0 \le t \le 1), \\ &ax(0) - bx'(0) = \alpha[x], \qquad cx(1) + dx'(1) = \beta[x]. \end{aligned} \tag{1.1}$$

Here $p, q, r : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are given functions, and $\alpha, \beta : C[0, 1] \to \mathbb{R}$ are linear functionals which are expressed by Riemann–Stieltjes integrals. The well-known multi-point BVPs are a special case of the problem (1.1).

Many important contributions to this problem have been given during the last 20 years by Webb [5–16], and Webb with Infante [2–4, 17–22]. While there is a vast literature on continuously differentiable solutions, considerably less is known on solutions with derivatives of bounded variation, although such solutions (e.g., monotone or convex solutions) have some interest in applications. An exception is the recent paper [1], where the authors prove, under suitable hypotheses, the existence of a continuous solution of bounded variation of the equation

$$x(t) = \alpha[x]v(t) + \beta[x]w(t) + \lambda \int_0^1 k(t, s)g(s, x(s))\, ds \qquad (0 \le t \le 1), \tag{1.2}$$

building on a variant of Krasnosel'skij's fixed point principle. We will study a similar equation and look for solutions with derivatives of bounded variation.

So in this paper we are going to consider the classical space BV equipped with the usual norm

$$\|x\|_{BV} = |x(0)| + \mathrm{Var}(x; [0, 1]), \tag{1.3}$$

where $\mathrm{Var}(x; [0, 1])$ denotes the total Jordan variation of x on the interval $[0, 1]$, as well as the higher order space

$$BV^m := \{x \in BV : x', x'', \dots, x^{(m)} \in BV\},$$

equipped with the natural norm

$$\|x\|_{BV^m} = |x(0)| + \sum_{k=1}^{m} \|x^{(k)}\|_{BV}.$$

Observe that there is a peculiarity in the spaces BV^m for $m \ge 1$. Given $x \in BV^m$, the derivative $x^{(m)}$ belongs to BV and so can have only removable discontinuities or jumps; however, the well-known Darboux intermediate value theorem excludes such discontinuities. So the inclusion $BV^m \subseteq C^m$ holds, although the analogous "zero level" inclusion $BV \subseteq C$ is of course far from being true.

We will also need the space AC^m of all functions which have absolutely continuous derivatives up to order m, equipped with the norm inherited from BV^m. By the classical Vitali–Banach–Zaretskij theorem, the relation with the other spaces is then given by

$$AC^m \subset BV^m \subset C^m \ (m \ge 1), \qquad AC \subset BV \cap C \subset C, \tag{1.4}$$

where all inclusions are strict. In what follows, we will look for solutions $x \in AC^{m-1}$ of an m-th order nonlinear differential equation with nonlocal boundary conditions, with a particular emphasis on examples which illustrate how far our sufficient solvability conditions are from being necessary. If there are more than one sufficient condition we will also show their independence, in the sense that none of them implies the others.

2 Boundary value problems with BV data

To begin with, let us discuss the second order equation

$$x''(t) + \lambda g(t, x(t)) = 0 \qquad (0 \le t \le 1), \tag{2.1}$$

subject to the coupled boundary conditions

$$x(0) = \alpha[x], \qquad x(1) = \beta[x], \tag{2.2}$$

where $\alpha, \beta : BV \to \mathbb{R}$ are given linear functionals. This means that we take $p(t) = q(t) \equiv 0$, $r(t) \equiv 1$, $a = c = 1$, and $b = d = 0$ in (1.1). Occasionally, we will also consider more general data. Throughout this paper we suppose that the nonlinearity g in (2.1) satisfies the three hypotheses

(H1) $g(\cdot, u)$ is measurable for all $u \in \mathbb{R}$;

(H2) for each $R > 0$ there exists $a_R \in L_\infty[0, 1]$ such that $|g(t, u)| \le a_R(t)$ for $0 \le t \le 1$ and $|u| \le R$;

(H3) $g(t, \cdot) \in C(\mathbb{R})$ for almost all $t \in [0, 1]$.

In the sequel we refer to the problem (2.1)/(2.2) by the symbol (BVP). In order to solve this problem, we consider along with (BVP) the Hammerstein integral equation

$$x(t) = Ax(t) + \lambda \int_0^1 \kappa(t, s) g(s, x(s))\, ds \qquad (0 \le t \le 1), \tag{2.3}$$

where

$$\kappa(t, s) = \begin{cases} s(1-t) & \text{for } 0 \le s \le t \le 1, \\ t(1-s) & \text{for } 0 \le t < s \le 1 \end{cases}$$

is the usual Green's function of the second order derivative, and A is a linear operator (to be specified below) from BV into itself. The bridge between (2.3) and our (BVP) is built by our first result

Proposition 2.1. *Let $A : BV \to BV$ be defined by*

$$Ax(t) := (1-t)\alpha[x] + t\beta[x] \qquad (0 \le t \le 1). \tag{2.4}$$

Then the following holds.

(a) Every function $x \in BV$ solving (2.3) belongs to AC^1 and solves (BVP) almost everywhere on $[0, 1]$.

(b) If, in addition, g is continuous on $[0, 1] \times \mathbb{R}$, then every solution x of (2.3) is of class C^2 and solves (BVP) everywhere on $[0, 1]$.

(c) Conversely, if $x \in AC^1$ solves (BVP) almost everywhere on $[0, 1]$, then x is a solution of the integral equation (2.3).

Proof. (a) Assume that (2.3) is satisfied for some $x \in BV$ and some $\lambda \in \mathbb{R}$. First observe that $h(s) := g(s, x(s))$ belongs to L_∞ because of our hypotheses (H1)/(H2)/(H3). Moreover, the function $\varphi : [0, 1] \to \mathbb{R}$ defined by

$$\varphi(t) := \int_0^1 \kappa(t, s) g(s, x(s))\, ds = (1-t) \int_0^t s h(s)\, ds - t \int_1^t (1-s) h(s)\, ds$$

belongs to AC with

$$\varphi'(t) = -\int_0^t sh(s)\,ds - \int_1^t (1-s)h(s)\,ds$$

for almost all $t \in [0,1]$. But since the right hand side is again in AC we conclude that $\varphi \in AC^1$. Moreover,

$$\varphi''(t) = -th(t) - (1-t)h(t) = -h(t)$$

for almost all $t \in [0,1]$. In addition, by the definition (2.4) of A the function Ax is affine and hence of class C^2 with $(Ax)'' = 0$. From (2.3) it follows that

$$x(t) = Ax(t) + \lambda\varphi(t) \qquad (0 \le t \le 1);$$

in particular, this shows that (2.1) holds indeed almost everywhere in $[0,1]$. Moreover, since $\varphi(0) = \varphi(1) = 0$, $Ax(0) = \alpha[x]$, and $Ax(1) = \beta[x]$ the first part of the proof is complete.

(b) If, in addition, g is continuous, then so must be x'' which means that x is of class C^2 and solves (BVP) everywhere on $[0,1]$.

(c) Putting $h(t) = g(t, x(t))$ as before and integrating (2.1) twice over $[0,t]$ we obtain

$$x(t) = x(0) + x'(0)t - \lambda \int_0^t (t-s)h(s)\,ds.$$

Evaluating this at $t = 1$ yields

$$x'(0) = \beta[x] - \alpha[x] + \lambda \int_0^1 (1-s)h(s)\,ds$$

which gives the desired result. □

Observe that the problem discussed in Proposition 2.1 is of the form (1.2) with $v(t) = 1-t$ and $w(t) = t$. The inclusions (1.4) suggest that we cannot expect the solution of (2.3) to lie in BV^2, unless the function g is continuous.

Proposition 2.1 shows that the problem of solving (BVP) may be reduced to finding solutions $x \in BV$ of the Hammerstein equation (2.3). Of course, the structure of (2.3) suggests to use fixed point theorems, such as the Banach–Caccioppoli contraction mapping principle, the Schauder fixed point principle, or the Krasnosel'skij fixed point principle which is a combination of both. To this end, we have to make sure that the two functionals $\alpha, \beta \in BV^*$ behave in such a way that the norm $\|A^n\|_{BV\to BV}$ of the iterates A^n of the operator (2.4) shrinks below 1 for some $n \in \mathbb{N}$, and the integral operator in (2.3) is compact. Two conditions which fulfill the first requirement are given in the following

Theorem 2.2. *Assume that the two functionals $\alpha, \beta \in BV^*$ satisfy one of the conditions*

$$\|\alpha\|_{BV^*} + \|\alpha - \beta\|_{BV^*} < 1 \tag{2.5}$$

or

$$\alpha[e_0] = \beta[e_0] = 0, \qquad |\alpha[e_1] - \beta[e_1]| < 1, \tag{2.6}$$

where

$$e_k(t) := t^k \qquad (0 \le t \le 1,\ k = 0, 1, 2, \ldots). \tag{2.7}$$

Then for each $R > 0$ there is some $\rho > 0$ such that (BVP) has, for fixed $\lambda \in (-\rho, \rho)$, a solution $x \in AC^1$ satisfying $\|x\|_{BV} \le R$. If, in addition, g is continuous on $[0,1] \times \mathbb{R}$, then every such solution is of class C^2.

Proof. Define A as in Proposition 2.1, that is, $Ax = \alpha[x](e_0 - e_1) + \beta[x]e_1$. Since α and β are supposed to be bounded and linear, so is A. We show for either of the two options (2.5) and (2.6) that there is some $n \in \mathbb{N}$ such that $\|A^n\|_{BV\to BV} < 1$. Once this is done, standard solvability results for (2.3) give the claim. By Proposition 2.1, the solution x belongs to AC^1, has the correct boundary values according to (2.2), and satisfies (2.1) almost everywhere. If g is continuous, it follows easily from Proposition 2.1 (b) that x is then of class C^2.

So we claim that $\|A^n\|_{BV\to BV} < 1$ for some $n \in \mathbb{N}$ provided that α and β satisfy (2.5) or (2.6). Let us start with (2.5). For any $x \in BV$ we have

$$\begin{aligned}\|Ax\|_{BV} &= \|\alpha[x]e_0 + (\beta[x] - \alpha[x])e_1\|_{BV} \le \|e_0\|_{BV}\big|\alpha[x]\big| + \|e_1\|_{BV}\big|\beta[x] - \alpha[x]\big| \\ &\le \|\alpha\|_{BV^*}\|x\|_{BV} + \|\alpha - \beta\|_{BV^*}\|x\|_{BV},\end{aligned}$$

since $\|e_0\|_{BV} = \|e_1\|_{BV} = 1$. Consequently,

$$\|A\|_{BV\to BV} \le \|\alpha\|_{BV^*} + \|\alpha - \beta\|_{BV^*} < 1,$$

by (2.5), showing that A is a contraction. In this case, we may take $n = 1$.

We now assume that α and β satisfy option (2.6). Note that in this case, $Ae_0 = 0$. By induction, we first prove that the iterates of A are given by

$$A^{n+2}x = (\alpha[e_1](e_0 - e_1) + \beta[e_1]e_1)(\beta[e_1] - \alpha[e_1])^n(\beta[x] - \alpha[x]) \tag{2.8}$$

for $x \in BV$ and $n \in \mathbb{N}_0$, where we set $0^0 := 1$. First, using (2.6) we get

$$\begin{aligned}A(Ax) &= A(\alpha[x](e_0 - e_1) + \beta[x]e_1) = \alpha[x]A(e_0 - e_1) + \beta[x]Ae_1 \\ &= \alpha[x](\alpha[e_0](e_0 - e_1) + \beta[e_0]e_1) + (\beta[x] - \alpha[x])(\alpha[e_1](e_0 - e_1) + \beta[e_1]e_1) \\ &= (\alpha[e_1](e_0 - e_1) + \beta[e_1]e_1)(\beta[x] - \alpha[x]),\end{aligned}$$

and this is (2.8) for $n = 0$. Moreover,

$$\begin{aligned}\beta[Ax] - \alpha[Ax] &= (\beta - \alpha)[\alpha[x](e_0 - e_1) + \beta[x]e_1] \\ &= (\beta - \alpha)[e_0 - e_1]\alpha[x] + (\beta - \alpha)[e_1]\beta[x] = (\beta[e_1] - \alpha[e_1])(\beta[x] - \alpha[x]).\end{aligned}$$

From this we deduce that if (2.8) has been proved for some $n \in \mathbb{N}_0$, then

$$\begin{aligned}A^{n+3}x = A^{n+2}(Ax) &= (\alpha[e_1](e_0 - e_1) + \beta[e_1]e_1)(\beta[e_1] - \alpha[e_1])^n(\beta[Ax] - \alpha[Ax]) \\ &= (\alpha[e_1](e_0 - e_1) + \beta[e_1]e_1)(\beta[e_1] - \alpha[e_1])^{n+1}(\beta[x] - \alpha[x]).\end{aligned}$$

By induction, (2.8) is established. As a consequence we get for $n \ge 2$

$$\|A^n\|_{BV\to BV} \le \|\alpha[e_1](e_0 - e_1) + \beta[e_1]e_1\|_{BV}\big|\beta[e_1] - \alpha[e_1]\big|^{n-2}(\|\alpha\|_{BV^*} + \|\beta\|_{BV^*}).$$

Since $\big|\beta[e_1] - \alpha[e_1]\big| < 1$, by (2.6), and this is the only term depending on n, we find some $n \in \mathbb{N}$ such that $\|A^n\|_{BV\to BV} < 1$ as claimed. □

To illustrate the applicability of Theorem 2.2, we give now two examples. In the first example condition (2.5) works, but (2.6) does not, while in the second example condition (2.6) works, but (2.5) does not. Recall that we impose throughout the hypotheses (H1)/(H2)/(H3) on g.

Example 2.3. Consider the BVP

$$\begin{cases} x''(t) + \lambda g(t, x(t)) = 0 \quad (0 \le t \le 1), \\ x(0) = \frac{1}{7}x\left(\frac{1}{2}\right) + \frac{1}{6}x\left(\frac{2}{3}\right), \\ x(1) = \frac{1}{7}x\left(\frac{1}{4}\right) + \frac{1}{6}x\left(\frac{4}{5}\right). \end{cases} \tag{2.9}$$

The functionals

$$\alpha[x] := \frac{1}{7}x\left(\frac{1}{2}\right) + \frac{1}{6}x\left(\frac{2}{3}\right), \qquad \beta[x] := \frac{1}{7}x\left(\frac{1}{4}\right) + \frac{1}{6}x\left(\frac{4}{5}\right)$$

are obviously linear and bounded on BV and satisfy

$$\left|\alpha[x]\right| \le \frac{1}{7}\|x\|_\infty + \frac{1}{6}\|x\|_\infty \le \frac{13}{42}\|x\|_{BV}$$

where $\|\cdot\|_\infty$ denotes the supremum norm, and

$$\begin{aligned} \left|\alpha[x] - \beta[x]\right| &= \left|\frac{1}{7}\left[x\left(\frac{1}{2}\right) - x\left(\frac{1}{4}\right)\right] + \frac{1}{6}\left[x\left(\frac{2}{3}\right) - x\left(\frac{4}{5}\right)\right]\right| \\ &\le \frac{1}{7}\operatorname{Var}(x;[0,1]) + \frac{1}{6}\operatorname{Var}(x;[0,1]) \le \frac{13}{42}\|x\|_{BV}. \end{aligned}$$

Consequently,

$$\|\alpha\|_{BV^*} + \|\alpha - \beta\|_{BV^*} \le \frac{13}{21} < 1,$$

which means that α and β satisfy option (2.5) of Theorem 2.2. We conclude that (2.9) has for small $|\lambda|$ an AC^1-solution. On the other hand, α and β do not satisfy option (2.6), as $\alpha[e_0] = \beta[e_0] = 13/42 \ne 0$.

Example 2.4. Consider the BVP

$$\begin{cases} x''(t) + \lambda g(t, x(t)) = 0 \quad (0 \le t \le 1), \\ x(0) = 3x\left(\frac{1}{2}\right) - 3x\left(\frac{2}{3}\right), \\ x(1) = 2x\left(\frac{1}{4}\right) - 2x\left(\frac{4}{5}\right). \end{cases} \tag{2.10}$$

The functionals

$$\alpha[x] := 3x\left(\frac{1}{2}\right) - 3x\left(\frac{2}{3}\right), \qquad \beta[x] := 2x\left(\frac{1}{4}\right) - 2x\left(\frac{4}{5}\right)$$

are obviously linear and bounded on BV and satisfy

$$\alpha[e_0] = \beta[e_0] = 0, \qquad \left|\alpha[e_1] - \beta[e_1]\right| = \left|\frac{3}{2} - 2 - \frac{1}{2} + \frac{8}{5}\right| = \frac{3}{5} < 1,$$

which means that α and β satisfy option (2.6) of Theorem 2.2. We conclude that (2.10) has for small $|\lambda|$ an AC^1-solution. On the other hand, α and β do not satisfy option (2.5), because

$$\|\chi_{[0,1/2]}\|_{BV} = 1 + 1 = 2, \qquad \alpha[\chi_{[0,1/2]}] = 3,$$

and so $\|\alpha\|_{BV^*} \ge 3/2 > 1$.

3 A refinement of Theorem 2.2

The preceding two examples show that the crucial conditions (2.5) and (2.6) in Theorem 2.2 are independent. As one could expect, there exist BVPs where neither (2.5) nor (2.6) can be used. Here is a simple example.

Example 3.1. Consider the BVP

$$\begin{cases} x''(t) + \lambda g(t, x(t)) = 0 \quad (0 \le t \le 1), \\ x(0) = x\left(\frac{1}{3}\right) + x\left(\frac{2}{3}\right), \\ x(1) = -\frac{1}{2}x\left(\frac{1}{3}\right) - \frac{1}{2}x\left(\frac{2}{3}\right). \end{cases} \tag{3.1}$$

The functionals

$$\alpha[x] := x\left(\tfrac{1}{3}\right) + x\left(\tfrac{2}{3}\right), \qquad \beta[x] := -\tfrac{1}{2}x\left(\tfrac{1}{3}\right) - \tfrac{1}{2}x\left(\tfrac{2}{3}\right)$$

are obviously linear and bounded on BV. However, $\alpha[e_0] = 2 \neq 0$, so option (2.6) cannot be used. The same relation shows that $\|\alpha\|_{BV^*} \ge 2$, and so option (2.5) cannot be used either.

In view of Example 3.1 the question arises how to generalize the ideas of Theorem 2.2 in order to cover a larger range of applications. Due to the special structure of the linear operator (2.4) it is possible to give an exact formula for its spectral radius. For this purpose we prove now an abstract result about the spectral radius of an even slightly more general class of operators which might be of interest on its own.

Proposition 3.2. *Let $(X, \|\cdot\|)$ be a Banach space, $v, w \in X$ fixed, and $\alpha, \beta \in X^*$. Define $A : X \to X$ by*

$$Ax := \alpha[x]v + \beta[x]w \qquad (x \in X). \tag{3.2}$$

Then the matrix

$$\mathcal{A} := \begin{pmatrix} \alpha[v] & \beta[v] \\ \alpha[w] & \beta[w] \end{pmatrix} \in \mathbb{R}^{2\times 2} \tag{3.3}$$

and the operator A have the same spectral radius.

Proof. We first show that $\mathfrak{R}(A) \le \mathfrak{R}(\mathcal{A})$, where $\mathfrak{R}$ denotes the spectral radius, by means of the classical Gel'fand formula. The iterates of A can be written in the form

$$A^n x = \alpha_n[x]v + \beta_n[x]w \quad (x \in X),$$

where $\alpha_n, \beta_n \in X^*$ satisfy for all $x \in X$ the linear recursions

$$\alpha_1[x] := \alpha[x], \qquad \alpha_{n+1}[x] = \alpha_n[v]\alpha[x] + \alpha_n[w]\beta[x] \tag{3.4}$$

and

$$\beta_1[x] := \beta[x], \qquad \beta_{n+1}[x] = \beta_n[v]\alpha[x] + \beta_n[w]\beta[x]. \tag{3.5}$$

Indeed, once the formula for A^n has been established, we get

$$\begin{aligned} A^{n+1}x = A^n(Ax) &= \alpha_n[Ax]v + \beta_n[Ax]w \\ &= (\alpha_n[v]\alpha[x] + \alpha_n[w]\beta[x])v + (\beta_n[v]\alpha[x] + \beta_n[w]\beta[x])w \\ &= \alpha_{n+1}[x]v + \beta_{n+1}[x]w. \end{aligned}$$

Plugging v and w for x into the recursion formulas (3.4) and (3.5) we see that the four numbers $\alpha_n[v]$, $\alpha_n[w]$, $\beta_n[v]$ and $\beta_n[w]$ in turn satisfy the matrix recursions $B_{n+1} = \mathcal{A}B_n$, where

$$B_k := \begin{pmatrix} \alpha_k[v] & \beta_k[v] \\ \alpha_k[w] & \beta_k[w] \end{pmatrix}.$$

Thus, $B_1 = \mathcal{A}$ and, more generally, $B_k = \mathcal{A}^k$. Setting

$$M := \max \left\{ \|v\| \, \|\alpha\|_{X^*}, \|v\| \, \|\beta\|_{X^*}, \|w\| \, \|\alpha\|_{X^*}, \|w\| \, \|\beta\|_{X^*} \right\},$$

our recursion for A^{n+1} implies

$$\begin{aligned} \|A^{n+1}\|_{X\to X} &\le \|v\| \left(|\alpha_n[v]| \, \|\alpha\|_{X^*} + |\alpha_n[w]| \, \|\beta\|_{X^*} \right) + \|w\| \left(|\beta_n[v]| \, \|\alpha\|_{X^*} + |\beta_n[w]| \, \|\beta\|_{X^*} \right) \\ &\le M \left(|\alpha_n[v]| + |\alpha_n[w]| + |\beta_n[v]| + |\beta_n[w]| \right) \le 2M\|B_n\|_\infty = 2M\|\mathcal{A}^n\|_\infty, \end{aligned}$$

where $\|\cdot\|_\infty$ denotes the row sum norm of a matrix. Taking the n-th root in this estimate, Gel'fand's formula yields

$$\mathfrak{R}(A) = \lim_{n\to\infty} \|A^{n+1}\|_{X\to X}^{1/n} \le \lim_{n\to\infty} (2M\|\mathcal{A}^n\|_\infty)^{1/n} = \mathfrak{R}(\mathcal{A}).$$

We now prove the reverse estimate and distinguish the two cases when the set $\{v, w\}$ is linearly dependent or linearly independent in X.

<u>1st case</u>: Assume $w = \lambda v$ for some $\lambda \in \mathbb{R}$. In this case the matrix (3.3) reads

$$\mathcal{A} = \begin{pmatrix} \alpha[v] & \beta[v] \\ \lambda\alpha[v] & \lambda\beta[v] \end{pmatrix},$$

so $\mathfrak{R}(\mathcal{A}) = |\alpha[v] + \lambda\beta[v]|$. Moreover, the functional $\gamma := \alpha + \lambda\beta \in X^*$ satisfies

$$Ax = \gamma[x]v, \quad A^2x = \gamma[v]\gamma[x]v,$$
$$A^3x = \gamma[v]^2\gamma[x]v, \; \ldots, \; A^nx = \gamma[v]^{n-1}\gamma[x]v$$

for all $n \in \mathbb{N}$ and $x \in X$. In case $v = o$ we also have $w = o$, hence $\mathfrak{R}(A) = \mathfrak{R}(\mathcal{A}) = 0$. We therefore assume $v \neq o$. If $\gamma[x] = 0$ for all $x \in X$ we have $\alpha = -\lambda\beta$ which implies, on the one hand, $Ax = 0$, hence $\mathfrak{R}(A) = 0$, and

$$\mathcal{A} = \begin{pmatrix} -\lambda\beta[v] & \beta[v] \\ -\lambda^2\beta[v] & \lambda\beta[v] \end{pmatrix} = \beta[v] \begin{pmatrix} -\lambda & 1 \\ -\lambda^2 & \lambda \end{pmatrix}$$

hence $\mathfrak{R}(\mathcal{A}) = 0$, on the other. So suppose that there is some $y \in X$ with $\|y\| = 1$ and $\gamma[y] \neq 0$. Then from our recursion formula for the iterates of A we conclude that

$$\|A^n\|_{X\to X} \ge \|A^ny\| = |\gamma[v]|^{n-1} |\gamma[y]| \|v\|.$$

Consequently,

$$\mathfrak{R}(A) = \lim_{n\to\infty} \|A^n\|_{X\to X}^{1/n} \ge |\gamma[v]| = |\alpha[v] + \lambda\beta[v]| = \mathfrak{R}(\mathcal{A})$$

as claimed.

<u>2nd case</u>: Assume $w \neq \mu v$ for all $\mu \in \mathbb{R}$. We use the fact that the spectral radius of an operator $A : X \to X$ on a real space X coincides with the spectral radius of its complexification $A_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$. Recall that $X_{\mathbb{C}} := \{x + iy : x, y \in X\}$ is equipped with the norm

$$\|x + iy\|_{X_{\mathbb{C}}} := \max_{0 \leq t \leq 2\pi} \|(\cos t)x + (\sin t)y\|,$$

and $A_{\mathbb{C}}$ is defined by $A_{\mathbb{C}}(x + iy) := Ax + iAy$. Similarly, the functionals α and β are complexified by putting

$$\alpha_{\mathbb{C}}[x + iy] := \alpha[x] + i\alpha[y], \qquad \beta_{\mathbb{C}}[x + iy] := \beta[x] + i\beta[y].$$

Note that $\|A_{\mathbb{C}}\|_{X_{\mathbb{C}} \to X_{\mathbb{C}}} = \|A\|_{X \to X}$, $\|\alpha_{\mathbb{C}}\|_{X^*_{\mathbb{C}}} = \|\alpha\|_{X^*}$, and $\|\beta_{\mathbb{C}}\|_{X^*_{\mathbb{C}}} = \|\beta\|_{X^*}$. The relation (3.2) translates then into complexifications in the form

$$A_{\mathbb{C}} z = \alpha_{\mathbb{C}}[z]v + \beta_{\mathbb{C}}[z]w \qquad (z \in X_{\mathbb{C}}).$$

Let now $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{A}^T$ with eigenvector $u = (u_1, u_2) \in \mathbb{C}^2$. This means that $u^T \mathcal{A} = \lambda u^T$, i.e. in components,

$$\alpha[v]u_1 + \alpha[w]u_2 = \lambda u_1, \qquad \beta[v]u_1 + \beta[w]u_2 = \lambda u_2.$$

Since $\{v, w\}$ is linearly independent in X, by hypothesis, we find $x, y \in X$ such that

$$Ax = \mathrm{Re}(u_1)v + \mathrm{Re}(u_2)w, \qquad Ay = \mathrm{Im}(u_1)v + \mathrm{Im}(u_2)w.$$

The element $z := x + iy \in X_{\mathbb{C}}$ satisfies then

$$A_{\mathbb{C}} z = A_{\mathbb{C}}(x + iy) = Ax + iAy = vu_1 + wu_2.$$

But from $u = (u_1, u_2) \neq (0, 0)$ we conclude that $A_{\mathbb{C}} z \neq o$, hence

$$\begin{aligned} A_{\mathbb{C}}(A_{\mathbb{C}} z) &= \alpha_{\mathbb{C}}[A_{\mathbb{C}} z]v + \beta_{\mathbb{C}}[A_{\mathbb{C}} z]w = (u_1\alpha[v] + u_2\alpha[w])v + (u_1\beta[v] + u_2\beta[w])w \\ &= \lambda(u_1 v + u_2 w) = \lambda A_{\mathbb{C}} z. \end{aligned}$$

Since $Az_{\mathbb{C}} \neq o$, we conclude that $A_{\mathbb{C}} z \in X_{\mathbb{C}}$ is an eigenvector of $A_{\mathbb{C}}$ corresponding to the eigenvalue λ. This implies that $\mathfrak{R}(A) \geq \mathfrak{R}(\mathcal{A}^T) = \mathfrak{R}(\mathcal{A})$ which completes the proof. □

Let us illustrate Proposition 3.2 by two simple examples, the first being one-dimensional, the second infinite dimensional, which we collect in the following

Example 3.3. The simplest case is of course $X = \mathbb{R}$. Then we have $Ax = (v\alpha + w\beta)x$, where x, v, w, α, and β are all real numbers, and A represents a straight line with slope $v\alpha + w\beta$. Since $A^n x = (v\alpha + w\beta)^n x$, the linear map A has the spectral radius $|v\alpha + w\beta|$. On the other hand, the matrix (3.3) is here

$$\mathcal{A} = \begin{pmatrix} \alpha[v] & \beta[v] \\ \alpha[w] & \beta[w] \end{pmatrix} = \begin{pmatrix} v\alpha & v\beta \\ w\alpha & w\beta \end{pmatrix}$$

which has the two eigenvalues 0 and $v\alpha + w\beta$, and therefore the same spectral radius as A.

A slightly less trivial example reads as follows. In the space $X = C[0, 1]$, let A be given by (3.2), where $v(t) \equiv 1$, $\alpha[x] := x(0)$, $w(t) := t$, and $\beta[x] := x(1)$. A trivial calculation shows then that

$$A^n x(t) = x(0) + ((n-1)x(0) + x(1))t, \qquad \|A^n\|_{X \to X} = n + 1.$$

Consequently, the linear operator A has spectral radius 1. On the other hand, the matrix (3.3) is here

$$\mathcal{A} = \begin{pmatrix} \alpha[v] & \beta[v] \\ \alpha[w] & \beta[w] \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which has the double eigenvalue 1, and therefore the same spectral radius as A.

The following refinement of Theorem 2.2 is now an immediate consequence of Proposition 3.2.

Theorem 3.4. *Let $\alpha, \beta \in BV^*$ be bounded linear functionals satisfying*

$$\mathfrak{R}(\mathcal{A}) < 1, \tag{3.6}$$

where $\mathcal{A}$ denotes the matrix (3.3), $\mathfrak{R}(\mathcal{A})$ its spectral radius, and

$$v(t) := e_1(1-t) = 1-t, \qquad w(t) := e_1(t) = t.$$

Then for each $R > 0$ there is some $\rho > 0$ such that (BVP) has, for fixed $\lambda \in (-\rho, \rho)$, a solution $x \in AC^1$ satisfying $\|x\|_{BV} \leq R$. If, in addition, g is continuous on $[0,1] \times \mathbb{R}$, then every such solution is of class C^2.

Proof. The argument is similar as in the proof of Theorem 2.2. Accordingly, we only have to show that the operator A in (3.2) satisfies $\|A^n\|_{BV \to BV} < 1$ for some $n \in \mathbb{N}$. But this is clear, since (3.6) in combination with Proposition 3.2 yields $\mathfrak{R}(A) < 1$. $\square$

We point out that Theorem 2.2 is completely covered by Theorem 3.4. Indeed, in the proof of Theorem 2.2 we have shown that each of the hypotheses (2.5) or (2.6) implies that $\mathfrak{R}(A) < 1$, and so also $\mathfrak{R}(\mathcal{A}) < 1$, by Proposition 3.2, with $\mathcal{A}$ given by (3.3). Moreover, Theorem 3.4 has several advantages. First, it does not use the operator norm $\|\cdot\|_{BV \to BV}$, but the spectral radius, which is invariant when passing to an equivalent norm. For example, if we replace the norm (1.3) by the (larger, but equivalent) norm

$$|||x|||_{BV} = \|x\|_\infty + \mathrm{Var}(x;[0,1]) = \sup_{0 \leq t \leq 1} |x(t)| + \mathrm{Var}(x;[0,1]),$$

we must impose in Theorem 2.2, instead of (2.5), the stronger condition

$$|||\alpha|||_{BV^*} + 2|||\alpha - \beta|||_{BV^*} < 1,$$

because in this norm we have $|||e_k|||_{BV} = 2$. Second, condition (3.6) is easier to verify than the conditions imposed in Theorem 2.2. Third, Theorem 3.4 covers more cases than Theorem 2.2, as we show now by means of an example.

Example 3.5. Consider again the BVP (3.1) from Example 3.1. As we have seen there, neither (2.5) nor (2.6) applies to this BVP. On the other hand, taking into account the form of the functionals α and β and the definition of v and w used in Theorem 3.4 we get here

$$\begin{cases} \alpha[v] = v\left(\frac{1}{3}\right) + v\left(\frac{2}{3}\right) = 1, \\ \beta[v] = -\frac{1}{2}v\left(\frac{1}{3}\right) - \frac{1}{2}v\left(\frac{2}{3}\right) = -\frac{1}{2} \\ \alpha[w] = w\left(\frac{1}{3}\right) + w\left(\frac{2}{3}\right) = 1, \\ \beta[w] = -\frac{1}{2}w\left(\frac{1}{3}\right) - \frac{1}{2}w\left(\frac{2}{3}\right) = -\frac{1}{2}. \end{cases}$$

So in this case the matrix $\mathcal{A}$ has the eigenvalues 0 and $1/2$, which shows that Theorem 3.4 applies, while Theorem 2.2 does not.

We may summarize our discussion as follows. In all examples discussed so far we imposed, similarly as in (1.1), boundary conditions of the form

$$x(0) = ax(\sigma_1) - bx(\sigma_2), \quad x(1) = cx(\tau_1) + dx(\tau_2), \tag{3.7}$$

where $\sigma_1, \sigma_2, \tau_1, \tau_2 \in (0,1)$ are fixed. Theorem 3.4 applies to equation (2.1) with these boundary conditions if and only if

$$\mathfrak{R}(\mathcal{M}) < 1, \tag{3.8}$$

where $\mathcal{M} = \mathcal{M}(a,b,c,d,\sigma_1,\sigma_2,\tau_1,\tau_2)$ is the matrix

$$\mathcal{M} = \begin{pmatrix} a(1-\sigma_1) - b(1-\sigma_2) & c(1-\tau_1) + d(1-\tau_2) \\ a\sigma_1 - b\sigma_2 & c\tau_1 + d\tau_2 \end{pmatrix}. \tag{3.9}$$

For instance, in Example 3.1 we have $a = 1$, $b = -1$, $c = d = -1/2$, $\sigma_1 = \tau_1 = 1/3$, and $\sigma_2 = \tau_2 = 2/3$, which gives

$$\mathcal{M} = \begin{pmatrix} 1 & -1/2 \\ 1 & -1/2 \end{pmatrix}$$

and implies the solvability of (3.1), as we have seen in Example 3.5. On the other hand, since condition $\mathfrak{R}(\mathcal{M}) < 1$ is both necessary and sufficient, we may easily construct a BVP which is not covered even by Theorem 3.4.

Example 3.6. Consider the BVP

$$\begin{cases} x''(t) + \lambda g(t,x(t)) = 0 & (0 \le t \le 1), \\ x(0) = ax\left(\frac{1}{2}\right), \quad x(1) = cx\left(\frac{1}{2}\right). \end{cases} \tag{3.10}$$

The functionals

$$\alpha[x] := ax\left(\tfrac{1}{2}\right), \qquad \beta[x] := cx\left(\tfrac{1}{2}\right)$$

are obviously linear and bounded on BV. Since $b = d = 0$, $\sigma_1 = \tau_1 = 1/2$, the matrix (3.9) is here

$$\mathcal{M} = \frac{1}{2}\begin{pmatrix} a & c \\ a & c \end{pmatrix}.$$

Since this matrix has spectral radius $|a+c|/2$, Theorem 3.4 applies to the BVP (3.10) if and only if $-2 < a + c < 2$.

We point out that condition (3.8) is necessary for the applicability of Theorem 3.4, but *not* for the existence of a solution $x \in AC^1$ of (BVP). This is illustrated by the following

Example 3.7. Consider the BVP

$$\begin{cases} x''(t) - 2(1+2t^2)x(t) = 0 & (0 \le t \le 1), \\ x(0) = e^{-1/4}x\left(\frac{1}{2}\right), \quad x(1) = e^{3/4}x\left(\frac{1}{2}\right). \end{cases} \tag{3.11}$$

Obviously, the nonlinearity $g(t,u) = (2+4t^2)u$ satisfies (H1)/(H2)/(H3). In the notation of (3.7) we have here $a = e^{-1/4}$, $c = e^{3/4}$, $b = d = 0$, and $\sigma_1 = \tau_1 = 1/2$. Consequently, the matrix (3.9) reads

$$\mathcal{M} = \frac{1}{2}\begin{pmatrix} e^{-1/4} & e^{3/4} \\ e^{-1/4} & e^{3/4} \end{pmatrix}$$

which has spectral radius

$$\mathfrak{R}(\mathcal{M}) = \frac{1+e}{2e^{1/4}} > 1.$$

So Theorem 3.4, let alone Theorem 2.2, does not apply. Nevertheless, it is easy to check that $x(t) := e^{t^2}$ is an (even analytic) solution of the boundary value problem (3.11).

4 Integral type boundary conditions

Theorem 3.4 applies not only to "pointwise" boundary conditions like (3.7), but also to "global" boundary conditions of the form

$$x(0) = \int_0^1 k_0(s)x(s)\,ds, \qquad x(1) = \int_0^1 k_1(s)x(s)\,ds, \tag{4.1}$$

where $k_0, k_1 \in L_1$ are given. The functionals α and β are defined here by the integrals in (4.1), and so Theorem 3.4 applies if and only if $\mathfrak{R}(\mathcal{M}) < 1$, where $\mathcal{M} = \mathcal{M}(k_0, k_1)$ is the matrix

$$\mathcal{M} = \begin{pmatrix} \displaystyle\int_0^1 k_0(s)(1-s)\,ds & \displaystyle\int_0^1 k_1(s)(1-s)\,ds \\ \displaystyle\int_0^1 k_0(s)s\,ds & \displaystyle\int_0^1 k_1(s)s\,ds \end{pmatrix}. \tag{4.2}$$

We illustrate this by another simple example which contains a free parameter $c \in \mathbb{R}$.

Example 4.1. Consider the BVP

$$\begin{cases} x''(t) + \lambda g(t, x(t)) = 0 \quad (0 \le t \le 1), \\ x(0) = \displaystyle\int_0^1 x(s)\,ds, \quad x(1) = c\displaystyle\int_0^1 x(s)\,ds, \end{cases} \tag{4.3}$$

where g satisfies (H1)/(H2)/(H3). Here we have $k_0(s) \equiv 1$ and $k_1(s) \equiv c$, so the matrix $\mathcal{M} = \mathcal{M}(1, c)$ becomes

$$\mathcal{M} = \begin{pmatrix} \displaystyle\int_0^1 k_0(s)(1-s)\,ds & \displaystyle\int_0^1 k_1(s)(1-s)\,ds \\ \displaystyle\int_0^1 k_0(s)s\,ds & \displaystyle\int_0^1 k_1(s)s\,ds \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & c \\ 1 & c \end{pmatrix}.$$

Since this matrix has spectral radius $(c+1)/2$, we may guarantee the solvability of problem (4.3) for small $|\lambda|$ if $-3 < c < 1$.

5 A higher order problem

The theory developed in the preceding sections may be applied to other similar boundary value problems than those we have considered in the examples so far. For instance, we can modify our constructions to cover a third order problem like

$$\begin{cases} x'''(t) + \lambda g(t, x(t)) = 0 \quad (0 \le t \le 1), \\ x'(0) = \alpha[x], \quad x'(1) = \beta[x] \end{cases} \tag{5.1}$$

with $\alpha, \beta \in BV^*$ as before. We do not state a formal theorem, since we do not want the reader to get drowned in too many technicalities, but just sketch an outline of the idea, because the arguments are similar as those used before.

We are looking for solutions $x \in AC^2$ that satisfy the differential equation in (5.1) almost everywhere in $[0, 1]$ and have the correct boundary values $x'(0) = \alpha[x]$ and $x'(1) = \beta[x]$. In order to find such a solution we solve the integral equation

$$x(t) = Ax(t) + \lambda \int_0^t \int_0^1 \kappa(\tau, s) g(s, x(s))\,ds\,d\tau \qquad (0 \le t \le 1) \tag{5.2}$$

in the space BV, where κ is the same Green's function as before, and the linear operator $A : BV \to BV$ is again given by (3.2), where now

$$v(t) := -\frac{1}{2}e_2(1-t) = -\frac{1}{2}(1-t)^2, \qquad w(t) := \frac{1}{2}e_2(t) = \frac{1}{2}t^2.$$

For $x \in AC^2$ the outer integral in (5.2) defines a differentiable function. Similarly as in Proposition 2.1 one may show that any function $x \in BV$ satisfying (5.2) is a solution in AC^2 to the boundary value problem (5.1), and vice versa. Note that for the first derivative of a solution x of (5.2) we have

$$x'(t) = (1-t)\alpha[x] + t\beta[x] + \lambda \int_0^1 \kappa(t,s)g(s,x(s))\,ds \qquad (0 \le t \le 1), \tag{5.3}$$

and so indeed $x'(0) = \alpha[x]$ and $x'(1) = \beta[x]$.

Now, in order to solve (5.2) we can use Fubini's Theorem to reduce the double integral to a single one and transform the integral equation into

$$x(t) = Ax(t) + \lambda \int_0^1 \hat{\kappa}(t,s)g(s,x(s))\,ds \qquad (0 \le t \le 1), \tag{5.4}$$

where

$$\hat{\kappa}(t,s) := \int_0^t \kappa(\tau,s)\,d\tau = \begin{cases} \frac{1}{2}s(2t-t^2-s) & \text{for } 0 \le s \le t \le 1, \\ \frac{1}{2}t^2(1-s) & \text{for } 0 \le t \le s \le 1. \end{cases}$$

Consequently, under the hypotheses of Theorem 3.4 (with v and w as above), we may solve (5.2) and therefore also (5.1) exactly as we solved (BVP). Instead of going into details, let us close this section with an example.

Example 5.1. Consider the third order BVP

$$\begin{cases} x'''(t) - 4t(2t^2+3)x(t) = 0 \quad (0 \le t \le 1), \\ x'(0) = 0, \quad x'(1) = 2e^{3/4}x\left(\frac{1}{2}\right). \end{cases} \tag{5.5}$$

Here the integral equation (5.4) is

$$x(t) = e^{3/4}x\left(\tfrac{1}{2}\right)t^2 + 2\int_0^t s^2(2t-t^2-s)(2s^2+3)x(s)\,ds + 2t^2\int_t^1 (1-s)s(2s^2+3)x(s)\,ds.$$

A somewhat cumbersome, but straightforward calculation shows that $x(t) = e^{t^2}$ is a solution. However, if we are only interested in the existence of a solution without constructing it explicitly, we may use Proposition 2.1 and calculate the spectral radius of the matrix

$$\mathcal{A} = \begin{pmatrix} \alpha[v] & \beta[v] \\ \alpha[w] & \beta[w] \end{pmatrix} = \frac{e^{3/4}}{4}\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix},$$

which turns out to be $e^{3/4}/4 < 1$. So in contrast to Example 3.7 we may now apply Theorem 3.4.

6 Initial value problems with BV data

To conclude, let us briefly discuss the second order equation (2.1), but now subject to the uncoupled initial conditions

$$x(0) = \alpha[x], \qquad x'(0) = \beta[x], \tag{6.1}$$

where $\alpha, \beta : BV \to \mathbb{R}$ are given linear functionals. Here we do not repeat all the results which are parallel to those for boundary value problems, but rather point out the differences. In the sequel we refer to the problem (2.1)/(6.1) by the symbol (IVP).

In order to solve this problem, we consider along with (IVP) the Hammerstein–Volterra integral equation

$$x(t) = Ax(t) + \lambda \int_0^t \nu(t,s)g(s,x(s))\,ds \qquad (0 \le t \le 1), \tag{6.2}$$

where the Volterra kernel is given by

$$\nu(t,s) = \begin{cases} s-t & \text{for } 0 \le s \le t \le 1, \\ 0 & \text{for } 0 \le t < s \le 1, \end{cases}$$

and $A : BV \to BV$ is a linear operator. The following is then a perfect analogue to Proposition 2.1.

Proposition 6.1. *Let $A : BV \to BV$ be defined by*

$$Ax(t) := \alpha[x] + t\beta[x] \qquad (0 \le t \le 1). \tag{6.3}$$

Then the following holds:

(a) Every function $x \in BV$ solving (6.2) belongs to AC^1 and solves (IVP) almost everywhere on $[0,1]$.

(b) If, in addition, g is continuous on $[0,1] \times \mathbb{R}$, then every solution x of (6.2) is of class C^2 and solves (IVP) everywhere on $[0,1]$.

(c) Conversely, if $x \in AC^1$ solves (IVP) almost everywhere on $[0,1]$, then x is a solution of the integral equation (6.2).

The proof is very similar to that of Proposition 2.1, with the difference that we now define the function $\varphi : [0,1] \to \mathbb{R}$ by

$$\varphi(t) := \int_0^1 \nu(t,s)g(s,x(s))\,ds = \int_0^t (s-t)h(s)\,ds$$

and use the fact that $\varphi \in AC^1$ with $\varphi(0) = \varphi'(0) = 0$.

The sufficient condition (2.5) imposed in Theorem 2.2 becomes now even easier: since Ax is, for fixed $x \in BV$, a straight line joining the points $(0, \alpha[x])$ and $(1, \alpha[x] + \beta[x])$, we can calculate its BV norm explicitly and obtain

$$\|Ax\|_{BV} = |Ax(0)| + \mathrm{Var}(Ax;[0,1]) = |\alpha[x]| + |\beta[x]| \le (\|\alpha\|_{BV^*} + \|\beta\|_{BV^*})\|x\|_{BV}.$$

Thus, the estimate

$$\|\alpha\|_{BV^*} + \|\beta\|_{BV^*} < 1 \tag{6.4}$$

which is parallel to (2.5) now guarantees that $\|A\|_{BV\to BV} < 1$ and makes it possible to apply Krasnosl'skij's fixed point principle to (6.2) for sufficiently small $|\lambda|$.

Of course, as in Section 2 we could easily find specific IVPs to illustrate the applicability of (6.4). Instead, it is more interesting to compare (2.5) and (6.4). It is tempting to think that (2.5) implies (6.4), or vice versa. But no such implication is true, as the following two examples show.

Example 6.2. Define two functionals $\alpha, \beta \in BV^*$ by

$$\alpha[x] := \beta[x] := \tfrac{1}{2}x\left(\tfrac{1}{2}\right).$$

Then $\|\alpha\|_{BV^*} = \|\beta\|_{BV^*} = 1/2$ and $\|\alpha - \beta\|_{BV^*} = 0$. Thus, condition (2.5) is fulfilled, while condition (6.4) is violated.

Example 6.3. On the other hand, if we define $\alpha, \beta \in BV^*$ by

$$\alpha[x] := \tfrac{1}{3}x\left(\tfrac{1}{2}\right), \qquad \beta[x] := -\tfrac{1}{3}x\left(\tfrac{1}{2}\right),$$

it is easy to see that condition (2.5) is violated, while condition (6.4) is fulfilled.

We now jump to Theorem 3.4 and see how it looks like in the setting of (IVP). Since the structure of the linear operator A in (6.3) is covered by Proposition 3.2, we have a general method to calculate the spectral radius of A. Accordingly, the following analogue to Theorem 3.4 holds true.

Theorem 6.4. *Let $\alpha, \beta \in BV^*$ be bounded linear functionals satisfying (3.6), where $\mathcal{A}$ denotes the matrix (3.3) for $v := e_0$ and $w := e_1$. Then for each $R > 0$ there is some $\rho > 0$ such that (IVP) has, for fixed $\lambda \in (-\rho, \rho)$, a solution $x \in AC^1$ satisfying $\|x\|_{BV} \le R$. If, in addition, g is continuous on $[0,1] \times \mathbb{R}$, then every such solution is of class C^2.*

Since the argument is similar, we skip the proof of Theorem 6.4. Instead, let us go back to Example 6.2, where condition (6.4) fails. Even worse, it is clear that $Ax = \alpha[x]e_0 + \beta[x]e_1 = \alpha[x](e_0 + e_1)$ cannot be a contraction in BV, because $\|Ae_0\|_{BV} = 1$. However, we have

$$\mathfrak{R}\begin{pmatrix} \alpha[e_0] & \beta[e_0] \\ \alpha[e_1] & \beta[e_1] \end{pmatrix} = \mathfrak{R}\begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 1/4 \end{pmatrix} = \frac{3}{4} < 1,$$

and so Theorem 6.4 tells us that the IVP in Example 6.2 has as a solution $x \in AC^1$ for small $|\lambda|$.

Finally, let us look at an initial value condition which corresponds to the very general boundary condition (3.7). Its analogue has the form

$$x(0) = ax(\sigma_1) - bx(\sigma_2), \qquad x'(0) = cx(\tau_1) + dx(\tau_2), \tag{6.5}$$

where $\sigma_1, \sigma_2, \tau_1, \tau_2 \in (0,1)$ are fixed. Theorem 6.4 applies to equation (2.1) with these initial conditions if and only if

$$\mathfrak{R}(\mathcal{N}) < 1, \tag{6.6}$$

where $\mathcal{N} = \mathcal{N}(a, b, c, d, \sigma_1, \sigma_2, \tau_1, \tau_2)$ is the matrix

$$\mathcal{N} = \begin{pmatrix} a - b & c + d \\ a\sigma_1 - b\sigma_2 & c\tau_1 + d\tau_2 \end{pmatrix}. \tag{6.7}$$

In the next example we show that neither of the conditions $\mathfrak{R}(\mathcal{M}) < 1$ or $\mathfrak{R}(\mathcal{N}) < 1$ implies the other, where $\mathcal{M}$ is given by (3.9).

Example 6.5. Let $\sigma_1 := 1/3$, $\sigma_2 := 2/3$, and $c = d := 0$ which means that we consider both the BVP

$$\begin{cases} x''(t) + \lambda g(t, x(t)) = 0 \quad (0 \le t \le 1), \\ x(0) = ax\left(\frac{1}{3}\right) - bx\left(\frac{2}{3}\right), \quad x(1) = 0, \end{cases} \tag{6.8}$$

and simultaneously the IVP

$$\begin{cases} x''(t) + \lambda g(t, x(t)) = 0 \quad (0 \le t \le 1), \\ x(0) = ax\left(\frac{1}{3}\right) - bx\left(\frac{2}{3}\right), \quad x'(0) = 0. \end{cases} \tag{6.9}$$

Then

$$\mathcal{M} = \frac{1}{3}\begin{pmatrix} 2a - b & 0 \\ a - 2b & 0 \end{pmatrix}, \qquad \mathcal{N} = \frac{1}{3}\begin{pmatrix} 3a - 3b & 0 \\ a - 2b & 0 \end{pmatrix}.$$

The matrix $\mathcal{M}$ has spectral radius $|2a - b|/3$, the matrix $\mathcal{N}$ has spectral radius $|a - b|$. Consequently, for $a := -1/2$ and $b := -5/2$ we have $\mathfrak{R}(\mathcal{M}) = 1/2$, but $\mathfrak{R}(\mathcal{N}) = 2$ (which ensures the solvability of (6.8), but not of (6.9)). For $a := 11/2$ and $b := 5$, however, it is exactly the other way round.

At this point the same warning as in Section 3 is in order. Condition (6.6) is necessary and sufficient for the applicability of Theorem 6.4, but only sufficient for the solvability of (IVP). This is illustrated by our final

Example 6.6. Consider the IVP

$$\begin{cases} x''(t) + 4\left(4t^2 x(t) + \sqrt{1 - x(t)^2}\right) \quad (0 \le t \le 1), \\ x(0) = \sqrt{2}x\left(\sqrt{\pi/8}\right), \quad x'(0) = 0. \end{cases} \tag{6.10}$$

Clearly, the nonlinearity $g(t, u) = 4(4t^2 u + \sqrt{1 - u^2})$ satisfies (H1)/(H2)/(H3). In the notation of (6.5) we have here $a = \sqrt{2}$, $b = c = d = 0$, and $\sigma_1 = \sqrt{\pi/8}$. Consequently, the matrix (6.7) reads

$$\mathcal{N} = \begin{pmatrix} \sqrt{2} & 0 \\ \sqrt{\pi}/2 & 0 \end{pmatrix} \tag{6.11}$$

which has spectral radius

$$\mathfrak{R}(\mathcal{N}) = \sqrt{2} > 1,$$

so Theorem 6.4 does not apply. Nevertheless, it is easy to check that $x(t) := \cos(2t^2)$ is an (even analytic) solution of the initial value problem (6.10).

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Existence results for a clamped beam equation with integral boundary conditions

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. In this paper we investigate the existence of positive solutions of fourth-order non autonomous differential equations with integral boundary conditions, the nonlinearity is a continuous function that depends on the spatial variable and its the second-order derivative. The approach relies an extension of Krasnoselskii's fixed point theorem in a cone. Some examples are given to illustrate our results.

Keywords: Green's functions, Fourth-order boundary value problem, integral boundary conditions, positive solutions, extension of Krasnoselskii's fixed point theorem in a cone.

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1 Introduction

Fourth-order boundary value problems with integral boundary conditions arises in the mathematical modeling of viscoelastic and inelastic flows, thermos-elasticity, deformation of beams and plate deflection theory [12,14,22].

In [2], Cabada and Enguiça characterized the inverse positive character of operator $u^{(4)} + M\,u$ coupled with the, so called, clamped beam boundary conditions

$$u^{(4)}(t) + Mu(t) = \sigma(t), \quad t \in I := [0,1] \tag{1.1}$$

$$u(0) = u(1) = u'(0) = u'(1) = 0. \tag{1.2}$$

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Using oscillation theory [23], on [2] are obtained the exact values on the real parameter $M \in (-m_1^4, m_0^4)$, for which the related Green's function g_M is strictly positive in $(0,1) \times (0,1)$. To be concise, $m_1 \cong 4.73004$ is the first positive root of equation

$$\cos m \cosh m = 1,$$

and $-m_1^4$ coincides with the first negative eigenvalue of operator $u^{(4)}$ coupled to boundary conditions (1.2).

Moreover, $m_0 \approx 5.553$ is the smaller positive solution of equation

$$\tanh \frac{m}{\sqrt{2}} = \tan \frac{m}{\sqrt{2}}, \tag{1.3}$$

and, as it is showed at [4], m_0^4 is the first positive eigenvalue of operator $u^{(4)}$ coupled to boundary conditions $u(0) = u'(0) = u''(0) = u(1) = 0$.

These results have been extended in [7] (and further in [8]) for any n-th order linear differential operator.

The existence of positive solutions for nonlinear problems are deduced by using the upper and lower solutions method and fixed point theorems in cones. In those cases, the nonlinearity depends only on the function u. For these problems the dependence on the second derivative of their nonlinearity has taken less attention.

In this work we will study the existence of positive solution of a more general fourth order problem related to clamped beam:

$$u^{(4)}(t) + Mu(t) = f(t, u(t), u''(t)), \quad t \in I, \tag{1.4}$$

subject to the perturbed functional boundary conditions:

$$u(1) = u'(0) = u'(1) = 0,\ u(0) = \lambda \int_0^1 u(s)\nu(s)\, ds. \tag{1.5}$$

Where $M \in (-m_1^4, 4\pi^4)$, $\nu \in L^1(I)$ is a positive weight function a.e. on $(0,1)$ and λ is a positive parameter bounded from above by a constant that will be introduced later. We suppose that the function f satisfy the following regularity assumption

(H_0) $f : I \times [0, \infty) \times \mathbb{R} \to [0, \infty)$ is a continuous function.

Equation (1.4) models the stationary states of the deflection of an elastic beam. The boundary conditions (1.5) can be thought of as having the end at 1 clamped, and having some mechanism at end 0 that controls the displacement according to feedback from devices measuring the displacements along parts of the beam.

This paper is a continuation of the work done in [5] for problem

$$u^{(4)}(t) + Mu(t) + f(t, u(t)) = 0, \quad t \in I,$$

subject to the perturbed functional boundary conditions:

$$u(0) = u'(0) = u''(0) = 0,\ u(1) = \lambda \int_0^1 u(s)\, ds.$$

A standard approach to study positive solutions of a boundary value problem such as (1.4)–(1.5) consists of finding the corresponding Green's function G_M and seek solutions as fixed points of the Hammerstein integral equations with kernel G_M. The majority of methods are based on classical fixed point index theory and Krasnoselskii's fixed point theorem in a cone. The majority of authors work in a suitable cone K in a Banach space which is made using the property of Green's function. Sometimes the Green's function associated to this integral equation can change its sign. In theses cases, the authors should work in a cone smaller than K (see [17–19, 21]). The construction of a such cone requires more concise properties of the Green's function (see [3,6,13]).

We note that in our problem, the nonlinearity f depends on the second order derivatives. Using the classical Krasnoselskii's expansion/contraction theorem, we need to study the sign of the second order derivative of the Green's function and look for a nonnegative function ϕ such that

$$(C_1) \qquad \left|\frac{\partial^2 G_M}{\partial t^2}(t,s)\right| \leq \phi(s), \quad (t,s) \in I \times I,$$

and

$$(C_2) \qquad \frac{\partial^2 G_M}{\partial t^2}(t,s) \geq c\,\phi(s), \quad (t,s) \in [a,b] \times I,$$

for some $[a,b] \subset I$ and $c \in (0,1)$.

In our case, the explicit form of second derivative of Green's function $\frac{\partial^2 G_M}{\partial t^2}$ is very complicated and the previous inequalities ((C_1) and (C_2)) become hard to be checked. So, we apply an extension of Krasnoselskii's fixed point theorem that was used in [15,16,20,24]. With this result, we do not need to prove the inequalities (C_1) and (C_2). Here we need only the conditions (C_1) and (C_2) for the Green's function G_M. As far as we know, Problem (1.4)–(1.5) have not been previously studied. At the end of this paper, some examples are given to show that the theoretical results can be computed.

This paper is organized as follows. In Section 2, we introduce some basic definitions and lemmas to prove our main results and through this section we prove that the Green's function associated to (1.1), (1.5) satisfies some suitable properties. In Section 3, we show the existence of at least one positive solution. In section 4, some examples are presented to illustrate our main results.

2 Preliminaries and Green's function properties

In this section we introduce some preliminary results which will be used along the paper. First, we provide some background definitions cited from cone theory in Banach spaces. After that, we introduce some definitions and properties of the Green's function G_M related to problem (1.1), (1.5).

Definition 2.1. Let E be a real Banach space. A nonempty convex closed set $P \subset E$ is said to be a cone provided that

(i) $\alpha u \in P$ for all $u \in P$ and all $\alpha \geq 0$;

(ii) $u, -u \in P$ implies $u = 0$.

In the sequel, we enunciate the a fixed point theorem due to Guo and Ge [16].

Lemma 2.2 ([16, Theorem 2.1])**.** *Let E be a Banach space and $P \subset E$ a cone. Suppose $\alpha, \beta : E \to [0,\infty)$ are two continuous convex functionals satisfying*

$$\alpha(\mu u) = |\mu|\alpha(u), \quad \beta(\mu u) = |\mu|\beta(u), \quad u \in E,\ \mu \in \mathbb{R}$$

and $\|u\| \le N \max\{\alpha(u), \beta(u)\}$, *for* $u \in E$ *and* $\alpha(u_1) \le \alpha(u_2)$ *for* $u_1, u_2 \in P$, $u_1 \le u_2$, *where* $N > 0$ *is a constant.*

Let $r_2 > r_1 > 0$, $L > 0$ *be constants and* $\Omega_i = \{u \in E : \alpha(u) < r_i, \beta(u) < L\}$, $i = 1, 2$. *be two bounded open sets in* E. *Set* $D_i = \{u \in E : \alpha(u) = r_i\}$. *Assume that* $T : P \to P$ *is a completely continuous operator satisfying*

(C_1) $\alpha(Tu) < r_1$, $u \in D_1 \cap P$; $\alpha(Tu) > r_2$, $u \in D_2 \cap P$,

(C_2) $\beta(Tu) < L$, $u \in P$,

(C_3) *there is a* $p \in (\Omega_2 \cap P) \setminus \{0\}$ *such that* $\alpha(p) \neq 0$ *and* $\alpha(u + \mu p) \ge \alpha(u)$ *for all* $u \in P$ *and* $\mu \ge 0$.

Then T *has at least one fixed point in* $(\Omega_2 \setminus \overline{\Omega_1}) \cap P$.

Moreover, we enunciate the following result concerning the expression of the Green's function g_M, related to the linear Problem (1.1), (1.5). The proof can be found in [1, 2]. To this end, we introduce the following condition:

$$M < 0 \quad \text{and} \quad \cos\left(\sqrt[4]{-M}\right)\cosh\left(\sqrt[4]{-M}\right) = 1. \tag{2.1}$$

Lemma 2.3. *Let* $\sigma \in C(I)$ *and* $M \in \mathbb{R}$. *Then problem* (1.1)–(1.2) *has a unique solution if and only if* (2.1) *does not hold.*

In such a case, it is given by the following expression:

$$u(t) = \int_0^1 g_M(t,s)\sigma(s)\,ds.$$

Here, for $M = -m^4 < 0$, *we have*

$$g_M(t,s) = \begin{cases} g_1(t,s,m) & \text{if } 0 \le s \le t \le 1 \\ g_1(s,t,m) & \text{if } 0 \le t \le s \le 1, \end{cases}$$

with

$$\begin{aligned}
g_1(t,s,m) = {} & \frac{1}{8m^3\left((1+e^{2m})\cos(m) - 2e^m\right)} \\
& \times \Big\{ e^{-m(4s+t)}\left(-2e^{mt}\cos(mt) + e^{2mt} + 1\right)\Big(\left(e^{5ms} - e^{m(3s+2)}\right)\cos(m) \\
& + e^{3sm+m} - e^{5sm+m} + e^{4ms}\left(-1 + e^{2m}\right)\cos(m - ms) + e^{5ms}\sin(m) \\
& + e^{m(3s+2)}\sin(m) - 2e^{4sm+m}\sin(ms) - e^{4ms}\sin(m - ms) - e^{4sm+2m}\sin(m - ms)\Big) \\
& - 2e^{m(s-t)} + 2e^{m(t-s)}\left((1+e^{2m})\cos(m) - 2e^m\right) \\
& + e^{-m(4s+t)}\Big(\left(e^{5ms} + e^{m(3s+2)}\right)\cos(m) - e^{3sm+m} - e^{5sm+m} - 2e^{4sm+m}\cos(ms) \\
& + e^{4ms}\cos(m - ms) + e^{4sm+2m}\cos(m - ms) - e^{5ms}\sin(m) + e^{m(3s+2)}\sin(m) \\
& + e^{4ms}\sin(m - ms) - e^{4sm+2m}\sin(m - ms)\Big)\left(2e^{mt}\sin(mt) - e^{2mt} + 1\right) \\
& - 4\sin(m(t-s))\left((1+e^{2m})\cos(m) - 2e^m\right)\Big\}.
\end{aligned}$$

If $M=0$, it is given by

$$g_0(t,s) = -\frac{1}{6}\begin{cases} s^2(t-1)^2(2ts+s-3t) & \text{if } 0 \le s \le t \le 1, \\ t^2(s-1)^2(2ts+t-3s) & \text{if } 0 < t \le s \le 1. \end{cases}$$

Moreover, when $M=m^4>0$ it follows the expression

$$g_M(t,s) = \begin{cases} g_2(t,s,m) & \text{if } 0 \le s \le t \le 1 \\ g_2(s,t,m) & \text{if } 0 \le t \le s \le 1, \end{cases}$$

$$\begin{aligned} g_2(t,s,m) = {} & \frac{e^{-\frac{m(-6+3s+t)}{\sqrt{2}}}}{2\sqrt{2}m^3\left(1+e^{2\sqrt{2}m}+2e^{\sqrt{2}m}\left(-2+\cos\left(\sqrt{2}m\right)\right)\right)}\Bigg\{-2\left(-1+e^{\sqrt{2}mt}\right) \\ & \times\left(\left(e^{\sqrt{2}m(-2+s)}-e^{2\sqrt{2}m(-1+s)}\right)\cos\left(\frac{m(-2+s)}{\sqrt{2}}\right)+\left(-e^{\sqrt{2}m(-2+s)}+e^{2\sqrt{2}m(-1+s)}\right)\right. \\ & \times\cos\left(\frac{ms}{\sqrt{2}}\right)+\left(e^{\sqrt{2}m(-2+s)}-e^{\sqrt{2}m(-1+s)}+e^{2\sqrt{2}m(-1+s)}-e^{\sqrt{2}m(-3+2s)}\right) \\ & \left.\times\sin\left(\frac{ms}{\sqrt{2}}\right)\right)\sin\left(\frac{mt}{\sqrt{2}}\right)+\left(\left(e^{\sqrt{2}m(-2+s)}+e^{2\sqrt{2}m(-1+s)}\right)\cos\left(\frac{m(-2+s)}{\sqrt{2}}\right)\right. \\ & +\left(-2e^{\sqrt{2}m(-2+s)}+e^{\sqrt{2}m(-1+s)}-2e^{2\sqrt{2}m(-1+s)}+e^{\sqrt{2}m(-3+2s)}\right)\cos\left(\frac{ms}{\sqrt{2}}\right) \\ & +\left(-e^{\sqrt{2}m(-2+s)}+e^{2\sqrt{2}m(-1+s)}\right)\sin\left(\frac{m(-2+s)}{\sqrt{2}}\right)+\left(e^{\sqrt{2}m(-1+s)}-e^{\sqrt{2}m(-3+2s)}\right) \\ & \left.\times\sin\left(\frac{ms}{\sqrt{2}}\right)\right)\left(\left(-1+e^{\sqrt{2}mt}\right)\cos\left(\frac{mt}{\sqrt{2}}\right)-\left(1+e^{\sqrt{2}mt}\right)\sin\left(\frac{mt}{\sqrt{2}}\right)\right)\Bigg\}. \end{aligned}$$

Using the expressions given in Lemma 2.3, coupled to the definition of a Green's function [8] and, as a particular case of [8, Theorem 2.14 and Theorem 5.1], we deduce the following properties for function g_M:

Corollary 2.4. *Assuming that condition (2.1) does not hold. Then, function g_M, defined in Lemma 2.3, satisfies the following properties:*

1. *g_M is symmetric, that is $g_M(t,s)=g_M(s,t)$, for all $t,s\in I$.*
2. $g_M(0,s)=\frac{\partial g_M}{\partial t}(0,s)=g_M(1,s)=\frac{\partial g_M}{\partial t}(1,s)=0,$ *for all $s\in I$.*
3. $g_M(t,1)=\frac{\partial g_M}{\partial s}(t,1)=g_M(t,0)=\frac{\partial g_M}{\partial s}(t,0)=0,$ *for all $t\in I$.*

   Moreover, if $M\in(-m_1^4,m_0^4)$ the following inequalities are fulfilled:
4. *$g_M(t,s)>0$ for all $t,s\in(0,1)$.*
5. $\frac{\partial^2 g_M}{\partial t^2}(0,s)>0$ *and* $\frac{\partial^2 g_M}{\partial t^2}(1,s)>0,$ *for all $s\in(0,1)$.*
6. $\frac{\partial^2 g_M}{\partial s^2}(t,1)>0$ *and* $\frac{\partial^2 g_M}{\partial s^2}(t,0)>0,$ *for all $t\in(0,1)$.*

To obtain the expression of the solution of Problem (1.1),(1.5), we must study the solution of a suitable non-homogeneous boundary value problem as follows

Lemma 2.5 ([2, Theorem 3.12]). *The following problem:*

$$\begin{cases} u^{(4)}(t) + Mu(t) = 0, & t \in I, \\ u(1) = u'(0) = u'(1) = 0, \\ u(0) = 1, \end{cases} \tag{2.2}$$

has no solution if and only (2.1) *holds.*

In any other case, it has a unique solution, denoted by w_M, which is given by the following expression:

$$w_M(t) = \begin{cases} \dfrac{\cos\left(\frac{mt}{\sqrt{2}}\right)\cosh\left(\frac{m(t-2)}{\sqrt{2}}\right) - \sin\left(\frac{mt}{\sqrt{2}}\right)\sinh\left(\frac{m(t-2)}{\sqrt{2}}\right)}{\cos\left(\sqrt{2}m\right) + \cosh\left(\sqrt{2}m\right) - 2} \\ \quad + \dfrac{\left(\cos(\frac{m(t-2)}{\sqrt{2}}) - 2\cos(\frac{mt}{\sqrt{2}})\right)\cosh\left(\frac{mt}{\sqrt{2}}\right)}{\cos\left(\sqrt{2}m\right) + \cosh\left(\sqrt{2}m\right) - 2} \\ \quad + \dfrac{\sin\left(\frac{m(t-2)}{\sqrt{2}}\right)\sinh\left(\frac{mt}{\sqrt{2}}\right)}{\cos\left(\sqrt{2}m\right) + \cosh\left(\sqrt{2}m\right) - 2} & \text{if } m > 0 \text{ and } M = m^4, \\ (t-1)^2(1+2t) & \text{if } M = 0, \\ \dfrac{-\cos(m-mt) + \cosh(m)(\cos(mt) - \cosh(mt))}{2\cos(m)\cosh(m) - 2} \\ \quad + \dfrac{\cos(m)\cosh(mt) - \sin(mt)\sinh(m)}{2\cos(m)\cosh(m) - 2} \\ \quad + \dfrac{(\sin(m) + \sinh(m))\sinh(mt))}{2\cos(m)\cosh(m) - 2} & \text{if } m > 0 \text{ and } M = -m^4. \end{cases} \tag{2.3}$$

In [2, Theorem 3.12] it is proved that if $M > 0$, then $w_M(t) > 0$ for all $t \in [0,1)$ if and only $M \in (0, 4\pi^4]$. It is obvious that $w_0(t) > 0$ for all $t \in [0,1)$.

To study the sign in the negative case, $M = -m^4$, we must introduce the concept of disconjugate equation given in [10].

Definition 2.6. Let $a_k \in C^{n-k}(I)$ for $k = 1, \ldots, n$. The general n-th order linear differential equation $u^{(n)}(t) + a_1(t)\, u^{(n-1)}(t) + \cdots + a_{n-1}(t)\, u'(t) + a_n(t)\, u(t) = 0$ defined on any arbitrary interval $[a,b]$ is said to be disconjugate on an interval $J \subset [a,b]$ if every non trivial solution has, at most, $n-1$ zeros on J, multiple zeros being counted according to their multiplicity.

Moreover, we use the characterization for an equation to be disconjugate given in [9, Theorem 2.1] for a general nth order linear equation. Next, we enunciate the particular case for operator $u^{(4)} + M\, u$.

Lemma 2.7. *The linear equation $u^{(4)}(t) + M\, u(t) = 0$ is disconjugate on the interval I if and only if $M \in (-m_1^4, m_0^4)$.*

As a consequence, due to the continuity of the expression of w_M with respect to M, since $w_0''(1) = 6$, we have that if there is some $\bar{M} \in (-m_1^4, 0)$ for which $w_{\bar{M}}$ takes some negative values on $(0,1)$, then it must exists $M^* \in (\bar{M}, 0)$ such that one of the two following situations holds:

There is $t_0 \in (0,1)$ such that $w_{M^*}(t_0) = w'_{M^*}(t_0) = w_{M^*}(1) = w'_{M^*}(1) = 0$,

which contradicts Lemma 2.7, or

$$w_{M^*}(1) = w'_{M^*}(1) = w''_{M^*}(1) = 0.$$

But, in this last case, we have that

$$w''_{-m^4}(1) = \frac{m^2(\cos(m) - \cosh(m))}{\cos(m)\cosh(m) - 1},$$

which never takes the value zero for $m > 0$.

Therefore, if $M \in (-m_1^4, 0)$ then $w_M(t) > 0$ for all $t \in [0,1)$.

From the expression of $w''_M(1)$ we have that $w_M < 0$ in a neighborhood of $t = 1$ for M smaller and close enough to $-m_1^4$.

Now, suppose that there is some $M_1 < -m_1^4$ for wich $w_{M_1} > 0$ on $[0,1)$. Let $-m_1^4 < M_2 < 0$, we have that for all $t \in [0,1)$, the following property is fulfilled:

$$w^{(4)}_{M_2}(t) - w^{(4)}_{M_1}(t) = -M_2\,(w_{M_2} - w_{M_1})(t) - (M_2 - M_1)w_{M_1}(t) < -M_2\,(w_{M_2} - w_{M_1})(t).$$

Now, since $w_{M_2} - w_{M_1}$ satisfies the boundary conditions (1.2), from Corollary 2.4, we deduce that $0 < w_{M_2} < w_{M_1}$ on $(0,1)$. But this contradicts the fact that

$$\lim_{M \to -m_1^{4+}} \{w_M(t)\} = +\infty, \quad \text{for all } t \in (0,1).$$

So, we have proved the following result:

Lemma 2.8. $w_M > 0$ *on* $[0,1)$ *if and only if* $M \in (-m_1^4, 4\pi^4)$.

Now, by denoting

$$C_M = \int_0^1 w_M(\tau)\nu(\tau)\,d\tau, \tag{2.4}$$

we are in a position to obtain the explicit expression of the Green's function related to the equation (1.1) coupled to boundary conditions (1.5). The result is the following.

Lemma 2.9. *Let* $\sigma \in L^1(I)$, $\lambda > 0$ *and* $M \in \mathbb{R}$ *be such that* (2.1) *does not hold. Then problem*

$$\begin{cases} u^{(4)}(t) + Mu(t) = \sigma(t), \quad t \in I, \\ u'(0) = u(1) = u'(1) = 0, \\ u(0) = \lambda \displaystyle\int_0^1 u(s)\nu(s)\,ds \end{cases} \tag{2.5}$$

has a unique solution if and only if

$$\lambda\, C_M \neq 1.$$

In such a case, it is given by the following expression

$$u_M(t) = \int_0^1 G_M(t,s)\sigma(s)\,ds$$

where

$$G_M(t,s) = g_M(t,s) + \frac{\lambda w_M(t)}{1 - \lambda C_M} \int_0^1 g_M(\tau,s)\nu(\tau)\,d\tau, \tag{2.6}$$

w_M *and* C_M *are defined in* (2.3) *and* (2.4) *respectively and* g_M *is showed in Lemma* 2.3.

Proof. Since (2.1) does not hold, we have that Problems (1.1)–(1.2) and (2.2) are uniquely solvable. Let v_M and w_M be the unique solutions of each problem respectively. Then, it is clear that

$$u_M(t) = v_M(t) + \lambda\, w_M(t) \int_0^1 u_M(s)\nu(s)\, ds$$

is the unique solution of problem (2.5).

As a consequence, for all $t \in I$, the following equalities are fulfilled:

$$u_M(t) = \int_0^1 g_M(t,s)\sigma(s)\, ds + \lambda w_M(t) \int_0^1 u_M(s)\nu(s)\, ds. \tag{2.7}$$

Let $A_M = \int_0^1 u_M(\tau)\nu(\tau)\, d\tau$, then, from the previous equality, we deduce that

$$A_M = \int_0^1 \int_0^1 g_M(\tau,s)\nu(\tau)\sigma(s)\, ds\, d\tau + \lambda A_M \int_0^1 w_M(\tau)\nu(\tau)\, d\tau$$

or, which is the same,

$$A_M = \frac{\displaystyle\int_0^1 \sigma(s) \int_0^1 g_M(\tau,s)\nu(\tau)\, d\tau\, ds}{1 - \lambda \int_0^1 w_M(\tau)\nu(\tau)\, d\tau}.$$

Replacing this value in (2.7), we arrive at the following expression for function u_M:

$$u_M(t) = \int_0^1 g_M(t,s)\sigma(s)\, ds + \lambda w_M(t) \frac{\displaystyle\int_0^1 \sigma(s) \int_0^1 g_M(\tau,s)\nu(\tau)\, d\tau\, ds}{1 - \lambda \int_0^1 w_M(\tau)\nu(\tau)\, d\tau},$$

and the proof is concluded. □

Assuming that (2.1) does not hold, let z_M be the unique solution of the following boundary value problem:

$$z^{(4)}(t) + Mz(t) = \nu(t) \quad t \in I, \quad z(0) = z(1) = z'(0) = z'(1) = 0, \tag{2.8}$$

which is given by the following expression

$$z_M(t) = \int_0^1 g_M(t,s)\nu(s)\, ds.$$

Moreover, if $M \in (-m_1^4, m_0^4)$, since $\nu(t) > 0$ a.e. $t \in (0,1)$, from Corollary 2.4, we have that $z_M(t) > 0$ for all $t \in (0,1)$, $z''_M(0) > 0$ and $z''_M(1) > 0$.

We point out that, by direct computations, it is possible to obtain the explicit expression of function z_M for any particular choice of function ν.

A careful analysis of the Green's function G_M allows us to deduce the following result:

Theorem 2.10. *Let $G_M(t,s)$ be the Green's function related to problem (1.1), (1.5) given by expression (2.6). Then if $M \in (-m_1^4, 4\pi^4)$ and $\lambda \in (0, 1/C_M)$ we have that $G_M(t,s) > 0$ for all $(t,s) \in (0,1) \times (0,1)$. Moreover there exist $R > 0$ and $h \in C(I)$, such that $h(1) = 0$ and $h > 0$ on $[0,1)$, for which the following inequalities are fulfilled:*

$$h(t)\, \frac{\lambda}{1 - \lambda\, C_M} z_M(s) \le G_M(t,s) \le R\, \frac{\lambda}{1 - \lambda\, C_M} z_M(s), \text{ for all } (t,s) \in I \times I. \tag{2.9}$$

Proof. First, notice that $4\pi^4 < m_0^4$. So, since $M \in (-m_1^4, 4\pi^4)$ we have, from Corollary 2.4, that $g_M > 0$ on $(0,1)\times(0,1)$ and, as a direct consequence of $\lambda \in (0, 1/C_M)$ and the fact that $w_M > 0$ on $[0,1)$ for all $M \in (-m_0^4, 4\pi^4)$ (Lemma 2.8), we conclude, from (2.6), that $G_M(t,s) > 0$ for all $(t,s) \in (0,1)\times(0,1)$.

Now, we denote by

$$\varphi(t,s) = \frac{G_M(t,s)}{G_M(0,s)} = \frac{1-\lambda C_M}{\lambda}\,\frac{g_M(t,s)}{\displaystyle\int_0^1 g_M(s,r)v(r)\,dr} + w_M(t). \tag{2.10}$$

It is clear that function φ is continuous on $[0,1]\times(0,1)$, $\varphi(0,s) = 1$ and $\varphi(1,s) = 0$ for all $s \in I$.

Using the properties of g_M showed in Lemma 2.3 and those of z_M previously explained, by means of L'Hôpital's rule, we deduce, for all $t \in (0,1)$:

$$\lim_{s\to 0^+} \frac{g_M(t,s)}{\displaystyle\int_0^1 g_M(s,r)v(r)\,dr} = \lim_{s\to 0^+} \frac{g_M(t,s)}{z_M(s)} = \lim_{s\to 0^+} \frac{\frac{\partial^2 g_M}{\partial s^2}(t,s)}{z_M''(s)} = \frac{\frac{\partial^2 g_M}{\partial s^2}(t,0)}{z_M''(0)} > 0.$$

Thus,

$$\lim_{s\to 0^+} \varphi(t,s) = \frac{1-\lambda C_M}{\lambda}\left(\frac{\frac{\partial^2 g_M}{\partial s^2}(t,0)}{z_M''(0)}\right) + w_M(t) := l_1(t) > 0 \quad \text{for all } t \in [0,1).$$

Analogously, if $t \in (0,1)$, we have

$$\lim_{s\to 1^-} \frac{g_M(t,s)}{\displaystyle\int_0^1 g_M(s,r)v(r)\,dr} = \lim_{s\to 1^-} \frac{g_M(t,s)}{z_M(s)} = \lim_{s\to 1^-} \frac{\frac{\partial^2 g_M}{\partial s^2}(t,s)}{z_M''(s)} = \frac{\frac{\partial^2 g_M}{\partial s^2}(t,1)}{z_M''(1)} > 0$$

and

$$\lim_{s\to 1^-} \varphi(t,s) = \frac{1-\lambda C_M}{\lambda}\left(\frac{\frac{\partial^2 g_M}{\partial s^2}(t,1)}{z_M''(1)}\right) + w_M(t) := l_2(t) > 0 \quad \text{for all } t \in [0,1).$$

The limits $l_1(t)$ and $l_2(t)$ exist and are finite, so φ has removable discontinuities at $s = 0, 1$, and we can extend it to a function $\tilde{\varphi} \in C(I \times I)$.

Therefore $h(t) = \min_{s\in[0,1]} \tilde{\varphi}(t,s)$ is a continuous function such that

$$h(1) = 0 \quad \text{and} \quad 0 < h(t) \le \tilde{\varphi}(t,s) \le R \text{ for all } (t,s) \in [0,1)\times[0,1],$$

where $R = \max_{(t,s)\in I\times I} \tilde{\varphi}(t,s)$. □

Corollary 2.11. *Let $G_M(t,s)$ be Green's function related to problem (1.1), (1.5) given by expression (2.6). Then if $M \in (-m_1^4, 4\pi^4)$ and $\lambda \in (0, 1/C_M)$ we have that for all positive constant $\delta \in (0,1)$ there exists $\gamma(\delta) \in (0,1)$ for which the following inequality is fulfilled:*

$$\gamma(\delta)\,\frac{\lambda}{1-\lambda\,C_M} z_M(s) \le G_M(t,s), \quad \text{for all } (t,s) \in [0,\delta]\times I. \tag{2.11}$$

Proof. The result follows from the fact that function h is continuous on I and strictly positive on $[0,1)$. □

3 Existence of positive solutions

In this section, we are concerned with the existence of positive solutions of the boundary value problem (1.4)–(1.5). Firstly, we shall give a result of completely continuous operator. Then, we shall derive the existence results. Consider the vectorial space

$$E = \{u \in C^2(I); \quad u'(0) = u'(1) = 0\}$$

with the weighted norm $\|u\| = \|u\|_\infty + \|u''\|_\infty$.

Since, for any $u \in E$, and all $t \in I$, it is satisfied that

$$u'(t) = \int_0^t u''(s)\, ds.$$

We deduce that

$$\|u\| \le \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty \le \|u\|_\infty + 2\,\|u''\|_\infty \le 2\,\|u\|,$$

we have that $\|\cdot\|$ is an equivalent norm to the usual one in E. As consequence, E is a Banach Space with the weighted norm $\|\cdot\|$.

The following result is a direct consequence of the results showed in previous sections.

Let T the operator from E to E defined by

$$(T\,u)(t) = \int_0^1 G_M(t,s) f(s, u(s), u''(s))\, ds. \tag{3.1}$$

Lemma 3.1. *Assume that f satisfies condition (H_0), then, $u \in C^2(I)$ is a solution of (1.4)–(1.5) if and only if u is a fixed point of operator T defined on (3.1).*

Now, by considering function h and constant R, obtained in Theorem 2.10, we look for the fixed points of operator T at the following cone,

$$K = \left\{ u \in C^2(I) \text{ and } u(t) \ge \frac{h(t)}{R}\, \|u\|_\infty \text{ for all } t \in I \right\}. \tag{3.2}$$

Lemma 3.2. *If condition (H_0) is fulfilled, then operator $T : K \to K$, defined in (3.1), is completely continuous.*

Proof. From the non-negativeness of functions f and G_M we deduce that $(T\,u)(t) \ge 0$ for all $t \in I$ and $u \in K$. Using that $G_M \in C^2(I \times I)$, from the continuity of function f we deduce the completely continuous character of operator T as a direct application of Arzelà–Ascoli Theorem [11].

Let $u \in K$, by (2.9), we have that the following inequalities are fulfilled for all $t \in I$

$$\begin{aligned}
(T\,u)(t) &= \int_0^1 G_M(t,s) f(s, u(s), u''(s))\, ds \\
&\ge h(t)\, \frac{\lambda}{1 - \lambda\, C_M} \int_0^1 z_M(s)\, f(s, u(s), u''(s))\, ds \\
&\ge \frac{h(t)}{R} \int_0^1 \max_{t \in I} \{G_M(t,s)\} f(s, u(s), u''(s))\, ds \\
&\ge \frac{h(t)}{R} \max_{t \in I} \int_0^1 G_M(t,s) f(s, u(s), u''(s))\, ds \\
&= \frac{h(t)}{R} \|T\,u\|_\infty.
\end{aligned}$$

Moreover, from Corollary 2.4, (2), we have that

$$(T\,u)'(0) = (T\,u)'(1) = 0,$$

and, as a consequence, $T\,u \in K$ for all $u \in K$ and the proof is complete. $\square$

In the sequel, for any pair δ, γ satisfying (2.11) we introduce the following cone as follows:

$$K_\gamma^\delta = \left\{ u \in K \text{ and } \min_{t\in[0,\delta]} u(t) \geq \frac{\gamma}{R}\|u\|_\infty \right\}. \tag{3.3}$$

As in the proof of Lemma 3.2, one can verify the following result.

Lemma 3.3. *Assuming condition (H_0), we have that $T(K_\gamma^\delta) \subset K_\gamma^\delta$.*

Define the convex functionals $\alpha(u) = \|u\|_\infty$, $\beta(u) = \|u''\|_\infty$. Then, we have that

$$\|u\| \leq 2\max\{\alpha(u), \beta(u)\},$$

$$\alpha(\mu u) = |\mu|\alpha(u), \;\; \beta(\mu u) = |\mu|\beta(u), \quad u \in E, \; \mu \in \mathbb{R},$$

and since for all $u \in K$, it is satisfied that $u \geq 0$ on I, we have that if $u_1, u_2 \in K$ are such that $u_1 \leq u_2$ on I, then $\alpha(u_1) \leq \alpha(u_2)$.

In the following, we introduce the positive constants:

$$m = \max_{t\in I} \int_0^\delta G_M(t,s)\,ds, \tag{3.4}$$

$$M_1 = \max_{t\in I} \int_0^1 G_M(t,s)\,ds \tag{3.5}$$

and

$$M_2 = \max_{t\in I} \int_0^1 \left| \frac{\partial^2 G_M}{\partial t^2}(t,s) \right| ds. \tag{3.6}$$

We suppose that there are $L > b > \frac{\gamma}{R}b > c > 0$ such that f satisfies the following growth conditions:

(H_1) $f(t,u,v) < \frac{c}{M_1}$, for $(t,u,v) \in I \times [0,c] \times [-L,L]$,

(H_2) $f(t,u,v) \geq \frac{b}{m}$, for $(t,u,v) \in I \times [\frac{\gamma}{R}b, b] \times [-L,L]$,

(H_3) $f(t,u,v) < \frac{L}{M_2}$, for $(t,u,v) \in I \times [0,b] \times [-L,L]$.

Theorem 3.4. *Assume that conditions (H_0)–(H_3) are fulfilled. Then the boundary value problem* (1.4)–(1.5) *has at least one positive solution u satisfying*

$$c < \|u\|_\infty < b, \quad \|u''\|_\infty < L.$$

Proof. Take

$$\Omega_1 = \{u \in E : \|u\|_\infty < c, \|u''\|_\infty < L\}, \quad \Omega_2 = \{u \in E : \|u\|_\infty < b, \|u''\|_\infty < L\}$$

two boundary open sets in E, and

$$D_1 = \{u \in E : \|u\|_\infty = c\}, \quad D_2 = \{u \in E : \|u\|_\infty = b\}.$$

As in [16], we define the following double truncated continuous function as follows:

$$f^*(t,u,v) = \begin{cases} f(t,u,v) & \text{if } (t,u,v) \in I \times [0,b] \times \mathbb{R}, \\ f(t,b,v) & \text{if } (t,u,v) \in I \times [b,\infty) \times \mathbb{R}, \end{cases}$$

and

$$f_1(t,u,v) = \begin{cases} f^*(t,u,-L) & \text{if } (t,u,v) \in I \times [0,\infty) \times (-\infty,-L], \\ f^*(t,u,v) & \text{if } (t,u,v) \in I \times [0,\infty) \times [-L,L], \\ f^*(t,u,L) & \text{if } (t,u,v) \in I \times [0,\infty) \times [L,\infty). \end{cases}$$

As a direct consequence, we have that f_1 satisfies the following properties:

(H_1^1) $f_1(t,u,v) < \frac{c}{M_1}$, for $(t,u,v) \in I \times [0,c] \times \mathbb{R}$,

(H_2^1) $f_1(t,u,v) \geq \frac{b}{m}$, for $(t,u,v) \in I \times [\frac{\gamma}{R}b,\infty) \times \mathbb{R}$,

(H_3^1) $f_1(t,u,v) < \frac{L}{M_2}$, for $(t,u,v) \in I \times [0,\infty) \times \mathbb{R}$.

Now, we define the operator

$$(T_1\, u)(t) = \int_0^1 G_M(t,s) f_1(s,u(s),u''(s))\, ds,$$

whose fixed points coincide with the solutions of problem

$$u^{(4)}(t) + Mu(t) = f_1(t,u(t),u''(t)), \quad t \in I, \tag{3.7}$$

coupled to boundary conditions (1.5).

As in Lemmas 3.2 and 3.3 it is not difficult to verify that $T_1 : K_\gamma^\delta \to K_\gamma^\delta$ is a completely continuous operator.

Let $p = \frac{1}{2}b \in (\Omega_2 \cap K_\gamma^\delta) \setminus \{0\}$. It is easy to see that $\alpha(u+\mu p) \geq \alpha(u)$ for all $u \in K_\gamma^\delta$ and $\mu \geq 0$.

In view of (H_1) and $\alpha(u) = c$, $u \in D_1 \cap K_\gamma^\delta$, we have that

$$\alpha(T_1 u) = \max_{t\in I} \left| \int_0^1 G_M(t,s) f_1(s,u(s),u''(s))\, ds \right| < \max_{t\in I} \int_0^1 G_M(t,s) \frac{c}{M_1}\, ds \leq c.$$

Hence, $\alpha(T_1 u) < c$.

Therefore, using (H_2) and the fact that $u(s) \geq \frac{\gamma}{R}\alpha(u)$ for all $s \in [0,\delta]$, we have for all $u \in D_2 \cap K_\gamma^\delta$ the following inequality is fulfilled

$$\alpha(T_1 u) = \max_{t\in I} \left| \int_0^1 G_M(t,s) f_1(s,u(s),u''(s))\, ds \right| > \max_{t\in I} \int_0^\delta G_M(t,s) \frac{b}{m}\, ds \geq b.$$

Hence, $\alpha(T_1 u) > b$.

$$\beta(T_1 u) = \max_{t\in I} \left| \int_0^1 \frac{\partial^2 G_M}{\partial t^2}(t,s) f_1(s,u(s),u''(s))\, ds \right| < \max_{t\in I} \int_0^1 \left| \frac{\partial^2 G_M}{\partial t^2}(t,s) \right| \frac{L}{M_2}\, ds \leq L.$$

Hence, $\beta(T_1 u) < L$.

Therefore, u is a positive solution for the boundary value problem (3.7), (1.5) satisfying

$$c < \|u\|_\infty < b, \quad \|u''\|_\infty < L.$$

From the definition of function f_1, we conclude that the obtained solutions are also solutions of (1.4)–(1.5) and the proof is complete. □

4 Examples

In the sequel, we will obtain the different bounds and results for the particular case when $M = 0$ and $v(t) = 1$ for all $t \in I$. That is, we want to prove the existence of positive solutions of the problem:

$$L_0 u(t) = u^{(4)}(t) = f(t, u(t), u''(t)), \quad t \in [0,1] \tag{4.1}$$

subject to the boundary conditions:

$$u(1) = u'(0) = u'(1) = 0, \quad u(0) = \lambda \int_0^1 u(s)\, ds. \tag{4.2}$$

It is immediate to verify that

$$C_0 := \int_0^1 (t-1)^2(1+2t)dt = \frac{1}{2}.$$

As a consequence: $0 < \lambda < 2$.

Now, let us obtain the correspondent δ, γ and R. The expression of the related Green's function is given in Lemma 2.3.

Using the notation in Theorem 2.10, we have

$$\tilde{\varphi}(t,s) = \begin{cases} \phi_1(t,s) & \text{if } 0 < s < t < 1, \\ \psi_1(t) & \text{if } s = 0, \\ \psi_2(t) & \text{if } s = 1, \\ \phi_2(t,s) & \text{if } 0 < t \le s < 1. \end{cases}$$

So we have

$$\phi_1(t,s) = \frac{(-1+t)^2\left(-4(s+2st) - 4t(-3+\lambda) + \lambda + s^2(1+2t)\lambda\right)}{(-1+s)^2\lambda},$$

$$\phi_2(t,s) = \frac{2t^3(-2+\lambda) + 2st^2(-3+2t)(-2+\lambda) + s^2(-1+t)^2(1+2t)\lambda}{s^2\lambda},$$

$$\psi_1(t) = \frac{(-1+t)^2(-4t(-3+\lambda)+\lambda)}{\lambda}$$

and

$$\psi_2(t) = 1 + \frac{t^2(12 - 9\lambda + 4t(-3+2\lambda))}{\lambda}.$$

It is clear that

$$\frac{\partial \tilde{\varphi}}{\partial s}(t,s) = \begin{cases} \frac{\partial \phi_1(t,s)}{\partial s} & \text{if } 0 < s < t < 1, \\ 0 & \text{if } s = 0 \text{ or } s = 1, \\ \frac{\partial \phi_2(t,s)}{\partial s} & \text{if } 0 < t \le s < 1, \end{cases}$$

where

$$\frac{\partial \phi_1(t,s)}{\partial s} = -\frac{2(-1+t)^2(1+s+2(-2+s)t)(-2+\lambda)}{(-1+s)^3\lambda}$$

and

$$\frac{\partial \phi_2(t,s)}{\partial s} = -\frac{2t^2(-3s+2(1+s)t)(-2+\lambda)}{s^3\lambda}.$$

Let $\alpha_1(t) = \frac{4t-1}{2t+1}$ and $\alpha_2(t) = \frac{2t}{3-2t}$, it is obvious that

$$\frac{\partial \phi_1(t,s)}{\partial s} = 0 \text{ if and only if } s = \alpha_1(t)$$

and

$$\frac{\partial \phi_2(t,s)}{\partial s} = 0 \text{ if and only if } s = \alpha_2(t).$$

- If $t \in [0, \frac{1}{4}]$, in this case $\phi_1(t,\cdot)$ is decreasing on $[0,t]$ and $\phi_2(t,\cdot.)$ is decreasing on $[t,1]$. In this case for all $t \in [0, \frac{1}{4}]$, $\max_{s\in I} \widetilde{\varphi}(t,s) = \psi_1(t)$ and $h(t) = \min_{s\in I} \widetilde{\varphi}(t,s) = \psi_2(t)$.
- If $t \in [\frac{1}{4}, \frac{1}{2}]$, $\alpha_1(t) \in [0,t]$ in this case $\phi_1(t,\cdot)$ is increasing on $[0,\alpha_1(t)]$ and it is decreasing on $[\alpha_1(t),t]$ and $\phi_2(t,\cdot.)$ is decreasing on $[t,1]$. Then for all $t \in [\frac{1}{4}, \frac{1}{2}]$ we have

$$\max_{s\in I} \widetilde{\varphi}(t,s) = \phi_1(t,\alpha_1(t)) = \frac{(-1+t)(1+2t)(-2+4t(-1+\lambda)-\lambda)}{2\lambda} \tag{4.3}$$

  and

$$h(t) = \min_{s\in I} \widetilde{\varphi}(t,s) = \min\{\psi_1(t),\psi_2(t)\} = \psi_2(t). \tag{4.4}$$

- If $t \in [\frac{1}{2}, \frac{3}{4}]$, $\alpha_2(t) \in [t,1]$ in this case $\phi_2(t,\cdot.)$ is increasing on $[t,\alpha_2(t)]$ and is decreasing on $[\alpha_2(t),1]$ and $\phi_1(t,\cdot)$ is increasing on $[0,t]$. Then for all $t \in [\frac{1}{2}, \frac{3}{4}]$ we have

$$\max_{s\in I} \widetilde{\varphi}(t,s) = \phi_2(t,\alpha_2(t)) = \frac{2\lambda + t(-3+2t)(-6+4t+3\lambda)}{2\lambda} \tag{4.5}$$

  and

$$h(t) = \min_{s\in I} \widetilde{\varphi}(t,s) = \min\{\psi_1(t),\psi_2(t)\} = \psi_1(t). \tag{4.6}$$

- If $t \in [\frac{3}{4}, 1]$, in this case $\phi_1(t,\cdot)$ is increasing on $[0,t]$ and $\phi_2(t,\cdot.)$ is increasing on $[t,1]$. In this case for all $t \in [\frac{3}{4}, 1]$, $\max_{s\in I} \widetilde{\varphi}(t,s) = \psi_2(t)$ and $h(t) = \psi_1(t)$.

In conclusion we obtain

$$\max_{s\in I} \widetilde{\varphi}(t,s) = \begin{cases} \psi_1(t) & \text{if } t \in \left[0, \frac{1}{4}\right], \\ \phi_1(t,\alpha_1(t)) & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ \phi_2(t,\alpha_2(t)) & \text{if } t \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ \psi_2(t) & \text{if } t \in \left[\frac{3}{4}, 1\right], \end{cases}$$

and

$$\min_{s\in I} \widetilde{\varphi}(t,s) = \begin{cases} \psi_2(t) & \text{if } t \in \left[0, \frac{1}{2}\right], \\ \psi_1(t) & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Let $R_1 = \max_{t\in\left[\frac{1}{4},\frac{1}{2}\right]} \phi_1(t,\alpha_1(t))$, $R_2 = \max_{t\in\left[\frac{1}{2},\frac{3}{4}\right]} \phi_2(t,\alpha_2(t))$, $R_3 = \max_{t\in\left[0,\frac{1}{4}\right]} \psi_1(t)$ and $R_4 = \max_{t\in\left[\frac{3}{4},1\right]} \psi_2(t)$. We deduce that $R = \max_{(t,s)\in I\times I} \widetilde{\varphi}(t,s) = \max\{R_1,R_2,R_3,R_4\}$ and

$$\gamma = \min_{t\in[0,\delta]} h(t) = \begin{cases} \min\{1,\psi_2(\delta)\} & \text{if } \delta \in \left(0, \frac{1}{2}\right], \\ \min\{1,\psi_1(\delta)\} & \text{if } \delta \in \left[\frac{1}{2}, 1\right). \end{cases}$$

Choosing $\delta = 0.9$ and $\lambda = 1$. By computation we obtain

$R = \max\{1.6875, 1, 1.6875, 1\} = 1.6875$, $\gamma = \min_{t\in[0,\delta]} h(t) = 0.082$, $\frac{\gamma}{R} = 0.0485926$.

By simple calculation, we have that $M_2 = \max_{t\in I} \int_0^1 \left|\frac{\partial^2 G_0}{\partial t^2}(t,s)\right| ds \approx 0.1$, $m = \max_{t\in I} \int_0^\delta G_0(t,s)\, ds \approx 0.00417006$ and $M_1 = \max_{t\in I} \int_0^1 G_0(t,s)\, ds \approx 0.0042$.

Example 4.1. Let

$$f(t,u,v) = \frac{t}{100} + 4.71241u + 0.000416894\, u^3 + (0.00521618 + 0.000125066\, u^2)\frac{|v|}{90}.$$

Choosing $b = 60$, $\frac{\gamma}{R}b = 2.91556$, $\frac{\gamma}{Rm} = 11.6527$, $c = 0.5$ and $L = 400$. By simple calculation, f satisfy (H_0) and we have that:

$$f(t,u,v) \le 2.38958 < \frac{c}{M_1} = 119.048 \quad \text{for all } (t,u,v) \in I \times [0,c] \times [-L,L],$$
$$f(t,u,v) \ge 13.7496 > \frac{\gamma}{Rm} = 11.6527 \quad \text{for all } (t,u,v) \in I \times \left[\frac{\gamma}{R}b, b\right] \times [-L,L]$$

and

$$f(t,u,v) \le 374.828 < \frac{L}{M_2} = 4000 \quad \text{for all } (t,u,v) \in I \times [0,b] \times [-L,L].$$

With the use of Theorem 3.4, the boundary value problem (1.4)–(1.5) has at least one positive solution u satisfying

$$0.5 < \|u\|_\infty < 60, \qquad \|u''\|_\infty < 400.$$

Example 4.2. Let

$$f(t,u,v) = a(t)u + b(t)u^3 + c(t)|v|^\alpha, \ \alpha \in (0,1)$$

where

$$a(t) = \begin{cases} t+10 & \text{if } t \in [0,\frac{1}{2}], \\ -t+11 & \text{if } t \in [\frac{1}{2},\frac{3}{4}], \\ 3t+8 & \text{if } t \in [\frac{3}{4},1], \end{cases} \qquad b(t) = \begin{cases} e^t & \text{if } t \in [0,\frac{1}{2}], \\ 2e^{\frac{1}{2}}t & \text{if } t \in [\frac{1}{2},1] \end{cases}$$

and

$$c(t) = \begin{cases} \left(\dfrac{10^{-3}}{9}\right)^\alpha \dfrac{1}{2^\alpha} & \text{if } t \in [0,\frac{1}{2}], \\ \left(\dfrac{10^{-3}}{9}\right)^\alpha t^\alpha & \text{if } t \in [\frac{1}{2},1]. \end{cases}$$

For this we have, for all $t \in [0,1]$, $10 \le a(t) \le 11$, $1 \le b(t) \le 2e^{\frac{1}{2}}$ and $\left(\frac{10^{-3}}{9}\right)^\alpha \frac{1}{2^\alpha} \le c(t) \le \left(\frac{10^{-3}}{9}\right)^\alpha$. Choosing $c = 1$, $b = 30$, $\frac{\gamma}{R}b = 1.457778$ and $L = 9 \times 10^3$. By simple calculation, we have that:

$$f(t,u,v) \le 15.2974 < \frac{c}{M_1} = 238.09 \quad \text{for all } (t,u,v) \in I \times [0,c] \times [-L,L],$$
$$f(t,u,v) \ge 17.6757 > \frac{\gamma}{Rm} = 11.6527 \quad \text{for all } (t,u,v) \in I \times \left[\frac{\gamma}{R}b, b\right] \times [-L,L]$$

and

$$f(t,u,v) \le 89361.9486 < \frac{L}{M_2} = 90000 \quad \text{for all } (t,u,v) \in I \times [0,b] \times [-L,L].$$

With the use of Theorem 3.4, the boundary value problem (1.4)–(1.5) has at least one positive solution u satisfying

$$1 < \|u\|_\infty < 30, \qquad \|u''\|_\infty < 9 \times 10^3.$$

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On the double layer potential ansatz for the n-dimensional Helmholtz equation with Neumann condition

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. In the present paper we consider the Neumann problem for the n-dimensional Helmholtz equation. In particular we deal with the problem of representability of the solutions by means of double layer potentials.

Keywords: Helmholtz equation, potential theory, integral representations.

2020 Mathematics Subject Classification: 35J57, 31B10, 35C15.

1 Introduction

Some time ago one of the authors proposed a method for treating the boundary integral equation of the first kind arising when you impose the Dirichlet condition for Laplace equation to a simple layer potential [2]. This method hinges on the theory of reducible operators and on the theory of differential forms, it does not use the theory of pseudodifferential operators and could be considered as an extension to higher dimensions of Muskhelishvili's method (see [3]). Later, this approach was extended to different BVPs for several partial differential equations and systems in simply and multiple connected domains (see [5] and the references therein).

Recently we have showed how to use this approach to solve the Dirichlet problem for the n-dimensional Helmholtz equation by means of a simple layer potential [6]. The aim of the present paper is to continue that investigation, showing how our method could be used to solve the Neumann problem for the same equation by means of a double layer potential. To this end, we make use of some fundamental results given by Colton and Kress in their celebrated monograph [7], in particular on the description of the traces on the boundary of eigensolutions of Dirichlet or Neumann problems. Colton and Kress proved their results in

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spaces of continuous functions on a C^2 boundary. As already remarked in [6], the same results can be established under more general assumptions by nowadays standard arguments in potential theory (see, e.g., [10]). In particular, they hold in L^p spaces on a Lyapunov boundary. When we consider their results, we shall always refer to them under these more general hypotheses.

Differently from [7], here we consider the Neumann problem with data in $L^p(\Sigma)$ and we obtain that the solution can be represented as a double layer potential with density in the Sobolev space $W^{1,p}(\Sigma)$.

We shall consider domains in $\mathbb{R}^n$, with $n \geq 3$. In principle our method could be applied also for $n = 2$ with some appropriate modifications, as to change fundamental solution and radiation condition (see [7, pp. 106–107]).

The paper is organized as follows. After summarizing notations and definitions in Section 2, we collect some preliminary results in Section 3. We mention that we prove a regularity result for the eigensolutions of a certain integral equation (see Proposition 3.2) without using the usual regularity properties of the double layer potential (see [8] for recent results in this direction and for an extensive bibliography). Our approach seems to be simpler and it is a consequence of some of our previous results on Laplace equation.

In the short Section 4 we recall the main result we have obtained in [6] for the Dirichlet problem. Section 5 is devoted to the main result of the present paper: we prove that the Neumann problem with data in $L^p(\Sigma)$ $(1 < p < \infty)$ can be represented by a double layer potential with density in $W^{1,p}(\Sigma)$ if and only if the data satisfies some necessary orthogonality conditions.

2 Notations and definitions

From now on Ω will be a bounded domain (open connected set) of $\mathbb{R}^n$ $(n \geq 3)$ whose boundary Σ is a Lyapunov hypersurface (i.e. Σ has a uniformly Hölder continuous normal field of some exponent $\lambda \in (0, 1]$), and such that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected; $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ denotes the outwards unit normal vector at the point $x = (x_1, \ldots, x_n) \in \Sigma$. The Euclidean norm for elements of $\mathbb{R}^n$ is denoted by $|\cdot|$.

Now fix $1 < p < \infty$. By $L^p(\Sigma)$ we denote the space of p-integrable complex-valued functions defined on Σ. By $L^p_h(\Sigma)$ we mean the space of the differential forms of degree $h \geq 1$ whose components belong to $L^p(\Sigma)$.

The Sobolev space $W^{1,p}(\Sigma)$ can be defined as the space of functions in $L^p(\Sigma)$ such that their weak differential belongs to $L^p_1(\Sigma)$.

If u is an h-form in Ω, the symbol du denotes the differential of u, while $*u$ denotes the dual Hodge form. Finally, we write $\underset{\Sigma}{*}w = w_0$ if w is an $(n-1)$-form on Σ and $w = w_0 d\sigma$.

Besides the theory of differential forms, the method we use hinges on the theory of reducible operators. Here we recall that, given two Banach spaces E and F, a continuous linear operator $S : E \to F$ can be reduced on the left if there exists a continuous linear operator $S' : F \to E$ such that $S'S = I + T$, I being the identity operator on E and T a compact operator on E. An operator S reducible on the right can be defined analogously. If S can be reduced (on the left or right), then its range is closed and, as a consequence the equation $S\alpha = \beta$ admits a solution if and only if $\langle \gamma, \beta \rangle = 0$, for any $\gamma \in F^*$ such that $S^*\gamma = 0$, where S^* is the adjoint of S. For more details we refer the readers, e.g., to [9] or [11].

We consider the n-dimensional Helmholtz equation

$$\Delta u + k^2 u = 0 \tag{2.1}$$

where $k \in \mathbb{C} \setminus \{0\}$, $\mathrm{Im}(k) \geq 0$, $u : \Omega \to \mathbb{C}$, and Δ is the Laplace operator. The fundamental solution of (2.1) is given by

$$\Phi(x) = \frac{i}{4}\left(\frac{k}{2\pi|x|}\right)^{(n-2)/2} H^{(1)}_{(n-2)/2}(k|x|)$$

where $H^{(1)}_{\mu}$ is the Hankel function of the first kind of order μ (see, e.g., [1, p. 42]). In what follows it will be useful to consider the auxiliary function

$$h(x) = \Phi(x) - s(x) \quad (x \in \mathbb{R}^n \setminus \{0\})\,,$$

where s is the fundamental solution of $-\Delta$, i.e. for $n \geq 3$ and $x \in \mathbb{R}^n \setminus \{0\}$,

$$s(x) = \frac{1}{(n-2)\omega_n}|x|^{2-n} \qquad \left(\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}\right).$$

We observe that (see [12, Lemma A.5, p. 571])

$$|\nabla h(x)| \leq c|x|^{3-n}, \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}. \tag{2.2}$$

Hence, from (2.2), and recalling that $|\nabla s(x)| \leq c_1|x|^{1-n}$, immediately we get

$$|\nabla \Phi(x)| \leq c_2|x|^{1-n}\,. \tag{2.3}$$

Moreover,

$$\left|\frac{\partial^2 h(x)}{\partial x_j \partial x_l}\right| \leq c|x|^{2-n}, \quad \forall\, x \in \mathbb{R}^n \setminus \{0\},\ j,l = 1,\dots,n. \tag{2.4}$$

As we shall see, we are interested to solve the Neumann problem related to the Helmholtz equation (2.1) in the class of potentials defined as follows.

Definition 2.1. We say that a function w belongs to the space $\mathcal{D}^p$ if and only if there exists $\psi \in W^{1,p}(\Sigma)$ such that w can be represented by means of a double layer potential with density ψ, i.e.

$$w(x) = \int_\Sigma \psi(y)\,\frac{\partial \Phi}{\partial \nu_y}(x-y)\,d\sigma_y, \quad x \in \Omega\,.$$

We also recall the following class of functions used in [6].

Definition 2.2. We say that a function u belongs to the space $\mathcal{S}^p$ if and only if there exists $\varphi \in L^p(\Sigma)$ such that u can be represented by means of a simple layer potential with density φ, i.e.

$$u(x) = \int_\Sigma \varphi(y)\,\Phi(x-y)\,d\sigma_y, \quad x \in \Omega\,. \tag{2.5}$$

We shall distinguish by indices $+$ and $-$ the nontangential limit obtained by approaching the boundary Σ from $\mathbb{R}^n \setminus \overline{\Omega}$ and Ω, respectively (see, e.g. [10, p. 293]).

We remark that by $\langle f, g\rangle$ we denote the bilinear form

$$\int_\Sigma f\,g\,d\sigma\,.$$

3 Preliminary results

Let us introduce the integral operators:

$$K : L^p(\Sigma) \to L^p(\Sigma)\,, \quad K\varphi(x) = 2\int_\Sigma \varphi(y)\frac{\partial\Phi}{\partial\nu_y}(x-y)\,d\sigma_y$$

and its adjoint

$$K^* : L^q(\Sigma) \to L^q(\Sigma)\,, \quad K^*\psi(x) = 2\int_\Sigma \psi(y)\frac{\partial\Phi}{\partial\nu_x}(x-y)\,d\sigma_y\,.$$

where $1 < p < \infty$ and $\frac{1}{p}+\frac{1}{q}=1$. K and K^* are adjoint operators with respect to the duality

$$\langle\psi, K\varphi\rangle = \langle K^*\psi, \varphi\rangle.$$

Moreover, K and K^* are compact operators because of (2.3).

Here, we are interested in the kernels of the operators $I \pm K$ and $I \pm K^*$, where I is the identity operator on the relevant Lebesgue space. To this end, let us denote by $\mathcal{U}_0$ the space of solutions of

$$\begin{cases} u \in C^{1,\lambda}(\overline{\Omega}) \cap C^2(\Omega), & \\ \Delta u + k^2 u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial\nu} = 0 & \text{on } \Sigma \end{cases}$$

and by $\mathcal{V}_0$ the space of solutions of

$$\begin{cases} u \in C^{1,\lambda}(\overline{\Omega}) \cap C^2(\Omega), & \\ \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma\,. \end{cases}$$

Note that $\mathcal{U}_0 = \{0\}$ (resp. $\mathcal{V}_0 = \{0\}$) whenever k^2 is not an interior Neumann eigenvalue (resp. an interior Dirichlet eigenvalue).

It is known that (see [7, Theorem 3.17])

$$\mathcal{N}(I+K) = \{u_{|\Sigma} : u \in \mathcal{U}_0\} \tag{3.1}$$

and that (see [7, Theorem 3.22])

$$\mathcal{N}(I-K^*) = \left\{\frac{\partial v}{\partial\nu}\Big|_\Sigma : v \in \mathcal{V}_0\right\}\,. \tag{3.2}$$

Let $\dim\mathcal{N}(I+K) = m_N$ and $\dim\mathcal{N}(I-K) = m_D$. Note that, $m_N = 0$ if k^2 is not an interior Neumann eigenvalue, while $m_D = 0$ whenever k^2 is not an interior Dirichlet eigenvalue.

Moreover

$$\dim\mathcal{N}(I+K) = \dim\mathcal{N}(I+K^*) \quad\text{and}\quad \dim\mathcal{N}(I-K) = \dim\mathcal{N}(I-K^*)\,.$$

We have also the following lemma.

Lemma 3.1. $\mathcal{N}(I \pm K) \perp \mathcal{N}(I \mp K^*)$.

Proof. If $\alpha \in \mathcal{N}(I \pm K)$ and $\beta \in \mathcal{N}(I \mp K^*)$, then

$$\langle \alpha, \beta \rangle = \langle \mp K\alpha, \beta \rangle = \mp \langle \alpha, K^*\beta \rangle = -\langle \alpha, \beta \rangle \,,$$

and hence $\langle \alpha, \beta \rangle = 0$. $\square$

The next proposition shows that the functions in $\mathcal{N}(I-K)$ belong to the Sobolev space $W^{1,p}(\Sigma)$. As said in the introduction, this result could be deduced by regularizing properties of the double layer potential, but here we use a different approach which seems to be simpler.

Proposition 3.2. *Let $\zeta \in L^p(\Sigma)$ be a solution of the equation $\zeta - K\zeta = 0$. Then ζ belongs to $W^{1,p}(\Sigma)$.*

Proof. Since $\zeta \in \mathcal{N}(I-K)$, the potential

$$v(x) = \int_\Sigma \zeta(y) \frac{\partial \Phi}{\partial \nu_y}(x-y)\, d\sigma_y$$

satisfies the condition $v_- = 0$ on Σ.

We can write the equation $\zeta - K\zeta = 0$ as

$$-\frac{1}{2}\zeta(x) + \int_\Sigma \zeta(y) \frac{\partial s}{\partial \nu_y}(x-y)\, d\sigma_y = T(x),$$

where

$$T(x) = -\int_\Sigma \zeta(y) \frac{\partial h}{\partial \nu_y}(x-y)\, d\sigma_y \,.$$

Thanks to (2.2) and (2.4), the function T belongs to $W^{1,p}(\Sigma)$. Therefore the harmonic function

$$a(x) = \int_\Sigma \zeta(y) \frac{\partial s}{\partial \nu_y}(x-y)\, d\sigma_y$$

satisfies the boundary condition $a = T$ on Σ. As proved in [2], the function a can be represented as a simple layer potential with density $A \in L^p(\Sigma)$:

$$a(x) = \int_\Sigma A(y)\, s(x-y)\, d\sigma_y \,.$$

This implies that there exists the normal derivative $\partial a/\partial \nu$ almost everywhere on Σ and it belongs to $L^p(\Sigma)$ (see [2, pp. 182–183]). It follows that the function ζ satisfies the condition

$$\frac{\partial}{\partial \nu_x} \int_\Sigma \zeta(y) \frac{\partial s}{\partial \nu_y}(x-y)\, d\sigma_y = \frac{\partial a}{\partial \nu}(x)$$

on Σ.

Thanks to [4, p.29] we can say that there exists a solution $\zeta_0 \in W^{1,p}(\Sigma)$ of this equation, since the right-hand side has zero mean value on Σ. Therefore

$$\frac{\partial}{\partial \nu_x} \int_\Sigma (\zeta(y) - \zeta_0(y)) \frac{\partial s}{\partial \nu_y}(x-y)\, d\sigma_y = 0$$

on Σ and the potential

$$\int_\Sigma (\zeta(y) - \zeta_0(y)) \frac{\partial s}{\partial \nu_y}(x-y)\, d\sigma_y$$

has to be constant in Ω. It follows $\zeta = \zeta_0 + c$ and this completes the proof. $\square$

In the following theorem we collect some useful results contained in [7, Theorems 3.18 and 3.23].

Theorem 3.3.

(i) Let $\{\lambda_1, \dots, \lambda_{m_N}\}$ *be a basis of* $\mathcal{N}(I + K^*)$ *and define*

$$u_j(x) = \int_\Sigma \lambda_j(y)\Phi(x-y)\, d\sigma_y \quad x \in \mathbb{R}^n \setminus \Sigma, j = 1, \dots, m_N .$$

Then

$$\lambda_j = -\frac{\partial u_j}{\partial \nu^+} \quad on\ \Sigma, j = 1, \dots, m_N ,$$

and the functions

$$\rho_j = -\overline{u}_{j,+} \quad on\ \Sigma, j = 1, \dots, m_N$$

form a basis of $\mathcal{N}(I + K)$.

Moreover, the determinant of the matrix $(\langle \rho_j, \lambda_l \rangle)_{j,l=1,\dots,m_N}$ *is nonzero.*

(ii) Let $\{\zeta_1, \dots, \zeta_{m_D}\}$ *be a basis of* $\mathcal{N}(I - K)$ *and define*

$$v_j(x) = \int_\Sigma \zeta_j(y) \frac{\partial \Phi}{\partial \nu_y}(x-y)\, d\sigma_y \quad x \in \mathbb{R}^n \setminus \Sigma, j = 1, \dots, m_D .$$

Then

$$\zeta_j = v_{j,+} \quad on\ \Sigma, j = 1, \dots, m_D ,$$

and the functions

$$\mu_j = \frac{\partial v_j}{\partial \nu^+} \quad on\ \Sigma, j = 1, \dots, m_D \tag{3.3}$$

form a basis of $\mathcal{N}(I - K^*)$.

Moreover, the determinant of the matrix $(\langle \mu_j, \zeta_l \rangle)_{j,l=1,\dots,m_D}$ *is nonzero.*

Remark 3.4. Thanks to the Lyapunov property of the double layer potential (see [7, Theorem 2.21]), (3.3) is equivalent to

$$\mu_j = \frac{\partial v_j}{\partial \nu^-} \quad \text{on } \Sigma, j = 1, \dots, m_D . \tag{3.4}$$

4 The Dirichlet problem

In this section we describe the main lines of the method applied in [6] to the Dirichlet problem

$$\begin{cases} u \in S^p, \\ \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Sigma, f \in W^{1,p}(\Sigma) . \end{cases} \tag{4.1}$$

First, we imposed the boundary condition to (2.5), obtaining

$$\int_\Sigma \varphi(y)\Phi(x-y)\, d\sigma_y = f(x), \quad x \in \Sigma . \tag{4.2}$$

Then, taking the exterior differential d of both sides of the integral equation of the first kind (4.2), we get the singular integral equation

$$S\varphi(x) = df(x), \quad \text{a.e. } x \in \Sigma, \tag{4.3}$$

where

$$S\varphi(x) = \int_\Sigma \varphi(y) d_x[\Phi(x-y)] d\sigma_y .$$

The singular integral operator $S : L^p(\Sigma) \to L^p_1(\Sigma)$ can be reduced on the left by the singular integral operator $J' : L^p_1(\Sigma) \longrightarrow L^p(\Sigma)$ defined as

$$J'\psi(z) = \underset{\Sigma}{*} \int_\Sigma \psi(x) \wedge d_z[s_{n-2}(z,x)], \quad z \in \Sigma,$$

with

$$s_{n-2}(x,y) = \sum_{j_1 < \ldots < j_{n-2}} s(x-y) dx^{j_1} \ldots dx^{j_{n-2}} dy^{j_1} \ldots dy^{j_{n-2}}$$

being the Hodge double $(n-2)$-form (see [6, Theorem 2]).

Therefore, the range of S is closed and equation (4.3) has a solution $\varphi \in L^p(\Sigma)$ if and only if

$$\int_\Sigma \gamma \wedge df = 0$$

for every $\gamma \in W^{1,q}_{n-2}(\Sigma)$ $(q = p/(p-1))$(*) such that $d\gamma = \frac{\partial v}{\partial \nu} d\sigma$, for all $v \in \mathcal{V}_0$ (see [6, Theorem 4]).

Using the above results, we proved the representability theorem for the Dirichlet problem via simple layer potentials, rewritten here in a new form.

Theorem 4.1. *Let $f \in W^{1,p}(\Sigma)$. There exists a solution of (4.1) if and only if f satisfies the compatibility conditions*

$$\int_\Sigma f \mu_j \, d\sigma = 0 \quad \textit{for every } j = 1, \ldots, m_D . \tag{4.4}$$

Proof. From [6, Theorem 5] it follows that there exists a solution of (4.1) if and only if f satisfies the compatibility conditions

$$\int_\Sigma f \frac{\partial v}{\partial \nu} d\sigma = 0 \quad \text{for all } v \in \mathcal{V}_0 . \tag{4.5}$$

Conditions (4.5) and (4.4) are equivalent because of (3.2), Theorem 3.3-(ii), and (3.4). □

5 The Neumann problem

In this section we consider the Neumann problem

$$\begin{cases} w \in \mathcal{D}^p, \\ \Delta w + k^2 w = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \nu} = g & \text{on } \Sigma, \end{cases} \tag{5.1}$$

(*) By $W^{1,q}_{n-2}(\Sigma)$ we denote the space of differential forms of degree $n-2$ whose coefficients belong to $W^{1,q}(\Sigma)$.

where $g \in L^p(\Sigma)$ satisfies

$$\int_\Sigma gu\, d\sigma = 0, \quad \forall\, u \in \mathcal{U}_0 . \tag{5.2}$$

Observe that conditions (5.2) are necessary for the solvability of the problem (5.1) because of Green's formulas.

Moreover, conditions (5.2) can be rewritten as

$$\int_\Sigma g\rho_j\, d\sigma = 0, \quad j = 1, \dots, m_N . \tag{5.3}$$

We begin by stating some preliminary results.

Proposition 5.1. *Consider $u \in \mathcal{S}^p$ with density $\varphi \in L^p(\Sigma)$ and let $W_0 \in \mathcal{D}^p$ with density u:*

$$W_0(x) = \int_\Sigma u(y) \frac{\partial \Phi}{\partial \nu_y}(x-y)\, d\sigma_y, \quad x \in \Omega .$$

Then

$$\frac{\partial W_0}{\partial \nu}(x) = -\frac{1}{4}\varphi(x) + \frac{1}{4}K^{*2}\varphi(x) . \tag{5.4}$$

for almost every $x \in \Sigma$.

Proof. First observe that u solves equation (2.1), and hence (see [7, Theorem 3.1])

$$u(x) = \int_\Sigma \left\{ \Phi(x-y)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Phi}{\partial \nu_y}(x-y) \right\} d\sigma_y, \quad x \in \Omega .$$

Moreover, for u the following jump relation holds (see [7, Theorem 2.19])

$$\frac{\partial u}{\partial \nu^-}(x) = \lim_{\substack{y \to x \\ y \in \nu_x^-}} \frac{\partial u}{\partial \nu}(y) = \frac{1}{2}\,\varphi(x) + \int_\Sigma \varphi(y)\frac{\partial \Phi}{\partial \nu_x}(x-y)\, d\sigma_y,$$

almost everywhere on Σ. We have also

$$\begin{aligned}
\frac{\partial W_0}{\partial \nu}(x) &= \frac{\partial}{\partial \nu}\left\{ -u(x) + \int_\Sigma \Phi(x-y)\frac{\partial u}{\partial \nu}(y) d\sigma_y \right\} \\
&= -\frac{\partial u}{\partial \nu}(x) + \frac{\partial}{\partial \nu_x}\int_\Sigma \Phi(x-y)\frac{\partial u}{\partial \nu}(y) d\sigma_y \\
&= \left(\frac{1}{2} - 1\right)\frac{\partial u}{\partial \nu}(x) + \int_\Sigma \frac{\partial u}{\partial \nu}(y)\frac{\partial \Phi}{\partial \nu_x}(x-y)\, d\sigma_y \\
&= -\frac{1}{2}\left\{ \frac{1}{2}\varphi(x) + \int_\Sigma \varphi(y)\frac{\partial \Phi}{\partial \nu_x}(x-y)\, d\sigma_y \right\} \\
&\quad + \int_\Sigma \left\{ \frac{1}{2}\varphi(y) + \int_\Sigma \varphi(z)\frac{\partial \Phi}{\partial \nu_y}(y-z)\, d\sigma_z \right\} \frac{\partial \Phi}{\partial \nu_x}(x-y)\, d\sigma_y \\
&= -\frac{1}{4}\varphi(x) + \int_\Sigma \varphi(z)\, d\sigma_z \int_\Sigma \frac{\partial \Phi}{\partial \nu_y}(y-z)\frac{\partial \Phi}{\partial \nu_x}(x-y)\, d\sigma_y.
\end{aligned}$$

Hence formula (5.4) is proved. □

Lemma 5.2. *The Fredholm equation*

$$-\varphi + K^{*2}\varphi = 4g , \tag{5.5}$$

where $g \in L^p(\Sigma)$, admits a solution $\varphi \in L^p(\Sigma)$ if and only if conditions

$$\int_\Sigma g\rho_j \, d\sigma = 0, \quad j = 1, \dots, m_N \tag{5.6}$$

and

$$\int_\Sigma g\zeta_i \, d\sigma = 0, \quad i = 1, \dots, m_D \tag{5.7}$$

are satisfied.

Proof. Assume that (5.6) and (5.7) are satisfied and rewrite equation (5.5) as

$$(I + K^*)(-I + K^*)\varphi = 4g.$$

Observe that the equation $(I + K^*)\gamma = 4g$ admits a solution because of (5.6). Denote by γ_0 such a solution and consider

$$(-I + K^*)\varphi = \gamma_0. \tag{5.8}$$

The last equation is solvable if and only if $\langle \gamma_0, \zeta_i \rangle = 0$ for every $\zeta_i \in \mathcal{N}(I - K)$, $i = 1, \dots, m_D$. We have

$$\langle \gamma_0, \zeta_i \rangle = \langle \gamma_0, K\zeta_i \rangle = \langle K^*\gamma_0, \zeta_i \rangle = -\langle \gamma_0, \zeta_i \rangle + \langle 4g, \zeta_i \rangle,$$

and then, thanks to (5.7),

$$\langle \gamma_0, \zeta_i \rangle = \langle 2g, \zeta_i \rangle = 0, \quad i = 1, \dots, m_D.$$

This shows that there exists a solution φ of (5.8). Therefore φ satisfies (5.5).

Conversely, if φ is such that (5.5) holds, we have

$$(-I + K^*)(I + K^*)\varphi = 4g.$$

In particular, $4g \in \mathcal{R}(I - K^*) = \mathcal{N}(I - K)^\perp$, and then conditions (5.7) are satisfied. On the other hand, $(I + K^*)(-I + K^*)\varphi = 4g$, hence $4g \in \mathcal{R}(I + K^*) = \mathcal{N}(I + K)^\perp$, and then all conditions in (5.6) hold. □

Lemma 5.3. *Given $\psi \in W^{1,p}(\Sigma)$ there exist $\varphi \in L^p(\Sigma)$ and $c_1, \dots, c_{m_D} \in \mathbb{C}$ such that*

$$\psi(x) = \int_\Sigma \varphi(y)\Phi(x - y) \, d\sigma_y + \sum_{i=1}^{m_D} c_i \zeta_i(x), \quad x \in \Sigma. \tag{5.9}$$

The vector $(c_1, \dots, c_{m_D})$ is the unique solution of the system

$$\sum_{i=1}^{m_D} c_i \langle \zeta_i, \mu_j \rangle = \langle \psi, \mu_j \rangle, \quad j = 1, \dots, m_D. \tag{5.10}$$

Proof. Let $\psi \in W^{1,p}(\Sigma)$. In view of Proposition 3.2 the function $\psi - \sum_{i=1}^{m_D} c_i \zeta_i$ belongs to $W^{1,p}(\Sigma)$ for any $c_1, \dots, c_{m_D}$. Thanks to Theorem 4.1, there exists $\varphi \in L^p(\Sigma)$ satisfying (5.9) if and only if

$$\int_\Sigma \left(\psi - \sum_{i=1}^{m_D} c_i \zeta_i \right) \mu_j d\sigma = 0, \quad j = 1, \dots, m_D,$$

that is, $(c_1, \dots, c_{m_D})$ is solution of system (5.10). Note that the constants $c_1, \dots, c_{m_D}$ are uniquely determined since the determinant of the matrix $(\langle \mu_j, \zeta_l \rangle)_{j,l=1,\dots,m_D}$ is nonzero (see Theorem 3.3). □

Theorem 5.4. *There exists a solution of* (5.1) *if and only if* g *satisfies* (5.2).

Proof. Assume that g satisfies (5.2). Let $(c_1, \dots, c_{m_D})$ be the solution of the system

$$\sum_{i=1}^{m_D} c_i \int_\Sigma \mu_i \zeta_j d\sigma = \int_\Sigma g \zeta_j d\sigma\,, \quad j = 1, \dots, m_D \tag{5.11}$$

and consider the potential

$$w(x) = \int_\Sigma \left(\int_\Sigma \varphi(z) \Phi(y-z) d\sigma_z \right) \frac{\partial \Phi}{\partial \nu_y}(x-y) d\sigma_y + \sum_{i=1}^{m_D} c_i \int_\Sigma \zeta_i(y) \frac{\partial \Phi}{\partial \nu_y}(x-y)\, d\sigma_y\,, \quad x \in \Omega\,,$$

where $\varphi \in L^p(\Sigma)$ has to be determined. By imposing the boundary condition we obtain

$$\begin{aligned} &\frac{\partial}{\partial \nu_x} \int_\Sigma \left(\int_\Sigma \varphi(z) \Phi(y-z) d\sigma_z \right) \frac{\partial \Phi}{\partial \nu_y}(x-y) d\sigma_y + \sum_{i=1}^{m_D} c_i \frac{\partial}{\partial \nu_x} \int_\Sigma \zeta_i(y) \frac{\partial \Phi}{\partial \nu_y}(x-y)\, d\sigma_y \\ &= -\frac{1}{4}\varphi(x) + \frac{1}{4} K^{*2} \varphi(x) + \sum_{i=1}^{m_D} c_i \mu_i(x) = g(x)\,, \quad x \in \Sigma\,, \end{aligned}$$

because of (5.4), (3.3), and (3.4). Then w satisfies the boundary conditions if and only if

$$-\varphi + K^{*2}\varphi = 4 \left(g - \sum_{i=1}^{m_D} c_i \mu_i \right) \quad \text{on } \Sigma\,.$$

By virtue of Lemma 5.2, there exists a solution $\varphi \in L^p(\Sigma)$ of this equation if and only if

$$\int_\Sigma \left(g - \sum_{i=1}^{m_D} c_i \mu_i \right) \rho_j d\sigma = 0, \quad j = 1, \dots, m_N \tag{5.12}$$

and

$$\int_\Sigma \left(g - \sum_{i=1}^{m_D} c_i \mu_i \right) \zeta_j d\sigma = 0, \quad j = 1, \dots, m_D\,. \tag{5.13}$$

Conditions (5.12) are satisfied because

$$\int_\Sigma \left(g - \sum_{i=1}^{m_D} c_i \mu_i \right) \rho_j d\sigma = - \sum_{i=1}^{m_D} c_i \int_\Sigma \mu_i \rho_j d\sigma = 0$$

thanks to (5.3) and Lemma 3.1. On the other hand, conditions (5.13) hold in view of (5.11).

Conversely, let $w \in \mathcal{D}^p$ be a solution of (5.1) with density $\psi \in W^{1,p}(\Sigma)$. From Lemma 5.3, ψ can be written as in (5.9). Therefore,

$$-\varphi + K^{*2}\varphi + 4 \sum_{i=1}^{m_D} c_i \mu_i = 4g \quad \text{on } \Sigma\,.$$

Now we consider $u \in \mathcal{U}_0$. From (3.1), $u|_\Sigma \in \mathcal{N}(I+K)$ and, from Lemma 3.1, $\int_\Sigma \mu_i u\, d\sigma = 0$.

On the other hand, $-\varphi + K^{*2}\varphi \in \mathcal{R}(I+K^*) = \mathcal{N}(I+K)^\perp$, and hence we have that $\int_\Sigma (-\varphi + K^{*2}\varphi) u\, d\sigma = 0$.

Accordingly,

$$\int_\Sigma 4gu\, d\sigma = \int_\Sigma (-\varphi + K^{*2}\varphi) u\, d\sigma + 4 \sum_{i=1}^{m_D} c_i \int_\Sigma \mu_i u\, d\sigma = 0$$

and condition (5.2) is fulfilled. □

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Decaying positive global solutions of second order difference equations with mean curvature operator

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. A boundary value problem on an unbounded domain, associated to difference equations with the Euclidean mean curvature operator is considered. The existence of solutions which are positive on the whole domain and decaying at infinity is examined by proving new Sturm comparison theorems for linear difference equations and using a fixed point approach based on a linearization device. The process of discretization of the boundary value problem on the unbounded domain is examined, and some discrepancies between the discrete and the continuous cases are pointed out, too.

Keywords: second order nonlinear difference equations, Euclidean mean curvature operator, boundary value problems, decaying solutions, recessive solutions, comparison theorems.

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1 Introduction

In this paper we study the boundary value problem (BVP) on the half-line for difference equation with the Euclidean mean curvature operator

$$\Delta\left(a_k \frac{\Delta x_k}{\sqrt{1+(\Delta x_k)^2}}\right) + b_k F(x_{k+1}) = 0, \tag{1.1}$$

subject to the conditions

$$x_m = c, \quad x_k > 0, \quad \Delta x_k \le 0, \quad \lim_{k\to\infty} x_k = 0, \tag{1.2}$$

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where $m \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$, $k \in \mathbb{Z}_m := \{k \in \mathbb{Z} : k \geq m\}$ and $c \in (0, \infty)$.

Throughout the paper the following conditions are assumed:

(H$_1$) The sequence a satisfies $a_k > 0$ for $k \in \mathbb{Z}_m$ and

$$\sum_{j=m}^{\infty} \frac{1}{a_j} < \infty.$$

(H$_2$) The sequence b satisfies $b_k \geq 0$ for $k \in \mathbb{Z}_m$ and

$$\sum_{j=m}^{\infty} b_j \sum_{i=j}^{\infty} \frac{1}{a_i} < \infty.$$

(H$_3$) The function F is continuous on $\mathbb{R}$, $F(u)u > 0$ for $u \neq 0$, and

$$\lim_{u \to 0+} \frac{F(u)}{u} < \infty. \tag{1.3}$$

When modeling real life phenomena, boundary value problems for second order differential equations play important role. The BVP (1.1)–(1.2) originates from the discretization process for searching radial solutions, which are globally positive and decaying, for PDE with Euclidean mean curvature operator. By globally positive solutions we mean solutions which are positive on the whole domain $\mathbb{Z}_m$. The Euclidean mean curvature operator arises in the study of some fluid mechanics problems, in particular capillarity-type phenomena for compressible and incompressible fluids.

Recently, discrete BVPs, associated to equation (1.1), have been widely studied, both in bounded and unbounded domains, see, e.g., [2] and references therein. Many of these papers can be seen as a finite dimensional variant of results established in the continuous case. For instance, we refer to [5–7, 21] for BVPs involving mean curvature operators in Euclidean and Minkowski spaces, both in the continuous and in the discrete case. Other results in this direction are in [8, 9], in which the multiplicity of solutions of certain BVPs involving the p-Laplacian is examined. Finally, in [12, 14] for second order equations with p-Laplacian the existence of globally positive decaying Kneser solutions, that is solutions x such that $x_n > 0$, $\Delta x_n < 0$ for $n \geq 1$ and $\lim_{n\to\infty} x_n = 0$, is examined.

Several approaches have been used in literature for treating the above problems. Especially, we refer to variational methods [22], the critical point theory [9] and fixed point theorems on cones [24, 25].

Here, we extend to second order difference equations with Euclidean mean curvature some results on globally positive decaying Kneser solutions stated in [12] for equations with p-Laplacian and $b_n < 0$.

This paper is motivated also by [13], in which BVPs for differential equation with the Euclidean mean curvature operator on the half-line $[1, \infty)$ have been studied subjected to the boundary conditions $x(1) = 1$ and $\lim_{t\to\infty} x(t) = 0$. The study in [13] is accomplished by using a linearization device and some properties of principal solutions of certain disconjugate second-order linear differential equations. Here, we consider the discrete setting of the problem studied in [13]. However, the discrete analogue presented here requires different technique. This is caused by a different behavior of decaying solutions as well as by peculiarities of the discrete setting which lead to a modified fixed point approach. Jointly with this, we prove new Sturm comparison theorems and new properties of recessive solutions for linear difference equations. Our existence result is based on a fixed point theorem for operators defined in a Fréchet space by a Schauder's linearization device. This method is originated in [10], later extended to the discrete case in [20], and recently developed in [15]. This tool does not require the explicit form of the fixed point operator T and simplifies the check of the topological properties of T on the unbounded domain, since these properties become an immediate consequence of *a-priori* bounds for an associated linear equation. These bounds are obtained in an implicit form by means of the concepts of recessive solutions for second order linear equations. The main properties and results which are needed in our arguments, are presented in Sections 2 and 3. In Section 4 the solvability of the BVP (1.1)–(1.2) is given, by assuming some implicit conditions on sequences a and b. Several effective criteria are given, too. These criteria are obtained by considering suitable linear equations which can be viewed as Sturm majorants of the auxiliary linearized equation. In Section 5 we compare our results with those stated in the continuous case in [13]. Throughout the paper we emphasize some discrepancies, which arise between the continuous case and the discrete one.

2 Discrete versus continuous decay

Several properties in the discrete setting have no continuous analogue. For instance, for a positive sequence x we always have

$$\frac{\Delta x_k}{x_k} = \frac{x_{k+1}}{x_k} - 1 > -1.$$

In the continuous case, obviously, this does not occur in general, and the decay can be completely different. For example, if $x(t) = e^{-2t}$ then $x'(t)/x(t) = -2$ for all t. Further, the ratio x'/x can be also unbounded from below, as the function $x(t) = \mathrm{e}^{-\mathrm{e}^t}$ shows.

Another interesting observation is the following. If two positive continuous functions x, y satisfy the inequality

$$\frac{x'(t)}{x(t)} \le M\frac{y'(t)}{y(t)}, \quad t \ge t_0,$$

then there exists $K > 0$ such that $x(t) \le Ky^M(t)$ for $t \ge t_0$. This is not true in the discrete case, as the following example illustrates.

Example 2.1. Consider the sequences x, y given by

$$x_k = \frac{1}{2^{2^k}}, \quad y_k = \frac{1}{2^{2^{k+2}}}.$$

Then

$$\frac{x_{k+1}}{x_k} = \frac{1}{2^{2^k}}, \quad \frac{y_{k+1}}{y_k} = \frac{1}{2^{2^{k+2}}},$$

and

$$\frac{\Delta x_k}{x_k} = \frac{1}{2^{2^k}} - 1 \le \frac{1}{2} - 1 = -\frac{1}{2} \le \frac{1}{2}\left(\frac{1}{2^{2^{k+2}}} - 1\right) = \frac{1}{2}\frac{\Delta y_k}{y_k}.$$

On the other hand, the inequality $x_k \le Ky_k^{1/2}$ is false for every value of $K > 0$. Indeed,

$$\frac{x_k}{\sqrt{y_k}} = \frac{2^{2^{k+1}}}{2^{2^k}} = 2^{2^k}$$

which is clearly unbounded.

The situation in the discrete case is described in the following two lemmas.

Lemma 2.2. *Let x, y be positive sequences on $\mathbb{Z}_m$ such that $M \in (0,1)$ exists, satisfying*

$$\frac{\Delta x_k}{x_k} \le M \frac{\Delta y_k}{y_k} \tag{2.1}$$

for $k \in \mathbb{Z}_m$. Then $1 + M\Delta y_k / y_k > 0$ for $k \in \mathbb{Z}_m$, and

$$x_k \le x_m \prod_{j=m}^{k-1} \left(1 + M \frac{\Delta y_j}{y_j}\right).$$

Proof. First of all note that, from $M \in (0,1)$ and the positivity of y, we have

$$1 + M \frac{\Delta y_k}{y_k} = 1 + M \frac{y_{k+1}}{y_k} - M > 0, \quad k \in \mathbb{Z}_m.$$

From (2.1) we get

$$\frac{x_{k+1}}{x_k} \le 1 + M \frac{\Delta y_k}{y_k},$$

and taking the product from m to $k-1$, $k > m$, we obtain

$$\frac{x_k}{x_m} = \frac{x_{m+1}}{x_m} \frac{x_{m+2}}{x_{m+1}} \cdots \frac{x_k}{x_{k-1}} \le \prod_{j=m}^{k-1} \left(1 + M \frac{\Delta y_j}{y_j}\right). \qquad \square$$

From the classical theory of infinite products (see for instance [19]) the infinite product $P = \prod_{k=m}^{\infty}(1 + q_k)$ of real numbers is said to *converge* if there is $N \in \mathbb{Z}_m$ such that $1 + q_k \neq 0$ for $k \ge N$ and

$$P_n = \prod_{k=N}^{n} (1 + q_k)$$

has a *finite and nonzero* limit as $n \to \infty$.

In case $-1 < q_k \le 0$, $\{P_n\}$ is a positive nonincreasing sequence, thus P being *divergent* (not converging to a nonzero number) means that

$$\lim_{n\to\infty} \prod_{k=N}^{n} (1 + q_k) = 0. \tag{2.2}$$

Moreover, the convergence of P is equivalent to the convergence of the series $\sum_{k=N}^{\infty} \ln(1 + q_k)$ and this is equivalent to the convergence of the series $\sum_{k=N}^{\infty} q_k$. Indeed, if $\sum_{k=m}^{\infty} q_k$ is convergent, then $\lim_{k\to\infty} q_k = 0$ and hence,

$$\lim_{k\to\infty} \frac{\ln(1 + q_k)}{q_k} = 1,$$

i.e., $\ln(1 + q_k) \sim q_k$ as $k \to \infty$. Since summing preserves asymptotic equivalence, we get that $\sum_{k=m}^{\infty} \ln(1 + q_k)$ converges. Similarly, we obtain the opposite direction.

Therefore, in case $-1 < q_k \le 0$, (2.2) holds if and only if $\sum_{k=N}^{\infty} q_k$ diverges to $-\infty$.

The following holds.

Lemma 2.3. *Let y be a positive nonincreasing sequence on $\mathbb{Z}_m$ such that $\lim_{k\to\infty} y_k = 0$. Then, for any $M \in (0,1)$,*

$$\lim_{k\to\infty} \prod_{j=m}^{k} \left(1 + M \frac{\Delta y_j}{y_j}\right) = 0.$$

Proof. From the theory of infinite products it is sufficient to show that

$$\sum_{j=m}^{\infty} \frac{\Delta y_j}{y_j} = -\infty. \tag{2.3}$$

We distinguish two cases:
1) there exists $N > 0$ such that $y_{k+1}/y_k \geq N$ for $k \in \mathbb{Z}_m$;
2) $\inf_{k\in\mathbb{Z}_m} y_{k+1}/y_k = 0$.

As for the former case, from the Lagrange mean value theorem, we have

$$-\Delta \ln y_k = -\frac{\Delta y_k}{\xi_k} \leq -\frac{\Delta y_k}{y_{k+1}} = -\frac{\Delta y_k}{y_k} \cdot \frac{y_k}{y_{k+1}} \leq -\frac{\Delta y_k}{N y_k},$$

where ξ_k is such that $y_{k+1} \leq \xi_k \leq y_k$ for $k \in \mathbb{Z}_m$. Summing the above inequality from m to $n-1$, $n > m$, we get

$$\ln y_m - \ln y_n \leq -\frac{1}{N} \sum_{j=m}^{n-1} \frac{\Delta y_j}{y_j}.$$

Since $\lim_{n\to\infty} y_n = 0$, letting $n \to \infty$ we get (2.3).

Next we deal with the case $\inf_{k\in\mathbb{Z}_m} y_{k+1}/y_k = 0$. This is equivalent to

$$\liminf_{k\to\infty} \frac{\Delta y_k}{y_k} = \liminf_{k\to\infty} \frac{y_{k+1}}{y_k} - 1 = -1,$$

which implies (2.3), since $\sum_{j=m}^{k} \Delta y_j / y_j$ is negative nonincreasing. □

3 A Sturm-type comparison theorem for linear equations

The main idea of our approach is based on an application of a fixed point theorem and on global monotonicity properties of recessive solutions of linear equations. To this goal, in this section we prove a new Sturm-type comparison theorem for linear difference equations.

Consider the linear equation

$$\Delta(r_k \Delta y_k) + p_k y_{k+1} = 0, \tag{3.1}$$

where $p_k \geq 0$ and $r_k > 0$ on $\mathbb{Z}_m$. We say that a solution y of equation (3.1) has a generalized zero in n if either $y_n = 0$ or $y_{n-1} y_n < 0$, see e.g. [1,3]. A (nontrivial) solution y of (3.1) is said to be *nonoscillatory* if $y_k y_{k+1} > 0$ for all large k. Equation (3.1) is said to be *nonoscillatory* if all its nontrivial solutions are nonoscillatory. It is well known that, by the Sturm type separation theorem, the nonoscillation of (3.1) is equivalent to the existence of a nonoscillatory solution see e.g. [2, Theorem 1.4.4], [3].

If (3.1) is nonoscillatory, then there exists a nontrivial solution u, uniquely determined up to a constant factor, such that

$$\lim_{k\to\infty} \frac{u_k}{y_k} = 0,$$

where y denotes an arbitrary nontrivial solution of (3.1), linearly independent of u. Solution u is called *recessive solution* and y a *dominant solution*, see e.g. [4]. Recessive solutions can be characterized in the following ways (both these properties are proved in [4]):

(i) A solution u of (3.1) is recessive if and only if

$$\sum_{j=m}^{\infty} \frac{1}{r_j u_j u_{j+1}} = \infty.$$

(ii) For a recessive solution u of (3.1) and any linearly independent solution y (i.e. dominant solution) of (3.1), one has

$$\frac{\Delta u_k}{u_k} < \frac{\Delta y_k}{y_k} \quad \text{eventually.} \tag{3.2}$$

Along with equation (3.1) consider the equation

$$\Delta(R_k \Delta x_k) + P_k x_{k+1} = 0 \tag{3.3}$$

where $P_k \geq p_k \geq 0$ and $0 < R_k \leq r_k$ on $\mathbb{Z}_m$; equation (3.3) is said to be a *Sturm majorant* of (3.1).

From [2, Lemma 1.7.2], it follows that if (3.3) is nonoscillatory, then (3.1) is nonoscillatory as well. In this section we always assume that (3.3) is nonoscillatory.

The following two propositions are slight modifications of results in [16]. They are preparatory to the main comparison result.

Proposition 3.1 ([16, Lemma 2])**.** *Let x be a positive solution of (3.3) on $\mathbb{Z}_m$ and y be a solution of (3.1) such that $y_m > 0$ and $r_m \Delta y_m / y_m \geq R_m \Delta x_m / x_m$. Then*

$$y_k > 0 \quad \text{and} \quad \frac{r_k \Delta y_k}{y_k} \geq \frac{R_k \Delta x_k}{x_k}, \quad \text{for } k \in \mathbb{Z}_m.$$

Moreover, if $y, \bar{y}$ are solutions of (3.1) such that $y_k > 0$, $k \in \mathbb{Z}_m$, and $\bar{y}_m > 0$, $\Delta \bar{y}_m / \bar{y}_m > \Delta y_m / y_m$, then

$$\bar{y}_k > 0 \quad \text{and} \quad \frac{\Delta \bar{y}_k}{\bar{y}_k} > \frac{\Delta y_k}{y_k}, \quad \text{for } k \in \mathbb{Z}_m.$$

Proposition 3.2 ([16, Theorem 3])**.** *If a recessive solution v of (3.1) has a generalized zero in $N \in \mathbb{Z}_m$ and has no generalized zero in (N, ∞), then any solution of (3.3) has a generalized zero in $(N-1, \infty)$.*

The following lemma is an improved version of [16, Theorem 1].

Lemma 3.3. *Let u, v be recessive solutions of (3.1) and (3.3), respectively, satisfying $u_k > 0, v_k > 0$ for $k \in \mathbb{Z}_m$. Then*

$$\frac{r_k \Delta u_k}{u_k} \leq \frac{R_k \Delta v_k}{v_k} \quad \text{for } k \in \mathbb{Z}_m. \tag{3.4}$$

Proof. By contradiction, assume that there exists $N \in \mathbb{Z}_m$ such that $r_N \Delta u_N / u_N > R_N \Delta v_N / v_N$. Let y be a solution of (3.1) satisfying $y_N > 0$ and $r_N \Delta y_N / y_N = R_N \Delta v_N / v_N$. Then $r_N \Delta y_N / y_N < r_N \Delta u_N / u_N$, (which implies that y is linearly independent with u) and from Proposition 3.1 we get $y_k > 0$, $\Delta y_k / y_k < \Delta u_k / u_k$ for $k \in \mathbb{Z}_N$, which contradicts (3.2). □

Lemma 3.4. *Let x be a positive solution of (3.3) on $\mathbb{Z}_m$. Then there exists a recessive solution u of (3.1), which is positive on $\mathbb{Z}_m$.*

Proof. Let u be a recessive solution of (3.1), whose existence is guaranteed by nonoscillation of majorant equation (3.3). By contradiction, assume that there exists $N \in \mathbb{Z}_m$ such that

$$u_N \neq 0, \quad u_N u_{N+1} \leq 0.$$

Then u cannot have a generalized zero in $(N+1, \infty)$. Indeed, if u has a generalized zero in $M \in \mathbb{Z}_{N+2}$, then by the Sturm comparison theorem on a finite interval (see e.g., [2, Theorem 1.4.3], [3, Theorem 1.2]), every solution of (3.3) has a generalized zero in $(N, M]$, which is a contradiction with the positivity of x. Applying now Proposition 3.2, we get that any solution of (3.3) has a generalized zero in (N, ∞) which again contradicts the positivity of x on $\mathbb{Z}_m$. □

The next theorem is, in fact, the main statement of this section and it plays an important role in the proof of Theorem 4.1.

Theorem 3.5. *Let x be a positive solution of* (3.3) *on $\mathbb{Z}_m$. Then there is a recessive solution u of* (3.1), *which is positive on $\mathbb{Z}_m$ and satisfies*

$$\frac{r_k \Delta u_k}{u_k} \leq \frac{R_k \Delta x_k}{x_k}, \quad k \in \mathbb{Z}_m. \tag{3.5}$$

In addition, if x is decreasing (nonincreasing) on $\mathbb{Z}_m$, then u is decreasing (nonincreasing) on $\mathbb{Z}_m$.

Proof. Let x be a positive solution of (3.3) on $\mathbb{Z}_m$. From Lemma 3.4, there exist a recessive solution u of (3.1) and a recessive solution v of (3.3), which are both positive on $\mathbb{Z}_m$. We claim that

$$\frac{\Delta v_k}{v_k} \leq \frac{\Delta x_k}{x_k} \quad \text{for } k \in \mathbb{Z}_m. \tag{3.6}$$

Indeed, suppose by contradiction that there is $N \in \mathbb{Z}_m$ such that $\Delta x_N / x_N < \Delta v_N / v_N$. Then, in view of Proposition 3.1, $\Delta x_k / x_k < \Delta v_k / v_k$ for $k \in \mathbb{Z}_N$, which contradicts (3.2). Combining (3.6) and (3.4), we obtain (3.5). The last assertion of the statement is an immediate consequence of (3.5). □

Taking $p = P$ and $r = R$ in Theorem 3.5, we get the following corollary.

Corollary 3.6. *If* (3.3) *has a positive decreasing (nonincreasing) solution on $\mathbb{Z}_m$, then there exists a recessive solution of* (3.3) *which is positive decreasing (nonincreasing) on $\mathbb{Z}_m$.*

We close this section by the following characterization of the asymptotic behavior of recessive solutions which will be used later.

Lemma 3.7. *Let*

$$\sum_{j=m}^{\infty} \frac{1}{r_j} < \infty \quad \text{and} \quad \sum_{j=m}^{\infty} p_j \sum_{i=j+1}^{\infty} \frac{1}{r_i} < \infty.$$

Then (3.1) *is nonoscillatory. Moreover, for every $d \neq 0$,* (3.1) *has an eventually positive, nonincreasing recessive solution u, tending to zero and satisfying*

$$\lim_{k \to \infty} \frac{u_k}{\sum_{j=k}^{\infty} r_j^{-1}} = d.$$

Proof. It follows from [11, Lemma 2.1 and Corollary 3.6]. More precisely, the result [11, Lemma 2.1] guarantees $\lim_{k\to\infty} r_k \Delta u_k = -d < 0$. Now, from the discrete L'Hospital rule, we get

$$\lim_{k \to \infty} \frac{u_k}{\sum_{j=k}^{\infty} r_j^{-1}} = \lim_{k \to \infty} \frac{\Delta u_k}{-r_k^{-1}} = d.$$ □

4 Main result: solvability of BVP

Our main result is the following.

Theorem 4.1. *Let (H_i), i=1,2,3, be satisfied and*

$$L_c = \sup_{u\in(0,c]} \frac{F(u)}{u}. \tag{4.1}$$

If the linear difference equation

$$\Delta\left(\frac{a_k}{\sqrt{1+c^2}}\,\Delta z_k\right) + L_c b_k z_{k+1} = 0, \tag{4.2}$$

has a positive decreasing solution on $\mathbb{Z}_m$, then BVP (1.1)–(1.2) has at least one solution.

Effective criteria, ensuring the existence of a positive decreasing solution of (4.2), are given at the end of this section.

From this theorem and its proof we get the following.

Corollary 4.2. *Let (H_i), $i = 1, 2, 3$, be satisfied. If (4.2) has a positive decreasing solution on $\mathbb{Z}_m$ for $c = c_0 > 0$, then (1.1)–(1.2) has at least one solution for every $c \in (0, c_0]$.*

To prove Theorem 4.1, we use a fixed point approach, based on the Schauder–Tychonoff theorem on the Fréchet space

$$\mathbb{X} = \{u : \mathbb{Z}_m \to \mathbb{R}\}$$

of all sequences defined on $\mathbb{Z}_m$, endowed with the topology of pointwise convergence on $\mathbb{Z}_m$. The use of the Fréchet space $\mathbb{X}$, instead of a suitable Banach space, is advantageous especially for the compactness test. Even if this is true also in the continuous case, in the discrete case the situation is even more simple, since any bounded set in $\mathbb{X}$ is relatively compact from the discrete Arzelà–Ascoli theorem. We recall that a set $\Omega \subset \mathbb{X}$ is bounded if the sequences in Ω are equibounded on every compact subset of $\mathbb{Z}_m$. The compactness test is therefore very simple just owing to the topology of $\mathbb{X}$, while in discrete Banach spaces can require some checks which are not always immediate.

Notice that, if $\Omega \subset \mathbb{X}$ is bounded, then $\Omega^\Delta = \{\Delta u,\ u \in \Omega\}$ is bounded, too. This is a significant discrepancy between the continuous and the discrete case; such a property can simplify the solvability of discrete boundary value problems associated to equations of order two or higher with respect to the continuous counterpart because *a-priori* bounds for the first difference

$$\Delta x_n = x_{n+1} - x_n$$

are a direct consequences of *a-priori* bounds for x_n, and similarly for higher order differences.

In [20, Theorem 2.1], the authors proved an existence result for BVPs associated to functional difference equations in Fréchet spaces (see also [20, Corollary 2.6], [15, Theorem 4] and remarks therein). That result is a discrete counterpart of an existence result stated in [10, Theorem 1.3] for the continuous case, and reduces the problem to that of finding good *a-priori* bounds for the unknown of a auxiliary linearized equation.

The function

$$\Phi(v) = \frac{v}{\sqrt{1+v^2}}$$

can be decomposed as

$$\Phi(v) = vJ(v),$$

where J is a continuous function on $\mathbb{R}$ such that $\lim_{v\to 0} J(v) = 1$. This suggests the form of an auxiliary linearized equation. Using the same arguments as in the proof of [20, Theorem 2.1], with minor changes, we have the following.

Theorem 4.3. *Consider the (functional) BVP*

$$\begin{cases} \Delta(a_n \Delta x_n J(\Delta x_n)) = g(n,x), & n \in \mathbb{Z}_m, \\ x \in S, \end{cases} \tag{4.3}$$

where $J : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{Z}_m \times \mathbb{X} \to \mathbb{R}$ are continuous maps, and S is a subset of $\mathbb{X}$.

Let $G : \mathbb{Z}_m \times \mathbb{X}^2 \to \mathbb{R}$ be a continuous map such that $G(k,q,q) = g(k,q)$ for all $(k,q) \in \mathbb{Z}_m \times \mathbb{X}$. If there exists a nonempty, closed, convex and bounded set $\Omega \subset \mathbb{X}$ such that:

a) for any $u \in \Omega$ the problem

$$\begin{cases} \Delta(a_n J(\Delta u_n)\Delta y_n) = G(n,y,u), & n \in \mathbb{Z}_m, \\ y \in S, \end{cases} \tag{4.4}$$

has a unique solution $y = T(u)$;

b) $T(\Omega) \subset \Omega$;

c) $\overline{T(\Omega)} \subset S$,

then (4.3) has at least one solution.

Proof. We briefly summarize the main arguments, for reader's convenience, which are a minor modification of the ones in [20, Theorem 2.1].

Let us show that the operator $T : \Omega \to \Omega$ is continuous with relatively compact image. The relatively compactness of $T(\Omega)$ follows immediately from b), since Ω is bounded. To prove the continuity of T in Ω, let $\{u^j\}$ be a sequence in Ω, $u^j \to u^\infty \in \Omega$, and let $v^j = T(u^j)$. Since $T(\Omega)$ is relatively compact, $\{v^j\}$ admits a subsequence (still indicated with $\{v^j\}$) which is convergent to v^∞, with $v^\infty \in S$ from c). Since J, G are continuous on their domains, we obtain

$$0 = \Delta(a_n J(\Delta u_n^j)\Delta v_n^j) - G(n, v^j, u^j) \to \Delta(a_n J(\Delta u_n^\infty)\Delta v_n^\infty) - G(n, v^\infty, u^\infty)$$

as $j \to \infty$. The uniqueness of the solution of (4.4) implies $v^\infty = T(u^\infty)$, and therefore T is continuous. By the Schauder–Tychonoff fixed point theorem, T has at least one fixed point in Ω, which is clearly a solution of (4.3). □

Proof of Theorem 4.1. Let z be the recessive solution of (4.2) such that $z_m = c$, $z_k > 0$, $\Delta z_k \le 0$, $k \in \mathbb{Z}_m$; the existence of a recessive solution with these properties follows from Corollary 3.6. Further, from Lemma 3.7, we have $\lim_{k\to\infty} z_k = 0$.

Define the set Ω by

$$\Omega = \left\{ u \in \mathbb{X} : 0 \le u_k \le c \prod_{j=m}^{k-1} \left(1 + M\frac{\Delta z_j}{z_j}\right), k \in \mathbb{Z}_m \right\},$$

where $\mathbb{X}$ is the Fréchet space of all real sequences defined on $\mathbb{Z}_m$, endowed with the topology of pointwise convergence on $\mathbb{Z}_m$, and $M = 1/\sqrt{1+c^2} \in (0,1)$. Clearly Ω is a closed, bounded and convex subset of $\mathbb{X}$.

For any $u \in \Omega$, consider the following BVP

$$\begin{cases} \Delta\left(\dfrac{a_k}{\sqrt{1+(\Delta u_k)^2}}\Delta y_k\right) + b_k\tilde{F}(u_{k+1})y_{k+1} = 0, & k \in \mathbb{Z}_m, \\ y \in S \end{cases} \tag{4.5}$$

where

$$\tilde{F}(v) = \frac{F(v)}{v} \quad \text{for } v > 0, \quad \tilde{F}(0) = \lim_{v\to 0^+} \frac{F(v)}{v}$$

is continuous on $\mathbb{R}^+$, due to assumption (1.3), and

$$S = \left\{ y \in \mathbb{X} : y_m = c,\, y_k > 0, \Delta y_k \le 0 \text{ for } k \in \mathbb{Z}_m,\, \sum_{j=m}^{\infty} \frac{1}{a_j y_j y_{j+1}} = \infty \right\}.$$

Since $0 \le u_k \le c$, for every $u \in \Omega$, we have $-c \le \Delta u_k \le c$, and so $(\Delta u_k)^2 \le c^2$. Therefore,

$$\frac{1}{\sqrt{1+(\Delta u_k)^2}} \ge \frac{1}{\sqrt{1+c^2}}$$

for every $u \in \Omega$ and $k \in \mathbb{Z}_m$. Further $\tilde{F}(u_{k+1}) \le L_c$ for $u \in \Omega$, and hence (4.2) is Sturm majorant for the linear equation in (4.5). Let $\widehat{y} = \widehat{y}(u)$ be the recessive solution of the equation in (4.5) such that $\widehat{y}_m = c$. Then $\widehat{y}$ is positive nonincreasing on $\mathbb{Z}_m$ by Theorem 3.5, and, in view of $\widehat{y}_m = c$ and the uniqueness of recessive solutions up to the constant factor, $\widehat{y}$ is the unique solution of (4.5). Define the operator $\mathcal{T} : \Omega \to \mathbb{X}$ by $(\mathcal{T}u)_k = \widehat{y}_k$ for $u \in \Omega$.

From Theorem 3.5, we get

$$\frac{a_k \Delta\widehat{y}_k}{\widehat{y}_k} \le \frac{a_k \Delta\widehat{y}_k}{\widehat{y}_k\sqrt{1+(\Delta u_k)^2}} \le \frac{a_k M \Delta z_k}{z_k} \le 0,$$

which implies $\Delta\widehat{y}_k/\widehat{y}_k \le M\Delta z_k/z_k$, $k \in \mathbb{Z}_m$. By Lemma 2.2,

$$\widehat{y}_k \le c\prod_{j=m}^{k-1}\left(1 + M\frac{\Delta z_j}{z_j}\right), \quad k \in \mathbb{Z}_m,$$

which yields $\mathcal{T}(\Omega) \subseteq \Omega$.

Next we show that $\overline{\mathcal{T}(\Omega)} \subseteq S$. Let $\overline{y} \in \overline{\mathcal{T}(\Omega)}$. Then there exists $\{u^j\} \subset \Omega$ such that $\{\mathcal{T}u^j\}$ converges to $\overline{y}$ (in the topology of $\mathbb{X}$). It is not restrictive to assume $\{u^j\} \to \bar{u} \in \Omega$ since Ω is compact. Since $\mathcal{T}u^j =: \widehat{y}^j$ is the (unique) solution of (4.5), we have $\widehat{y}^j_m = c$, $\widehat{y}^j_k > 0$ and $\Delta\widehat{y}^j_k \le 0$ on $\mathbb{Z}_m$ for every $j \in \mathbb{N}$. Consequently, $\overline{y}_m = c$, $\overline{y}_k \ge 0$, $\Delta\overline{y}_k \le 0$ for $k \in \mathbb{Z}_m$. Further, since $\tilde{F}$ is continuous, $\overline{y}$ is a solution of the equation in (4.5) for $u = \overline{u}$. Suppose now that there is $T \in \mathbb{Z}_m$ such that $\overline{y}_T = 0$. Then clearly $\Delta\overline{y}_T = 0$ and by the global existence and uniqueness of the initial value problem associated to any linear equation, we get $\overline{y} \equiv 0$ on $\mathbb{Z}_m$, which contradicts to $\overline{y}_m = c > 0$. Thus $\overline{y}_k > 0$ for all $k \in \mathbb{Z}_m$.

We have just to prove that $\sum_{j=m}^{\infty}(a_j\overline{y}_j\overline{y}_{j+1})^{-1} = \infty$. In view of Lemma 3.7, there exists $N > 0$ such that $\overline{y}_k \le N\sum_{j=k}^{\infty} a_j^{-1}$ on $\mathbb{Z}_m$. Noting that

$$\Delta\left(\frac{1}{\sum_{j=k}^{\infty} a_j^{-1}}\right) = \frac{1}{a_k\sum_{j=k}^{\infty} a_j^{-1}\sum_{j=k+1}^{\infty} a_j^{-1}},$$

we obtain

$$\sum_{j=m}^{k-1} \frac{1}{a_j \overline{y}_j \overline{y}_{j+1}} \geq \sum_{j=m}^{k-1} \frac{1}{N^2 a_j \sum_{i=j}^{\infty} a_i^{-1} \sum_{i=j+1}^{\infty} a_i^{-1}} = \frac{1}{N^2} \sum_{j=m}^{k-1} \Delta \left(\frac{1}{\sum_{i=j}^{\infty} a_i^{-1}} \right)$$
$$= \frac{1}{N^2} \left(\frac{1}{\sum_{j=k}^{\infty} a_j^{-1}} - \frac{1}{\sum_{j=m}^{\infty} a_j^{-1}} \right) \to \infty \quad \text{as } k \to \infty.$$

Thus $\overline{y} \in S$, i.e., $\overline{\mathcal{T}(\Omega)} \subseteq S$. By applying Theorem 4.3, we obtain that the problem

$$\begin{cases} \Delta \left(a_k \dfrac{\Delta x_k}{\sqrt{1+(\Delta x_k)^2}} \right) + b_k F(x_{k+1}) = 0, \quad k \in \mathbb{Z}_m, \\ x \in S \end{cases}$$

has at least a solution $\bar{x} \in \Omega$. From the definition of the set Ω,

$$\overline{x}_k \leq c \prod_{j=m}^{k-1} \left(1 + M \frac{\Delta z_j}{z_j} \right)$$

and since $M \in (0,1)$ and $\lim_{k\to\infty} z_k = 0$, we have

$$\lim_{k\to\infty} \prod_{j=m}^{k-1} \left(1 + M \frac{\Delta z_j}{z_j} \right) = 0$$

by Lemma 2.3. Thus $\bar{x}_k \to 0$ as $k \to \infty$, and $\bar{x}$ is a solution of the BVP (1.1)–(1.2). □

Proof of Corollary 4.2. Assume that (4.2) has a positive decreasing solution for $c = c_0 > 0$, and let $c_1 \in (0, c_0)$. Then equation (4.2) with $c = c_0$ is a Sturm majorant of (4.2) with $c = c_1$, and from Theorem 3.5, equation (4.2) with $c = c_1$ has a positive decreasing solution. The application of Theorem 4.1 leads to the existence of a solution of (1.1)–(1.2) for $c = c_1$. □

Effective criteria for the solvability of BVP (1.1)–(1.2) can be obtained by considering as a Sturm majorant of (4.2) any linear equation that is known to have a global positive solution.

In the continuous case, a typical approach to obtaining global positivity of solutions for equation

$$(t^2 y')' + \gamma y = 0, \quad t \geq 1, \tag{4.6}$$

where $0 < \gamma \leq 1/4$, is based on the Sturm theory. In virtue of the transformation $x = t^2 y'$, this equation is equivalent to the Euler equation

$$x'' + \frac{\gamma}{t^2} x = 0, \quad t \geq 1, \tag{4.7}$$

whose general solutions are well-known.

In the discrete case, various types of Euler equations are considered in the literature, see, e.g. [18, 23] and references therein. It is somehow problematic to find a solution for some natural forms of discrete Euler equations in the self-adjoint form (3.1).

Here our aim is to deal with solutions of Euler type equations.

Lemma 4.4. *The equation*

$$\Delta\big((k+1)^2 \Delta x_k\big) + \frac{1}{4} x_{k+1} = 0 \tag{4.8}$$

has a recessive solution which is positive decreasing on $\mathbb{N}$.

Proof. Consider the sequence

$$y_k = \prod_{j=1}^{k-1} \frac{2j+1}{2j}, \quad k \geq 1,$$

with the usual convention $\prod_{j=1}^{0} u_j = 1$. One can verify that y is a positive increasing solution of the equation

$$\Delta^2 y_k + \frac{1}{2(k+1)(2k+1)} y_{k+1} = 0 \tag{4.9}$$

on $\mathbb{N}$.

Set $x_k = \Delta y_k$. Then x is a positive decreasing solution of the equation

$$\Delta(2(k+1)(2k+1)\Delta x_k) + x_{k+1} = 0 \tag{4.10}$$

on $\mathbb{N}$. Obviously,

$$2(k+1)(2k+1) \leq 4(k+1)^2, \quad k \geq 1,$$

thus (4.10) is a Sturm majorant of (4.8). By Theorem 3.5, (4.8) has a recessive solution which is positive decreasing on $\mathbb{N}$. □

Equation (4.8) can be understand as the reciprocal equation to the Euler difference equation

$$\Delta^2 u_k + \frac{1}{4(k+1)^2} u_{k+1} = 0, \tag{4.11}$$

i.e., these equations are related by the substitution relation $u_k = d\Delta x_k$, $d \in \mathbb{R}$, where u satisfies (4.11) provided x is a solution of (4.8). The form of (4.11) perfectly fits the discretization of the differential equation (4.7) with $\gamma = 1/4$, using the usual central difference scheme.

Corollary 4.5. *Let (H_i), $i = 1, 2, 3$, be satisfied and L_c be defined by (4.1). The BVP (1.1)–(1.2) has at least one solution if there exists $\lambda > 0$ such that for $k \geq 1$*

$$a_k \geq 4\lambda(k+1)^2, \quad \sqrt{1+c^2}\, L_c\, b_k \leq \lambda. \tag{4.12}$$

Proof. Consider the equation (4.8). By Lemma 4.4, it has a positive decreasing solution on $\mathbb{N}$. The same trivially holds for the equivalent equation

$$\Delta\big(4\lambda(k+1)^2 \Delta x_k\big) + \lambda x_{k+1} = 0. \tag{4.13}$$

Since (4.12) holds, (4.13) is a Sturm majorant of (4.2), and by Theorem 3.5, equation (4.2) has a positive decreasing solution on $\mathbb{N}$. Now the conclusion follows from Theorem 4.1. □

Remark. Note that the sequence b does not need to be bounded. For example, consider as a Sturm majorant of (4.2) the equation

$$\Delta\big(\lambda k 2^{k+1} \Delta x_k\big) + \lambda 2^{k+1} x_{k+1} = 0, \quad k \geq 0.$$

One can check that this equation has the solution $x_k = 2^{-k}$. This leads to the conditions

$$a_k \geq \lambda k 2^{k+1}, \quad \sqrt{1+c^2}\, L_c\, b_k \leq \lambda 2^{k+1} \quad \text{for } k \geq 0$$

ensuring the solvability of the BVP (1.1)–(1.2).

Another criteria can be obtained by considering the equation

$$\Delta\big(\lambda k^3 \Delta x_k\big) + \lambda \frac{k^2+3k+1}{k+2} x_{k+1} = 0, \quad k \ge 1$$

having the solution $x_k = 1/k$. This comparison with (4.2) leads to the conditions

$$a_k \ge \lambda k^3, \quad \sqrt{1+c^2}\, L_c\, b_k \le \lambda \frac{k^2+3k+1}{k+2} \quad \text{for } k \ge 1\,.$$

The following example illustrates our result.

Example 4.6. Consider the BVP

$$\begin{cases} \Delta\big((k+1)^2 \Phi(\Delta x_k)\big) + \dfrac{|\sin k|}{4\sqrt{2}\,k}\, x_{k+1}^3 = 0, \quad k \ge 1, \\ x_1 = c, \quad x_k > 0, \quad \Delta x_k \le 0, \quad \lim\limits_{k\to\infty} x_k = 0. \end{cases} \tag{4.14}$$

We have $L_c = c^2$, $a_k = (k+1)^2$, and $b_k = \frac{|\sin k|}{4\sqrt{2}\,k}$. Conditions in (4.12) are fulfilled for any $c \in (0,1]$ when taking $\lambda = 1/4$. Indeed,

$$a_k = (k+1)^2 = 4\lambda(k+1)^2$$

and

$$\sqrt{1+c^2} L_c b_k = \sqrt{1+c^2} c^2 b_k \le \sqrt{2} b_k \le \frac{1}{4}|\sin k| \le \frac{1}{4} = \lambda.$$

Corollary 4.5 now guarantees solvability of the BVP (4.14) for any $c \in (0,1]$.

5 Comments and open problems

It is interesting to compare our discrete BVP with the continuous one investigated in [13]. Here the BVP for the differential equation with the Euclidean mean curvature operator

$$\begin{cases} \left(a(t) \dfrac{x'}{\sqrt{1+x'^2}}\right)' + b(t)F(x) = 0, \quad t \in [1,\infty), \\ x(1) = 1,\ x(t) > 0,\ x'(t) \le 0 \quad \text{for } t \ge 1,\ \lim\limits_{t\to\infty} x(t) = 0, \end{cases} \tag{P}$$

has been considered. Sometimes solutions of differential equations satisfying the condition

$$x(t) > 0, \quad x'(t) \le 0\,, \quad t \in [1,\infty),$$

are called *Kneser solutions* and the problem to find such solution is called *Kneser problem*.

The problem (P) has been studied under the following conditions:

(C_1) The function a is continuous on $[1,\infty)$, $a(t) > 0$ in $[1,\infty)$, and

$$\int_1^\infty \frac{1}{a(t)}\, dt < \infty.$$

(C_2) The function b is continuous on $[1,\infty)$, $b(t) \ge 0$ and

$$\int_1^\infty b(t) \int_t^\infty \frac{1}{a(s)} ds dt < \infty.$$

(C_3) The function F is continuous on $\mathbb{R}$, $F(u)u > 0$ for $u \neq 0$, and such that

$$\limsup_{u \to 0^+} \frac{F(u)}{u} < \infty. \tag{5.1}$$

The main result for solvability of (P) is the following. Note that the principal solution for linear differential equation is defined similarly as the recessive solution, see e.g. [13, 17].

Theorem 5.1 ([13, Theorem 3.1])**.** *Let (C_i), $i = 1, 2, 3$, be verified and*

$$L = \sup_{u \in (0,1]} \frac{F(u)}{u}.$$

Assume

$$\alpha = \inf_{t \geq 1} a(t)A(t) > 1,$$

where

$$A(t) = \int_t^\infty \frac{1}{a(s)}\, ds.$$

If the principal solution z_0 of the linear equation

$$\big(a(t)z'\big)' + \frac{\alpha}{\sqrt{\alpha^2 - 1}} L\, b(t) z = 0, \quad t \geq 1,$$

is positive and nonincreasing on $[1, \infty)$, then the BVP (P) has at least one solution.

It is worth to note that the method used in [13] does not allow that $\alpha = 1$ and thus Theorem 5.1 is not immediately applicable when $a(t) = t^2$. In [13] there are given several effective criteria for the solvability of the BVP (P) which are similar to Corollary 4.5. An example, which can be viewed as a discrete counterpart, is the above Example 4.6.

Open problems.

(1) The comparison between Theorem 4.1 for the discrete BVP and Theorem 5.1 for the continuous one, suggests to investigate the BVP (1.1)–(1.2) on times scales.

(2) In [13], the solvability of the continuous BVP has been proved under the weaker assumption (5.1) posed on F. This is due to the fact that the set Ω is defined using a precise lower bound which is different from zero. It is an open problem if a similar estimation from below can be used in the discrete case and assumption (1.3) can be replaced by (5.1).

(3) Similar BVPs concerning the existence of Kneser solutions for difference equations with p-Laplacian operator are considered in [12] when $b_k < 0$ for $k \in \mathbb{Z}^+$. It should be interesting to extend the solvability of the BVP (1.1)–(1.2) to the case in which the sequence b is negative and in the more general situation when the sequence b is of indefinite sign.

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Influence of singular weights on the asymptotic behavior of positive solutions for classes of quasilinear equations

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. Main objective of this paper is to study positive decaying solutions for a class of quasilinear problems with weights. We consider one dimensional problems on an interval which may be finite or infinite. In particular, when the interval is infinite, unlike the known cases in the history where constant weights force the solution not to decay, we discuss singular weights in the diffusion and reaction terms which produce positive solutions that decay to zero at infinity. We also discuss singular weights that lead to positive solutions not satisfying Hopf's boundary lemma. Further, we apply our results to radially symmetric solutions to classes of problems in higher dimensions, say in an annular domain or in the exterior region of a ball. Finally, we provide examples to illustrate our results.

Keywords: quasilinear problems, singular weights, asymptotic behavior, decaying positive solutions.

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1 Introduction

We consider the following quasilinear Dirichlet problem with weights

$$\begin{cases} -\left(\rho(t)|u'(t)|^{p-2}u'(t)\right)' = \sigma(t)f(t,u(t)), \quad t \in (a,b), \\ \lim\limits_{t\to a^+} u(t) = \lim\limits_{t\to b^-} u(t) = 0\,, \end{cases} \tag{1.1}$$

with $p > 1$, $\rho = \rho(t)$ and $\sigma = \sigma(t)$, $t \in (a,b)$ are positive weight functions that are measurable and finite everywhere in (a,b), where $-\infty \le a < b \le \infty$ and $f = f(t,s) : (a,b) \times \mathbb{R} \to \mathbb{R}$

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is continuous. Here we allow the weights ρ and σ to be singular (details are forthcoming in Section 2).

Study of the one dimensional model, such as (1.1), is often helpful to capture the qualitative behavior of the solution in the presence of the weights ρ and σ. Moreover, they provide insights for study of more complex models in higher dimension. Therefore, in this paper we present a careful analysis of the one dimensional problem (1.1), and at the end, also apply the obtained results to study the radially symmetric solutions to a class of problems in the higher dimensional case.

In Section 2, we formulate basic assumptions on weight functions ρ and σ and introduce an appropriate functional setting to study (1.1). In Section 3, we prove a general sub- and supersolution result, Theorem 3.1, using monotone iteration methods. In Section 4 and Section 5, we study two auxiliary problems, solutions of which are used in the construction of sub- and supersolution in order to apply Theorem 3.1. In particular, main results of Section 4 are Theorem 4.3 and Theorem 4.4, and similarly main results of Section 5 are Theorem 5.2 and Theorem 5.3. The asymptotic estimates derived in these theorems are utilized in the construction of a well ordered pair of sub- and supersolution. We obtain rather sharp decay estimates of the first eigenfunction of the p-Laplacian operator with weights in Section 4. These estimates are expressed in terms of the singularity or the degeneracy of the weight ρ, and are of independent interest. In Section 6, we consider the special case $(a, b) = (1, +\infty)$ and weight functions ρ and σ to be of "power type behavior" both near 1 and near $+\infty$. Corollary 6.2 is the special case of Theorem 4.3 and Theorem 4.4, where the asymptotics are expressed in terms of the powers of these weight functions ρ and σ. Similarly, Corollary 6.3 is the special case of Theorem 5.2 and Theorem 5.3. In Section 7, we consider an application of our one dimensional results obtained thus far to a radially symmetric Dirichlet problem for quasilinear PDEs on annular type domains or exterior domains in $\mathbb{R}^N$. In these cases, PDEs transform to special cases of (1.1) with $a > 0$ and $b \leq +\infty$. Therefore, we can reformulate the previous existence result, Corollary 7.2, and asymptotic analysis, Corollaries 7.3–7.6. Two illustrative examples are provided in Section 8. In particular, first we consider a special form of (1.1), and under appropriate assumptions on f, we construct a suitable pair of sub- and supersolution to guarantee the existence of a positive solution with prescribed decay rate at a and b, see Theorem 8.1. Second, we consider an analogous radially symmetric Dirichlet problem for a class of quasilinear PDEs, see Theorem 8.3. When the weights, ρ and σ, have power type behavior, we show that for certain powers, our positive solution cannot satisfy the Hopf maximum principle at the boundary, see Remark 8.5.

2 Notation and functional setting

Let $p > 1$, $p' = \frac{p}{p-1}$ and, $\rho = \rho(t)$ and $\sigma = \sigma(t)$, $t \in (a, b)$ be positive weight functions that are measurable and finite everywhere in (a, b), where $-\infty \leq a < b \leq \infty$. We define the following spaces which will be used throughout the paper. Let

$Y := L^p(a, b; \sigma)$ be the set of all measurable functions $u = u(t)$ in (a, b) satisfying

$$\|u\|_Y := \|u\|_{p,\sigma} = \left(\int_a^b \sigma(t)|u(t)|^p \mathrm{d}t \right)^{\frac{1}{p}} < +\infty;$$

$C_0^\infty(a, b)$ be the set of all smooth functions with a compact support in (a, b);

$X := W_0^{1,p}(a,b;\rho)$ be the closure of $C_0^\infty(a,b)$ with respect to the norm

$$\|u\|_X := \|u\|_{1,p,\rho} = \left(\int_a^b \rho(t)|u'(t)|^p \mathrm{d}t\right)^{\frac{1}{p}} ;$$

$X_L := W_L^{1,p}(a,b;\rho)$ be the set of all functions $u = u(t)$ in (a,b) such that for every compact interval $I \subset (a,b)$, u is absolutely continuous on I, $\lim_{t\to a^+} u(t) = 0$ and $\|u\|_X < \infty$;

$X_R := W_R^{1,p}(a,b;\rho)$ is defined analogously, except requiring $\lim_{t\to b^-} u(t) = 0$.

Properties of function spaces:

If $\sigma \in L^1_{\mathrm{loc}}(a,b)$, then $C_0^\infty(a,b)$ is dense in Y. If $\sigma^{1-p'} \in L^1_{\mathrm{loc}}(a,b)$, then Y is a uniformly convex Banach space. If $\rho^{1-p'} \in L^1_{\mathrm{loc}}(a,b)$, then X, X_L, X_R are uniformly convex Banach spaces, and $\rho \in L^1_{\mathrm{loc}}(a,b)$ implies that $X = X_R \cap X_L$. See [8,11] and [13] for details.

Next two theorems establish sufficient conditions for continuous and compact embeddings between the above defined weighted Sobolev and Lebesgue spaces. The proofs can be found in the book [13, Chapter 1].

Proposition 2.1. *Let*

$$\sup_{a<t<b} \left(\int_t^b \sigma(\tau)\mathrm{d}\tau\right)\left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} < \infty . \tag{2.1}$$

Then $X_L, X \hookrightarrow Y$ (continuous embedding). Let

$$\sup_{a<t<b} \left(\int_a^t \sigma(\tau)\mathrm{d}\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} < \infty . \tag{2.2}$$

Then $X_R, X \hookrightarrow Y$.

Proposition 2.2. *Let*

$$\lim_{\substack{t\to a^+\\ t\to b^-}} \left(\int_t^b \sigma(\tau)\mathrm{d}\tau\right)\left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0 . \tag{2.3}$$

Then $X_L, X \hookrightarrow\hookrightarrow Y$ (compact embedding). Let

$$\lim_{\substack{t\to a^+\\ t\to b^-}} \left(\int_a^t \sigma(\tau)\mathrm{d}\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0 . \tag{2.4}$$

Then $X_R, X \hookrightarrow\hookrightarrow Y$.

Unless specified otherwise, we always assume that ρ and σ satisfy either (2.3) or (2.4).

For the sake of brevity, we use the same notation for all generic positive constants. In order to avoid confusion, the reader is kindly asked to check the exact meaning of these constants separately in every section.

3 Monotone iterations

A function $u \in X$ is called a weak solution of (1.1) if the integral identity

$$\int_a^b \rho(t)|u'(t)|^{p-2}u'(t)\phi'(t)\mathrm{d}t = \int_a^b \sigma(t)f(t,u(t))\phi(t)\mathrm{d}t \tag{3.1}$$

holds for all test functions $\phi \in X$ with both integrals in (3.1) being finite.

In fact, if ρ and σ are continuous functions in (a,b) then a weak solution $u \in X$ of (1.1) is regular in the following sense (see [9]):

$$\left.\begin{array}{l} u \in C^1(a,b),\ \rho|u'|^{p-2}u' \in C^1(a,b),\ \text{the equation (1.1) holds} \\ \text{at every point and the boundary conditions are satisfied} \end{array}\right\}. \tag{3.2}$$

A function $\underline{u} \in X$, such that $\underline{u} \in C^1(a,b)$, $\rho|\underline{u}'|^{p-2}\underline{u}' \in C^1(a,b)$, is called a subsolution of (1.1), if for all $t \in (a,b)$ we have

$$-\left(\rho(t)|\underline{u}'(t)|^{p-2}\underline{u}'(t)\right)' \le \sigma(t)f(t,\underline{u}(t)), \quad t \in (a,b).$$

A supersolution $\overline{u} \in X$ of (1.1) is defined analogously with the reverse inequality. Note that $\underline{u}, \overline{u} \in X$ implies that

$$\lim_{t\to a^+} \underline{u}(t) = \lim_{t\to b^-} \underline{u}(t) = \lim_{t\to a^+} \overline{u}(t) = \lim_{t\to b^-} \overline{u}(t) = 0.$$

We state the following existence theorem.

Theorem 3.1. *Let $\underline{u}, \overline{u} \in X$ be sub- and supersolutions of (1.1) respectively, and $\underline{u} \le \overline{u}$ in (a,b). Assume that there exist constants $C_0 > 0$ and $\eta > 0$ such that the following hold:*

(H1) *$|f(t,s)| \le C_0|s|^{p-1}$ for all $t \in (a,b)$ and all $s \in \mathbb{R}$;*

(H2) *the function $s \mapsto f(t,s) + \eta|s|^{p-2}s$ is increasing on the interval $\left[\min_{t\in(a,b)} \underline{u}(t), \max_{t\in(a,b)} \overline{u}(t)\right]$ for all $t \in (a,b)$.*

Then there exist a minimal weak solution $u_{\min}$ and a maximal weak solution $u_{\max}$ of (1.1) such that

$$\underline{u} \le u_{\min} \le u_{\max} \le \overline{u} \quad \text{in } (a,b).$$

Proof. Let $F(z)(t) := \sigma(t)\left(f(t,z(t)) + \eta|z(t)|^{p-2}z(t)\right)$, $z \in Y$. By (H1), Hölder's inequality and the continuity of the Nemytskii operator, $F : Y \to X^*$ (the dual of X) is a continuous map. For $z \in Y$, consider the following quasilinear Dirichlet problem

$$\begin{cases} -\left(\rho(t)|u'(t)|^{p-2}u'(t)\right)' + \eta\sigma(t)|u(t)|^{p-2}u(t) = F(z)(t), & t \in (a,b), \\ \lim\limits_{t\to a^+} u(t) = \lim\limits_{t\to b^-} u(t) = 0. \end{cases} \tag{3.3}$$

Then (3.3) has a unique weak solution $u \in X$. Indeed, (3.3) understood in the weak sense is equivalent to the operator equation

$$J_\eta(u) = F(z) \tag{3.4}$$

where $J_\eta : X \to X^*$ is strictly monotone, continuous and weakly coercive operator. Therefore (3.4) has a unique solution (see [5, Sec. 12.3]) and hence (3.3) has a unique weak solution.

By [8, Lemma 3.3], $J_\eta^{-1} : X^* \to X$ is continuous. Therefore, $T := J_\eta^{-1} \circ F : Y \to X$ is continuous and by the compact embedding $X \hookrightarrow\hookrightarrow Y$, $T : Y \to Y$ is also compact. It is straight forward to check that $u = T(u)$ if and only if $u \in X$ is a weak solution of problem (1.1).

To complete the proof, we show that T is order preserving (monotone increasing) operator on the order interval $[\underline{u}, \overline{u}] \subset X$, and $\underline{u} \le T(\underline{u})$ and $\overline{u} \ge T(\overline{u})$, i.e., $\underline{u}$ and $\overline{u}$ are sub- and supersolutions of T, respectively, see [10, Section 6.3].

Indeed, let $z_1, z_2 \in Y$ satisfying $\underline{u} \le z_1 \le z_2 \le \overline{u}$, and let $u_i = T(z_i)$, $i = 1, 2$. Then

$$\begin{aligned} -\Big[\big(\rho(t)|u_2'(t)|^{p-2}u_2'(t)\big)' - \big(\rho(t)|u_1'(t)|^{p-2}u_1'(t)\big)'\Big] + \eta\sigma(t)\left[|u_2(t)|^{p-2}u_2(t) - |u_1(t)|^{p-2}u_1(t)\right] \\ = \sigma(t)\left(f(t, z_2(t)) + \eta|z_2(t)|^{p-2}z_2(t)\right) - \sigma(t)\left(f(t, z_1(t)) + \eta|z_1(t)|^{p-2}z_1(t)\right) \ge 0 \end{aligned} \tag{3.5}$$

in (a, b), by the assumption (H2). We claim $u_1 \le u_2$ in (a, b). Suppose not. Then by continuity of u_1 and u_2, there is a nonempty open interval $(a_1, b_1) \subseteq (a, b)$ such that $u_2(t) < u_1(t)$, $t \in (a_1, b_1)$, $\lim_{t \to a_1, b_1}(u_2(t) - u_1(t)) = 0$. Now, multiply (3.5) in (a_1, b_1) by $u_2 - u_1$, integrate from a_1 to b_1, perform integration by parts in the first two integrals and use $\lim_{t \to a_1, b_1}(u_2(t) - u_1(t)) = 0$ to get

$$\begin{aligned} \int_{a_1}^{b_1} \rho(t)\left(|u_2'(t)|^{p-2}u_2'(t) - |u_1'(t)|^{p-2}u_1'(t)\right)'(u_2'(t) - u_1'(t))\mathrm{d}t \\ + \eta\int_{a_1}^{b_1} \sigma(t)\left(|u_2(t)|^{p-2}u_2(t) - |u_1(t)|^{p-2}u_1(t)\right)(u_2(t) - u_1(t))\mathrm{d}t \le 0\,. \end{aligned}$$

This contradicts the fact that $s \mapsto |s|^{p-2}s$ is strictly increasing. Hence $u_1 \le u_2$. A similar argument as above yields $\underline{u} \le T(\underline{u})$ and $\overline{u} \ge T(\overline{u})$. Hence Theorem 3.1 holds. □

In the next two sections, we investigate special forms of (1.1) whose solutions are used in the construction of an ordered pair of sub- and supersolution in Section 8.

4 Asymptotic analysis of principal eigenfunction

We consider the following quasilinear eigenvalue problem with weights

$$\begin{cases} -\left(\rho(t)|u'(t)|^{p-2}u'(t)\right)' = \lambda\sigma(t)|u(t)|^{p-2}u(t), & t \in (a, b), \\ \lim_{t \to a^+} u(t) = \lim_{t \to b^-} u(t) = 0\,. \end{cases} \tag{4.1}$$

We define eigenvalues and eigenfunctions associated with (4.1) in the usual way.

Taking advantage of the compact embedding, $X \hookrightarrow\hookrightarrow Y$, from Proposition 2.2, we can construct a sequence of variational eigenvalues and corresponding eigenfunctions of (4.1) using the Lusternik–Schnirelman "inf-sup" argument provided ρ and σ satisfy (2.3) or/and (2.4). In particular, we have the following assertions concerning the principal eigenvalue λ_1 and associated principal eigenfunction $\varphi_1 \in X$.

Proposition 4.1. *Let (2.3) or (2.4) hold. Then*

$$\lambda_1 := \inf_{\substack{u \neq 0 \\ u \in X}} \frac{\int_a^b \rho(t)|u'(t)|^p \mathrm{d}t}{\int_a^b \sigma(t)|u(t)|^p \mathrm{d}t} > 0$$

is the principal eigenvalue of (4.1)*, and the infimum is achieved at a unique* $\varphi_1 \in X, \varphi_1 > 0$ *in* (a,b)*,* $\|\varphi_1\|_Y = 1$*. Moreover, if* ρ *and* σ *are continuous weight functions,* φ_1 *enjoys regularity properties* (3.2).

The proof follows from standard arguments, see for example, [1–3, 8, 12, 14].

Remark 4.2. It follows from Rolle's theorem, from the positivity of φ_1 and from the equation

$$\left(\rho(t)|\varphi_1'(t)|^{p-2}\varphi_1'(t)\right)' = -\lambda_1\sigma(t)\varphi_1^{p-1}(t) \quad (<0), \quad t \in (a,b)\,, \tag{4.2}$$

that there exist $\tilde{a}, \tilde{b} \in (a,b), \tilde{a} \leq \tilde{b}$, such that $\varphi_1'(\tilde{a}) = \varphi_1'(\tilde{b}) = 0$, $\varphi_1'(t) > 0$ for all $t \in (a,\tilde{a})$ and $\varphi_1'(t) < 0$ for all $t \in (\tilde{b}, b)$. Notice that it is possible to have $\tilde{a} = \tilde{b}$. This is the case, when, e.g., $\rho = \sigma = 1$ and $-\infty < a < b < +\infty$.

For certain classes of reaction terms f, the principal eigenfunction φ_1 or its suitable modifications very often serve as positive subsolutions to problem (1.1). To establish the ordering between subsolution and supersolution, behavior of subsolution near the boundary of the domain plays a crucial rule. Therefore, the goal of this section is to study asymptotic properties of $\varphi_1(t)$ as $t \to a^+$ and $t \to b^-$.

Theorem 4.3. *Let* ρ *and* σ *be continuous in* (a,b) *and,* $\tilde{a}$ *be as in* Remark 4.2*. Further, assume*

(i) *there exist* $c > 0$, $\varepsilon \in (0, p-1)$ *such that for all* $t \in (a,\tilde{a})$

$$\left(\int_t^b \sigma(\tau)\mathrm{d}\tau\right)\left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{\varepsilon} \leq c \tag{4.3}$$

and

(ii)

$$\lim_{t\to b^-}\left(\int_t^b \sigma(\tau)\mathrm{d}\tau\right)\left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0\,. \tag{4.4}$$

Then there exist $\bar{a} \in (a,\tilde{a}), c_1, c_2, \tilde{c}_2 > 0$ *such that for all* $t \in (a,\bar{a})$ *we have*

$$c_1\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau \leq \varphi_1(t) \leq c_2\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau\,, \tag{4.5}$$

and

$$c_1\rho^{1-p'}(t) \leq \varphi_1'(t) \leq \tilde{c}_2\rho^{1-p'}(t)\,. \tag{4.6}$$

Theorem 4.4. *Let* ρ *and* σ *be continuous in* (a,b) *and,* $\tilde{b}$ *be as in* Remark 4.2*. Further, assume*

(i) *there exist* $d > 0$, $\varepsilon \in (0, p-1)$ *such that for all* $t \in (\tilde{b}, b)$

$$\left(\int_a^t \sigma(\tau)\mathrm{d}\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{\varepsilon} \leq d \tag{4.7}$$

and

(ii)

$$\lim_{t\to a^+}\left(\int_a^t \sigma(\tau)\mathrm{d}\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0\,. \tag{4.8}$$

Then there exist $\overline{b} \in (\tilde{b}, b), d_1, d_2, \tilde{d}_2 > 0$ *such that for all* $t \in (\overline{b}, b)$ *we have*

$$d_1 \int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau \le \varphi_1(t) \le d_2 \int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau \tag{4.9}$$

and

$$d_1\rho^{1-p'}(t) \le -\varphi_1'(t) \le \tilde{d}_2\rho^{1-p'}(t)\,. \tag{4.10}$$

Remark 4.5. Condition (4.3) implies that for any $t \in (a, b)$ we have

$$\sigma \in L^1(t, b) \quad \text{and} \quad \rho^{1-p'} \in L^1(a, t)\,.$$

Similarly, condition (4.7) implies that for any $t \in (a, b)$ we have

$$\sigma \in L^1(a, t) \quad \text{and} \quad \rho^{1-p'} \in L^1(t, b)\,.$$

Remark 4.6. $\varepsilon < p - 1$ implies that (4.3) and (4.4) yield

$$\lim_{t\to a^+} \left(\int_t^b \sigma(\tau)\mathrm{d}\tau\right) \left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0\,. \tag{4.11}$$

Similarly, (4.7) and (4.8) yield

$$\lim_{t\to b^-} \left(\int_a^t \sigma(\tau)\mathrm{d}\tau\right) \left(\int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0\,. \tag{4.12}$$

Since (4.4) and (4.11) are nothing but (2.3), the assumptions of Theorem 4.3 guarantee that $\varphi_1 \in X$ exists, it is well defined, and satisfies the properties specified in Proposition 4.1. Also, since (4.8) and (4.12) are nothing but (2.4), similar conclusion can be drawn for Theorem 4.4 as well.

Remark 4.7. Estimate (4.9) can be found in [7] but its proof contains small gaps. Most gaps are filled in [6] for weights associated with the radial symmetric PDE case, cf. Section 7 of this paper. For completeness, we provide very careful and detailed proof for the general case of weights ρ and σ near the left end point $a \ge -\infty$ of the interval (a, b). The case of the right end point $b \le +\infty$ is similar.

Proof of Theorem 4.3. Let $\varphi_1 \in X$ be the normalized ($\|\varphi_1\|_Y = 1$) and positive principal eigenfunction, the existence of which follows from Proposition 4.1.

We first establish inequalities in (4.6). Integrating (4.2) from $\tau \in (a, \tilde{a})$ to $\tilde{a}$ and using Remark 4.2, we get,

$$\rho(\tau)|\varphi_1'(\tau)|^{p-2}\varphi_1'(\tau) = -\lambda_1 \int_{\tilde{a}}^{\tau} \sigma(\theta)\varphi_1^{p-1}(\theta)\mathrm{d}\theta\,,$$

and hence

$$\varphi_1'(\tau) = \lambda_1^{p'-1}\rho^{1-p'}(\tau) \left(\int_\tau^{\tilde{a}} \sigma(\theta)\varphi_1^{p-1}(\theta)\mathrm{d}\theta\right)^{p'-1}\,. \tag{4.13}$$

Choose $\overline{a} \in (a, \tilde{a})$. Then

$$c_1 := \lambda_1^{p'-1} \left(\int_{\overline{a}}^{\tilde{a}} \sigma(\theta)\varphi_1^{p-1}(\theta)\mathrm{d}\theta\right)^{p'-1} \le \lambda_1^{p'-1} \left(\int_{\overline{a}}^{\tilde{a}} \sigma(\theta)\mathrm{d}\theta\right)^{\frac{1}{p(p-1)}} \left(\int_a^b \sigma(\theta)\varphi_1^p(\theta)\mathrm{d}\theta\right)^{\frac{1}{p}} < \infty\,.$$

Thus for $t \in (a, \overline{a})$, we get from (4.13)

$$\varphi_1'(t) \geq \lambda_1^{p'-1} \rho^{1-p'}(t) \left(\int_{\overline{a}}^{\tilde{a}} \sigma(\theta) \varphi_1^{p-1}(\theta) \mathrm{d}\theta \right)^{p'-1} = c_1 \rho^{1-p'}(t),$$

establishing the left inequality in (4.6).

We assume for a moment that the right inequality in (4.5) holds and derive from here the right inequality in (4.6). Indeed, using the right inequality from (4.5) in (4.13), for $\tau \in (a, \overline{a})$, we get

$$\begin{aligned}
\varphi_1'(\tau) &\leq c_2 \lambda_1^{1-p'} \rho^{1-p'}(\tau) \left(\int_\tau^{\tilde{a}} \sigma(\theta) \left(\int_a^\theta \rho^{1-p'}(\theta_1) \mathrm{d}\theta_1 \right)^{p-1} \mathrm{d}\theta \right)^{p'-1} \\
&\overset{(4.3)}{\leq} c^{\frac{1}{\varepsilon}} c_2 \lambda_1^{1-p'} \rho^{1-p'}(\tau) \left(\int_\tau^{\tilde{a}} \sigma(\theta) \left(\int_\theta^b \sigma(\theta_1) \mathrm{d}\theta_1 \right)^{-\frac{p-1}{\varepsilon}} \mathrm{d}\theta \right)^{p'-1} \\
&= \frac{c^{\frac{1}{\varepsilon}} c_2 \lambda_1^{1-p'} \rho^{1-p'}(\tau)}{\left(\frac{p-1}{\varepsilon} - 1 \right)^{p'-1}} \left(\int_\tau^{\tilde{a}} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\int_\theta^b \sigma(\theta_1) \mathrm{d}\theta_1 \right)^{1-\frac{p-1}{\varepsilon}} \mathrm{d}\theta \right)^{p'-1} \\
&= \frac{c^{\frac{1}{\varepsilon}} c_2 \lambda_1^{1-p'} \rho^{1-p'}(\tau)}{\left(\frac{p-1}{\varepsilon} - 1 \right)^{p'-1}} \left[\left(\int_{\tilde{a}}^b \sigma(\theta_1) \mathrm{d}\theta_1 \right)^{1-\frac{p-1}{\varepsilon}} - \left(\int_\tau^b \sigma(\theta_1) \mathrm{d}\theta_1 \right)^{1-\frac{p-1}{\varepsilon}} \right]^{p'-1} \\
&\leq \frac{c^{\frac{1}{\varepsilon}} c_2 \lambda_1^{1-p'}}{\left(\frac{p-1}{\varepsilon} - 1 \right)^{p'-1}} \left(\int_{\tilde{a}}^b \sigma(\theta_1) \mathrm{d}\theta_1 \right)^{\frac{1}{p-1} - \frac{1}{\varepsilon}} \rho^{1-p'}(\tau) = \tilde{c}_2 \rho^{1-p'}(\tau),
\end{aligned}$$

where

$$\tilde{c}_2 := \frac{c^{\frac{1}{\varepsilon}} c_2 \lambda_1^{1-p'}}{\left(\frac{p-1}{\varepsilon} - 1 \right)^{p'-1}} \left(\int_{\tilde{a}}^b \sigma(\theta_1) \mathrm{d}\theta_1 \right)^{\frac{1}{p-1} - \frac{1}{\varepsilon}} < \infty.$$

The right inequality in (4.6) follows.

Next, we prove the left inequality in (4.5). For $t \in (a, \overline{a})$, we integrate (4.13) from a to t, we get

$$\begin{aligned}
\varphi_1(t) &= \int_a^t \varphi'{}_1(\tau) \mathrm{d}\tau = \lambda_1^{p'-1} \int_a^t \rho^{1-p'}(\tau) \left(\int_\tau^{\tilde{a}} \sigma(\theta) \varphi_1^{p-1}(\theta) \mathrm{d}\theta \right)^{p'-1} \mathrm{d}\tau \\
&\geq \lambda_1^{p'-1} \left(\int_{\overline{a}}^{\tilde{a}} \sigma(\theta) \varphi_1^{p-1}(\theta) \mathrm{d}\theta \right)^{p'-1} \left(\int_a^t \rho^{1-p'}(\tau) \mathrm{d}\tau \right) = c_1 \int_a^t \rho^{1-p'}(\tau) \mathrm{d}\tau
\end{aligned}$$

and the left inequality of (4.5) follows.

It remains to prove the right inequality in (4.5). This is the most profound part of the proof. We choose $t \in (a, \overline{a})$ and integrate (4.13) from a to t. Then applying Hölder's inequality and using $\left(\int_a^b \sigma(\theta)\varphi_1^p(\theta)\mathrm{d}\theta\right)^{\frac{1}{p}} = \|\varphi_1\|_Y = 1$, we get

$$\begin{aligned}
\varphi_1(t) &= \lambda_1^{p'-1}\int_a^t \rho^{1-p'}(\tau)\left(\int_\tau^{\tilde{a}} \sigma(\theta)\varphi_1^{p-1}(\theta)\mathrm{d}\theta\right)^{p'-1}\mathrm{d}\tau \\
&\le \lambda_1^{p'-1}\int_a^t \rho^{1-p'}(\tau)\left(\int_\tau^{\tilde{a}} \sigma(\theta)\varphi_1^{p}(\theta)\mathrm{d}\theta\right)^{\frac{1}{p}}\left(\int_\tau^{\tilde{a}} \sigma(\theta)\mathrm{d}\theta\right)^{\frac{p'-1}{p}}\mathrm{d}\tau \\
&\le \lambda_1^{p'-1}\left(\int_a^b \sigma(\theta)\varphi_1^{p}(\theta)\mathrm{d}\theta\right)^{\frac{1}{p}}\int_a^t \rho^{1-p'}(\tau)\left(\int_\tau^{\tilde{a}} \sigma(\theta)\mathrm{d}\theta\right)^{\frac{p'-1}{p}}\mathrm{d}\tau \\
&= \lambda_1^{p'-1}\int_a^t \rho^{1-p'}(\tau) I_1^{p'-1}(\tau)\mathrm{d}\tau\,, \qquad (4.14)
\end{aligned}$$

where

$$I_1(\tau) := \left(\int_\tau^{\tilde{a}} \sigma(\theta)\mathrm{d}\theta\right)^{\frac{1}{p}}.$$

We integrate (4.13) again from a to $t \in (a, \overline{a})$ and use (4.14) to get

$$\begin{aligned}
\varphi_1(t) &= \lambda_1^{p'-1}\int_a^t \rho^{1-p'}(\tau)\left(\int_\tau^{\tilde{a}} \sigma(\theta)\varphi_1^{p-1}(\theta)\mathrm{d}\theta\right)^{p'-1}\mathrm{d}\tau \\
&\le \lambda_1^{p'-1}\int_a^t \rho^{1-p'}(\tau)\left(\int_\tau^{\tilde{a}} \sigma(\theta)\left(\lambda_1^{p'-1}\int_a^\theta \rho^{1-p'}(\theta_1) I_1^{p'-1}(\theta_1)\mathrm{d}\theta_1\right)^{p-1}\mathrm{d}\theta\right)^{p'-1}\mathrm{d}\tau \\
&= k_2\int_a^t \rho^{1-p'}(\tau) I_2^{p'-1}(\tau)\mathrm{d}\tau\,,
\end{aligned}$$

where $k_2 := \lambda_1^{(p'-1)+(p'-1)^2(p-1)}$ and

$$I_2(\tau) := \int_\tau^{\tilde{a}} \sigma(\theta)\left(\int_a^\theta \rho^{1-p'}(\theta_1) I_1^{p'-1}(\theta_1)\mathrm{d}\theta_1\right)^{p-1}\mathrm{d}\theta\,.$$

By induction, for $n = 3, 4, \ldots$, we get

$$\varphi_1(t) \le k_n\int_a^t \rho^{1-p'}(\tau) I_n^{p'-1}(\tau)\mathrm{d}\tau\,, \qquad (4.15)$$

where $k_n := \lambda_1^{(p'-1)+(n-1)(p'-1)^2(p-1)}$ and

$$I_n(\tau) := \int_\tau^{\tilde{a}} \sigma(\theta)\left(\int_a^\theta \rho^{1-p'}(\theta_{n-1}) I_{n-1}^{p'-1}(\theta_{n-1})\mathrm{d}\theta_{n-1}\right)^{p-1}\mathrm{d}\theta\,.$$

It suffices to show that there exist $K > 0$ and $n_0 \in \mathbb{N}$, such that for all $\tau \in (a, \overline{a})$ we actually have

$$I_{n_0}(\tau) \le K\,. \qquad (4.16)$$

Indeed, once (4.16) is established, then (4.15) and (4.16) would imply the right inequality in (4.5) with $c_2 := k_{n_0} K^{p'-1} > 0$. Therefore, we concentrate on the proof of (4.16) with certain $K > 0$ and $n_0 \in \mathbb{N}$.

We start with the estimate of I_2 (we will denote by $a_1, a_2, \ldots$ the generic positive constants).

$$I_2(\tau) = \int_\tau^{\tilde{a}} \sigma(\theta)\left(\int_a^\theta \rho^{1-p'}(\theta_1) I_1^{p'-1}(\theta_1)\mathrm{d}\theta_1\right)^{p-1}\mathrm{d}\theta$$

$$
\begin{aligned}
&= \int_\tau^{\tilde a} \sigma(\theta) \left(\int_a^\theta \rho^{1-p'}(\theta_1) \left(\int_{\theta_1}^{\tilde a} \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p(p-1)}} \mathrm{d}\theta_1 \right)^{p-1} \mathrm{d}\theta \\
&\le \int_\tau^{\tilde a} \sigma(\theta) \left(\int_a^\theta \rho^{1-p'}(\theta_1) \left(\int_{\theta_1}^{b} \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p(p-1)}} \mathrm{d}\theta_1 \right)^{p-1} \mathrm{d}\theta \\
&\overset{(4.3)}{\le} a_1 \int_\tau^{\tilde a} \sigma(\theta) \left(\int_a^\theta \rho^{1-p'}(\theta_1) \left(\int_{a}^{\theta_1} \rho^{1-p'}(\tau_1) \mathrm{d}\tau_1 \right)^{-\frac{\varepsilon}{p(p-1)}} \mathrm{d}\theta_1 \right)^{p-1} \mathrm{d}\theta \\
&= a_1 \int_\tau^{\tilde a} \sigma(\theta) \left(\int_a^\theta \frac{\mathrm{d}}{\mathrm{d}\theta_1} \left(\frac{1}{1-\frac{\varepsilon}{p(p-1)}} \right) \left(\int_{a}^{\theta_1} \rho^{1-p'}(\tau_1) \mathrm{d}\tau_1 \right)^{1-\frac{\varepsilon}{p(p-1)}} \mathrm{d}\theta_1 \right)^{p-1} \mathrm{d}\theta \\
&= a_1 \int_\tau^{\tilde a} \sigma(\theta) \left(\frac{1}{1-\frac{\varepsilon}{p(p-1)}} \left(\int_{a}^{\theta} \rho^{1-p'}(\tau_1) \mathrm{d}\tau_1 \right)^{1-\frac{\varepsilon}{p(p-1)}} \right)^{p-1} \mathrm{d}\theta \\
&\le a_2 \int_\tau^{\tilde a} \sigma(\theta) \left(\int_{a}^{\theta} \rho^{1-p'}(\tau_1) \mathrm{d}\tau_1 \right)^{p-1-\frac{\varepsilon}{p}} \mathrm{d}\theta \,.
\end{aligned} \tag{4.17}
$$

Notice that the last inequality holds thanks to $\varepsilon < p(p-1)$. It follows from (4.3) that

$$
\left(\int_a^\theta \rho^{1-p'}(\tau_1) \mathrm{d}\tau_1 \right)^{p-1-\frac{\varepsilon}{p}} \le a_3 \left(\int_\theta^b \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}-\frac{p-1}{\varepsilon}} . \tag{4.18}
$$

Therefore (4.17) and (4.18) yield

$$
\begin{aligned}
I_2(\tau) &\le a_4 \int_\tau^{\tilde a} \sigma(\theta) \left(\int_\theta^b \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}-\frac{p-1}{\varepsilon}} \mathrm{d}\theta \\
&= a_4 \int_\tau^{\tilde a} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\frac{-1}{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} \right) \left(\int_\theta^b \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} \mathrm{d}\theta \\
&= \frac{a_4}{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} \left(\left(\int_\tau^b \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} - \left(\int_{\tilde a}^b \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} \right) .
\end{aligned} \tag{4.19}
$$

We may assume, without loss of generality, that

$$
\varepsilon \ne \frac{p}{p+1}(p-1) \quad \text{i.e.,} \quad \frac{1}{p} + \frac{\varepsilon - p + 1}{\varepsilon} \ne 0 \,.
$$

Therefore, one of the following two cases occurs.

Case 1: $\varepsilon < \frac{p}{p+1}(p-1)$, i.e., $\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon} < 0$. Then it follows from (4.19) that there exists $K > 0$ such that

$$
I_2(\tau) \le -\frac{a_4}{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} \left(\int_{\tilde a}^b \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} \le K ,
$$

i.e., (4.16) holds with $n_0 = 2$ and the proof is complete.

Case 2: $\varepsilon > \frac{p}{p+1}(p-1)$, i.e., $\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon} > 0$. Then it follows from (4.19) that

$$
I_2(\tau) \le \frac{a_4}{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} \left(\int_\tau^b \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} = a_5 \left(\int_\tau^b \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} . \tag{4.20}
$$

We continue our iterations:

$$
\begin{aligned}
I_3(\tau) &= \int_\tau^{\tilde a} \sigma(\theta)\left(\int_a^\theta \rho^{1-p'}(\theta_2) I_2^{p'-1}(\theta_2)\mathrm{d}\theta_2\right)^{p-1}\mathrm{d}\theta\\
&\overset{(4.20)}{\le} a_6 \int_\tau^{\tilde a} \sigma(\theta)\left(\int_a^\theta \rho^{1-p'}(\theta_2)\left(\int_{\theta_2}^b \sigma(\tau_1)\mathrm{d}\tau_1\right)^{\frac{1}{p(p-1)}+\frac{\varepsilon-p+1}{\varepsilon(p-1)}}\mathrm{d}\theta_2\right)^{p-1}\mathrm{d}\theta\\
&\overset{(4.3)}{\le} a_7 \int_\tau^{\tilde a} \sigma(\theta)\left(\int_a^\theta \rho^{1-p'}(\theta_2)\left(\int_a^{\theta_2} \rho^{1-p'}(\tau_1)\mathrm{d}\tau_1\right)^{-\frac{\varepsilon}{p(p-1)}-\frac{\varepsilon-p+1}{p-1}}\mathrm{d}\theta_2\right)^{p-1}\mathrm{d}\theta\\
&= a_7 \int_\tau^{\tilde a} \sigma(\theta)\left(\int_a^\theta \frac{1}{1-\frac{\varepsilon}{p(p-1)}-\frac{\varepsilon-p+1}{\varepsilon(p-1)}}\frac{\mathrm{d}}{\mathrm{d}\theta_2}\left(\int_a^{\theta_2} \rho^{1-p'}(\tau_1)\mathrm{d}\tau_1\right)^{1-\frac{\varepsilon}{p(p-1)}-\frac{\varepsilon-p+1}{p-1}}\mathrm{d}\theta_2\right)^{p-1}\mathrm{d}\theta\\
&= a_7 \int_\tau^{\tilde a} \sigma(\theta)\left(\frac{1}{1-\frac{\varepsilon}{p(p-1)}-\frac{\varepsilon-p+1}{\varepsilon(p-1)}}\left(\int_a^{\theta} \rho^{1-p'}(\tau_1)\mathrm{d}\tau_1\right)^{1-\frac{\varepsilon}{p(p-1)}-\frac{\varepsilon-p+1}{p-1}}\right)^{p-1}\mathrm{d}\theta\\
&\le a_8 \int_\tau^{\tilde a} \sigma(\theta)\left(\int_a^{\theta} \rho^{1-p'}(\tau_1)\mathrm{d}\tau_1\right)^{2p-2-\frac{\varepsilon}{p}-\varepsilon}\mathrm{d}\theta\,.
\end{aligned}
$$

Notice that $\varepsilon \in (0, p-1)$ and $p > 1$ yield the last inequality thanks to $\varepsilon < \frac{2p}{p+1}(p-1)$, i.e., $1-\frac{\varepsilon}{p(p-1)}-\frac{\varepsilon-p+1}{p-1} > 0$. It follows from (4.3) that

$$
\left(\int_a^\theta \rho^{1-p'}(\tau_1)\mathrm{d}\tau_1\right)^{2p-2-\frac{\varepsilon}{p}-\varepsilon} \le a_9\left(\int_\theta^b \sigma(\tau_1)\mathrm{d}\tau_1\right)^{\frac{1}{p}+1-\frac{1}{\varepsilon}(2p-2)}.
$$

Therefore,

$$
\begin{aligned}
I_3(\tau) &\le a_{10}\int_\tau^{\tilde a}\sigma(\theta)\left(\int_\theta^b \sigma(\tau_1)\mathrm{d}\tau_1\right)^{\frac{1}{p}+1-\frac{1}{\varepsilon}(2p-2)}\mathrm{d}\theta\\
&= a_{10}\int_\tau^{\tilde a}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{-1}{\frac{1}{p}+2\frac{\varepsilon-p+1}{\varepsilon}}\right)\left(\int_\theta^b \sigma(\tau_1)\mathrm{d}\tau_1\right)^{\frac{1}{p}+2\frac{\varepsilon-p+1}{\varepsilon}}\mathrm{d}\theta\\
&= \frac{a_{10}}{\frac{1}{p}+2\frac{\varepsilon-p+1}{\varepsilon}}\left(\left(\int_\tau^b \sigma(\tau_1)\mathrm{d}\tau_1\right)^{\frac{1}{p}+2\frac{\varepsilon-p+1}{\varepsilon}}-\left(\int_{\tilde a}^b \sigma(\tau_1)\mathrm{d}\tau_1\right)^{\frac{1}{p}+2\frac{\varepsilon-p+1}{\varepsilon}}\right).
\end{aligned}
\tag{4.21}
$$

Without loss of generality, we may assume $\varepsilon \ne \frac{2p}{2p+1}(p-1)$. Therefore, we distinguish between two cases again.

Case 1: $\varepsilon < \frac{2p}{2p+1}(p-1)$ i.e., $\frac{1}{p}+2\frac{\varepsilon-p+1}{\varepsilon} < 0$. Then it follows from (4.21) that there exists $K > 0$ such that

$$
I_3(\tau) \le -\frac{a_{10}}{\frac{1}{p}+2\frac{\varepsilon-p+1}{\varepsilon}}\left(\int_{\tilde a}^b \sigma(\tau_1)\mathrm{d}\tau_1\right)^{\frac{1}{p}+2\frac{\varepsilon-p+1}{\varepsilon}} \le K,
$$

i.e., (4.16) holds with $n_0 = 3$ and the proof is complete.

Case 2: $\varepsilon > \frac{2p}{2p+1}(p-1)$ i.e., $\frac{1}{p}+2\frac{\varepsilon-p+1}{\varepsilon} > 0$. Then it follows from (4.21) that

$$
I_3(\tau) \le \frac{a_{10}}{\frac{1}{p}+2\frac{\varepsilon-p+1}{\varepsilon}}\left(\int_\tau^b \sigma(\tau_1)\mathrm{d}\tau_1\right)^{\frac{1}{p}+2\frac{\varepsilon-p+1}{\varepsilon}}
$$

and we continue iterations.

Repeating the argument n times, we may assume without loss of generality, that $\varepsilon \neq \frac{np}{np+1}(p-1)$. We have then two different cases.

Case 1: $\varepsilon < \frac{np}{np+1}(p-1)$ i. e., $\frac{1}{p} + n\frac{\varepsilon-p+1}{\varepsilon} < 0$. Then there exists $K > 0$ such that

$$I_{n+1}(\tau) \leq -\frac{a_{11}}{\frac{1}{p} + n\frac{\varepsilon-p+1}{\varepsilon}} \left(\int_{\tilde{a}}^{b} \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p} + n\frac{\varepsilon-p+1}{\varepsilon}} \leq K,$$

i.e., (4.16) holds with $n_0 = n$ and the proof is complete.

Case 2: $\varepsilon > \frac{np}{np+1}(p-1)$ i.e., $\frac{1}{p} + n\frac{\varepsilon-p+1}{\varepsilon} > 0$. Then

$$I_{n+1}(\tau) \leq \frac{a_{11}}{\frac{1}{p} + n\frac{\varepsilon-p+1}{\varepsilon}} \left(\int_{\tau}^{b} \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p} + n\frac{\varepsilon-p+1}{\varepsilon}}$$

and we continue iterations.

Notice that for a given $\varepsilon \in (0, p-1)$, the second case does not occur after finite number of steps due to $\lim_{n\to\infty} \frac{np}{np+1} = 1$. Therefore the proof is complete after a finite number of iterations. This completes the proof of Theorem 4.3. $\square$

The proof of Theorem 4.4 follows by using analogous arguments.

5 Asymptotic analysis of an auxiliary function

A suitable multiple of the solution $e = e(t)$ of the auxiliary Dirichlet problem

$$\begin{cases} -\left(\rho(t)|u'(t)|^{p-2}u'(t)\right)' = \sigma(t), & t \in (a,b), \\ \lim\limits_{t\to a^+} u(t) = \lim\limits_{t\to b^-} u(t) = 0 \end{cases} \tag{5.1}$$

with $\sigma \in X^*$ serves as a positive supersolution of the problem (1.1). If we interpret (5.1) in the weak sense, then it is equivalent to the operator equation

$$J(u) = \sigma \tag{5.2}$$

where $J : X \to X^*$ is strictly monotone, continuous and weakly coercive operator. Therefore, there exists unique $e = e(t) \in X$ which is a solution of (5.2) and hence a weak solution of (5.1). Moreover, when $\sigma = \sigma(t)$ and $\rho = \rho(t)$ are continuous in (a,b) then the solution e enjoys regularity properties (3.2) of Section 3.

Moreover, since $\sigma > 0$ in (a,b), it follows from (5.1) that $e(t) > 0$ in (a,b). In addition, there exist $\tilde{a}_e, \tilde{b}_e \in (a,b), \tilde{a}_e \leq \tilde{b}_e$ such that $e'(\tilde{a}_e) = e'(\tilde{b}_e) = 0$, $e'(t) > 0$ for all $t \in (a, \tilde{a}_e)$ and $e'(t) < 0$ for all $t \in (\tilde{b}_e, b)$.

Remark 5.1. Notice that $\sigma \in L^1(a,b)$ is a sufficient condition for $\sigma \in X^*$. Also observe that $\sigma \in L^1(a,b)$ implies that (4.3) and (4.7) hold for an arbitrary $\varepsilon \in (0, p-1)$.

The following assertion is a counterpart of Theorem 4.3.

Theorem 5.2. *Let σ, ρ be continuous in (a,b), $\sigma \in L^1(a,b)$ and $\rho^{1-p'} \in L^1(a,t)$ for any $t \in (a,b)$. Let $\tilde{a}_e$ be associated with $e = e(t)$, and $\varepsilon \in (0, p-1)$. Then there exist $\bar{a}_e \in (a, \tilde{a}_e), c_1, c_2, \tilde{c}_2 > 0$ such that for all $t \in (a, \bar{a}_e)$, we have*

$$c_1 \int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau \le e(t) \le c_2 \left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau \right)^{1-\frac{\varepsilon}{p-1}} \tag{5.3}$$

and

$$c_1 \rho^{1-p'}(t) \le e'(t) \le \tilde{c}_2 \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau \right)^{1-\frac{\varepsilon}{p-1}}. \tag{5.4}$$

Similarly, the following assertion is a counterpart of Theorem 4.4.

Theorem 5.3. *Let σ, ρ be continuous in (a,b), $\sigma \in L^1(a,b)$ and $\rho^{1-p'} \in L^1(t,b)$ for any $t \in (a,b)$. Let $\tilde{b}_e$ be associated with $e = e(t)$, and $\varepsilon \in (0, p-1)$. Then there exist $\bar{b}_e \in (\tilde{b}_e, b), d_1, d_2, \tilde{d}_2 > 0$ such that for all $t \in (\bar{b}_e, b)$, we have*

$$d_1 \int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau \le e(t) \le d_2 \left(\int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau \right)^{1-\frac{\varepsilon}{p-1}} \tag{5.5}$$

and

$$d_1 \rho^{1-p'}(t) \le -e'(t) \le \tilde{d}_2 \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau \right)^{1-\frac{\varepsilon}{p-1}}. \tag{5.6}$$

Proof of Theorem 5.2. It follows by directly integrating the equation in (5.1) from $\tilde{a}_e$ to $t \in (a, \bar{a}_e)$ with $\bar{a}_e < \tilde{a}_e$ that

$$e'(t) = \rho^{1-p'}(t) \left(\int_t^{\tilde{a}_e} \sigma(\tau)\mathrm{d}\tau \right)^{p'-1} \ge c_1 \rho^{1-p'}(t), \tag{5.7}$$

with $c_1 := \left(\int_{\bar{a}_e}^{\tilde{a}_e} \sigma(\tau)\mathrm{d}\tau \right)^{p'-1}$, i.e., the left inequality in (5.4) holds. Now, integrating the equality in (5.7) from a to $t \in (a, \bar{a}_e)$ yields

$$\begin{aligned} e(t) &= \int_a^t e'(\tau)\mathrm{d}\tau = \int_a^t \rho^{1-p'}(\tau) \left(\int_\tau^{\tilde{a}_e} \sigma(\theta)\mathrm{d}\theta \right)^{p'-1} \mathrm{d}\tau \\ &\ge \left(\int_{\bar{a}_e}^{\tilde{a}_e} \sigma(\theta)\mathrm{d}\theta \right)^{p'-1} \int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau = c_1 \int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau \end{aligned}$$

and the left inequality in (5.3) follows.

In view of Remark 5.1, the condition (4.3) is satisfied for any $\varepsilon \in (0, p-1)$. For $\varepsilon \in (0, p-1)$ arbitrary, and for $t \in (a, \bar{a}_e)$, we have

$$\begin{aligned} e'(t) &= \rho^{1-p'}(t) \left(\int_t^{\tilde{a}_e} \sigma(\tau)\mathrm{d}\tau \right)^{p'-1} \le \rho^{1-p'}(t) \left(\int_t^b \sigma(\tau)\mathrm{d}\tau \right)^{p'-1} \\ &\overset{(4.3)}{\le} c^{p'-1} \rho^{1-p'}(t) \left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau \right)^{-\varepsilon(p'-1)} = \tilde{c}_2 \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau \right)^{1-\frac{\varepsilon}{p-1}}, \end{aligned} \tag{5.8}$$

where $\tilde{c}_2 := \frac{c^{p'-1}}{1-\frac{\varepsilon}{p-1}}$. Thus the right inequality in (5.4) holds. Finally, integrating (5.8) from a to $t \in (a, \bar{a}_e)$, we establish the right inequality in (5.3). Indeed,

$$e(t) = \int_a^t e'(\tau)\mathrm{d}\tau \le \int_a^t \tilde{c}_2 \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\int_a^\tau \rho^{1-p'}(\theta)\mathrm{d}\theta \right)^{1-\frac{\varepsilon}{p-1}} \mathrm{d}\tau = c_2 \left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau \right)^{1-\frac{\varepsilon}{p-1}},$$

where $c_2 := c^{p'-1}$. The proof of Theorem 5.2 is complete. □

The proof of Theorem 5.3 is similar.

6 Weight functions of special type

Here we consider the case $a = 1$, $b = +\infty$ and the following pair of continuous weight functions ρ and σ defined on $(1, +\infty)$:

$$\rho(t) = \begin{cases} (t-1)^{\alpha_1}, & t \in (1,2)\,, \\ 1, & t \in [2,10]\,, \\ \left(\frac{10}{t}\right)^{\alpha_\infty}, & t \in (10,+\infty)\,, \end{cases} \quad \text{and} \quad \sigma(t) = \begin{cases} (t-1)^{\beta_1}, & t \in (1,2)\,, \\ 1, & t \in [2,10]\,, \\ \left(\frac{10}{t}\right)^{\beta_\infty}, & t \in (10,+\infty)\,. \end{cases} \tag{6.1}$$

The weight functions ρ and σ have "power type behavior" prescribed by α_1 and β_1 near $a = 1$ and by α_∞ and β_∞ near $b = +\infty$. The following assertion is an immediate consequence of (6.1), (2.3) and (2.4).

Lemma 6.1. *Condition* (2.3) *holds if and only if*

$$\alpha_1 < \min\{\beta_1 + p, p-1\} \quad \textit{and} \quad \beta_\infty > \max\{\alpha_\infty + p, 1\}\,. \tag{6.2}$$

Condition (2.4) *holds if and only if*

$$\beta_1 > \max\{\alpha_1 - p, -1\} \quad \textit{and} \quad \alpha_\infty < \min\{\beta_\infty - p, 1-p\}\,. \tag{6.3}$$

In particular,

$$(6.2) \Rightarrow X_L, X \hookrightarrow\hookrightarrow Y \quad \textit{and} \quad (6.3) \Rightarrow X_R, X \hookrightarrow\hookrightarrow Y\,.$$

In this section, we discuss an application of Theorems 4.3, 4.4, 5.2 and 5.3. At first, we concentrate on assumptions (4.3) and (4.7) and interpret an asymptotic behavior of φ_1 given by (4.5), (4.6), (4.9) and (4.10) in terms of $\alpha_1, \alpha_\infty, \beta_1$ and β_∞.

Corollary 6.2. *Let us assume that* (6.2) *holds and $\varphi_1 \in X$ be the principal eigenfunction of* (4.1) *with ρ and σ given by* (6.1)*. Then there exist $\bar{a} > 1, c_1, \tilde{c}_1, c_2, \tilde{c}_2 > 0$ such that for all $t \in (1, \bar{a})$ we have*

$$c_1(t-1)^{1-\frac{\alpha_1}{p-1}} \le \varphi_1(t) \le c_2(t-1)^{1-\frac{\alpha_1}{p-1}}$$

and

$$\tilde{c}_1(t-1)^{-\frac{\alpha_1}{p-1}} \le \varphi_1'(t) \le \tilde{c}_2(t-1)^{-\frac{\alpha_1}{p-1}}\,.$$

Similarly, assume that (6.3) *holds. Then there exist $\bar{b} > 1, d_1, \tilde{d}_1, d_2, \tilde{d}_2 > 0$ such that for all $t \in (\bar{b}, +\infty)$ we have*

$$d_1 t^{1+\frac{\alpha_\infty}{p-1}} \le \varphi_1(t) \le d_2 t^{1+\frac{\alpha_\infty}{p-1}}$$

and

$$\tilde{d}_1 t^{\frac{\alpha_\infty}{p-1}} \le -\varphi_1'(t) \le \tilde{d}_2 t^{\frac{\alpha_\infty}{p-1}}\,.$$

Proof. The proof consists of verifying the assumptions of Theorem 4.3 and Theorem 4.4 in the case of the weight functions, ρ and σ, given by (6.1). Indeed, if we assume (6.2) then we distinguish between two cases. In the case $\beta_1 \ge -1$ the condition (4.3) holds with arbitrary $\varepsilon \in (0, p-1)$, and in the case $\beta_1 < -1$ we can take any $\varepsilon \in \left(\frac{(p-1)(\beta_1+1)}{\alpha_1 - p + 1}, p-1\right)$. Similarly, if we assume (6.3), condition (4.7) holds with arbitrary $\varepsilon \in (0, p-1)$ in the case $\beta_\infty \ge 1$, and any $\varepsilon \in \left(\frac{(p-1)(1-\beta_\infty)}{1-p-\alpha_\infty}, p-1\right)$ in the case $\beta_\infty < 1$. $\square$

Secondly, we discuss the asymptotic behavior of solution e of (5.1). Notice that in order to guarantee $\sigma \in L^1(1,+\infty)$, we must assume $\beta_1 > -1$ and $\beta_\infty > 1$. Then condition (6.2) reduces to

$$\alpha_1 < p-1 \quad \text{and} \quad \beta_\infty > \max\{\alpha_\infty + p, 1\} \tag{6.4}$$

and condition (6.3) reduces to

$$\beta_1 > \max\{\alpha_1 - p, -1\} \quad \text{and} \quad \alpha_\infty < 1-p\,. \tag{6.5}$$

Corollary 6.3. *Let us assume that (6.4) holds and $e \in X$ is a weak solution of (5.1) with ρ and σ given by (6.1). Let $\varepsilon \in (0, p-1)$ be arbitrary. Then there exist $\overline{a}_e > 1$, $c_1, \tilde{c}_1, c_2, \tilde{c}_2 > 0$ such that for all $t \in (1, \overline{a}_e)$ we have*

$$c_1(t-1)^{1-\frac{\alpha_1}{p-1}} \le e(t) \le c_2(t-1)^{\left(1-\frac{\alpha_1}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)}$$

and

$$\tilde{c}_1(t-1)^{-\frac{\alpha_1}{p-1}} \le e'(t) \le \tilde{c}_2(t-1)^{\left(1-\frac{\alpha_1}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)-1}.$$

Similarly, assume that (6.5) holds, and $\varepsilon \in (0, p-1)$ is arbitrary. Then there exist $\overline{b}_e > 1$, d_1, $\tilde{d}_1$, d_2, $\tilde{d}_2 > 0$ such that for all $t \in (\overline{b}_e, +\infty)$ we have

$$d_1 t^{1+\frac{\alpha_\infty}{p-1}} \le e(t) \le d_2 t^{\left(1+\frac{\alpha_\infty}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)}$$

and

$$\tilde{d}_1 t^{\frac{\alpha_\infty}{p-1}} \le -e'(t) \le \tilde{d}_2 t^{\left(1+\frac{\alpha_\infty}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)-1}.$$

Remark 6.4. With obvious modifications we can derive analogous assertions if $a, b \in \mathbb{R}$ (i.e., (a,b) is a bounded interval), $a = -\infty, b \in \mathbb{R}$ (i.e., $(a,b) = (-\infty, b)$ is bounded above) and $a = -\infty, b = +\infty$ (i.e., $(a,b) = \mathbb{R}$).

7 Application to partial differential equations

In this section, we will apply the one dimensional results obtained thus far to study the radially symmetric solutions to a class of quasilinear PDEs satisfying Dirichlet boundary conditions. Our results in this section are valid in various domains in $\mathbb{R}^N$ with $N \ge 2$ such as $B_R := \left\{x \in \mathbb{R}^N : |x| < R\right\} \subset \mathbb{R}^N$ where B_R is a ball if $R < +\infty$ and entire $\mathbb{R}^N$ if $R = +\infty$, or

$$A_{R_1}^{R_2} := \left\{x \in \mathbb{R}^N : R_1 < |x| < R_2\right\} \quad \text{for } 0 < R_1 < R_2 \le +\infty$$

where $A_{R_1}^{R_2}$ is an annular domain if $R_2 < +\infty$ and an exterior domain if $R_2 = +\infty$.

Here we focus on radially symmetric solutions to the boundary value problem:

$$\begin{cases} -\operatorname{div}\left(v(|x|)|\nabla u(|x|)|^{p-2}\nabla u(|x|)\right) = w(|x|)f(|x|, u(|x|)), & x \in A_{R_1}^{R_2} \\ u(x) = 0, \quad x \in \partial A_{R_1}^{R_2}, \end{cases} \tag{7.1}$$

where v and w are positive continuous weight functions. After substitution $r = |x|$, the above problem transforms to

$$\begin{cases} -\left(r^{N-1}v(r)|u'(r)|^{p-2}u'(r)\right)' = r^{N-1}w(r)f(r, u(r)), & r \in (R_1, R_2), \\ \lim\limits_{r\to R_1} u(r) = \lim\limits_{r\to R_2} u(r) = 0\,, \end{cases} \tag{7.2}$$

where $f : (R_1, R_2) \times \mathbb{R} \to \mathbb{R}$ is as in Section 3. The problem (7.2) corresponds to (1.1) with the following change of notation:

$$t = r, \quad a = R_1, \quad b = R_2, \quad \rho(t) = r^{N-1}v(r), \quad \sigma(t) = r^{N-1}w(r)\,.$$

We say that a radially symmetric function $u = u(|x|), x \in A_{R_1}^{R_2}$, is a weak solution of problem (7.1) if the function $u = u(r)$, $r \in (R_1, R_2)$, is a weak solution of problem (7.2) in the sense mentioned at the beginning of Section 3. Similarly, using corresponding notions from Section 3, we can define radially symmetric sub- and supersolutions to (7.1).

Natural spaces to study the radially symmetric solutions to problem (7.1) are Sobolev and Lebesgue spaces X and Y of all radially symmetric functions with norms depending on v and w, respectively. More precisely, let $v^{1-p'}, w^{1-p'}, v, w \in L^1_{\text{loc}}(A_{R_1}^{R_2})$. Then X and Y are uniformly convex Banach spaces and $C_0^\infty(A_{R_1}^{R_2})$ is dense in both X and Y. A radial function $u \in Y$ if and only if $u = u(r)$ is a measurable function in (R_1, R_2) satisfying

$$\|u\|_Y = \left(\int_{R_1}^{R_2} r^{N-1}w(r)|u(r)|^p \mathrm{d}r\right)^{\frac{1}{p}} < \infty\,.$$

Similarly, a radial function $u \in X$ if and only if $u = u(r)$ is absolutely continuous on every compact subinterval of (R_1, R_2), $\lim\limits_{r \to R_1} u(r) = \lim\limits_{r \to R_2} u(r) = 0$ and

$$\|u\|_X = \left(\int_{R_1}^{R_2} r^{N-1}v(r)|u'(r)|^p \mathrm{d}r\right)^{\frac{1}{p}} < \infty\,.$$

Obvious change of the notation in (2.1)–(2.4) leads to the following sufficient conditions for continuous and compact embeddings $X \hookrightarrow Y$ and $X \hookrightarrow\hookrightarrow Y$, respectively.

Proposition 7.1.

(A) Let either

$$\sup_{R_1<r<R_2} \left(\int_r^{R_2} \tau^{N-1}w(\tau)\mathrm{d}\tau\right)\left(\int_{R_1}^r \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} < \infty$$

or

$$\sup_{R_1<r<R_2} \left(\int_{R_1}^r \tau^{N-1}w(\tau)\mathrm{d}\tau\right)\left(\int_r^{R_2} \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} < \infty$$

hold. Then $X \hookrightarrow Y$.

(B) Let either

$$\lim_{r\to R_1,R_2} \left(\int_r^{R_2} \tau^{N-1}w(\tau)\mathrm{d}\tau\right)\left(\int_{R_1}^r \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0 \tag{7.3}$$

or

$$\lim_{r\to R_1,R_2} \left(\int_{R_1}^r \tau^{N-1}w(\tau)\mathrm{d}\tau\right)\left(\int_r^{R_2} \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0 \tag{7.4}$$

hold. Then $X \hookrightarrow\hookrightarrow Y$.

As a consequence of this compact embedding, the following result follows from Theorem 3.1.

Corollary 7.2. *Let $\underline{u} \in X$ and $\overline{u} \in X$ be subsolution and supersolution of (7.2), respectively, and $\underline{u} \le \overline{u}$ in (R_1, R_2). Let $f : (R_1, R_2) \times \mathbb{R} \to \mathbb{R}$ be as in Section 3. Then there exist a minimal weak solution $u_{\min} \in X$ and a maximal weak solution $u_{\max} \in X$ of (7.2) which satisfy $\underline{u} \le u_{\min} \le u_{\max} \le \overline{u}$ in (R_1, R_2).*

Next, let us consider the eigenvalue problem

$$\begin{cases} -\left(r^{N-1}v(r)|u'(r)|^{p-2}u'(r)\right)' = \lambda r^{N-1}w(r)|u(r)|^{p-2}u(r), & r \in (R_1, R_2), \\ \lim\limits_{r\to R_1} u(r) = \lim\limits_{r\to R_2} u(r) = 0\,. \end{cases} \tag{7.5}$$

Under the assumption (7.3) or (7.4) the principal eigenvalue of (7.5),

$$\lambda_1 := \inf_{\substack{u\neq 0\\ u\in X}} \frac{\int_{R_1}^{R_2} r^{N-1}v(r)|u'(r)|^p \mathrm{d}r}{\int_{R_1}^{R_2} r^{N-1}w(r)|u(r)|^p \mathrm{d}r} > 0$$

is achieved at a unique $\varphi_1 \in X, \varphi_1 > 0$ in (R_1, R_2) and $\|\varphi_1\|_Y = 1$. Asymptotic estimates of φ_1 for $r \to R_1$ and $r \to R_2$ follow from Theorem 4.3 and Theorem 4.4. Indeed, Let $R_1 < \tilde{R}_1 \le \tilde{R}_2 < R_2$ be such that $\varphi_1'(\tilde{R}_1) = \varphi_1'(\tilde{R}_2) = 0$ and $\varphi_1'(r) > 0$ in $(R_1, \tilde{R}_1)$ and $\varphi_1'(r) < 0$ in $(\tilde{R}_2, R_2)$. The existence of $\tilde{R}_1$ and $\tilde{R}_2$ are explained in Remark 4.2. Then due to Theorem 4.3 and Theorem 4.4, we have:

Corollary 7.3. *Let $c > 0$, $\varepsilon \in (0, p-1)$ be such that for all $r \in (R_1, \tilde{R}_1)$*

$$\left(\int_r^{R_2} \tau^{N-1}w(\tau)\mathrm{d}\tau\right)\left(\int_{R_1}^r \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau\right)^{\varepsilon} \le c$$

and

$$\lim_{r\to R_2}\left(\int_r^{R_2} \tau^{N-1}w(\tau)\mathrm{d}\tau\right)\left(\int_{R_1}^r \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0\,.$$

Then there exist $\overline{R}_1 \in (R_1, \tilde{R}_1), c_1, c_2, \tilde{c}_2 > 0$ such that for all $r \in (R_1, \overline{R}_1)$ we have

$$c_1 \int_{R_1}^r \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau \le \varphi_1(r) \le c_2 \int_{R_1}^r \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau$$

and

$$c_1 r^{\frac{1-N}{p-1}}v^{1-p'}(r) \le \varphi_1'(r) \le \tilde{c}_2 r^{\frac{1-N}{p-1}}v^{1-p'}(r)\,.$$

Let $d > 0$, $\varepsilon \in (0, p-1)$ be such that for all $r \in (\tilde{R}_2, R_2)$

$$\left(\int_{R_1}^r \tau^{N-1}w(\tau)\mathrm{d}\tau\right)\left(\int_r^{R_2} \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau\right)^{\varepsilon} \le d$$

and

$$\lim_{r\to R_1}\left(\int_{R_1}^r \tau^{N-1}w(\tau)\mathrm{d}\tau\right)\left(\int_r^{R_2} \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0\,.$$

Then there exist $\overline{R}_2 \in (\tilde{R}_2, R_2), d_1, d_2, \tilde{d}_2 > 0$ such that for all $r \in (\overline{R}_2, R_2)$ we have

$$d_1 \int_r^{R_2} \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau \le \varphi_1(r) \le d_2 \int_r^{R_2} \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau$$

and

$$d_1 r^{\frac{1-N}{p-1}}v^{1-p'}(r) \le -\varphi_1'(r) \le \tilde{d}_2 r^{\frac{1-N}{p-1}}v^{1-p'}(r)\,.$$

Next, we consider the case $R_1 = 1$, $R_2 = +\infty$, i.e., $A_{R_1}^{R_2} = \overline{B}_1^c$ is exterior of unit ball centered at the origin. Let us consider continuous radial weights v and w defined on $(1, +\infty)$ as follows:

$$v(r) = \begin{cases} (r-1)^{\alpha_1}, & r \in (1,2), \\ 1, & r \in [2,10], \\ \left(\frac{10}{r}\right)^{\alpha_\infty}, & r \in (10, +\infty); \end{cases} \qquad w(r) = \begin{cases} (r-1)^{\beta_1}, & r \in (1,2), \\ 1, & r \in [2,10], \\ \left(\frac{10}{r}\right)^{\beta_\infty}, & r \in (10, +\infty). \end{cases} \tag{7.6}$$

Similarly to Section 6, we can now reformulate the sufficient conditions in Proposition 7.1. We also express conditions stated in Corollary 7.3 in terms of $\alpha_1, \alpha_\infty, \beta_1$ and β_∞. Clearly, now also the dimension $N \geq 2$ will be involved in these conditions. Indeed, condition (7.3) holds if and only if

$$\alpha_1 < \min\{\beta_1 + p, p-1\} \text{ and } \beta_\infty > \max\{\alpha_\infty + p, N\} \tag{7.7}$$

and condition (7.4) holds if and only if

$$\beta_1 > \max\{\alpha_1 - p, -1\} \text{ and } \alpha_\infty < \min\{\beta_\infty - p, N - p\}. \tag{7.8}$$

In particular, the compact embedding $X \hookrightarrow\hookrightarrow Y$ holds if either (7.7) or (7.8) holds. Since v and w are continuous, $\varphi_1(r)$ is regular in the sense of (3.2) from Section 3.

Next, we formulate asymptotic behavior of φ_1, see Corollary 7.3, in the language of powers $\alpha_1, \alpha_\infty, \beta_1$ and β_∞.

Corollary 7.4. *If (7.7) holds, then there exist $\overline{R}_1 > 1, c_1, \tilde{c}_1, c_2, \tilde{c}_2 > 0$ such that for all $r \in (1, \overline{R}_1)$ we have*

$$c_1(r-1)^{1-\frac{\alpha_1}{p-1}} \leq \varphi_1(r) \leq c_2(r-1)^{1-\frac{\alpha_1}{p-1}}$$

and

$$\tilde{c}_1(r-1)^{-\frac{\alpha_1}{p-1}} \leq \varphi_1'(r) \leq \tilde{c}_2(r-1)^{-\frac{\alpha_1}{p-1}}.$$

If (7.8) holds, then there exist $\overline{R}_2 > 1, d_1, \tilde{d}_1, d_2, \tilde{d}_2 > 0$ such that for all $r \in (\overline{R}_2, +\infty)$ we have

$$d_1 r^{1+\frac{\alpha_\infty+1-N}{p-1}} \leq \varphi_1(r) \leq d_2 r^{1+\frac{\alpha_\infty+1-N}{p-1}}$$

and

$$\tilde{d}_1 r^{\frac{\alpha_\infty+1-N}{p-1}} \leq -\varphi_1'(r) \leq \tilde{d}_2 r^{\frac{\alpha_\infty+1-N}{p-1}}.$$

While the asymptotics near 1 corresponds to the asymptotics in the first part of Corollary 6.2, the asymptotics near $+\infty$ is affected by an additional term "r^{N-1}".

Similarly, we can study the asymptotic properties of the weak solution $e(r)$ to the following auxiliary problem

$$\begin{cases} -\left(r^{N-1}v(r)|u'(r)|^{p-2}u'(r)\right)' = r^{N-1}w(r), & r \in (R_1, R_2), \\ \lim\limits_{r\to R_1} u(r) = \lim\limits_{r \to R_2} u(r) = 0. \end{cases} \tag{7.9}$$

In fact, we can formulate an analogue of Theorem 5.2 and Theorem 5.3.

Corollary 7.5. *Let $r^{N-1}w(r) \in L^1(R_1, R_2)$. Given $\varepsilon \in (0, p-1)$ arbitrary, there exist $\overline{R}_1^e \in (R_1, R_2)$, c_1, c_2, $\tilde{c}_2 > 0$ such that for all $r \in (R_1, \overline{R}_1^e)$ we have*

$$c_1 \int_{R_1}^r \tau^{\frac{1-N}{p-1}} v^{1-p'}(\tau)\mathrm{d}\tau \leq e(r) \leq c_2 \left(\int_{R_1}^r \tau^{\frac{1-N}{p-1}} v^{1-p'}(\tau)\mathrm{d}\tau\right)^{1-\frac{\varepsilon}{p-1}}$$

and

$$c_1 r^{\frac{1-N}{p-1}} v^{1-p'}(r) \le e'(r) \le \tilde{c}_2 \frac{\mathrm{d}}{\mathrm{d}r}\left(\int_{R_1}^{r} \tau^{\frac{1-N}{p-1}} v^{1-p'}(\tau)\mathrm{d}\tau\right)^{1-\frac{\varepsilon}{p-1}}.$$

Similarly, given $\varepsilon \in (0, p-1)$ *arbitrary, there exist* $\overline{R}_2^e \in (R_1, R_2)$, d_1, d_2, $\tilde{d}_2 > 0$ *such that for all* $r \in (\overline{R}_2^e, R_2)$ *we have*

$$d_1 \int_r^{R_2} \tau^{\frac{1-N}{p-1}} v^{1-p'}(\tau)\mathrm{d}\tau \le e(r) \le d_2 \left(\int_r^{R_2} \tau^{\frac{1-N}{p-1}} v^{1-p'}(\tau)\mathrm{d}\tau\right)^{1-\frac{\varepsilon}{p-1}}$$

and

$$d_1 r^{\frac{1-N}{p-1}} v^{1-p'}(r) \le -e'(r) \le \tilde{d}_2 \frac{\mathrm{d}}{\mathrm{d}r}\left(\int_r^{R_2} \tau^{\frac{1-N}{p-1}} v^{1-p'}(\tau)\mathrm{d}\tau\right)^{1-\frac{\varepsilon}{p-1}}.$$

If v and w are given by (7.6) then $w \in L^1(\overline{B}_1^c)$ requires $\beta_1 > -1$ and $\beta_\infty > N$. In particular, (7.7) reduces to

$$\alpha_1 < p-1 \quad \text{and} \quad \beta_\infty > \max\{\alpha_\infty + p, N\} \tag{7.10}$$

and (7.8) reduces to

$$\beta_1 > \max\{\alpha_1 - p, -1\} \quad \text{and} \quad \alpha_\infty < N-p\,. \tag{7.11}$$

Note that (4.3) holds for arbitrary $\varepsilon \in (0, p-1)$ in this special case. Then the asymptotic estimates for e and e' read as follows.

Corollary 7.6. *Given* $\varepsilon \in (0, p-1)$ *arbitrary, there exist* $\overline{R}_1^e > 1, c_1, \tilde{c}_1, c_2, \tilde{c}_2 > 0$ *such that for all* $r \in (1, \overline{R}_1^e)$ *we have*

$$c_1(r-1)^{1-\frac{\alpha_1}{p-1}} \le e(r) \le c_2(r-1)^{\left(1-\frac{\alpha_1}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)}$$

and

$$\tilde{c}_1(r-1)^{-\frac{\alpha_1}{p-1}} \le e'(r) \le \tilde{c}_2(r-1)^{\left(1-\frac{\alpha_1}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)-1}.$$

Similarly, given $\varepsilon \in (0, p-1)$ *arbitrary, there exist* $\overline{R}_2^e > 1$, $d_1, \tilde{d}_1, d_2, \tilde{d}_2 > 0$ *such that for all* $r \in (\overline{R}_2^e, +\infty)$ *we have*

$$d_1 r^{1+\frac{\alpha_\infty+1-N}{p-1}} \le e(r) \le d_2 r^{\left(1+\frac{\alpha_\infty+1-N}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)} \tag{7.12}$$

and

$$\tilde{d}_1 r^{\frac{\alpha_\infty+1-N}{p-1}} \le -e'(r) \le \tilde{d}_2 r^{\left(1+\frac{\alpha_\infty+1-N}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)-1}.$$

Remark 7.7. Let us emphasize the importance of asymptotic estimates presented above. We will utilize them later for constructing an ordered pair of sub- and supersolution for problem (7.1). Since modifications of φ_1 and e will serve as a subsolution and a supersolution, respectively, the estimates above will allow to compare the resulting subsolution and a supersolution near the finite boundary and near infinity.

Remark 7.8. We compare our results for (7.9) with $R_2 = +\infty$ and corresponding results of Bidaut-Véron and Pohozaev [4, Prop. 2.6, (ii)]. Let $N > p$. Consider v and w as given in (7.6) with $\alpha_1 = \alpha_\infty = \beta_1 = 0$ and $\beta_\infty > N$. Then the left inequality in (7.12) coincides with the lower estimate from [4], the first inequality in (2.34). Let $N \le p$. The second inequality in (2.34) from [4] implies that any possible nonnegative weak solution of equation in (7.9) cannot decay to zero as $r \to +\infty$, i.e., (7.9) does not have a weak solution. On the other hand, choosing now $\alpha_1 = \beta_1 = 0$, $\alpha_\infty < N - p$, $\beta_\infty > N$, problem (7.9) has a positive weak solution satisfying decay asymptotic estimates presented above. This says that a sufficiently singular diffusion coefficient $v(r)$ could guarantee the existence of a weak solution having prescribed decay at infinity.

8 Examples

We will discuss some examples to demonstrate our general existence result from Theorem 3.1 and the use of asymptotics obtained for the eigenfunction φ_1 and the auxiliary function e in Section 4 and Section 5, respectively. For simplicity, we consider $f(t,s) = f(s)$, where $f : [0, +\infty) \to \mathbb{R}$ is C^1 and satisfies the following additional assumptions:

(H3) there exists a constant $K > 0$ such that $\lim_{s\to 0} \frac{f(s)}{s^{p-1}} = K$;

(H4) there exists $r_0 > 0$ such that $f(s)(r_0 - s) > 0$ for all $s > 0$, $s \neq r_0$.

We observe that since f is C^1, (H3)–(H4) imply that f satisfies (H1)–(H2).

We consider the following one dimensional quasilinear problem

$$\begin{cases} -\left(\rho(t)|u'(t)|^{p-2}u'(t)\right)' = \lambda\sigma(t)f(u(t)), \quad t \in (1, +\infty), \\ \lim\limits_{t\to 1^+} u(t) = \lim\limits_{t\to +\infty} u(t) = 0 \end{cases} \tag{8.1}$$

where $\lambda > 0$ is a parameter.

Then we prove the following result.

Theorem 8.1. *Let the weight functions ρ and σ be as in (6.1) with $\alpha_1, \beta_1, \alpha_\infty, \beta_\infty$ satisfying (6.4) and (6.5). Let $p > 1$ and (H3)–(H4) hold. Then for any $\lambda > \frac{\lambda_1}{K}$, there exist a minimal weak solution $u_{\min}$ and a maximal weak solution $u_{\max}$ of (8.1). Moreover, given $\varepsilon \in (0, p-1)$, there exist constants $C > 1$, $C_1, C_2 > 0$ such that for all $t \in (1, C)$ we have*

$$C_1(t-1)^{1-\frac{\alpha_1}{p-1}} \le u_{\min}(t) \le u_{\max}(t) \le C_2(t-1)^{\left(1-\frac{\alpha_1}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)}.$$

Similarly, given $\varepsilon \in (0, p-1)$, there exist constants $D > 1$, $D_1, D_2 > 0$ such that for all $t \in (D, +\infty)$ we have

$$D_1 t^{1+\frac{\alpha_\infty}{p-1}} \le u_{\min}(t) \le u_{\max}(t) \le D_2 t^{\left(1+\frac{\alpha_\infty}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)}.$$

Proof. In order to apply Theorem 3.1, we construct a suitable pair of well ordered sub- and supersolution of (8.1). We will first construct a positive supersolution of (8.1) with the help of the auxiliary function $e > 0$, weak solution of (5.1). Let $\overline{u} := (\lambda A_0)^{\frac{1}{p-1}} e$, where $A_0 := \sup_{s\ge 0} f(s) > 0$. Then

$$-\left(\rho(t)|\overline{u}'(t)|^{p-2}\overline{u}'(t)\right)' = \lambda A_0 \sigma(t) \ge \lambda\sigma(t)f(\overline{u}).$$

Now we construct a positive subsolution of (8.1) using the eigenfunction $\varphi_1 > 0$ corresponding to the principal eigenvalue λ_1 of (4.1). Note that continuity, and decay properties, (4.5) and (4.9), of the eigenfunction φ_1 imply that $\|\varphi_1\|_\infty < +\infty$. First, we consider a function

$$G(s) := \lambda_1 s^{p-1} - \lambda f(s) \quad \text{for } s \ge 0.$$

Using hypothesis (H3), we see that $G(s) = \lambda_1 s^{p-1} - \lambda K s^{p-1} - o(s^{p-1})$. Let $\lambda > \frac{\lambda_1}{K}$ be fixed. Then there exists $s_\lambda > 0$ such that for any $s \in (0, s_\lambda)$, we have $G(s) < 0$. For $m \le \frac{s_\lambda}{\|\varphi_1\|_\infty}$, we show that $\underline{u} := m\varphi_1$ is a subsolution of (8.1). Indeed, it follows from the discussion above and the fact that $\sigma(t) > 0$ in $(1, +\infty)$

$$-\left(\rho(t)|\underline{u}'(t)|^{p-2}\underline{u}'(t)\right)' = \lambda_1 \sigma(t) m^{p-1}\varphi_1^{p-1} \le \lambda\sigma(t) f(m\varphi_1) = \lambda\sigma(t) f(\underline{u}) .$$

Now using the decay estimates in Corollary 6.2 of the eigenfunction φ_1 and Corollary 6.3 of the auxiliary function e at the end points of the interval $(1, +\infty)$, we can adjust the constant $m \approx 0$ so that $\underline{u} \le \overline{u}$ in $(1, +\infty)$. Then by Theorem 3.1, there exist a minimal weak solution u_{min} and a maximal weak solution $u_{\max}$ of (8.1) such that

$$0 < \underline{u} \le u_{\min} \le u_{\max} \le \overline{u} \quad \text{in } (1, +\infty) ,$$

and enjoy the regularity properties (3.2). This completes the proof. □

Remark 8.2. We observe that the rates of decay of positive weak solutions obtained in Theorem 8.1 are independent of the nonlinearity f.

Next, we consider radially symmetric positive solutions of the following PDE in dimension $N > 1$

$$\begin{cases} -\text{div}\left(v(|x|)|\nabla u(|x|)|^{p-2}\nabla u(|x|)\right) = \lambda w(|x|) f(u(|x|)), \quad x \in A_1^{+\infty} \subset \mathbb{R}^N, \\ u(x) = 0, \quad x \in \partial A_1^{+\infty} , \end{cases} \tag{8.2}$$

where $\lambda > 0$ is a parameter, f is as above, and $A_1^{+\infty} = \overline{B}_1^c$ is the exterior of a unit ball. We obtain the counterpart of Theorem 8.1 below.

Theorem 8.3. *Let the weight functions v and w be as in (7.6) with $\alpha_1, \beta_1, \alpha_\infty, \beta_\infty$ satisfying (7.10) and (7.11). Let $p > 1$ and (H3)–(H4) hold. Then for any $\lambda > \frac{\lambda_1}{K}$, there exist a minimal weak solution $u_{\min}$ and a maximal weak solution $u_{\max}$ of (8.2). Moreover, given $\varepsilon \in (0, p-1)$, there exist constants $C > 1$, $C_1, C_2 > 0$ such that for all $|x| \in (1, C)$ we have*

$$C_1 \left(|x| - 1\right)^{1 - \frac{\alpha_1}{p-1}} \le u_{\min}(|x|) \le u_{\max}(|x|) \le C_2 \left(|x| - 1\right)^{\left(1 - \frac{\alpha_1}{p-1}\right)\left(1 - \frac{\varepsilon}{p-1}\right)} . \tag{8.3}$$

Similarly, given $\varepsilon \in (0, p-1)$, there exist constants $D > 1$, $D_1, D_2 > 0$ such that for all $|x| \in (D, +\infty)$ we have

$$D_1 |x|^{1 + \frac{\alpha_\infty + 1 - N}{p-1}} \le u_{\min}(|x|) \le u_{\max}(|x|) \le D_2 |x|^{\left(1 + \frac{\alpha_\infty + 1 - N}{p-1}\right)\left(1 - \frac{\varepsilon}{p-1}\right)} .$$

Proof. Substituting $r = |x|$, (8.2) transforms to

$$\begin{cases} -\left(r^{N-1} v(r)|u'(r)|^{p-2}u'(r)\right)' = \lambda r^{N-1} w(r) f(u(r)), \quad r \in (1, +\infty), \\ \lim\limits_{r \to 1^+} u(r) = \lim\limits_{r \to +\infty} u(r) = 0 . \end{cases} \tag{8.4}$$

Observe that (8.4) is a special case of (8.1) with $\rho(t) = t^{N-1} v(t)$ and $\sigma(t) = t^{N-1} w(t)$ for $t \in (1, +\infty)$. Then the proof follows by repeating the constructions in the proof of Theorem 8.1. □

Remark 8.4. We observe again that the rates of decay of positive weak solutions obtained in Theorem 8.3 are independent of the nonlinearity f. However, the decay rate at infinity depends on the dimension $N > 1$.

Remark 8.5. Notice that it follows from (7.10) that $\alpha_1 < p-1$. If $\alpha_1 \in (0, p-1)$ then $1 - \frac{\alpha_1}{p-1} < 1$ and hence the left inequality in (8.3) yields that $\frac{\partial u}{\partial \vec{n}} = +\infty$ on ∂B_1, where $\vec{n}$ denotes the outer unit normal vector of ∂B_1. On the other hand, if $\alpha_1 < 0$ then we can choose $\varepsilon \in (0, p-1)$ so that $(1 - \frac{\alpha_1}{p-1})(1 - \frac{\varepsilon}{p-1}) > 1$ and then the right inequality in (8.3) yields that $\frac{\partial u}{\partial \vec{n}} = 0$ on ∂B_1. Therefore, if $\alpha_1 \in (-\infty, 0) \cup (0, p-1)$, any weak solution u of (8.2) violates the Hopf maximum principle on ∂B_1, cf. [15, Thm. 5].

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Three point boundary value problems for ordinary differential equations, uniqueness implies existence

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. We consider a family of three point $n-2,1,1$ conjugate boundary value problems for nth order nonlinear ordinary differential equations and obtain conditions in terms of uniqueness of solutions imply existence of solutions. A standard hypothesis that has proved effective in uniqueness implies existence type results is to assume uniqueness of solutions of a large family of $n-$point boundary value problems. Here, we replace that standard hypothesis with one in which we assume uniqueness of solutions of large families of two and three point boundary value problems. We then close the paper with verifiable conditions on the nonlinear term that in fact imply global uniqueness of solutions of the large family of three point boundary value problems.

Keywords: uniqueness implies existence, nonlinear interpolation, ordinary differential equations, three point boundary value problems.

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1 Introduction

In a seminal paper, [23], Lasota and Opial proved that for second order ordinary differential equations, global existence and uniqueness of solutions of initial value problems and uniqueness of solutions of two point conjugate (Dirichlet) boundary value problems implies existence of solutions of two point conjugate boundary value problems. A vast study of problems referred to as uniqueness implies existence for higher order ($n-$th order) nonlinear problems was initiated. Following this work many related results were obtained; see for example, [3,8,9,15,19,21,22,24]. Henderson and many different co-authors have obtained analogous results for nonlocal boundary value problems, [2,14,16], for example, as well as boundary value problems for finite difference equations [11–13] for example, and boundary value problems

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for dynamic equations on time scales [17, 18], for example. Recently, these types of results were gathered in the monograph [4].

The results for n-th order problems, referred to above, all assumed a baseline unique solvability criterion for n-point Dirichlet type boundary conditions (n-point conjugate type boundary conditions.) Recently, the authors [5] revisited these uniqueness implies existence arguments with the baseline of a unique solvability criterion for two-point $n-1,1$ conjugate type boundary conditions. In this paper, we continue to develop the ideas initiated in [5] and begin with a baseline of unique solvability for two-point $n-1,1$ conjugate type boundary conditions and unique solvability criterion for two-point $n-2,1,1$ conjugate type boundary conditions.

Let $n \geq 2$ denote an integer and let $a < T_1 < T_2 < T_3 < b$. Let $a_i \in \mathbb{R}$, $i = 1, \dots, n$. Throughout this work, we shall consider the ordinary differential equation

$$y^{(n)}(t) = f(t, y(t), \dots, y^{(n-1)}(t)), \quad t \in [T_1, T_3], \tag{1.1}$$

where $f : (a, b) \times \mathbb{R}^n \to \mathbb{R}$, or the ordinary differential equation

$$y^{(n)}(t) = f(t, y(t)), \quad t \in [T_1, T_3], \tag{1.2}$$

where $f : (a, b) \times \mathbb{R} \to \mathbb{R}$. We shall consider three point boundary value problems for either (1.1) or (1.2) with the boundary conditions, for $j \in \{1, 2\}$,

$$y^{(i-1)}(T_1) = a_i, \quad i = 1, \dots, n-2, \quad y(T_2) = a_{n-1}, \quad y^{(j-1)}(T_3) = a_n, \tag{1.3}$$

and we shall consider two point boundary value problems for either (1.1) or (1.2) with the boundary conditions, for $j \in \{1, 2\}$,

$$y^{(i-1)}(T_1) = a_i, \quad i = 1, \dots, n-1, \quad y^{(j-1)}(T_2) = a_n. \tag{1.4}$$

For expository reasons only we state the $n-$point conjugate boundary conditions,

$$y(T_i) = a_i, \quad i \in \{1, \dots, n\}, \tag{1.5}$$

where $a < T_1 < \cdots < T_n < b$.

The intent of this work is to show that under the assumptions of uniqueness of solutions of the boundary value problems (1.1), (1.3) and of the boundary value problems (1.1), (1.4), then there exists a solution of the boundary value problem (1.1) with boundary conditions (1.3) in the case $j = 1$.

With respect to (1.1), common assumptions for the types of results that we consider are:

(A) $f(t, y_1, \dots, y_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous;

(B) Solutions of initial value problems for (1.1) are unique and extend to (a, b);

With respect to (1.2), the assumptions (A) and (B) are replaced, respectively, by

(A') $f(t, y) : (a, b) \times \mathbb{R} \to \mathbb{R}$ is continuous;

(B') Solutions of initial value problems for (1.2) are unique and extend to (a, b).

There are two main purposes of this work. The first purpose is to obtain uniqueness of solutions for the boundary value problems (1.1), (1.3) and (1.1), (1.4) implies existence of solutions for the family of two-point boundary value problems (1.1), (1.3) in the case $j = 1$, and the primary tool will be a modification of the original sequential compactness argument provided by Lasota and Opial [23]. The second purpose is to obtain verifiable hypotheses that imply the uniqueness of solutions for the boundary value problems (1.2), (1.3) and (1.2), (1.4); hence, as a corollary, these verifiable hypotheses imply existence of solutions for the family of two-point boundary value problems (1.1), (1.3) in the case $j = 1$. And as it turns out, the existence will be global in $T_2 < T_3 < b$.

In Section 2, we remind the reader of a generalized mean value theorem for higher order derivatives that is commonly used in interpolation theory. It is this generalized mean value theorem that allows the Lasota and Opial argument [23] to be modified. Then in Section 3, we shall consider the general ordinary differential equation (1.1) with the boundary conditions (1.3) or (1.4). It is in Section 3 where we carry out the first main purpose of this work; in particular we produce hypotheses such that uniqueness of solutions for the boundary value problems (1.1), (1.3) and (1.1), (1.4) implies existence of solutions for the family of two-point boundary value problems (1.1), (1.3) in the case $j = 1$.

To implement the results in the literature cited above or likewise for the main result in Section 3, bounds on $T_3 - T_1$ are often required so that the contraction mapping principle can be employed to obtain the appropriate uniqueness criteria. This has led to the concept of best interval lengths for Lipschitz equations [6, 10, 20]. So in Section 4, to carry out the second purpose of this work to produce verifiable hypotheses, we consider the ordinary differential equation (1.2) with boundary conditions (1.3) or (1.4) and we assume f satisfies a Lipschitz condition in y. We construct Green's functions and estimates so that the contraction mapping principle can apply. Then in Section 5, we impose monotonicity hypotheses on f (in addition to the Lipschitz assumption) to produce the verifiable hypotheses to fulfill the second purpose of the article. In doing so, we obtain a type of global uniqueness implies existence result as will be discussed further in Section 5.

We state three further common assumptions, two of which are used throughout the paper.

(C) Solutions of the $n-$point boundary value problems (1.1), (1.5) are unique if they exist.

(D) Solutions of the two-point boundary value problems (1.1), (1.4) are unique if they exist.

(E) Solutions of the three point boundary value problems (1.1), (1.3) are unique if they exist.

We do not assume Condition (C) in this work; we state it to clearly see the contrast between this work and those cited in the first paragraph.

2 A review of divided differences

Lasota and Opial [23] literally employed the mean value theorem to construct a sequential compactness argument for the the second order conjugate boundary value problem. To modify that construction, we introduce a divided difference construction that is employed to derive an error bound for interpolating polynomials. An extension of the mean value theorem is the result. For the sake of self containment, we provide the following details. We refer the reader to the text by Conte and de Boor [1]. Let $t_0, \dots, t_i$ denote $i + 1$ distinct real numbers and let

$z : \mathbb{R} \to \mathbb{R}$. Define $z[t_l] = z(t_l)$, $l = 0, \ldots, i$ and if $t_l, \ldots, t_{k+1}$ denote $k - l + 2$ distinct points, define

$$z[t_l, \ldots, t_{k+1}] = \frac{z[t_{l+1}, \ldots, t_{k+1}] - z[t_l, \ldots, t_k]}{t_{k+1} - t_l}.$$

The following theorem is obtained by repeated applications of Rolle's theorem to the difference of z and the polynomial that interpolates z at the $i + 1$ distinct points $t_0, \ldots, t_i$; a proof can be found in [1, Theorem 2.2].

Theorem 2.1. *Assume $z(t)$ is a real-valued function, defined on $[a, b]$ and i times differentiable in (a, b). If $t_0, \ldots, t_i$ are $i + 1$ distinct points in $[a, b]$, then there exists*

$$c \in (\min\{t_0, \ldots, t_i\}, \max\{t_0, \ldots, t_i\})$$

such that

$$z[t_0, \ldots, t_i] = \frac{z^{(i)}(c)}{i!}.$$

In Section 3, we shall set $h > 0$ and choose $t_0 = T, t_1 = T + h, \ldots, t_i = T + ih$ to be equally spaced. In this setting

$$z[T, T + h, \ldots, T + ih] = \frac{\sum_{l=0}^{i} (-1)^{i-l} \binom{i}{l} z(T + lh)}{i! h^i}.$$

For example, if $i = 1$, Theorem 2.1 is the mean value theorem and if $i = 2$, there exists $c \in (T, T + 2h)$ such that

$$\frac{z(T) - 2z(T + h) + z(T + 2h)}{2! h^2} = \frac{z''(c)}{2!}.$$

So, in general there exists $c \in (T_1, T_1 + ih)$ such that

$$\frac{\sum_{l=0}^{i} (-1)^{i-l} \binom{i}{l} z(T + ih)}{h^i} = z^{(i)}(c). \tag{2.1}$$

3 Uniqueness of solutions implies existence of solutions

In this section we consider the families of boundary value problems (1.1), (1.3) and (1.1), (1.4). We shall provide two preliminary results, Lemma 3.1 and Theorem 3.3, one addressing the continuous dependence of solutions of (1.1) on initial conditions and another addressing the continuous dependence of solutions of (1.1) on two point boundary conditions.

We state the first lemma without proof. See [7, page 14].

Lemma 3.1. *Assume that with respect to (1.1), Conditions (A) and (B) are satisfied. Then, given a solution y of (1.1), given $t_0 \in (a, b)$, given any compact interval $[c, d] \subset (a, b)$, and given $\epsilon > 0$, there exists $\delta > 0$ such that if z is a solution of (1.1) satisfying $|y^{(i-1)}(t_0) - z^{(i-1)}(t_0)| < \delta$, $i = 1, \ldots, n$, then $|y^{(i-1)}(t) - z^{(i-1)}(t)| < \epsilon$, $i = 1, \ldots, n$, for all $t \in [c, d]$.*

For the sake of self-containment, we also state the Brouwer invariance of domain theorem.

Theorem 3.2. *If $\mathcal{U} \subset \mathbb{R}^k$ is open, $\phi : \mathcal{U} \to \mathbb{R}^k$ is one-to-one and continuous on $\mathcal{U}$, then ϕ is a homeomorphism and $\phi(\mathcal{U})$ is open in $\mathbb{R}^k$.*

In [5], the authors employed the Brouwer invariance of domain theorem to prove continuous dependence of solutions on the boundary conditions (1.4); in particular, they proved the following theorem.

Theorem 3.3. *Assume that with respect to* (1.1) *Conditions (A), (B), and (D) are satisfied. Let* $j \in \{1,2\}$.

(i) Given any $a < T_1 < T_2 < b$, *and any solution* y *of* (1.1), *there exists* $\epsilon > 0$ *such that if* $|T_{11} - T_1| < \epsilon$, $|y^{(i-1)}(T_1) - y_{i1}| < \epsilon$, $i = 1, \ldots, n-1$, *and* $|T_{21} - T_2| < \epsilon$ $|y^{(j-1)}(T_2) - y_{n1}| < \epsilon$, *then there exists a solution* z *of* (1.1) *such that* $z^{(i-1)}(T_{11}) = y_{l1}$, $i = 1, \ldots, n-1$, $z^{(j-1)}(T_{21}) = y_{n1}$.

(ii) If $T_{1k} \to T_1$, $T_{2k} \to T_2$, $y_{ik} \to y_i$, $i = 1, \ldots, n$ *and* z_k *is a sequence of solutions of* (1.1) *satisfying* $z_k^{(i-1)}(T_{1k}) = y_{ik}$, $i = 1, \ldots, n-1$, $z_k^{(j-1)}(T_{2k}) = y_{nk}$, *then for each* $i \in \{1, \ldots, n\}$, $z_k^{(i-1)}$ *converges uniformly to* $y^{(i-1)}$ *on compact subintervals of* (a, b).

Here, we shall employ the Brouwer invariance of domain theorem to prove continuous dependence of solutions on the boundary conditions (1.3).

Theorem 3.4. *Assume that with respect to* (1.1) *Conditions (A), (B), and (E) are satisfied. Let* $j \in \{1,2\}$.

(i) Given any $a < T_1 < T_2 < T_3 < b$, *and any solution* y *of* (1.1), *there exists* $\epsilon > 0$ *such that if* $|T_{11} - T_1| < \epsilon$, $|y^{(i-1)}(T_1) - y_{i1}| < \epsilon$, $i = 1, \ldots, n-2$, $|T_{21} - T_2| < \epsilon$, *and* $|T_{31} - T_3| < \epsilon$, $|y(T_2) - y_{(n-1)1}| < \epsilon$, $|y(T_3) - y_{n1}| < \epsilon$, *then there exists a solution* z *of* (1.1) *such that* $z^{(i-1)}(T_{11}) = y_{l1}$, $i = 1, \ldots, n-2$, $z(T_{21}) = y_{(n-1)1}$, *and* $z^{(j-1)}(T_{31}) = y_{n1}$.

(ii) If $T_{1k} \to T_1$, $T_{2k} \to T_2$, $T_{3k} \to T_3$, $y_{ik} \to y_i$, $i = 1, \ldots, n$ *and* z_k *is a sequence of solutions of* (1.1) *satisfying* $z_k^{(i-1)}(T_{1k}) = y_{ik}$, $i = 1, \ldots, n-2$, $z_k(T_{2k}) = y_{(n-1)k}$, $z_k^{(j-1)}(T_{3k}) = y_{nk}$, *then for each* $i \in \{1, \ldots, n\}$, $z_k^{(i-1)}$ *converges uniformly to* $y^{(i-1)}$ *on compact subintervals of* (a, b).

Proof. Let $j \in \{1,2\}$. Define $\mathcal{U} \subset \mathbb{R}^{n+3}$ to be the open set

$$\mathcal{U} = \{(T_1, T_2, T_3, c_1, \ldots, c_n) : a < T_1 < T_2 < T_3 < b, c_i \in \mathbb{R}, i = 1, \ldots, n\}.$$

Let $t_0 \in (a, b)$. Define $\phi : \mathcal{U} \to \mathbb{R}^{n+3}$ by

$$\phi(T_1, T_2, T_3, c_1, \ldots, c_n) = (T_1, T_2, T_3, y(T_1), \ldots, y^{(n-3)}(T_1), y(T_2), y^{(j-1)}(T_3)),$$

where y is the unique solution of (1.1) satisfying the initial conditions $y^{(i-1)}(t_0) = c_i$, $i = 1, \ldots, n$. Then by Lemma 3.1, ϕ is continuous on $\mathcal{U}$.

To see that ϕ is a $1-1$ map on $\mathcal{U}$ let

$$(t_1, t_2, t_3, c_1, \ldots, c_n), (s_1, s_2, s_3, d_1, \ldots, d_n) \in \mathcal{U}$$

and assume

$$\phi(t_1, t_2, t_3, c_1, \ldots, c_n) = \phi(s_1, s_2, s_3, d_1, \ldots, d_n).$$

By the definition of ϕ, $t_i = s_i$, $i = 1, 2, 3$. It follows by Condition (E) that $c_i = d_i$, $i = 1, \ldots, n$, since if y, z are solutions of (1.1) and $y^{(i-1)}(T_1) = z^{(i-1)}(T_1)$, $i = 1, \ldots, n-2$, $y(T_2) = z(T_2)$, $y^{(j-1)}(T_3) = z^{(j-1)}(T_3)$, then $y \equiv z$ on (a, b); in particular, $c_i = y^{(i-1)}(t_0) = z^{(i-1)}(t_0) = d_i$, $i = 1, \ldots, n$. Apply Brouwer's invariance of domain theorem to obtain that $\phi(\mathcal{U})$ is open in $\mathbb{R}^{n+3}$ which proves (i), and to obtain that ϕ^{-1} is continuous on $\mathcal{U}$ which proves (ii). $\square$

Finally we state the uniqueness implies existence theorem proved by the authors in [5].

Theorem 3.5. *Assume that with respect to* (1.1)*, Conditions (A), (B), and (D) are satisfied. Then for each* $a < T_1 < T_2 < b$, $a_i \in \mathbb{R}$, $i = 1, \dots, n$, *the two point boundary value problem* (1.1), (1.4) *has a solution.*

We are now in a position to adapt the method of Lasota and Opial [23] and show that the uniqueness of solutions of the boundary value problems (1.1), (1.3) and (1.1), (1.4) implies the existence of solutions of the boundary value problem (1.1), (1.3) for $j = 1$.

Theorem 3.6. *Assume that with respect to* (1.1)*, Conditions (A), (B), (D) and (E) are satisfied. Then for each* $a < T_1 < T_2 < T_3 < b$, $a_i \in \mathbb{R}$, $i = 1, \dots, n$, *then for* $j = 1$, *the three point boundary value problem* (1.1), (1.3) *has a solution.*

Proof. Let $m \in \mathbb{R}$ and denote by $y(t; m)$ the solution of the two-point boundary value problem (1.1), with boundary conditions

$$y^{(i-1)}(T_1; m) = a_i, \quad i = 1, \dots, n-2, \quad y^{(n-2)}(T_1; m) = m, \quad y(T_2) = a_{n-1}.$$

Let

$$\Omega = \{p \in \mathbb{R} : \text{ there exists } m \in \mathbb{R} \text{ with } y(T_3; m) = p\}.$$

So the theorem is proved by showing $\Omega = \mathbb{R}$. By Theorem 3.5, $\Omega \neq \emptyset$, so the theorem is proved by showing Ω is opened and closed. That Ω is open follows from Theorem 3.4.

To show Ω is closed, let p_0 denote a limit point of Ω and without loss of generality let p_k denote a strictly increasing sequence of reals in Ω converging to p_0. Assume $y(T_3; m_k) = p_k$ for each $k \in \mathbb{N}_1$. It follows by the uniqueness of solutions, Condition (E), that

$$y^{(j-1)}(t; m_{k_1}) \neq y^{(j-1)}(t; m_{k_2}), \quad t \in (T_2, b), \tag{3.1}$$

for each $j \in \{1, 2\}$, if $k_1 < k_2$ and in particular,

$$y(t; m_1) < y(t; m_k) \quad t \in (T_2, b), \tag{3.2}$$

for each k.

Either $y'(T_3; m_k) \leq 0$ infinitely often or $y'(T_3; m_k) \geq 0$ infinitely often. Relabel if necessary and assume $y'(T_3; m_k) \leq 0$ or $y'(T_3; m_k) \geq 0$ for each k. Finally note that (3.1) implies that we may assume $y'(T_3; m_k) < 0$ or $y'(T_3; m_k) > 0$ for each k.

We first assume the case $y'(T_3; m_k) < 0$ for each k. Find $T_3 < T_4 < b$ such that $y'(t; m_1) \leq 0$, for $t \in [T_3, T_4]$. Then $y(t; m_1)$ is decreasing on $[T_3, T_4]$. By (3.2), if $t \in [T_3, T_4]$ and $k \geq 1$, then

$$L = y(T_4; m_1) \leq y(t; m_1) \leq y(t; m_k). \tag{3.3}$$

Fix k and find $T_3 < T_{4k} \leq T_4$ such that $y'(t; m_k) < 0$ on $[T_3, T_{4k}]$. Then $y(t; m_k)$ is decreasing on $[T_3, T_{4k}]$; in particular

$$L \leq y(T_{4k}; m_1) < y(T_{4k}; m_k) \leq y(t; m_k) \leq y(T_3; m_k) \leq p_0 \tag{3.4}$$

for $t \in [T_3, T_{4k}]$.

The observation employed by Lasota and Opial [23] is

$$0 > \frac{y(T_{4k}; m_k) - y(T_3; m_k)}{T_{4k} - T_3} \geq \frac{L - p_0}{T_{4k} - T_3} \geq \frac{L - p_0}{T_4 - T_3} = K_1. \tag{3.5}$$

Apply the mean value theorem (or (2.1) in the case $i = 1$ to the left hand side of (3.5), to see that

$$S_{k1} = \{t \in [T_3, T_{4k}] : K_1 - 1 \leq y'(t; m_k) < 0\} \neq \emptyset;$$

by the continuity of $y'(t; m_k)$, there exists a closed interval of positive length,

$$I_1 = [T_{3k1}, T_{4k1}] \subset S_{k1} \subset [T_3, T_{4k}].$$

To outline an induction argument in i, the order of the derivative $y^{(i-1)}$, set $h = \frac{T_{4k1} - T_{3k1}}{2}$ and consider

$$\frac{y(T_{3k1}; m_k) - 2y(T_{3k1} + h; m_k) + y(T_{3k1} + 2h; m_k)}{h^2}.$$

Then, continuing to observe that $y(t, m_k)$ is decreasing on I_1,

$$\frac{y(T_{31}; m_k) - 2y(T_{31} + h) + y(T_{31} + 2h)}{h^2} \geq \frac{2(L - p_0)}{h^2} = \frac{2^3(L - p_0)}{(T_{4k1} - T_{3k1})^2} \geq \frac{2^3(L - p_0)}{(T_4 - T_3)^2} = K_2$$

and

$$\frac{y(T_{31}; m_k) - 2y(T_{31} + h) + y(T_3 + 2h)}{h^2} \leq \frac{2(p_0 - L)}{h^2} \leq -K_2.$$

In particular,

$$\left| \frac{y(T_{31}; m_k) - 2y(T_{31} + h) + z(T_{31} + 2h)}{h^2} \right| \leq K_2.$$

Apply (2.1) in the case $i = 2$ and the set

$$S_{k2} = \{t \in [T_{3k1}, T_{4k1}] : |y''(t; m_k)| \leq -K_2 + 1\} \neq \emptyset$$

and contains a closed interval of positive length

$$I_2 = [T_{3k2}, T_{4k2}] \subset S_{k2} \subset [T_{3k1}, T_{43k1}] \subset [T_3, T_4].$$

The induction hypothesis is then, for $i \in \{2, \ldots n - 2\}$ assume there exist $T_{3ki} < T_{4ki}$ such that $I_i = [T_{3ki}, T_{4ki}] \subset [T_{3k(i-1)}, T_{4k(i-1)}] \subset [T_3, T_4]$ and

$$|y^{(i)}(t; m_k)| \leq -K_i + 1, \quad t \in I_i$$

where

$$K_i = \frac{i^i 2^{i-1}(L - p_0)}{(T_4 - T_3)^i}.$$

Set $h = \frac{T_{4ki} - T_{3ki}}{i+1}$. Then,

$$\left| \frac{\sum_{l=0}^{i+1} (-1)^{i+1-l} \binom{i+1}{l} y(T_{3ki} + lh)}{h^{i+1}} \right| \geq \frac{(i+1)^{i+1} 2^i (L - p_0)}{(T_{4ki} - T_{3ki})^{i+1}} \geq \frac{(i+1)^{i+1} 2^i (L - p_0)}{(T_4 - T_3)^{i+1}} = -K_{i+1}.$$

Apply (2.1) in the case $i + 1$ and the set,

$$S_{k(i+1)} = \{t \in [T_{3ki}, T_{4ki}] : |y^{(i+1)}(t; m_k)| \leq -K_{i+1} + 1\} \neq \emptyset$$

and contains a closed interval of positive length

$$I_{i+1} = [T_{3(i+1)}, T_{4(i+1)}] \subset [T_{3i}, T_{4i}] \subset [T_3, T_4].$$

Recall, k is fixed. For this fixed k, choose $t_k \in I_{n-1}$. Then

$$(t_k, y(t_k; m_k), y'(t_k; m_k), \dots, y^{(n-1)}(t_k; m_k)) \in [T_3, T_4] \times [L, p_0] \times \Pi_{i=1}^{n-1}[-K_i - 1, K_i + 1].$$

The set on the righthand side is a compact subset of $\mathbb{R}^{n+1}$ and independent of k. Perform this process for each k and generate a sequence

$$\{(t_k, y(t_k; m_k), y'(t_k; m_k), \dots, y^{(n-1)}(t_k; m_k))\}_{k=1}^{\infty} \subset [T_3, T_4] \times [L, p_0] \times \Pi_{i=1}^{n-1}[-K_i - 1, K_i + 1].$$

In particular, there exists a convergent subsequence (relabeling if necessary)

$$\{(t_k, y(t_k; m_k), y'(t_k; m_k), \dots, y^{(n-1)}(t_k; m_k))\} \to (t_0, c_1, \dots, c_n)$$

where $t_0 \in [T_3, T_4]$. Since $t_0 \in (a, b)$ and by the continuous dependence of solutions of initial value problems, Lemma 3.1, $y(t; m_k)$ converges in $C^{n-1}[T_1, T_3]$ to a solution, say $z(t)$, of the initial value problem (1.1), with initial conditions, $y^{(i-1)}(t_0) = c_i$, $i = 1, \dots, n$. Thus, $p_0 = z(T_3)$ which implies $p_0 \in \Omega$ and Ω is closed. This completes the proof if $y'(T_3; m_k) < 0$ for each k.

If $y'(T_3; m_k) > 0$ for each k, find $T_2 < T_4 < T_3$ such that $y'(t; m_1) \geq 0$, for $t \in [T_4, T_3]$. Then

$$L = y(T_4; m_1) < y(T_4; m_k) \leq y(t; m_k) \leq p_0, \quad T_4 \leq t \leq T_3,$$

and the above argument can be modified to apply on $[T_4, T_3]$. This completes the proof. □

4 Local uniqueness of solutions

In this section, we state conditions on $f(t, y)$ such that solutions of a boundary value problem (1.2), (1.3) are unique, if they exist, for $T_3 - T_1$ sufficiently small. The ideas here are not new and the result we state is standard, but the estimates that are employed are possibly new and the construction is provided for the sake of self containment. Assume that $f : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous and that there exists a positive constant, P such that

$$|f(t, y) - f(t, z)| \leq P|y - z| \tag{4.1}$$

for all $(t, y), (t, z) \in (a, b) \times \mathbb{R}$.

We require specific estimates for the Green's function for the boundary value problem (1.2), (1.3) for each $j = 1, 2$.

For $j = 1$, the Green's function, $G(1; t, s)$ for the boundary value problem (1.2), (1.3) has the following representation. If $T_1 \leq s \leq T_2$,

$$G(1; t, s) = \begin{cases} \frac{(t-T_1)^{n-2}}{(T_3-T_2)(n-1)!}\left[\frac{(T_2-s)^{n-1}(t-T_3)}{(T_2-T_1)^{n-2}} + \frac{(T_3-s)^{n-1}(T_2-t)}{(T_3-T_1)^{n-2}}\right], & T_1 \leq t \leq s \leq T_2, \\ \frac{(t-T_1)^{n-2}}{(T_3-T_2))(n-1)!}\left[\frac{T_2-s)^{n-1}(t-T_3)}{(T_2-T_1)^{n-2}} + \frac{(T_3-s)^{n-1}(T_2-t)}{(T_3-T_1)^{n-2}}\right] + \frac{(t-s)^{n-1}}{(n-1)!}, & T_1 \leq s \leq t \leq T_3, \end{cases}$$

and if $T_2 \leq s \leq T_3$,

$$G(1; t, s) = \begin{cases} \frac{(t-T_1)^{n-2}(T_3-s)^{n-1}(T_2-t)}{(T_3-T_2)(T_3-T_1)^{n-2}(n-1)!}, & T_1 \leq t \leq s \leq T_2, \\ \frac{(t-T_1)^{n-2}(T_3-s)^{n-1}(T_2-t)}{(T_3-T_2)(T_3-T_1)^{n-2}(n-1)!} + \frac{(t-s)^{n-1}}{(n-1)!}, & T_1 \leq s \leq t \leq T_3. \end{cases}$$

The Green's function is constructed in the following way. If (1.1) or (1.2) is a nonhomogenous linear equation, then the general solution is

$$y(t) = \sum_{i=1}^{n} c_i (t - T_1)^{i-1} + \int_{T_1}^{t} \frac{(t-s)^{n-1}}{(n-1)!} f(s) ds.$$

The homogeneous boundary conditions at T_1 imply $c_i = 0$, $i = 1, \ldots, n-2$. The homogeneous boundary conditions at T_2 and T_3 imply

$$\begin{cases} 0 = c_{n-1} + c_n (T_2 - T_1) + \int_{T_1}^{T_2} \frac{(T_2-s)^{n-1}}{(T_2-T_1)^{n-2}(n-1)!} f(s) ds, \\ 0 = c_{n-1} + c_n (T_3 - T_1) + \int_{T_1}^{T_3} \frac{(T_3-s)^{n-1}}{(T_3-T_1)^{n-2}(n-1)!} f(s) ds. \end{cases}$$

We now seek a bound on $|G(1;t,s)|$ on $[T_1, T_3] \times [T_1, T_3]$. The term $(T_3 - T_2)$ in the common denominator is apparently problematic. We provide algebraic details to show the term is not problematic. First note that if $T_1 \leq s$, then usual calculus methods imply that the function

$$h(\alpha) = \frac{(\alpha - s)^{n-1}}{(\alpha - T_1)^{n-2}}$$

is increasing in α for $s \leq \alpha$. In particular,

$$\frac{(T_2 - s)^{n-1}}{(T_2 - T_1)^{n-2}} < \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}}.$$

If $T_1 \leq t \leq T_2$,

$$\begin{aligned} \frac{(T_2 - s)^{n-1}}{(T_2 - T_1)^{n-2}}(t - T_3) &> \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}}(t - T_3) \\ &= \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}}(t - T_2) + \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}}(T_2 - T_3). \end{aligned}$$

So,

$$\frac{(T_2 - s)^{n-1}}{(T_2 - T_1)^{n-2}}(t - T_3) + \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}}(T_2 - t) > \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}}(T_2 - T_3).$$

Similarly, if $T_2 \leq t \leq T_3$,

$$\frac{(T_2 - s)^{n-1}}{(T_2 - T_1)^{n-2}}(t - T_3) + \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}}(T_2 - t) < \frac{(T_2 - s)^{n-1}}{(T_2 - T_1)^{n-2}}(T_2 - T_3).$$

Keeping in mind that the function $h(\alpha)$ is increasing we have, for $T_1 \leq s \leq T_2$, $T_1 \leq t \leq T_3$,

$$\left| \frac{(T_2 - s)^{n-1}}{(T_2 - T_1)^{n-2}}(t - T_3) + \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}}(T_2 - t) \right| \leq \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}}(T_3 - T_2). \tag{4.2}$$

Now with the help of (4.2) it is now clear to see that

$$|G(1;t,s)| \leq \frac{2(T_3 - T_1)^{n-1}}{(n-1)!}, \quad (t,s) \in [T_1, T_2] \times [T_1, T_2]. \tag{4.3}$$

For $j = 2$, to construct the Green's function, $G(2;t,s)$, we solve a similar system of two equations to compute c_{n-1} and c_n for the boundary value problem (1.2), (1.4) and obtain the following representation. Let $D = (T_3 - T_1) + (n-2)(T_3 - T_2)$. Define

$$g(t,s) = \frac{(T_2 - s)^{n-1}}{(n-1)!(T_2 - T_1)^{n-2}}(-(n-1)(T_3 - T_1) + (n-2)(t - T_1)) + \frac{(T_3 - s)^{n-2}}{(n-2)!(T_3 - T_1)^{n-3}}(T_2 - t).$$

If $T_1 \le s \le T_2$,

$$G(2;t,s) = \begin{cases} \frac{(t-T_1)^{n-2} g(t,s)}{D}, & T_1 \le t \le s \le T_2, \\ \frac{(t-T_1)^{n-2} g(t,s)}{D} + \frac{(t-s)^{n-1}}{(n-1)!}, & T_1 \le s \le t \le T_3, \end{cases}$$

and if $T_2 \le s \le T_3$,

$$G(2;t,s) = \begin{cases} \frac{(t-T_1)^{n-2}(T_3-s)^{n-2}}{D(n-2)!(T_3-T_1)^{n-3}}(T_2 - t), & T_1 \le t \le s \le T_2, \\ \frac{(t-T_1)^{n-2}(T_3-s)^{n-2}}{D(n-2)!(T_3-T_1)^{n-3}}(T_2 - t) + \frac{(t-s)^{n-1}}{(n-1)!}, & T_1 \le s \le t \le T_3. \end{cases}$$

Now the term $T_3 - T_2$ in D is not problematic since $D > T_3 - T_1$.

To bound $|G(2;t,s)|$, we keep in mind that $h(\alpha)$ is increasing and write

$$|-(n-1)(T_3 - T_1) + (n-2)(t - T_1)| = |(n-2)(t - T_3) - (T_3 - T_1)| \le (n-1)(T_3 - T_1).$$

Then,

$$\left| \frac{(T_2 - s)^{n-1}}{(n-1)!(T_2 - T_1)^{n-2}}(-(n-1)(T_3 - T_1) + (n-2)(t - T_1)) \right| \le \frac{(T_3 - T_1)^{n-1}}{(n-2)!}$$

and

$$\left| \frac{(T_3 - s)^{n-2}}{(n-2)!(T_3 - T_1)^{n-3}}(T_2 - t) \right| \le \frac{(T_3 - T_1)^{n-1}}{(n-2)!}.$$

Thus,

$$|G(2;t,s)| \le \frac{(2n-1)(T_3 - T_1)^{n-1}}{(n-1)!}, \quad (t,s) \in [T_1, T_2] \times [T_1, T_2]. \tag{4.4}$$

For each $a < T_1 < T_2 < T_3 < b$, consider the usual Banach space $C[T_1, T_3]$ with norm

$$\|y\| = \max_{T_1 \le t \le T_3} |y(t)|.$$

For each $j \in \{1, 2\}$, define the fixed point operator $T(j;\cdot) : C[T_1, T_3] \to C[T_1, T_3]$ by

$$T(j;y)(t) = p_{cj}(t) + \int_{T_1}^{T_3} G(j;t,s) f(s, y(s)) ds,$$

where p_{cj} denotes the $n-1$ order polynomial satisfying the boundary conditions (1.3). Then (4.1), (4.3) and (4.4) are readily employed to see that if $y, z \in C[T_1, T_3]$, then for $T_1 \le t \le T_3$,

$$\begin{aligned} |T(j;y)(t) - T(j;z)(t)| &\le \int_{T_1}^{T_3} |G(j;t,s)||f(s, y(s) - f(s, z(s))| ds \\ &\le \max \left\{ \frac{2(T_3 - T_1)^n}{(n-1)!}, \frac{(2n-1)(T_3 - T_1)^n}{(n-1)!} \right\} P\|y - z\|. \end{aligned} \tag{4.5}$$

Choose

$$\delta = \left(\frac{(n-1)!}{(2n-1)P} \right)^{\frac{1}{n}} = \min \left\{ \left(\frac{(n-1)!}{2P} \right)^{\frac{1}{n}}, \left(\frac{(n-1)!}{(2n-1)P} \right)^{\frac{1}{n}} \right\}$$

and assume $|T_3 - T_1| < \delta$. Then the each fixed point map $T(j; \cdot)$ for $j \in \{1,2\}$ is a contraction map on $C[T_1, T_3]$.

Theorem 4.1. *Assume that $f : (a,b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous and that there exists positive constant P such that f satisfies (4.1) for all $(t,y), (t,z) \in (a,b) \times \mathbb{R}^n$. Assume $|T_3 - T_1| < \delta$ where*

$$\delta = \left(\frac{(n-1)!}{(2n-1)P} \right)^{\frac{1}{n}}.$$

Then for each $j \in \{1,2\}$ there exists a unique solution of the boundary value problem (1.2), (1.3).

The following information about the boundary value problem (1.2), (1.4) will be required in the next section so we state it here. For each $j \in \{1,2\}$, it was shown in [5] that the corresponding Green's function $\mathcal{G}(j; t, s)$ for the boundary value problem (1.2), (1.4) has the following representation and satisfies the following estimate:

$$\mathcal{G}(j;t,s) = \begin{cases} -\frac{(t-T_1)^{n-1}(T_2-s)^{n-j}}{(n-1)!(T_2-T_1)^{n-j}}, & T_1 \leq s \leq t \leq T_2, \\ -\frac{(t-T_1)^{n-1}(T_2-s)^{n-j}}{(n-1)!(T_2-T_1)^{n-j}} + \frac{(t-s)^{n-1}}{(n-1)!}, & T_1 \leq s \leq t \leq T_2. \end{cases} \tag{4.6}$$

Note that for each $i \in \{1, \ldots, n\}$,

$$|\mathcal{G}(j;t,s)| \leq \frac{2|(T_2 - T_1)|^{n-1}}{(n-1)!}, \quad (t,s) \in [T_1, T_2] \times [T_1, T_2]. \tag{4.7}$$

5 A type of global uniqueness of solutions implies existence of solutions for $n = 3$

In this section we consider the boundary value problem (1.2), (1.3) or the boundary value problem (1.2), (1.4), for $j \in \{1,2\}$ in the specific case that $n = 3$. We assume f continues to satisfy a Lipschitz condition in y; we shall also impose a new monotonicity condition on f. We shall assume that f is monotone decreasing in y for $t \in (T_1, T_2)$ and that f is monotone increasing in y for $t \in (T_2, T_3)$. Since the monotonicity of f depends on T_2, beginning with Theorem 5.2 we shall assume that T_2 is fixed and f is a function of $(T_2; t, y)$. For sake of exposition, we shall also assume that T_1 is fixed.

For $j \in \{1,2\}$, we first briefly address the local uniqueness of solutions for the boundary value problem, (1.2), (1.4). Continuing in the framework of the contraction mapping principle, employ the Banach space $\mathcal{B} = C[T_1, T_2]$ with the usual supremum norm. Then the fixed point operator

$$\mathcal{T}(j;y)(t) = p_c(t) + \int_{T_1}^{T_2} \mathcal{G}(j;t,s) f(s, y(s)) ds,$$

maps $\mathcal{B}$ into $\mathcal{B}$ if f is continuous and fixed points are 3 times continuously differentiable. By the estimates obtained in the preceding section, if each operator $T(j; y)$ is a contraction map, then each operator $\mathcal{T}(j; y)$ is a contraction map.

Theorem 5.1. *Assume that $f : (a,b) \times \mathbb{R} \to \mathbb{R}$ is continuous and assume there exists a positive constant, P, such that*

$$|f(t,y) - f(t,z)| \le P|y - z|$$

for all $(t,y),(t,z) \in (a,b) \times \mathbb{R}$. Assume $a < T_1 < T_2 < T_3 < b$ and and $T_3 - T_1 < \delta$ where

$$\delta = \left(\frac{(3-1)!}{(6-1)P} \right)^{\frac{1}{3}}.$$

Then for each $j \in \{1,2\}$ there exists a unique solution of the boundary value problem (1.2), (1.3) and there exists a unique solution of the boundary value problem (1.2), (1.4).

In the next result, we assume, in addition, that f is increasing in y and we prove a type of global uniqueness of solutions of the boundary value problem (1.2), (1.3). By global, we mean that although there is a constraint on $T_2 - T_1$, there is no local constraint on $T_3 - T_2$.

Theorem 5.2. *Assume $a < T_1 < T_2 < b$ and assume T_1 and T_2 are fixed. Assume that $f : (a,b) \times \mathbb{R} \to \mathbb{R}$ is continuous and assume there exists a positive constant, P, such that*

$$|f(t,y) - f(t,z)| \le P|y - z|$$

for all $(t,y),(t,z) \in (a,b) \times \mathbb{R}$. Assume $a < T_1 < T_2 < T_3 < b$. Set

$$\delta = \left(\frac{(3-1)!}{(6-1)P} \right)^{\frac{1}{3}}$$

and assume $T_2 - T_1 < \delta$. Assume

$$\begin{aligned} f(t,y) &\ge f(t,z), \quad t \in (T_1,T_2], \quad y < z, \\ f(t,y) &\le f(t,z), \quad t \in [T_2,b), \quad y < z. \end{aligned} \tag{5.1}$$

Then solutions of the boundary value problem (1.2), (1.3) are unique if they exist.

Proof. Assume for the sake of contradiction that y_1 and y_2 are distinct solutions of the boundary value problem (1.2), (1.3). We first argue that there exists $T_4 \in (T_1,T_2) \cup (T_2,T_3)$ such that $(y_1 - y_2)(T_4) = 0$. So, for the sake of contradiction, assume $y_1 - y_2$ is of constant sign on $(T_1,T_2) \cup (T_2,T_3)$ and without loss of generality assume $(y_1 - y_2)(t) > 0$ for $T_2 < t < T_3$. Set $u(t) = (y_1 - y_2)(t)$ and so $u(t) > 0$ on (T_2,T_3).

To obtain the contradiction, we shall consider multiple cases.

First assume $u(t) < 0$ on (T_1,T_2). Then by (5.1), $u'''(t) > 0$ on $(T_1,T_2) \cup (T_2,T_3)$. Thus, u'' is monotone increasing on (T_1,T_3). Apply Rolle's theorem to u which satisfies $u(T_1) = 0, u(T_2) = 0, u(T_3) = 0$ to obtain T_{11}, T_{12} and T_{21} satisfying

$$T_1 < T_{11} < T_2 < T_{12} < T_3, \quad T_{11} < T_{21} < T_{12}$$

such that

$$u'(T_{1i}) = 0, \quad i = 1,2, \quad u''(T_{21}) = 0.$$

Since u'' is monotone, there are no other roots of u'' or u' in (T_1,T_3). Since u'' is monotone increasing, $u''(t) > 0$ for $T_{21} < t$, this in turn implies u' is increasing for $T_{21} < t$. As $T_{21} < T_{12}$, this implies $u'(T_3) > 0$ which contradicts the hypothesis $u(t) > 0$ on (T_2,T_3).

Second, assume $u(t) > 0$ on (T_1, T_2). So now, $u'''(t) < 0$ on (T_1, T_2), $u'''(t) > 0$ on (T_2, T_3). In particular, u'' is decreasing on (T_1, T_2) and increasing on (T_2, T_3). We know u'' has at least one root in $[T_1, T_3]$ by Rolle's theorem and u'' has at most two roots in $[T_1, T_3]$ by the monotonicity property we have just observed on u''. Three more cases to consider are introduced.

Assume u'' has precisely one root, $T_{21} \in [T_1, T_3]$. By Rolle's theorem, $T_{21} \in [T_1, T_3]$ and u' has precisely two roots, T_{11}, T_{12} in $[T_1, T_3]$ satisfying

$$T_1 < T_{11} < T_{21} < T_{12} < T_3.$$

Since u'' is decreasing on (T_1, T_2) and increasing on (T_2, T_3) it must be the case that $T_{21} \leq T_2$. (If $T_{21} = T_2$, then T_{21} is a repeated root. The argument works here too, so we are not counting multiplicity in the assumption u'' has precisely one root.) In particular, $u''(t) > 0$, on $[T_1, T_{21})$. Thus u' is increasing on $[T_1, T_{21})$ and $u'(T_{11}) = 0$, where $T_{11} < T_{21}$. From here, we conclude $u'(T_1) < 0$. This yields a contradiction because it is assumed that $u(t) > 0$ on (T_1, T_2).

We now come to the possibility that u'' has two distinct roots, $T_{21} < T_{22}$ in $[T_1, T_3]$. By Rolle's theorem, either $T_{11} < T_{21}$ or $T_{22} < T_{21}$. These are the final two cases to consider.

Assume $T_{11} < T_{21}$. Now T_{11} has been generated by Rolle's theorem and u'' has no roots in $(T_1, T_{21}]$. So we can conclude that $u'(T_1) \neq 0$. So u'' is decreasing on (T_1, T_2) again implies $u''(t) > 0$ on $[T_1, T_{21})$. This in turn implies u' is increasing on (T_1, T_{21}) and so $u'(t) < 0$ on $[T_1, T_{11})$. We conclude that $u'(T_1) < 0$ contradicts $u(t) > 0$ on (T_1, T_2).

For the final case, assume $T_{22} < T_{12}$. Due due the monotone nature of u'' it is clear that $u''(t) > 0$ on $(T_1, T_{21}) \cup (T_{22}, T_3)$ and $u''(t) < 0$ on (T_{21}, T_{22}). (It could be the case that $T_1 = T_{21}$. In this case, $u''(t) < 0$ on (T_1, T_{22}) and $u''(t) > 0$ on (T_{22}, T_3).) Regardless, $u''(t) > 0$ on (T_{22}, T_3), which implies u' is increasing on (T_{22}, T_3). Finally, $T_{22} < T_{12}$ implies $u'(T_3) > 0$. This produces the final contradiction since it is assume throughout that $u(t) > 0$ on (T_1, T_2).

Thus there exists $T_4 \in (T_1, T_2) \cup (T_2, T_3)$ such that $y_1(T_4) = y_2(T_4)$. It is clear by Theorem 4.1 in the case $n = 3$ and the hypothesis $|T_2 - T_1| < \delta$ that $T_4 \notin (T_1, T_2)$. So, $T_4 \in (T_2, T_3)$.

Let

$$S = \{t \in (T_2, T_3) : (y_1 - y_2)(t) = 0\}.$$

We have just shown $S \neq \emptyset$. Let $\tau = \inf S$. If $\tau > T_2$, argue that $(y_1 - y_2)(\tau) = 0$. This follows by continuity if τ is a limit point of S and by definition if τ is an isolated point of S. Thus if $\tau > T_1$, y_1 and y_2 are distinct solutions of a boundary value problem (1.2), (1.3) for $T_3 = \tau$. Apply the argument that employed four cases to conclude there exists $T_4 \in (T_2, \tau)$ such that $(y_1 - y_2)(T_3) = 0$; in particular, the assumption that $\tau = \inf S > T_1$ is false.

So, $\inf S = T_2$. Find $T \in S$ such that $0 < T - T_1 < \delta$. Then Theorem 5.1 implies $y_1 \equiv y_2$ on $[T_1, T]$. Now Condition (B) implies $y_1 \equiv y_2$ on (a, b). □

We close the article with a corollary, which represents the main result addressing the second purpose of this work.

Corollary 5.3. *Assume $a < T_1 < T_2 < b$ and assume T_1 and T_2 are fixed. Assume that $f : (a, b) \times \mathbb{R} \to \mathbb{R}$ is continuous and assume there exists a positive constant, P, such that*

$$|f(t, y) - f(t, z)| \leq P|y - z|$$

for all $(t, y), (t, z) \in (a, b) \times \mathbb{R}$. Assume $a < T_1 < T_2 < T_3 < b$. Set

$$\delta = \left(\frac{(3-1)!}{(6-1)P} \right)^{\frac{1}{3}}$$

and assume $T_2 - T_1 < \delta$. *Assume* f *satisfies* (5.1). *Assume that with respect to* (1.2), *Conditions* (A') *and* (B') *are satisfied. Then for* $j = 1$, *and for each* $T_2 < T_3 < b$, *the three point boundary value problem* (1.2), (1.3) *has a solution.*

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Positive solutions for second-order differential equations with singularities and separated integral boundary conditions

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. We study the existence of positive solutions for second-order differential equations with separated integral boundary conditions. The nonlinear part of the equation involves the derivative and may be singular for the second and third space variables. The result ensures existence of a positive solution when the parameters are in certain ranges. The proof depends on general properties of the associated Green's function and the Krasnosel'skii–Guo fixed point theorem applied to a perturbed Hammerstein integral operator. Both numerical and analytical examples are constructed to illustrate applications of the theorem to a group of equations. The result generalizes previous work.

Keywords: fixed point, Green's function, Hammerstein integral operator, positive solution, singular boundary value problem.

2020 Mathematics Subject Classification: 34B10, 34B16, 34B18.

1 Introduction

We are interested in the following singular Boundary Value Problem (BVP) for second-order differential equations with non-local boundary conditions involving integrals:

$$\begin{cases} u''(t) + f(t, u(t), u'(t)) = 0, \quad t \in [0, 1], \\ \theta u(0) - \alpha u'(0) = \int_0^1 g_1(s)u(s)ds, \\ \gamma u(1) + \beta u'(1) = \int_0^1 g_2(s)u(s)ds, \end{cases} \tag{1.1}$$

where the parameters $\theta, \alpha, \beta > 0, \gamma \geq 0$. The nonlinear function f is continuous, non-negative on $[0, 1] \times (0, \infty) \times (0, \infty)$ and may be singular at zero on its space variables.

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When $\gamma = 0$, BVP (1.1) reduces to the problem studied in [17]. It also includes the anti-symmetric boundary conditions $u(0) = u'(0), u(1) = -u'(1)$ [13]. The local singular BVP studied in [16] is a special case of the boundary conditions of (1.1) when $\gamma = 0$, and $g_1 = g_2 = 0$ as well. Similar boundary conditions have been studied for fractional differential equations in [2] which assumed that $\theta = \gamma$ and $\alpha = -\beta$.

In the study of BVPs and their applications, nonlocal boundary conditions usually involve discrete multi-point boundary conditions. Previously, the following three-point BVPs have been extensively studied [3–5, 10, 12]:

$$u(0) = 0, \quad u(1) = \alpha u(\eta),$$

or

$$u'(0) = 0, \quad u(1) = \alpha u(\eta),$$

where $0 < \eta < 1$, α is a parameter. Later, the boundary conditions were further extended to involve integrals and functionals [7–9, 13–15]. In particular, in [14], existence of multiple positive solutions for nonlocal BVPs involving various integral conditions were obtained for the case that the nonlinear function f does not involve the first-order derivative. On the other side, results on non-existence of positive solutions for different types of nonlocal BVPs were discussed in [11].

Our main result on the existence of positive solutions of BVP (1.1) is proved by using the similar techniques that were applied in [17] and originally developed by Webb and Infante [13]. The idea is to restrict the singular function f to a subset $[0,1] \times [\rho_1, \infty) \times [\rho_2, \infty)$ of $[0,1] \times (0, \infty) \times (0, \infty)$, where $\rho_1, \rho_2 > 0$ are properly selected such that problem (1.1) can be converted to the following perturbed Hammerstein integral operator of the form

$$Fu(t) = \int_0^1 G(t,s) f(s, u(s), u'(s)) ds + r(t)\eta[u] + w(t)\xi[u], \tag{1.2}$$

where $\eta[u]$ and $\xi[u]$ are positive linear functionals on $C[0,1]$, r and w satisfy certain upper bound conditions. Then existence of a positive solution for problem (1.1) is equivalent to a fixed point problem for the operator F.

For convenience, we give the following definition of an order cone P in a Banach space and the well-known Krasnosel'skii–Guo fixed point theorem on a cone P that will be applied to prove the existence result in Section 3.

Definition 1.1. A cone P in a Banach space X is a closed convex set such that $\lambda x \in P$ for every $x \in P$ and for all $\lambda \geq 0$, and satisfying $P \cap (-P) = \{0\}$.

For any $r > 0$, we denote $\Omega_r = \{x \in X : \|x\| < r\}$ and $\partial\Omega_r = \{x \in X : \|x\| = r\}$.

Theorem 1.1 (Krasnosel'skii–Guo [6])**.** Let $T : P \to P$ be a compact map. Assume that there exist two positive constants r, R with $r \neq R$ such that

$$\|Tu\| \leq \|u\| \text{ for every } u \in P \text{ with } \|u\| = r,$$

and

$$\|Tu\| \geq \|u\| \text{ for every } u \in P \text{ with } \|u\| = R.$$

Then there exists $u_0 \in P$ such that $Tu_0 = u_0$ and $\min\{r, R\} \leq \|u_0\| \leq \max\{r, R\}$.

In Section 2, we first prove some properties of the Green's function G in (1.2) that are essential in the construction of the cone for our proof.

2 Preliminaries

Let $h_1(t) = \gamma(1-t) + \beta$, $h_2(t) = \alpha + \theta t$ and $m = \theta\gamma + \theta\beta + \alpha\gamma$. The following assumption ensures that problem (1.1) is non-resonant [5]:

(H1) $\left(m - \int_0^1 g_1(s)h_1(s)ds\right)\left(m - \int_0^1 g_2(s)h_2(s)ds\right) - \int_0^1 g_1(s)h_2(s)ds \int_0^1 g_2(s)h_1(s)ds \neq 0.$

This condition implies that BVP (2.1) has only the trivial solution:

$$\begin{cases} u''(t) = 0, \quad t \in [0,1], \\ \theta u(0) - \alpha u'(0) = \int_0^1 g_1(s)u(s)ds, \\ \gamma u(1) + \beta u'(1) = \int_0^1 g_2(s)u(s)ds. \end{cases} \tag{2.1}$$

Under condition (H1), BVP (1.1) can be converted to a fixed point problem for the nonlinear operator F in (1.2), where G is the Green's function of the problem

$$\begin{cases} u''(t) + y(t) = 0, \quad t \in [0,1], \\ \theta u(0) - \alpha u'(0) = 0, \\ \gamma u(1) + \beta u'(1) = 0, \end{cases} \tag{2.2}$$

r and w are the unique solutions of

$$\begin{cases} u''(t) = 0, \quad t \in [0,1], \\ \theta u(0) - \alpha u'(0) = 1, \\ \gamma u(1) + \beta u'(1) = 0, \end{cases} \tag{2.3}$$

and

$$\begin{cases} u''(t) = 0, \quad t \in [0,1], \\ \theta u(0) - \alpha u'(0) = 0, \\ \gamma u(1) + \beta u'(1) = 1, \end{cases} \tag{2.4}$$

respectively. By calculation, we can find that $r(t) = \frac{h_1(t)}{m}$, $w(t) = \frac{h_2(t)}{m}$, and

$$G(t,s) = \begin{cases} \dfrac{h_2(s)h_1(t)}{m}, & 0 \le s \le t \le 1, \\ \dfrac{h_2(t)h_1(s)}{m}, & 0 \le t \le s \le 1. \end{cases} \tag{2.5}$$

Condition (H1) is equivalent to

$$\left(1 - \int_0^1 g_1(s)r(s)ds\right)\left(1 - \int_0^1 g_2(s)w(s)ds\right) - \int_0^1 g_1(s)w(s)ds \int_0^1 g_2(s)r(s)ds \neq 0. \tag{2.6}$$

Lemma 2.1. Let $\Phi(s) = G(s,s)$, then

$$c_0\Phi(s) \le G(t,s) \le \Phi(s), \quad \text{for } 0 < t, s < 1,$$

where

$$c_0 = \begin{cases} \frac{\alpha}{\alpha+\theta}, & \gamma = 0 \ \text{ or } \left(\gamma \neq 0, \text{ and } \frac{\beta}{\gamma} - \frac{\alpha}{\theta} \ge 1\right), \\ \frac{\beta}{\beta+\gamma}, & \gamma \neq 0, \ \frac{\beta}{\gamma} - \frac{\alpha}{\theta} \le -1, \\ \frac{\alpha\beta}{(\alpha+\theta)(\gamma+\beta)}, & \gamma \neq 0, \ -1 < \frac{\beta}{\gamma} - \frac{\alpha}{\theta} < 1. \end{cases} \tag{2.7}$$

Proof: For both cases of $0 \le s \le t \le 1$ and $0 \le t \le s \le 1$, we can easily verify that $G(t,s) \le G(s,s)$ from the inequalities: $h_1(t) \le h_1(s)$ for $0 \le s \le t \le 1$ and $h_2(t) \le h_2(s)$ for $0 \le t \le s \le 1$. Now consider

$$c_0 G(s,s) = \frac{c_0(-\theta\gamma s^2 + (\gamma\theta + \beta\theta - \alpha\gamma)s + \alpha(\gamma+\beta))}{m}.$$

(1) If $\gamma = 0$,

$$c_0 G(s,s) = \frac{c_0(\theta s + \alpha)}{\theta} < \frac{c_0(\theta + \alpha)}{\theta} = \frac{\alpha}{\theta} \le \frac{(\alpha + \theta t \text{ (or } s))}{\theta} = G(t,s).$$

(2) If $\gamma \neq 0$, let $h(s) := -\theta\gamma s^2 + (\gamma\theta + \beta\theta - \alpha\gamma)s + \alpha(\gamma+\beta)$. Then h has the critical point:

$$s_0 = \frac{1}{2} + \frac{1}{2}\left(\frac{\beta}{\gamma} - \frac{\alpha}{\theta}\right).$$

Assume that $\frac{\beta}{\gamma} - \frac{\alpha}{\theta} \ge 1$, $\max\{h(s), s \in [0,1]\} = h(1)$,

$$\begin{aligned} c_0 G(s,s) &\le \frac{c_0(\alpha+\theta)\beta}{m} \le \frac{c_0(\alpha+\theta)(\beta + \gamma(1 - t \text{ (or } s)))}{m} \\ &\le \frac{(\alpha + \theta s \text{ (or } t))(\gamma + \beta - \gamma t \text{ (or } s))}{m} = G(t,s). \end{aligned}$$

On the other hand, if $\frac{\beta}{\gamma} - \frac{\alpha}{\theta} \le -1$, $\max\{h(s), s \in [0,1]\} = h(0)$,

$$\begin{aligned} c_0 G(s,s) &\le \frac{c_0\alpha(\gamma+\beta)}{m} \le \frac{c_0(\alpha + \theta s \text{ (or } t))(\gamma+\beta)}{m} \\ &\le \frac{(\alpha + \theta s \text{ (or } t))(\gamma + \beta - \gamma t \text{ (or } s))}{m} = G(t,s). \end{aligned}$$

In the case of $-1 < \frac{\beta}{\gamma} - \frac{\alpha}{\theta} < 1$, we have $-\alpha\gamma < \gamma\theta - \beta\theta$,

$$\max\{h(s), s \in [0,1]\} = \alpha(\gamma+\beta) + \frac{(\gamma\theta + \beta\theta - \alpha\gamma)^2}{4\theta\gamma} < \alpha(\gamma+\beta) + \theta\gamma.$$

Therefore,

$$\begin{aligned} c_0 h(s) < c_0(\alpha(\gamma+\beta) + \theta\gamma) &= \frac{\alpha}{\alpha+\theta}\left(\alpha\beta + \frac{\beta\theta\gamma}{\gamma+\beta}\right) \\ &< \alpha\beta < (\alpha + \theta s \text{ (or } t))(\gamma + \beta - \gamma t \text{ (or } s)), \end{aligned}$$

and

$$c_0 G(s,s) < \frac{(\alpha + \theta s \text{ (or } t))(\gamma + \beta - \gamma t \text{ (or } s))}{m} = G(t,s).$$

□

The following simple property of the constant c_0 will be useful in the sequel.

Property 2.2. Let c_0 be defined as (2.7). Then $c_0 \le \min\left\{\frac{\alpha}{\alpha+\theta}, \frac{\beta}{\gamma+\beta}\right\}$.

Proof: If $\gamma = 0$, it is true since $c_0 = \frac{\alpha}{\alpha+\theta} < 1$. It is also clear for the case of $\gamma \neq 0$ and $-1 < \frac{\beta}{\gamma} - \frac{\alpha}{\theta} < 1$. Assume that $\gamma \neq 0$ and $\frac{\beta}{\gamma} - \frac{\alpha}{\theta} \ge 1$. Then $\frac{\beta}{\gamma} > \frac{\alpha}{\theta}$ implies $\frac{\gamma}{\beta+\gamma} < \frac{\theta}{\alpha+\theta}$. Hence $c_0 = \frac{\alpha}{\alpha+\theta} < \frac{\beta}{\beta+\gamma}$. Similarly, it can be shown that $c_0 = \frac{\beta}{\beta+\gamma} < \frac{\alpha}{\alpha+\theta}$ with the assumption of $\frac{\beta}{\gamma} - \frac{\alpha}{\theta} \le -1$. □

3 Main result

Let $C^1[0,1]$ be the Banach space of continuously differentiable functions with the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$ and $\|u\|_\infty = \max\{|u(t)| : t \in [0,1]\}$. Following similar approaches of [13, 17], we consider the BVP for $\widetilde{f}$, the restriction of f on $[0,1] \times [\rho_1, \infty] \times [\rho_2, \infty]$, where $\rho_1 > 0$, $\rho_2 > 0$:

$$\begin{cases} u''(t) + \widetilde{f}(t, u(t), u'(t)) = 0, \quad t \in [0,1], \\ \theta u(0) - \alpha u'(0) = \int_0^1 g_1(s)u(s)ds, \\ \gamma u(1) + \beta u'(1) = \int_0^1 g_2(s)u(s)ds. \end{cases} \tag{3.1}$$

If u_0 is a positive solution of the regular BVP (3.1), then $u_0(t) \geq \rho_1 > 0$ and $u_0'(t) \geq \rho_2$, so u_0 is a positive solution of (1.1). In addition to (H1), we introduce more assumptions on function $\widetilde{f}$ and the coefficients θ, α, γ and β that appear in (3.1). Let

$$l_1 = \int_0^1 g_1(s)ds, \qquad l_2 = \int_0^1 g_2(s)ds,$$

and $m = \theta(\gamma+\beta) + \alpha\gamma$ as defined in Section 2. Assume there exist $0 < r < R$ and $K, k > 0$ such that:

(H2) $c_0 \min\left\{1, \frac{\alpha}{\theta}\right\} r \geq \rho_1$ and $c \min\left\{1, \frac{\alpha}{\theta}\right\} r \geq \rho_2$;

(H3) $\frac{\widetilde{f}(t,u,v)}{R} \leq K \leq \frac{2(m-\beta l_1 - \alpha l_2 - \theta l_2)}{\theta\gamma + 2(\theta+\alpha)\beta}$ for $(t,u,v) \in [0,1] \times \left[Rc_0 \min\left\{1, \frac{\alpha}{\theta}\right\}, R\right] \times \left[Rc \min\left\{1, \frac{\alpha}{\theta}\right\}, R\right]$;

(H4) $\frac{\widetilde{f}(t,u,v)}{r} \geq k \geq \frac{2(m - c_0\beta \min\left\{1,\frac{\alpha}{\theta}\right\} l_1 - c_0(\alpha+\theta)\min\left\{1,\frac{\alpha}{\theta}\right\} l_2)}{(2\alpha+\theta)\beta}$ for $(t,u,v) \in [0,1] \times \left[rc_0 \min\left\{1, \frac{\alpha}{\theta}\right\}, r\right] \times \left[rc \min\left\{1, \frac{\alpha}{\theta}\right\}, r\right]$;

(H5) $(\theta l_2 - \gamma l_1) \min\left\{1, \frac{\alpha}{\theta}\right\} r - RK\gamma(\theta+\alpha) > 0$.

Of conditions (H2)–(H4), the constant c is defined as

$$c := \frac{-RK\gamma(\theta+\alpha) + (\theta l_2 - \gamma l_1)\min\left\{1, \frac{\alpha}{\theta}\right\} r}{(\alpha+\theta)(\gamma+\beta)RK + [(\gamma+\beta)l_1 + (\alpha+\theta)l_2]\, R}.$$

Since

$$\theta l_2 - \gamma l_1 < (\gamma+\beta)l_1 + (\alpha+\theta)l_2, \ \min\left\{1, \frac{\alpha}{\theta}\right\} r < R,$$

We have

$$(\theta l_2 - \gamma l_1)\min\left\{1, \frac{\alpha}{\theta}\right\} r - RK\gamma(\theta+\alpha) < (\alpha+\theta)(\gamma+\beta)RK + [(\gamma+\beta)l_1 + (\alpha+\theta)l_2]\, R.$$

Condition (H5) implies that $0 \leq c < 1$.

Theorem 3.1. Under the assumptions (H1)–(H5), the regular BVP (3.1) has a positive solution u satisfying

$$c_0 \min\left\{1, \frac{\alpha}{\theta}\right\} r \leq u(t) \leq R,$$

and

$$c \min\left\{1, \frac{\alpha}{\theta}\right\} r \leq u'(t) \leq R.$$

Proof: Similar as (1.2), we consider

$$(\widetilde{F}u)(t) = \int_0^1 G(t,s)\widetilde{f}(s,u(s),u'(s))ds + r(t)\int_0^1 g_1(s)u(s)ds + w(t)\int_0^1 g_2(s)u(s)ds \tag{3.2}$$

and its derivative

$$\begin{aligned}(\widetilde{F}u)'(t) = &\int_0^t \frac{-\gamma(\alpha+\theta s)}{m}\widetilde{f}(s,u(s),u'(s))ds + \int_t^1 \frac{\theta(\gamma+\beta-\gamma s)}{m}\widetilde{f}(s,u(s),u'(s))ds \\ &- \frac{\gamma}{m}\int_0^1 g_1(s)u(s)ds + \frac{\theta}{m}\int_0^1 g_2(s)u(s)ds.\end{aligned}$$

Define the cone P of $C^1[0,1]$ as

$$P = \left\{u \in C^1[0,1] : u(0) \geq \frac{\alpha}{\theta}\|u'\|_\infty,\ u'(t) \geq c\|u\|_\infty,\ u(t) \geq c_0\|u\|_\infty,\ t \in [0,1]\right\}. \tag{3.3}$$

Notice that the constant c in P involves the upper bound RK and the lower bound rk of $\widetilde{f}$ on the closed subsets. If $u \in P$, then

$$u(t) \geq c_0\|u\|_\infty \geq c_0 u(0) \geq c_0\frac{\alpha}{\theta}\|u'\|_\infty.$$

Hence

$$\begin{aligned}u(t) &\geq \max\left\{c_0\|u\|_\infty,\ c_0\frac{\alpha}{\theta}\|u'\|_\infty\right\} \\ &\geq c_0\min\left\{1,\frac{\alpha}{\theta}\right\}\max\left\{\|u\|_\infty,\|u'\|_\infty\right\} = c_0\min\left\{1,\frac{\alpha}{\theta}\right\}\|u\|.\end{aligned}$$

Also

$$u'(t) \geq c\|u\|_\infty \geq c\frac{\alpha}{\theta}\|u'\|_\infty.$$

Therefore

$$u'(t) \geq c\max\left\{\|u\|_\infty,\ \frac{\alpha}{\theta}\|u'\|_\infty\right\} \geq c\min\left\{1,\frac{\alpha}{\theta}\right\}\|u\|.$$

Let

$$\Omega_1 = \{u \in C^1[0,1] : \|u\| < r\} \quad \textit{and} \quad \Omega_2 = \{u \in C^1[0,1] : \|u\| < R\}.$$

We show that $\widetilde{F} : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$. If $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$, then $\widetilde{F}u \in C^1[0,1]$, and

$$\begin{aligned}c_0\|\widetilde{F}u\|_\infty &\leq c_0\int_0^1 G(s,s)\widetilde{f}(s,u(s),u'(s))ds + c_0\frac{\beta+\gamma}{m}\int_0^1 g_1(s)u(s)ds \\ &\quad + c_0\frac{\alpha+\theta}{m}\int_0^1 g_2(s)u(s)ds \\ &\leq \int_0^1 G(t,s)\widetilde{f}(s,u(s),u'(s))ds + \frac{\beta+\gamma(1-t)}{m}\int_0^1 g_1(s)u(s)ds \\ &\quad + \frac{\alpha+\theta t}{m}\int_0^1 g_2(s)u(s)ds \\ &= \widetilde{F}u(t).\end{aligned} \tag{3.4}$$

Next, conditions (H3) and (H4) imply that $f(t,u,v) \leq RK$ for $(t,u,v) \in [0,1]\times[Rc_0\min\{1,\frac{\alpha}{\theta}\},R]\times[Rc\min\{1,\frac{\alpha}{\theta}\},R]$ and $f(t,u,v) \geq rk$ for $(t,u,v) \in [0,1]\times\left[rc_0\min\left\{1,\frac{\alpha}{\theta}\right\},r\right]\times\left[rc\min\left\{1,\frac{\alpha}{\theta}\right\},r\right]$.

For $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$, we obtain that

$$
\begin{aligned}
(\widetilde{F}u)'(t) = & \int_0^t \frac{-\gamma(\alpha+\theta s)}{m}\widetilde{f}(s,u(s),u'(s))ds + \int_t^1 \frac{\theta(\gamma+\beta-\gamma s)}{m}\widetilde{f}(s,u(s),u'(s))ds \\
& - \frac{\gamma}{m}\int_0^1 g_1(s)u(s)ds + \frac{\theta}{m}\int_0^1 g_2(s)u(s)ds \\
\geq & RK\frac{-\gamma(\alpha+\theta)}{m} + rk\int_t^1 \frac{\theta(\gamma+\beta-\gamma s)}{m}ds + \int_0^1 \frac{-\gamma g_1(s)+\theta g_2(s)}{m}u(s)ds \\
\geq & RK\frac{-\gamma(\alpha+\theta)}{m} + rk\frac{\theta(\gamma+\beta)}{m}(1-t) - rk\frac{\theta\gamma}{2m}(1-t^2) + \frac{\theta l_2 - \gamma l_1}{m} r\min\left\{1,\frac{\alpha}{\theta}\right\} \\
= & \frac{rk\theta\gamma}{2m}t^2 - \frac{rk\theta(\gamma+\beta)}{m}t - RK\frac{-\gamma(\alpha+\theta)}{m} + \frac{rk\theta(\gamma+\beta)}{m} - rk\frac{\theta\gamma}{2m} \\
& + \frac{\theta l_2-\gamma l_1}{m} r\min\left\{1,\frac{\alpha}{\theta}\right\} = H(t).
\end{aligned} \tag{3.5}
$$

If $\gamma = 0$, then

$$
H(t) = -\frac{rk\theta\beta}{m}t + \frac{rk\theta\beta}{m} + \frac{\theta l_2}{m} r\min\left\{1,\frac{\alpha}{\theta}\right\}.
$$

H is decreasing for $t \in [0,1]$. The minimum occurs at $t = 1$. When $\gamma \neq 0$, H is a quadratic function with the critical point $\frac{\gamma+\beta}{\gamma} > 1$. For $t \in [0,1]$, the minimum also occurs at $t = 1$. Hence

$$
(\widetilde{F}u)'(t) \geq H(1) = \frac{-RK\gamma(\alpha+\theta) + (\theta l_2 - \gamma l_1)r\ min\{1,\frac{\alpha}{\theta}\}}{m}. \tag{3.6}
$$

On the other hand,

$$
\begin{aligned}
c\|\widetilde{F}u\|_\infty & \leq c\int_0^1 \frac{(\alpha+\theta)(\gamma+\beta)}{m}\widetilde{f}(s,u(s),u'(s))ds + c\frac{\gamma+\beta}{m}\int_0^1 g_1(s)u(s)ds \\
& \quad + c\frac{\alpha+\theta}{m}\int_0^1 g_2(s)u(s)ds \\
& \leq c\left[RK\frac{(\alpha+\theta)(\gamma+\beta)}{m} + \frac{(\gamma+\beta)R}{m}l_1 + \frac{(\alpha+\theta)R}{m}l_2\right] \\
& = \frac{-RK\gamma(\theta+\alpha) + (\theta l_2 - \gamma l_1)\min\left\{1,\frac{\alpha}{\theta}\right\} r}{m}.
\end{aligned} \tag{3.7}
$$

From (3.6) and (3.7), we have

$$
(\widetilde{F}u)'(t) \geq c\|\widetilde{F}u\|_\infty \geq 0, \quad t \in [0,1]. \tag{3.8}
$$

The non-negative property of $(\widetilde{F}u)'$ ensures that

$$
\begin{aligned}
\frac{\alpha}{\theta}\|(\widetilde{F}u)'\|_\infty & = \frac{\alpha}{\theta}\max_{t\in[0,1]}(\widetilde{F}u)'(t) \\
& \leq \int_0^1 \frac{\alpha(\gamma+\beta-\gamma s)}{m}\widetilde{f}(s,u(s),u'(s))ds - \frac{\alpha\gamma}{\theta m}\int_0^1 g_1(s)u(s)ds + \frac{\alpha}{m}\int_0^1 g_2(s)u(s)ds \\
& \leq \int_0^1 \frac{\alpha(\gamma+\beta-\gamma s)}{m}\widetilde{f}(s,u(s),u'(s))ds + \frac{\gamma+\beta}{m}\int_0^1 g_1(s)u(s)ds + \frac{\alpha}{m}\int_0^1 g_2(s)u(s)ds \\
& = \widetilde{F}u(0).
\end{aligned} \tag{3.9}
$$

Combining (3.4), (3.8) and (3.9), we obtain that $\widetilde{F}$ maps $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ to P.

Next, for $u \in P \cap \partial\Omega_2$, $\|u\| = R$,

$$Rc_0 \min\left\{1, \frac{\alpha}{\theta}\right\} \le u(t) \le R \quad \text{and} \quad Rc \min\left\{1, \frac{\alpha}{\theta}\right\} \le u'(t) \le R.$$

$$\begin{aligned}
\|\widetilde{F}u(t)\|_\infty &= \widetilde{F}u(1) \\
&= \int_0^1 G(1,s)\widetilde{f}(s,u(s),u'(s))ds + \frac{\beta}{m}\int_0^1 g_1(s)u(s)ds \\
&\quad + \frac{\alpha+\theta}{m}\int_0^1 g_2(s)u(s)ds \\
&\le KR\int_0^1 G(1,s)ds + \frac{\beta R}{m}l_1 + \frac{(\alpha+\theta)R}{m}l_2 \\
&= KR\left(\frac{\alpha\beta}{m} + \frac{\theta\beta}{2m}\right) + \frac{\beta R l_1 + (\alpha+\theta)R l_2}{m},
\end{aligned}$$

and

$$\begin{aligned}
\|(\widetilde{F}u)'(t)\|_\infty &\le \int_0^1 \frac{\theta(\gamma+\beta-\gamma s)}{m}\widetilde{f}(s,u(s),u'(s))ds - \frac{\gamma}{m}\int_0^1 g_1(s)u(s)ds \\
&\quad + \frac{\theta}{m}\int_0^1 g_2(s)u(s)ds \\
&\le KR\int_0^1 \frac{\alpha(\gamma+\beta-\gamma s)}{m}ds + \frac{\theta l_2}{m}R \\
&= \frac{KR(\frac{\theta\gamma}{2}+\theta\beta)+\theta l_2}{m}.
\end{aligned}$$

Thus, (H3) implies

$$\begin{aligned}
\|(\widetilde{F}u)(t)\| &= \max\left\{\|Fu\|_\infty, \|(\widetilde{F}u)'\|_\infty\right\} \\
&\le \frac{K(\frac{\theta\gamma}{2}+\theta\beta+\alpha\beta)+\beta l_1+(\alpha+\theta)l_2}{m}R \\
&\le R = \|u\|.
\end{aligned}$$

For $u \in P \cap \partial\Omega_1$, $\|u\| = r$,

$$rc_0 \min\left\{1, \frac{\alpha}{\theta}\right\} \le u(t) \le r \quad \text{and} \quad rc \min\left\{1, \frac{\alpha}{\theta}\right\} \le u'(t) \le r.$$

From (H4), we obtain

$$\begin{aligned}
&\|\widetilde{F}u\| \ge \|\widetilde{F}u\|_\infty \\
&\ge \int_0^1 G(1,s)\widetilde{f}(s,u(s),u'(s))ds + \frac{\beta}{m}\int_0^1 g_1(s)u(s)ds + \frac{\alpha+\theta}{m}\int_0^1 g_2(s)u(s)ds \\
&\ge kr\int_0^1 G(1,s)ds + \frac{\beta rc_0\min\left\{1,\frac{\alpha}{\theta}\right\}l_1}{m} + \frac{(\alpha+\theta)rc_0\min\left\{1,\frac{\alpha}{\theta}\right\}l_2}{m} \\
&\ge kr\left(\frac{\alpha\beta}{m}+\frac{\theta\beta}{2m}\right) + \frac{\beta rc_0\min\left\{1,\frac{\alpha}{\theta}\right\}l_1 + (\alpha+\theta)rc_0\min\left\{1,\frac{\alpha}{\theta}\right\}l_2}{m} \\
&= \frac{k(\alpha\beta+\frac{\theta\beta}{2}) + (\beta\min\left\{1,\frac{\alpha}{\theta}\right\}l_1 + (\alpha+\theta)\min\left\{1,\frac{\alpha}{\theta}\right\}l_2)c_0}{m}r \\
&\ge r = \|u\|.
\end{aligned}$$

It can be shown that $\widetilde{F}$ is compact on $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ following the standard arguments. Theorem 1.1 ensures that $\widetilde{F}$ has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. □

4 Examples

We construct two examples to illustrate applications of Theorem 3.1. Example 4.1 represents a group of BVPs satisfying the conditions of Theorem 3.1 but results of [17] can not be applied. Example 4.3 shows that it is possible for BVPs satisfying all conditions of Theorem 3.1 to have multiple solutions including negative solutions.

Example 4.1. Consider the boundary value problem

$$\begin{cases} u''(t) + (0.5t+1)\left(\frac{0.01}{u(t)} + \frac{0.0001}{u'(t)}\right) = 0, \quad t \in [0,1], \\ u(0) - u'(0) = \int_0^1 g_1(s)u(s)ds, \\ 0.01u(1) + 4u'(1) = \int_0^1 g_2(s)u(s)ds, \end{cases} \tag{4.1}$$

where the parameters $\alpha = \theta = 1$, $\beta = 4$, $\gamma = 0.01$. Let g_1, g_2 be selected such that $l_1 = \frac{1}{16}$ and $l_2 = 1$. We can find that $c_0 = 0.5$, $m = 4.02$. For example, for $g_1(s) = \frac{s}{8}$, $g_2(s) = 2s$, we can verify that (H1) is true. Let $R = 2$ and $r = 0.1$. Condition (H3) is satisfied if

$$\frac{f(t,u,v)}{2} \leq K < 0.22 < \frac{m - \beta l_1 - (\alpha+\theta)l_2}{\frac{\theta\gamma}{2} + \theta\beta + \alpha\beta} < 0.23,$$

for $(t,u,v) \in [0,1] \times [1,2] \times [2c,2]$, where $c = \frac{-0.04K+0.09375}{16.04K+4.50125}$. Since c is decreasing with respect to K, by calculation, we have $c \geq 0.011$. From

$$\frac{1}{2}(0.5t+1)\left(\frac{0.01}{u} + \frac{0.0001}{v}\right) < 0.01, \quad \text{for } (t,u,v) \in [0,1] \times [1,2] \times [2c,2],$$

we know that (H3) and (H5) are valid for $K \in [0.01, 0.22]$.

To find k satisfying condition (H4), we calculate that

$$0.5 > \frac{m - c_0\left(\beta \min\left\{1, \frac{\alpha}{\theta}\right\} l_1 + (\alpha+\theta)\min\left\{1, \frac{\alpha}{\theta}\right\} l_2\right)}{\frac{\theta\beta}{2} + \alpha\beta} > 0.49.$$

As $\frac{f(t,u,v)}{r} \geq 1.01$ for $(t,u,v) \in [0,1] \times [0.05, 0.1] \times [0.1c, 0.1]$, (H4) is satisfied for $k \in [0.50, 1.01]$. By Theorem (3.1), BVP (3.8) has a positive solution $u \in C^1[0,1]$ such that $0.05 \leq \rho_1 \leq u(t) \leq 2$ and $0.0011 \leq \rho_2 \leq u'(t) \leq 2$.

Remark 4.2. More generally, for all $\alpha = \theta, \beta = 4, \gamma = 0.01, l_2 = 1$, the calculation of Example 4.1 works as long as l_1 is small enough. The extreme case is $g_1(s) = 0$. Then the first boundary condition is reduced to $u(0) - u'(0) = 0$. We can verify that $c_0 = 0.5, m = 4.02\alpha$. Select the same values of R and r as that of Example 4.1, we can find the intervals for $K \in [0.01, 0.25]$ and $k \in [0.51, 1.01]$. The solution and its derivative are still in the same range as obtained in Example 4.1.

Example 4.3. The following problem is in the form of BVP (1.1):

$$\begin{cases} u''(t) + \frac{\ln(t+2)}{10^4(t+2)^2u(t)} + \frac{1}{10^4(t+2)^3u'(t)} + \frac{0.98u'(t)}{t+2} = 0, \\ u(0) - u'(0) = \xi_1 \int_0^1 su(s)ds, \\ 10^{-4}u(1) + u'(1) = \xi_2 \int_0^1 su(s)ds, \end{cases} \tag{4.2}$$

where $\theta = \alpha = 1$, $\gamma = 10^{-4}$, $\beta = 1$, $g_1(s) = \xi_1 s$ and $g_2(s) = \xi_2 s$. Select $\xi_1 = \frac{4\ln 2 - 2}{3 - \ln 729 + \ln 256}$, $\xi_2 = \frac{3\ln 3 + 10^4}{7500(3 - \ln 729 + \ln 256)}$, then $0.1979 < l_1 < 0.1980$, $0.3413 < l_2 < 0.3414$. It is easy to find that $c_0 = \frac{1}{2}$, and $m = 1.0002$. Let $r = 0.02$, $R = 1$, we can verify that all conditions (H1)-(H5) are satisfied. In fact, equation (4.2) is exact, we can find that $u_1(t) = 0.1\ln(t+2)$ and $u_2(t) = -0.1\ln(t+2)$ are two solutions of problem (4.2). This shows the existence of multiple and negative solutions.

Different from Example 4.3, Example 4.1 cannot be solved analytically. In order to validate this result of Example 4.1, we use the sinc-collocation numerical method based on the derivative interpolation to obtain a numerical solution of BVP (4.1). The sinc-collocation method is a highly efficient numerical technique with exponential rate of convergence. The details of the approach can be found in [1]. The numerical algorithm is coded in Python. The graphs of the obtained solutions u and u' for both cases of $g_1(s) = 0$ and $g_1(s) = \frac{s}{8}$ are depicted in Figures 4.1 and 4.2 respectively. Clearly they all satisfy the bounds obtained from Example 4.1.

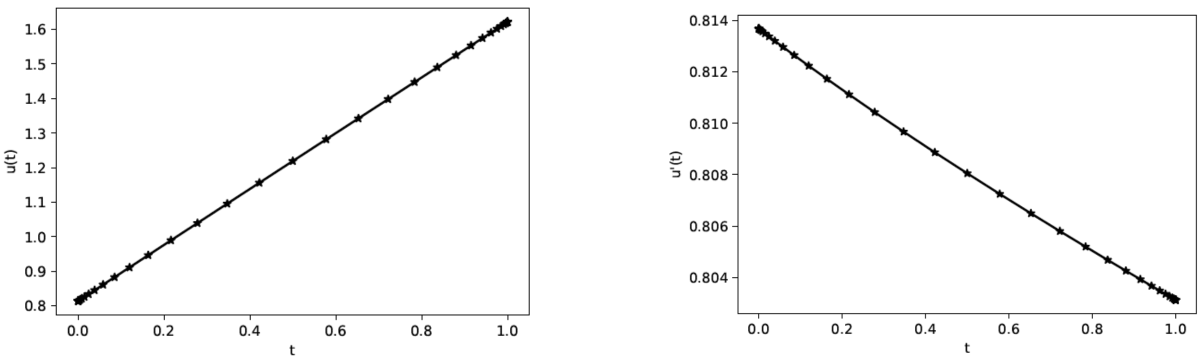

Figure 4.1: Numerical solution of BVP (4.1) ($g_1 = 0$, $g_2 = 2s$)

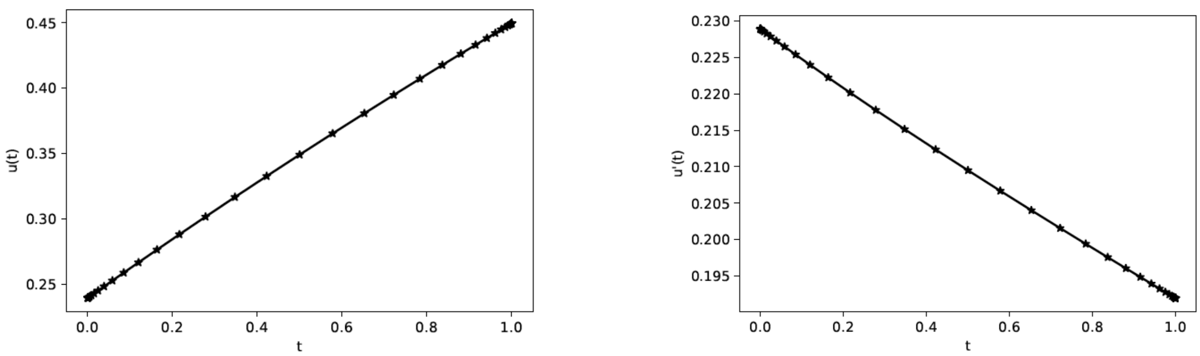

Figure 4.2: Numerical solution of BVP (4.1) ($g_1 = \frac{s}{8}$, $g_2 = 2s$)

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On the global attractor of delay differential equations with unimodal feedback not satisfying the negative Schwarzian derivative condition

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. We study the size of the global attractor for a delay differential equation with unimodal feedback. We are interested in extending and complementing a dichotomy result by Liz and Röst, which assumed that the Schwarzian derivative of the nonlinear feedback is negative in a certain interval. Using recent stability results for difference equations, we obtain a stability dichotomy for the original delay differential equation in the situation wherein the Schwarzian derivative of the nonlinear term may change sign. We illustrate the applicability of our results with several examples.

Keywords: delay differential equations, difference equations, global attractor.

2020 Mathematics Subject Classification: 39B82, 39A30.

1 Introduction

The nonlinear delay differential equation

$$x'(t) = -\mu\big(x(t) - f(x(t-\tau))\big), \quad t > 0, \tag{1.1}$$

with $\mu, \tau > 0$ and $f \colon I \subset \mathbb{R} \to I$, has been widely studied in the literature because of its multiple applications in, for example, biology, physics or economics [1,2,14,17]. In the case of f being monotone, the dynamics are well understood, see [8,9,18] and references therein. In particular, it is known that chaotic dynamics cannot occur [15]. The natural generalization of

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the previous case, in which f changes monotonicity once, is more complicated and may lead to chaotic behaviour [10].

In this paper, f is assumed to be unimodal. More specifically, we impose that the following condition holds for f.

(U) $f\colon (a,b) \subset \mathbb{R} \to (a,b)$ is differentiable, with $-\infty \le a < b \le +\infty$; satisfies that there is a unique x_* such that $f'(x) > 0$ if $a \le x < x_*$, $f'(x_*) = 0$, and $f'(x) < 0$ if $x_* < x < b$; and that there exists $K \in (x_*, b)$ such that $f(K) = K$, $f(x) > x$ for $x \in (a,K)$, and $f(x) < x$ for $x \in (K,b)$.

Notice that if condition **(U)** holds, then K is the unique *fixed point* for the map f, i.e. $f(K) = K$, and therefore the constant function $x(t) = K$ is a positive equilibrium of the delay equation (1.1). Moreover, we empshasise that assuming that the fixed point K belongs to (x_*, b) is not restrictive for our purpose of studying the asymptotic behaviour of equation (1.1), since if K belongs to the interval (a, x_*), then all the solutions of the delay equation are known to converge to K; see, for example [16].

Whenever condition **(U)** holds, we denote the image by f of the point where the maximum of f is attained and the image by f of this maximum by β and α, respectively, that is,

$$\beta := f(x_*) \quad \text{and} \quad \alpha := f(\beta). \tag{1.2}$$

With the notation in (1.2), we introduce an additional assumption on f.

(L) Condition **(U)** holds and $f(f(x_*)) > x_*$.

A well-known approach for investigating equation (1.1) comprises studying the behaviour of the related difference equation

$$x_{n+1} = f(x_n), \quad x_0 \in (a,b), \tag{1.3}$$

see, for example, [7, 13]. Using that approach and taking advantage of the properties of unimodal maps it is possible to show that if **(L)** holds, then for any solution $x(t)$ of (1.1) with initial condition in $C([-\tau,0],(a,b))$ there exists $t_0 > 0$ such that $x(t) \in [\alpha, \beta]$ for $t \ge t_0$; and we informally say that the interval $[\alpha,\beta]$ contains the global attractor of the equation (1.1). Thus, if **(L)** holds, then the interval $[\alpha,\beta]$ contains the global attractor of (1.1) independently of the delay τ. Moreover, complicated dynamics cannot occur for equation (1.1) since the ω-limit set of any solution is the positive equilibrium $\{K\}$ or a periodic orbit. We refer the reader to [16] for a proof of these results in the particular case of $(a,b) = (0,+\infty)$.

The interval $[\alpha,\beta]$ might not be the sharpest, that is, it might have a proper subinterval which contains the global attractor of (1.1). Therefore, an interesting problem stated in [16] is to try to estimate this sharpest attracting interval—or even better to calculate it—when condition **(L)** holds. Here, we deal with such a problem.

In [11], Liz and Röst consider the same problem and showed that when f satisfies **(L)** and has negative Schwarzian derivative, then the sharpest interval containing the attractor of equation (1.1) can be determined and the following dichotomy result holds.

Theorem 1.1 (Theorem 6 in [11]). *Assume that condition **(L)** holds and, further, that f satisfies the following condition.*

(S) *f is three times differentiable and $(Sf)(x) < 0$ on the interval $[\alpha, \beta]$, where Sf denotes the Schwarzian derivative of f, defined by*

$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 .$$

Then exactly one of the following holds:

1. *$f'(K) \geq -1$ and the global attractor of (1.1) for all values of the delay τ is $\{K\}$.*

2. *$f'(K) < -1$ and the sharpest invariant and attracting interval containing the global attractor of (1.1) for all values of the delay τ is $[\bar{\alpha}, \bar{\beta}]$, where $\{\bar{\alpha}, \bar{\beta}\}$ is the unique nontrivial 2-cycle (i.e., $\bar{\alpha} = f(\bar{\beta})$ and $\bar{\beta} = f(\bar{\alpha})$) of the map f in $[\alpha, \beta]$.*

Remark 1.2. We note that Theorem 1.1 as stated above is, in fact, a slightly generalized version of Theorem 6 in [11], which follows from the ideas in [11]. Specifically, the result by Liz and Röst is stated for the particular case in which $a = 0$ and $b = +\infty$. Moreover, their condition (U) imposes $f''(x) > 0$ on $(0, x_*)$, but this is just to guarantee that f has a unique positive fixed point. We note that under the conditions in [11], $f\colon [0, +\infty) \to [0, +\infty)$ is continuous and satisfies $f(0) = 0$ and $f(x) > 0$ for $x > 0$, hence, the restriction to the open interval $(0, +\infty)$ that we consider in Theorem 1.1 is well defined. Finally, note that the initial condition in [11] is a nonzero and nonnegative real function on $[-\tau, 0]$ and consequently all the solutions are strictly positive for $t > 0$, as remarked there. Hence, there is no loss of generality in assuming the initial condition to be strictly positive as we do here.

As the authors of [11] highlighted, the function f appearing in important examples of equation (1.1), including the Mackey–Glass and Nicholson's blowflies models [6, 12], does have negative Schwarzian derivative. Nevertheless, it is not hard to find situations where **(S)** does not hold and, therefore, Theorem 1.1 is not applicable. Hence, it is interesting to look for results extending and complementing Theorem 1.1.

In order to obtain such results, without the assumption that f has negative Schwarzian derivative, we instead take advantage of a consequence of **(L)**, namely, that $f_{|(\alpha,\beta)}$ is strictly decreasing. In this case, the recent method presented in [4, 5] for studying the dynamics of difference equations is applicable, and we employ it to establish a dichotomy result for (1.1) by studying (1.3). Proposition 2.6 is the key technical ingredient for the difference equations we consider, and our main result is Theorem 3.2. The latter result has the same conclusions of Theorem 1.1, but different hypotheses.

Interestingly, the proof of Proposition 2.6 uses the second inequality in the Hermite–Hadamard inequality for a strictly convex function $h\colon [a, b] \to \mathbb{R}$,

$$h\left(\frac{a+b}{2}\right) < \frac{1}{b-a}\int_a^b h(x)dx < \frac{h(a)+h(b)}{2}, \tag{1.4}$$

to show that certain function, intimately linked to the dynamics of the difference equation, is strictly increasing. Whereas, for guaranteeing that such a function has a strict global minimum—enough for obtaining the first conclusion in Theorem 1.1—one needs to invoke the first inequality in (1.4).

In recognition of the current special volume, Professor Webb is a world-expert on the use of topological tools in the study of nonlinear problems. Indirectly, fixed point index theory applied to the study of differential equations plays a role in this paper. Indeed, this is one of the tools used by Mallet-Paret and Nussbaum in [13] to prove the existence of slowly oscillating periodic solutions. The properties of those slowly oscillating periodic solutions underpin [11, Proposition 5], which we invoke in the proof of our main result.

The rest of the paper is organized as follows. The next section contains the preliminaries: notation and some stability results for difference equations. Section 3 contains our main results. Finally, last section of the paper includes some examples to illustrate these main results and compare them with Theorem 1.1.

2 Preliminaries

2.1 Notation

As usual $\mathbb{N}$ and $\mathbb{R}$ denote the positive integers (natural numbers) and real numbers, respectively. Furthermore, $\mathbb{R}_+ := \{r \in \mathbb{R} : r \geq 0\}$.

Let $I \subset \mathbb{R}$ be an interval (bounded or unbounded) and f a continuous map from I to itself. We denote

$$f^{(0)} := \mathrm{id}, \quad f^{(n+1)} = f \circ f^{(n)}, \quad n \geq 1,\ n \in \mathbb{N},$$

with id denoting the identity map; i.e., $\mathrm{id}(x) = x$ for all $x \in I$.

We let $\mathcal{C}^n(J, I)$ denote the space of functions $\xi \colon J \to I$ with n continuous derivatives and, to simplify the notation, $\mathcal{C}^n((a,b),\mathbb{R})$ is denoted by $\mathcal{C}^n(a,b) := \mathcal{C}^n((a,b),\mathbb{R})$ when no confusion is possible. We do not explicitly indicate the domains of the identity and constant functions. They are assumed to be the largest sets for which the corresponding expressions make sense.

We say that $x(t;\xi)$ is a solution of the differential equation (1.1) with $f \in \mathcal{C}(\mathbb{R})$ and initial condition $\xi \in \mathcal{C}([-\tau,0],\mathbb{R})$ if $x(\cdot;\xi) \in \mathcal{C}([-\tau,+\infty),\mathbb{R})$, $x(\cdot;\xi)_{|(0,+\infty)} \in \mathcal{C}^1(0,+\infty)$, it satisfies the differential equation (1.1) for $t > 0$ and $x(s;\xi) = \xi(s)$ for $s \in [-\tau, 0]$. The method of steps, e.g., see [17], shows that there exists a unique solution of the differential equation (1.1) for any $f \in \mathcal{C}(\mathbb{R})$ and initial condition $\xi \in \mathcal{C}([-\tau,0],\mathbb{R})$. Moreover, if $f \in \mathcal{C}(I,I)$ and $\xi \in \mathcal{C}([-\tau,0],I)$, then the invariance principle (see [7, Theorem 2.1]) guarantees that the unique solution of (1.1) satisfies $x(t;\xi) \in I$ for all $t \in [-\tau,+\infty)$.

2.2 Stability of difference equations

In this section, we study stability properties of the difference equation

$$y_{n+1} = y_n + g(y_n), \quad y_0 \in \mathrm{dom}\, g\,, \tag{2.1}$$

with $g \in \mathfrak{G}$, where

$$\mathfrak{G} := \bigcup_{-\infty \leq a < b \leq \infty} \mathfrak{G}(a,b)\,,$$

and

$$\mathfrak{G}(a,b) := \{g \in \mathcal{C}^1(a,b) : a < \mathrm{id} + g < b,\ g' < 0,\ 0 \in g((a,b))\}\,.$$

Here $g\big((a,b)\big)$ denotes the image of (a,b) under g. It is clear that for each $g \in \mathfrak{G}_*$ the difference equation (2.1) is well defined and there exists a unique $y_g \in (a,b)$ such that $g(y_g) = 0$. In particular, y_g is a unique equilbrium of (2.1). We use the usual definitions of stability, local asymptotic stability and global asymptotic stability for the equilibrium y_g of the difference equation (2.1). From now on, G.A.S. stands for globally asymptotically stable and L.A.S. for locally asymptotically stable. We state what we understand by a global repeller.

Definition 2.1. We say that y_g is a *a global repeller* for the difference equation (2.1) if the sequence $((\mathrm{id}+g)^{(n)}(y))_n$ has no accumulation points in (a,b) for any $y \in (a,b) \setminus y_g$.

Next, we define a function to study the stability properties of the equilibrium of (2.1), which was introduced in [5].

Definition 2.2. For each $g \in \mathfrak{G}$, set $b_g := \min\{-\inf g, \sup g\}$. The function $\sigma_g\colon (-b_g, b_g) \to (0,+\infty)$ is defined by

$$\sigma_g(u) = \begin{cases} \frac{g^{-1}(-u)-g^{-1}(u)}{u} & \text{if } u \neq 0\,, \\ \frac{-2}{g'(y_g)} & \text{if } u = 0\,. \end{cases}$$

The following remark will be very useful in this section.

Remark 2.3. Since $(g^{-1})'$ is continuous, σ_g satisfies

$$\sigma_g(u) = \frac{1}{u}\int_{-u}^{u} -(g^{-1})'(s)ds \quad \forall\, u \in (0, b_g).$$

The function σ_g is continuous, even and positive. Moreover, $y \in \operatorname{dom} g \setminus y_g$ satisfies $(\mathrm{id}+g)^{(2)}(y) = y$ if, and only if, $u = g^{-1}(y) \in \operatorname{dom}\sigma_g$ satisfies $\sigma(u) = \sigma(-u) = 1$, see [5]. In other words, the nontrivial period-2 solutions of (2.1) correspond to the symmetric intersections of the graph of σ_g with the the graph of the constant function with value 1.

Our next result shows that the stability properties of y_g are intimately linked to the relative position of the function σ_g with respect to the constant function with value 1.

Theorem 2.4. *Let $g \in \mathfrak{G}$. The following statements hold for the unique equilibrium y_g of (2.1):*

a) y_g is L.A.S. if $\sigma_g(0) > 1$, and it is unstable if $\sigma_g(0) < 1$.

b) y_g is G.A.S. if, and only if, $\sigma_g(u) > 1$ for all $u \in (-b_g, b_g) \setminus \{0\}$.

c) If $\sigma_g(u) \geq 1$ for all u in a neighbourhood of $u = 0$, then y_g is stable.

d) If $\sigma_g(u) < 1$ for all u in a punctured neighbourhood of $u = 0$, then y_g is unstable.

e) y_g is a global repeller if, and only if, $\sigma_g(u) < 1$ for all $u \in (-b_g, b_g) \setminus \{0\}$.

f) If $\sigma_g(u) > 1$ for all u in a punctured neighbourhood of $u = 0$, then y_g is L.A.S.

Proof. The proof of statements a)–d) can be found in [5]. Similar ideas can be used to prove statements e) and f). Indeed, the reader just needs to reverse the inequalities in the proof of b) and to invoke [5, Proposition 4.d] to obtain the proof of e); whereas reversing the inequalities in d) and invoking [5, Proposition 3.a] gives the proof of f). □

Our next result illustrates how Theorem 2.4 can be used to obtain sufficient conditions for the (in)stability of the equilibrium y_g of the difference equation (2.1).

Proposition 2.5. *Let $g \in \mathfrak{G}$. The following statements hold.*

a) If $g'(y) < -2$ for all $y \in (a,b)\backslash\{y_g\}$, then y_g is a global repeller for (2.1).

b) If $g'(y) > -2$ for all $y \in (a,b)\backslash\{y_g\}$, then y_g is G.A.S. for (2.1).

Proof. To prove statement a), we argue that

$$\sigma_g(u) < 1 \quad \forall\, u \in (0, b_g)\,, \tag{2.2}$$

and invoke statement e) of Theorem 2.4, combined with the property that σ_g is an even function.

For which purpose, recalling that

$$(g^{-1})'(u) = \frac{1}{g'(g^{-1}(u))} \quad \forall\, u \in (-b_g, b_g)\,,$$

our hypothesis on g in statement a) implies that

$$-(g^{-1})'(u) < \frac{1}{2} \quad \forall\, u \in (-b_g, b_g)\backslash\{0\}\,.$$

Therefore, recalling Remark 2.3,

$$\sigma_g(u) = \frac{1}{u}\int_{-u}^{u} -(g^{-1})'(s)ds < 1 \quad \forall\, u \in (0, b_g)\,,$$

and so (2.2) holds.

The proof of statement b) is similar, and argue that

$$\sigma_g(u) > 1 \quad \forall\, u \in (0, b_g)\,,$$

which, when combined with statement b) of Theorem 2.4, proves the claim. □

Proposition 2.6. *Let $g \in \mathfrak{G}$. The following statements hold.*

a) If $(g^{-1})'$ is strictly convex or strictly concave, then the difference equation (2.1) has at most one nontrivial period-2 solution.

b) If $(g^{-1})'$ is strictly convex and $g'(y_g) \leq -2$, then y_g is a global repeller for (2.1).

c) If $(g^{-1})'$ is strictly concave and $g'(y_g) \geq -2$, then y_g is G.A.S. for (2.1).

Noting that

$$(g^{-1})'''(u) = \frac{3(g''(y))^2 - g'(y)g'''(y)}{(g'(y))^5} \quad \forall\, u = g(y),\ y \in (a, b)\,,$$

a sufficient condition for strict convexity (concavity) of $(g^{-1})'$ in the case that $g \in \mathcal{C}^3(\text{dom } g)$ is that

$$3(g'')^2 - g'g''' \tag{2.3}$$

is negative (positive).

Proof of Proposition 2.6. We claim that if g^{-1} is strictly convex (concave), then the function σ_g is strictly decreasing (increasing) on the interval $(0, b_g)$. Assuming this, strict monotonicity of σ_g implies that there is at most one solution of $1 = \sigma_g(u)$ in $(0, b_g)$, and so invoking the properties of σ_g recalled after Definition 2.2, we conclude that (2.1) has at most one nontrivial period-2 solution, proving statement a).

Thus, if $\left(g^{-1}\right)'$ is strictly convex, then Remark 2.3 and an application of the second Hermite–Hadamard inequality in (1.4) yields

$$\begin{aligned} u\frac{d}{du}\sigma_g(u) &= u\frac{d}{du}\left(\frac{-1}{u}\int_{-u}^{u}(g^{-1})'(s)ds\right) \\ &= \frac{1}{u}\int_{-u}^{u}(g^{-1})'(s)ds - \left((g^{-1})'(-u)+(g^{-1})'(u)\right) < 0 \quad \forall\, u\in(0,b_g)\,, \end{aligned}$$

that is, $\sigma_g' < 0$ and so σ_g is strictly decreasing on $(0,b_g)$.

Analogously, if $(g^{-1})'$ is strictly concave, then $-(g^{-1})'$ is strictly convex and so

$$\begin{aligned} -u\frac{d}{du}\sigma_g(u) &= u\frac{d}{du}\left(\frac{-1}{u}\int_{-u}^{u}-(g^{-1})'(s)ds\right) \\ &= \frac{1}{u}\int_{-u}^{u}-(g^{-1})'(s)ds - \left(-(g^{-1})'(-u)+-(g^{-1})'(u)\right) < 0 \quad \forall\, u\in(0,b_g)\,. \end{aligned}$$

that is, $\sigma_g' > 0$. We conclude that σ_g is strictly increasing on $(0,b_g)$. The proof of statement a) is complete.

Under the hypotheses in statement b), that (2.2) holds is clear upon noting that $\sigma_g(0) = -2/g'(y_g) \le 1$ and that we have just shown in proof of statement a) that σ_g is strictly increasing on the interval $(0,b_g)$. Invoking statement e) of Theorem 2.4 completes the proof of statement b).

Reasoning analogous to that used in the proof of statement b) proves statement c), and so we omit the details. □

Remark 2.7.

(i) If $\left(g^{-1}\right)'$ is strictly convex, then using the first Hermite–Hadamard inequality in (1.4) gives

$$\sigma_g(u) = \frac{-1}{u}\int_{-u}^{u}(g^{-1})'(s)ds < -2(g^{-1})'(0) = \frac{-2}{g'(y_g)} = \sigma_g(0) \quad \forall\, u\in(0,b_g)\,,$$

and, consequently, σ_g attains a global maximum at 0. Hence, statement b) in Proposition 2.6 may be proven by statement e) of Theorem 2.4 directly together with the first inequality in the Hermite–Hadamard inequality, instead of the second inequality as was done above. A similar comment is valid for statement c) of Proposition 2.6.

(ii) Assume that $b_g = +\infty$. Since

$$\left(g^{-1}\right)'(u) = \frac{1}{g'(g^{-1}(u))} < 0 \quad \forall\, u\in(-\infty,+\infty)\,,$$

as g is strictly decreasing, it follows that $-\left(g^{-1}\right)'(u) > 0$. In particular, if $\left(g^{-1}\right)'$ is convex in $\mathbb{R}$, then $-\left(g^{-1}\right)'$ is concave and positive in $\mathbb{R}$, and hence must be constant. Therefore, $\left(g^{-1}\right)'$ cannot be strictly convex. This implies that (2.3) cannot be negative.

In light of the above, when $b_g = +\infty$, statement b) of Proposition 2.6 cannot be applied.

We finish the section by showing how Theorem 2.4 and Proposition 2.6 can be used to study a particular type of positive difference equation via topological conjugacy. We define

$$\mathfrak{C} := \bigcup_{-\infty\le a<b\le\infty} \mathcal{C}(a,b) \qquad \text{and} \qquad \mathfrak{C}_+ := \bigcup_{0\le a<b\le\infty} \mathcal{C}_+(a,b),$$

with $\mathcal{C}_+(J) = \{d \in \mathcal{C}(J) : d > 0\}$, and define $\mathfrak{T}\colon \mathfrak{C}_+ \to \mathfrak{C}$ by $\mathfrak{T}(d) = \ln \circ d \circ \exp$. Clearly, $\mathfrak{T}$ is bijective, with inverse $\mathfrak{T}^{-1}\colon \mathfrak{C} \to \mathfrak{C}_+$ given by $\mathfrak{T}^{-1}(g) = \exp \circ g \circ \ln$. Define

$$\mathfrak{D} := \mathfrak{T}^{-1}(\mathfrak{G}) = \bigcup_{0 \le a < b \le \infty} \mathfrak{D}(a,b)\,,$$

with

$$\mathfrak{D}(a,b) := \left\{d \in \mathcal{C}^1(a,b) : a < \mathrm{id} \cdot d < b,\ d' < 0,\ 1 \in d\big((a,b)\big)\right\},$$

and consider the difference equation

$$x_{n+1} = x_n d(x_n), \quad x_0 \in \operatorname{dom} d\,, \tag{2.4}$$

where $d \in \mathfrak{D}$. Note that for each $d \in \mathfrak{D}$ there exists a unique $x_d \in \operatorname{dom} d$ such that $d(x_d) = 1$, and consequently x_d is an equilibrium of (2.4).

A routine calculation shows that $x = (x_n)$ is a solution of (2.1) if, and only if, $z = e^x$ is a solution of (2.4), where g and d are related by $d = \mathfrak{T}^{-1}(g)$. Therefore, stability properties of (2.4) may be studied by applying Theorem 2.4 and Proposition 2.6 to the transformed version (2.4).

3 Sharpest interval containing the attractor

We will make use of the following result (see [7, Theorems 2.2 and 2.3]).

Lemma 3.1. *If there exists an interval $I_0 \subset I$ such that*

$$\inf I_0 \le \liminf_{n\to+\infty} f^{(n)}(x) \le \limsup_{n\to+\infty} f^{(n)}(x) \le \sup I_0 \quad \forall\, x \in I\,,$$

then the solutions of (1.1) satisfy

$$\inf I_0 \le \liminf_{t\to+\infty} x(t,\xi) \le \limsup_{t\to+\infty} x(t,\xi) \le \sup I_0 \quad \forall\, \tau > 0,\ \forall\, \xi \in \mathcal{C}([-\tau,0],I)\,.$$

In particular, if K is G.A.S. for the difference equation (1.3), then

$$\lim_{t\to+\infty} x(t;\xi) = K \quad \forall\, \tau > 0,\ \forall\, \xi \in \mathcal{C}([-\tau,0],I)\,.$$

The following theorem is the main result of this paper. It provides a partial answer to the problem of finding the sharpest attracting interval for the delay-differential equation (1.1) under condition **(L)** by establishing a dichotomy, in the flavour of that of Theorem 1.1.

Theorem 3.2. *Assume that **(L)** holds, that f is three times differentiable and satisfies*

$$3(f'')^2 - (f'-1)f''' > 0\,, \tag{3.1}$$

on the interval (α,β). Then exactly one of the following holds:

1. *$f'(K) \ge -1$ and the global attractor of (1.1) for all values of the delay τ is $\{K\}$.*

2. *$f'(K) < -1$ and the sharpest invariant and attracting interval containing the global attractor of (1.1) for all values of the delay τ is $[\bar{\alpha},\bar{\beta}]$, where $\{\bar{\alpha},\bar{\beta}\}$ is the unique nontrivial 2-cycle of the map f in $[\alpha,\beta]$.*

Proof. Using condition **(L)**, it is not hard, but tedious since several cases need to be considered, to see that for any $x_0 \in I$ there exists $n \in \mathbb{N}$ such that $f^{(n)}(x_0) \in (\alpha, \beta)$, and $f([\alpha, \beta]) \subset [\alpha, \beta)$.

Define $g := f - \mathrm{id}$. We claim that g belongs to $\mathfrak{G}(\alpha, \beta)$. To see this, note that g is strictly decreasing in $[\alpha, \beta]$ since f is. Also, note that

$$g(x) + x = f(x) \in (\alpha, \beta) \quad \forall\, x \in (\alpha, \beta)$$

and, since $f([\alpha, \beta]) \subset [\alpha, \beta)$,

$$f(\beta) - \beta < 0 < f(\alpha) - \alpha,$$

so $0 \in g\big((\alpha, \beta)\big)$, and we have that $g \in \mathfrak{G}(\alpha, \beta)$.

Assume first that $f'(K) \geq -1$. Using that for any $x_0 \in I$ there exists $n \in \mathbb{N}$ such that $f^{(n)}(x_0) \in (\alpha, \beta)$ and invoking the second part of Lemma 3.1, it is enough to show that K is G.A.S. for the difference equation (2.1). And this follows from the second part of Proposition 2.6 after noting that $g'(K) \geq -2$, because $f'(K) \geq -1$, and that the function in (2.3) is positive, because (3.1) holds.

Assume now that $f'(K) < -1$. Since $f([\alpha, \beta]) \subset [\alpha, \beta)$, by a celebrated result of Coppel [3], f has at least one nontrivial 2-cycle $\{\bar{\alpha}, \bar{\beta}\}$ with $[\bar{\alpha}, \bar{\beta}] \subsetneq [\alpha, \beta]$. Moreover, by Proposition 2.6, it is the unique nontrivial 2-cycle contained in $[\alpha, \beta]$.

Next, invoking [11, Lemma 2], $[\bar{\alpha}, \bar{\beta}]$ is an attracting and forward invariant interval for the map f. Therefore, by Lemma 3.1, the interval $[\bar{\alpha}, \bar{\beta}]$ contains the global attractor of (1.1). Finally, using [11, Proposition 5] we see that any closed subinterval of $[\bar{\alpha}, \bar{\beta}]$ does not contain the global attractor of (1.1) for all $\tau > 0$ because we can find slowly oscillating periodic solutions of (1.1) taking values as close as desired to $\bar{\alpha}$ and $\bar{\beta}$. $\square$

It is interesting to note that the previous result is based on rewriting the difference equation (1.3) in the form (2.1). In Theorem 3.2, we have used the natural choice $g = f - \mathrm{id}$. However, this transformation is not the unique and any topologically conjugate difference equation of (1.1) belonging to model (2.1) will give a different condition on f for the validity of the dichotomy. In particular, if f is positive and $x \mapsto f(x)/x$ is decreasing, then we obtain the following result from the topological conjugacy described at the end of Section 2.

Proposition 3.3. *Assume that* **(L)** *holds, that $d(x) := f(x)/x$ is three times differentiable with $d' < 0$, and that*

$$3(g'')^2 - g'g''' > 0\,,$$

on the interval $(\ln \alpha, \ln \beta)$, where $g := \ln \circ d \circ \exp$. Then the conclusions of Theorem 3.2 hold.

4 Examples

This section provides several examples demonstrating the applicability of Theorem 3.2 and Proposition 3.3. The first example shows that Theorem 3.2 can be applied in situations where Theorem 1.1 can not.

Example 4.1. Consider equation (1.1) with $f\colon (0,1) \to (0,1)$ given by

$$f(x) = \frac{19}{20} x(1-x)(5 - 4x + 2x^3). \tag{4.1}$$

The graph of f is plotted in Panel A in Figure 4.1. Using Sturm's Theorem, it is easy to see that neither f nor $f - 1$ have any real roots in the open interval $(0, 1)$. Moreover, $f(1/2) =$

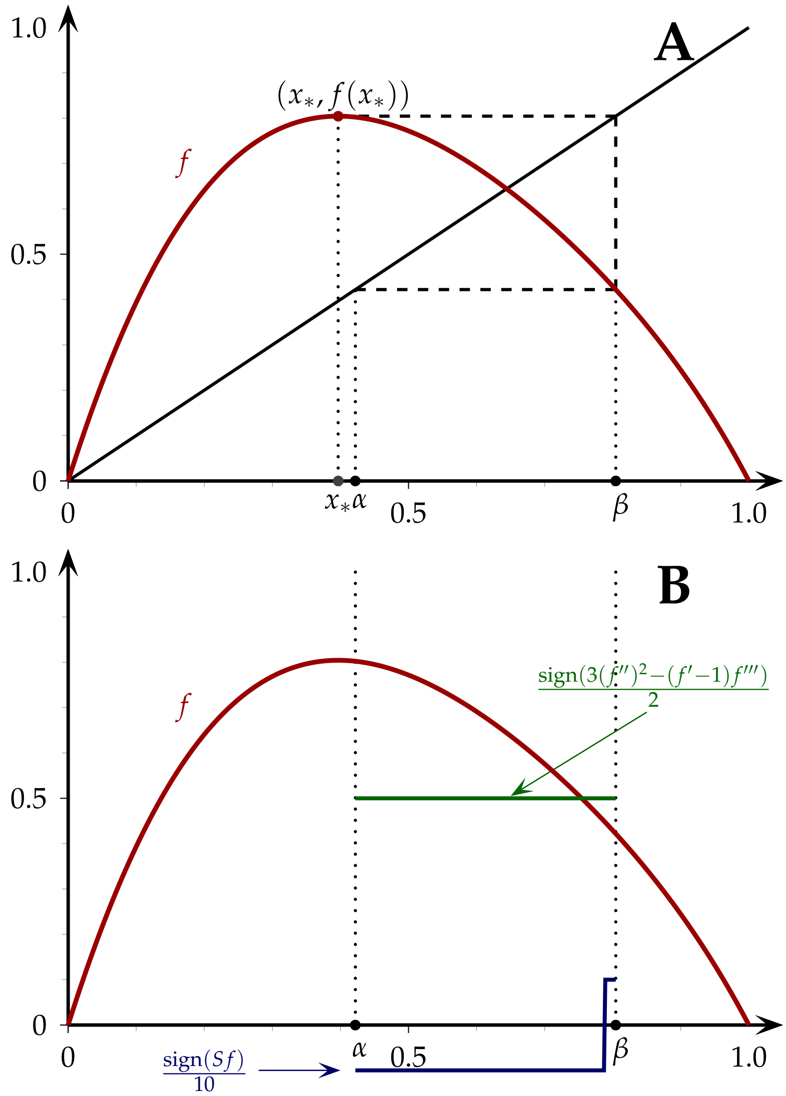

Figure 4.1: Panel A shows the graph of $f(x) = \frac{19}{20}x(1-x)(5-4x+2x^3)$. Observe that condition **(U)** holds. Also note that $f(f(x_*)) > x_*$ and condition **(L)** holds. Panel B shows, in the interval $[\alpha, \beta]$, the graphs of scaled versions of the sign function composed with, respectively, the Schwarzian derivative of f and $3(f'')^2 - (f'-1)f'''$. Observe that the sign of $3(f'')^2 - (f'-1)f'''$ remains positive, meanwhile Sf changes sign in the interval $[\alpha, \beta]$.

$247/320 \in (0,1)$. Hence, f is well-defined. On the other hand, $f'(x) = -\frac{19}{20}(10x^4 - 8x^3 - 12x^2 + 18x - 5)$ and so $f'(0) = \frac{19}{4} > 1$. Moreover, invoking again Sturm's Theorem, f' has exactly one real root x_* (which one can calculate explicitly since f' is a polynomial of degree 4) in the interval $(0,1)$. At $x_* \approx 0.3966$ the function f attains a local maximum because $f(0^+) = f(1^-) = 0$. Solving the equation $f(x) = x$, we find that f has a unique solution $K \in (0,1)$, which again can be explicitly calculated, with $K \approx 0.6441$; and so $x_* < K$. Thus, f satisfies the unimodal condition **(U)** with $a = 0$ and $b = 1$. Observe in Panel A in Figure 4.1 that condition **(L)** holds for f because $x_* < \alpha = f(f(x_*))$.

Panel B in Figure 4.1 illustrates that Theorem 1.1 cannot be used to study the behaviour of equation (1.1) with f given by (4.1). Indeed, we observe that the condition **(S)** is violated, i.e.,

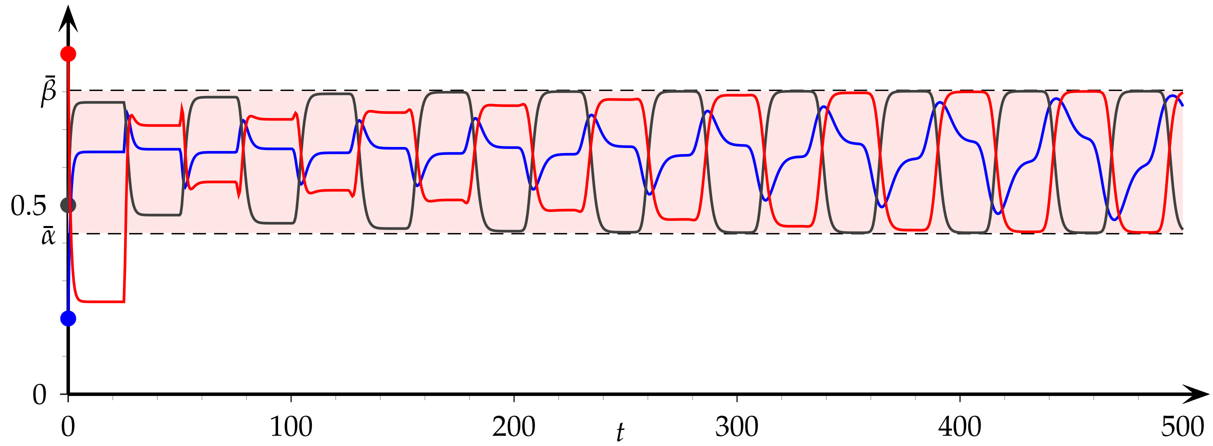

Figure 4.2: The figure shows three different solutions of equation 1.1 with $\mu = 1$, $\tau = 25$ and f as in (4.1). The initial condition is a constant function $\xi \in \mathcal{C}([-\tau, 0], \mathbb{R})$, namely, in the blue curve $\xi = 0.2$, the red curve $\xi = 0.9$, and the black curve $\xi = 0.5$. The pink region is determined by the 2-cycle $\bar{\alpha}, \bar{\beta}$. Observe how the three solutions are asymptotically trapped in this region.

the Schwarzian derivative, Sf, is not negative in the interval $[\alpha, \beta]$. In contrast, the function $3(f'')^2 - (f' - 1)f'''$ has positive sign (again this is easily verified using Sturm's Theorem in the interval $[0, 1]$, which contains the interval $[\alpha, \beta]$). Thus, f satisfies the assumptions of Theorem 3.2.

Since $f'(K) \approx -1.1390$, invoking Theorem 3.2 we conclude that the sharpest invariant and attracting interval containing the attractor of equation (1.1) for all values of the delay τ is determined by the unique nontrivial 2-cycle $\{\bar{\alpha}, \bar{\beta}\}$ of f in the interval $[\alpha, \beta]$. Numerically, we find that $\bar{\alpha} \approx 0.4269$ and $\bar{\beta} \approx 0.8013$.

In Figure 4.2, we plot three solutions of equation (1.1) with f as in (4.1), $\mu = 1$, $\tau = 25$ and different constant initial conditions. Observe that all the solutions asymptotically take values in the interval determined by the 2-cycle $\{\bar{\alpha}, \bar{\beta}\}$ as the result predicts. Moreover, observe that as $t \to \infty$ the solutions oscillate in a range that it is close to the length of the interval $[\bar{\alpha}, \bar{\beta}]$. $\diamondsuit$

The next example shows that Proposition 3.3 can be applied in situations where the assumptions in Theorem 1.1, and in Theorem 3.2, do not hold.

Example 4.2. Consider equation (1.1) with $f\colon (0, 1) \to (0, 1)$ given by

$$f(x) = \frac{3}{10} x \left(1 - \frac{1}{10} \ln(x)\right)^{15}. \tag{4.2}$$

Differentiating, we have

$$f'(x) = \frac{3}{10}\left(1 - \frac{\ln(x)}{10}\right)^{15} - \frac{9}{20}\left(1 - \frac{\ln(x)}{10}\right)^{14} = -\frac{3(\ln(x) - 10)^{14}(\ln(x) + 5)}{10^{16}}.$$

Therefore, f has a critical point at $x_* = \mathrm{e}^{-5} \in (0, 1)$. Moreover,

$$f''(x) = -\frac{9(\ln(x) - 10)^{13}(\ln(x) + 4)}{2 \cdot 10^{15} x},$$

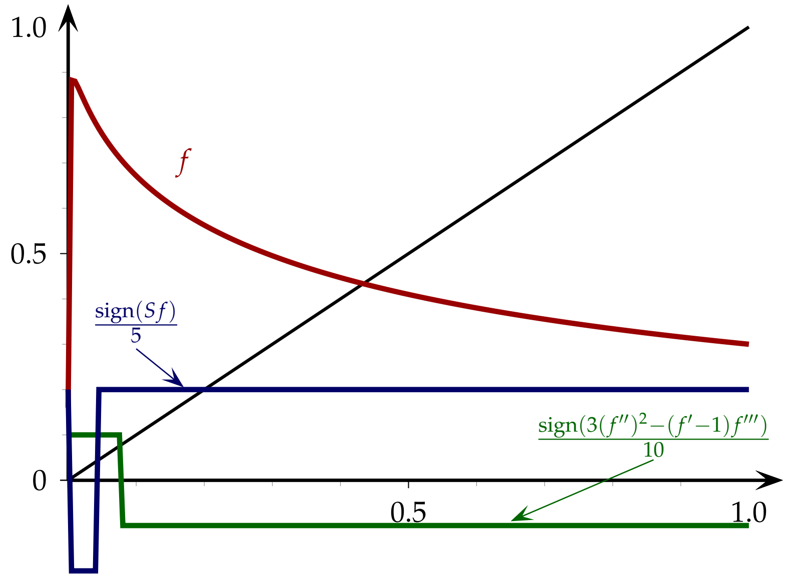

Figure 4.3: Graphs of the function $f(x) = \frac{3}{10}x(1 - \frac{1}{10}\ln(x))^{15}$ (red curve), and the graphs of scaled versions of the sign function composed with, respectively, the Schwarzian derivative of f (blue curve) and $3(f'')^2 - (f' - 1)f'''$ (green curve). Observe that at the fixed point K, Sf is positive and $3(f'')^2 - (f' - 1)f'''$ is negative. Since $x_* \in [\alpha, \beta]$, the assumptions of Theorem 1.1 and Theorem 3.2 are not satisfied.

and so $f''(x_*) < 0$. Noting that $f(0^+) = 0$ and $f(1^-) = 3/10$, we conclude that f is well-defined and unimodal in the interval $(0, 1)$. Now, note that f is convex in the interval $(0, \mathrm{e}^{-4})$ and $\lim_{x\to 0^+} f(x)/x = +\infty$. Consequently, f has a unique fixed point K in the interval $(0, 1)$ and it satisfies $x_* < K$. This shows that **(U)** hold for (4.2)

Next, we obtain that

$$\beta = f(x_*) = \frac{43046721\mathrm{e}^{-5}}{327680} \approx 0.8852\,,$$

and

$$\alpha = f(f(x_*)) = \frac{129140163\mathrm{e}^{-5}}{3276800}\left(1 - \frac{1}{10}\ln\left(\frac{43046721\mathrm{e}^{-5}}{327680}\right)\right)^{15} \approx 0.3185\,.$$

Recalling that $x_* = \mathrm{e}^{-5}$, we have that condition **(L)** holds. In this case, neither Theorem 1.1 nor Theorem 3.2 can be used because the Schwarzian derivative and the function $3(f'')^2 - (f' - 1)f'''$ do not satisfy the sign restrictions in the interval $[\alpha, \beta]$, cf. Figure 4.3.

Nevertheless, Proposition 3.3 holds. We need to verify that

$$d(x) = \frac{f(x)}{x} = \frac{3}{10}\left(1 - \frac{1}{10}\ln(x)\right)^{15}$$

is decreasing, which is trivial, and $g(x) = \ln \circ d \circ \exp$ satisfies $3(g'')^2 - g'g''' > 0$ in the interval $[\ln\alpha, \ln\beta]$. Deriving, we obtain

$$3(g''(x))^2 - g'(x)g'''(x) = \frac{225}{(x-10)^4}, \tag{4.3}$$

and $3(g''(x))^2 - g'(x)g'''(x)$ is positive in the interval $[\ln\alpha, \ln\beta]$.

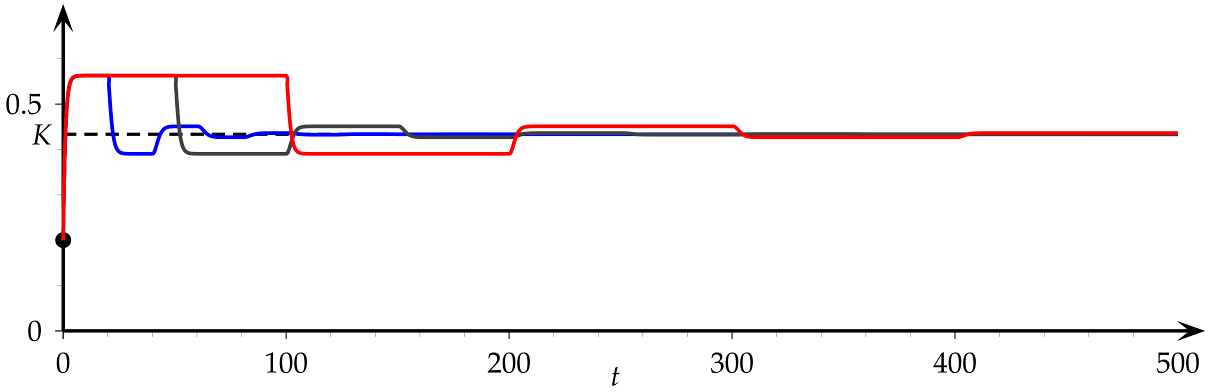

Figure 4.4: The figure shows three solutions of equation 1.1 with $\mu = 1$ and f as in (4.2). The initial condition is a constant function $\xi \in \mathcal{C}([-\tau, 0], \mathbb{R})$, namely, $\xi = 0.2$, but the delay τ is different. For the blue curve we fixed $\tau = 20$, for the black curve $\tau = 50$, and for the red curve $\tau = 100$. Observe how independently of τ the three solutions tend to K.

Computing the derivative of f at its fixed point K, we obtain that this derivative is greater than -1 (approx. -0.3843). By Proposition 3.3 for any initial condition $\xi \in \mathcal{C}([-\tau, 0], (0, 1))$ the solutions of (1.1) tend to K as t tends to $+\infty$, with independence of the size of the delay $\tau > 0$ and the value of $\mu > 0$ as Figure 4.4 illustrates. $\diamondsuit$

Probably, the most famous representatives of equation (1.1) are the Nicholson's blowflies equation and the Mackey–Glass equation. In the Nicholson's blowflies equation f is given by

$$f(x) = \frac{1}{\mu} x e^{-x}, \tag{4.4}$$

whereas in the Mackey–Glass equation f is given by

$$f(x) = \frac{1}{\mu} \frac{ax}{1 + x^b}, \quad a > 0,\ b \geq 1. \tag{4.5}$$

Both (4.4) and (4.5) have negative Schwarzian derivative, and therefore Theorem 1.1 can be used to study them. This was illustrated in [11, Section 3] with a couple of examples. We notice that Proposition 3.3 can be used to obtain the same conclusions as in those examples. Indeed, $d(x) = f(x)/x$ is decreasing both for (4.4) and (4.5). Therefore, to invoke Proposition 3.3 we need to check that $g(x) = \ln \circ d \circ \exp$ satisfies $3(g'')^2 - g'g''' > 0$ in the interval $(\ln \alpha, \ln \beta)$. The following examples show that the inequality holds not only in the interval $(\ln \alpha, \ln \beta)$ but in the whole $\mathbb{R}$.

Example 4.3. Nicholson's blowflies equation. In this case, $g(x) = \ln(1/\mu) - e^x$ and trivially

$$3(g'')^2 - g'g''' = 2e^{2x} > 0\,. \qquad \diamondsuit$$

Example 4.4. The Mackey–Glass equation. In this case, $g(x) = \ln(a/\mu) - \ln(1 + e^{bx})$ and after some straightforward calculations we obtain that

$$3(g'')^2(x) - g'(x)g'''(x) = \frac{b^4 e^{2bx}(2 + e^{bx})}{(1 + e^{bx})^4} > 0\,. \qquad \diamondsuit$$

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Ergodic limits for inhomogeneous evolution equations

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. Let u satisfy an inhomogeneous wave equation such as

$$u''(t) + A^2u(t) = h(t), \qquad u(0) = f, \quad u'(0) = g.$$

We show that in many cases, the limit as $t \to \infty$ of $\frac{1}{t}\int_0^t u(s)ds$ exists, and can be calculated explicitly.

Keywords: ergodic theory, inhomogeneous wave equations, uniformly bounded groups, asymptotics of linear evolution equations.

2020 Mathematics Subject Classification: Primary: 35B40, 47A35, 47D03; Secondary: 37A30, 47D06, 47D09.

1 Introduction

The mean ergodic theorem (MET) deals with the asymptotic behavior of semigroups governing

$$\frac{du}{dt} = Au, \qquad u(0) = f \tag{1.1}$$

and cosine functions governing

$$\frac{d^2u}{dt^2} = A^2u, \qquad u(0) = f, \quad u'(0) = 0. \tag{1.2}$$

The conclusion is that the unique mild solution u of (1.1) and of (1.2) both satisfy

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t u(s)ds \tag{1.3}$$

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exists and equals Pf, where P is a suitable projection onto the null space of A. Of course, some hypotheses are necessary, including the uniform boundedness of the solution semigroup or cosine function.

Our goal here is to obtain analogous results for solutions of the corresponding inhomogeneous problems

$$\frac{du}{dt} = Au + h(t), \qquad u(0) = f, \tag{1.4}$$

$$\frac{d^2u}{dt^2} = A^2u + h(t), \qquad u(0) = f, \quad \frac{du}{dt}(0) = g. \tag{1.5}$$

For (1.5) the ergodic limits do not always exist.

2 First order equations

Let A generate a uniformly bounded strongly continuous (or (C_0)) group $\{e^{tA} : t \in \mathbb{R}\} \subset L(X)$ on a Banach space X. For $f \in X$ and $h \in L^1(\mathbb{R}, X)$, the unique mild solution of (1.4) is given by the strongly continuous function

$$u(t) = e^{tA}f + \int_0^t e^{(t-s)A}h(s)ds, \qquad t \in \mathbb{R}. \tag{2.1}$$

For background on semigroups and cosine functions, see e.g. Goldstein [4]. The mild solution u is a strong solution in $C^1(\mathbb{R}, X)$ provided $f \in D(A)$ and either $h \in C^1(\mathbb{R}, X)$ or both h and Ah belong to $C(\mathbb{R}, X)$. We will assume $h \in L^1(\mathbb{R}, X)$ (or maybe $h \in L^1(\mathbb{R}^+, X)$, $\mathbb{R}^+ = [0, \infty)$ since we study (1.3)).

Let $X_0 := N(A) + \overline{R(A)}$, with N and R denoting null space and range, respectively. For $f \in N(A)$, $e^{tA}f = f$ for all $t \in \mathbb{R}$, while for $f = Ag \in R(A)$,

$$\frac{1}{t}\int_0^t e^{sA}f ds = \frac{1}{t}\int_0^t \frac{d}{ds}(e^{sA}g)ds = \frac{e^{tA}g - g}{t} \to 0$$

as $t \to \infty$, whence $N(A) \cap \overline{R(A)} = \{0\}$. Then the MET says that $\frac{1}{t}\int_0^t e^{sA}f ds \to Pf$ (strong convergence) as $t \to \infty$, for all $f = f_1 + f_2 \in N(A) + \overline{R(A)} =: X_0$ and $Pf = f_1$ where P is the projection of X_0 onto $N(A) = \overline{N(A)}$ along $\overline{R(A)}$.

Note that P is bounded because

$$\|P\| \le \sup_{t \in \mathbb{R}} \|e^{tA}\| = M < \infty.$$

Also, $X_0 = X$ if X is reflexive. Moreover P is an orthogonal projection if $X = H$ is a Hilbert space and $M = 1$, i.e., $\{e^{tA} : t \in \mathbb{R}\}$ is a (C_0) unitary group. For the final term in (2.1),

$$\int_0^t e^{(t-s)A}h(s)ds = e^{tA}\int_0^t e^{-sA}h(s)ds,$$

$$k(t) := \int_0^t e^{-sA}h(s)ds \to \int_0^\infty e^{-sA}h(s)ds =: k_0 \tag{2.2}$$

as $t \to \infty$, and

$$\left\| e^{tA}\left[\int_0^t e^{-sA}h(s)ds - k_0\right]\right\| = \left\| e^{tA}\int_t^\infty e^{-sA}h(s)ds\right\| \to 0 \tag{2.3}$$

as $t \to \infty$ by the uniform boundedness of $\{e^{tA}\}$ and (2.2). Thus

$$\frac{1}{\tau}\int_0^\tau \left(\int_0^t e^{(t-s)A}h(s)ds\right) dt = \frac{1}{\tau}\int_0^\tau e^{tA}k_0 dt + o(1)$$

converges as $\tau \to \infty$ to Pk_0 by the MET and (2.3).

This proves

Theorem 2.1. *Let $\{e^{tA} : t \in \mathbb{R}\}$ be a uniformly bounded (C_0) group on X, let $X_0 = N(A) + \overline{R(A)}$, and P_0 be the (bounded) projection of X_0 onto $N(A)$ along $\overline{R(A)}$. Let $h \in L^1(\mathbb{R}, X)$. Let u, given by* (2.1)*, be the unique mild solution of* (1.4)*. Then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t u(s)ds = P(f + k_0)$$

where P is the projection of X_0 onto $N(A)$ along $\overline{R(A)}$ and

$$k_0 = \int_0^\infty e^{-sA}h(s)ds.$$

3 Second order case

In 1963, W. Littman [6] showed that the initial value problem for the wave equation $\frac{\partial^2 u}{\partial t^2} = \Delta u$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$ is wellposed (in the sense of existence, uniqueness and continuous dependence on the initial conditions) on a space based on $L^p(\mathbb{R}^n)$ iff $p = 2$ when $n \geq 2$. Earlier, K. Friedrichs had pointed out that wave propagation was intimately related to energy considerations, so again, Hilbert space was the optimal context for the study of waves. Still, some special equations can be studied in an L^p context, so we start this section in Hilbert space and later consider Banach spaces as well.

Let B generate a uniformly bounded (C_0) group on a Hilbert space $H_1 = (H, \langle\cdot,\cdot\rangle)$. Then there is as equivalent inner product $\langle\langle\cdot,\cdot\rangle\rangle$ such that on $H_2 = (H, \langle\langle\cdot,\cdot\rangle\rangle)$, B is a skewadjoint operator. This 1947 result is due to B. Sz.-Nagy [7]; cf. also [4]. Thus there is a bijective bounded linear operator $V : H_1 \to H_2$ with bounded inverse such that

$$e^{tB}|_{H_1} = V^{-1}(e^{tB}|_{H_2})V$$

and $\{e^{tB}|_{H_2} : t \in \mathbb{R}\}$ is a (C_0) unitary group on H_2. Then the P in Theorem 2.1 is an orthogonal projection in the H_2 context.

The selfadjoint operator $L = iB$ on H_2 determines the cosine function C given by

$$C(t) = \cos(tL) = \frac{1}{2}(e^{itL} + e^{-itL}), \qquad t \in \mathbb{R} \tag{3.1}$$

(see p.118 of [4]). The corresponding sine function can be defined by

$$\sin(tL) = \frac{1}{2i}(e^{itL} - e^{-itL}), \qquad t \in \mathbb{R}.$$

By a (now commonly accepted) abuse of notation, we define the modified sine function $S(t)$ (and omit the adjective "modified") by

$$S(t) = \frac{1}{2i}(e^{itL} - e^{-itL})L^{-1} \tag{3.2}$$

provided L is injective. But since $\frac{\sin(\lambda)}{\lambda} \to 1$ as $\lambda \to 0$, we can use the spectral theorem and the functional calculus to define $S(t)$ by (3.2) on $R(L)$ and $S(t) = tP$ on $N(A)$, because $v(t) = S(t)g$ is the unique solution of

$$v'' + L^2 v = 0, \qquad v(0) = 0, \quad v'(0) = g$$

for $g \in N(L)$. It is easy to see that

$$S(t)f = \int_0^t C(s)f ds, \tag{3.3}$$

and this can be used to define $S(t) \in L(H_2)$ for $t \in \mathbb{R}$. The unique mild solution of

$$u'' + L^2 u = h(t), \qquad u(0) = f, \quad u'(0) = g \tag{3.4}$$

is given by

$$u(t) = C(t)f + S(t)g + \int_0^t S(t-s)h(s)ds. \tag{3.5}$$

It is a strong $C^2(\mathbb{R}, H_2)$ solution provided $f \in D(L^2)$, $g \in D(L)$ and $h \in C^1(\mathbb{R}, H_2)$.

Now suppose $A = iL$ generates a uniformly bounded (C_0) group on a Banach space X. Then (3.1) and (3.3) define C and S, and (3.5) gives the unique mild solution of (3.4).

Now let A be as in Theorem 2.1, so that (3.4) becomes

$$u'' = A^2 u + h(t), \qquad u(0) = f, \quad u'(0) = g. \tag{3.6}$$

We next state the analogue of Theorem 2.1 for second order equations.

Theorem 3.1. *Let A, X_0, P be as in Theorem 2.1. Let u, defined by (3.5), be the unique mild solution of (3.6), where we assume $(1+t)h(t) \in L^1(\mathbb{R}^+, X)$, $f \in D(A)$ and $g \in X_0$. Let $k_1 = \int_0^\infty Ph(s)ds \in N(A)$. If $k_1 \neq -Pg$, then*

$$\lim_{t\to\infty} \left\| \frac{1}{t} \int_0^t u(s)ds \right\| = \infty,$$

so that the ergodic limit $\lim_{t\to\infty} \frac{1}{t} \int_0^t u(s)ds$ fails to exist. If $k_1 = -Pg$, $k_0 = \int_0^\infty sPh(s)ds$ and if

$$\lim_{t\to\infty} t \left(Pg + \int_0^t Ph(s)ds \right) = k_2 \in N(A) \tag{3.7}$$

exists, then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t u(s)ds = Pf + k_2 - k_0.$$

Proof. The unique mild solution of (3.6) is

$$u(t) = \sum_{j=1}^{3} u_j(t) := C(t)f + S(t)g + \int_0^t S(t-s)h(s)ds. \tag{3.8}$$

By the MET for cosine functions,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t u_1(s)ds = Pf.$$

Now assume $(I-P)g, (I-P)h(s) \in R(A)$ for each $s \geq 0$. Then

$$u_2(t) = S(t)Ag_1 = \frac{1}{2}(e^{tA} - e^{-tA})g_1$$

and
$$\frac{1}{t}\int_0^t u_2(t)dt \to 0$$
as $t \to \infty$ by the MET for semigroups. Furthermore, we can approximate $(I-P)u_3(t)$ in $L^1(\mathbb{R}^+, X)$ by a sequence of the form
$$\int_0^t S(t-s)A\tilde{h}_n(s)ds$$
where $\tilde{h}_n(s) \in D(A)$ and $\tilde{h}_n \in L^1(\mathbb{R}^+, X)$. We omit writing the subscript n. Then
$$\begin{aligned}\int_0^t S(t-s)A\tilde{h}(s)ds &= \int_0^t \frac{1}{2}\left(e^{(t-s)A} - e^{(s-t)A}\right)\tilde{h}(s)ds \\ &= \frac{1}{2}\left[e^{tA}\int_0^t e^{-sA}\tilde{h}(s)ds - e^{-tA}\int_0^t e^{sA}\tilde{h}(s)ds\right] \\ &= \frac{1}{2}\left(e^{tA}l_- - e^{-tA}l_+\right) + o(1)\end{aligned}$$
as $t \to \infty$ where
$$l_\pm = \int_0^\infty e^{\mp sA}\tilde{h}(s)ds \in \overline{R(A)}.$$
Then
$$\begin{aligned}\frac{1}{\tau}\int_0^\tau\int_0^t S(t-s)A\tilde{h}(s)ds &= \frac{1}{2\tau}\int_0^\tau \left(e^{tA}l_- - e^{-tA}l_+\right)dt + o(1) \\ &\to 0\end{aligned}$$
by the MET for semigroups. This completes the portion of the proof dealing with $(I-P)u(t)$. Now we consider $Pu(t)$, using (3.8). Then
$$\begin{aligned}Pu(t) &= C(t)Pf + S(t)Pg + \int_0^t S(t-s)Ph(s)ds \\ &= C(t)Pf + tPg + \int_0^t (t-s)Ph(s)ds\end{aligned}$$
since $S(t) = tP$ on $N(A)$. Next
$$\frac{1}{t}\int_0^t Pu_1(s)ds = \frac{1}{t}\int_0^t C(s)Pfds \to Pf$$
as $t \to \infty$, and
$$\begin{aligned}w(t) := Pu_2(t) + Pu_3(t) &= tPg + t\int_0^t Ph(s)ds - \int_0^t sPh(s)ds \\ &= t\left(Pg + \int_0^t Ph(s)ds\right) - \int_0^\infty sPh(s)ds + o(1) \qquad (3.9)\end{aligned}$$
as $t \to \infty$. Let
$$k_1 = \int_0^\infty Ph(s)ds, \qquad k_0 = \int_0^\infty sPh(s)ds. \qquad (3.10)$$
If $Pg + k_1 \neq 0$, then $\|w(t)\| \to \infty$ as $t \to \infty$, whence
$$\left\|\frac{1}{t}\int_0^t w(s)ds\right\| \to \infty, \quad \text{as } t \to \infty.$$

Thus
$$\left\|\frac{1}{t}\int_0^t u(s)ds\right\| \to \infty, \quad \text{as } t\to\infty.$$

Now suppose $Pg + \int_0^\infty Ph(s)ds = 0$ and

$$\lim_{t\to\infty} t\Big(Pg + \int_0^t Ph(s)ds\Big) = k_2 \in N(A)$$

exists in X. Then
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t Pu(s)ds = P(f+l) = Pf + k_2 - k_0$$

by (3.9), (3.10). Theorem 3.1 now follows. □

4 Examples

We conclude with some examples. The first is the Wentzell wave equation on a bounded domain Ω in $\mathbb{R}^n$.

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u, \qquad x\in\Omega,\ t\in\mathbb{R}, \tag{4.1}$$

with initial conditions
$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x) \tag{4.2}$$

and dynamic boundary conditions

$$\frac{\partial^2 u}{\partial t^2} - \beta\frac{\partial u}{\partial n} - \gamma u + q\beta\Delta_{LB}u = 0, \qquad x\in\Omega,\ t\in\mathbb{R}, \tag{4.3}$$

where Ω is a $C^{2+\varepsilon}$ bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, $\varepsilon > 0$, $0 < \beta \in C^1(\partial\Omega)$, $0 \le \gamma \in C(\partial\Omega)$, $q \in [0,\infty)$, and Δ_{LB} is the Laplace–Beltrami operator on $\partial\Omega$. Assuming (4.1) holds for $x \in \partial\Omega$, then one can replace $\frac{\partial^2 u}{\partial t^2}$ by $\mathrm{tr}(\Delta u)$ in (4.3) and (4.3) then becomes a Wentzell boundary condition

$$\mathrm{tr}(\Delta u) - \beta\frac{\partial u}{\partial n} - \gamma u + q\beta\Delta_{LB}u = 0$$

on $\partial\Omega$. Let
$$X_2 = L^2(\Omega, dx) \oplus L^2(\partial\Omega, \frac{dS}{\beta(x)}),$$

$$S_0 = \begin{bmatrix} \Delta & 0 \\ -\beta\frac{\partial}{\partial n} & -\gamma + q\beta\Delta_{LB} \end{bmatrix},$$

$D(S_0) = \left\{U = \left[\begin{smallmatrix} u \\ \mathrm{tr}(u)\end{smallmatrix}\right] =: u \in C^2(\overline{\Omega})\right\}$, $S_1 = \overline{S_0}$. Then $S_1 = S_1^* \ge \varepsilon I$ on X_2 for some $\varepsilon > 0$, and

$$\frac{\partial^2 U}{\partial t^2} + S_1 U = h(x,t)$$

is the inhomogeneous Wentzell wave equation corresponding to (4.1)–(4.3). See [1–3].

The operator S_1 has a compact resolvent and has an orthonormal basis $\{\varphi_k\}_{k=0}^\infty$ of eigenfunctions corresponding to eigenvalues $0 < \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$, with λ_0 a simple eigenvalue and $\varphi_0 > 0$ in Ω, the "ground state eigenfunction". Now let $A = i(S_1 - \lambda_0)^{\frac{1}{2}}$, so that iA

is selfadjoint on X_2 and $N(A) = \mathrm{span}\{\varphi_0\}$, a one dimensional space. For $F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in X_2$, PF is the constant function with value $\langle F, \varphi_0 \rangle_{X_2} = \int_\Omega f_1(x)\varphi_0(x)dx + \int_{\partial\Omega} f_2(x)\varphi_0(x)\frac{dS}{\beta(x)}$. Theorem 3.1 applies. The initial condition $\frac{\partial u}{\partial t}(0) = g \in X_2$ is in $\overline{R(A)}$ iff $\langle g, \varphi_0 \rangle_{X_2} = 0$. The ergodic limits of Theorem 3.1 will all exist if the limit (3.7) exists, that is,

$$\lim_{t\to\infty} t \left(\langle g, \varphi_0 \rangle_{X_2} + \int_0^t \langle h(s), \varphi_0 \rangle_{X_2} ds \right) \tag{4.4}$$

exists. Since $\int_0^\infty \langle h(s), \varphi_0 \rangle_{X_2} ds$ exists, the existence of (4.4) means, when

$$\int_0^\infty \langle h(s), \varphi_0 \rangle_{X_2} ds = -\langle g, \varphi_0 \rangle_{X_2},$$

that the integral in (4.4) converges fast enough as $t \to \infty$.

For non Hilbert space examples, we look at the one dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + h(x,t), \qquad u(x,0) = f(x), \quad u_t(x,0) = g(x) \tag{4.5}$$

for $x, t \in \mathbb{R}$. Let $w \in BUC(\mathbb{R})$ be a weight function which satisfies $0 < \varepsilon \le w(x) \le \frac{1}{\varepsilon} < \infty$ for all $x \in \mathbb{R}$. Let $X_p = L^p(\mathbb{R}, w(x)dx)$, $X_\infty = BUC_w(\mathbb{R})$ with norm $\|f\|_{w_\infty} = \sup_{x\in\mathbb{R}} |f(x)|w(x)$.

Let $A = \frac{d}{dx}$, $e^{tA}f(x) = f(x+t)$. The unique mild solution of (4.5) in X_p, $1 \le p \le \infty$, is

$$u(x,t) = \frac{1}{2}\Big(f(x+t) + f(x-t)\Big) + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds + \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} h(r,x)drds.$$

Then A generates a uniformly (C_0) group on X_p which is not isometric if $w \neq$ constant.

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Impulsive boundary value problems for nonlinear implicit Caputo-exponential type fractional differential equations

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. This paper deals with existence and uniqueness of solutions to a class of impulsive boundary value problem for nonlinear implicit fractional differential equations involving the Caputo-exponential fractional derivative. The existence results are based on Schaefer's fixed point theorem and the uniqueness result is established via Banach's contraction principle. Two examples are given to illustrate the main results.

Keywords: boundary value problem, Caputo-exponential fractional derivative, implicit fractional differential equations, existence, fixed point, impulses.

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1 Introduction

The fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer orders. Fractional differential equations arise in various fields of science and engineering. Indeed, we can find numerous applications in control theory of dynamical systems, chaotic dynamics, fractals, optics, and signal processing, fluid flow, viscoelasticity, polymer science, rheology, physics, chemistry, biology, astrophysics, cosmology, thermodynamics, mechanics, and other fields. For further details and applications, see, for example, [8,24,28,29]. For some fundamental results on the theory of fractional calculus and fractional ordinary and partial differential equations, we refer to the reader to the books [1,2,21,25,35], the articles [5,6,17], and the references therein.

Impulsive differential equations describe observed evolution processes of several real world phenomena in a natural manner, and exhibit several new phenomena such as noncontinuability and merging of solutions, rhythmical beating, etc. Dynamic processes associated with

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sudden changes in their states are governed by impulsive differential equations. This theory models many phenomena in control theory, population dynamics, medicine, and economics. Recently, fractional differential equations with impulse effects have also received considerable attention, for example, the monographs by Abbas *et al.* [3] Benchohra *et al.* [13], Lakshmikantham *et al.* [26], Samoilenko and Perestyuk [30], and the papers of Benchohra *et al.* [9,16,19], Chang *et al.* [20], Henderson *et al.* [23], and Wang *et al.* [32], as well as the references cited therein.

On the other hand, boundary value problems for fractional differential equations have received considerable attention because they occur in the mathematical modeling of a variety of physical processes; see for example [6,7,11,12,34]. In [10,14,15,18], the authors give existence and uniqueness results for some classes of implicit fractional order differential equations.

Recently, in [27, 31] the authors introduce the exponential fractional calculus and give some existence and uniqueness results for solutions of initial and boundary value problems for fractional differential equations involving Caputo-exponential fractional derivatives (as defined in the next section).

The main goal of this paper is to study existence and uniqueness results for solutions to a more general class of impulsive boundary value problem (BVP for short) given by the following nonlinear implicit fractional-order differential equation:

$$ {}^{e}_{c}D^{\alpha}_{t_k}\varpi(t) = f(t,\varpi(t),\, {}^{e}_{c}D^{\alpha}_{t_k}\varpi(t)), \quad \text{for each } t \in J_k \subseteq J,\ k = 0,1,\ldots,m, \tag{1.1}$$

$$\Delta\varpi|_{t=t_k} = I_k\left(\varpi\left(t_k^-\right)\right), \quad k = 1,\ldots,m, \tag{1.2}$$

$$c_1\varpi(a) + c_2\varpi(b) = c_3, \tag{1.3}$$

where $a = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, ${}^{e}_{c}D^{\alpha}_{a^+}$ denotes the Caputo-exponential fractional derivative of order α, $0 < \alpha \le 1$, $J = [a,b]$, $J_0 = [a,t_1]$, $J_k = (t_k,t_{k+1}]$, $k = 1,2,\ldots,m$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, c_1, c_2, c_3 are real constants with $c_1 + c_2 \neq 0$, $\Delta\varpi|_{t=t_k} = \varpi\left(t_k^+\right) - \varpi\left(t_k^-\right)$, and $\varpi\left(t_k^+\right) = \lim_{h\to 0^+}\varpi(t_k + h)$ and $\varpi\left(t_k^-\right) = \lim_{h\to 0^-}\varpi(t_k + h)$ represent the right and left hand limits of $\varpi(t)$ at $t = t_k$, respectively.

The present paper is organized as follows. In Section 2, some notations are introduced and we recall some preliminary concepts about Caputo-exponential fractional derivatives and some auxiliary results. In Section 3, two results on the impulsive boundary value problem (1.1)–(1.3) are presented: the first one is based on the Banach contraction principle and the second one on Schaefer's fixed point theorem. In the last section, we give two examples to illustrate the applicability of our main results.

2 Preliminaries

In this section, we introduce notations, definitions, and lemmas that are useful in the next section. Let $J := [a,b]$ such that $a < b$. By $C := C(J,\mathbb{R})$ we denote the Banach space of all continuous functions ϖ from J into $\mathbb{R}$ with the supremum norm

$$\|\varpi\|_\infty = \sup_{t\in J}|\varpi(t)|.$$

As usual, $AC(J)$ denote the space of absolutely continuous function from J into $\mathbb{R}$. We denote by $AC^n_e(J)$ the space

$$AC^n_e(J) := \left\{\varpi : J \to \mathbb{R} : {}^{e}D^{n-1}\varpi(t) \in AC(J),\ {}^{e}D = e^{-t}\frac{d}{dt}\right\},$$

where $n = [\alpha] + 1$, with $[\alpha]$ the integer part of α.

In particular, if $0 < \alpha \leq 1$, then $n = 1$ and $AC_e^1(J) := AC_e(J)$.

Definition 2.1 ([27, 31]). The exponential fractional integral of order $\alpha > 0$ of a function $h \in L^1(J, E)$ is defined by

$$({}^eI_a^\alpha h)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \left(e^t - e^s\right)^{\alpha-1} h(s) e^s ds, \quad \text{for each } \ t \in J,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt, \quad \xi > 0.$$

Definition 2.2 ([27,31]). Let $\alpha > 0$ and $h \in AC_e^n(J)$. The exponential fractional derivatives of Caputo type of order α is defined by

$$({}_c^eD_a^\alpha h)(t) := \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(e^t - e^s\right)^{n-\alpha-1} \left(e^{-s} \frac{d}{ds}\right)^n h(s) \frac{ds}{e^{-s}}, \quad \text{for each } t \in J,$$

where $n = [\alpha] + 1$. In particular, if $\alpha = 0$, then

$$\left({}_c^eD_{(\cdot)}^0 h\right)(t) := h(t).$$

Lemma 2.3 ([27,31]). *Let $\alpha > 0$, $n = [\alpha] + 1$, and $h \in AC_e^n(J)$. Then we have the formula*

$${}^eI_a^\alpha ({}_c^eD_a^\alpha h)(t) = h(t) - \sum_{k=0}^{n-1} \frac{(e^s - e^a)^k}{k!} {}^eD^k h(a).$$

Lemma 2.4. *Let $\alpha > 0$, and $h \in AC_e^n(J)$. Then the differential equation*

$${}_c^eD_a^\alpha h(t) = 0$$

has the solution

$$h(t) = \eta_0 + \eta_1(e^s - e^a) + \eta_2(e^s - e^a)^2 + \ldots + \eta_{n-1}(e^s - e^a)^{n-1},$$

where $\eta_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, and $n = [\alpha] + 1$.

Lemma 2.5. *Let $\alpha > 0$, and $h \in AC_e^n(J)$. Then*

$${}^eI_a^\alpha \left({}_c^eD_a^\alpha h\right)(t) = h(t) + \eta_0 + \eta_1(e^s - e^a) + \eta_2(e^s - e^a)^2 + \ldots + \eta_{n-1}(e^s - e^a)^{n-1},$$

for some $\eta_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, and $n = [\alpha] + 1$.

Theorem 2.6 ([22] (Banach's fixed point theorem)). *Let C be a non-empty closed subset of a Banach space X; then any contraction mapping F of C into itself has a unique fixed point.*

Theorem 2.7 ([22] (Schaefer's fixed point theorem)). *Let X be a Banach space and $\Theta : X \to X$ be a completely continuous operator. If the set*

$$\varepsilon = \{\varpi \in X : \varpi = \lambda\Theta\varpi, \textit{ for some } \lambda \in (0, 1)\}$$

is bounded, then Θ has fixed point.

3 Main results

Consider the set of functions

$$PC(J,\mathbb{R}) = \{\varpi : J \to \mathbb{R} \mid \varpi \in C\left((t_k,t_{k+1}],\mathbb{R}\right),\ k=0,\dots,m, \text{ and there exist } \varpi(t_k^+) \text{ and } \varpi(t_k^-),\ k=1,\dots,m, \text{ with } \varpi(t_k^-)=\varpi(t_k)\}.$$

This set, together with the norm

$$\|\varpi\|_{PC} = \sup_{t\in J}|\varpi(t)|,$$

is a Banach space. Let $J_0 = [a,t_1]$ and $J_k = (t_k,t_{k+1}]$ for $k=1,\dots,m$.

Now, let us start by defining what we mean by a solution of the problem (1.1)–(1.3).

Definition 3.1. A function $\varpi \in PC\left(J,\mathbb{R}\right)\cap\left(\cup_{k=0}^{m}AC_e\left(J_k,\mathbb{R}\right)\right)$ is said to be a solution of (1.1)–(1.3) if ϖ satisfies the equation ${}_c^eD_{a^+}^{\alpha}\varpi(t) = f(t,\varpi(t),{}_c^eD_{a^+}^{\alpha}\varpi(t))$, on J_k and the conditions

$$\Delta\varpi|_{t=t_k} = I_k\left(\varpi\left(t_k^-\right)\right), \quad \text{for } k=1,\dots,m,$$

$$c_1\varpi(a)+c_2\varpi(b)=c_3.$$

To prove the existence of solutions to (1.1)–(1.3), we need the following auxiliary lemmas.

Lemma 3.2. *Let $0<\alpha\le 1$ and let $\varphi : J\to\mathbb{R}$ be continuous. A function ϖ is a solution of the integral equation*

$$\varpi(t)=\begin{cases} \frac{-1}{c_1+c_2}\Big[c_2\sum_{i=1}^{m}I_i\left(\varpi\left(t_i^-\right)\right)+c_2\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}\frac{\varphi(s)}{\Gamma(\alpha)}e^sds \\ \quad +c_2\int_{t_m}^{b}\left(e^b-e^s\right)^{\alpha-1}\frac{\varphi(s)}{\Gamma(\alpha)}e^sds-c_3\Big]+\int_a^t\left(e^t-e^s\right)^{\alpha-1}\frac{\varphi(s)}{\Gamma(\alpha)}e^sds, & \text{if } t\in[a,t_1], \\ \\ \frac{-1}{c_1+c_2}\Big[c_2\sum_{i=1}^{m}I_i\left(\varpi\left(t_i^-\right)\right)+c_2\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}\frac{\varphi(s)}{\Gamma(\alpha)}e^sds \\ \quad +c_2\int_{t_m}^{b}\left(e^b-e^s\right)^{\alpha-1}\frac{\varphi(s)}{\Gamma(\alpha)}e^sds-c_3\Big]+\sum_{i=1}^{k}I_i\left(\varpi\left(t_i^-\right)\right) & \text{if } t\in(t_k,t_{k+1}], \\ \quad +\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}\frac{\varphi(s)}{\Gamma(\alpha)}e^sds+\int_{t_k}^{t}\left(e^t-e^s\right)^{\alpha-1}\frac{\varphi(s)}{\Gamma(\alpha)}e^sds, \end{cases} \tag{3.1}$$

where $k=1,\dots,m$, if and only if, ϖ is a solution of the fractional BVP

$${}_c^eD_{t_k}^{\alpha}\varpi(t)=\varphi(t), \quad t\in J_k, \tag{3.2}$$

$$\Delta\varpi|_{t=t_k}=I_k\left(\varpi\left(t_k^-\right)\right), \quad \textit{for } k=1,\dots,m, \tag{3.3}$$

$$c_1\varpi(a)+c_2\varpi(b)=c_3. \tag{3.4}$$

Proof. Assume that ϖ satisfies (3.2)–(3.4). If $t\in[a,t_1]$, then

$${}_c^eD_a^{\alpha}\varpi(t)=\varphi(t).$$

By Lemma 2.5,

$$\varpi(t)=\eta_0+{}^{e}I_a^{\alpha}\varphi(t)=\eta_0+\frac{1}{\Gamma(\alpha)}\int_a^t\left(e^t-e^s\right)^{\alpha-1}\varphi(s)e^s ds.$$

If $t\in(t_1,t_2]$, then by Lemma 2.5 we obtain

$$\begin{aligned}\varpi(t)&=\varpi\left(t_1^+\right)+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t\left(e^t-e^s\right)^{\alpha-1}\varphi(s)e^s ds\\&=\Delta\varpi|_{t=t_1}+\varpi\left(t_1^-\right)+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t\left(e^t-e^s\right)^{\alpha-1}\varphi(s)e^s ds\\&=I_1\left(\varpi\left(t_1^-\right)\right)+\left[\eta_0+\frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left(e^{t_1}-e^s\right)^{\alpha-1}\varphi(s)e^s ds\right]\\&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t\left(e^t-e^s\right)^{\alpha-1}\varphi(s)e^s ds\\&=\eta_0+I_1\left(\varpi\left(t_1^-\right)\right)+\frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left(e^{t_1}-e^s\right)^{\alpha-1}\varphi(s)e^s ds\\&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t\left(e^t-e^s\right)^{\alpha-1}\varphi(s)e^s ds.\end{aligned}$$

If $t\in(t_2,t_3]$, then by Lemma 2.5 we have

$$\begin{aligned}\varpi(t)&=\varpi\left(t_2^+\right)+\frac{1}{\Gamma(\alpha)}\int_{t_2}^t\left(e^t-e^s\right)^{\alpha-1}\varphi(s)e^s ds\\&=\Delta\varpi|_{t=t_2}+\varpi\left(t_2^-\right)+\frac{1}{\Gamma(\alpha)}\int_{t_2}^t\left(e^t-e^s\right)^{\alpha-1}\varphi(s)e^s ds\\&=I_2\left(\varpi\left(t_2^-\right)\right)+\left[\eta_0+I_1\left(\varpi\left(t_1^-\right)\right)+\frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left(e^{t_1}-e^s\right)^{\alpha-1}\varphi(s)e^s ds\right.\\&\quad\left.+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(e^{t_2}-e^s\right)^{\alpha-1}\varphi(s)e^s ds\right]+\frac{1}{\Gamma(\alpha)}\int_{t_2}^t\left(e^t-e^s\right)^{\alpha-1}\varphi(s)e^s ds\\&=\eta_0+\left[I_1\left(\varpi\left(t_1^-\right)\right)+I_2\left(\varpi\left(t_2^-\right)\right)\right]+\left[\frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left(e^{t_1}-e^s\right)^{\alpha-1}\varphi(s)e^s ds\right.\\&\quad\left.+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(e^{t_2}-e^s\right)^{\alpha-1}\varphi(s)e^s ds\right]+\frac{1}{\Gamma(\alpha)}\int_{t_2}^t\left(e^t-e^s\right)^{\alpha-1}\varphi(s)e^s ds.\end{aligned}$$

Repeating this process, the solution $\varpi(t)$ for $t\in(t_k,t_{k+1}]$, where $k=1,\ldots,m$, can be written as

$$\begin{aligned}\varpi(t)=\eta_0&+\sum_{i=1}^{k}I_i\left(\varpi\left(t_i^-\right)\right)+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}\varphi(s)e^s ds\\&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^t\left(e^t-e^s\right)^{\alpha-1}\varphi(s)e^s ds.\end{aligned}$$

It is clear that

$$\varpi(a)=\eta_0$$

and

$$\begin{aligned}\varpi(b) = \eta_0 + \sum_{i=1}^{m} I_i\left(\varpi\left(t_i^-\right)\right) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_m}^{b} \left(e^b - e^s\right)^{\alpha-1} \varphi(s) e^s ds.\end{aligned}$$

Hence, by applying the boundary conditions $c_1\varpi(a) + c_2\varpi(b) = c_3$, we see that

$$\begin{aligned}c_3 = \eta_0(c_1 + c_2) + c_2 \sum_{i=1}^{m} I_i\left(\varpi\left(t_i^-\right)\right) + \frac{c_2}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \\ + \frac{c_2}{\Gamma(\alpha)} \int_{t_m}^{b} \left(e^b - e^s\right)^{\alpha-1} \varphi(s) e^s ds.\end{aligned}$$

Then,

$$\begin{aligned}\eta_0 = \frac{-1}{c_1 + c_2} \Bigg[c_2 \sum_{i=1}^{m} I_i\left(\varpi\left(t_i^-\right)\right) + \frac{c_2}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \\ + \frac{c_2}{\Gamma(\alpha)} \int_{t_m}^{b} \left(e^b - e^s\right)^{\alpha-1} \varphi(s) e^s ds - c_3 \Bigg].\end{aligned}$$

Thus, if $t \in (t_k, t_{k+1}]$, where $k = 1, \ldots, m$, then

$$\begin{aligned}\varpi(t) = \frac{-1}{c_1 + c_2} \Bigg[c_2 \sum_{i=1}^{m} I_i\left(\varpi\left(t_i^-\right)\right) + \frac{c_2}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s\right)^{\alpha-1} \varphi(s) e^s ds \\ + \frac{c_2}{\Gamma(\alpha)} \int_{t_m}^{b} \left(e^b - e^s\right)^{\alpha-1} \varphi(s) e^s ds - c_3 \Bigg] + \sum_{i=1}^{k} I_i\left(\varpi\left(t_i^-\right)\right) \\ + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s\right)^{\alpha-1} \varphi(s) e^s ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(e^t - e^s\right)^{\alpha-1} \varphi(s) e^s ds.\end{aligned}$$

Conversely, assume that ϖ satisfies the impulsive fractional integral equation (3.1). If $t \in [a, t_1]$ then $c_1\varpi(a) + c_2\varpi(b) = c_3$, and using the fact that ${}^e_cD^\alpha_a$ is the left inverse of ${}^eI^\alpha_a$ gives

$${}^e_cD^\alpha_a \varpi(t) = \varphi(t), \quad \text{for each } t \in [a, t_1].$$

If $t \in (t_k, t_{k+1}]$ for $k = 1, \ldots, m$, then, by using the fact that ${}^e_cD^\alpha_{t_k} C = 0$, where C is a constant, and ${}^e_cD^\alpha_{t_k}$ is the left inverse of ${}^eI^\alpha_{t_k}$, we have

$${}^e_cD^\alpha_{t_k} \varpi(t) = \varphi(t), \quad \text{for each } t \in (t_k, t_{k+1}].$$

Also, we can easily show that

$$\Delta\varpi|_{t=t_k} = I_k\left(\varpi\left(t_k^-\right)\right), \quad k = 1, \ldots, m. \qquad \square$$

Now, we state and prove our first existence result for the problem (1.1)–(1.3); it is based on the Banach contraction principle. The following hypotheses will be used in the sequel.

(H1) The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) There exist constants $k_1 > 0$ and $0 < k_2 < 1$ such that

$$|f(t,\varpi_1,\omega_1) - f(t,\varpi_2,\omega_2)| \leq k_1|\varpi_1 - \varpi_2| + k_2|\omega_1 - \omega_2|,$$

for any $\varpi_1, \varpi_2, \omega_1, \omega_2 \in \mathbb{R}$ and $t \in J$.

(H3) There exists a constant $\xi > 0$ such that

$$|I_k(\varpi_1) - I_k(\varpi_2)| \leq \xi|\varpi_1 - \varpi_2|,$$

for each $\varpi_1, \varpi_2 \in \mathbb{R}$ and $k = 1, 2, \ldots, m$.

Set

$$\gamma = \frac{k_1}{1-k_2}, \quad \mu_1 = \frac{|c_2|}{|c_1+c_2|} + 1 \quad \text{and} \quad \mu_2 = \frac{\gamma\,(m+1)\left(e^b - e^a\right)^\alpha}{\Gamma\,(\alpha+1)}.$$

Theorem 3.3. *Assume that (H1)–(H3) are satisfied. If*

$$\mu_1\,(m\xi + \mu_2) < 1, \tag{3.5}$$

then the boundary value problem (1.1)–(1.3) *has a unique solution on* J.

Proof. To transform the problem (1.1)–(1.3) into a fixed point problem, consider the operator $\Theta : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ defined by

$$\begin{aligned}\Theta(\varpi)(t) = {} & \frac{-1}{c_1+c_2}\left[c_2\sum_{i=1}^{m} I_i\left(\varpi\left(t_i^-\right)\right) + \frac{c_2}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}\varphi(s)e^s ds\right.\\ & \left. + \frac{c_2}{\Gamma(\alpha)}\int_{t_m}^{b}\left(e^b-e^s\right)^{\alpha-1}\varphi(s)e^s ds - c_3\right] + \sum_{a<t_k<t} I_k\left(\varpi\left(t_k^-\right)\right)\\ & + \frac{1}{\Gamma(\alpha)}\sum_{a<t_k<t}\int_{t_{k-1}}^{t_k}\left(e^{t_k}-e^s\right)^{\alpha-1}\varphi(s)e^s ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(e^t-e^s\right)^{\alpha-1}\varphi(s)e^s ds,\end{aligned} \tag{3.6}$$

where $\varphi \in C(J,\mathbb{R})$ satisfies

$$\varphi(t) = f(t,\varpi(t),\varphi(t)).$$

It is clear that solutions of problem (1.1)–(1.3) are the fixed points of the operator Θ. Now, for $\varpi_1, \varpi_2 \in PC(J,\mathbb{R})$ and for each $t \in J$, we have

$$\begin{aligned}|\Theta(\varpi_1)(t) - \Theta(\varpi_2)(t)| \leq {} & \frac{|c_2|}{|c_1+c_2|}\left[\sum_{i=1}^{m}\left|I_i\left(\varpi_1\left(t_i^-\right)\right) - I_i\left(\varpi_2\left(t_i^-\right)\right)\right|\right.\\ & + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}e^s|\varphi_1(s)-\varphi_2(s)|ds\\ & \left. + \frac{1}{\Gamma(\alpha)}\int_{t_m}^{b}\left(e^b-e^s\right)^{\alpha-1}e^s|\varphi_1(s)-\varphi_2(s)|ds\right]\\ & + \sum_{a<t_k<t}\left|I_k\left(\varpi_1\left(t_k^-\right)\right) - I_k\left(\varpi_2\left(t_k^-\right)\right)\right|\\ & + \frac{1}{\Gamma(\alpha)}\sum_{a<t_k<t}\int_{t_{k-1}}^{t_k}\left(e^{t_k}-e^s\right)^{\alpha-1}e^s|\varphi_1(s)-\varphi_2(s)|ds\\ & + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(e^t-e^s\right)^{\alpha-1}e^s|\varphi_1(s)-\varphi_2(s)|ds,\end{aligned}$$

where φ_1, $\varphi_2 \in C(J,\mathbb{R})$ are such that

$$\varphi_1(t) = f(t,\varpi_1(t),\varphi_1(t)) \quad \text{and} \quad \varphi_2(t) = f(t,\varpi_2(t),\varphi_2(t)).$$

By (H2), we have

$$\begin{aligned}|\varphi_1(s)-\varphi_2(s)| &= |f(t,\varpi_1(t),\varphi_1(t)) - f(t,\varpi_2(t),\varphi_2(t))| \\ &\le k_1|\varpi_1(t)-\varpi_2(t)| + k_2|\varphi_1(t)-\varphi_2(t)|,\end{aligned}$$

so

$$|\varphi_1(s)-\varphi_2(s)| \le \gamma\ |\varpi_1(s)-\varpi_2(s)|. \tag{3.7}$$

Hence, for each $t \in J$,

$$\begin{aligned}|\Theta(\varpi_1)(t)-\Theta(\varpi_2)(t)| \le{}& \frac{|c_2|}{|c_1+c_2|}\left[\sum_{k=1}^{m}\xi\left|\varpi_1\left(t_k^-\right)-\varpi_2\left(t_k^-\right)\right|\right. \\ &+\frac{\gamma}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(e^{t_k}-e^s\right)^{\alpha-1}e^s\,|\varpi_1(s)-\varpi_2(s)|\,ds \\ &\left.+\frac{\gamma}{\Gamma(\alpha)}\int_{t_m}^{b}\left(e^b-e^s\right)^{\alpha-1}e^s\,|\varpi_1(s)-\varpi_2(s)|\,ds\right] \\ &+\sum_{i=1}^{m}\xi\left|\varpi_1\left(t_i^-\right)-\varpi_2\left(t_i^-\right)\right| \\ &+\frac{\gamma}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(e^{t_k}-e^s\right)^{\alpha-1}e^s\,|\varpi_1(s)-\varpi_2(s)|\,ds \\ &+\frac{\gamma}{\Gamma(\alpha)}\int_{t_k}^{t}\left(e^t-e^s\right)^{\alpha-1}e^s\,|\varpi_1(s)-\varpi_2(s)|\,ds \\ \le{}& \frac{|c_2|}{|c_1+c_2|}\left[m\xi+\frac{\gamma m\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\gamma\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)}\right]\|\varpi_1-\varpi_2\|_{PC} \\ &+\left[m\xi+\frac{\gamma m\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\gamma\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)}\right]\|\varpi_1-\varpi_2\|_{PC} \\ ={}& \left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left[m\xi+\frac{\gamma(m+1)\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)}\right]\|\varpi_1-\varpi_2\|_{PC}.\end{aligned}$$

Thus,

$$\|\Theta(\varpi_1)-\Theta(\varpi_2)\|_{PC} \le \mu_1\left(m\xi+\mu_2\right)\|\varpi_1-\varpi_2\|_{PC}.$$

By (3.5), the operator Θ is a contraction. Hence, by Banach's contraction principle, Θ has a unique fixed point that is a unique solution of (1.1)–(1.3). □

Our second existence result is based on Schaefer's fixed point theorem (Theorem 2.7 above). Let us introduce the following condition:

(H4) There exist constants $\widetilde{\xi}$, $\widetilde{I} > 0$ such that

$$|I_k(\varpi)| \le \widetilde{\xi}|\varpi| + \widetilde{I},$$

for each $\varpi \in \mathbb{R}$ and $k = 1,2,\dots,m$.

Notice that (H4) is weaker than condition (H3).

Theorem 3.4. *Assume that conditions (H1), (H2), and (H4) hold. If*

$$\mu_1 \left(m\widetilde{\xi} + \mu_2 \right) < 1, \tag{3.8}$$

then the problem (1.1)–(1.3) *has at least one solution on* J.

Proof. We shall use Schaefer's fixed point theorem to prove that Θ, defined by (3.6), has at least one fixed point on J. The proof will be given in several steps.

Step 1: Θ *is continuous.* Let $\{v_n\}$ be a sequence such that $v_n \to v$ in $PC(J, \mathbb{R})$. Then, for each $t \in J$,

$$\begin{aligned} |\Theta(v_n)(t) - \Theta(v)(t)| \le \frac{|c_2|}{|c_1 + c_2|} & \left[\sum_{i=1}^{m} \left| I_i \left(v_n \left(t_i^- \right) \right) - I_i \left(v \left(t_i^- \right) \right) \right| \right. \\ & + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(e^{t_i} - e^s \right)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| ds \\ & \left. + \frac{1}{\Gamma(\alpha)} \int_{t_m}^{b} \left(e^b - e^s \right)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| ds \right] \\ & + \sum_{a < t_k < t} \left| I_k \left(v_n \left(t_k^- \right) \right) - I_k \left(v \left(t_k^- \right) \right) \right| \\ & + \frac{1}{\Gamma(\alpha)} \sum_{a < t_k < t} \int_{t_{k-1}}^{t_k} \left(e^{t_k} - e^s \right)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(e^t - e^s \right)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| ds, \end{aligned} \tag{3.9}$$

where $\varphi_n, \varphi \in C(J, E)$ satisfy

$$\varphi_n(t) = f\left(t, v_n(t), \varphi_n(t)\right) \quad \text{and} \quad \varphi(t) = f\left(t, v(t), \varphi(t)\right).$$

By (H2), we have

$$\begin{aligned} |\varphi_n(t) - \varphi(t)| &= |f\left(t, v_n(t), \varphi_n(t)\right) - f\left(t, v(t), \varphi(t)\right)| \\ &\le k_1 \left| v_n(t) - v(t) \right| + k_2 \left| \varphi_n(t) - \varphi(t) \right|. \end{aligned}$$

Then,

$$|\varphi_n(t) - \varphi(t)| \le \ \gamma \ |v_n(t) - v(t)|.$$

Since $v_n \to v$, we have $\varphi_n(t) \to \varphi(t)$ as $n \to \infty$ for each $t \in J$. Let $\delta > 0$ be such that, for each $t \in J$, we have $|\varphi_n(t)| \le \delta$ and $|\varphi(t)| \le \delta$. Then,

$$\begin{aligned} (e^t - e^s)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| &\le (e^t - e^s)^{\alpha-1} e^s [|\varphi_n(s)| + |\varphi(s)|] \\ &\le 2\delta (e^t - e^s)^{\alpha-1} e^s \end{aligned}$$

and

$$\begin{aligned} (e^{t_k} - e^s)^{\alpha-1} e^s |\varphi_n(s) - \varphi(s)| &\le (e^{t_k} - e^s)^{\alpha-1} e^s [|\varphi_n(s)| + |\varphi(s)|] \\ &\le 2\delta (e^{t_k} - e^s)^{\alpha-1} e^s. \end{aligned}$$

For each $t \in J$, the functions $s \to 2\delta(e^t - e^s)^{\alpha-1}e^s$ and $s \to 2\delta(e^{t_k} - e^s)^{\alpha-1}e^s$ are integrable on $[a,t]$. Then, the Lebesgue dominated convergence theorem and (3.9) imply that

$$|\Theta(v_n)(t) - \Theta(v)(t)| \to 0 \quad \text{as } n \to \infty,$$

and so

$$\|\Theta(u_n) - \Theta(u)\|_{PC} \to 0 \quad \text{as } n \to \infty.$$

Therefore, Θ is continuous.

Step 2: Θ *maps bounded sets into bounded sets in* $PC(J,\mathbb{R})$. It suffices to show that for any $\overline{\delta} > 0$, there exists a positive constant $\overline{\ell}$ such that, for any $v \in B_{\overline{\delta}} = \left\{v \in PC(J,\mathbb{R}) : \|v\|_{PC} \leq \overline{\delta}\right\}$, we have $\|\Theta(v)\|_{PC} \leq \overline{\ell}$. Now for each $t \in J$,

$$\begin{aligned}
|\Theta(v)(t)| \leq{}& \frac{|c_2|}{|c_1+c_2|}\left[\sum_{i=1}^{m}\left|I_i\left(v\left(t_i^-\right)\right)\right| + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}e^s\,|\varphi(s)|\,ds\right.\\
&\left.+\frac{1}{\Gamma(\alpha)}\int_{t_m}^{b}\left(e^b-e^s\right)^{\alpha-1}e^s\,|\varphi(s)|\,ds\right] + \frac{|c_3|}{|c_1+c_2|} + \sum_{a<t_k<t}\left|I_k\left(v\left(t_k^-\right)\right)\right|\\
&+\frac{1}{\Gamma(\alpha)}\sum_{a<t_k<t}\int_{t_{k-1}}^{t_k}\left(e^{t_k}-e^s\right)^{\alpha-1}e^s\,|\varphi(s)|\,ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(e^t-e^s\right)^{\alpha-1}e^s\,|\varphi(s)|\,ds,
\end{aligned} \tag{3.10}$$

where $\varphi \in C(J,\mathbb{R})$ satisfies

$$\varphi(t) = f\left(t, v(t), \varphi(t)\right).$$

By (H2), for each $t \in J$ we have

$$\begin{aligned}
|\varphi(t)| &= |f\left(t,v(t),\varphi(t)\right) - f\left(t,0,0\right) + f\left(t,0,0\right)|\\
&\leq |f\left(t,v(t),\varphi(t)\right) - f\left(t,0,0\right)| + |f\left(t,0,0\right)|\\
&\leq k_1\,|v| + k_2\,|\varphi(t)| + \widetilde{f}.
\end{aligned}$$

Thus,

$$|\varphi(t)| \quad \leq \quad \gamma\,|v| + \tfrac{\widetilde{f}}{1-k_2}. \tag{3.11}$$

From this and (3.10), for any $v \in B_{\overline{\delta}}$, we have

$$\begin{aligned}
|\Theta(v)(t)| \leq{}& \frac{|c_2|}{|c_1+c_2|}\left[m\left(\widetilde{\xi}\,|v| + \widetilde{I}\right) + m\left(\gamma\,|v| + \frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)}\right.\\
&\left.+\left(\gamma\,|v| + \frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)}\right] + \frac{|c_3|}{|c_1+c_2|} + m\left(\widetilde{\xi}\,|v| + \widetilde{I}\right)\\
&+m\left(\gamma\,|v| + \frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)} + \left(\gamma\,|v| + \frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)}\\
={}& \left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left[m\left(\widetilde{\xi}\,|v| + \widetilde{I}\right) + \left(\gamma\,|v| + \frac{\widetilde{f}}{1-k_2}\right)\frac{(m+1)\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)}\right] + \frac{|c_3|}{|c_1+c_2|}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left[m\left(\widetilde{\xi}\overline{\delta}+\widetilde{I}\right)+\left(\gamma\overline{\delta}+\frac{\widetilde{f}}{1-k_2}\right)\frac{(m+1)\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)}\right]+\frac{|c_3|}{|c_1+c_2|}\\
&=\mu_1\left[m\left(\widetilde{\xi}\overline{\delta}+\widetilde{I}\right)+\left(\overline{\delta}+\frac{\widetilde{f}}{k_1}\right)\mu_2\right]+\frac{|c_3|}{|c_1+c_2|}\\
&=: \overline{\ell},
\end{aligned}
$$

which implies that $\|\Theta(v)\|_{PC} \leq \overline{\ell}$.

Step 3: Θ *maps bounded sets into equicontinuous sets in* $PC(J,\mathbb{R})$. Let τ_1, $\tau_2 \in J$ with $\tau_1 < \tau_2$, $B_{\overline{\delta}}$ be a bounded set in $PC(J,\mathbb{R})$ as in Step 2, and let $v \in B_{\overline{\delta}}$. Then, we have

$$
\begin{aligned}
&|\Theta(v)(\tau_2)-\Theta(v)(\tau_1)|\\
&\quad\leq \frac{1}{\Gamma(\alpha)}\int_a^{\tau_1}\left|\left[(e^{\tau_2}-e^s)^{\alpha-1}-(e^{\tau_1}-e^s)^{\alpha-1}\right]e^s\right||\varphi(s)|\,ds\\
&\qquad+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\left|(e^{\tau_2}-e^s)^{\alpha-1}e^s\right||\varphi(s)|\,ds+\sum_{\tau_1<t_k<\tau_2}\left|I_k\left(v\left(t_k^-\right)\right)\right|\\
&\qquad+\frac{1}{\Gamma(\alpha)}\sum_{\tau_1<t_k<\tau_2}\int_{t_{k-1}}^{t_k}\left|\left(e^{t_k}-e^s\right)^{\alpha-1}e^s\right||\varphi(s)|\,ds\\
&\quad\leq\left(\gamma|v|+\frac{\widetilde{f}}{(1-k_2)}\right)\frac{1}{\Gamma(\alpha+1)}\left[(e^{\tau_1}-e^a)^{\alpha}-(e^{\tau_2}-e^a)^{\alpha}\right.\\
&\qquad\left.+2\left(e^{\tau_2}-e^{\tau_1}\right)^{\alpha}\right]+(\tau_2-\tau_1)\left[\left(\widetilde{\xi}|v|+\widetilde{I}\right)+\left(\gamma|v|+\frac{\widetilde{f}}{(1-k_2)}\right)\frac{\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)}\right]\\
&\quad\leq\left(\gamma\overline{\delta}+\frac{\widetilde{f}}{(1-k_2)}\right)\frac{1}{\Gamma(\alpha+1)}\left[(e^{\tau_1}-e^a)^{\alpha}-(e^{\tau_2}-e^a)^{\alpha}\right.\\
&\qquad\left.+2\left(e^{\tau_2}-e^{\tau_1}\right)^{\alpha}\right]+(\tau_2-\tau_1)\left[\left(\widetilde{\xi}\overline{\delta}+\widetilde{I}\right)+\left(\gamma\overline{\delta}+\frac{\widetilde{f}}{(1-k_2)}\right)\frac{\left(e^b-e^a\right)^{\alpha}}{\Gamma(\alpha+1)}\right].
\end{aligned}
$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of the steps 1 to 3 together with the Ascoli–Arzelà theorem, we conclude that $\Theta : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is completely continuous.

Step 4: *A priori bounds.* It remain to show that the set

$$\varepsilon = \{v \in PC(J,\mathbb{R}) : v = \lambda\Theta(v), \text{ for some } \lambda \in (0,1)\}$$

is bounded. Let $v \in \varepsilon$; then $v = \lambda\Theta(v)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$, we have

$$
\begin{aligned}
v(t)=&\frac{-\lambda}{c_1+c_2}\left[c_2\sum_{i=1}^{m}I_i\left(v\left(t_i^-\right)\right)+\frac{c_2}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(e^{t_i}-e^s\right)^{\alpha-1}\varphi(s)e^s ds\right.\\
&\left.+\frac{c_2}{\Gamma(\alpha)}\int_{t_m}^{b}\left(e^b-e^s\right)^{\alpha-1}\varphi(s)e^s ds-c_3\right]+\lambda\sum_{a<t_k<t}I_k\left(v\left(t_k^-\right)\right)\\
&+\frac{\lambda}{\Gamma(\alpha)}\sum_{a<t_k<t}\int_{t_{k-1}}^{t_k}\left(e^{t_k}-e^s\right)^{\alpha-1}\varphi(s)e^s ds+\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}\left(e^t-e^s\right)^{\alpha-1}\varphi(s)e^s ds.
\end{aligned}
$$

From (3.11) and (H4), for each $t \in J$, we obtain

$$\begin{aligned}
|v(t)| \leq & \frac{|c_2|}{|c_1+c_2|}\left[m\left(\widetilde{\xi}\,|v|+\widetilde{I}\right)+m\left(\gamma\,|v|+\frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}\right.\\
&\left.+\left(\gamma\,|v|+\frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}\right]+\frac{|c_3|}{|c_1+c_2|}+m\left(\widetilde{\xi}\,|v|+\widetilde{I}\right)\\
&+m\left(\gamma\,|v|+\frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}+\left(\gamma\,|v|+\frac{\widetilde{f}}{1-k_2}\right)\frac{\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}\\
=&\left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left[m\left(\widetilde{\xi}\,|v|+\widetilde{I}\right)+\left(\gamma\,|v|+\frac{\widetilde{f}}{1-k_2}\right)\frac{(m+1)\left(e^b-e^a\right)^\alpha}{\Gamma(\alpha+1)}\right]+\frac{|c_3|}{|c_1+c_2|}\\
\leq&\left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left(m\widetilde{\xi}+\frac{\gamma\,(m+1)\left(e^b-e^a\right)^\alpha}{\Gamma\,(\alpha+1)}\right)|v|\\
&+\left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left(m\widetilde{I}+\frac{\widetilde{f}\,(m+1)\left(e^b-e^a\right)^\alpha}{(1-k_2)\,\Gamma\,(\alpha+1)}\right)+\frac{|c_3|}{|c_1+c_2|}\\
\leq&\,\mu_1\left(m\widetilde{\xi}+\mu_2\right)|v|+\mu_1\left(m\widetilde{I}+\frac{\widetilde{f}\mu_2}{k_1}\right)+\frac{|c_3|}{|c_1+c_2|}.
\end{aligned}$$

Thus,

$$\left[1-\mu_1\left(m\widetilde{\xi}+\mu_2\right)\right]\|v\|_{PC}\leq\mu_1\left(m\widetilde{I}+\frac{\widetilde{f}\mu_2}{k_1}\right)+\frac{|c_3|}{|c_1+c_2|}.$$

By using condition (3.8), it follows that

$$\|v\|_{PC}\leq\frac{\left[\mu_1\left(m\widetilde{I}+\frac{\widetilde{f}\mu_2}{k_1}\right)+\frac{|c_3|}{|c_1+c_2|}\right]}{\left[1-\mu_1\left(m\widetilde{\xi}+\mu_2\right)\right]}=:\overline{M}.$$

This shows that the set ε is bounded. As a consequence of Schaefer's fixed point theorem, Θ has at least one fixed point which in turn is a solution of (1.1)–(1.3). □

Remark 3.5. Often times using different techniques of proof for the same type of result necessitates requiring different hypotheses. It interesting to point out here that we have also been able to obtain both Theorems 3.3 and 3.4 above with no changes in conditions by using the Nonlinear Alternative of Leray–Schauder type.

Remark 3.6. Our results for the boundary value problem (1.1)–(1.3) remain true for the following cases:

- Initial value problem: $c_1 = 1,\ c_2 = 0$ and c_3 arbitrary.
- Terminal value problem: $c_1 = 0,\ c_2 = 1$ and c_3 arbitrary.
- Anti-periodic problem: $c_1 = c_2 \neq 0$ and $c_3 = 0$.

However, our results are not applicable to the periodic problem, i.e., the case $c_1 = 1$, $c_2 = -1$, and $c_3 = 0$.

4 Examples

In this section, we will give two examples to illustrate our main results.

Example 4.1. Consider the impulsive boundary value problem for the nonlinear implicit fractional differential equation

$$ {}^{e}_{c}D^{\frac{1}{2}}_{t_k}\varpi(t) = \frac{e^{-\sqrt{t+9}}\sin t}{7\left(t^2+1\right)\left(\sqrt{3}+|\varpi(t)|+\left|{}^{e}_{c}D^{\frac{1}{2}}_{t_k}\varpi(t)\right|\right)}, \quad \text{for each } t \in J_0 \cup J_1, \tag{4.1} $$

$$ \Delta\varpi|_{t=\frac{\pi}{2}} = \frac{2\left|\varpi(\frac{\pi}{2}^-)\right|}{3+\left|\varpi(\frac{\pi}{2}^-)\right|}, \tag{4.2} $$

$$ \varpi(0)+\varpi(\pi) = 13, \tag{4.3} $$

where $J_0 = \left[0, \frac{\pi}{2}\right]$, $J_1 = (\frac{\pi}{2}, \pi]$, $m = 1$, $\alpha = \frac{1}{2}$, $a = 0$, $b = \pi$, $c_1 = c_2 = 1$, $c_3 = 13$,

$$ f\left(t, \varpi, \omega\right) = \frac{e^{-\sqrt{t+9}}\sin t}{7\left(t^2+1\right)\left(\sqrt{3}+|\varpi|+|\omega|\right)}, $$

and

$$ I_1(\varpi) = \frac{|\varpi|}{19+|\varpi|}. $$

Now, for each $t \in [0, \pi]$ and for any ϖ_1, ϖ_2, ω_1, $\omega_2 \in \mathbb{R}$, we can show that

$$ \left|f\left(t, \varpi_1, \omega_1\right) - f\left(t, \varpi_2, \omega_2\right)\right| \leq \frac{1}{21e^3}\left(|\varpi_1-\varpi_2|+|\omega_1-\omega_2|\right) $$

and

$$ |I_1(\varpi_1) - I_1(\varpi_2)| \leq \frac{1}{19}|\varpi_1-\varpi_2|. $$

Thus, for $k_1 = k_2 = \frac{1}{21e^3}$ and $\xi = \frac{1}{19}$ we have that

$$ \begin{aligned} \mu_1\left(m\xi+\mu_2\right) &= \left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left[m\xi + \frac{k_1\left(m+1\right)\left(e^b-e^a\right)^{\alpha}}{\left(1-k_2\right)\Gamma(\alpha+1)}\right] \\ &= \frac{3}{2}\left[\frac{1}{19}+\frac{\frac{2\sqrt{e^{\pi}-1}}{21e^3}}{\left(1-\frac{1}{21e^3}\right)\Gamma(\frac{3}{2})}\right] \\ &= \frac{3}{2}\left[\frac{1}{19}+\frac{4\sqrt{e^{\pi}-1}}{\left(21e^3-1\right)\sqrt{\pi}}\right] \\ &\approx 0.1168003443 \\ &< 1. \end{aligned} $$

Hence, conditions (H1)–(H3) and (3.5) are satisfied. As a consequence of Theorem 3.3, the problem (4.1)–(4.3) has a unique solution on $[0, \pi]$.

Example 4.2. Consider the problem

$$ {}^{e}_{c}D^{\frac{1}{2}}_{t_k}\varpi(t) = \frac{e^{-\sqrt{t+16}}\left(2+|\varpi(t)|+\left|{}^{e}_{c}D^{\frac{1}{2}}_{t_k}\varpi(t)\right|\right)}{179\left(t^2+1\right)\left(1+|\varpi(t)|+\left|{}^{e}_{c}D^{\frac{1}{2}}_{t_k}\varpi(t)\right|\right)}, \quad \text{for each } t \in J_0 \cup J_1, \tag{4.4} $$

$$\Delta\varpi|_{t=\frac{1}{4}} = \frac{5\left|\varpi(\frac{1}{4}^{-})\right|+1}{20+\left|\varpi(\frac{1}{4}^{-})\right|}, \tag{4.5}$$

$$\varpi(0) = -\varpi(1), \tag{4.6}$$

where $J_0 = \left[0, \frac{1}{4}\right]$, $J_1 = (\frac{1}{4}, 1]$, $m = 1$, $\alpha = \frac{1}{2}$, $a = 0$, $b = 1$, $c_1 = c_2 = 1$, $c_3 = 0$,

$$f(t,\varpi,\omega) = \frac{e^{-\sqrt{t+16}}\,(2+|\varpi|+|\omega|)}{179\,(t^2+1)\,(1+|\varpi|+|\omega|)}, \quad \text{for each } t \in J_0 \cup J_1,$$

and

$$I_1(\varpi) = \frac{5\,|\varpi|+1}{20+|\varpi|}.$$

Now, for each $t \in [0,1]$ and for any $\varpi_1, \varpi_2, \omega_1, \omega_2 \in \mathbb{R}$, we can show that

$$|f(t,\varpi_1,\omega_1) - f(t,\varpi_2,\omega_2)| \leq \frac{1}{179e^4}\,(|\varpi_1-\varpi_2| + |\omega_1-\omega_2|)$$

and

$$|I_1(\varpi)| \leq \frac{1}{4}\,|\varpi| + \frac{1}{20}.$$

Thus, for $k_1 = k_2 = \frac{1}{179e^4}$ and $\widetilde{xi} = \frac{1}{4}$, we have

$$\begin{aligned}
\mu_1\left(m\widetilde{\xi}+\mu_2\right) &= \left(\frac{|c_2|}{|c_1+c_2|}+1\right)\left[m\xi + \frac{k_1\,(m+1)\,(e^b-e^a)^{\alpha}}{(1-k_2)\,\Gamma(\alpha+1)}\right] \\
&= \frac{3}{2}\left[\frac{1}{4} + \frac{\frac{2\sqrt{e-1}}{179e^4}}{(1-\frac{1}{179e^4})\,\Gamma(\frac{3}{2})}\right] \\
&= \frac{3}{2}\left[\frac{1}{4} + \frac{4\sqrt{e-1}}{(179e^4-1)\,\sqrt{\pi}}\right] \\
&= \frac{3}{8} + \frac{4\sqrt{e-1}}{(179e^4-1)\,\sqrt{\pi}} \\
&\approx 0.3753057 \\
&< 1.
\end{aligned}$$

Hence, conditions (H1), (H2), (H4), and (3.8) are satisfied, so by Theorem 3.4 the problem (4.4)–(4.6) has at least one solution on $[0,1]$.

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Minimal travelling wave speed and explicit solutions in monostable reaction-diffusion equations

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. We investigate the connection between the existence of an explicit travelling wave solution and the travelling wave with minimal speed in a scalar monostable reaction-diffusion equation.

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1 Introduction

In this short paper we investigate the somewhat puzzling connection between the existence of an explicit travelling wave solution and the travelling wave with minimal speed in a monostable reaction-diffusion equation. More precisely, there are examples in the literature (see below) where the explicitly computable travelling wave solution is the solution with minimal speed. Moreover, for parameter-dependent problems with a parameter-dependent family of explicit solutions, there are many cases where in fact there is a switching between the minimal speed being given by this explicit solution for some parameters, while for others it is given by the so-called linear speed, defined as the minimal value for which the problem linearised about the unstable steady state has a suitable eigenvalue. For a particular set of equations, of a type encountered in applications, we formulate sufficient conditions for each of these phenomena to occur.

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The plan of the paper is as follows. In this section, we introduce scalar monostable reaction-diffusion equations, define what we mean by a minimal speed, and discuss the linear (pulled) and the non-linear (pushed) regimes.

In Section 2, we define the set of exactly solvable equations and prove a result connecting the minimal wave speed and the speed of an explicit travelling wave solution.

Finally, in Section 3 we consider conditions for the exchange of minimality between the linear minimal speed and the speed of an explicit travelling wave solution.

Our proofs exploit two main tools: the variational principle due to Hadeler and Rothe [4] and the integrability characterisations of the minimal speed proved by Lucia, Muratov and Novaga in [6].

We consider reaction-diffusion equations of the form

$$u_t = u_{xx} + f(u, \beta), \tag{1.1}$$

where $\beta \in \mathbb{R}$ is a parameter, and f is a monostable nonlinearity, *i.e.*,

$$f(0,\beta) = f(1,\beta) = 0, \quad f'(0,\beta) > 0, \quad f'(1,\beta) < 0, \quad f(u,\beta) > 0 \quad \text{for } u \in (0,1).$$

In the travelling wave frame $z = x - ct$, $c \geq 0$, setting $U(z) = u(x,t)$, and denoting derivatives with respect to z by primes, (1.1) becomes

$$-cU' = U'' + f(U). \tag{1.2}$$

We seek **monotone fronts** connecting 1 and 0, *i.e.*, solutions $U(z)$ of (1.2) with $U'(z) < 0$ and

$$\lim_{z \to -\infty} U(z) = 1 \quad \text{and} \quad \lim_{z \to \infty} U(z) = 0.$$

Linearisation around the rest point with $U = 0$ shows that there cannot be any monotone fronts connecting 1 and 0 for $c < c_l := 2\sqrt{f'(0)}$. Phase plane analysis shows that there exists $c_{\min} \geq c_l$ such that there exists a monotone front for all $c \geq c_{\min} \geq c_l$. Determining $c_{\min}$ is often of interest in applications, see e.g. [2] for a discussion.

Definition 1.1. If $c_{\min} = c_l$, we say that we are in the case of *linear selection mechanism* ("pulled case") and if $c_{\min} > c_l$, of *nonlinear selection mechanism* ("pushed case").

Frequently, the basis of analysis of monotone fronts in the scalar monostable case (1.2) is the following construction: As $U(z)$ is a monotone solution, its derivative is a well-defined function of U. Set $F(U) := -U'$. Note that $F(U)$ is non-negative. Also, $F(0) = F(1) = 0$. Now,

$$F(U)' = (-U')' = -U''.$$

On the other hand, by the chain rule,

$$F(U)' = \frac{dF}{dU}U' = -\frac{dF}{dU}F.$$

Hence the problem of solving $U'' + cU' + f(U) = 0$ with the conditions that $\lim_{z\to-\infty} U(z) = 1$ and $\lim_{z\to\infty} U(z) = 0$ is equivalent to solving

$$F\frac{dF}{dU} - cF + f(U) = 0, \quad F(0) = F(1) = 0. \tag{1.3}$$

Using this construction, we have the Hadeler–Rothe variational principle [4]:

$$c_{\min} = \inf_{g\in\mathcal{G}} \sup_{0<U<1} \left\{ g'(U) + \frac{f(U)}{g(U)} \right\}, \tag{1.4}$$

where

$$\mathcal{G} = \left\{ g \in C^1([0,1]) \mid g(U) > 0 \text{ for } 0 < U < 1,\ g(0) = 0,\ g'(0) > 0 \right\}. \tag{1.5}$$

2 Exact solvability

We are interested in the situation when (1.2) has a solution $U(z)$ that can be determined by quadratures. A sufficient condition is:

Lemma 2.1. *The travelling wave equation of* (1.3) *with speed* $c = A(\beta)/\sqrt{B(\beta)}$ *is solvable by quadratures if* f *can be written in the form*

$$f(u,\beta) = h(u)\Big(A(\beta) - B(\beta)h'(u)\Big), \qquad h \in C^1([0,1]), \tag{2.1}$$

where $h(0) = h(1) = 0$, $h(u) \geq 0$, $h'(0) > 0$ *(without loss of generality* $h'(0) = 1$*),* $A(\beta) > B(\beta) > 0$*, and for all* $u \in [0,1]$, $A(\beta) - B(\beta)h'(u) > 0$.

Proof. In this case a solution of (1.3) is $F(U) = \gamma h(U)$ with

$$\gamma = \sqrt{B(\beta)}, \tag{2.2}$$

from which U and c can be computed by quadratures. □

We introduce notation for the speeds of the explicit fronts in Lemma 2.1:

$$c_{nl}(\beta) := \frac{A(\beta)}{\sqrt{B(\beta)}}. \tag{2.3}$$

We will describe as the **solvable case** the situation in which the nonlinearity $f(u,\beta)$ satisfies the conditions of Lemma 2.1. In the solvable case, we have that

$$c_l = 2\sqrt{A(\beta) - B(\beta)}. \tag{2.4}$$

Note that the fact that $A(\beta) > B(\beta)$ follows from the conditions of Lemma 2.1.

Of course, by the definition of minimal speed, we always have that

$$c_{\min}(\beta) \leq c_{nl}(\beta) = \frac{A(\beta)}{\sqrt{B(\beta)}}. \tag{2.5}$$

3 Minimality exchange

In this section, for a nonlinearity $f(u,\beta)$ of solvable type, we investigate conditions under which there exists a value β^*, such that for values β to one side of β^*, $c_{\min}(\beta) = c_l(\beta)$, and for values of β to the other side of β^*, $c_{\min}(\beta) = c_{nl}(\beta)$, so that at β^* minimality is exchanged between $c_l(\beta)$ and $c_{nl}(\beta)$. This is what we call a **minimality exchange**. Examples, two of which we outline below, are discussed in [4, 6] and the isotropic case of [2], which is also investigated in [3,8].

First note that for a minimality exchange, the graphs of $c_l(\beta)$ and $c_{nl}(\beta)$ must clearly intersect. Therefore the equation

$$2\sqrt{A(\beta)-B(\beta)} = \frac{A(\beta)}{\sqrt{B(\beta)}}$$

must have a solution, which is equivalent to demanding the existence of β^* such that $A(\beta^*) = 2B(\beta^*)$.

Hence, for instance, in any equation (1.1) with solvable $f(u,\beta)$ such that $A(\beta) = 2B(\beta)+1$, there can never be a minimality exchange between the linear and the nonlinear speeds.

Before continuing with the analysis, we present two concrete examples of minimality exchange. In [4, Eq. (27)], Hadeler and Rothe consider the nonlinearity

$$f(u,\beta) = u(1-u)(1+\beta u), \qquad \beta \geq -1,$$

which can be put into the framework of Lemma 2.1 by setting $h(u) = u(1-u)$, so that

$$f(u,\beta) = h(u)(A(\beta) - B(\beta)h'(u)),$$

where

$$A(\beta) = 1 + \frac{\beta}{2}, \qquad B(\beta) = \frac{\beta}{2}.$$

The solution of $A(\beta) = 2B(\beta)$ is therefore $\beta^* = 2$, the nonlinear speed is

$$c_{nl}(\beta) = \frac{2+\beta}{\sqrt{2\beta}},$$

and it is shown in [4] that a minimality exchange occurs at $\beta = \beta^*$, with $c_{\min}(\beta) = c_l(\beta)$ for $\beta < \beta^*$ and $c_{\min}(\beta) = c_{nl}(\beta)$ for $\beta > \beta^*$.

Our second example is given by the isotropic case of [2], where

$$f(u,\beta) = \frac{\sin(\pi u)}{2\pi}\left[1 - \beta\cos(\pi u)\right],$$

which fits into the framework of Lemma 2.1 by setting $h(u) = \frac{\sin(\pi u)}{\pi}$, so that

$$f(u,\beta) = h(u)(A(\beta) - B(\beta)h'(u)),$$

where

$$A(\beta) = \frac{1}{2}\ B(\beta) = \frac{\beta}{2}.$$

The equation $A(\beta) = 2B(\beta)$ then has solution $\beta^* = \frac{1}{2}$, the nonlinear speed is

$$c_{nl}(\beta) = \frac{1}{\sqrt{2\beta}},$$

and it is proved in [2, 3] that here too, a minimality exchange occurs at $\beta = \beta^*$, again with $c_{\min}(\beta) = c_l(\beta)$ for $\beta < \beta^*$ and $c_{\min}(\beta) = c_{nl}(\beta)$ for $\beta > \beta^*$.

We now establish our general results, starting with a sufficient condition for nonlinear selection.

Lemma 3.1. *For all β such that $A(\beta) < 2B(\beta)$, $c_{\min}(\beta) = c_{nl}(\beta)$.*

Proof. For any $c > 0$, denote by $H_c^1(\mathbb{R})$ the completion of $C_0^\infty(\mathbb{R})$ with respect to the norm

$$\|u\|_{1,c} = \|u\|_c + \|u_x\|_c, \quad \text{where} \quad \|u\|_c^2 = \int_{\mathbb{R}} e^{cx} u^2(x)\, dx.$$

If $U(z)$ is an explicit travelling front with $-U' = F(U) = \gamma h(U)$, we have

$$\lim_{z\to\infty} \frac{U'(z)}{U(z)} = \lim_{z\to\infty} -\gamma \frac{h(U(z))}{U(z)} = -\gamma\, h'(0) = -\gamma.$$

Hence for those values of the parameter β for which $c_{nl}(\beta) < 2\gamma$, $U \in H^1_{c_{nl}(\beta)}(\mathbb{R})$ and hence for such β, by Corollary 2.7 of [6] (see also Proposition 2 of [2]), $c(\beta)$ is the (nonlinear) minimal wave speed. The claim then follows by (2.2) and (2.3). □

We note that this lemma can also be obtained by the methods of [1]. To formulate our next results, we set

$$L = \max_{u\in(0,1)} h'(u) \geq 1.$$

We adapt some arguments from [2].

Proposition 3.2. *If $A(\beta) > 2LB(\beta)$,*

$$c_{\min}(\beta) \leq 2\sqrt{L}\sqrt{A(\beta) - LB(\beta)}, \tag{3.1}$$

and in particular,

$$c_{\min}(\beta) \neq c_{nl}(\beta).$$

Proof. Recall from Hadeler and Rothe [4] (see also [2], equation (11)) that

$$c_{\min}(\beta) = \inf_{g\in\Lambda} \sup_{U\in(0,1)} \left\{ g'(U) + \frac{f(U,\beta))}{g(U)} \right\}, \tag{3.2}$$

where

$$\Lambda = \left\{ g \in C^1([0,1]) : g(U) > 0 \text{ if } U \in (0,1),\ g(0) = 0,\ g'(0) > 0 \right\}. \tag{3.3}$$

Hence taking $g(U) = \nu h(U)$, $\nu > 0$, yields that

$$c_{\min}(\beta) \leq \inf_{\nu>0} \sup_{U\in(0,1)} \left\{ \nu h'(U) + \frac{A(\beta)}{\nu} - \frac{B(\beta)}{\nu} h'(U) \right\}.$$

To understand

$$\sup_{U\in(0,1)} \left\{ \left(\nu - \frac{B(\beta)}{\nu}\right) h'(U) + \frac{A(\beta)}{\nu} \right\},$$

there are two cases:

(i) $\nu^2 \leq B(\beta)$: Then

$$\sup_{U\in(0,1)} \left\{ \left(\nu - \frac{B(\beta)}{\nu}\right) h'(U) + \frac{A(\beta)}{\nu} \right\} = \frac{A(\beta) - lB(\beta)}{\nu} + l\nu,$$

which is monotone decreasing in ν, so

$$\inf_{\nu\leq\sqrt{B(\beta)}} \sup_{U\in(0,1)} \left\{ \left(\nu - \frac{B(\beta)}{\nu}\right) h'(U) + \frac{A(\beta)}{\nu} \right\} = \frac{A(\beta)}{\sqrt{B(\beta)}}.$$

(Note that this recovers the estimate (2.5) for $c_{\min}(\beta)$.)

(ii) $\nu^2 \geq B$: Then

$$\sup_{U\in(0,1)} \left\{ \left(\nu - \frac{B(\beta)}{\nu}\right) h'(U) + \frac{A(\beta)}{\nu} \right\} = \frac{A(\beta) - LB(\beta)}{\nu} + L\nu := q(\nu).$$

Since $A(\beta) - B(\beta)h'(u) > 0$ for all $u \in [0,1]$, it follows that $A(\beta) - LB(\beta) > 0$. So differentiating $q(\nu)$ gives that its global minimum for $\nu \in (0,\infty)$ occurs at

$$\nu_0 := \sqrt{\frac{A(\beta) - LB(\beta)}{L}}.$$

There are two possibilities: (a) If

$$\frac{A(\beta) - LB(\beta)}{L} \leq B(\beta),$$

the function $q(\nu)$ reaches its minimum over $[\sqrt{B(\beta)}, \infty)$ at the point $\nu = \sqrt{B(\beta)}$, so that

$$\inf_{\nu\geq\sqrt{B(\beta)}} \sup_{U\in(0,1)} \left\{ \left(\nu - \frac{B(\beta)}{\nu}\right) h'(U) + \frac{A(\beta)}{\nu} \right\} = \frac{A(\beta)}{\sqrt{B(\beta)}},$$

in which case we again just recover the estimate (2.5) for $c_{qmin}(\beta)$.

(b) On the other hand, if

$$\frac{A(\beta) - LB(\beta)}{L} > B(\beta),$$

that is, $A(\beta) > 2LB(\beta)$, we have that

$$c_{\min}(\beta) \leq \inf_{\nu>\sqrt{B}} \sup_{U\in(0,1)} \left\{ \left(\nu - \frac{B(\beta)}{\nu}\right) h'(U) + \frac{A(\beta)}{\nu} \right\} = q(\nu_0) = 2\sqrt{L}\sqrt{A(\beta) - LB(\beta)}. \quad (3.4)$$

Comparison of $q(\nu_0)$ in (3.4) with $c_{nl}(\beta)$ then shows that $c_{\min}(\beta) \neq c_{nl}(\beta)$ if $A(\beta) > 2LB(\beta)$. □

Now we can formulate sufficient conditions for minimality exchange. Below we say that a solution β^* of the equation $A(\beta) = 2B(\beta)$ is **non-degenerate** if the graphs of the functions $A(\cdot)$ and $2B(\cdot)$ intersect transversely at β^*. The following result applies in all the examples in [2, 4] mentioned above and covers the general case when $h(u)$ is concave and there is a non-degenerate solution to $A(\beta) = 2B(\beta)$.

Theorem 3.3. *Suppose there is a non-degenerate solution β^* to the equation $A(\beta) = 2B(\beta)$. Then if*

$$L = h'(0) = 1,$$

there is a minimality exchange at $\beta = \beta^*$.

Proof. Since if $A(\beta) < 2B(\beta)$ we have that $c_{\min}(\beta) = c_{nl}(\beta)$ by Lemma 3.1, and since by (3.4) with $L = 1$, for all $A(\beta) > 2B(\beta)$, $c_{\min}(\beta) = c_l(\beta)$, non-degeneracy of the solution β^* of $A(\beta) = 2B(\beta)$ implies that there is an exchange of minimality at β^*. □

Theorem 3.3 fully characterises minimality exchange when $L = 1$, that is, when $h'(u)$ attains its supremum L at $u = 0$, which holds in particular when h is concave. If $L > 1$, however, the situation is less clear. Lemma 3.1 clearly still implies that $c_{\min}(\beta) = c_{nl}(\beta) > c_l(\beta)$, so in particular nonlinear selection holds, if $A(\beta) < 2B(\beta)$, and linear selection holds, with $c_{\min}(\beta) = c_{nl}(\beta) = c_l(\beta)$ if $A(\beta) = 2B(\beta)$, but whether it is possible to have again nonlinear selection for some β with $A(\beta) > 2B(\beta)$, either with the minimal speed corresponding to the explicit solution or another value, is not obvious. The estimate (3.4) only applies when $A(\beta) > 2LB(\beta)$, and even in that range, (3.4) is no longer sufficient to imply linear selection if $L > 1$.

In Theorem 3.6 below, we present a result complementary to Theorem 3.3 that makes no assumption on h beyond the hypotheses in Lemma 2.1, but instead imposes monotonicity conditions on the dependence of A and B on β. This yields a partial answer to what happens when $L > 1$ and $A(\beta) > 2B(\beta)$. We begin with the following preliminary result, based on [6, Theorem 2.8], which forms the basis for the alternative sufficient condition for minimality exchange in Theorem 3.6.

Lemma 3.4. *Suppose that $A(\beta)$ and $B(\beta)$ are each non-decreasing in β, and $A(\beta) - B(\beta)$ is non-increasing in β. If $c_{\min}(\beta_1) > c_l(\beta_1)$ and $\beta_2 > \beta_1$, then*

$$c_{\min}(\beta_2) > c_l(\beta_2).$$

that is, if nonlinear selection holds for some β_1, nonlinear selection also holds for any $\beta_2 > \beta_1$,

Proof. We draw on Theorem 2.8 of Lucia, Muratov and Novaga [6], which says that $c_{\min}(\beta) > c_l(\beta)$ if and only if there exists $c > c_l(\beta)$ and $u \in H^1_c(\mathbb{R})$ such that

$$\Phi^\beta_c[u] := \int_{\mathbb{R}} e^{cx} \left(\frac{1}{2} u_x^2 - \int_0^u f(s,\beta)\, ds \right) dx \;\leq\; 0, \tag{3.5}$$

where $H^1_c(\mathbb{R})$ is as defined in the proof of Lemma 3.1.

First note that it follows from [6, Theorem 2.8] that since $c_{\min}(\beta_1) > c_l(\beta_1)$, there exists $c > c_l(\beta_1)$ and $u \in H^1_c(\mathbb{R})$ such that $\Phi^{\beta_1}_c[u] \leq 0$. Then

$$\begin{aligned}
\Phi^{\beta_1}_c[u] &= \int_{\mathbb{R}} e^{cx} \left(\frac{1}{2} u_x^2 - \int_0^u f(s,\beta_1) ds \right) dx \\
&= \int_{\mathbb{R}} e^{cx} \left(\frac{1}{2} u_x^2 - \int_0^u h(s)(A(\beta_1) - B(\beta_1) h'(s)) ds \right) dx \\
&= \int_{\mathbb{R}} e^{cx} \left(\frac{1}{2} u_x^2 - A(\beta_1) \int_0^u h(s)\, ds - \frac{B(\beta_1)}{2} h(u)^2 \right) dx \\
&\leq 0,
\end{aligned}$$

as $h(0) = 0$, and since $\beta_2 > \beta_1$ and $A(\cdot)$ and $B(\cdot)$ are non-decreasing, we have $A(\beta_2) \geq A(\beta_1)$ and $B(\beta_2) \geq B(\beta_1)$, so that

$$\Phi^{\beta_2}_c[u] \;\leq\; \Phi^{\beta_1}_c[u] \;\leq\; 0,$$

since $h(s) > 0$ for $0 < s < 1$. Moreover, $A(\cdot) - B(\cdot)$ is non-increasing, so

$$c_l(\beta_2) = 2\sqrt{A(\beta_2) - B(\beta_2)} \leq 2\sqrt{A(\beta_1) - B(\beta_1)} = c_l(\beta_1),$$

and hence

$$c > c_l(\beta_1) \geq c_l(\beta_2).$$

Thus $c > c_l(\beta_2)$ and $\Phi^{\beta_2}_c[u] \leq 0$, and hence [6, Theorem 2.8] implies that $c_{\min}(\beta_2) > c_l(\beta_2)$. □

The following is an immediate consequence of Lemma 3.4.

Corollary 3.5. *Suppose that $A(\beta)$ and $B(\beta)$ are each non-decreasing in β, and that $A(\beta) - B(\beta)$ is non-increasing in β. If $c_{\min}(\beta_2) = c_l(\beta_2)$ for some β_2 and $\beta_1 < \beta_2$, then $c_{\min}(\beta_1) = c_l(\beta_1)$.*

We can now prove our second set of sufficient conditions for minimality exchange.

Theorem 3.6. *Suppose that $A(\beta)$ and $B(\beta)$ are each non-decreasing in β, and $A(\beta) - B(\beta)$ is non-increasing in β. If there is a non-degenerate solution β^* to the equation $A(\beta) = 2B(\beta)$, then there is a minimality exchange at $\beta = \beta^*$, with $c_{\min}(\beta) = c_l(\beta)$ for $\beta \leq \beta^*$ and $c_{\min}(\beta) = c_{nl}(\beta) > c_l(\beta)$ for $\beta > \beta^*$.*

Proof. Note first that $A(\beta) - 2B(\beta = [A(\beta) - B(\beta)] - B(\beta)$ is non-increasing in β, so since the graphs of $A(\cdot)$ and $2B(\cdot)$ intersect transversally at β^*, it follows that $A(\beta) > 2B(\beta)$ when $\beta < \beta^*$, whereas $A(\beta) < 2B(\beta)$ when $\beta > \beta^*$. Lemma 3.1 then implies that $c_{\min}(\beta) = c_{nl}(\beta)$ when $\beta > \beta^*$, whereas Corollary 3.5 implies that linear selection holds when $\beta < \beta^*$. □

Note that for the two concrete examples of minimality exchange discussed in Section 3, both Theorem 3.3 and Theorem 3.6 apply.

An example of a solvable problem for which Theorem 3.6 applies but Theorem 3.3 does not, is given by taking $A = 1$, $B = \beta/2$ and $h(u) = e^{2u}u(1-u)$, which is not concave. Then $L = 1.52218$, $c_l = \sqrt{4-2\beta}$, $c_{nl} = \sqrt{2/\beta}$, $c_l(\beta) = c_{nl}(\beta)$ at $\beta^* = 1$, and Theorem 3.6 ensures that there is minimality exchange at $\beta^* = 1$.

4 Conclusions

In this article we have focussed on a class of parameter-dependent monostable reaction-diffusion equations with explicit travelling-wave solutions and used this class to explore the phenomenon of minimality exchange, when the minimal wave speed switches from a linearly determined value to the speed of the explicitly determined front as a parameter changes. Two alternative sets of sufficient conditions for minimality exchange are proved, in Theorems 3.3 and 3.6. Why there should be such an exchange, not only from linear selection to nonlinear selection, but to nonlinear selection given by an *explicit* solution, is quite puzzling at first sight. Our framework here provides insight into why minimality exchange of this type occurs, and includes concrete examples from [2–4, 6]. The proofs draw on various tools for determining whether there is linear or nonlinear selection - in particular, ideas developed previously in the special case of an isotropic liquid-crystal model [2], as well as general results from [4, 6]. Some additional interesting material about minimal wave speeds is given in [3, Section 10.1.1], including Theorem 10.12, which provides sufficient criteria that can be used to identify cases when a given explicit solution has the minimal wave speed, and the examples that follow.

As suggested by the anonymous referee, instead of considering in (2.1) a nonlinearity parameterised by β, as was also done in [4, 6, 8] and in many examples in [3], our methods could have been used to treat a two-parameter system $f(u, A, B) = h(u)(A - Bh'(u))$ to map out domains of linear and nonlinear speed selection in the (A, B) plane.

We have treated one class of parameter-dependent solvable equations that includes important special cases, but clearly there are many further solvability results for explicit travelling-wave solutions in the literature. See, for instance, [3, Chapter 13] and [7]. In addition, the change of variables $G := 1/F$ converts (1.3) into an Abel equation, for which certain classes of explicit solutions can be found using tools such as the Chiellini integrability condition and the Lemke transformation (see, for example, [5] and the references therein). It would be interesting to expand and develop the approach introduced here to cover a larger range of explicit solutions to obtain further insight into the mechanisms for minimality exchange.

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Existence of periodic solutions of pendulum-like ordinary and functional differential equations

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. The equation

$$x''(t) = a(t, x(t)) + b(t, x) + d(t, x)e(x'(t))$$

is considered, where $a : \mathbb{R}^2 \to \mathbb{R}$, $b, d : \mathbb{R} \times C(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$, $e : \mathbb{R} \to \mathbb{R}$ are continuous, and a, b, d are T-periodic with respect to t. Using the Leray–Schauder degree theory we prove that a sign condition, in which a dominates b, is sufficient for the existence of a T-periodic solution. The main theorem is applied to the equation of the forced damped pendulum.

Keywords: Leray–Schauder degree, forced damped pendulum.

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1 Introduction

Second order differential equations of the type

$$x'' = h(t, x, x')$$

are basic models in mechanics: h is the resultant force acting on the system. When h is T-periodic with respect to t then it is an important problem to find conditions for the existence of T-periodic answer, T-periodic motions of the system. A simple model is the periodically forced damped mathematical pendulum

$$x'' + g(t, x, x') + a \sin x = e(t), \tag{1.1}$$

where e is T-periodic, g is T-periodic with respect to t and satisfies the following Nagumo-type condition: there exists a constant C such that every possible solution x of (1.1) satisfying $\sup_{[0,T]} |x| < 3\pi/2$ has the property $|x'(t)| < C$ $(t \in \mathbb{R})$. H. W. Knobloch [8] proved that if

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$\sup_{[0,T]} |e| < a$, then equation (1.1) has T-periodic solutions. J. Mawhin and M. Willem [10,12] extended this result to more general equations.

In the practice many important technical models connected with the pendulum are described by more general differential equations than (1.1). As particular cases we will consider in detail the mathematical pendulum with periodically vibrating suspension point and a functional differential equation model. The equations cannot be handled by Knobloch's or by Mawhin's and Willem's extensions. We extend the Leray–Schauder method for more general pendulum-like equations, i.e., differential equations containing a main part satisfying the same sign condition as the sine function in the pendulum equation but admitting also periodic perturbations.

In this paper we introduce a wide class of pendulum-like differential equations admitting a variety of perturbations including ordinary and functional terms even with unbounded delays. The proof of the existence of periodic solutions is based upon the Leray–Schauder continuation method [5,6,9,10].

2 The main theorem and its proof

For a fixed $T > 0$ we will use the standard notations:

$$C := \{\varphi : \mathbb{R} \to \mathbb{R} \mid \varphi \text{ is continuous}\};$$
$$C^1 := \{\psi : \mathbb{R} \to \mathbb{R} \mid \psi \text{ is continuously differentiable}\};$$
$$C_T := \{\varphi \in C : \varphi \text{ is } T\text{-periodic}\}, \qquad C_T^1 := \{\psi \in C^1 : \psi \text{ is } T\text{-periodic}\}.$$

If $\varphi \in C$ is bounded, $\psi \in C^1$, and ψ, ψ' are bounded on $\mathbb{R}$, then define

$$\|\varphi\|_0 := \sup_{t\in\mathbb{R}} |\varphi(t)|, \qquad \|\psi\|_1 := \max\left\{\sup_{t\in\mathbb{R}} |\psi(t)|;\ \sup_{t\in\mathbb{R}} |\psi'(t)|\right\}.$$

Consider the equation

$$x''(t) = a(t, x(t)) + b(t, x) + d(t, x)e(x'(t)), \tag{2.1}$$

where functions $a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $b, d : \mathbb{R} \times C \to \mathbb{R}$; $e : \mathbb{R} \to \mathbb{R}$ are continuous, and $e(0) = 0$. Moreover, we suppose that for every fixed $\overline{u} \in \mathbb{R}$, $\overline{\varphi} \in C$ functions $t \mapsto a(t, \overline{u}), b(t, \overline{\varphi}), d(t, \overline{\varphi})$ are T-periodic.

Functions a, b, d, e generate the following operators:

$$\begin{aligned}
A &: C \to C, && \varphi \mapsto A\varphi, && (A\varphi)(t) := a(t, \varphi(t));\\
B &: C \to C, && \varphi \mapsto B\varphi, && (B\varphi)(t) := b(t, \varphi);\\
D &: C \to C, && \varphi \mapsto D\varphi, && (D\varphi)(t) := d(t, \varphi);\\
D_e &: C^1 \to C, && \psi \mapsto D_e\psi, && (D_e\psi)(t) := d(t, \psi)e(\psi'(t)).
\end{aligned}$$

For $R > 0$, $S > 0$ given we define the subset

$$C_T(-R, S) := \{\varphi \in C_T : -R \le \varphi(t) \le S \ (t \in \mathbb{R})\}.$$

By the use of the notations $f : \mathbb{R} \times C^1 \to \mathbb{R}$, $F : C^1 \to C$,

$$f(t, \psi) := a(t, \psi(t)) + b(t, \psi) + d(t, \psi)e(\psi'(t)),$$
$$F\psi := f(\cdot, \psi) = a(\cdot, \psi(\cdot)) + b(\cdot, \psi) + d(\cdot, \psi)e(\psi'(\cdot)) = A\psi + B\psi + D_e\psi$$

equation (2.1) can be rewritten in the shortened form

$$x''(t) = f(t,x) = Fx(t). \tag{2.2}$$

Theorem 2.1. *Suppose that there exist positive constants R, S and a continuous nondecreasing function $\phi : (0,\infty) \to (0,\infty)$ such that*

(i)
$$a(t,S) > \sup\{|b(t,\varphi)| : \varphi \in C_T(-R,S)\} =: \beta_{-R,S}(t),$$
$$a(t,-R) < -\beta_{-R,S}(t) \qquad (t \in \mathbb{R});$$

(ii) operators B and D map bounded sets of C_T into bounded sets of C_T;

(iii)
$$\int_1^\infty \frac{u}{\phi(u)}\,\mathrm{d}u = \infty, \qquad |e(u)| \le \phi(|u|) \qquad (u \in \mathbb{R})$$

hold.

Then there exists a T-periodic solution $x \in C_T(-R,S)$ of (2.1).

Proof. We use the Leray–Schauder degree for completely continuous perturbation of the identity operator [5, 6, 9, 10, 13]. We suppose that the reader is familiar with the definition of the Brouwer degree and the Leray–Schauder degree and their most basic properties (see, e.g., [4]).

Now we sketch the main steps of the proof. We find an open bounded set $\Omega \subset C_T^1$ and a family of mappings $M_\lambda : \overline{\Omega} \to C_T^1$ ($\lambda \in [0,1]$) having the following properties:

(a) if x is a fixed point of M_1 in Ω, then x is the desired periodic solution of (2.1), i.e., $x \in C_T(-R,S)$, and x is a solution of (2.1);

(b) the function
$$M^* : \overline{\Omega} \times [0,1] \to C_T^1, \quad M^*(\psi,\lambda) = M_\lambda \psi$$
is completely continuous;

(c) if $\varphi \in \partial\Omega$ and $\lambda \in [0,1]$, then $\varphi \ne M_\lambda \varphi$;

(d) if $I : C \to C$ is the identity operator and $d[I - M_\lambda, \Omega, 0]$ denotes the Leray–Schauder degree of M_λ with respect to Ω, then $d[I - M_0, \Omega, 0] \ne 0$.

Then an application of basic theorems of the theory of the Leray–Schauder degree yields the assertion of the theorem.

For the definition of $\Omega \subset C_T^1$ we need a Nagumo-type result [13] for the family of equations

$$x''(t) = \lambda f(t,x) \qquad (\lambda \in [0,1]) \tag{2.3}$$

associated with (2.2).

Lemma 2.2. *Suppose that conditions (i)–(iii) in Theorem 2.1 are satisfied. Then there is a $K > 1$ such that for any $\lambda \in [0,1]$ and for an arbitrary solution $x \in C_T(-R,S)$ of (2.3) the inequality*

$$|x'(t)| \le K - 1 \qquad (t \in \mathbb{R})$$

holds.

Proof. Consider an arbitrary solution $x \in C_T(-R,S)$ of (2.1). By conditions (ii) and (iii) there exist constants K_1 and K_2 independent of $\lambda \in [0,1]$ and the solution x such that

$$|x''(t)| \le \max\{|a(s,u)| : 0 \le s \le T, -R \le u \le S\} + K_1 + K_2\phi(|x'(t)|) \qquad (0 \le t \le T).$$

Let us define

$$\tilde{\phi}(v) := K_1 + K_2\phi(v) \qquad (v > 0).$$

Then

$$\frac{v}{\tilde{\phi}(v)} \ge \frac{1}{2K_2}\frac{v}{\phi(v)},$$

provided that $\phi(v) \ge K_1/K_2$. The Nagumo–Hartman Lemma [7, Lemma XII. 5.1] and condition (iii) of the theorem imply the existence of the desired K. □

Now we can define the basic set Ω and the homotopy mapping M_λ for the Leray–Schauder degree. Let K be the constant associated with R, S by Lemma 2.2 and consider the set

$$\Omega := \Omega_{R,S,K} := \left\{\psi \in C_T^1 : -R < \psi(t) < S, |\psi'(t)| < K \quad (t \in [0,T])\right\}. \tag{2.4}$$

This set is open and bounded in C_T^1.

To define the family of mappings $M_\lambda : \overline{\Omega} \to C_T^1$ $(\lambda \in [0,1])$ we need further notation. The mean value operator $P : C_T \to C_T$ is defined by

$$(P\varphi)(t) := \frac{1}{T}\int_0^T \varphi(t)\,\mathrm{d}t \qquad (\varphi \in C_T).$$

Introduce the subspace $C_{T,I-P} := \{\varphi \in C_T : P\varphi = 0\}$ and the operator of the primitivation $H : C_{T,I-P} \to C_{T,I-P} \cap C_T^1$ by

$$(H\varphi)(t) := \int_0^t \varphi(s)\,\mathrm{d}s - \frac{1}{T}\int_0^T \left(\int_0^t \varphi(s)\,\mathrm{d}s\right)\mathrm{d}t.$$

It is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(H\left(I-P\right)\varphi\right)(t) = \varphi(t) - P\varphi \qquad (\varphi \in C_T). \tag{2.5}$$

Now for $\lambda \in [0,1]$ we define the mapping:

$$M_\lambda : \overline{\Omega} \to C_T^1, \qquad M_\lambda\psi := M^*(\psi,\lambda), \tag{2.6}$$

where

$$M^* : C_T^1 \times [0,1] \to C_T^1, \qquad M^*(\psi,\lambda) := P\psi - PF\psi + \lambda H^2(I-P)F\psi. \tag{2.7}$$

Property (a) is a consequence of the following lemma.

Lemma 2.3. *For $\lambda \in (0,1]$ a function $\psi \in C_T^1$ is a fixed point of M_λ, i.e., $\psi = M_\lambda\psi$ if and only if ψ is a T-periodic solution of* (2.3).

Function $\psi \in C_T^1$ is a fixed point of M_0 if and only if

$$\psi = P\psi \quad \text{and} \quad PFP\psi = 0. \tag{2.8}$$

Proof. Suppose that $\lambda \in (0,1]$ is fixed, and $\psi \in C_T^1$ is a fixed point of M_λ:

$$\psi = P\psi - PF\psi + \lambda H^2(I-P)F\psi. \tag{2.9}$$

Applying functional P to both sides we get $PF\psi = 0$. By (2.9) ψ is two times differentiable and we obtain $\psi''(t) = \lambda f(t,\psi)$ ($t \in \mathbb{R}$), which means that ψ is a solution of (2.3).

On the other hand, if ψ is a T-periodic solution of (2.3) then

$$P\psi'' = \frac{1}{T}\int_0^T \psi''(t)\,\mathrm{d}t = \frac{1}{T}\big(\psi'(T) - \psi'(0)\big) = 0,$$

consequently $PF\psi = 0$, and we can write

$$\psi''(t) = \lambda\{f(t,\psi) - PF\psi\}.$$

Integrating this equality we obtain

$$\psi'(t) = \psi'(0) + \lambda\int_0^t (f(s,\psi) - PF\psi)\,\mathrm{d}s,$$

which, together with the definition of H, gives

$$\psi' = \psi'(0) + \frac{\lambda}{T}\int_0^T \left(\int_0^t (f(s,\psi) - PF\psi)\,\mathrm{d}s\right)\mathrm{d}t + \lambda H(I-P)F\psi.$$

Apply functional P to both sides of this equality. Since $P\psi' = 0$ we have

$$\psi'(0) + \frac{\lambda}{T}\int_0^T \left(\int_0^t (f(s,\psi) - PF\psi)\,\mathrm{d}s\right)\mathrm{d}t = 0,$$

therefore $\psi' = \lambda H(I-P)F\psi$. Integration yields

$$\psi = \text{const.} + \lambda H^2(I-P)F\psi.$$

From the definition of H there follows const. $= P\psi$, which, together with $PF\psi = 0$, shows that ψ is a fixed point of M_λ, i.e., (2.9) holds.

Now we turn to the proof of the second statement of the lemma concerning the case $\lambda = 0$. Suppose that $\psi \in C_T^1$ is a fixed point of $M_0 = P - PF$, i.e.,

$$\psi = P\psi - PF\psi. \tag{2.10}$$

Obviously, $\psi = P\psi$ and, consequently, (2.8) holds.

On the other hand, if (2.8) holds, then

$$\psi = P\psi = P\psi + PFP\psi = P\psi + PF\psi = M_0\psi.$$

In other words, ψ is a fixed point of M_0. □

Step (b) is contained in the following lemma.

Lemma 2.4. *Under the conditions of Theorem 2.1 function M^* is completely continuous on the set $\overline{\Omega} \times [0,1]$, provided that the norm $|||\cdot|||$ in $\overline{\Omega} \times [0,1]$ is defined by*

$$|||(\psi,\lambda)||| := \|\psi\|_1 + |\lambda| \qquad ((\psi,\lambda) \in \overline{\Omega} \times [0,1]).$$

Proof. The continuity of M^* follows from the conditions on a, b, d, e. In fact, to this property it is enough to prove the continuity of $F : C_T^1 \to C_T$. Obviously, $A, B, D : C_T^1 \to C_T$ are continuous. For $D_e : C_T^1 \to C_T$, let us fix a $\overline{\psi} \in C_T^1$ and consider the sets

$$Q := \left\{ \psi \in C_T^1 : \|\psi - \overline{\psi}\|_1 \le 1 \right\} \subset C_T^1,$$
$$Q_1 := \left\{ v \in \mathbb{R} : \min_{[0,T]} \overline{\psi}'(t) - 1 \le v \le \max_{[0,T]} \overline{\psi}'(t) + 1 \right\} \subset \mathbb{R}.$$

There are constants K_0, K_1 such that

$$|d(t,\psi)| \le K_0 \quad \text{if } \|\psi - \overline{\psi}\|_1 \le 1,\ 0 \le t \le T,$$
$$|e(\overline{\psi}'(t))| \le K_1 \quad \text{if } 0 \le t \le T.$$

Let $\varepsilon > 0$ be arbitrary. Function e is uniformly continuous on Q_1, and D is continuous at $\overline{\psi}$. Therefore there is a δ $(0 < \delta < 1)$ such that$\|\psi - \overline{\psi}\|_1 < \delta$ and $v_1, v_2 \in Q_1$, $|v_1 - v_2| < \delta$ imply

$$\|D\psi - D\overline{\psi}\|_0 < \frac{\varepsilon}{2K_1}, \qquad |e(v_1) - e(v_2)| < \frac{\varepsilon}{2K_0}.$$

If $\|\psi - \overline{\psi}\|_1 < \delta$, then

$$\begin{aligned} &|d(t,\psi)e(\psi'(t)) - d(t,\overline{\psi})e(\overline{\psi}'(t))| \\ &\quad \le |d(t,\psi)||e(\psi'(t)) - e(\overline{\psi}'(t))| + |d(t,\psi) - d(t,\overline{\psi})||e(\overline{\psi}'(t))| \\ &\quad \le K_0 \frac{\varepsilon}{2K_0} + K_1 \frac{\varepsilon}{2K_1} = \varepsilon, \end{aligned}$$

i.e., D_e is continuous.

Finally, we prove that M^* maps $\overline{\Omega} \times [0,1]$ into a precompact set in C^1. It is easy to see that $\|H\varphi\|_1 \le (2T+1)\|\varphi\|_0$ $(\varphi \in C_{T,I-P})$. Continuity of a, e and condition (ii) in Theorem 2.1 imply the existence of K_2, K_3 such that

$$|||(\psi,\lambda)||| \le K_2, \qquad \|F\psi\|_0 \le K_3 \qquad ((\psi,\lambda) \in \overline{\Omega} \times [0,1]).$$

Therefore

$$\begin{aligned} \|M^*(\psi,\lambda)\|_0 &\le \|\psi\|_0 + \|F\psi\|_0 + 2(2T+1)^2\|F\psi\|_0 \\ &\le K_2 + (1 + 2(2T+1)^2)K_3 \qquad ((\psi,\lambda) \in \overline{\Omega} \times [0,1]). \end{aligned}$$

On the other hand,

$$\begin{aligned} \|M^*(\psi,\lambda)'\|_0 &\le \|\lambda H(I-P)F\psi\|_0 \\ &\le 2(2T+1)\|F\psi\|_0 \le 2(2T+1)K_3, \\ \|M^*(\psi,\lambda)''\|_0 &\le \|\lambda(F\psi - PF\psi)\|_0 \\ &\le 2\|F\psi\|_0 \le K_3 \qquad ((\psi,\lambda) \in \overline{\Omega} \times [0,1]), \end{aligned}$$

consequently the elements of $M^*(\overline{\Omega} \times [0,1]) \subset C_T^1$ are uniformly bounded and equicontinuous. By the Arzelà–Ascoli Theorem [7, Selection Theorem I.2.3] $M^*(\overline{\Omega} \times [0,1])$ is precompact. □

In general, step (c) is the biggest challenge in proofs of Leray–Schauder type; it depends most strongly on the specialities of the differential equation.

Lemma 2.5. *Under the conditions of Theorem 2.1, if $\psi \in \overline{\Omega}$ is a fixed point of M_λ for some $\lambda \in [0,1]$, then $\psi \notin \partial\Omega$.*

Proof. Suppose that the statement is not true, i.e., $\psi \in \partial\Omega$. If $\lambda \in (0,1]$, then by Lemma 2.3 ψ is a solution of (2.3). According to Lemma 2.2 there exists at least one $\tau \in [0,T)$ such that the function $t \mapsto r(t) := \psi^2(t)$ $(t \in \mathbb{R})$ has a total maximum at $t = \tau$, therefore $r'(\tau) = \psi'(\tau) = 0$, $r''(\tau) \le 0$, and either $\psi(\tau) = S$ or $\psi(\tau) = -R$. Condition (i) implies that either

$$\begin{aligned} r''(\tau) &= 2\psi(\tau)\psi''(\tau) = 2\lambda\psi(\tau)\{a(\tau,\psi(\tau)) + b(\tau,\psi)\} \\ &\ge 2\lambda|\psi(\tau)|\{a(\tau,\psi(\tau))\operatorname{sign}(\psi(\tau)) - \beta_{-R,S}(\tau)\} \\ &= 2\lambda S\{a(\tau,S) - \beta_{-R,S}(\tau)\} > 0, \end{aligned} \tag{2.11}$$

or

$$r''(\tau) \ge 2\lambda R\{-a(\tau,-R) - \beta_{-R,S}(\tau)\} > 0. \tag{2.12}$$

Both of them contradict $r''(\tau) \le 0$.

If $\lambda = 0$, then from (2.8) we know that $\psi(t) \equiv \psi_0 = \text{const.}$ and

$$m(\psi_0) := \frac{1}{T}\int_0^T (a(t,\psi_0) + b(t,\psi_0))\,\mathrm{d}t = 0. \tag{2.13}$$

On the other hand, we also know that either $\psi_0 = S$ or $\psi_0 = -R$. In the first case from condition (i) we get

$$|a(t,\psi_0) + b(t,\psi_0)| > a(t,S) - \beta_{-R,S}(t) > 0 \qquad (t \in \mathbb{R}), \tag{2.14}$$

which contradicts (2.13). The second case is similar. □

Lemma 2.6. *Under conditions of Theorem 2.1,*

$$d[I - M_0, \Omega, 0] = d[m, (-R,S), 0], \tag{2.15}$$

and the Brower degree on the right-hand side is equal to 1.

Proof. (2.15) is a consequence of (2.8). By virtue of condition (i) we have

$$\begin{aligned} m(-R) &= \frac{1}{T}\int_0^T (a(t,-R) + b(t,-R))\,\mathrm{d}t \\ &< \frac{1}{T}\int_0^T (a(t,-R) + \beta_{-R,S}(t))\,\mathrm{d}t < 0, \end{aligned}$$

$$\begin{aligned} m(S) &= \frac{1}{T}\int_0^T (a(t,S) + b(t,S))\,\mathrm{d}t \\ &< \frac{1}{T}\int_0^T (a(t,S) - \beta_{-R,S}(t))\,\mathrm{d}t > 0. \end{aligned}$$

But $d[m,(-R,S),0]$ depends only on $m(-R)$ and $m(S)$, and for the linear function connecting $m(-R)$ and $m(S)$ the degree is equal to 1, so $d[m,(-R,S),0] = 1$. □

Lemmas 2.3–2.4–2.5 make it possible to apply the theorem of invariance of the Leray–Schauder degree with respect to homotopy to the mapping M^* defined by (2.7), consequently

$$d[I - M_1, \Omega, 0] = d[m, (-R,S), 0] = 1.$$

On the basis of the Kronecker Existence Theorem [13] and Lemma 2.3 this means that (2.1) has a T-periodic solution $x \in C_T(-R,S)$. □

3 Applications

3.1 The forced mathematical pendulum with vibrating suspension point

The mathematical pendulum is one of the most important model equations in the nonlinear mechanics (see, e.g., [2]). When it is under the action of an outer periodic force then its motions are described by the equation

$$\varphi'' + \frac{g}{l}\sin\varphi = q(t) \tag{3.1}$$

where φ denotes the angle between the direction vertically downward and the rod of the pendulum measured anticlockwise, l is the length of the rod, g denotes the constant of gravity, and $q : \mathbb{R} \to \mathbb{R}$ is a T-periodic continuous function. A great number of papers have been devoted to the problem of finding T-periodic solutions of the equation (see an excellent history and literature in [11]). H. W. Knobloch [8], using the degree theory and taking also some damping, proved that the equation

$$\varphi'' + |\varphi'|\varphi' + \frac{g}{l}\sin\varphi = q(t) \tag{3.2}$$

has at least one T-periodic solution, provided that

$$\|q\|_\infty := \max_{[0,T]} |q(t)| < \frac{g}{l}. \tag{3.3}$$

Using the same technique, J. Mawhin and M. Willem [12] could guarantee multiple periodic solutions.

In the technical practice it often happens that the suspension point of the rod is vibrating in the plane of the motions of the pendulum. Consider now the case of the vibration

$$x_0(t) = Ue_1\cos\omega t, \qquad y_0(t) = Ue_2\sin\omega t \qquad (t \in \mathbb{R}),$$

where the x-axis is directed vertically downward, $U > 0$ is the amplitude, $\omega := m\pi/T$ is the frequency of the vibration; $m \in \mathbb{N}$ and the unit vector $(e_1, e_2) \in \mathbb{R}^2$ are fixed. It can be seen that Lagrange's equation of motion of the second kind has the form

$$\begin{aligned}\varphi'' &- \frac{U}{l}\omega\sin\omega t(e_1\cos\varphi + e_2\sin\varphi)\varphi' \\ &+ \left(\frac{g}{l} + \frac{U}{l}\omega^2 e_1\cos\omega t\right)\sin\varphi - \frac{U}{l}\omega^2 e_2\cos\omega t\cos\varphi \\ &= b_1(t,\varphi) - d(t,\varphi)e(\varphi').\end{aligned} \tag{3.4}$$

Here the force function $b_1 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous, the function $b_1(\cdot, u)$ is T-periodic, $d : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $e : \mathbb{R} \to \mathbb{R}$ are continuous, $d(\cdot,\varphi)$ is T-periodic, and $e(0) = 0$. Introduce the notation

$$V := \max\left\{|b_1(t,u)| : 0 \le t \le T, \frac{\pi}{2} \le u \le \frac{3\pi}{2}\right\}.$$

Corollary 3.1. *Suppose that there exists a continuous function $\phi : (0,\infty) \to (0,\infty)$ ($\phi(r) \ge r$) such that the condition (iii) in Theorem 2.1 is satisfied. If*

$$U\omega^2 + Vl < g, \tag{3.5}$$

then equation (3.4) has a T-periodic solution φ such that $\pi/2 \le \varphi(t) \le 3\pi/2$ ($t \in \mathbb{R}$).

Proof. In the new variable $\theta := \varphi - \pi$ equation (3.4) has the form

$$\begin{aligned}\theta'' = & -\frac{U}{l}\omega \sin \omega t(e_1 \cos\theta + e_2 \sin\theta)\theta' + \left(\frac{g}{l} + \frac{U}{l}\omega^2 e_1 \cos\omega t\right)\sin\theta \\ & -\frac{U}{l}\omega^2 e_2 \cos\omega t \cos\theta + b_1(t, \theta+\pi) - d(t,\theta+\pi)e(\theta').\end{aligned} \tag{3.6}$$

There are constants c_1, c_2 such that

$$\left|\frac{U}{l}\omega \sin\omega t(e_1\cos\theta + e_2\sin\theta)\theta'\right| + |d(t,\theta+\pi)e(\theta')| \le c_1|\theta'| + c_2\phi(|\theta'|) \le (c_1+c_2)\phi(|\theta'|),$$

so condition (iii) in Theorem 2.1 is satisfied. We can choose $a(t,u) := (g/l)\sin u$, $R := \pi/2$, $S := 3\pi/2$. Then $\beta_{-R,S}(t) \equiv V$ and we apply Theorem 2.1 to equation (3.6) to get the corollary. □

Condition (3.5) can be considered as a generalization of (3.3) to (3.6). In Knobloch's special case (3.5) gives (3.3).

3.2 A second order integro-differential equation with unbounded delay

Consider the equation

$$x''(t) = a(t, x(t)) + \int_{-\infty}^{\infty} k(t,s)x(s)\,\mathrm{d}s + d_1(t, x_t)e(x'(t)) + p(t), \quad (t \in \mathbb{R}) \tag{3.7}$$

where $k : \mathbb{R}^2 \to \mathbb{R}$ is continuous, $k(t+T, s+T) \equiv k(t,s)$ $(t, s \in \mathbb{R})$, $d_1 : \mathbb{R} \times C((-\infty, 0]; \mathbb{R}) \to \mathbb{R}$ is continuous, $d_1(t+T, \chi) \equiv d_1(t,\chi)$ $(\chi \in C((-\infty,0];\mathbb{R}))$, $p : \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic. We used the standard notation $x_t(\tau) := x(t+\tau)$ $(t \in \mathbb{R}, \tau \le 0)$.

Equation (3.7) can be considered as a perturbation of the pendulum equation (3.1). As we will see in the following corollary, function sin will be replaced by a function a satisfying a sign condition like the sine function and dominating the other terms in the equation. By example of (3.7) we would like to illuminate that our main result Theorem 2.1 is robust in the sense that it makes possible a variety of applications where different types of equations appear such as functional differential equations even with unbounded delays. Actually, such equations can occur among others in mechanics (see, e.g., [1, 4.3. Examples]) and population dynamics [3].

The following corollary is a direct consequence of Theorem 2.1.

Corollary 3.2. *Suppose that there exists a continuous function $\phi : (0,\infty) \to (0,\infty)$ such that the condition (iii) in Theorem 2.1 is satisfied. If there are positive constants R, S such that*

$$\begin{aligned} a(t,S) &> \max\{R,S\}\int_{-\infty}^{\infty}|k(t,s)|\,\mathrm{d}s + \|p\|_0 =: \beta_{-R,S}(t), \\ a(t,-R) &< -\beta_{-R,S}(t) \qquad (t \in \mathbb{R}),\end{aligned} \tag{3.8}$$

and d_1 transforms every bounded set contained in $\mathbb{R} \times C((0,\infty]; \mathbb{R})$ into a bounded set of $\mathbb{R}$, then there exists a T-periodic solution $x \in C_T(-R, S)$ of (3.7).

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*

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Fisher–Kolmogorov type perturbations of the mean curvature operator in Minkowski space

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. We provide a complete description of the existence/non-existence and multiplicity of distinct pairs of nontrivial solutions to the problem with Minkowski operator

$$-\text{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \lambda u(1-a|u|^q) \quad \text{in } \Omega, \;\; u|_{\partial\Omega}=0, \quad (a \geq 0 < q),$$

when $\lambda \in (0,\infty)$, in terms of the spectrum of the classical Laplacian. Beforehand, we obtain multiplicity of solutions for parameterized and non-parameterized Dirichlet problems involving odd perturbations of this operator. The approach relies on critical point theory for convex, lower semicontinuous perturbations of C^1-functionals.

Keywords: Minkowski operator, Fisher–Kolmogorov nonlinearities, Krasnoselskii's genus, critical point.

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1 Introduction and preliminaries

In this paper we deal with the Dirichlet boundary value problem

$$\begin{cases} -\mathcal{M}(u) = \lambda g(u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with boundary $\partial\Omega$ of class C^2, $\lambda > 0$ is a real parameter, $g : \mathbb{R} \to \mathbb{R}$ is an odd continuous function and $\mathcal{M}$ stands for the mean curvature operator in Minkowski space:

$$\mathcal{M}(u) = \text{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right).$$

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Problems involving the operator $\mathcal{M}$ are originated in differential geometry and relativity. These are related to maximal and constant mean curvature spacelike hypersurfaces (spacelike submanifolds of codimension one in the flat Minkowski space $\mathbb{L}^{N+1} := \{(x,t) : x \in \mathbb{R}^N,\ t \in \mathbb{R}\}$ endowed with the Lorentzian metric $\sum_{j=1}^{N}(dx_j)^2 - (dt)^2$, where (x,t) are the canonical coordinates in $\mathbb{R}^{N+1}$) having the property that the trace of the extrinsic curvature is zero, respectively, constant. On the other hand, assuming that a spacelike hypersurface in $\mathbb{L}^{N+1}$ is the graph of a smooth function $u : \Omega \to \mathbb{R}$ with Ω a domain in $\{(x,t) : x \in \mathbb{R}^N, t = 0\} \simeq \mathbb{R}^N$, the (strictly) spacelike condition implies $|\nabla u| < 1$ and u satisfies an equation of type

$$\mathcal{M}(u) = H(x,u) \quad \text{in } \Omega,$$

where H is a prescribed mean curvature function. If H is continuous and bounded, it has been shown in [4] that the above equation subjected to a Dirichlet condition has at least one solution. More recently, the existence of additional solutions, such as of mountain pass type, was obtained in [5,6] and the existence of Filippov type solutions for discontinuous Dirichlet problems involving the operator $\mathcal{M}$ was established in [7]. For other recent developments of the subject, we refer the reader to [2,3,9–11,15,16] and the references therein.

As in [10], by a *solution* of (1.1) we mean a function $u \in C^{0,1}(\overline{\Omega})$, such that $\|\nabla u\|_\infty < 1$, which vanishes on $\partial\Omega$ and satisfies

$$\int_\Omega \frac{\nabla u \cdot \nabla w}{\sqrt{1 - |\nabla u|^2}}\, dx = \lambda \int_\Omega g(u) w \, dx, \tag{1.2}$$

for every $w \in W_0^{1,1}(\Omega)$. Here and below, $\|\cdot\|_\infty$ stands for the usual sup-norm on $L^\infty(\Omega)$. As shown in [10, Remark 2], if u is a solution of (1.1), in the sense of the previous definition, then $u \in W^{2,r}(\Omega)$ for all finite $r \geq 1$ and satisfies the equation a.e. in Ω. Reciprocally, since, for $p > N$, one has

$$W^{2,p}(\Omega) \subset C^1(\overline{\Omega}) \subset W^{1,\infty}(\Omega) = C^{0,1}(\overline{\Omega}),$$

it is straightforward to check that if a function $u \in W^{2,p}(\Omega)$ for some $p > N$, with $\|\nabla u\|_\infty < 1$ satisfies the equation a.e. in Ω and vanishes on $\partial\Omega$, then it is a solution of (1.1).

This study is mainly motivated by the result obtained in [17] concerning the multiplicity of T-periodic solutions for the equation with relativistic operator:

$$-\left(\frac{u'}{\sqrt{1 - |u'|^2}}\right)' = \lambda g_1(u) \quad \text{in } [0,T]; \tag{1.3}$$

by g_a we denote the Fisher–Kolmogorov type nonlinearity $g_a(t) = t(1 - a|t|^q)$, $\forall\, t \in \mathbb{R}$ ($a \geq 0 < q$). This type of nonlinearities was originally motivated by models in biological population dynamics and led to the reaction-diffusion equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = u(1 - u^2),$$

referred to as *the classical Fisher–Kolmogorov equation* [12,13,18]. Also, higher-order equations of type

$$u^{iv} - pu'' = u(q(t) - r(t)u^2), \qquad \text{(with } q, r \text{ positive functions)}$$

which corresponds, if $p > 0$, to *the extended Fisher–Kolmogorov equations* are models for phase transitions and other bistable phenomena (see e.g. [8,20–23,27]). So, in [17, Theorem 2.1] it is shown that if $\lambda > 4\pi^2 m^3/T^2$ for some $m \geq 2$, then equation (1.3) subjected to periodic boundary conditions has at least $m-1$ distinct pairs of non-constant solutions. By comparison, in the case of the Dirichlet problem for the parametrized equation

$$-\mathcal{M}(u) = \lambda g_a(u) \quad \text{in } \Omega,$$

we obtain (see Theorem 2.5) a complete description of the existence/non-existence and multiplicity of distinct pairs of nontrivial solutions when $\lambda \in (0, \infty)$, in terms of the eigenvalues of the classical $-\Delta$. It is worth to point out that the multiplicity part of the result relies on a Clark type theorem for the general problem (1.1) (see Theorem 2.2). Moreover, this theorem enables us to derive existence of finitely or infinitely many solutions to Dirichlet problems for non-parametrized equations having the form

$$-\mathcal{M}(u) = f(u) \quad \text{in } \Omega,$$

with odd continuous $f : \mathbb{R} \to \mathbb{R}$, by controlling the asymptotic behavior of the primitive of f near the origin (see Corollary 2.3).

We conclude this introductory part by briefly recalling some notions and results in the frame of Szulkin's critical point theory [26], which will be needed in the sequel. Let $(Y, \|\cdot\|)$ be a real Banach space and $\mathcal{I} : Y \to (-\infty, +\infty]$ be a functional of the type

$$\mathcal{I} = \mathcal{F} + \psi, \tag{1.4}$$

where $\mathcal{F} \in C^1(Y, \mathbb{R})$ and $\psi : Y \to (-\infty, +\infty]$ is convex, lower semicontinuous and proper (i.e., $D(\psi) := \{u \in Y : \psi(u) < +\infty\} \neq \emptyset$). A point $u \in Y$ is said to be *a critical point* of $\mathcal{I}$ if $u \in D(\psi)$ and if it satisfies the inequality

$$\langle \mathcal{F}'(u), v - u \rangle + \psi(v) - \psi(u) \geq 0 \quad \forall\, v \in D(\psi).$$

It is straightforward to see that each local minimum of $\mathcal{I}$ is necessarily a critical point of $\mathcal{I}$ [26, Proposition 1.1]. A sequence $\{u_n\} \subset D(\psi)$ is called a (PS)-*sequence* if $\mathcal{I}(u_n) \to c \in \mathbb{R}$ and

$$\langle \mathcal{F}'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - u_n\| \quad \forall\, v \in D(\psi),$$

where $\varepsilon_n \to 0$. The functional $\mathcal{I}$ is said *to satisfy the* (PS) *condition* if any (PS)-sequence has a convergent subsequence in Y.

Let Σ be the collection of all symmetric subsets of $Y \setminus \{0\}$ which are closed in Y. The *genus* (Krasnoselskii) of a nonempty set $A \in \Sigma$ is defined as being the smallest integer k with the property that there exists an odd continuous mapping $h : A \to \mathbb{R}^k \setminus \{0\}$; in this case we write $\gamma(A) = k$. If such an integer does not exist, $\gamma(A) = +\infty$. Also, if $A \in \Sigma$ is homeomorphic to S^{k-1} ($k-1$ dimension unit sphere in the Euclidean space $\mathbb{R}^k$) by an odd homeomorphism, then $\gamma(A) = k$ (see e.g. [25, Corollary 5.5]). For properties and more details of the notion of genus we refer the reader to [24, 25]. Denoting by $\Gamma \subset 2^Y$ the collection of all nonempty compact symmetric subsets of Y, considered with the Hausdorff–Pompeiu distance, we set

$$\Gamma_j := \mathrm{cl}\{A \in \Gamma : \; 0 \notin A, \; \gamma(A) \geq j\}.$$

The following is an immediate consequence of [26, Theorem 4.3].

Theorem 1.1. *Let $\mathcal{I}$ be of type* (1.4) *with $\mathcal{F}$ and ψ even. Also, suppose that $\mathcal{I}$ is bounded from below, satisfies the (PS) condition and $\mathcal{I}(0) = 0$. If*

$$\inf_{A \in \Gamma_m} \sup_{v \in A} \mathcal{I}(v) < 0,$$

then the functional $\mathcal{I}$ has at least m distinct pairs of nontrivial critical points.

2 Main results

Using the ideas from [5], we introduce the variational formulation for problem (1.1). Accordingly, let

$$K_0 := \{u \in W^{1,\infty}(\Omega) : \|\nabla u\|_\infty \le 1,\ u|_{\partial\Omega} = 0\}.$$

The convex set K_0 is compact in $C(\overline{\Omega})$ [5, Lemma 2.2]. The functional $\Psi : C(\overline{\Omega}) \to (-\infty, +\infty]$ defined by

$$\Psi(u) = \begin{cases} \int_\Omega [1 - \sqrt{1 - |\nabla u|^2}]dx, & \text{for } u \in K_0, \\ +\infty, & \text{for } u \in C(\overline{\Omega}) \setminus K_0 \end{cases}$$

is convex and lower semicontinuous [5, Lemma 2.4]. Also, it is easy to see that

$$\Psi(u) \le \int_\Omega |\nabla u|^2, \quad \forall\, u \in K_0. \tag{2.1}$$

Let the C^1-functional $\mathcal{G}_\lambda : C(\overline{\Omega}) \to \mathbb{R}$ be given by

$$\mathcal{G}_\lambda(u) = -\lambda \int_\Omega G(u)dx,$$

where

$$G(t) = \int_0^t g(\tau)d\tau.$$

Then, the energy functional $I_\lambda : C(\overline{\Omega}) \to (-\infty, +\infty]$ associated to problem (1.1) is

$$I_\lambda = \Psi + \mathcal{G}_\lambda$$

and it has the structure required by Szulkin's critical point theory. Also, by the compactness of $K_0 \subset C(\overline{\Omega})$ it is easy to see that I_λ satisfies the (PS) condition.

From [5, Theorem 2.1], one has the following:

Proposition 2.1. *If a function $u_\lambda \in C(\overline{\Omega})$ is a critical point of I_λ, then it is a solution of problem* (1.1). *Moreover, I_λ is bounded from below and attains its infimum at some $u_\lambda \in K_0$, which is a critical point of I_λ and hence, a solution of* (1.1).

We briefly recall some classical spectral aspects of the operator $-\Delta$ in the Sobolev space $H_0^1(\Omega)$ - which is seen as being endowed with the usual scalar product

$$(u, v)_1 = \int_\Omega \nabla u \cdot \nabla v\, dx, \qquad \text{for all } u, v \in H_0^1(\Omega).$$

A real number $\lambda^\Delta \in \mathbb{R}$ is called an *eigenvalue* of $-\Delta$ in $H_0^1(\Omega)$, if problem

$$\begin{cases} -\Delta u = \lambda^\Delta u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

has a nontrivial weak solution φ, i.e. there exists $\varphi \in H_0^1(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \nabla\varphi \cdot \nabla v\, dx = \lambda^\Delta \int_\Omega \varphi v\, dx, \quad \text{for all } v \in H_0^1(\Omega).$$

The solution φ is called *eigenfunction* corresponding to the eigenvalue λ^Δ. It is known that there exists a sequence of eigenvalues $0 < \lambda_1^\Delta < \lambda_2^\Delta \le \cdots \le \lambda_j^\Delta \le \cdots$ (going to $+\infty$) and a sequence of corresponding eigenfunctions $\{\varphi_j\}_{j\in\mathbb{N}}$ defining an orthonormal basis of $H_0^1(\Omega)$. Also, since $\partial\Omega$ is of class C^2 one has that each eigenfunction φ_j belongs to $H^2(\Omega)$ and by a bootstrap argument combining a standard regularity result [14, Theorem 9.15] and the Sobolev embedding theorem [1, Theorem 4.12] we get that φ_j actually belongs to $W^{2,p}(\Omega)$ with some $p > N$. Therefore, φ_j belongs to $C^1(\overline{\Omega})$ and hence $|\nabla\varphi_j| \in C(\overline{\Omega})$ for all $j \in \mathbb{N}$.

Theorem 2.2. *If $\lambda > 2\lambda_m^\Delta$ for some $m \in \mathbb{N}$ and*

$$\liminf_{t\to 0+} \frac{2G(t)}{t^2} \ge 1, \tag{2.2}$$

then problem (1.1) has at least m distinct pairs of nontrivial solutions.

Proof. We apply Theorem 1.1 with $Y = C(\overline{\Omega})$ and $\mathcal{I} = I_\lambda$. Set

$$c_1(m) := \left(\sum_{j=1}^m \|\nabla\varphi_j\|_\infty^2\right)^{\frac{1}{2}} \quad \text{and} \quad c_2(m) := \left(\sum_{j=1}^m \|\varphi_j\|_\infty^2\right)^{\frac{1}{2}}.$$

Since $\lambda > 2\lambda_m^\Delta$, we can choose $\varepsilon \in (0,1)$ so that $\lambda > 2\lambda_m^\Delta/(1-\varepsilon)$ and by virtue of (2.2), there exists $\delta > 0$ such that

$$2G(t) \ge (1-\varepsilon)t^2 \quad \text{as } |t| \le \delta. \tag{2.3}$$

Consider the finite dimensional space

$$X_m := \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_m\},$$

equipped with the norm

$$\|\alpha_1\varphi_1 + \cdots + \alpha_m\varphi_m\|_{X_m} = \left(\alpha_1^2 + \cdots + \alpha_m^2\right)^{\frac{1}{2}}.$$

and let $A_m(\rho)$ be the subset of $C(\overline{\Omega})$ defined by

$$A_m(\rho) := \{v \in X_m : \ \|v\|_{X_m} = \rho\},$$

where ρ is a positive number $\le \min\left\{\frac{1}{c_1(m)}, \frac{\delta}{c_2(m)}\right\}$. Then, it is easy to see that the odd mapping $H : A_m(\rho) \to S^{m-1}$ defined by

$$H\left(\sum_{k=1}^m \alpha_k\varphi_k\right) = \left(\frac{\alpha_1}{\rho}, \dots, \frac{\alpha_m}{\rho}\right)$$

is a homeomorphism between $A_m(\rho)$ and S^{m-1} and so, $\gamma(A_m(\rho)) = m$. Hence, $A_m(\rho) \in \Gamma_m \subset 2^{C(\overline{\Omega})}$.

Let $v = \sum_{k=1}^m \alpha_k\varphi_k \in A_m(\rho)$. Clearly, $v|_{\partial\Omega} = 0$ and we have

$$|\nabla v| \le \sum_{k=1}^m |\alpha_k||\nabla\varphi_k| \le \left(\sum_{k=1}^m \alpha_k^2\right)^{1/2}\left(\sum_{k=1}^m |\nabla\varphi_k|^2\right)^{1/2} \le \rho c_1(m).$$

Therefore, as ρ was chosen $\leq 1/c_1(m)$, one get $\|\nabla v\|_\infty \leq 1$, meaning that $v \in K_0$. On the other hand, using that $\{\varphi_j\}_{j\in\mathbb{N}}$ is orthonormal in $H_0^1(\Omega)$, one has

$$\int_\Omega v^2 dx \geq \frac{\rho^2}{\lambda_m^\Delta} \quad \text{and} \quad \int_\Omega |\nabla v|^2 dx = \rho^2. \tag{2.4}$$

Then, from

$$|v| \leq \left(\sum_{k=1}^m \alpha_k^2\right)^{1/2} \left(\sum_{k=1}^m |\varphi_k|^2\right)^{1/2} \leq \rho c_2(m) \leq \delta,$$

together with (2.1), (2.3) and (2.4), we estimate I_λ as follows

$$\begin{aligned} I_\lambda(v) = \Psi(v) + \mathcal{G}_\lambda(v) &\leq \int_\Omega |\nabla v|^2 dx - \frac{\lambda}{2}(1-\varepsilon)\int_\Omega v^2 dx \\ &\leq \rho^2\left(1 - \frac{\lambda(1-\varepsilon)}{2\lambda_m^\Delta}\right) = \rho^2 \frac{2\lambda_m^\Delta - \lambda(1-\varepsilon)}{2\lambda_m^\Delta} < 0. \end{aligned}$$

This yields

$$\inf_{A\in\Gamma_m} \sup_{v\in A} \mathcal{I}_\lambda(v) \leq \sup_{v\in A_m(\rho)} \mathcal{I}_\lambda(v) < 0$$

and, since I_λ is bounded from below, the proof is accomplished by Theorem 1.1 and Proposition 2.1. $\square$

The above theorem can be applied to derive multiplicity of nontrivial solutions for autonomous non-parameterized Dirichlet problems having the form

$$\begin{cases} -\mathcal{M}(u) = f(u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.5}$$

where the mapping $f : \mathbb{R} \to \mathbb{R}$ is odd and continuous. We set $F(t) = \int_0^t f(\tau)d\tau$ $(t \in \mathbb{R})$.

Corollary 2.3.

(i) *If*

$$\liminf_{t\to 0+} \frac{F(t)}{t^2} > \lambda_m^\Delta \tag{2.6}$$

for some $m \in \mathbb{N}$, then problem (2.5) has at least m distinct pairs of nontrivial solutions.

(ii) *If*

$$\lim_{t\to 0+} \frac{F(t)}{t^2} = +\infty, \tag{2.7}$$

then problem (2.5) has infinitely many distinct pairs of nontrivial solutions.

Proof. (i) By (2.6), there exists $\overline{\lambda}$ such that

$$\liminf_{t\to 0+} \frac{2F(t)}{t^2} \geq \overline{\lambda} > 2\lambda_m^\Delta$$

and the result follows from Theorem 2.2 with $g(t) = f(t)/\overline{\lambda}$.

(ii) This is immediate from (i) and (2.7). $\square$

Example 2.4.

(i) For any $m \in \mathbb{N}$ and $\varepsilon > 0$, problem

$$\begin{cases} -\mathcal{M}(u) = 2(\lambda_m^{\Delta} + \varepsilon) \sin u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

has at least m distinct pairs of nontrivial solutions.

(ii) If $\alpha \in (0,1)$, then problem

$$\begin{cases} -\mathcal{M}(u) = |u|^{\alpha-1} u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

has infinitely many distinct pairs of nontrivial solutions.

Now, we study existence/non-existence and multiplicity of nontrivial solutions for Dirichlet problems involving Fisher-Kolmogorov nonlinearities:

$$\begin{cases} -\mathcal{M}(u) = \lambda u(1 - a|u|^q) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.8}$$

where $a \geq 0$ and $q > 0$ are constants. Notice, in this case one has

$$G(t) = \frac{t^2}{2} - a\frac{|t|^{q+2}}{q+2}, \quad \forall\, t \in \mathbb{R} \tag{2.9}$$

and

$$I_\lambda(u) = \Psi(u) - \lambda \int_\Omega \left[\frac{u^2}{2} - a\frac{|u|^{q+2}}{q+2} \right] dx, \quad u \in C(\overline{\Omega}). \tag{2.10}$$

The next theorem will invoke the constant

$$a_\Omega := \frac{\operatorname{diam}(\Omega)}{2},$$

where $\operatorname{diam}(\Omega)$ stands for the diameter of Ω. Using the mean value theorem, it is straightforward to check that any solution u of a problem of type (1.1) satisfies

$$\|u\|_\infty < a_\Omega. \tag{2.11}$$

Theorem 2.5.

(i) If $\lambda > 2\lambda_m^{\Delta}$, for some $m \geq 2$, then problem (2.8) has at least m distinct pairs of nontrivial solutions.

(ii) If $\lambda > \lambda_1^{\Delta}$, then problem (2.8) has at least one pair of nontrivial solutions $(u_\lambda, -u_\lambda)$, with u_λ a minimizer of the corresponding I_λ. In addition, if $a \in [0, a_\Omega^{-q})$, one may suppose that $u_\lambda > 0$ on Ω.

(iii) If $\lambda \in (0, \lambda_1^{\Delta}]$, the only solution of (2.8) is the trivial one.

Proof. (i) This follows from Theorem 2.2 and (2.9).

(ii) Let $\varphi_1 > 0$ be an eigenfunction of $-\Delta$ in $H_0^1(\Omega)$ corresponding to the first eigenvalue λ_1^Δ and set

$$\psi_1 := \frac{\varphi_1}{\|\nabla\varphi_1\|_\infty}.$$

As $\varphi_1 \in C^1(\overline{\Omega})$, it is clear that $\psi_1 \in K_0 \setminus \{0\}$. Since

$$\lambda_1^\Delta = \frac{\displaystyle\int_\Omega |\nabla\psi_1|^2\,dx}{\displaystyle\int_\Omega \psi_1^2\,dx},$$

we have (as observed in [19]):

$$\lim_{t\to 0^+} \frac{\displaystyle\int_\Omega \left[1-\sqrt{1-|t\nabla\psi_1|^2}\right]dx}{\displaystyle\frac{1}{2}\int_\Omega (t\psi_1)^2 dx} = \lim_{t\to 0^+} \frac{\displaystyle\int_\Omega \frac{t|\nabla\psi_1|^2}{\sqrt{1-|t\nabla\psi_1|^2}}\,dx}{\displaystyle t\int_\Omega \psi_1^2\,dx} = \lambda_1^\Delta. \tag{2.12}$$

Now, let $\lambda > \lambda_1^\Delta$ and let us fix some $\varepsilon > 0$ with $\lambda_1^\Delta < \lambda - \varepsilon$. On account of (2.12), there exists $t_{\lambda,\varepsilon} \in (0,1)$ such that

$$\frac{\displaystyle\int_\Omega \left[1-\sqrt{1-|t\nabla\psi_1|^2}\right]dx}{\displaystyle\frac{1}{2}\int_\Omega (t\psi_1)^2 dx} < \lambda - \varepsilon, \quad \forall\, t \in (0, t_{\lambda,\varepsilon}). \tag{2.13}$$

Next, from (2.13) and taking $t^*_{\lambda,\varepsilon} \in (0, t_{\lambda,\varepsilon})$ with

$$\lambda a \frac{(t^*_{\lambda,\varepsilon}\psi_1(x))^q}{q+2} < \frac{\varepsilon}{2}, \quad \forall\, x \in \overline{\Omega},$$

we estimate I_λ in (2.10) as follows

$$\begin{aligned}
I_\lambda(t^*_{\lambda,\varepsilon}\psi_1) &= \Psi(t^*_{\lambda,\varepsilon}\psi_1) - \lambda\int_\Omega \left[\frac{(t^*_{\lambda,\varepsilon}\psi_1)^2}{2} - a\frac{(t^*_{\lambda,\varepsilon}\psi_1)^{q+2}}{q+2}\right]dx \\
&= \int_\Omega \left[1-\sqrt{1-|\nabla(t^*_{\lambda,\varepsilon}\psi_1)|^2}\right]dx - \lambda\int_\Omega \left[\frac{(t^*_{\lambda,\varepsilon}\psi_1)^2}{2} - a\frac{(t^*_{\lambda,\varepsilon}\psi_1)^{q+2}}{q+2}\right]dx \\
&< \frac{\lambda-\varepsilon}{2}\int_\Omega (t^*_{\lambda,\varepsilon}\psi_1)^2 dx - \frac{\lambda}{2}\int_\Omega (t^*_{\lambda,\varepsilon}\psi_1)^2 dx + \lambda\int_\Omega a\frac{(t^*_{\lambda,\varepsilon}\psi_1)^{q+2}}{q+2}\,dx \\
&= \int_\Omega (t^*_{\lambda,\varepsilon}\psi_1)^2 \left[\lambda a\frac{(t^*_{\lambda,\varepsilon}\psi_1)^q}{q+2} - \frac{\varepsilon}{2}\right]dx < 0 = I_\lambda(0).
\end{aligned}$$

From Proposition 2.1 we infer that, if $\lambda > \lambda_1^\Delta$, the even functional I_λ attains its infimum at some $u_\lambda \in K_0 \setminus \{0\}$, hence problem (2.8) has a pair of nontrivial solutions $(u_\lambda, -u_\lambda)$. Since $|u_\lambda|$ is still a minimizer of I_λ, it also solves (2.8) and, taking into account (2.11), we obtain

$$-\mathcal{M}(|u_\lambda|) = \lambda|u_\lambda|(1-a|u_\lambda|^q) \geq \lambda|u_\lambda|\left(1-a\,a_\Omega^q\right).$$

Then, since $|u_\lambda| > 0$ in a subset of Ω having positive measure, from [11, Lemma 2.6] it follows that actually $|u_\lambda| > 0$ in the whole Ω.

(iii) Assume, by contradiction, that for such a λ, a function u is a nontrivial solution of (2.8). On account of (1.2), one gets

$$\lambda \int_\Omega u^2(1 - a|u|^q)\, dx = \int_\Omega \frac{|\nabla u|^2}{\sqrt{1 - |\nabla u|^2}}\, dx \geq \int_\Omega |\nabla u|^2\, dx \geq \lambda_1^\Delta \int_\Omega u^2\, dx. \tag{2.14}$$

If $a > 0$, we have

$$0 > -\lambda a \int_\Omega |u|^{q+2}\, dx \geq (\lambda_1^\Delta - \lambda) \int_\Omega u^2\, dx \geq 0,$$

i.e. a contradiction. In the case $a = 0$, if $\lambda < \lambda_1^\Delta$, as above we obtain the contradiction

$$0 \geq (\lambda_1^\Delta - \lambda) \int_\Omega u^2\, dx > 0.$$

Also, if $\lambda = \lambda_1^\Delta$, from (2.14) (with $a = 0$) we have that

$$\int_\Omega |\nabla u|^2 \left(\frac{1}{\sqrt{1 - |\nabla u|^2}} - 1 \right) dx = 0,$$

or,

$$\int_\Omega \frac{|\nabla u|^4}{\left(1 + \sqrt{1 - |\nabla u|^2}\right) \sqrt{1 - |\nabla u|^2}}\, dx = 0$$

which, since $u \in C^1(\overline{\Omega})$, implies $|\nabla u| = 0$ on $\overline{\Omega}$. It follows that u is constant and then, as $u \in K_0$, we infer that $u \equiv 0$ – a contradiction. Hence, (2.8) has only the trivial solution provided that $\lambda \in (0, \lambda_1^\Delta]$ and the proof is now complete. □

Remark 2.6. (i) It is worth noticing that in the particular case $a = 0$, Theorem 2.5 recovers and improves the main result of paper [19], which states that problem

$$\begin{cases} -\mathcal{M}(u) = \lambda u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

has a nontrivial solution iff $\lambda > \lambda_1^\Delta$ and for such a λ, a nontrivial solution can be chosen to be nonnegative on Ω and to minimize the corresponding I_λ.

(ii) In Theorem 2.5 it is assumed: if $m = 1$, $\lambda > \lambda_m^\Delta$, and if $m > 1$, $\lambda > 2\lambda_m^\Delta$, instead of $\lambda > \lambda_m^\Delta$. This comes from the fact that in Theorem 2.2 we were not able to prove that $\lambda > 2\lambda_m$ can be replaced by the weaker condition $\lambda > \lambda_m^\Delta$. Actually, at the moment it is not clear that this can be done under assumption (2.2) – this remains an open problem. Nevertheless, it is worth to point out that Theorem 2.2 yields the following: problem (1.1) has at least m ($\in \mathbb{N}$) distinct pairs of nontrivial solutions if $\lambda > \lambda_m^\Delta$ and

$$\liminf_{t \to 0+} \frac{G(t)}{t^2} \geq 1. \tag{2.15}$$

To see this, rewrite the equation in (1.1) as

$$-\mathcal{M}(u) = 2\lambda \tilde{g}(u) \quad \text{in } \Omega,$$

with $\tilde{g}(u) = g(u)/2$ and apply Theorem 2.2. In this form this seems to allow in Theorem 2.5 the more natural assumption $\lambda > \lambda_m^\Delta$, instead of $\lambda > 2\lambda_m^\Delta$, for $m > 1$. However, this cannot be applied to problem (2.8) since G defined in (2.9) does not satisfy (2.15).

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A coincidence problem for a second-order semi-linear differential equation

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. In this paper, we study a class of problems at resonance for a general second-order linear operator $Lu = u'' + p(t)u' + q(t)u$. We impose abstract functional conditions and derive several criteria for the existence of a solution for every resonance scenario.

Keywords: functional condition, semi-linear differential equation, resonance.

2020 Mathematics Subject Classification: 34B10, 34B15.

1 Introduction

We consider the semi-linear equation

$$u''(t) + p(t)u'(t) + q(t)u(t) = f(t, u(t), u'(t)), \quad a.e.\ t \in (0, 1), \tag{1.1}$$

subject to the linear functional conditions

$$F_1(u) = 0, \qquad F_2(u) = 0, \tag{1.2}$$

where F_1 and F_2 are continuous linear functionals on $C^1[0, 1]$.

One of the early works that stimulated interest to applications of the coincidence degree theory to non-local boundary value problems was the paper by Feng and Webb [3]. Our work is motivated by [3] and [2]. In [2], the authors studied the resonant functional problem

$$u''(t) = f(t, u(t), u'(t)), \quad \text{a.e. } t \in (0, 1), \tag{1.3}$$

$$B_1(u) = 0, \qquad B_2(u) = 0, \tag{1.4}$$

where f is Carathéodory, B_1 and B_2 are continuous linear functionals on $C^1[0, 1]$. Imposing $B_1(t)B_2(1) = B_2(t)B_1(1)$, the problem (1.3), (1.4) is at resonance of dimension one or two. An existence result was obtained for every possible resonance scenario.

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In order to apply the coincidence degree approach of Mawhin and many other methods of functional analysis in ordinary semi-linear differential equations, one relies on the knowledge of a fundamental solution set. In all known to us papers based on these methods, the linear operator L, such as $Lu = (pu')'$ in [7], can be "inverted" by the reduction of order method. The method developed here can be also applied to fractional order problems, that is, when L is an integro-differential operator such as the Riemann–Liouville, Caputo fractional derivatives and their numerous generalizations. Since we deal with a linear differential operator that, in general, does not admit the reduction of order, this work is also a generalization of many results such as [1, 6–8]. Moreover, if the boundary conditions, or, for that matter multi-point conditions, or even linear conditions involving Riemann–Stieltjes integrals are chosen, a specific resonance is "fixed". Obviously, in this case, one would only hope to study one or very few resonance conditions per paper. We believe a more productive approach would yield a formalism for solving a class of problems.

In our setting, the problem is abstract since we deal with a large class of general second-order linear differential operators whose fundamental solution set is $\{\phi_1, \phi_2\}$. Not only our work is an abstract generalization of many results in that respect but also due to the functional conditions (1.2) studied here, which certainly include (1.6). In fact, as in [2], we study every "geometric" scenario of resonance. In particular, in [2], the authors considered (1.1) with $p(t) = q(t) = 0$ subject to (1.2). Thus, the present work extends the results of [2], as well.

In [7], the author considered several resonance cases in the framework of the generalized Sturm–Liouville boundary value problem

$$(p(t)u'(t))' - q(t)u(t) = f\left(t, \int_0^t u(s)\,ds, u'(t)\right), \qquad t \in (0,1), \tag{1.5}$$

$$au(0) - bp(0)u'(0) = \mu_1 u(\xi), \qquad cu(1) + dp(1)u'(1) = \mu_2 u(\xi), \tag{1.6}$$

where $a, b, c, d \in \mathbb{R}$, $0 < \xi < 1$, and f is continuous and

$$\mu_1\left(c\int_\xi^1 \frac{1}{p(s)}\,ds + d\right) + \mu_2\left(a\int_0^\xi \frac{1}{p(s)}\,ds + b\right) = ad + bc + ac\int_0^1 \frac{1}{p(s)}\,ds. \tag{1.7}$$

By means of a "shift" operator, a resonant problem can be converted to a non-resonant problem [5] and, thus, need not be studied as a coincidence equation $Lu = Nu$. In [7], the problem is not at resonance if

$$L_0 u(t) = (p(t)u'(t))' - q(t)u(t).$$

Considering

$$Lu(t) = (p(t)u'(t))' = q(t)u(t) + f\left(t, \int_0^t u(s)\,ds, u'(t)\right), \qquad t \in (0,1),$$

the equation (1.7) becomes a resonance condition. The advantage here is that the fundamental solution set of L is easy to obtain while for L_0 we only know that it exists but, in general, there is no hope to obtain it explicitly. It is also worth mentioning that whenever a criterion for the existence of a solution to the coincidence equation $Lu = f(t, u, u')$ is obtained, it can always, with a little effort, be extended to $Lu = f(t, u, T_1(u), u', T_2(u'))$, where T_1 and T_2 are bounded operators such as the primitive of $u(t)$ in (1.5), on a suitable functional space. Indeed, the projection scheme needed to apply the coincidence degree approach to these equations is exactly the same, and the only difference is in the "growth" condition on the function f.

In [7], the author introduces a convenience assumption

$$(c\mu_1 - a\mu_2)\int_0^{\xi}\frac{s}{p(s)}\,ds + c(a-\mu_1)\int_0^1\frac{s}{p(s)}\,ds + d(a-\mu_1) \neq 0. \tag{1.8}$$

In order to guarantee that the projector Q is well-defined, conditions similar to (1.8) have been imposed in many papers (e.g., see the references in [2] and the remarks therein). In our work, we construct the projection scheme so that Q is well-defined without relying on such "convenience" assumptions that are rather restrictive and simply unnecessary.

In this section, we state the preliminaries and the result due to Mawhin [4] used, in the second section, to obtain a solution of (1.1), (1.2).

In order to develop our method, we need to make several basic assumptions. Of course, we assume that the fundamental solution set $\{\phi_1, \phi_2\}$ is known. We would like to consider a solution of (1.1) in classical spaces and make use of the representation

$$u(t) = \int_0^t k(t,s)Lu(s)\,ds + l_1(u)\phi_1(t) + l_2(u)\phi_2(t), \tag{1.9}$$

where

$$k(t,s) = \frac{\phi_1(s)\phi_2(t) - \phi_2(s)\phi_1(t)}{W(\phi_1,\phi_2)(s)}, \quad l_1(u) = \frac{W(u,\phi_2)(0)}{W(\phi_1,\phi_2)(0)}, \quad l_2(u) = \frac{W(\phi_1,u)(0)}{W(\phi_1,\phi_2)(0)}, \tag{1.10}$$

where

$$\Phi(\phi_1,\phi_2)(t) = \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{bmatrix}$$

and $W(\phi_1,\phi_2)(t) = \det\Phi(\phi_1,\phi_2)(t) = \phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t)$ is the Wronskian of the fundamental solution set on $[0,1]$. Our approach relies on the boundedness of $W(\phi_1,\phi_2)(t)$ and $W(\phi_1,\phi_2)(0) \neq 0$. So, the following would fulfill our wishes:

(L) $p, q \in C[0,1]$, $\gamma_1 = \max_{t,s\in[0,1]} |k(t,s)|$, $\gamma_2 = \sup_{t,s\in[0,1]} \left|\frac{\partial}{\partial t}k(t,s)\right|$, $\gamma = \max\{\gamma_1,\gamma_2\}$.

It should be mentioned that the assumption on p can be weakened, which would force one to use weighted norms.

Introduce $X = C^1[0,1]$, $\|u\|_X = \max\{\|u\|_0, \|u'\|_0\}$, where $\|u\|_0 = \max_{t\in[0,1]} |u(t)|$. The next standing assumption concerns the linear functions in (1.2):

(F) $F_i : X \to \mathbb{R}$, $|F_i(u)| \leq \rho_i\|u\|_X$, where $\rho_i > 0$, $i = 1,2$, $F_1(\phi_1) = \alpha a$, $F_1(\phi_2) = \alpha b$, $F_2(\phi_1) = a$, $F_2(\phi_2) = b$, $\alpha, a, b \in \mathbb{R}$, $a^2 + b^2 \neq 0$.

Under this assumption the differential operator in (1.1) is not invertible and the functional problem is said to be at resonance. Furthermore, in order to claim that all possible resonance cases have been considered, we also need to study the case $a = b = 0$, which is only briefly discussed in Section 2.

Definition 1.1. Let X and Z be normed spaces. A linear mapping $L\colon \operatorname{dom} L \subset X \to Z$ is called a Fredholm mapping if the following two conditions hold:

(i) $\ker L$ has a finite dimension, and

(ii) $\operatorname{Im} L$ is closed and has a finite co-dimension.

If L is a Fredholm mapping, its (Fredholm) *index* is the integer $\operatorname{Ind} L = \dim\ker L - \operatorname{codim}\operatorname{Im} L$.

Since we work with a Fredholm mapping of index zero, it follows from Definition 1.1 that there exist continuous projectors $P\colon X \to X$ and $Q\colon Z \to Z$ such that

$$\operatorname{Im} P = \ker L, \quad \ker Q = \operatorname{Im} L, \quad X = \ker L \oplus \ker P, \quad Z = \operatorname{Im} L \oplus \operatorname{Im} Q \tag{1.11}$$

and that the mapping

$$L|_{\operatorname{dom} L \cap \ker P}\colon \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$$

is invertible. The inverse of $L|_{\operatorname{dom} L \cap \ker P}$ we denote by $K_P\colon \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$. The generalized inverse of L denoted by $K_{P,Q}\colon Z \to \operatorname{dom} L \cap \ker P$ is defined by $K_{P,Q} = K_P(I - Q)$.

Definition 1.2. Let $L\colon \operatorname{dom} L \subset X \to Z$ be a Fredholm mapping, E be a metric space, and $N\colon E \to Z$ be a mapping. We say that N is L-compact on E if $QN\colon E \to Z$ and $K_{P,Q}N\colon E \to X$ are compact on E. In addition, we say, that N is L-completely continuous if it is L-compact on every bounded $E \subset X$.

Let $Z = L^1[0,1]$ with the Lebesgue norm denoted by $\|\cdot\|_1$. Consider the mapping $L\colon \operatorname{dom} L \subset X \to Z$ with

$$\operatorname{dom} L = \{u \in X : u' \in AC[0,1],\ u \text{ satisfies } (1.2)\}$$

defined by

$$Lu(t) = u''(t) + p(t)u'(t) + q(t)u(t).$$

Define the mapping $N\colon X \to Z$ by

$$Nu(t) = f(t, u(t), u'(t)).$$

Thus, (1.1), (1.2) is converted into the coincidence equation $Lu = Nu$ whose solution will be shown to exist by applying the following theorem due to Mawhin [4, Theorem IV.13].

Theorem 1.3. *Let $\Omega \subset X$ be open and bounded, L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lu \neq \lambda Nu$ for every $(u, \lambda) \in ((\operatorname{dom} L \setminus \ker L) \cap \partial\Omega) \times (0,1)$;

(ii) $Nu \notin \operatorname{Im} L$ for every $u \in \ker L \cap \partial\Omega$;

(iii) $\deg(JQN|_{\ker L \cap \partial\Omega}, \Omega \cap \ker L, 0) \neq 0$, with $Q\colon Z \to Z$ a continuous projector such that $\ker Q = \operatorname{Im} L$ and $J\colon \operatorname{Im} Q \to \ker L$ is any isomorphism.

Then the equation $Lu = Nu$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

Lemma 1.4. *The mapping $L\colon \operatorname{dom} L \subset X \to Z$ is a Fredholm mapping of index zero.*

Proof. By (F), it is clear that $\ker L = \{c(-b\phi_1 + a\phi_2) : c \in \mathbb{R}\} \cong \mathbb{R}$. For convenience, let

$$Tg(t) = \int_0^t k(t,s) g(s)\, ds. \tag{1.12}$$

We claim that $\operatorname{Im} L = \{g \in Z : (F_1 - \alpha F_2)Tg = 0\}$. Now, $g \in \operatorname{Im} L$ if there exists $u \in \operatorname{dom} L$ such that $Lu = g$. Recalling (1.9), that is, $u = Tg + l_1(u)\phi_1 + l_2(u)\phi_2$, we have, by (F),

$$F_1(u) = F_1(Tg) + \alpha(l_1(u)a + l_2(u)b) = 0, \qquad F_2(u) = F_2(Tg) + l_1(u)a + l_2(u)b = 0.$$

It follows, $\operatorname{Im} L \subset \{g \in Z : (F_1 - \alpha F_2)Tg = 0\}$.

Let $g \in \{g \in Z : (F_1 - \alpha F_2)Tg = 0\}$. Define

$$u = Tg - \frac{F_2(Tg)}{a^2 + b^2}(a\phi_1 + b\phi_2).$$

Then

$$Lu = LTg - \frac{F_2(Tg)}{a^2 + b^2}(aL\phi_1 + bL\phi_2) = g.$$

Also,

$$F_1(u) = F_1(Tg) - \frac{F_2(Tg)}{a^2 + b^2}(aF_1(\phi_1) + bF_1(\phi_2)) = F_1(Tg) - \alpha F_2(Tg) = 0$$

and, similarly, $F_2(u) = 0$. That is, $u \in \operatorname{dom} L$, so $g \in \operatorname{Im} L$. We have

$$\{g \in Z : (F_1 - \alpha F_2)Tg = 0\} \subset \operatorname{Im} L.$$

Therefore, $\{g \in Z : (F_1 - \alpha F_2)Tg = 0\} = \operatorname{Im} L$.

We show that there exists $h \in Z$ such that $(F_1 - \alpha F_2)Th \neq 0$. Let $F = F_1 - \alpha F_2$. By (F), $F(\phi_1) = F(\phi_2) = 0$. Since F_1 and F_2 are linearly independent on X, there exists $u_0 \in X$ such that $F(u_0) \neq 0$. Since F is continuous on X, for $\epsilon > 0$, there exists a polynomial p such that $\|p - u_0\|_X < \epsilon$ and $F(p) \neq 0$. Set $h = Lp \in Z$. Again, recall (1.9). Then $F(Th) = F(TLp) = F(p - l_1(p)\phi_1 - l_2(p)\phi_2) = F(p) - l_1(p)F(\phi_1) - l_2(p)F(\phi_2) = F(p) \neq 0$. Since T and F are linear, we may assume, without loss of generality, that $(F_1 - \alpha F_2)Th = 1$. Define $Q : Z \to Z$ by

$$Qg(t) = (F_1 - \alpha F_2)(Tg)h(t) = (F_1 - \alpha F_2)\left(\int_0^t k(t,s)g(s)\,ds\right)h(t).$$

Since $Qh(t) = (F_1 - \alpha F_2)(Th)h(t) = h(t)$, then $Q^2 q = Qg$, $g \in Z$. It is obvious that $Q : Z \to Z$ is a continuous map and $Z = \ker Q \oplus \operatorname{Im} Q$, $\operatorname{Im} Q = \{ch : c \in \mathbb{R}\}$ with $\dim \operatorname{Im} Q = 1$, and $\ker Q = \operatorname{Im} L$.

Define $P, \tilde{P}, P_0 : X \to X$ by

$$\begin{aligned} Pu(t) &= \frac{-bW(u,\phi_2)(0) + aW(\phi_1,u)(0)}{(a^2+b^2)W(\phi_1,\phi_2)(0)}(-b\phi_1(t) + a\phi_2(t)) \\ &= \frac{-bl_1(u) + al_2(u)}{a^2+b^2}(-b\phi_1(t) + a\phi_2(t)), \end{aligned} \tag{1.13}$$

$$\begin{aligned} \tilde{P}u(t) &= \frac{aW(u,\phi_2)(0) + bW(\phi_1,u)(0)}{(a^2+b^2)W(\phi_1,\phi_2)(0)}(a\phi_1(t) + b\phi_2(t)) \\ &= \frac{al_1(u) + bl_2(u)}{a^2+b^2}(a\phi_1(t) + b\phi_2(t)), \end{aligned} \tag{1.14}$$

and

$$P_0(t) = \frac{W(u,\phi_2)(0)}{W(\phi_1,\phi_2)(0)}\phi_1(t) + \frac{W(\phi_1,u)(0)}{W(\phi_1,\phi_2)(0)}\phi_2(t) = l_1(u)\phi_1(t) + l_2(u)\phi_2(t), \tag{1.15}$$

where the second expression of each map is obtained using (1.10). Since

$$P\phi_1 = -\frac{b}{a^2+b^2}(-b\phi_1 + a\phi_2), \quad P\phi_2 = \frac{a}{a^2+b^2}(-b\phi_1 + a\phi_2),$$

then $P(-b\phi_1 + a\phi_2) = -b\phi_1 + a\phi_2$. Therefore, $P^2 = P$, $X = \ker P \oplus \operatorname{Im} P$, where $\operatorname{Im} P = \{c(-b\phi_1 + a\phi_2) : c \in \mathbb{R}\} = \ker L$. Similarly, $\tilde{P}^2 = \tilde{P}$, $X = \ker \tilde{P} \oplus \operatorname{Im} \tilde{P}$, where $\operatorname{Im} \tilde{P} =$

$\{c(a\phi_1 + b\phi_2) : c \in \mathbb{R}\}$. Moreover, $P_0^2 = P_0$, $X = \ker P_0 \oplus \operatorname{Im} P_0$, where $\operatorname{Im} P_0 = \{c_1\phi_1 + c_2\phi_2 : c_1, c_2 \in \mathbb{R}\}$. Finally,

$$P + \tilde{P} = P_0 \tag{1.16}$$

and $P\tilde{P} = \tilde{P}P = 0$ on X.

Since the relationships (1.11) hold, the projectors P and Q are exact. In summary, L is a Fredholm mapping of index zero. □

The next two results provide the generalized inverse of L and its norm-estimates. Recall (1.12).

Lemma 1.5. *If the map $K_P : Z \to X$ is defined by*

$$K_p g = -\frac{1}{a^2 + b^2} F_2(Tg)(a\phi_1 + b\phi_2) + Tg, \tag{1.17}$$

then $LK_P g = g$, $g \in Z$, and $K_p Lu = u$, $u \in \operatorname{dom} L \cap \ker P$.

Proof. It is easy to see that $LK_P g = g$, $g \in Z$. Let $u \in \operatorname{dom} L \cap \ker P$ and $g = Lu$. Using (1.9) and (1.15),

$$Tg = u - l_1(u)\phi_1 - l_2(u)\phi_2 = u - P_0 u.$$

Then $F_2(Tg) = F_2(u) - l_1(u)F_2(\phi_1) - l_2(u)F_2(\phi_2) = -al_1(u) - bl_2(u)$ since $u \in \operatorname{dom} L$. As a result,

$$K_P Lu = \frac{al_1(u) + bl_2(u)}{a^2 + b^2}(a\phi_1 + b\phi_2) + u - P_0 u = \tilde{P}u + u - P_0 u = u - Pu = u$$

by (1.16) and since $u \in \ker P$. □

Obviously,

$$\|Tg\|_0 \le \gamma_1 \|g\|_1, \quad \|(Tg)'\|_0 \le \gamma_2 \|g\|_1, \quad \|Tg\|_X \le \gamma \|g\|_1.$$

Also, $|F_2(Tg)| \le \rho_2 \|Tg\|_X \le \gamma\rho_2 \|g\|_1$. Hence,

$$\|K_P g\|_0 \le \frac{\rho_2 \|a\phi_1 + b\phi_2\|_0}{(a^2 + b^2)} \|Tg\|_X + \|Tg\|_0 \le \left(\frac{\rho_2 \gamma \|a\phi_1 + b\phi_2\|_0}{a^2 + b^2} + \gamma_1 \right) \|g\|_1,$$

$$\|(K_P g)'\|_0 \le \frac{\rho_2 \|a\phi_1' + b\phi_2'\|_0}{(a^2 + b^2)} \|Tg\|_X + \|(Tg)'\|_0 \le \left(\frac{\rho_2 \gamma \|a\phi_1' + b\phi_2'\|_0}{a^2 + b^2} + \gamma_2 \right) \|g\|_1.$$

The estimates on the generalized inverse are summarized in the next result.

Lemma 1.6. *The map $K_P : Z \to X$ satisfies*

(a) $\|K_P g\|_0 \le A\|g\|_1$, *where*

$$A = \frac{\rho_2 \gamma \|a\phi_1 + b\phi_2\|_0}{a^2 + b^2} + \gamma_1,$$

(b) $\|(K_P g)'\|_0 \le B\|g\|_1$, *where*

$$B = \frac{\rho_2 \gamma \|a\phi_1' + b\phi_2'\|_0}{a^2 + b^2} + \gamma_2,$$

(c) $\|K_P g\|_X \le \|K_P\| \|g\|_1$, *where* $\|K_P\| = \max\{A, B\}$.

2 Main results

Assume that the following conditions on the function $f(t, x_1, x_2)$ are satisfied:

(H_1) there exists a constant $M_0 > 0$ such that, for each $u \in \operatorname{dom} L \setminus \ker L$ with $|u(t)| + |u'(t)| > M_0$, $t \in [0,1]$, we have $QNu(t) \neq 0$,

(H2) there exist functions $\delta_0, \delta_1, \delta_2 \in L^1[0,1]$ such that, for all $(x_1, x_2) \in \mathbb{R}^2$ and *a.e.* $t \in [0,1]$,

$$|f(t, x_1, x_2)| \leq \delta(t) + \delta_1(t)|x_1| + \delta_2(t)|x_2|.$$

(H3) there exists a constant $M_1 > 0$ such that if $|c| > M_1$, then $c(F_1 - \alpha F_2)(TNu_c) > 0$, where $u_c = c(-b\phi_1 + a\phi_2)$.

In the next result, $\|\Phi^{-1}(\phi_1, \phi_2)(t)\|$ is the matrix norm compatible with the norm $\max\{|a_1|, |a_2|\}$ of a vector $[a_1, a_2]^T \in \mathbb{R}^2$.

Theorem 2.1. *If* (L), (F), $(H1)$–$(H3)$ *hold, then the functional problem* (1.1), (1.2) *has at least one solution provided*

$$D_1(\|\delta_1\|_1 + \|\delta_2\|_1) < 1, \tag{2.1}$$

where

$$\begin{aligned} D_1 = \max\Big\{&\gamma_1 + \gamma \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1, \phi_2)(t)\|(\|\phi_1\|_0 + \|\phi_2\|_0), \\ &\gamma_2 + \gamma \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1, \phi_2)(t)\|(\|\phi_1'\|_0 + \|\phi_2'\|_0)\Big\}. \end{aligned}$$

Proof. Let $\Omega_1 = \{u \in \operatorname{dom} L \setminus \ker L\colon Lu = \lambda Nu,\ \lambda \in (0,1)\}$. If $u \in \Omega_1$, it follows, from (H_1), that there exists $t_0 \in [0,1]$ such that $|u(t_0)|, |u'(t_0)| \leq M_0$. Now,

$$u = \lambda TNu + l_1(u)\phi_1 + l_2(u)\phi_2, \quad u' = \lambda(TNu)' + l_1(u)\phi_1' + l_2(u)\phi_2'. \tag{2.2}$$

Thus,

$$\begin{bmatrix} l_1(u) \\ l_2(u) \end{bmatrix} = \Phi^{-1}(\phi_1, \phi_2)(t_0) \begin{bmatrix} u(t_0) - \lambda TNu(t_0) \\ u'(t_0) - \lambda(TNu)'(t_0) \end{bmatrix}.$$

In what follows, C_i, $i = 1, \dots, 5$, are positive constants whose exact values are ignored. Hence,

$$\begin{aligned} |l_1(u)|, |l_2(u)| &= \max\{|l_1(u)|, |l_2(u)|\} \\ &= \|\Phi^{-1}(\phi_1, \phi_2)(t_0)\| \max\{|u(t_0) - \lambda TNu(t_0)|, |u'(t_0) - \lambda(TNu)'(t_0)|\} \\ &\leq \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1, \phi_2)(t)\| \max\{|u(t_0)| + \lambda|TNu(t_0)|, |u'(t_0)| + \lambda|(TNu)'(t_0)|\} \\ &\leq \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1, \phi_2)(t)\| \max\{M_0 + \lambda|TNu(t_0)|, M_0 + \lambda|(TNu)'(t_0)|\} \\ &< \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1, \phi_2)(t)\| \max\{M_0 + \gamma_1\|Nu\|_1, M_0 + \gamma_2\|Nu\|_1\} \\ &= C_1 + \gamma \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1, \phi_2)(t)\| \|Nu\|_1. \end{aligned}$$

We have

$$\begin{aligned} \|u\|_0 &\leq \gamma_1\|Nu\|_1 + |l_1(u)|\|\phi_1\|_0 + |l_2(u)|\|\phi_2\|_0 \\ &< C_2 + \left(\gamma_1 + \gamma \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1, \phi_2)(t)\|(\|\phi_1\|_0 + \|\phi_2\|_0)\right) \|Nu\|_1 \end{aligned}$$

and, similarly,

$$\|u'\|_0 < C_3 + \left(\gamma_2 + \gamma \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1,\phi_2)(t)\|(\|\phi_1'\|_0 + \|\phi_2'\|_0)\right) \|Nu\|_1.$$

Hence,

$$\begin{aligned}\|u\|_X < C_4 + \max\{\gamma_1 + \gamma \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1,\phi_2)(t)\|(\|\phi_1\|_0 + \|\phi_2\|_0),\\ \gamma_2 + \gamma \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1,\phi_2)(t)\|(\|\phi_1'\|_0 + \|\phi_2'\|_0)\}\|Nu\|_1.\end{aligned}$$

By (H_2), $\|Nu\|_1 \le \|\delta_0\|_1 + \|\delta_1\|_1\|u\|_0 + \|\delta_2\|_1\|u'\|_0 \le \|\delta_0\|_1 + (\|\delta_1\|_1 + \|\delta_2\|_1)\|u\|_X$, so

$$\|u\|_X < C_5 + D_1(\|\delta_1\|_1 + \|\delta_2\|_1)\|u\|_X$$

for all $u \in \Omega_1$. In view of the inequality (2.1), Ω_1 is bounded.

Define $\Omega_2 = \{u \in \ker L\colon Nu \in \operatorname{Im} L\}$. Then $u = c(-b\phi_1 + a\phi_2)$ for some $c \in \mathbb{R}$. Since $Nu \in \operatorname{Im} L = \ker Q$, $(F_1 - \alpha F_2)TNu = 0$. By (H_3), $|c| \le M_1$, that is, Ω_2 is bounded.

Define $J : Z \to X$ by

$$Jg(t) = (F_1 - \alpha F_2)(Tg)(-b\phi_1(t) + a\phi_2(t)).$$

Recall the characterization of $\operatorname{Im} Q$ in the proof of Lemma 1.4. Since $J(ch)(t) = c(F_1 - \alpha F_2)(Th)(-b\phi_1 + a\phi_2) = c(-b\phi_1 + a\phi_2)$, $J\colon \operatorname{Im} Q \to \ker L$ is an isomorphism.

Let $\Omega_3 = \{u \in \ker L\colon \lambda u + (1-\lambda)JQNu = 0, \lambda \in [0,1]\}$. Let $u \in \Omega_3$ be denoted by $u_c = c(-b\phi_1 + a\phi_2)$. Then $\lambda u + (1-\lambda)JQNu = 0$ implies $\lambda c + (1-\lambda)(F_1 - \alpha F_2)TNu_c = 0$. If $\lambda = 0$, then $JQNu_c = 0$, that is, $u \in \Omega_2$, which is bounded. If $\lambda = 1$, then $c = 0$. If $\lambda \in (0,1)$, then, by (H_2),

$$0 < \lambda c^2 = -(1-\lambda)c(F_1 - \alpha F_2)TNu_c < 0,$$

which is a contradiction. Thus, Ω_3 is bounded.

Let Ω be open and bounded such that $\cup_{i=1}^{3}\overline{\Omega}_i \subset \Omega$. Then the assumptions (i) and (ii) of Theorem 1.3 are fulfilled. It is a routine exercise to show that the mapping N is L-compact on $\overline{\Omega}$. Lemma 1.4 states that L if Fredholm of index zero. We now demonstrate that the third assumption of Theorem 1.3 is verified.

We apply the degree property of invariance under a homotopy to

$$H(u,\lambda) = \lambda Iu + (1-\lambda)JQNu, \quad (u,\lambda) \in X \times [0,1].$$

If $u \in \ker L \cap \partial\Omega$, then

$$\begin{aligned}\ker(JQN|_{\ker L\cap\partial\Omega}, \Omega \cap \ker L, 0) &= \ker(H(\cdot,0), \Omega\cap\ker L, 0)\\ &= \ker(H(\cdot,1), \Omega\cap\ker L, 0)\\ &= \ker(I, \Omega\cap\ker L, 0)\\ &\neq 0,\end{aligned}$$

that is, the assumption (iii) of Theorem 1.3 is checked and the proof is completed. □

It is worth mentioning that the inequality in (H_3) may be reversed since the proof will carry over with a slight modification.

We will replace (H_1) of Theorem 2.1 with

(H_4) there exists a constant $M_0 > 0$ such that, for each $u \in \operatorname{dom} L \setminus \ker L$ with $|u(t)| > M_0$, $t \in [0,1]$, we have $QNu(t) \neq 0$.

Theorem 2.2. *If (L), (F), (H_2)–(H_4) hold, then the boundary value problem (1.1), (1.2) has at least one solution provided $-b\phi_1(t) + a\phi_2(t) \neq 0$ on $[0,1]$, and*

$$D_2(\|\delta_1\|_1 + \|\delta_2\|_1) < 1, \tag{2.3}$$

where

$$D_2 = \frac{A\|-b\phi_1 + a\phi_2\|_X}{\min_{t\in[0,1]} |-b\phi_1(t) + a\phi_2(t)|} + \|K_P\|.$$

Proof. As in the proof of Theorem 2.1, let $\Omega_1 = \{u \in \operatorname{dom} L \setminus \ker L\colon Lu = \lambda Nu,\ \lambda \in (0,1)\}$. For $u \in \Omega_1$, it follows from (H_4) that there exists $t_0 \in [0,1]$ such that $|u(t_0)| \le M_0$.
Remark: Note that it does not follow from (H_4) that $|u'(t_0)| \le M_0$, so we cannot apply the approach taken in the proof of Theorem 2.1 to the present case. Likewise, the inequality $|u(t_0)| \le M_0$ can not be obtained from (H_5) of Theorem 2.3, which will not allows us to apply the argument of Theorem 2.1. For this reason, here and in the proof of Theorem 2.3 we rely on $u = Pu + (I-P)u$.

Consider $u \in \Omega_1$ and $u = u_1 + u_2$, $u_1 = Pu \in \operatorname{Im} P = \ker L$, $u_2 = (I-P)u = K_P L u = \lambda K_P N u$. We have, by Lemma 1.6,

$$\|u_2\|_0 < A\|Nu\|_1, \quad \|u_2\|_X < \|K_P\|\|Nu\|_1. \tag{2.4}$$

Now, $u_1 = u - u_2$, so that $|Pu(t_0)| = |u_1(t_0)| \le |u(t_0)| + |u_2(t_0)| < M_0 + A\|Nu\|_1$. We have

$$|u_1(t_0)| = \frac{|-bl_1(u) + al_2(u)|}{a^2+b^2}|-b\phi_1(t_0) + a\phi_2(t_0)| < M_0 + A\|Nu\|_1.$$

In particular,

$$\frac{|-bl_1(u) + al_2(u)|}{a^2+b^2} \le \frac{M_0 + A\|Nu\|_1}{\min_{t\in[0,1]} |-b\phi_1(t) + a\phi_2(t)|}.$$

Hence,

$$\begin{aligned} \|u_1\|_X = \|Pu\|_X &\le \frac{|-bl_1(u) + al_2(u)|}{a^2+b^2}\|-b\phi_1 + a\phi_2\|_X \\ &\le \frac{\|-b\phi_1 + a\phi_2\|_X}{\min_{t\in[0,1]} |-b\phi_1(t) + a\phi_2(t)|}(M_0 + A\|Nu\|_1). \end{aligned} \tag{2.5}$$

Combining (2.5) and (2.4), we conclude

$$\begin{aligned} \|u\|_X &\le \|u_1\|_X + \|u_2\|_X \\ &< C_1 + \left(\frac{A\|-b\phi_1 + a\phi_2\|_X}{\min_{t\in[0,1]} |-b\phi_1(t) + a\phi_2(t)|} + \|K_P\|\right)\|Nu\|_1 \\ &< C_2 + \left(\frac{A\|-b\phi_1 + a\phi_2\|_X}{\min_{t\in[0,1]} |-b\phi_1(t) + a\phi_2(t)|} + \|K_P\|\right)(\|\delta_1\|_1 + \|\delta_2\|_1)\|u\|_X \\ &< C_2 + D_2(\|\delta_1\|_1 + \|\delta_2\|_1)\|u\|_X. \end{aligned}$$

Therefore, by (2.3), Ω_1 is bounded. The rest of the proof is identical to that of Theorem 2.1. □

The next result relies on the assumption

(H_5) there exists a constant $M_0 > 0$ such that, for each $u \in \operatorname{dom} L \setminus \ker L$ with $|u'(t)| > M_0$, $t \in [0,1]$, we have $QNu(t) \neq 0$.

Theorem 2.3. *If* (L), (F), (H_2), (H_3), *and* (H_5) *hold, then the boundary value problem* (1.1), (1.2) *has at least one solution provided* $-b\phi_1'(t) + a\phi_2'(t) \neq 0$ *on* $[0,1]$, *and*

$$D_3(\|\delta_1\|_1 + \|\delta_2\|_1) < 1, \tag{2.6}$$

where

$$D_3 = \frac{B\|-b\phi_1(t) + a\phi_2\|_X}{\min_{t\in[0,1]} |-b\phi_1'(t) + a\phi_2'(t)|} + \|K_P\|.$$

Proof. Again, let $\Omega_1 = \{u \in \operatorname{dom} L \setminus \ker L \colon Lu = \lambda Nu,\ \lambda \in (0,1)\}$ and $u \in \Omega_1$. By (H_5), there exists $t_0 \in [0,1]$ such that $|u'(t_0)| \leq M_0$.

As in the proof of Theorem 2.2, choose $u \in \Omega_1$, where $u = u_1 + u_2$, $u_1 = Pu \in \operatorname{Im} P = \ker L$, $u_2 = (I-P)u = K_P L u = \lambda K_P N u$. We have, by Lemma 1.6,

$$\|u_2'\|_0 < B\|Nu\|_1, \qquad \|u_2\|_X < \|K_P\|\|Nu\|_1. \tag{2.7}$$

Since $u_1 = u - u_2$, then $|(Pu)'(t_0)| = |u_1'(t_0)| \leq |u'(t_0)| + |u_2'(t_0)| < M_0 + B\|Nu\|_1$. We have

$$|u_1'(t_0)| = \frac{|-bl_1(u) + al_2(u)|}{a^2+b^2} |-b\phi_1'(t_0) + a\phi_2'(t_0)| < M_0 + A\|Nu\|_1.$$

For $u \in \Omega_1$, we have

$$\frac{|-bl_1(u) + al_2(u)|}{a^2+b^2} \leq \frac{M_0 + B\|Nu\|_1}{\min_{t\in[0,1]} |-b\phi_1'(t) + a\phi_2'(t)|}.$$

We infer

$$\begin{aligned}
\|u_1\|_X = \|Pu\|_X &\leq \frac{|-bl_1(u) + al_2(u)|}{a^2+b^2} \|-b\phi_1 + a\phi_2\|_X \\
&\leq \frac{\|-b\phi_1 + a\phi_2\|_X}{\min_{t\in[0,1]} |-b\phi_1'(t) + a\phi_2'(t)|}(M_0 + B\|Nu\|_1).
\end{aligned} \tag{2.8}$$

Applying (2.7) and (2.8), we deduce

$$\begin{aligned}
\|u\|_X &\leq \|u_1\|_X + \|u_2\|_X \\
&< C_1 + \left(\frac{B\|-b\phi_1 + a\phi_2\|_X}{\min_{t\in[0,1]} |-b\phi_1'(t) + a\phi_2'(t)|} + \|K_P\| \right) \|Nu\|_1 \\
&< C_2 + \left(\frac{B\|-b\phi_1 + a\phi_2\|_X}{\min_{t\in[0,1]} |-b\phi_1'(t) + a\phi_2'(t)|} + \|K_P\| \right) (\|\delta_1\|_1 + \|\delta_2\|_1)\|u\|_X \\
&< C_2 + D_3(\|\delta_1\|_1 + \|\delta_2\|_1)\|u\|_X.
\end{aligned}$$

Therefore, Ω_1 is bounded in view of (2.6). The rest of the proof replicates those of the previous theorems. $\square$

Note that the preceding results depend on $a^2 + b^2 \neq 0$ and deal with such resonance conditions that $\dim \ker L = 1$. If $a = b = 0$, then $\dim \ker L = 2$ and the projector P is simply P_0. We can find linearly independent $h_1, h_2 \in Z$ such that $\operatorname{Im} Q = \{c_1 h_1 + c_2 h_2 : c_1, c_2 \in \mathbb{R}\}$. Moreover, the generalized inverse has a simple form, namely, $K_P g = Tg$. Finally, we observe that the method of proof of Theorem 2.1 applies directly to this case.

Note that (1.3), (1.4) is a special case of (1.1), (1.2), that is, the former serves as an example of the latter. In conclusion, we present an example that cannot be so cheaply obtained.

Consider

$$Lu(t) = u''(t) - u(t) = \kappa(1 + 2\sin u'(t) + u(t)), \quad \text{a.e. } t \in (0,1), \tag{2.9}$$

where $\kappa \neq 0$, and

$$F_1(u) = u(0) - u(1) = 0, \qquad F_2(u) = u'(0) + u'(1) = 0. \tag{2.10}$$

In this case, $\phi_1(t) = e^t$ and $\phi_2(t) = e^{-t}$ with $W(\phi_1, \phi_2)(t) = -2$, $k(t,s) = \sinh(t-s)$. The equation (1.9) becomes

$$u(t) = \int_0^t \sinh(t-s) Lu(s)\, ds + u'(0)\sinh t + u(0)\cosh t.$$

Then $F_1(\phi_1) = 1 - e$, $F_1(\phi_2) = 1 - e^{-1}$, $F_2(\phi_1) = 1 + e$, $F_2(\phi_2) = -1 - e^{-1}$, that is, we have (F) with $a = 1 + e$, $b = -1 - e^{-1}$, and $\alpha = \frac{1-e}{1+e}$. Hence,

$$\ker L = \{c(-b\phi_1(t) + a\phi_2(t)) : c \in \mathbb{R}\} = \{c(e^t + e^{1-t}) : c \in \mathbb{R}\}.$$

Note that $-b\phi_1(t) + a\phi_2(t) \neq 0$ on $[0,1]$.

We also derive

$$\begin{aligned}(F_1 - \alpha F_2)Tg &= -\int_0^1 \sinh(1-s) g(s)\, ds + \frac{1-e}{1+e}\int_0^1 \cosh(1-s) g(s)\, ds \\ &= -\int_0^1 \left(\sinh(1-s) + \frac{e-1}{e+1}\cosh(1-s)\right) g(s)\, ds.\end{aligned}$$

In particular,

$$\begin{aligned}\operatorname{Im} L &= \{g \in Z : (F_1 - \alpha F_2)Tg = 0\} \\ &= \left\{g \in Z : \int_0^1 \left(\sinh(1-s) + \frac{e-1}{e+1}\cosh(1-s)\right) g(s)\, ds = 0\right\}.\end{aligned}$$

Introduce, for convenience,

$$\mathcal{K}(s) = -\sinh(1-s) - \frac{e-1}{e+1}\cosh(1-s) < 0$$

on $[0,1]$. As a result, if $|u(t)| > M_0 = 4$, we have

$$(F_1 - \alpha F_2)TNu = \kappa \int_0^1 \mathcal{K}(s)(1 + 2\sin u'(s) + u(s))\, ds \neq 0.$$

Hence (H_4) holds. It is also easy to find $M_1 > 0$ such that $|c| > M_1$ implies $c(F_1 - \alpha F_2)TNu_c \neq 0$. Indeed,

$$c(F_1 - \alpha F_2)TNu_c = c\kappa \int_0^1 \mathcal{K}(s)(1 + 2\sin u_c'(s))\, ds + c^2\kappa \int_0^1 \mathcal{K}(s)(-b\phi_1(s) + a\phi_2(s))\, ds,$$

where the first integral is bounded in c and the second integral is a constant. Thus, if $|c|$ is large enough, the assumption (H_3) is fulfilled.

Obviously, if $|\kappa|$ is small enough, then also (2.1) holds. Indeed,

$$|\kappa(1 + 2\sin u'(t) + u(t))| \le |\kappa| + 2|\kappa||u'(t)| + |\kappa||u(t)|,$$

that is, $\|\delta_1\|_1 = 2|\kappa|$ and $\|\delta_2\|_1 = |\kappa|$ can be made small enough to fulfill (H_3) by choosing a sufficiently small $|\kappa|$. By Theorem 2.2, the problem (2.9), (2.10) has a solution. Finally, since $-b\phi_1'(1/2) + a\phi_2'(1/2) = 0$, Theorem 2.3 cannot be applied to this particular problem at resonance.

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Periodic solutions with long period for the Mackey–Glass equation

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. The limiting version of the Mackey–Glass delay differential equation $x'(t) = -ax(t) + bf(x(t-1))$ is considered where a, b are positive reals, and $f(\xi) = \xi$ for $\xi \in [0, 1)$, $f(1) = 1/2$, and $f(\xi) = 0$ for $\xi > 1$. For every $a > 0$ we prove the existence of an $\varepsilon_0 = \varepsilon_0(a) > 0$ so that for all $b \in (a, a + \varepsilon_0)$ there exists a periodic solution $p = p(a, b) : \mathbb{R} \to (0, \infty)$ with minimal period $\omega(a, b)$ such that $\omega(a, b) \to \infty$ as $b \to a+$. A consequence is that, for each $a > 0$, $b \in (a, a + \varepsilon_0(a))$ and sufficiently large n, the classical Mackey–Glass equation $y'(t) = -ay(t) + by(t-1)/[1 + y^n(t-1)]$ has an orbitally asymptotically stable periodic orbit, as well, close to the periodic orbit of the limiting equation.

Keywords: Mackey–Glass equation, periodic solution, limiting nonlinearity, discontinuous right-hand side, long period.

2020 Mathematics Subject Classification: 34K13, 34K39, 34K06.

1 Introduction

The Mackey–Glass equation

$$y'(t) = -ay(t) + b\frac{y(t-\tau)}{1 + y^n(t-\tau)}$$

with positive parameters a, b, τ, n was proposed to model blood production and destruction in the study of oscillation and chaos in physiological control systems by Mackey and Glass [13]. This simple-looking differential equation with a single delay attracted the attention of many mathematicians since its hump-shaped nonlinearity causes entirely different dynamics compared to the case where the nonlinearity is monotone. See [16] for a similar equation. There exist several rigorous mathematical results, numerical and experimental studies on the Mackey–Glass equation showing convergence of the solutions, oscillations with different

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characteristics, and the complexity of the dynamics, see e.g. [1,3,6,7,9,15,17–19,22,23]. Despite the intense research, the dynamics is not fully understood yet.

The recent paper [2] studies the classical Mackey–Glass delay differential equation

$$y'(t) = -ay(t) + bf_n(y(t-1)) \tag{E_n}$$

where a, b, n are positive reals, $f_n(\xi) = \xi/[1+\xi^n]$ for $\xi \geq 0$, $\tau = 1$ can be assumed by rescaling the time. [2] constructs stable periodic solutions of (E_n) for some $b > a > 0$ and large n. The periodic solutions can have complicated shapes, see [2]. A limiting version of (E_n) plays a key role in the construction. The function $f(\xi) = \lim_{n\to\infty} f_n(\xi)$ is given by $f(\xi) = \xi$ for $\xi \in [0,1)$, $f(1) = 1/2$, and $f(\xi) = 0$ for $\xi > 1$. The equation

$$x'(t) = -ax(t) + bf(x(t-1)) \tag{E_∞}$$

is called the limiting Mackey–Glass equation.

Let $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ denote the set of real numbers, complex numbers and positive integers, respectively. Let C be the Banach space $C([-1,0],\mathbb{R})$ equipped with the norm $\|\varphi\| = \max_{s\in[-1,0]} |\varphi(s)|$. For a continuous function $u : I \to \mathbb{R}$ defined on an interval I, and for $t, t-1 \in I$, $u_t \in C$ is given by $u_t(s) = u(t+s)$, $s \in [-1,0]$. Introduce the subsets

$$\begin{aligned} C^+ &= \{\psi \in C : \psi(s) > 0 \text{ for all } s \in [-1,0]\}, \\ C_r^+ &= \left\{\psi \in C^+ : \psi^{-1}(c) \text{ is finite for all } c \in (0,1]\right\} \end{aligned}$$

of C where $\psi^{-1}(c) = \{s \in [-1,0] : \psi(s) = c\}$. C^+ and C_r^+ are metric spaces with the metric $d(\varphi,\psi) = \|\varphi - \psi\|$.

A solution of equation (E_n) on $[-1,\infty)$ with initial function $\psi \in C^+$ is a continuous function $y : [-1,\infty) \to \mathbb{R}$ so that $y_0 = \psi$, the restriction $y|_{(0,\infty)}$ is differentiable, and equation (E_n) holds for all $t > 0$. The solutions are easily obtained from the variation-of-constants formula for ordinary differential equations on successive intervals of length one,

$$y(t) = e^{-a(t-k)}y(k) + b\int_k^t e^{-a(t-s)} f_n(y(s-1))\, ds \tag{1.1}$$

where $k \in \mathbb{N} \cup \{0\}$, $k \leq t \leq k+1$. Hence it is well known that each $\psi \in C^+$ uniquely determines a solution $y = y^{n,\psi} : [-1,\infty) \to \mathbb{R}$ with $y_0^{n,\psi} = \psi$, and $y^{n,\psi}(t) > 0$ for all $t \geq 0$.

For equation (E_∞) with the discontinuous f, we use formula (1.1) with f instead of f_n to define solutions. A solution of equation (E_∞) with initial function $\varphi \in C^+$ is a continuous function $x = x^\varphi : [-1,t_\varphi) \to \mathbb{R}$ with some $0 < t_\varphi \leq \infty$ such that $x_0 = \varphi$, the map $[0,t_\varphi) \ni s \mapsto f(x(s-1)) \in \mathbb{R}$ is locally integrable, and

$$x(t) = e^{-a(t-k)}x(k) + b\int_k^t e^{-a(t-s)} f(x(s-1))\, ds \tag{1.2}$$

holds for all $k \in \mathbb{N} \cup \{0\}$ and $t \in [0,t_\varphi)$ with $k \leq t \leq k+1$.

It is easy to show that, for any $\varphi \in C^+$, there is a unique solution x^φ of equation (E_∞) on $[-1,\infty)$. However, comparing solutions with initial functions $\varphi > 1$, $\varphi \equiv 1$, one sees that there is no continuous dependence on initial data in C^+. Therefore we restrict our attention to the subset C_r^+ of C^+. The choice of C_r^+ as a phase space guarantees not only continuous dependence on initial data, but also allows to compare certain solutions of equations (E_∞) and (E_n) for large n. This is not used here, but it is important in [2]. [2] proves that for each $\varphi \in C_r^+$

there is a unique maximal solution $x^\varphi : [-1, \infty) \to \mathbb{R}$ of equation (E_∞). The maximal solution x^φ satisfies $x_t^\varphi \in C_r^+$ for all $t \geq 0$; and if $t > 0$ and $x^\varphi(t-1) \neq 1$, then x^φ is differentiable at t, and equation (E_∞) holds at t.

One of the main results of [2] is as follows.

Theorem 1.1. *If the parameters $b > a > 0$ are given so that*

(H) equation (E_∞) has an ω-periodic solution $p : \mathbb{R} \to \mathbb{R}$ with the following properties:

(i) $p(0) = 1$, $p(t) > 1$ for all $t \in [-1, 0)$,

(ii) $(p(t), p(t-1)) \neq (1, a/b)$ for all $t \in [0, \omega]$

holds then there exists an $n_* \geq 4$ such that, for all $n \geq n_*$, equation (E_n) has a periodic solution $p^n : \mathbb{R} \to \mathbb{R}$ with period $\omega^n > 0$ so that the periodic orbits

$$\mathcal{O}^n = \{p_t^n : t \in [0, \omega^n]\}$$

are hyperbolic, orbitally stable, exponentially attractive with asymptotic phase, moreover, $\omega^n \to \omega$, $\operatorname{dist}\{\mathcal{O}^n, \mathcal{O}\} \to 0$ *as $n \to \infty$, where $\mathcal{O} = \{p_t : t \in [0, \omega]\}$.*

[2] shows that in case b is large comparing to a, namely $b > \max\{ae^a, e^a - e^{-a}\}$, then (H) is satisfied. In addition, by using a rigorous computer-assisted technique, [2] gives parameter values a, b such that (H) is valid, and the obtained stable periodic orbits for the Mackey–Glass equation may have complicated structures.

[2] remarks that (H) holds if $b > a > 0$ and b is sufficiently close to a, and refers to this work for the proof. The aim of this paper is to prove this fact, namely the following result.

Theorem 1.2. *For every $a > 0$ there exists an $\varepsilon_0 = \varepsilon_0(a) > 0$ such that for the parameters a, b with $b \in (a, a + \varepsilon_0)$ condition (H) holds.*

In particular, for the periodic solution $p = p(a, b)$ of equation (E_∞) the minimal period $\omega = \omega(a, b)$ satisfies $\omega > 5$, and there exists a $\sigma = \sigma(a, b) \in (4, \omega - 1)$ so that

$$0 < p(t) < 1 \text{ for all } t \in (0, \sigma); \; p(t) > 1 \text{ for all } t \in (\sigma, \omega).$$

Moreover, if $a > 0$ is fixed and $(b_k)_{k=1}^\infty$ is a sequence in $(a, a + \varepsilon_0(a))$, $\lim_{k\to\infty} b_k = a$ then $\sigma(a, b_k) \to \infty$, $\omega(a, b_k) \to \infty$ as $k \to \infty$.

Theorems 1.1 and 1.2 immediately imply the following result for equation (E_n).

Theorem 1.3. *For each $a > 0$ there exists an $\varepsilon_0 = \varepsilon_0(a) > 0$ such that for every $b \in (a, a + \varepsilon_0)$ there exists an $n^* = n^*(a, b) \geq 4$ so that, for all $n \geq n^*$, equation (E_n) has a periodic solution $p^n : \mathbb{R} \to \mathbb{R}$ with minimal period $\omega^n(a, b)$ so that the periodic orbits*

$$\mathcal{O}^n = \{p_t^n : t \in [0, \omega^n]\}$$

are hyperbolic, orbitally stable, exponentially attractive with asymptotic phase. Moreover, if $(b_k)_{k=1}^\infty$ is a sequence in $(a, a + \varepsilon_0(a))$ with $\lim_{k\to\infty} b_k = a$, $n_k > n^*(a, b_k)$ then $\omega^n(a, b_k) \to \infty$ as $k \to \infty$.

Note that the papers [8] by Karakostas et al. and [5] by Gopalsamy et al. give conditions for the global attractivity of the unique positive equilibrium of (E_n) for $b > a > 0$, and n is below a certain constant given in terms of a, b. Theorem 1.3 requires n to be large.

Section 2 contains the proof of Theorem 1.2. The proof requires the study of a special solution of a linear autonomous delay differential equation. If $\varphi \in C_r^+$ is any function such that $\varphi(s) > 1$ for $s \in [-1, 0)$ and $\varphi(0) = 1$ then the unique solution $x = x^{\varphi}$ of equation (E_∞) satisfies $x(t) = e^{-at}$ for $t \in [0, 1]$. In order to find a periodic solution of (E_∞) as stated in Theorem 1.2 we consider the linear autonomous equation

$$u'(t) = -au(t) + bu(t-1)$$

for $t > 1$ with $u(t) = e^{-at}$, $t \in [0, 1]$. If we find a $T > 0$ such that $u(t) < 1$ for $t \in (0, T)$, $u(T) = 1$, $u(t) > 1$ for $t \in (T, T+1]$, then it is straightforward to see that $x(t) = u(t)$ for all $t \in [0, T+1]$. Then, equation (E_∞) gives $x'(t) = -ax(t)$ for all $t > T+1$ as long as $x(t-1) > 1$. Hence there exists an $\omega > T+1$ with $x(\omega) = 1$ and $x(t) > 1$ for all $t \in (T, \omega)$. By the fact $f(\xi) = 0$ for $\xi > 1$, the solution x does not change on $[0, \infty)$ if φ is replaced by x_ω, and consequently $x(t) = x(t+\omega)$ follows for all $t \geq -1$. Therefore the proof of Theorem 1.2 is reduced to the existence of a $T > 0$ with $u(t) < 1$ for $t \in (0, T)$, $u(T) = 1$, $u(t) > 1$ for $t \in (T, T+1]$. Property (H)(ii) is guaranteed by $u'(T) > 0$.

We remark that the use of a limiting equation in order to study nonlinear delay differential equations when the nonlinearity is close to its limiting function is not new. We refer to the papers [10–12, 21, 24–26] where the limiting step function reduces the search of periodic solutions to a finite dimensional problem. The limiting Mackey–Glass nonlinearity f is not a step function. The introduction of the limiting Mackey–Glass equation does not reduce the search for periodic solutions to a finite dimensional problem, nevertheless it can simplify it. The paper [14] considered the limiting Mackey–Glass nonlinearity to construct periodic solutions for an equation different from (E_n). The result of [14] is analogous to the case when b is large comparing to a, mentioned above for the Mackey–Glass equation.

2 The proof of Theorem 1.2

The proof is divided into eight steps. The desired periodic solution of equation (E_∞) will be an ω-periodic extension of a function $w : [0, \omega] \to \mathbb{R}$. We construct w in the remaining part of this section.

Step 1. Let $a > 0$ be fixed, and consider the characteristic function

$$h : \mathbb{C} \times \mathbb{R} \ni (z, \varepsilon) \mapsto z + a - (a + \varepsilon)e^{-z} \in \mathbb{C}$$

of the linear delay differential equation $v'(t) = -av(t) + (a+\varepsilon)v(t-1)$. By $h(0, 0) = 0$, $D_1 h(0, 0) = 1 + a$, and $D_2 h(0, 0) = -1$, the Implicit Function Theorem can be applied to get that there are $\varepsilon_1 \in (0, \min\{a, 1/4\})$, $r_1 \in (0, 1)$ and a C^1-smooth map $\lambda_0 : (-\varepsilon_1, \varepsilon_1) \to \mathbb{C}$ such that $\lambda_0(0) = 0$, $h(\lambda_0(\varepsilon), \varepsilon) = 0$, and $(\lambda_0(\varepsilon), \varepsilon)$ is the unique solution of $h(z, \varepsilon) = 0$ in the set $\{z \in \mathbb{C} : |z| < r_1\} \times (-\varepsilon_1, \varepsilon_1)$. Since a and ε are real in the equation $h(z, \varepsilon) = 0$, (z, ε) is a solution together with $(\bar{z}, \varepsilon)$. Then, by uniqueness, it follows that $\lambda_0(\varepsilon) \in \mathbb{R}$, $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$.

Chapter XI of [4] applies to get that the zeros of the characteristic function $h(z, \varepsilon)$ for $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ are $\lambda_0(\varepsilon) \in \mathbb{R}$ and a sequence of pairs $\left(\lambda_j(\varepsilon), \overline{\lambda_j(\varepsilon)}\right)_{j=1}^{\infty}$ with

$$\lambda_0(\varepsilon) > \operatorname{Re} \lambda_1(\varepsilon) > \operatorname{Re} \lambda_2(\varepsilon) > \cdots > \operatorname{Re} \lambda_j(\varepsilon) \to -\infty \quad \text{as} \quad j \to \infty$$

and

$$\operatorname{Im} \lambda_j \in \big((2j-1)\pi, 2j\pi\big) \quad (j \in \mathbb{N}).$$

If $\varepsilon = 0$ then $\lambda_0(0) = 0$, and consequently $\operatorname{Re} \lambda_1(0) < 0$. Fix $c \in (0, a)$ so that

$$\operatorname{Re} \lambda_1(0) < -2c.$$

Notice that the choice of c depends only on a.

Differentiating the equation $h(\lambda_0(\varepsilon),\varepsilon) = 0$ with respect to ε we obtain $\lambda_0'(0) = 1/(1+a)$, and thus

$$\lambda_0(\varepsilon) = \frac{\varepsilon}{1+a} + \eta(\varepsilon)$$

with a function $\eta : (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}$ satisfying $\lim_{\varepsilon\to 0} \eta(\varepsilon)/\varepsilon = 0$. Applying the above representation for $\lambda_0(\varepsilon)$, we assume (in addition to the above properties of ε_1) that ε_1 is so small that

$$\lambda_0(\varepsilon) < \frac{2\varepsilon}{1+2a} \quad \text{for all } \varepsilon \in (0, \varepsilon_1), \tag{2.1}$$

where the equality $2\varepsilon/(1+2a) = \varepsilon/(1+a) + \varepsilon/[(1+a)(1+2a)]$ shows that this is possible.

By Rouché's theorem [20] there exists an $\varepsilon_2 \in (0, \varepsilon_1)$ such that

$$\operatorname{Re} \lambda_1(\varepsilon) < -2c \quad \text{for all } \varepsilon \in [0, \varepsilon_2].$$

In particular, $h(z,\varepsilon) \neq 0$ on the line $\{-c + is : s \in \mathbb{R}\}$ for all $\varepsilon \in [0, \varepsilon_2]$.

Step 2. For $\varepsilon \in (0, \varepsilon_2)$ consider the unique solution $v : [-1, \infty) \to \mathbb{R}$ of the linear equation

$$v'(t) = -av(t) + (a+\varepsilon)v(t-1) \qquad (t > 0) \tag{2.2}$$

with initial function $v_0(s) = e^{-a(s+1)}$, $-1 \leq s \leq 0$. Remark that v and λ_0 depend on ε as well. Taking the Laplace transform of both sides of (2.2) and expressing the Laplace transform $\mathcal{L}(v)(z)$ of v,

$$\mathcal{L}(v)(z) = \frac{1}{h(z,\varepsilon)} \left[e^{-a} + (a+\varepsilon) \frac{1 - e^{-(z+a)}}{z+a} \right]$$

is obtained where the right hand side can be written as $F(z,\varepsilon) = F_1(z) + F_2(z,\varepsilon)$ with

$$F_1(z) = \frac{e^{-a}}{z+a}, \qquad F_2(z,\varepsilon) = \frac{a+\varepsilon}{(z+a)h(z,\varepsilon)}.$$

According to Chapter I of [4], by taking the inverse Laplace transform, function v can be written as

$$v(t) = e^{\lambda_0 t} \operatorname{Res}_{\lambda_0} F(z,\varepsilon) + \frac{1}{2\pi} e^{-ct} \lim_{T\to\infty} \int_{-T}^{T} e^{ist} F(-c+is, \varepsilon)\, ds \quad (t > 0).$$

As $F_1(z)$ is holomorphic in a neighborhood of λ_0, one finds $\operatorname{Res}_{\lambda_0} F(z,\varepsilon) = \operatorname{Res}_{\lambda_0} F_2(z,\varepsilon)$. By using that $h(z,\varepsilon)$ has a simple zero at λ_0, and $\lambda_0 + a = (a+\varepsilon)e^{-\lambda_0}$, we get

$$\operatorname{Res}_{\lambda_0} F(z,\varepsilon) = \frac{a+\varepsilon}{(\lambda_0 + a)\, D_1 h(\lambda_0, \varepsilon)} = \frac{a+\varepsilon}{(\lambda_0+a)(1 + (a+\varepsilon)e^{-\lambda_0})} = \frac{e^{\lambda_0}}{1+a+\lambda_0}.$$

For $t \geq 1$, integration by parts leads to

$$\int_{-T}^{T} e^{ist} F_1(-c+is)\, ds = \left[\frac{e^{ist}}{it} \frac{e^{-a}}{a-c+is} \right]_{s=-T}^{s=T} + \int_{-T}^{T} \frac{e^{ist}}{it} \frac{ie^{-a}}{(a-c+is)^2}\, ds.$$

Thus

$$\left| \lim_{T\to\infty} \int_{-T}^{T} e^{ist} F_1(-c+is)\, ds \right| \leq \int_{-\infty}^{\infty} \left| \frac{e^{ist}}{it} \frac{ie^{-a}}{(a-c+is)^2} \right| ds \leq K_1$$

with
$$K_1 = 2\int_0^\infty \frac{e^{-a}}{(a-c)^2+s^2}\,ds.$$
Let $s_0 = 2(a+1)e^c$. The continuous function $(s,\varepsilon) \mapsto h(-c+is,\varepsilon) \in \mathbb{C}$ is nonzero on the set $[-s_0,s_0]\times[0,\varepsilon_2]$. So there exists $k>0$ such that $|F_2(-c+is,\varepsilon)| \le k$ on the compact set $[-s_0,s_0]\times[0,\varepsilon_2]$. If $|s|\ge s_0$, $\varepsilon\in[0,\varepsilon_2]$ then, by the choice of s_0,
$$\begin{aligned}|h(-c+is,\varepsilon)| &\ge |a-c+is| - |(a+\varepsilon)e^{c-is}| \ge \left[(a-c)^2+s^2\right]^{1/2} - (a+1)e^c\\ &\ge \frac{1}{2}\left[(a-c)^2+s^2\right]^{1/2}.\end{aligned}$$
Consequently
$$\begin{aligned}\left|\lim_{T\to\infty}\int_{-T}^{T} e^{ist}F_2(-c+is,\varepsilon)\,ds\right| &\le \int_{-\infty}^{\infty}|F_2(-c+is,\varepsilon)|\,ds\\ &\le 2\int_0^{s_0} k\,ds + 2\int_{s_0}^\infty \frac{a+1}{(1/2)[(a-c)^2+s^2]}\,ds\\ &= K_2\end{aligned}$$
with
$$K_2 = 2ks_0 + 4\int_{s_0}^\infty \frac{(a+1)}{(a-c)^2+s^2}\,ds.$$
Notice that both K_1 and K_2 are independent of $\varepsilon\in(0,\varepsilon_2)$.

Summarizing the above estimations we obtain that
$$v(t) = \frac{e^{\lambda_0(t+1)}}{1+a+\lambda_0} + \hat{r}(t) \qquad (t\ge 1)$$
for some continuous function $\hat{r}:[1,\infty)\to\mathbb{R}$ satisfying
$$|\hat{r}(t)| \le \hat{K}e^{-ct} \qquad (t\ge 1)$$
with $\hat{K} = (K_1+K_2)/(2\pi)$. Note that $\hat{r}$ depends on ε, however $\hat{K}$ and c are independent of ε.

Step 3. For $\varepsilon\in(0,\varepsilon_2)$ define the function $u:[0,\infty)\to\mathbb{R}$ by $u(t)=v(t-1)$, $t\ge 0$. Then $u(t)=e^{-at}$ for $t\in[0,1]$, u is differentiable on $(1,\infty)$ and satisfies
$$u'(t) = -au(t) + (a+\varepsilon)u(t-1) \qquad (t>1). \tag{2.3}$$
Moreover, defining $r(t)=\hat{r}(t-1)$ for $t\ge 2$, $K=\hat{K}e^c$, u has the representation
$$u(t) = \frac{e^{\lambda_0 t}}{1+a+\lambda_0} + r(t) \qquad (t\ge 2) \tag{2.4}$$
with the continuous function $r:[2,\infty)\to\mathbb{R}$ satisfying
$$|r(t)| \le Ke^{-ct} \qquad (t\ge 2). \tag{2.5}$$

From equation (2.3)
$$\begin{aligned}u(t) &= e^{-a(t-1)}u(1) + \int_1^t (a+\varepsilon)e^{-a(t-s)}e^{-a(s-1)}\,ds\\ &= e^{-at}\left[1+(a+\varepsilon)e^a(t-1)\right] \qquad (t\in[1,2])\end{aligned}$$

and

$$u'(t) = e^{-at}\left[-a - a(a+\varepsilon)e^{a}(t-1) + (a+\varepsilon)e^{a}\right] \qquad (t \in (1,2]).$$

Define

$$t_0 = t_0(\varepsilon) = 1 + \frac{1}{a} - \frac{1}{(a+\varepsilon)e^{a}}.$$

Choose $\varepsilon_3 \in (0, \varepsilon_2]$ so that

$$\varepsilon_3 < \frac{a}{1-a}\left(e^{-a} - 1 + a\right)$$

provided $a \in (0,1)$, and let $\varepsilon_3 = \varepsilon_2$ if $a \geq 1$.

Suppose $\varepsilon \in (0, \varepsilon_3)$. Then $t_0 = t_0(\varepsilon) \in (1,2)$ is the unique zero of u' in $(1,2)$, and it is easy to see that

$$\max_{t\in[1,2]} u(t) = u(t_0) = e^{-at_0}\left[1 + (a+\varepsilon)e^{a}(t_0 - 1)\right] = \frac{a+\varepsilon}{a}\exp\left[\frac{ae^{-a}}{a+\varepsilon} - 1\right]. \tag{2.6}$$

Step 4. In this step we show the following

CLAIM:

(i) For each $k \in \mathbb{N}$

$$\max_{t\in[k+1,k+2]} u(t) \leq \left(1 + \frac{\varepsilon}{a}\right)\max_{t\in[k,k+1]} u(t),$$

and

(ii) for each $N \in \mathbb{N}$

$$\max_{t\in[N+1,N+2]} u(t) \leq \left(1 + \frac{\varepsilon}{a}\right)^{N}\max_{t\in[1,2]} u(t).$$

Let $k \in \mathbb{N}$ be given. If $\max_{t\in[k+1,k+2]} \leq \max_{t\in[k,k+1]} u(t)$ then the stated inequality obviously holds for k. If $\max_{t\in[k+1,k+2]} u(t) > \max_{t\in[k,k+1]} u(t)$, then there exists a $t_1 \in (k+1, k+2]$ such that $u'(t_1) \geq 0$ and $u(t_1) = \max_{t\in[k+1,k+2]} u(t)$. Equation (2.3) at $t = t_1$ and $u'(t_1) \geq 0$ imply the inequality $-au(t_1) + (a+\varepsilon)u(t_1 - 1) \geq 0$. Hence

$$\max_{t\in[k+1,k+2]} u(t) = u(t_1) \leq \frac{a+\varepsilon}{a}u(t_1 - 1) \leq \left(1 + \frac{\varepsilon}{a}\right)\max_{t\in[k,k+1]} u(t),$$

that is, the stated inequality is satisfied. This proves (i).

A repeated application of (i) gives (ii):

$$\begin{aligned}\max_{t\in[N+1,N+2]} u(t) &\leq \left(1 + \frac{\varepsilon}{a}\right)\max_{t\in[N,N+1]} u(t) \leq \left(1 + \frac{\varepsilon}{a}\right)^{2}\max_{t\in[N-1,N]} u(t)\\ &\leq \cdots \leq \left(1 + \frac{\varepsilon}{a}\right)^{N}\max_{t\in[1,2]} u(t).\end{aligned}$$

Step 5. Choose $\xi_0 \in (\exp(e^{-a} - 1), 1)$. The function

$$(0,\infty) \ni \varepsilon \mapsto \frac{a+\varepsilon}{a}\exp\left[\frac{ae^{-a}}{a+\varepsilon} - 1\right] \in \mathbb{R}$$

strictly increases and its limit is $\exp(e^{-a} - 1)$ as $\varepsilon \to 0+$. Therefore there exists an $\varepsilon_4 \in (0, \varepsilon_3)$ such that

$$\frac{a+\varepsilon}{a}\exp\left[\frac{ae^{-a}}{a+\varepsilon} - 1\right] < \xi_0$$

for all $\varepsilon \in (0, \varepsilon_4)$.

By the equality (2.6) in Step 3 and the choice of ε_4, for all $\varepsilon \in (0, \varepsilon_4)$, the inequality $\max_{t\in[1,2]} u(t) < \xi_0$ holds. Then by the CLAIM in Step 4

$$\max_{t\in[1,N+2]} u(t) < \left(1 + \frac{\varepsilon}{a}\right)^N \xi_0 \tag{2.7}$$

follows for all $N \in \mathbb{N}$.

For a given $N \in \mathbb{N}$, from (2.7) one gets

$$\max_{t\in[1,N+2]} u(t) < 1$$

provided $\varepsilon \in (0, \varepsilon_4)$ is so small that

$$\varepsilon < a\left[(1/\xi_0)^{1/N} - 1\right]. \tag{2.8}$$

Step 6. Let $N \in \mathbb{N} \setminus \{1, 2\}$ be given. We look for a condition on $\varepsilon \in (0, \varepsilon_4)$ to guarantee

$$u'(t) > 0 \qquad \text{for all } t > N. \tag{2.9}$$

Equation (2.3) gives that

$$au(t) < (a + \varepsilon)u(t-1) \qquad \text{for all } t > N \tag{2.10}$$

is sufficient to yield (2.9). By the representation (2.4) condition (2.10) is equivalent to

$$\frac{a}{1 + a + \lambda_0} e^{\lambda_0 t}\left[\left(1 + \frac{\varepsilon}{a}\right) e^{-\lambda_0} - 1\right] > ar(t) - (a + \varepsilon)r(t-1) \qquad (t > N),$$

that is

$$\left(1 + \frac{\varepsilon}{a}\right) e^{-\lambda_0} - 1 > \frac{1 + a + \lambda_0}{a} e^{-\lambda_0 t}\left[ar(t) - (a + \varepsilon)r(t-1)\right] \qquad (t > N).$$

From $\varepsilon < 1$, $0 < \lambda_0(\varepsilon) < 1$ and (2.5) one obtains

$$\begin{aligned}
&\frac{1 + a + \lambda_0}{a} e^{-\lambda_0 t}\left[ar(t) - (a + \varepsilon)r(t-1)\right] \\
&\quad < \frac{(a+2)(2a+1)}{a} K e^{-c(t-1)} \\
&\quad < \frac{(a+2)(2a+1)}{a} K e^{c} e^{-cN} \qquad (t > N).
\end{aligned}$$

Recall that, by the choice of ε_1 in Step 1,

$$\lambda_0(\varepsilon) < \frac{2\varepsilon}{2a+1}.$$

Hence

$$e^{-\lambda_0(\varepsilon)} > 1 - \lambda_0(\varepsilon) > 1 - \frac{2\varepsilon}{2a+1}.$$

Thus, by using $\varepsilon_1 < 1/4$ as well,

$$\begin{aligned}
\left(1 + \frac{\varepsilon}{a}\right) e^{-\lambda_0(\varepsilon)} - 1 &> \left(1 + \frac{\varepsilon}{a}\right)\left(1 - \frac{2\varepsilon}{2a+1}\right) - 1 \\
&= \frac{\varepsilon - 2\varepsilon^2}{a(2a+1)} > \frac{\varepsilon}{2a(2a+1)}.
\end{aligned}$$

Consequently, (2.9) holds if, in addition to $\varepsilon \in (0, \varepsilon_4)$,

$$\varepsilon > \xi_1 e^{-cN} \tag{2.11}$$

with $\xi_1 = 2(a+2)(2a+1)^2 K e^c$.

Step 7. In order to satisfy conditions (2.8) and (2.11) simultaneously consider $a\left[(1/\xi_0)^{1/N} - 1\right]$ and $\xi_1 e^{-cN}$. By L'Hospital's rule

$$\lim_{N\to\infty} \frac{\xi_1 e^{-cN}}{a\left[(1/\xi_0)^{1/N} - 1\right]} = 0.$$

Therefore there exists an integer $N_0 > 2$ such that

$$\frac{\xi_1 e^{-cN}}{a\left[(1/\xi_0)^{1/(N+1)} - 1\right]} < 1 \quad \text{for all integers } N \geq N_0. \tag{2.12}$$

Define $\varepsilon_* \in (0, \varepsilon_4)$ so that

$$\varepsilon_* < a\left[(1/\xi_0)^{1/N_0} - 1\right].$$

Let $\varepsilon \in (0, \varepsilon_*)$ be fixed. By $\varepsilon < \varepsilon_*$ and $\lim_{N\to\infty} a\left[(1/\xi_0)^{1/N} - 1\right] = 0$ there exists a maximal integer $N(\varepsilon) \geq N_0$ so that

$$\varepsilon < a\left[(1/\xi_0)^{1/N(\varepsilon)} - 1\right]. \tag{2.13}$$

The maximality of $N(\varepsilon) \geq N_0$ and inequality (2.12) imply

$$\xi_1 e^{-cN(\varepsilon)} < a\left[(1/\xi_0)^{1/(N(\varepsilon)+1)} - 1\right] \leq \varepsilon.$$

Therefore, we arrive at the inequality

$$\xi_1 e^{-cN(\varepsilon)} < \varepsilon < a\left[(1/\xi_0)^{1/N(\varepsilon)} - 1\right], \tag{2.14}$$

that is, for every $\varepsilon \in (0, \varepsilon_*)$ inequalities (2.11) and (2.8) hold with $N = N(\varepsilon)$.

Step 8. By Steps 5–7, for each $\varepsilon \in (0, \varepsilon_*)$ there exists an integer $N = N(\varepsilon) > 2$ such that the unique continuous function $u = u(\varepsilon) : [0, \infty) \to \mathbb{R}$ satisfying $u(t) = e^{-at}$ for $t \in [0, 1]$, and equation (2.3) on $(1, \infty)$ has the properties

$$\begin{aligned} 1 = u(0) > u(t) > 0 &\quad \text{for all } t \in (0, N+2), \\ u'(t) > 0 &\quad \text{for all } t > N, \\ u(t) \to \infty &\quad \text{as } t \to \infty. \end{aligned} \tag{2.15}$$

The last property is clear from $\lambda_0(\varepsilon) > 0$, (2.4) and (2.5).

From (2.15) it follows that there exits a unique $\sigma(\varepsilon) > N(\varepsilon) + 2 > 4$ so that $u(\sigma(\varepsilon)) = 1$ and $u'(\sigma(\varepsilon)) > 0$. From $u'(\sigma(\varepsilon)) > 0$ it is clear that $u(\sigma(\varepsilon) - 1) \neq a/(a+\varepsilon)$. The maximality of $N(\varepsilon)$ in inequality (2.13) implies that $N(\varepsilon) \to \infty$, $\sigma(\varepsilon) \to \infty$ as $\varepsilon \to 0+$.

Let $\omega(\varepsilon) = \sigma(\varepsilon) + 1 + (1/a)\log u(\sigma(\varepsilon) + 1) > 5$. Define the function $w : [0, \omega(\varepsilon)] \to \mathbb{R}$ by

$$w(t) = \begin{cases} u(t) & \text{if } t \in [0, \sigma(\varepsilon) + 1], \\ u(\sigma(\varepsilon) + 1) e^{-a(t - \sigma(\varepsilon) - 1)} & \text{if } t \in [\sigma(\varepsilon) + 1, \omega(\varepsilon)]. \end{cases}$$

Then $w(t) > 1$ for all $t \in (\sigma, \omega)$, and $w(\omega) = 1$. Let $p : \mathbb{R} \to \mathbb{R}$ be the $\omega(\varepsilon)$-periodic extension of w to $\mathbb{R}$.

For the fixed $a > 0$ set $\varepsilon_0 = \varepsilon_*$. Observe that c, K, and consequently ξ_0, ξ_1, depend only on a. Then relation (2.12) shows that N_0 is also a function of a . Therefore, ε_0 depends only on a.

If $b \in (a, a + \varepsilon_0)$ then the above constructed $p(\varepsilon)$ with $\varepsilon = b - a \in (0, \varepsilon_*)$ is clearly an $\omega(\varepsilon)$-periodic solution of equation (E_∞) satisfying (H). Setting $\omega(a, b) = \omega(\varepsilon)$ and $\sigma(a, b) = \sigma(\varepsilon)$, we see that all statements of Theorem 1.2 are satisfied, and the proof is complete.

The typical shape of the periodic solutions obtained in this paper for (E_∞) is shown in Figure 2.1 with $a = 9$, $b = 9.7$.

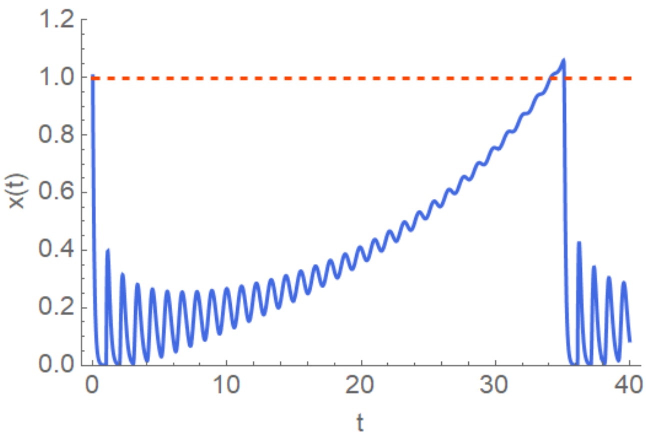

Figure 2.1: The periodic solution of (E_∞) for $a = 9$, $b = 9.7$

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Compactness of Riemann–Liouville fractional integral operators

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. We obtain results on compactness of two linear Hammerstein integral operators with singularities, and apply the results to give new proof that Riemann–Liouville fractional integral operators of order $\alpha \in (0,1)$ map $L^p(0,1)$ to $C[0,1]$ and are compact for each $p \in \left(\frac{1}{1-\alpha}, \infty\right]$. We show that the spectral radii of the Riemann–Liouville fractional operators are zero.

Keywords: linear Hammerstein integral operator, Riemann–Liouville fractional integral operator, compactness, spectral radius.

2020 Mathematics Subject Classification: Primary: 26A33; Secondary: 45A05, 45D05.

1 Introduction

Riemann–Liouville left-sided and right-sided fractional integral operators of order $\alpha \in (0,1)$, denoted by $I_{0^+}^{1-\alpha}$ and $I_{1^-}^{1-\alpha}$, respectively, are two special linear Volterra integral operators with the kernel

$$k(x,y) = \frac{1}{|x-y|^\alpha}. \tag{1.1}$$

The kernel k is singular at each (x,x) and the singularities often make it difficult to study problems such as continuity and compactness of these operators defined in subspaces of $L^1(0,1)$.

It is well known that $I_{0^+}^{1-\alpha}$ is bounded from $L^p(0,1)$ to $L^p(0,1)$ for each $p \in [1,\infty]$, and from $L^p(0,1)$ to $C[0,1]$ for each $p \in \left(\frac{1}{1-\alpha}, \infty\right]$, see [6, Theorem 2.6], [11, Theorem 12], [20, Theorem 3.6] and [23, Proposition 3.2 (1) and (3)]. It is implicitly proved in [6, Theorem 6.1] that $I_{0^+}^{1-\alpha}$ is compact from $C[0,1]$ to $C[0,1]$ and in [22, Theorem 4.8] that $I_{0^+}^{1-\alpha}$ is compact from $D \subset C[0,1] \to P$, where D is a subset of $C[0,1]$ and P is the standard positive cone in $C[0,1]$.

In this paper, we prove that both $I_{0^+}^{1-\alpha}$ and $I_{1^-}^{1-\alpha}$ are compact from $L^p(0,1)$ to $C[0,1]$ for each $p \in \left(\frac{1}{1-\alpha}, \infty\right]$. This allows one to study the existence of solutions of the initial or

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boundary value problems for nonlinear fractional differential equations with discontinuous nonlinearities by applying the fixed point theorems or fixed point index theories. We refer to [2–6, 8–10, 13, 15, 18, 19, 21–27] for the study of these nonlinear problems.

To study the compactness of $I_{0^+}^{1-\alpha}$, we first study compactness of the following two linear Hammerstein integral operators L and $\mathscr{L}$:

$$Lv(x) = \int_0^1 k(x,y)v(y)\,dy \quad \text{for each } x \in [0,1], \tag{1.2}$$

and

$$\mathscr{L}v(x) = \int_0^1 \sigma(x,y)k(x,y)v(y)\,dy \quad \text{for each } x \in [0,1], \tag{1.3}$$

where $k : [0,1] \times [0,1] \setminus \mathscr{D} \to \mathbb{R}$ has singularities in a subset

$$\mathscr{D} = \{(x,y) : x \in [0,1], y \in D(x)\}$$

to be defined in Section 2, and $\sigma(x,y) = \operatorname{sgn}(x-y)$. It is not trivial to prove compactness of these operators due to the singularities of k on $\mathscr{D}$.

Under suitable assumptions on k, we prove that both L and $\mathscr{L}$ map $L^p(0,1)$ to $C[0,1]$ and are compact for $p \in [1,\infty]$. In particular, when $\mathscr{D} = \{(x,x) : x \in [0,1]\}$, we show that $I_{0^+}^{1-\alpha}$ and $I_{1^-}^{1-\alpha}$ are proportional to the sum and substraction of the two operators L_α and $\mathscr{L}_\alpha$, respectively, where L_α and $\mathscr{L}_\alpha$ are the two operators L and $\mathscr{L}$ with the kernel k defined in (1.1). When $p \in \left(\frac{1}{1-\alpha},\infty\right]$, these relations are used to derive compactness of $I_{0^+}^{1-\alpha}$ and $I_{1^-}^{1-\alpha}$ from the compactness of L_α and $\mathscr{L}_\alpha$. As applications of compactness of $I_{0^+}^{1-\alpha}$, we show that the spectral radius of $I_{0^+}^{1-\alpha}$ is 0, and $I_{0^+}^{1-\alpha}$ has no eigenfunctions.

2 Compactness of linear integral operators

In this section, we study the following two linear Hammerstein integral operators

$$Lv(x) = \int_0^1 k(x,y)v(y)\,dy \quad \text{for each } x \in [0,1], \tag{2.1}$$

where the kernel k is allowed to have singularities on $[0,1] \times [0,1]$ and

$$\mathscr{L}v(x) = \int_0^1 \sigma(x,y)k(x,y)v(y)\,dy \quad \text{for each } x \in [0,1], \tag{2.2}$$

where $\sigma : [0,1] \times [0,1] \to \mathbb{R}$ is defined by

$$\sigma(x,y) = \operatorname{sgn}(x-y) = \begin{cases} 1 & \text{if } y < x, \\ 0 & \text{if } y = x, \\ -1 & \text{if } x < y. \end{cases} \tag{2.3}$$

Unless stated otherwise, $p, q \in [1,\infty]$ are the conjugate indices, that is, they satisfy the following condition:

$$1/p + 1/q = 1, \tag{2.4}$$

where if $p = \infty$, then $q = 1$ and if $p = 1$, then $q = \infty$.

We denote by $L^p[0,1]$ and $L^p_+[0,1]$ the Banach space of functions for which the pth power of the absolute values are Lebesgue integrable with the norm $\|\cdot\|_{L^p(0,1)}$, and its positive cone, respectively, and by $C[0,1]$ the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ with the maximum norm denoted by $\|\cdot\|_{C[0,1]}$ or $\|\cdot\|$.

Let X, Y be Banach spaces. Recall that a linear map $L : X \to Y$ is said to be compact if L is continuous and $\overline{L(S)}$ is compact for each bounded subset $S \subset X$.

Assume that for each $x \in [0,1]$, there exists a subset $D(x)$ of $[0,1]$ satisfying $\text{meas}(D(x)) = 0$. Let

$$\mathscr{D} = \{(x,y) : x \in [0,1],\ y \in D(x)\}.$$

It is easy to verify that $(x,y) \in [0,1] \times [0,1] \setminus \mathscr{D}$ if and only if $x \in [0,1]$ and $y \in [0,1] \setminus D(x)$.

Theorem 2.1. *Let $p, q \in [1,\infty]$ satisfy (2.4). Assume that $k : [0,1] \times [0,1] \setminus \mathscr{D} \to \mathbb{R}$ satisfies the following conditions.*

(i) For each $x \in [0,1]$, $k(x,\cdot) : [0,1] \setminus D(x) \to \mathbb{R}$ satisfies $k(x,\cdot) \in L^q(0,1)$.

(ii) For each $\tau \in [0,1]$, $\lim_{x\to\tau} \|k(x,\cdot) - k(\tau,\cdot)\|_{L^q(0,1)} = 0$.

Then the map L defined in (2.1) maps $L^p(0,1)$ to $C[0,1]$ and is compact.

Proof. Let $v \in L^p(0,1)$. By the condition (i) we have

$$|Lv(x)| = \left|\int_0^1 k(x,y)v(y)\,dy\right| \le \|k(x,\cdot)\|_{L^q(0,1)}\|v\|_{L^p(0,1)} < \infty \quad \text{for each } x \in [0,1]$$

and Lv is well defined on $[0,1]$. For $\tau, x \in [0,1]$, we have

$$\begin{aligned} |Lv(x) - Lv(\tau)| &\le \int_0^1 |k(x,y) - k(\tau,y)||v(y)|\,dy \\ &\le \|k(x,\cdot) - k(\tau,\cdot)\|_{L^q(0,1)}\|v\|_{L^p(0,1)}. \end{aligned} \tag{2.5}$$

It follows from the condition (ii) that $Lv \in C[0,1]$ for $v \in L^p(0,1)$ and the first part of the result holds.

We define a map $\mathscr{K} : [0,1] \to L^q(0,1)$ by

$$\mathscr{K}(x) = k(x,\cdot).$$

Then the conditions (i) and (ii) are equivalent to the fact that $\mathscr{K} : [0,1] \to L^q(0,1)$ is continuous. Hence, $\|\mathscr{K}(\cdot)\|_{L^q(0,1)} : [0,1] \to \mathbb{R}_+$ is continuous, and thus

$$M_1 := \max\{\|k(x,\cdot)\|_{L^q(0,1)} : x \in [0,1]\} < \infty.$$

Let $S \subset L^p(0,1)$ be a bounded set in $L^p(0,1)$. Then

$$M_2 := \max\{\|v\|_{L^p(0,1)} : v \in S\} < \infty.$$

Hence, for $v \in S$ and $x \in [0,1]$,

$$|Lv(x)| \le \int_0^1 |k(x,y)||v(y)|\,dy \le \|k(x,\cdot)\|_{L^q(0,1)}\|v\|_{L^p(0,1)} \le M_1M_2 < \infty.$$

Hence, $\|Lv\|_{C[0,1]} \le M_1M_2$ and $L(S)$ is bounded in $C[0,1]$. By (2.5), we have for $v \in S$ and $\tau, x \in [0,1]$,

$$|Lv(x) - Lv(\tau)| \le M_2\|k(x,\cdot) - k(\tau,\cdot)\|_{L^q(0,1)}.$$

It follows from the condition (ii) that $L(S)$ is equicontinuous. By the Ascoli–Arzelà Theorem, $\overline{L(S)}$ is compact.

Let $\{v_n\} \subset L^p(0,1)$ and $v \in L^p(0,1)$ such that $\|v_n - v\|_{L^p(0,1)} \to 0$. Then we have for $x \in [0,1]$,

$$|Lv_n(x) - Lv(x)| \leq \left| \int_0^1 |k(x,y)||v_n(y) - v(y)|\, dy \right| \leq \|k(x,\cdot)\|_{L^q(0,1)} \|v_n - v\|_{L^p(0,1)}$$

and

$$\|Lv_n - Lv\|_{C[0,1]} \leq M_1 \|v_n - v\|_{L^p(0,1)} \to 0.$$

Hence, $L : L^p(0,1) \to C[0,1]$ is continuous and thus, is compact. □

The compactness result of Theorem 2.1 with $q = 1$ is closely related to [17, Lemma 2.1].

Lemma 2.2. *Assume that* $k : [0,1] \times [0,1] \setminus \{(x,x) : x \in [0,1]\} \to \mathbb{R}$ *satisfies the following condition.*

(H) *There exists* $q \in [1,\infty]$ *such that for each* $x \in [0,1]$, $k(x,\cdot) \in L^q(0,1)$.

Then the following assertions hold.

(1) *If* $q \in [1,\infty)$ *and*

$$\lim_{x\to\tau} \|k(x,\cdot) - k(\tau,\cdot)\|_{L^q(0,1)} = 0 \quad \text{for some } \tau \in [0,1], \tag{2.6}$$

then

$$\lim_{x\to\tau} \|\sigma(x,\cdot)k(x,\cdot) - \sigma(\tau,\cdot)k(\tau,\cdot)\|_{L^q(0,1)} = 0. \tag{2.7}$$

(2) *If* $q \in [1,\infty]$ *and* $k(x,y) \geq 0$ *for* $x,y \in [0,1]$ *with* $x \neq y$ *and then* (2.7) *implies* (2.6).

Proof. (1) Let $q \in [1,\infty)$. Let $x,\tau,y \in [0,1]$ with $x \neq y$ and $\tau \neq y$. If $\tau < x$, then

$$\sigma(x,y) - \sigma(\tau,y) = \begin{cases} 0 & \text{if } y < \tau, \\ 2 & \text{if } \tau < y < x, \\ 0 & \text{if } x < y. \end{cases} \tag{2.8}$$

If $x < \tau$, then

$$\sigma(x,y) - \sigma(\tau,y) = \begin{cases} 0 & \text{if } y < x, \\ -2 & \text{if } x < y < \tau, \\ 0 & \text{if } x < \tau < y. \end{cases} \tag{2.9}$$

For $x,\tau,y \in [0,1]$ with $x \neq y$ and $\tau \neq y$, let

$$\Phi(x,\tau,y) = \sigma(x,y)k(x,y) - \sigma(\tau,y)k(\tau,y).$$

Then

$$\begin{aligned} |\Phi(x,\tau,y)|^q &\leq \Big[|\sigma(x,y)||k(x,y) - k(\tau,y)| + |k(\tau,y)||\sigma(x,y) - \sigma(\tau,y)|\Big]^q \\ &\leq \Big[|k(x,y) - k(\tau,y)| + |k(\tau,y)||\sigma(x,y) - \sigma(\tau,y)|\Big]^q \\ &\leq |k(x,y) - k(\tau,y)|^q + |k(\tau,y)|^q|\sigma(x,y) - \sigma(\tau,y)|^q. \end{aligned} \tag{2.10}$$

Assume that (1) holds. If $x, \tau, y \in [0,1]$ with $x \neq y$, $\tau \neq y$ and $\tau < x$, then by (2.8) and (2.10), we have

$$\begin{aligned}\int_0^1 |\Phi(x,\tau,y)|^q &\leq \int_0^1 |k(x,y)-k(\tau,y)|^q + |k(\tau,y)|^q|\sigma(x,y)-\sigma(\tau,y)|^q\,dy \\ &= \int_0^1 |k(x,y)-k(\tau,y)|^q\,dy + \int_0^1 |k(\tau,y)|^q|\sigma(x,y)-\sigma(\tau,y)|^q\,dy \\ &= \int_0^1 |k(x,y)-k(\tau,y)|^q\,dy + \int_\tau^x |k(\tau,y)|^q|\sigma(x,y)-\sigma(\tau,y)|^q\,dy \\ &= \int_0^1 |k(x,y)-k(\tau,y)|^q\,dy + 2^q\int_\tau^x |k(\tau,y)|^q\,dy.\end{aligned}$$

This, together with the condition (H) implies

$$\lim_{x\to\tau^+}\int_0^1 |\Phi(x,\tau,y)|^q \leq \lim_{x\to\tau^+}\int_0^1 |k(x,y)-k(\tau,y)|^q\,dy + 2^q\lim_{x\to\tau^+}\int_\tau^x |k(\tau,y)|^q\,dy = 0$$

and $\lim_{x\to\tau^+}\int_0^1 |\Phi(x,\tau,y)|^q = 0$. Similarly, if $x < \tau$, by using (2.9) and (2.10), we have $\lim_{x\to\tau^-}\int_0^1 |\Phi(x,\tau,y)|^q = 0$. It follows that (2.7) holds.

(2) Let $q \in [1,\infty]$. Since $k(x,y) \geq 0$ for $x, y \in [0,1]$ with $x \neq y$,

$$\Big|\sigma(x,y)k(x,y)\Big| = k(x,y) \quad \text{for } x, y \in [0,1] \text{ with } x \neq y.$$

Hence, we have for $x, \tau, y \in [0,1]$ with $x \neq y$ and $\tau \neq y$,

$$\begin{aligned}\Big|k(x,y)-k(\tau,y)\Big| &= \Big||\sigma(x,y)k(x,y)| - |\sigma(\tau,y)k(\tau,y)|\Big| \\ &\leq \Big|\sigma(x,y)k(x,y) - \sigma(\tau,y)k(\tau,y)\Big|.\end{aligned} \tag{2.11}$$

If $q = \infty$, then by (2.11), we have

$$\lim_{x\to\tau}\|k(x,\cdot)-k(\tau,\cdot)\|_{L^\infty(0,1)} \leq \lim_{x\to\tau}\|\sigma(x,\cdot)k(x,\cdot)-\sigma(\tau,\cdot)k(\tau,\cdot)\|_{L^\infty(0,1)}. \tag{2.12}$$

If $q \in [1,\infty)$, then by (2.11), we have

$$\begin{aligned}\Big|k(x,y)-k(\tau,y)\Big|^q &= \Big||\sigma(x,y)k(x,y)| - |\sigma(\tau,y)k(\tau,y)|\Big|^q \\ &\leq \Big|\sigma(x,y)k(x,y) - \sigma(\tau,y)k(\tau,y)\Big|^q\end{aligned}$$

and

$$\lim_{x\to\tau}\|k(x,\cdot)-k(\tau,\cdot)\|_{L^q(0,1)} \leq \lim_{x\to\tau}\|\sigma(x,\cdot)k(x,\cdot)-\sigma(\tau,\cdot)k(\tau,\cdot)\|_{L^q(0,1)}. \tag{2.13}$$

By (2.7), (2.12) and (2.13), we see that (2.6) holds. □

By Theorem 2.1 and Lemma 2.2, we obtain the following results.

Theorem 2.3. *Let $q \in [1,\infty)$ and $p \in (1,\infty]$ satisfy (2.4). Assume that $k : [0,1]\times[0,1]\setminus\{(x,x) : x \in [0,1]\} \to \mathbb{R}_+$ satisfies the conditions (i) and (ii) of Theorem 2.1. Then the maps L defined in (2.1) and $\mathscr{L}$ defined in (2.2) map $L^p(0,1)$ to $C[0,1]$ and are compact.*

Proof. By Theorem 2.1 with $\mathscr{D} = \{(x, x) : x \in [0, 1]\}$, L maps $L^p(0, 1)$ to $C[0, 1]$ and is compact. By (H), $\sigma(x, \cdot)k(x, \cdot) \in L^q(0, 1)$. By Lemma 2.2 (1), Theorem 2.1 (ii) implies (2.7) holds for each $\tau \in [0, 1]$. Hence, σk satisfies the conditions (i) and (ii) of Theorem 2.1. It follows from Theorem 2.1 that $\mathscr{L}$ maps $L^p(0, 1)$ to $C[0, 1]$ and is compact. □

Theorem 2.4. *Assume that* $k : [0, 1] \times [0, 1] \setminus \{(x, x) : x \in [0, 1]\} \to \mathbb{R}_+$ *satisfies the following conditions.*

(i) *For each* $x \in [0, 1]$, $k(x, \cdot) \in L^\infty(0, 1)$.

(ii) *For each* $\tau \in [0, 1]$, $\lim_{x \to \tau} \|\sigma(x, \cdot)k(x, \cdot) - \sigma(\tau, \cdot)k(\tau, \cdot)\|_{L^\infty(0,1)} = 0$.

Then the maps L *defined in* (2.1) *and* $\mathscr{L}$ *defined in* (2.2) *map* $L^1(0, 1)$ *to* $C[0, 1]$ *and are compact.*

Proof. By the condition (i), $\sigma(x, \cdot)k(x, \cdot) \in L^\infty(0, 1)$. This, together with the condition (ii), shows that σk satisfies the conditions (i) and (ii) of Theorem 2.1 with $\mathscr{D} = \{(x, x) : x \in [0, 1]\}$. It follows from Theorem 2.1 that $\mathscr{L}$ maps $L^1(0, 1)$ to $C[0, 1]$ and is compact. By Lemma 2.2 (2), the condition (ii) implies that k satisfies Theorem 2.1 (ii). Hence, k satisfies Theorem 2.1 (i) and (ii) with $q = \infty$. It follows from Theorem 2.1 that L maps $L^1(0, 1)$ to $C[0, 1]$ and is compact. □

As applications of the above results, we study the following two specific linear Hammerstein integral operators:

$$L_\alpha v(x) = \int_0^1 \frac{1}{|x - y|^\alpha} v(y)\, dy \quad \text{for each } x \in [0, 1] \tag{2.14}$$

and

$$\mathscr{L}_\alpha v(x) = \int_0^1 \frac{\sigma(x, y)}{|x - y|^\alpha} v(y)\, dy \quad \text{for each } x \in [0, 1], \tag{2.15}$$

where $\alpha \in (0, 1)$.

We first prove the following result.

Lemma 2.5.

(1) *If* $\alpha \in (0, 1)$, *then*

$$\int_0^1 \frac{1}{|x - y|^\alpha}\, dy = \frac{1}{1 - \alpha}\left[x^{1-\alpha} + (1 - x)^{1-\alpha}\right] \quad \textit{for each } x \in [0, 1]$$

and

$$\int_0^1 \frac{\sigma(x, y)}{|x - y|^\alpha}\, dy = \frac{1}{1 - \alpha}\left[x^{1-\alpha} - (1 - x)^{1-\alpha}\right] \quad \textit{for each } x \in [0, 1].$$

(2) *If* $\alpha \in [1, \infty)$, *then* $\int_0^1 \frac{1}{|x-y|^\alpha}\, dy = \infty$ *for each* $x \in [0, 1]$.

Proof. (1) Let $\alpha \in (0, 1)$ and $x \in [0, 1]$. Then

$$\int_0^1 \frac{1}{|x - y|^\alpha}\, dy = \int_0^x \frac{1}{(x - y)^\alpha}\, dy + \int_x^1 \frac{1}{(y - x)^\alpha}\, dy = \frac{x^{1-\alpha}}{1 - \alpha} + \frac{(1 - x)^{1-\alpha}}{1 - \alpha}.$$

Similarly, the second equality holds.

(2) Let $\alpha \in [1,\infty)$, $x \in \left[0,\frac{1}{2}\right]$ and $z = y - x$ for $y \in [0,1]$. Then $x \le 1-x$ and

$$\int_0^1 \frac{1}{|x-y|^\alpha}\,dy = \int_{-x}^{1-x} \frac{1}{|z|^\alpha}\,dz \ge \int_{-x}^{x} \frac{1}{|z|^\alpha}\,dz = 2\int_0^x \frac{1}{z^\alpha}\,dz = \infty.$$

Let $x \in \left(\frac{1}{2},1\right]$ and let $z = y - x$ for $y \in [0,1]$. Then $-x < -(1-x)$ and

$$\int_0^1 \frac{1}{|x-y|^\alpha}\,dy = \int_{-x}^{1-x} \frac{1}{|z|^\alpha}\,dz \ge \int_{-(1-x)}^{1-x} \frac{1}{|z|^\alpha}\,dz = 2\int_0^{1-x} \frac{1}{z^\alpha}\,dz = \infty. \qquad \square$$

The following result gives an application of Theorem 2.3.

Theorem 2.6. *Let $\alpha \in (0,1)$ and $p \in \left(\frac{1}{1-\alpha},\infty\right]$. Then the maps L_α defined in (2.14) and $\mathscr{L}_\alpha$ defined in (2.15) map $L^p(0,1)$ to $C[0,1]$ and are compact.*

Proof. Let $p \in \left(\frac{1}{1-\alpha},\infty\right]$ and $q \in \left[1,\frac{1}{\alpha}\right)$ satisfy (2.4). We define $k : [0,1]\times[0,1] \setminus \{(x,x) : x \in [0,1]\} \to \mathbb{R}$ by

$$k(x,y) = \frac{1}{|x-y|^\alpha}. \tag{2.16}$$

Since $\alpha q \in (0,1)$, by Lemma 2.5 (1), we have for each $x \in [0,1]$,

$$\int_0^1 |k(x,y)|^q\,dy = \int_0^1 \frac{1}{|x-y|^{\alpha q}}\,dy < \infty.$$

Hence, $k(x,\cdot) \in L^q(0,1)$ for each $x \in [0,1]$ and Theorem 2.2 (i) holds. Let $\tau \in (0,1)$ and $\delta_1 \in \left(0,\min\{\tau,1-\tau\}\right)$. Let

$$\varepsilon \in \left(0,\frac{2^{4-\alpha q}\delta_1^{1-\alpha q}}{1-\alpha q}\right), \quad \delta_\varepsilon = \left[\frac{\varepsilon(1-\alpha q)}{2^{4-\alpha q}}\right]^{\frac{1}{1-\alpha q}} \quad \text{and } \delta \in (0,\delta_\varepsilon).$$

Then $\delta < \delta_\varepsilon < \delta_1 < \frac{1}{2}$. For $x,y \in [0,1]$ with $x \ne y$ and $y \ne \tau$, let

$$k(x,y) - k(\tau,y) = \frac{1}{|x-y|^\alpha} - \frac{1}{|\tau-y|^\alpha}.$$

Let $D_1 = \{x \in [0,1] : |x-\tau| \le \delta\} \times [0,\tau-\delta_\varepsilon] \cup [\tau+\delta_\varepsilon,1]$. Then D_1 is closed and

$$|x-y| \ge |y-\tau| - |x-\tau| \ge \delta_\varepsilon - \delta > 0 \quad \text{for } (x,y) \in D_1.$$

Hence, $k : D_1 \to \mathbb{R}$ is uniformly continuous on D_1. Let $\sigma \in \left(0,\frac{1}{2(1-2\delta_\varepsilon)}\right)$. It follows that there exists $\delta^* \in (0,\delta)$ such that when $|x-\tau| < \delta^*$,

$$|k(x,y) - k(\tau,y)|^q < \sigma\varepsilon \quad \text{for } y \in [0,\tau-\delta_\varepsilon] \cup [\tau+\delta_\varepsilon,1].$$

Hence, when $|x-\tau| < \delta^*$, we have

$$\begin{aligned}&\int_0^{\tau-\delta_\varepsilon} |k(x,y)-k(\tau,y)|^q\,dy + \int_{\tau+\delta_\varepsilon}^1 |k(x,y)-k(\tau,y)|^q\,dy \\ &\le \sigma\varepsilon(\tau-\delta_\varepsilon) + \sigma\varepsilon(1-\tau-\delta_\varepsilon) = \sigma\varepsilon(1-2\delta_\varepsilon) < \frac{\varepsilon}{2}\end{aligned}$$

and

$$\int_{\tau-\delta_\varepsilon}^{\tau+\delta_\varepsilon} |k(x,y)|^q\,dy = \int_{\tau-\delta_\varepsilon}^{\tau+\delta_\varepsilon} \frac{1}{|x-y|^{\alpha q}}\,dy = \int_{\tau-x-\delta_\varepsilon}^{\tau-x+\delta_\varepsilon} \frac{1}{|u|^{\alpha q}}\,du$$
$$\le \int_{-(\delta^*+\delta_\varepsilon)}^{\delta^*+\delta_\varepsilon} \frac{1}{|u|^{\alpha q}}\,du = 2\int_0^{\delta^*+\delta_\varepsilon} \frac{1}{u^{\alpha q}}\,du = \frac{2(\delta^*+\delta_\varepsilon)^{1-\alpha q}}{1-\alpha q} < \frac{2(2\delta_\varepsilon)^{1-\alpha q}}{1-\alpha q}$$
$$= \frac{2^{2-\alpha q}}{1-\alpha q}\frac{\varepsilon(1-\alpha q)}{2^{4-\alpha q}} = \frac{\varepsilon}{4}.$$

This implies that when $|x-\tau| < \delta^*$,

$$\int_0^1 |k(x,y)-k(\tau,y)|^q\,dy$$
$$= \int_0^{\tau-\delta_\varepsilon} |k(x,y)-k(\tau,y)|^q\,dy + \int_{\tau+\delta_\varepsilon}^1 |k(x,y)-k(\tau,y)|^q\,dy + \int_{\tau-\delta_\varepsilon}^{\tau+\delta_\varepsilon} |k(x,y)-k(\tau,y)|^q\,dy$$
$$< \frac{\varepsilon}{2} + \int_{\tau-\delta_\varepsilon}^{\tau+\delta_\varepsilon} \left(|k(x,y)|+|k(\tau,y)|\right)^q dy$$
$$< \frac{\varepsilon}{2} + \int_{\tau-\delta_\varepsilon}^{\tau+\delta_\varepsilon} |k(x,y)|^q\,dy + \int_{\tau-\delta_\varepsilon}^{\tau+\delta_\varepsilon} |k(\tau,y)|^q\,dy$$
$$= \frac{\varepsilon}{2} + \int_{\tau-\delta_\varepsilon}^{\tau+\delta_\varepsilon} \frac{1}{|x-y|^{\alpha q}}\,dy + \int_{\tau-\delta_\varepsilon}^{\tau+\delta_\varepsilon} \frac{1}{|\tau-y|^{\alpha q}}\,dy < \frac{\varepsilon}{2}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4} = \varepsilon.$$

Hence,

$$\lim_{x\to\tau} \int_0^1 |k(x,y)-k(\tau,y)|^q\,dy = 0$$

and Theorem 2.2 (ii) holds. The proofs are similar if $\tau = 0$ or $\tau = 1$. The result follows from Theorem 2.3. □

3 Compactness of Riemann–Liouville fractional integral operators

Let $a, b \in \mathbb{R}$ with $a < b$ and $\varphi : [a,b] \to \mathbb{R}$ be a measurable function. The Riemann–Liouville left-sided fractional integral operator of order $\alpha \in (0,\infty)$ is defined by

$$I^\alpha_{a^+}\varphi(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\varphi(y)}{(x-y)^{1-\alpha}}\,dy \quad \text{for each } x \in [a,b], \tag{3.1}$$

provided the Lebesgue integral on the right side of (3.1) exists for almost every (a.e.) $x \in [a,b]$, and Γ is the standard Gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx,$$

see [6, p. 13], [14, p. 69] and [20, p. 33]. Similarly, the Riemann–Liouville right-sided fractional integral operator of order $\alpha \in (0,\infty)$ is defined by

$$I^\alpha_{b^-}\varphi(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{\varphi(y)}{(y-x)^{1-\alpha}}\,dy \quad \text{for each } x \in [a,b], \tag{3.2}$$

provided the Lebesgue integral on the right side of (3.2) exists for a.e. $x \in [a,b]$. Hardy and Littlewood [11] called these integrals 'right- handed' integral 'with origin a', and 'left-handed' integral 'with origin b', respectively.

Note that in (3.1), we still use the symbol $I^{\alpha}_{a+}\varphi(x)$ to denote the Lebesgue integral on the right side of (3.2) at x even when the integral does not exist at x. Hence, we treat (3.1) to hold for each $x \in [a,b]$. Similarly, we treat (3.2) to hold for each $x \in [a,b]$.

It is well known that both I^{α}_{a+} and I^{α}_{b-} map $L^1(0,1)$ to $L^1(0,1)$, see [6, Theorem 2.1], [14, Lemma 2.1], [20, Theorem 2.6], [14, Lemma 2.1] and [20, Theorem 2.6].

We only use I^{α}_{0+} and I^{α}_{1-} to denote the following operators:

$$I^{\alpha}_{0+}\varphi(x) := \frac{1}{\Gamma(\alpha)}\int_0^x \frac{\varphi(y)}{(x-y)^{1-\alpha}}\,dy \quad \text{for each } x \in [0,1], \tag{3.3}$$

and

$$I^{\alpha}_{1-}\varphi(x) := \frac{1}{\Gamma(\alpha)}\int_x^1 \frac{\varphi(y)}{(y-x)^{1-\alpha}}\,dy \quad \text{for each } x \in [0,1]. \tag{3.4}$$

We give the following relationships among the above operators given in (3.1), (3.2), (3.3) and (3.4). They are well known to experts, but we give the proofs for completeness because we have not found them anywhere else.

Proposition 3.1. *Let $\varphi \in L^1(a,b)$ and let*

$$t(x) = (1-x)a + xb \quad \textit{for each } x \in [0,1]. \tag{3.5}$$

and

$$(\varphi(t))(x) = \varphi(t(x)) \quad \textit{for a.e. } x \in [0,1]. \tag{3.6}$$

Then the following assertions hold.

(1) *If $(I^{\alpha}_{a+}\varphi)(t(x))$ exists for some $x \in [0,1]$, then*

$$(I^{\alpha}_{a+}\varphi)(t(x)) = (b-a)^{\alpha}(I^{\alpha}_{0+}(\varphi(t))(x). \tag{3.7}$$

(2) *If $(I^{\alpha}_{b-}\varphi)(t(x))$ exists for some $x \in [0,1]$, then*

$$(I^{\alpha}_{b-}\varphi)(t(x)) = (b-a)^{\alpha}(I^{\alpha}_{1-}(\varphi(t))(x). \tag{3.8}$$

Proof. Let $\varphi \in L^1(a,b)$ and let $y = t(x)$ for $x \in [0,1]$. Then

$$\int_a^b |\varphi(y)|\,dy = (b-a)\int_0^1 |\varphi(t(x))|\,dx = (b-a)\int_0^1 |(\varphi(t))(x)|\,dx$$

This implies $\varphi(t) \in L^1(0,1)$.

(1) Assume that $(I^{\alpha}_{a+}\varphi)(t(x))$ exists for some $x \in [0,1]$. Then by (3.1), we have

$$(I^{\alpha}_{a+}\varphi)(t(x)) = \frac{1}{\Gamma(\alpha)}\int_a^{t(x)} \frac{\varphi(s)}{(t(x)-s)^{1-\alpha}}\,ds. \tag{3.9}$$

Let $s = t(y) = (1-y)a + yb$ for $y \in [0,x]$. Then

$$\begin{aligned}\int_a^{t(x)} \frac{\varphi(s)}{(t(x)-s)^{1-\alpha}}\,ds &= \int_0^x \frac{\varphi(t(y))}{[(b-a)(x-y)]^{1-\alpha}}\,(b-a)dy \\ &= (b-a)^{\alpha}\int_0^x \frac{(\varphi(t))(y)}{(x-y)^{1-\alpha}}\,dy.\end{aligned}$$

This, together with (3.9), implies (3.7).

(2) Assume that $(I^\alpha_{b-}\varphi)(t(x))$ exists for some $x \in [0,1]$. Then by (3.2), we have

$$(I^\alpha_{b-}\varphi)(t(x)) = \frac{1}{\Gamma(\alpha)} \int_{t(x)}^{b} \frac{\varphi(s)}{(s-t(x))^{1-\alpha}}\, ds. \tag{3.10}$$

Let $s = t(y) = (1-y)a + yb$ for $y \in [x,b]$. Then

$$\begin{aligned}\int_{t(x)}^{b} \frac{\varphi(s)}{(s-t(x))^{1-\alpha}}\, ds &= \int_x^b \frac{\varphi(t(y))}{[(b-a)(y-x)]^{1-\alpha}}(b-a)dy \\ &= (b-a)^\alpha \int_x^b \frac{(\varphi(t))(y)}{(y-x)^{1-\alpha}}\, dy.\end{aligned}$$

This, together with (3.10), implies (3.8). □

Proposition 3.2. *Let $\varphi \in L^1(a,b)$ and let*

$$t(x) = a + b - x \quad \textit{for each } x \in [a,b] \tag{3.11}$$

and

$$(\varphi(t))(x) = \varphi(t(x)) \quad \textit{for a.e. } x \in [a,b]. \tag{3.12}$$

If $(I^\alpha_{b-}\varphi)(t(x))$ exists for some $x \in [a,b]$, then

$$(I^\alpha_{b-}\varphi)(t(x)) = (I^\alpha_{a+}\varphi(t))(x). \tag{3.13}$$

Proof. Assume that $(I^\alpha_{b-}\varphi)(t(x))$ exists for some $x \in [a,b]$. Then by (3.2), we have

$$(I^\alpha_{b-}\varphi)(t(x)) = \frac{1}{\Gamma(\alpha)} \int_{t(x)}^{b} \frac{\varphi(s)}{(s-t(x))^{1-\alpha}}\, ds. \tag{3.14}$$

Let $s = t(y) = a + b - y$ for $y \in [a,x]$. Then

$$\int_{t(x)}^{b} \frac{\varphi(s)}{(s-t(x))^{1-\alpha}}\, ds = \int_a^x \frac{\varphi(t(y))}{(x-y)^{1-\alpha}}\, dy = \int_a^x \frac{(\varphi(t))(y)}{(x-y)^{1-\alpha}}\, dy.$$

This, together with (3.14), implies (3.13). □

Proposition 3.3. *Let $\varphi \in L^1(a,b)$ and let*

$$t^*(x) = xa + (1-x)b \quad \textit{for each } x \in [0,1] \tag{3.15}$$

and

$$(\varphi(t^*))(x) = \varphi(t^*(x)) \quad \textit{for a.e. } x \in [0,1]. \tag{3.16}$$

If $(I^\alpha_{b-}\varphi)(t^*(x))$ exists for some $x \in [0,1]$, then

$$(I^\alpha_{b-}\varphi)(t^*(x)) = (b-a)^\alpha (I^\alpha_{0+}\varphi(t^*))(x) \quad \textit{for each } x \in [a,b].$$

Proof. Note that $t^*(x) = t(1-x)$ for each $x \in [0,1]$, where t is the same as in (3.5). By (3.8), we have for each $x \in [0,1]$,

$$\begin{aligned}(I^\alpha_{b-}\varphi)(t^*(x)) &= (I^\alpha_{b-}\varphi)(t(1-x)) = (b-a)^\alpha (I^\alpha_{1-}(\varphi(t)))(1-x) \\ &= \frac{(b-a)^\alpha}{\Gamma(\alpha)} \int_{1-x}^{1} \frac{\varphi(t(y))}{(y-(1-x))^{1-\alpha}}\, dy = \frac{(b-a)^\alpha}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t(1-z))}{((1-z)-(1-x))^{1-\alpha}}\, dz \\ &= \frac{(b-a)^\alpha}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t^*(z))}{(x-z)^{1-\alpha}}\, dz = (b-a)^\alpha (I^\alpha_{0+}\varphi(t^*))(x)\end{aligned}$$

and the result holds. □

By Proposition 3.3, we obtain the following result.

Corollary 3.4. *Assume that $v \in L^1(0,1)$ satisfies that $(I_{1^-}^{1-\alpha}v)(x)$ exists for some $x \in [0,1]$. Then*

$$(I_{1^-}^{1-\alpha}v)(x) = (I_{0^+}^{1-\alpha}v^*)(1-x),$$

where $v^*(s) = v(1-s)$ for a.e. $a \in [0,1]$.

To apply the results in Section 2, in the following we always assume $\alpha \in (0,1)$ and consider the following Riemann–Liouville fractional integral operators:

$$I_{0^+}^{1-\alpha}v(x) := \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{v(y)}{(x-y)^\alpha}\,dy \quad \text{for each } x \in [0,1] \tag{3.17}$$

and

$$I_{1^-}^{1-\alpha}v(x) := \frac{1}{\Gamma(1-\alpha)} \int_x^1 \frac{v(y)}{(y-x)^\alpha}\,dy \quad \text{for each } x \in [0,1], \tag{3.18}$$

where $v \in L^1(0,1)$.

Now, we prove that if $p \in \left(\frac{1}{1-\alpha}, \infty\right)$, then both $I_{0^+}^{1-\alpha}$ and $I_{1^-}^{1-\alpha}$ map $L^p(0,1)$ to $C[0,1]$ and are compact.

Theorem 3.5. *Let $p \in \left(\frac{1}{1-\alpha}, \infty\right]$. Then the following assertions hold.*

(1) *The maps $I_{0^+}^{1-\alpha}$ and $I_{1^-}^{1-\alpha}$ map $L^p(0,1)$ to $C[0,1]$ and are compact.*

(2) *For each $v \in L^p(0,1)$, $I_{0^+}^{1-\alpha}v(0) = I_{1^-}^{1-\alpha}v(1) = 0$.*

(3) *For each $x \in [0,1]$,*

$$I_{0^+}^{1-\alpha}\hat{1}(x) = \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} \quad \textit{and } I_{1^-}^{1-\alpha}\hat{1}(x) = \frac{(1-x)^{1-\alpha}}{\Gamma(2-\alpha)}, \tag{3.19}$$

where $\hat{1}(x) \equiv 1$ for each $x \in [0,1]$.

Proof. (1) Let $v \in L^p(0,1)$ and $x \in [0,1]$. Then

$$\begin{aligned} \mathscr{L}_\alpha v(x) &= \int_0^x \frac{1}{(x-y)^\alpha} v(y)\,dy - \int_x^1 \frac{1}{(y-x)^\alpha} v(y)\,dy \\ &= \Gamma(1-\alpha)\left[(I_{0^+}^{1-\alpha}v)(x) - (I_{1^-}^{1-\alpha}v)(x)\right] \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} L_\alpha v(x) &= \int_0^x \frac{1}{(x-y)^\alpha} v(y)\,dy + \int_x^1 \frac{1}{(y-x)^\alpha} v(y)\,dy \\ &= \Gamma(1-\alpha)\left[(I_{0^+}^{1-\alpha}v)(x) + (I_{1^-}^{1-\alpha}v)(x)\right]. \end{aligned} \tag{3.21}$$

By (3.20) and (3.21), we have for $x \in [0,1]$,

$$I_{0^+}^{1-\alpha}v(x) = \frac{1}{2\Gamma(1-\alpha)}\left[L_\alpha v(x) + \mathscr{L}_\alpha v(x)\right] \tag{3.22}$$

and

$$I_{1^-}^{1-\alpha}v(x) = \frac{1}{2\Gamma(1-\alpha)}\left[L_\alpha v(x) - \mathscr{L}_\alpha v(x)\right]. \tag{3.23}$$

The results follow from Theorem 2.6.

(2) For $v \in L^p(0,1)$, by (2.14) and (2.15), it is easy to see that

$$L_\alpha v(0) = -\mathscr{L}_\alpha v(0) \quad \text{and} \quad L_\alpha v(1) = \mathscr{L}_\alpha v(1).$$

This, together with (3.22) and (3.23), implies

$$(I_{0^+}^{1-\alpha})v(0) = I_{1^-}^{1-\alpha} v(1) = 0.$$

(3) By Lemma 2.5 (1) and (3.20) and (3.21) with $v = \hat{1}$, we see that (3.19) holds. □

Remark 3.6. In a personal communication, Professor J. R. L. Webb informed me that there is another known proof of compactness of $I_{0^+}^{1-\alpha}$ which I reproduce below. By [11, Theorem 12] (or [6, Theorem 2.6], [20, Theorem 3.6] and [23, Proposition 3.2 (3)]), $I_{0^+}^{1-\alpha}$ maps $L^p(0,1)$ to the Hölder space $C^{0,\beta}$, and the Hölder space $C^{0,\beta}$ with the norm

$$\|u\|_{0,\beta} := \max_{x\in[0,1]} |u(x)| + \sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\beta}$$

is compactly imbedded in $C[0,1]$, where $\beta = 1-\alpha-\frac{1}{p}$. Indeed, if $\{u_n\}$ is a bounded sequence in $C^{0,\beta}[0,1]$, say $\|u_n\|_{0,\beta} \le M < \infty$, then we have for $x \neq y$,

$$|u_n(x) - u_n(y)| = \frac{|u_n(x)-u_n(y)|}{|x-y|^\beta}|x-y|^\beta \le M|x-y|^\beta,$$

so $\{u_n\}$ is bounded and equicontinuous, and hence relatively compact in $C[0,1]$ by the Ascoli–Arzelà theorem.

Remark 3.7. By Proposition 3.4, $I_{1^-}^{1-\alpha} : L^p(0,1) \to C[0,1]$ is compact for each $p \in \left(\frac{1}{1-\alpha}, \infty\right]$. By (3.22), (3.23) and compactness of $I_{0^+}^{1-\alpha}$ and $I_{1^-}^{1-\alpha}$ obtained in Remark 3.6, we see that Theorem 3.5 (1) is equivalent to Theorem 2.6.

By Propositions 3.1, 3.2 and 3.3, we see that all the theorems which are proved for one operator of the operators: $I_{a^+}^\alpha$, $I_{b^-}^\alpha$, $I_{0^+}^\alpha$ and $I_{1^-}^\alpha$, will apply, with the obvious changes, to the others. Therefore, in the following, we only consider the operator $I_{0^+}^{1-\alpha}$.

As an application of continuity and compactness of $I_{0^+}^{1-\alpha}$, we prove the following result on the eigenfunctions and spectral radius of $I_{0^+}^{1-\alpha}$.

Theorem 3.8. *Let $p \in \left(\frac{1}{1-\alpha}, \infty\right]$. Then the following assertions hold.*

(1) *If there exist $\varphi \in L_+^p(0,1)$ and $\mu \in (0,\infty)$ such that*

$$\varphi(x) = \mu I_{0^+}^{1-\alpha}\varphi(x) \quad \text{for a.e. } x \in [0,1]. \tag{3.24}$$

Then $\varphi(x) = 0$ for each $x \in [0,1]$.

(2) *$r(I_{0^+}^{1-\alpha}) = 0$, where $r(I_{0^+}^{1-\alpha})$ is the spectral radius of $I_{0^+}^{1-\alpha}$.*

Proof. Let P be the standard positive cone in $C[0,1]$, that is,

$$P = \{u \in C[0,1] : u(x) \ge 0 \quad \text{for } x \in [0,1]\}. \tag{3.25}$$

Then P is a total and normal cone in $C[0,1]$.

(1) By Theorem 3.5 (1), $I_{0+}^{1-\alpha}\varphi \in P$. By (3.24), $\varphi \in P$. By (3.24) and weakly singular Gronwall inequality [12, Lemma 7.1.1] or ([6, Lemma 6.19], [7, Lemma 4.3] and [22, Theorem 3.2]), we have $\varphi(x) = 0$ for a.e. $x \in [0,1]$. Since $\varphi \in C[0,1]$, $\varphi(x) = 0$ for each $x \in [0,1]$.

(2) By Theorem 3.5 (1), for $p \in \left(\frac{1}{1-\alpha}, \infty\right]$, the operator $I_{0+}^{1-\alpha}$ maps P to P and is compact. If $r(I_{0+}^{1-\alpha}) > 0$, it would follow from Krein–Rutman theorem (see [1, Theorem 3.1] or [16]) that there exists an eigenvector $\varphi \in P \setminus \{0\}$ such that

$$I_{0+}^{1-\alpha}\varphi(x) = r(I_{0+}^{1-\alpha})\varphi(x) \quad \text{for each } x \in [0,1].$$

By the result (i), we obtain $\varphi(x) = 0$ for each $x \in [0,1]$, which contradicts the fact $\varphi \in P \setminus \{0\}$. □

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Multiplicity of nodal solutions for fourth order equation with clamped beam boundary conditions

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. In this paper, we study the global structure of nodal solutions of

$$\begin{cases} u''''(x) = \lambda h(x) f(u(x)), & 0 < x < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases}$$

where $\lambda > 0$ is a parameter, $h \in C([0,1], (0,\infty))$, $f \in C(\mathbb{R})$ and $sf(s) > 0$ for $|s| > 0$. We show the existence of S-shaped component of nodal solutions for the above problem. The proof is based on the bifurcation technique.

Keywords: clamped beam, fourth order equations, connected component, nodal solutions, bifurcation.

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1 Introduction

The deformations of an elastic beam whose both ends clamped are described by the fourth order problem

$$\begin{aligned} u''''(x) &= \lambda h(x) f(u(x)), \qquad x \in (0,1), \\ u(0) &= u(1) = u'(0) = u'(1) = 0, \end{aligned} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $f \in C(\mathbb{R})$, $f(0) = 0$, $sf(s) > 0$ for all $s \neq 0$ and $h \in C([0,1], (0,\infty))$.

Existence and multiplicity of solutions of (1.1) have been extensively studied by several authors [1,3,6,10,11,14,18,21,22]. For examples, Agarwal and Chow [1] studied the existence of solutions of (1.1) by contraction mapping and iterative methods. Cabada and Enguiça [3] developed the method of lower and upper solutions to show the existence and multiplicity of solutions. Pei and Chang [14] proved the existence of symmetric positive solutions by using

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a monotone iterative technique. Yao [21], Zhai, Song and Han [22] established the existence and multiplicity of solutions via the fixed point theorem in cone.

Recently, Sim and Tanaka [19] were concerned with the existence of three positive solutions for the p-Laplacian problem

$$\begin{cases} -(|y'|^{p-2}y')' = \lambda a(x)f(y), & x \in (0,1), \\ y(0) = y(1) = 0, \end{cases}$$

by employing a bifurcation technique, where the nonlinearity f is asymptotic linear near 0 and sublinear near ∞. They obtained an S-shaped unbounded continuum (which grows to the right from the initial point, to the left at some point and to the right near $\lambda = \infty$). The proof of their main result heavily depends on the Sturm comparison theorem [20]. For other related results on the existence and multiplicity of solutions of fourth order problems, see Li and Gao [12] and Li [13].

Motivated by the above work, we shall study the existence of S-shaped unbounded continua of nodal solutions of fourth order problems (1.1). However, it seems hard to follow this argument in [19, Lemma 3.2] directly for fourth order problem since the Sturm comparison theorem is not available for the fourth order problems, and the nodal solution of (1.1) is not concave down in $[0,1]$.

Let $Y = C[0,1]$ with the norm

$$\|u\|_\infty = \max_{t\in[0,1]} |u(t)|.$$

Let $E = \{u \in C^3[0,1] : \ u(0) = u(1) = u'(0) = u'(1) = 0\}$ with the norm

$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty, \|u'''\|_\infty\}.$$

Let S_k^+ denote the set of functions in E which have exactly $k-1$ simple zeros in $(0,1)$ and are positive near $t = 0$, and set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. They are disjoint and open in E. Finally, let $\Phi_k^\pm = \mathbb{R} \times S_k^\pm$ and $\Phi_k = \mathbb{R} \times S_k$.

We shall make use of the following assumptions

(A1) $h \in C[0,1]$ with $0 < h_* \le h(x) \le h^*$ on $[0,1]$ for some $h_*, h^* \in (0,\infty)$;

(A2) $f : \mathbb{R} \to \mathbb{R}$ is non-decreasing, and there exists $s_0 > 0$ such that

$$f_* := \inf_{0<s\le s_0} \frac{f(s)}{s} < \sup_{0<s\le s_0} \frac{f(s)}{s} =: f^*$$

with

$$0 < f_* < f^* < \infty;$$

(A3) there exist $\alpha > 0$, $f_0 := \lim_{|s|\to 0} \frac{f(s)}{s} \in (0,\infty)$ and $f_1 > 0$ such that

$$\lim_{|s|\to 0} \frac{f(s) - f_0 s}{s|s|^\alpha} = -f_1;$$

(A4) $f(0) = 0$, $sf(s) > 0$ for $s \ne 0$, $f_\infty := \lim_{|s|\to\infty} \frac{f(s)}{s} = 0$.

Remark 1.1. Typical modal of f which satisfies (A3) is the following

$$\hat{f}(s) = \begin{cases} 2s - s^2, & s \geq 0, \\ 2s + s^2, & s < 0, \end{cases}$$

where $f_0 = 2,\ f_1 = 1$ and $\alpha = 1$.

The rest of the paper is organized as follows. In Section 2, we state and prove several preliminary results on the nodal solutions (λ, u) of (1.1) with $\|u\|_\infty = s_0$ and state a method of lower and upper solutions due to Cabada [3]. In Section 3, we state our main result and show the existence of bifurcation from some eigenvalue for the corresponding problem according to the standard argument and the rightward direction of bifurcation. Section 4 is devoted to show the change of direction of bifurcation. Finally in Section 5 we show an *a-priori* bound of solutions for (1.1) and complete the proof of Theorem 3.2.

2 Preliminaries

The following result is a special case of Leighton and Nehari [11, Theorem 5.2]

Lemma 2.1. *Let $p, p_1 : [a,b] \to (0,\infty)$ be two continuous functions with*

$$p(x) \leq p_1(x), \qquad x \in [a,b]. \tag{2.1}$$

Let

$$y'''' - p(x)y = 0, \qquad x \in [a,b], \tag{2.2}$$
$$y_1'''' - p_1(x)y_1 = 0, \qquad x \in [a,b]. \tag{2.3}$$

If

$$y(a) = y_1(a) = y(b) = y_1(b) = 0$$

and the number of zeros of $y(x)$ and $y_1(x)$ in $[a,b]$ is denoted by n and n' $(n \geq 4)$, respectively, then

$$n' \geq n - 1.$$

Lemma 2.2. *Let $k \geq 4$ and $\nu \in \{+,-\}$. Let (A2) hold. If*

$$h_* f_* > 0, \tag{2.4}$$

then there exists $\Lambda > 0$, such that for any solution $(\lambda, u) \in \mathbb{R}^+ \times S_k^\nu$ of (1.1) with $\|u\|_\infty = s_0$, one has

$$\lambda \leq \Lambda := \frac{\gamma_{k+2}}{h_* f_*}, \tag{2.5}$$

where γ_{k+2} is the $(k+2)$-th eigenvalue of the linear problem

$$\begin{aligned} &v'''' = \gamma v(x), \qquad x \in (0,1), \\ &v(0) = v(1) = v'(0) = v'(1) = 0, \end{aligned} \tag{2.6}$$

which is simple, and its corresponding eigenfunction ϕ_{k+2} has $k+1$ zeros in $(0,1)$.

Proof. Assume on the contrary that $\lambda > \Lambda$. Combining this with $\Lambda := \frac{\gamma_{k+2}}{h_* f_*}$ and using

$$u''''(x) = \lambda h(x)\frac{f(u(x))}{u(x)}u(x), \qquad x \in (0,1),$$
$$u(0) = u(1) = u'(0) = u'(1) = 0$$

and the fact

$$\lambda h(x)\frac{f(u(x))}{u(x)} > \gamma_{k+2}, \qquad x \in [0,1],$$

it deduces that $u \in S^{\nu}_{j+1}$ for some $j \geq k+1$. However, this contradicts the fact $u \in S^{\nu}_k$. □

Lemma 2.3. *Let*

$$M := \max\{\lambda h(x)f(s) : x \in [0,1],\, s \in [0,s_0],\, 0 \leq \lambda \leq \Lambda\}. \tag{2.7}$$

Then for any solution $(\eta, u) \in \mathbb{R}^+ \times S^+_k$ *of* (1.1) *with* $\|u\|_\infty = s_0$, *one has*

$$\|u'\|_\infty \leq M. \tag{2.8}$$

Proof. It follows from the equation in (1.1) and (2.7) that

$$\|u''''\|_\infty \leq M,$$

which together with the boundary value conditions in (1.1) imply the desired result. □

Let

$$0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$$

be the zeros on u in $[0,1]$. Let x_j be such that

$$|u(x_j)| = \max\{|u(t)| : t \in [t_j, t_{j+1}]\}, \qquad j \in \{0,1,\cdots,k-1\}.$$

Lemma 2.4. *Let*

$$|u(x_{j_0})| = \|u\|_\infty = s_0.$$

Then

$$t_{j_0+1} - t_{j_0} \geq \frac{2s_0}{M}.$$

Proof. We only deal with the case that $u(x_{j_0}) = \|u\|_\infty = s_0$. The other can be treated by the similar method.

Consider the lines

$$y - u(x_{j_0}) = M(t - x_{j_0}), \qquad y - u(x_{j_0}) = -M(t - x_{j_0}).$$

They intersect on the horizontal axis at

$$\left(x_{j_0} - \frac{u(x_{j_0})}{M}, 0\right), \qquad \left(x_{j_0} + \frac{u(x_{j_0})}{M}, 0\right),$$

respectively. Thus, it follows from this and (2.8) that

$$\left(x_{j_0} - \frac{u(x_{j_0})}{M}, x_{j_0} + \frac{u(x_{j_0})}{M}\right) \subset (t_{j_0}, t_{j_0+1}).$$ □

Lemma 2.5. *Let (λ, u) be a S_k^ν-solution with $\|u\|_\infty = s_0$. Let*

$$|u(x_0)| = \max_{t_{j_0} \le x \le t_{j_0+1}} |u(x)|.$$

Then

$$\left[x_0 - \frac{s_0}{M}, x_0 + \frac{s_0}{M}\right] \subset (t_{j_0}, t_{j_0+1}),$$

$$\min\left\{|u(t)| : t \in \left[x_0 - \frac{s_0}{2M}, x_0 + \frac{s_0}{2M}\right]\right\} \ge \frac{1}{2}\|u\|_\infty. \tag{2.9}$$

Proof. We only deal with the case $u(x_0) > 0$. The other case can be treated by the similar way.

Using the fact

$$u(t) \ge u(x_0) + M(t - x_0), \qquad t \in \left[x_0 - \frac{s_0}{2M}, x_0\right],$$
$$u(t) \ge u(x_0) - M(t - x_0), \qquad t \in \left[x_0, x_0 + \frac{s_0}{2M}\right],$$

and the similar argument in the proof of Lemma 2.4, we may get the desired result. □

Definition 2.6. We say that $\alpha \in C^4[a,b]$ is a lower solution of

$$\begin{aligned} y'''' &= g(x,y), \qquad x \in (a,b), \\ y(a) &= y(b) = y'(a) = y'(b) = 0, \end{aligned} \tag{2.10}$$

if

$$\begin{aligned} &\alpha''''(x) \le g(x, \alpha(x)), \\ &\alpha(a) \le 0, \quad \alpha(b) \le 0, \quad \alpha'(a) \le 0, \quad \alpha'(b) \ge 0. \end{aligned} \tag{2.11}$$

We say that $\beta \in C^4[a,b]$ is an upper solution of (2.10) if β satisfies the reversed inequalities of the definition of lower solution.

Let us consider the following inequality that will appear later:

$$g(x, \alpha(x)) \le g(x,u) \le g(x, \beta(x)), \qquad \alpha(x) \le u \le \beta(x). \tag{2.12}$$

Lemma 2.7 (Cabada [3, Theorem 4.2])**.** *Suppose that $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and α, β are respectively a lower and an upper solution of (2.10). If $\alpha \le \beta$ and (2.12) holds, then there exists a solution $u(x)$ of (2.10) such that*

$$\alpha(x) \le u(x) \le \beta(x), \qquad x \in [a,b].$$

3 Rightward bifurcation

Let μ_k be the k-th eigenvalue of

$$\begin{aligned} y'''' &= \mu h(x) y, \qquad x \in (0,1), \\ y(0) &= y(1) = y'(0) = y'(1) = 0. \end{aligned}$$

Then its corresponding eigenfunction φ_k has exactly $k-1$ simple zeros in $(0,1)$, see Elias [8, Corollary 2 and Theorems 1 and 3].

To state our main result, we need to make the following assumption which will guarantee that any S_k^ν-solution u with $\|u\|_\infty = s_0$ implies $\lambda < \frac{\mu_k}{f_0}$, see the proof of Lemma 4.2 below.

(A5) Let $k \geq 4$ and
$$\frac{\mu_k}{f_0} h_* \min_{|s|\in[\frac{1}{2}s_0, s_0]} \frac{f(s)}{s} > \chi_3,$$
where χ_k is the k-th eigenvalue of
$$\begin{aligned} &y'''' = \chi y, \qquad x \in \left(0, \frac{s_0}{M}\right), \\ &y(0) = y\left(\frac{s_0}{M}\right) = y'(0) = y'\left(\frac{s_0}{M}\right) = 0. \end{aligned} \tag{3.1}$$

Remark 3.1. As we mentioned above, to show the existence of three nodal solutions, we shall employ a bifurcation technique. Indeed, under (A3) we have an unbounded connected component which is bifurcating from μ_k/f_0. Conditions (A1), (A3) and (A4) push the direction of bifurcation to the right near $u = 0$. Since Conditions (A5) and (A4) mean that $f(s)/s$ is large enough in $[s_0/2, s_0]$ and sublinear near ∞, respectively, it is natural to expect that the bifurcation curve (λ, u) grows to the right from the initial point $(\mu_k/f_0, 0)$, to the left at some point and to the right near $\lambda = \infty$.

Arguing the shape of bifurcation we have the following

Theorem 3.2. *Assume that (A1)–(A5) hold. Let $\nu \in \{+,-\}$. Then there exist $\lambda_* \in (0, \mu_k/f_0)$ and $\lambda^* > \mu_k/f_0$, such that*

(i) (1.1) *has at least one S_k^ν-solution if $\lambda = \lambda_*$;*

(ii) (1.1) *has at least two S_k^ν-solutions if $\lambda_* < \lambda \leq \mu_k/f_0$;*

(iii) (1.1) *has at least three S_k^ν-solutions if $\mu_k/f_0 < \lambda < \lambda^*$;*

(iv) (1.1) *has at least two S_k^ν-solutions if $\lambda = \lambda^*$;*

(v) (1.1) *has at least one S_k^ν-solution if $\lambda > \lambda^*$.*

In the rest of this section, we show a global bifurcation phenomena from the trivial branch with the rightward direction of bifurcation. Rewriting (1.1) by
$$\begin{aligned} &u''''(x) = \lambda h(x) f_0 u(x) + \lambda h(x)[f(u(x)) - f_0 u(x)], \qquad x \in (0,1), \\ &u(0) = u(1) = u'(0) = u'(1) = 0, \end{aligned} \tag{3.2}$$
and using Dancer [7, Theorem 2] and following the similar arguments in the proof of [5, Theorem 3.2], we have

Lemma 3.3. *Assume that (A1)–(A4) hold. Then for each $\nu \in \{+,-\}$, there exists an unbounded continuum C_k^ν which is bifurcating from $(\mu_k/f_0, 0)$ for* (1.1)*. Moreover, if $(\lambda, u) \in C_k^\nu$, then u is a S_k^ν-solution for* (1.1)*.*

Lemma 3.4. *Assume that (A1)–(A4) hold. Let u be a S_k^ν-solution of* (1.1)*. Then there exists a constant $C > 0$ independent of u such that*
$$\|u'\|_\infty \leq \lambda C \|u\|_\infty.$$

Proof.

$$u(x) = \lambda \int_0^1 G(x,s)h(s)f(u(s))ds, \qquad x \in [0,1].$$

The Green function G can be explicitly given by

$$G(x,s) = \frac{1}{6}\begin{cases} x^2(1-s)^2[(s-x)+2(1-x)s], & 0 \le x \le s \le 1, \\ s^2(1-x)^2[(x-s)+2(1-s)x], & 0 \le s \le x \le 1, \end{cases} \tag{3.3}$$

see Cabada and Enguiça [3]. Thus

$$u'(x) = \lambda \int_0^1 G_x(x,s)h(s)f(u(s))ds, \qquad x \in [0,1], \tag{3.4}$$

Noticing that (A3) and (A4) imply that

$$|f(s)| \le f^\diamond |s|, \qquad s \in \mathbb{R} \tag{3.5}$$

for some $f^\diamond > 0$, it follows from (3.3), (3.4) and the fact

$$G(x,s) \le 1/4, \quad (x,s) \in [0,1]\times[0,1]; \qquad |G_x(x,t)| \le 1, \quad (x,t) \in [0,1]\times[0,1]$$

that

$$|u'(x)| \le \lambda f^\diamond \int_0^1 h(t)dt\|u\|_\infty, \qquad x \in [0,1]. \qquad \square$$

By the same method used in the proof of [19, Lemma 3.3], with obvious changes, we may get the following

Lemma 3.5. *Assume that (A1)–(A4) hold. Let (λ_n, u_n) be a sequence of S_k^ν-solutions to (1.1) which satisfies $\|u_n\|_\infty \to 0$ and $\lambda_n \to \mu_k/f_0$. Let $\varphi_k \in S_k^\nu$ be the eigenfunction corresponding to μ_k which satisfies $\|\varphi_k\|_\infty = 1$. Then there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that $u_n/\|u_n\|_\infty$ converges uniformly to φ_k on $[0,1]$.*

Lemma 3.6. *Assume that (A1)–(A4) hold. Then there exists $\delta > 0$ such that $(\lambda, u) \in C_k^\nu$ and $|\lambda - \mu_k/f_0| + \|u\|_\infty \le \delta$ imply $\lambda > \mu_k/f_0$.*

Proof. We only deal with the case that $\nu = +$. The other case can be treated by the similar method.

Assume to the contrary that there exists a sequence $\{(\beta_n, u_n)\}$ such that $(\beta_n, u_n) \in C_k^+$, $\beta_n \to \mu_k/f_0$, $\|u_n\|_\infty \to 0$ and $\beta_n \le \mu_k/f_0$. By Lemma 3.5, there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that $u_n/\|u_n\|_\infty$ converges uniformly to φ_k on $[0,1]$. Multiplying the equation of (1.1) with $(\lambda, u) = (\beta_n, u_n)$ by u_n and integrating it over $[0,1]$, we obtain

$$\beta_n \int_0^1 h(x)f(u_n(x))u_n(x)dx = \int_0^1 |u_n''(x)|^2 dx,$$

and accordingly,

$$\beta_n \int_0^1 h(x)\frac{f(u_n(x))}{\|u_n\|_\infty}\frac{u_n(x)}{\|u_n\|_\infty}dx = \int_0^1 \frac{|u_n''(x)|^2}{\|u_n\|_\infty^2}dx. \tag{3.6}$$

From Lemma 3.5, after taking a subsequence and relabeling if necessary, $u_n/\|u_n\|_\infty$ converges to φ_k in $C[0,1]$.

$$\int_0^1 |\varphi_k''(x)|^2 dx = \mu_k \int_0^1 h(x)|\varphi_k(x)|^2 dx,$$

it follows that

$$\beta_n \int_0^1 h(x)\frac{f(u_n(x))}{\|u_n\|_\infty}\frac{u_n(x)}{\|u_n\|_\infty}dx = \mu_k \int_0^1 h(x)\frac{|u_n(x)|^2}{\|u_n\|_\infty^2}dx - \zeta(n),$$
$$\beta_n \int_0^1 h(x)f(u_n(x))u_n(x)dx = \mu_k \int_0^1 h(x)|u_n(x)|^2dx - \zeta(n)\|u_n\|_\infty^2$$

with a function $\zeta : \mathbb{N} \to \mathbb{R}$ satisfying

$$\lim_{n\to\infty} \zeta(n) = 0. \tag{3.7}$$

That is

$$\begin{aligned}\int_0^1 h(x)&\frac{f(u_n(x)) - f_0u_n(x)}{|u_n(x)|^\alpha u_n(x)}\left|\frac{u_n(x)}{\|u_n\|_\infty}\right|^{2+\alpha} dx\\ &= \frac{\beta_n}{\|u_n\|_\infty^\alpha}\left[\frac{\mu_k - f_0}{\beta_n}\int_0^1 h(x)\left|\frac{u_n(x)}{\|u_n\|_\infty}\right|^2 dx - \zeta(n)\right].\end{aligned} \tag{3.8}$$

Lebesgue's dominated convergence theorem and condition (A3) imply that

$$\int_0^1 h(x)\frac{f(u_n(x)) - f_0u_n}{|u_n(x)|^\alpha u_n(x)}\left|\frac{u_n(x)}{\|u_n\|_\infty}\right|^{2+\alpha} dx \to -f_1\int_0^1 h(x)|\varphi_k|^{2+\alpha}dx < 0$$

and

$$\int_0^1 h(x)\left|\frac{u_n(x)}{\|u_n\|_\infty}\right|^2 dx \to \int_0^1 h(x)|\varphi_k|^2dx > 0.$$

This contradicts with $\beta_n \le \mu_k/f_0$. □

4 Direction turn of bifurcation

In this section, we will show that

$$C_k^\nu \cap \{(\lambda, w) : (\lambda, w) \in (\mu_k/f_0, \infty) \times E \text{ with } \|w\|_\infty = s_0\} = \emptyset.$$

In other word, there exists a "barrier strip" for C_k^ν. From Lemmas 2.4–2.5, we obtain

Lemma 4.1. *Assume that (A1)–(A4) hold. Let u be a S_k^ν-solution of (1.1) with $\|u\|_\infty = s_0$. Then there exists $I_u := (\alpha_u, \beta_u)$, such that*

$$\begin{aligned}&u(\alpha_u) = u(\beta_u) = 0,\\ &\beta_u - \alpha_u \ge \frac{2s_0}{M},\\ &|u| > 0 \ \text{ in } I_u, \quad \|u\|_\infty = u(t_0) \quad \text{for some } t_0 \in (\alpha_u, \beta_u).\end{aligned} \tag{4.1}$$

$$\frac{1}{2}\|u\|_\infty \le |u(x)| \le \|u\|_\infty, \qquad x \in \left[x_0 - \frac{s_0}{2M}, x_0 + \frac{s_0}{2M}\right] =: [a, b]. \tag{4.2}$$

Lemma 4.2. *Assume that (A1)–(A5) hold. Let $(\lambda, u) \in C_k^\nu$ be such that $\|u\|_\infty = s_0$. Then $\lambda < \mu_k/f_0$.*

Proof. Let u be a S_k^ν-solution of (1.1) with $\|u\|_\infty = s_0$. By Lemma 4.1,

$$\frac{1}{2}\|u\|_\infty \le |u(x)| \le \|u\|_\infty, \qquad x \in [a, b] = J_u. \tag{4.3}$$

Note that u is a solution of

$$u''''(x) = \lambda h(x)\frac{f(u(x))}{u(x)}u(x), \qquad x \in J_u.$$

Assume on the contrary that

$$\lambda \geq \mu_k/f_0. \tag{4.4}$$

Then for $x \in J_u$, we have from (A5) that

$$\lambda h(x)\frac{f(u(x))}{u(x)} \geq \frac{\mu_k}{f_0}h_* \min_{s\in[s_0/2,s_0]}\frac{f(s)}{s} > \chi_3, \qquad x \in J_u. \tag{4.5}$$

Take

$$\begin{aligned} \beta(t) &:= u(t), \qquad t \in J_u; \\ \alpha(t) &:= \epsilon\psi_1(t), \qquad t \in J_u, \end{aligned}$$

where ψ_k is the eigenfunction corresponding to the k-th eigenvalue r_k of the problem

$$\begin{aligned} \psi'''' &= r\psi(t), \qquad t \in (a,b), \\ \psi(a) = \psi(b) &= \psi'(a) = \psi'(b) = 0, \end{aligned} \tag{4.6}$$

and $\psi_1(t) > 0$ in (a,b). Since the equations in (3.1) and (4.6) are autonomous,

$$r_1 = \chi_1. \tag{4.7}$$

We claim that

$$\beta'(a) > 0, \qquad \beta'(b) < 0.$$

In fact, let us denote

$$\tilde{\gamma}(x) := \lambda h(x)\frac{f(u(x))}{u(x)} > 0 \quad \text{for } x \in (0,1),$$

and

$$\tilde{\gamma}(0) := \lambda h(0)f_0, \qquad \tilde{\gamma}(1) := \lambda h(1)f_0.$$

Then $\tilde{\gamma} \in C^0[0,1]$ since $f_0 = \lim_{s\to 0} f(s)/s$ exists by (A3). Now, the claim can be easily deduced from Bari and Rynne [2, Lemma 2.1] and Elias [8] and the facts

$$u'''' = \lambda h(x)\frac{f(u(x))}{u(x)}u(x), \qquad x \in (0,1).$$

Obviously, β is an upper solution of

$$\begin{aligned} z''''(x) &= \lambda h(x)\frac{f(u(x))}{u(x)}z(x), \qquad a < x < b, \\ z(a) &= z(b) = z'(a) = z'(b) = 0. \end{aligned} \tag{4.8}$$

From (4.5) and (4.7), it is follows that

$$(\epsilon\psi_1(x))'''' = r_1(\epsilon\psi_1(x)) = \chi_1(\epsilon\psi_1(x)) < \chi_3(\epsilon\psi_1(x)) < \lambda h(x)\frac{f(u(x))}{u(x)}(\epsilon\psi_1(x)), \qquad x \in (a,b).$$

So, α is a lower solution of (4.8).

We may take $\epsilon > 0$ is so small that

$$\alpha(x) \leq \beta(x), \qquad x \in (a,b).$$

Therefore, it follows from Cabada [3, Theorem 4.2] that there exists a solution $y(x)$ of (4.8) such that

$$\alpha(x) \leq y(x) \leq \beta(x). \tag{4.9}$$

On the other hand, $\|y\|_\infty \leq \|u\|_\infty = s_0$ implies that the weight function in (4.8) satisfies

$$\lambda h(x)\frac{f(u(x))}{u(x)} > \chi_3, \qquad x \in (a,b).$$

Combining this with the facts $\psi_3(x-a)$ has exactly two simple zeros in (a,b) and

$$y'''' = \lambda h(x)\frac{f(u(x))}{u(x)}y(x), \qquad x \in (a,b),$$

and using Lemma 2.1, it deduces that y has a zero in (a,b). However, this contradicts (4.9). □

5 Second turn and proof of Theorem 3.2

In this section, we shall give a-priori estimate and finalize the proof of Theorem 3.2.

Lemma 5.1. *Assume that (A1)–(A4) hold. Let (λ, u) be a S_k^ν-solution of (1.1). Then there exists $\lambda_* > 0$ such that $\lambda \geq \lambda_*$.*

Proof. Lemma 3.4 implies that (3.2) holds for some constant $C > 0$, which is independent of u. Let $\|u\|_\infty = u(x_0)$. From (3.2) it follows that

$$\|u\|_\infty = |u(x_0)| \leq \int_0^{x_0} |u'(x)|dx \leq \lambda C\|u\|_\infty,$$

that is, $\lambda \geq C^{-1}$. □

Lemma 5.2. *Assume that (A1)–(A4) hold. Let $J = [a_1, b_1]$ be a compact interval in $(0,\infty)$. Then for given $\nu \in \{+,-\}$, there exists $M_J > 0$ such that for all $\lambda \in J$, all possible S_k^ν-solutions u of (1.1) satisfy*

$$\|u\|_\infty \leq M_J. \tag{5.1}$$

Proof. By (A4), we have that for any $\sigma > 0$, there exists $C_\sigma > 0$, such that

$$|f(s)| \leq C_\sigma + \sigma|s|. \tag{5.2}$$

This together with (3.3) imply

$$\begin{aligned}
|u(x)| &= \lambda\left|\int_0^1 G(x,s)h(s)f(u(s))ds\right| \\
&\leq \lambda\left|\int_0^1 G(x,s)h(s)(C_\sigma + \sigma|u(s)|)ds\right| \\
&\leq b_1\left|\int_0^1 G(x,s)h^*(C_\sigma + \sigma|u(s)|)ds\right| \\
&\leq C_1 + \sigma C_2\|u\|_\infty,
\end{aligned} \tag{5.3}$$

where

$$C_1 := b_1 h^* C_\sigma \max\{G(x,s) : (x,s) \in [0,1] \times [0,1]\},$$
$$C_2 := b_1 h^* \max\{G(x,s) : (x,s) \in [0,1] \times [0,1]\}.$$

Take σ so small that $\sigma C_2 < 1$. Then it follows from (5.3) that

$$\|u\|_\infty \le \frac{C_1}{1-\sigma C_2} =: M_J. \qquad \square$$

Lemma 5.3. *Assume that (A1)–(A4) hold. Let C_k^ν be as in Lemma 3.3. Then there exists $\{(\lambda_n, u_n)\}$ such that $(\lambda_n, u_n) \in C_k^\nu$, $\lambda_n \to \infty$ as $n \to \infty$ and $\|u_n\|_\infty \to \infty$.*

Proof. We only deal with the case $\nu = +$. The case $\nu = -$ can be treated by the similar method.

Since C_k^+ is unbounded, there exists $\{(\lambda_n, u_n)\}$ solutions of (1.1) such that $\{(\lambda_n, u_n)\} \subset C_k^+$ and $|\lambda_n| + \|u_n\|_\infty \to \infty$. Lemma 5.1 implies that $\lambda_n > 0$.

Assume on the contrary that there exists sequence $\{(\lambda_n, u_n)\}$ with

$$\|u_n\|_\infty \le M_1, \qquad \forall n \in \mathbb{N}.$$

Then $\lambda_n \to \infty$, and

$$u_n'''' = \lambda_n h(x) \frac{f(u_n)}{u_n} u_n. \tag{5.4}$$

Since

$$h_* \min_{0<s\le M_1} \frac{f(s)}{s} \ge \delta_0 > 0, \tag{5.5}$$

Since $u'''' = 0$ is disconjugate in $[0,1]$ and

$$\lambda_n h(x) \frac{f(u_n)}{u_n} \to \infty \quad \text{uniformly for } x \in [0,1],$$

it follows from the proof of [8, Lemma 4] (see also the remarks in the final paragraph on [8, p. 43], or the proof of Rynne [17, Lemma 3.7]) that u_n has more than k zeros in any given subinterval $I_* \subseteq [0,1]$ if n is large enough. However, this contradicts the fact $u \in S_k^+$. $\square$

Proof of Theorem 3.2. Let C_k^ν be as in Lemma 3.3. We only deal with C_k^+ since the case C_k^- can be treated similarly.

By Lemma 3.6, C_k^+ is bifurcating from $(\mu_k/f_0, 0)$ and goes rightward. Let (λ_n, u_n) be as in Lemma 5.3. Then there exists $(\lambda_0, u_0) \in C_k^+$ such that $\|u_0\|_\infty = s_0$. Lemma 4.2 implies that $\lambda_0 < \mu_k/f_0$.

By Lemmas 3.6, 4.2 and 5.2, it follows that for $\epsilon > 0$ small enough, C_k^+ passes through some points $(\mu_k/f_0 - \epsilon, v_1)$ and $(\mu_k/f_0 + \epsilon, v_2)$ with

$$\|v_1\|_\infty < s_0 < \|v_2\|_\infty.$$

By Lemmas 3.6, 4.2 and 5.2 again, there exist $\underline{\lambda}$ and $\bar{\lambda}$ which satisfy $0 < \underline{\lambda} < \mu_k/f_0 < \bar{\lambda}$ and both (i) and (ii):

(i) if $\lambda \in (\mu_k/f_0, \bar{\lambda}]$, then there exist u and v such that (λ, u), $(\lambda, v) \in C_k^+$ and

$$\|u\|_\infty < \|v\|_\infty < s_0;$$

(ii) if $\underline{\lambda} < \mu_k/f_0$ and $\lambda \in [\underline{\lambda}, \mu_k/f_0]$, then there exist u and v such that (λ, u), $(\lambda, v) \in C_k^+$ and $\|u\|_\infty < s_0 < \|v\|_\infty$.

Define

$$\lambda^* = \sup\{\bar{\lambda} : \bar{\lambda} \text{ satisfies (i)}\}, \qquad \lambda_* = \inf\{\underline{\lambda} : \underline{\lambda} \text{ satisfies (ii)}\}.$$

Then by the standard argument, (1.1) has a S_k^+-solution at $\lambda = \lambda_*$ and $\lambda = \lambda^*$, respectively. Since C_k^+ passes through $(\mu_k/f_0 + \epsilon, v_2)$ and (λ_n, u_n), Lemmas 4.2 and 5.2 show that, for each $\lambda > \mu_k/f_0$, there exists w such that $(\lambda, w) \in C_k^+$ and $\|w\|_\infty > s_0$. This completes the proof. □

Remark 5.4. Let $\rho > 1$ be a positive parameter. Let $g_1 \in C([4,\infty),(0,\infty))$ and $g_2 \in C([1,2],(0,\infty))$ such that

$$g_1(4) = 4\rho + 2, \quad \lim_{|s|\to\infty} \frac{g_1(s)}{s} = 0, \quad g_2(1) = 1, \quad g_2(2) = 2 + 2\rho.$$

Let

$$\hat{f}(s) = \begin{cases} g_1(s), & s \in [4,\infty), \\ \rho s + 2, & s \in [2,4), \\ g_2(s), & s \in (1,2), \\ 2s - s^2, & s \in [0,1], \end{cases}$$

and

$$\tilde{f}(s) = \begin{cases} \hat{f}(s), & s \in [0,\infty), \\ -\hat{f}(-s), & s \in [-\infty, 0). \end{cases}$$

Then $\tilde{f}$ satisfies $(A4)$ and $(A3)$ with $\tilde{f}_0 = 2, \tilde{f}_1 = 1, \alpha = 1$. If we take $s_0 = 4$ and $h(x) \equiv 1$ in $[0,1]$, then $(A5)$ can be rewritten as

$$\frac{\mu_k}{2}\left(\rho + \frac{1}{2}\right) > \chi_3.$$

In order to compute χ_3, we may use (2.5) and (2.7) to find Λ and M, and then use (3.1) to find χ_3. In fact,

$$\chi_3 = \mu_3 \left(\frac{M}{s_0}\right)^4, \qquad \mu_3 \doteq (10.9956)^4 \doteq 14617.6.$$

Therefore, Theorem 3.2 can be used to deal with the case $f = \tilde{f}$ and $h \equiv 1$ if ρ large enough.

Remark 5.5. We may study the oscillating global continua of positive solutions of (1.1) under the conditions

(A6) there exist two positive constant γ^+, γ^- and a sequence $\{\xi_k\} \subset (0,\infty)$ with

$$\xi_{2j-1} < \xi_{2j} < \xi_{2j} < \xi_{2j+1}, \qquad \xi_{2j-1} < \frac{1}{24}\xi_{2j}, \qquad j = 1,2,\dots; \tag{5.6}$$

such that

$$\frac{f(s)}{s} < \frac{f_0}{(\lambda_1 + \gamma^+ f_0)\int_0^1 \max\{G(t,s) : t \in [0,1]\} h(s)ds}, \qquad s \in (0, \xi_{2j-1}], \tag{5.7}$$

$$\frac{f(s)}{s} > \frac{f_0}{(\lambda_1 - \gamma^- f_0)\eta_0^2 \int_{1/4}^{3/4} \min\{G(t,s) : t \in [1/4,3/4]\} h(s)ds}, \qquad s \in \left[\frac{1}{24}\xi_{2j}, \xi_{2j}\right]. \tag{5.8}$$

Together $f_0 \in (0,\infty)$ with the facts that

$$G(t,s) \geq \frac{1}{24} G(j(s),s), \qquad (t,s) \in \left[\frac{1}{4}, \frac{3}{4}\right],$$

where

$$j(s) = \begin{cases} \frac{1}{3-2s}, & 0 \le s \le \frac{1}{2}, \\ \frac{2s}{1+2s}, & \frac{1}{2} \le s \le 1. \end{cases}$$

By the similar argument in Rynne [16], we may get that for all $\lambda \in \big(\frac{\lambda_1}{f_0} - \gamma^-, \frac{\lambda_1}{f_0} + \gamma^+\big)$, (1.1) has infinitely many positive solutions. Obviously, (5.8) is similar to (A5).

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Semi-linear impulsive higher order boundary value problems

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. This paper considers two-point higher order impulsive boundary value problems, with a strongly nonlinear fully differential equation with an increasing homeomorphism. It is stressed that the impulsive effects are defined by very general functions, that can depend on the unknown function and its derivatives, till order $n-1$.

The arguments are based on the lower and upper solutions method, together with Leray–Schauder fixed point theorem. An application, to estimate the bending of a one-sided clamped beam under some impulsive forces, is given in the last section.

Keywords: higher order boundary value problems, generalized impulsive conditions, upper and lower solutions, fixed point theory.

2020 Mathematics Subject Classification: 34B37, 34B10, 34B15.

1 Introduction

In this article we study the two point boundary value problem composed by the one-dimensional ϕ-Laplacian equation

$$(\phi(u^{(n-1)}(t)))' + q(t)f(t, u(t), \dots, u^{(n-1)}(t)) = 0, \qquad t \in [a,b] \backslash \{t_1, \dots, t_m\}, \tag{1.1}$$

where ϕ is an increasing homeomorphism such that $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$, $q \in L^\infty[a,b]$ is a positive function and $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function, together with the boundary conditions

$$u^{(j)}(a) = A_j, \quad u^{(n-1)}(b) = B, \qquad j = 0, 1, \dots, n-2, \quad A_j, B \in \mathbb{R}, \tag{1.2}$$

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and the impulsive conditions

$$\begin{aligned} \Delta u^{(i)}(t_k) &= I_{i,k}(u(t_k), u'(t_k), \ldots, u^{(n-1)}(t_k)), \qquad i = 0, 1, \ldots, n-2, \\ \Delta \phi(u^{(n-1)}(t_k)) &= I_{n-1,k}(u(t_k), u'(t_k), , \ldots, u^{(n-1)}(t_k)), \end{aligned} \tag{1.3}$$

being $\Delta u^{(i)}(t_k) = u^{(i)}(t_k^+) - u^{(i)}(t_k^-)$, $i = 0, 1, \ldots, n-1$, $k = 1, 2, \ldots, m$, $I_{i,k} \in C(\mathbb{R}^n, \mathbb{R})$, and t_k fixed points such that $a = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = b$.

Impulsive boundary value problems have been studied by many authors where it is highlighted the huge possibilities of applications to phenomena where a sudden variation happens. Indeed, these types of jumps occur in different areas such as population dynamics, engineering, control, and optimization theory, medicine, ecology, biology and biotechnology, economics, pharmacokinetics, and many other fields.

From a large number of items existent in the literature on classical impulsive differential problems, we mention, for instance, [1, 2, 17, 19–21] and the references therein. The most applied arguments are based on critical point theory and variational methods [18, 22, 26], fixed point theory on cones [6, 29], bifurcation results [13, 15], and upper and lower solutions techniques suggested in [4, 5, 12, 14].

In the last years, p-Laplacian and ϕ-Laplacian operators have been applied to semi-linear, quasi-linear, and strongly nonlinear differential equations, in singular and regular cases, increasing the range of theoretical and practical applications, as it can be seen, for example, in [3, 11, 25, 27, 28, 30] and in their references. However, impulsive problems with this type of nonlinear differential equations are scarce.

In [16], the third order differential equation

$$(\phi(u''(t)))' + q(t) f(t, u(t), u'(t), u''(t)) = 0, \qquad t \in [a, b] \backslash \{t_1, \ldots, t_n\},$$

is studied, where ϕ is an increasing homeomorphism, $q \in C([a, b])$ with $q > 0$, $f \in C([a, b] \times \mathbb{R}^3, \mathbb{R})$, the two-point boundary conditions

$$u(a) = A, \quad u'(a) = B, \quad u''(b) = C, \qquad A, B, C \in \mathbb{R},$$

and the impulsive effects are given by

$$\begin{aligned} \Delta u(t_k) &= I_{1k}(t_k, u(t_k), u'(t_k)), \\ \Delta u'(t_k) &= I_{2k}(t_k, u(t_k), u'(t_k), u''(t_k)), \\ \Delta \phi(u''(t_k)) &= I_{3k}(t_k, u(t_k), u'(t_k), u''(t_k)), \end{aligned}$$

where $k = 1, 2, \ldots, n$, $I_{1k} \in C([a, b] \times \mathbb{R}^2, \mathbb{R})$, and $I_{ik} \in C([a, b] \times \mathbb{R}^3, \mathbb{R})$, $i = 2, 3$.

In this work, we found a method that allows generalizing the above results to higher-order boundary value problems with impulsive functions, depending not only on the unknown function but also on its derivatives till order $n-1$. To best of our knowledge, it is the first time, where such nonlinear higher-order problems are considered with this type of generalized impulsive functions.

This paper is organized in the following way: Section 2 contains the functional framework, some definitions and an explicit form for the solution of the associated homogeneous problem. Section 3 presents the main existence and localization theorem obtained via lower and upper solutions technique and a fixed point theorem. The last section gives a technique to estimate the bending of a one-sided clamped beam under some impulsive forces and how it can be obtained some qualitative data about its variation.

2 Definitions and preliminary results

Let

$$PC^{n-1}[a,b] = \left\{ \begin{array}{l} u : u \in C^{n-1}([a,b];\mathbb{R}) \text{ for } t \neq t_k, u^{(i)}(t_k) = u^{(i)}(t_k^-), u^{(i)}(t_k^+) \\ \quad \text{exists for } k = 1,2,\ldots,m, \text{ and } i = 0,1,\ldots,n-1 \end{array} \right\}.$$

Denote $X := PC^{n-1}[a,b]$. Then X is a Banach space with norm

$$\|u\|_X = \max\{\|u^{(i)}\|_\infty,\ i = 0,1,\ldots,n-1\},$$

where

$$\|w\|_\infty = \sup_{a \le t \le b} |w(t)|.$$

Defining $J := [a,b]$ and $J' = J \backslash \{t_1,\ldots,t_m\}$, for a solution u of problem (1.1)–(1.3) one should consider $u(t) \in E$, where

$$E := PC^{n-1}(J) \cap C^n(J').$$

Next lemma provides a uniqueness result for a linear problem related to (1.1)–(1.3).

Lemma 2.1. *For $v \in PC[a,b]$, the problem composed by the differential equation*

$$(\phi(u^{(n-1)}(t)))' + v(t) = 0 \tag{2.1}$$

together with conditions (1.2), (1.3), has a unique solution given by

$$\begin{aligned} u(t) = &\sum_{i=0}^{n-2} \left(\left[A_i + \sum_{k\,:\,t_k<t} I_{i,k}\left(t_k, u(t_k),\ldots,u^{(n-1)}(t_k)\right)\right] \frac{(t-a)^{n-2-i}}{(n-2-i)!} \right) \\ &+ \int_a^t \frac{(t-s)^{n-2}}{(n-2)!} \phi^{-1} \left(\phi(B) + \int_s^b v(r)dr - \sum_{k\,:\,t_k>s} I_{n-1,k}\left(t_k, u(t_k),\ldots,u^{(n-1)}(t_k)\right) \right) ds. \end{aligned}$$

Proof. Integrating the differential equation (2.1) for $t \in (t_m, b]$ we get, by (1.2),

$$\phi\left(u^{(n-1)}(t)\right) = \phi(B) + \int_{t_m}^b v(s)ds. \tag{2.2}$$

By integration of (2.1) for $t \in (t_{m-1}, t_m]$ one has by (2.2)

$$\begin{aligned} \phi\left(u^{(n-1)}(t)\right) &= \int_t^{t_m} v(s)ds - I_{n-1,m}\left(t_m, u(t_m),\ldots,u^{(n-1)}(t_m)\right) + \phi\left(u^{(n-1)}\left(t_m^+\right)\right) \\ &= \phi(B) - I_{n-1,m}\left(t_m, u(t_m),\ldots,u^{(n-1)}(t_m)\right) + \int_t^b v(s)ds \end{aligned}$$

and so,

$$u^{(n-1)}(t) = \phi^{-1}\left(\phi(B) - I_{n-1,m}\left(t_m, u(t_m),\ldots,u^{(n-1)}(t_m)\right) + \int_t^b v(s)ds\right).$$

Therefore, for $t \in [a,b]$, we have

$$u^{(n-1)}(t) = \phi^{-1}\left(\phi(B) + \int_t^b v(s)ds - \sum_{k\,:\,t_k>t} I_{n-1,k}\left(t_k, u(t_k),\ldots,u^{(n-1)}(t_k)\right)\right). \tag{2.3}$$

Integrating (2.3), for $t \in [a, t_1]$,

$$u^{(n-2)}(t) = A_{n-2} + \int_a^t \left(\phi^{-1} \left(\phi(B) + \int_s^b v(r)dr - \sum_{k\,:\,t_k>s} I_{n-1,k} \left(t_k, u(t_k), \ldots, u^{(n-1)}(t_k) \right) \right) \right) ds.$$

By integration of (2.3) on $(t_1, t_2]$ and (1.3)

$$\begin{aligned} u^{(n-2)}(t) = {} & I_{n-2,1} \left(t_1, u(t_1), \ldots, u^{(n-1)}(t_1) \right) \\ & + \int_{t_1}^t \left(\phi^{-1} \left(\phi(B) + \int_s^b v(r)dr - \sum_{k\,:\,t_k>s} I_{n-1,k} \left(t_k, u(t_k), \ldots, u^{(n-1)}(t_k) \right) \right) \right) ds. \end{aligned}$$

Therefore, for $t \in [a, b]$,

$$\begin{aligned} u^{(n-2)}(t) = {} & \sum_{k\,:\,t_k<t} \left(I_{n-2,k} \left(t_k, u(t_k), \ldots, u^{(n-1)}(t_k) \right) \right) + A_{n-2} \\ & + \int_a^t \left(\phi^{-1} \left(\phi(B) + \int_s^b v(r)dr - \sum_{k\,:\,t_k>s} I_{n-1,k} \left(t_k, u(t_k), \ldots, u^{(n-1)}(t_k) \right) \right) \right) ds. \end{aligned}$$

Following the same method, by iterate integrations and (1.3), we obtain for $t \in [a, b]$

$$\begin{aligned} u(t) = {} & \sum_{i=0}^{n-2} \left(\left[A_i + \sum_{k\,:\,t_k<t} I_{i,k} \left(t_k, u(t_k), \ldots, u^{(n-1)}(t_k) \right) \right] \frac{(t-a)^{n-2-i}}{(n-2-i)!} \right) \\ & + \int_a^t \frac{(t-s)^{n-2}}{(n-2)!} \phi^{-1} \left(\phi(B) + \int_s^b v(s)ds - \sum_{k\,:\,t_k>s} I_{n-1,k} \left(t_k, u(t_k), \ldots, u^{(n-1)}(t_k) \right) \right) ds. \end{aligned}$$

□

Lower and upper solutions will play a key role in our method, and they are defined as it follows:

Definition 2.2. A function $\alpha(t) \in E$ with $\phi(\alpha^{(n-1)}(t)) \in PC^1[a, b]$ is a lower solution of problem (1.1), (1.2), (1.3) if

$$\begin{cases} \left(\phi \left(\alpha^{(n-1)}(t) \right) \right)' + q(t) f \left(t, \alpha(t), \alpha'(t), \ldots, \alpha^{(n-1)}(t) \right) \geq 0 \\ \alpha^{(j)}(a) \leq A_j, \; j = 0, 1, \ldots, n-2, \\ \alpha^{(n-1)}(b) \leq B \\ \Delta\alpha^{(i)}(t_k) \leq I_{i,k}(\alpha(t_k), \ldots, \alpha^{(n-1)}(t_k)), \; i = 0, 1, \ldots, n-3, \\ \Delta\alpha^{(n-2)}(t_k) > I_{n-2,k}(\alpha(t_k), \ldots, \alpha^{(n-1)}(t_k)) \\ \Delta\phi(\alpha^{(n-1)}(t_k)) > I_{n-1,k}(\alpha(t_k), \ldots, \alpha^{(n-1)}(t_k)), \end{cases} \tag{2.4}$$

for $k = 1, 2, \ldots, m$.

A function $\beta(t) \in E$ such that $\phi(\beta^{(n-1)}(t)) \in PC^1[a, b]$ is an upper solution of (1.1)–(1.3) if it satisfies the opposite inequalities.

To control the derivative $u^{(n-1)}(t)$ we will apply the Nagumo condition:

Definition 2.3. An L^1-Carathéodory function $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ satisfies a Nagumo condition related to a pair of functions $\gamma, \Gamma \in E$, with $\gamma^{(i)}(t) \leq \Gamma^{(i)}(t)$, for $i = 0, 1, \ldots, n-2$, and $t \in [a,b]$, if there exists a function $\psi : C([0,+\infty),]0,+\infty))$ such that

$$|f(t, x_0, x_1, \ldots, x_{n-1})| \leq \psi(|x_{n-1}|), \quad \text{for all } (t, x_0, x_1, \ldots, x_{n-1}) \in S \tag{2.5}$$

with

$$S := \{(t, x_0, x_1, \ldots, x_{n-1}) \in [a,b] \times \mathbb{R}^n : \gamma^{(i)}(t) \leq x_i \leq \Gamma^{(i)}(t), i = 0, 1, \ldots, n-2\},$$

and

$$\int_{\phi(\mu)}^{+\infty} \frac{ds}{\psi(\phi^{-1}(s))} > \int_a^b q(s)ds, \tag{2.6}$$

where

$$\mu := \max_{k=0,1,2,\ldots,m} \left\{ \left| \frac{\Gamma^{(n-2)}(t_{k+1}) - \gamma^{(n-2)}(t_k)}{t_{k+1} - t_k} \right|, \left| \frac{\gamma^{(n-2)}(t_{k+1}) - \Gamma^{(n-2)}(t_k)}{t_{k+1} - t_k} \right| \right\}.$$

From the Nagumo condition we deduce an *a priori* estimation for $u^{(n-1)}(t)$:

Lemma 2.4. *If the L^1-Carathéodory function $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ satisfies a Nagumo condition in the set S, referred to the functions γ and Γ, then there is $N \geq \mu > 0$ such that every solution u of the differential equation (1.1) verifies $\|u^{(n-1)}\|_\infty \leq N$.*

Proof. Let $u(t)$ be a solution of (1.1) such that

$$\gamma^{(i)}(t) \leq u^{(i)}(t) \leq \Gamma^{(i)}(t), \quad \text{for } i = 0, 1, \ldots, n-2 \text{ and } t \in [a,b].$$

By the Mean Value Theorem, there exists $\eta_0 \in (t_k, t_{k+1})$ with

$$u^{(n-1)}(\eta_0) = \frac{u^{(n-2)}(t_{k+1}) - u^{(n-2)}(t_k)}{t_{k+1} - t_k}, \quad \text{with } k = 0, 1, 2, \ldots, m.$$

Moreover,

$$\begin{aligned} -N \leq -\mu \leq \frac{\gamma^{(n-2)}(t_{k+1}) - \Gamma^{(n-2)}(t_k)}{t_{k+1} - t_k} &\leq u^{(n-1)}(\eta_0) \\ \leq \frac{\Gamma^{(n-2)}(t_{k+1}) - \gamma^{(n-2)}(t_k)}{t_{k+1} - t_k} &\leq \mu \leq N. \end{aligned} \tag{2.7}$$

If $\left|u^{(n-1)}(t)\right| \leq N$ for every $t \in [a,b]$, the proof is complete.

On the contrary, assume that there is $\tau \in [a,b]$ such that $\left|u^{(n-1)}(\tau)\right| > N$. Consider the case where $u^{(n-1)}(\tau) > N$. Therefore there is η_1 such that $u^{(n-1)}(\eta_1) = N$. Suppose, without loss of generality, that $\eta_0 < \eta_1$. So,

$$u^{(n-1)}(t) > 0 \quad \text{and} \quad u^{(n-1)}(\eta_0) \leq u^{(n-1)}(t) \leq N, \quad \text{for } t \in [\eta_0, \eta_1].$$

So

$$|\phi(u^{(n-1)}(t))| = |q(t) f(t, u(t), \ldots, u^{(n-1)}(t))| \leq q(t) |\psi(u^{(n-1)}(t))|, \quad \text{for } t \in [\eta_0, \eta_1],$$

and

$$\int_{\phi(u^{(n-1)}(\eta_0))}^{\phi(N)} \frac{ds}{\psi(\phi^{-1}(s))} \leq \int_{\eta_0}^{\eta_1} \frac{|(\phi(u^{(n-1)}(t))'|}{\psi(u^{(n-1)}(t))} dt$$
$$= \int_{\eta_0}^{\eta_1} \frac{|q(t)f(t,u(t),\ldots,u^{(n-1)}(t))|}{\psi(u^{(n-1)}(t))} dt \leq \int_{\eta_0}^{\eta_1} q(t)dt < \int_a^b q(t)dt.$$

As $u^{(n-1)}(\eta_0) \leq \mu < N$, by the monotony of ϕ,

$$\phi(u^{(n-1)}(\eta_0)) \leq \phi(\mu)$$

and, by (2.6),

$$\int_{\phi(u^{(n-1)}(\eta_0))}^{\phi(N)} \frac{ds}{\psi(\phi^{-1}(s))} \geq \int_{\phi(\mu)}^{\phi(N)} \frac{ds}{\psi(\phi^{-1}(s))} > \int_a^b q(t)dt$$

which leads to a contradiction.

The other cases, that is, $u^{(n-1)}(\tau) > N$ with $\eta_1 < \eta_0$, and $u^{(n-1)}(\tau) < -N$ with $\eta_0 < \eta_1$ or $\eta_1 < \eta_0$, follow the same arguments to obtain a contradiction.

Therefore $|u^{(n-1)}(t)| \leq N$, for $t \in [a,b]$. □

Forward, in our method, we will use the following lemma, given in [23]:

Lemma 2.5. *For $v, w \in C(I)$ such that $v(x) \leq w(x)$, for every $x \in I$, define*

$$q(x,u) = \max\{v, \min\{u,w\}\}.$$

Then, for each $u \in C^1(I)$ the next two properties hold:

(a) $\frac{d}{dx}q(x,u(x))$ *exists for a.e.* $x \in I$.

(b) If $u, u_m \in C^1(I)$ and $u_m \to u$ in $C^1(I)$ then

$$\frac{d}{dx}q(x,u_m(x)) \to \frac{d}{dx}q(x,u(x)) \quad \text{for a.e. } x \in I.$$

We recall the classical Schauder's fixed point theorem:

Theorem 2.6. *Let M be a nonempty, closed, bounded and convex subset of a Banach space X, and suppose that $T : M \to M$ is a compact operator. Then T as at least one fixed point in M.*

3 Existence and localization result

The main result is an existence and localization theorem, as it provides not only the existence of solutions but also some of its qualitative properties.

Theorem 3.1. *Suppose that there are α and β lower and upper solutions, respectively, of problem (1.1)–(1.3) such that*

$$\alpha^{(n-2)}(t) \leq \beta^{(n-2)}(t), \quad \text{for } t \in [a,b].$$

Assume that the L^1-Carathéodory function $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ satisfies a Nagumo condition, related to α and β, and verifies

$$f(t,\alpha(t),\ldots,\alpha^{(n-3)}(t)),y,z) \leq f(t,x_0,\ldots,x_{n-1}) \leq f(t,\beta(t),\ldots,\beta^{(n-3)}(t),y,z), \tag{3.1}$$

for $\alpha^{(i)}(t) \leq x_i \leq \beta^{(i)}(t)$, *for* $i = 0, \dots, n-3$, *and fixed* $(y, z) \in \mathbb{R}^2$.
Moreover, if the impulsive functions satisfy

$$I_{j,k}(\alpha(t_k), \dots, \alpha^{(n-1)}(t_k)) \leq I_{j,k}(x_0, \dots, x_{n-1}) \leq I_{j,k}(\beta(t_k), \dots, \beta^{(n-1)}(t_k)), \tag{3.2}$$

for $j = 0, \dots, n-3$, $\alpha^{(i)}(t_k) \leq x_i \leq \beta^{(i)}(t_k)$, *for* $i = 0, 1, \dots, n-2$, $k = 1, 2, \dots, m$, *and*

$$\begin{aligned} I_{n-2,k}(\alpha(t_k), \dots, \alpha^{(n-3)}(t_k), y, z) &\geq I_{n-2,k}(x_0, \dots, x_{n-1}) \\ &\geq I_{n-2,k}(\beta(t_k), \dots, \beta^{(n-3)}(t_k), y, z) \end{aligned} \tag{3.3}$$

for $\alpha^{(i)}(t) \leq x_i \leq \beta^{(i)}(t)$, *for* $i = 0, \dots, n-3$, *and fixed* $(y, z) \in \mathbb{R}^2$, *then problem* (1.1)–(1.3) *has at least one solution* $u \in E$, *such that*

$$\alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t) \quad \text{for } i = 0, 1, \dots, n-2 \text{ and } -N \leq u^{(n-1)}(t) \leq N,$$

for $t \in [a, b]$ *and* N *given by* (2.7).

Proof. Define the continuous functions δ_i, for $i = 0, 1, \dots, n-2$,

$$\delta_i(t, u^{(i)}(t)) = \begin{cases} \beta^{(i)}(t), & u^{(i)}(t) \geq \beta^{(i)}(t) \\ u^{(i)}(t), & \alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t) \\ \alpha^{(i)}(t), & u^{(i)}(t) \leq \alpha^{(i)}(t) \end{cases}$$

and consider the following modified and perturbed equation

$$\begin{aligned} (\phi(u^{(n-1)}(t)))' + q(t) f\left(t, \delta_0(t, u(t)), \dots, \delta_{n-2}(t, u^{(n-2)}(t)), \frac{d}{dt}\left(\delta_{n-2}(t, u^{(n-2)}(t))\right)\right) \\ + \frac{\delta_{n-2}(t, u^{(n-2)}(t)) - u^{(n-2)}(t)}{1 + |u^{(n-2)}(t) - \delta_{n-2}(t, u^{(n-2)}(t))|} = 0, \end{aligned} \tag{3.4}$$

coupled with boundary conditions (1.2) and the truncated impulsive conditions, for $i = 0, 1, \dots, n-2$,

$$\begin{aligned} \Delta u^{(i)}(t_k) &= I_{i,k}\left(\begin{array}{c} \delta_0(t_k, u(t_k)), \dots, \delta_{n-2}(t_k, u^{(n-2)}(t_k), \\ \frac{d}{dt}(\delta_{n-2}(t_k, u^{(n-2)}(t_k))) \end{array} \right) := I^*_{i,k}(t_k), \\ \Delta \phi(u^{(n-1)}(t)) &= I_{n-1,k}\left(\begin{array}{c} \delta_0(t_k, u(t_k)), \dots, \delta_{n-2}(t_k, u^{(n-2)}(t_k)), \\ \frac{d}{dt}(\delta_{n-2}(t_k, u^{(n-2)}(t_k))) \end{array} \right) := I^*_{n-1,k}(t_k). \end{aligned} \tag{3.5}$$

Define the operator $T : E \to E$ by

$$\begin{aligned} T(u)(t) := \sum_{i=0}^{n-2} \left(\left[A_i + \sum_{k\,:\,t_k < t} I^*_{i,k} \right] \frac{(t-a)^{n-2-i}}{(n-2-i)!} \right) \\ + \int_a^t \frac{(t-s)^{n-2}}{(n-2)!} \phi^{-1} \left(\phi(B) + \int_s^b v(s) ds - \sum_{k\,:\,t_k > s} I^*_{n-1,k} \right) ds. \end{aligned}$$

By Lemma 2.1, it is clear that the fixed points of T, u_*, are solutions of the initial problem (1.1)-(3.5), if they verify

$$\alpha^{(i)}(t) \leq u_*^{(i)}(t) \leq \beta^{(i)}(t), \quad \text{for } t \in [a, b] \text{ and } i = 0, 1, \dots, n-2.$$

As T is compact, by Schauder's fixed point theorem, T has a fixed point $u \in E$, which is a solution of (3.4), (1.2), (3.5). To prove that this solution verifies

$$\alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t), \quad \text{for } t \in [a,b], \text{ and } i = 0, 1, \ldots, n-2,$$

suppose, by contradiction, that, for $i = n-2$, there is $t \in [a,b]$ such that

$$u^{(n-2)}(t) > \beta^{(n-2)}(t).$$

Define $\zeta \in [a,b]$ as

$$\sup_{t\in[a,b]} (u^{(n-2)}(t) - \beta^{(n-2)}(t)) := u^{(n-2)}(\zeta) - \beta^{(n-2)}(\zeta)) > 0. \tag{3.6}$$

By (1.2) and Definition 2.2, $u^{(n-2)}(a) - \beta^{(n-2)}(a) \leq 0$, then $\zeta \neq a$. On the other hand $u^{(n-1)}(b) - \beta^{(n-1)}(b) < 0$ and then $\zeta \neq b$, by (3.6).

Therefore $\zeta \in]a,b[$.

Case 1: Assume that there is $p \in \{1, 2, \ldots, m\}$ such that $\zeta \in (t_p, t_{p+1})$.

Consider $\epsilon > 0$ small enough such that

$$u^{(n-2)}(t) - \beta^{(n-2)}(t) > 0 \quad \text{and} \quad u^{(n-1)}(t) - \beta^{(n-1)}(t) \leq 0, \quad \text{for } t \in (\zeta, \zeta+\epsilon). \tag{3.7}$$

Therefore, by (3.1) and (3.7), for all $t \in (\zeta, \zeta+\epsilon)$, we have the following contradiction

$$\begin{aligned}
0 &\geq \phi\left(u^{(n-1)}(t)\right)' - \phi\left(\beta^{(n-1)}(t)\right)' \\
&\geq -q(t) f\left(t, \delta_0(t, u(t)), \ldots, \delta_{n-2}(t, u^{(n-2)}(t)), \frac{d}{dt}\left(\delta_{n-2}(t, u^{(n-2)}(t))\right)\right) \\
&\quad - \frac{\delta_{n-2}\left(t, u^{(n-2)}(t)\right) - u^{(n-2)}(t)}{1 + \left|u^{(n-2)}(t) - \delta_{n-2}\left(t, u^{(n-2)}(t)\right)\right|} + q(t) f\left(t, \beta(t), \ldots, \beta^{(n-1)}(t)\right) \\
&= -q(t) f\left(t, \delta_0(t, u(t)), \ldots, \delta_{n-3}(t, u(t)), \beta^{(n-2)}(t), \beta^{(n-1)}(t)\right) \\
&\quad - \frac{\beta^{(n-2)}(t) - u^{(n-2)}(t)}{1 + \left|u^{(n-2)}(t) - \beta^{(n-2)}(t)\right|} + q(t) f\left(t, \beta(t), \ldots, \beta^{(n-1)}(t)\right) \\
&\geq -q(t) f\left(t, \beta(t), \ldots, \beta^{(n-1)}(t)\right) - \frac{\beta^{(n-2)}(t) - u^{(n-2)}(t)}{1 + \left|u^{(n-2)}(t) - \beta^{(n-2)}(t)\right|} \\
&\quad + q(t) f\left(t, \beta(t), \ldots, \beta^{(n-1)}(t)\right) = \frac{u^{(n-2)}(t) - \beta^{(n-2)}(t)}{1 + \left|u^{(n-2)}(t) - \beta^{(n-2)}(t)\right|} > 0.
\end{aligned}$$

Case 2: Consider that there exists $k \in \{1, 2, \ldots, m\}$ such that, or

$$\max_{t\in[a,b]} \left(u^{(n-2)}(t) - \beta^{(n-2)}(t)\right) := u^{(n-2)}(t_k^-) - \beta^{(n-2)}(t_k^-) > 0 \tag{3.8}$$

or

$$\sup_{t\in[a,b]} \left(u^{(n-2)}(t) - \beta^{(n-2)}(t)\right) := u^{(n-2)}(t_k^+) - \beta^{(n-2)}(t_k^+) > 0. \tag{3.9}$$

If (3.8) holds, then

$$\Delta\left(u^{(n-2)}(t) - \beta^{(n-2)}(t)\right) \leq 0$$

and, by (3.3) and Definition 2.2, we have the contradiction

$$\begin{aligned}0 &\geq \Delta u^{(n-2)}(t_k) - \Delta\beta^{(n-2)}(t_k) = I^*_{n-2,k} - \Delta\beta^{(n-2)}(t_k)\\ &= I_{n-2,k}\left(\delta_0(t,u(t)),\ldots,\delta_{n-3}\left(t,u(t)\right),\beta^{(n-2)}(t),\beta^{(n-1)}(t)\right) - \Delta\beta^{(n-2)}(t_k)\\ &\geq I_{n-2,k}\left(t_k,\beta(t_k),\ldots,\beta^{(n-1)}(t_k)\right) - \Delta\beta^{(n-2)}(t_k) > 0.\end{aligned}$$

Consider now (3.9). So, there is $\epsilon > 0$ such that, for $t \in (t_k, t_k + \epsilon)$,

$$u^{(n-1)}(t) - \beta^{(n-1)}(t) \leq 0,$$

and the arguments follow by the same technique as in Case 1, to have

$$u^{(n-2)}(t) \leq \beta^{(n-2)}(t), \qquad \forall t \in [a,b].$$

To prove that $u^{(n-2)}(t) \geq \alpha^{(n-2)}(t)$, $\forall t \in [a,b]$, the method is similar. Therefore

$$\alpha^{(n-2)}(t) \leq u^{(n-2)}(t) \leq \beta^{(n-2)}, \quad \text{for } t \in [a,b].$$

Integrating the first inequality in $[a,t_1]$, we have

$$\begin{aligned}\alpha^{(n-3)}(t) &\leq u^{(n-3)}(t) - u^{(n-3)}(a) + \alpha^{(n-3)}(a)\\ &= u^{(n-3)}(t) - A_{n-3} + \alpha^{(n-3)}(a) \leq u^{(n-3)}(t).\end{aligned} \tag{3.10}$$

For $t \in (t_1,t_2]$, by (3.2) and (3.10),

$$\begin{aligned}\alpha^{(n-3)}(t) &\leq u^{(n-3)}(t) - u^{(n-3)}(t_1^+) + \alpha^{(n-3)}(t_1^+)\\ &\leq u^{(n-3)}(t) - I^*_{n-3,1}(t_1) - u^{(n-3)}(t_1)\\ &\quad + I_{n-3,1}\left(t_1,\alpha(t_1),\ldots,\alpha^{n-1}(t_1)\right) + \alpha^{(n-3)}(t_1)\\ &\leq u^{(n-3)}(t) - I^*_{n-3,1}(t_1) + I_{n-3,1}\left(t_1,\alpha(t_1),\ldots,\alpha^{(n-1)}(t_1)\right)\\ &\leq u^{(n-3)}(t).\end{aligned}$$

Applying this method for each interval $(t_k, t_{k+1}]$, $k = 2,\ldots,m$, we obtain

$$\alpha^{(n-3)}(t) \leq u^{(n-3)}(t), \qquad \forall t \in [a,b],$$

and, by the same technique,

$$\beta^{(n-3)}(t) \geq u^{(n-3)}(t), \qquad \forall t \in [a,b].$$

By iteration of these arguments, we conclude

$$\alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t), \quad \text{for } i = 0,1,\ldots,n-2, \text{ and } t \in [a,b].$$

The estimation $\left|u^{(n-1)}(t)\right| \leq N$ is a trivial consequence of Lemma 2.4. □

4 Estimation for the bending of one-sided clamped beam under impulsive effects

Problems related to beam structures and especially beams that support some forces as impulses, are part of a vast field of investigation in boundary value problems theory, see, for example, [7–10, 24].

In this application we consider a model to describe the bending of a beam with length $L > 1$, given by the fourth-order equation

$$\frac{EI}{A}u^{(4)}(x) + \frac{3}{2}\sqrt[3]{u'(x)}|u''(x)| - ku(x) - \gamma u'''(x) = 0, \quad \text{for } x \in \left]0, L\right[, \tag{4.1}$$

where $E > 0$ is the Young modulus, $I > 0$ the mass moment of inertia, $A > 0$ the cross section area, $k > 0$ the tension of a spring force vertically applied on the beam, and $\gamma > 0$ the shear force coefficient.

At the end points the behavior of the beam is given by the following boundary conditions

$$u(0) = 0, \quad u'(0) = 1, \quad u''(0) = 0, \quad u'''(L) = 0, \tag{4.2}$$

meaning that the beam is clamped on the left end side.

For clearance, we consider only one moment of impulse which occurs at $t_1 = 1$. The impulsive effects are given by generalized functions with dependence on the unknown function itself, and on several derivatives till order three,

$$\begin{aligned} \Delta u(1) &= u(1) + u'(1) - 2u''(1) - u'''(1) \\ \Delta u'(1) &= u(1) + u'(1) - 2u''(1) - u'''(1) \\ \Delta u''(1) &= -u(1) - u'(1) + u''(1) + 5u'''(1) - 1 \\ \Delta u'''(1) &= u(1) - u'(1) + u''(1) + u'''(1) - 1. \end{aligned} \tag{4.3}$$

This problem (4.1)-(4.3) is a particular case of (1.1)-(1.3) with $[a, b] = [0, L]$, $n = 4$,

$$f(x, y_0, y_1, y_2, y_3) = \frac{A}{EI}\left(\frac{3}{2}\sqrt[3]{y_1}|y_2| - ky_0 - \gamma y_3\right), \tag{4.4}$$

$\phi(w) = w, q(t) \equiv 1, m = 1$, $t_1 = 1$, and the impulsive functions given by

$$\begin{aligned} I_{0,1}(w_0, w_1, w_2, w_3) &= w_0 + w_1 - 2w_2 - w_3 \\ I_{1,1}(w_0, w_1, w_2, w_3) &= w_0 + w_1 - 2w_2 - w_3 \\ I_{2,1}(w_0, w_1, w_2, w_3) &= -w_0 - w_1 + w_2 + 5w_3 - 1 \\ I_{3,1}(w_0, w_1, w_2, w_3) &= w_0 - w_1 + w_2 + w_3 - 1. \end{aligned}$$

As a numeric example we can consider $A = 1$, $EI = 1$, $k = 1$, $\gamma = 6$, $L = 2$. In this case, the continuous functions

$$\alpha(x) = 0, \quad \beta(x) = \frac{x^3}{6} + x^2 + x, \quad \text{for } x \in [0, 2],$$

are, respectively, lower and upper solutions of problem (4.1)–(4.3), according to Definition 3.6.

In fact, for $\alpha(x) \equiv 0$ the inequalities are trivially satisfied and for β, we have,

$$\beta(0) = 0, \quad \beta'(0) = 1, \quad \beta''(0) = 2 > 0, \quad \beta'''(2) = 1 > 0,$$
$$\Delta\beta(1) = 0 \geq \beta(0) + \beta'(0) - 2\beta''(0) - \beta'''(0) = -\frac{4}{3},$$
$$\Delta\beta'(1) = 0 \geq \beta(0) + \beta'(0) - 2\beta''(0) - \beta'''(0) = -\frac{4}{3},$$
$$\Delta\beta''(1) = 0 < -\beta(0) - \beta'(0) + \beta''(0) + 5\beta'''(0) - 1 = \frac{4}{3}$$
$$\Delta\beta'''(1) = 0 < \beta(0) - \beta'(0) + \beta''(0) + \beta'''(0) - 1 = \frac{5}{3}.$$

The nonlinear part $f(x, y_0, y_1, y_2, y_3)$, given by (4.4), verifies a Nagumo condition on the set

$$S_* = \left\{ \begin{array}{l} (t, y_0, y_1, y_2, y_3) \in [0,2] \times \mathbb{R}^n : 0 \leq y_0 \leq \frac{x^3}{6} + x^2 + x, \\ \qquad 0 \leq y_1 \leq \frac{x^2}{2} + 2x + 1, 0 \leq y_2 \leq x + 2 \end{array} \right\}$$

with

$$\mu = \max\left\{ |\beta''(2)|, |\beta''(0)|, |\beta''(1)| \right\} = 4,$$
$$\psi(|y_3|) := |y_3| + \frac{22}{3},$$

and

$$\int_{\mu}^{+\infty} \frac{ds}{s + \frac{22}{3}} = +\infty > \int_0^L 1\, ds = L.$$

Moreover, f is nondecreasing on y_0 and, by Theorem 3.1, there exists a solution $u(x)$ of problem (4.1)–(4.3) such that

$$\alpha^{(i)}(x) \leqslant u^{(i)}(x) \leqslant \beta^{(i)}(x), \qquad i = 0, 1, 2, \text{ for } x \in [0,2],$$

that is

$$0 \leq u(x) \leq \frac{x^3}{6} + x^2 + x,$$
$$0 \leq u'(x) \leq \frac{x^2}{2} + 2x + 1,$$
$$0 \leq u''(x) \leq x + 2, \quad \text{for } x \in [0,2].$$

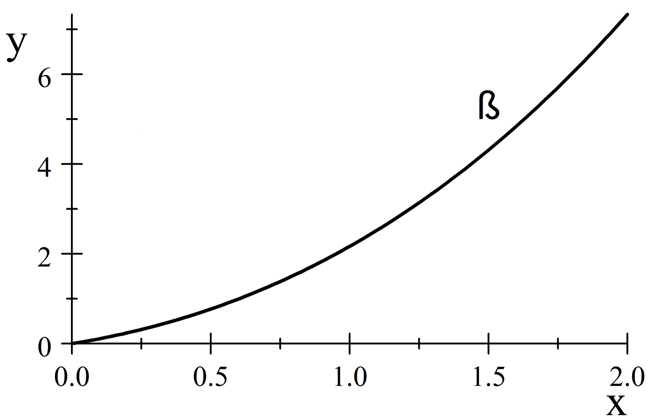

Figure 4.1: Strip of u localization.

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Implicit elliptic equations via Krasnoselskii–Schaefer type theorems

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. Existence of solutions to the Dirichlet problem for implicit elliptic equations is established by using Krasnoselskii–Schaefer type theorems owed to Burton–Kirk and Gao–Li–Zhang. The nonlinearity of the equations splits into two terms: one term depending on the state, its gradient and the elliptic principal part is Lipschitz continuous, and the other one only depending on the state and its gradient has a superlinear growth and satisfies a sign condition. Correspondingly, the associated operator is a sum of a contraction with a completely continuous mapping. The solutions are found in a ball of a Lebesgue space of a sufficiently large radius established by the method of *a priori* bounds.

Keywords: implicit elliptic equation, fixed point, Krasnoselskii theorem for the sum of two operators.

2020 Mathematics Subject Classification: 35J60, 47H10, 47J05.

1 Introduction

Krasnoselskii's fixed point theorem for the sum of two operators [12] – a typical hybrid fixed point result – has been used to prove the existence of solutions for many classes of problems when the associated operators do not comply to a pure fixed point principle. Its hybrid character is given by a combination of the Banach and Schauder fixed point theorems.

Theorem 1.1 (Krasnoselskii)**.** *Let D be a bounded closed convex nonempty subset of a Banach space $(X, |\cdot|)$ and let A, B be two operators such that*

(i) *$A : D \to X$ is a contraction;*

(ii) *$B : D \to X$ is continuous with $B(D)$ relatively compact;*

(iii) *$A(x) + B(y) \in D$ for every $x, y \in D$.*

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Then the operator $A+B$ has at least one fixed point, i.e., there exists $x \in D$ such that $x = A(x) + B(x)$.

There are many extensions of Krasnoselskii's theorem in several directions, for single and multi-valued mappings, self and non-self mappings, for generalized contractions and generalized compact-type operators, see for example [2, 5, 6, 10, 14, 18].

The strong invariance condition (iii) is required by the similar condition from Schauder's fixed point theorem. The last one is removed and replaced with the Leray–Schauder boundary condition by Schaefer's fixed point theorem [17].

Theorem 1.2 (Schaefer). *Let D_R be the closed ball centered at the origin and of radius R of a Banach space X, and let $N : D_R \to X$ be continuous with $N(D_R)$ relatively compact. If*

$$\lambda N(x) \neq x \quad \text{for all } x \in \partial D_R,\ \lambda \in (0,1), \tag{1.1}$$

then N has at least one fixed point.

There are known hybrid theorems of Krasnoselskii type that combine Banach's contraction principle with Schaefer's fixed point theorem. Such a result is owed to Burton and Kirk [6].

Theorem 1.3 (Burton–Kirk). *Let D_R be the closed ball centered at the origin and of radius R of a Banach space X, and let A, B be operators such that*

(j) *$A : X \to X$ is a contraction;*

(jj) *$B : D_R \to X$ is continuous with $B(D_R)$ relatively compact;*

(jjj) *$x \neq \lambda A\left(\frac{1}{\lambda}x\right) + \lambda B(x)$ for all $x \in \partial D_R$ and $\lambda \in (0,1)$.*

Then the operator $A+B$ has at least one fixed point, i.e., there exists $x \in D_R$ such that $x = A(x) + B(x)$.

A similar result is owed to Gao, Li and Zhang [11].

Theorem 1.4 (Gao–Li–Zhang). *Let D_R be the closed ball centered at the origin and of radius R of a Banach space X, and let A, B be operators such that*

(h) *$A : X \to X$ is a contraction;*

(hh) *$B : D_R \to X$ is continuous with $B(D_R)$ relatively compact;*

(hhh) *$x \neq A(x) + \lambda B(x)$ for all $x \in \partial D_R$ and $\lambda \in (0,1)$.*

Then the operator $A+B$ has at least one fixed point, i.e., there exists $x \in D_R$ such that $x = A(x) + B(x)$.

In proof, the difference between Theorem 1.3 and Theorem 1.4 consists in the homotopy that is considered. In the first case, the homotopy is $\lambda(I-A)^{-1}B$, while in the second case, it is $(I-A)^{-1}\lambda B$.

Obviously, if A is identically zero, then both results by Burton–Kirk and Gao–Li–Zhang reduce to Schaefer's theorem.

Remark 1.5 (Method of *a priori* bounds)**.** In applications, usually both operators A, B are defined on the whole space X and a ball D_R as required by condition (jjj) of Theorem 1.3 and (hhh) of Theorem 1.4 exists if the set of all solutions for $\lambda \in (0,1)$ of the equations

$$x = \lambda A\left(\frac{1}{\lambda}x\right) + \lambda B(x)$$

and

$$x = A(x) + \lambda B(x),$$

respectively, is bounded in X.

The aim of this paper is to give an application of the previous Krasnoselskii–Schaefer type theorems to the Dirichlet problem for implicit elliptic equations. Such equations have been intensively studied in the literature, see for example [7, 9]. Our result extends and complements previous contributions in this direction such as those in [4, 13, 15, 16].

We conclude the Introduction by some basic notions and results from the linear theory of partial differential equations [3, 16].

We shall work in the Sobolev space $H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is open bounded, endowed with the energetic norm

$$|u|_{H_0^1} = |\nabla u|_{L^2} = \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}.$$

Its dual space is $H^{-1}(\Omega)$ and the pairing of a functional $v \in H^{-1}(\Omega)$ and a function $u \in H_0^1(\Omega)$ is denoted by (v, u). We identify $L^2(\Omega)$ to its dual and thus we have $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$. Then, in particular, for $v \in L^2(\Omega)$, one has

$$(v, u) = (v, u)_{L^2} = \int_\Omega uv, \quad u \in H_0^1(\Omega).$$

Recall that the operator $(-\Delta)^{-1}$ is an isometry between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, so

$$|v|_{H^{-1}} = \left|(-\Delta)^{-1}v\right|_{H_0^1}, \quad v \in H^{-1}(\Omega).$$

Also, the embedding $H_0^1(\Omega) \subset L^p(\Omega)$ holds and is continuous for $1 \leq p \leq 2^* = 2n/(n-2)$, and the same happens for the embedding $L^q(\Omega) \subset H^{-1}(\Omega)$ if $q \geq (2^*)' = 2n/(n+2)$. These embeddings are compact for $p < 2^*$ and $q > (2^*)'$, respectively.

2 Application

We discuss here the Dirichlet problem for implicit nonlinear elliptic equations,

$$\begin{cases} -\Delta u = f(x, u, \nabla u, \Delta u) + g(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ is open bounded $(n \geq 3)$; $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy the Carathéodory conditions.

To give sense to the composition $f(x, u, \nabla u, \Delta u)$, we need to look for solutions $u \in H_0^1(\Omega)$ such that Δu is a function. More exactly we shall require that $\Delta u \in L^q(\Omega)$ for a given number $q \geq (2^*)'$.

If we let $v := -\Delta u$, then the equation becomes

$$v = f\Big(x, (-\Delta)^{-1}v, \nabla(-\Delta)^{-1}v, -v\Big) + g\Big(x, (-\Delta)^{-1}v, \nabla(-\Delta)^{-1}v\Big).$$

As noted above, this equation will be solved in a Lebesgue space $L^q(\Omega)$ with $q \geq (2^*)'$. We assume in addition that $q \leq 2$, which implies $L^2(\Omega) \subset L^q(\Omega)$.

Let $A, B : L^q(\Omega) \to L^q(\Omega)$ be given by

$$A(v) = f\Big(\cdot, (-\Delta)^{-1}v, \nabla(-\Delta)^{-1}v, -v\Big)$$
$$B(v) = g\Big(\cdot, (-\Delta)^{-1}v, \nabla(-\Delta)^{-1}v\Big).$$

Clearly we need some additional conditions on f and g to guarantee that the two operators are well-defined from $L^q(\Omega)$ to itself, and then, wishing to apply Theorem 1.3 or Theorem 1.4 we have to guarantee that A is a contraction, and B is completely continuous.

We begin by a technical lemma concerning the embedding constants. By an *embedding constant* for a continuous embedding $X \subset Y$ of two Banach spaces $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$, we mean a number $c > 0$ such that

$$|x|_Y \leq c|x|_X \quad \text{for every} \quad x \in X.$$

Note that if c is an embedding constant for the inclusion $X \subset Y$, then c is also an embedding constant for the dual inclusion $Y' \subset X'$. Indeed, for any $u \in Y'$, one has

$$|u|_{X'} = \sup_{\substack{x\in X\\ x\neq 0}} \frac{|(u,x)|}{|x|_X} \leq \sup_{\substack{x\in X\\ x\neq 0}} \frac{|(u,x)|}{c^{-1}|x|_Y} \leq c \sup_{\substack{x\in Y\\ x\neq 0}} \frac{|(u,x)|}{|x|_Y} = c|u|_{Y'}.$$

Recall that, according to the Poincaré inequality, the best (smallest) embedding constant for the inclusions $H_0^1(\Omega) \subset L^2(\Omega)$ and $L^2(\Omega) \subset H^{-1}(\Omega)$ is $1/\sqrt{\lambda_1}$, where λ_1 is the first eigenvalue of the Dirichlet problem for the operator $-\Delta$.

Lemma 2.1. *Let $(2^*)' \leq q \leq 2$ and let c_1, c_2, c_3 be embedding constants for the inclusions*

$$H_0^1(\Omega) \subset L^q(\Omega), \quad L^2(\Omega) \subset L^q(\Omega), \quad L^q(\Omega) \subset H^{-1}(\Omega). \tag{2.2}$$

Then one may consider

$$c_2 = c_1\sqrt{\lambda_1}, \quad c_3 = \frac{1}{c_1\lambda_1}.$$

Proof. From $H_0^1(\Omega) \subset L^2(\Omega) \subset L^q(\Omega)$, if $u \in H_0^1(\Omega)$, one has

$$|u|_{L^q} \leq c_2|u|_{L^2} \leq \frac{c_2}{\sqrt{\lambda_1}}|u|_{H_0^1},$$

hence $c_1 = c_2/\sqrt{\lambda_1}$, or $c_2 = c_1\sqrt{\lambda_1}$. To prove the second equality, let $u \in H_0^1(\Omega)$. On the one hand, using twice Poincaré's inequality, we have

$$|u|_{H^{-1}} \leq \frac{1}{\sqrt{\lambda_1}}|u|_{L^2} \leq \frac{1}{\lambda_1}|u|_{H_0^1},$$

and on the other hand,

$$|u|_{H^{-1}} \leq c_3|u|_{L^q} \leq c_1c_3|u|_{H_0^1}.$$

Hence $c_1c_3 = 1/\lambda_1$. $\square$

The next lemma guarantees that the operator A is a contraction.

Lemma 2.2. *Assume that there exist constants $a, b, c \geq 0$ such that*

$$|f(x,y,z,w) - f(x,\overline{y},\overline{z},\overline{w})| \leq a|y-\overline{y}| + b|z-\overline{z}| + c|w-\overline{w}|$$

for all $y, \overline{y}, w, \overline{w} \in \mathbb{R}$; $z, \overline{z} \in \mathbb{R}^n$ and a.a. $x \in \Omega$. Also assume that $f(\cdot, 0, 0, 0) \in L^2(\Omega)$. If

$$l := \frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}} + c < 1,$$

then A is a contraction on the space $L^q(\Omega)$ for any $q \in [1,2]$.

Proof. From the basic result about Nemytskii's operator (see, a.e., [16]), we have that A maps $L^q(\Omega)$ to itself. Let $v, w \in L^q(\Omega)$. Then using the embedding constants for the inclusions (2.2) and the relationships between them given by Lemma 2.1, we have

$$\begin{aligned}
|A(v) - A(w)|_{L^q} &\leq a\left|(-\Delta)^{-1}(v-w)\right|_{L^q} + b\left|\nabla(-\Delta)^{-1}(v-w)\right|_{L^q} + c|v-w|_{L^q} \\
&\leq ac_1\left|(-\Delta)^{-1}(v-w)\right|_{H_0^1} + bc_2\left|\nabla(-\Delta)^{-1}(v-w)\right|_{L^2} + c|v-w|_{L^q} \\
&= ac_1|v-w|_{H^{-1}} + bc_2\left|(-\Delta)^{-1}(v-w)\right|_{H_0^1} + c|v-w|_{L^q} \\
&= (ac_1 + bc_2)|v-w|_{H^{-1}} + c|v-w|_{L^q} \\
&\leq ((ac_1 + bc_2)c_3 + c)|v-w|_{L^q} \\
&= \left(\frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}} + c\right)|v-w|_{L^q}. \qquad \square
\end{aligned}$$

Furthermore, we have the following result about the complete continuity of the operator B on the space $L^q(\Omega)$.

Lemma 2.3. *Assume that there exist constants $a_0, b_0 \geq 0$; $\alpha \in [1, 2^*/(2^*)')$, $\beta \in [1, 2/(2^*)')$; and function $h \in L^2(\Omega)$ such that*

$$|g(x,y,z)| \leq a_0|y|^\alpha + b_0|z|^\beta + h(x) \tag{2.3}$$

for all $y \in \mathbb{R}$, $z \in \mathbb{R}^n$ and a.a. $x \in \Omega$. Then the operator $B : L^q(\Omega) \to L^q(\Omega)$ is well-defined and completely continuous for

$$q = \min\left\{\frac{2^*}{\alpha}, \frac{2}{\beta}\right\}. \tag{2.4}$$

Proof. First note that the restrictions on α and β imply that q given by (2.4) satisfies $(2^*)' < q \leq 2$.

Now the operator B is the composition NPJ of three operators

$$\begin{aligned}
&J : L^q(\Omega) \to H^{-1}(\Omega), && J(v) = v \\
&P : H^{-1}(\Omega) \to L^{2^*}(\Omega) \times L^2(\Omega; \mathbb{R}^n), && P(v) = \left((-\Delta)^{-1}v,\ \nabla(-\Delta)^{-1}v\right) \\
&N : L^{2^*}(\Omega) \times L^2(\Omega; \mathbb{R}^n) \to L^q(\Omega), && N(u,v) = g(\cdot, u, v).
\end{aligned}$$

Here J is completely continuous since the embedding $L^q(\Omega) \subset H^{-1}(\Omega)$ is compact $\left(q > (2^*)'\right)$, and obviously, the linear operator P is bounded. Next we show that N is well-defined, continuous and bounded (maps bounded sets into bounded sets). According to the basic result about Nemytskii's operator, this happens if we have a growth condition on g of the form

$$|g(x, w_1, w_2)| \leq a_0 |w_1|^{\frac{2^*}{q}} + b_0 |w_2|^{\frac{2}{q}} + h_0(x) \quad (w_1 \in \mathbb{R},\ w_2 \in \mathbb{R}^n,\ \text{a.a. } x \in \Omega) \tag{2.5}$$

with $a_0, b_0 \in \mathbb{R}_+$ and $h_0 \in L^q(\Omega)$. From (2.4), we have

$$1 \leq \alpha \leq \frac{2^*}{q}, \quad 1 \leq \beta \leq \frac{2}{q}.$$

Then the exponents α, β in (2.3) can be replaced by the larger ones $2^*/q$ and $2/\beta$ and thus the growth condition (2.3) implies (2.5), with a suitable function h_0 that incorporates h. Hence N has the desired properties.

The above properties of the operators J, P and N imply that B is well-defined and completely continuous from $L^q(\Omega)$ to itself. □

It remains to find *a priori* bounds of the solutions as required by Remark 1.5.

Lemma 2.4. *Under the assumptions of Lemmas 2.2 and 2.3, if in addition g satisfies the sign condition*

$$y g(x, y, z) \leq 0 \tag{2.6}$$

for all $y \in \mathbb{R}$, $z \in \mathbb{R}^n$ and a.a. $x \in \Omega$, then the sets of solutions of the equations

$$v = \lambda A\left(\frac{1}{\lambda} v\right) + \lambda B(v) \quad (\lambda \in (0,1)) \tag{2.7}$$

and of the equations

$$v = A(v) + \lambda B(v) \quad (\lambda \in (0,1)) \tag{2.8}$$

are bounded in $L^q(\Omega)$.

Proof. We shall prove the statement for the family of equations (2.7). The proof is similar for (2.8).

Step 1: We first prove the boundedness of the solutions in $H^{-1}(\Omega)$. Let $v \in L^q(\Omega)$ be any solution of (2.7). Since $v \in H^{-1}(\Omega)$, we may write

$$\left(v, (-\Delta)^{-1} v\right) = \lambda \left(A\left(\frac{1}{\lambda} v\right), (-\Delta)^{-1} v \right) + \lambda \left(B(v), (-\Delta)^{-1} v \right). \tag{2.9}$$

On the left side we have $\left|(-\Delta)^{-1} v\right|_{H_0^1}^2$ which is equal to $|v|_{H^{-1}}^2$. Also, from (2.6) we have

$$\left(B(v), (-\Delta)^{-1} v\right) = \int_\Omega g\left(x, (-\Delta)^{-1} v, \nabla (-\Delta)^{-1} v\right) (-\Delta)^{-1} v \leq 0.$$

Next, using the Lipschitz property of f, and denoting $\gamma_0 := |f(\cdot,0,0,0)|_{L^2}$ we obtain

$$
\begin{aligned}
&\lambda\left(A\left(\frac{1}{\lambda}v\right),(-\Delta)^{-1}v\right)\\
&\quad= \lambda\int_\Omega f\left(x,\frac{1}{\lambda}(-\Delta)^{-1}v,\frac{1}{\lambda}\nabla(-\Delta)^{-1}v,-\frac{1}{\lambda}v\right)(-\Delta)^{-1}v\\
&\quad\le \int_\Omega\left(a\left|(-\Delta)^{-1}v\right|+b\left|\nabla(-\Delta)^{-1}v\right|+c|v|+|f(x,0,0,0)|\right)\left|(-\Delta)^{-1}v\right|\\
&\quad\le a\left|(-\Delta)^{-1}v\right|^2_{L^2}+b\left|\nabla(-\Delta)^{-1}v\right|_{L^2}\left|(-\Delta)^{-1}v\right|_{L^2}\\
&\qquad+c\int_\Omega|v|\left|(-\Delta)^{-1}v\right|+\gamma_0\left|(-\Delta)^{-1}v\right|_{L^2}\\
&\quad\le \frac{a}{\lambda_1}\left|(-\Delta)^{-1}v\right|^2_{H^1_0}+\frac{b}{\sqrt{\lambda_1}}\left|(-\Delta)^{-1}v\right|^2_{H^1_0}\\
&\qquad+c\int_\Omega|v|\left|(-\Delta)^{-1}v\right|+\frac{1}{\sqrt{\lambda_1}}\gamma_0\left|(-\Delta)^{-1}v\right|_{H^1_0}\\
&\quad= \frac{a}{\lambda_1}|v|^2_{H^{-1}}+\frac{b}{\sqrt{\lambda_1}}|v|^2_{H^{-1}}+c\int_\Omega|v|\left|(-\Delta)^{-1}v\right|+\frac{1}{\sqrt{\lambda_1}}\gamma_0|v|_{H^{-1}}.
\end{aligned}
$$

Since

$$\int_\Omega|v|\left|(-\Delta)^{-1}v\right| = \left(v,s(-\Delta)^{-1}v\right),$$

where function s has only two values ±1 giving the sign of $v(-\Delta)^{-1}v$, we then have

$$\int_\Omega|v|\left|(-\Delta)^{-1}v\right| \le |v|_{H^{-1}}\left|s(-\Delta)^{-1}v\right|_{H^1_0} = |v|_{H^{-1}}\left|(-\Delta)^{-1}v\right|_{H^1_0} = |v|^2_{H^{-1}}.$$

It follows that

$$\lambda\left(A\left(\frac{1}{\lambda}v\right),(-\Delta)^{-1}v\right) \le \left(\frac{a}{\lambda_1}+\frac{b}{\sqrt{\lambda_1}}+c\right)|v|^2_{H^{-1}}+d|v|_{H^{-1}},$$

where $d=\gamma_0/\sqrt{\lambda_1}$. Thus (2.9) gives

$$|v|^2_{H^{-1}} \le l|v|^2_{H^{-1}}+d|v|_{H^{-1}}$$

which based on $l<1$ implies that

$$|v|_{H^{-1}} \le C_1, \tag{2.10}$$

where $C_1 = d/(1-l)$ does not depend on λ.

Step 2. $|B(v)|_{L^q} \le C_2$ for some constant C_2. Indeed, one has

$$|B(v)|_{L^q} \le a_0\left|\left|(-\Delta)^{-1}v\right|^\alpha\right|_{L^q}+b_0\left|\left|\nabla(-\Delta)^{-1}v\right|^\beta\right|_{L^q}+|h|_{L^q} \tag{2.11}$$

Furthermore, since $\alpha q \le 2^*$, we have the continuous embedding $H^1_0(\Omega)\subset L^{\alpha q}(\Omega)$, and so for some constant $\bar{c}$, we have

$$\left|\left|(-\Delta)^{-1}v\right|^\alpha\right|_{L^q} = \left|(-\Delta)^{-1}v\right|^\alpha_{L^{\alpha q}} \le \bar{c}\left|(-\Delta)^{-1}v\right|^\alpha_{H^1_0} = \bar{c}|v|^\alpha_{H^{-1}}. \tag{2.12}$$

Similarly, since $\beta q \leq 2$, we have

$$\begin{aligned} \left| \left| \nabla(-\Delta)^{-1} v \right|^{\beta} \right|_{L^q} &= \left| \nabla(-\Delta)^{-1} v \right|^{\beta}_{L^{\beta q}} \leq \bar{\bar{c}} \left| \nabla(-\Delta)^{-1} v \right|^{\beta}_{L^2} \\ &= \bar{\bar{c}} \left| (-\Delta)^{-1} v \right|^{\beta}_{H_0^1} = \bar{\bar{c}} |v|^{\beta}_{H^{-1}}. \end{aligned} \tag{2.13}$$

Now (2.10)–(2.13) lead to the conclusion at Step 2.

Step 3. $|v|_{L^q} \leq C$ for some constant C. Indeed, if $\gamma = |f(\cdot,0,0,0)|_{L^q}$, then one has

$$|v|_{L^q} \leq \lambda \left| A\left(\frac{1}{\lambda} v\right) \right|_{L^q} + \lambda |B(v)|_{L^q} \leq l|v|_{L^q} + \gamma + |B(v)|_{L^q}.$$

Hence

$$|v|_{L^q} \leq \frac{1}{1-l} (|B(v)|_{L^q} + \gamma),$$

which together with the result at Step 2 gives the conclusion with $C = (C_2 + \gamma)/(1-l)$. □

The above lemmas together with Theorem 1.3 (or alternatively, Theorem 1.4) and Remark 1.5 allow us to state the following existence result.

Theorem 2.5. *If f and g satisfy the conditions in Lemmas 2.2–2.4, then problem (2.1) has at least one solution $u \in H_0^1(\Omega)$ with $\Delta u \in L^q(\Omega)$, where $q = \min\{2^*/\alpha,\ 2/\beta\}$.*

Remark 2.6. The sign condition (2.6) can be replaced by

$$yg(x,y,z) \leq \sigma y^2$$

for all $y \in \mathbb{R}$, $z \in \mathbb{R}^n$ and a.a. $x \in \Omega$, for some $\sigma < (1-l)\lambda_1$.

Remark 2.7. If $g(x,y,z)$ has a linear growth in y, z with constants a_0 and b_0, and

$$\frac{a+a_0}{\lambda_1} + \frac{b+b_0}{\sqrt{\lambda_1}} + c < 1,$$

then the conclusion of Theorem 2.5 can be obtain using Krasnoselskii's theorem, without a sign condition on g. This happens, since in this case, it is possible to find a ball of $L^q(\Omega)$ of a sufficiently large radius such that the strong invariance condition of Krasnoselskii's theorem is fulfilled.

Finally we would like to mention that the result can be adapted to a general elliptic operator replacing the Laplacian, and the technique is possible to be used for treating other classes of implicit differential equations.

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Existence results for a class of p–q Laplacian semipositone boundary value problems

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. Let Ω be a bounded domain in $\mathbb{R}^N$; $N > 1$ with a smooth boundary or $\Omega = (0,1)$. We study positive solutions to the boundary value problem of the form:

$$\begin{aligned} -\Delta_p u - \Delta_q u &= \lambda f(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $q \in [2, p)$, λ is a positive parameter, and $f : [0,\infty) \mapsto \mathbb{R}$ is a class of C^1, non-decreasing and p-sublinear functions at infinity (i.e. $\lim_{t\to\infty} \frac{f(t)}{t^{p-1}} = 0$) that are negative at the origin (semipositone). We discuss the existence of positive solutions for $\lambda \gg 1$. Further, when $p = 4, q = 2$, $\Omega = (0,1)$ and $f(s) = (s+1)^\gamma - 2$; $\gamma \in (0,3)$, we provide the exact bifurcation diagram for positive solutions. In particular, we observe two positive solutions for a finite range of λ and a unique positive solution for $\lambda \gg 1$.

Keywords: p–q Laplacian, semipositone problems, positive solutions.

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1 Introduction

In [3], authors discussed results which imply the existence of positive solutions for $\lambda \gg 1$ for the boundary value problem:

$$\begin{aligned} -\Delta_p u &= \lambda f(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $p > 1$, Ω is a bounded domain in $\mathbb{R}^N$; $N > 1$ with a smooth boundary, λ is a positive parameter, and $\Delta_s u = \operatorname{div} |\nabla u|^{s-2}\nabla u)$; $s > 1$, and $f : [0,\infty) \to \mathbb{R}$ satisfies:

(H1) f is C^1, non-decreasing, p-sublinear at infinity $\big($i.e. $\lim_{t\to\infty} \frac{f(t)}{t^{p-1}} = 0\big)$,

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(**H2**) $f(0) < 0$,

(**H3**) $\lim_{t\to\infty} f(t) = \infty$.

In the literature, such problems where $f(0) < 0$, are referred as semipositone problems. It is well known that establishing the existence of a positive solution for semipositone problems are challenging, see [1,4,9,10] and references therein.

In recent years, there has been considerable interest to study boundary value problems involving the p–q Laplacian operator ($-\Delta_p - \Delta_q$, $q \in (1, p)$), for examples, see [2,5,8,11] and the references therein. Such operators often occur in the mathematical modelling of chemical reactions and plasma physics. In this article, we extend this study of p–q Laplacian boundary value problem for a class of semipositone reaction terms. Namely, we study the boundary value problem

$$\begin{aligned} -\Delta_p u - \Delta_q u &= \lambda f(u) \quad \text{in } \Omega, \\ u &= 0 \qquad\quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

for $q \in [2, p)$. We establish the following result.

Theorem 1.1. *Assume* (**H1**), (**H2**) *hold and there exists $A > 0, \sigma > 0$ such that*

$$f(s) \geq As^{\sigma}, \quad \textit{for } s \gg 1.$$

Then (1.2) *has a positive solution for $\lambda \gg 1$.*

Remark 1.2. It is easy to see that (1.2) does not admit any positive solution for $\lambda \approx 0$. This follows due to the p-sublinear condition at infinity which implies there exists a $M > 0$ such that $f(s) \leq Ms^{p-1}, \forall s > 0$. Hence, if u is a positive solution, multiplying (1.2) by u and integrating we obtain

$$\int_{\Omega} |\nabla u|^p dx + \int_{\Omega} |\nabla u|^q dx \leq \lambda M \int_{\Omega} |u|^p dx$$

which implies

$$\lambda \geq \left(\frac{1}{M}\right) \left(\frac{\int_{\Omega} |\nabla u|^p \, dx}{\int_{\Omega} |u|^p \, dx}\right) \geq \frac{\lambda_{1,p}}{M},$$

where $\lambda_{1,p} > 0$ is the principal eigenvalue of $-\Delta_p$ on Ω with Dirichlet boundary condition.

We will use the method of sub-super solutions to establish Theorem 1.1. We will adapt and extend the ideas used in [3] to construct a crucial positive sub-solution.

Finally, for the case when $\Omega = (0, 1)$, $p = 4$ and $q = 2$, namely to the two-point boundary value problem:

$$\begin{aligned} -[(u')^3]' - [(u')]' &= \lambda f(u) \qquad \text{in } (0,1), \\ u(0) = 0 &= u(1) \end{aligned} \tag{1.3}$$

with $f(s) = (s + 1)^r - 2$; $r \in (0, 3)$, we will provide exact bifurcation diagrams for positive solutions in Section 4. Bifurcation diagrams we obtained are of the form given in Figure 1.1. Note that this bifurcation diagram implies the existence of two positive solutions for certain finite range of λ and a unique positive solution for $\lambda \gg 1$.

The rest of the paper is organized as follows. In Section 2, we will recall some important results that are required for the development of this article. Section 3 is dedicated to the proof of Theorem 1.1, and Section 4 is devoted to obtaining the bifurcation diagram of positive solutions to (1.3).

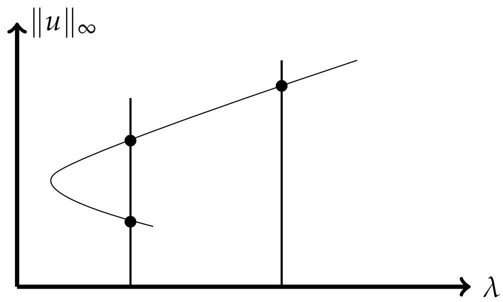

Figure 1.1: Bifurcation diagram for positive solutions to (1.3)

2 Preliminaries

In this section, we recall some results concerning a sub-super solution method for p–q Laplacian boundary value problem. First, by a weak solution of (1.2) we mean a function $u \in W_0^{1,p}(\Omega)$ which satisfies:

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\phi + \int_\Omega |\nabla u|^{q-2}\nabla u.\nabla\phi = \lambda\int_\Omega f(u)\phi, \qquad \forall\phi\in C_0^\infty(\Omega).$$

However, in this paper, we in fact study $C^1(\overline{\Omega})$ solution. Next, by a sub-solution (super solution) of (1.2) we mean a function $v \in W^{1,p}(\Omega)\cap C^1(\overline{\Omega})$ such that $v \leq (\geq)\, 0$ on $\partial\Omega$ and satisfies:

$$\int_\Omega |\nabla v|^{p-2}\nabla v.\nabla\phi + \int_\Omega |\nabla v|^{q-2}\nabla v.\nabla\phi \leq (\geq)\ \lambda\int_\Omega f(v)\phi, \qquad \forall\phi\in C_0^\infty(\Omega),\ \phi\geq 0 \quad \text{in } \Omega.$$

Then the following sub-super solution result holds.

Lemma 2.1. *Let ψ, z be sub and super solutions of (1.2) respectively such that $\psi \leq z$ in Ω. Then (1.2) has a solution $u \in C^1(\overline{\Omega})$ such that $\psi \leq u \leq z$.*

Proof. We refer to Corollary 1 of [6] for the proof. □

3 Proof of Theorem 1.1

In this section, we use sub-super solution method to prove Theorem 1.1. We adapt and extend the ideas used in [3] to construct a crucial positive sub-solution.

Construction of a sub-solution : Let λ_1 be the principal eigenvalue and $\phi_1 \in C^\infty(\overline{\Omega})$ be the corresponding eigenfunction of

$$\begin{aligned} -\Delta\phi_1 &= \lambda_1\phi_1 && \text{in } \Omega,\\ \phi_1 &= 0 && \text{on } \partial\Omega \end{aligned}$$

such that $\phi_1 > 0$ in Ω and $\|\phi_1\|_\infty = 1$. Then $\Delta_p\phi_1$, $\Delta_q\phi_1$ are in $L^\infty(\Omega)$, since $2 \leq q < p$. Further, by Hopf's lemma $|\nabla\phi_1| > 0$ on $\partial\Omega$. Now we consider

$$\psi = \lambda^r\phi_1^\beta, \quad \text{where } \beta = \frac{p}{p-1} \text{ and } r \in \left(\frac{1}{p-1}, \frac{1}{p-1-\sigma}\right).$$

Note that without loss of generality we can assume $\sigma < q-1$. Then, for $s = p, q$,

$$-\Delta_s\psi = \lambda^{r(s-1)}\beta^{s-1}\phi_1^{(\beta-1)(s-1)}[-\Delta_s\phi_1] - \lambda^{r(s-1)}\beta^{s-1}(\beta-1)(s-1)\frac{|\nabla\phi_1|^s}{\phi_1^{s-\beta(s-1)}}.$$

Note that $s-\beta(s-1) = 0$ when $s = p$ and $s-\beta(s-1) > 0$ when $s = q$. Also, $|\nabla\phi_1| > 0$ on $\partial\Omega$, $\phi_1 = 0$ on $\partial\Omega$ and $\phi_1 \in C^\infty(\overline{\Omega})$. Therefore, by continuity, there exists a δ neighborhood of $\partial\Omega$, say $\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) \le \delta\}$ such that

$$-\Delta_s\psi < 0 \quad \text{in } \Omega_\delta \tag{3.1}$$

for $s = p, q$. Further, since $\Delta_p\phi_1 \in L^\infty(\Omega)$ we see that $\exists\ \epsilon_p > 0$ (independent of λ) such that

$$-\Delta_p\psi \le -\lambda^{r(p-1)}\epsilon_p \quad \text{in } \Omega_\delta.$$

As $r(p-1) > 1$, for $\lambda \gg 1$ it follows that

$$-\Delta_p\psi \le -\lambda^{r(p-1)}\epsilon_p \le \lambda f(0) \le \lambda f(\psi) \quad \text{in } \Omega_\delta.$$

Hence, by (3.1) for $\lambda \gg 1$ we have

$$-\Delta_p\psi - \Delta_q\psi \le \lambda f(\psi) \quad \text{in } \Omega_\delta. \tag{3.2}$$

Next let $\mu > 0$ be such that $\phi_1^\beta \ge \mu$ in $\Omega \setminus \Omega_\delta$ and $M_s > 0$ $(s = p, q)$ be such that $-\Delta_s\psi \le M_s\lambda^{r(s-1)}$ in Ω. Since $r < \frac{1}{s-1-\sigma}$ $(s = p, q)$, it follows that for $\lambda \gg 1$ we have

$$\begin{aligned}-\Delta_s\psi \le M_s\lambda^{r(s-1)} &\le \left(\frac{\lambda A}{2}\right)(\lambda^r\mu)^\sigma \\ &\le \left(\frac{\lambda}{2}\right)f(\psi) \quad \text{in } \Omega\setminus\Omega_\delta.\end{aligned}$$

Thus, for $\lambda \gg 1$, we obtain

$$-\Delta_p\psi - \Delta_q\psi \le \lambda f(\psi) \quad \text{in } \Omega\setminus\Omega_\delta. \tag{3.3}$$

Combining (3.2) and (3.3), for $\lambda \gg 1$ we see that

$$-\Delta_p\psi - \Delta_q\psi \le \lambda f(\psi) \ \text{ in } \Omega. \tag{3.4}$$

Therefore, ψ is a sub-solution of (1.2) when $\lambda \gg 1$.

Construction of a super solution: Let $R > 0$ be such that $\overline{\Omega} \subseteq B_R(0)$, where $B_R(0)$ is the open ball of radius R centered at origin. Now consider

$$\eta(r) = \frac{1-(\frac{r}{R})^{p'}}{p'} \quad \text{on } B_R,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Notice that $0 \le \eta \le 1$. Also for $0 \le r \le R$,

$$\eta'(r) = -\frac{r^{p'-1}}{R^{p'}},$$

$$-\Delta_s\eta = -\left(|\eta'(r)|^{s-2}\eta'(r)\right)' = \left(\frac{r^{(p'-1)(s-1)}}{R^{p'(s-1)}}\right)' \ge 0 \quad \text{in } B_R, \tag{3.5}$$

for $s = p, q$. In particular,

$$-\Delta_p \eta = \frac{1}{R^p}. \tag{3.6}$$

Now let $Z = M(\lambda)\eta$, where $M(\lambda) \gg 1$ so that $\frac{[M(\lambda)]^{p-1}}{f(M(\lambda))} \geq \lambda R^p$. Note that this is possible by (**H1**). Then, using that f is non-decreasing, (3.5) and (3.6) we have

$$-\Delta_p Z - \Delta_q Z \geq -\Delta_p Z = \frac{M(\lambda)^{p-1}}{R^p} \geq \lambda f(M(\lambda)) \geq \lambda f(Z) \text{ in } B_R. \tag{3.7}$$

Clearly $Z \geq 0$ on $\partial\Omega$ and hence it is a super solution of (1.2).

Proof of Theorem 1.1. Let ψ be a sub-solution of (1.2) for $\lambda \gg 1$ (as constructed in (3.4)). Then, we can construct a super solution Z of (1.2) (as constructed in (3.7)). Further, since $Z > 0$ in $\overline{\Omega}$, we can choose $M(\lambda) \gg 1$ such that $Z \geq \psi$ in $\overline{\Omega}$. Hence by Lemma 2.1, (1.2) has a positive solution $u_\lambda \in [\psi, Z]$ for $\lambda \gg 1$ and Theorem 1.1 is proven. $\square$

4 Bifurcation diagram for positive solutions to (1.3)

Here we adapt and extend the method used by Laetsch in [7] where he studied the boundary value problem: $-u'' = \lambda f(u); (0,1)$, $u(0) = 0 = u(1)$. First we note that since (1.3) is autonomous, any positive solution u must be symmetric about $x = \frac{1}{2}$, increasing on $(0, \frac{1}{2})$, and decreasing on $(\frac{1}{2}, 1)$. Let $u(\frac{1}{2}) = \rho$ (say).

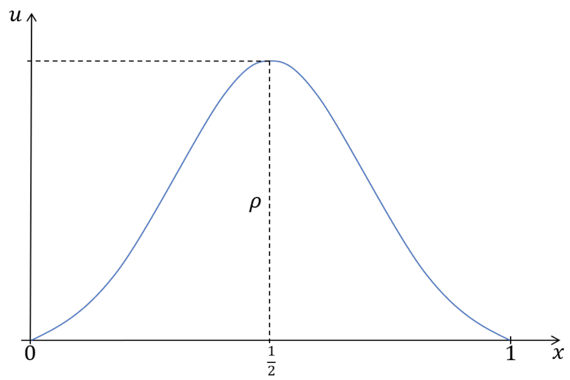

Figure 4.1: Shape of a positive solution to (1.3)

Now multiplying (1.3) by u' and integrating we obtain

$$-\frac{3}{4}[(u')^4]' - \frac{1}{2}[(u')^2]' = \lambda(F(u))' \quad \text{in } (0,1)$$

where $F(s) = \int_0^s f(z)dz$. Further integrating we obtain

$$3[u'(x)]^4 + 2[u'(x)]^2 = 4\lambda[F(\rho) - F(u(x))] \quad \text{in } [0, \tfrac{1}{2}]$$

and hence

$$u'(x) = \frac{\sqrt{[1 + 12\lambda(F(\rho) - F(u(x)))]^{\frac{1}{2}} - 1}}{\sqrt{3}} \quad \text{in } [0, \tfrac{1}{2}]. \tag{4.1}$$

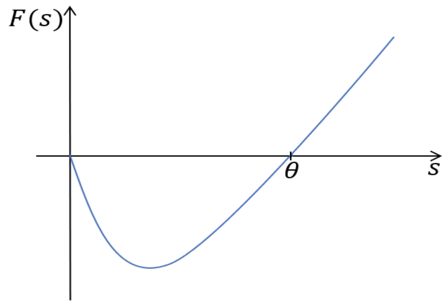

Figure 4.2: Shape of a function F

Noting that $u'(0) = \frac{\sqrt{[1+12\lambda F(\rho)]^{\frac{1}{2}}-1}}{\sqrt{3}}$, it is easy to see that ρ must be greater or equal to θ where θ is the position zero of F. Integrating (4.1) we get

$$\int_0^{u(x)} \frac{ds}{\sqrt{\left[1+12\lambda(F(\rho)-F(s))\right]^{\frac{1}{2}}-1}} = \frac{x}{\sqrt{3}} \quad \text{in } [0,\tfrac{1}{2}), \tag{4.2}$$

and setting $x \to (\frac{1}{2})^-$ we obtain

$$G(\lambda,\rho) = \int_0^{\rho} \frac{ds}{\sqrt{\left[1+12\lambda(F(\rho)-F(s))\right]^{\frac{1}{2}}-1}} = \frac{1}{2\sqrt{3}}. \tag{4.3}$$

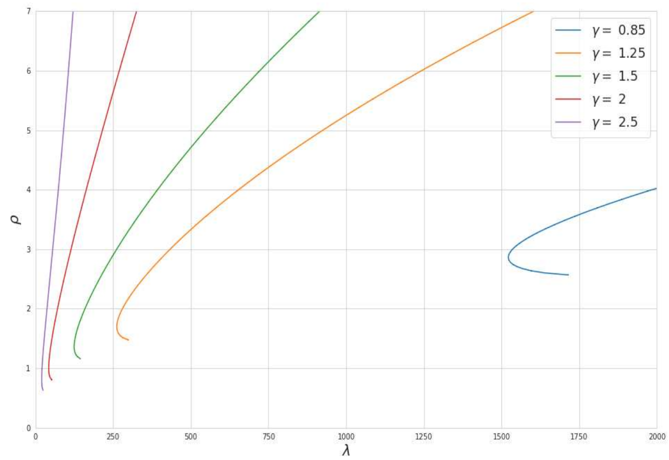

Figure 4.3: Bifurcation diagrams for (1.3) when $f(s) = (s+1)^{\gamma} - 2$; $\gamma = 0.85, 1.25, 1.5, 2.0, 2.5$.

It can be shown that that for $\lambda > 0$ and $\rho \geq \theta$, $G(\lambda,\rho)$ is well defined. Further, if $\lambda > 0$, $\rho \geq \theta$ satisfies (4.3), then (4.2) yields a C^2 function $u : [0,\frac{1}{2}) \to [0,\rho)$ such that $u(x) \to \rho$ as $x \to (\frac{1}{2})^-$. Extending this function on $[0,1]$ so that $u(\frac{1}{2}) = \rho$, and it is symmetric about $x = \frac{1}{2}$, it can be shown that it will be a positive solution of (1.3). Hence the bifurcation diagram for positive solutions to (1.3) is given by:

$$S = \left\{(\lambda,\rho) \mid \lambda > 0, \rho \geq \theta \;\&\; G(\lambda,\rho) = \tfrac{1}{2\sqrt{3}}\right\}. \tag{4.4}$$

Now, when $f(s) = (s+1)^{\gamma} - 2$; $\gamma \in (0,3)$, we compute S using *Mathematica*. In particular, here are the bifurcation diagrams we obtained for $\gamma = 0.85, 1.25, 1.5, 2.0$ and 2.5 (see Figure 4.3).

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New class of practically solvable systems of difference equations of hyperbolic-cotangent-type

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. The systems of difference equations

$$x_{n+1}=\frac{u_nv_{n-2}+a}{u_n+v_{n-2}},\quad y_{n+1}=\frac{w_ns_{n-2}+a}{w_n+s_{n-2}},\quad n\in\mathbb{N}_0,$$

where a, u_0, w_0, v_j, s_j $j=-2,-1,0$, are complex numbers, and the sequences u_n, v_n, w_n, s_n are x_n or y_n, are studied. It is shown that each of these sixteen systems is practically solvable, complementing some recent results on solvability of related systems of difference equations.

Keywords: system of difference equations, general solution, solvability of difference equations, hyperbolic-cotangent-type system of difference equations.

2020 Mathematics Subject Classification: 39A45.

1 Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, be the sets of natural, whole, real and complex numbers respectively, and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. If $p,q\in\mathbb{Z}$ and $p\le q$, then $j=\overline{p,q}$ is a notation for $j=p,p+1,\dots,q$.

First important results on solvability of difference equations and systems belong to de Moivre [5–7], D. Bernoulli [3], Euler [9], Lagrange [15] and Laplace [16]. They found a few methods for solving linear difference equations with constant coefficients, as well as methods for solving some linear difference equations with nonconstant coefficients and some nonlinear difference equations. Many books containing basic methods for solving difference equations and systems have appeared since (see, e.g., [4, 11–13, 18, 19, 21, 22]). It is interesting to note

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that many difference equations and systems have naturally appeared as some mathematical models for problems in combinatorics, population dynamics and other branches of sciences (see, e.g., [5–7, 11–13, 15–17, 20, 21, 31, 49]). The fact that it is difficult to find new methods for solving difference equations and systems has influenced on a lack of considerable interest in the topic for a long time. Use of computers seems renewed some interest in the topic in the last two decades.

During the '90s has started some interest in concrete difference equations and systems. Papaschinopoulos and Schinas have influenced on the study of such systems (see, e.g., [23–28, 32, 33]). Work [29] is on solvability, whereas [24–26, 28, 32, 33] can be regarded as ones on solvability in a wider sense, since they are devoted to finding invariants of the systems studied therein. Beside their study, have appeared several papers by some other authors which essentially rediscovered some known results. These facts motivated us to study the solvability of difference equations and systems (see, e.g., [1, 34–48] and many other related references therein).

Let $k, l \in \mathbb{N}_0$, $a \in \mathbb{R}$ (or $\mathbb{C}$), and

$$z_{n+1} = \frac{z_{n-k}z_{n-l} + a}{z_{n-k} + z_{n-l}}, \quad n \in \mathbb{N}_0. \tag{1.1}$$

Equation (1.1) have been studied by several authors. Convergence of positive solutions to the equation follows from a result in [14] (see [2]). For some generalizations of the result in [14], see [8] and [27]. The fact that equation (1.1) resembles the hyperbolic-cotangent sum formula has been a good hint for solvability of the equation. Some special cases of the equation were studied in [30]. In [43] was presented a natural way for showing solvability of the equation.

The following systems

$$x_{n+1} = \frac{u_{n-k}v_{n-l} + a}{u_{n-k} + v_{n-l}}, \quad y_{n+1} = \frac{w_{n-k}s_{n-l} + a}{w_{n-k} + s_{n-l}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $k, l \in \mathbb{N}_0$, $a, u_{-j}, w_{-j}, v_{-j'}, s_{-j'} \in \mathbb{C}$, $j = \overline{0,k}$, $j' = \overline{0,l}$, and u_n, v_n, w_n, s_n are x_n or y_n, are natural extensions of equation (1.1) (for studying the systems in the form, we have been also motivated by [34]).

The case $k = 0$, $l = 1$, was studied in [47] and [48], and also in [41] where we presented another method. We have also shown therein the theoretical solvability of the systems in (1.2). The case $k = 1$, $l = 2$, has been recently studied in [40]. Here we study practical solvability of the systems in (1.2) in the case $k = 0$ and $l = 2$, continuing our research in [40, 41, 43, 47, 48]. We use and combine some methods from these, as well as the following works: [35–39, 42, 46]. The investigation of the case has been announced in [41].

2 Main results

First we mention two lemmas. The first one belongs to Lagrange (see, e.g., [10, 13, 46]), while the second one should be folklore (for a proof see [40]), and have been applied for several times recently (see, e.g., [38, 39, 46]).

Lemma 2.1. *Let t_l, $l = \overline{1,m}$, be the roots of $p_m(t) = \alpha_m t^m + \cdots + \alpha_1 t + \alpha_0$, $\alpha_m \neq 0$, and assume that $t_l \neq t_j$, when $l \neq j$. Then*

$$\sum_{l=1}^{m} \frac{t_l^j}{p_m'(t_l)} = 0, \quad j = \overline{0, m-2}, \quad \textit{and} \quad \sum_{l=1}^{m} \frac{t_l^{m-1}}{p_m'(t_l)} = \frac{1}{\alpha_m}.$$

Lemma 2.2. *Consider the equation*

$$x_n = a_1x_{n-1} + a_2x_{n-2} + \cdots + a_mx_{n-m}, \quad n \ge l, \tag{2.1}$$

where $l \in \mathbb{Z}$, $a_j \in \mathbb{C}$, $j = \overline{1,m}$, $a_m \neq 0$. Let t_k, $k = \overline{1,m}$, be the roots of $q_m(t) = t^m - a_1t^{m-1} - a_2t^{m-2} - \cdots - a_m$, and assume that $t_k \neq t_s$, when $k \neq s$.

Then, the solution to equation (2.1) satisfying the initial conditions

$$x_{j-m} = 0, \quad j = \overline{l, l+m-2}, \quad \text{and} \quad x_{l-1} = 1, \tag{2.2}$$

is

$$x_n = \sum_{k=1}^{m} \frac{t_k^{n+m-l}}{q_m'(t_k)}, \tag{2.3}$$

for $n \ge l - m$.

We transform the systems in (1.2) with $k = 0$ and $l = 2$ to some more suitable ones. We have

$$x_{n+1} \pm \sqrt{a} = \frac{(u_n \pm \sqrt{a})(v_{n-2} \pm \sqrt{a})}{u_n + v_{n-2}} \quad \text{and} \quad y_{n+1} \pm \sqrt{a} = \frac{(w_n \pm \sqrt{a})(s_{n-2} \pm \sqrt{a})}{w_n + s_{n-2}},$$

for $n \in \mathbb{N}_0$, and consequently

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{u_n+\sqrt{a}}{u_n-\sqrt{a}} \cdot \frac{v_{n-2}+\sqrt{a}}{v_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{w_n+\sqrt{a}}{w_n-\sqrt{a}} \cdot \frac{s_{n-2}+\sqrt{a}}{s_{n-2}-\sqrt{a}}, \tag{2.4}$$

for $n \in \mathbb{N}_0$.

Hence, the following systems are studied

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{2.5}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{2.6}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{2.7}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{2.8}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{2.9}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{2.10}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{2.11}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{2.12}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{2.13}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\cdot\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{2.14}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{2.15}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{2.16}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\cdot\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{2.17}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\cdot\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{2.18}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{2.19}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{2.20}$$

for $n \in \mathbb{N}_0$.

Let

$$\zeta_n=\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\quad\text{and}\quad \eta_n=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}},$$

then

$$x_n=\sqrt{a}\,\frac{\zeta_n+1}{\zeta_n-1}\quad\text{and}\quad y_n=\sqrt{a}\,\frac{\eta_n+1}{\eta_n-1}, \tag{2.21}$$

and the systems (2.5)–(2.20) respectively become

$$\zeta_{n+1}=\zeta_n\zeta_{n-2},\quad \eta_{n+1}=\zeta_n\zeta_{n-2}, \tag{2.22}$$
$$\zeta_{n+1}=\zeta_n\zeta_{n-2},\quad \eta_{n+1}=\eta_n\zeta_{n-2}, \tag{2.23}$$
$$\zeta_{n+1}=\zeta_n\zeta_{n-2},\quad \eta_{n+1}=\zeta_n\eta_{n-2}, \tag{2.24}$$
$$\zeta_{n+1}=\zeta_n\zeta_{n-2},\quad \eta_{n+1}=\eta_n\eta_{n-2}, \tag{2.25}$$
$$\zeta_{n+1}=\eta_n\zeta_{n-2},\quad \eta_{n+1}=\zeta_n\zeta_{n-2}, \tag{2.26}$$
$$\zeta_{n+1}=\eta_n\zeta_{n-2},\quad \eta_{n+1}=\eta_n\zeta_{n-2}, \tag{2.27}$$
$$\zeta_{n+1}=\eta_n\zeta_{n-2},\quad \eta_{n+1}=\zeta_n\eta_{n-2}, \tag{2.28}$$
$$\zeta_{n+1}=\eta_n\zeta_{n-2},\quad \eta_{n+1}=\eta_n\eta_{n-2}, \tag{2.29}$$
$$\zeta_{n+1}=\zeta_n\eta_{n-2},\quad \eta_{n+1}=\zeta_n\zeta_{n-2}, \tag{2.30}$$
$$\zeta_{n+1}=\zeta_n\eta_{n-2},\quad \eta_{n+1}=\eta_n\zeta_{n-2}, \tag{2.31}$$
$$\zeta_{n+1}=\zeta_n\eta_{n-2},\quad \eta_{n+1}=\zeta_n\eta_{n-2}, \tag{2.32}$$
$$\zeta_{n+1}=\zeta_n\eta_{n-2},\quad \eta_{n+1}=\eta_n\eta_{n-2}, \tag{2.33}$$
$$\zeta_{n+1}=\eta_n\eta_{n-2},\quad \eta_{n+1}=\zeta_n\zeta_{n-2}, \tag{2.34}$$
$$\zeta_{n+1}=\eta_n\eta_{n-2},\quad \eta_{n+1}=\eta_n\zeta_{n-2}, \tag{2.35}$$
$$\zeta_{n+1}=\eta_n\eta_{n-2},\quad \eta_{n+1}=\zeta_n\eta_{n-2}, \tag{2.36}$$
$$\zeta_{n+1}=\eta_n\eta_{n-2},\quad \eta_{n+1}=\eta_n\eta_{n-2}, \tag{2.37}$$

for $n \in \mathbb{N}_0$.

To study the systems we use some ideas in [35–40, 42, 46]. The case $a = 0$ is simple (see [41]). Hence, it is omitted.

2.1 System (2.22)

First, note that

$$\zeta_n = \eta_n, \quad n \in \mathbb{N}. \tag{2.38}$$

Let

$$a_1 = 1, \quad b_1 = 0, \quad c_1 = 1, \tag{2.39}$$

then

$$\zeta_n = \zeta_{n-1}^{a_1}\zeta_{n-2}^{b_1}\zeta_{n-3}^{c_1}, \quad n \in \mathbb{N}. \tag{2.40}$$

Use of (2.40) implies

$$\zeta_n = (\zeta_{n-2}\zeta_{n-4})^{a_1}\zeta_{n-2}^{b_1}\zeta_{n-3}^{c_1} = \zeta_{n-2}^{a_1+b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{a_1} = \zeta_{n-2}^{a_2}\zeta_{n-3}^{b_2}\zeta_{n-4}^{c_2},$$

for $n \geq 2$, where

$$a_2 := a_1 + b_1, \quad b_2 := c_1, \quad c_2 := a_1.$$

Assume

$$\zeta_n = \zeta_{n-k}^{a_k}\zeta_{n-k-1}^{b_k}\zeta_{n-k-2}^{c_k}, \tag{2.41}$$

$$a_k = a_{k-1} + b_{k-1}, \quad b_k = c_{k-1}, \quad c_k = a_{k-1}, \tag{2.42}$$

for a $k \geq 2$ and $n \geq k$.

If we use (2.40) in (2.41), we obtain

$$\begin{aligned}\zeta_n &= (\zeta_{n-k-1}\zeta_{n-k-3})^{a_k}\zeta_{n-k-1}^{b_k}\zeta_{n-k-2}^{c_k}\\ &= \zeta_{n-k-1}^{a_k+b_k}\zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{a_k}\\ &= \zeta_{n-k-1}^{a_{k+1}}\zeta_{n-k-2}^{b_{k+1}}\zeta_{n-k-3}^{c_{k+1}},\end{aligned}$$

where

$$a_{k+1} := a_k + b_k, \quad b_{k+1} := c_k, \quad c_{k+1} := a_k.$$

In this way, by using induction, we proved that (2.41) and (2.42) hold for every $2 \leq k \leq n$.

From (2.39) and (2.42) we have

$$a_n = a_{n-1} + a_{n-3}, \tag{2.43}$$

not only for $n \geq 4$, but even for all $n \in \mathbb{Z}$, and

$$a_0 = 1, \quad a_{-1} = a_{-2} = 0, \quad a_{-3} = 1, \quad a_{-4} = 0. \tag{2.44}$$

By taking $k = n$ in (2.41), and employing (2.42) and (2.43), it follows that

$$\zeta_n = \zeta_0^{a_n}\zeta_{-1}^{b_n}\zeta_{-2}^{c_n} = \zeta_0^{a_n}\zeta_{-1}^{a_{n-2}}\zeta_{-2}^{a_{n-1}}, \tag{2.45}$$

not only for $n \in \mathbb{N}$, but even for $n \geq -2$.

Combining (2.38) and (2.45), we have

$$\eta_n = \zeta_0^{a_n}\zeta_{-1}^{a_{n-2}}\zeta_{-2}^{a_{n-1}}, \tag{2.46}$$

for $n \in \mathbb{N}$.

Now note that the characteristic polynomial

$$P_3(\lambda) = \lambda^3 - \lambda^2 - 1 = 0 \tag{2.47}$$

is associated with (2.43), and it has three different roots, say λ_j, $j = \overline{1,3}$. They are routinely found [10].

By using Lemma 2.2, we see that

$$a_n = \sum_{j=1}^{3} \frac{\lambda_j^{n+2}}{P_3'(\lambda_j)}, \quad n \in \mathbb{Z}, \tag{2.48}$$

is the solution to (2.43) satisfying the initial conditions $a_{-2} = a_{-1} = 0$ and $a_0 = 1$.

From (2.21), (2.45) and (2.46), the following corollary follows.

Corollary 2.3. *If $a \neq 0$, then the general solution to (2.5) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n \geq -2,$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n \in \mathbb{N},$$

where a_n is given by (2.48).

2.2 System (2.23)

First note that (2.45) holds, and that

$$\eta_n = \eta_{n-1}\zeta_{n-3}, \quad n \in \mathbb{N}. \tag{2.49}$$

By using (2.45) in (2.49), we obtain

$$\begin{aligned}\eta_n &= \eta_0 \prod_{j=1}^{n} \zeta_{j-3} \\ &= \eta_0 \prod_{j=1}^{n} \zeta_0^{a_{j-3}} \zeta_{-1}^{a_{j-5}} \zeta_{-2}^{a_{j-4}} \\ &= \eta_0 \zeta_0^{\sum_{j=1}^{n} a_{j-3}} \zeta_{-1}^{\sum_{j=1}^{n} a_{j-5}} \zeta_{-2}^{\sum_{j=1}^{n} a_{j-4}},\end{aligned} \tag{2.50}$$

for $n \in \mathbb{N}_0$.

Employing (2.43) and (2.44), it follows that

$$\sum_{j=1}^{n} a_{j-3} = \sum_{j=1}^{n} (a_j - a_{j-1}) = a_n - 1, \tag{2.51}$$

$$\sum_{j=1}^{n} a_{j-5} = \sum_{j=1}^{n} (a_{j-2} - a_{j-3}) = a_{n-2}, \tag{2.52}$$

$$\sum_{j=1}^{n} a_{j-4} = \sum_{j=1}^{n} (a_{j-1} - a_{j-2}) = a_{n-1}, \tag{2.53}$$

for $n \in \mathbb{N}_0$.

From (2.50)–(2.53), it follows that

$$\eta_n = \eta_0 \zeta_0^{a_n - 1} \zeta_{-1}^{a_{n-2}} \zeta_{-2}^{a_{n-1}}, \quad n \in \mathbb{N}_0. \tag{2.54}$$

From (2.21), (2.45) and (2.54), the following corollary follows.

Corollary 2.4. *If $a \neq 0$, then the general solution to (2.6) is*

$$x_n = \sqrt{a}\, \frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n} \left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}} \left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n} \left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}} \left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}} - 1}, \quad n \geq -2,$$

$$y_n = \sqrt{a}\, \frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right) \left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n-1} \left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}} \left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}} + 1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right) \left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n-1} \left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}} \left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}} - 1}, \quad n \in \mathbb{N}_0,$$

where a_n is given by (2.48).

2.3 System (2.24)

First note that (2.45) holds, and that

$$\eta_n = \zeta_{n-1} \eta_{n-3},$$

for $n \in \mathbb{N}$, that is,

$$\eta_{3n+i} = \zeta_{3n-1+i} \eta_{3(n-1)+i}, \tag{2.55}$$

for $n \in \mathbb{N}$, $i = -2, -1, 0$.

From (2.45) and (2.55), we have

$$\begin{aligned} \eta_{3n} &= \eta_0 \prod_{j=1}^{n} \zeta_{3j-1} \\ &= \eta_0 \prod_{j=1}^{n} \zeta_0^{a_{3j-1}} \zeta_{-1}^{a_{3j-3}} \zeta_{-2}^{a_{3j-2}} \\ &= \eta_0 \zeta_0^{\sum_{j=1}^{n} a_{3j-1}} \zeta_{-1}^{\sum_{j=1}^{n} a_{3j-3}} \zeta_{-2}^{\sum_{j=1}^{n} a_{3j-2}}, \end{aligned} \tag{2.56}$$

for $n \in \mathbb{N}_0$,

$$\begin{aligned} \eta_{3n+1} &= \eta_{-2} \prod_{j=0}^{n} \zeta_{3j} \\ &= \eta_{-2} \prod_{j=0}^{n} \zeta_0^{a_{3j}} \zeta_{-1}^{a_{3j-2}} \zeta_{-2}^{a_{3j-1}} \\ &= \eta_{-2} \zeta_0^{\sum_{j=0}^{n} a_{3j}} \zeta_{-1}^{\sum_{j=0}^{n} a_{3j-2}} \zeta_{-2}^{\sum_{j=0}^{n} a_{3j-1}}, \end{aligned} \tag{2.57}$$

for $n \geq -1$, and

$$\begin{aligned} \eta_{3n+2} &= \eta_{-1} \prod_{j=0}^{n} \zeta_{3j+1} \\ &= \eta_{-1} \prod_{j=0}^{n} \zeta_0^{a_{3j+1}} \zeta_{-1}^{a_{3j-1}} \zeta_{-2}^{a_{3j}} \\ &= \eta_{-1} \zeta_0^{\sum_{j=0}^{n} a_{3j+1}} \zeta_{-1}^{\sum_{j=0}^{n} a_{3j-1}} \zeta_{-2}^{\sum_{j=0}^{n} a_{3j}}, \end{aligned} \tag{2.58}$$

for $n \ge -1$.

Employing (2.43) and (2.44), it follows that

$$\sum_{j=1}^{n} a_{3j-3} = \sum_{j=1}^{n}(a_{3j-2} - a_{3j-5}) = a_{3n-2}, \tag{2.59}$$

$$\sum_{j=1}^{n} a_{3j-2} = \sum_{j=1}^{n}(a_{3j-1} - a_{3j-4}) = a_{3n-1}, \tag{2.60}$$

$$\sum_{j=1}^{n} a_{3j-1} = \sum_{j=1}^{n}(a_{3j} - a_{3j-3}) = a_{3n} - 1, \tag{2.61}$$

$$\sum_{j=0}^{n} a_{3j-2} = \sum_{j=0}^{n}(a_{3j-1} - a_{3j-4}) = a_{3n-1}, \tag{2.62}$$

$$\sum_{j=0}^{n} a_{3j-1} = \sum_{j=0}^{n}(a_{3j} - a_{3j-3}) = a_{3n} - 1, \tag{2.63}$$

$$\sum_{j=0}^{n} a_{3j} = \sum_{j=0}^{n}(a_{3j+1} - a_{3j-2}) = a_{3n+1}, \tag{2.64}$$

$$\sum_{j=0}^{n} a_{3j+1} = \sum_{j=0}^{n}(a_{3j+2} - a_{3j-1}) = a_{3n+2}, \tag{2.65}$$

Use of (2.59)–(2.65) in (2.56)–(2.58), yield

$$\eta_{3n} = \eta_0 \zeta_0^{a_{3n}-1} \zeta_{-1}^{a_{3n-2}} \zeta_{-2}^{a_{3n-1}}, \tag{2.66}$$

for $n \in \mathbb{N}_0$,

$$\eta_{3n+1} = \eta_{-2} \zeta_0^{a_{3n+1}} \zeta_{-1}^{a_{3n-1}} \zeta_{-2}^{a_{3n}-1}, \tag{2.67}$$

for $n \ge -1$, and

$$\eta_{3n+2} = \eta_{-1} \zeta_0^{a_{3n+2}} \zeta_{-1}^{a_{3n}-1} \zeta_{-2}^{a_{3n+1}}, \tag{2.68}$$

for $n \ge -1$.

From (2.21), (2.45), (2.66)–(2.68), the following corollary follows.

Corollary 2.5. *If $a \neq 0$, then the general solution to (2.7) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n \ge -2,$$

$$y_{3n} = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{3n}-1}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{3n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{3n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{3n}-1}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{3n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{3n-1}}-1}, \quad n \in \mathbb{N}_0,$$

$$y_{3n+1} = \sqrt{a}\,\frac{\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{3n+1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{3n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{3n}-1}+1}{\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{3n+1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{3n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{3n}-1}-1}, \quad n \ge -1,$$

$$y_{3n+2} = \sqrt{a}\,\frac{\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{3n+2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{3n}-1}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{3n+1}}+1}{\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{3n+2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{3n}-1}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{3n+1}}-1}, \quad n \ge -1,$$

where sequence a_n is given by (2.48).

2.4 System (2.25)

Note that (2.45) holds, and that

$$\eta_n = \eta_0^{a_n}\eta_{-1}^{a_{n-2}}\eta_{-2}^{a_{n-1}},$$

for $n \geq -2$.

From this and (2.21) the following corollary follows.

Corollary 2.6. *If $a \neq 0$, then the general solution to (2.8) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1},$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1},$$

for $n \geq -2$, where a_n is given by (2.48).

2.5 System (2.26)

The relations in (2.26) yield

$$\zeta_n = \zeta_{n-2}\zeta_{n-3}\zeta_{n-4}, \quad n \geq 2. \tag{2.69}$$

Let

$$b_1 = c_1 = d_1 = 1, \quad e_1 = 0, \tag{2.70}$$

then (2.69) can be written as

$$\zeta_n = \zeta_{n-2}^{b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}, \quad n \geq 2. \tag{2.71}$$

Use of (2.69) in (2.71) yield

$$\begin{aligned}\zeta_n &= \zeta_{n-2}^{b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\\ &= (\zeta_{n-4}\zeta_{n-5}\zeta_{n-6})^{b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\\ &= \zeta_{n-3}^{c_1}\zeta_{n-4}^{b_1+d_1}\zeta_{n-5}^{b_1+e_1}\zeta_{n-6}^{b_1}\\ &= \zeta_{n-3}^{b_2}\zeta_{n-4}^{c_2}\zeta_{n-5}^{d_2}\zeta_{n-6}^{e_2}\end{aligned}$$

for $n \geq 4$, where

$$b_2 := c_1, \quad c_2 := b_1 + d_1, \quad d_2 := b_1 + e_1, \quad e_2 := b_1.$$

Suppose that

$$\zeta_n = \zeta_{n-k-1}^{b_k}\zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k} \tag{2.72}$$

and

$$b_k = c_{k-1}, \quad c_k = b_{k-1} + d_{k-1}, \quad d_k = b_{k-1} + e_{k-1}, \quad e_k = b_{k-1}, \tag{2.73}$$

for a $k \geq 2$ and $n \geq k+2$.

Employing (2.69) in (2.72), it follows that

$$\begin{aligned}\zeta_n &= \zeta_{n-k-1}^{b_k}\zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\\ &= (\zeta_{n-k-3}\zeta_{n-k-4}\zeta_{n-k-5})^{b_k}\zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\\ &= \zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{b_k+d_k}\zeta_{n-k-4}^{b_k+e_k}\zeta_{n-k-5}^{b_k}\\ &= \zeta_{n-k-2}^{b_{k+1}}\zeta_{n-k-3}^{c_{k+1}}\zeta_{n-k-4}^{d_{k+1}}\zeta_{n-k-5}^{e_{k+1}},\end{aligned}$$

where

$$b_{k+1} := c_k, \quad c_{k+1} := b_k + d_k, \quad d_{k+1} := b_k + e_k, \quad e_{k+1} := b_k,$$

for a $k \ge 2$ and every $n \ge k+3$. Hence, (2.72) and (2.73) really hold for $2 \le k \le n-2$.

From (2.70) and (2.73) we have

$$b_n = b_{n-2} + b_{n-3} + b_{n-4}, \tag{2.74}$$

not only for $n \ge 5$, but also for every $n \in \mathbb{Z}$, and that

$$b_0 = 0, \quad b_{-1} = 1, \quad b_{-2} = b_{-3} = b_{-4} = 0, \quad b_{-5} = 1.$$

Letting $k = n-2$ in (2.72), it follows that

$$\begin{aligned}\zeta_n &= \zeta_1^{b_{n-2}}\zeta_0^{c_{n-2}}\zeta_{-1}^{d_{n-2}}\zeta_{-2}^{e_{n-2}}\\ &= (\eta_0\zeta_{-2})^{b_{n-2}}\zeta_0^{c_{n-2}}\zeta_{-1}^{d_{n-2}}\zeta_{-2}^{e_{n-2}}\\ &= \eta_0^{b_{n-2}}\zeta_0^{c_{n-2}}\zeta_{-1}^{d_{n-2}}\zeta_{-2}^{b_{n-2}+e_{n-2}}\\ &= \eta_0^{b_{n-2}}\zeta_0^{b_{n-1}}\zeta_{-1}^{b_{n-3}+b_{n-4}}\zeta_{-2}^{b_{n-2}+b_{n-3}},\end{aligned} \tag{2.75}$$

for $n \ge -2$.

By using (2.75) in the second equation in (2.26), it follows that

$$\begin{aligned}\eta_n &= \zeta_{n-1}\zeta_{n-3}\\ &= \eta_0^{b_{n-3}+b_{n-5}}\zeta_0^{b_{n-2}+b_{n-4}}\zeta_{-1}^{b_{n-3}+b_{n-4}}\zeta_{-2}^{b_{n-2}+b_{n-3}},\end{aligned} \tag{2.76}$$

for $n \in \mathbb{N}_0$.

The characteristic polynomial associated with equation (2.74) is

$$P_4(\lambda) = \lambda^4 - \lambda^2 - \lambda - 1.$$

Since

$$P_4(\lambda) = (\lambda+1)(\lambda^3 - \lambda^2 - 1),$$

we have that three roots of P_4, coincide with the roots, λ_j, $j = \overline{1,3}$, of polynomial (2.47), whereas $\lambda_4 = -1$.

Lemma 2.2 shows that the solution to (2.74) satisfying the initial conditions $b_{-4} = b_{-3} = b_{-2} = 0$ and $b_{-1} = 1$, is

$$b_n = \sum_{j=1}^{4} \frac{\lambda_j^{n+4}}{P_4'(\lambda_j)}, \quad n \in \mathbb{Z}. \tag{2.77}$$

From (2.21), (2.75) and (2.76), the following corollary follows.

Corollary 2.7. *If $a \neq 0$, then the general solution to (2.9) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-3}+b_{n-4}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-2}+b_{n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-3}+b_{n-4}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-2}+b_{n-3}}-1},$$

for $n \geq -2$, and

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-3}+b_{n-5}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-2}+b_{n-4}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-3}+b_{n-4}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-2}+b_{n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-3}+b_{n-5}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-2}+b_{n-4}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-3}+b_{n-4}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-2}+b_{n-3}}-1},$$

for $n \in \mathbb{N}_0$, where b_n is given by (2.77).

2.6 System (2.27)

Clearly, we have

$$\zeta_n = \eta_n, \quad n \in \mathbb{N},$$

from which along with (2.27), it follows that

$$\zeta_{n+1} = \zeta_n \zeta_{n-2}, \quad n \in \mathbb{N}.$$

Hence, by using (2.45) it follows that

$$\begin{aligned}\zeta_n &= \zeta_1^{a_{n-1}}\zeta_0^{a_{n-3}}\zeta_{-1}^{a_{n-2}}\\ &= (\eta_0\zeta_{-2})^{a_{n-1}}\zeta_0^{a_{n-3}}\zeta_{-1}^{a_{n-2}}\\ &= \eta_0^{a_{n-1}}\zeta_0^{a_{n-3}}\zeta_{-1}^{a_{n-2}}\zeta_{-2}^{a_{n-1}},\end{aligned} \tag{2.78}$$

for $n \geq -1$, where a_n is the solution to (2.43) satisfying the initial conditions $a_{-2} = a_{-1} = 0$ and $a_0 = 1$. Hence

$$\eta_n = \eta_0^{a_{n-1}}\zeta_0^{a_{n-3}}\zeta_{-1}^{a_{n-2}}\zeta_{-2}^{a_{n-1}}, \tag{2.79}$$

for $n \in \mathbb{N}$.

From (2.21), (2.78) and (2.79), the following corollary follows.

Corollary 2.8. *If $a \neq 0$, then the general solution to (2.10) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1},$$

for $n \geq -1$, and

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1},$$

for $n \in \mathbb{N}$, where a_n is given by (2.48).

2.7 System (2.28)

From (2.28) we easily get

$$\zeta_n = \zeta_{n-2}\zeta_{n-3}^2\zeta_{n-6}^{-1}, \quad n \geq 4. \tag{2.80}$$

Let

$$b_1 = 1, \quad c_1 = 2, \quad d_1 = e_1 = 0, \quad f_1 = -1, \quad g_1 = 0. \tag{2.81}$$

From this and (2.80), we have

$$\begin{aligned}
\zeta_n &= \zeta_{n-2}^{b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\zeta_{n-6}^{f_1}\zeta_{n-7}^{g_1} \\
&= (\zeta_{n-4}\zeta_{n-5}^2\zeta_{n-8}^{-1})^{b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\zeta_{n-6}^{f_1}\zeta_{n-7}^{g_1} \\
&= \zeta_{n-3}^{c_1}\zeta_{n-4}^{b_1+d_1}\zeta_{n-5}^{2b_1+e_1}\zeta_{n-6}^{f_1}\zeta_{n-7}^{g_1}\zeta_{n-8}^{-b_1} \\
&= \zeta_{n-3}^{b_2}\zeta_{n-4}^{c_2}\zeta_{n-5}^{d_2}\zeta_{n-6}^{e_2}\zeta_{n-7}^{f_2}\zeta_{n-8}^{g_2},
\end{aligned}$$

for $n \geq 6$, where

$$b_2 := c_1, \;\; c_2 := b_1 + d_1, \;\; d_2 := 2b_1 + e_1, \;\; e_2 := f_1, \;\; f_2 := g_1, \;\; g_2 := -b_1.$$

Assume

$$\zeta_n = \zeta_{n-k-1}^{b_k}\zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\zeta_{n-k-5}^{f_k}\zeta_{n-k-6}^{g_k}, \tag{2.82}$$

for a $k \geq 2$ and all $n \geq k+4$, and

$$\begin{aligned}
&b_k = c_{k-1}, \quad c_k = b_{k-1} + d_{k-1}, \quad d_k = 2b_{k-1} + e_{k-1}, \\
&e_k = f_{k-1}, \quad f_k = g_{k-1}, \quad g_k = -b_{k-1}.
\end{aligned} \tag{2.83}$$

We have

$$\begin{aligned}
\zeta_n &= \zeta_{n-k-1}^{b_k}\zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\zeta_{n-k-5}^{f_k}\zeta_{n-k-6}^{g_k} \\
&= (\zeta_{n-k-3}\zeta_{n-k-4}^2\zeta_{n-k-7}^{-1})^{b_k}\zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\zeta_{n-k-5}^{f_k}\zeta_{n-k-6}^{g_k} \\
&= \zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{b_k+d_k}\zeta_{n-k-4}^{2b_k+e_k}\zeta_{n-k-5}^{f_k}\zeta_{n-k-6}^{g_k}\zeta_{n-k-7}^{-b_k} \\
&= \zeta_{n-k-2}^{b_{k+1}}\zeta_{n-k-3}^{c_{k+1}}\zeta_{n-k-4}^{d_{k+1}}\zeta_{n-k-5}^{e_{k+1}}\zeta_{n-k-6}^{f_{k+1}}\zeta_{n-k-7}^{g_{k+1}},
\end{aligned}$$

for $n \geq k+5$, where

$$\begin{aligned}
&b_{k+1} = c_k, \quad c_{k+1} = b_k + d_k, \quad d_{k+1} = 2b_k + e_k, \\
&e_{k+1} = f_k, \quad f_{k+1} = g_k, \qquad g_{k+1} = -b_k.
\end{aligned}$$

Hence (2.82) and (2.83) really hold when $2 \leq k \leq n-4$.

From (2.81) and (2.83) we have

$$b_n = b_{n-2} + 2b_{n-3} - b_{n-6}, \tag{2.84}$$

not only for $n \geq 7$, but for all $n \in \mathbb{Z}$, and

$$b_0 = 0, \quad b_{-1} = 1, \quad b_{-j} = 0, \; j = \overline{2,6}, \quad b_{-7} = -1, \quad b_{-8} = 0.$$

By taking $k = n-4$ in (2.82), it follows that

$$\begin{aligned}\zeta_n &= \zeta_3^{b_{n-4}}\zeta_2^{c_{n-4}}\zeta_1^{d_{n-4}}\zeta_0^{e_{n-4}}\zeta_{-1}^{f_{n-4}}\zeta_{-2}^{g_{n-4}}\\ &= (\eta_0\eta_{-1}\zeta_0\zeta_{-2})^{b_{n-4}}(\eta_{-2}\zeta_0\zeta_{-1})^{c_{n-4}}(\eta_0\zeta_{-2})^{d_{n-4}}\zeta_0^{e_{n-4}}\zeta_{-1}^{f_{n-4}}\zeta_{-2}^{g_{n-4}}\\ &= \zeta_0^{b_{n-4}+c_{n-4}+e_{n-4}}\zeta_{-1}^{c_{n-4}+f_{n-4}}\zeta_{-2}^{b_{n-4}+d_{n-4}+g_{n-4}}\eta_0^{b_{n-4}+d_{n-4}}\eta_{-1}^{b_{n-4}}\eta_{-2}^{c_{n-4}}\\ &= \zeta_0^{b_{n-1}-b_{n-4}}\zeta_{-1}^{b_{n-3}-b_{n-6}}\zeta_{-2}^{b_{n-2}-b_{n-5}}\eta_0^{b_{n-2}}\eta_{-1}^{b_{n-4}}\eta_{-2}^{b_{n-3}},\end{aligned} \tag{2.85}$$

for $n \geq -2$.

Using (2.85) in the first equation in (2.28), we obtain

$$\begin{aligned}\eta_n &= \zeta_{n+1}/\zeta_{n-2}\\ &= \zeta_0^{b_n-2b_{n-3}+b_{n-6}}\zeta_{-1}^{b_{n-2}-2b_{n-5}+b_{n-8}}\zeta_{-2}^{b_{n-1}-2b_{n-4}+b_{n-7}}\eta_0^{b_{n-1}-b_{n-4}}\eta_{-1}^{b_{n-3}-b_{n-6}}\eta_{-2}^{b_{n-2}-b_{n-5}}\\ &= \zeta_0^{b_{n-2}}\zeta_{-1}^{b_{n-4}}\zeta_{-2}^{b_{n-3}}\eta_0^{b_{n-1}-b_{n-4}}\eta_{-1}^{b_{n-3}-b_{n-6}}\eta_{-2}^{b_{n-2}-b_{n-5}},\end{aligned} \tag{2.86}$$

for $n \geq -2$.

The characteristic polynomial associated with equation (2.84) is

$$P_6(t) = t^6 - t^4 - 2t^3 + 1 = (t^3 - t^2 - 1)(t^3 + t^2 - 1).$$

Let t_j, $j = \overline{1,6}$, be its roots. Clearly $t_j = \lambda_j$, $j = \overline{1,3}$, (the roots of polynomial (2.47)), whereas the other three roots of P_6 are the roots of the polynomial $t^3 + t^2 - 1$.

Thus, the solution to (2.84) satisfying the initial conditions $b_{-j} = 0$, $k = \overline{2,6}$, and $b_{-1} = 1$, is

$$b_n = \sum_{j=1}^{6} \frac{t_j^{n+6}}{P_6'(t_j)}, \quad n \in \mathbb{Z}. \tag{2.87}$$

From (2.21), (2.85) and (2.86), the following corollary follows.

Corollary 2.9. *If $a \neq 0$, then the general solution to (2.11) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\beta_{n-1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{\beta_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{\beta_{n-2}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-4}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\beta_{n-1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{\beta_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{\beta_{n-2}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-4}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-3}}-1},$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-4}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\beta_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{\beta_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{\beta_{n-2}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-4}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\beta_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{\beta_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{\beta_{n-2}}-1},$$

for $n \geq -2$, where the sequence b_n is given by (2.87) and $\beta_n := b_n - b_{n-3}$.

2.8 System (2.29)

This system is obtained from (2.24) by interchanging letters ζ and η.

Hence, we have

$$\zeta_{3n} = \zeta_0\eta_0^{a_{3n}-1}\eta_{-1}^{a_{3n-2}}\eta_{-2}^{a_{3n-1}}, \quad n \in \mathbb{N}_0, \tag{2.88}$$

$$\zeta_{3n+1} = \zeta_{-2}\eta_0^{a_{3n+1}}\eta_{-1}^{a_{3n-1}}\eta_{-2}^{a_{3n}-1}, \quad n \geq -1, \tag{2.89}$$

$$\zeta_{3n+2} = \zeta_{-1}\eta_0^{a_{3n+2}}\eta_{-1}^{a_{3n}-1}\eta_{-2}^{a_{3n+1}}, \quad n \geq -1, \tag{2.90}$$

$$\eta_n = \eta_0^{a_n}\eta_{-1}^{a_{n-2}}\eta_{-2}^{a_{n-1}}, \quad n \geq -2. \tag{2.91}$$

From (2.21), (2.88)–(2.91), the following corollary follows.

Corollary 2.10. *If $a \neq 0$, then the general solution to (2.12) is*

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n \geq -2,$$

$$x_{3n} = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{3n}-1}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{3n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{3n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{3n}-1}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{3n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{3n-1}}-1}, \quad n \in \mathbb{N}_0,$$

$$x_{3n+1} = \sqrt{a}\,\frac{\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{3n+1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{3n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{3n}-1}+1}{\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{3n+1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{3n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{3n}-1}-1}, \quad n \geq -1,$$

$$x_{3n+2} = \sqrt{a}\,\frac{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{3n+2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{3n}-1}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{3n+1}}+1}{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{3n+2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{3n}-1}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{3n+1}}-1}, \; n \geq -1,$$

where sequence a_n is given by (2.48).

2.9 System (2.30)

From (2.30), we have

$$\zeta_n = \zeta_{n-1}\zeta_{n-4}\zeta_{n-6}, \tag{2.92}$$

for $n \geq 4$.

Let

$$a_1 = 1, \quad b_1 = c_1 = 0, \quad d_1 = 1, \quad e_1 = 0, \quad f_1 = 1, \tag{2.93}$$

then

$$\zeta_n = \zeta_{n-1}^{a_1}\zeta_{n-2}^{b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\zeta_{n-6}^{f_1}, \quad n \geq 4. \tag{2.94}$$

From (2.92) and (2.94), it follows that

$$\begin{aligned}
\zeta_n &= \zeta_{n-1}^{a_1}\zeta_{n-2}^{b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\zeta_{n-6}^{f_1}\\
&= (\zeta_{n-2}\zeta_{n-5}\zeta_{n-7})^{a_1}\zeta_{n-2}^{b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\zeta_{n-6}^{f_1}\\
&= \zeta_{n-2}^{a_1+b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{a_1+e_1}\zeta_{n-6}^{f_1}\zeta_{n-7}^{a_1}\\
&= \zeta_{n-2}^{a_2}\zeta_{n-3}^{b_2}\zeta_{n-4}^{c_2}\zeta_{n-5}^{d_2}\zeta_{n-6}^{e_2}\zeta_{n-7}^{f_2},
\end{aligned}$$

for $n \geq 5$, where

$$a_2 := a_1 + b_1, \quad b_2 := c_1, \quad c_2 := d_1, \quad d_2 := a_1 + e_1, \quad e_2 := f_1, \quad f_2 := a_1.$$

Similar to the case of equation (2.80) it is shown

$$\zeta_n = \zeta_{n-k}^{a_k}\zeta_{n-k-1}^{b_k}\zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\zeta_{n-k-5}^{f_k}, \tag{2.95}$$

for a $k \geq 2$ and $n \geq k+3$, and that

$$\begin{aligned}
a_k &= a_{k-1} + b_{k-1}, \quad b_k = c_{k-1}, \quad c_k = d_{k-1},\\
d_k &= a_{k-1} + e_{k-1}, \quad e_k = f_{k-1}, \quad f_k = a_{k-1}.
\end{aligned} \tag{2.96}$$

From (2.93) and (2.96), we have

$$a_n = a_{n-1} + a_{n-4} + a_{n-6}, \tag{2.97}$$

not only for $n \ge 7$, but for all $n \in \mathbb{Z}$, and that

$$a_0 = 1, \quad a_{-j} = 0,\ j = \overline{1,5}, \quad a_{-6} = 1, \quad a_{-7} = 0, \quad a_{-8} = -1.$$

Letting $k = n-3$ in (2.95), it follows that

$$\begin{aligned}
\zeta_n &= \zeta_3^{a_{n-3}} \zeta_2^{b_{n-3}} \zeta_1^{c_{n-3}} \zeta_0^{d_{n-3}} \zeta_{-1}^{e_{n-3}} \zeta_{-2}^{f_{n-3}} \\
&= (\zeta_0\eta_0\eta_{-1}\eta_{-2})^{a_{n-3}} (\zeta_0\eta_{-1}\eta_{-2})^{b_{n-3}} (\zeta_0\eta_{-2})^{c_{n-3}} \zeta_0^{d_{n-3}} \zeta_{-1}^{e_{n-3}} \zeta_{-2}^{f_{n-3}} \\
&= \zeta_0^{a_{n-3}+b_{n-3}+c_{n-3}+d_{n-3}} \zeta_{-1}^{e_{n-3}} \zeta_{-2}^{f_{n-3}} \eta_0^{a_{n-3}} \eta_{-1}^{a_{n-3}+b_{n-3}} \eta_{-2}^{a_{n-3}+b_{n-3}+c_{n-3}} \\
&= \zeta_0^{a_n} \zeta_{-1}^{a_{n-5}} \zeta_{-2}^{a_{n-4}} \eta_0^{a_{n-3}} \eta_{-1}^{a_{n-2}} \eta_{-2}^{a_{n-1}},
\end{aligned} \tag{2.98}$$

for $n \ge -2$.

If in the second equation in (2.30) is used (2.98), we get

$$\begin{aligned}
\eta_n &= \zeta_{n-1}\zeta_{n-3} \\
&= \zeta_0^{a_{n-1}+a_{n-3}} \zeta_{-1}^{a_{n-6}+a_{n-8}} \zeta_{-2}^{a_{n-5}+a_{n-7}} \eta_0^{a_{n-4}+a_{n-6}} \eta_{-1}^{a_{n-3}+a_{n-5}} \eta_{-2}^{a_{n-2}+a_{n-4}},
\end{aligned} \tag{2.99}$$

for $n \ge -2$.

The characteristic polynomial

$$\widetilde{P}_6(t) = t^6 - t^5 - t^2 - 1 = (t^3 - t^2 - 1)(t^3 + 1)$$

associated with equation (2.97) has the roots

$$t_1 = \lambda_1, \quad t_2 = \lambda_2, \quad t_3 = \lambda_3, \quad t_4 = -1, \quad t_{5,6} = e^{\pm i\frac{\pi}{3}},$$

where λ_j, $j = \overline{1,3}$, are the roots of polynomial (2.47).

Hence, the solution to equation (2.97) satisfying the initial conditions $a_{-j} = 0$, $j = \overline{1,5}$, and $a_0 = 1$ is

$$a_n = \sum_{j=1}^{6} \frac{t_j^{n+5}}{\widetilde{P}_6'(t_j)}, \quad n \in \mathbb{Z}. \tag{2.100}$$

From (2.21), (2.98) and (2.99), the following corollary follows.

Corollary 2.11. *If $a \ne 0$, then the general solution to (2.13) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-4}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-4}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1},$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-6}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-5}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-4}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-2}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-6}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-5}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-4}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-2}}-1},$$

for $n \ge -2$, where the sequence a_n is given by (2.100) and $b_n = a_n + a_{n-2}$.

2.10 System (2.31)

From (2.31), we have

$$\zeta_n = \zeta_{n-1}^2 \zeta_{n-2}^{-1} \zeta_{n-6}, \tag{2.101}$$

for $n \geq 4$.

Let

$$a_1 = 2, \quad b_1 = -1, \quad c_1 = d_1 = e_1 = 0, \quad f_1 = 1, \tag{2.102}$$

then

$$\zeta_n = \zeta_{n-1}^{a_1} \zeta_{n-2}^{b_1} \zeta_{n-3}^{c_1} \zeta_{n-4}^{d_1} \zeta_{n-5}^{e_1} \zeta_{n-6}^{f_1}, \tag{2.103}$$

for $n \geq 4$.

Employing (2.101) in (2.103), it follows that

$$\begin{aligned}
\zeta_n &= \zeta_{n-1}^{a_1} \zeta_{n-2}^{b_1} \zeta_{n-3}^{c_1} \zeta_{n-4}^{d_1} \zeta_{n-5}^{e_1} \zeta_{n-6}^{f_1} \\
&= (\zeta_{n-2}^2 \zeta_{n-3}^{-1} \zeta_{n-7})^{a_1} \zeta_{n-2}^{b_1} \zeta_{n-3}^{c_1} \zeta_{n-4}^{d_1} \zeta_{n-5}^{e_1} \zeta_{n-6}^{f_1} \\
&= \zeta_{n-2}^{2a_1+b_1} \zeta_{n-3}^{-a_1+c_1} \zeta_{n-4}^{d_1} \zeta_{n-5}^{e_1} \zeta_{n-6}^{f_1} \zeta_{n-7}^{a_1} \\
&= \zeta_{n-2}^{a_2} \zeta_{n-3}^{b_2} \zeta_{n-4}^{c_2} \zeta_{n-5}^{d_2} \zeta_{n-6}^{e_2} \zeta_{n-7}^{f_2},
\end{aligned}$$

for $n \geq 5$, where

$$a_2 := 2a_1 + b_1, \quad b_2 := -a_1 + c_1, \quad c_2 := d_1, \quad d_2 := e_1, \quad e_2 := f_1, \quad f_2 := a_1.$$

Similar to equation (2.80), it is shown that

$$\zeta_n = \zeta_{n-k}^{a_k} \zeta_{n-k-1}^{b_k} \zeta_{n-k-2}^{c_k} \zeta_{n-k-3}^{d_k} \zeta_{n-k-4}^{e_k} \zeta_{n-k-5}^{f_k}, \tag{2.104}$$

and

$$\begin{aligned}
&a_k = 2a_{k-1} + b_{k-1}, \quad b_k = -a_{k-1} + c_{k-1}, \quad c_k = d_{k-1}, \\
&d_k = e_{k-1}, \quad e_k = f_{k-1}, \quad f_k = a_{k-1},
\end{aligned} \tag{2.105}$$

for a $2 \leq k \leq n-3$.

From (2.102) and (2.105) we have

$$a_n = 2a_{n-1} - a_{n-2} + a_{n-6}, \tag{2.106}$$

not only for $n \geq 7$, but for all $n \in \mathbb{Z}$, and that

$$a_0 = 1, \quad a_{-j} = 0, \ j = \overline{1,5}, \quad a_{-6} = 1, \quad a_{-7} = 0.$$

For $k = n-3$, from (2.104), we have

$$\begin{aligned}
\zeta_n &= \zeta_3^{a_{n-3}} \zeta_2^{b_{n-3}} \zeta_1^{c_{n-3}} \zeta_0^{d_{n-3}} \zeta_{-1}^{e_{n-3}} \zeta_{-2}^{f_{n-3}} \\
&= (\zeta_0 \eta_0 \eta_{-1} \eta_{-2})^{a_{n-3}} (\zeta_0 \eta_{-1} \eta_{-2})^{b_{n-3}} (\zeta_0 \eta_{-2})^{c_{n-3}} \zeta_0^{d_{n-3}} \zeta_{-1}^{e_{n-3}} \zeta_{-2}^{f_{n-3}} \\
&= \zeta_0^{a_{n-3}+b_{n-3}+c_{n-3}+d_{n-3}} \zeta_{-1}^{e_{n-3}} \zeta_{-2}^{f_{n-3}} \eta_0^{a_{n-3}} \eta_{-1}^{a_{n-3}+b_{n-3}} \eta_{-2}^{a_{n-3}+b_{n-3}+c_{n-3}} \\
&= \zeta_0^{a_n - a_{n-1}} \zeta_{-1}^{a_{n-5}} \zeta_{-2}^{a_{n-4}} \eta_0^{a_{n-3}} \eta_{-1}^{a_{n-2}-a_{n-3}} \eta_{-2}^{a_{n-1}-a_{n-2}},
\end{aligned} \tag{2.107}$$

for $n \geq -2$.

From (2.31) and (2.107), it follows that

$$\begin{aligned}\eta_n &= \zeta_{n+3}/\zeta_{n+2}\\ &= \zeta_0^{a_{n+3}-2a_{n+2}+a_{n+1}}\zeta_{-1}^{a_{n-2}-a_{n-3}}\zeta_{-2}^{a_{n-1}-a_{n-2}}\eta_0^{a_n-a_{n-1}}\eta_{-1}^{a_{n+1}-2a_n+a_{n-1}}\eta_{-2}^{a_{n+2}-2a_{n+1}+a_n}\\ &= \zeta_0^{a_{n-3}}\zeta_{-1}^{a_{n-2}-a_{n-3}}\zeta_{-2}^{a_{n-1}-a_{n-2}}\eta_0^{a_n-a_{n-1}}\eta_{-1}^{a_{n-5}}\eta_{-2}^{a_{n-4}},\end{aligned} \tag{2.108}$$

for $n \ge -2$ ((2.108) is also obtained from (2.107) due to the symmetry of system (2.31)).

The characteristic polynomial associated with (2.106) is

$$\widehat{P}_6(t) = t^6 - 2t^5 + t^4 - 1 = (t^3 - t^2 - 1)(t^3 - t^2 + 1).$$

Let $\widetilde{t}_j$, $j = \overline{1,6}$, be the roots of polynomial $\widehat{P}_6$. Then, the solution to (2.106) such that $a_{-j} = 0$, $j = \overline{1,5}$, and $a_0 = 1$, is

$$a_n = \sum_{j=1}^{6} \frac{\widetilde{t}_j^{\,n+5}}{\widehat{P}_6'(\widetilde{t}_j)}, \quad n \in \mathbb{Z}. \tag{2.109}$$

From this and (2.21) the following corollary follows.

Corollary 2.12. *If $a \ne 0$, then the general solution to (2.14) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\Delta a_{n-1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-4}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{\Delta a_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{\Delta a_{n-2}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\Delta a_{n-1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-4}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{\Delta a_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{\Delta a_{n-2}}-1},$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{\Delta a_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{\Delta a_{n-2}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\Delta a_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-4}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{\Delta a_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{\Delta a_{n-2}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\Delta a_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-4}}-1},$$

for $n \ge -2$, where the sequence a_n is given by (2.109) and $\Delta a_n = a_{n+1} - a_n$.

2.11 System (2.32)

From (2.32) we see that

$$\zeta_n = \eta_n, \quad n \in \mathbb{N},$$

implying that

$$\zeta_n = \zeta_{n-1}\zeta_{n-3},$$

for $n \ge 4$.

Employing (2.45) it follows that

$$\begin{aligned}\zeta_n &= \zeta_3^{a_{n-3}}\zeta_2^{a_{n-5}}\zeta_1^{a_{n-4}}\\ &= (\zeta_0\eta_0\eta_{-1}\eta_{-2})^{a_{n-3}}(\zeta_0\eta_{-1}\eta_{-2})^{a_{n-5}}(\zeta_0\eta_{-2})^{a_{n-4}}\\ &= \zeta_0^{a_{n-1}}\eta_0^{a_{n-3}}\eta_{-1}^{a_{n-2}}\eta_{-2}^{a_{n-1}},\end{aligned} \tag{2.110}$$

for $n \in \mathbb{N}$, where a_n is the solution to equation (2.43) such that $a_{-1} = a_{-2} = 0$ and $a_0 = 1$, and consequently

$$\eta_n = \zeta_0^{a_{n-1}}\eta_0^{a_{n-3}}\eta_{-1}^{a_{n-2}}\eta_{-2}^{a_{n-1}}, \tag{2.111}$$

for $n \in \mathbb{N}_0$.

From (2.21), (2.110) and (2.111), the following corollary follows.

Corollary 2.13. *If $a \neq 0$, then the general solution to (2.15) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1},$$

for $n \in \mathbb{N}$, and

$$y_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1},$$

for $n \in \mathbb{N}_0$, where a_n is given by (2.48).

2.12 System (2.33)

This system is get from (2.23) by interchanging letters ζ and η. Hence, we have

$$\eta_n = \eta_0^{a_n}\eta_{-1}^{a_{n-2}}\eta_{-2}^{a_{n-1}}$$

for $n \geq -2$, and

$$\zeta_n = \zeta_0\eta_0^{a_n-1}\eta_{-1}^{a_{n-2}}\eta_{-2}^{a_{n-1}},$$

for $n \in \mathbb{N}_0$.

From this and (2.21) the following corollary follows.

Corollary 2.14. *If $a \neq 0$, then the general solution to (2.16) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n-1}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n-1}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n \geq -2,$$

where a_n is given by (2.48).

2.13 System (2.34)

From (2.34) we have

$$\zeta_n = \zeta_{n-2}\zeta_{n-4}^2\zeta_{n-6}, \tag{2.112}$$

for $n \geq 4$.

Let

$$a_1 := 1, \quad b_1 := 2, \quad c_1 := 1. \tag{2.113}$$

Then, from (2.112) and (2.113), it follows that

$$\zeta_{2n} = \zeta_{2n-2}^{a_1}\zeta_{2n-4}^{b_1}\zeta_{2n-6}^{c_1}, \tag{2.114}$$

for $n \geq 2$.

Employing (2.112) in (2.114), it follows that

$$\begin{aligned}\zeta_{2n} &= \zeta_{2n-2}^{a_1}\zeta_{2n-4}^{b_1}\zeta_{2n-6}^{c_1}\\ &= (\zeta_{2n-4}\zeta_{2n-6}^{2}\zeta_{2n-8})^{a_1}\zeta_{2n-4}^{b_1}\zeta_{2n-6}^{c_1}\\ &= \zeta_{2n-4}^{a_1+b_1}\zeta_{2n-6}^{2a_1+c_1}\zeta_{2n-8}^{a_1}\\ &= \zeta_{2n-4}^{a_2}\zeta_{2n-6}^{b_2}\zeta_{2n-8}^{c_2},\end{aligned}$$

for $n \geq 3$, where

$$a_2 := a_1 + b_1, \quad b_2 := 2a_1 + c_1, \quad c_2 := a_1.$$

Similar to equation (2.40), it follows that

$$\zeta_{2n} = \zeta_{2(n-k)}^{a_k}\zeta_{2(n-k-1)}^{b_k}\zeta_{2(n-k-2)}^{c_k}, \tag{2.115}$$

and

$$a_k = a_{k-1} + b_{k-1}, \quad b_k = 2a_{k-1} + c_{k-1}, \quad c_k = a_{k-1}, \tag{2.116}$$

for a $k \geq 2$ and all $n \geq k+1$.

From (2.113) and (2.116) we have

$$a_n = a_{n-1} + 2a_{n-2} + a_{n-3}, \tag{2.117}$$

and

$$a_0 = 1, \quad a_{-1} = 0, \quad a_{-2} = 0, \quad a_{-3} = 1.$$

Letting $k = n-1$ in (2.115), it follows that

$$\begin{aligned}\zeta_{2n} &= \zeta_{2}^{a_{n-1}}\zeta_{0}^{b_{n-1}}\zeta_{-2}^{c_{n-1}}\\ &= (\zeta_0\zeta_{-2}\eta_{-1})^{a_{n-1}}\zeta_0^{a_n-a_{n-1}}\zeta_{-2}^{a_{n-2}}\\ &= \zeta_0^{a_n}\zeta_{-2}^{a_{n-1}+a_{n-2}}\eta_{-1}^{a_{n-1}},\end{aligned} \tag{2.118}$$

for $n \geq -1$.

Similarly is get

$$\begin{aligned}\zeta_{2n-1} &= \zeta_{3}^{a_{n-2}}\zeta_{1}^{b_{n-2}}\zeta_{-1}^{c_{n-2}}\\ &= (\zeta_{-1}\eta_0^2\eta_{-2})^{a_{n-2}}(\eta_0\eta_{-2})^{a_{n-1}-a_{n-2}}\zeta_{-1}^{a_{n-3}}\\ &= \zeta_{-1}^{a_{n-2}+a_{n-3}}\eta_0^{a_{n-1}+a_{n-2}}\eta_{-2}^{a_{n-1}},\end{aligned} \tag{2.119}$$

for $n \in \mathbb{N}_0$.

Combining (2.34) and (2.118), it follows that

$$\begin{aligned}\eta_{2n-1} &= \zeta_{2n-2}\zeta_{2n-4}\\ &= \zeta_0^{a_{n-1}+a_{n-2}}\zeta_{-2}^{a_{n-2}+2a_{n-3}+a_{n-4}}\eta_{-1}^{a_{n-2}+a_{n-3}}\\ &= \zeta_0^{a_{n-1}+a_{n-2}}\zeta_{-2}^{a_{n-1}}\eta_{-1}^{a_{n-2}+a_{n-3}},\end{aligned}$$

for $n \in \mathbb{N}_0$, whereas by combining (2.34) and (2.119), it follows that

$$\begin{aligned}\eta_{2n} &= \zeta_{2n-1}\zeta_{2n-3}\\ &= \zeta_{-1}^{a_{n-2}+2a_{n-3}+a_{n-4}}\eta_0^{a_{n-1}+2a_{n-2}+a_{n-3}}\eta_{-1}^{a_{n-1}+a_{n-2}}\\ &= \zeta_{-1}^{a_{n-1}}\eta_0^{a_n}\eta_{-2}^{a_{n-1}+a_{n-2}},\end{aligned}$$

for $n \ge -1$.

The characteristic polynomial associated with (2.117) is

$$\widehat{P}_3(t) = t^3 - t^2 - 2t - 1.$$

Let t_j, $j = \overline{1,3}$, be the roots of the polynomial. Then, the solution to (2.117) such that $a_{-2} = a_{-1} = 0$ and $a_0 = 1$, is

$$a_n = \sum_{j=1}^{3} \frac{t_j^{n+2}}{\widehat{P}_3'(t_j)}, \quad n \in \mathbb{Z}. \tag{2.120}$$

From this and (2.21) the following corollary follows.

Corollary 2.15. *If $a \neq 0$, then the general solution to (2.17) is*

$$x_{2n} = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}+a_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}+a_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-1}}-1}, \qquad n \ge -1$$

$$x_{2n-1} = \sqrt{a}\,\frac{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}+a_{n-3}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-1}+a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}+a_{n-3}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-1}+a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n \in \mathbb{N}_0$$

$$y_{2n} = \sqrt{a}\,\frac{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}+a_{n-2}}+1}{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}+a_{n-2}}-1}, \qquad n \ge -1$$

$$y_{2n-1} = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-1}+a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}+a_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-1}+a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}+a_{n-3}}-1}, \quad n \in \mathbb{N}_0,$$

where the sequence a_n is given by (2.120).

2.14 System (2.35)

This system is get from (2.30) by interchanging letters ζ and η. Hence, we have

$$\zeta_n = \eta_0^{a_{n-1}+a_{n-3}}\eta_{-1}^{a_{n-6}+a_{n-8}}\eta_{-2}^{a_{n-5}+a_{n-7}}\zeta_0^{a_{n-4}+a_{n-6}}\zeta_{-1}^{a_{n-3}+a_{n-5}}\zeta_{-2}^{a_{n-2}+a_{n-4}},$$

for $n \ge -2$, and

$$\eta_n = \eta_0^{a_n}\eta_{-1}^{a_{n-5}}\eta_{-2}^{a_{n-4}}\zeta_0^{a_{n-3}}\zeta_{-1}^{a_{n-2}}\zeta_{-2}^{a_{n-1}},$$

for $n \ge -2$.

From this and (2.21) the following corollary follows.

Corollary 2.16. *If $a \neq 0$, then the general solution to (2.18) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-6}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-5}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-4}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-2}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-6}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-5}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-4}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-2}}-1},$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-4}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-4}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1},$$

for $n \ge -2$, where the sequence a_n is given by (2.100) and $b_n = a_n + a_{n-2}$.

2.15 System (2.36)

This system is obtained from (2.26) by interchanging letters ζ and η. Hence, we have

$$\begin{aligned}\zeta_n &= \zeta_0^{b_{n-3}+b_{n-5}}\eta_0^{b_{n-2}+b_{n-4}}\eta_{-1}^{b_{n-3}+b_{n-4}}\eta_{-2}^{b_{n-2}+b_{n-3}}, \quad n\in\mathbb{N}_0,\\ \eta_n &= \zeta_0^{b_{n-2}}\eta_0^{b_{n-1}}\eta_{-1}^{b_{n-3}+b_{n-4}}\eta_{-2}^{b_{n-2}+b_{n-3}}, \quad n\geq -2.\end{aligned}$$

From this and (2.21) the following corollary follows.

Corollary 2.17. *If $a\neq 0$, then the general solution to (2.19) is*

$$x_n=\sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-3}+b_{n-5}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-2}+b_{n-4}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-3}+b_{n-4}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-2}+b_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-3}+b_{n-5}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-2}+b_{n-4}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-3}+b_{n-4}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-2}+b_{n-3}}-1},$$

for $n\in\mathbb{N}_0$, and

$$y_n=\sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-3}+b_{n-4}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-2}+b_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-3}+b_{n-4}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-2}+b_{n-3}}-1},$$

for $n\geq -2$, where b_n is given by (2.77).

2.16 System (2.37)

System (2.37) is get from system (2.22) by interchanging letters ζ and η only. Hence, we have

$$\begin{aligned}\zeta_n &= \eta_0^{a_n}\eta_{-1}^{a_{n-2}}\eta_{-2}^{a_{n-1}}, \quad n\in\mathbb{N},\\ \eta_n &= \eta_0^{a_n}\eta_{-1}^{a_{n-2}}\eta_{-2}^{a_{n-1}}, \quad n\geq -2.\end{aligned}$$

From this and (2.21) the following corollary follows.

Corollary 2.18. *If $a\neq 0$, then the general solution to (2.20) is*

$$\begin{aligned}x_n &= \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n\in\mathbb{N},\\ y_n &= \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n\geq -2,\end{aligned}$$

where a_n is given by (2.48).

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Qualitative properties and global bifurcation of solutions for a singular boundary value problem

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. This paper deals with a singular, nonlinear Sturm–Liouville problem of the form $\{A(x)u'(x)\}' + \lambda u(x) = f(x, u(x), u'(x))$ on $(0, 1)$ where A is positive on $(0, 1]$ but decays quadratically to zero as x approaches zero. This is the lowest level of degeneracy for which the problem exhibits behaviour radically different from the regular case. In this paper earlier results on the existence of bifurcation points are extended to yield global information about connected components of solutions.

Keywords: singular Sturm–Liouville problem, global bifurcation, Hadamard differentiable mapping.

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1 Introduction

The aim of this paper is to investigate the set of solutions of the boundary value problem,

$$-\{A(x)u'(x)\}' + V(x)u(x) + n(x, u'(x)) + g(x, u(x)) = \lambda u(x) \quad \text{for } 0 < x < 1, \tag{1.1}$$

$$u(1) = 0 \quad \text{and} \quad \int_0^1 A(x)u'(x)^2 dx < \infty, \tag{1.2}$$

for an unknown function u such that $u \in C^1((0, 1])$ and Au' is absolutely continuous on the compact subintervals of $(0, 1]$. The differential equation is singular at $x = 0$ because we suppose that the coefficient A satisfies the following condition.

(A) $A \in C([0, 1])$ with $A(x) > 0$ for $x > 0$ and $\lim_{x\to 0} \frac{A(x)}{x^2} = a > 0$.

Hence there exist constants $C_2 \geq C_1 > 0$ such that $C_1x^2 \leq A(x) \leq C_2x^2$ for all $x \in [0, 1]$.

As we have shown in previous work on the problem in [31], this level of degeneracy leads to behaviour that does not occur for regular problems nor problems with weaker degeneracy.

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For example, solutions can become unbounded as x tends to zero and there may be no bifurcation at a simple eigenvalue of the linearisation lying below the essential spectrum. For a more detailed presentation of the critical character of quadratic degeneracy we refer to [33] concerning the analogous elliptic problem in higher dimensions. Other aspects of criticality have been emphasised in some work on the asymptotic behaviour of solutions for a porous medium equation with degeneracy [17,18]. In the stability analysis for the parabolic problem associated with the higher dimensional analogue of (1.1)(1.2) it is shown in [32] that the principle of linearised stability can fail at the stationary solution $u \equiv 0$ when the degeneracy is critical. For subcritical degeneracy, i.e. when $\liminf_{x\to 0} x^{-d}A(x) > 0$ for some $d < 2$, global bifurcation of positive stationary solutions and their stability are proved in [20] for a parabolic problem corresponding to the higher dimensional analogue of (1.1)(1.2).

Before proceeding to describe other aspects of the problem some information about the lower order terms in (1.1) is necessary. The potential V in (1.1) is bounded and has a well-defined limit as $x \to 0$.

(V) $V \in L^\infty(0,1)$ and there exists $V_0 \in \mathbb{R}$ such that $\lim_{z\to 0} \|V - V_0\|_{L^\infty(0,z)} = 0$.

The nonlinear terms n and g are of higher order in the sense that

$$\lim_{s\to 0} \frac{n(x,s)}{s} = \lim_{s\to 0} \frac{g(x,s)}{s} = 0 \quad \text{for all } x \in (0,1) \tag{1.3}$$

and they satisfy some additional conditions introduced in Subection 2.2. Under these hypotheses $u \equiv 0$ is a solution of (1.1)(1.2) and the parameter $\lambda \in \mathbb{R}$ is treated as an eigenvalue. The sense in which the equation (1.1) is satisfied is made precise in Section 2.3. In this form the problem has been studied in some detail in [31,33] and Section 2 contains the conclusions from those papers that are needed here.

In view of (1.3) the linearisation of (1.1) is the singular Sturm–Liouville problem

$$-\{A(x)u'(x)\}' + V(x)u(x) = \lambda u(x), \quad \text{where } u \in L^2(0,1) \text{ and } u(1) = 0, \tag{1.4}$$

and its spectrum is discussed in Section 2.4. It is in the limit point case at $x = 0$ when (A) and (V) are satisfied but

$$\lim_{x\to 0} A(x)u'(x) = 0, \tag{1.5}$$

appears as a natural boundary condition. In fact, it is noted in Section 2 that the expression $-(Au')' + Vu$ defines a self-adjoint operator, $S_A + V$, acting in $L^2(0,1)$ with domain

$$D_A = \{u \in L^2(0,1) : (Au')' \in L^2(0,1) \text{ and } u(1) = 0\}$$

and all elements of D_A satisfy (1.2) and (1.5). The eigenvalues of $S_A + V$ are all simple and its essential spectrum is the interval $[\frac{a}{4} + V_0, \infty)$. In Section 2.4 some special cases treated in [33] are recalled showing that $S_A + V$ may or may not have eigenvalues.

The main results of this paper give information about the global behaviour of components of solutions $(\lambda, u) \in \mathbb{R} \times D_A$ of the singular problem (1.1)(1.2) in the spirit of the regular case treated in [7,24]. This involves confronting two principal difficulties arising from the degeneracy. First of all, the presence of a non-trivial essential spectrum of the linearisation indicates that the problem cannot be reduced to an equation for a compact perturbation of the identity. Secondly, previous work on the existence of bifurcation points for problem (1.1)(1.2) has shown that, under reasonable assumptions about the nonlinear terms, Fréchet differentiability at the trivial solution $u \equiv 0$ cannot be obtained. Indeed, there are cases in [31,33] where there is no bifurcation at an eigenvalue of $S_A + V$ lying below its essential spectrum, a situation which could not occur if the nonlinearity were Fréchet differentiable at $u \equiv 0$.

The conclusions obtained here concerning problem (1.1)(1.2) are established by following what has become a standard path since the classic paper by Rabinowitz [23]. First of all an abstract result is formulated under hypotheses that accommodate the two main difficulties just mentioned. This result is then applied to the boundary value problem and the nodal properties of solutions are used to sharpen the information about components of solutions given by the abstract theory.

Let X and Y be real Banach spaces and consider a mapping $F : \mathbb{R} \times X \to Y$ having the properties that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$ and $F(\lambda, \cdot) : X \to Y$ is at least Hadamard differentiable at 0. For the equation $F(\lambda, u) = 0$, local results concerning bifurcation at isolated singular points of the derivative $D_u F(\lambda, 0)$ were established in [29] using the Brouwer degree after reduction to a finite dimensional space. In a similar setting global conclusions about connected components of solutions have been obtained recently in [34] using a topological degree for continuous perturbations of C^1-Fredholm maps constructed by Benevieri, Calamai and Furi [3,4], combined with techniques from in [29]. In these contributions a considerable amount of rather specialised terminology is required in order the formulate the hypotheses. The class of admissible perturbations for the existence of the degree defined in [3,4] is specified using notions related to the Kuratowski measure of non-compactness and the conditions for bifurcation involve the parity of the path $\lambda \mapsto D_u F(\lambda, 0)$ as defined by Fitzpatrick and Pejsachowicz, [15]. Instead of recalling these results in their fully generality with all the requisite terminology, we formulate two special cases concerning equations of a simpler form in Hilbert space. With the exception of Hadamard and w-Hadamard differentiability which are defined in Section 4.1, these results can be stated using only well-known concepts and problem (1.1)(1.2) can be dealt with in this context.

The Hilbert space theory, as set out in Section 4, is applied to problem (1.1)(1.2) in Section 5. As for regular Sturm–Liouville problems, the nodal properties of solutions and comparison principles for self-adjoint operators can be used to refine the conclusions coming directly from the abstract theory. However the strong degeneracy of equation (1.1) at $x = 0$ means that the behaviour of solutions as $x \to 0$ requires some care and various aspects of this are investigated in Section 3, generalizing results of a similar nature in [31]. As special cases of the main results in Section 5, hypotheses are provided under which the following somewhat unusual phenomena occur. Consider problem (1.1)(1.2) with $n \equiv 0$. Given any $n \in \mathbb{N}$, there are coefficients A and V such that the linearisation (1.4) has exactly n simple eigenvalues $\lambda_1 < \lambda_2, \ldots. < \lambda_n$ below its essential spectrum which is $[m_e, \infty)$ where $m_e = \frac{a}{4} + V_0$.

(1) For any $k \in \{1, \ldots, n\}$ there is a class of nonlinearities with $g(x,s)s \leq 0$ for all $(x,s) \in (0,1) \times \mathbb{R}$ for which an unbounded component of non-trivial solutions bifurcates from $(\lambda_i, 0)$ for each $i \leq k$, but there is no bifurcation from $(\lambda_i, 0)$ for $i > k$. (See Remark 5.4.)

(2) There is another class of nonlinearities with $g(x,s)s \geq 0$ for all $(x,s) \in (0,1) \times \mathbb{R}$ for which a component $\mathcal{C}_i$ of non-trivial solutions bifurcates from $(\lambda_i, 0)$ for every $i \in \{1, ..., n\}$ and $\{\lambda : (\lambda, u) \in \mathcal{C}_i\} = [\lambda_i, m_e)$. If $(\lambda, u) \in \mathcal{C}_i$ with λ near λ_i, $u \in C^1((0,1]) \cap L^\infty(0,1)$, whereas for λ near m_e, $u \in C^1((0,1])$ but $u(x) \to \infty$ as $n \to \infty$. (See Remark 5.6.)

Many references to problems of the type studied here can be found in the papers [17,18, 20,31,33] and, as shown in an appendix in [31], several other types of equation can be reduced to the form (1.1) by a change of variable. The radially symmetric version of the analogous problem in higher dimensions can also be transformed to (1.1)(1.2). Following what was done in [13] for a closely related case, this is mentioned in [31] and more details are given in Section 6.6 of [33] where local results on bifurcation are formulated.

The line of research pursued here on bifurcation for problems like (1.1)(1.2) was stimulated by the unusual behaviour revealed in [28] concerning the buckling of a critically tapered rod which is modelled by an equation having the same kind of degeneracy. Using variational methods it is shown in [28] that an unbounded curve of positive solutions bifurcates from the lowest point Λ of the spectrum of the linearisation, even if it is not an eigenvalue. In fact, bifurcation occurs at every point in the interval $[\Lambda, \infty)$. For the same problem, global bifurcation at all eigenvalues lying below the essential spectrum was established in collaboration with G. Vuillaume [35, 36] using a topological approach. In this buckling problem the full nonlinear equation involves a compact perturbation of the identity but it is not Fréchet differentiable at the trivial solution and its linearisation is not a compact perturbation of the identity. In work with G. Evéquoz [13, 14] on a more general class of degenerate problems a variational method was used show that bifurcation can occur at points which are not necessarily eigenvalues of the linearisation and singular behaviour of the bifurcating solutions was demonstrated in the radially symmetric case. Some of the abstract results on bifurcation for non-Fréchet differentiable problems are summarised in [30] together with references to applications to uniformly elliptic equations on $\mathbb{R}^N$.

2 A class of singular boundary value problems

Throughout this section it is assumed that the function A satisfies condition (A). The first step is to define the domain of a positive self-adjoint operator, S_A, in $L^2(0,1)$ associated with the singular differential operator $-(Au')'$ and the boundary condition $u(1) = 0$. In addition to noting some crucial properties of functions in this domain, D_A, it is also necessary to investigate the domain, H_A, of the positive, self-adjoint square-root, $S_A^{\frac{1}{2}}$. Although the set D_A depends upon A, it turns out that H_A is the same set for all coefficients satisfying condition (A). Most of the results mentioned in this section are proved in [31].

2.1 The spaces D_A and H_A

From the results in Section 2 of [31] the set D_A can be defined as

$$D_A = \{u \in C^1((0,1]) \cap L^2(0,1) : (Au')' \in L^2(0,1) \text{ and } u(1) = 0\},$$

where $(Au')'$ is the generalized derivative on $(0,1)$ of the continuous function Au'. It is also shown in [31] that

$$S_A : D_A \subset L^2(0,1) \to L^2(0,1) \quad \text{with} \quad S_A u = -(Au')' \quad \text{for } u \in D_A$$

is a self-adjoint operator having the following properties. See Lemmas 2.1 and 2.2 and Corollary 2.3 in [31].

(D1) $(S_A u, v)_{L^2} = \int_0^1 Au'v'\,dx$ for all $u, v \in D_A$.

(D2) $(S_A u, u)_{L^2} \geq \frac{C_1}{4}\|u\|_{L^2}^2$ and $\|u\|_{L^2} \leq \frac{2}{\sqrt{C_1}}\|A^{\frac{1}{2}}u'\|_{L^2} \leq \frac{4}{C_1}\|S_A u\|_{L^2}$ for all $u \in D_A$.

(D3) $S_A : D_A \to L^2$ is an isomorphism and $S_A^{-1}w = \int_x^1 \frac{1}{A(y)}[\int_0^y w(z)\,dz]\,dy$ for all $w \in L^2(0,1)$.

Henceforth, $L^2 = L^2(0,1)$ and $a \geq C_1 \equiv \inf\left\{\frac{A(x)}{x^2} : 0 < x \leq 1\right\} > 0$ by (A). By (D2), $\|S_A u\|_{L^2}$ defines a norm on D_A that is equivalent to the graph norm of S_A. Elements of D_A enjoy the following properties which are proved in Lemmas 2.4 and 2.5 in [31].

(P1) $x^{\frac{3}{2}}u'(x) \to 0$ as $x \to 0$ and $\|x^{\frac{3}{2}}u'\|_{L^\infty} \leq \frac{1}{C_1}\|S_A u\|_{L^2}$ for all $u \in D_A$.

(P2) $x^{\frac{1}{2}}u(x) \to 0$ as $x \to 0$ and $\|x^{\frac{1}{2}}u\|_{L^\infty} \leq \frac{1}{\sqrt{C_1}}\|A^{\frac{1}{2}}u'\|_{L^2} \leq \frac{2}{C_1}\|S_A u\|_{L^2}$ for all $u \in D_A$.

By condition (A) and (P1), $A(x)u'(x) \to 0$ as $x \to 0$ for all $u \in D_A$ showing that (1.5) is a natural boundary condition for the operator S_A. If $u \in D_A$ and $u(z) = 0$ for some $z \in (0,1]$, it follows from (P1) and (P2) that

$$\int_0^z [S_A u(x)]u(x)\,dx = \int_0^z A(x)u'(x)^2 dx. \tag{2.1}$$

Let

$$H = \left\{u \in L^2(0,1) : \int_0^1 x^2 u'(x)^2 dx < \infty\right\}$$

where u' is the generalized derivative of u on $(0,1)$. If $u \in H$, its restriction to $(\eta,1)$ belongs to the usual Sobolev space $H^1((\eta,1))$ for all $\eta \in (0,1)$ and so, with the usual abuse to terminology, we can consider that $u \in C((0,1])$. The space H_A is defined by

$$H_A = \{u \in H : u(1) = 0\} = \left\{u \in L^2(0,1) : \int_0^1 A(x)u'(x)^2 dx < \infty \text{ and } u(1) = 0\right\}.$$

It is a Hilbert space for the norm defined by $\|u\|_A = \|A^{\frac{1}{2}}u'\|_{L^2}$ and the corresponding scalar product is denoted by

$$\langle u, v\rangle_A = \int_0^1 A(x)u'(x)v'(x)\,dx \quad \text{for } u, v \in H_A.$$

Denoting the unique positive, self-adjoint square root of S_A by $S_A^{\frac{1}{2}} : D(S_A^{\frac{1}{2}}) \subset L^2(0,1) \to L^2(0,1)$, it is also shown in [31] that $H_A = D(S_A^{\frac{1}{2}})$. In particular, D_A is a dense subspace of $(H_A, \|\cdot\|_A)$ and so (D1), (D2) and (P2) imply the following properties.

(H1) $\|u\|_{L^2} \leq \frac{2}{\sqrt{C_1}}\|u\|_A$ and $\|u\|_A = \|S_A^{\frac{1}{2}}u\|_{L^2}$ for all $u \in H_A$.

(H2) $x^{\frac{1}{2}}u(x) \to 0$ as $x \to 0$ and $\|x^{\frac{1}{2}}u\|_{L^\infty} \leq \frac{1}{\sqrt{C_1}}\|u\|_A$ for all $u \in H_A$.

Using (H1) with $A(x) = x^2$ and a simple rescaling, we have that

$$\int_0^z u(x)^2 dx \leq 4\int_0^z x^2 u'(x)^2 dx \quad \text{if } u \in H_A \text{ and } u(z) = 0 \text{ for some } z \in (0,1]. \tag{2.2}$$

By (P1) and (H2),

$$\int_0^1 [S_A u(x)]v(x)\,dx = \int_0^1 A(x)u'(x)v'(x)\,dx \quad \text{for all } u \in D_A \text{ and } v \in H_A. \tag{2.3}$$

The following compactness property is justified in Remark 2.2 in [31].

(H3) If $\{u_n\} \subset H_A$ is a sequence converging weakly to u in H_A, $\{u_n\} \subset C([\eta,1])$ and it converges uniformly to u on $[\eta,1]$ for all $\eta \in (0,1)$.

2.2 Properties of the nonlinearities

The Nemytskii operator associated with a Caratheodory function $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ will be denoted by $\tilde{f}$. Thus $\tilde{f}(u)(x) = f(x, u(x))$ for a measurable function $u : (0,1) \to \mathbb{R}$ and $x \in (0,1)$.

We now formulate the assumptions which will be used to deal with the nonlinear terms in equation (1.1). They ensure that the corresponding operators are well-defined and map the spaces D_A and H_A into $L^2(0,1)$. For the continuity and differentiability properties of these operators it is understood that D_A and H_A are considered with the norms $\|S_A\|_{L^2}$ and $\|u\|_A$, respectively.

(F) $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

- (i) $\lim_{s\to 0} \frac{f(x,s)}{s} = 0$ for all $x \in (0,1)$,
- (ii) for some $\ell \in [0,\infty)$, $|f(x,s) - f(x,t)| \le \ell |s-t|$ for all $x \in (0,1)$ and $s,t \in \mathbb{R}$.

For a function satisfying condition (F), let

$$\ell_f = \sup\left\{ \frac{|f(x,s) - f(x,t)|}{|s-t|} : 0 < x < 1 \text{ and } s \neq t \right\}. \tag{2.4}$$

The next result refers to Hadamard and w-Hadamard differentiability of a mapping. The definitions of these notions are recalled in Section 4.1.

Proposition 2.1. *Let condition (F) be satisfied by a function f.*

(i) Then the associated Nemytskii operator maps $L^2 = L^2(0,1)$ into itself and $\tilde{f} : L^2 \to L^2$ is uniformly Lipschitz continuous with

$$\|\tilde{f}(u) - \tilde{f}(v)\|_{L^2} \le \ell_f \|u - v\|_{L^2} \quad \text{for all } u, v \in L^2 \tag{2.5}$$

Furthermore, $\tilde{f} : L^2 \to L^2$ is Gâteaux differentiable at 0 with derivative 0.

(ii) $\tilde{f} : L^2 \to L^2$ is Hadamard differentiable at 0 and $\tilde{f} : H_A \to L^2$ is w-Hadamard differentiable at 0 with derivative 0.

(iii) In addition to condition (F) suppose that there is a constant α with the property that, for all $\delta > 0$, there exist $x(\delta) \in (0,1)$ and $M(\delta)$ such that $|f(x,s) - \alpha s| \le M(\delta) + \delta|s|$ for all $(x,s) \in (0, x(\delta)) \times \mathbb{R}$. Then the mapping $\tilde{f} - \alpha I : H_A \to L^2$ is compact.

Proof. For parts (i) and (ii) see Lemma 3.1 in [31]. Part (iii) appears as Lemma 4.3 (b) in [34]. □

Remark 2.2. Since D_A is continuously embedded in L^2, $\tilde{f} : D_A \to L^2$ is also Hadamard differentiable at 0. However, it is important to emphasise that an assumption like (F) does not imply Fréchet differentiability of $\tilde{f}$ at 0, even when $f \in C^\infty([0,1] \times \mathbb{R})$. For example, it is shown in Example 3.1 in [31] that when $f(x,s) = h(s)$, where $h \in C^\infty(\mathbb{R})$ with $h(0) = h'(0) = 0$ and $\sup_{s\in\mathbb{R}} |h'(s)| < \infty$, condition (F) is satisfied but $\tilde{f} : D_A \to L^2$ is Fréchet differentiable at 0 if and only if $h \equiv 0$.

Fréchet differentiability of $\tilde{f}$ does hold provided that the function $f(x,s)$ decays in an appropriate way as $x \to 0$, as stipulated in the following condition.

(E) $f = \sum_{i=1}^{k} f_i$ where for each i, $f_i : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

- (i) $f_i(x, \cdot) \in C^1(\mathbb{R})$ and $f_i(x,0) = 0$ for all $x \in (0,1)$,
- (ii) there exist K_i and $\alpha_i > \frac{\sigma_i}{2}$ such that $|\partial_s f_i(x,s)| \leq K_i x^{\alpha_i} |s|^{\sigma_i}$ for all $x \in (0,1)$ and $s \in \mathbb{R}$ where $0 < \sigma_1 < \cdots < \sigma_k$.

For a function f satisfying condition (E), let $\mathfrak{C}_f(s) = \sum_{i=1}^{k} s^{\sigma_i}$ for $s \geq 0$ and note that for $s, t \geq 0$,

$$\min\{1, t^{\sigma_k}\}\mathfrak{C}_f(s) \leq \min\{t^{\sigma_1}, t^{\sigma_k}\}\mathfrak{C}_f(s) \leq \mathfrak{C}_f(ts) \leq \max\{t^{\sigma_1}, t^{\sigma_k}\}\mathfrak{C}_f(s) \leq \max\{1, t^{\sigma_k}\}\mathfrak{C}_f(s). \tag{2.6}$$

It follows from (E) and property (H2) that for all $u \in H_A$ and $x \in (0,1)$,

$$\left|\frac{f_i(x, u(x))}{u(x)}\right| \leq K_i x^{\alpha_i - \frac{\sigma_i}{2}} [x^{\frac{1}{2}} |u(x)|]^{\sigma_i} \leq K_i C_1^{-\sigma_i/2} \|u\|_A^{\sigma_i} x^{\alpha_i - \frac{\sigma_i}{2}} \quad \text{if } u(x) \neq 0. \tag{2.7}$$

and

$$|f_i(x, u(x))u(x)| \leq K_i C_1^{-\sigma_i/2} \|u\|_A^{\sigma_i} x^{\alpha_i - \frac{\sigma_i}{2}} u(x)^2. \tag{2.8}$$

Hence, setting $\nu = \min\{\alpha_i - \frac{\sigma_i}{2} : 1 \leq i \leq k\}$, there exists a constant C such that

$$|f(x, u(x))| \leq C x^{\nu} \mathfrak{C}_f(\|u\|_A)|u(x)| \quad \text{for all } u \in H_A \text{ and } x \in (0,1). \tag{2.9}$$

Thus $\tilde{f}(u) \in L^2$ for all $u \in H_A$ and the next result shows that condition (E) ensures that $\tilde{f} : H_A \to L^2$ is both continuously Fréchet differentiable on H_A and compact.

Proposition 2.3. *Let f satisfy the condition (E). Then $\tilde{f} \in C^1(H_A, L^2)$ and there is a constant $C > 0$ such that $\|\tilde{f}'(u)\|_{B(H_A, L^2)} \leq C\mathfrak{C}_f(\|u\|_A)$. Furthermore, the mapping $\tilde{f} : H_A \to L^2$ is compact.*

Proof. Continuous differentiability is established in Lemma 3.2 in [31]. Compactness is easily proved using the estimate (2.9) on the interval $(0, \eta)$ and property (H3) on $[\eta, 1]$ for $\eta \in (0,1)$ in the same way as in Lemma 4.5 of [32] which deals with a similar situation in higher dimensions. □

Remark 2.4. For $u, v \in H_A$, $\|\tilde{f}(u) - \tilde{f}(v)\|_{L^2} \leq C\mathfrak{C}_f(\|u\|_A + \|v\|_A)\|u - v\|_A$ and, in particular, $\|\tilde{f}(u)\|_{L^2} \leq C\mathfrak{C}_f(\|u\|_A)\|u\|_A$ for all $u \in H_A$. It also follows from this lemma that $\tilde{f} \in C^1(D_A, L^2)$ and there is a constant C such that $\|\tilde{f}'(u)\|_{B(D_A, L^2)} \leq C\mathfrak{C}_f(\|S_A u\|_{L^2})$ for all $u \in D_A$.

We now turn to the nonlinear term in equation (1.1) containing u'. Recalling that $D_A \subset C^1((0,1])$ a mapping N is defined on D_A by setting $N(u)(x) = n(x, u'(x)) = \tilde{n}(u')(x)$ where $n : (0,1) \times \mathbb{R} \to \mathbb{R}$. The following condition ensures that N maps D_A into L^2.

(M) $n = \sum_{i=1}^{j} n_i$ where for each i, $n_i : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

- (i) $n_i(x, \cdot) \in C^1(\mathbb{R})$ and $n_i(x,0) = 0$ for all $x \in (0,1)$,
- (ii) there exist $K_i > 0$ and $\beta_i > \frac{3\gamma_i}{2} + 1$ such that $|\partial_s n_i(x,s)| \leq K_i x^{\beta_i} |s|^{\gamma_i}$ for all $x \in (0,1)$ and $s \in \mathbb{R}$ where $0 < \gamma_1 < \cdots < \gamma_j$.

For a function n satisfying condition (M), let $\mathfrak{D}_n(s) = \sum_{i=1}^{j} s^{\gamma_i}$ for $s \geq 0$.

It follows from (M) and property (P1) that for all $u \in D_A$ and $x \in (0,1)$,

$$|n_i(x,u'(x))| \leq K_i x^{\beta_i - \frac{3\gamma_i}{2} - 1}|xu'(x)||x^{\frac{3}{2}}u'(x)|^{\gamma_i} \leq K_i C_1^{-\gamma_i/2}\|S_A u\|_{L^2}^{\gamma_i} x^{\beta - \frac{\gamma_i}{2} - 1}|xu'(x)| \tag{2.10}$$

and hence there is a constant K such that

$$|n_i(x,u'(x))u(x)| \leq K\|S_A u\|_{L^2}^{\gamma_i} x^{\beta_i - \frac{3\gamma_i}{2} - 1}\{u(x)^2 + x^2 u'(x)^2\}. \tag{2.11}$$

Setting $\nu = \min\{\beta_i - \frac{3\gamma_i}{2} - 1 : 1 \leq i \leq j\}$ it follows from (2.10) that there exists a constant C such that

$$|n(x,u'(x))| \leq C x^{\nu} \mathfrak{D}_n(\|S_A u\|_{L^2})|xu'(x)| \quad \text{for all } u \in D_A \text{ and } x \in (0,1).$$

Hence $N(u) \in L^2$ for all $u \in D_A$ and the main properties of the mapping $N : D_A \to L^2$ are given in the next result.

Proposition 2.5. *When n satisfies the condition (M), $N \in C^1(D_A, L^2)$ with $N'(u)v = \partial_s n(\cdot, u')v'$ for all $u, v \in D_A$ and there is a constant $C > 0$ such that $\|N'(u)\|_{B(D_A,L^2)} \leq C\mathfrak{D}_n(\|S_A u\|_{L^2})$. Furthermore, the mapping $N : D_A \to L^2$ is compact.*

Proof. See Lemma 3.4 in [31] and Lemma 4.3 (a) in [34]. □

2.3 Solutions of problem (1.1)(1.2) and bifurcation points

In dealing with problem (1.1)(1.2) from now on it will be assumed that the following condition is satisfied.

(S) The coefficients A and V satisfy conditions (A) and (V). The function n satisfies condition (M) and g can be written as $g_1 + g_2$ where g_1 satisfies condition (F) and g_2 satisfies condition (E).

Under the assumption (S) it follows from Propositions 2.1 to 2.5 that a continuous mapping $F : \mathbb{R} \times D_A \to L^2$ is defined by

$$F(\lambda, u) = S_A u + Vu + N(u) + \tilde{g}(u) - \lambda u, \tag{2.12}$$

provided that D_A is considered with a norm equivalent to the graph norm of S_A. By property (D2), all elements of D_A satisfy (1.2).

Definition 2.6. Henceforth, a solution of problem (1.1)(1.2) is defined to be an element $(\lambda, u) \in \mathbb{R} \times D_A$ such that $F(\lambda, u) = 0$, where F is given by (2.12).

Clearly $(\lambda, 0)$ is a solution for all $\lambda \in \mathbb{R}$ and

$$\mathcal{E} = \{(\lambda, u) \in \mathbb{R} \times D_A : F(\lambda, u) = 0 \text{ and } u \not\equiv 0\} \tag{2.13}$$

denotes the set of all non-trivial solutions of problem (1.1)(1.2). We recall that for $u \in D_A$, $u \in C^1((0,1])$ and, setting $v = Au'$, we also have that v is absolutely continuous on $[0,1]$, as noted at the beginning of Section 2 in [31]. If (λ, u) is a solution of (1.1)(1.2), $v'(x) = f(\lambda, x, u(x), v(x))$ for almost all $x \in (0,1)$ where $f(\lambda, x, p, q) = [V(x) - \lambda]p + n(x, q/A(x)) + g(x, p)$ for $x \in (0,1]$ and $p, q \in \mathbb{R}$. Thus, when A is not differentiable on $(0,1)$, equation (1.1) is satisfied in the sense of a quasi-differential equation, that is

$$(u(x), v(x))' = (v(x)/A(x), f(\lambda, x, u(x), v(x))) \quad \text{for almost all } x \in (0,1). \tag{2.14}$$

(See III.10.1 in [10] and Chapter 2 of [25], for example.) For any given $\eta \in (0,1]$ and $(p_0, q_0) \in \mathbb{R}$, assumption (S) ensures that there exist $L > 0$ and $\delta \in (0, \eta)$ such that

$$\frac{|q_1 - q_2|}{A(x)} \leq L|q_1 - q_2| \text{ and } |f(\lambda, x, p_1, q_1) - f(\lambda, x, p_2, q_2)| \leq L\|(p_1, q_1) - (p_2, q_2)\|$$

for $x \in [\eta - \delta, 1]$ and $\|(p_i, q_i) - (p_0, q_0)\| < \delta$ for $i = 1$ and 2. Hence for any $x_0 > 0$, local existence and uniqueness of the solution of the initial value problem $u(x_0) = p_0$, $v(x_0) = q_0$ for (2.14) hold by standard arguments applied to the equivalent integral equation. (See Chapter 2 of [6], for example.) In particular, if (λ, u) is a solution of (1.1)(1.2) and $u(x_0) = u'(x_0) = 0$ for some $x_0 \in (0,1]$, then $u(x) = 0$ for all $x \in (0,1]$ and it follows that, if $(\lambda, u) \in \mathcal{E}$, then u has a finite number of zeros in any compact subinterval of $(0,1]$ and that they are all simple zeros.

Having clarified what is meant by a solution of problem (1.1)(1.2), we now turn to the notion of bifurcation point.

Definition 2.7. A real number μ is called a bifurcation point for problem (1.1)(1.2) if and only if $(\mu, 0) \in \overline{\mathcal{E}}$ where $\overline{\mathcal{E}}$ denotes the closure of $\mathcal{E}$ in the space $\mathbb{R} \times D_A$ and D_A is considered with the norm $u \mapsto \|S_A u\|_{L^2}$.

To explore the content of this definition, consider a sequence $\{(\lambda_n, u_n)\}$ in $\mathcal{E}$ such that $\lambda_n \to \mu$ and $\|S_A u_n\|_{L^2} \to 0$ as $n \to \infty$. By properties (P1) and (P2) in Section 2.1 this implies that $\|x^{1/2} u_n\|_{L^\infty} \to 0$ and $\|x^{3/2} u_n'\|_{L^\infty} \to 0$ as $n \to \infty$. Hence, $\{u_n\}$ and $\{u_n'\}$ converge uniformly to zero on all compact subintervals of $(0,1]$, but not necessarily on $(0,1]$. However, by (D2) we do have that $\|u_n\|_{L^2} + \|u_n\|_A \to 0$ as $n \to \infty$. The results in Section 5 provide sufficient conditions for a number μ to be a bifurcation point and under their hypotheses the functions u_n have only a finite number of zeros in $(0,1]$. It follows from this and Proposition 3.5(ii) that there exists $n_0 \in \mathbb{N}$ such that $\lim_{x \to 0} u_n(x) = \pm\infty$ for all $n \geq n_0$ if $\mu \in (V_0 + J_s(g_1), \frac{a}{4} + V_0)$. Further details of situations where this phenomenon occurs are given in Section 5.

The assumption (S) and Propositions 2.1 to 2.5 also imply that for all $\lambda \in \mathbb{R}$ the mapping $F(\lambda, \cdot) : D_A \to L^2$ defined by (2.12) is Hadamard differentiable at 0 with $D_u F(\lambda, 0) = S_A + V \in B(D_A, L^2)$. Hence we expect that bifurcation theory for problem (1.1)(1.2) will require some information about the spectrum of the operator $S_A + V$.

2.4 Spectral theory of the linearization

Conditions (A) and (V) are supposed to be satisfied throughout this subsection. Here we summarize the main features of the self-adjoint operator $S = S_A + V : D(S) = D(S_A) \subset L^2(0,1) \to L^2(0,1)$ that are established in [31] and [33]. More precisely, properties (S1) and (S3) are part of Theorem 4.1 in [33] and (S5) is justified by the discussion preceding Theorem 6.11 in [33]. Property (S2) follows from the comments after Definition 2.6 about solutions of (2.14) with n and g equal to zero. In the same way, properties (S4) and (S6) are special cases of Lemma 3.1 and Proposition 3.5, although similar conclusions also appear in [31,33]. Recall that

$$\sigma(S) = \{\lambda \in \mathbb{R} : S - \lambda I : D(S) \to L^2(0,1) \text{ is not an isomorphism}\}$$

and

$$\sigma_e(S) = \{\lambda \in \mathbb{R} : S - \lambda I : D(S) \to L^2(0,1) \text{ is not a Fredholm operator}\}.$$

Let

$$m = \inf \sigma(S) \quad \text{and} \quad m_e = \inf \sigma_e(s).$$

(S1) $\sigma_e(S) = [\frac{a}{4} + V_0, \infty)$ and $\frac{C_1}{4} + \operatorname{ess\,inf} V \leq m \leq m_e = \frac{a}{4} + V_0$.

(S2) All eigenvalues of S are simple and eigenfunctions have only simple zeros in $(0, 1]$.

(S3) If $m < m_e$, it is an eigenvalue having an eigenfunction ϕ with $\phi(x) > 0$ for $0 < x < 1$.

(S4) If u is an eigenfunction for an eigenvalue in the interval $(-\infty, m_e)$, then u has only a finite number of zeros in $(0, 1]$.

(S5) If the eigenvalues are numbered in increasing order with $m = \mu_1 < \mu_2$ etc. and if $\mu_k < m_e$, then an eigenfunction for μ_k has exactly k zeros in $(0, 1]$.

(S6) If μ is an eigenvalue in $(-\infty, V_0)$ its eigenfunction is bounded on $(0, 1)$ whereas, if $V_0 < \mu < m_e$ it has an eigenfunction ϕ with $\phi(x) \to \infty$ as $x \to 0$.

There are cases where S has no eigenvalues, for example, when $A(x) = x^2$ and $V(x) \equiv 0$. More generally, if A and V have the additional properties that A and $V \in C^1((0, 1])$ with

$$\lim_{x \to 0} A'(x)/x = 2a, \quad \lim_{x \to 0} xV'(x) = 0 \quad \text{and} \quad A(x)/x^2 \text{ and } V \text{ non-decreasing on } (0, 1),$$

then S has no eigenvalues. See Corollary 3.9.

The following special cases, which are treated in Section 4.2 of [33], together with the usual comparison principle for self-adjoint operators, provide examples of situations where S does have eigenvalues in $(-\infty, m_e)$.

Example 2.8. Let $A(x) = x^2$ and for some $\tau \in (0, 1)$ and $L > 0$, let

$$V(x) = 0 \quad \text{for } 0 < x < \tau \quad \text{and} \quad V(x) = -L \quad \text{for } \tau < x < 1.$$

Then $\sigma_e(S) = [\frac{1}{4}, \infty)$ and S has no eigenvalues in this interval.

If $\sqrt{L} \ln \frac{1}{\tau} \leq \frac{\pi}{2}$, S has no eigenvalues.

If $(n - \frac{1}{2})\pi < \sqrt{L} \ln \frac{1}{\tau} \leq (n + \frac{1}{2})\pi$ for some positive integer n, then S has exactly n eigenvalues in $(-\infty, \frac{1}{4})$.

The explicit form of the eigenfunctions and estimates for the eigenvalues are also given in [33].

Example 2.9. For $0 < x < 1$, let $A(x) = x^2$ and $V(x) = -(\frac{n\pi s}{2})^2 x^s$ where $s \in (0, \infty)$ and n is a positive integer.

Then $\sigma_e(S) = [\frac{1}{4}, \infty)$ and S has at least n eigenvalues in $(-\infty, \frac{1}{4})$. In fact, $\mu_n = \frac{1}{4}(1 - \frac{s^2}{4})$ is the n-th eigenvalue and $\phi(x) = x^{-\frac{1}{2}(1+\frac{s}{2})} \sin(n\pi x^{\frac{s}{2}})$ is an eigenfunction for μ_n.

Example 2.10. For $\tau \in (0, 1)$, let

$$A(x) = x^2 \quad \text{for } 0 \leq x \leq \tau \quad \text{and} \quad A(x) = \tau^2 \quad \text{for } \tau < x \leq 1.$$

Then $\sigma_e(S_A) = [\frac{1}{4}, \infty)$. If $\tau \geq \frac{2}{2+\pi}$, S_A has no eigenvalues whereas if $\frac{2}{2+(4n+1)\pi} \leq \tau < \frac{2}{2+(4n-3)\pi}$ for a positive integer n, then S_A has exactly n eigenvalues in $(-\infty, \frac{1}{4})$.

The explicit form of the eigenfunctions and estimates for the eigenvalues are also given in [33].

The operator $S = S_A + V$ is always bounded below and for some proofs it is useful to make a shift so that it becomes positive. For any $c > -m$, the operator $S_c \equiv S + cI$ with domain $D(S_c) = D(S) = D(S_A)$ has many properties similar to those of S_A. It is positive definite and self-adjoint. The graph norms of S and S_c are equivalent to the norm defined by $\|S_A u\|_{L^2}$ on $D(S_A)$. Furthermore, the domain of its positive, self-adjoint square root, $S_c^{\frac{1}{2}}$, is H_A and $\|\cdot\|_A$ is equivalent to the graph norm of $S_c^{\frac{1}{2}}$ on H_A. See Section 4.3 of [33] for more details.

The proof of Theorem 5.5 uses some facts about the spectrum of the self-adjoint operator $W \in B(L^2, L^2)$ defined by $W = I - (\lambda + c - \alpha)S_c^{-1}$ where $\alpha \geq 0$, $c > \max\{0, \alpha - \operatorname{ess\,inf} V\}$ and $\alpha - c < \lambda < m_e$. Note that $c > \alpha - m$ by property (S1) so $\alpha - c < m \leq m_e$ and $c > -m$. Hence $S_c : D_A \to L^2$ is an isomorphism and $S_c^{-1} \in B(L^2, L^2)$ is injective but not surjective. Hence $1 \in \sigma_e(W)$ and it is easy to check that

$$\sigma(W) = \{1\} \cup \left\{1 - \frac{\lambda + c - \alpha}{\mu + c} : \mu \in \sigma(S)\right\} \quad \text{and} \quad \sigma_e(W) = \{1\} \cup \left\{1 - \frac{\lambda + c - \alpha}{\mu + c} : \mu \in \sigma_e(S)\right\}.$$

Since $\lambda + c - \alpha > 0$, it follows that $1 - \frac{\lambda+c-\alpha}{\mu+c}$ is an increasing function of μ and hence

$$\inf \sigma(W) = \frac{m + \alpha - \lambda}{m + c} \quad \text{and} \quad 0 < \inf \sigma_e(W) = \frac{m_e + \alpha - \lambda}{m_e + c} < 1. \tag{2.15}$$

3 Qualitative properties of solutions

As noted in Section 2.3, solutions of (1.1)(1.2) have only a finite number of zeros in any compact subinterval in $(0,1]$ and all zeros are simple. Most of the results in this section concern the behaviour of solutions as x approaches the singular point $x = 0$. Some integral identities also lead to conclusions about the non-existence of non-trivial solutions of (1.1)(1.2) and the absence of eigenvalues of the operator $S_A + V$. Earlier work on the properties of solutions for a related problem can be found in the paper [5] by Caldiroli and Musina which deals with equations of the form $-\{\omega(x)u'(x)\}' = f(u(x))$ under a variety of assumptions about the decay of $\omega(x)$ as $x \to 0$.

3.1 Nodal properties of solutions

The first results in this part provide conditions under which solutions of (1.1)(1.2) have a finite number of zeros in $(0,1]$. For a function $u \in C((0,1])$ having only a finite number of zeros in $(0,1]$ the number of zeros in $(0,1]$ will be denoted by $\sharp(u)$. Under the hypotheses of Corollary 3.3 this number is locally constant on $\mathcal{E}$.

Lemma 3.1. *Let condition (S) be satisfied.*

(i) Given $\delta > 0$ and $C > 0$ there exists $\eta \in (0,1)$ such that $u(x) \neq 0$ for $x \in (0,\eta]$ whenever $(\lambda, u) \in \mathcal{E}$ with $\lambda \leq m_e - \ell_{g_1} - \delta$ and $\|S_A u\|_{L^2} \leq C$.

(ii) If there exists $z \in (0,1)$ such that either $g(x,s)s \geq 0$ for all $(x,s) \in (0,z) \times \mathbb{R}$, or $g_1(x,s)s \geq 0$ for all $(x,s) \in (0,z) \times \mathbb{R}$, then the conclusion holds for $\lambda \leq m_e - \delta$ and $\|S_A u\|_{L^2} \leq C$.

Proof. (i) Fix δ and C as in the statement of the lemma. By (F), (2.9) and (2.11), there exist a constant $D > 0$ and an exponent $\nu > 0$ for which the following inequalities hold for all

$x \in (0,1)$ and all $u \in D_A$ with $\|S_A u\|_{L^2} \le C$.

$$\widetilde{g_1}(u)(x)u(x) \ge -\ell_{g_1} u(x)^2, \tag{3.1}$$
$$\widetilde{g_2}(u)(x)u(x) \ge -Dx^{\nu} u(x)^2, \tag{3.2}$$
$$N(u)(x)u(x) \ge -Dx^{\nu}\{u(x)^2 + x^2 u'(x)^2\}. \tag{3.3}$$

Set $\varepsilon = \min\left\{\frac{a}{2}, \frac{\delta}{4}\right\}$ and then choose $\eta \in (0,1)$ such that, for $0 < x \le \eta$,

$$A(x) \ge (a-\varepsilon)x^2, \qquad V(x) \ge V_0 - \varepsilon \quad \text{and} \quad Dx^{\nu} \le \varepsilon.$$

Consider $(\lambda, u) \in \mathcal{E}$ with $\lambda \le m_e - \ell_{g_1} - \delta$ and $\|S_A u\|_{L^2} \le C$. If $u(z) = 0$ for some $z \in (0, \eta]$, then using (2.1) and (2.2) we have

$$0 = \int_0^z A(x)u'(x)^2 + V(x)u(x)^2 + N(u)(x)u(x) + \widetilde{g}(u)(x)u(x) - \lambda u(x)^2 dx \tag{3.4}$$

$$\begin{aligned} &\ge \int_0^z (a-\varepsilon)x^2 u'(x)^2 + [V_0 - \varepsilon]u(x)^2 \\ &\qquad - \varepsilon\{u(x)^2 + x^2u'(x)^2\} - \ell_{g_1}u(x)^2 - \varepsilon u(x)^2 - \lambda u(x)^2 dx \end{aligned} \tag{3.5}$$

$$\ge \int_0^z \frac{a-2\varepsilon}{4}u(x)^2 + u(x)^2\left\{V_0 - 3\varepsilon - \ell_{g_1} - \lambda\right\} dx = \int_0^z u(x)^2 \left\{m_e - \ell_{g_1} - \lambda - \frac{7}{2}\varepsilon\right\} dx \tag{3.6}$$

$$\ge \int_0^z u(x)^2\left\{\delta - \frac{7}{2}\varepsilon\right\} dx \ge \frac{\varepsilon}{2}\int_0^z u(x)^2 dx > 0. \tag{3.7}$$

From this contradiction we may conclude that u has no zeros in the interval $(0, z]$.

(ii) In this case the term $\ell_{g_1}u(x)^2$ in (3.5) and (3.6) and be dropped and (3.7) holds for $\lambda \le m_e - \delta$. □

Lemma 3.2. *For $\eta \in (0,1)$, $C^1_\eta \equiv \{u \in C^1([\eta,1]) : u(1) = 0\}$ with norm $\|u\|_\eta = \max\{|u'(x)| : \eta \le x \le 1\}$ is a Banach space.*

(i) Setting $P_\eta u(x) = u(x)$ for $u \in D_A$ and $x \in [\eta, 1]$, $P_\eta \in B(D_A, C^1_\eta)$ is compact.

(ii) If $u \in C^1_\eta$ has exactly n zeros in $(\eta, 1]$ all of which are simple and $u(\eta) \neq 0$, there exists $\delta > 0$ such that for all $v \in C^1_\eta$ with $\|u - v\|_\eta < \delta$, v has exactly n zeros in $(\eta, 1]$ all of which are simple and $v(\eta) \neq 0$.

Proof. (i) By the definition of D_A, $P_\eta(D_A) \subset C^1_\eta$. Let $\{u_n\}$ be a bounded sequence in D_A and let $v_n = (P_\eta u_n)'$. By the Ascoli–Arzelà Theorem, it suffices to show that the sequence $\{v_n\}$ is uniformly bounded and equi-continuous on $[\eta, 1]$. By property (D3) of S_A,

$$v_n(x) = -\frac{1}{A(x)}\int_0^x w_n(y)\, dy \quad \text{for } \eta \le x \le 1$$

where $w_n = S_A u_n$ and $\{w_n\}$ is a bounded sequence in $L^2(0,1)$. Let $M = \sup \|w_n\|_{L^2}$. Then, since $A(x) \ge C_1 x^2$ on $[0,1]$,

$$|v_n(x)| \le \frac{1}{C_1 x^2} x^{\frac{1}{2}} \left\{\int_0^x w_n(y)^2 dy\right\}^{\frac{1}{2}} \le \frac{M}{C_1 \eta^{\frac{3}{2}}} \quad \text{for } \eta \le x \le 1$$

and for $\eta \le x \le z \le 1$,

$$|v_n(x) - v_n(z)| \le \frac{1}{A(x)} \int_x^z |w_n(y)|\, dy + \left| \frac{1}{A(x)} - \frac{1}{A(z)} \right| \int_0^x |w_n(y)|\, dy$$
$$\le \frac{M(z-x)^{\frac{1}{2}}}{C_1 \eta^2} + \frac{M|A(z) - A(x)|}{C_1^2 \eta^{\frac{7}{2}}}.$$

It follows that $\{v_n\}$ has a subsequence converging in $C([\eta, 1])$ and consequently that $P_\eta : D_A \to C_\eta^1$ is a compact operator.

(ii) This is an easy exercise. The details are given in Lemma 3.1 of [36], for example. □

Corollary 3.3. *Suppose that condition (S) is satisfied and that $(\lambda, u) \in \mathcal{E}$ has the property that there exist $\delta > 0$ and $\eta \in (0,1)$ such that, for all $(\xi, v) \in \mathcal{E}$ with $|\xi - \lambda| + \|S_A(u - v)\|_{L^2} < \delta$, v has no zeros in the interval $(0, \eta]$. Then there exists $\varepsilon > 0$ such that $\sharp(v) = \sharp(u)$ for all $(\xi, v) \in \mathcal{E}$ with $|\xi - \lambda| + \|S_A(v - u)\|_{L^2} < \varepsilon$.*

Proof. By Lemma 3.2 (i), $P_\eta \in B(D_A, C_\eta^1)$ and so the conclusion follows from Lemma 3.2 (ii). □

For $z \in (0,1)$ let

$$E(z) = \left\{ (x,s) \in (0,1) \times \mathbb{R} : 0 < x < z \text{ and } |s| < x^{-\frac{1}{2}} \ln \frac{1}{x} \right\}$$

and

$$D(z) = \left\{ (x,s) \in (0,1) \times \mathbb{R} : 0 < x < z \text{ and } z^{-\frac{1}{2}} \ln \frac{1}{z} < |s| < x^{-\frac{1}{2}} \ln \frac{1}{x} \right\}.$$

Then, for a Carathéodory function $g : (0,1) \times \mathbb{R} \to \mathbb{R}$, let

$$J_i(g) = \lim_{z \to 0} \operatorname*{ess\,inf}_{0<x<z} \inf \left\{ \frac{g(x,s)}{s} : 0 < |s| < x^{-\frac{1}{2}} \ln \frac{1}{x} \right\} \tag{3.8}$$

$$J_s(g) = \lim_{z \to 0} \operatorname*{ess\,sup}_{0<x<z} \sup \left\{ \frac{g(x,s)}{s} : 0 < |s| < x^{-\frac{1}{2}} \ln \frac{1}{x} \right\} \tag{3.9}$$

$$I_i(g) = \lim_{z \to 0} \operatorname*{ess\,inf}_{0<x<z} \inf \left\{ \frac{g(x,s)}{s} : z^{-\frac{1}{2}} \ln \frac{1}{z} < |s| < x^{-\frac{1}{2}} \ln \frac{1}{x} \right\} \tag{3.10}$$

$$I_s(g) = \lim_{z \to 0} \operatorname*{ess\,sup}_{0<x<z} \sup \left\{ \frac{g(x,s)}{s} : z^{-\frac{1}{2}} \ln \frac{1}{z} < |s| < x^{-\frac{1}{2}} \ln \frac{1}{x} \right\}. \tag{3.11}$$

When dealing with solutions of (1.1)(1.2) these quantities lead to the following properties which will be exploited in Proposition 3.5.

Lemma 3.4. *Let condition (S) be satisfied.*

(i) Then $-\ell_{g_1} \le J_i(g_1) = J_i(g) \le 0 \le J_s(g) = J_s(g_1) \le \ell_{g_1}$ and $J_i(g) \le I_i(g_1) = I_i(g) \le I_s(g_1) = I_s(g) \le J_s(g)$. If g_1 also satisfies the compactness condition in Proposition 2.1 then $I_i(g) = I_s(g) = \alpha$.

(ii) If $(\lambda, u) \in \mathcal{E}$, there exists $z \in (0,1)$ such that $(x, u(x)) \in E(z)$ for all $x \in (0, z)$. Setting

$$B_u(x) = \frac{g(x, u(x))}{u(x)} \text{ if } u(x) \neq 0 \text{ and } B_u(x) = 0 \text{ if } u(x) = 0, \tag{3.12}$$

$J_i(g_1) \le \liminf_{x \to 0} B_u(x) \le \limsup_{x \to 0} B_u(x) \le J_s(g_1)$. If either $u(x) \to \infty$ or $u(x) \to -\infty$ as $x \to 0$, then $I_i(g_1) \le \liminf_{x \to 0} B_u(x) \le \limsup_{x \to 0} B_u(x) \le I_s(g_1)$.

Here $\limsup_{x\to 0} B_u(x) = \lim_{x\to 0} \operatorname{ess\,sup}_{0<y<x} B_u(y)$ and similarly for the lim inf.

Proof. (i) Since $g(x,s)/s \to 0$ as $s \to 0$ for all $x \in (0,1)$, $J_s(g) \geq 0 \geq J_i(g)$. Furthermore, $-\ell_{g_1} \leq g_1(x,s)/s \leq \ell_{g_1}$ for $x \in (0,1)$ and $s \neq 0$ so $-\ell_{g_1} \leq I_i(g_1) \leq I_s(g_1) \leq \ell_{g_1}$.

Let f be a function satisfying condition (E)(ii) for $\alpha > \sigma/2 > 0$. Then, for $0 < x < 1$ and $0 < |s| < x^{-\frac{1}{2}} \ln \frac{1}{x}$,

$$\left|\frac{f(x,s)}{s}\right| \leq Kx^{\alpha}|s|^{\sigma} \leq Kx^{\alpha-\frac{\sigma}{2}}\left(\ln\frac{1}{x}\right)^{\sigma}$$

and hence

$$\lim_{z\to 0} \operatorname*{ess\,sup}_{0<x<z} \sup\left\{\left|\frac{f(x,s)}{s}\right| : 0 < |s| < x^{-\frac{1}{2}}\ln\frac{1}{x}\right\} = 0.$$

Since $g - g_1$ is a finite sum of functions of this type and $D(z) \subset E(z)$ the conclusions follow.

(ii) By property (P2) there exists a constant K such that $x^{\frac{1}{2}}|u(x)| \leq K$ for $0 < x < 1$. Hence there exists $z_0 \in (0,1)$ such that $(x, u(x)) \in E(z)$ for $0 < x < z < z_0$. Furthermore, if $|u(x)| \to \infty$ as $x \to 0$, for all $z \in (0, z_0)$, there exists $\delta_z < z$ such that $(x, u(x)) \in D(z)$ for $0 < x < \delta_z$. The conclusions in part (ii) are easily deduced from these observations. □

We can now establish a number of results concerning the behaviour of a solution of (1.1)(1.2) as $x \to 0$. They generalise and improve similar conclusions in Theorem 5.1 of [31].

Proposition 3.5. *Let condition (S) be satisfied and $n \equiv 0$.*

(i) If $\lambda < m_e + J_i(g_1)$ and $(\lambda, u) \in \mathcal{E}$, there exists $\eta \in (0,1)$ such that u has no zeros in the interval $(0, \eta]$.

(ii) If $\lambda > V_0 + J_s(g_1)$ and $(\lambda, u) \in \mathcal{E}$, then either u has a sequence of zeros converging to 0 or $\lim_{x\to 0} u(x) = \pm\infty$.

(iii) If $\lambda > \max\{V_0 + J_s(g_1), m_e + I_s(g_1)\}$ and $(\lambda, u) \in \mathcal{E}$, then u has a sequence of zeros converging to 0.

(iv) If $\lambda < V_0 + I_i(g_1)$ and $(\lambda, u) \in \mathcal{E}$, then $u \in L^{\infty}(0,1)$.

Remark 3.6. Since $m_e + J_s(g_1) \geq \max\{V_0 + J_s(g_1), m_e + I_s(g_1)\}$ it follows from part (iii) that u has a sequence of zeros converging to 0 if $\lambda > m_e + J_s(g_1)$ and $(\lambda, u) \in \mathcal{E}$.

Taking $g \equiv 0$, Proposition 3.5 gives the following information about an eigenfunction, ϕ, of $S_A + V$ associated with an eigenvalue λ. For $\lambda < m_e$ it has a finite number of zeros whereas for $\lambda > m_e$ it has infinitely many zeros. If $\lambda < V_0$, ϕ is bounded on $(0,1)$ and if $V_0 < \lambda < m_e$, $\phi(x) \to \pm\infty$ as $x \to 0$.

Proof. Recall that $m_e = \frac{a}{4} + V_0$ and, for $(\lambda, u) \in \mathcal{E}$, set $B(\lambda, u)(x) = \lambda - V(x) - B_u(x)$.

Part (i) This can be proved in the same way as Lemma 3.1 since, given any $\varepsilon > 0$, there exists $\eta \in (0,1)$ such that, for $0 < x < \eta$, $A(x) \geq (a - \varepsilon)x^2$, $V(x) \geq V_0 - \varepsilon$ and $g(x, u(x))u(x) = B_u(x)u(x)^2 \geq \{J_i(g_1) - \varepsilon\}u(x)^2$. It suffices to repeat the estimates (3.4) to (3.7) with minor adjustments.

Part (ii) Consider $(\lambda, u) \in \mathcal{E}$ and suppose that u has only a finite number of zeros in $(0,1)$. Since $u \in C((0,1])$ there exists $\eta > 0$ such that either $u > 0$ on $(0, \eta]$ or $u < 0$ on $(0, \eta]$.

Suppose that $u > 0$ on $(0, \eta]$. By property (D3) of D_A,

$$A(x)u'(x) = -\int_0^x B(\lambda, u)(y)u(y)\, dy \quad \text{for } 0 < x \leq \eta.$$

Define $\varepsilon > 0$ by $3\varepsilon = \lambda - V_0 - J_s(g_1)$. By condition (V) and Lemma 3.4, η can be chosen so that $V(x) \le V_0 + \varepsilon$ and $B_u(x) \le J_s(g_1) + \varepsilon$ for $0 < x < \eta$. Then for $0 < y < \eta$, $B(\lambda, u)(y) \ge \lambda - V_0 - J(g_1) - 2\varepsilon = \varepsilon$ and so

$$A(x)u'(x) \le -\int_0^x \varepsilon u(y)\,dy < 0 \quad \text{for } 0 < x < \eta,$$

from which it follows that u is decreasing on $(0, \eta)$ and consequently, $A(x)u'(x) \le -\varepsilon u(\eta)x$ for $0 < x \le \eta$. By condition (A),

$$u(\eta) - \lim_{x\to 0} u(x) \le -\varepsilon u(\eta) \int_0^\eta \frac{y}{A(y)}\,dy = -\infty,$$

proving that $u(x) \to \infty$ as $x \to 0$.

The case where $u < 0$ on $(0, \eta]$ can be dealt with in the same way.

Part (iii) Choose $\gamma > \frac{1}{4}$ such that $\lambda > \gamma a + V_0 + I_s(g_1)$ and then define $\varepsilon > 0$ by $(3+\gamma)\varepsilon = \lambda - \gamma a - V_0 - I_s(g_1)$.

There exists $\eta \in (0,1)$ such that, for $0 < x < \eta$, $A(x) \le (a+\varepsilon)x^2$ and $V(x) \le V_0 + \varepsilon$.

Suppose that u has only a finite number of zeros. By part (ii), $u(x) \to \pm\infty$ as $x \to 0$ and so, referring to Lemma 3.4 and reducing η, we may suppose that $B_u(x) \le I_s(g_1) + \varepsilon$ for $0 < x < \eta$. Then, for $0 < x < \eta$,

$$B(\lambda, u)(x) \ge \lambda - V_0 - I_s(g_1) - 2\varepsilon = \gamma a + (3+\gamma)\varepsilon - 2\varepsilon = \gamma(a+\varepsilon) + \varepsilon.$$

The function w defined by $w(x) = x^{-1/2}\sin\left(\sqrt{\gamma - \frac{1}{4}}\ln x\right)$ for $x > 0$ satisfies the equation $-(x^2 w'(x))' = \gamma w(x)$, which can be written as

$$-(C(x)w'(x))' = Dw(x) \quad \text{where} \quad C(x) = (a+\varepsilon)x^2 \quad \text{and} \quad D = \gamma(a+\varepsilon).$$

On the interval $(0,\eta)$, $-(Au')' = B(\lambda, u)u$, $A \le C$, $B(\lambda, u) > D$ and w has an infinite sequence of zeros converging to 0. Hence, by the Sturm comparison theorem, u also has a sequence of zeros in $(0, \eta)$ converging to 0. (For the type of coefficients appearing here, the comparison can be established using Picone's identity by the arguments in 10.31 of [19].) This proves part (iii).

Part (iv) Let $\varepsilon > 0$ be defined by $3\varepsilon = V_0 + I_i(g_1) - \lambda$. There exist $\eta \in (0,1)$ and $S > 0$ such that, for $0 < x < \eta$, $V(x) \ge V_0 - \varepsilon$ and, if $|u(x)| > S$, $B_u(x) \ge I_i(g_1) - \varepsilon$ by Lemma 3.4. Then

$$B(\lambda, u)(x) \le \lambda - V_0 - I_i(g_1) + 2\varepsilon = -\varepsilon \quad \text{on } \omega \equiv \{x \in (0,\eta) : |u(x)| \ge S\}.$$

Let $T = \max\{S, \max_{\eta \le x \le 1} |u(x)|\}$ and $\omega^+ = \{x \in (0,1) : u(x) > T\}$. Then $\omega^+ \subset \omega$ and $(u-T)^+ \in H_A$. Hence $\operatorname{supp}(u-T)^+ \subset \omega^+$ and by (2.3),

$$\begin{aligned} 0 \le \int_{\omega^+} A(u')^2 dx &= \int_0^1 Au'[(u-T)^+]' dx = \int_0^1 S_A u(u-T)^+ dx = \int_0^1 B(\lambda, u)u(u-T)^+ dx \\ &= \int_{\omega^+} B(\lambda, u)u(u-T)\,dx. \end{aligned}$$

But $B(\lambda, u) \le -\varepsilon$ and $u(u-T)^+ > 0$ on the open set ω^+ which must be empty, since otherwise the final integral would be negative. This proves that $u(x) \le T$ on $(0,1]$.

A similar argument using $\omega^- = \{x \in (0,1) : u(x) < -T\}$ and $(u+T)^-$ shows that $u(x) \ge -T$ on $(0,1]$, completing the proof of part (iv). $\square$

3.2 Integral identities and their consequences

The following identities involving solutions of (1.1)(1.2) lead to new information about their behaviour as $x \to 0$ and also to some conditions under which non-trivial solutions do not exist.

Proposition 3.7. *In addition to the assumption (S) with $n \equiv 0$, suppose that the following condition is satisfied.*

(T1) There exists $\delta \in (0,1]$ such that A and $V \in C^1((0,\delta])$ with $\lim_{x\to 0} \frac{A'(x)}{2x} = a$ and $\lim_{x\to 0} xV'(x) = 0$. Also g_1 and $f_i \in C^1((0,\delta) \times \mathbb{R})$ where $g_2 = \sum_{i=1}^k f_i$ with

$$|x\partial_x g_1(x,s)| \leq K|s| \quad \text{and} \quad |x\partial_x f_i(x,s)| \leq K_i x^{\sigma_i/2}|s|^{1+\sigma_i} \quad \text{for } (x,s) \in (0,\delta) \times \mathbb{R},$$

where σ_i is given by assumption (E).

Set

$$\Phi(x,s) = \int_0^s g(x,t)\, dt \quad \text{for } (x,s) \in (0,1) \times \mathbb{R}.$$

Suppose that $u(z) = 0$ for some $z \in (0,\delta]$. Then

$$\int_0^z [A - xA'](u')^2 + (\lambda - V - xV')u^2 - 2\{\Phi(x,u) + x\partial_x\Phi(x,u)\}\, dx = zA(z)u'(z)^2, \tag{3.13}$$

$$\int_0^z A(u')^2 + (V-\lambda)u^2 + g(x,u)u\, dx = 0, \tag{3.14}$$

$$\int_0^z [2A - xA'](u')^2 - xV'u^2 + g(x,u)u - 2\{\Phi(x,u) + x\partial_x\Phi(x,u)\}\, dx = zA(z)u'(z)^2. \tag{3.15}$$

Proof. This result is a slight generalization of Lemma 5.2 in [31] and Theorem 7.7 in [33]. The proof requires only minor modifications to the arguments used in these references. □

From the identity (3.13) we can derive an variant of part (i) of Proposition 3.5.

Corollary 3.8. *Under the assumptions (S) with $n \equiv 0$ and (T1), if $(\lambda,u) \in \mathcal{E}$ and $\lambda < m_e + 2\liminf_{x\to 0} \inf_{s\neq 0} \frac{\Phi(x,s)+x\partial_x\Phi(x,s)}{s^2}$, there exists $\eta \in (0,1)$ such that $u(x) \neq 0$ for $x \in (0,\eta]$.*

Proof. By the assumptions about the coefficients A and V, $[A(x) - xA'(x)]/x^2 \to -a$ and $V(x) + xV'(x) \to V_0$ as $x \to 0$. For (λ,u) as in the statement, first choose $\varepsilon > 0$ such that

$$\varepsilon < a \quad \text{and} \quad \lambda + 4\varepsilon < m_e + 2\liminf_{x\to 0} \inf_{s\neq 0} \frac{\Phi(x,s) + x\partial_x\Phi(x,s)}{s^2},$$

and then, for δ as in (T1), choose $\eta \in (0,\delta)$ such that for $0 < x \leq \eta$,

$$\frac{A(x) - xA'(x)}{x^2} < -a + \varepsilon, V(x) + xV'(x) > V_0 - \varepsilon$$

and

$$\inf_{s\neq 0} \frac{\Phi(x,s) + x\partial_x\Phi(x,s)}{s^2} > \liminf_{x\to 0} \inf_{s\neq 0} \frac{\Phi(x,s) + x\partial_x\Phi(x,s)}{s^2} - \varepsilon.$$

Suppose now that $u(z) = 0$ for some $z \in (0,\eta]$. By (2.2),

$$\int_0^z [A(x) - xA'(x)]u'(x)^2 dx < -(a-\varepsilon)\int_0^z x^2 u'(x)^2 dx \leq -\frac{a-\varepsilon}{4}\int_0^z u(x)^2 dx$$

and hence (3.13) yields

$$zA(z)u'(z)^2 < \int_0^z \left\{ \lambda - \frac{a-\varepsilon}{4} - V_0 + \varepsilon - 2\liminf_{x\to 0}\inf_{s\neq 0} \frac{\Phi(x,s)+x\partial_x\Phi(x,s)}{s^2} + 2\varepsilon \right\} u(x)^2 dx$$
$$\leq -\frac{3}{4}\varepsilon \int_0^z u(x)^2 dx < 0.$$

Since $A(z) > 0$ and $u'(z) \neq 0$, this is false and so $u(z) \neq 0$ for all $z \in (0, \eta]$. □

Unlike the other results concerning the behaviour of solutions as $x \to 0$ the identity (3.15) yields information without placing any restriction on λ.

Corollary 3.9. *In addition to the assumptions (S) with $n \equiv 0$ and (T1), suppose that*

(T2) there exists $\eta \in (0, \delta]$ such that $\frac{A(x)}{x^2}$ and $V(x)$ are non-decreasing functions of x on $(0, \eta)$ and

$$g(x,s)s \leq 2\{\Phi(x,s) + x\partial_x\Phi(x,s)\} \text{ for } (x,s) \in (0,\eta) \times \mathbb{R}. \tag{3.16}$$

Then for any $(\lambda, u) \in \mathcal{E}$, $u(x) \neq 0$ for $0 < x \leq \eta$ and consequently, $\lambda \leq \max\{V_0 + J_s(g_1), m_e + I_s(g_1)\}$ by part (iii) of Proposition 3.5.

Since $u(1) = 0$ for all $u \in D_A$, if (T1) and (T2) are satisfied with $\delta = \eta = 1$, $\mathcal{E} = \emptyset$ and, taking $g \equiv 0$, the operator $S = S_A + V$ has no eigenvalues.

Proof. If $(\lambda, u) \in \mathcal{E}$ and $u(z) = 0$ for some $z \in (0, \eta]$, then $u'(z) \neq 0$ and $zA(z)u'(z)^2 > 0$. But the hypotheses imply that $2A(x) - xA'(x) \leq 0$ and $V'(x) \geq 0$ on $(0, \eta)$ so (3.15) implies that $zA(z)u'(z)^2 \leq 0$, a contradiction. Hence $u(x) \neq 0$ for $x \in (0, \eta]$. □

Remark 3.10. Consider a function g having the properties required in conditions (S) and (T1). For $(x,s) \in (0,\delta) \times \mathbb{R}$,

$$\Phi(x,s) = \int_0^s \frac{g(x,t)}{t} t\, dt = \frac{1}{2}\left\{ g(x,s)s - \int_0^s t^2 \partial_t \left[\frac{g(x,t)}{t}\right] dt \right\}$$

and so

$$g(x,s)s - 2\{\Phi(x,s) + x\partial_x\Phi(x,s)\} = \int_0^s t \left\{ t\partial_t \left[\frac{g(x,t)}{t}\right] - 2x\partial_x \left[\frac{g(x,t)}{t}\right] \right\} dt.$$

Hence, condition (3.16) is satisfied provided that there exists $\eta \in (0, \delta]$ such that

$$s\partial_s \left[\frac{g(x,s)}{s}\right] \leq 2x\partial_x \left[\frac{g(x,s)}{s}\right] \quad \text{for all } x \in (0,\eta) \text{ and } s \neq 0.$$

A stronger, but more transparent, sufficient condition for (3.16) to hold is

$$s\partial_s \left[\frac{g(x,s)}{s}\right] \leq 0 \leq \partial_x \left[\frac{g(x,s)}{s}\right] \quad \text{for all } x \in (0,\eta) \text{ and } s \neq 0. \tag{3.17}$$

Note that since condition (S) implies that $g(x,s)/s \to 0$ as $s \to 0$ for all $x \in (0,1)$, (3.17) can only be satisfied in cases where $g(x,s)/s \leq 0$ for all $x \in (0, \eta]$ and $s \neq 0$.

4 Global bifurcation in Hilbert space

In this section two results about global bifurcation of solutions for equations in Hilbert space are formulated as Theorems 4.7 and 4.10. They are deduced from recent work in [34] on equations of a more general type in Banach space. It seems worthwhile deriving the special cases given here because their statement avoids a series of not so standard notions which are required for the form treated in [34], but which are not needed here. Of course, the notions in question inevitably appear in the proofs of Theorems 4.7 and 4.10 which amount to verifying that the hypotheses of Theorems 3.4 and 3.5 in [34] are satisfied.

4.1 Preliminaries

In preparation for the subsequent discussion some notation is fixed and a few definitions are recalled.

Let X and Y be two real Banach spaces. As usual, the space of all bounded linear operators from X into Y will be denoted by $B(X,Y)$ and, for $T \in B(X,Y)$, $\|T\| = \sup\{\|Tu\| : u \in X \text{ and } \|u\| = 1\}$.

$\mathrm{Iso}(X,Y) = \{T \in B(X,Y) : T : X \to Y \text{ is an isomorphism }\}$

$\Phi_0(X,Y) = \{T \in B(X,Y) : T : X \to Y \text{ is a Fredholm operator of index } 0\}$

For $(\lambda,u) \in \mathbb{R} \times X$, $\|(\lambda,u)\| = |\lambda| + \|u\|$ and, for $\Omega \subset \mathbb{R} \times X$, $\Omega_\lambda = \{u \in X : (\lambda,u) \in \Omega\}$ and $p(\Omega) = \{\lambda \in \mathbb{R} : \Omega_\lambda \neq \emptyset\}$.

When U and V are subsets of the same Banach space $d(U,V) = \inf\{\|u-v\| : u \in U \text{ and } v \in V\}$ and if $U = \{u\}$ is a singleton, $d(u,V) = d(\{u\},V)$. The boundary of U is denoted by ∂U.

Consider now a Hilbert space $(H, (\cdot,\cdot), \|\cdot\|)$ and a self-adjoint operator $L : D(L) \subset H \to H$ acting in H. The space $D(L)$ equipped with its graph norm, $(\|u\|^2 + \|Lu\|^2)^{1/2}$, is a Hilbert space and $L \in B(D(L),H)$. The spectrum and essential spectrum of L are defined by

$\sigma(L) = \{\lambda \in \mathbb{R} : L - \lambda I \notin \mathrm{Iso}(D(L),H)\}$ and $\sigma_e(L) = \{\lambda \in \mathbb{R} : L - \lambda I \notin \Phi_0(D(L),H)\}$.

When $L \in B(H,H)$ is self-adjoint, $\sigma(L)$ is bounded and $r_e(L) = \max\{|\lambda| : \lambda \in \sigma_e(L)\}$ denotes the radius of its essential spectrum.

Proposition 4.1. *For two bounded self-adjoint operators A and B on a real Hilbert space H, $\inf \sigma_e(A+B) \geq \inf \sigma_e(A) + \inf \sigma_e(B)$.*

Proof. Without further mention it is understood that all the operators introduced in this proof are bounded and self-adjoint. Let $a = \inf \sigma_e(A)$ and $b = \inf \sigma_e(B)$. Choose any $\xi < a+b$ and set $\varepsilon = (a+b-\xi)/2$. Let $T = A - (a-\varepsilon)I$ and $S = B - (b-\varepsilon)I$.

Then $\inf \sigma_e(T) = \varepsilon > 0$ and, from the spectral theory of A, there exists $\eta > 0$ such that T can be written as $D + C$ where $(Du,u) \geq \eta\|u\|^2$ for all $u \in H$ and C has finite rank. (In the notation of Proposition 3.1 in [11], it suffices to take $\eta = \min\{\inf \sigma(T_+), -\sup \sigma(T_-)\}$, $C = 2TP_- - \eta P_0$ and $D = T - C$.) Similarly, $S = E + C_1$ where $(Eu,u) \geq \eta\|u\|^2$ for all $u \in H$ and C_1 has finite rank. For $u \in H$ this yields

$$([A + B - \xi I - C - C_1]u, u) = ([D + E + (a + b - 2\varepsilon - \xi)I]u, u) \geq 2\eta\|u\|^2.$$

Hence $\|[A + B - \xi I - C - C_1]u\| \geq 2\eta\|u\|$ from which it follows that the self-adjoint operator $A + B - \xi I - C - C_1 \in \mathrm{Iso}(H,H)$ and consequently that $A + B - \xi I \in \Phi_0(H,H)$ since $C + C_1$ is compact. This proves that $\inf \sigma_e(A+B) \geq a+b$. $\square$

As pointed out in Remark 2.2, for the simplest types of functions satisfying condition (F), the associated Nemytskii operator is not Fréchet differentiable. The results in Sections 4.2 and 4.3 deal with bifurcation in Hilbert space where differentiability at the trivial solution holds in some weaker sense. To avoid confusion with variants appearing elsewhere the relevant definitions are now recalled in the form used in this paper.

Consider a mapping $G : U \subset X \to Y$ where X and Y are real Banach spaces and U is an open subset of X.

Definition 4.2. The mapping G is said to be Gâteaux differentiable at $u \in U$ if there exists an operator $T \in B(X, Y)$ such that, for all $v \in X$,

$$\left\| \frac{G(u + tv) - G(u)}{t} - Tv \right\| \to 0 \quad \text{as } t \to 0 \text{ in } \mathbb{R}.$$

This notion is quite standard as are variants in which T is not required to be linear. (See [16].) The next definition is less well-known.

Definition 4.3. The mapping G is said to be w-Hadamard differentiable at $u \in U$ if there exists an operator $T \in B(X, Y)$ having the following property. For every $v \in X$,

$$\frac{G(u + t_n v_n) - G(u)}{t_n} \rightharpoonup Tv \text{ weakly in } Y \text{ as } n \to \infty \text{ for all sequences } \{t_n\} \subset \mathbb{R}\backslash\{0\} \text{ and}$$
$$\{v_n\} \subset X \text{ such that } t_n \to 0 \text{ and } v_n \rightharpoonup v \text{ weakly in } X \text{ as } n \to \infty.$$

It was named in this way in [11, 12] where it seems to have been used for the first time in discussing bifurcation, but variants can be found in [2, 21]. The terminology was chosen to reflect the analogy with the better known notion of Hadamard differentiability. (See [16].)

Definition 4.4. The mapping G is said to be Hadamard differentiable at $u \in U$ if there exists an operator $T \in B(X, Y)$ such that, for all $v \in X$,

$$\left\| \frac{G(u + t_n v_n) - G(u)}{t_n} - Tv \right\| \to 0 \text{ as } n \to \infty \text{ for all sequences } \{t_n\} \subset \mathbb{R}\backslash\{0\} \text{ and}$$
$$\{v_n\} \subset X \text{ such that } t_n \to 0 \text{ and } \|v_n - v\| \to 0 \text{ as } n \to \infty.$$

In all these definitions, the linear operator T is unique, if it exists. Furthermore, if G is differentiable at u in more than one sense, the operator T is the same in all cases and it will be denoted by $G'(u)$. If $F : \mathbb{R} \times X \to Y$ and $G = F(\lambda, \cdot)$, $G'(u)$ will be denoted by $D_u F(\lambda, u)$. It is easy to see that Hadamard differentiability at u implies Gâteaux differentiability at u. Also Fréchet differentiability at u implies differentiability in the sense of all three definitions but none of these notions implies Fréchet differentiability.

Example 4.5. Consider a function $f \in C^1(\mathbb{R}, \mathbb{R})$ such that $f(0) = 0$ and $K \equiv \sup\{|f'(s)| : s \in \mathbb{R}\} < \infty$. Since $|f(s)| \leq K|s|$ for all $s \in \mathbb{R}$, $f(u(\cdot)) \in L^2(0, 1)$ whenever $u \in L^2(0, 1)$ and the associated Nemytskii operator $\tilde{f} : L^2 \to L^2$ is uniformly Lipschitz continuous. In Example 2.3 of [11] and the subsequent remark it is shown that, for all $u \in L^2$, $\tilde{f} : L^2 \to L^2$ is Gâteaux differentiable, w-Hadamard differentiable and Hadamard differentiable at u. On the other hand, if there exists even one element $u \in L^2$ at which $\tilde{f} : L^2 \to L^2$ is Fréchet differentiable, then $f : \mathbb{R} \to \mathbb{R}$ must be linear.

Section 4.3 deals with bifurcation for a problem that is w-Hadamard at the trivial solution, whereas Gâteaux differentiability is assumed in Section 4.2. However in Section 4.2 the problem is also required to be Lipschitz continuous in an open neighbourhood of the trivial solution and this together with Gâteaux differentiability implies Hadamard differentiability at the trivial solution. In fact, the situation treated in Section 4.2 is based on previous work [29] relying heavily on Hadamard differentiability.

Both cases treated here concern bifurcation for an equation $F(\lambda, u) = 0$ at a point μ where $F : \mathbb{R} \times X \to Y$ and $L(\mu) \equiv D_u F(\mu, 0) \in \Phi_0(X, Y)$. In fact, X and Y are Hilbert spaces with $X \subset Y$ and $L(\mu) : X \subset Y \to Y$ is a self-adjoint operator acting in Y. In case 2, $\sigma_e(L(\mu)) \subset (0, \infty)$ but $F(\mu, \cdot) : X \to Y$ need not be Lipschitz continuous, whereas case 1 covers situations where μ may be in a gap in $\sigma_e(L(\mu))$, provided that $d(0, \sigma_e(L(\mu)))$ is sufficiently large relative to the Lipschitz modulus of $F(\mu, \cdot) - L(\mu)$.

4.2 Global bifurcation, case 1

Let $(Y, (\cdot, \cdot), \|\cdot\|)$ be a real Hilbert space and X a subspace of Y that is the domain of some self-adjoint operator acting in Y . Recall from Proposition 5.4 of [29] that the graph norms of all such operators on X are equivalent and let $\|\cdot\|_X$ denote one of these norms. Then $(X, \|\cdot\|_X)$ is a Hilbert space, $\|u\|_Y \leq \|u\|_X$ for all $u \in X$ and X is dense in Y. In this part we consider equations of the form

$$M(u) = \lambda u \quad \text{for } (\lambda, u) \in \mathbb{R} \times X, \tag{4.1}$$

where $M = M_1 + M_2 : X \to Y$ has the following properties.

(m1) $M_1 \in C^1(X, Y)$, $M_1(0) = 0$, $M_1'(0) : X \subset Y \to Y$ is a self-adjoint operator acting in Y and the remainder $R_1 \equiv M_1 - M_1'(0) : X \to Y$ is compact.

(m2) $M_2 : X \to Y$ is Gâteaux differentiable at 0 with $M_2'(0) = 0$ and $M_2(0) = 0$. Furthermore,

$$\ell \equiv \sup \left\{ \frac{\|M_2(u) - M_2(v)\|_Y}{\|u - v\|_Y} : u, v \in X \text{ and } u \neq v \right\} < \infty.$$

Remark 4.6. By (m2), M_2 could be extended to a uniformly Lipschitz continuous mapping of Y into itself. Since X is continuously embedded in Y, $M_2 : X \to Y$ is also uniformly Lipschitz continuous. It follows from these assumptions that $M = M_1 + M_2 : X \to Y$ is locally Lipschitz continuous on X and Gâteaux differentiable at 0 with $M'(0) = M_1'(0)$. In connection with the hypotheses for case 2, it should be noted that (m2) implies that $M_2 : X \to Y$ is Hadamard differentiable at 0 and that (m1) and (m2) imply that $M : X \to Y$ also has this property. However, condition (m2) does not imply that $M_2 : X \to Y$ is w-Hadamard differentiable at 0.

Let $d_\ell = \{\lambda \in \mathbb{R} : d(\lambda, \sigma_e(M'(0))) > \ell\}$ and, for $\mu \in d_\ell$, let $J_\mu(\ell)$ denote the maximal interval in d_ℓ containing μ.

Let $\mathcal{E} = \{(\lambda, u) \in \mathbb{R} \times X : M(u) = \lambda u \text{ and } u \neq 0\}$ denote the set of non-trivial solutions of (4.1) and let $\overline{\mathcal{E}}$ denote its closure in $\mathbb{R} \times X$. The assumptions (m1) and (m2) imply that $(\lambda, 0)$ is a solution of (4.1) for all $\lambda \in \mathbb{R}$ and $\overline{\mathcal{E}} \backslash \mathcal{E} \subset \mathbb{R} \times \{0\}$. A real number μ is a bifurcation point for equation (4.1) if and only if $(\mu, 0) \in \overline{\mathcal{E}}$.

Theorem 4.7. *Consider equation (4.1) under the assumptions (m1) and (m2). Suppose that $\mu \in d_\ell$ and let $U = J_\mu(\ell) \times X$.*

(1) If $\ker\{M'(0)-\mu I\}=\{0\}$*,* μ *is not a bifurcation point for equation (4.1).*

(2) If μ *is an eigenvalue of odd multiplicity of* $M'(0)$ *it is a bifurcation point for equation (4.1). The connected component* $\mathcal{D}_\mu$ *of* $\overline{\mathcal{E}}\cap U$ *containing* $(\mu,0)$ *has at least one of the following properties.*

(a) $\{|\lambda|+\|u\|_X : (\lambda,u)\in\mathcal{D}_\mu\}=[0,\infty)$.

(b) $d(p(\mathcal{D}_\mu),\sigma_e(M'(0))=\ell$.

(c) $\mathcal{D}_\mu\cap[J_\mu(\ell)\backslash\{\mu\}]\times\{0\}\neq\emptyset$.

(3) If $\ker\{M'(0)-\mu I\}=\operatorname{span}\{\phi\}$ *where* $\|\phi\|_Y=1$ *and* $\{(\lambda_n,u_n)\}\subset\mathcal{E}$ *is such that* $\lambda_n\to\mu$ *and* $\|u_n\|_X\to 0$*, then there exists* $n_0\in\mathbb{N}$ *such that for all* $n\geq n_0$*,* $u_n=(u_n,\phi)\{\phi+w_n\}$ *where* $(w_n,\phi)=0$ *and* $\|w_n\|_X\to 0$ *as* $n\to\infty$*.*

Remark 4.8. The hypotheses of Theorem 4.7 are similar to those of Corollary 6.11 in [29]. Apart from the fact that they hold on all of X instead of a ball centred at the origin, the compactness of R_1 is added. Parts (1) and (3) are already established in Corollary 6.11 of [29] but part (2) provides new global information. If conditions (m1) and (m2) are satisfied and $M_2 : X\to Y$ is Fréchet differentiable at 0, part (2) of Theorem 4.7 could be deduced from Theorem 1.1 in [27] which was itself based on Theorem 1.6 in [26]. Those results used Nussbaum's degree [22] for k-set contractions and they were applied to a a class of Sturm–Liouville problems on the interval $(0,\infty)$ in [26,27]. If, in addition, $M_2 : X\to Y$ is continuously differentiable on an open neighbourhood of 0, the conclusion in part (3) can be strengthened using the standard result about bifurcation at a simple eigenvalue [8].

Proof. (1) This follows from part (i) of Corollary 6.11 in [29].

(2) It follows from part (ii) of Corollary 6.11 in [29] that μ is a bifurcation point. We suppose now that $\mathcal{D}_\mu$ does not have properties (a) and (b) and use Theorem 3.4 in [34] to show that it must satisfy (c). From the assumption that $\mathcal{D}_\mu$ is bounded it follows that $I_\mu\equiv[\inf p(\mathcal{D}_\mu),\sup p(\mathcal{D}_\mu)]$ is a compact interval and then $d(p(\mathcal{D}_\mu),\sigma_e(M'(0))\neq\ell$ means that there exists $k>\ell$ such that $I_\mu\subset J_\mu(k)$. Hence $d(\mathcal{D}_\mu,\partial(J_\mu(k)\times X))>0$.

The hypotheses of Theorem 3.4 in [34] involve the essential conditioning number, $\gamma(M'(0)-\lambda I)$. By Corollary 5.6 in [29], for all $\lambda\notin\sigma_e(M'(0))$ and all $\varepsilon>0$,

$$\gamma(M'(0)-\lambda I)\leq\frac{1}{d(\lambda,\sigma_e(M'(0)))}+\varepsilon K_\lambda,$$

provided that the graph norm of $\varepsilon M'(0)$ is used on X and

$$K_\lambda=\max\left\{1,\frac{|p|}{\lambda-p},\frac{|q|}{q-\lambda}\right\},$$

where (p,q) is the maximal interval in $\mathbb{R}\backslash\sigma_e(M'(0))$ containing λ. If either $p=-\infty$ or $q=\infty$, the corresponding ratio is replaced by 1. We require this estimate for $\lambda\in J_\mu(k)$ and for λ in this interval it is easy to check that $K_\lambda\leq K\equiv\max\left\{1,\frac{|p|}{k},\frac{|q|}{k}\right\}$, with the same convention concerning the cases $p=-\infty$ and $q=\infty$. Thus, for $\lambda\in J_\mu(k)$,

$$\gamma(M'(0)-\lambda I)\leq\frac{1}{k}+\varepsilon K.$$

For the rest of this proof, choose and fix $\varepsilon>0$ such that $\varepsilon K<\frac{1}{\ell}-\frac{1}{k}$ and let $\|\cdot\|_X$ denote the graph norm of $\varepsilon M'(0)$. (If $\ell=0$, $M_2\equiv 0$ and any $\varepsilon>0$ is acceptable.) We now have

$$d(\lambda,\sigma_e(M'(0)))>k\quad\text{and}\quad\gamma(M_1'(0)-\lambda I)=\gamma(M'(0)-\lambda I)<\frac{1}{\ell}\quad\text{for all }\lambda\in J_\mu(k).\tag{4.2}$$

Setting $F(\lambda,u) = M(u) - \lambda u$, we aim to show that the hypotheses of Theorem 3.4 in [34] are satisfied with

$$U = J_\mu(\ell) \times X, \Omega = J_\mu(k) \times X, G(\lambda,u) = M_1(u) - \lambda u \quad \text{and} \quad K(\lambda,u) = M_2(u).$$

Using the notation of [34], let $\mathcal{S} = \{(\lambda,u) \in U : F(\lambda,u) = 0 \text{ and } u \neq 0\}$. Then $\mathcal{S} = \mathcal{E} \cap U$ and it is easy to check that $\overline{\mathcal{E}} \cap U$ coincides with the closure of $\mathcal{S}$ in U. Hence $\mathcal{D}_\mu = \mathcal{C}_\mu(U,F)$ in the notation of the Introduction in [34].

Clearly condition (D0) in [34] is satisfied with $J(\Omega) = J_\mu(k)$. Furthermore, $G \in C^1(\Omega,Y)$ and $D_u G(\lambda,u) = M_1'(u) - \lambda I = R_1'(u) + M_1'(0) - \lambda I$. Since $R_1 \in C^1(X,Y)$ and $R_1 : X \to Y$ is compact, it follows from Proposition 8.2 in [9] that $R_1'(u) : X \to Y$ is compact for all $u \in X$. Hence $D_u G(\lambda,u) \in \Phi_0(X,Y)$ if and only if $M_1'(0) - \lambda I \in \Phi_0(X,Y)$. But $J_\mu(k) \cap \sigma_e(M_1'(0)) = \emptyset$ so $M_1'(0) - \lambda I \in \Phi_0(X,Y)$ for all $\lambda \in J_\mu(k)$ and hence condition (D1) in [34] is satisfied. It is an immediate consequence of (m2) that K satisfies condition (D2) with $D_u K(\lambda,0) = 0$ and furthermore

$$\|K(\lambda,u) - K(\lambda,v)\| \leq \ell \|u - v\|_X \quad \text{for all } u,v \in X.$$

In the notation of [34] for the measure of non-compactness, $\alpha(K(\lambda,\cdot),V) \leq \ell$ for every bounded subset V of X for which $\alpha(V)$ is positive. On the other hand, by the compactness of $R_1 : X \to Y$, (4.2) and Proposition 2.1(iv) in [34], for all $\lambda \in J_\mu(k)$,

$$\begin{aligned} \omega(M_1 - \lambda I, V) &= \omega(R_1 + M_1'(0) - \lambda I, V) \geq \omega(M_1'(0) - \lambda I, V) - \alpha(R_1, V) && (4.3)\\ &= \omega(M_1'(0) - \lambda I, V) \geq 1/\gamma(M_1'(0) - \lambda I) > \ell \geq \alpha(K(\lambda,\cdot),V), && (4.4) \end{aligned}$$

which shows that condition (D3) in [34] is also satisfied.

Setting $L(\lambda) = D_u F(\lambda,0)$ and $\rho(\lambda,u) = K(\lambda,u) - D_u K(\lambda,0)$ as in [34], we have $L(\lambda) = M'(0) - \lambda I \in \Phi_0(X,Y)$, $L_X(\rho,\lambda) \leq \ell$ and $\Delta_r(\rho,\lambda) = 0$ for all $\lambda \in J_\mu(k)$ and $r > 0$. It follows from (4.2) and (4.4) that the conditions (3.15) and (3.16) in [34] are satisfied. Finally, using Criterion I in Section 5.2 of [29], the local parity, $\sigma(L,\mu)$ of the path L at the isolated singular point μ is -1 since $M'(0)$ is self-adjoint and μ has odd multiplicity. At this point, it follows from Theorem 3.4 in [34] that $\mathcal{D}_\mu$ has at least one of the following properties.

(i) $\{|\lambda| + \|u\|_X : (\lambda,u) \in \mathcal{D}_\mu\} = [0,\infty)$.

(ii) $d(\mathcal{D}_\mu, \partial\Omega) = 0$.

(iii) $\mathcal{D}_\mu \cap [\mathbb{R}\backslash\{\mu\}] \times \{0\} \neq \emptyset$.

Recall that we are assuming that $\mathcal{D}_\mu$ does not have the properties (a) and (b) and that Ω has been chosen so that $d(\mathcal{D}_\mu, \partial\Omega) > 0$. Hence $\mathcal{D}_\mu$ must have property (iii) and this implies property (c) since $\mathcal{D}_\mu \subset J_\mu(\ell) \times X$ by definition.

(3) This follows from part (iii) of Corollary 6.11 in [29].

□

4.3 Global bifurcation, case 2

In this part we deal with an equation of the form

$$M(u) = \lambda T(u) \quad \text{for } (\lambda,u) \in \mathbb{R} \times H, \tag{4.5}$$

where $(H, (\cdot,\cdot), \|\cdot\|)$ is a real Hilbert space. The mappings $M = M_1 + M_2$ and T have the following properties.

(W0) $T \in B(H,H)$ is a self-adjoint operator and $(Tu, u) > 0$ for $u \in H \backslash \{0\}$.

(W1) $M_1 \in C^1(H,H)$ with $M_1(0) = 0$ and $M_1'(0)$ is self-adjoint. Furthermore, the remainder $R_1 = M_1 - M_1'(0) : H \to H$ is a compact operator.

(W2) $M_2 \in C(H,H)$ with $M_2(0) = 0$. The mapping $M_2 : H \to H$ is compact and w-Hadamard differentiable at 0 with $M_2'(0)$ self-adjoint. Furthermore,

$$\liminf_{\|u\|\to 0} \frac{(R_2(u), u)}{\|u\|^2} \geq 0, \quad \text{where } R_2 = M_2 - M_2'(0).$$

Remark 4.9. The properties in (W2) do not imply that $M_2'(0) : H \to H$ is a compact linear operator. Since $M_2(0) = 0$ it follows from the w-Hadamard differentiability of M_2 at 0 that

$$\lim_{t\to 0} \frac{(R_2(tu), tu)}{\|tu\|^2} = 0 \quad \text{for all } u \in H\backslash\{0\}$$

and so (W2) implies that

$$\liminf_{\|u\|\to 0} \frac{(R_2(u), u)}{\|u\|^2} = 0.$$

By (W1),

$$\lim_{\|u\|\to 0} \frac{(R_1(u), u)}{\|u\|^2} = 0$$

since $\|R_1(u)\|/\|u\| \to 0$ as $\|u\| \to 0$ and so, when (W1) is satisfied, the assumption about the lim inf in (W2) is equivalent to

$$\liminf_{\|u\|\to 0} \frac{(M(u) - M'(0)u, u)}{\|u\|^2} \geq 0. \tag{4.6}$$

Let $\mathcal{E} = \{(\lambda, u) \in \mathbb{R} \times H : M(u) = \lambda T(u) \text{ and } u \neq 0\}$ denote the set of non-trivial solutions of (4.5) and let $\overline{\mathcal{E}}$ denote its closure in $\mathbb{R} \times H$. As in case 1, μ is a bifurcation point for (4.5) if and only if $(\mu, 0) \in \overline{\mathcal{E}}$.

Theorem 4.10. *Under the hypotheses (W0) to (W2), let J be an open interval such that $\inf \sigma_e(M_1'(0) - \lambda T) > r_e(M_2'(0))$ for all $\lambda \in J$. Then $\inf \sigma_e(M'(0) - \lambda T) > 0$ for $\lambda \in J$.*

Consider a point $\mu \in J$ and let $U = J \times H$.

(1) If $\ker\{M'(0) - \mu T\} = \{0\}$, μ is not a bifurcation point for equation (4.5).

(2) If $\dim \ker\{M'(0) - \mu T\}$ is odd, μ is a bifurcation point for the equation (4.5). In fact, the connected component $\mathcal{D}_\mu$ of $\overline{\mathcal{E}} \cap U$ containing $(\mu, 0)$ has at least one of the following properties.

(a) $\{|\lambda| + \|u\|_X : (\lambda, u) \in \mathcal{D}_\mu\} = [0, \infty)$.

(b) $d(p(\mathcal{D}_\mu), \partial J) = 0$.

(c) $\mathcal{D}_\mu \cap [J\backslash\{\mu\}] \times \{0\} \neq \emptyset$.

(3) Suppose that $\ker\{M'(0) - \mu T\} = \operatorname{span}\{\phi\}$ where $\|\phi\| = 1$ and that $\{(\lambda_n, u_n)\} \subset \mathcal{E}$ is such that $\lambda_n \to \mu$ and $\|u_n\| \to 0$ as $n \to \infty$. Let $v_n = u_n/\|u_n\|$. Then there exist a subsequence and $c \in \mathbb{R}\backslash\{0\}$ for which $v_{n_k} \rightharpoonup c\phi$ weakly in H as $n_k \to \infty$.

Remark 4.11. By Proposition 4.1, for all $\lambda \in J$

$$\begin{aligned}\inf \sigma_e(M'(0) - \lambda T) &\geq \inf \sigma_e(M_1'(0) - \lambda T) + \inf \sigma_e(M_2'(0)) \\ &\geq \inf \sigma_e(M_1'(0 - \lambda T)) - r_e(M_2'(0)) > 0.\end{aligned}$$

In particular, $M'(0) - \lambda T$ and $M_1'(0) - \lambda T \in \Phi_0(H, H)$ for all $\lambda \in J$.

Proof. Parts (1) and (2) will be deduced from Lemma 3.3 and Theorem 3.5 in [34]. With this in mind, let $F(\lambda, u) = M(u) - \lambda Tu$ for $(\lambda, u) \in U$. Then $\mathcal{S} \equiv \{(\lambda, u) \in U : F(\lambda, u) = 0 \text{ and } u \neq 0\} = \mathcal{E} \cap U$ and it is easy to check that $\overline{\mathcal{E}} \cap U$ coincides with the closure of $\mathcal{S}$ in U. Hence in the notation of the Introduction in [34], $\mathcal{D}_\mu = \mathcal{C}_\mu(U, F)$. Setting

$$\Omega = U = J \times H, G(\lambda, u) = M_1(u) - \lambda Tu \quad \text{and} \quad K(\lambda, u) = M_2(u),$$

we consider first the hypotheses of Lemma 3.3 in [34].

By (W0) to (W2), $F \in C(U, H)$ and $F(\lambda, \cdot) : H \to H$ is w-Hadamard differentiable at 0 with $L(\lambda) \equiv D_u F(\lambda, 0) = M'(0) - \lambda T$ for all $\lambda \in J$. The remainder $R(\lambda, u) = F(\lambda, u) - D_u F(\lambda, 0)u = R_1(u) + R_2(u)$ is independent of λ so the quantity $\Delta_r(F, \lambda) \to 0$ as $r \to 0$ for all $\lambda \in J$. By (W1), (W2) and (4.6),

$$\liminf_{\|u\| \to 0} \frac{(R(\lambda, u), u)}{\|u\|^2} = \liminf_{\|u\| \to 0} \frac{(R_2(u), u)}{\|u\|^2} \geq 0.$$

As noted in Remark 4.11 , $\inf \sigma_e(L(\lambda)) > 0$ and since by Remark 3.2 in [34], $w_l(L(\lambda)) = \inf \sigma_e(L(\lambda))$, it follows that condition (3.14)(a) in [34] is satisfied at $\lambda \in J$ whenever $\ker\{M'(0) - \lambda T\} = \{0\}$. The conclusion in part (1) is now justified by Lemma 3.3(ii) in [34].

For part (2) we use Theorem 3.5 in [34], noting first of all that (D0) is satisfied and that by (W1), $G \in C^1(\Omega, H)$ with $D_u G(\lambda, u) = M_1'(u) - \lambda T = R_1'(u) + M_1'(0) - \lambda T$. By Proposition 8.2 in [9], (W1) also implies that, for all $u \in H$, $R_1'(u) \in B(H, H)$ is compact and so $D_u G(\lambda, u) \in \Phi_0(H, H)$ for all $\lambda \in J$ by Remark 4.11. This proves that condition (d1) in [34] is satisfied and condition (d2) is an immediate consequence of hypothesis (W2). For (d3), consider a bounded subset V of H for which the set-measure of non-compactness, $\alpha(V)$, is positive. Then in the notation of [34], for all $\lambda \in J$,

$$\begin{aligned}\omega(G(\lambda, \cdot), V) &= \omega(R_1 + M_1'(0) - \lambda T, V) \geq \omega(M_1'(0) - \lambda T, V) - \alpha(R_1, V) \\ &= \omega(M_1'(0) - \lambda T) \geq \inf \sigma_e(M_1'(0) - \lambda T) > 0,\end{aligned}$$

by the compactness of R_1 and Remark 2.1 in [34]. Since $\alpha(M_2, V) = 0$ by the compactness of M_2, this shows that condition (d3) in [34] is satisfied. Furthermore, referring again to Remark 2.1 in [34], for $\lambda \in J$,

$$\alpha(D_u K(\lambda, 0)) = \alpha(M_2'(0)) \leq r_e(M_2'(0)) < \inf \sigma_e(M_1'(0 - \lambda T)) \leq \omega(D_u G(\lambda, 0)).$$

Also $\alpha_0(K(\lambda, \cdot)) = 0$ by the compactness of M_2. Hence condition (3.16) in [34] is satisfied because $\rho(\lambda, u) \equiv K(\lambda, u) - D_u K(\lambda, 0)u$ does not depend upon λ.

We have already noted in Remark 4.11 that $L(\lambda) = M'(0) - \lambda T \in \Phi_0(H, H)$ for all $\lambda \in J$. If $u \in \ker L(\lambda)$ and $L'(\lambda)u = -Tu \in \text{range } L(\lambda) = [\ker L(\lambda)]^\perp$, it follows that $(Tu, u) = 0$ and hence $u = 0$ by (W0). Using Criterion I in [29] for the calculation of the local parity, $\sigma(L, \lambda)$, of the path L across λ we find that $\sigma(L, \lambda) = (-1)^n$ where $n = \dim \ker L(\lambda)$. By Remark 3.2 in

[34] and Remark 4.11, $w_l(L(\lambda)) = \inf \sigma_e(M'(0) - \lambda T) > 0$ for all $\lambda \in J$ and so, as in the proof of part (1), (W1), (W2) and (4.6) imply that condition (3.18)(a) in [34] is satisfied.

The conclusion in part (2) now follows from Theorem 3.5 in [34].

(3) For the sequence $\{(\lambda_n, u_n)\}$ in the statement let $t_n = \|u_n\|$. Then $t_n \to 0$ and $u_n = t_n v_n$. Passing to a subsequence, we suppose henceforth that $v_n \rightharpoonup v$ weakly in H. Since $M(0) = 0$ and M is w-Hadamard differentiable at zero it follows that $M(u_n)/\|u_n\| = M(t_n v_n)/t_n \rightharpoonup M'(0)v$ weakly in H as $n \to \infty$. Hence

$$L(\mu)v_n = \{M'(0) - \mu T\}v_n = M'(0)v_n - \frac{M(u_n)}{\|u_n\|} + (\lambda_n - \mu)Tv_n \rightharpoonup 0 \quad \text{weakly in } H \tag{4.7}$$

since $M(u_n) = \lambda_n T u_n$, $\lambda_n \to \mu$ and $M'(0)v_n \rightharpoonup M'(0)v$ weakly in H as $n \to \infty$. This implies that $L(\mu)v = 0$ and so $v = c\phi$ for some $c \in \mathbb{R}$.

If $c = 0$, $v_n \rightharpoonup 0$ weakly in H and so, in the notation of Section 3 of [34], $\{v_n\} \subset \Sigma$ from which it follows that

$$\liminf_{n\to\infty} (L(\mu)v_n, v_n) \geq w_l(L(\mu)) = \inf \sigma_e(L(\mu))$$

by Remark 3.2 in [34], where $\inf \sigma_e(L(\mu)) > 0$ by Remark 4.11. On the other hand from (4.7) and (4.6) we obtain

$$\begin{aligned}\liminf_{n\to\infty} (L(\mu)v_n, v_n) &= \liminf_{n\to\infty} \left\{\frac{(M'(0)u_n - M(u_n), u_n)}{\|u_n\|^2} + (\lambda_n - \mu)(Tv_n, v_n)\right\} \\ &= \liminf_{n\to\infty} \frac{(M'(0)u_n - M(u_n), u_n)}{\|u_n\|^2} \leq 0,\end{aligned}$$

contradicting the earlier conclusion. Hence $c \neq 0$. □

5 Global bifurcation for the boundary value problem

Under the assumption (S) formulated in Section 2.3, Theorems 4.7 and 4.10 will be used to obtain conclusions about the bifurcation of solutions for problem (1.1)(1.2) in the sense of Definition 2.7. The first result is based upon Theorem 4.7 and it deals with that happens for λ in the interval $(-\infty, m_e - \ell_{g_1})$ where $m_e = \inf \sigma_e(S_A + V) = \frac{a}{4} + V_0$ and ℓ_{g_1} is the best Lipschitz for the part g_1 of g which satisfies condition (F). It follows from Theorem 4.7 that there is global bifurcation at every eigenvalue of $S_A + V$ in the interval $(-\infty, m_e - \ell_{g_1})$. Corollary 5.3 deals with a special case, where $n \equiv 0$ and $g(x,s)s \leq 0$ for $(x,s) \in (0,1) \times \mathbb{R}$, in which it can be shown that there may be no bifurcation at eigenvalues of $S_A + V$ lying in the interval $(m_e - \ell_{g_1}, \infty)$. The situation where $g(x,s)s \geq 0$ for $(x,s) \in (0,1) \times \mathbb{R}$ is quite different and bifurcation at all eigenvalues of $S_A + V$ in the interval $(-\infty, m_e)$ can be proved using Theorem 4.10. This case is treated in Section 5.1.

Throughout this section $\mathcal{E}$ denotes the set of all non-trivial solutions of (1.1)(1.2) in $\mathbb{R} \times D_A$ as defined in Section 2.3 and D_A is considered with a norm that is equivalent to the graph norm, $\|\cdot\|_S$, of $S = S_A + V$. Of course, the conclusions do not depend upon the choice of norm. It is often convenient to use the norm defined by $\|S_A u\|_{L^2}$ but the proof of Theorem 5.5 is based on a different choice. The nodal properties of solutions established in Section 3 are used to show that possibility (c) in Theorems 4.7 and 4.10 does not occur.

Theorem 5.1. *Let the assumption (S) be satisfied and consider $\mu \in (-\infty, m_e - \ell_{g_1})$.*

(A) *If μ is a bifurcation point for problem (1.1)(1.2), then μ is an eigenvalue of the self-adjoint operator $S = S_A + V$.*

(B) *If μ is an eigenvalue of S then μ is a bifurcation point for (1.1)(1.2) and the component $\mathcal{C}_\mu$ of $\overline{\mathcal{E}} \cap (-\infty, m_e - \ell_{g_1}) \times D_A$ containing $(\mu, 0)$ has at least one of the following properties.*

(i) $\{|\lambda| + \|u\|_S : (\lambda, u) \in \mathcal{C}_\mu\} = [0, \infty)$.

(ii) $\sup\{\lambda : (\lambda, u) \in \mathcal{C}_\mu\} = m_e - \ell_{g_1}$.

(C) *If μ is the $k-th$ eigenvalue of S, then $\sharp(u) = k$ for all $(\lambda, u) \in \mathcal{C}_\mu \cap \mathcal{E}$, where $\sharp(u)$ denotes the number of zeros of u in $(0, 1]$ and $\mathcal{C}_\mu \cap \mathbb{R} \times \{0\} = \{(\mu, 0)\}$.*

Remark 5.2. If assumption (S) is satisfied and $n \equiv 0$, then for $(\lambda, u) \in \mathcal{E}$,

$$\|S_A u\|_{L^2} \leq (|\lambda| + \|V\|_{L^\infty} + \ell_{g_1})\|u\|_{L^2} + C\mathfrak{C}_{g_2}(\|u\|_A)\|u\|_A$$

by Remark 2.4. In this case property (i) in the conclusion can be replaced by $\{|\lambda| + \|u\|_A : (\lambda, u) \in \mathcal{C}_\mu\} = [0, \infty)$ and if in addition, $g_2 \equiv 0$, it can be replaced by $\{|\lambda| + \|u\|_{L^2} : (\lambda, u) \in \mathcal{C}_\mu\} = [0, \infty)$.

Proof. The first step in this proof is to observe that the hypotheses of Theorem 4.7 are satisfied for the equation $F(\lambda, u) = 0$ where F is defined by (2.12). For this we take $Y = L^2$ and $X = D_A$ equipped with the norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_S$, respectively, and set

$$M_1(u) = Su + N(u) + \widetilde{g_2}(u) \quad \text{and} \quad M_2(u) = \widetilde{g_1}(u) \quad \text{for } u \in X.$$

From assumption (S) and Propositions 2.1, 2.3 and 2.5 it follows that the conditions (m1)and (m2) are satisfied with $M'(0) = M_1'(0) = S$ and $R_1 = N + \widetilde{g_2}$. In the notation of Theorem 4.7, $J_\mu(\ell) = (-\infty, m_e - \ell_{g_1})$. From Theorem 4.7, we obtain immediately part (A) and that, if $\mathcal{C}_\mu$ has neither property (i) nor (ii), then there exists an eigenvalue ξ of S in $(-\infty, m_e - \ell_{g_1})\backslash\{\mu\}$ such that $(\xi, 0) \in \mathcal{C}_\mu$ and hence $\mathcal{C}_\xi = \mathcal{C}_\mu$. To show that this third situation does not occur it suffices to prove part (C).

(C) Since $\mathcal{C}_\mu \subset (-\infty, m_e - \ell_{g_1}) \times D_A$, it follows from Lemma 3.1(i) that u has only a finite number of zeros in $(0, 1]$ if $(\lambda, u) \in \mathcal{C}_\mu \cap \mathcal{E}$. Setting $Z(\lambda, u) = \sharp(u)$ for $(\lambda, u) \in \mathcal{C}_\mu \cap \mathcal{E}$, Corollary 3.3 shows that $Z : \mathcal{C}_\mu \cap \mathcal{E} \to \mathbb{N}$ is continuous. Consider now a point $(\xi, 0) \in \mathcal{C}_\mu$. It follows from Lemma 3.1(i) that there exist an open ball B in $\mathbb{R} \times D_A$, centred at $(\xi, 0)$, and $\eta \in (0, 1)$ such that $u(x) \neq 0$ for $0 < x \leq \eta$ if $(\lambda, u) \in B \cap \mathcal{E}$. By part (A), ξ is an eigenvalue of S and an associated eigenfunction ϕ_ξ with $\|\phi_\xi\|_{L^2} = 1$ has a finite number of zeros $\sharp(\phi_\xi)$ in $(0, 1]$ by property (S4) in Section 2.4. Hence η can be chosen so that $\phi_\xi(x) \neq 0$ for $0 < x \leq \eta$. Suppose that there is a sequence $\{(\lambda_n, u_n)\} \subset B \cap \mathcal{E}$ such that $\lambda_n \to \xi$ and $\|u_n\|_S \to 0$ as $n \to \infty$ and, for all $n \in \mathbb{N}$, $\sharp(u_n) \neq \sharp(\phi_\xi)$. By part (3) of Theorem 4.7 we can suppose that $u_n = (u_n, \phi_\xi)\{\phi_\xi + w_n\}$ where $\|w_n\|_S \to 0$ as $n \to \infty$ and, for all n, $(u_n, \phi_\xi) \neq 0$ since $(\lambda_n, u_n) \in \mathcal{E}$. In the notation of Lemma 3.2, $\|P_\eta w_n\|_\eta \to 0$ as $n \to \infty$ and it follows from Lemma 3.2 that there exists n_0 such that $\sharp(\phi_\xi + w_n) = \sharp(\phi_\xi)$ for all $n \geq n_0$, since $\phi_\xi + w_n$ like u_n has no zeros in the interval $(0, \eta]$ because $(\lambda_n, u_n) \in B \cap \mathcal{E}$ and $(u_n, \phi_\xi) \neq 0$. But this implies that $\sharp(u_n) = \sharp(\phi_\xi)$ for all $n \geq n_0$, contradicting the choice of the sequence $\{(\lambda_n, u_n)\}$. Hence there exists an open neighbourhood U_ξ of $(\xi, 0)$ in $\mathbb{R} \times D_A$ such that $Z(\lambda, u) = \sharp(\phi_\xi)$ for all $(\lambda, u) \in U_\xi \cap \mathcal{E}$. Setting $Z(\xi, 0) = \sharp(\phi_\xi)$ for all $(\xi, 0) \in \mathcal{C}_\mu$ we have now proved that $Z : \mathcal{C}_\mu \to \mathbb{N}$ is continuous and hence constant by the connectedness of $\mathcal{C}_\mu$. Since μ is a bifurcation point, it follows that $Z(\lambda, u) = \sharp(\phi_\mu)$ for all $(\lambda, u) \in \mathcal{C}_\mu$. This establishes part (C). $\square$

The following special case sheds some light on the restriction to the interval $(-\infty, m_e - \ell_{g_1})$ in Theorem 5.1. It uses the conditions (T1) and (T2) introduced in Section 3.2 and the quantities defined in (3.8) to (3.11).

Corollary 5.3. *Suppose that conditions (S), (T1) and (T2) are satisfied with $n \equiv 0$ and $g(x,s)s \leq 0$ for all $(x,s) \in (0,1) \times \mathbb{R}$. Let $\Theta \equiv \max\{V_0, m_e + I_s(g_1)\}$. Then $-\ell_{g_1} \leq I_s(g_1) \leq J_s(g_1) = 0$ and $m_e - \ell_{g_1} \leq \Theta \leq m_e$.*

(A) A point $\mu \in (-\infty, m_e - \ell_{g_1})$ is a bifurcation point for problem (1.1)(1.2) if and only if it is an eigenvalues of S. When it is an eigenvalue, the component $\mathcal{C}_\mu$ of $(-\infty, m_e - \ell_{g_1}) \times D_A$ containing $(\mu, 0)$ is a subset of $(-\infty, \mu] \times D_A$ and $\{|\lambda| + \|u\|_A : (\lambda, u) \in \mathcal{C}_\mu\} = [0, \infty)$. If μ is the k-th eigenvalue of S, $\sharp(u) = k$ for all $(\lambda, u) \in \mathcal{C}_\mu \cap \mathcal{E}$.

(B) There are no bifurcation points for (1.1)(1.2) in the interval (Θ, ∞) since $\mathcal{E} \cap (\Theta, \infty) \times D_A = \emptyset$.

Remark 5.4. If $I_s(g_1) = I_i(g_1) = -\ell_{g_1}$ and $\ell_{g_1} \leq \frac{a}{4}$, then $\Theta = m_e - \ell_{g_1}$.

As an example, suppose that $g_1(x,s) = -r(x)k(s)$ for $(x,s) \in (0,1) \times \mathbb{R}$ where the functions r and k satisfy the following conditions.

(R) $r \in C^1([0,1])$ with $r'(x) \leq 0$ for $0 \leq x \leq 1$, $r(0) > 0$ and $r(1) \geq 0$.

(K) $k \in C^1(\mathbb{R})$ is odd, convex on $[0, \infty)$ and $k'(0) = 0 < k'(\infty) \equiv \lim_{s \to \infty} k'(s) < \infty$.

Then $I_s(g_1) = I_i(g_1) = -\ell_{g_1} = -r(0)k'(\infty)$ and $\Theta = m_e - r(0)k'(\infty)$ if $r(0)k'(\infty) \leq \frac{a}{4}$.

The assumptions (R) and (K) also imply that the function $g_1(x,s) = -r(x)k(s)$ satisfies condition (3.17). Hence, taking $g = g_1$ and S to be as in Example 2.8 or 2.10 we obtain situations where all the hypotheses of Corollary 5.3 are satisfied and $\sigma(S) = \{\lambda_i : 1 \leq i \leq n\} \cup [\frac{1}{4}, \infty)$ where $\lambda_1 > 0$ and $\lambda_i < \lambda_{i+1} < \frac{1}{4} = m_e$ for $1 \leq i \leq n-1$. The quantity Θ is now $\frac{1}{4} - r(0)k'(\infty)$ and it can be placed anywhere in the interval $(0, \frac{1}{4})$ by adjusting $r(0)k'(\infty)$. When $\Theta \notin \{\lambda_i : 1 \leq i \leq n\}$, λ_i is a bifurcation point if and only if $\lambda_i < \Theta$.

Proof. (A) By Theorem 5.1 it suffices to show that $\lambda \leq \mu$ for all $(\lambda, u) \in \mathcal{C}_\mu$. This will be done using the standard comparison principle for the eigenvalues of self-adjoint operators. (See Theorems 1.2 and 1.3 in Chapter XI of [10], for example.) Let $(\lambda, u) \in \mathcal{C}_\mu \cap \mathcal{E}$ and set $W = W_1 + W_2$ where

$$W_i(x) = \frac{g_i(x, u(x))}{u(x)} \quad \text{if} \quad u(x) \neq 0 \quad \text{and} \quad W_i(x) = 0 \quad \text{if} \quad u(x) = 0 \quad \text{for } i = 1, 2.$$

By assumption (F) for g_1 and (2.9) for g_2, $W_i \in L^\infty(0,1)$ for $i = 1$ and 2 and hence $S + W_1$, $S + W_2$ and $S + W : D_A \subset L^2 \to L^2$ are all self-adjoint operators. By (2.9) and Lemma 2.7 in [31], multiplication by W_2 defines a compact mapping from D_A into L^2 and so $\sigma_e(S + W) = \sigma_e(S + W_1)$. But $W_1(x) \geq -\ell_{g_1}$ on $(0,1)$ so $\inf \sigma_e(S + W_1) \geq \inf \sigma_e(S) - \ell_{g_1} = m_e - \ell_{g_1}$ showing that $\inf \sigma_e(S + W) \geq m_e - \ell_{g_1}$. Also $\mu < m_e - \ell_{g_1}$ is the k-th eigenvalue of S and so it follows that $\lambda_k \leq \mu$ where λ_k is the k-th eigenvalue of $S + W$ since $W(x) \leq 0$ on $(0,1)$. But $Su + Wu = \lambda u$ and $u \not\equiv 0$ since $(\lambda, u) \in \mathcal{E}$ and so $\lambda < m_e$ is an eigenvalue of $S + W$ with u as an eigenfunction. We claim that $\lambda = \lambda_k$ since u has exactly k zeros in $(0,1]$. This is a standard property of regular Strum–Liouville problems and it continues to hold in the present singular situation. A proof is given in Appendix A of [36] for the case $V = W = 0$ but it can be extended to the general case $V + W \in L^\infty(0,1)$ with only notational changes. This being so the proof of part (A) is now complete since $\lambda = \lambda_k \leq \mu$.

(B) From Corollary 3.9 and part (iv) of Proposition 3.5, $\mathcal{E} \cap (\Theta, \infty) \times D_A = \emptyset$. $\square$

5.1 The case where $n \equiv 0$ and $g(x,s)s \geq 0$

When $n \equiv 0$ and $g(x,s)s \geq 0$, Theorem 4.10 can be used to deal with problem (1.1)(1.2) instead of Theorem 4.7. This has the advantage that the size of the Lipschitz constant for g_1 no longer plays a role and so the restriction to the interval $(-\infty, m_e - \ell_{g_1})$ in Theorem 5.1 can be avoided.

Theorem 5.5. *Suppose that assumption (S) is satisfied with $n \equiv 0$ and that the function $g = g_1 + g_2$ has the following additional properties.*

(a) $g(x,s)s \geq 0$ for all $(x,s) \in (0,1) \times \mathbb{R}$.

(b) For some $\alpha_{g_1} \geq 0$ and all $\delta > 0$ there exist $x(\delta) \in (0,1)$ and $M(\delta)$ such that $|g_1(x,s) - \alpha_{g_1}s| \leq M(\delta) + \delta|s|$ for $(x,s) \in (0, x(\delta)) \times \mathbb{R}$.

Consider $\mu \in (-\infty, m_e)$.

(A) If μ is a bifurcation point for problem (1.1)(1.2) then μ is an eigenvalue of $S = S_A + V$.

(B) If μ is the k-th eigenvalue of S then μ is a bifurcation point for (1.1)(1.2) and the component $\mathcal{C}_\mu$ of $\overline{\mathcal{E}} \cap (-\infty, m_e) \times D_A$ containing $(\mu, 0)$ is a subset of $[\mu, m_e) \times D_A$ and $\sharp(u) = k$ for all $(\lambda, u) \in \mathcal{C}_\mu \cap \mathcal{E}$. It has at least one of the following properties.

(i) $\{\|u\|_A : (\lambda, u) \in \mathcal{C}_\mu\} = [0, \infty)$.

(ii) $\sup\{\lambda : (\lambda, u) \in \mathcal{C}_\mu\} = m_e$.

Proof. For $(\lambda, u) \in \mathcal{E}$,

$$\lambda \int_0^1 u^2 dx = \int_0^1 (Su)u + \tilde{g}(u)u \, dx \geq m \int_0^1 u^2 dx, \tag{5.1}$$

showing that $\mathcal{E} \subset [m, \infty) \times D_A$.

Choose $c > \max\{0, \alpha_{g_1} - \operatorname{ess\,inf} V\}$. Then, by property (S1) in Section 2.4, $m + c > \frac{C_1}{4} + \alpha_{g_1} > 0$ and $S_c \equiv S + c$ is a positive self-adjoint operator with $D(S_c) = D_A$ as discussed at the end of Section 2.4. In particular, $\|S_c u\|_{L^2} \geq (m+c)\|u\|_{L^2}$ for all $u \in D_A$ and $\|u\|_c \equiv \|S_c u\|_{L^2}$ defines a norm, $\|\cdot\|_c$ which is equivalent to the graph norm of S on D_A. Furthermore, $D(S_c^{\frac{1}{2}}) = H_A$ and $\|S_c^{\frac{1}{2}} u\|_{L^2} \geq (m+c)^{1/2}\|u\|_{L^2}$. For $u \in D_A$,

$$\|u\|_A^2 = \int_0^1 A|\nabla u|^2 dx \leq \int_0^1 A|\nabla u|^2 + Vu^2 + cu^2 dx = \int_0^1 (S_c u)u \, dx = \|S_c^{\frac{1}{2}} u\|_{L^2}^2 \tag{5.2}$$

$$\leq \|u\|_A^2 + \|V + c\|_{L^\infty}\|u\|_{L^2}^2 \leq K_c^2 \|u\|_A^2, , \quad \text{where } K_c = \left(1 + \frac{4\|V + c\|_{L^\infty}}{C_1}\right)^{1/2} \tag{5.3}$$

by property (H1) in Section 2.1. Hence $\|u\|_A \leq \|S_c^{\frac{1}{2}} u\|_{L^2} \leq K_c \|u\|_A$ for all $u \in D_A$ and, since D_A is a dense subspace of H_A, these inequalities hold for all $u \in H_A$.

Since $S_c^{-\frac{1}{2}} \in B(L^2, H_A)$ and $\tilde{g} \in C(H_A, L^2)$ by Propositions 2.1 and 2.3, a continuous mapping $f : \mathbb{R} \times L^2 \to L^2$ is defined by

$$f(\lambda, v) = v + S_c^{-\frac{1}{2}} \tilde{g}(S_c^{-\frac{1}{2}} v) - (\lambda + c) S_c^{-1} v \quad \text{for } (\lambda, v) \in \mathbb{R} \times L^2. \tag{5.4}$$

If $f(\lambda, v) = 0$, $v \in H_A$ and consequently $u = S_c^{-\frac{1}{2}} v \in D(S_c) = D_A$ with $F(\lambda, u) = 0$, where F is defined in (2.12). Setting

$$\mathcal{S} = \{(\lambda, v) \in \mathbb{R} \times L^2 : f(\lambda, v) = 0 \text{ and } v \neq 0\},$$

it follows easily that

$$\mathcal{E} = \left\{ \left(\lambda, S_c^{-\frac{1}{2}} v \right) : (\lambda, v) \in \mathcal{S} \right\} \quad \text{and so} \quad \mathcal{S} \subset [m, \infty) \times H_A \quad \text{by (5.1)}. \tag{5.5}$$

The rest of this proof involves discussing first bifurcation for the equation $f(\lambda, v) = 0$ and then deducing the desired conclusion about $F(\lambda, u) = 0$ from this.

Step 1. With $H = L^2$, equation (5.4) has the form (4.5) if we set

$M_1(v) = v - (c - \alpha_{g_1}) S_c^{-1} v + S_c^{-\frac{1}{2}} \widetilde{g}_2(S_c^{-\frac{1}{2}} v)$, $\quad M_2(v) = S_c^{-\frac{1}{2}} [\widetilde{g}_1 - \alpha_{g_1}](S_c^{-\frac{1}{2}} v)$ and $\quad Tv = S_c^{-1} v$

for $v \in L^2$. We aim to show that the hypotheses of Theorem 4.10 are satisfied on the interval $J = (\alpha_{g_1} - c, m_e)$. We have already shown that $\alpha_{g_1} - c < m$ so $J \neq \emptyset$.

From the choice of c we have that $T \in B(L^2, L^2)$ is a positive self-adjoint operator with $0 = \inf \sigma(T) < \sup \sigma_e(T) = (m_e + c)^{-1} \leq \sup \sigma(T) = (m + c)^{-1} = \|T\|$. If $(Tv, v)_{L^2} = 0$ and $u = S_c^{-1} v$, $0 = (u, S_c u)_{L^2} \geq (m + c)\|u\|_{L^2}^2$ so $u = 0$ and hence $v = 0$. Thus condition (W0) is satisfied and $0 \in \sigma_e(T)$.

By Proposition 2.3, $\widetilde{g}_2 \in C^1(H_A, L^2)$ and so $M_1 \in C^1(L^2, L^2)$ since $S_c^{-\frac{1}{2}} \in B(L^2, H_A)$. Also $M_1'(v) = I - (c - \alpha_{g_1}) S_c^{-1} + S_c^{-\frac{1}{2}} \widetilde{g}_2'(S_c^{-\frac{1}{2}} v) S_c^{-\frac{1}{2}}$ for all $v \in L^2$ and, in particular $M_1'(0) = I - (c - \alpha_{g_1}) T$ is self-adjoint. Furthermore, $M_1 - M_1'(0) = S_c^{-\frac{1}{2}} \widetilde{g}_2(S_c^{-\frac{1}{2}} \cdot) : L^2 \to L^2$ is compact, since $\widetilde{g}_2 : H_A \to L^2$ is compact by Proposition 2.3 and $S_c^{-\frac{1}{2}} \in B(L^2, H_A)$. Thus condition (W1) is satisfied. In the same way it follows easily for Proposition 2.1 that $M_2 \in C(L^2, L^2)$ is compact and w-Hadamard differentiable at 0 with $M_2'(0) = -\alpha_{g_1} T$. For $v \in L^2$, we now have that $M(v) - M'(0)v = S_c^{-\frac{1}{2}} \widetilde{g}(S_c^{-\frac{1}{2}} v)$ and so

$$(M(v) - M'(0)v, v) = \int_0^1 [S_c^{-\frac{1}{2}} \widetilde{g}(S_c^{-\frac{1}{2}} v)] v \, dx = \int_0^1 \widetilde{g}(S_c^{-\frac{1}{2}} v) S_c^{-\frac{1}{2}} v \, dx \geq 0,$$

since $S_c^{-\frac{1}{2}} : L^2 \to L^2$ is self-adjoint. In view of (4.6), this shows that condition (W2) is satisfied.

Since $\lambda + c - \alpha_{g_1} > 0$ for all $\lambda \in J$, it follows from (2.15) that

$$\inf \sigma_e(M_1'(0) - \lambda T) = \inf \sigma_e(I - (\lambda + c - \alpha_{g_1}) T) = \frac{m_e - \lambda + \alpha_{g_1}}{m_e + c}$$

whereas $r_e(M_2'(0)) = r_e(\alpha_{g_1} T) = \frac{\alpha_{g_1}}{m_e + c}$. Thus we see that $\inf \sigma_e(M_1'(0) - \lambda T) > r_e(M_2'(0))$ for $\lambda \in J = (\alpha_{g_1} - c, m_e)$. Let $U = J \times L^2$.

We have now verified that the hypotheses of Theorem 4.10 are satisfied in the present context and so $M'(0) - \lambda T \in \Phi_0(L^2, L^2)$ for all $\lambda \in J$ and $J \cap \sigma_e(S) = \emptyset$. Since $M'(0) - \lambda T = S_c^{-\frac{1}{2}} [S - \lambda I] S_c^{-\frac{1}{2}}$, it follows that $\dim \ker[M'(0) - \lambda T] = \dim \ker[S - \lambda I]$. Recalling that $\mathcal{S} \subset [m, \infty) \times H_A$ by (5.5) and that $\inf J < m$, it now follows from Theorem 4.10 that $\mu < m_e$ is a bifurcation point for the equation $f(\lambda, v) = 0$ if and only if $\mu \in \sigma(S)$. Furthermore, when $\mu \in \sigma(S) \cap (-\infty, m_e)$ the component $\mathcal{D}_\mu$ of $\overline{\mathcal{S}} \cap (J \times L^2)$ containing $(\mu, 0)$ has at least one of the properties (a), (b) and (c) in part (2) of Theorem 4.10. Since $\inf J < m \leq \inf p(\mathcal{D}_\mu)$ these properties can be replaced by

(i′) $\{\|v\|_{L^2} : (\lambda, v) \in \mathcal{D}_\mu\} = [0, \infty)$.

(ii′) $\sup p(\mathcal{D}_\mu) = m_e$.

(iii′) $\mathcal{D}_\mu = \mathcal{D}_\nu$ for some $\nu \in \sigma(S) \cap J$ where $\nu \neq \mu$.

Step 2. It has already been observed that $\mathcal{E} = H(\mathcal{S})$ where $H(\lambda, v) = (\lambda, S_c^{-\frac{1}{2}} v)$ for $(\lambda, v) \in$

$\mathbb{R} \times L^2$ and that $\mathcal{S} \subset [m, \infty) \times H_A$. We now show that $H : \mathcal{S} \cup [\mathbb{R} \times \{0\}] \to \mathcal{E} \cup [\mathbb{R} \times \{0\}]$ is a homeomorphism for the metrics induced by $\| \cdot \|_{L^2}$ on L^2 and $\| \cdot \|_c$ on D_A. Clearly, H is a bijection with $H^{-1}(\lambda, u) = (\lambda, S_c^{\frac{1}{2}} u)$. For $(\lambda, v), (\mu, w) \in \mathcal{S} \cup [\mathbb{R} \times \{0\}]$,

$$\begin{aligned}
\|H(\lambda, v) - H(\mu, w)\|_{\mathcal{E}} &= |\lambda - \mu| + \|S_c^{-\frac{1}{2}}(v - w)\|_c = |\lambda - \mu| + \|S_c^{\frac{1}{2}}(v - w)\|_{L^2} \\
&= |\lambda - \mu| + \|(\lambda + c)S_c^{-\frac{1}{2}} v - \tilde{g}(S_c^{-\frac{1}{2}} v) - (\mu + c)S_c^{-\frac{1}{2}} w + \tilde{g}(S_c^{-\frac{1}{2}} w)\|_{L^2} \\
&\leq |\lambda - \mu|(1 + \|S_c^{-\frac{1}{2}} v\|_{L^2}) + (|\mu| + c + \ell_{g_1})\|S_c^{-\frac{1}{2}}(v - w)\|_{L^2} \\
&\quad + C\mathfrak{C}_{g_2}(\|S_c^{-\frac{1}{2}} v\|_A + \|S_c^{-\frac{1}{2}} w\|_A)\|S_c^{-\frac{1}{2}}(v - w)\|_A \quad \text{(by Remark 2.4)} \\
&\leq |\lambda - \mu|(1 + (m + c)^{-\frac{1}{2}}\|v\|_{L^2}) + \{(m + c)^{-\frac{1}{2}}(|\mu| + c + \ell_{g_1}) \\
&\quad + C\mathfrak{C}_{g_2}(\|v\|_{L^2} + \|w\|_{L^2})\}\|v - w\|_{L^2} \quad \text{(by (5.2))},
\end{aligned}$$

showing that H is continuous. For the continuity of H^{-1}, consider $(\lambda, u), (\mu, z) \in \mathcal{E} \cup [\mathbb{R} \times \{0\}]$. Then

$$\begin{aligned}
\|H^{-1}(\lambda, u) - H^{-1}(\mu, z)\|_{\mathcal{S}} &= |\lambda - \mu| + \|S_c^{\frac{1}{2}}(u - z)\|_{L^2} \leq |\lambda - \mu| + (m + c)^{-\frac{1}{2}}\|S_c(u - z)\|_{L^2} \\
&= |\lambda - \mu| + (m + c)^{-\frac{1}{2}}\|u - z\|_c,
\end{aligned}$$

as required. At this point we can now assert that $\mathcal{C}_\mu = H(\mathcal{D}_\mu)$ and hence that $\mathcal{C}_\mu$ has at least one of the following properties.

(i') $\{\|S_c^{\frac{1}{2}} u\|_{L^2} : (\lambda, u) \in \mathcal{C}_\mu\} = [0, \infty)$.

(ii') $\sup p(\mathcal{C}_\mu) = m_e$.

(iii') $\mathcal{C}_\mu = \mathcal{C}_\nu$ for some $\nu \in \sigma(S) \cap J$ where $\nu \neq \mu$.

Recalling that $\|S_c^{\frac{1}{2}} u\|_{L^2}$ and $\|u\|_A$ define equivalent norms on H_A, it now suffices to show that property (iii') cannot occur.

Step 3. The proof that $\sharp(u) = k$ for all $(\lambda, u) \in \mathcal{C}_\mu \cap \mathcal{E}$ is essentially the same as for part (C) of Theorem 5.1, using part (ii) of Lemma 3.1 instead of part (i). The only difference occurs in showing that if $(\xi, 0) \in \mathcal{C}_\mu$, there is an open neighbourhood U_ξ of $(\xi, 0)$ in $\mathbb{R} \times D_A$ such that $Z(\lambda, u) = \sharp(\phi_\xi)$ for all $(\lambda, u) \in U_\xi \cap \mathcal{E}$, where ϕ_ξ is a normalised eigenfunction of S associated with ξ. To prove this we again argue by contradiction, supposing that there is a sequence $(\lambda_n, u_n) \in \mathcal{E}$ such that $\lambda_n \to \xi$ and $\|u_n\|_c \to 0$ as $n \to \infty$ and, for all $n \in \mathbb{N}$, $\sharp(u_n) \neq \sharp(\phi_\xi)$. Setting $v_n = S_c^{\frac{1}{2}} u_n$ and $\psi_\xi = S_c^{\frac{1}{2}} \phi_\xi$, we have that $(\lambda_n, v_n) \in \mathcal{S}$, $\|v_n\|_{L^2} \to 0$ as $n \to \infty$ and $M'(0)\psi_\xi = \xi T \psi_\xi$.

Setting $w_n = v_n / \|v_n\|_{L^2}$, it follows from part (C) of Theorem 4.10 that by passing to a further subsequence we can suppose that $w_n \rightharpoonup d\psi_\xi$ weakly in L^2 as $n \to \infty$ where the constant d is not equal to zero. Since $S_c^{-\frac{1}{2}} \in B(L^2, H_A)$, this implies that $S_c^{-\frac{1}{2}} w_n \rightharpoonup d S_c^{-\frac{1}{2}} \psi_\xi$ weakly in H_A. By Propositions 2.1 and 2.3, $\tilde{g} : H_A \to L^2$ is w-Hadamard differentiable at 0 with $\tilde{g}'(0) = 0$. Hence

$$\frac{\tilde{g}(S_c^{-\frac{1}{2}} v_n)}{\|v_n\|_{L^2}} = \frac{\tilde{g}(\|v_n\|_{L^2} S_c^{-\frac{1}{2}} w_n)}{\|v_n\|_{L^2}} \rightharpoonup 0 \quad \text{weakly in } L^2 \text{ as } n \to \infty.$$

But $(\lambda_n, v_n) \in \mathcal{S}$ for all n and so

$$S_c^{\frac{1}{2}} w_n = (\lambda_n + c) S_c^{-\frac{1}{2}} w_n - \frac{\tilde{g}(S_c^{-\frac{1}{2}} v_n)}{\|v_n\|_{L^2}} \rightharpoonup (\xi + c) d S_c^{-\frac{1}{2}} \psi_\xi = (\xi + c) d \phi_\xi \quad \text{weakly in } L^2.$$

Let $(\cdot,\cdot)_c$ denote the scalar product associated with the norm $\|\cdot\|_c$ on D_A. We now have that $S_c^{-\frac{1}{2}} w_n \in D_A$ for all n and, for all $u \in D_A$,

$$(S_c^{-\frac{1}{2}} w_n, u)_c = (S_c^{\frac{1}{2}} w_n, S_c u)_{L^2} \to (\xi + c) d (\phi_\xi, S_c u)_{L^2} = d (S_c \phi_\xi, S_c u)_{L^2} = d(\phi_\xi, u)_c \quad \text{as } n \to \infty.$$

Thus $q_n u_n = S_c^{-\frac{1}{2}} w_n \rightharpoonup d\phi_\xi$ weakly in D_A as $n \to \infty$, where $q_n = \|v_n\|_{L^2} > 0$ for all n. Recalling that $\{(\lambda_n, u_n)\} \subset \mathcal{E}$ with $\lambda_n \to \xi$ and $\|u_n\|_c \to 0$ as $n \to \infty$, it follows from Lemma 3.1(ii) that there exists $\eta \in (0,1)$ such that $u_n(x) \neq 0$ for $0 < x \leq \eta$ and all n. By property (S3) in Section 2.4 we can choose η so that $\phi_\xi(x) \neq 0$ for $0 < x \leq \eta$. By part (i) of Lemma 3.2, $\|P_\eta(q_n u_n) - P_\eta(d\phi_\xi)\|_\eta \to 0$ as $n \to \infty$. It now follows from part (ii) of Lemma 3.2 that there exists n_0 such that $\sharp(q_n u_n) = \sharp(d\phi_\xi)$ for all $n \geq n_0$. Since $\sharp(q_n u_n) = \sharp(u_n)$ and $\sharp(d\phi_\xi) = \sharp(\phi_\xi)$, this contradicts the initial choice of the sequence $\{(\lambda_n, u_n)\}$ and establishes the continuity of the mapping Z at $(\xi, 0)$.

As in the proof of Theorem 5.1 we can now conclude that $\sharp(u) = \sharp(\phi_\mu) = k$ for all $(\lambda, u) \in \mathcal{C}_\mu \cap \mathcal{E}$ and consequently that property (iii′) does not occur.

To complete the proof it only remains to show that $\lambda \geq \mu$ for all $(\lambda, u) \in \mathcal{C}_\mu$. This can be done using the comparison principle self-adjoint operators just as in the proof of Corollary 5.3. Note that in this case, $W \geq 0$ on $(0,1)$ so $\inf \sigma_e(S + W) \geq \inf \sigma_e(S) = m_e$. □

Under some additional assumptions an "a priori" bound for solutions in a component $\mathcal{C}_\mu$ can be established and hence $p(\mathcal{C}_\mu) = [\mu, m_e)$.

Remark 5.6. Recall from Lemma 3.4 that assumption (b) of Theorem 5.5 implies that $I_i(g_1) = I_s(g_1) = \alpha_{g_1}$. Hence if $(\lambda, u) \in \mathcal{C}_\mu$ with $\lambda < V_0 + \alpha_{g_1}$, then $u \in L^\infty(0,1)$ by Proposition 3.5(iv). But u has only a finite number of zeros in $(0,1]$ if $(\lambda, u) \in \mathcal{C}_\mu$ and so it follows from Proposition 3.5(ii) that $\lim_{x\to 0} u(x) = \pm\infty$ if $\lambda > V_0 + J_s(g_1)$. Note that $V_0 + J_s(g_1) < m_e$ provided that $J_s(g_1) < \frac{a}{4}$. The next result exhibits a situation where $p(\mathcal{C}_\mu) = [\mu, m_e)$ and hence, if $\mu < V_0 + \alpha_{g_1}$ and $J_s(g_1) < \frac{a}{4}$, the behaviour of solutions in $\mathcal{C}_\mu$ changes as λ increases. If $(\lambda, u) \in \mathcal{C}_\mu$ with λ near μ, $u \in L^\infty(0,1)$ whereas for $\lambda \in (V_0 + J_s(g_1), m_e)$, $\lim_{x\to 0} u(x) = \pm\infty$. If $g_1(x,s) = r(x)k(s)$ where the functions r and k satisfy the conditions (R) and (K) introduced in Remark 5.4, $J_s(g_1) = I_i(g_1) = \ell_{g_1} = \alpha_{g_1} = r(0)k'(\infty)$ and the transition occurs when λ crosses $V_0 + r(0)k'(\infty)$ if $\mu < V_0 + r(0)k'(\infty)$ and $r(0)k'(\infty) < \frac{a}{4}$. Both cases $u(x) \to \infty$ and $u(x) \to -\infty$ as $x \to 0$ occur since k is odd and hence $\mathcal{C}_\mu = \{(\lambda, -u) : (\lambda, u) \in \mathcal{C}_\mu\}$. Noting that $k(s)/s$ is non-decreasing on $(0,\infty)$ with $\lim_{s\to\infty} k(s)/s = k'(\infty)$, condition (3) in Theorem 5.7 and condition (3′) in Proposition 5.8 will be satisfied in this case if $k'(\infty) > \operatorname{ess\,sup}_{0<x<1} \frac{V_0 - V(x)}{r(x)}$.

Let $t_+ = \max\{0, t\}$ for $t \in \mathbb{R}$. Observe that, since $V \in L^\infty(0,1)$, condition (2) in the following result only involves the behaviour of $V(x)$ as $x \to 0$. Assumptions (1) and (2) are satisfied in Examples 2.8 and 2.9.

Theorem 5.7. *In addition to the hypotheses of Theorem 5.5 suppose that the following conditions are satisfied.*

(1) $A \in C^1((0,1))$ *and* $\{x^{\frac{1}{2}} c(x)\}' \geq 0$ *for* $0 < x < 1$ *where* $c(x) = \frac{A(x)}{x^2} - a$.

(2) $\int_0^1 x^{-1}[V_0 - V(x)]_+ dx < \infty$.

(3) There exist $K_2 > K_1 > 0$ *such that* $V_0 \leq V(x) + \frac{g(x,s)}{s}$ *for all* $x \in (0,1)$ *and* $K_1 \leq x^{\frac{1}{2}}|s| \leq K_2$.

For every eigenvalue μ *of* S *in* $(-\infty, m_e)$, $\sup p(\mathcal{C}_\mu) = m_e$ *where* $\mathcal{C}_\mu$ *is defined in Theorem 5.5.*

Proof. Let us suppose that $m_e - \sup p(\mathcal{C}_\mu) = \eta > 0$. In view of Theorem 5.5 it suffices to deduce from this that $\sup\{\|u\|_A : (\lambda, u) \in \mathcal{C}_\mu\} < \infty$.

Step 1. We claim that if $(\lambda, u) \in \mathcal{C}_\mu$, then $|u(x)| < K_1 x^{-\frac{1}{x}}$ for all $x \in (0,1)$ where $K_1 > 0$ is given by assumption (3). To justify this assertion, choose $K \in (K_1, K_2)$ and let $U = \{(\lambda, u) \in \mathcal{C}_\mu : x^{\frac{1}{2}}|u(x)| < K \text{ for all } x \in (0,1]\}$. We now show that U is both open and closed in $\mathcal{C}_\mu$.

If $(\lambda, u) \in U$, it follows that $u(1) = 0$ and, from property (P2) in Section 2.1, $x^{\frac{1}{2}}u(x) \to 0$ as $x \to 0$. Hence there exists $\varepsilon > 0$ such that $x^{\frac{1}{2}}|u(x)| \leq K - \varepsilon$ for all $x \in (0,1]$. Referring again to property (P2), there exists $\delta > 0$ such that $\|x^{\frac{1}{2}}(v-u)\|_{L^\infty} < \varepsilon/2$ for $v \in D_A$ with $\|S_A(v-u)\|_{L^2} < \delta$ and hence

$$x^{\frac{1}{2}}|v(x)| \leq x^{\frac{1}{2}}|u(x)| + x^{\frac{1}{2}}|v(x) - u(x)| < K - \varepsilon/2.$$

This proves that U is an open subset of $\mathcal{C}_\mu$.

To prove that it is also a closed subset of $\mathcal{C}_\mu$ consider $(\lambda, u) \in \mathcal{C}_\mu$ and a sequence $\{(\lambda_n, u_n)\}$ in U such that $\lambda_n \to \lambda$ and $\|S_A(u_n - u)\|_{L^2} \to 0$ as $n \to \infty$. By (P2), $u_n(x) \to u(x)$ for all $x \in (0,1]$ as $n \to \infty$ and so $x^{\frac{1}{2}}|u(x)| \leq K$ for $0 < x \leq 1$. Suppose that $\sup_{0<x\leq 1} x^{\frac{1}{2}}u(x) = K$ and let (p,q) be a maximal interval such that $x^{\frac{1}{2}}u(x) > K_1$. Since $\lim_{x\to 0} x^{\frac{1}{2}}u(x) = u(1) = 0$ we have $0 < p < q < 1$ and, setting $v(x) = K_1 x^{-\frac{1}{2}}$, $u(p) = v(p)$, $u'(p) \geq v'(p)$, $u(q) = v(q)$ and $u'(q) \leq v'(q)$ since $u, v \in C^1([p,q])$. Hence

$$\begin{aligned}\int_p^q (Au')'v - (Av')'u\,dx &= A[u'v - v'u]|_p^q \\ &= A(q)v(q)[u'(q) - v'(q)] - A(p)v(p)[u'(p) - v'(p)] \leq 0.\end{aligned}$$

But it is easy to check that assumption (1) implies that $-(Av')' \geq \frac{a}{4}v$ on $(0,1)$. Since $u(x) > 0$ on (p,q) this yields

$$\int_p^q (Au')'v - (Av')'u\,dx \geq \int_p^q u(x)v(x)\left\{V(x) + \frac{g(x,u(x))}{u(x)} - \lambda + \frac{a}{4}\right\} dx,$$

where

$$V(x) + \frac{g(x,u(x))}{u(x)} - \lambda + \frac{a}{4} = m_e - \lambda + V(x) - V_0 + \frac{g(x,u(x))}{u(x)} \geq m_e - \lambda,$$

by assumption (3) because $K_1 < x^{\frac{1}{2}}u(x) \leq K < K_2$ on (p,q). This implies that

$$\int_p^q (Au')'v - (Av')'u\,dx \geq (m_e - \lambda)\int_p^q u(x)v(x)\,dx > 0$$

since $\lambda \leq \sup p(\mathcal{C}_\mu) \leq m_e - \eta$, contradicting the previous conclusion.
Hence $\sup_{0<x\leq 1} x^{\frac{1}{2}}u(x) < K$.

A similar argument shows that $x^{\frac{1}{2}}u(x) > -K$ for $0 < x \leq 1$ and so $(\lambda, u) \in U$, proving that U is a closed subset of $\mathcal{C}_\mu$.

Clearly $(\mu, 0) \in U$ and we have now shown that U is both open and closed in $\mathcal{C}_\mu$. Since $\mathcal{C}_\mu$ is connected this means that $U = \mathcal{C}_\mu$ and hence $|u(x)| < Kx^{-\frac{1}{2}}$ for all $x \in (0,1]$ and $(\lambda, u) \in \mathcal{C}_\mu$. This completes Step 1.

Step 2. Here we prove that $\|u\|_A^2 \leq \frac{K_1^2}{\varepsilon} \int_0^1 x^{-1}[V_0 - V(x)]_+ \, dx$ for all $(\lambda, u) \in \mathcal{C}_\mu$ where $\varepsilon = \frac{1}{2} \min\left\{1, \frac{4\eta}{a}\right\}$ and $\eta = m_e - \sup p(\mathcal{C}_\mu)$.

For any $(\lambda, u) \in \mathcal{C}_\mu$,

$$\begin{aligned}
\varepsilon\|u\|_A^2 &= \int_0^1 A(u')^2 dx - (1-\varepsilon)\|u\|_A^2 \leq \int_0^1 A(u')^2 dx - (1-\varepsilon)\int_0^1 ax^2(u')^2 dx \\
&\leq \int_0^1 A(u')^2 dx - (1-\varepsilon)\frac{a}{4}\int_0^1 u^2 dx
\end{aligned}$$

by property (H1) in Section 2.1 since assumption (1) implies that $A(x) \geq ax^2$ for $0 \leq x \leq 1$. But $g(x,s)s \geq 0$ for all $(x,s) \in (0,1) \times \mathbb{R}$ so

$$\int_0^1 A(u')^2 dx = \int_0^1 [\lambda - V(x)]u(x)^2 - g(x, u(x))u(x)\, dx \leq \int_0^1 (\lambda - V)u^2 dx.$$

Hence

$$\begin{aligned}
\varepsilon\|u\|_A^2 &\leq \int_0^1 \left\{\lambda - V(x) - (1-\varepsilon)\frac{a}{4}\right\} u(x)^2 dx = \int_0^1 \left\{\lambda - m_e + V_0 - V(x) + \frac{a\varepsilon}{4}\right\} u(x)^2 dx \\
&\leq \int_0^1 [V_0 - V(x)]_+ u(x)^2 dx \leq K_1^2 \int_0^1 x^{-1}[V_0 - V(x)]_+ \, dx
\end{aligned}$$

by Step 1 since $\lambda - m_e + \frac{a\varepsilon}{4} \leq -\eta + \frac{a\varepsilon}{4} \leq 0$.

From assumption (2) it now follows that $\sup\{\|u\|_A : (\lambda, u) \in \mathcal{C}_\mu\} < \infty$ if $\sup p(\mathcal{C}_\mu) < m_e$. The conclusion follows from Theorem 5.5. □

After strengthening assumption (3) the arguments used to prove Theorem 5.7 yield an "a priori" bound for all solutions of (1.1)(1.2) with $\lambda \leq m_e - \eta$ for some $\eta > 0$, not just those in the components $\mathcal{C}_\mu$.

Proposition 5.8. *Suppose that condition (S) is satisfied with $n \equiv 0$ and that $g(x,s)s \geq 0$ for all $(x,s) \in (0,1) \times \mathbb{R}$. Assume also that the following conditions are satisfied.*

(1) $A \in C^1((0,1))$ and $\{x^{\frac{1}{2}}c(x)\}' \geq 0$ for $0 < x < 1$ where $c(x) = \frac{A(x)}{x^2} - a$.

(2) $\int_0^1 x^{-1}[V_0 - V(x)]_+ dx < \infty$.

(3′) There exists $K > 0$ such that $V_0 \leq V(x) + \frac{g(x,s)}{s}$ for all $x \in (0,1)$ and $x^{\frac{1}{2}}|s| \geq K$.

Then, for every $\eta > 0$,

$$\|u\|_A^2 \leq \frac{K^2}{\delta(\eta)} \int_0^1 x^{-1}[V_0 - V(x)]_+ dx \quad \text{for all } (\lambda, u) \in \mathcal{E}_\eta \equiv \mathcal{E} \cap (-\infty, m_e - \eta) \times D_A,$$

where $\delta(\eta) = \min\{1, \frac{4\eta}{a}\}$. By Remark 5.2, this implies an "a priori" bound for $\|S_A u\|_{L^2}$ also.

Proof. Fix $\eta > 0$ and then take any $\varepsilon \in (0, \delta(\eta))$. Let $v(x) = Kx^{-\frac{1}{2}}$ where K is given by condition (3′).

The argument used to prove that U is a closed subset of $\mathcal{C}_\mu$ in the proof of Theorem 5.7 shows that $|u(x)| \leq v(x)$ for all $(\lambda, u) \in \mathcal{E}_\eta$ and all $x \in (0,1)$ when condition (3) is replaced by (3′). The desired conclusion is then obtained by repeating Step 2 of that proof. □

Remark 5.9. The results in this section improve previous conclusions in Theorem 4.5(ii) of [31] about bifurcation at eigenvalues of S in the interval $(-\infty, m_e)$, even at the local level, when $n \equiv 0$ and $g(x,s)s \geq 0$. However, they do not give a complete description of all bifurcation points in this case since, as shown in Theorem 4.5(iii), bifurcation can occur at points in $[m_e, \infty)$ which are not eigenvalues of S. See also Section 6.3 of [33] for generalisations to higher dimensions.

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On the solvability of some discontinuous functional impulsive problems

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. This paper deals with impulsive problems consisting of second order differential equation with impulsive effects depending implicitly on the solution and with rather general nonlocal boundary conditions. The arguments are based on the lower and upper solutions method and a fixed point theorem.

Keywords: functional problems, generalized impulsive conditions, upper and lower solutions, fixed point theory, boundary value problem.

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1 Introduction

Impulsive problems have been object of growing and constant interest, mainly because they provide adequate mathematical tools to describe evolution processes with sudden changes, and to model real phenomena in science, as, for instance, population and biological dynamics, biotechnology and ecology, engineering and industrial robotic, etc. As a result, differential equations with impulses have been recently studied by many authors. They employed various methods and techniques, such as, bifurcation theory [16, 17], method of lower and upper solutions [9, 14, 23, 24], fixed point theorems and fixed point index in cones [11, 12, 32], critical point theory and variational methods [22, 30, 33]. For contributions to general and classical theory we refer to e.g. [1, 13, 25].

Problems with implicit impulse conditions depending both on values of the solution and its derivative at the points of the impulse action have been considered by several authors (see [3, 4, 15, 19, 20] and the references therein). In particular, we refer to [18] dealing with the

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problem

$$\begin{gathered} u''(t) = f(t, u(t), u'(t)) \quad \text{a.e. on } [0, \infty), \\ \Delta u(t_k) = I_{0k}(t_k, u(t_k), u'(t_k)), \quad \Delta u'(t_k) = I_{1k}(t_k, u(t_k), u'(t_k)) \quad \text{for } k \in \mathbb{N}, \\ u(0) = A, \quad u'(\infty) = B, \end{gathered}$$

where $\{t_k\}$ is a sequence of points in $(0, \infty)$ such that $t_k < t_{k+1}$ for $k \in \mathbb{N}$ and $\lim_{k \to +\infty} t_k = \infty$; $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is an L^1-Carathéodory function;

$$u'(\infty) := \lim_{t \to \infty} u'(t), \; \Delta u^{(i)}(t) := u^{(i)}(t+) - u^{(i)}(t-) \quad \text{for } t \in (0, \infty)$$

and $i \in \{0, 1\}$; $A, B \in \mathbb{R}$ and $I_{ik} : (0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ are continuous for $i \in \{0, 1\}$ and $k \in \mathbb{N}$. The arguments included Green's functions, Schauder's fixed point theorem and, to have the compactness of the representing operator, also the equiconvergence both at ∞, and at each impulse moment t_k.

Similarly, functional boundary conditions generalize local boundary data and encompass a broad spectrum of conditions where global information on the unknown function is given, including integral and nonlocal conditions, advanced or delay data, maximum or minimum arguments, among others. Existence, nonexistence and multiplicity results for general boundary conditions were studied, for example, in [2, 5, 8, 26–29], for scalar differential equations and, in [6], for coupled systems of differential equations.

Our idea in this paper is to combine both techniques, applied in the papers mentioned above, in the study of impulsive problems with impulse effects depending both on the unknown function and on its first derivative and with nonlocal boundary conditions. In particular, our aim is to get results on the existence of solutions to the boundary value problem

$$-u''(t) = f(t, u(t), u'(t)) \quad \text{a.e. on } [0, 1], \tag{1.1}$$

$$\Delta u(t) = I_{0k}(t, u(t), u'(t)), \quad \Delta u'(t) = I_{1k}(t, u(t), u'(t)) \quad \text{if } t = t_k \in D, \tag{1.2}$$

$$L_0(u(0), u(1), u'(0), u) = 0, \quad L_1(u(0), u(1), u'(1), u) = 0, \tag{1.3}$$

where $m \in \mathbb{N}$, $D = \{t_1, \ldots, t_m\} \subset (0, 1)$, $t_1 < \cdots < t_m$, $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathédory function, $I_{0k}, I_{1k} : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ with $k \in \{1, \ldots, m\}$ and $L_0, L_1 : \mathbb{R}^3 \times \mathcal{PC}_D \to \mathbb{R}$ satisfy conditions given below, $\mathcal{PC}_D$ is the space of piecewise continuous functions defined below and the symbol Δ has a usual meaning, i.e. $\Delta v(t) = v(t+) - v(t-)$ for any $t \in [0, 1]$ and any function $v : [0, 1] \to \mathbb{R}$ such that both limits in the above formula are defined and have finite values.

As far as we know, nonlocal boundary conditions together with impulsive effects of the types (1.3) and (1.2) are treated in this paper for the first time. This was enabled due to the implemented technique: lower and upper solutions method together with a proper truncation argument. Let us emphasize that, on the contrary to e.g. periodic problem, for (1.1)–(1.3) no a priori estimate of the derivative of the sought solution is available.

The paper is organized as follows: in Section 2 the general framework is established and the basic definitions are introduced. In Section 3 we present our main result: existence and localization theorem and its proof. Last section provides a nontrivial example illustrating the power of our main result.

2 Preliminaries

For a given function $v : [0,1] \to \mathbb{R}$ and points $t \in (0,1]$ and $s \in [0,1)$, the symbols $v(t-)$ and $v(s+)$ stand respectively for the corresponding one-sided limits

$$v(t-) := \lim_{\tau \to t-} v(\tau) \text{ and } v(s+) := \lim_{\tau \to s+} v(\tau)$$

whenever these limits exist and have finite values. In such a case, for $t \in (0,1)$, we write $\Delta v(t) = v(t+) - v(t-)$. Note that the functions such that $v(t-) \in \mathbb{R}$ for all $t \in (0,1]$ and $v(s+) \in \mathbb{R}$ for all $s \in [0,1)$ are usually called regulated functions. The space $\mathcal{G}$ of such functions is known to be a Banach space with respect to the supremum norm

$$\|v\| = \|v\|_\infty := \sup_{t\in[0,1]} |v(t)| \quad \text{for } v \in \mathcal{G}.$$

For basic properties of regulated functions, see e.g. [7], [10] or [21]. For our purposes, the following compactness criterion for subspaces of $\mathcal{G}$ will be essential (cf. [7] or [21, Lemma 4.3.4 and Corollary 4.3.7]).

Theorem 2.1 (Fraňková). *A given subset B of the space $\mathcal{G}$ of regulated functions is relatively compact if and only if*

- *B is the set of equi-regulated functions, i.e. for every $\varepsilon > 0$ there is a division $\{\alpha_0 < \ldots < \alpha_n\}$ of the interval $[0,1]$ such that for every $v \in B$, $j \in \{1,\ldots,n\}$ and $t,s \in (\alpha_{j-1},\alpha_j)$ we have $|v(t) - v(s)| < \varepsilon$*

and

- *the set $\{v(t) : v \in B\} \subset \mathbb{R}$ is bounded for each $t \in [0,1]$.*

In what follows, the symbol D stands for the fixed set $D = \{t_1,\ldots,t_m\}$ of points of impulses in the open interval $(0,1)$ ordered in such a way that $0 < t_1 < \cdots < t_m < 1$. It will be helpful to denote also $t_0 = 0$ and $t_{m+1} = 1$. The symbols $\mathcal{PC}_D$ and $\mathcal{PC}^1_D$ then denote respectively the corresponding sets of functions piecewise continuous on $[0,1]$ or with a derivative piecewise continuous on $[0,1]$. More precisely, $\mathcal{PC}_D$ is the set of all functions $u : [0,1] \to \mathbb{R}$ continuous at every $t \in [0,1] \setminus D$, continuous from the left at every $t \in D$ and having, in addition, finite right limits $u(s+)$ for all $s \in D$. Obviously, when equipped with usual algebraic operations, the space $\mathcal{PC}_D$ is a closed subspace of the Banach space $\mathcal{G}$ of regulated functions. Therefore, it is also a Banach space (with respect to the supremum norm). Analogously, $\mathcal{PC}^1_D$ is the set of all functions $u \in \mathcal{PC}_D$ having a finite derivative $u'(t)$ at each $t \in [0,1] \setminus D$, while u' is continuous at each $t \in [0,1] \setminus D$ and, in addition, it has finite limits $u'(t-)$ and $u'(s+)$ for all $t,s \in D$. For a given $u \in \mathcal{PC}^1_D$, by u' we always mean a function which coincides with the derivative of u on $(0,1) \setminus D$ and is extended to the whole interval $[0,1]$ by the prescriptions

$$u'(0) = u'(0+), \qquad u'(1) = u'(1-) \quad \text{and} \quad u'(t) = u'(t-) \quad \text{if } t \in D.$$

Of course, for such an extension of the derivative we have $u' \in \mathcal{PC}_D$ whenever $u \in \mathcal{PC}^1_D$. It is easy to verify that both the mappings

$$u \in \mathcal{PC}^1_D \to (u', u(0), \Delta u(t_1), \ldots, \Delta u(t_m)) \in \mathcal{PC}_D \times \mathbb{R}^{m+1},$$

and

$$(v, d_0, d_1, \ldots, d_m) \in \mathcal{PC}_D \times \mathbb{R}^{m+1}$$
$$\to u(t) = d_0 + \int_0^t v(s)\,\mathrm{d}s + \sum_{k=1}^m d_k\,\chi_{(t_k,t]}(t) \in \mathcal{PC}_D^1,$$

where $\chi_M(t) = 1$ if $t \in M$ and $\chi_M(t) = 0$ if $t \notin M$, are continuous with respect to the norms

$$\|u\|_{\mathcal{PC}^1} := \|u\|_\infty + \|u'\|_\infty \text{ on } \mathcal{PC}_D^1$$

and

$$\|(v, d_0, d_1, \ldots, d_m)\| = \|v\|_\infty + \sum_{k=0}^m |d_k| \text{ on } \mathcal{PC}_D \times \mathbb{R}^{m+1}$$

and provide a one-to-one correspondence between the spaces $\mathcal{PC}_D^1$ and $\mathcal{PC}_D \times \mathbb{R}^{m+1}$. As a consequence, $\mathcal{PC}_D^1$ is a Banach space when equipped with the norm $\|\cdot\|_{\mathcal{PC}^1}$. This together with Theorem 2.1 leads directly to the following compactness criterion for subsets of $\mathcal{PC}_D^1$.

Corollary 2.2. *A subset B of the space $\mathcal{PC}_D^1$ is relatively compact if and only if*

- *the set $\{u' : u \in B\}$ is equi-regulated*

and

- *for a given $t \in [0,1]$, the set $\{u(t) : u \in B\} \subset \mathbb{R}$ is bounded.*

Solutions to our problem (1.1)–(1.3) will be understood in the Carathéodory sense as functions with piecewise absolutely continuous derivatives. More precisely, the symbol $\mathcal{AC}_D^1$ stands for the set of functions $u \in \mathcal{PC}_D^1$ having first derivatives absolutely continuous on each subinterval (t_{k-1}, t_k) with $k \in \{1, \ldots, m+1\}$ and solutions to (1.1)–(1.3) are defined as follows.

Definition 2.3. By a solution u of problem (1.1)–(1.3) we understand a function $u \in \mathcal{AC}_D^1$ satisfying equation (1.1) a.e. on $[0,1]$ together with conditions (1.2) and (1.3).

Throughout the paper we consider the following assumptions:

(A) $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ of (1.1) satisfies the Carathéodory conditions, i.e.

- $f(\cdot, x, y)$ is Lebesgue integrable for all $(x, y) \in \mathbb{R}^2$,
- $f(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^2$ for a.e. $t \in [0,1]$,
- for each $\rho > 0$ there is a function μ_ρ Lebesgue integrable on $[0,1]$ and such that $|f(t, x, y)| \le \mu_\rho(t)$ for a.e. $t \in [0,1]$ and all $x, y \in \mathbb{R}$ such that $|x| \le \rho$ and $|y| \le \rho$.

(B) $L_0, L_1 : \mathbb{R}^3 \times \mathcal{PC}_D \to \mathbb{R}$ and $I_{0k}, I_{1k} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous for all $k \in \{1, \ldots, m\}$.

Important tools for our proofs will be associated lower and upper solutions given by the following definition.

Definition 2.4. A function $\alpha \in \mathcal{AC}^1_D$ is a lower solution of (1.1)–(1.3) if

$$\begin{cases} -\alpha''(t) \le f(t,\alpha(t),\alpha'(t)) & \text{a.e. on } [0,1], \\ \Delta\alpha(t_k) \le I_{0k}(t_k,\alpha(t_k),\alpha'(t_k)) & \text{for } k \in \{1,\dots,m\}, \\ \Delta\alpha'(t_k) > I_{1k}(t_k,\alpha(t_k),\alpha'(t_k)) & \text{for } k \in \{1,\dots,m\}, \\ L_0(\alpha(0),\alpha(1),\alpha'(0),\alpha) \ge 0, \\ L_1(\alpha(0),\alpha(1),\alpha'(1),\alpha) \ge 0, \end{cases} \tag{2.1}$$

while a function $\beta \in \mathcal{AC}^1_D$ is an upper solution of (1.1)–(1.3) if

$$\begin{cases} -\beta''(t) \ge f(t,\beta(t),\beta'(t)) & \text{a.e. on } t \in [0,1], \\ \Delta\beta(t_k) \ge I_{0k}(t_k,\beta(t_k),\beta'(t_k)) & \text{for } k \in \{1,\dots,m\}, \\ \Delta\beta'(t_k) < I_{1k}(t_k,\beta(t_k),\beta'(t_k)) & \text{for } k \in \{1,\dots,m\}, \\ L_0(\beta(0),\beta(1),\beta'(0),\beta) \le 0, \\ L_1(\beta(0),\beta(1),\beta'(1),\beta) \le 0. \end{cases} \tag{2.2}$$

The following lemma enables us to construct the operator representation of our problem. Its proof is obvious and can be left to readers.

Lemma 2.5. *Linear problem*

$$-u''(t) = h(t) \quad \textit{for a.e. } t \in [0,1],$$
$$\Delta u(t_k) = C_k,\ \Delta u'(t_k) = D_k \quad \textit{for } k \in \{1,\dots,m\},$$
$$u(0) = A, \quad u'(1) = B$$

has a unique solution for any h Lebesgue integrable on $[0,1]$, $A, B \in \mathbb{R}$, $C_k, D_k \in \mathbb{R}$ $(k \in \{1,\dots,m\})$. This solution is given by

$$u(t) = A + B\,t + \int_0^1 G(t,s)\,h(s)\,ds + \sum_{k=1}^m C_k\,\chi_{(t_k,1]}(t) + \sum_{k=1}^m D_k\,(t-t_k)\,\chi_{(t_k,1]}(t) - t\sum_{k=1}^m D_k\,,$$

where

$$G(t,s) = \begin{cases} s & \textit{if } 0 \le s \le t \le 1, \\ t & \textit{if } 0 \le t \le s \le 1, \end{cases}$$

is the Green function associated to the homogeneous problem

$$-u''(t) = 0, \quad u(0) = 0,\ u'(1) = 0.$$

Remark 2.6. In what follows, the following evident estimate

$$\max\left\{ \sup_{t,s\in[0,1]} \left|G(t,s)\right|,\ \sup_{t,s\in[0,1]} \left|\frac{\partial G}{\partial t}(t,s)\right| \right\} = 1 \tag{2.3}$$

will be useful.

Our main existence tool will be the Schauder fixed point theorem (see e.g. [31, Theorem 2.A]).

Theorem 2.7 (Schauder)**.** *Let B be a nonempty, closed, bounded and convex subset of a Banach space X and let $T : B \to X$ be a compact operator mapping B into B. Then T has at least one fixed point in B.*

3 Main result

First, we will construct a proper auxiliary problem and its operator representation. To this aim, the existence of associated lower and upper solutions will be needed. Thus, we will make use of the following assumption.

(C) Problem (1.1)–(1.3) possesses a pair α, β of a lower and an upper solutions such that

$$\alpha(t) \leq \beta(t) \text{ and } \alpha'(t) \leq \beta'(t) \text{ for } t \in [0,1]. \tag{3.1}$$

Then, for $t \in [0,1]$ and $x, y, w \in \mathbb{R}$, define

$$\delta_0(t,w) = \begin{cases} \alpha(t) & \text{if } w < \alpha(t), \\ w & \text{if } w \in [\alpha(t), \beta(t)], \\ \beta(t) & \text{if } w > \beta(t), \end{cases} \tag{3.2}$$

$$\delta_1(t,w) = \begin{cases} \alpha'(t) & \text{if } w < \alpha'(t), \\ w & \text{if } w \in [\alpha'(t), \beta'(t)], \\ \beta'(t) & \text{if } w > \beta'(t), \end{cases} \tag{3.3}$$

and

$$\widetilde{f}(t,x,y) = f(t, \delta_0(t,x), \delta_1(t,y)) + \frac{\delta_1(t,y) - y}{1 + |y - \delta_1(t,y)|}, \tag{3.4}$$

and consider the following auxiliary problem

$$\begin{cases} -u''(t) = \widetilde{f}(t, u(t), u'(t)) \quad \text{for } t \in [0,1] \setminus D, \\ \Delta u(t_k) = I_{0k}\big(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k))\big), \\ \Delta u'(t_k) = I_{1k}\big(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k))\big), \\ u(0) = \delta_0\big(0, u(0) + L_0(u(0), u(1), u'(0), u)\big), \\ u'(1) = \delta_1\big(1, u'(1) + L_1(u(0), u(1), u'(1), u)\big). \end{cases} \tag{3.5}$$

Finally, we define

$$\begin{cases} (Tu)(t) = \delta_0\big(0, u(0) + L_0(u(0), u(1), u'(0), u)\big) \\ \quad +\delta_1\big(1, u'(1) + L_1(u(0), u(1), u'(1), u)\big)\, t \\ \quad + \sum_{k=1}^{m} \big[I_{0k}(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k)))\big]\chi_{(t_k,1]}(t) \\ \quad + \sum_{k=1}^{m} \big[I_{1k}(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k)))\,(t - t_k)\big]\chi_{(t_k,1]}(t) \\ \quad -t \sum_{k=1}^{m} I_{1k}(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k))) \\ \quad + \int_0^1 G(t,s)\, \widetilde{f}(s, u(s), u'(s))\, \mathrm{d}s \quad \text{for } u \in \mathcal{PC}^1_D \text{ and } t \in [0,1]. \end{cases} \tag{3.6}$$

The relationship between the operator T and the auxiliary problem (3.5) is described by the following assertion.

Proposition 3.1. *A function $u \in \mathcal{PC}^1_D$ is a solution to* (3.5) *if and only if it is a fixed point of the operator T given by* (3.6)*.*

Proof. From the construction of the operator T it is clear that any fixed point of T has piecewise absolutely continuous derivative, more precisely it belongs to the set $\mathcal{AC}^1_D$. Moreover, having in mind Lemma 2.5 we easily verify that u solves problem (3.5) if and only if it is a fixed point of T. □

Remark 3.2. If for a given $\rho > 0$ the function μ_ρ has a meaning from (A), then having in mind definitions (3.2)–(3.4), we can see that the following estimate of $\widetilde{f}$ is true:

$$|\widetilde{f}(t,x,y)| \le \mu_{r_0}(t) + 1 \quad \text{for a.e. } t \in [0,1] \text{ and all } x, y \in \mathbb{R},$$

where

$$r_0 = \max\big\{\|\alpha\|_\infty, \|\beta\|_\infty, \|\alpha'\|_\infty, \|\beta'\|_\infty\big\}. \tag{3.7}$$

As a result, we may put

$$\mu_\rho(t) = \mu_{r_0}(t) \quad \text{for all } \rho \ge r_0 \text{ and a.e. } t \in [0,1]. \tag{3.8}$$

Next, we will find conditions ensuring the solvability of problem (3.5).

Proposition 3.3. *Let assumptions* (A)–(C) *hold. Then problem* (3.5) *has at least one solution $\bar{u} \in \mathcal{PC}^1_D$.*

Proof. We will prove that the operator T satisfies the assumptions of the Schauder fixed point theorem (Theorem 2.7).

For better transparency, this proof is divided into several steps.

Step 1. *We will show that the operator T maps $\mathcal{PC}^1_D$ into $\mathcal{PC}^1_D$.*

Clearly, $Tu \in \mathcal{PC}_D$ for every $u \in \mathcal{PC}_D^1$. Furthermore, differentiating the relation (3.6), we get

$$\begin{cases} (Tu)'(t) = \delta_1\big(1, u'(1) + L_1(u(0), u(1), u'(1), u)\big) \\ \quad + \sum_{k=1}^{m} \big[I_{1k}(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k)))\big]\chi_{(t_k,1]}(t) \\ \quad - \sum_{k=1}^{m} I_{1k}(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k))) + \int_0^1 \frac{\partial G}{\partial t}(t,s)\, \widetilde{f}(s, u(s), u'(s))\, ds \end{cases} \tag{3.9}$$

for $u \in \mathcal{PC}_D^1$ and $t \in [0,1] \setminus D$, wherefrom, taking into account the properties of the Green function G, we deduce immediately that $Tu \in \mathcal{PC}_D^1$ for each $u \in \mathcal{PC}_D^1$.

Step 2. *Let $t \in [0,1]$ and a bounded subset B of $\mathcal{PC}_D^1$ be given. We will show that the set $(TB)(t) = \big\{(Tu)(t) : u \in B\big\}$ is then bounded subset of $\mathbb{R}$.*

Choose an arbitrary $t \in [0,1]$ and let $\|u\|_{\mathcal{PC}^1} = \|u\|_\infty + \|u'\|_\infty \le \rho < \infty$ for every $u \in B$. Our aim is to find a uniform estimate for elements of $(TB)(t)$.

First, by (3.2) and (3.3) we have

$$\big|\delta_0\big(0, u(0) + L_0(u(0), u(1), u'(0), u)\big)\big| \le \max\big\{|\alpha(0)|, |\beta(0)|\big\}$$

and

$$\big|\delta_1\big(0, u'(1) + L_1(u(0), u(1), u'(1), u)\big)\, t\big| \le \max\big\{|\alpha'(1)|, |\beta'(1)|\big\}.$$

Further, due to continuity of I_{0k} and I_{1k}, for an arbitrary $k \in \{1.\dots,m\}$ we get

$$\big|I_{0k}(t_k, \delta_0(t_k, u(t_k), u'(t_k)))\, \chi_{(t_k,1]}(t)\big| \le M_{0k} := \max_{(x,y)\in Q_k} |I_{0k}(t_k, x, y)| < \infty$$

and

$$\big|I_{1k}(t_k, \delta_0(t_k, u(t_k), u'(t_k)))\, (t - t_k)\, \chi_{(t_k,1]}(t)\big| \le M_{1k} := \max_{(x,y)\in Q_k} |I_{1k}(t_k, x, y)| < \infty,$$

where $Q_k = [\alpha(t_k), \beta(t_k)] \times [\alpha'(t_k), \beta'(t_k)]$.

Finally, by (2.3) we have $|G(t,s)| \le 1$ for $t, s \in [0,1]$ and consequently by the third point of (A) and by the definition (3.4) of $\widetilde{f}$ we have

$$\left| \int_0^1 G(t,s)\, \widetilde{f}(s, u(s), u'(s))\, ds \right| \le \int_0^1 (\mu_\rho(s) + 1)\, ds.$$

To summarize, the relation

$$\begin{cases} |(Tu)(t)| \le \max\big\{|\alpha(0)|, |\beta(0)|\big\} + \max\big\{|\alpha'(1)|, |\beta'(1)|\big\} \\ \qquad\qquad + M_0 + 2\, M_1 + \int_0^1 (\mu_\rho(s) + 1)\, ds < \infty, \end{cases}$$

where

$$M_0 = \sum_{k=1}^{m} M_{0k} \quad \text{and} \quad M_1 = \sum_{k=1}^{m} M_{1k},$$

holds for any $u \in B$. This proves our claim.

Step 3. *Let B be a bounded subset of $\mathcal{PC}_D^1$. We will show that the set $\big\{(Tu)' : u \in B\big\}$ is equi-regulated.*

Let $B \subset \mathcal{PC}^1_D$ be bounded and let $\rho > 0$ be such that $B \subset B_\rho = \{u \in \mathcal{PC}^1_D : \|u\|_{\mathcal{PC}^1} \le \rho\}$. Further, let $\varepsilon > 0$ be given and let $[s,t] \subset (t_{\ell-1}, t_\ell)$ for some $\ell \in \{1,\dots,m+1\}$. Then

$$\begin{aligned}
&\sum_{k=1}^{m} \big[I_{1k}(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k)))\big] \chi_{(t_k,1]}(t) \\
&\qquad = \sum_{k=\ell-1}^{m} \big[I_{1k}(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k)))\big] \chi_{(t_k,1]}(t) \\
&\qquad = \sum_{k=\ell-1}^{m} \big[I_{1k}(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k)))\big] \chi_{(t_k,1]}(s) \\
&\qquad = \sum_{k=1}^{m} \big[I_{1k}(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k)))\big] \chi_{(t_k,1]}(s)
\end{aligned}$$

and

$$(Tu)'(t) - (Tu)'(s) = \int_0^1 \left(\frac{\partial G}{\partial t}(t,\tau) - \frac{\partial G}{\partial t}(s,\tau)\right) f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau$$

for all $u \in B$. Further, since

$$\frac{\partial G}{\partial t}(t,\tau) - \frac{\partial G}{\partial t}(s,\tau) = \begin{cases} 0, & \text{if } 0 \le \tau < s < 1 \text{ or } 0 < t < \tau \le 1, \\ 1, & \text{if } 0 < s < \tau < t < 1, \end{cases}$$

it follows that

$$\begin{aligned}
\big|(Tu)'(t) - (Tu)'(s)\big| &\le \int_s^t \left|\frac{\partial G}{\partial t}(t,\tau) - \frac{\partial G}{\partial t}(s,\tau)\right| (\mu_\rho(\tau) + 1)\,\mathrm{d}\tau \\
&\le \int_s^t (\mu_\rho(\tau) + 1)\,\mathrm{d}\tau \quad \text{for all } u \in B
\end{aligned}$$

and hence

$$\big|(Tu)'(t) - (Tu)'(s)\big| < \varepsilon \quad \text{for all } u \in B \text{ whenever} \quad \int_s^t (\mu_\rho(\tau) + 1)\,\mathrm{d}\tau < \varepsilon.$$

Therefore, any refinement $\{\alpha_0, \dots, \alpha_n\}$ of $\{t_0, \dots, t_{m+1}\}$ which is such that

$$\int_{\alpha_{j-1}}^{\alpha_j} (\mu_\rho(\tau) + 1)\,\mathrm{d}\tau < \varepsilon \quad \text{for all } j \in \{1, \dots, n\}$$

satisfies the requirements from the definition of equi-regulatedness contained in Theorem 2.1. Consequently, the set $\{(Tu)' : u \in B\}$ is equi-regulated and this completes the proof of our claim.

Step 4. *We will construct a nonempty, closed, bounded and convex subset B of $\mathcal{PC}^1_D$ such that $T\,B \subset B$.*

Let $B \subset \mathcal{PC}^1_D$ be bounded and let $\rho > 0$ be such that $B \subset B_\rho = \{u \in \mathcal{PC}^1_D : \|u\|_{\mathcal{PC}^1} \le \rho\}$. Recall that by Step 2 we have

$$\begin{aligned}
\|Tu\|_\infty \le{}& \max\{|\alpha(0)|, |\beta(0)|\} + \max\{|\alpha'(1)|, |\beta'(1)|\} \\
&+ M_0 + 2\,M_1 + \int_0^1 (\mu_\rho(s) + 1)\,\mathrm{d}s \quad \text{for all } u \in B_\rho.
\end{aligned}$$

Similarly, from (3.9) we deduce that the inequality

$$|(Tu)'(t)| \leq \max\{|\alpha'(1)|, |\beta'(1)|\} + 2\,M_1 + \int_0^1 (\mu_\rho(s)+1)\,\mathrm{d}s$$

holds for any $u \in B$ and any $t \in [0,1]$, i.e.

$$\|(Tu)'\|_\infty \leq \max\{|\alpha'(1)|, |\beta'(1)|\} + 2\,M_1 + \int_0^1 (\mu_\rho(s)+1)\,\mathrm{d}s$$

for all $u \in B_\rho$. Hence, with respect to (3.8), we conclude that

$$\|Tu\|_{\mathcal{PC}^1} = \|Tu\|_\infty + \|(Tu)'\|_\infty \leq \varkappa(r_0) \quad \text{for all } u \in B_\rho \text{ and } \rho \geq r_0$$

where

$$\left\{\begin{aligned} \varkappa(\rho) &:= \max\{|\alpha(0)|, |\beta(0)|\} + 2\max\{|\alpha'(1)|, |\beta'(1)|\} \\ &\qquad\qquad + M_0 + 4\,M_1 + 2\int_0^1 (\mu_\rho(s)+1)\,\mathrm{d}s \quad \text{for } \rho > 0. \end{aligned}\right. \tag{3.10}$$

Now, if we put $R = \max\{r_0, \varkappa(r_0)\}$ and $B = B_R$, then the inequality $\|Tu\|_{\mathcal{PC}^1} \leq R$ will be true for all $u \in B$. This proves our claim.

To summarize, by Steps 1–3 and Corollary 2.2, the operator T is compact in $\mathcal{PC}^1_D$ and, by Step 4, it maps the nonempty, closed, bounded and convex set $B = B_R$ into itself. By Theorem 2.7 it follows that T has a fixed point $\bar{u} \in B$ which is a solution of (3.5) according to Proposition 3.1. □

Now we can formulate our main result. It provides sufficient conditions for the existence of at least one solution of problem (1.1)–(1.3), as well as its localization.

Theorem 3.4. *Let the assumptions of Proposition 3.3 be satisfied. Furthermore, suppose:*

$$\left\{\begin{aligned} & f(t,\alpha(t),\alpha'(t))) \leq f(t,x,\alpha'(t)) \\ & \quad \textit{for a.e. } t \in [0,1] \textit{ and } x \in [\alpha(t),\beta(t)], \\ & f(t,x,\beta'(t)) \leq f(t,\beta(t),\beta'(t))) \\ & \quad \textit{for a.e. } t \in [0,1] \textit{ and } x \in [\alpha(t),\beta(t)], \end{aligned}\right. \tag{3.11}$$

$$\left\{\begin{aligned} & I_{0k}(t_k,\alpha(t_k),\alpha'(t_k)) \leq I_{0k}(t_k,x,y) \leq I_{0k}(t_k,\beta(t_k),\beta'(t_k)) \\ & \quad \textit{for } (x,y) \in [\alpha(t_k),\beta(t_k)] \times [\alpha'(t_k),\beta'(t_k)] \textit{ and } k \in \{1,\ldots,m\}, \end{aligned}\right. \tag{3.12}$$

$$\left\{\begin{aligned} & I_{1k}(t_k,x,\alpha'(t_k)) \leq I_{1k}(t_k,\alpha(t_k),\alpha'(t_k)) \\ & \quad \textit{for } x \geq \alpha(t_k) \textit{ and } k \in \{1,\ldots,m\}, \\ & I_{1k}(t_k,x,\beta'(t_k)) \geq I_{1k}(t_k,\beta(t_k),\beta'(t_k)) \\ & \quad \textit{for } x \leq \beta(t_k) \textit{ and } k \in \{1,\ldots,m\}, \end{aligned}\right. \tag{3.13}$$

$$\left\{\begin{aligned} & L_0(\beta(0),y,z,u) \leq L_0(\beta(0),\beta(1),\beta'(1),\beta) \\ & \quad \textit{for } y \leq \beta(1),\ z \leq \beta'(1) \textit{ and } u \in \mathcal{PC}_D \textit{ such that } u \leq \beta \textit{ on } [0,1], \\ & L_0(\alpha(0),y,z,u) \geq L_0(\alpha(0),\alpha(1),\alpha'(1),\alpha) \\ & \quad \textit{for } y \geq \alpha(1),\ z \geq \alpha'(1) \textit{ and } u \in \mathcal{PC}_D \textit{ such that } u \geq \alpha \textit{ on } [0,1] \end{aligned}\right. \tag{3.14}$$

and

$$\begin{cases} L_1(x,y,\beta'(1),u) \le L_1(\beta(0),\beta(1),\beta'(1),\beta) \\ \quad \text{for } x \le \beta(0),\ y \le \beta(1) \text{ and } u \in \mathcal{PC}_D \text{ such that } u \le \beta \text{ on } [0,1], \\ L_1(x,y,\alpha'(1),u) \ge L_0(\alpha(0),\alpha(1),\alpha'(1),\alpha) \\ \quad \text{for } x \ge \alpha(0),\ y \ge \alpha(1) \text{ and } u \in \mathcal{PC}_D \text{ such that } u \ge \alpha \text{ on } [0,1]. \end{cases} \tag{3.15}$$

Then problem (1.1)–(1.3) *has at least one solution u such that*

$$\alpha(t) \le u(t) \le \beta(t) \quad \text{and} \quad \alpha'(t) \le u'(t) \le \beta'(t) \quad \text{for } t \in [0,1].$$

Proof. By Proposition 3.3 the auxiliary problem (3.5) has a solution $\bar{u}$ such that $\|\bar{u}\|_{\mathcal{PC}^1} \le R$, where $R = \max\{r_0, \varkappa(r_0)\} > 0$ is given by (3.7) and (3.10). Thus, it remains to show that $\bar{u}$ satisfies the following set of inequalities

$$\alpha(t) \le \bar{u}(t) \le \beta(t) \quad \text{for } t \in [0,1], \tag{3.16}$$

$$\alpha'(t) \le \bar{u}'(t) \le \beta'(t) \quad \text{for } t \in [0,1] \tag{3.17}$$

and

$$\alpha(0) \le \bar{u}(0) + L_0(\bar{u}(0),\bar{u}(1),\bar{u}'(0),\bar{u}) \le \beta(0), \tag{3.18}$$

$$\alpha'(1) \le \bar{u}'(1) + L_1(\bar{u}(0),\bar{u}(1),\bar{u}'(1),\bar{u}) \le \beta'(1). \tag{3.19}$$

Indeed, in such a case, in view of (3.2) and (3.3), the relations

$$\begin{aligned} \widetilde{f}(t,\bar{u}(t),\bar{u}'(t)) &= f(t,\bar{u}(t),\bar{u}'(t)), \\ \delta_0\big(0,\bar{u}(0) + L_0(\bar{u}(0),\bar{u}(1),\bar{u}'(0),\bar{u})\big) &= \bar{u}(0) + L_0(\bar{u}(0),\bar{u}(1),\bar{u}'(0),\bar{u}), \\ \delta_1\big(1,\bar{u}'(1) + L_1(\bar{u}(0),\bar{u}(1),\bar{u}'(1),\bar{u})\big) &= \bar{u}'(1) + L_1(\bar{u}(0),\bar{u}(1),\bar{u}'(1),\bar{u}), \\ I_{0k}\big(t_k,\delta_0(t_k,\bar{u}(t_k)),\delta_1(t_k,\bar{u}'(t_k))\big) &= I_{0k}\big(t_k,\bar{u}(t_k)),\bar{u}'(t_k))\big) \end{aligned}$$

and

$$I_{1k}\big(t_k,\delta_0(t_k,\bar{u}(t_k)),\delta_1(t_k,\bar{u}'(t_k))\big) = I_{1k}\big(t_k,\bar{u}(t_k)),\bar{u}'(t_k))\big)$$

are true for all $t \in [0,1]$ and $k \in \{1,\dots,m\}$. Therefore, it follows immediately that then $\bar{u}$ is the desired solution of the given problem (1.1)–(1.3).

• **ad** (3.17): Suppose that there is a $\bar{t} \in [0,1]$ such that

$$\bar{u}'(\bar{t}) - \beta'(\bar{t}) = \max_{t\in[0,1]} \big(\bar{u}'(t) - \beta'(t)\big) > 0. \tag{3.20}$$

As, by the definition (3.3) of δ_1 and by the last relation in (3.5) we have

$$\bar{u}'(1) = \delta_1(1,\bar{u}'(1) + L_1(\bar{u}(0),\bar{u}(1),\bar{u}'(1),\bar{u})) \le \beta'(1),$$

it follows that $\bar{t} < 1$.

Assume that $\bar{t} \in [0,1)\backslash D$. Then either $\bar{t} \in [0,t_1)$ or $\bar{t} \in (t_{k-1},t_k)$ for some $k \in \{2,\dots,m+1\}$. In both cases there is a $\Delta > 0$ such that $\bar{t} + \Delta < t_k$ and $\bar{u}'(s) - \beta'(s) > 0$ for all $s \in [\bar{t},\bar{t}+\Delta]$.

In particular, $\delta_1(s,\bar{u}(s)) = \beta'(s)$ for $s \in [\bar{t}, \bar{t}+\Delta]$. Now, using (3.11) and the first inequality in (2.2), we will deduce for $t \in [\bar{t}, \bar{t}+\Delta]$

$$\begin{aligned}
0 &\geq \big(\bar{u}'(t) - \beta'(t)\big) - \big(\bar{u}'(\bar{t}) - \beta'(\bar{t})\big) = \int_{\bar{t}}^{t} \big(\bar{u}''(s) - \beta''(s)\big)\,\mathrm{d}s \\
&= \int_{\bar{t}}^{t} \Big(-f(s,\delta_0(s,\bar{u}(s)),\delta_1(s,\bar{u}'(s))) - \frac{\delta_1(s,\bar{u}'(s)) - \bar{u}'(s)}{|\bar{u}'(s) - \delta_1(s,\bar{u}'(s))| + 1} - \beta''(s)\Big)\,\mathrm{d}s \\
&= \int_{\bar{t}}^{t} \Big(- f(s,\delta_0(s,\bar{u}(s)),\beta'(s))) + \frac{\bar{u}'(s) - \beta'(s)}{\bar{u}'(s) - \beta'(s) + 1} - \beta''(s)\Big)\,\mathrm{d}s \\
&> \int_{\bar{t}}^{t} \Big(- f(s,\delta_0(s,\bar{u}(s)),\beta'(s))) - \beta''(s)\Big)\,\mathrm{d}s \\
&\geq \int_{\bar{t}}^{t} \Big(- f(s,\beta(s),\beta'(s)) - \beta''(s)\Big)\,\mathrm{d}s \geq 0,
\end{aligned}$$

a contradiction, of course.

It remains to consider the possibility that there exists $k \in \{1,2,...,m\}$ such that either (3.20) with $\bar{t} = t_k$ or

$$\bar{u}'(t_k+) - \beta'(t_k+) = \sup_{t\in[0,1]} \big(\bar{u}'(t) - \beta'(t)\big) > 0$$

holds. The latter case leads to a contradiction by arguments analogous to those used above. So, let (3.20) with $\bar{t} = t_k$ for some $k \in \{1,2,...,m\}$ be the case. In particular, we have

$$\bar{u}'(t_k+) - \beta'(t_k+) \leq \bar{u}'(t_k) - \beta'(t_k), \;\; \delta_1(t_k,\bar{u}'(t_k)) = \beta'(t_k)$$

and $\Delta\bar{u}'(t_k) \leq \Delta\beta'(t_k)$, i.e.

$$\begin{aligned}
0 &\geq \Delta\bar{u}'(t_k) - \Delta\beta'(t_k) = I_{1k}(t_k,\delta_0(t_k,\bar{u}(t_k)),\delta_1(t_k,\bar{u}'(t_k))) - \Delta\beta'(t_k) \\
&= I_{1k}(t_k,\delta_0(t_k,\bar{u}(t_k)),\beta'(t_k)) - \Delta\beta'(t_k).
\end{aligned}$$

Thanks to (3.13), Definition 2.4 (cf. the third line in (2.2)) and the third line in (3.5) this leads to a contradiction

$$\begin{aligned}
0 &\geq I_{1k}(t_k,\delta_0(t_k,\bar{u}(t_k)),\delta_1(t_k,\bar{u}'(t_k))) - \Delta\beta'(t_k) \\
&\geq I_{1k}(t_k,\delta_0(t_k,\bar{u}(t_k)),\beta'(t_k)) - \Delta\beta'(t_k). \geq I_{1k}(t_k,\beta(t_k),\beta'(t_k)) - \Delta\beta'(t_k) > 0.
\end{aligned}$$

This means that $\bar{u}'(t) \leq \beta'(t)$ holds for $t \in [0,1]$.

Similarly, we can prove that also $\alpha'(t) \leq \bar{u}'(t)$ holds for $t \in [0,1]$. This completes the proof of (3.17).

• **ad** (3.16): Integrating the inequality $\alpha'(t) \leq \bar{u}'(t)$ over $[0,t]$ for $t \in (0,t_1]$, we get

$$\alpha(t) - \alpha(0) \leq \bar{u}(t) - \bar{u}(0) \quad \text{for } t \in [0,t_1]. \tag{3.21}$$

Further, as $\bar{u}(0) = \delta_0(0,\bar{u}(0) + L_0(\bar{u}(0),\bar{u}(1),\bar{u}'(0),\bar{u})) \geq \alpha(0)$, it follows that

$$\alpha(t) \leq \bar{u}(t) + \alpha(0) - \bar{u}(0) \leq \bar{u}(t) \quad \text{for} \;\; t \in [0,t_1].$$

Now, let $k \in \{1,\ldots,m\}$ be such that

$$\alpha(t) \leq \bar{u}(t) \quad \text{for} \;\; t \in [0,t_k]. \tag{3.22}$$

Analogously to (3.21) we derive

$$\alpha(t) - \alpha(t_k+) \le \bar{u}(t) - \bar{u}(t_k+) \quad \text{for } t \in (t_k, t_{k+1}]$$

and, with respect to the second line in (3.5), we get

$$\begin{aligned}\alpha(t) &\le \bar{u}(t) + \alpha(t_k+) - \bar{u}(t_k+)\\ &= \bar{u}(t) + \alpha(t_k+) - I_{0k}(t_k, \delta_0(t_k, \bar{u}(t_k)), \delta_1(t_k, \bar{u}'(t_k))) - \bar{u}(t_k)\end{aligned}$$

for $t \in (t_k, t_{k+1}]$. Furthermore, having in mind that

$$\alpha(t_k) \le \delta_0(t_k, \bar{u}(t_k)) \le \beta(t_k) \quad \text{and} \quad \alpha'(t_k) \le \delta_1(t_k, \bar{u}'(t_k)) \le \beta'(t_k)$$

due to (3.2) and (3.3), we can use (3.12), the second line in (2.1) and hypothesis (3.22) to deduce that

$$\begin{aligned}\alpha(t) &= \bar{u}(t) + \alpha(t_k+) - I_{0k}(t_k, \delta_0(t_k, \bar{u}(t_k)), \delta_1(t_k, \bar{u}'(t_k))) - \bar{u}(t_k)\\ &\le \bar{u}(t) + \alpha(t_k+) - I_{0k}(t_k, \alpha(t_k), \alpha'(t_k)) - \bar{u}(t_k)\\ &\le \bar{u}(t) + \alpha(t_k) - \bar{u}(t_k) \le \bar{u}(t) \quad \text{for } \ t \in (t_k, t_{k+1}].\end{aligned}$$

By induction principle, we can conclude that $\alpha(t) \le \bar{u}(t)$ holds on the whole interval $[0,1]$. Similarly we can prove that $\bar{u}(t) \le \beta(t)$ on $[0,1]$. This completes the proof of (3.16).

• **ad** (3.18): Suppose that

$$\bar{u}(0) + L_0(\bar{u}(0), \bar{u}(1), \bar{u}'(0), \bar{u}) > \beta(0).$$

Then by (3.5) and (3.2)

$$\bar{u}(0) = \delta_0(0, \bar{u}(0) + L_0(\bar{u}(0), \bar{u}(1), \bar{u}'(0), \bar{u}) = \beta(0)$$

and using the monotonicity type condition (3.14) we obtain

$$\begin{aligned}0 &< \bar{u}(0) + L_0(\bar{u}(0), \bar{u}(1), \bar{u}'(0), \bar{u}) - \beta(0)\\ &= L_0(\beta(0), \bar{u}(1), \bar{u}'(0), \bar{u}) \le L_0(\beta(0), \beta(1), \beta'(0), \beta) \le 0,\end{aligned}$$

a contradiction. Hence, it must be

$$\bar{u}(0) + L_0(\bar{u}(0), \bar{u}(1), \bar{u}'(0), \bar{u}) \le \beta(0).$$

Similarly, we would show that

$$\alpha(0) \le \bar{u}(0) + L_0(\bar{u}(0), \bar{u}(1), \bar{u}'(0), \bar{u}),$$

is true, as well. Thus, the relations (3.18) are true.

• **ad** (3.19): Suppose that

$$\bar{u}'(1) + L_1(\bar{u}(0), \bar{u}(1), \bar{u}'(1), \bar{u}) > \beta'(1).$$

Then by (3.5) and (3.3)

$$\bar{u}'(1) = \delta_1(1, \bar{u}'(1) + L_1(\bar{u}(0), \bar{u}(1), \bar{u}'(1), \bar{u}) = \beta'(1).$$

Furthermore, the monotonicity type condition (3.15) together with (2.2) yield the following contradiction:

$$\begin{aligned}0 &< \bar{u}'(1) + L_1(\bar{u}(0), \bar{u}(1), \bar{u}'(1), \bar{u}) - \beta'(1)\\ &= L_1(\bar{u}(0), \bar{u}(1), \beta'(1), \bar{u}) \leq L_1(\beta(0), \beta(1), \beta'(1), \beta) \leq 0.\end{aligned}$$

Consequently, it has to be

$$\bar{u}'(1) + L_1(\bar{u}(0), \bar{u}(1), \bar{u}'(1), \bar{u}) \leq \beta'(1).$$

Similarly it can be shown that

$$\alpha'(1) \leq \bar{u}'(1) + L_1(\bar{u}(0), \bar{u}(1), \bar{u}'(1), \bar{u}).$$

This completes the proof of (3.19).

To summarize, all the relations (3.16)–(3.19) are true and hence the fixed point $\bar{u}$ of T is a solution of the given problem (1.1)–(1.3). □

4 Example

To illustrate the range of applications of our main result, let us consider problem (1.1)–(1.3), with $m = 1$, $D = \{t_1\} = \{\frac{1}{2}\}$,

$$f(t,x,y) = \begin{cases} 0.001\,[(t-2)\,y^3 + x] & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 0.001\,[(t-6)\,y^3 + x] & \text{if } \frac{1}{2} < t \leq 1, \end{cases}$$

$$I_{01}(\tfrac{1}{2}, x, y) = 0.1\left[\tfrac{3}{2} + \tfrac{1}{3}\,x + y^3\right], \quad I_{11}(\tfrac{1}{2}, x, y) = 0.1\left[\tfrac{1}{2} - \tfrac{1}{4}\,x + y^3\right],$$

$$L_0(x,y,z,u) = -\,x + \tfrac{1}{6}\Big(z + \sup_{t\in[0,1]} u(t)\Big), \; L_0(x,y,z,u) = -\,2y - z + \int_0^1 u(t)\,\mathrm{d}t.$$

It is easy to verify that (A), (B) are satisfied. Furthermore, the functions

$$\alpha(t) = -(t+1) \;\text{ for } t \in [0,1] \quad \text{and} \quad \beta(t) = \begin{cases} t+1 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ t+4 & \text{if } \frac{1}{2} < t \leq 1 \end{cases}$$

are lower and upper solutions of the given problem and conditions (3.1), (3.11), (3.12), (3.13), (3.14) and (3.15) hold. Therefore, our Theorem 3.4 ensures the existence of its solution $u_* \in \mathcal{PC}^1_D$ such that

$$\alpha(t) \leq u_*(t) \leq \beta(t) \;\text{ and }\; -1 \leq u_*'(t) \leq 1.$$

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Preface

It is a true privilege and a great pleasure for me to present the special volume of the *Electronic Journal of Qualitative Theory of Differential Equations* dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday.

Jeff is an outstanding figure in the realm of nonlinear analysis and he is the author of 110 research papers and one textbook.

Jeff completed his doctoral studies in 1970 at the University of Sussex, under the supervision of David Edmunds. After a year at the Istituto per le Applicazioni del Calcolo in Rome he returned to Sussex for a two-year postdoc with David. In 1973 he started as lecturer at the University of Glasgow and was later promoted to reader and then to full professor. Jeff retired from the University of Glasgow in 2009.

Jeff spent some sabbatical leaves at other institutions: Indiana University (1978), Tulane University (1982), Rutgers University (1994) and University of Calabria (2003). He has also made shorter visits to several universities in Europe and North America.

While in Glasgow, Jeff supervised nine PhD students and one MSc student. Most of them continued their paths in academia and are now well respected researchers in their fields.

The main strands of Jeff's research are the following: contractive-type maps, degree and fixed point index theories, strongly nonlinear elliptic problems, A-proper maps, nonlocal boundary value problems for ODEs, maximum principles and Gronwall type inequalities.

Along the years, Jeff has done substantial work for the mathematical community: he has done, and continues to do, refereeing work for many journals, acted as a reviewer for *Zentralblatt für Mathematik* and *Mathematical Reviews*, and did editorial work for *Fixed Point Theory and Applications*, *Glasgow Mathematical Journal*, *Bulletin*, *Journal* and *Proceedings of the London Mathematical Society*, *Nonlinear Analysis TMA*, *Proceedings of the Royal Society of Edinburgh Sect. A* and the AIMS book series *Differential Equations & Dynamical Systems*. He is currently, jointly with Tibor Krisztin, Editor-in-Chief of the *Electronic Journal of Qualitative Theory of Differential Equations*. He has also organised many special sessions at conferences.

His work was recognised in 1984 when he was elected a Fellow of the Royal Society of Edinburgh. In 2014 he was named as one of Thomson Reuters Highly Cited Researchers and one of 2014 The World's Most Influential Scientific Minds. Jeff has been a keynote or main speaker in a number of international conferences.

In the present volume we publish 26 papers of very high quality by former students, friends and colleagues of Jeff. The contributions focus on various aspects (e.g. solvability, stability, entropy) of different classes of equations: ordinary, discrete, delay, fractional, impulsive, functional, partial differential equations, integral and evolution equations. All the manuscripts faced a rigorous refereeing process. I am confident that the readers of this special volume will enjoy it and will find inspiration for their future research. I wish to thank the authors for their quality contributions, the co-editors of this issue Alberto Cabada, Attila Dénes, Paul Eloe, Ábel Garab, John R. Graef, Tibor Krisztin and Patrizia Pucci for their devoted work, and I am grateful to all the referees for their careful work. Last but not least, I wish to thank Jeff for his contributions to mathematics and to the mathematical community.

Many happy returns, Professor Webb!

Gennaro Infante, Cosenza, 15th February 2021